Naked singularities in cylindrical collapse of counterrotating dust shells

Brien C. Nolan*
School of Mathematical Sciences, Dublin City University, Glasnevin, Dublin 9, Ireland
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Solutions describing the gravitational collapse of asymptotically flat cylindrical and prolate shells of (null) dust are shown to admit globally naked singularities.

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We consider two models of cylindrical collapse, both based on the cylindrically symmetric line element

$$ds^2 = -\exp[-\alpha(v)]du dv + dz^2 + r^2(u,v) d\phi^2. \quad (1)$$

u,v are null coordinates and we take $r=(v-u)/2 \geqslant 0$. Then u,v label outgoing and ingoing null hypersurfaces, respectively; we take both u and v to increase into the future. The energy-momentum tensor is in the form of null dust with the Ricci tensor given by

$$R_{ab} = -\frac{\alpha'(v)}{4r} k_a k_b, \quad k_a = \nabla_a v.$$

Then the strong, weak and dominant energy conditions are all equivalent to

$$\alpha' \leqslant 0, \quad (2)$$

which we assume henceforth. The matter content of the space-time may be described as infalling null dust. There is a curvature singularity along $r=0$. Since the solution is Petrov type N with a pure radiation energy-momentum tensor, all curvature invariants vanish. It has been shown in [1] that this is a parallel propagated curvature singularity. Additionally, scalars such as $R_{ab}w^a w^b$ diverge at $r=0$ for any unit time-like vector w^a. At fixed z on the singularity, we have

$$ds^2 = -e^{-\alpha(v)} dv^2,$$

so the singularity is time-like.

The causal nature of the singularity is further elucidated by studying the null geodesics in the 2-space $z=$const, $\phi=$const. The geodesic equations yield (for future-pointing geodesics)

$$\dot{v} = k^2 \exp[\alpha(v)], \quad (3)$$

$$u = u_0 + u_1 \tau, \quad (4)$$

where the overdot represents differentiation with respect to the affine parameter τ, $u_1 \geqslant 0$ and k are constants. The null condition is $u_1 k^2 = 0$; both constants cannot vanish simultaneously.

If $k=0$, then $u_1>0$. We see that $u \to -\infty, r \to +\infty$ as $\tau \to -\infty$. As τ increases, r decreases to 0 at parameter time $\tau = (v_0 - u_0)/u_1 > 0$. So these geodesics are past complete and terminate in the future at the singularity $r=0$ in finite parameter time.

If $u_1=0$, then $k^2>0$. We calculate

$$\ddot{v} = \alpha'(v)\dot{v}^2,$$

so $\tau \to v(\tau)$ is increasing and concave [according to the energy condition (2)]. Hence v cannot reach $+\infty$ in finite parameter time. Either $v \to \infty$ as $\tau \to \infty$, or v is bounded above. In the latter case, monotonicity implies that v has a limit v_F as τ increases. Then either $v \to v_F$ as $\tau \to \infty$, or $v = v_F$ at some $\tau = \tau_F$ with $\dot{v}(\tau_F) = 0$. This implies $\alpha(v_F) = -\infty$, and by monotonicity, $\alpha'(v_F) = -\infty$. In this case, there is a singularity along $v = v_F$. In the other cases, the geodesics are future complete. Future null infinity $\mathcal{J}^+$ is at $v = v_F$ or $v = +\infty$, depending on whether or not v is bounded above as $\tau \to \infty$. In the former case, r has a finite limit at $\mathcal{J}^+$, which will be different along different geodesics.

v decreases as τ decreases. Either r decreases to 0, or the geodesic meets a singularity at $v = v_P$ whereat $\dot{v} \to \infty$. This occurs if and only if $\alpha(v_P) = +\infty$. Again, monotonicity implies that $\alpha'(v_P) = -\infty$.

So there are four different possible global structures, depending on whether or not there exist values v_P, v_F for which $\alpha(v_{P,F}) = +\infty, -\infty$. The conformal diagrams are given in Figs. 1–4. Note that when $\mathcal{J}^\pm$ exists and corresponds to $r \to \infty$, the space-time is asymptotically flat at that surface in the following sense: all superenergy terms [2] such as $R_{ab}w^a w^b, C_{abcd}w^a w^b w^c w^d$ (w^a null or unit time-like) vanish in the limit as the surface is approached *for fixed values of z*. Along any time-like geodesic, parallel propagated components of the Riemann tensor will also vanish in

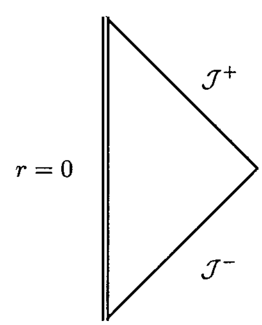

FIG. 1. Conformal diagram for the case when α is bounded on $\mathbb{R}$. v lies in the range $(-\infty, +\infty)$. Singularities are indicated with a double line.

*Electronic address: brien.nolan@dcu.ie

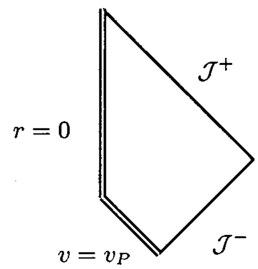

FIG. 2. Conformal diagram for the case when α is bounded below on $\mathbb{R}$, but there exists $v_P \in \mathbb{R}$ such that $\alpha \to +\infty$ as $v \downarrow v_P$. v lies in the range $(v_P, +\infty)$.

this limit [1]. Since the space-times represent infinite cylinders, they cannot be asymptotically flat in directions along which z becomes infinite. This feature can be removed; see below.

We note that to obtain space-times filled with *outgoing* null dust, we define $U=-v, V=-u, \beta(U)=\alpha(v)$ and redesignate the future direction as the reverse of that used above. The line element is

$$ds^2 = -\exp[-\beta(U)]dUdV + dz^2 + R^2 d\phi^2, \qquad (5)$$

where $R=(V-U)/2$. The Ricci tensor is given by

$$R_{ab} = \frac{\beta'(U)}{4r} \nabla_a U \nabla_b U,$$

and the energy condition now reads

$$\beta'(U) \geq 0. \qquad (6)$$

The relevant conformal diagrams are obtained by inverting those of Figs. 1–4.

In order to make the connection with cosmic censorship, i.e. to study collapse for an initially regular configuration, we replace the interior of a past light cone of a point on $r=0$ with a portion of Minkowski space. The resulting space-times are asymptotically flat and evolve from regular initial data. They may be considered the cylindrical version of Vaidya space-time. Writing the line element of Minkowski space-time in the cylindrical form

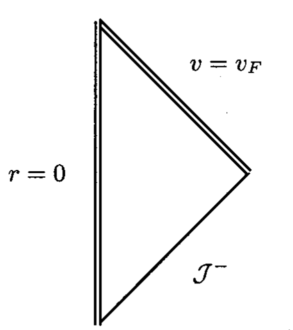

FIG. 3. Conformal diagram for the case when α is bounded above on $\mathbb{R}$, but there exists $v_F \in \mathbb{R}$ such that $\alpha \to -\infty$ as $v \uparrow v_F$. v lies in the range $(-\infty, v_F)$.

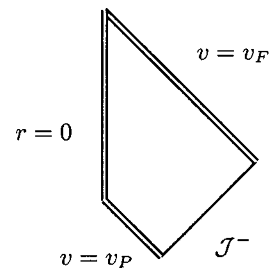

FIG. 4. Conformal diagram for the case when there exists $v_P, v_F \in \mathbb{R}$ such that $\alpha \to +\infty$ as $v \downarrow v_P$ and $\alpha \to -\infty$ as $v \uparrow v_F$. v lies in the range (v_P, v_F).

$$ds^2 = -d\bar{u}d\bar{v} + d\bar{z}^2 + \frac{1}{4}(\bar{v}-\bar{u})^2 d\bar{\phi}^2,$$

we obtain a smooth matching across an ingoing null hypersurface by taking $\bar{x}^a = x^a$, matching across $v=0$ (a translation may be required to guarantee that 0 lies in the domain of α and that $v_P < 0 < v_F$ if such exist) and using the boundary condition $\alpha(0)=0$ [3]. We paste the portion $v<0$ of Minkowski space-time to the portion $v \geq 0$ of the space-time with line element (1).

Two possibilities arise and are shown in Figs. 5 and 6, respectively. Both contain locally naked singularities. Figure 5 corresponds to space-times modeling the gravitational collapse of cylindrical null dust which are asymptotically flat at fixed z and which arise from regular initial data. The singularity in this case is globally naked.

A slightly different model has been studied in [4,5] and [6]. Here, the exterior region is taken to be filled with *outgoing* null dust, and is matched across a time-like shell of (ordinary) dust to a flat interior. One can interpret this as a collapsing cylindrical shell emitting both gravitational radiation (the solution is Petrov type N) and massless partilces. It was shown in [5] and [6] that this shell can be constructed in such a manner that the collapse proceeds to $r=0$ without any trapped or marginally trapped surfaces appearing either on the shell or in the exterior geometry. This leads one to suspect that the resulting singularity may be naked; the present

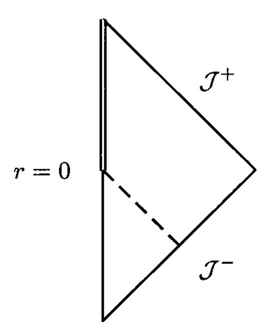

FIG. 5. Conformal diagram in the u-v plane for the case when α is bounded below on $\mathbb{R}$. The dashed line is $v=0$; the portion to the past of $v=0$ is flat.

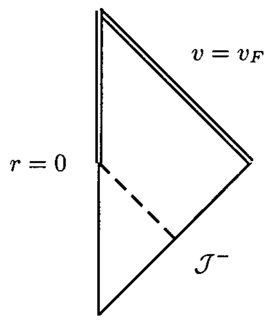

FIG. 6. Conformal diagram for the case when $\alpha \to +\infty$ as $v \uparrow v_F > 0$. The dashed line is $v=0$; the portion to the past of $v=0$ is flat.

results confirm this. The shell is constructed as follows. The metric inside the shell is taken to be flat:

$$ds^2 = -dt^2 + dr^2 + dz^2 + r^2 d\phi^2.$$

The exterior geometry is given by Eq. (5). Define T by $U = T - R$, $V = T + R$, and take the shell to be at $R = R_0(T) = r_0(t)$. Demanding continuity of the metric yields a relation for t in terms of T. The surface energy-momentum tensor of the shell generally takes the form of an imperfect fluid; following [4], we impose the condition that there are stresses only in the azimuthal direction. This choice of matter admits the interpretation of a thin shell of dust particles in which counter-rotation of one-half of the particles results in zero net angular momentum, and is referred to as a "shell of counter-rotating dust" [4,6,7]. The field equations for the shell yield the following differential equation for $R_0(T)$ which includes the terms $\beta(U_0), \beta'(U_0)$, where $U_0 = T - R_0(T)$:

$$R_0'' = (1 - R_0'^2)\left\{ \frac{\Delta^{1/2}}{R_0} + \frac{R_0' - \Delta^{1/2}}{\Delta^{1/2} - 1}\frac{\beta'}{2}(1 - R_0') \right\}, \qquad (7)$$

where

$$\Delta = R_0'^2 + e^{-\beta}(1 - R_0'^2).$$

(An erroneous version of this equation was given in [4] and was corrected in [6].)

The solutions studied in [4,5] and [6] are similar to the following simple example. For our present purposes, it suffices to show that there exists a function $\beta(U)$ and a solution $R_0(T)$ of Eq. (7) with the following properties: $R_0(T)$ decreases monotonically from a finite positive value to $R_0 = 0$ in finite proper time; $|R_0'| < 1$; $\beta'(U) \geqslant 0$; $\Delta > 0$ along the solution; a suitable energy condition is satisfied by the shell. In order to do this, we make the choice $R_0(T) = a - bT$ where a, $b > 0$ and $b < 1$. Then Eq. (7) may be written as the following equation for $y(\tau) \coloneqq \beta(T - R_0(T))$ where $\tau = -\ln|a - bT|$:

$$\frac{dy}{d\tau} = \frac{2}{b}\frac{\Delta - \Delta^{1/2}}{b + \Delta^{1/2}}. \qquad (8)$$

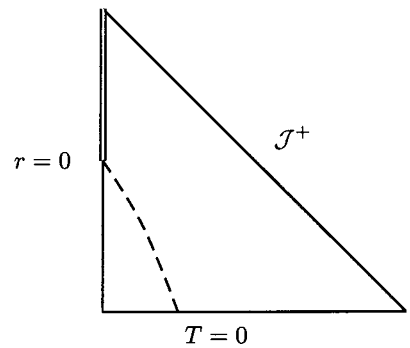

FIG. 7. Conformal diagram in the U-V plane for the collapse of a shell of counter-rotating dust particles. The dashed curve represents the shell; space-time is flat in the interior of the shell. The exterior corresponds to the time reversal of Fig. 1 or 3.

The right-hand side here is a C^1 function of y, and so this equation admits a unique solution through any point (τ_0, y_0). If we choose data with $y < 0$, we see that y increases towards $y = 0$ asymptotically as $\tau \to \infty$. Such a solution cannot reach $y = 0$ in a finite time since $y \equiv 0$ is the unique solution through any point $(\tau, y) = (\tau_0, 0)$. In other words, there exist solutions of Eq. (7) for which R_0 decreases to zero in finite time $T = a/b$, with β satisfying $\beta_0 \leqslant \beta \leqslant 0$ on $T \in [0, a/b]$. The shell has density ρ and azimuthal stress p_ϕ satisfying

$$\rho = p_\phi = e^{\beta/2}(1 - R_0'^2)^{-1/2} R_0^{-1}(\Delta^{1/2} - 1),$$

which is clearly positive for the solution constructed, and the other conditions mentioned above are clearly satisfied. We note that Gleiser's general analysis shows that there are numerous possibilities which lead to the outcome exemplified by this case [6].

To see that the resulting singularity is globally naked, we note that since $\beta(U_0) \leqslant 0$ during the collapse, β cannot diverge to $+\infty$ before the shell undergoes complete collapse and hence the conformal diagram for the matched space-time must be of the form shown in either Fig. 7 or Fig. 8. When there is no future singularity U_F, i.e. when Fig. 7 applies, the space-times give more examples of cylindrical collapse resulting in a globally naked singularity. The space-time is asymptotically flat at $\mathcal{J}^+$ ($r \to \infty$ at fixed U) for fixed z.

The problem of the lack of asymptotic flatness at large z can be dealt with as follows. The space-time with line element (1) can be matched smoothly across any surface $z = \text{const}$ with the spherically symmetric space-time with the line element

$$ds^2 = -\exp[-\alpha(v)]du\,dv + r^2(u,v)(d\theta^2 + \sin^2\theta\, d\phi^2), \qquad (9)$$

provided the matching is done across the equator $\theta = \pi/2$ of the spherically symmetric space-time. Doing this at two different values of z and choosing normals in the appropriate directions corresponds to replacing the infinite cylinder with a finite cylinder bounded by hemispherical caps [8]. The Ricci tensor of Eq. (9) has nonvanishing components

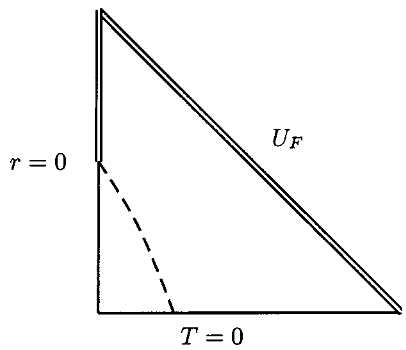

FIG. 8. Conformal diagram in the U-V plane for collapse of a shell of counter-rotating dust particles. The dashed curve represent the shell; space-time is flat in the interior of the shell. The exterior corresponds to the time reversal of Fig. 2 or 4 with $U_F=-v_P$. The possible past singularity is ignored.

$$R_{vv}=-\frac{\alpha'}{4r}, \quad R_{\theta\theta}=\csc^2\theta R_{\phi\phi}=1-e^{\alpha}.$$

The models of Figs. 5 and 6 used the boundary condition $\alpha(0)=0$, and the energy conditions for the cylinder gave $\alpha'(v)\leqslant 0$. Using these, it is straightforward to show that all energy conditions (weak, strong, dominant) are satisfied in the hemispheres. A similar conclusion holds for the off-shell regions of the models of Figs. 7 and 8 using the condition $\beta\leqslant 0$ found above. The collapsing shell of matter now has a finite cylindrical body and hemispherical caps. We take its equation of motion to be given by the same equation $r=r_0[T(\tau)]$ in both the cylindrical and spherical regions (this is in line with our "recycling" of the cylindrical metric functions in the spherical regions). It is straightforward to show that the surface energy-momentum tensor of the shell in the cylindrical region is given by [4–6]

$$S_{ab}=\rho\delta_a^\tau\delta_b^\tau+p_z\delta_a^z\delta_b^z+r_0^2(p_z+\rho)\delta_a^\phi\delta_b^\phi.$$

Then it transpires that the surface energy-momentum tensor in the spherical region is

$$S_{ab}=2\rho\delta_a^\tau\delta_b^\tau+p_zr_0^2\delta_a^\theta\delta_b^\theta+p_zr_0^2\sin^2\theta\delta_a^\phi\delta_b^\phi.$$

Imposing the "counterrotating dust" condition $p_z=0$ shows that the collapsing shell is composed of dust with no tangential stresses in the hemispheres. In particular, the energy conditions in the hemispheres are inherited from the energy conditions in the cylindrical region. Asymptotic flatness in the hemispherical regions is guaranteed by asymptotic flatness at fixed z in the cylindrical region. The properties of the singularities which we have examined depends only on the geometry of the Lorentzian 2-space $z=$const, $\phi=$const of Eq. (1). Since this 2-space is identical to the 2-space $\theta=$const, $\phi=$const of Eq. (9), the singularity structure in the cylindrical region applies throughout the entire space-time.

Finally, we note the unsuitability of these models for studying the hoop conjecture. Since the hemispheres may be attached at any pair of values of z, the resulting object may have an arbitrary degree of prolateness, from zero (spherical) to infinity (infinite cylinder). Thus the lack of occurrence of an apparent horizon (i.e. trapped surfaces) relates only to the particular choice of metric functions and not to the asphericity of the collapsing object.

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