EXPONENTIAL ASYMPTOTIC STABILITY FOR LINEAR VOLTERRA EQUATIONS

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Abstract

This note studies the exponential asymptotic stability of the zero solution of the linear Volterra equation

$$\dot{x}(t) = Ax(t) + \int_0^t K(t-s)x(s)\,ds$$

by extending results in the paper of Murakami "Exponential Asymptotic Stability for scalar linear Volterra Equations", Differential and Integral Equations, 4, 1991. In particular, when K is integrable and has entries which do not change sign, and the equation has a uniformly asymptotically stable solution, exponential asymptotic stability can be identified by an exponential decay condition on the entries of K.

1. Introduction

In this note, we study the exponential asymptotic stability of the zero solution of the linear Volterra equation

$$\dot{x}(t) = Ax(t) + \int_0^t K(t-s)x(s)\,ds, \quad t \geq 0 \tag{1.1}$$

where the solution vector x is in $\mathbf{R}^n$, A is a constant $n \times n$ matrix, and K is an $n \times n$ matrix function with integrable entries. The question as to whether the exponential asymptotic stability of the zero solution of (1.1) is necessarily implied by its uniform asymptotic stability was posed in the survey of Corduneanu and Lakshmikantham [2]. As remarked in that article, the question is a reasonable one as the right hand side of (1.1) is time homogeneous, and in view of the fact that uniform asymptotic stability and exponential asymptotic stability are equivalent concepts for linear functional differential equations with bounded delay.

The paper of Murakami [6], on which our study is based, throws considerable light on the nature of the relationship between uniform asymptotic stability and exponential asymptotic stability of the zero solution of (1.1), detailing the connection between the two types of stability for scalar equations of type (1.1).

The first theorem of Murakami's paper [Theorem 1] neatly characterises sufficient conditions under which uniform asymptotic stability of the zero solution will imply its exponential asymptotic stability. Specifically, he shows that when the kernel satisfies an exponential decay condition, the two concepts coincide. Indeed, the result can be readily generalised to Volterra equations in an arbitrary number of finite dimensions.

The second result in the paper is a very striking one. The author proves [Theorem 2, [6]] that when the kernel is integrable and does not change sign on $\mathbf{R}^+$, and the zero solution is exponentially asymptotically stable, then the kernel satisfies an exponential decay condition. Thus, for scalar equations of type (1.1), exponential asymptotic stability will only arise if the kernel decays exponentially [Theorem 3, [6]].

As is mentioned at the end of Murakami's paper, it is reasonable to conjecture that this result can be extended to systems with an arbitrary number of finite dimensions. The proof of this conjecture is the subject of the following note. We mirror Theorem 3 in [6] by showing for a system in which the entries of the matrix kernel K do not change sign that the exponential and uniform asymptotic stability of the zero solution are equivalent if and only if all the entries of K satisfy an exponential decay criterion.

2. Preliminary Material

In this section, we introduce some notation, give definitions and allude to some key results employed in this article.

Let $\mathbf{R} = (-\infty, \infty)$ and $\mathbf{R}^+ = [0, \infty)$. Let $M_n(\mathbf{R})$ be the space of $n \times n$ matrices with real entries. We say that the function $K : \mathbf{R}^+ \to M_n(\mathbf{R})$ is in $L^1(\mathbf{R}^+)$ if each of its entries is a scalar Lebesgue integrable function. Denote by $C(J; \mathbf{R}^n)$ the space of continuous $\mathbf{R}^n$-valued functions on the compact interval $J \subset \mathbf{R}$. For $\phi \in C([0, t_0]; \mathbf{R}^n)$, define $|\phi|_{t_0} = \sup\{\|\phi(t)\|_1 \, : \, 0 \le t \le t_0\}$, where $\|x\|_1$ is the sum of the absolute values of the entries of the vector $x \in \mathbf{R}^n$. For $K = (k_{ij}) \in M_n(\mathbf{R})$, we define $\|K\|_N = \sum_{i=1}^n \sum_{j=1}^n |k_{ij}|$. Note that this defines a norm on a finite dimensional linear space, so that all other norms on this space are equivalent to $\| \cdot \|_N$. We denote by $\mathbf{C}$ the set of complex numbers, and the real part of $s \in \mathbf{C}$ by $\Re s$. If $f : \mathbf{R}^+ \to M_n(\mathbf{R})$, we can define the Laplace transform of f at $s \in \mathbf{C}$ to be

$$\hat{f}(s) = \int_0^\infty f(t) e^{-st} \, dt.$$

If $\alpha \in \mathbf{R}$ and $\int_0^\infty \|f(t)\|_N e^{-\alpha t} \, dt < \infty$, then $\hat{f}(s)$ exists and is continuous in s for $\Re s \ge \alpha$, and analytic on $\Re s > -\alpha$. See for example, Churchill, p.171 [1], or Widder [7].

Consider the linear Volterra equation

$$\dot{x}(t) = Ax(t) + \int_0^t K(t-s) x(s) \, ds, \quad t \ge 0 \tag{2.1}$$

where A, $K(t) \in M_n(\mathbf{R})$, K is continuous on $\mathbf{R}$ and $K \in L^1(\mathbf{R}^+)$. For any $t_0 \ge 0$ and $\phi \in C([0, t_0], \mathbf{R}^n)$, there is a unique $\mathbf{R}^n$-valued function $x(t)$, which satisfies (2.1) on $[t_0, \infty)$ and for which $x(t) = \phi(t)$ for $t \in [0, t_0]$. We denote such a solution by $x(t; t_0, \phi)$. The function $x(t) \equiv 0$ is a solution of (2.1) and is called the zero solution of (2.1).

Consider now the matrical equation

$$X'(t) = AX(t) + \int_0^t K(t-s)X(s)\,ds, \quad t \geq 0 \tag{2.2}$$

with $X(0) = I_n$, where I_n is the identity matrix in $M_n(\mathbf{R})$. The unique $X(t) \in M_n(\mathbf{R})$ which satisfies (2.2) is called the resolvent, or principal matrix solution for (2.1).

We further recall the various standard notions of stability of the zero solution required for our analysis. The zero solution of (2.1) is said to be *uniformly stable* (US), if, for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $t_0 \in \mathbf{R}^+$ and $\phi \in C([0,t_0],\mathbf{R}^n)$ with $|\phi|_{t_0} < \delta(\varepsilon)$ implies $\|x(t;t_0,\phi)\|_1 < \varepsilon$ for all $t \geq t_0$. The zero solution is said to be *uniformly asymptotically stable* (UAS) if is US and there exists $\delta > 0$ with the following property: for each $\varepsilon > 0$ there exists a $T(\varepsilon) > 0$ such that $t_0 \in \mathbf{R}^+$ and $\phi \in C([0,t_0],\mathbf{R}^n)$ with $|\phi|_{t_0} < \delta$ implies $\|x(t,t_0,\phi)\|_1 < \varepsilon$ for all $t \geq t_0 + T(\varepsilon)$. The zero solution of (2.1) is said to be *exponentially asymptotically stable* (Ex AS), if there exists $C, \alpha > 0$ such that

$$\|x(t;t_0,\phi)\|_1 \leq Ce^{-\alpha(t-t_0)}|\phi|_{t_0}, \quad t \geq t_0 \geq 0$$

for any $\phi \in C([0,t_0],\mathbf{R}^n)$.

These definitions are standard: the reader may refer to Miller [5]. Existence and uniqueness are covered in [3].

The properties of the resolvent X are deeply linked to the stability of the zero solution of (2.1). It is shown in [5] that the zero solution of (2.1) is UAS if and only if $X \in L^1(\mathbf{R}^+)$. Moreover, in [4], it is shown that $X \in L^1(\mathbf{R}^+)$ if and only if

$$H(s) = sI_n - A - \hat{K}(s) \tag{2.3}$$

satisfies $\det H(s) \neq 0$ for $\Re s \geq 0$.

3. Results

If the zero solution of (2.1) is UAS, $X \in L^1(\mathbf{R}^+)$, so $\hat{X}(s)$ exists and is continuous in s for $\Re s \geq 0$. Taking Laplace transforms both sides of (2.2), we get $H(s)\hat{X}(s) = I_n$ for $\Re s > 0$. By the continuity of $\hat{X}$, H, we have

$$H(s)\hat{X}(s) = I_n \quad \Re s \geq 0. \tag{3.1}$$

Thus $\hat{X}(s) = H^{-1}(s)$, for $\Re s \geq 0$, as $\det H(s) \neq 0$ for $\Re s \geq 0$.

Theorem 1 in [6] says that if

$$\int_0^\infty |K(t)|e^{\gamma t}\,dt < \infty, \text{ for some } \gamma > 0 \tag{3.2}$$

and the zero solution of the scalar equation (2.1) is uniformly asymptotic stable, then the zero solution is also Ex AS.

It is mentioned in [6] that the proof can be generalised, with minor modifications, to establish a corresponding result for systems.

Theorem 3.1 (Murakami). *If the zero solution of (2.1) is UAS, and the kernel K satisfies*

$$\int_0^\infty \|K(t)\|_N e^{\gamma t}\, dt < \infty \quad \textit{for some } \gamma > 0$$

then the zero solution is Ex AS.

We now turn to a partial converse of Theorem 3.1, which is strongly motivated by Theorem 2 in [6], and whose proof it follows very closely. The hypotheses of that theorem can be extracted from the following by setting the dimension $n = 1$ and considering the scalar case. Indeed, the proof of our Proposition 3.2 is exactly that of Theorem 2 in [6] with $n = 1$. However, Murakami did not succeed in obtaining the extension to finite dimensions. In view of this gap, the nuances involved in passing from the scalar to the general finite dimensional case, and to achieve a coherent exposition, we show the details here.

Theorem 3.2. *Let $K \in L^1(\mathbf{R}^+)$ and suppose that none of the entries of $K(t) \in M_n(\mathbf{R})$ change sign on $\mathbf{R}^+$. If $\|X(t)\|_N \le Ce^{-\beta t}$, for some $C, \beta > 0$, then there exists $\gamma > 0$ such that*

$$\int_0^\infty \|K(t)\|_N e^{\gamma t}\, dt < \infty. \tag{3.3}$$

Proof. Let $0 < \alpha < \beta$. Clearly, by hypothesis, $\hat{X}(s)$ exists and is analytic in s for $\Re s > -\alpha$ and continuous for $\Re s \ge -\alpha$. Patently, (3.1) holds. Since the zero solution is UAS, $\det H(0) \ne 0$, so $\det \hat{X}(0) \ne 0$. Since $\hat{X}(s)$ is continuous at $s = 0$, and the determinant of a matrix is a continuous function of its entries, $s \mapsto \det \hat{X}(s)$ is continuous at $s = 0$. Therefore, there exists an open neighbourhood of 0 (U', say) such that $\det \hat{X}(s) \ne 0$ for $s \in U'$: thus $\hat{X}^{-1}(s)$ exists on U'. Since the entries of $\hat{X}(s)$ are analytic on $U := U' \cap \{s : \Re s > -\alpha\}$, so must be the entries of $\hat{X}^{-1}(s)$. Thus

$$F(s) = sI_n - A - \hat{X}^{-1}(s)$$

is analytic on U, and satisfies

$$\hat{K}(s) = F(s), \quad \Re s \ge 0. \tag{3.4}$$

The proof of the result now follows by a sequence of contradictions: we posit that $t^n K(t) \notin L^1(\mathbf{R}^+)$ for $n = 1, 2, \ldots$ in turn, and show that each of these hypotheses introduces a contradiction. The existence of the "moments" of K, together with the analyticity of F and (3.4) enables us to prove (3.3).

We claim $tK(t) \in L^1(\mathbf{R}^+)$. If this is false, there exists $T > 1$ such that

$$\int_0^T \|K(t)\|_N (t-1)\, dt > M$$

where $M = \|F'(0)\|_N$. Next, it is possible to construct $\delta_T = \min(1, e^{-T})$, so that if

$0 < h < \delta_T$ then

$$\frac{1 - e^{-ht}}{h} \geq t - 1, \quad t \in [0, T]. \tag{3.5}$$

Thus for $0 < h < \delta_T$, we have

$$\begin{aligned}
\left\| \frac{F(h) - F(0)}{h} \right\|_N &= \left\| \frac{\hat{K}(h) - \hat{K}(0)}{h} \right\|_N \\
&= \left\| \int_0^\infty K(t) \frac{1 - e^{-ht}}{h} \, dt \right\|_N \\
&= \sum_{i=1}^n \sum_{j=1}^n \left| \int_0^\infty k_{ij}(t) \frac{1 - e^{-ht}}{h} \, dt \right| \\
&= \sum_{i=1}^n \sum_{j=1}^n \int_0^\infty |k_{ij}(t)| \frac{1 - e^{-ht}}{h} \, dt,
\end{aligned}$$

using the fact that $k_{ij}(t)$ has the same sign on $\mathbf{R}^+$ at the last step. From (3.5), for $0 < h < \delta_T$, we thus have

$$\left\| \frac{F(h) - F(0)}{h} \right\|_N \geq \sum_{i=1}^n \sum_{j=1}^n \int_0^T |k_{ij}(t)|(t-1) \, dt = \int_0^T \|K(t)\|_N (t-1) \, dt.$$

Using the continuity of the norm $\| \cdot \|_N$, we obtain

$$M = \|F'(0)\|_N = \lim_{h \to 0^+} \left\| \frac{F(h) - F(0)}{h} \right\|_N \geq \int_0^T \|K(t)\|_N (t-1) \, dt > M,$$

a contradiction. Thus $tK(t) \in L^1(\mathbf{R}^+)$.

Next, since $1 - x \leq e^{-x}$ for $x \geq 0$, for $h > 0$, $s \geq 0$ and $t \in \mathbf{R}^+$, we have

$$\left| \frac{1 - e^{-ht}}{h} e^{-st} k_{ij}(t) \right| = t \left| \frac{1 - e^{-ht}}{ht} \right| |e^{-st}| |k_{ij}(t)| \leq t |k_{ij}(t)|. \tag{3.6}$$

Thus, using the Dominated Convergence Theorem, in conjunction with (3.4) and (3.6), we get

$$\begin{aligned}
\lim_{h \to 0^+} \frac{F_{ij}(s+h) - F_{ij}(s)}{h} &= \lim_{h \to 0^+} \frac{\hat{k}_{ij}(s+h) - \hat{k}_{ij}(s)}{h} \\
&= \lim_{h \to 0^+} \int_0^\infty \frac{e^{-ht} - 1}{h} e^{-st} k_{ij}(t) \, dt \\
&= \int_0^\infty -t e^{-st} k_{ij}(t) \, dt,
\end{aligned}$$

so $F_{ij}(s) = -\int_0^\infty t e^{-st} k_{ij}(t) \, dt$ for $s \geq 0$. Therefore $F'(s) = -\int_0^\infty t e^{-st} K(t) \, dt$.

By applying the argument for $K(t)$ in the foregoing paragraphs to $tK(t)$, we obtain

$$t^2K(t) \in L^1(\mathbf{R}^+), \qquad F''(s) = \int_0^\infty t^2 e^{-st}K(t)\,dt, \quad s \geq 0,$$

and repeating this procedure gives

$$t^nK(t) \in L^1(\mathbf{R}^+), \qquad F^{(n)}(s) = (-1)^n \int_0^\infty t^n e^{-st}K(t)\,dt, \quad s \geq 0 \tag{3.7}$$

for $n = 1, 2, \ldots$. Since F is analytic on U, we have that

$$\sum_{m=0}^{\infty} \frac{F_{ij}^{(m)}(0)}{m!} s^m$$

is absolutely convergent on the disc $D = \{z \in \mathbf{C} : |z| < \gamma\}$, for some $\gamma > 0$ and all $i, j = 1, \ldots n$. Thus, by (3.7), we have

$$\infty > \sum_{m=0}^{\infty} \frac{|F_{ij}^{(m)}(0)|}{m!}\gamma^m = \sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \int_0^\infty t^m |k_{ij}(t)|\,dt, \tag{3.8}$$

and so, using Fatou's Lemma

$$\begin{aligned}
\int_0^\infty e^{\gamma t}\|K(t)\|_N\,dt &= \int_0^\infty e^{\gamma t} \sum_{i=1}^n \sum_{j=1}^n |k_{ij}(t)|\,dt = \sum_{i=1}^n \sum_{j=1}^n \int_0^\infty e^{\gamma t}|k_{ij}(t)|\,dt \\
&= \sum_{i=1}^n \sum_{j=1}^n \int_0^\infty \sum_{m=0}^{\infty} \frac{(\gamma t)^m}{m!}|k_{ij}(t)|\,dt \\
&= \sum_{i=1}^n \sum_{j=1}^n \left[\sum_{m=0}^{\infty} \frac{\gamma^m}{m!} \int_0^\infty t^m|k_{ij}(t)|\,dt\right] < \infty,
\end{aligned}$$

by (3.8), and the proof is complete. ■

We now have all the ingredients to prove Theorem 3.3.

Theorem 3.3. *Suppose that $K \in L^1(\mathbf{R}^+)$ and that the entries of K do not change sign on $\mathbf{R}^+$. If the zero solution of (1.1) is uniformly asymptotically stable, then the following are equivalent:*

(i) The zero solution of (1.1) is exponentially asymptotically stable,

(ii) The principal matrix solution of (1.1), $X(t)$, satisfies $\|X(t)\| \leq Ce^{-\alpha t}$ for some $C, \alpha > 0$,

(iii) There exists $\gamma > 0$ such that

$$\int_0^\infty \|K(t)\|e^{\gamma t}\,dt < \infty,$$

where $\|\cdot\|$ is any matrix norm.

PROOF. Since $M_n(\mathbf{R})$ is a finite dimensional linear space, norm equivalence in this space means that the result need only be established for the special choice of norm $\|\cdot\|_N$.

Clearly, (i) implies (ii) trivially. That (ii) implies (iii) is the subject of Theorem 3.2. (iii) implies (i) is simply Theorem 3.1. Hence (i)-(iii) are equivalent. ■

The proof of Theorem 3.3 shows how Theorem 3 in [6] can be extended to the non-scalar case, a problem which was proposed in [6].

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