Even perturbations of the self-similar Vaidya space-time

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We study even parity metric and matter perturbations of all angular modes in self-similar Vaidya space-time. We focus on the case where the background contains a naked singularity. Initial conditions are imposed, describing a finite perturbation emerging from the portion of flat space-time preceding the matter-filled region of space-time. The most general perturbation satisfying the initial conditions is allowed to impinge upon the Cauchy horizon (CH), where the perturbation remains finite: There is no "blue-sheet" instability. However, when the perturbation evolves through the CH and onto the second future similarity horizon of the naked singularity, divergence necessarily occurs: This surface is found to be unstable. The analysis is based on the study of individual modes following a Mellin transform of the perturbation. We present an argument that the full perturbation remains finite after resummation of the (possibly infinite number of) modes.

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I. INTRODUCTION AND SUMMARY

The cosmic censorship hypothesis (CCH) was proposed by Penrose and states that naked singularities will not evolve from generic initial data (see Wald [1] for detailed discussion). The motivation is obvious: Information leaving the singularity is impossible to predict and all physical laws in the region affected by the naked singularity will break down. While the CCH seems intuitively reasonable, there are some notable counterexamples, for instance, the Reissner-Nordström and Kerr timelike singularities, certain classes of self-similar perfect fluid [2] and dust [3] solutions, and the self-similar scalar field [4]. In general, self-similar spherically symmetric (SSSS) space-times are a rich source of counterexamples to the CCH.

These naked singularities occur in space-times with very high degrees of symmetry, and, thus, perhaps the naked singularities are due to these unphysical restraints. We seek to move away from this symmetry by perturbing the space-time and examining if the singularity remains naked; that is, we look for stability.

In the Reissner-Nordström space-time, Penrose [5] noticed that the Cauchy horizon (CH), that surface marking the boundary between regions where the singularity is and is not visible to observers, would be a surface of infinite blueshift. Building on this, Chandrasekhar and Hartle [6] found that metric perturbations originating from the exterior have a flux that observers crossing the CH will measure as infinite. Thus, the CH has a "blue-sheet" instability, which prevents observers from crossing the CH and observing the singularity. This result has been firmly established in further work (see [7,8]). This suggests the Cauchy horizon is perhaps unstable in other possible counterexamples to cosmic censorship. We will examine this mechanism of instability in SSSS space-times.

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In a previous paper [9], the authors have shown that all self-similar spherically symmetric space-times which admit a Cauchy horizon, and thus a naked singularity, are stable when we inject a scalar field; that is, the flux of the scalar field as measured by a timelike observer crossing the Cauchy horizon remains finite. This paper extends on that work by considering metric and matter perturbations of SSSS space-times. In related work, Miyamoto and Harada [10] showed that the CH in the general class of space-times mentioned above is unstable at the semiclassical level.

For the perturbations, we will use the formalism of Gerlach and Sengupta [11]. They describe gauge-invariant metric and matter perturbations of spherically symmetric space-times by a $2 + 2$ split and multipole decomposition. This formalism has been used for a variety of backgrounds, for example, Schwarzschild [12], timelike dust [13], and perfect fluid [14]. As we must specify the matter content, we have begun by considering the self-similar spherically symmetric null dust (i.e. self-similar Vaidya) space-time. This space-time is of interest because it provides a mathematically simple (self-similar and spherically symmetric) example of an exact solution of Einstein's equation (that is, the metric can be written explicitly in terms of elementary functions) with an energy-momentum tensor obeying the dominant energy condition that admits a naked singularity for an easily specified set of parameters. Waugh and Lake [15] showed the stability of the Cauchy horizon in Vaidya space-time at the level of the eikonal approximation.

In the following section, we define the background space-time and give the conditions for a naked singularity. We also introduce the Mellin transform and point out how the partial differential perturbation equations can be expressed as parametrized ordinary differential equations: The perturbation is then given by the inverse Mellin transform (resummation) of the solutions of the ordinary differential equations (ODEs). The analysis of Secs. III, IV, and V deals with these ODEs and their solutions (the "modes"). In Sec. III, we describe the gauge-invariant formalism of Gerlach and Sengupta. While metric perturbations are primarily useful in modeling gravitational radiation, which manifests in quadrupolar moments and above (see, for example, [16]), for completeness we will consider all multipole moments of the perturbation.

In Sec. IV, we begin specifying our initial data. For completeness we must take into account perturbations arising from the region of flat space preceding the matter-filled Vaidya space-time. We will denote the boundary between these two regions $\mathcal{N}$ and define $\mathcal{N}$ as $x = 0$, where $x < 0$ is flat space-time and $x > 0$ is Vaidya space-time; here x is a naturally occurring similarity variable. Our initial/boundary conditions are (i) finiteness of the perturbations on the axis $r = 0$ in the Minkowski region, (ii) finiteness on $\mathcal{N}$ as $x \uparrow 0$, (iii) finiteness on $\mathcal{N}$ as $x \downarrow 0$, and (iv) continuous matching across $x = 0$.

When we solve the perturbation equations for multipole mode number $l \geq 2$, we find a two-parameter family of solutions in the portion of Minkowski space-time. To satisfy initial condition (i), we must set one of these parameters to zero. This specifies the type of solutions which will solve initial condition (ii). The second initial condition restricts the acceptable range of the mode number s that arises from the Mellin transform of the perturbation equations; that is, we can consider only $s \geq 2$, $s \neq l + m$, where $m \in \mathbb{N}$. In Sec. V, we solve the perturbation equations in the Vaidya region, and we find a four-parameter family of solutions. However, to solve initial condition (iii), we must further restrict the range of s to $s > 2$, $s \notin \mathbb{Z}$. To satisfy initial condition (iv), one of the parameters is fixed and thus we have a three-parameter family of solutions.

This class of perturbations satisfying the initial conditions is then allowed to evolve onto the Cauchy horizon, and there it is found that the flux of the perturbation is finite. Thus, the Cauchy horizon is stable under metric and matter perturbations ($l \geq 2$) and does not display the blue-sheet instability shown by Reissner-Nordström space-time. Interestingly, when the perturbations are allowed to evolve through the Cauchy horizon and onto the second future similarity horizon (see below), here the flux of the perturbations will diverge, meaning that this horizon is *unstable* under metric and matter perturbations ($l \geq 2$).

For the dipole mode $l = 1$, we reproduce the familiar result that the perturbation in the Minkowski region is pure gauge; that is, we can always find a coordinate system in which the $l = 1$ perturbation vanishes. In the Vaidya region there is an $l = 1$ perturbation which cannot be gauged away, and this evolves through the Vaidya space-time without divergence.

The $l = 0$ mode is spherically symmetric, and we can use uniqueness results: In the Minkowski space-time the $l = 0$ perturbation generates a Schwarzschild space-time, which has to have its mass set to zero to satisfy initial condition (i). In the Vaidya region, after a spherically symmetric perturbation we again have Vaidya space-time, merely with a new mass function and null coordinate.

In Sec. VI, we consider the problem of resummation: Does finiteness of the individual modes for an allowed range of the parameter s imply finiteness of the full perturbation given by the inverse Mellin transform over s of the modes? We present what we consider a plausibility argument that the answer to this question is "yes": Acomplete rigorous proof is beyond the scope of the present paper. We conclude with a brief discussion in Sec. VII. We use the conventions of Wald [1] and set $c = G = 1$.

II. SELF-SIMILAR VAIDYA SPACE-TIME

The matter model we consider is the in-falling null dust or Vaidya model. Its metric is found from the Schwarzschild metric by replacing the constant mass term m with a function of the advanced (ingoing) Bondi coordinate v:

$$g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{m(v)}{r}\right)dv^2 + 2dvdr + r^2 d\Omega^2, \tag{2.1}$$

where $d\Omega^2$ is the line element of the unit 2-sphere. By setting m linear, that is, $m = \lambda v$, the space-time becomes self-similar; that is, there is a homothetic Killing vector field

$$\vec{\zeta} = v\frac{\partial}{\partial v} + r\frac{\partial}{\partial r}, \tag{2.2}$$

such that $\mathcal{L}_{\vec{\zeta}} g_{\mu\nu} = 2g_{\mu\nu}$. Thus, we can introduce the similarity coordinate $x = v/r$, and in (x, r) coordinates the self-similar Vaidya metric and matter tensors become

$$g_{\mu\nu}dx^\mu dx^\nu = r^2(-1 + \lambda x)dx^2 + 2r(1 - x + \lambda x^2)dxdr + x(2 - x + \lambda x^2)dr^2 + r^2 d\Omega^2, \tag{2.3}$$

$$t_{\mu\nu}dx^\mu dx^\nu = \rho \ell_\mu \ell_\nu dx^\mu dx^\nu = \frac{\lambda}{8\pi}dx^2 + \frac{\lambda x}{8\pi r}dxdr + \frac{\lambda x^2}{8\pi r^2}dr^2, \tag{2.4}$$

where $\rho = \dot{m}(v)/8\pi r^2 = \lambda/8\pi r^2$ is the energy density, $\ell_\mu = -\partial_\mu v$ is the ingoing null direction, and Minkowski space-time is recovered in the limit $\lambda \rightarrow 0$.

The matter field is switched on at $v = x = 0$ (which we will call the "threshold" and is denoted $\mathcal{N}$ in Fig. 1), the region $v < 0$ being flat. When the matter collapses to the origin of coordinates, a singularity forms. We can describe the global structure of the space-time by analyzing the causal nature of the similarity coordinate x. Null homothetic lines are zeros of g_{rr} given in (2.3), that is, $x(\lambda x^2 - x + 2)$; thus, $x = 0$ is the null homothetic line pointing radially inward to the singularity at the scaling origin (its past null cone, $\mathcal{N}$). If there are other positive real zeros of

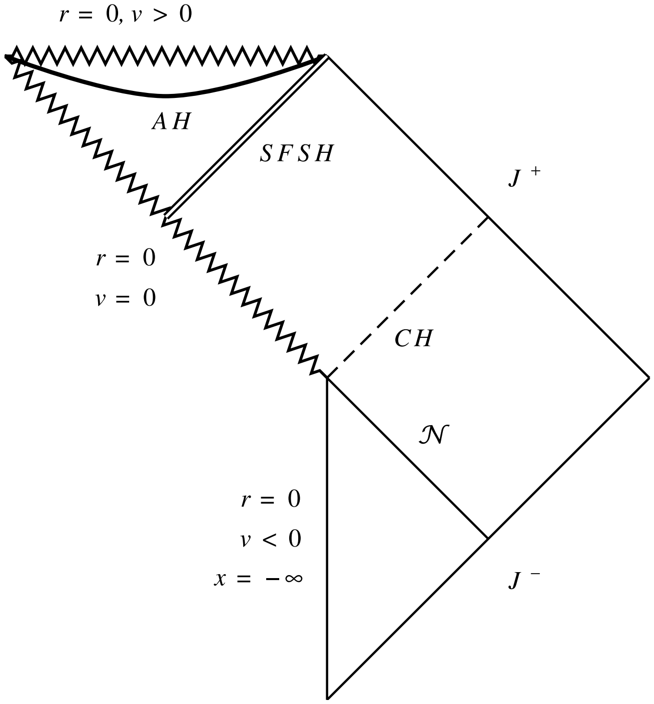

FIG. 1. Conformal diagram for Vaidya space-time admitting a globally naked singularity. The region preceding $\mathcal{N}$ is a portion of Minkowski space-time, and the region to the future of $\mathcal{N}$ is filled with null dust. There are three similarity horizons at which the similarity coordinate x is null: $x = 0$ denoted $\mathcal{N}$, $x = x_c$ shown as a dashed line, and $x = x_e$ shown as a double line. We identify $x = x_c$ as the Cauchy horizon and will call $x = x_e$ the SFSH. The apparent horizon given by $x = 1/\lambda$ is shown as a bold curve.

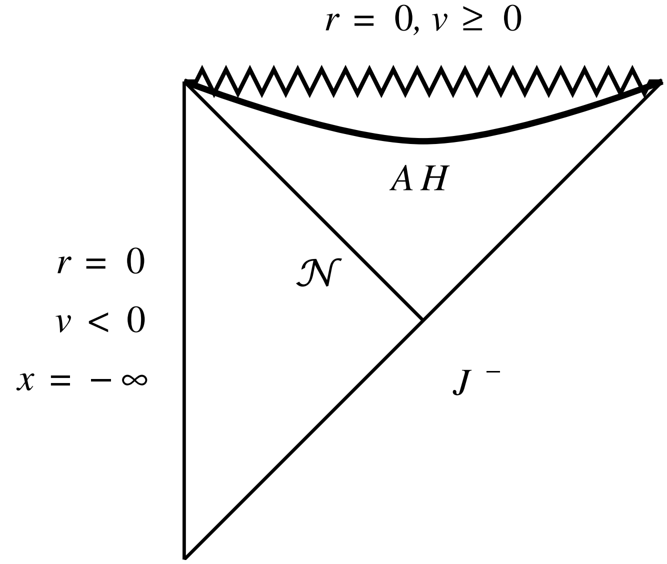

FIG. 2. Conformal diagram for Vaidya space-time with censored singularity, corresponding to the case $\lambda > 1/8$. A spacelike singularity forms at $r = 0$ for $v \geq 0$.

$\lambda x^2 - x + 2$, then these represent null homothetic lines emanating to future null infinity which meet the singularity in the past. The lowest of these zeros is thus the first null geodesic to leave the singularity and escape to future null infinity: the Cauchy horizon. It is given by

$$x = \frac{1}{2\lambda}(1 - \sqrt{1 - 8\lambda}) \equiv x_c \tag{2.5}$$

and exists for $0 < \lambda < 1/8$. Subsequent zeros are additional "similarity horizons": They mark the transition of x from timelike to spacelike or vice versa. For the self-similar Vaidya space-time, there is one more similarity horizon,

$$x = \frac{1}{2\lambda}(1 + \sqrt{1 - 8\lambda}) \equiv x_e. \tag{2.6}$$

For $0 < \lambda < 1/8$ these similarity horizons are distinct and the singularity is globally naked. In the notation of Carr and Gundlach [17], the causal structure is rTfSfTpSs and is shown in Fig. 1. For $\lambda = 1/8$ these horizons coincide and the singularity is instantaneously (marginally) naked; we will not consider this case in this paper. For $\lambda > 1/8$ a black hole forms; see Fig. 2.

The second similarity horizon is, in the purely self-similar Vaidya case, the last null geodesic to leave the singularity and escape to future null infinity and, thus, can be called the event horizon. However, to have an asymptotically flat model, we can match across $v = v_+ > 0$ with the exterior Schwarzschild space-time; in this case $x = x_e$ would not be the event horizon. Thus, we will call $x = x_e$ the second future similarity horizon (SFSH).

Our use of the coordinates (x, r) is motivated by the fact that they are global, important surfaces are easily defined in terms of the x coordinate, and the entire space-time (Minkowski and Vaidya) is covered by $r \in [0, \infty)$. Then using the Mellin transform the globally defined partial differential perturbation equations reduce to ordinary differential equations; this occurs precisely because our space-time is self-similar. The Mellin transform is defined by

$$G(x; s) = \mathcal{M}[g(x, r)](r \to s) := \int_0^\infty g(x, r) r^{s-1} dr, \tag{2.7}$$

with $s \in \mathbb{C}$. In coordinates (x, r), partial derivatives with respect to r appear only in the perturbation equations in the form $r\frac{\partial}{\partial r}$, and so the Mellin transform of these equations involves only derivatives with respect to x and coefficients independent of r. This amounts to replacing $g(x, r)$ with $r^s G(x; s)$, in the same way that Laplace transforms amount to replacing $g(x, r)$ with $e^{sr} G(x; s)$. The solution is recovered by performing an inverse Mellin transform, i.e. by integrating over the valid range of s. See Sec. VI for a discussion of the validity of this procedure and details on the Mellin transform. The following analysis is carried out for what we will refer to as the modes, i.e. the parametrized (by complex s) functions $G(x;s)$.

III. PERTURBATION FORMALISM

Next we review the formalism of Gerlach and Sengupta [11]. We exploit the spherical symmetry of the background by performing a $2+2$ split of the space-time into M^2 (manifold spanned by time and radial coordinates) and S^2 (unit 2-sphere) and then decomposing the angular part of the perturbation in terms of spherical harmonics.

First, some notation: Greek indices represent coordinates on the four-dimensional space-time, capital Latin indices coordinates on M^2, lowercase Latin indices S^2, we define covariant derivatives so:

$$g_{\mu\nu;\lambda} = 0, \qquad g_{AB|C} = 0, \qquad g_{ab:c} = 0, \tag{3.1}$$

and a comma defines an ordinary partial derivative.

A. Angular decomposition

We write a nonspherical metric perturbation

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}(t, r, \theta, \phi), \tag{3.2}$$

where the overtilde denotes background quantities; and similarly for the matter perturbation

$$t_{\mu\nu} = \tilde{t}_{\mu\nu} + \Delta t_{\mu\nu}(t, r, \theta, \phi). \tag{3.3}$$

From the spherical harmonics we can construct bases for vectors,

$$Y_{,a}; \qquad S_a = \epsilon_a{}^b Y_{,b}, \tag{3.4}$$

and tensors,

$$Y\gamma_{ab}; \qquad Z_{ab} = Y_{,a:b} + \tfrac{1}{2}l(l+1)Y\gamma_{ab}; \qquad S_{(a:b)}, \tag{3.5}$$

where summation over the mode numbers l, m is implied. Using these, we decompose the perturbation in terms of scalar, vector, and tensor objects defined on M^2 times scalar, vector, and tensor bases defined on S^2.

When we compute the linearized Einstein equations for a perturbed space-time decomposed in this way, we find they naturally decouple into two sectors, even and odd. Gundlach and Martín-García's [14] definition is the most straightforward: That sector whose bases are in even powers of $\epsilon_a{}^b$ is called even (or polar or spheroidal); that sector whose bases are in odd powers of $\epsilon_a{}^b$ is called odd (or axial or toroidal). In this paper, we will consider only the even sector.

We write the (even) metric perturbation as

$$h_{\mu\nu} = \begin{pmatrix} h_{AB}Y & h_A Y_{,a} \\ \text{Symm} & r^2(KY\gamma_{ab} + GY_{,a:b}) \end{pmatrix}, \tag{3.6}$$

and the (even) matter perturbation as

$$\Delta t_{\mu\nu} = \begin{pmatrix} \Delta t_{AB}Y & \Delta t_A Y_{,a} \\ \text{Symm} & r^2\Delta t^1 Y\gamma_{ab} + \Delta t^2 Z_{ab} \end{pmatrix}. \tag{3.7}$$

Thus, the metric perturbation is defined by a symmetric two-tensor, a two-vector, and two scalars,

$$h_{AB}, \qquad h_A, \qquad G, \qquad K, \tag{3.8}$$

and the matter perturbation is defined by the same,

$$\Delta t_{AB}, \qquad \Delta t_A, \qquad \Delta t^1, \qquad \Delta t^2. \tag{3.9}$$

When we write down Einstein's equations for this metric and matter tensor, we identify the left- and right-hand side coefficients of the scalar, vector, and tensor bases given in (3.4) and (3.5), and these are our evolution equations for the perturbation. The great simplification is that these equations are in terms of two-dimensional objects h_{AB}, etc., and their derivatives defined on the manifold M^2. This makes calculating the perturbation equations much easier as the connections used in calculating derivatives, e.g. $h_{A|B}$, are those defined on the background.

It is important to note that for $l = 0$ all of the basis objects given in (3.4) and (3.5) vanish except for $Y\gamma_{ab}$ and for $l = 1$ Z_{ab} and $S_{(a:b)}$ vanish. Thus, some equations do not exist in these cases, and we must consider $l = 0$ and $l = 1$ separately from the more general $l \geq 2$ case.

B. Gauge invariance

Two space-times are identical if they differ only by a diffeomorphism [1] (we take the passive view of a diffeomorphism as a coordinate transformation). There is a danger that if you add a "perturbation"

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}, \tag{3.10}$$

you are in fact still looking at the same space-time after undergoing a coordinate transformation, rather than after being perturbed in a physically meaningful way. To escape this problem, we must interest ourselves only in those objects which do not change under an infinitesimal coordinate (gauge) transformation. These are called gauge invariants and are the true measure of a physically meaningful perturbation. (More precisely, these are identification gauge-invariant; see Stewart and Walker [18] for details.)

The vector field generating a gauge transformation has an even/odd parity decomposition as

$$\boldsymbol{\xi} = \xi_A Y dx^A + \xi Y_{,a}\, dx^a. \tag{3.11}$$

The gauge change induced on the tensor $h_{\mu\nu}$ is

$$h_{\mu\nu} \to \bar{h}_{\mu\nu} = h_{\mu\nu} + \mathcal{L}_{\vec{\xi}}\tilde{g}_{\mu\nu}, \tag{3.12}$$

where an overbar represents gauge transformed objects and where

$$\mathcal{L}_{\tilde{\xi}}\tilde{g}_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \quad (3.13)$$

with ∇_μ the covariant derivative associated with the *background* metric. Thus, we can write down how all of the perturbation objects in $h_{\mu\nu}$ and $\Delta t_{\mu\nu}$ transform [11].

Now we construct the gauge invariants; that is, we take linear combinations of (3.8) and (3.9) to form objects which do not change under a gauge transformation. These are

$$\left.\begin{aligned} k_{AB} &= h_{AB} - (p_{A|B} + p_{B|A}) \\ k &= K - 2\tilde{v}^A p_A \end{aligned}\right\}\text{(metric)} \quad (3.14a)$$

$$\left.\begin{aligned} T_{AB} &= \Delta t_{AB} - \tilde{t}_{AB|C}p^C - \tilde{t}_A{}^C p_{C|B} - \tilde{t}_B{}^C p_{C|A} \\ T_A &= \Delta t_A - \tilde{t}_A{}^C p_C - r^2(\tilde{t}^a{}_a/4)G_{,A} \\ T^1 &= \Delta t^1 - (p^C/r^2)(r^2\tilde{t}^a{}_a/2)_{,C} + l(l+1)(\tilde{t}^a{}_a/4)G \\ T^2 &= \Delta t^2 - (r^2\tilde{t}^a{}_a/2)G \end{aligned}\right\}\text{(matter)}, \quad (3.14b)$$

where $p_A = h_A - \frac{1}{2}r^2 G_{,A}$ and $\tilde{v}^A = r^{|A}/r$. The perturbation evolution equations are then recast entirely in terms of the gauge invariants. We give these equations in Appendix A.

There is an especially useful gauge choice we can make, called the Regge-Wheeler or longitudinal gauge. This consists of transforming to a specific coordinate system via

$$\boldsymbol{\xi} = \left(h_A - \frac{r^2}{2}G_{,A}\right)Y dx^A + \frac{r^2}{2}GY_{,a}\,dx^a, \quad (3.15)$$

in which $h_A = G = 0 = p_A$. Since we are measuring gauge invariants, we are free to make this transformation, the benefit being that in this gauge the bare perturbations (3.8) and (3.9) and the gauge invariants match.

When $l = 0, 1$, we cannot construct a set of gauge-invariant objects such as those given above for the same reason that there are fewer equations in these sectors (see Appendix A): the vanishing of some or all of the bases given in (3.4) and (3.5). Thus, for $l = 1$ modes we can at best construct only partially gauge-invariant objects, and for $l = 0$ all remnants of gauge invariance are lost.

Finally, we must consider what to measure on the relevant surfaces in order to test for stability. Following Chandrasekhar [19], we use the Weyl scalars to measure the flux of energy of the perturbations. For a detailed discussion on how the Weyl scalars relate to the Zerilli, Regge-Wheeler, Moncrief, etc., scalars, see Lousto [20]. Stewart and Walker [18] showed that the only Weyl scalars which are both identification gauge-invariant, which is the sense described above, and tetrad gauge-invariant (independent of the choice of null tetrad with which the Weyl scalars are defined) are the Petrov type-N terms. Furthermore, they are only tetrad and identification gauge-invariant if the background is type-D or conformally flat. Since Vaidya space-time is spherically symmetric, and therefore type-D, this means there are two fully (tetrad and identification) gauge-invariant Weyl scalars (for $l \geq 2$), $\delta\Psi_0$ and $\delta\Psi_4$. These scalars represent pure transverse gravitational waves propagating in the radial inward (respectively, outward) null directions of a spherically symmetric background.

With a particular choice of null tetrad, these scalars can be given as

$$\begin{aligned} \delta\Psi_0 &= \frac{1}{2r^2}\tilde{\ell}^A\tilde{\ell}^B k_{AB}(\tilde{w}^a\tilde{w}^b Y_{:ab}), \\ \delta\Psi_4 &= \frac{1}{2r^2}\tilde{n}^A\tilde{n}^B k_{AB}(\tilde{w}^{*a}\tilde{w}^{*b} Y_{:ab}), \end{aligned} \quad (3.16)$$

where $\tilde{\ell}^\mu$, $\tilde{n}^\mu$, $\tilde{m}^\mu = r^{-1}\tilde{w}^\mu(\theta,\phi)$ and $\tilde{m}^{*\mu}$ are a null tetrad of the background and the $*$ represents complex conjugation (see, for example, Nolan [21]). Tetrad gauge invariance of these objects must be carefully interpreted. In the type-D background, there is an obvious choice of null tetrad: we take $\tilde{\ell}^\mu, \tilde{n}^\mu$ to be the principal null directions of the Weyl tensor and take $\tilde{m}^\mu$ and its conjugate to be unit spacelike vectors orthogonal to $\tilde{\ell}^\mu, \tilde{n}^\mu$ to complete the tetrad. Then $\delta\Psi_0$ and $\delta\Psi_4$ are identification gauge-invariant and also tetrad gauge-invariant with respect to any *infinitesimal* Lorentz transformation of the tetrad and also with respect to any finite null rotation that leaves the directions of $\tilde{\ell}^\mu, \tilde{n}^\mu$ fixed. However, these scalars *are not* preserved under the *finite* boost rotations

$$\tilde{\ell}^\mu \to A\tilde{\ell}^\mu, \qquad \tilde{n}^\mu \to A^{-1}\tilde{n}^\mu, \qquad \tilde{m} \to e^{i\omega}\tilde{m},$$
$$A > 0, \qquad \omega \in \mathbb{R},$$

under which they rescale as

$$\delta\Psi_0 \to A^2\delta\Psi_0, \qquad \delta\Psi_4 \to A^{-2}\delta\Psi_4.$$

For the sake of boundary conditions, however, we must take this scale covariance into account (see Beetle and Burko [22] for a generalized discussion), and thus to first order our "master" function will be [21]

$$\delta P_{-1} = |\delta\Psi_0\delta\Psi_4|^{1/2}. \quad (3.17)$$

For $l = 0, 1$ the angular part of $\delta\Psi_0, \delta\Psi_4$ is zero, and, thus, these scalars vanish in these sectors.

IV. MINKOWSKI REGION

For completeness our analysis must include the region of flat space preceding the Vaidya region. In our global, self-similar coordinates (x, r), Minkowski space-time is given by

$$g_{\mu\nu}dx^\mu dx^\nu = -r^2dx^2 + 2r(1-x)dxdr + x(2-x)dr^2 + r^2d\Omega^2, \qquad (4.1)$$
$$t_{\mu\nu}dx^\mu dx^\nu = 0.$$

We seek to identify only that class of perturbations which are finite on the portion of the axis $r = 0, v < 0$, and on $\mathcal{N}$, the surface marking the transition between flat space-time and Vaidya space-time, which we call the threshold. $\mathcal{N}$ is the past null cone of the origin $(x, r) = (0, 0)$. We consider the case where the perturbed space-time is still empty; that is, the matter perturbation is zero. Thus, the right-hand side of all perturbation equations is zero.

A. $l \geq 2$ modes, Minkowski

The perturbation equations for Minkowski background are

$$2\tilde{v}^C(k_{AB|C} - k_{CA|B} - k_{CB|A} - \tilde{g}_{AB}k_{CD}{}^{|D} + \tilde{g}_{AB}k_{CD}\tilde{v}^D) - \frac{l(l+1)}{r^2}k_{AB} - \tilde{g}_{AB}\left(2k_{,C}{}^{|C} - \frac{(l-1)(l+2)}{r^2}k\right) + 2(\tilde{v}_A k_{,B} + \tilde{v}_B k_{,A} + k_{,A|B}) = 0,$$
$$(\tilde{v}^C\tilde{v}^D + \tilde{v}^{C|D})k_{CD} = 0, \qquad (4.2)$$
$$k_{,A} - k_{AC}{}^{|C} = 0,$$
$$k_A{}^A = 0,$$

where $\tilde{v}^A = r^{|A}/r$.

We write these equations in (x, r) coordinates and perform a Mellin transform (2.7) over r. There are four unknowns, which, after the Mellin transform, we denote

$$k_{AB} = \begin{pmatrix} r^{s+1}A_s(x) & r^sB_s(x) \\ r^sB_s(x) & r^{s-1}C_s(x) \end{pmatrix}, \qquad k = r^{s-1}K_s(x). \qquad (4.3)$$

We define a new variable $D = B - xA$, where for brevity we have dropped the subscript s. The scalars we wish to measure then become

$$\delta\Psi_0 = -2r^{s-3}D, \qquad \delta\Psi_4 = \tfrac{1}{2}r^{s-3}(2A + D), \qquad \delta P_{-1} = |\delta\Psi_0\delta\Psi_4|^{1/2}. \qquad (4.4)$$

We can decouple the system to get an ODE in D,

$$x(x-2)D'' + 2(1 + s + x - sx)D' - (l + l^2 + s - s^2)D = 0, \qquad (4.5)$$

which is hypergeometric,

$$z(1-z)D''(z) + (\gamma - (\alpha + \beta + 1)z)D'(z) - \alpha\beta D(z) = 0, \qquad (4.6)$$

with $\alpha = 1 + l - s$, $\beta = -l - s$, $\gamma = -1 - s$, and $z = x/2$. There are two solutions near the axis $x = -\infty$ [23], denoted by the subscript A,

$$_{\mathrm{A}}D_1 = (-z)^{s-l-1}{}_2F_1(1 + l - s, 3 + l; 2 + 2l; z^{-1}), \qquad (4.7)$$

$$_{\mathrm{A}}D_2 = -\ln(-z)\,_{\mathrm{A}}D_1 + (-z)^{s+l}(1 + O(z^{-1})), \qquad (4.8)$$

since $\alpha - \beta = 2l + 1 \in \mathbb{Z}$.

If we form the general solution

$$D = d_{1\mathrm{A}}D_1 + d_{2\mathrm{A}}D_2, \qquad (4.9)$$

and from this solutions for A, B, C, and K, we find to leading order near $x = -\infty$ $(r = 0)$,

$$\delta\Psi_{0,4} \sim d_1r^{l-2}(1 + O(r)) + d_2r^{-l-3}(1 + O(r)). \qquad (4.10)$$

Thus, in order to make $\delta\Psi_{0,4}$, and hence δP_{-1}, regular at the axis, we need to set $d_2 = 0$ in the general solution for D.

Now we allow only the first solution for D to evolve up to the past null cone of the origin $\mathcal{N}$, the threshold. When we do so, we use the nature of the hypergeometric equation to write the acceptable solution at the regular axis as a linear combination of the two solutions on $\mathcal{N}$. That is to say, near $x = 0$ this solution has the form

$$_{\mathrm{A}}D_1 = d_{3\mathrm{T}}D_1 + d_{4\mathrm{T}}D_2, \qquad (4.11)$$

where $_{\mathrm{T}}D_1$, $_{\mathrm{T}}D_2$ are two naturally arising linearly independent solutions of the hypergeometric equation near $x = 0$. In finding these solutions, the relation between α, β, and γ is key so we must consider two cases: $s \in \mathbb{Z}$ and $s \notin \mathbb{Z}$.

The more straightforward case is when $s \notin \mathbb{Z}$. Then $1 - \gamma \notin \mathbb{Z}$ and we can take

$$_{\mathrm{T}}D_1 = F(\alpha, \beta; \gamma; z), \qquad _{\mathrm{T}}D_2 = z^{1-\gamma}F(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; z), \qquad (4.12)$$

and (4.11) holds with [24]

$$d_3 = \frac{\Gamma(1-\gamma)\Gamma(\alpha-\beta+1)}{\Gamma(1-\beta)\Gamma(\alpha-\gamma+1)},\quad d_4 = -\frac{\Gamma(\gamma)\Gamma(1-\gamma)\Gamma(\alpha-\beta+1)}{\Gamma(2-\gamma)\Gamma(\gamma-\beta)\Gamma(\alpha)}e^{i\pi(\gamma-1)}. \tag{4.13}$$

Again, the solutions for A, B, C, and K can be recovered from these expressions for D.

When we calculate the scalars due to ${}_{\mathrm{T}}D_1$, ${}_{\mathrm{T}}D_2$ near $x = 0$ (and away from the singularity are $r = 0$), we find

$$\delta\Psi_0 \sim d_3(1+O(x)) + d_4 x^{s+2}(1+O(x)),\quad \delta\Psi_4 \sim d_3(1+O(x)) + d_4 x^{s-2}(1+O(x)). \tag{4.14}$$

These two scalars and δP_{-1} will be finite on $x = 0$ iff

$$\mathrm{Re}\,(s) > 2, \qquad s \notin \mathbb{Z}. \tag{4.15}$$

We can also see this with a general argument: The other singular point of the hypergeometric equation is at $z = 1$, or $x = 2$. This is the future null cone of the origin in Minkowski space-time. We would expect solutions to be regular here, too, and from the transformation

$${}_2F_1(\alpha,\beta;\gamma;z) = (1-z)^{\gamma-\alpha-\beta}{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma;z), \tag{4.16}$$

we require $\mathrm{Re}(\gamma-\alpha-\beta) = \mathrm{Re}(s-2) > 0$.

The case $s \in \mathbb{Z}$ is cumbersome and we will only summarize the results here. This case is again split according to the sign of $1-\gamma \in \mathbb{Z}$; however, there are no solutions for $1-\gamma \le 0$ which are finite on $\mathcal{N}$, and, thus, we present here only the solutions for $1-\gamma > 0$ ($s+2>0$).

If $\gamma = 1-m$, where m is a natural number, and α, β are different from the numbers $0, -1, -2, \ldots, 1-m$, then there are two linearly independent solutions near $x = 0$ given by [25]

$$\begin{aligned} {}_{\mathrm{T}}D_1 &= z^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;z),\\ {}_{\mathrm{T}}D_2 &= \ln(z)\,{}_{\mathrm{T}}D_1 - \sum_{n=1}^{m}\frac{(n-1)!(-m)_n}{(\gamma-\alpha)_n(\gamma-\beta)_n}z^{m-n}\\ &\quad + \sum_{n=0}^{\infty}\frac{(\alpha+m)_n(\beta+m)_n}{(1+m)_n n!}[h^*(n)-h^*(0)]z^{m-n}, \end{aligned} \tag{4.17}$$

where

$$\begin{aligned} h^*(n) &= \psi(\alpha+m+n) + \psi(\beta+m+n)\\ &\quad - \psi(1+m+n) - \psi(1+n), \end{aligned} \tag{4.18}$$

and $\psi = \psi^{(0)} = \Gamma'/\Gamma$ is the Digamma function. The scalars $\delta\Psi_{0,4}$ and P_{-1} will be finite on $x = 0$ due to these solutions if $\mathrm{Re}(s) \ge 2$.

If, however, α or β is equal to one of the numbers $0, -1, -2, \ldots, 1-m$, then the solution given above loses meaning; this will occur for $s = l + m$, where m is a natural number (≥ 1). In this case, two linearly independent solutions are

$${}_{\mathrm{T}}D_1 = F(\alpha,\beta;\gamma;z), \tag{4.19}$$

$${}_{\mathrm{T}}D_2 = F(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;z). \tag{4.20}$$

The scalars due to these solutions *will* diverge on $x = 0$. Therefore, we must *not* consider the modes $s = l + m$ when summing to find the general solution. This does not challenge the generality of the result, however, as all multipole mode numbers l (≥ 2) will still be counted.

Thus, we have found the largest class of perturbations that will have finite flux on the axis and on the threshold: All $s \in \mathbb{C}$ such that $\mathrm{Re}(s) \ge 2$, except for $s = l + m$, where m is a natural number. These solutions will be matched across $x = 0$ into the Vaidya space-time and then allowed to evolve up to the Cauchy horizon.

B. $l = 1$ mode, Minkowski

While the result in this sector is well known, for the sake of completeness we briefly give the analysis in terms of the formalism defined above.

In this sector, not all the perturbation equations apply, and we can define only partially gauge-invariant objects. Looking at (3.6), for $l = 1$ we find $Y\gamma_{ab} = -Y_{,a:b}$ and, thus,

$$h_{\mu\nu}dx^\mu dx^\nu = \ldots + r^2(K-G)Y\gamma_{ab}dx^a dx^b. \tag{4.21}$$

If we let $H = K - G$, then the gauge transformation of H generated by $\xi_\mu dx^\mu = \xi_A Y dx^A + \xi Y_{,a}\,dx^a$ is

$$\overline{H} - H = \overline{K} - K - (\overline{G} - G) = 2\xi/r^2 - 2\tilde{v}^A\xi_A. \tag{4.22}$$

Now we rename H as K, and we have effectively set $G = 0$ with K now transforming as H given above. Thus, $p_A = h_A$, and we are left with this sensitivity to the angular part of the gauge transformation,

$$k_{AB} \to k_{AB} + [r^2(\xi/r^2)_{,A}]_{|B} + [r^2(\xi/r^2)_{,B}]_{|A}, \tag{4.23}$$

$$k \to k + 2\xi/r^2 + 2\tilde{v}^A r^2(\xi/r^2)_{,A}. \tag{4.24}$$

We can, however, use this to our advantage. Following Sarbach and Tiglio [12], we look to transform into a coordinate system in which ${k_A}^A = 0$. To do this, we choose ξ such that

$$[r^2(\xi/r^2)_{,A}]^{|A} = -{k_A}^A. \tag{4.25}$$

Then we are free to make further gauge transformations, provided

$$[r^2(\xi/r^2)_{,A}]^{|A} = 0. \tag{4.26}$$

Thus, we can reinstate the second scalar perturbation equation, ${k_A}^A = 0$, as a gauge choice.

Using this, we split the tensor perturbation equation into its trace and trace-free parts. Then the set of equations for $l = 1$ is given by

$$2\tilde{v}^C(k_{AB|C} - k_{CA|B} - k_{CB|A}) - \frac{2}{r^2}k_{AB} + 2(\tilde{v}^A k_{,B} + \tilde{v}^B k_{,A} + k_{,A|B}) - \tilde{g}_{AB}k_{,C}{}^{|C} = 0, \qquad (4.27)$$
$$2\tilde{v}^C\tilde{v}^D k_{CD} = k_{,C}{}^{|C} + 2\tilde{v}^C k_{,C}, \qquad k_{AC}{}^{|C} = k_{,A},$$

where the first equation is the trace-free part of the tensor equation, and the second is its trace.

Now we must consider for what to solve the equations. The scalars $\delta\Psi_{0,4}$ given before have an angular dependence which is zero for $l = 1$, and, thus, they vanish in this sector. The other options for true scalars to measure are $k_A{}^A$ and k; however, we have chosen a gauge in which the trace of k_{AB} is zero, and, thus, the only scalar left to measure is k. We will show this scalar is pure gauge; that is, there is enough residual gauge freedom to transform into a gauge in which $k = 0$. In this gauge, all the components of k_{AB} are also zero, and thus this perturbation sector is empty.

For convenience, we look at the perturbation equations in orthogonal coordinates,

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2, \qquad (4.28)$$

and in this coordinate system we label components

$$k_{AB} = \begin{pmatrix} A(t,r) & B(t,r) \\ B(t,r) & A(t,r) \end{pmatrix}, \qquad (4.29)$$

since k_{AB} is symmetric and trace-free. Now when we look at the perturbation equations, the trace equation gives A, and one of the trace-free equations gives B, in terms of k and its derivatives,

$$A(t,r) = \frac{r}{2}(2k_{,r} + r(k_{,rr} - k_{,tt})),$$
$$B(t,r) = \frac{r}{2}(2k_{,t} + r^2(k_{,tt} - k_{,rr})_{,t}). \qquad (4.30)$$

The remaining equations give

$$r(\Box k)_{,r} = -4(\Box k), \qquad (4.31)$$

which we solve as

$$\Box k = f(t)r^{-4}. \qquad (4.32)$$

k should satisfy this equation with initial data $k(0,r) = \alpha(r)$, $\dot{k}(0,r) = \beta(r)$ satisfying $\alpha, \beta \in C^1$, and $k(t < 0, r = 0)$ finite. This implies $f(t) = 0$, and, thus, k solves the homogeneous wave equation

$$\Box k = 0. \qquad (4.33)$$

k, however, is not gauge-invariant. Looking at (4.24), we see $k \to \bar{k} = k + \eta$, where $\eta = 2\xi/r^2 + 2\tilde{v}^A r^2(\xi/r^2)_{,A}$. Thus,

$$\Box k \to \Box\bar{k} = \Box k + \Box\eta, \qquad (4.34)$$

but $\Box\eta = 0$ [provided (4.26) holds], and, thus, k and η satisfy the same equation. Therefore, we can choose $\eta = -k$, and, thus, we can always transform to a gauge in which $k = 0$. In this gauge we also have $k_{AB} = 0$, and, thus, the entire $l = 1$ perturbation is pure gauge.

C. $l = 0$ mode, Minkowski

The $l = 0$ mode represents a spherically symmetric perturbation. As we are considering zero matter perturbation, we have the perturbed space-time as spherically symmetric vacuum, and thus Birkhoff's theorem applies; that is, the perturbed space-time is Schwarzschild. We will recover the specifics using our perturbation formalism described above.

For $l = 0$, $Y_{,a} = 0$ and, thus, $h_A = G = 0$. We cannot form gauge invariants and, thus, use K and h_{AB}, which are fully dependent on gauge transformations $\xi_\mu dx^\mu = \xi_A dx^A$ as

$$h_{AB} \to h_{AB} - (\xi_{A|B} + \xi_{B|A}), \qquad K \to K - 2\tilde{v}^A\xi_A. \qquad (4.35)$$

We can transform to a gauge in which $K = h^A{}_A = 0$ by choosing ξ_A such that

$$\xi_A{}^{|A} = \tfrac{1}{2}h^A{}_A, \qquad \tilde{v}^A\xi_A = \tfrac{1}{2}K. \qquad (4.36)$$

In (t, r) coordinates, this means making a gauge transformation $\xi_r = \frac{1}{2}rK$ and $\xi_{t,t} = \xi_{r,r} - \frac{1}{2}h_A{}^A$. Further transformations preserving $K = h^A{}_A = 0$ must be of the form $\xi_A = f(r)\delta^t_A$ for any arbitrary function $f(r)$. Thus, of the remaining perturbation terms, $A(t,r)$ is gauge-invariant, whereas $B(t,r) \to B(t,r) - f'(r)$, where

$$h_{AB} = \begin{pmatrix} A(t,r) & B(t,r) \\ B(t,r) & A(t,r) \end{pmatrix}. \qquad (4.37)$$

The perturbation equations reduce to $A_{,t} = B_{,t} = 0$ and $rA_{,r} = -A$. Since B is a function of r alone, we can choose $f(r)$ to set $B = 0$. Thus, the only perturbation term which cannot be gauged away is

$$A = \frac{c}{r}. \qquad (4.38)$$

Renaming the constant $c = 2m$, and noting $(1 - 2m/r)^{-1} \approx (1 + 2m/r)$ for m small (which is the case in this linear model), we have recovered the Schwarzschild line element. There is an intrinsic singularity on the axis $r = 0$, and, thus, this solution contradicts our initial data; therefore, there is no $l = 0$ perturbation.

V. VAIDYA REGION

In this section, we describe metric and matter perturbations of the Vaidya solution given in (2.3) and (2.4). Initial conditions for this problem are that any perturbations coming from the region of flat space preceding the threshold should be finite on, and match continuously across, this surface. As before, we will need to split the analysis into the $l = 0$, $l = 1$, and $l \geq 2$ cases.

A. $l \geq 2$ modes, Vaidya

We consider the case where the perturbed space-time is that of a null dust and, thus, has matter tensor

$$t_{\mu\nu} = (\tilde{\rho} + \delta\rho)(\tilde{\ell}_\mu + \delta\ell_\mu)(\tilde{\ell}_\nu + \delta\ell_\nu), \tag{5.1}$$

where $\tilde{\rho} + \delta\rho$ is the energy density and $\tilde{\ell}_\mu + \delta\ell_\mu$ is the null vector of the perturbed space-time. As before, we decompose these perturbation objects in terms of the spherical harmonics. Since we are measuring the gauge invariants given in (3.14), we can examine in any gauge we choose. For convenience, we will use the Regge-Wheeler (RW) gauge given in (3.15). Thus, we can write the matter gauge invariants as

$$T_{AB} = \begin{pmatrix} r^2\delta\rho - \frac{\lambda\partial_x\Gamma}{4\pi r} & rx\delta\rho - \frac{\lambda\partial_r\Gamma}{8\pi r} - \frac{\lambda x\partial_x\Gamma}{8\pi r^2} \\ \text{Symm} & x^2\delta\rho - \frac{\lambda x\partial_r\Gamma}{4\pi r^2} \end{pmatrix},$$
$$T_A = \begin{pmatrix} \frac{-\lambda\Gamma}{8\pi r} \\ \frac{-\lambda x\Gamma}{8\pi r^2} \end{pmatrix}, \qquad T^1 = T^2 = 0. \tag{5.2}$$

We define Γ as the perturbation in the null coordinate; that is, we can define the null coordinate of the perturbed space-time as $V = v - \Gamma(x, r)Y(\theta, \phi)$. It follows from the conservation equations that the null vector of the perturbed space-time is $\tilde{\ell}_\mu + \delta\ell_\mu = -\partial_\mu V$.

We write out the perturbation equations as given in Appendix A and, as before, perform a Mellin transform over r. There are now six unknowns,

$$k_{AB} = \begin{pmatrix} r^{s+1}A_s(x) & r^s B_s(x) \\ r^s B_s(x) & r^{s-1}C_s(x) \end{pmatrix}, \qquad k = r^{s-1}K_s(x),$$
$$\Gamma = r^s G_s(x), \qquad \delta\rho = r^{s-3}M_s(x). \tag{5.3}$$

A constraint equation defining $\delta\rho$ decouples and so we are left with five unknowns in the evolution system. We use the scalar equation (A1d) to remove one of the metric variables, and so we can reduce the set of equations to a four-dimensional first order linear system. We give this system in Appendix B.

As before (dropping the subscript s), we define the variable $D = B - xA$, and the scalars to measure become

$$\delta\Psi_0 = \frac{2r^{s-3}D}{\lambda x - 1}, \qquad \delta\Psi_4 = \frac{r^{s-3}(2A + (1 - \lambda x)D)}{2},$$
$$\delta P_{-1} = |\delta\Psi_0\delta\Psi_4|^{1/2}. \tag{5.4}$$

From our system of equations we can decouple a second order ordinary differential equation for D,

$$D''(x) + p(x)D'(x) + q(x)D(x) = 0, \tag{5.5}$$

where

$$p(x) = -\frac{s+1}{x} + \frac{2\lambda}{1 - \lambda x} + \frac{s - 3 - (s-6)\lambda x}{\lambda x^2 - x + 2}, \tag{5.6}$$

$$q(x) = \frac{(\lambda x - 1)^2(l + l^2 + s^2(\lambda x - 1)) + \lambda(x - 2 + 6\lambda x - \lambda x^2) - s(\lambda x - 1)(1 + \lambda(-2 + x(2\lambda x - 3)))}{x(\lambda x - 1)^2(\lambda x^2 - x + 2)}. \tag{5.7}$$

The regular singular point at $x = 0$ is the past null cone of the origin of coordinates and is the surface over which we move from flat space-time to Vaidya space-time (the threshold). When $\lambda < 1/8$, the space-time has the structure given in Fig. 1, and there are two surfaces which are regular singular points of the above equation: The first is the Cauchy horizon at $x = x_c$ and the second is the SFSH $x = x_e$, the first and second zeros of $\lambda x^2 - x + 2$, respectively. Finally, $x_a = 1/\lambda$ is a regular singular point describing the apparent horizon.

We will begin the analysis by examining the threshold $x = 0$ and ensuring that our initial conditions are met. As the analysis will differ depending on whether s is an integer or not, we will consider these two cases separately. Then we will allow the acceptable solutions to evolve up to the Cauchy horizon and then on to the SFSH.

1. Threshold, $s \notin \mathbb{Z}$

We can use the method of Frobenius to describe solutions near regular singular points, and we present these using the P-symbol notation [23]. For the first four columns in the P symbol, the first row's entry denotes the location of the regular singular point; the second and third rows' denote the leading order exponents of two infinite power series solutions at that point. If these exponents do not differ by an integer, then these series are two linearly independent solutions for D near that point; if the exponents do differ by an integer, then a logarithmic term may be introduced. Thus, for $\lambda < 1/8$,

$$P\begin{Bmatrix} 0 & x_c & x_e & x_a & \\ 0 & 0 & 0 & 1 & ;x \\ s+2 & \frac{1}{2}(s - 4 + \frac{s}{\sqrt{1-8\lambda}}) & \frac{1}{2}(s - 4 - \frac{s}{\sqrt{1-8\lambda}}) & 2 & \end{Bmatrix}. \tag{5.8}$$

As this is a second order equation coming from a fourth order system, there must be two solutions with $D = 0$. We find these by setting $D = 0$ in the system, which simplifies the equations greatly. Thus, we can find the set of exact solutions corresponding to $D = 0$, which we call solutions III and IV and are valid everywhere and irrespective of whether s is an integer or not:

$$\begin{array}{lcc} & \text{III} & \text{IV} \\ A = & 2g_0\lambda x^s & (s-1)k_0x^{s-2} - \frac{(l^2+l-2)}{2}k_0x^{s-1} \\ D = & 0 & 0 \\ K = & 0 & k_0x^{s-1} \\ G = & g_0x^s & 0, \end{array} \tag{5.9}$$

where k_0, g_0 are arbitrary constants. The full set of solutions is found using system methods, as this approach has the benefit of solving for all four solutions at once, with the required accuracy found by simply taking further terms in the series. These system methods are described in Appendix C, and we briefly outline their application here: The system of perturbation equations can be written in the form

$$Y' = \frac{1}{x^2}\left(J + \sum_{m=1}^{\infty} A_m x^m\right)Y, \tag{5.10}$$

where $Y = (A, D, K, G)^T$ and $J \neq 0$. Thus, $x = 0$ is an irregular singular point. We find that J has eigenvalue 0 multiplicity four. J cannot be diagonalized and we use Theorem C.4 given in Appendix C to remove off-diagonal terms, which effectively reduces the order of the singularity to a regular singular point. Now the leading order coefficient matrix has eigenvalues $0, s, s, s-2$, and so we apply Theorem C.2 twice to reduce the eigenvalues to 0 and $s-2$ multiplicity three. Finally, we apply Theorem C.1 to obtain the following.

To leading order, we find a full set of linearly independent solutions with asymptotic behavior (as $x \longrightarrow 0$)

$$\begin{array}{lcccc} & \text{I} & \text{II} & \text{III} & \text{IV} \\ A = & O(1) & O(x^{s-2}) & O(x^s) & O(x^{s-2}) \\ D = & O(1) & O(x^{s+2}) & 0 & 0 \\ K = & O(1) & O(x^{s+1}) & 0 & O(x^{s-1}) \\ G = & O(1) & O(x^{s+2}) & O(x^s) & 0. \end{array} \tag{5.11}$$

Solutions I and II correspond with the Minkowski solutions (matching across $x = 0$ is dealt with in the next subsection), and III and IV are the $D = 0$ solutions given in (5.9).

Taking A and D to be a linear combination of these solutions, we calculate the leading order of the scalars near $x = 0$ as

$$\begin{aligned} \delta\Psi_0 &\sim O(1) + O(x^{s+2}), \\ \delta\Psi_4 &\sim O(1) + O(x^{s-2}) + O(x^s), \end{aligned} \tag{5.12}$$

(for brevity we have left out the constants of combination). Thus, for our master function δP_{-1} to be finite on the threshold due to the Vaidya solutions, we find we must maintain the same constraint as that coming from the flat space solutions, that is,

$$\mathrm{Re}\,(s) > 2, \qquad s \notin \mathbb{Z}. \tag{5.13}$$

2. Threshold, $s \in \mathbb{Z}$

When s is an integer, the system methods break down for the following reason: The eigenvalues of the leading order coefficient matrix of the regular singular point at $x = 0$ are 0 and $s-2$ multiplicity three. These differ by an integer, and, thus, they must be repeatedly reduced until they are equal. However, each time we reduce an eigenvalue, we must diagonalize the leading order coefficient matrix, which prevents us from simply reducing the eigenvalue (the unspecified number) $s-2$ times.

Instead, we use the ordinary differential equation for D to write the solutions near $x = 0$ as, using the fact that $s + 2 > 0$ (since the flat space solutions constrained $s \geq 2$),

$$D = d_5 \sum_{m=0}^{\infty} A_m x^{m+s+2} + d_6\left\{k \ln x \sum_{m=0}^{\infty} A_m x^{m+s+2} + \sum_{m=0}^{\infty} B_m x^m\right\}, \tag{5.14}$$

with d_5, d_6 constants and k possibly zero.

We use this solution as an inhomogeneous term to solve for the other variables by integration, and we find

$$G \sim g_0x^s - d_1\sum_{m=0}^{\infty}\frac{A_m x^{m+s+2}}{m+2} - d_2\left\{\sum_{m=0}^{\infty}\frac{B_m x^m}{m-s} + k\sum_{m=0}^{\infty}\frac{A_m x^{m+s+2}}{m+2}\left(\ln x - \frac{1}{m+2}\right)\right\}, \tag{5.15}$$

$$\begin{aligned} K \sim k_0x^{s-1} &+ d_1\left\{\sum_{m=0}^{\infty}\frac{A_m x^{m+s+1}}{m+2}(-2(m+s+2) + x(1-s-2\lambda))\right\} + d_2\left\{\sum_{m=0}^{\infty} B_m x^m\left(\frac{1-s-2\lambda}{m-s+1} - \frac{2m}{(m-s)x}\right) + k\sum_{m=0}^{\infty} A_m x^{m+s+1}\left(\frac{-2}{m+2} \right.\right. \\ &- \frac{2(m+s+2)}{m+2}\left(\ln x - \frac{1}{m+2}\right) \\ &\left.\left. + \frac{x(1-s-2\lambda)}{m+3}\left(\ln x - \frac{1}{m+3}\right)\right)\right\}, \end{aligned} \tag{5.16}$$

$$A \sim (s + l^2 + l - 1)x^{-1}D + 2\lambda G + (s^2 - 1)x^{-1}K - sK'. \tag{5.17}$$

Since both D and K have $O(1)$ terms, we see there is an A solution which diverges on the threshold like x^{-1}. This divergent term cannot be switched off, for the following reason:

On the axis there were two solutions for D, which we denoted

$${}_{\mathrm{A}}D_1 \sim x^{s-l-1}, \qquad {}_{\mathrm{A}}D_2 \sim x^{s+l}. \tag{5.18}$$

The scalars $\delta\Psi_{0,4}$ due to the first solution went like r^{l-2}, whereas the second solution gave $\delta\Psi_{0,4} \sim r^{-l-3}$. Thus, we needed to switch off the divergent term in the general solution, $D|_{x=-\infty} = d_{1A}D_1 + d_{2A}D_2$, by setting $d_2 = 0$.

Now when this solution was allowed to evolve to the threshold, $D|_{x=0} = d_{3T}D_1 + d_{4T}D_2$, the constants $d_3, d_4 \neq 0$ were fixed [see (4.13)]. To match across $x = 0$, we must have (since we are using a global coordinate system)

$$D^M|_{x=0} = D^V|_{x=0}, \tag{5.19}$$

where M denotes solutions coming from Minkowski space-time and V denotes solutions coming from Vaidya space-time. Thus, we require

$$\lim_{x\uparrow 0}(d_{3T}D_1^M + d_{4T}D_2^M) = \lim_{x\downarrow 0}(d_{5T}D_1^V + d_{6T}D_2^V). \tag{5.20}$$

From (4.12) and (4.17), we see the solutions from Minkowski space-time are $O(1)$, $O(x^{s+2})$. When $s \in \mathbb{Z} \geq 2$, we see from (5.14) the solutions from Vaidya space-time are also $O(1)$, $O(x^{s+2})$, and, thus, to match continuously across $x = 0$ we cannot switch off the $O(1)$ D solution.

Thus, when we calculate A, and hence $\delta\Psi_4$, there will be divergence as $x \downarrow 0$ due to this solution. This does not happen when $s \notin \mathbb{Z}$, since there is no divergent A solution when $\mathrm{Re}(s) > 2$.

There were four constraints for initial data: (i) finite flux on the axis, (ii) finite flux on the threshold when approached from flat space, (iii) finite flux on the threshold when approached from Vaidya space-time, and (iv) continuous matching across $x = 0$. The most general class of perturbations which satisfies all of these conditions are those with

$$\mathrm{Re}\,(s) > 2, \qquad s \notin \mathbb{Z}. \tag{5.21}$$

3. Cauchy horizon

When $\lambda < 1/8$, the Cauchy horizon is a regular singular point of the system given in Appendix B. Its leading order coefficient matrix has eigenvalues 0 multiplicity three and

$$\sigma \equiv \frac{1}{2}\left(s - 4 + \frac{s}{\sqrt{1-8\lambda}}\right). \tag{5.22}$$

When σ is not an integer, we can use the system methods outlined in Appendix C. Applying Theorem C.1, we find solutions with asymptotic behavior

$$\begin{array}{lcccc} & \mathrm{I} & \mathrm{II} & \mathrm{III} & \mathrm{IV} \\ A = & O(1) & O(w^{\sigma}) & O(1) & O(1) \\ D = & O(1) & O(w^{\sigma}) & 0 & 0 \\ K = & O(w) & O(w^{\sigma+1}) & 0 & O(1) \\ G = & O(w) & O(w^{\sigma+1}) & O(1) & 0 \end{array} \tag{5.23}$$

as $w \to 0$, where $w = x - x_c$ [for consistency, see (5.8)].

Now we make an important observation: Since

$$0 < \sqrt{1-8\lambda} < 1 \tag{5.24}$$

for $0 < \lambda < 1/8$, therefore

$$\sigma = \frac{1}{2}\left(s - 4 + \frac{s}{\sqrt{1-8\lambda}}\right) > \frac{1}{2}(2s-4), \tag{5.25}$$

and thus $\mathrm{Re}(\sigma) > 0$ for $\mathrm{Re}(s) > 2$. Alternatively, we can say

$$\sigma = s - 2 + O(\lambda), \tag{5.26}$$

where each coefficient of λ^n is positive, and, thus, again $\mathrm{Re}(\sigma) > 0$ for $\mathrm{Re}(s) > 2$. Thus, each solution for A and D as given in (5.23) is at most $O(1)$ near $x = x_c$; all the solutions for A and D which are series beginning with w, w^{σ}, or $w^{\sigma+1}$ will decrease to zero as we approach the Cauchy horizon.

Since

$$\delta\Psi_0 = \frac{2r^{s-3}D}{\lambda x - 1}, \qquad \delta\Psi_4 = \frac{r^{s-3}(2A + (1-\lambda x)D)}{2}, \tag{5.27}$$

and A and D near the Cauchy horizon are a linear combination of $O(1)$ solutions, the scalars $\delta\Psi_{0,4}$ representing the flux of the perturbation, and hence the scalar δP_{-1}, will be finite on the Cauchy horizon $x = x_c$. Thus, when $\sigma \notin \mathbb{Z}$, the Cauchy horizon is stable under metric and matter perturbations.

However, for each value of the parameter $\lambda < 1/8$, there will be a mode number s such that $\sigma \in \mathbb{Z}$, and, thus, we must also consider this case. From (5.8), we see a general solution for D near $w = x - x_c = 0$ can be written as

$$D = d_3 \sum_{m=0}^{\infty} A_m w^{m+\sigma} + d_4\left\{k \ln w \sum_{m=0}^{\infty} A_m w^{m+\sigma} + \sum_{m=0}^{\infty} B_m w^m\right\}, \tag{5.28}$$

where d_3, d_4 are constants and k can be zero. Since we are considering $\lambda > 0$ ($\lambda = 0$ being vacuum space-time) and $\mathrm{Re}(s) > 2$, we have $\sigma \geq 1$ if $\sigma \in \mathbb{Z}$. Now we use this solution for D as an inhomogeneous term to integrate the perturbation equations. Near $w = 0$, we find a four-parameter set of solutions:

$$K \sim k_0 + \frac{x_c^{-1}(x_c - x_e)}{1 - \lambda x_c} \int w D' dw - x_c^{-1}(x_c + 3 - n) \times \int D dw,$$
$$G \sim g_0 + \frac{1}{x_c(\lambda x_c - 1)} \int D dw, \tag{5.29}$$
$$A \sim \frac{1}{x_c(1 - \lambda x_c)} D + 2\lambda G + \frac{1}{2} x_c^{-1}[\lambda x_c - x_c(l^2 + l - 2) + 6]K - \frac{1}{2}(\lambda x_c^2 + 4)K',$$

where k_0, g_0 are constants, a prime denotes differentiation with respect to w, and

$$\int D dw = d_3 \sum_{m=0}^{\infty} \frac{A_m w^{m+\sigma+1}}{m + \sigma + 1} + d_4 \bigg\{ k \sum_{m=0}^{\infty} \frac{A_m w^{m+\sigma+1}}{m + \sigma + 1} \times \bigg(\ln w - \frac{1}{m + \sigma + 1} \bigg) + \sum_{m=0}^{\infty} \frac{B_m w^{m+1}}{m + 1} \bigg\},$$
$$\int w D' dw = d_3 \sum_{m=0}^{\infty} \frac{A_m (m + \sigma) w^{m+\sigma+1}}{m + \sigma + 1} + d_4 \bigg\{ k \sum_{m=0}^{\infty} \frac{A_m w^{m+\sigma+1}}{m + \sigma + 1} \bigg(1 + (m + \sigma) \ln w - \frac{m + \sigma}{m + \sigma + 1} \bigg) + \sum_{m=0}^{\infty} \frac{B_m m w^{m+1}}{m + 1} \bigg\}.$$

Since $\sigma \geq 1$ and $\lim_{w\to 0} w^\sigma \ln w = 0$, we see all of these variables A, D, K, G, and thus the scalars $\delta\Psi_{0,4}$ and δP_{-1}, are again finite in the limit $w \to 0$.

Thus, the full set of perturbations which are finite on the axis and on the threshold $\mathcal{N}$ will evolve up to the Cauchy horizon and beyond *without* their flux diverging. Therefore, in the case of self-similar null dust there is a naked singularity whose Cauchy horizon is stable under metric and matter perturbations.

4. Second future similarity horizon

Now something interesting happens when we allow the solution to evolve past the Cauchy horizon and on to the next singular surface, the SFSH given by $x_e = \frac{1}{2\lambda}(1 + \sqrt{1 - 8\lambda})$. The first scalar depends only on D,

$$\delta\Psi_0 = \frac{2r^{s-3}}{\lambda x - 1} D, \tag{5.30}$$

and the solutions for D near $x = x_e$ can be found directly from (5.8) as

$$D = d_1 \sum_{m=0}^{\infty} A_m (x - x_e)^m + d_2 \sum_{m=0}^{\infty} B_m (x - x_e)^{m+\varsigma}, \tag{5.31}$$

where

$$\varsigma = \frac{1}{2}\bigg(s - 4 - \frac{s}{\sqrt{1 - 8\lambda}} \bigg). \tag{5.32}$$

Since $0 < \sqrt{1 - 8\lambda} < 1$, we see ς will always be negative for $\mathrm{Re}(s) \geq 2$. Thus, there is a class of solutions which are finite on the axis, finite on the threshold $\mathcal{N}$, finite on the Cauchy horizon, and then finally diverge on the SFSH. We emphasize that this instability is due to $x = x_e$ being a similarity horizon of the space-time and not an event horizon.

B. $l = 1$ mode, Vaidya

In this sector, we can define only partially gauge-invariant objects. As in Minkowski space-time, Sec. III B, the metric perturbation objects are gauge sensitive to $\xi_\mu dx^\mu = \xi_A Y dx^A + \xi Y_{,a} dx^a$ as

$$k_{AB} \to k_{AB} + [r^2(\xi/r^2)_{,A}]_{|B} + [r^2(\xi/r^2)_{,B}]_{|A}, \tag{5.33}$$

$$k \to k + 2\xi/r^2 + 2\tilde{v}^A r^2(\xi/r^2)_{,A}. \tag{5.34}$$

We examine the case where the perturbed space-time is that of null dust, and, thus, the bare matter perturbations are as given in (5.2),

$$\Delta t_{AB} = \begin{pmatrix} r^2\delta\rho - \frac{\lambda\partial_x\Gamma}{4\pi r} & rx\delta\rho - \frac{\lambda\partial_r\Gamma}{8\pi r} - \frac{\lambda x\partial_x\Gamma}{8\pi r^2} \\ \mathrm{Symm} & x^2\delta\rho - \frac{\lambda x\partial_r\Gamma}{4\pi r^2} \end{pmatrix},$$
$$\Delta t_A = \begin{pmatrix} \frac{-\lambda\Gamma}{8\pi r} \\ \frac{-\lambda x\Gamma}{8\pi r^2} \end{pmatrix}, \qquad \Delta t^1 = \Delta t^2 = 0. \tag{5.35}$$

Since $p_A = h_A$ in this sector, we can transform into the equivalent of the Regge-Wheeler gauge by choosing

$$\xi_A = h_A - r^2(\xi/r^2)_{|A}, \tag{5.36}$$

the benefit being that in this gauge $p_A = 0$ and, therefore, $T_{AB} = \Delta t_{AB}$, etc. Thus, we can express the right-hand side of the perturbation equations given in Appendix A in terms of $\delta\rho, \Gamma$.

Further transformations maintain this condition provided $\xi_A = -r^2(\xi/r^2)_{|A}$. Importantly, this fixes ξ_A while keeping ξ completely free.

T_{AB} and T_A are not gauge-invariant; they are sensitive to gauge transformations as

$$\overline{T}_{AB} - T_{AB} = \tilde{t}_{AB|C} r^2(\xi/r^2)^{|C} + \tilde{t}_{CB}(r^2(\xi/r^2)^{|C})_{|A} + \tilde{t}_{CA}(r^2(\xi/r^2)^{|C})_{|B},$$
$$\overline{T}_A - T_A = -\tilde{t}_{AB} r^2(\xi/r^2)^{|B}. \tag{5.37}$$

The vanishing of T^1 and T^2 is gauge-invariant.

Now we look at the perturbation equations in (x, r) coordinates. As in the $l \geq 2$ sector, we perform a Mellin transformation of the equations, which is equivalent to parametrizing the perturbation components as in (5.3),

$$k_{AB} = \begin{pmatrix} r^{s+1}A_s(x) & r^s B_s(x) \\ r^s B_s(x) & r^{s-1}C_s(x) \end{pmatrix}, \qquad k = r^{s-1}K_s(x),$$
$$\Gamma = r^s G_s(x), \qquad \delta\rho = r^{s-3}M_s(x). \tag{5.38}$$

Again, dropping the subscript s, we define the new variable $D = B - xA$.

We exploit the gauge freedom and transform into a gauge in which $k_A{}^A = 0$, by choosing ξ such that

$$[r^2(\xi/r^2)_{,A}]^{|A} = -k_A{}^A. \tag{5.39}$$

Then we are allowed make further transformations which preserve RW gauge and $k_A{}^A = 0$ provided

$$[r^2(\xi/r^2)_{,A}]^{|A} = 0, \qquad \xi_A + r^2(\xi/r^2)_{|A} = 0. \tag{5.40}$$

Thus, we recover the perturbation equation (A1d), which was not valid in the $l = 1$ sector, as a gauge choice. As before, the set of perturbation equations in this gauge reduces to one constraint equation defining $\delta\rho$, a first order system in A, D, K, and G, and we can decouple a second order ordinary differential equation for D.

There is some gauge freedom in ξ left, and we will use this to gauge away k. Let us formalize this: Let ξ satisfy gauge conditions $L_1\xi = 0$; let k satisfy perturbation field equation $L_2 k = 0$; and let k transform as $k \rightarrow k + L_3\xi$. Since $L_2L_3\xi = 0$ subject to $L_1\xi = 0$, there is a gauge in which $k = 0$.

In the coordinate system (x, r), where we perform a Mellin transform over r such that $\xi = r^{s-1}\xi_s$, the above equations are

$$L_1\xi = -(-1+s)(-x\lambda + s(-1+x\lambda))\xi_s(x) + (-2+2x-3x^2\lambda + 2s(1-x+x^2\lambda))\xi_s'(x) + x(-2+x-x^2\lambda)\xi_s''(x),$$
$$L_2\xi = \xi_s'''(x) + p_1(x)\xi_s''(x) + q_1(x)\xi_s'(x) + r_1(x)\xi_s(x), \qquad L_3\xi = (s + x\lambda - sx\lambda)\xi_s(x) + (1-x+x^2\lambda)\xi_s'(x), \tag{5.41}$$

where

$$p_1(x) = \frac{6+5(-2+x)x-4x(-3+x+3x^2)\lambda + x^3(24+7x)\lambda^2 + n^2(-1+x\lambda)(4+3x(-1+x\lambda))}{x(1+n-x+x(3-n+x)\lambda)(2+x(-1+x\lambda))} + \frac{n(2+x(1-16\lambda+x(-3+\lambda(16+x(6-(17+3x)\lambda)))))}{x(1+n-x+x(3-n+x)\lambda)(2+x(-1+x\lambda))},$$

$$q_1(x) = \frac{(2-n)(-2n(1+n)+(1+n)(-1+3n)x+(1-3n)x^2) - 2x(3+n(-1+(-3+n)n)-x-(-3+n)n(-4+3n)x)}{x^2(1+n-x+x(3-n+x)\lambda)(2+x(-1+x\lambda))} + \frac{(-2x(5+3(-3+n)n)x^2)\lambda + (-1+n)x^3(-30-8x+n(19-3n+3x))\lambda^2}{x^2(1+n-x+x(3-n+x)\lambda)(2+x(-1+x\lambda))},$$

$$r_1(x) = -\frac{(-1+n)((2-n)(1+n)(n-x)+2(3+n^3x+x(5+x)+2n(1+x^2)-n^2(1+x(4+x)))\lambda)}{x^2(1+n-x+x(3-n+x)\lambda)(2+x(-1+x\lambda))} + \frac{(-1+n)((-3+n)n(-4+n-x)x^2\lambda^2)}{x^2(1+n-x+x(3-n+x)\lambda)(2+x(-1+x\lambda))}.$$

A direct consequence of $k = 0$ is that $D = 0$. Thus, we are in a gauge in which $p_A = k_A{}^A = k = D = 0$, and to remain in this gauge we are allowed further transformations provided

$$[r^2(\xi/r^2)_{,A}]^{|A} = 0, \qquad 2\xi/r^2 + 2\tilde{v}^A r^2(\xi/r^2)_{,A} = 0. \tag{5.42}$$

The only ξ which satisfies both these constraints is $\xi = 0$. Therefore, there is no remaining gauge freedom and so the remaining perturbation variables are gauge-invariant.

Thus, we have found a one-parameter family of solutions,

$$k_{AB} = \begin{pmatrix} 2g_0\lambda r^{s+1}x^s & 2g_0\lambda r^s x^{s+1} \\ 2g_0\lambda r^s x^{s+1} & 2g_0\lambda r^{s-1}x^{s+2} \end{pmatrix},$$
$$T_{AB} = \begin{pmatrix} \frac{\lambda g_0(s+x)r^{s-1}x^{s-1}}{4\pi} & \frac{\lambda g_0(s+x)r^{s-2}x^s}{4\pi} \\ \frac{\lambda g_0(s+x)r^{s-2}x^s}{4\pi} & \frac{\lambda g_0(s+x)r^{s-3}x^{s+1}}{4\pi} \end{pmatrix}, \tag{5.43}$$
$$T_A = \begin{pmatrix} \frac{-\lambda g_0 r^{s-1}x^s}{8\pi} \\ \frac{-\lambda g_0 r^{s-2}x^{s+1}}{8\pi} \end{pmatrix}, \qquad T^1 = T^2 = k = 0.$$

To match continuously with the empty Minkowski region preceding $x = 0$, we require $\mathrm{Re}(s) \geq 1$. These solutions will evolve without divergence through the rest of the space-time.

Note the perturbations will vanish if $g_0 = 0$; that is to say, if we had considered only metric perturbations, and no matter perturbations, we would have returned an empty sector. Note also the sector is empty in the Minkowski limit $\lambda \to 0$.

C. $l = 0$ mode, Vaidya

When $l = 0$, then $Y_{,a} = 0$ and our perturbations are

$$h_{\mu\nu} = \begin{pmatrix} h_{AB} & 0 \\ 0 & r^2 K \gamma_{ab} \end{pmatrix}, \qquad \Delta t_{\mu\nu} = \begin{pmatrix} \Delta t_{AB} & 0 \\ 0 & r^2 \Delta t^1 \gamma_{ab} \end{pmatrix}. \tag{5.44}$$

We will use the coordinates (v, r, θ, ϕ), where v is the null coordinate of the background. Thus, the metric of the background is $\tilde{g}_{\mu\nu}dx^\mu dx^\nu = -(1 - \frac{\lambda v}{r})dv^2 + 2dvdr + r^2\gamma_{ab}dx^a dx^b$.

Since the matter tensor has the form (5.1) and $\tilde{\ell}_\mu$ has no angular dependence, we find $\Delta t^1 = 0$. We can describe the ingoing radial null geodesic of the perturbed space-time as $\tilde{\ell}_\mu + \delta\ell_\mu = -\nabla_\mu V$, where $V = v + \Gamma(v, r)$ is the null coordinate of the perturbed space-time. Our unknowns, therefore, are h_{AB}, K, $\delta\rho$, and Γ.

We cannot construct gauge invariants in the $l = 0$ sector, and, thus, we exploit the remaining gauge freedom to set some variables equal to zero. The perturbation variables are gauge dependent as

$$h_{AB} \to h_{AB} - (\xi_{A|B} + \xi_{B|A}), \qquad K \to K - 2\tilde{v}^A \xi_A,$$
$$\Delta t_{AB} \to \Delta t_{AB} - \tilde{t}_{AB|C}\xi^C - \tilde{t}_{CB}\xi^C{}_{|A} - \tilde{t}_{CA}\xi^C{}_{|B}. \tag{5.45}$$

As before, we transform into a gauge in which $K = h^A{}_A = 0$. To do this, we choose ξ_A such that [in the (v, r) coordinate system where $\dot{\theta}$ and θ' denote differentiation with respect to v and r, respectively]

$$\xi_v + \left(1 - \frac{\lambda v}{r}\right)\xi_r = \frac{1}{2} rK, \qquad \dot{\xi}_r = \frac{1}{2} h^A{}_A. \tag{5.46}$$

Then we are free to make further gauge transformations which preserve this condition provided

$$\xi_v + \left(1 - \frac{\lambda v}{r}\right)\xi_r = 0, \qquad \dot{\xi}_r = 0. \tag{5.47}$$

Now we look at the field equations in this gauge (we will let $h_{vv} = A$, $h_{vr} = h_{rv} = B$, and $h_{rr} = C$). The first is

$$\dot{C} = 0, \tag{5.48}$$

and, thus, $C = C(r)$. When we perform a gauge transformation on this quantity subject to (5.45) and (5.47), we find $C \to C - 2\xi_r'$, but since ξ_r is an arbitrary function of r, this means we can choose a gauge in which $C = 0$ (and, thus, $B = 0$ since $h^A{}_A = 0$). Thus, we have transformed to a gauge in which $K = h^A{}_A = B = C = 0$, and to remain in this gauge we are allowed further gauge transformations of the form $\xi_A = c_0(-(1 - \frac{\lambda v}{r})\delta_A^v + \delta_A^r)$, with c_0 an arbitrary constant.

The remaining perturbation equations in this gauge are

$$rA'' + 2A' = 0, \qquad rA' + A + \lambda\Gamma' = 0,$$
$$\frac{1}{r^2}\left(1 - \frac{\lambda v}{r}\right)(rA' + A) + \frac{1}{r}\dot{A} - \frac{2\lambda}{r^2}\dot{\Gamma} = 8\pi\delta\rho. \tag{5.49}$$

That $\ell_\mu + \delta\ell_\mu$ must be null and geodesic gives $\Gamma' = 0$, and hence

$$\Gamma = \alpha(v), \qquad A = \frac{\beta(v)}{r}, \qquad \delta\rho = \frac{1}{8\pi r^2}(\dot{\beta} - 2\lambda\dot{\alpha}). \tag{5.50}$$

Further gauge transformations give

$$A \to A - 2\frac{\lambda}{r}c_0, \qquad \Delta t_{AB} \to \Delta t_{AB}, \tag{5.51}$$

and, thus, these remaining perturbation quantities cannot be gauged away.

What we have shown here is essentially a uniqueness result: All the above perturbations can be generated by a perturbation in the mass function and the null vector. The metric and matter tensors for spherically symmetric null dust are given by

$$g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{m(v)}{r}\right)dv^2 + 2dvdr + r^2 d\Omega^2,$$
$$t_{\mu\nu}dx^\mu dx^\nu = \frac{\dot{m}(v)}{8\pi r^2}\ell_\mu \ell_\nu, \tag{5.52}$$

where before perturbation $m(v) = \lambda v$ and $\ell_\mu = -\partial_\mu v$, and after perturbation $m(v) = \lambda v + \beta(v) - 2\lambda c_0$ and $\ell_\mu = -\partial_\mu(v + \alpha(v))$. The terms α, β, c_0 are arbitrary, and, therefore, there is no reason to suspect divergence on the Cauchy horizon or elsewhere.

VI. RESUMMATION

In this section, we review some properties of the Mellin transform (see e.g. [26,27]) and present a plausibility argument that finiteness of the modes (as seen above at the Cauchy horizon) implies finiteness of the full perturbation found by resumming the modes. This argument is based on a study of the Klein-Gordon equation (or wave equation) in Vaidya space-time and its correspondence to the wave equation in Minkowski space-time.

We recall that the line element of self-similar Vaidya space-time may be written as

$$ds^2 = r^2(-1 + \lambda x)dx^2 + 2r(1 - x + \lambda x^2)dxdr + x(2 - x + \lambda x^2)dr^2 + r^2 d\Omega^2, \qquad (6.1)$$

with the coordinate ranges $r \in [0, \infty)$, $x \in (-\infty, \infty)$. We note that Minkowski space-time corresponds to taking $\lambda = 0$. Then the wave equation (or, more accurately, the partial differential equation satisfied by the lth multipole moment $\phi = \phi_l$ of the Klein-Gordon field) is

$$-x(2 - x + \lambda x^2)\phi_{,xx} + 2(1 - x + \lambda x^2)r\phi_{,rx} + (1 - \lambda x)r^2\phi_{,rr} - \lambda x^2\phi_{,x} + (2 - \lambda x)r\phi_{,r} - l(l + 1)\phi = 0. \qquad (6.2)$$

The Mellin transform of this equation yields a parametrized ODE. The first step is to define the Mellin transform of the field ϕ (note that the arrow indicates the variable with respect to which the Mellin transform is taken):

$$P(x; t) = \mathcal{M}[\phi(x, r)](r \to t) := \int_0^\infty \phi(x, r)r^{t-1}dr, \qquad (6.3)$$

where $t \in \mathbb{C}$ and satisfies $\tau_1 \leq \text{Re}(t) \leq \tau_2$, the constants $\tau_{1,2}$ being defined by the condition that the integral converges for these values of t. The field $\phi(x, r)$ is recovered via the inverse Mellin transform:

$$\phi(x, r) = \mathcal{M}^{-1}[P(x; t)](t \to r) = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} P(x; t)r^{-t}dt, \qquad (6.4)$$

where the inversion contour is vertical and lies in the strip $\tau_1 \leq \text{Re}(t) \leq \tau_2$. We should note here that, as the resummation is over terms of the form $P(x; t)r^{-t}$, the role played by t in the present section corresponds to the role played by $-s$ in Secs. II, III, IV, and V. See, in particular, the last paragraph of Sec. II.

Integrating by parts immediately yields the following results:

Lemma 6.1.—

$$\mathcal{M}[r\phi_{,r}(x, r)](r \to t) = -t\mathcal{M}[\phi(x, r)](r \to t) = -tP(x; t), \qquad (6.5)$$

provided

$$\lim_{r\to 0} r^t\phi(x, r) = \lim_{r\to\infty} r^t\phi(x, r) = 0. \qquad (6.6)$$

Lemma 6.2.—

$$\mathcal{M}[r^2\phi_{,rr}(x, r)](r \to t) = t(t + 1)\mathcal{M}[\phi(x, r)](r \to t) = t(t + 1)P(x; t), \qquad (6.7)$$

provided (6.6) holds and

$$\lim_{r\to 0} r^{t+1}\phi_{,r}(x, r) = \lim_{r\to\infty} r^{t+1}\phi_{,r}(x, r) = 0. \qquad (6.8)$$

Assuming that these conditions hold for values of t on a strip of the complex plane, we can take the Mellin transform of (6.2) to obtain the ODE

$$-x(2 - x + \lambda x^2)P'' - (2t(1 - x) + (1 + 2t)\lambda x^2)P' + (t^2 - t - l(l + 1) - t^2\lambda x)P = 0, \qquad (6.9)$$

where the prime here and throughout refers to differentiation with respect to argument. This equation has analytic coefficients (and, hence, analytic solutions) everywhere except at infinity and at the singular points $x = 0, x_c, x_e$, the roots (in increasing order) of $x(2 - x + \lambda x^2) = 0$. Note that in Minkowski space-time, there is no third singular point x_e. These are all regular singular points of the equation, and, thus, the standard Frobenius theory can be used to study the global behavior of solutions of (6.9) (see, for example, [28] or any textbook on linear differential equations). We recall that $x = x_0 =$ constant is a spacelike hypersurface for $x_0 \in (0, x_c)$ and that $x = 0, x_c$ are null hypersurfaces.

For Minkowski space-time, $\lambda = 0$ and (6.9) is a hypergeometric differential equation, and we can give an essentially complete account of the problem at hand. We proceed to do so in order to clarify the nature of this problem and our putative solution. So let $\lambda = 0$ in (6.9) and, for convenience of comparison with the standard text by Bateman [24], let $x = 2z$ (all results quoted below are taken from this reference). Then (6.9) is the hypergeometric equation

$$z(1 - z)u'' + (c - (a + b + 1)z)u' - abu = 0, \qquad (6.10)$$

where $u(z) = P(x)$, $a = t + l$, $b = t - l - 1$, and $c = t$. We will assume that $t \notin \mathbb{Z}$. This can be assumed without loss of generality by a deformation of the inversion contour.

The past and future null cones of the origin then correspond to the singular points $z = 0, 1$, respectively, of this equation. We encounter here a slight difficulty. As a null hypersurface, $x = z = 0$ cannot be an initial data surface for Eq. (6.2). We expect that this will translate into $z = 0$ failing to be a "good" initial point for the ODE (6.10). However, our overall aim is to argue that finiteness of the field on $x = 0$ along with finiteness of the modes at the future null cone (Cauchy horizon) is sufficient to imply finiteness of the field at the future null cone. We can connect these two by determining their respective connections to Cauchy data on a spacelike hypersurface, for example, $x = 1(z = 1/2)$. So consider the Cauchy data

$$\alpha(r) = \phi|_{x=1}, \qquad \frac{\beta(r)}{2} = \phi_{,x}|_{x=1}.$$

We assume that α, β satisfy unspecified differentiability and integrability conditions that, in particular, allow us to calculate the Mellin transforms

$$a(t) = \mathcal{M}[\alpha(r)](r \to t), \qquad b(t) = \mathcal{M}[\beta(r)](r \to t)$$

on some strip $\tau_1 \leq \mathrm{Re}(t) \leq \tau_2$ of the complex plane (the factor 1/2 is given for later convenience). These then yield the initial data for the Mellin transform $P(x; t)$ of ϕ at $x = 1$, i.e. for $u(z)$ at $z = 1/2$:

$$u(\tfrac{1}{2}) = a(t), \qquad u'(\tfrac{1}{2}) = b(t).$$

We emphasize that our assumptions on α, β imply the existence of the inverse Mellin transform of a, b taken over a vertical contour in $\tau_1 \leq \mathrm{Re}(t) \leq \tau_2$.

The next step is to determine the solution for u at $z = 0$ and at $z = 1$ in terms of $a(t)$ and $b(t)$. We use the following pairs of linearly independent solutions at these two points (again following the notation of Bateman [24]). At $z = 0$ we use

$$u_1(z; t) = (1 - z)^{1-t} F(-l, l + 1; t; z),$$
$$u_5(z; t) = z^{1-t} F(l + 1, -l; 2 - t; z),$$

and at $z = 1$ we use

$$u_2(z; t) = z^{1-t} F(l + 1, -l; t; 1 - z),$$
$$u_6(z; t) = (1 - z)^{1-t} F(-l, l + 1; 2 - t; 1 - z),$$

where

$$F(a, b; c; z) = {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n,$$

with

$$(a)_0 = 1, \qquad (a)_n = a(a + 1) \cdots (a + n - 1), \qquad n = 1, 2, 3, \ldots$$

is the standard hypergeometric function. Note then that all the hypergeometric functions present are in fact polynomials, and so u_5 and u_6 are finite sums of powers of z and $1 - z$, respectively. u_1 and u_2 are infinite series in z and $1 - z$, respectively, both with radius of convergence at least 1. We will concentrate on the solution at $z = 0$. The general solution of (6.10) at $z = 0$ is

$$u(z; t) = c_1(t) u_1(z; t) + c_5(t) u_5(z; t), \tag{6.11}$$

giving

$$u'(z; t) = c_1(t) u_1'(z; t) + c_5(t) u_5'(z; t). \tag{6.12}$$

Note that existence of $u_5|_{z=0}$ and $u_5'|_{z=0}$ requires $\mathrm{Re}(t) \leq 0$: Ruling out the integer case, we take $\mathrm{Re}(t) < 0$. As $z = 1/2$ is within the radius of convergence of the series solutions, we have

$$c_1(t) u_1(\tfrac{1}{2}; t) + c_5(t) u_2(\tfrac{1}{2}; t) = a(t),$$
$$c_1(t) u_1'(\tfrac{1}{2}; t) + c_5(t) u_2'(\tfrac{1}{2}; t) = b(t).$$

Solving for $c_1(t), c_2(t)$ gives

$$c_1(t) = \frac{2^{1-2t}}{1 - t}\left(a(t) u_5'\left(\frac{1}{2}; t\right) - b(t) u_5\left(\frac{1}{2}; t\right)\right),$$
$$c_5(t) = \frac{2^{1-2t}}{1 - t}\left(-a(t) u_1'\left(\frac{1}{2}; t\right) + b(t) u_1\left(\frac{1}{2}; t\right)\right),$$

where we have used Abel's formula for the Wronskian which arises as a determinant in solving for c_1, c_5. We can obtain similar expressions for $c_2(t), c_6(t)$, the coefficients of $u_2(z; t), u_6(z; t)$ in the general solution for u at $z = 1$.

Our next step is to address the following question. Do the conditions on $a(t), b(t)$ [i.e. existence of the inverse Mellin transform in the strip $\tau_1 \leq \mathrm{Re}(t) \leq \tau_2$] imply the existence of the inverse Mellin transforms of

$$P(0; t) = c_1(t) u_1(0; t) + c_5(t) u_5(0, t),$$
$$P'(0; t) = \tfrac{1}{2}(c_1(t) u_1'(0; t) + c_5(t) u_5'(0, t)),$$

and of the corresponding expressions at $x = 1$? By linearity, this is true if and only if it is separately true for a and for b. We consider only the case $a \neq 0, b = 0$: The reverse case is similar. Then noting that $u_1(0; t) = 1$ and that $u_5(0; t) = u_5'(0; t) = 0$ [since $\mathrm{Re}(t) < 0$], we have

$$t_0(t) := P(0; t) = \frac{2^{1-2t}}{1 - t} u_5'\left(\frac{1}{2}; t\right) a(t) = \sum_{k=0}^{\infty} c_{k,l} \frac{2^{-2t}}{(t - 1)(t - 2) \cdots (t - k)} a(t),$$

where

$$c_{k,l} = (-1)^k \frac{2}{k!} (l + 1)_k (-l)_k, \qquad k = 0, 1, \ldots.$$

The rational function of t can be written as the sum of inverse linear terms using

$$\frac{1}{(t-1)(t-2)\cdots(t-k)} = \sum_{j=1}^{k} \frac{(-1)^{k-j}}{(k-j)!(j-1)!} \frac{1}{t-j}.$$

We now point out two properties of the Mellin transform that will allow us to perform the required inversion.

Lemma 6.3.—If

$$G(t) = \mathcal{M}[\gamma(r)](r \to t),$$

then

$$k^t G(t) = \mathcal{M}\left[\gamma\left(\frac{r}{k}\right)\right](r \to t)$$

for $0 < k \in \mathbb{R}$, provided all relevant integrals converge.

Proof.—This is immediate from the formula (6.4) for the inverse Mellin transform.

Lemma 6.4.—If

$$G(t) = \mathcal{M}[\gamma(r)](r \to t),$$

then

$$\frac{G(t)}{k+t} = \mathcal{M}\left[-r^k \int_0^r \frac{\gamma(y)}{y^{k+1}} dy\right](r \to t)$$

for $k \in \mathbb{R}$, provided all relevant integrals converge.

Proof.—This follows from Lemma 6.1.

Using these results, we could give a closed form expression for the inverse Mellin transform of $t_0(t)$ in terms of integrals of $\alpha(r)$, the inverse Mellin transform of $a(t)$. These will have the structure of a (terminating) series, whose kth term consists of a sum of k terms, the jth of which has the form

$$d_{j,k}\left(\frac{r}{4}\right)^{-j} \int_0^{r/4} \alpha(y) y^{j-1} dy \tag{6.13}$$

for some constants $d_{j,k}$. Note that the $r/4$ arises from the term 2^{-2t} in $t_0(t)$ and by using Lemma 6.3. Since there is only a finite number of terms present, the only questions regarding convergence relate to the existence of the integrals (6.13). This existence follows from the conditions laid down on the initial data function $\alpha(r)$. In exactly the same way, we can deduce the existence of the inverse Mellin transform of

$$t_1(t) = P(2; t) = u(1; t) = \frac{2^{1-2t}}{t-1} u_6'\left(\frac{1}{2}; t\right) a(t)$$

and of the terms corresponding to $t_0(t)$ and $t_1(t)$ that arise by taking $a(t) = 0, b(t) \neq 0$.

To conclude the discussion for Minkowski space-time, we take a slightly different point of view, where we treat $u(0; t) = c_1(t) u_1(0; t) + c_5(t) u_5(0; t)$ and $u'(0; t) = c_1(t) u_1'(0; t) + c_5(t) u_5'(0; t)$ as the fundamental data for the problem. Choosing $c_1(t), c_5(t)$ so that the inverse Mellin transforms of these terms exist on a strip of the complex plane implies (by the work done above) that the inverses of u and u' also exist on the spacelike hypersurface $z = 1/2$. Then verifying that the restriction on t implies finiteness of the basis solutions u_2, u_6 at $z = 1$, we can finally conclude that the inverse Mellin transforms of P and P' exist at the future null cone $z = 1$.

The thrust of this argument is that the "scattering coefficients" $c_i(t)$ *do not* affect resummability of the solution and, so, that finiteness of the basis solutions u_2, u_6 at $z = 1$ is a necessary and also *sufficient* condition for finiteness of the field ϕ at the future null cone.

Apart from actually verifying these assertions rigorously, the last step in our argument involves showing that the scenario summarized in the two preceding paragraphs carries over to Vaidya space-time. The crucial step is to consider the t dependence of pairs of fundamental solutions of the ODE at the two crucial singular points: (p_1^N, p_2^N) at the past null cone and (p_1^C, p_2^C) at the Cauchy horizon (future null cone). We consider first solutions at $x = 0$.

The standard form of (6.9) at $x = 0$ is

$$x^2 P'' + x q(x; t) P' + r(x; t) P = 0,$$

where

$$q(x; t) = \frac{2t - 2tx + (1 + 2t)\lambda x^2}{2 - x + \lambda x^2},$$

$$r(x; t) = -\frac{(t^2 - t - l(l+1) - t^2 \lambda x) x}{2 - x + \lambda x^2}.$$

Note that these are analytic at $x = 0$ and can be written in the form

$$q(x; t) = \sum_{k=0}^{\infty} q_k(t) x^k, \qquad r(x; t) = \sum_{k=0}^{\infty} r_k(t) x^k,$$

where convergence of the series is guaranteed for $x \in [0, x_c)$. The indicial equation is

$$I(\nu) := \nu(\nu - 1) + q_0 \nu + r_0 = \nu(\nu - 1 + t) = 0,$$

with solutions $\nu = 0, 1 - t \notin \mathbb{Z}$. Thus, we have the linearly independent solutions

$$p_1^N(x; t) = \sum_{k=0}^{\infty} a_k(t) x^k, \qquad a_0 = 1,$$

$$p_2^N(x; t) = |x|^{1-t} \sum_{k=0}^{\infty} b_k(t) x^k, \qquad b_0 = 1.$$

In each case, the radius of convergence of the series is at least x_c. Thus, this representation of a fundamental set of solutions is valid for all $x \in [0, x_c)$. The recurrence relations for the $a_k(t)$ are

$$a_k(t) = -\frac{1}{I(k)} \sum_{j=0}^{k-1} (jq_{k-j}(t) + r_{k-j}(t))a_j(t), \qquad k \geq 1,$$

and the recurrence relations for the b_k are

$$b_k(t) = -\frac{1}{I(k+1-t)} \sum_{j=0}^{k-1} ((j+1-t)q_{k-j}(t) + r_{k-j}(t))b_j(t), \quad k \geq 1.$$

We wish to determine the t dependence of these coefficients. Noting that the coefficients $q_k(t)$ and $r_k(t)$ are, respectively, linear and quadratic in t and that $I(\nu)$ is linear in t, it is straightforward to prove that

$$a_k(t) = \frac{\mathcal{P}_{2k}(t)}{\mathcal{P}_k(t)}, \qquad b_k = \frac{\mathcal{P}_{2k}(t)}{\mathcal{P}_k(t)},$$

where $\mathcal{P}_k(t)$ is used to represent an arbitrary polynomial of degree k which may be different in different formulas and within a single formula. Then division in the ring of polynomials yields

$$a_k = \mathcal{P}_k(t) + \frac{\mathcal{P}_{k-1}(t)}{\mathcal{P}_k(t)},$$

with a similar result for b_k. From here it is clear that we can write more explicitly

$$\begin{aligned} a_k &= \mathcal{P}_k(t) + \frac{\mathcal{P}_{k-1}(t)}{t(t+1)\cdots(t+k-1)} \\ &= \mathcal{P}_k(t) + \sum_{j=0}^{k-1} \frac{a_k^j}{t-j}, \\ b_k &= \mathcal{P}_k(t) + \frac{\mathcal{P}_{k-1}(t)}{(2-t)(3-t)\cdots(k+1-t)} \\ &= \mathcal{P}_k(t) + \sum_{j=0}^{k-1} \frac{b_k^j}{j+1-t}, \end{aligned}$$

where the a_k^j and b_k^j are independent of t.

Before proceeding, we note another property of the Mellin transform that will allow us to deal with the inversion of terms arising from the presence of the $\mathcal{P}_k(t)$ in the coefficients $a_k(t)$.

Lemma 6.5.—If

$$G(t) = \mathcal{M}[\gamma(r)](r \to t),$$

then, provided the relevant integrals converge,

$$s^n G(t) = \mathcal{M}[D^n \gamma(r)](r \to t),$$

where the differential operator is defined by $D = -r\frac{\partial}{\partial r}$.

Proof.—This follows by induction and by using Lemma 6.1.

There is an immediate corollary:

Corollary 6.1.—If

$$G(t) = \mathcal{M}[\gamma(r)](r \to t),$$

and $p(t) = p_0 + p_1 t + \cdots + p_n t^n$ is a polynomial in t, then

$$p(t)G(t) = \mathcal{M}[(p_0 + p_1 D + \cdots + p_n D^n)\gamma(r)](r \to t).$$

Note that the preceding analysis applies equally well to the case $\lambda = 0$, i.e. to the wave equation in Minkowski space-time. Thus, the solutions u_1, u_5 can equally well be written as combinations of p_1^N, p_2^N. We emphasize this as we now have solutions in Minkowski space-time that involve infinite series rather than polynomials.

The solutions p_i^N, $i = 1, 2$ "scatter" to a naturally arising fundamental set of linearly independent solutions p_1^C, p_2^C defined at $x = x_c$ (where $x_c = 2$ in Minkowski space-time). Indeed, by writing (6.9) in standard form at $x = x_c$ and determining the roots of the indicial equation thereat, we can write explicitly

$$p_1^C(x; t) = \sum_{n=0}^{\infty} A_k(t)(x - x_c)^n, \tag{6.14}$$

$$p_2^C(x; t) = |x - x_c|^{\nu_2} \sum_{n=0}^{\infty} B_k(t)(x - x_c)^n, \tag{6.15}$$

where

$$\nu_2 = \frac{(1 - 8\lambda + \sqrt{1 - 8\lambda})(1 - t)}{2 - 16\lambda} = 1 - t + O(\lambda)$$

and both series converge in some neighborhood of $x = x_c$. We note that $\text{Re}(\nu_2) > \text{Re}(1 - t)$, and so the earlier restriction $\text{Re}(t) < 0$ implies that $\text{Re}(\nu_2) > 1$, and so both solutions p_i^C, $i = 1, 2$ are finite at the Cauchy horizon. (This applies quite generally in self-similar collapse to a naked singularity when the dominant energy condition holds: see [9].) As these solutions can be written as *different* $\mathbb{C}$-linear combinations of p_i^N, $i = 1, 2$, this implies that the series representations of p_i^N, $i = 1, 2$ *must both con-*

verge at $x = x_c$. This is of crucial importance to our argument, as we can now determine the field $\phi(x, r)$ at the future null cone $x = x_c$ by carrying out (i.e. checking convergence of) the inverse Mellin transform of the sum of the convergent series

$$P(x_c; t) = a(t)p_1^N(x_c; t) + p_2^N(x_c; t). \tag{6.16}$$

Here $a(t), b(t)$ are functions whose inverse Mellin transforms $\alpha(r), \beta(r)$ exist for inversion contours lying in some strip of the complex plane. α, β constitute Cauchy data for ϕ and satisfy unspecified differentiability and integrability conditions.

As we have seen, this inverse Mellin transform will involve an infinite series of terms of the form

$$\alpha_k(r) = \mathcal{P}_k(D)\alpha(r) + \sum_{c_i} k_i r^{c_i} \int_0^r \frac{\alpha(y)}{y^{c_i-1}} dy,$$

where D is the differential operator introduced above. There will be similar terms arising from the contribution by $\beta(r)$, the inverse Mellin transform of $b(t)$. Existence of these individual terms can be guaranteed by imposing differentiability and integrability conditions on the initial data functions $\alpha(r), \beta(r)$. (Assuming differentiability of arbitrarily high order should not be a significant constraint here, as linearity would allow us to work in distributions or to use a density argument to generalize from analytic functions to more interesting spaces. We also note that the sign of the c_i in the α_k above will not be of particular relevance, as for either case of this sign, either the multiplicative prefactor or the divisor in the integrand will have a mollifying effect.) Hence, our problem boils down to deducing convergence of series of the form $\sum_{k=0}^{\infty} \alpha_k(r)$. Here is another crucial point: Our analysis of the problem in Minkowski space-time from a slightly different point of view *guarantees* that such series must converge in the case $\lambda = 0$.

Now the coefficients of these series are generated by the coefficients of the series in (6.16). The only difference between Minkowski space-time and Vaidya space-time is the value of λ [$\lambda = 0$ and $\lambda \in (0, 1/8)$, respectively]. For $\lambda = 0$, the coefficients of the series of complex numbers (6.16) guarantees convergence in $\mathbb{C}$: These coefficients generate coefficients of an infinite series $\sum(c_k\alpha_k(r) + d_k\beta_k(r))$ in a certain unspecified function space—call it $\mathcal{F}$—that, as we have argued, guarantee convergence in this space. *Convergence of this series depends only on the coefficients inherited from* (6.16). To conclude our argument, we maintain that this pattern is repeated when $\lambda > 0$: The coefficients of the $\mathbb{C}$-convergent series (6.16) generate coefficients of series of functions in a certain unspecified function space that must then also converge in this function space.

We conclude this section by summarizing the argument. Restricting the values of t to allow only a finite flux at the regular axis and at the past null cone is sufficient in the present case to guarantee finiteness of the Mellin transform $P(x; t)$ of the field ϕ at the future null cone. To determine if ϕ itself is finite at the future null cone, we must calculate the inverse Mellin transform of P. The analytic form of the line element of Vaidya space-time allows us to determine the general form of this inverse: It involves an infinite series of finite sums of derivatives and integrals of the initial data for ϕ. As these finite sums converge, convergence of the full series of functions in $\mathcal{F}$ depends only on the constant coefficients of the series: These coefficients are generated by the coefficients in the convergent $\mathbb{C}$-series representation of $P(x; t)$ at $x = x_c$. In Minkwoski space-time, the coefficients $\{p_k(0)\}_{k=0}^{\infty}$ of the series $P(x_c; t)$ of complex numbers produces a convergent series representation for $\phi|_{x=x_c}$ in $\mathcal{F}$. We claim that in Vaidya space-time, the same will happen: The coefficients $\{p_k(\lambda)\}_{k=0}^{\infty}$ of the series $P(x_c; t)$ of complex numbers produce a convergent series representation for $\phi|_{x=x_c}$ in $\mathcal{F}$.

This last paragraph constitutes our argument that, in order to determine finiteness of the field at the Cauchy horizon, it is sufficient to check finiteness of the modes thereat. There clearly remains a good deal to prove, but the analysis above suggests a way of doing this: The main thing that requires checking is the convergence of the finite sums $\alpha_k(r)$ and of the overall series $\sum \alpha_k$. Finally, we note that, although this argument has been presented for only the wave equation, it should generalize to any set of linear equations, as, for example, the perturbation equations considered in this paper.

VII. CONCLUSIONS

We have considered metric and matter perturbations of all angular modes falling on the Cauchy horizon formed by the naked singularity arising from the collapse of a self-similar null dust. There is no class of perturbation which satisfies the initial conditions and gives a divergent flux on the Cauchy horizon; thus, there is no blue-sheet instability as is seen in, for example, the Reissner-Nordström naked singularity. (We have shown rigorously that the modes are finite at the Cauchy horizon, and given a plausibility argument that the full perturbation itself, found by resumming over the modes, is finite.) Interestingly, the second future similarity horizon of the self-similar null dust space-time *is* unstable for perturbations with multipole modes $l \geq 2$.

The question of uniqueness of the perturbation to the future of the Cauchy horizon has not been addressed, but this should not affect the divergence encountered at the SFSH. If we consider the wave equation as studied in Sec. VI as a paradigm, we note that the solution (6.15) and its derivative vanish at the Cauchy horizon for the allowed range of t [$\mathrm{Re}(t) < 0$]. Thus, an arbitrary additional amount of p_2^C could be added to the solution for $x > x_c$ while preserving continuity and differentiability. However, in Minkowski space-time a selection procedure tells us the correct addition to make, based on reflection through the (still) regular axis. It is not clear that one can apply a similar argument in Vaidya space-time, where one encounters a singularity at $r = 0$ to the future of the future null cone (Cauchy horizon): Indeed, this is the central problem caused by a naked singularity, and the very issue that the cosmic censor seeks to render irrelevant. Whether establishing uniqueness is possible or not, some amount of both independent solutions p_i^C, $i = 1, 2$ will persist in $x > x_c$, leading to divergence at $x = x_e$.

It is worrysome from the point of view of the cosmic censorship hypothesis that this naked singularity persists. However, this worry is tempered by the fact that an instability is encountered at the SFSH of the space-time; the naked singularity survives only for a finite time. It is then of interest to consider the lifetime of this naked singularity and what effects it might have on the space-time. It is also of interest to know how generic this short-lived stability of the naked singularity is. In the terminology of Carr and Gundlach [17], this second future similarity horizon is a ‘‘splash,’’ whereas the other similarity horizons are ‘‘fans.’’ We speculate that finiteness of the flux of perturbations on a fan-type similarity horizon, and divergence on a splash-type similarity horizon, may be a general feature of self-similar spherically symmetric space-times; the issue is currently being studied.

The results here also have some bearing on the issue of stability in critical collapse. In perfect fluid collapse, the critical space-times are continuously self-similar, i.e. admit a homothetic killing vector field (see e.g. [29]). By definition, the critical space-times possess a single (spherically symmetric) unstable mode for perturbations injected along the homothetic surfaces $x =$ constant and followed up to the singularity. In our variables, this corresponds to a mode of the form $r^s Q(x)$, with the limit $r \to 0$ taken along $x =$ constant and with $x < x_c$. Then $\mathrm{Re}(s) < 0$ corresponds to instability. In the present paper, the limiting behavior of the perturbation has been studied in the approach to the first and second future similarity horizons in Vaidya space-time. Carrying out a similar calculation for the critical fluid space-times may reveal very different stability properties to that shown by critical collapse studies.

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APPENDIX A: PERTURBATION EQUATIONS IN TERMS OF GAUGE INVARIANTS

We give here the full set of perturbation equations for the gauge invariants defined in (3.14). Note that not all equations apply for $l = 0, 1$ modes.

$$
\begin{aligned}
&2\tilde{v}^C(k_{AB|C} - k_{CA|B} - k_{CB|A} + 2\tilde{g}_{AB}k_{CD}{}^{|D}) - \left(\frac{l(l+1)}{r^2} + \tilde{G}_C{}^C + \tilde{G}_a{}^a + 2\tilde{\mathcal{R}}\right)k_{AB} \\
&+\tilde{g}_{AB}\left(\frac{l(l+1)}{r^2} + \frac{1}{2}(\tilde{G}_C{}^C + \tilde{G}_a{}^a) + \tilde{\mathcal{R}}\right)k_D{}^D - 2\tilde{g}_{AB}\tilde{v}^C k_D{}^D{}_{|C} \\
&+\tilde{g}_{AB}(2\tilde{v}^{C|D} + 4\tilde{v}^C\tilde{v}^D - \tilde{G}^{CD})k_{CD} - \tilde{g}_{AB}\left(2k,_C{}^{|C} + 6\tilde{v}^C k,_C - \frac{(l-1)(l+2)}{r^2}k\right) \\
&\qquad + 2(\tilde{v}_A k,_B + \tilde{v}_B k,_A + k,_{A|B}) = -16\pi T_{AB} \qquad (l \geq 0), \qquad \text{(A1a)}
\end{aligned}
$$

$$
\begin{aligned}
&-(k,_C{}^{|C} + 2\tilde{v}^C k,_C + \tilde{G}_a{}^a k) + (k_{CD}{}^{|C|D} + 2\tilde{v}^C k_{CD}{}^{|D} + 2(\tilde{v}^{C|D} + \tilde{v}^C\tilde{v}^D)k_{CD}) \\
&\qquad -\left(k_C{}^C{}_{|D}{}^{|D} + \tilde{\mathcal{R}}k_C{}^C - \frac{l(l+1)}{2r^2}k_C{}^C\right) = -16\pi T^1 \qquad (l \geq 0), \qquad \text{(A1b)}
\end{aligned}
$$

$$
k,_A - k_{AC}{}^{|C} + k_C{}^C{}_{|A} - \tilde{v}_A k_C{}^C = -16\pi T_A \qquad (l \geq 1), \qquad \text{(A1c)}
$$

$$
k_A{}^A = -16\pi T^2 \qquad (l \geq 2). \qquad \text{(A1d)}
$$

Here $\tilde{v}^A = r^{|A}/r$, and $\tilde{G}_{\mu\nu}$ is the Einstein tensor of the background space-time. $\tilde{\mathcal{R}}$ is the Gaussian curvature of M^2, the manifold spanned by the time and radial coordinates, and, thus, equals half the Ricci scalar of M^2.

APPENDIX B: FIRST ORDER SYSTEM OF PERTURBATION EQUATIONS

The perturbation equations give rise to the first order system $Y' = M(x)Y$, where $Y = (A, D, K, G)^T$ and the coefficients of M are

$$
\begin{aligned}
M_{11} &= \frac{4 - 4x + 2(1+\lambda)x^2 - 3\lambda x^3 + \lambda^2 x^4 + 2s^2(2 - x + \lambda x^2) + s(-4 + 2x - \lambda x^3 + \lambda^2 x^4)}{x(2s + \lambda x^2)(2 - x + \lambda x^2)}, \\
M_{12} &= \frac{4(-1+x) - 2s^2 x(-1+x\lambda)^2 + l(2 + x(-1+x\lambda))(2 + x(-1+x\lambda)(2 + x(-1+x\lambda)))}{x^2(-1+x\lambda)(2s + x^2\lambda)(2 + x(-1+x\lambda))} \\
&\quad + \frac{l^2(2 + x(-1+x\lambda))(2 + x(-1+x\lambda)(2 + x(-1+x\lambda)))}{x^2(-1+x\lambda)(2s + x^2\lambda)(2 + x(-1+x\lambda))} \\
&\quad + \frac{-(s(-1+x\lambda)(4 + x(-2 + x(2 + \lambda(-2 + x(-5 + 3x\lambda))))))}{x^2(-1+x\lambda)(2s + x^2\lambda)(2 + x(-1+x\lambda))} \\
&\quad + \frac{x(8\lambda + x(-2 + \lambda(-12 + x(9 + \lambda(8 + 7x(-2 + x\lambda))))))}{x^2(-1+x\lambda)(2s + x^2\lambda)(2 + x(-1+x\lambda))}, \\
M_{13} &= \frac{(2 + (-2 + l + l^2)x)(2 + (-2+x)x) - x^2(-6 + (9 + 2l(1+l)(-1+x) - 4x)x)\lambda + x^4(3 + (-2 + l + l^2)x)\lambda^2}{x^2(2s + x^2\lambda)(2 + x(-1+x\lambda))} \\
&\quad + \frac{-2s^2(2 + x(-1+x\lambda)) + sx(2 - x(2 + \lambda(4 + x(-5 + 3x\lambda))))}{x^2(2s + x^2\lambda)(2 + x(-1+x\lambda))}, \\
M_{14} &= \frac{2\lambda(-4 + 2s(2 + x(-1+x\lambda)) - x(-2+x\lambda)(2 + x(-1+x\lambda)))}{x(2s + x^2\lambda)(2 + x(-1+x\lambda))}, \\
M_{21} &= \frac{2x(-1+x\lambda)}{(2s + x^2\lambda)(2 + x(-1+x\lambda))}, \\
M_{22} &= \frac{2 + l(1+l)(-2+x) - ((-1+s)(-6 + 4s - x) + 2l(-1+x) + 2l^2(-1+x))x\lambda}{(-1+x\lambda)(2s + x^2\lambda)(2 + x(-1+x\lambda))} \\
&\quad + \frac{x^2(6 + 2s(-3 + s - x) + (3 + l + l^2)x)\lambda^2 + (-2+s)x^4\lambda^3 + 2s(-2 + s + 2\lambda)}{(-1+x\lambda)(2s + x^2\lambda)(2 + x(-1+x\lambda))}, \\
M_{23} &= -\frac{(-2 + 2s - (-2 + l + l^2)x)(-1+x\lambda)}{(2s + x^2\lambda)(2 + x(-1+x\lambda))}, \\
M_{24} &= \frac{-4x\lambda(-1+x\lambda)}{(2s + x^2\lambda)(2 + x(-1+x\lambda))}, \\
M_{31} &= \frac{-2}{2s + x^2\lambda}, \\
M_{32} &= \frac{2 - 2s + l(1+l)(-2+x) - x(4 - 2s + l(1+l)x)\lambda}{x(-1+x\lambda)(2s + x^2\lambda)}, \\
M_{33} &= \frac{-2 + 2s^2 + sx^2\lambda - x(-2 + l + l^2 + x\lambda)}{2sx + x^3\lambda}, \\
M_{34} &= \frac{4\lambda}{2s + x^2\lambda}, \\
M_{41} &= 0, \\
M_{42} &= \frac{1}{-x + x^2\lambda}, \\
M_{43} &= 0, \\
M_{44} &= \frac{s}{x}.
\end{aligned}
\tag{B1}
$$

APPENDIX C: METHODS FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS WITH SINGULAR POINTS

For a first order system $Y' = M(x)Y$, we define p as the least number such that the system can be written near $x = 0$ as

$$Y' = \frac{1}{x^p}\left(J + \sum_{m=1}^{\infty} A_m x^m\right)Y, \tag{C1}$$

and near $x = \infty$ as

$$Y' = -x^{p-2}\left(J + \sum_{m=1}^{\infty} A_m x^{-m}\right)Y, \tag{C2}$$

with $J \neq 0$ and constant.

We classify singular points and describe the solution as:

***p* = 0, regular point.**—The solutions of a system of differential equations are at least as regular as the coefficients of the system of differential equations. For the purposes of this paper, it is enough to know that solutions near a regular point are themselves regular, as none of the surfaces of interest are regular points.

***p* = 1, regular singular point.**—Also known as a simple singularity or a singularity of the first kind. Here we distinguish solutions depending on whether the eigenvalues of J given above differ by an integer or not. If they do not, we apply Theorem C.1 immediately. If they do, we reduce those eigenvalues individually until they are equal using Theorem C.2 and then apply Theorem C.1 (see [28]).

Theorem C.1.—In the system (C1), if J has eigenvalues which do not differ by positive integers, then, in a disk around $x = 0$ not containing another singular point, (C1) has a fundamental matrix Φ of the form

$$\Phi = Px^J, \quad \text{where } P(x) = \sum_{m=0}^{\infty} x^m P_m, \qquad P_0 = I. \tag{C3}$$

Theorem C.2.—Let the distinct eigenvalues of J in (C1) be (disregarding multiplicity) $\mu_1, \ldots \mu_k$ ($k \leq N$, where N is the order of the system). There is a matrix $V(x)$ such that $Y = VQ$ transforms (C1) into

$$Q' = \frac{1}{x}\left(\hat{J} + \sum_{m=1}^{\infty} \hat{A}_m x^m\right)Q, \tag{C4}$$

where $\hat{J}$ has eigenvalues $\mu_1 - 1, \mu_2, \ldots, \mu_k$. V is given by a diagonal matrix with entries 1, except where the eigenvalue(s) to be reduced occurs, where there is an x.

***p* ≥ 2, irregular singular point.**—Also known as a nonsimple singularity or a singularity of the second kind. Here we distinguish solutions depending on whether one can diagonalize J given in (C2). If J has distinct eigenvalues, then J is diagonalizable and we apply Theorem C.3. If J has multiple eigenvalues and J can only be reduced to Jordan normal form, then we apply Theorem C.4 to remove off-diagonal terms (see [30]). When the eigenvalues are repeated zeros, this has the effect of reducing the order of the singularity, as happens in this paper at the threshold (see Sec. V).

There is a class of problems in between: Sometimes a matrix has multiple eigenvalues and yet can still be diagonalized. In this case, there is a straightforward theorem given by Ref. [30] if $A_1 = 0$ (as is sometimes the case when a high order equation is written as a first order system). If not, there is a very cumbersome solution given by Ref. [28].

Theorem C.3.—Let J have distinct eigenvalues $\mu_1, \ldots, \mu_N$. Then (C1) has a fundamental matrix

$$\Phi = Px^R e^H, \quad \text{where } P(x) = \sum_{m=0}^{\infty} x^m P_m, \qquad P_0 = I, \tag{C5}$$

R is a diagonal matrix of complex constants, and H is a matrix polynomial ($r = p - 2$)

$$H = \frac{x^{r+1}}{r+1}H_0 + \frac{x^r}{r}H_1 + \ldots + xH_r, \qquad H_i = \mathrm{diag}(\mu_1^{(i)}, \ldots, \mu_N^{(i)}), \qquad \mu_j^{(0)} = \mu_j. \tag{C6}$$

Theorem C.4.—For brevity's sake, we give the theorem only for $p = 2$ as this is the case that arises in this paper. We transform J to its Jordan normal form $\hat{J}$ and write the blocks of $\hat{J}$ as $\mu I + \rho E$, where E is the matrix with 1's along its superdiagonal and zeros elsewhere. For each block of $\hat{J}$, define the matrices

$$D = \mathrm{diag}(1, \rho x, \ldots, (\rho x)^{N-1}), \qquad B = \begin{pmatrix} 1 & 1 & 1/2! & \ldots & 1/(N-1)! \\ 0 & 1 & 1 & \ldots & 1/(N-2)! \\ & & \ddots & \ddots & \vdots \\ & & & & 1 \end{pmatrix}, \tag{C7}$$

where N is the order of the system. Then the transformation $Y = D^{-1}BW$ gives the system

$$W' = \left[\mu I + D'D^{-1} + B^{-1}D\left(\sum_{m=1}^{\infty} A_m x^{-m}\right)D^{-1}B\right]W, \tag{C8}$$

and the leading order coefficient matrix has had its off-diagonal terms removed.

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