

Einstein-Rosen waves and the self-similarity hypothesis in cylindrical symmetryTomohiro Harada,^{1,*} Ken-ichi Nakao,^{2,†} and Brien C. Nolan^{3,‡}¹*Department of Physics, Rikkyo University, Toshima, Tokyo 171-8501, Japan*²*Department of Physics, Osaka City University, Osaka 558-8585, Japan*³*School of Mathematical Sciences, Dublin City University, Glasnevin, Dublin 9, Ireland*

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The self-similarity hypothesis claims that in classical general relativity, spherically symmetric solutions may naturally evolve to a self-similar form in certain circumstances. In this context, the validity of the corresponding hypothesis in nonspherical geometry is very interesting as there may exist gravitational waves. We investigate self-similar vacuum solutions to the Einstein equation in the so-called whole-cylinder symmetry. We find that those solutions are reduced to part of the Minkowski spacetime with a regular or conically singular axis and with trivial or nontrivial topology if the homothetic vector is orthogonal to the cylinders of symmetry. These solutions are analogous to the Milne universe, but only in the direction parallel to the axis. Using these solutions, we discuss the nonuniqueness (and nonvanishing nature) of C energy and the existence of a cylindrical trapping horizon in Minkowski spacetime. Then, as we generalize the analysis, we find a two-parameter family of self-similar vacuum solutions, where the homothetic vector is not orthogonal to the cylinders in general. The family includes the Minkowski, the Kasner, and the cylindrical Milne solutions. The obtained solutions describe the interior to the exploding (imploding) shell of gravitational waves or the collapse (explosion) of gravitational waves involving singularities from nonsingular initial data in general. Since recent numerical simulations strongly suggest that one of these solutions may describe the asymptotic behavior of gravitational waves from the collapse of a dust cylinder, this means that the self-similarity hypothesis is naturally generalized to cylindrical symmetry.

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I. INTRODUCTION

In studying nonspherical vacuum gravitational fields, cylindrically symmetric systems have the advantages of being essentially $1 + 1$ dimensional, and, unlike the spherical case, possessing a dynamical degree of freedom in gravity, i.e., gravitational waves. Solutions in this system are discussed in [1] and are called Einstein-Rosen waves. These have played an important part in the history of gravitational wave research, principally in elucidating the reality of gravitational waves as carriers of energy [2]. Thus, several researchers have studied vacuum and non-vacuum cylindrical systems in an attempt to clarify the nature of (especially) nonspherical gravitational collapse, in a system that involves the essential nonlinearity of the gravitational field and the emission of gravitational waves but requires the analysis of partial differential equations with just one spatial dimension. See for example [3–17].

As in many other fundamental theories, self-similarity plays an important role in gravitation. This importance is encapsulated in Carr's self-similarity hypothesis [18,19], which originally asserts that in the cosmological context, spherically symmetric solutions of Einstein's equations evolve to a self-similar form. This evolution can be towards either an intermediate or an endstate. Later it was found

that this is actually the case in gravitational collapse. Examples include the collapse of a soft fluid sphere [20,21] and critical phenomena emerging in the spherical collapse of a variety of matter fields [22]. We should also note that spatially homogeneous cosmological models provide an example of tendency towards self-similar solutions in non spherically symmetric systems: see [23] for a review.

Spherically symmetric self-similar spacetimes have been extensively studied and are now well understood. A natural next step therefore is to consider the role of self-similarity in cylindrical systems. These arise as a special case of $G2$ spacetimes—i.e. spacetimes admitting a two-dimensional group of isometries, usually but not always Abelian. There have been several studies of self-similar $G2$ cosmological models—again see [23], but there has been little work heretofore on the role of self-similarity in cylindrical collapse. Indeed it is generally true that in the cylindrical case, results are fewer and farther between: the additional degrees of freedom that exist in the cylindrical case, which render the systems of equations encountered considerably more difficult, have not to date allowed the development of as clear a picture as we have in the spherical case (but see [12,24–26]). We note that Ref. [26] deals with self-similar cylindrical dust collapse.

The present work seeks to increase our understanding of the role of self-similarity in cylindrical gravitational collapse, and the veracity of the similarity hypothesis in this context. In particular, this work is motivated by numerical

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results obtained by two of the present authors (KN and TH) and their collaborators which show that self-similar cylindrically symmetric vacuum solutions appear to describe the asymptotic behavior of the (numerically obtained) gravitational field outside a collapsing hollow dust cylinder [27]. We present here a two-parameter family of self-similar *vacuum* solutions in cylindrical symmetry.

This paper is organized as follows. In Sec. II, we present formulations for vacuum spacetimes and self-similar spacetimes in the so-called whole-cylinder symmetry independently. In Sec. III, we solve the Einstein field equations for vacuum self-similar spacetimes and obtain exact solutions. The motivation for this is twofold. First, we wish to study self-similar cylindrical collapse of pure gravitational waves, as representing the simplest example of self-similar cylindrical collapse. The hope is that in some sense, this would provide the standard model for self-similar cylindrical collapse in the same way that the Schwarzschild solution provides the standard model of the (endstate of) spherically symmetric collapse. The second motivation is to determine the description of possible exteriors of collapsing self-similar, cylindrical matter. We recall also the overarching motivation for this study: to determine self-similar solutions that may act as (intermediate) attractors of more general cylindrically symmetric spacetimes. In fact we show that Einstein-Rosen waves with a homothetic vector orthogonal to the cylinders of symmetry are *flat* and reduce to part of the Minkowski spacetime with or without a conical singularity and with or without nontrivial topology. Moreover, we show that the above metric form of part of the Minkowski spacetime implies nontrivial (i.e. nonzero) *C* energy [3] and a trapping horizon. In Sec. IV, we naturally extend the solutions to more general class and find that these solutions are also self-similar where the homothetic vector is not orthogonal to the cylinders of symmetry in general. We consider the analytical extension of the solutions, analyze the global structure of the obtained spacetimes, and find that these describe interesting nonlinear dynamics of gravitational waves. We discuss the physical interpretation of these solutions. In Sec. V, we summarize the paper. We use the units, in which $G = c = 1$.

II. CYLINDRICAL SYMMETRY AND SELF SIMILARITY

A. Spacetimes in whole-cylinder symmetry

For cylindrically symmetric spacetimes, we assume that there are two commuting, spatial Killing vectors ($\xi_{(1)}$, $\xi_{(2)}$) such that the orthogonal space is integrable and the Killing coordinate φ , where $\xi_{(1)} = \partial/\partial\varphi$, is identified at 0 and 2π . We call $\xi_{(1)}$ and $\xi_{(2)}$ azimuthal and translational Killing vectors, respectively. Here we shall additionally assume that each of the two Killing vectors be hypersurface orthogonal, which is called whole-cylinder symmetry [3,4]

or the polarized case [10]. The circumferential radius ρ , the specific length ℓ , and the areal radius r are defined as

$$\rho^2 := \xi_{(1)a}\xi_{(1)}^a, \quad \ell^2 := \xi_{(2)a}\xi_{(2)}^a, \quad \text{and} \quad r := \rho\ell. \quad (2.1)$$

Note then that r is the areal radius of the orbits of the isometry group and so $r \geq 0$ with $r = 0$ at the axis. The line element in this class of spacetimes is given by [28]:

$$ds^2 = -2e^{2\gamma(u,v)-2\psi(u,v)}dudv + e^{-2\psi(u,v)}r^2(u,v)d\varphi^2 + e^{2\psi(u,v)}dz^2. \quad (2.2)$$

We note that this form of the line element is unchanged under rescalings of the null coordinates $u \rightarrow \bar{u}(u)$, $v \rightarrow \bar{v}(v)$. The regular axis condition [10,28] ensures the ratio of an infinitesimal circle around the axis to its diameter to be π .

The *C*-energy E is then defined as [3,8,10]

$$E := \frac{1}{8}(1 - \ell^{-2}\nabla^a r \nabla_a r). \quad (2.3)$$

This quantity is assumed to represent the line energy density enclosed inside the cylinder. A cylinder, which is a two-surface given by $v = \text{const}$ and $v = \text{const}$, is said to be trapped, marginally trapped, and untrapped if $\nabla^a r$ is timelike, null, and spacelike, respectively. In terms of the *C* energy, a cylinder is trapped, marginally trapped, and untrapped if $E > 1/8$, $E = 1/8$, and $E < 1/8$, respectively. A cylindrical trapping horizon is a hypersurface foliated by marginal cylinders.

The Einstein equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ reduce to

$$r_{,uu} + 2r\psi_{,u}^2 - 2\gamma_{,u}r_{,u} = 8\pi r T_{uu}, \quad (2.4)$$

$$r_{,vv} + 2r\psi_{,v}^2 - 2\gamma_{,v}r_{,v} = -8\pi r T_{vv}, \quad (2.5)$$

$$r_{,uv} = 8\pi r T_{uv}, \quad (2.6)$$

$$-2r^2 e^{-2\gamma}(\gamma_{,uv} + \psi_{,u}\psi_{,v}) = 8\pi T_{\varphi\varphi}, \quad (2.7)$$

$$e^{4\psi-2\gamma}(2r\psi_{,uv} + r_{,u}\psi_{,v} + r_{,v}\psi_{,u} - r\gamma_{,uv} - r_{,uv} - r\psi_{,u}\psi_{,v}) = 4\pi r T_{zz}, \quad (2.8)$$

where $T_{\mu\nu}$ is the stress-energy tensor for the matter fields.

B. Einstein-Rosen waves

Here we consider vacuum spacetimes in whole-cylinder symmetry. Solutions of the field equations in this case are called Einstein-Rosen waves [1]. For such a cylindrically symmetric vacuum spacetime, in which the line element is given by Eq. (2.2), Eq. (2.6) reduces to

$$r_{,uv} = 0, \quad (2.9)$$

implying

$$r = f(u) + g(v), \quad (2.10)$$

where f and g are arbitrary functions.

If $\nabla^a r$ is spacelike, we can choose f and g by rescaling u and v such that

$$r = \frac{v - u}{\sqrt{2}}. \quad (2.11)$$

Introducing the time and radial coordinates

$$t = \frac{v + u}{\sqrt{2}} \quad \text{and} \quad x = \frac{v - u}{\sqrt{2}}, \quad (2.12)$$

we obtain the metric in the form

$$ds^2 = e^{2(\gamma-\psi)}(-dt^2 + dx^2) + e^{-2\psi}x^2d\varphi^2 + e^{2\psi}dz^2. \quad (2.13)$$

We note that this line element, subject to the Einstein equations below, corresponds to the original Einstein-Rosen waves. That is, the Einstein-Rosen paper [1] deals exclusively with the case where $\nabla^a r$ is spacelike. For convenience, we will use the term to refer to any vacuum whole-cylinder symmetric solution of the Einstein equations.

The nontrivial components of the Einstein equations become the following simple set of partial differential equations:

$$-\psi_{,tt} + \psi_{,xx} + \frac{1}{x}\psi_{,x} = 0, \quad (2.14)$$

$$\gamma_{,x} = x(\psi_{,x}^2 + \psi_{,t}^2), \quad (2.15)$$

$$\gamma_{,t} = 2x\psi_{,x}\psi_{,t}. \quad (2.16)$$

Equation (2.7) is automatically satisfied due to Eqs. (2.15) and (2.16). The regular axis condition reduces to $\gamma \rightarrow 0$ as $x \rightarrow 0$ in this coordinate system.

If $\nabla^a r$ is timelike, we can choose f and g by rescaling u and v such that

$$r = \frac{v + u}{\sqrt{2}} = t. \quad (2.17)$$

Then, we get the metric in the form

$$ds^2 = e^{2(\gamma-\psi)}(-dt^2 + dx^2) + e^{-2\psi}t^2d\varphi^2 + e^{2\psi}dz^2. \quad (2.18)$$

The nontrivial components of the Einstein equations become the following simple set of ordinary differential equations:

$$-\psi_{,tt} + \psi_{,xx} - \frac{1}{t}\psi_{,t} = 0, \quad (2.19)$$

$$\gamma_{,t} = t(\psi_{,t}^2 + \psi_{,x}^2), \quad (2.20)$$

$$\gamma_{,x} = 2t\psi_{,t}\psi_{,x}. \quad (2.21)$$

We can see that the equations are the same as those for spacelike $\nabla^a r$ if we exchange t and x .

If $\nabla^a r$ is null, we can choose f and g such that

$$r = u. \quad (2.22)$$

The Einstein equations reduce to

$$\psi_{,uv} = 0, \quad (2.23)$$

$$\psi_{,v} = 0, \quad (2.24)$$

$$\gamma_{,u} = u\psi_{,u}^2. \quad (2.25)$$

$\psi = \psi(u)$ follows from the above.

C. Self-similar spacetimes in whole-cylinder symmetry

We now consider the case where the spacetime is self-similar as well as cylindrically symmetric: there is no *a priori* reason to suppose that this will lead to trivial solutions only. In other words, we assume that the spacetime admits a vector field \mathbf{v} which satisfies the following equation:

$$\mathcal{L}_{\mathbf{v}}g_{\mu\nu} = 2g_{\mu\nu}, \quad (2.26)$$

where $\mathcal{L}_{\mathbf{v}}$ denotes the Lie derivative along \mathbf{v} . We refer to the vector field \mathbf{v} and Eq. (2.26) as the homothetic vector and the homothetic equation, respectively.

We assume that \mathbf{v} has the following form:

$$\mathbf{v} = \alpha(u, v)\frac{\partial}{\partial u} + \beta(u, v)\frac{\partial}{\partial v}, \quad (2.27)$$

where \mathbf{v} is then assumed to be orthogonal to the cylinders of symmetry. We will refer to this as a cylindrical homothetic vector. We will mention the limitation of this assumption later. Then, the homothetic equations (2.26) yield $\alpha = \alpha(u)$ and $\beta = \beta(v)$. We can then generically make the coordinate transformation $\bar{u} = \bar{u}(u)$ and $\bar{v} = \bar{v}(v)$ satisfying the following relations:

$$\alpha(u)\frac{d\bar{u}}{du} = 2\bar{u} \quad \text{and} \quad \beta(v)\frac{d\bar{v}}{dv} = 2\bar{v}. \quad (2.28)$$

Then, we can have

$$\mathbf{v} = 2\bar{u}\frac{\partial}{\partial \bar{u}} + 2\bar{v}\frac{\partial}{\partial \bar{v}}, \quad (2.29)$$

where here and hereafter we omit bars for simplicity. The homothetic vector is timelike, spacelike, and null if uv is positive, negative, and zero, respectively. If $\alpha = 0$ or $\beta = 0$, the homothetic vector is null and we do not consider these special cases here.

It is then straightforward to show that the homothetic equations (2.26) lead to

$$e^{2\psi} = |u|e^{2P(\eta)}, \quad r = |u|S(\eta), \quad e^{2\gamma} = e^{2G(\eta)}, \quad (2.30)$$

where $\eta = v/u$ and P , S , and G are arbitrary functions.

Thus, we obtain the following standard form of the metric for whole-cylindrically symmetric self-similar spacetimes:

$$ds^2 = -2e^{2G(\eta)-2P(\eta)}|u|^{-1}dudv + e^{-2P(\eta)}|u|S^2(\eta)d\varphi^2 + e^{2P(\eta)}|u|dz^2. \quad (2.31)$$

We can substitute the above form into the Einstein equations (2.4), (2.5), (2.6), (2.7), and (2.8) and obtain a set of ordinary differential equations for P , G , and S . Fortunately, for the vacuum case, we can greatly simplify the system.

III. EINSTEIN-ROSEN WAVES WITH A CYLINDRICAL HOMOTHEIC VECTOR

A. One-parameter family of solutions

We consider self-similar vacuum solutions in this section. It is not trivial that the choice of f and g adopted in Sec. II B is compatible with the self-similarity introduced in Sec. II C. In fact, for r to be compatible with the self-similarity, i.e., Eqs. (2.30), f and g introduced in Eq. (2.10) for the vacuum solution must satisfy

$$f = C_1 u, \quad \text{and} \quad g = C_2 v, \quad (3.1)$$

where C_1 and C_2 are arbitrary constants. (We note that a trivial addition of a constant may be required to obtain the above form.) Hence, the choices of f and g given by Eqs. (2.11) and (2.17) are both compatible with self similarity.

For the moment we restrict ourselves to the case where r has a spacelike gradient. Now we can adopt (t, x) coordinates and by combining Eqs. (2.13), (2.30), and (2.31), we find the metric in the following form:

$$ds^2 = e^{2\Gamma(\xi)-2\Psi(\xi)}x^{-1}(-dt^2 + dx^2) + e^{-2\Psi(\xi)}xd\varphi^2 + e^{2\Psi(\xi)}xdz^2, \quad (3.2)$$

where

$$\xi = \frac{t}{x}. \quad (3.3)$$

(We recall that $x = r \geq 0$.) In this case, the Einstein equations reduce to a set of simple ordinary differential equations. Noting

$$\psi = \Psi(\xi) + \frac{1}{2}\ln x, \quad (3.4)$$

$$\gamma = \Gamma(\xi). \quad (3.5)$$

Equations (2.14), (2.15), and (2.16) reduce to the following ordinary differential equations:

$$(\xi^2 - 1)\Psi'' + \xi\Psi' = 0, \quad (3.6)$$

$$\xi\Gamma' = -\left(\xi\Psi' - \frac{1}{2}\right)^2 - \Psi'^2, \quad (3.7)$$

$$\Gamma' = -2\left(\xi\Psi' - \frac{1}{2}\right)\Psi', \quad (3.8)$$

where the prime denotes the derivative with respect to ξ . From Eqs. (3.7) and (3.8), we obtain

$$(\xi^2 - 1)\Psi'^2 - \frac{1}{4} = 0. \quad (3.9)$$

Therefore, $\xi^2 > 1$ and

$$\Psi' = \pm \frac{1}{2\sqrt{\xi^2 - 1}}. \quad (3.10)$$

This satisfies Eq. (3.6). We can integrate the above equation and obtain

$$\Psi = \frac{1}{2}\ln|\xi \pm \sqrt{\xi^2 - 1}| + \Psi_0, \quad (3.11)$$

where Ψ_0 is a constant of integration. Γ is obtained by substituting Eq. (3.10) into Eq. (3.8) and integrating the resultant equation. The result is

$$\Gamma = \frac{1}{2}\ln\left|\frac{1}{2}\left(\frac{\xi}{\sqrt{\xi^2 - 1}} \pm 1\right)\right| + \lambda, \quad (3.12)$$

where λ is a constant of integration. Getting back to the original metric functions ψ and γ , the solution is given by

$$\psi = \frac{1}{2}\ln|\xi \pm \sqrt{\xi^2 - 1}| + \Psi_0 + \frac{1}{2}\ln x, \quad (3.13)$$

$$\gamma = \frac{1}{2}\ln\left|\frac{1}{2}\left(\frac{\xi}{\sqrt{\xi^2 - 1}} \pm 1\right)\right| + \lambda. \quad (3.14)$$

We can assume $t > 0$ because the flip of the sign of t corresponds to the other branch of solutions. Note that we can set $\Psi_0 = 0$ by absorbing it into the coordinates as follows:

$$\begin{aligned} \tilde{t} &= e^{-2\Psi_0}t, & \tilde{x} &= xe^{-2\Psi_0}, \\ \tilde{\varphi} &= \varphi, & \tilde{z} &= e^{2\Psi_0}z. \end{aligned} \quad (3.15)$$

Up to this gauge parameter, the solutions are parametrized by λ .

For the timelike $\nabla^a r$ case, we can obtain the solution just by exchanging t and x in the solution for the spacelike $\nabla^a r$ case. This corresponds to the replacement of ξ with ξ^{-1} . The solution is therefore given by

$$\psi = \frac{1}{2}\ln|\xi^{-1} \pm \sqrt{\xi^{-2} - 1}| + \Psi_0 + \frac{1}{2}\ln|t|, \quad (3.16)$$

$$\gamma = \frac{1}{2}\ln\left|\frac{1}{2}\left(\frac{\xi^{-1}}{\sqrt{\xi^{-2} - 1}} \pm 1\right)\right| + \lambda, \quad (3.17)$$

where the solution is valid for $\xi^2 < 1$. We can assume $x > 0$ in this case.

B. Flatness and topology of the solutions

As indicated in the introduction, our aim in deriving the solutions of the previous subsection is to study the simplest case of self-similar cylindrical collapse. However it transpires that these solutions do not represent collapsing gravitational waves. The solutions are flat everywhere except along the axis, and thus the solutions correspond either to part of Minkowski spacetime, or to a line conical singularity in flat spacetime: the assumption of self-similarity of the gravitational waves rules out any other possibility. This is a nontrivial result regarding self-similar cylindrical collapse. Furthermore, the form of the flat spacetime metric that emerges demonstrates explicitly that there are cylindrical trapping horizons in Minkowski spacetime, and that C energy is (i) nonunique and (ii) nonzero in Minkowski spacetime. This seriously undermines the interpretation of C energy as the gravitational energy of a cylindrical spacetime.

We can explicitly show that all coordinate components of the Riemann curvature tensor vanish for the obtained solutions. This means that the solutions are flat. Here we show that these solutions are indeed part of the Minkowski spacetimes.

1. Untrapped case

We here assume that $\nabla^a r$ is spacelike, i.e., (t, x) corresponds to an untrapped cylinder. The regular axis condition implies $\Gamma \rightarrow 0$ as $\xi \rightarrow \infty$. From Eq. (3.12), this is possible only for the upper-sign solution with $\lambda = 0$. Hence, the solution becomes

$$\gamma = \frac{1}{2} \ln \left| \frac{1}{2} \left(\frac{\xi}{\sqrt{\xi^2 - 1}} + 1 \right) \right|. \quad (3.18)$$

The upper-sign solution for the different choice of λ gives a conical singularity with the ratio of an infinitesimal circle's circumference to its diameter $\pi e^{-\lambda}$ rather than π . We first concentrate on the upper-sign solution with $\lambda = 0$ below.

Choosing the coordinates where $\Psi_0 = 0$, we can write the metric as

$$ds^2 = \frac{1}{2\sqrt{t^2 - x^2}} (-dt^2 + dx^2) + |t \mp \sqrt{t^2 - x^2}| d\varphi^2 + |t \pm \sqrt{t^2 - x^2}| dz^2, \quad (3.19)$$

where the solution is valid only for $t^2 > x^2$. Assuming $t > 0$, through the coordinate transformations

$$T^2 = t + \sqrt{t^2 - x^2} \quad \text{and} \quad X^2 = t - \sqrt{t^2 - x^2}, \quad (3.20)$$

or

$$t = \frac{T^2 + X^2}{2} \quad \text{and} \quad x = TX, \quad (3.21)$$

we obtain the following metric for the upper-sign:

$$ds^2 = -dT^2 + dX^2 + X^2 d\varphi^2 + T^2 dz^2, \quad (3.22)$$

where $0 \leq X < T$ by construction. Through another coordinate transformation

$$\begin{aligned} \tau &= T \cosh z, & \zeta &= T \sinh z, \\ p &= X \cos \varphi, & \text{and} & \quad q = X \sin \varphi, \end{aligned} \quad (3.23)$$

we finally obtain the usual Minkowski spacetime in the standard Cartesian coordinates

$$ds^2 = -d\tau^2 + dp^2 + dq^2 + d\zeta^2, \quad (3.24)$$

where $\tau^2 > p^2 + q^2 + \zeta^2$ and hence the solution covers the inside of the light cone $\tau^2 = p^2 + q^2 + \zeta^2$. This region is shown as the dark shaded disk in Fig. 1, where the constant τ spacelike hypersurface is plotted. The z axis, i.e. $x = 0$ in (t, x, φ, z) coordinates, is transformed to the ζ axis, i.e., $p = q = 0$ in (τ, p, q, ζ) coordinates. The null hypersurface $t^2 = x^2$ in (t, x, φ, z) coordinates is transformed to the light cone $\tau^2 = p^2 + q^2 + \zeta^2$ in the standard Cartesian coordinates.

For the lower-sign solution, the metric is written as

$$ds^2 = -dT^2 + dX^2 + T^2 d\varphi^2 + X^2 dz^2, \quad (3.25)$$

where $0 \leq X < T$. Through another coordinate transformation

$$\begin{aligned} \tau &= T \cosh \varphi, & \zeta &= T \sinh \varphi, \\ p &= X \cos z, & \text{and} & \quad q = X \sin z, \end{aligned} \quad (3.26)$$

we finally obtain the metric

$$ds^2 = -d\tau^2 + dp^2 + dq^2 + d\zeta^2, \quad (3.27)$$

where $\tau^2 > p^2 + q^2 + \zeta^2$. This is also the Minkowski spacetime in the standard Cartesian coordinates but with

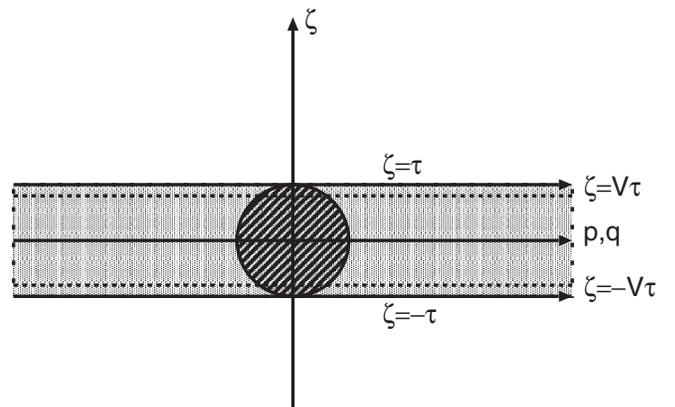


FIG. 1. The constant τ spacelike hypersurface is shown in the standard Cartesian coordinates (τ, p, q, ζ) . The dark shaded region is untrapped, while the light shaded region is trapped. The circle shows the light cone, which is a cylindrical trapping horizon. The region which is unshaded is not described by the cylindrical vacuum flat solutions. The dashed lines denote the timelike planes which are identified with $\zeta = 0$.

nontrivial topology. In the original (t, x, φ, z) coordinates, $\varphi = 2\pi$ is identified with $\varphi = 0$. This results in the identification between the two timelike hypersurfaces $\zeta = 0$ and $\zeta = V\tau$, where $V = \tanh 2\pi$. The latter timelike hypersurface is shown as a dashed line in Fig. 1. More precisely, the point $(\tau, p, q, 0)$ is identified with the point $(\tau/\sqrt{1-V^2}, p, q, \tau V/\sqrt{1-V^2})$. Then, there appears a timelike geodesic with infinite spatial acceleration in an approach to the spacelike line $\tau = \zeta = 0$, which will be described in detail in the appendix. The appendix also serves to clarify the nature of the topological identifications made here. On the other hand, a circle identification has not been imposed on z , implying that the two-dimensional pq plane consists of covering planes folded infinitely many times. The z axis is again transformed to the ζ -axis, i.e., $p = q = 0$.

We note that here the (original) cylinders of symmetry $(t, x) = (t_0, x_0)$ with t_0, x_0 both constant have the following representation in Minkowski coordinates (τ, p, q, ζ) :

$$\tau^2 - \zeta^2 = t_0 + \sqrt{t_0^2 - x_0^2}, \quad p^2 + q^2 = t_0 - \sqrt{t_0^2 - x_0^2}. \quad (3.28)$$

2. Trapped case

We then assume that $\nabla^a r$ is timelike, i.e., (t, x) corresponds to a trapped cylinder. Also for this case, choosing $\Psi_0 = 0$ and $\lambda = 0$, we obtain

$$ds^2 = \frac{1}{2\sqrt{x^2 - t^2}}(-dt^2 + dx^2) + |x \mp \sqrt{x^2 - t^2}|d\varphi^2 + |x \pm \sqrt{x^2 - t^2}|dz^2, \quad (3.29)$$

where the solution is valid only for $x^2 > t^2$. By implementing coordinate transformations similar to those used in the untrapped case, we obtain for the lower-sign solution

$$ds^2 = -dT^2 + dX^2 + X^2d\varphi^2 + T^2dz^2, \quad (3.30)$$

where $0 < T < X$. This is identical with the upper-sign solution for the untrapped region and hence transformed to the standard Cartesian coordinates (τ, p, q, ζ) , where the solution covers the region $\zeta^2 < \tau^2 < p^2 + q^2 + \zeta^2$. This corresponds to the intersection of the outside of light cone $\tau^2 = p^2 + q^2 + \zeta^2$ and the timelike portion sandwiched by two planes $\tau = \pm\zeta$. This region is shown as a light shaded region in Fig. 1. The topology of the spacetime is trivial.

For the upper-sign solution, we obtain

$$ds^2 = -dT^2 + dX^2 + T^2d\varphi^2 + X^2dz^2, \quad (3.31)$$

where $0 < T < X$. This is identical with the lower-sign solution for the untrapped region and hence transformed to the standard Cartesian coordinates (τ, p, q, ζ) , where the solution covers the region $\zeta^2 < \tau^2 < p^2 + q^2 + \zeta^2$, i.e., the intersection of the outside of the light cone and the

timelike portion sandwiched by two planes $\tau = \pm\zeta$. We should note that the topology is nontrivial because φ is circularly identified while z is not.

We note that in the trapped case the (original) cylinders of symmetry $(t, x) = (t_0, x_0)$ with t_0, x_0 both constant have the following representation in Minkowski coordinates (τ, p, q, ζ) :

$$\tau^2 - \zeta^2 = x_0 - \sqrt{x_0^2 - t_0^2}, \quad p^2 + q^2 = x_0 + \sqrt{x_0^2 - t_0^2}. \quad (3.32)$$

Hence, for $\lambda = 0$, the union of the upper-sign solution for the untrapped region and the lower-sign solution for the trapped region describes the timelike portion of the Minkowski spacetime sandwiched by two light planes when the two solutions are matched on the light cone $\tau^2 = p^2 + q^2 + \zeta^2$.

The union of the lower-sign solutions for the untrapped region and the upper-sign solution for the trapped region also describes the same region of the Minkowski spacetime when they are matched on the light cone $\tau^2 = p^2 + q^2 + \zeta^2$ but with nontrivial topology. The solutions with nonvanishing λ will have an additional conical singularity.

This solution without the conical singularity is quite analogous to the Milne universe solution, which is also part of the Minkowski spacetime. However, the present solution is somewhat different from the Milne universe in the following respect. Recall that observers with constant spatial coordinates run radially outward with a constant speed and they do so homogeneously and isotropically in the Minkowski spacetime. On the other hand, in the present solution, observers with constant spatial coordinates run with a constant speed but only in the direction parallel to the axis and do not in the two perpendicular ones. We shall call the present solution the cylindrical Milne solution in this paper.

C. C energy and trapping horizon in the minkowski spacetime

C energy was introduced in [3] as a tool with which cylindrical spacetimes may be analyzed. It has several interesting and useful features: It is covariant and is associated with a conserved flux vector; it has the correct Newtonian limit, the mass per specific length of the cylinder [10]; it is propagated by Einstein-Rosen waves. Thus it is a candidate for “the energy of whole-cylinder-symmetric spacetimes” (the phrase appears in quotation marks in [3], p. 251) and a later study refers to C energy as “gravitational energy per specific length” [10]. As a particular application, C energy has been used to investigate the fate of an infinitesimally thin cylindrical shell composed of counter-rotating dust particles by Apostolatos and Thorne [5], and later by one of the present author (KN) and his collaborators [17]: it should be stressed that the conclusions in these two papers do not agree with each

other although both rely on the properties of the C energy. A further criterion that should be satisfied by a candidate for gravitational energy of any form is that it should vanish in the absence of a gravitational field, i.e. in flat spacetime. It transpires however that the cylindrical representations of flat spacetime we have found above show that the C energy *does not* always vanish in this case.

For the Einstein-Rosen waves written in the forms of Eqs. (2.13) and (2.18), the C energy reduces to the following simple forms:

$$E = \frac{1}{8}(1 - e^{-2\gamma}) \quad \text{and} \quad E = \frac{1}{8}(1 + e^{-2\gamma}), \quad (3.33)$$

respectively. For simplicity, we discuss the cylindrical Milne solution with trivial topology, which is given by pasting the upper-sign solution of Eq. (3.19) and the lower-sign solution of Eq. (3.29) on the null hypersurface $t^2 = x^2$. For these metrics, we obtain respectively

$$E = \frac{1}{8} \frac{t - \sqrt{t^2 - x^2}}{t + \sqrt{t^2 - x^2}}, \quad \text{and} \quad E = \frac{1}{8} \frac{x + \sqrt{x^2 - t^2}}{x - \sqrt{x^2 - t^2}}. \quad (3.34)$$

This is rewritten in both cases as

$$E = \frac{1}{8} \frac{p^2 + q^2}{\tau^2 - \zeta^2}, \quad (3.35)$$

in terms of the standard Cartesian coordinates. Since these metrics are those for part of the Minkowski spacetime, it can have nonvanishing C energy.

However, if we write the metric of the Minkowski spacetime in the standard cylindrical coordinates, we have $\gamma = \psi = 0$ in Eq. (2.13) and hence $E = 0$. This result questions the physical interpretation of the C energy. Indeed, the trick is in the choice of the two commuting Killing vectors, or equivalently, the choice of the cylinders. If we take $(\xi_{(1)}, \xi_{(2)}) = (\partial/\partial\varphi, \partial/\partial z)$, then ρ , ℓ , and r are calculated as

$$\rho^2 = |t - \sqrt{t^2 - x^2}| = p^2 + q^2, \quad (3.36)$$

$$\ell^2 = |t + \sqrt{t^2 - x^2}| = \tau^2 - \zeta^2, \quad (3.37)$$

$$r^2 = x^2 = (\tau^2 - \zeta^2)(p^2 + q^2), \quad (3.38)$$

where $t^2 > x^2$ or $\tau^2 > p^2 + q^2 + \zeta^2$. Hence, the regular axis condition is satisfied and we obtain nontrivial C energy. Instead, if we take $(\xi_{(1)}, \xi_{(2)}) = (\partial/\partial\varphi, \partial/\partial\zeta)$, then we have $\rho^2 = p^2 + q^2$, $\ell = 1$, and $r^2 = p^2 + q^2$ and obtain vanishing C energy. We should also note that the z axis in the former is transformed to the ζ axis in the latter. This means that the definition of C energy is ambiguous for the same axis in the same cylindrically symmetric spacetime unless a pair of two commuting Killing vectors are fully specified. Thorne [3] has noted the lack of uniqueness in the definition of C energy in the case of unpolarized

cylindrical spacetimes, for which the Killing vectors $(\xi_{(1)}, \xi_{(2)})$ are not orthogonal. Nonuniqueness in the unpolarized case is related to the loss of invariance of the spacetime under reflections through any plane either containing the axis or perpendicular to it. However as we see in the present case, nonuniqueness can remain even in the polarized case when there is more than one choice of the azimuthal and translational Killing fields.

This clearly gives rise to a question about the interpretation of C energy as “gravitational energy per specific length” [10], given that it may be nonzero in the absence of a gravitational field. But as we have pointed out, C energy has many attractive and useful features, and so perhaps the most natural question to ask at this point is if there exists an alternative definition that would have the additional feature of vanishing for *any* cylindrical slicing of Minkowski spacetime. We hope to address this question in future work.

Although the uniqueness of the C energy may be recovered by *specifying* the pair of Killing vectors, it is still true that the null hypersurface $t^2 = x^2$ in the original coordinates or the light cone $\tau^2 = p^2 + q^2 + \zeta^2$ in the standard Cartesian coordinates gives a cylindrical trapping horizon. The inside of the light cone $\tau^2 > p^2 + q^2 + \zeta^2$ is untrapped, while the outside of the light cone, i.e., $\zeta^2 < \tau^2 < p^2 + q^2 + \zeta^2$ is trapped. The constant r hypersurfaces, given by Eq. (3.38), are shown in Fig. 2 on the constant τ hypersurface. On the other hand, with the pair $(\partial/\partial\varphi, \partial/\partial\zeta)$ of Killing vectors, the constant r hypersurfaces given by

$$p^2 + q^2 = r^2 \quad (3.39)$$

are all timelike. See also [29] for trapped surfaces in the Minkowski spacetime.

In cylindrically symmetric spacetimes, a trapping horizon is defined as a hypersurface foliated by marginally

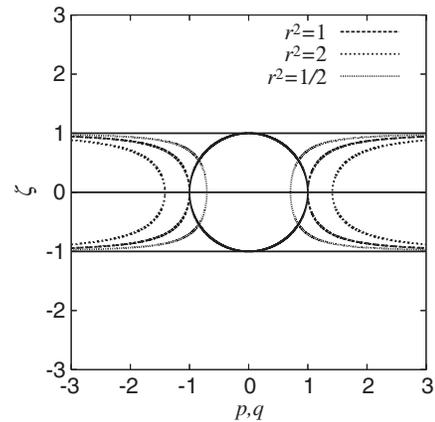


FIG. 2. The constant τ spacelike hypersurface is shown in the standard Cartesian coordinates (τ, p, q, ζ) , where τ is chosen to be unity. The short-dashed, dashed, and long dashed lines denote the intersection of the constant r hypersurfaces given by Eq. (3.38) with the constant τ hypersurface.

trapped cylinders and hence will not be closed in general. Since they are not closed, it will not imply the existence of spacetime singularity. Thus, this example is a lesson that we cannot reasonably identify a trapping horizon with a black hole horizon for cylindrically symmetric spacetimes [10,12]. On the other hand, we could have closed trapped surfaces if we change the identification. In that case, we may encounter a sort of singularity as described in the appendix.

IV. MORE GENERAL SELF-SIMILAR EINSTEIN-ROSEN WAVES

In the previous two sections, we showed that the only self-similar cylindrically symmetric vacuum spacetimes comprise flat spacetimes, possibly with line conical singularities along the axis. These are trivial examples of self-similar spacetimes, and so while these may act as (intermediate) asymptotic endstates of certain more general cylindrical configurations—and indeed a line conical singularity in an otherwise flat spacetime can result from the complete collapse of cylindrical null dust—it is clear that this does not lend any weight to the self-similarity hypothesis in cylindrical symmetry. It transpires however that part of the reason for this is that we have looked only at a quite restrictive class of self-similar cylindrical spacetimes. Dropping the assumption that the homothetic vector field is orthogonal to the cylinders of symmetry yields some interesting results which we describe here.

A. Two-parameter family of solutions

In this section, we concentrate on the spacelike $\nabla^a r$ case, where the region is untrapped. In this case, Eqs. (2.14), (2.15), and (2.16), give a complete set of governing equations. Among them, Eq. (2.14) gives the dynamics of ψ and the other equations determine the derivatives of γ . For self-similar solutions with a cylindrical homothetic vector, ψ is given by Eq. (3.4). Note that each of two terms on the right-hand side of Eq. (3.4) is a solution of Eq. (2.14)—provided that the first term Ψ is taken to be a solution of Eq. (3.6). Since Eq. (2.14) is linear, this means that the arbitrary linear combination of the two terms gives a solution of Eq. (2.14). Moreover, we can assume the similar form also for γ from Eq. (3.5). From this consideration, we assume the following form for ψ and γ :

$$\psi = a\tilde{\Psi}(\xi) + b\frac{1}{2}\ln|x|, \quad (4.1)$$

$$\gamma = c\tilde{\Gamma}(\xi) + d\frac{1}{2}\ln|x|. \quad (4.2)$$

From Eqs. (2.14), (2.15), and (2.16), we get the following ordinary differential equations for $\tilde{\Psi}$ and $\tilde{\Gamma}$:

$$(\xi^2 - 1)\tilde{\Psi}'' + \xi\tilde{\Psi}' = 0, \quad (4.3)$$

$$-c\xi\tilde{\Gamma}' + \frac{d}{2} = \left(a\xi\tilde{\Psi}' - \frac{b}{2}\right)^2 + a^2\tilde{\Psi}'^2, \quad (4.4)$$

$$c\tilde{\Gamma}' = -2\left(a\xi\tilde{\Psi}' - \frac{b}{2}\right)a\tilde{\Psi}'. \quad (4.5)$$

From Eq. (4.3), we get

$$\tilde{\Psi}' = \frac{\tilde{\Psi}_1}{\sqrt{|\xi^2 - 1|}}. \quad (4.6)$$

We can always assume $\tilde{\Psi}_1 = 1/2$ because of the factor a in Eq. (4.1). Eliminating $\tilde{\Gamma}'$ from Eqs. (4.4) and (4.5), using Eq. (4.6), and putting $\tilde{\Psi}_1 = 1/2$, we obtain

$$\text{sign}(1 - \xi^2)a^2 + b^2 = 2d. \quad (4.7)$$

Then, for $\xi^2 > 1$, the solution is given by

$$\psi = a\frac{1}{2}\ln|\xi + \sqrt{\xi^2 - 1}| + b\frac{1}{2}\ln x + \tilde{\Psi}_0, \quad (4.8)$$

$$\gamma = a\frac{1}{2}[-a\ln\sqrt{\xi^2 - 1} + b\ln|\xi + \sqrt{\xi^2 - 1}|] + d\frac{1}{2}\ln|x| + \tilde{\Gamma}_0, \quad (4.9)$$

where we omit the lower-sign solution because the sign can be absorbed into the sign of a .

To be more specific, we assume that the axis is regular or conically singular at least, which implies that γ approaches a finite value for $t > 0$ and $x \rightarrow 0$. This condition strongly restricts the parameters. This limit corresponds to $\xi \rightarrow \infty$ and $x \rightarrow 0$, where ξ^{-1} and x approach zero independently. Hence, the condition on the axis implies $a = b$, $d = 0$ and a finite value for $\tilde{\Gamma}_0$. Putting $a = b = 2\kappa$ and $\tilde{\Gamma}_0 = \lambda$, we obtain the following solution with a regular or conically singular axis:

$$\psi = \kappa[\ln(\xi + \sqrt{\xi^2 - 1}) + \ln|x|], \quad (4.10)$$

$$\gamma = 2\kappa^2 \ln \left| \frac{1}{2} \left(\frac{\xi}{\sqrt{\xi^2 - 1}} + 1 \right) \right| + \lambda, \quad (4.11)$$

where $\tilde{\Psi}_0$ is eliminated in use of the scaling freedom of t and x . These solutions are parametrized by κ and λ . Note that the solution is self-similar in the sense discussed so far if $\kappa = 1/2$ and the axis is regular if and only if $\lambda = 0$. As we will see later, the spacetime is nonflat except for $\kappa = 0$ and $1/2$.

For later convenience, we write down the line element explicitly both in (t, x, φ, z) and (T, X, φ, z) coordinates as

$$\begin{aligned}
ds^2 &= \frac{(t + \sqrt{t^2 - x^2})^{2\kappa(2\kappa-1)}}{2^{4\kappa^2}(t^2 - x^2)^{2\kappa^2}} e^{2\lambda}(-dt^2 + dx^2) \\
&\quad + \frac{x^2}{(t + \sqrt{t^2 - x^2})^{2\kappa}} d\varphi^2 + (t + \sqrt{t^2 - x^2})^{2\kappa} dz^2 \\
&= \frac{T^{4\kappa(2\kappa-1)}}{(T^2 - X^2)^{4\kappa^2-1}} e^{2\lambda}(-dT^2 + dX^2) \\
&\quad + \frac{X^2}{T^{2(2\kappa-1)}} d\varphi^2 + T^{4\kappa} dz^2, \tag{4.12}
\end{aligned}$$

where T and X are given by Eq. (3.20). The original domain of the solution is given by $0 \leq x < t < \infty$ and this is mapped to $0 \leq X < T < \infty$. It is clear that for $\kappa = 0$ the solution reduces to the Minkowski spacetime with a regular ($\lambda = 0$) or a conically singular ($\lambda \neq 0$) axis. Moreover, we can easily find that for $\kappa = -1/2$ this reduces to a Kasner solution with a regular ($\lambda = 0$) or a conically singular ($\lambda \neq 0$) axis.

B. Non cylindrical homothetic vector

To make it clear whether the solutions obtained above have some kind of self-similarity, we shall consider the scaling transformation $\bar{t} = At$, $\bar{x} = Ax$, $\bar{\varphi} = \varphi$, and $\bar{z} = z$. Because of this transformation, ψ and γ transform as follows:

$$\bar{\psi} = \psi + 2\kappa \ln A \quad \text{and} \quad \bar{\gamma} = \gamma. \tag{4.13}$$

Then, the metric components g_{tt} , g_{xx} , $g_{\varphi\varphi}$, and g_{zz} transform as follows:

$$\begin{aligned}
\bar{g}_{tt} &= A^{2(1-\kappa)} g_{tt}, & \bar{g}_{xx} &= A^{2(1-\kappa)} g_{xx}, \\
\bar{g}_{\varphi\varphi} &= A^{2(1-\kappa)} g_{\varphi\varphi}, & \text{and} & \quad \bar{g}_{zz} = A^{2\kappa} g_{zz}.
\end{aligned} \tag{4.14}$$

Therefore, for $\kappa \neq 1$, if we define a vector field \mathbf{v} as

$$\mathbf{v} := \frac{1}{1-\kappa} t \frac{\partial}{\partial t} + \frac{1}{1-\kappa} x \frac{\partial}{\partial x}, \tag{4.15}$$

we can write

$$\begin{aligned}
\mathcal{L}_{\mathbf{v}} g_{tt} &= 2g_{tt}, & \mathcal{L}_{\mathbf{v}} g_{xx} &= 2g_{xx}, \\
\mathcal{L}_{\mathbf{v}} g_{\varphi\varphi} &= 2g_{\varphi\varphi}, & \text{and} & \quad \mathcal{L}_{\mathbf{v}} g_{zz} = \frac{2\kappa}{1-\kappa} g_{zz}.
\end{aligned} \tag{4.16}$$

It is clear that if and only if $\kappa = 1/2$, the vector field \mathbf{v} is a homothetic vector.

However, if we instead scale the coordinates as $\bar{t} = At$, $\bar{x} = Ax$, $\bar{\varphi} = \varphi$, and $\bar{z} = A^{1-2\kappa} z$, the metric components transform as follows:

$$\bar{g}_{\mu\nu} = A^{2(1-\kappa)} g_{\mu\nu}. \tag{4.17}$$

For $\kappa \neq 1$, this implies that for the vector field \mathbf{w} given by

$$\mathbf{w} := \frac{1}{1-\kappa} t \frac{\partial}{\partial t} + \frac{1}{1-\kappa} x \frac{\partial}{\partial x} + \frac{1-2\kappa}{1-\kappa} z \frac{\partial}{\partial z}, \tag{4.18}$$

we obtain

$$\mathcal{L}_{\mathbf{w}} g_{\mu\nu} = 2g_{\mu\nu}. \tag{4.19}$$

Therefore, \mathbf{w} is a homothetic vector and hence the spacetime described by this solution is self-similar. The homothetic vector \mathbf{w} is not cylindrical if $\kappa \neq 1/2$.

For $\kappa = 1$, if we instead define \mathbf{W} as

$$\mathbf{W} := t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} - z \frac{\partial}{\partial z}, \tag{4.20}$$

we obtain

$$\mathcal{L}_{\mathbf{W}} g_{ab} = 0, \tag{4.21}$$

and therefore \mathbf{W} is a Killing vector. This is independent from the azimuthal and the translational Killing vectors.

V. PHYSICAL INTERPRETATION OF SELF-SIMILAR EINSTEIN-ROSEN WAVES

In the previous section we have derived a two-parameter family of self-similar solutions, where the homothetic vector is not cylindrical in general. As we have seen, the two-parameter family of solutions includes Minkowski ($\kappa = 0$), cylindrical Milne ($\kappa = 1/2$), and Kasner ($\kappa = -1/2$) solutions. In this section we present the physical interpretation of the family of solutions for general values of κ , based on the analysis of spacetime singularities, infinities, and analytical extensions. We show that these solutions describe physically interesting nonlinear phenomena of gravitational waves.

A. Curvature invariant, C-energy and areal radius

For the two-parameter family of solutions obtained here, the Kretschmann invariant $I := R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$ is calculated to give

$$\begin{aligned}
I &= 2^{4+8\kappa^2} \kappa^2 (1-2\kappa)^2 e^{-4\lambda} (t + \sqrt{t^2 - x^2})^{-2(4\kappa^2 - 2\kappa + 1)} \\
&\quad \times (t^2 - x^2)^{4\kappa^2 - 3/2} \times [(1+2\kappa)(1-\kappa)t \\
&\quad + (2-\kappa+2\kappa^2)\sqrt{t^2 - x^2}] \\
&= 2^6 \kappa^2 (1-2\kappa)^2 e^{-4\lambda} T^{-4(4\kappa^2 - 2\kappa + 1)} (T^2 - X^2)^{8\kappa^2 - 3} \\
&\quad \times [3T^2 - (4\kappa^2 - 2\kappa + 1)X^2]. \tag{5.1}
\end{aligned}$$

This is identically zero and hence the spacetime is flat if and only if $\kappa = 0$ or $1/2$. For $T^2 = X^2$ the invariant is diverging if $0 < \kappa^2 < 3/8$ and $\kappa^2 \neq 1/4$, while it is finite if $\kappa^2 \geq 3/8$ or $\kappa = 0, \pm 1/2$. The invariant is finite at the z -axis or $x = 0$ if $0 < t < \infty$. The invariant at the axis is vanishing even for $0 \leq t < \infty$ if $\kappa = 0, \kappa = 1/2$ or $\kappa > 1$, while it is diverging at $t = 0$ if $\kappa < 0, 0 < \kappa < 1/2$ and $1/2 < \kappa < 1$. Only for $\kappa = 1$, it is nonzero and finite for $0 \leq t < \infty$ on the axis. Since the solution is vacuum, the Riemann tensor reduces to the Weyl tensor. Hence, in a intuitive sense, we can regard the Kretschmann invariant as the field strength of the pure gravitational field and/or

gravitational waves, although this may be negative for $\kappa < -1/2$.

The C -energy for this solution is calculated to give

$$\begin{aligned} E &= \frac{1}{8} \left[1 - e^{-2\lambda} \left\{ \frac{1}{2} \left(\frac{t}{\sqrt{t^2 - x^2}} + 1 \right) \right\}^{-4\kappa^2} \right] \\ &= \frac{1}{8} \left[1 - e^{-2\lambda} \left(\frac{T^2 - X^2}{T^2} \right)^{4\kappa^2} \right]. \end{aligned} \quad (5.2)$$

$E = (1 - e^{-2\lambda})/8$ at the axis $x = 0$. This is nonzero if and only if $\lambda \neq 0$, i.e., at the conical singularity. Whether λ is zero or not, $E = 1/8$ for $t^2 = x^2$ if only $\kappa \neq 0$, suggesting a cylindrical trapping horizon. For $\kappa = 0$, the C -energy is identically given by $E = (1 - e^{-2\lambda})/8$ and hence constant in the whole region described by this solution. As already mentioned, E is vanishing for $\kappa = \lambda = 0$, but is not for $\kappa = 1/2$ and $\lambda = 0$, although both correspond to a flat geometry. The C energy is unchanged by flipping the sign of κ .

The areal radius r of the cylinder can be calculated as

$$r^2 = x^2 = T^2 X^2. \quad (5.3)$$

Hence, the $r = \text{const}$ surface is given by a hyperboloid in the TX plane.

B. Analytical extension and global structure

We recall that the original domain of the solutions is mapped to $0 \leq X < T < \infty$ in the TX plane. The ‘‘event’’ $t = x = 0$, or equivalently, $T = X = 0$ seems to be a *s.p.* (scalar polynomial) curvature singularity because of the divergence of the curvature invariant if $\kappa < 1$ and $\kappa \neq 0$, $1/2$. But this is a subtle issue. The proper time σ along a curve on which X vanishes and z is constant is given by

$$\sigma = e^\lambda \int T^{-2\kappa+1} dT. \quad (5.4)$$

We can easily see that σ is finite in an approach to $T = 0$ if and only if $\kappa < 1$. If and only if $\kappa \geq 1$, then $T = X = 0$ is at a timelike infinity and hence not within the physical spacetime. On the other hand, $(T, X) = (\infty, 0)$ is at a timelike infinity for $\kappa \leq 1$, while it is a spacetime singularity for $\kappa > 1$.

For $t^2 = x^2$, or equivalently, $T^2 = X^2$, the curvature invariant is diverging as long as $0 < \kappa^2 < 3/8$ and $\kappa^2 \neq 1/4$. We introduce the following null coordinates:

$$U = T - X \quad \text{and} \quad V = T + X. \quad (5.5)$$

Then, we have

$$\begin{aligned} ds^2 &= -e^{2\lambda} \left(\frac{V+U}{2} \right)^{4\kappa(2\kappa-1)} \frac{dUdV}{(UV)^{4\kappa^2-1}} \\ &+ \left(\frac{V+U}{2} \right)^{2(1-2\kappa)} \left(\frac{V-U}{2} \right)^2 d\varphi^2 + \left(\frac{V+U}{2} \right)^{4\kappa} dz^2. \end{aligned} \quad (5.6)$$

The original domain of the solutions is mapped to $0 < U \leq V < \infty$. Here we consider radial null geodesics along which U , φ , and z are constant. The geodesic equation for these null geodesics is given by

$$\frac{d}{d\omega} \left[\frac{(V+U)^{4\kappa(2\kappa-1)}}{(VU)^{4\kappa^2-1}} \frac{dV}{d\omega} \right] = 0, \quad (5.7)$$

where $U = \text{const}$ and ω is an affine parameter. Thus, we can find that $V = \infty$ is a null infinity for any κ . We can similarly find that $U = 0$ can be reached in a finite affine length along null geodesics with $V = \text{const}$ for $0 < \kappa^2 < 1/2$, while it is a null infinity for $\kappa^2 \geq 1/2$. Below we discuss the cases $\kappa^2 = 1/2$, $0 < \kappa^2 < 1/2$, and $\kappa^2 > 1/2$, separately.

I. $\kappa^2 = 1/2$

First, we consider the case of $\kappa^2 = 1/2$, and introduce

$$U = e^u \quad \text{and} \quad V = e^v. \quad (5.8)$$

Then, the metric (5.6) becomes

$$\begin{aligned} ds^2 &= -e^{2\lambda} \left(\frac{e^v + e^u}{2} \right)^{2(2\mp\sqrt{2})} dudv + \left(\frac{e^v + e^u}{2} \right)^{2(1\mp\sqrt{2})} \\ &\times \left(\frac{e^v - e^u}{2} \right)^2 d\varphi^2 + \left(\frac{e^v + e^u}{2} \right)^{\pm 2\sqrt{2}} dz^2 \end{aligned} \quad (5.9)$$

for $\kappa = \pm 1/\sqrt{2}$. The original domain is mapped to $-\infty < u \leq v < \infty$ and this describes a maximal extension because $v = \infty$ and $u = -\infty$ both correspond to null infinities. In the above coordinate system, there seems to be no singular point. But as already shown, the event $u = v = -\infty$ or $T = X = 0$ is a spacetime singularity. The conformal diagram of the solution is given in Fig. 3(d). There is no cylindrical trapping horizon and the whole spacetime is untrapped.

2. $0 < \kappa^2 < 1/2$

Next, we consider the case of $0 < \kappa^2 < 1/2$. In this case, we introduce u and v as

$$U = u^n \quad \text{and} \quad V = v^n, \quad (5.10)$$

where $n := 1/[2(1 - 2\kappa^2)] > 0$. Then, we have

$$\begin{aligned} ds^2 &= -[2(1 - 2\kappa^2)^2]^{-2} e^{2\lambda} \left(\frac{V+U}{2} \right)^{4\kappa(2\kappa-1)} dudv \\ &+ \left(\frac{V+U}{2} \right)^{2(1-2\kappa)} \left(\frac{V-U}{2} \right)^2 d\varphi^2 \\ &+ \left(\frac{V+U}{2} \right)^{4\kappa} dz^2. \end{aligned} \quad (5.11)$$

The original domain is mapped to $0 < u \leq v < \infty$. As we have shown, $v = \infty$ is a null infinity, while $u = 0$ is finite. For $0 < \kappa^2 < 3/8$ and $\kappa^2 \neq 1/4$, the Kretschmann invariant diverges at $u = 0$ and hence $u = 0$ corresponds to a null singularity. For $3/8 \leq \kappa^2 < 1/2$, $u = 0$ is an at least

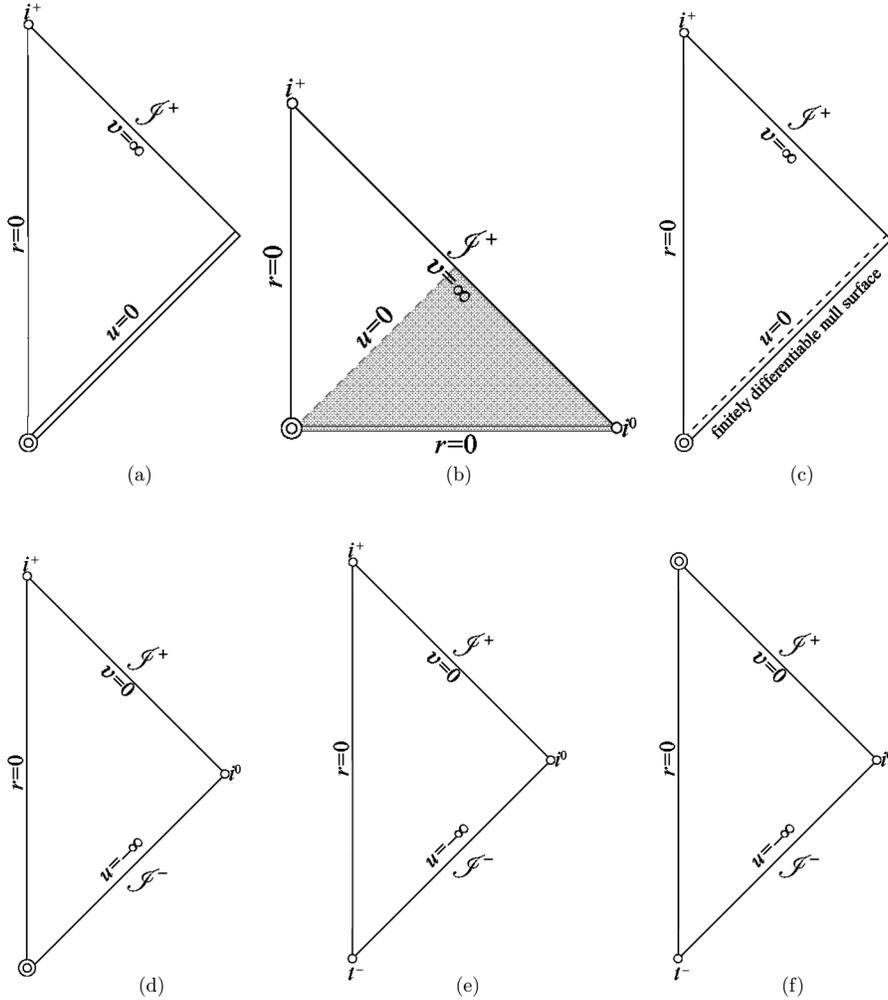


FIG. 3. The conformal diagrams of the two-parameter family of solutions for (a) $0 < \kappa^2 < 1/4$, $1/4 < \kappa^2 < 3/8$, (b) $n := 1/[2(1 - 2\kappa^2)] = 2, 3, 4, \dots$, (c) $n > 2$ but not an integer, (d) $\kappa \leq -1/\sqrt{2}$, $1/\sqrt{2} \leq \kappa < 1$, (e) $\kappa = 1$, and (f) $\kappa > 1$. The dashed lines denote cylindrical trapping horizons. The shaded and unshaded regions denote trapped and untrapped regions, respectively. The single circles denote timelike and spacelike infinities, while the double circles and double lines denote spacetime singularities. The solutions reduce to the Minkowski, the cylindrical Milne, and the Kasner solutions for $\kappa = 0, 1/2$, and $-1/2$, respectively. Choosing $\lambda \neq 0$ simply introduces a conically singular axis. For $3/8 \leq \kappa^2 < 1/2$ or $n \geq 2$, if n is an odd integer, there appears spacelike curvature singularity with $r = 0$ [see (b)]; if n is an even integer, it is replaced by noncurvature but quasiregular singularity [see (b)]; if n is not an integer, then the spacetime admits not an analytic but only a $C^{[n]}$ extension beyond the null surface $u = 0$ [see (c)], where $[n]$ is the largest integer which is no greater than n .

C^2 extendible null hypersurface, which coincides with a cylindrical trapping horizon, and we can discuss the extension beyond this hypersurface. To examine the affine length of the radial null geodesic with $u = 0$, we should consider

$$\frac{d}{d\omega} \left[(V + U)^{4\kappa(2\kappa-1)} \frac{dv}{d\omega} \right] = 0, \quad (5.12)$$

instead of Eq. (5.7). We can then find that the affine length is infinite to $v = \infty$ and finite to $v = 0$ even along the null geodesic with $u = 0$ for $3/8 \leq \kappa^2 < 1/2$.

If and only if n is a natural number, the extension beyond this surface can be analytic and we can naturally extend the

spacetime by Eq. (5.10). The following discussion depends on whether n is odd or even.

If $n = 2l + 1$ ($l = 1, 2, \dots$), the maximally extended domain is given by $\{0 < v < \infty \text{ and } -v < u \leq v\}$. On the surface $v = -u$, we have $T = 0$ and hence the Kretschmann invariant diverges. So this surface corresponds to a spacelike singularity. There is a cylindrical trapping horizon on $u = 0$. The region $\{0 < u \leq v < \infty\}$ is untrapped, while the region $\{0 < -u < v < \infty\}$ is trapped.

If $n = 2l$ ($l = 1, 2, \dots$), the maximally extended spacetime is apparently given by $-\infty < u \leq v < \infty$. $u = -\infty$ is a null infinity. $v = -u = \infty$ is a spacelike infinity. $u = v = 0$ is a spacetime singularity. $v = u = \pm\infty$ are both

timelike infinities. It is interesting to see the surface $v = -u$. Noting $T = (v^n + u^n)/2$ and $X = (v^n - u^n)/2$, we find that the Kretschmann invariant is finite there except for $u = v = 0$, while the areal radius r vanishes. It turns out that we need to pay close attention to this surface. To get an insight into this surface, we introduce t and x coordinates, where $u = t - x$ and $v = t + x$, so that we should focus on the surface $t = 0$. Near this surface, the metric can be written as

$$ds^2 \simeq [2(1 - 2\kappa^2)]^{-2} x^{2\kappa(2\kappa-1)/(1-2\kappa^2)} [e^{2\lambda}(-dt^2 + dx^2) + t^2 d\varphi^2] + x^{2\kappa/(1-2\kappa^2)} dz^2.$$

It follows from the identification between $\varphi = 0$ and $\varphi = 2\pi$ that there is a timelike geodesic on the $t\varphi$ plane with infinite spatial acceleration, as shown in the appendix, and hence the spacetime is geodesically incomplete in a sense that there is a geodesic which cannot be uniquely extended. Thus, there is no analytical extension beyond this surface. Then, the structure of the resultant spacetime is similar to the $n = 2l + 1$ case, except for that ‘‘singularity’’ on the surface $t = 0$ is only ‘‘quasiregular’’ in the sense of Ellis and Schmidt [30].

It should be noted that it is impossible to analytically extend the spacetime even beyond $u = 0$ even for $3/8 < \kappa^2 < 1/2$ if $n = 1/[2(1 - 2\kappa^2)]$ is not an integer. For this case, the functions U and V are at least twice differentiable but not C^∞ with respect to u at $u = 0$. The spacetime admits at most $C^{[n]}$ extension beyond the null surface $u = 0$, where $[n]$ is the largest integer which is no greater than n .

The conformal diagrams of the solutions for different values of κ are given in Figs. 3(a) and 3(b). As we can see in these figures, the case of integer n is particularly intriguing in the context of gravitational collapse because these solutions are self-similar, describe the collapse of gravitational waves and admit nonsingular initial data on a spacelike Cauchy surface containing both trapped and untrapped cylinders.

3. $\kappa^2 > 1/2$

Also in this case, we introduce u and v as

$$U = (-u)^n \quad \text{and} \quad V = (-v)^n, \quad (5.13)$$

where $n = 1/[2(1 - 2\kappa^2)] < 0$ and we put the negative signs to keep the increase of u and v corresponding to that of U and V , respectively. Then, the line element is given by exactly the same form as Eq. (5.11). The original domain is mapped to $-\infty < u \leq v < 0$. Since both $u = -\infty$ and $v = 0$ correspond to null infinities, the original domain describes the whole spacetime. Noting $T = [(-v)^n + (-u)^n]/2$ and $X = [(-v)^n - (-u)^n]/2$, we find that $T = X = 0$ corresponds to $v = u = -\infty$, while $(T, X) = (\infty, 0)$ corresponds to $v = u = 0$. The conformal diagrams of the solutions for different values of κ are

therefore given in Figs. 3(d)–3(f). There is no cylindrical trapping horizon and the whole spacetime is untrapped in this case.

In summary of this section, the two-parameter family of solutions describe a variety of global structures. They are classified in terms of κ , or equivalently, $n = [2(1 - 2\kappa^2)]^{-1}$. For $\kappa = 0, 1/2$ and $-1/2$, the solution reduces to the Minkowski, the cylindrical Milne, and the Kasner solutions, respectively. If $0 < \kappa^2 < 1/4$ or $1/4 < \kappa^2 < 3/8$, the spacetime describes the interior of the exploding (imploding) cylindrical shell of gravitational waves. For $3/8 \leq \kappa^2 < 1/2$, we have the following three cases: if $n = 3, 5, 7, \dots$, the spacetime describes the collapsing gravitational waves to a spacelike singularity or exploding gravitational waves from a spacelike curvature singularity; if $n = 2, 4, 6, \dots$, the spacetime structure is quite similar to the odd n case but the spacelike curvature singularity is replaced by a quasiregular one; if $n > 2$ is not an integer, the spacetime does not admit an analytic extension beyond the null surface. For $\kappa^2 \geq 1/2$ and $\kappa \neq 1$, the conformal diagram is similar to the Minkowski one except for that a singularity replaces a timelike infinity. For $\kappa = 1$, the conformal diagram is similar to the Minkowski one and the whole spacetime is regular.

It is interesting to note that the present analysis proceeds quite analogously to Wang’s [12] for cylindrically symmetric self-similar solutions with a massless scalar field in $(3 + 1)$ dimensions and hence Hirschmann’s *et al.* [31] for circularly symmetric self-similar solutions with a massless scalar field in $(2 + 1)$ dimensions although the system and the result are both different in detail. This follows from the fact that the governing system of partial differential equations are quite similar for these systems. It is also interesting to note that the present solution generically involves singularities, because one does not usually get a singularity from the collapse of cylindrical waves (global regularity of Einstein-Rosen waves: see e.g. Ashtekar *et al.* [9]). In order to generate the singularity, initial data on the Cauchy surface must be non asymptotically flat. So we have a sort of converse result: if we allow non asymptotically flat initial data, then a singularity can form solely from the collapse of gravitational waves. Furthermore, if trapped surfaces are initially absent, then they cannot form (see Thorne [3]). In the relevant cases [Fig. 3(b), integer $n \geq 2$], we have a nonsingular Cauchy surface which already contains trapped cylinders.

VI. CONCLUSIONS

In this paper, we have studied self-similar vacuum spacetimes in whole-cylinder symmetry: self-similar Einstein-Rosen waves. The primary motivation was to determine possible (intermediate) asymptotic endstates for more general Einstein-Rosen waves, and for other cylindrical systems. That is, we wish to study the self-similarity hypothesis in cylindrical symmetry, especially

in the context of gravitational collapse. There is a considerable body of evidence for the hypothesis in spherical collapse (see e.g. [19]) and there is also evidence for the hypothesis in the context of cosmological models: Hewitt *et al.* have shown that among a class of cylindrical inhomogeneous cosmological models, there are self-similar models which are asymptotic endstates for the general class [32].

Assuming a homothetic vector orthogonal to the cylinders of symmetry, we have obtained the standard form of the metric in cylindrically symmetric self-similar spacetimes. We have then applied this to the vacuum case and obtained solutions. In fact, the obtained solutions are all flat. We have explicitly shown that the spacetimes are part of the Minkowski spacetime with a regular or conically singular axis and with trivial or nontrivial topology. Although such spacetimes can emerge as the endstate of complete cylindrical gravitational collapse—for example, of cylindrical null dust [11], this cannot be interpreted as evidence for the self-similarity hypothesis as the spacetimes are flat and therefore only trivially self-similar.

Using the obtained self-similar expression for part of the Minkowski spacetime, we have argued that the C energy which is supposed to represent the gravitational energy per specific length of the cylindrically symmetric spacetime is subject to the choice of the translational Killing vector even if one chooses the same regular axis. We have also discussed that there exists a cylindrical trapping horizon in the Minkowski spacetime and that the notion of trapping horizons might not be useful for defining black holes in cylindrically symmetric spacetimes—at least, not in the case where the marginal two-surfaces foliating the horizon are cylinders.

Next, we have extended the analysis to the more general class of Einstein-Rosen waves, still respecting some kind of scaling behavior. Assuming a regular or conically singular axis, we have obtained a two-parameter family of non flat self-similar solutions, where the homothetic vector is not orthogonal to the cylinders in general. We have seen that the solution physically describes the interior of the exploding (imploding) shell of gravitational waves or the collapse (explosion) of gravitational waves depending on the parameter choice. Additionally, as a special case we have obtained a solution with a non azimuthal and non translational Killing vector which is not orthogonal to the cylinders. There is also a discrete subset of solutions which exhibit the collapse of gravitational waves developed from nonsingular initial data on a spacelike Cauchy surface.

We conclude that self-similar Einstein-Rosen waves can describe nontrivial dynamics of gravitational waves if and only if the homothetic vector is not orthogonal to the cylinders of symmetry. Although the original proposal for the self-similarity hypothesis in general relativity is restricted in spherical symmetry, it is also likely that some of these self-similar solutions can describe the

asymptotic behavior of more general solutions even in cylindrical symmetry. In fact, recent numerical simulations [27] strongly suggest that the asymptotic behavior of a dispersing gravitational wave within the null hypersurface $t^2 = x^2$ after the collapse of a dust cylinder is well approximated by a member of the family of solutions obtained here with $\kappa = -0.0206$ and $\lambda \neq 0$ (see Figs. 8 and 9 of [27]). The present paper clarifies that this asymptotic solution belongs to the family of self-similar Einstein-Rosen waves with a non cylindrical homothetic vector and a conical singularity and that this asymptotic solution corresponds to gravitational waves inside the exploding shell of gravitational waves as shown in Fig. 3(a). Hence, it would be reasonable to generalize the self-similarity hypothesis—including in the context of gravitational collapse—as follows: under certain physical circumstances, solutions will naturally evolve to a self-similar form not only in spherical symmetry but also in a variety of symmetry classes.

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APPENDIX A: TIMELIKE GEODESICS WITH INFINITE SPATIAL ACCELERATION IN THE PLANE SYMMETRIC CLOSED MILNE UNIVERSE

The line element of the two-dimensional closed Milne universe is given by

$$ds^2 = -dT^2 + T^2 d\varphi^2, \quad (\text{A1})$$

where $0 \leq \varphi < 2\pi$ and $\varphi = 0$ and $\varphi = 2\pi$ with the same T are identified with each other. This is obtained from Eq. (3.25) or (3.31) by omitting the two-dimensional plane part. The Milne universe is locally identical to the Minkowski spacetime. In fact, the coordinate transformation

$$\tau = T \cosh \varphi \quad \text{and} \quad \zeta = T \sinh \varphi, \quad (\text{A2})$$

leads to the line element of standard form of the Minkowski spacetime

$$ds^2 = -d\tau^2 + d\zeta^2. \quad (\text{A3})$$

In this (τ, ζ) -coordinate system, the curves $\zeta = 0$ and $\tau = \zeta/V$ are identified with each other, where $V = \tanh 2\pi$.

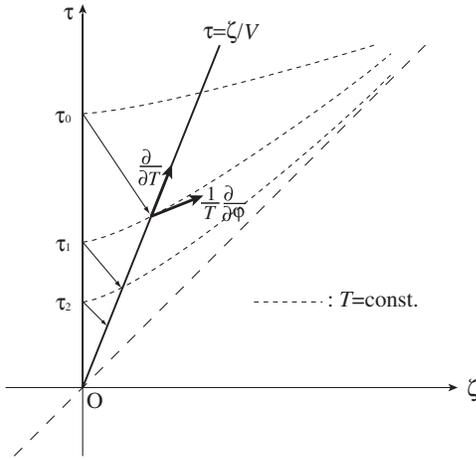


FIG. 4. The closed Milne universe in the Minkowski spacetime.

More precisely, $(\tau, 0)$ is identified with $(\tau/\sqrt{1-V^2}, \tau V/\sqrt{1-V^2})$, where and hereafter we use the standard Cartesian coordinates (τ, ζ) of the Minkowski spacetime. See Fig. 4.

We consider a past-directed timelike geodesic which starts from $p_0: (\tau, \zeta) = (\tau_0, 0)$ with a unit tangent vector

$$\mathbf{u} = \mathbf{u}_{(0)} := -\frac{1}{\sqrt{1-v_0^2}} \frac{\partial}{\partial \tau} + \frac{v_0}{\sqrt{1-v_0^2}} \frac{\partial}{\partial \zeta}, \quad (\text{A4})$$

where $0 < v_0 < 1$ is assumed. This geodesic and the curve $\tau = \zeta/V$ intersect at the event

$$\tilde{p}_0: (\tau, \zeta) = \left(\frac{v_0 \tau_0}{V + v_0}, \frac{V v_0 \tau_0}{V + v_0} \right). \quad (\text{A5})$$

This event is identified with

$$p_1: (\tau, \zeta) = (\tau_1, 0), \quad (\text{A6})$$

where

$$\tau_1 := \frac{v_0 \sqrt{1-v_0^2}}{V + v_0} \tau_0. \quad (\text{A7})$$

We can determine the tangent to the geodesic at p_1 as follows. In coordinates (T, φ) , the points $\tilde{p}_0: (T_0, 0)$ and $p_1: (T_0, 2\pi)$ are identified. We must also identify the unit vector fields

$$\frac{\partial}{\partial T} \Big|_{\tilde{p}_0} = \frac{\partial}{\partial T} \Big|_{p_1}, \quad \frac{1}{T} \frac{\partial}{\partial \varphi} \Big|_{\tilde{p}_0} = \frac{1}{T} \frac{\partial}{\partial \varphi} \Big|_{p_1}. \quad (\text{A8})$$

The coordinate transformation (A2) leads to

$$\frac{\partial}{\partial T} = \frac{\tau}{T} \frac{\partial}{\partial \tau} + \frac{\zeta}{T} \frac{\partial}{\partial \zeta}, \quad \frac{1}{T} \frac{\partial}{\partial \varphi} = \zeta \frac{\partial}{\partial \tau} + \tau \frac{\partial}{\partial \zeta}, \quad (\text{A9})$$

and the corresponding inverse relationship:

$$\frac{\partial}{\partial \tau} = \frac{\tau}{T} \frac{\partial}{\partial T} - \frac{\zeta}{T^3} \frac{\partial}{\partial \varphi}, \quad \frac{\partial}{\partial \zeta} = -\frac{\zeta}{T} \frac{\partial}{\partial T} + \frac{\tau}{T^3} \frac{\partial}{\partial \varphi}. \quad (\text{A10})$$

We can use this last equation to expand $u_{(0)}^\mu|_{\tilde{p}_0}$ in terms of $\frac{\partial}{\partial T}$ and $T^{-1} \frac{\partial}{\partial \varphi}$, invoke the identification (A8), revert to (τ, ζ) coordinates by using (A9), and hence obtain the unit tangent to the geodesic at p_1 in coordinates (τ, ζ) as

$$\mathbf{u}_{(1)} = -\frac{1}{\sqrt{1-v_1^2}} \frac{\partial}{\partial \tau} + \frac{v_1}{\sqrt{1-v_1^2}} \frac{\partial}{\partial \zeta}, \quad (\text{A11})$$

where

$$v_1 = \frac{v_0 + V}{1 + V v_0}. \quad (\text{A12})$$

It is also straightforward to show that in the transition from $(\tau_0, 0)$ to $(\tau_1, 0)$, a proper time of duration

$$s_0 := \frac{V \sqrt{1-v_0^2}}{v_0 + V} \tau_0$$

elapses along the geodesic.

Thus, this timelike geodesic goes through the points $(\tau, \zeta) = (\tau_1, 0), (\tau_2, 0), \dots$. We can derive the recursion relations

$$\tau_{i+1} = \frac{v_0 \sqrt{1-v_i^2}}{V + v_i} \tau_i, \quad (\text{A13})$$

$$v_{i+1} = \frac{v_i + V}{1 + V v_i}, \quad (\text{A14})$$

$$s_{i+1} = \frac{v_i \sqrt{1-v_i^2} \sqrt{1-V^2}}{v_i + 2V + v_i V^2} s_i. \quad (\text{A15})$$

Here we introduce a new variable

$$\delta_i := 1 - v_i, \quad (\text{A16})$$

and rewrite Eqs. (A13) and (A14) as

$$\tau_{i+1} = \frac{(1 - \delta_i) \sqrt{\delta_i(2 - \delta_i)}}{V + 1 - \delta_i} \tau_i, \quad (\text{A17})$$

$$\delta_{i+1} = \frac{1 - V}{1 + V(1 - \delta_i)} \delta_i. \quad (\text{A18})$$

From Eq. (A18), we have

$$\frac{\delta_{i+1}}{\delta_i} = \frac{1 - V}{1 + V(1 - \delta_i)}. \quad (\text{A19})$$

Since $0 < \delta_0 < 1$ and $0 < V < 1$, we have $0 < \delta_1 < \delta_0$ and hence $0 < \delta_{i+1} < \delta_i$. This implies

$$0 < \delta_{i+1} < \left[\frac{1-V}{1+V(1-\delta_0)} \right]^i \delta_0. \quad (\text{A20})$$

Therefore,

$$\lim_{i \rightarrow \infty} \delta_i = 0. \quad (\text{A21})$$

As for τ_i , since $0 < \delta_i < \delta_0 < 1$ for $i \geq 1$, we have from Eq. (A17)

$$0 < \frac{\tau_{i+1}}{\tau_i} = \frac{(1-\delta_i)\sqrt{\delta_i(2-\delta_i)}}{V+1-\delta_i} < \frac{1-\delta_i}{V+1-\delta_i} < 1. \quad (\text{A22})$$

Therefore,

$$\lim_{i \rightarrow \infty} \tau_i = 0. \quad (\text{A23})$$

Thus the three-velocity v of the timelike geodesic which approaches $p_\infty:(\tau, \zeta) = (0, 0)$ becomes asymptotically the speed of light. The spatial part of the past-directed timelike geodesic is infinitely accelerated in an approach to the origin $p_\infty:(\tau, \zeta) = (0, 0)$.

We note however that p_∞ lies at a finite time in the past along the history of the geodesic. The total proper time that elapses along the geodesic is given by

$$s = \sum_{i=0}^{\infty} s_i.$$

However,

$$\frac{s_{i+1}}{s_i} = \frac{v_i \sqrt{1-v_i^2} \sqrt{1-V^2}}{v_i + 2V + v_i V^2} \rightarrow 0, \quad i \rightarrow \infty,$$

so the series converges: $s < +\infty$.

Thus, there is no unique extension of the geodesics beyond $(\tau, \zeta) = (0, 0)$ and this behavior of geodesics is quite analogous to that around a conical singularity. This is solely due to the topological identification in φ and not related to the blow up of curvature. This corresponds to a quasiregular singularity defined by Ellis and Schmidt [30].

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