Naked Singularities in Self-Similar Gravitational Collapse: Stability Properties of the Cauchy Horizon

Emily M. Duffy B.Sc. MASt.
School of Mathematical Sciences
Dublin City University

Supervisor: Dr. Brien C. Nolan

A Thesis Submitted for the Degree of Doctor of Philosophy

September 2011

Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Doctor of Philosophy in Mathematics is entirely my own work, that I have exercised reasonable care to ensure that the work is original, and does not to the best of my knowledge breach any law of copyright, and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

Signed: ______________________ ID Number: ________________

Date: ________________________

Acknowledgements

I am very grateful to my supervisor, Dr. Brien Nolan, for his enthusiasm, his dedication and his insight into this subject, which proved invaluable throughout this research.

For thought-provoking and enlightening conversations, I would like to thank Mr Eoin Condron, Ms Órlaith Mannion and Dr. Marc Casals. I would also like to thank the staff and postgraduate students of the School of Mathematical Sciences for their support and friendship throughout my Ph.D.

I would like to thank my family and friends for all their love and support, especially Darran for all his love.

This research was funded by the Irish Research Council for Science, Engineering and Technology, grant number P02955.

Contents

Abstract

The background of this thesis is the cosmic censorship hypothesis, which states that the gravitational collapse of physically reasonable matter should not result in the formation of naked singularities. In the absence of a proof of this hypothesis, much effort has been directed towards examining spacetimes which contain naked singularities, in an attempt to determine the nature of the cosmic censor. One area of particular interest is the study of perturbations in naked singularity spacetimes. Should perturbations of a spacetime diverge on the Cauchy horizon associated with the naked singularity, then this spacetime can be ruled out as a serious counter-example to cosmic censorship. In this thesis we examine the behaviour of general linear perturbations of the class of self-similar Lemaître-Tolman-Bondi spacetimes which contain a naked singularity. The perturbations naturally split into two classes, odd and even parity, which we consider in turn. For the odd parity perturbation, we first identify a single gauge invariant scalar which describes the perturbation and obeys an inhomogeneous wave equation. We then show that a perturbation which evolves from initially regular data remains finite on the Cauchy horizon. Finiteness is demonstrated by considering the behaviour of suitable energy norms of the perturbation (and pointwise values of these quantities) on natural spacelike hypersurfaces. For the even parity perturbations, we first show that a particular average of the state variable describing the perturbations generically diverges at the Cauchy horizon. Using this, we show that the L^p-norm of the perturbations also diverges, for $1 \leq p \leq \infty$. This divergence has a characteristic form that depends only on the background spacetime. By combining these results with an extension of odd parity methods, as well as some theorems from real analysis, we can demonstrate that the perturbations generically diverge pointwise on the Cauchy horizon. A general perturbation is a sum of odd and even perturbations; our results therefore indicate that a general perturbation diverges on the Cauchy horizon. This result supports the cosmic censorship hypothesis.

List of Figures

Part I
Introduction and Background

Chapter 1

Introduction

We consider here the gravitational collapse of a massive body. The standard model of gravitational collapse states that if the mass of the body exceeds the Chandrasekhar limit, then once it begins to collapse it will continue to do so, eventually forming a singularity which is hidden behind an event horizon. Since the event horizon's formation preceded that of the singularity, the external universe is "shielded" from the singularity and will receive no matter or radiation originating from it [56]. Cosmic censorship aims to ensure the validity of this model by showing that (under certain conditions) an event horizon must always form.

In certain collapse scenarios the usual order of singularity and event horizon formation is reversed, so that the singularity is visible to the external universe. Such singularities are known as naked singularities and numerous examples, mostly in spherical symmetry, have been discovered (see Section 1.3.2 for details). These naked singularities are an undesirable aspect of gravitational collapse models, for the following reason. A singularity can be thought of as the boundary of a spacetime. In order to solve the hyperbolic equations controlling the behaviour of matter (and of spacetime itself) to the future of a singularity, one would need to provide boundary conditions on the singularity. However, it is impossible to determine what these conditions should be and so, naked singularities destroy the predictability of classical general relativity (see [27] for more details).

In response to the unwelcome existence of naked singularities, Roger Penrose formulated the cosmic censorship hypothesis [46] in 1969. Multiple rigorous statements of various versions of the hypothesis exist (see Section 1.3.1), but roughly speaking, it states that the gravitational collapse of a physically reasonable body should not result in the formation of a naked singularity. Thus far, the general statement of this hypothesis has resisted all attempts at a proof. The nature of the putative cosmic censor is unknown, but it is thought that the naked singularities which are present in various models may be attributed to one or more of

- unphysical symmetries, such as perfect spherical or cylindrical symmetry, or self-similarity;
- unphysical matter models, such as null dust or pressure-free perfect fluids;
- a non-generic choice of initial data.

We focus here on the first possibility, that the naked singularity is due to the unrealistic symmetry of the spacetime. One response to this line of reasoning is to examine the stability of the naked singularity spacetime to perturbations which do not share the symmetry of the background. In particular, we consider the behaviour of perturbations on the Cauchy horizon of the spacetime. The Cauchy horizon is a null hypersurface, corresponding to the first null ray emitted by the naked singularity. Equally, it can be thought of as a hypersurface which divides all observers into two classes, those who can see the naked singularity (that is, those within the future null cone of its past endpoint) and those who cannot (that is, those outside the future null cone of its past endpoint). Should perturbations diverge on the Cauchy horizon of a spacetime, then we may rule out that spacetime as a serious counterexample to cosmic censorship. The naked singularity would then be regarded as a single (non-typical) member of a whole class of spacetimes, in which the Cauchy horizon is replaced with a null singularity. We note that the behaviour of perturbations in the Reissner-Nordström spacetime illustrates this phenomenon [6]. In this spacetime, metric perturbations which arrive at the

Cauchy horizon from the exterior have an infinite flux there, as measured by observers crossing the horizon. One might expect that perturbations evolving through other spacetimes containing naked singularities would show similar divergent behaviour on the Cauchy horizon. Should perturbations of a given naked singularity spacetime remain finite on the Cauchy horizon, one can (in some cases) still rule out this spacetime as a serious counter-example to cosmic censorship if it displays one or more of the other defects mentioned above.

In this section, we introduce some important concepts which provide the background for this thesis. We first briefly summarize the notation we use in this thesis, before moving on to the theory of general relativity, a very broad subject from which we will present a few salient points. Far more general treatments can be found in [56] and [50]. We next discuss in more detail the hypothesis of cosmic censorship, as well as introducing some ideas from perturbation theory and the notion of self-similarity in Sections 1.4 and 1.5. We briefly mention previous work in the area of perturbations of self-similar spacetimes in Section 1.5.1. In Sections 1.6 and 1.7 we present some of the mathematical methods used in Chapters 4, 5 and 6. Finally, we give an overview of the work contained in the thesis and the layout of each part.

1.1 Notation

The notation $\mathcal{M}$ will be used to indicate a manifold; in cases where the dimension is important, $\mathcal{M}^4$ will be used, where the index indicates the dimension of the manifold. $\mathcal{S}^2$ will be used to refer to the two-sphere, and in Chapter 3, $\mathcal{M}^4 = \mathcal{M}^2 \times S^2$. A spacetime will be denoted $(\mathcal{M}, g)$, where g indicates the metric tensor. We shall use the signature $(-,+,+,+)$ for the metric.

For four vectors, we use the notation x^α, where α runs over $0, 1, 2, 3$. In Chapter 3, we will use x^A (where $A = 0, 1$) to refer to the first two components of x^α and x^a (where $a = 2, 3$) to refer to the last two components. We note the common alternative usage where $a = 1, 2, 3$ which we shall not use here.

Let f be some scalar function. $f_{;\alpha}$ indicates the full covariant derivative, whereas $f_{|A}$ indicates the covariant derivative on the submanifold $\mathcal{M}^2$. Similarly, $f_{:a}$ indicates the covariant derivative on the submanifold $\mathcal{S}^2$. The notation $\dot{f}$ will be reserved for $\dot{f} = \partial f/\partial z$ where z is the similarity variable. $\mathcal{L}$ will be used to indicate a Lie derivative. We will also use $f_{,x} = \frac{\partial f}{\partial x}$.

We shall use the notation $Y_l^m \equiv Y$ for the spherical harmonics. Their derivatives with respect to x^a will be denoted $\{Y_a := Y_{:a}\}$. We will label $\{S_a := \epsilon_a{}^b Y_b\}$ and $\{Z_{ab} := Y_{a:b} + \frac{l(l+1)}{2} Y \gamma_{ab}\}$ where γ_{ab} is the metric on a two-sphere. Occasionally the Bach bracket notation will be used for these derivatives. In this notation, for some tensor $B_{\mu\nu}$, $B_{(\mu\nu)} := \frac{1}{2}(B_{\mu\nu} + B_{\nu\mu})$ and $B_{[\mu\nu]} := \frac{1}{2}(B_{\mu\nu} - B_{\nu\mu})$.

In Chapter 3, we discuss perturbation theory and in general, given some quantity Q (which could be any rank of tensor), we shall denote the background part as $\bar{Q}$ and the perturbation as δQ, so that $Q = \bar{Q} + \delta Q$. There is one exception to this notation in Chapter 5, where we will use the notation $\bar{u}$ to indicate $\bar{u} = \int_{\mathbb{R}} \vec{u}\, dp$, but it should be clear from context when this notation is being used.

In Chapters 4, 5 and 6, we will make use of various functional norms. We shall denote the Euclidean norm of a vector $\vec{f}$, which depends on some variable x, as $|\vec{f}|$, and the L^2-norm of $\vec{f}$ will be denoted

$$||\vec{f}||_2 = \left(\int_{\mathbb{R}} |\vec{f}|^2 dx \right)^{1/2},$$

that is, the integral of the Euclidean norm squared. For a scalar quantity f, this of course reduces to $||f||_2 = \left(\int_{\mathbb{R}} |f|^2 dx\right)^{1/2}$. The general L^q-norm will be denoted

$$||\vec{f}||_q = \left(\int_{\mathbb{R}} |\vec{f}|^q \, dx \right)^{1/q}.$$

An immediate generalisation of this norm is the $\mathbb{H}^{1,2}$-norm, which is given by

$$||\vec{f}||_{(1,2)} = \left(\int_{\mathbb{R}} |\vec{f}|^2 + |\vec{f}_{,x}|^2 \, dx \right)^{1/2},$$

so that a function is in $\mathbb{H}^{1,2}$ if it and its first derivative are in L^2. For a vector $\vec{f}$ to be in the space $\mathbb{H}^{p,q}$, its first p derivatives must be in the space L^q. In other words, it must have a finite $\mathbb{H}^{p,q}$-norm, where the norm is given by

$$||\vec{f}||_{(p,q)} = \left(\int_{\mathbb{R}} \sum_{i=0}^{p} \left| \frac{d^i \vec{f}}{dx^i} \right|^q dx, \right)^{1/q},$$

where we take the $i = 0$ case to indicate the vector $\vec{f}$ itself. If we have a function $f(t, x)$ of two variables, then the $\mathbb{H}^{p,q}$-norm is given by

$$||\vec{f}||_{(p,q)}(t) = \left(\int_{\mathbb{R}} \sum_{i=0}^{p} \left| \frac{\partial^i \vec{f}}{\partial x^i} \right|^q dx, \right)^{1/q}.$$

In practice, the highest order norm we will make use of is the $\mathbb{H}^{3,2}$-norm.

We will make use of the common abbreviations “ODE” and “PDE” for ordinary and partial differential equations respectively. We use throughout units in which $G = c = 1$ and follow the conventions of [56] for the definition of the Riemann and Einstein tensors and the stress-energy tensor.

1.2 General Relativity

In this section, we briefly review some fundamental ideas from the general theory of relativity. See [56], [50] or [52] for more details.

1.2.1 Differentiable Manifolds

We begin with the notion of an n-dimensional manifold $\mathcal{M}$ and a chart ϕ which maps neighbourhoods U of the manifold to $\mathbb{R}^n$; that is, a chart on U is a one-to-one map,

$$\phi : U \to \phi(U) \subset \mathbb{R}^n.$$

$\phi(p) \in \mathbb{R}^n$ are the coordinates of a point $p \in \mathcal{M}$, usually written as $\phi(p) = (x^1(p), .., x^n(p))$.

It is not immediately clear how vectors should be defined on such manifolds. In order to define vectors, we will first need the concepts of a smooth function and a smooth curve. The definition of a smooth function is complicated by the fact that we do not yet have a notion of differentiation on the manifold $\mathcal{M}$. We therefore proceed by mapping the function back to $\mathbb{R}^n$, and then use the usual notion of smoothness defined on $\mathbb{R}^n$. A similar trick will allow us to define smooth curves.

So, let $f : \mathcal{M} \to \mathbb{R}$ be a real function on $\mathcal{M}$, let ϕ be some chart and define the function $F : \mathbb{R}^n \to \mathbb{R}$, $F := f \circ \phi^{-1}$. Then f is C^k if and only if F is C^k in the usual sense. Similarly, let $I = (a, b) \subset \mathbb{R}$ be an interval in $\mathbb{R}$. Then a smooth curve in $\mathcal{M}$ is a map $\lambda : I \to \mathcal{M}$ such that $\phi \circ \lambda : I \to \mathbb{R}^n$ is smooth in the usual sense.

We can now define the tangent vector to the curve λ at the point $p \in \mathcal{M}$ as a map from the set of smooth functions defined on a neighbourhood of p to $\mathbb{R}$. The tangent vector is given by

$$v_p := \dot{\lambda}_p : f \to \dot{\lambda}_p(f) = \frac{d}{dt}(f \circ \lambda)\bigg|_{t=0},$$

where t is the parameter along the curve λ and without loss of generality, we assume that the point p is at $t = 0$. We next define $\mathcal{T}_p(\mathcal{M})$ to be the set of tangent vectors at the point p. One can show that $\mathcal{T}_p(\mathcal{M})$ is a vector space with the same dimension as that of the manifold. If we introduce a coordinate basis $\{\partial/\partial x^\alpha\} := \{e_\alpha\}$ for the tangent space, then we can write $v_p \in \mathcal{T}_p(\mathcal{M})$ in terms of that basis as

$$v_p = v_p^\alpha \frac{\partial}{\partial x^\alpha},$$

where the v_p^α are the components of v_p in the basis $\{e_\alpha\}$. We use the usual Einstein summation convention, in which repeated indices are summed over all values.

There exists a space dual to $\mathcal{T}_p(\mathcal{M})$ which we denote $\mathcal{T}_p^*(\mathcal{M})$. $\mathcal{T}_p^*(\mathcal{M})$ is an n-dimensional space of linear maps $\sigma : \mathcal{T}_p(\mathcal{M}) \to \mathbb{R}$, whose members are called one-forms or covectors. If we introduce a basis $\{e^\alpha\} := \{dx^\alpha\}$ such that $e^\alpha e_\beta = \delta^\alpha_\beta$, then we can write $w_p \in \mathcal{T}_p^*(\mathcal{M})$ as

$$w_p = w^p_\alpha dx^\alpha,$$

where the w^p_α are the components of w_p in the basis $\{e^\alpha\}$. In what follows, we omit the subscript p indicating the point $p \in \mathcal{M}$ at which the tensor is defined.

In order to implement a change of basis from x^α to $x^{\alpha'}$, we introduce the Jacobians

$$X^{\alpha'}_{\ \beta} = \frac{\partial x^{\alpha'}}{\partial x^\beta}, \qquad\qquad X^{\alpha}_{\ \beta'} = \frac{\partial x^{\alpha}}{\partial x^{\beta'}},$$

such that $X^\alpha_{\ \beta'} X^{\beta'}_{\ \gamma} = \delta^\alpha_\gamma$. Then under a change of basis,

$$v^\alpha = X^\alpha_{\ \beta'} v^{\beta'}, \qquad\qquad w_\alpha = X^{\beta'}_{\ \alpha} w_{\beta'}.$$

Having considered the construction of vectors and covectors, we can immediately move on to tensors of arbitrary rank. Consider for example the map $S : \mathcal{T}_p(\mathcal{M}) \times \mathcal{T}_p(\mathcal{M}) \times \mathcal{T}^*_p(\mathcal{M}) \to \mathbb{R}$, which is linear in each of its arguments. It is a $\binom{1}{2}$-tensor with components $S_{\alpha\beta}{}^\gamma = S(e_\alpha, e_\beta, e^\gamma)$ in the basis $\{e^\alpha\}$. When acting on vectors X and Y, and a one-form Z, it produces a scalar $S(X, Y, Z)$ given by

$$S(X, Y, Z) = X^\alpha Y^\beta Z_\gamma S_{\alpha\beta}{}^\gamma.$$

Generalising from the vector and one-form cases, a transformation of coordinate basis can be implemented using

$$S_{\alpha\beta}{}^\gamma = X^{\eta'}_{\ \alpha} X^{\nu'}_{\ \beta} X^{\gamma}_{\ \mu'} S_{\eta'\nu'}{}^{\mu'}.$$

A tensor of rank $\binom{l}{m}$ is a map taking l vectors and m one-forms to $\mathbb{R}$ and changes of basis for such a tensor can be implemented in a similar way. We note that although we can write the components of tensors in any particular coordinate basis, tensors transform covariantly under a change of coordinate basis.

We now define the metric tensor, a symmetric, non-degenerate $\binom{0}{2}$-tensor g, such that $g : \mathcal{T}_p(\mathcal{M}) \times \mathcal{T}_p(\mathcal{M}) \to \mathbb{R}$. The condition of non-degeneracy means that for $X, Y \in \mathcal{T}_p(\mathcal{M})$, if $g(X, Y) = 0$ for all Y, then $X = 0$. The metric tensor can be used to raise and lower indices, so that for some vector v,

$v^\alpha = g^{\alpha\beta}v_\beta$. This metric tensor allows us to define the length ds of intervals through the relation

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta.$$

A Lorenztian metric is one for which at any point $p \in \mathcal{M}$, there exists a coordinate system in which the metric takes the form $\text{diag}(-1, 1, 1, 1)$, that is, the metric has signature $+2$. We define a spacetime to be a connected, Hausdorff manifold $\mathcal{M}$, on which a Lorentzian metric tensor g is defined for all points $p \in \mathcal{M}$. We shall denote a spacetime as $(\mathcal{M}, g)$.

We classify all vectors v^α as timelike, spacelike or null, as follows:

$$\begin{array}{lc} g_{\alpha\beta}v^\alpha v^\beta < 0 & \text{timelike} \\ g_{\alpha\beta}v^\alpha v^\beta > 0 & \text{spacelike} \\ g_{\alpha\beta}v^\alpha v^\beta = 0 & \text{null} \end{array}$$

We can easily extend this definition to curves by noting that a curve is timelike, spacelike or null if its tangent vector is timelike, spacelike or null respectively. Matter travels along timelike curves, while radiation travels along null curves.

1.2.2 Covariant Derivatives

We have yet to formulate a way to take derivatives on a manifold. The usual derivative is not invariant under changes of coordinate system. In order to construct a covariant derivative on the manifold $\mathcal{M}$, we require the introduction of one more concept, that of a linear connection. We define a linear connection ∇ on $\mathcal{M}$ to be a map sending smooth vector fields X and Y into a smooth vector field $\nabla_X Y$ such that

$$\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z, \qquad \nabla_{fX+Y} Z = f\nabla_X Z + \nabla_Y Z,$$

for any function $f : \mathcal{M} \to \mathbb{R}$. $\nabla_X Y$ is the covariant derivative of Y with respect to X. Then $\nabla Y : X \to \nabla_X Y$ is a linear map from $\mathcal{T}_p(\mathcal{M}) \to \mathcal{T}_p(\mathcal{M})$. Defining $\nabla_{e_\alpha} := \nabla_\alpha$, then the components of $\nabla_X Y$ are $\nabla_\alpha e_\beta = \Gamma^\gamma{}_{\alpha\beta} e_\gamma$, for some scalars $\Gamma^\gamma{}_{\alpha\beta}$, which are the components of the connection. By similar arguments, we can show that the components of the covariant derivative $\nabla_X Y$ are

$$Y^\alpha_{;\beta} = Y^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\gamma\beta} Y^\gamma,$$

where $Y^\alpha{}_{,\beta} = \partial Y^\alpha / \partial x^\beta$. Each term on the right hand side above does not transform as a tensor but their combination does (and is covariant). We will use the metric or Levi-Civita connection in which $\nabla g = 0$.

1.2.3 The Einstein Equations

Having defined tensors and a suitable form of differentiation on a manifold, we now consider how curvature may be defined. We begin with the Riemann tensor, which captures the failure of vectors to return to themselves after parallel transport along a closed curve. The components of the Riemann tensor are given by

$$R^\alpha_{\beta\gamma\delta} = \Gamma^\alpha_{\beta\delta,\gamma} - \Gamma^\alpha_{\beta\gamma,\delta} + \Gamma^\mu_{\beta\delta}\Gamma^\alpha_{\mu\gamma} - \Gamma^\mu_{\beta\gamma}\Gamma^\alpha_{\mu\delta}.$$

If we note that the components of the connection can be given as $\Gamma^\gamma_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha})$, then we see that the Riemann tensor involves second derivatives of the metric. We define the Ricci tensor to be the contraction of the Riemann tensor in the first and third indices, $R_{\alpha\beta} = R^\gamma_{\alpha\gamma\beta}$. Finally, the Ricci scalar is defined as $R = g^{\alpha\beta} R_{\alpha\beta}$. The Einstein tensor is given by

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}.$$

Before introducing the Einstein field equations, we briefly discuss the stress-energy tensor, also commonly known as the energy-momentum tensor. This is a $\binom{0}{2}$-tensor $T_{\alpha\beta}$ which describes the matter distribution present in a given spacetime. More precisely, it measures the flux of the α-component of momentum across a surface of constant x^β. We will make extensive use of the stress-energy for a perfect fluid,

$$T_{\alpha\beta} = (\rho + P) u_\alpha u_\beta + P g_{\alpha\beta},$$

where ρ is the energy density of the fluid, P its pressure and u_α the four-velocity of a fluid element.

We next provide a brief motivation for the particular form of the Einstein field equations, which relate the metric of a spacetime to its matter content. The Einstein equations are arrived at by aiming to satisfy the following requirements:

- the equations should reduce to Poisson's equation for the Newtonian field in the non-relativistic limit,
- the equations must introduce no preferred coordinate system (the principle of relativity),
- the equations should respect the conservation of stress-energy,

where the final statement is the covariant generalisation of the principle of conservation of energy. We begin with the analogue of the Einstein field equations in Newtonian theory, Poisson's equation,

$$\nabla^2\phi = 4\pi G\rho,$$

where ϕ is the gravitational potential, G is Newton's constant and ρ is the density of matter. Since Einstein's equations should generalise the above equation, we expect that the source term will involve density. However, density itself is not an invariant quantity, so we use instead the stress-energy tensor $T_{\alpha\beta}$ which includes the density.

The metric tensor is analogous to the Newtonian gravitational potential, so again, Poisson's equation suggests that we should look for a tensor which is second order in the metric for the left hand side of our equation. An immediate possibility is the Riemann tensor. However, if we postulate the field equations $R_{\alpha\beta} = \lambda T_{\alpha\beta}$ for some constant λ, we will find that conservation of stress-energy will produce an identity which contradicts the contracted Bianchi identity for the Riemann tensor. We can resolve this by choosing instead the Einstein tensor for the left hand side. The Einstein equations are therefore

$$G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}, \tag{1.1}$$

where the factor of $(8\pi G)/c^4$ (where G is Newton's constant of gravitation and c is the speed of light) is chosen so that these equations reduce to Newton's law of gravitation in the non-relativistic limit (we will normally use natural units in which $c = G = 1$). We note that the second of our three conditions is satisfied automatically, as our use of tensors to describe the spacetime and its matter content means that our equations are immediately covariant. Equations (1.1) are not the only equations which obey these three requirements, but they are the simplest and most widely accepted. The most common generalisation of (1.1) is to add a cosmological constant term so that we have instead $G_{\alpha\beta} + \Lambda g_{\alpha\beta} = (8\pi G/c^4)T_{\alpha\beta}$. We shall set $\Lambda = 0$ here.

In general these equations produce ten non-linear coupled partial differential equations (the symmetry of the Einstein tensor, $G_{\alpha\beta} = G_{\beta\alpha}$, reduces the number of equations from sixteen to ten). In Part II of this thesis we will impose self-similarity on our spacetime, which will reduce the Einstein equations to ordinary differential equations.

1.3 The Cosmic Censorship Hypothesis

1.3.1 Strong and Weak Cosmic Censorship

We next review the cosmic censorship hypothesis, which (roughly speaking) asserts that the gravitational collapse of physically reasonable matter should not result in the formation of a naked singularity. There are actually two forms of cosmic censorship, strong and weak cosmic censorship.

Before discussing these two forms, we review some definitions which we will use in this section.

- We begin with the notion of future null infinity, $\mathscr{J}^+$; roughly speaking, this is the set of points which are approached asymptotically by null rays which can escape to infinity.

- The Cauchy development of a hypersurface Σ refers to the set of points such that every inextendible causal curve through these points intersects Σ. That is (roughly speaking), it is the region of the manifold which can be predicted (or retrodicted) from data on Σ.

- We will also use the notion of global hyperbolicity; recall that a globally hyperbolic spacetime is a spacetime in which there exists a hypersurface Σ such that the Cauchy development of Σ is the manifold itself. Such a hypersurface Σ is known as a Cauchy surface.

- Consider a Cauchy surface Σ on which suitable initial data (namely the induced metric $h_{\mu\nu}$ and the extrinsic curvature $K_{\mu\nu}$) for the Einstein equations is defined. Then the maximal Cauchy development of Σ with this initial data is a spacetime $(\mathcal{M}, g)$ such that

  (1) $(\mathcal{M}, g)$ satisfies Einstein's equation,

  (2) $(\mathcal{M}, g)$ is globally hyperbolic with Cauchy surface Σ,

  (3) The induced metric and extrinsic curvature of Σ (as calculated using g) are $h_{\mu\nu}$ and $K_{\mu\nu}$,

  (4) Every other spacetime satisfying (1 - 3) can be mapped isometrically into a subset of $(\mathcal{M}, g)$.

  So roughly speaking, the maximal Cauchy development is the largest set of points which can be determined from initial data defined on Σ only.

- Finally, we give a rough definition of a complete future null infinity [14]; that is, future null infinity ($\mathcal{J}^+$) is complete if any null geodesic along $\mathcal{J}^+$ can be extended indefinitely relative to its affine parameter.

Naked singularities are free to send signals outwards towards the external universe. However, it is possible that such signals do not escape to future null infinity, but rather can only be detected by observers in the local vicinity of the singularity. Such singularities are called locally naked. On the other hand, the singularity may be able to send signals to future null infinity, in which case the singularity is called globally naked. The two forms of cosmic censorship relate to these two forms of naked singularity. The weak cosmic censorship hypothesis can be roughly stated in the following fashion.

Conjecture 1.3.1 *[56] The complete gravitational collapse of physically reasonable matter always results in the formation of a black hole rather than a naked singularity.*

To make this conjecture more precise, we must consider what conditions might be imposed on the stress-energy so as to ensure it describes physically reasonable matter. An immediate option is to impose one or more of the energy conditions (see [56] for a discussion of these) and typically, the dominant energy condition is imposed. This states that for all future directed, timelike k^μ, $-T^\mu_{\ \nu}k^\nu$ should be future directed and either timelike or null. Since this vector is the current density measured by an observer with velocity k^μ, we can roughly paraphrase this condition as saying that the speed of the energy flow of matter should never exceed the speed of light.

Furthermore, we impose suitable initial data for the Einstein equations on some Cauchy surface Σ in the form of the induced metric on Σ, $h_{\mu\nu}$, and the second fundamental form (also known as the extrinsic curvature) $K_{\mu\nu}$. Roughly speaking, we can think of the second fundamental form as being the "time derivative" of the metric, evaluated on Σ.

The weak cosmic censorship hypothesis arises in the context of gravitational collapse, and thus, we consider asymptotically flat initial data. The weak cosmic censorship hypothesis maintains that these data evolve to an asyptotically flat spacetime in which any singularities present are not visible from infinity.

Consider a spacetime $(\mathcal{M}, g)$ containing a globally naked singularity. This spacetime will have a Cauchy horizon which intersects future null infinity $\mathcal{J}^+$. The Cauchy horizon marks the future boundary of the maximal Cauchy development $\hat{\mathcal{M}}$ of a putative Cauchy surface Σ for the spacetime. Clearly, $\hat{\mathcal{M}}$ is extendible across the Cauchy horizon and consequently, $\mathcal{J}^+|_{\hat{\mathcal{M}}}$ is also extendible (to $\mathcal{J}^+|_{\mathcal{M}}$). This indicates that $\mathcal{J}^+|_{\hat{\mathcal{M}}}$ has finite affine length and so, is incomplete. On the other hand, if the singularity is censored, then such a null geodesic can be extended indefinitely, and thus, a black hole spacetime has a complete $\mathscr{I}^+$. Therefore, we can implement weak cosmic censorship by requiring the maximal Cauchy development to have a complete future null infinity. This leads us to a rigorous statement of weak cosmic censorship; see [57] for a detailed discussion of this conjecture.

Conjecture 1.3.2 *[57] Consider a* 3*-manifold* Σ*. Assume that nonsingular, asymptotically flat data* $(h_{\mu\nu}, K_{\mu\nu}, \Psi)$ *are assigned on* Σ *and that the Einstein equations are provided with a suitable matter source, represented by* Ψ*. Then the maximal Cauchy evolution of such data is generically a spacetime* $(\mathcal{M}, g)$ *which is asymptotically flat at future null infinity, with a complete* $\mathscr{I}^+$*.*

The strong cosmic censorship hypothesis states that the gravitational collapse of physically reasonable matter should not result in the formation of any naked singularities, that is, singularities visible either from future null infinity or from any other point. We can use the notion of global hyperbolicity to give a loose formulation of strong cosmic censorship as follows:

Conjecture 1.3.3 *[56] All physically reasonable spacetimes should be globally hyperbolic, that is, apart from a possible initial singularity, there should be no singularity which is ever visible to any observer.*

See [56] for a discussion of this conjecture. We note that a spacetime which contains a naked singularity is not globally hyperbolic. In fact, the Cauchy horizon, the first null ray emitted by the singularity, forms the edge of the domain of dependence of the hypersurface Σ. Roughly speaking, this means that the Cauchy horizon marks the point past which the spacetime becomes "unpredictable" due to the influence of the naked singularity.

The statement of strong cosmic censorship can be cast more precisely in terms of conditions on the matter content and the form of the Einstein equations. As before, we impose the dominant energy condition on the matter content of the spacetime and use a set $(\Sigma, h_{\alpha\beta}, K_{\alpha\beta})$ as initial data for Einstein's equation. We require that the Einstein-matter equations be put in the form of a second order, quasilinear, diagonal, hyperbolic system. The reason for this is that fundamental matter fields (for example, electromagnetism) are known to obey equations of this form (and we wish to use fundamental matter fields since otherwise any singularity which forms may be due to an unphysical matter model). Finally, we impose strong cosmic censorship by requiring that the maximal Cauchy development of $(\Sigma, h_{\alpha\beta}, K_{\alpha\beta})$ always yields an inextendible spacetime [1].

To see why inextendibility is important, recall the theorem of Choquet-Bruhat and Geroch [8] which, roughly speaking, tells us that to any initial data one can associate uniquely (up to a diffeomorphism) a maximal globally hyperbolic development of those data. However, we are not guaranteed that the resulting spacetime cannot be extended. In general, it might contain a Cauchy horizon past which it can be extended, and the extension need not be unique, that is, we may have a breakdown of predictability. Therefore, the statement of strong cosmic censorship essentially asserts that under certain conditions, the maximum globally hyperbolic development is inextendible, implying that there do not exist Cauchy horizons in this spacetime. For further details, see [56], [57] and [36].

Neither Conjecture 1.3.2 nor Conjecture 1.3.3 have been proven. There exists reasonable evidence to support Conjecture 1.3.2, in the form of special cases and examples, and no strong evidence either way for Conjecture 1.3.3. No attempt to prove a general version of cosmic censorship has been successful. The difficulty lies in the fact that cosmic censorship is a statement about the nature of solutions to Einstein's equations in quite general circumstances, but besides the singularity theorems of Hawking and Penrose, very little is known about global properties of such solutions. We next discuss various suggestions about what form the cosmic censor might take.

1.3.2 What is the Cosmic Censor?

There is a wide variety of models which exhibit a naked singularity. Most of these models are spherically symmetric, for example the dust and perfect fluid

[1] We note that while inextendibility of the maximum Cauchy development is essentially the condition required for strong cosmic censorship, this condition must be modified somewhat to take into account some special cases (the Kerr solution and Taub universe). We do not discuss this modification here; see [56] for details.

singularities [3], the Vaidya spacetime [42], the extremal Reissner-Nordström spacetimes [56] and the collapse of a massless scalar field [7]. Another important example of a naked singularity is provided by the extremal Kerr-Newman solution [56]. Naked singularities can also be observed in higher dimensional scenarios, such as the black string naked singularity [32]. They are found in spacetimes which are not asymptotically flat, such as an asymptotically anti-de Sitter spacetime with a Maxwell field and a scalar field acting under a particular potential [24]. Outside of spherical symmetry, one significant result is that of Shapiro and Teukolsky [51], who studied collisionless oblate and prolate spheroids and found that if the semimajor axis is sufficiently large, then a naked spindle singularity can form. Other significant results include the formation of naked singularities from the collapse of dust shells in cylindrical symmetry (see [37] and [31]) and the formation of naked singularities in the Einstein - massless scalar field system in axial symmetry [22].

The term "the cosmic censor" is used to refer to the phenomenon (or phenomena) which are thought to prevent the formation of naked singularities in physically reasonable spacetimes which evolve from general initial data. Many different suggestions as to the identity of the cosmic censor have been made; see [30], [57] and [12] for discussions of these. We point out some interesting suggestions here.

- We should certainly impose the condition that naked singularities must arise from the evolution of regular initial data; but examples satisfying this property abound (the perfect fluid, the Vaidya solution, the massless scalar field).

- An immediate option is to impose one of the energy conditions, and try to show that matter obeying this condition cannot form naked singularities. However, there are well known models (for example, the dust and perfect fluid collapse) where the matter obeys reasonable energy conditions and can still form naked singularities.

- It is tempting to reject any naked singularities which arise in "non-physical" models. Here we mean any models which do not accurately capture every feature of the real gravitational collapse of a star. For example, in this case the perfect fluid models would be counted as non-physical because they neglect viscosity and heat conduction. One way of implementing this condition is to require that the model does not form singularities in the absence of gravity. There are well known models (for example, the dust solution) in which singularities form in Minkowski space and these singularities (known as "matter singularities") therefore cannot be ascribed to a gravitational origin.

- It is sometimes thought that quantum gravity will solve the problem of cosmic censorship, since it is expected that it will somehow "smear out" the singularity. However, this is really not relevant to the problem of cosmic censorship, since the presence or absence of an event horizon is a purely classical phenomenon. Several authors (for example [30] and [57]) have pointed out that if cosmic censorship fails to hold, then it would be possible to directly observe the quantum gravitational regime. It has also been pointed out (see [27] for example) that explosive particle creation due to quantum effects late in the collapse may avert the formation of a naked singularity in some cases, but this can be interpreted as another manifestation of the problem of visible regions of extreme curvature.

- Choptuik [7] found that a spherically symmetric massless scalar field coupled to gravity can produce a naked singularity. However, to produce the naked singularity, one has to fine-tune the initial data, and nearby data produce either a black hole or the dissipation of the field. Therefore, the naked singularity cannot form as a result of the collapse of generic initial data. This is rather unphysical and we could therefore neglect any naked singularities which do not form from the collapse of generic initial data. We should note that there are also naked singularities which do form generically, for example the naked singularity in perfect fluid collapse.

- Finally, we could neglect any naked singularities which are unstable to perturbations. In this case, the formation of the singularity would be due to the exact symmetry (for example, self-similarity or spherical symmetry) of the background spacetime. In particular, we would require stability on the Cauchy horizon associated with the naked singularity. Should perturbations diverge on the Cauchy horizon, then we would expect the horizon to be replaced by a null singularity.

At present, the last two possibilities show the most promise. In summary, a serious counter-example to the cosmic censorship hypothesis would have to arise from a regular, generic choice of (asymptotically flat) initial data, would have to have a physically reasonable stress-energy tensor and would have to be stable to perturbations away from any background symmetry.

Thus far, possibly the strongest counter-example to cosmic censorship is the self-similar perfect fluid spacetime, which has a naked singularity for $0 < k \leq 0.0105$ [45], where k is the sound speed (squared) of the fluid. The matter model is a perfect fluid, which obeys reasonable energy conditions and the naked singularity forms from regular, generic initial data. Harada and Maeda [23] studied the behaviour of non-linear spherical perturbations in this model and determined that it displayed stability to such perturbations. The behaviour of non-spherical perturbations in this spacetime is not yet known.

There are two interesting spacetimes within which weak and strong cosmic censorship respectively have been proven, to which we now turn.

1.3.3 Cosmic Censorship in the EKG and Gowdy spacetimes

Weak cosmic censorship has been proven in the Einstein-Klein-Gordon spacetime. This spacetime is spherically symmetric and the matter model is a massless Klein-Gordon scalar field. This spacetime was studied analytically by Christodoulou [10], who found that under a particular set of conditions, there exist choices of initial data which give rise to naked singularities. In a later paper [11], it was shown that these naked singularities were non-generic.

Christodoulou made a choice of initial data, characterised by a function β, such that the future evolution of the initial data

(1) contains no singularities and a future complete $\mathcal{I}^+$, or

(2) contains a "normal" black hole, with accompanying event horizon, or

(3) obeys neither case (1) nor case (2) (this case includes the possibility of naked singularity formation).

Then with such a choice of initial data, there exists a continuous function g, such that for any real constant c, the spacetime evolving from initial data $\beta' = \beta + cg$, contains a "normal" black hole. This result indicates that the formation of a naked singularity relies on a choice of non-generic initial data, and that any slight perturbation of this initial data will cause the naked singularity to fail to form.

Another important example of cosmic censorship occurs in the Gowdy spacetimes, a spacetime with a two-dimensional isometry group with spacelike orbits (see [48] for a review of cosmic censorship in this spacetime). In this class of spacetimes, there exist spacetimes with inequivalent maximal extensions, so the question of whether or not cosmic censorship holds for this class is interesting. The Einstein equations with this symmetry reduce to a form amenable to Fuchsian analysis, which means that it is possible to determine the asymptotic behaviour of solutions to these equations in various directions. In two particular cases, the polarized Gowdy case and the T^3-case, it is possible to prove versions of strong cosmic censorship. In both cases, one can prove theorems asserting that under various conditions, the maximal Cauchy development of the prescribed initial data is inextendible.

It has been noted [57] that the main difficulty in studying cosmic censorship in general is that the mathematical sophistication necessary to prove some version of Conjecture 1.3.3 is not available. It follows that the study of specific examples may well be illuminating, and that the use of techniques such as perturbation theory in these spacetimes may be useful. We now discuss generally the methods of perturbation theory.

1.4 Perturbation Theory

As discussed in Section 1.2, in general relativity one deals with a metric $g_{\mu\nu}$ which describes the geometry of a spacetime. The initial task of perturbation theory is to develop a formalism with which to discuss perturbations of such a background metric. We introduce a perturbed metric g such that

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon h_{\mu\nu},$$

for $|\epsilon| \ll 1$, where $\bar{g}_{\mu\nu}$ is some background metric, and $h_{\mu\nu}$ is a perturbation. The main difficulty that arises with such a definition is the question of gauge invariance. Suppose we start with some background metric, and then introduce a perturbation, resulting in a new, perturbed metric. How are we to guarantee that there does not exist a coordinate system within which this new metric is identical to the original metric? This would indicate that the two metrics describe the same spacetime in two different coordinate systems.

This issue is handled by introducing gauge invariant perturbations. Such perturbations are guaranteed to preserve their form under a gauge transformation, thus ensuring that they have real physical meaning and are not artifacts of the choice of coordinate system. In particular, throughout this work we make use of a perturbation formalism due to Gerlach and Sengupta [19], which constructs explicitly gauge invariant linear perturbations for spherically symmetric spacetimes (see Chapter 3 for details).

1.5 Self-Similarity

In this work, we consider the self-similar Lemaître-Tolman-Bondi (LTB) spacetime [2]. There are two different classes of self-similarity, namely continuous or homothetic self-similarity (also known as self-similarity of the first kind) and discrete self-similarity (also called self-similarity of the second kind). We consider only continuous self-similarity, which we will refer to simply as self-similarity from here on. A spacetime displays self-similarity if it admits a homothetic Killing vector field, that is, a vector field $\vec{\xi}$ such that,

$$\mathcal{L}_{\vec{\xi}} g_{\mu\nu} = 2 g_{\mu\nu},$$

where the notation $\mathcal{L}_{\vec{\xi}} g_{\mu\nu}$ indicates the Lie derivative of the metric, taken in the direction of the vector field $\vec{\xi}$ (recall that, roughly speaking, the Lie derivative compares $g_{\mu\nu}$ at two different points along the integral curves of $\vec{\xi}$, and subtracts to construct a derivative. For the metric, the Lie derivative reduces to $\mathcal{L}_{\vec{\xi}} g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu}$). The choice of non-zero constant on the right hand side above is arbitrary, and can be fixed by rescaling $\vec{\xi}$.

Consider a spacetime $(\mathcal{M}^4, g_{\mu\nu})$. The manifold of a spherically symmetric spacetime can always be written as a product $\mathcal{M}^4 = \mathcal{M}^2 \times \mathcal{S}^2$, where $\mathcal{S}^2$ is the two-sphere. We will write the metric for such a spacetime using coordinates (t, r, θ, ϕ) where (t, r) are coordinates on the two-dimensional submanifold $\mathcal{M}^2$ and (θ, ϕ) are the two angles in the two-sphere.

The general form of a spherically symmetric metric can be written as

$$ds^2 = -e^{2\Phi}(t,r)dt^2 + e^{2\Psi}(t,r)dr^2 + R^2(t,r)d\Omega^2,$$

where $\Phi(t,r)$ and $\Psi(t,r)$ are arbitrary functions of t and r and $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the usual metric on a two-sphere. The imposition of self-similarity on this spacetime results in considerable simplification. In particular, we find that

$$\Phi(t,r) = \Phi(z), \qquad \Psi(t,r) = \Psi(z), \qquad R(t,r) = rS(z), \tag{1.2}$$

where $z = -t/r$. More generally, self-similarity can be thought of as a symmetry which causes physical quantities to depend only on dimensionless parameters (which are typically ratios of time and space variables such as z).

Self-similarity plays an important role in models of gravitational collapse, as in a variety of cases, the solution naturally takes on a self-similar form. Self-similar solutions are often important in describing the asymptotic behaviour of models, particularly cosmological or spatically homogeneous models [5]. More precisely, one often finds that self-similar models act as attractors in a dynamical systems treatment of the model. In fact, the similarity hypothesis of Carr states that under certain conditions (which have yet to be precisely defined) spherically symmetric spacetimes naturally develop into a self-similar form. Self-similarity is also important in studies of critical phenomena, as Type II critical solutions are continuously self-similar. For a discussion of self-similarity and its role in general relativity, see [4].

1.5.1 Perturbations of Self-Similar Spacetimes

In this section, we discuss some previous work on perturbations of self-similar spacetimes, both analytic and numerical, which is relevant to the content of this thesis.

Analytic work has been concentrated in the self-similar Vaidya and LTB spacetimes. In the self-similar Vaidya spacetime, this work indicated that linear perturbations remain finite on the Cauchy horizon. In the odd parity case [42], metric and matter perturbations were found to remain finite on the Cauchy horizon for all angular number l, where the term finite refers to certain integral energy norms which bound the growth of the perturbation. This result holds for a general choice of initial data. In the even parity case [40], both metric and matter perturbations were considered using a Mellin transform method. It was found that the modes of this transform remain finite on the Cauchy horizon, although they diverge on the second future similarity horizon (this corresponds to the last null ray emitted by the naked singularity). It was argued that the resummed Mellin transform modes which give the full perturbation also remain finite on the Cauchy horizon but a rigorous proof was not given.

Even parity perturbations of the self-similar LTB spacetime were also studied analytically, using both a Mellin transform and a Fourier mode decomposition [43]. Both matter and metric perturbations were studied. The Mellin transform reduces the linearised Einstein equations to ODEs with singular points, whose solutions can be studied. The initial data were chosen to be finite on the regular axis and on the past null cone of the naked singularity. It was found that the Mellin modes with this initial data remain finite on the Cauchy horizon (although as in the Vaidya spacetime they diverge on the second future similarity horizon). However, the issue of resumming the modes was not resolved and so this result does not fully answer the question of the stability of the self-similar LTB spacetime to even parity perturbations.

The behaviour of scalar waves in naked self-similar backgrounds [41] has also been considered. In particular, the behaviour of the multipoles of a minimally coupled scalar field on a spherically symmetric, self-similar background spacetime were studied. The only restriction on the matter model is that it obeys the weak energy condition. It was found that each multipole obeys a pointwise bound at the Cauchy horizon, has a finite L^2-norm and has a finite energy. The only possible divergence in the multipoles occurs at the singularity itself and not on the Cauchy horizon. This was interpreted as providing evidence that the naked singularity in these spacetimes is stable, although it was emphasised that this applies at linear order only.

The behaviour of linear non-spherical perturbations of the Roberts spacetime has also been studied [18]. This spacetime is continuously self-similar and spherically symmetric, and describes the gravitational collapse of a minimally coupled massless scalar field. It is used as a toy model for studies of the critical collapse of a massless scalar field. It was found that perturbations of this spacetime have no growing non-spherical perturbation modes.

Numerical work in this area has included studies of the behaviour of odd [25] and even [26] perturbations (for the $l = 2$ perturbation only) of the marginally bound LTB spacetime. In the odd parity case, the authors considered metric perturbations only and analysed them numerically with a choice of Gaussian initial data. It was found that the perturbations remain finite in the vicinity of the singularity. This was interpreted as evidence that the formation of this singularity is stable against odd parity modes of linear gravitational waves. In the even parity case [26], both metric and matter perturbations were considered and their equations of motion were numerically solved, again for the $l = 2$ mode only. It was found that all of the metric perturbations with one exception grow as the Cauchy horizon is approached. The energy flux crossing the Cauchy horizon is finite. This was interpreted as evidence that the Cauchy horizon is unstable to linear even parity perturbations. Taken as a whole, this work indicates that the naked marginally bound LTB spacetime is unstable to linear non-spherical perturbations.

Other numerical work includes the study of non-spherical linear perturbations of the Choptuik spacetime [34]. This is a discretely self-similar spacetime which acts as an attractor in the phase space of solutions to the Einstein - massless scalar field system; in this context, it is known as a critical solution (see [21] for a review of critical phenomena). It was found that all perturbations of this spacetime decay, with the $l = 2$ even parity perturbation decaying at the slowest rate. We note that the behaviour of perturbations was studied in the region between the regular centre and the past null cone only; in particular, their behaviour at the Cauchy horizon was not examined. The authors interpreted their result as evidence that the critical phenomena observed in scalar field collapse are expected to occur even in the presence of linear order perturbations away from spherical symmetry.

In this thesis, we consider the behaviour of non-spherical linear perturbations of the self-similar LTB spacetime. These perturbations decompose into odd and even parity perturbations (see Chapter 3 for details). The odd parity perturbations can be dealt with using techniques similar to those of [42] and [41], namely energy methods for symmetric hyperbolic systems. However, the even parity perturbations are considered using very different methods to those of [40] and [43]. Instead of using Mellin transforms and Fourier mode decompositions, we develop a series of techniques, starting with an extension of the odd parity energy methods, which do not involve introducing a mode decomposition which must then be resummed. We use an averaging technique, followed by methods for ODE systems with regular singular points (see Chapter 5 for details) to determine the behaviour of the L^p-norm of the perturbations. We then use energy methods to provide an initial bound on the perturbations, followed by an application of the method of characteristics to improve these bounds. Finally, we apply some theorems from real analysis to determine the pointwise (as opposed to the averaged) behaviour of the perturbation. We note that some of these ideas (in particular, the averaging technique) were first developed in [44], but the techniques used to determine the pointwise behaviour are new.

The development of these methods is significant, as they allow us to examine the even parity perturbations without introducing either a Mellin transform or a Fourier mode decomposition. This means that we can avoid entirely the problem of resumming the modes which result from these techniques. Additionally, these methods are in principle applicable to any self-similar spherically symmetric spacetime, and could in particular be applied to the self-similar perfect fluid spacetime. This is currently thought to be the strongest counter-example to cosmic censorship.

1.6 Theory of Ordinary and Partial Differential Equations

In Chapters 4, 5 and 6, we will make use of various methods and results from the theory of ordinary and partial differential equations which we present here.

1.6.1 L^p and Sobolev Spaces

We begin by defining the L^p-spaces and the Sobolev spaces, which we make extensive use of throughout this thesis. The L^p-spaces can be thought of as a useful generalisation of the familiar L^2-spaces, the set of square integrable functions. We define the L^p-norm of some function $f(x)$ as

$$||f||_p := \left[\int_\Omega |f|^p \, dx\right]^{\frac{1}{p}}$$

where $\Omega \subseteq \mathbb{R}$ (for our purposes, we will usually take $\Omega = \mathbb{R}$). The L^p-spaces consist of the set of all measurable functions with finite L^p-norm. We note that the space L^2 (a commonly used L^p-space) is the set of square integrable functions, that is, the set of measurable functions $f(x)$ such that

$$||f||_2 = \left[\int_\Omega |f|^2 dx\right]^{\frac{1}{2}} < \infty.$$

L^2 is a Hilbert space, in contrast with all other L^p-spaces. It is possible to define the L^∞-space as

$$L^\infty = \{f(x) : \operatorname{ess\,sup}_{x\in\Omega} |f(x)| < \infty\}.$$

So the space L^∞ contains those functions which are essentially bounded, that is, bounded up to a set of measure zero. A useful way of thinking about the L^p-spaces is to note that the space L^1 contains functions which blow up at isolated points but which must fall off sufficiently quickly at infinity. In contrast, the space L^∞ contains functions which may not blow up anywhere, but also do not need to fall off at infinity at all. So loosely speaking, for $p < q$, functions in L^p are can be more singular than those in L^q, but must also drop off at infinity more quickly.

We mention here a theorem which we will make use of in our study of the even parity perturbations. The L^p-embedding theorem states that for $1 \leq p < q \leq \infty$,

$$||f||_p \leq \mu(\Omega)^{\frac{1}{p}-\frac{1}{q}}||f||_q$$

where $\mu(\Omega)$ indicates the measure of the space Ω over which the L^p-spaces are defined. This theorem allows us to conclude that if $f \in L^q$, then $f \in L^p$, for $p < q$ (assuming that $\mu(\Omega) < \infty$). In this sense, the space L^p is "embedded" in the space L^q for $p < q$. For a general discussion of L^p spaces, see [1] or [35].

Having defined the L^p-spaces, we can move on to the definition of the Sobolev spaces. In order to define such spaces, we will need the idea of a weak derivative. Consider some test function $f \in C_0^\infty$. Then integration by parts produces

$$\int_{\mathbb{R}} g\frac{df}{dx}\,dx = -\int_{\mathbb{R}} \tilde{g}f\,dx, \tag{1.3}$$

where the boundary term vanishes due to the compact support of f. The function $\tilde{g}$ is said to be the weak derivative of some function g if it satisfies (1.3) for all test functions f. This can hold even if g is not differentiable; if g is differentiable then $\tilde{g} = dg/dx$. Weak derivatives therefore generalise the idea of the derivative, since a function may have weak derivatives of n-th order even if it is not n-times continuously differentiable (or indeed, not even continuous). We note that the n-th weak derivative is defined in an exactly analogous way. Weak derivatives arise from the need to have some notion of differentiability which is weaker than the usual one.

A Sobolev space, denoted $\mathbb{H}^{q,p}$, is defined as a space of functions whose weak derivatives, up to the q-th order, exist in some L^p-space. For example, consider the space $\mathbb{H}^{1,2}$. The first index indicates that we should go up to the first derivative (where the "zeroth derivative" is taken to be the function itself) and the second index indicates that these weak derivatives should be in the space L^2. Putting all this together, the space $\mathbb{H}^{1,2}$ is the set of all measurable functions f with finite $\mathbb{H}^{1,2}$-norm, that is

$$||f||_{(1,2)} = \left[\int_\Omega |f|^2 + \left| \frac{df}{dx} \right|^2 dx \right]^{\frac{1}{2}} < \infty.$$

Similarly, we can construct Sobolev norms for any order of Sobolev space by using the corresponding L^p-norms and the required number of weak derivatives. We next mention a result which we frequently use, Sobolev's inequality.

This inequality states that

$$|f(x)|^2 \leq \frac{1}{2} \int_{\mathbb{R}} |f|^2 + \left| \frac{df}{dx} \right|^2 dx,$$

for all $x \in \mathbb{R}$ and for a function $f(x) \in C_0^\infty(\mathbb{R}, \mathbb{R})$. The proof of this inequality can be found in [55]. We note that if we know that $f \in \mathbb{H}^{1,2}$, then this inequality allows us to conclude that $|f|^2$ is bounded almost everywhere. We finish by noting that similarly to the L^p-spaces, the space of test functions C_0^∞ is dense in the Sobolev spaces $\mathbb{H}^{q,p}$. For more detail on Sobolev spaces, one should refer to [1] or [35].

1.6.2 The Existence and Uniqueness Theorem

In the study of both the odd and even parity perturbations, we will use a result which provides for the existence of smooth and compactly supported solutions to partial differential equations which are symmetric hyperbolic, with a choice of smooth and compactly supported initial data. We consider the linear system

$$\frac{\partial \vec{u}}{\partial t} + \sum_{i=1}^{n} A_i(t,\vec{x}) \frac{\partial \vec{u}}{\partial x_i} + B(t,\vec{x})\vec{u} = \vec{f}(t,\vec{x}), \tag{1.4}$$

where $\vec{u}$ is some state m-vector, x_i labels the n spatial coordinates and the A_i and B are $m \times m$ matrices with elements in $C([0,T], C^1_B(\mathbb{R}^n))$. We write $\vec{x} = (x_1, x_2, \ldots, x_n)$ and for our purposes, $n = 1$ always. The interval $[0,T]$ is the range of our time coordinate t, and the notation C^1_B indicates those C^1-functions which are bounded. To make this system symmetric hyperbolic, we require the matrices A_i to be symmetric. We will further assume that $\vec{f}$ has components in $L^2((0,T), \mathbb{H}^{1,2}(\mathbb{R}^m))$.

For such systems, one can state an existence and uniqueness theorem which can be found in Chapter 12 of [35] ([47] also has a useful discussion of these methods).

Theorem 1.6.1 *The system (1.4) with initial data*

$$\vec{u}\Big|_{t=0} = \vec{g}(\vec{x}),$$

where $\vec{g} \in C^\infty_0(\mathbb{R}^n, \mathbb{R}^m)$, has a unique solution $\vec{u} \in C^\infty([0,T] \times \mathbb{R}^n, \mathbb{R}^m)$. For $t \in [0,T]$, $\vec{u}(\cdot, \vec{x}) : \mathbb{R}^n \to \mathbb{R}^m$ has compact support.

We will use this theorem to provide for the existence and uniqueness of solutions to the perturbation equations of motion prior to the Cauchy horizon. The proof of this theorem relies on the use of energy methods.

1.6.3 Regular Singular Points of ODEs

We now consider the theory of regular singular points in ordinary differential equations. A full exposition of this theory can be found in [58] and [13]. Such singular points will occur in our analysis of the even parity perturbations in Chapter 5.

Suppose we have an $n \times n$ matrix P which can be diagonalised. Then the Jordan canonical form of P is simply $J = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, where the λ_i are the eigenvalues of P. If P cannot be diagonalised, then the Jordan canonical form of P can be written as

$$J = \begin{pmatrix} J_1 & 0 & 0 & 0 & \dots \\ 0 & J_2 & 0 & 0 & \dots \\ 0 & 0 & J_3 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & J_n \end{pmatrix},$$

where the blocks J_i take the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & 0 & \dots \\ 0 & \lambda_i & 1 & 0 & \dots \\ 0 & 0 & \lambda_i & 0 & \dots \\ \dots & \dots & \dots & \dots & 1 \\ 0 & 0 & 0 & 0 & \lambda_i \end{pmatrix},$$

where λ_i is the i^{th} eigenvalue of P. We note also that there exists a matrix S, known as the similarity matrix, such that $J = S^{-1}PS$.

Now consider the equation

$$t\frac{d\vec{Y}}{dt} = A(t)\vec{Y}, \tag{1.5}$$

where $\vec{Y}$ is an m-dimensional state vector and A is an $m \times m$-dimensional matrix. We will assume that $A(t)$ is analytic at $t = 0$ so that $A(t)$ can be written in a power series as $A(t) = \sum_{r=0}^{\infty} A_r t^r$. $J = T^{-1}A(t = 0)T$ is the Jordan canonical form of $A(0)$. We can rewrite (1.5) as

$$\frac{d\vec{Y}}{dt} = \frac{1}{t}\left(A(0) + \sum_{r=1}^{\infty} A_r t^r\right)\vec{Y}.$$

We see that that we have a power t^{-1} on the right hand side. A singular point of this form is known as a regular singular point, or a singular point of the first kind. Solutions to equations of the form (1.5) near such points can be given. We could also have a higher power in the right hand side, for example t^{-p}, for $p > 1$. Such singular points are known as non-simple or singular points of the second kind and we will not discuss them further.

Before proceeding we will need the notion of a fundamental matrix. Consider the vector equation

$$\frac{d\vec{Y}}{dt} = A(t)\vec{Y}. \tag{1.6}$$

Suppose that we have an $m \times m$ matrix $V(t)$ which satisfies

$$\frac{dV}{dt} = A(t)V.$$

It is easy to see that each column of $V(t)$ will satisfy (1.6). Such a matrix $V(t)$ is known as a fundamental matrix if its columns are linearly independent, or equivalently, if $\det(V) \neq 0$. The columns of a fundamental matrix therefore provide a basis for solutions of (1.6). Using the idea of a fundamental matrix, we can state the following theorem.

Theorem 1.6.2 *Suppose that $A(t)$ is analytic at $t = 0$ and that no two eigenvalues of $A(0)$ differ by a positive integer. Then in a disc around $t = 0$ which does not contain another singular point, the differential equation (1.5) has a fundamental matrix solution of the form*

$$V = P(t)t^J,$$

where $J = T^{-1}A(0)T$ is the Jordan canonical form of $A(0)$. The matrix $P(t)$ can be written in a power series as $P(t) = \sum_{r=0}^{\infty} P_r x^r$ where $P_0 = \mathbb{I}$ and the coefficients P_m can be calculated by rational operations from the coefficients A_r in the power series for $A(t)$.

See [58] for the proof of this theorem and for the details of the matrix $P(x)$. We note that this theorem essentially provides us with m linearly independent solutions to (1.5). It is possible to deal with the case when the eigenvalues of J differ by positive integers; see [58] and [13] for details.

We will also need to deal with inhomogeneous equations of the form,

$$t\frac{d\vec{Y}}{dt} = A(t)\vec{Y} + \vec{B}(t),$$

and one can easily show that the particular solution is given by

$$V_P = V(t)\int_0^T V^{-1}(\tau)\vec{B}(\tau)\,d\tau,$$

where $V(t)$ is the solution to the homogeneous problem (1.5).

1.6.4 The Method of Characteristics

In Chapters 4 and 6, we will need to deal with solutions to transport equations in $1+1$ dimensions, that is, first order linear PDEs of the form

$$\frac{\partial x}{\partial t} + a(t,p)\frac{\partial x}{\partial p} + b(t,p)x = \Sigma(t,p).$$

In Chapters 4 and 6 we will have coefficients which depend on t only, that is

$$\frac{\partial x}{\partial t} + a(t)\frac{\partial x}{\partial p} + b(t)x = \Sigma(t,p). \tag{1.7}$$

We assume that $a(t)$ and $b(t)$ are analytic functions of t and that Σ is analytic in t and $\Sigma(\cdot,p) \in L^2(\mathbb{R},\mathbb{R})$. Solutions to transport equations can be arrived at using the method of characteristics, which we review here. For a general introduction to this method, see Chapter 1 of [35].

The characteristic curves of this equation are the solutions to the initial value problem

$$\frac{dp}{dt} = a(t), \qquad\qquad p(t_1) = \eta.$$

This has solution $p(t) = \pi(t) + \eta$ where $\pi(t) = -\int_{t_1}^{t} a(\tau)d\tau$. See Figure 1.1 for an example of typical characteristic curves (the details of the curves depends on the choice of $a(t)$). The fact that $\pi(t)$ is a function of t only tells us that the characteristic curves are all parallel (more precisely, their tangents at fixed t are parallel).

Along characteristics, (1.7) becomes

$$\frac{dx}{dt}(t,p(t)) + b(t)x(t,p(t)) = \Sigma(t,p(t)). \tag{1.8}$$

The integrating factor for this equation is given by $e^{\xi(t)}$ where $\xi(t) := -\int_{t_1}^{t} b(\tau)d\tau$ and the solution to (1.8) can be written

$$x(t,p) = e^{-\xi(t)}x_0(p - \pi(t)) - e^{-\xi(t)} \int_{t_1}^{t} e^{\xi(\tau)}\Sigma(\tau, p - \pi(t) + \pi(\tau))d\tau,$$

where $x(t_1, p) = x_0(p)$ is the initial data. In applications of this method in Chapters 4 and 6, we will make use of the integral form of the mean value theorem [49] and Fubini's theorem [54], which we state here for convenience.

Theorem 1.6.3 *Suppose that on the interval $x \in [a,b]$, $f(x)$ is a continuous function of x and $g(x)$ is positive and integrable. Then*

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx \tag{1.9}$$

for some $c \in [a,b]$.

Theorem 1.6.4 *Suppose we have a function $f(t,x)$ which is continuous on $t_1 \leq t \leq t_2$ and $-\infty \leq x_1 \leq x \leq x_2 \leq \infty$. Then*

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2} f(t,x)\,dx\,dt = \int_{x_1}^{x_2}\int_{t_1}^{t_2} f(t,x)\,dt\,dx.$$

That is, we can interchange the order of two integrals provided that the integrand is continuous with respect to both variables.

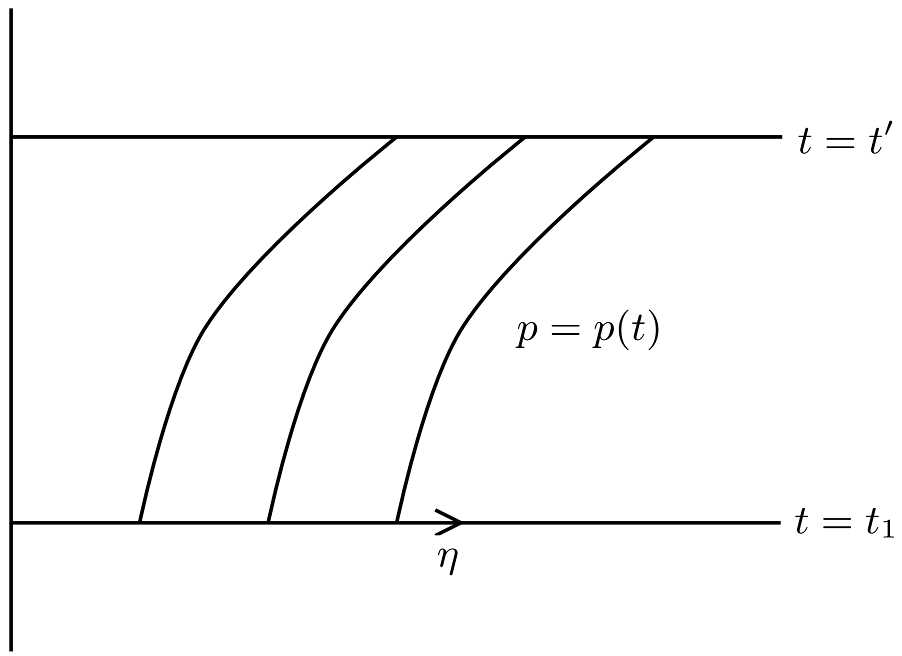

Figure 1.1: Characteristics. We show here an example of typical characteristic curves $p = p(t)$, along which (1.7) is solved.

Finally, we will also make use of the Cauchy-Schwarz inequality [1] in Chapters 4 and 6. This states that for functions $f(x), g(x) \in L^2(\mathbb{R}, \mathbb{R})$,

$$\left| \int_{\mathbb{R}} f(x)g(x)\, dx \right| \leq \frac{1}{2} \int_{\mathbb{R}} f^2(x) + g^2(x)\, dx.$$

1.7 Some Theorems from Real Analysis

In Chapter 6 we use some theorems from real analysis to complete our study of the behaviour of even parity perturbations. We present here those theorems and some useful background information. More details can be found in [33].

In order to present the following theorems, we must first review some terminology. A measure space (Ω, Σ, μ) is composed of a set Ω, together with a sigma-algebra Σ and a measure μ. A sigma-algebra is a set of subsets of Ω with the properties

- If $A \in \Sigma$, then $A^c \in \Sigma$, where A^c is the complement of A in Ω,
- If $A_1, A_2, \ldots$ is a countable family of sets in Ω then $\cup_{i=1}^{\infty} A_i \in \Sigma$,
- $\Omega \in \Sigma$.

A measure is a function μ which maps Σ into the nonnegative real numbers, such that

- The empty set has zero measure, $\mu(\Phi) = 0$,
- Countable additivity: If $A_1, A_2, \ldots$ is a sequence of disjoint sets in Σ then $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

Essentially one can think of the measure as providing a "weight" to each set in Σ. We note that if some property holds "μ-almost everywhere", this means that it fails at most on a set of measure zero. Having completed the preliminary definitions, we begin with the dominated convergence theorem of Lebesgue, which provides a set of conditions under which one may commute the taking of a limit with an integral.

Theorem 1.7.1 *Let $f^{(1)}, f^{(2)}, \ldots$ be a sequence of complex-valued summable functions on a measure space (Ω, Σ, μ). Suppose that these functions converge to a summable function f pointwise almost everywhere. If there exists a summable, non-negative function $G(x)$ on (Ω, Σ, μ) such that*

$$|f^{(i)}(x)| \leq G(x) \qquad \forall i = 1, 2, \ldots$$

then $|f(x)| \leq G(x)$ and

$$\lim_{i \to \infty} \int_\Omega f^{(i)}(x)\, \mu(dx) = \int_\Omega f(x)\, \mu(dx).$$

This theorem essentially provides a condition (namely the existence of some suitable dominating function $G(x)$) under which we may take the limit inside the integral sign. The proof of this theorem relies on Fatou's lemma, and can be found in [33].

When working with Sobolev and L^p-spaces, it is relatively easy to find sequences of the type required by this theorem. However, constructing a dominating function is a more difficult task, and we therefore present a useful theorem which allows us to find a dominating function for sequences in some particular L^p-space. The first part of this theorem actually states the completeness of L^p-spaces, while the second part identifies a subsequence of $f^{(i)}$ which is dominated.

Theorem 1.7.2 *Let $1 \leq p \leq \infty$ and let $f^{(i)}$, for $i = 1, 2, \ldots$ be a Cauchy sequence in $L^p(\Omega)$, that is, for each $\epsilon > 0$, there exists some N such that $||f^{(i)} - f^{(j)}||_p < \epsilon$ for $i, j > N$. Then there exists a unique function $f \in L^p$ such that $||f^{(i)} - f^{(j)}||_p \to 0$ as $i \to \infty$. That is, $f^{(i)}$ converges strongly to f in $L^p(\Omega)$.*

Furthermore, we can show there exists a subsequence $f^{(i_1)}, f^{(i_2)}, \ldots$ (with $i_1 < i_2 < \ldots$) such that

- *Domination: There exists a nonnegative function $F(x)$ such that $|f^{(i_k)}(x)| \leq F(x)$ $\forall k$ and μ-almost every x,*
- *Pointwise Convergence: $\lim_{k \to \infty} f^{(i_k)}(x) = f(x)$ for μ-almost every x.*

This theorem allows us to pull a dominated subsequence out of the sequence $f^{(i)}$, provided $f^{(i)}$ is Cauchy in an L^p-space. We are guaranteed that this subsequence tends to the same limit as the original sequence. The proof of this theorem relies on the fact that the sequence is Cauchy and on the monotone convergence theorem.

1.8 Layout of the Thesis

In Chapter 2, we discuss the self-similar LTB spacetime. We describe the background metric and the matter model and determine in Section 2.3 a criterion for nakedness of the singularity. In Chapter 3, we describe the perturbation formalism due to Gerlach and Sengupta which we use throughout.

In Chapter 4 we consider a particular subset of the perturbations, the odd parity perturbations. For angular number $l \geq 2$, we find a single gauge invariant scalar which fully describes the perturbation and state an existence and uniqueness theorem for solutions to its equation of motion prior to the Cauchy horizon in Section 4.3. We next use energy methods for symmetric hyperbolic systems to show that the perturbation remains finite throughout its evolution up to and on the Cauchy horizon. Here finiteness is measured in terms of *a priori* terms which bound the growth of the perturbation. We give an interpretation of this result in terms of the perturbed Weyl scalars in Section 4.5 and finally, we consider separately the $l = 1$ perturbation in Section 4.6 and show that a similar result pertains in this case.

In Chapter 5, we undertake a parallel discussion of the even parity perturbations. We first determine a fundamental system of equations governing their evolution in Section 5.2. We state an existence and uniqueness theorem for solutions to this system prior to the Cauchy horizon and in Section 5.3, we consider the behaviour of an average of the perturbation. Using this average, we establish that the L^p-norm of the perturbation (for $1 \leq p \leq \infty$) generically diverges on the Cauchy horizon. In Chapter 6, we return to the pointwise behaviour of the perturbation and establish a series of results which indicate that the perturbation diverges in a pointwise manner on the Cauchy horizon. This result is interpreted in terms of the perturbed Weyl scalars in

Section 6.2. Hence, we establish the linear instability of the naked singularity in this class of LTB spacetimes.

In Chapter 7 we summarise the work and discuss some areas for future research. In the appendix, we provide some technical details omitted from the main text.

Chapter 2

The Self-Similar LTB Spacetime

In this chapter, we discuss the background spacetimes used throughout this thesis, the self-similar Lemaître-Tolman-Bondi (LTB) spacetimes. We define the LTB spacetimes in Section 2.1, before imposing self-similarity in Section 2.2. Finally, we determine the condition required for the singularity in these spacetimes to be naked in Section 2.3.

2.1 The LTB Spacetime

The LTB spacetime is a spherically symmetric spacetime containing a pressure-free perfect fluid which undergoes an inhomogeneous collapse into a singularity. Under certain conditions this singularity can be naked. We will initially use comoving coordinates (t, r, θ, ϕ), in which the dust is stationary so that the dust velocity has a time component only. In these coordinates, the radius r labels each successive shell in the collapsing dust. The line element for such a spacetime can be written in comoving coordinates as

$$ds^2 = -dt^2 + e^{\nu}(t,r)dr^2 + R^2(t,r)d\Omega^2, \tag{2.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ and $R(t,r)$ is the physical radius of the dust. The stress-energy tensor of the dust can be written as

$$\bar{T}^{\mu\nu} = \bar{\rho}(t,r)\bar{u}^{\mu}\bar{u}^{\nu},$$

where $\bar{u}^\mu$ is the 4-velocity of the dust, that is, a future pointing, timelike unit vector field, which is tangential to the flow lines of the dust and satisfies $\bar{u}_\mu \bar{u}^\mu = -1$. $\bar{\rho}(t,r)$ is the rest mass density of the dust. In comoving coordinates, $\bar{u}^\mu = \delta^\mu_0$.

The background Einstein equations for the metric and stress energy in comoving coordinates immediately provide the following results,

$$e^{\nu/2} = \frac{\partial R/\partial r}{\sqrt{1+f(r)}}, \qquad \bar{\rho}(t,r) = \frac{\partial m(r)/\partial r}{4\pi \partial R/\partial r R^2}, \tag{2.2}$$
$$\left(\frac{\partial R}{\partial t}\right)^2 - \frac{2m(r)}{R} = f(r).$$

The function $m(r)$ is known as the Misner-Sharp mass and is a suitable mass measure for spherically symmetric spacetimes. The last equation in (2.2) has the form of a specific energy equation, which indicates that the function $f(r)$ can be interpreted as the total energy per unit mass of the dust. The background dynamics of the dust cloud can be determined by a choice of $m(r)$ (or a specification of the initial profile of $\rho(t,r)$) and a choice of $f(r)$.

Recall that a shell focusing singularity is a singularity which occurs when the physical radius $R(t,r)$ of the dust cloud vanishes, so that all the matter shells have been "focused" onto a single point. In this spacetime, a shell focusing singularity occurs on a surface of the form $t = t_{sf}(r)$, which includes the scaling origin $(t,r) = (0,0)$.

In spacetimes consisting of a collapsing cloud of matter, one can also encounter a shell crossing singularity, which occurs when two shells, labelled by particular values of the radius, r_1 and r_2, cross each other. More precisely, there are values r_1, r_2 and times t_A, t_B for which $R(t_A, r_1) < R(t_A, r_2)$ but $R(t_B, r_1) > R(t_B, r_2)$. No such singularity occurs in the spacetime under consideration here [16].

We immediately specialise to the marginally bound case by setting $f(r) = 0$.

2.2 Self-Similarity

We follow here the conventions of [4]. In comoving coordinates, the homothetic Killing vector field is given by $\vec{\xi} = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}$. When self-similarity is imposed on the metric and stress-energy tensor, we find that functions appearing in the metric, the dust density and the Misner-Sharp mass have the following scaling behaviour,

$$\nu(t,r) = \nu(z), \qquad R(t,r) = rS(z), \tag{2.3}$$

$$\bar{\rho}(t,r) = \frac{q(z)}{r^2}, \qquad m(r) = \lambda r, \tag{2.4}$$

where $z = -t/r$ is the similarity variable and λ is a constant (the case $\lambda = 0$ corresponds to flat spacetime). By combining (2.2), (2.3) and (2.4) we can find an expression for $\frac{\partial R}{\partial t}$,

$$\frac{\partial R}{\partial t} = -\frac{dS}{dz} = -\sqrt{\frac{2\lambda}{S}},$$

where we choose the negative sign for the square root, so that we are dealing with a collapse model. This can be immediately solved for $S(z)$,

$$S(z) = (az+1)^{2/3}, \tag{2.5}$$

where $a = 3\sqrt{\frac{\lambda}{2}}$ and we use the boundary conditions $R|_{t=0} = r$ and $\frac{\partial R}{\partial r}|_{t=0} = 1$. With this expression for $S(z)$ we can solve for $\frac{\partial R}{\partial r}$ explicitly. In (2.2) we convert $\frac{\partial R}{\partial r}$ to a derivative in (z,r) and find that

$$e^{\nu/2} = \frac{\partial R}{\partial r} = S - z\frac{dS}{dz} = (\frac{1}{3}az+1)(1+az)^{-1/3}. \tag{2.6}$$

We can also find an explicit form for the density function $q(z)$ by combining (2.2), (2.4) and (2.5),

$$q(z) = \frac{a^2}{6\pi(3+4az+a^2z^2)}. \tag{2.7}$$

We state the metric in (z,r) coordinates, for future use,

$$ds^2 = -r^2dz^2 + (e^{\nu}(z) - z^2)dr^2 - 2rzdrdz + R^2d\Omega^2. \tag{2.8}$$

We can also state the metric in terms of $S(z)$ only,

$$ds^2 = e^{2p}(-dz^2 + ((S - z\dot{S})^2 - z^2)dp^2 - 2zdzdp + S^2d\Omega^2), \tag{2.9}$$

where $p = \ln(r)$. In Sections 4.5 and 6.2 we will need the null directions of the self-similar LTB spacetime. In terms of (z,r) coordinates, the retarded null coordinate u and the advanced null coordinate v take the form

$$u = r\ \exp\left(-\int_z^{z_o} \frac{dz'}{f_+(z')}\right), \qquad v = r\ \exp\left(-\int_z^{z_o} \frac{dz'}{f_-(z')}\right), \tag{2.10}$$

where $f_\pm := \pm e^{\nu/2} + z$. In these coordinates, the metric takes the form

$$ds^2 = -\frac{t^2}{uv}(1 - e^{\nu}z^{-2})\,du\,dv + R^2(t,r)d\,\Omega^2.$$

In order to calculate the perturbed Weyl scalars, we will need the in- and outgoing null vectors, l^μ and n^μ. These vectors obey the normalisation $g_{\mu\nu}l^\mu n^\nu = -1$. A suitable choice is therefore

$$\vec{l} = \frac{1}{B(u,v)}\frac{\partial}{\partial u}, \qquad \vec{n} = \frac{\partial}{\partial v}, \tag{2.11}$$

where $B(u,v) = \frac{t^2}{2uv}\left(1 - \frac{e^{\nu(z)}}{z^2}\right)$. In what follows, we shall take a dot to indicate differentiation with respect to the similarity variable z, $\cdot = \frac{\partial}{\partial z}$.

2.3 Nakedness of the Singular Origin

We now consider the conditions required for the singularity at the scaling origin $(t,r) = (0,0)$ to be naked. As a necessary and sufficient condition for nakedness, the spacetime must admit causal curves which have their past endpoint on the singularity. It can be shown [38] that it is actually sufficient to consider only null geodesics with their past endpoints on the singularity, and without loss of generality, we restrict our attention to the case of radial null geodesics (RNGs). The equation which governs RNGs can be read off the metric (2.1),

$$\frac{dt}{dr} = \pm e^{\nu/2}.$$

Since we wish to consider outgoing RNGs we select the + sign. We can convert the above equation into an ODE in the similarity variable,

$$z + rz' = -e^{\nu/2}. \tag{2.12}$$

We look for constant solutions to this equation, which correspond to null geodesics that originate from the singularity. It can be shown that the existence of constant solutions to (2.12) is equivalent to the nakedness of the singularity. For constant solutions, we set the derivative of z to zero and combine (2.6) and (2.12) to find the following algebraic equation in z,

$$az^4 + \left(1 + \frac{a^3}{27}\right) z^3 + \left(\frac{a^2}{3}\right) z^2 + az + 1 = 0.$$

We wish to discover when this equation will have real solutions. This can easily be found using the polynomial discriminant for a quartic equation, which is negative when there are two real roots. In this case we have

$$D = \frac{1}{27}(-729 + 2808a^3 - 4a^6),$$

which is negative in the region $a < a^*$ where a^* is

$$a^* = \frac{3}{(2(26 + 15\sqrt{3}))^{1/3}} \approx 0.638...$$

This translates to the bound $\lambda \leq 0.09$. From (2.4), we can see that this result implies that singularities which are "not too massive" can be naked. See Figure 2.1 for a Penrose diagram of this spacetime.

Remark 2.3.1 In fact, one can find $D < 0$ in two ranges, namely $a < a^* \approx 0.64$ and $a > a^{**} \approx 8.89$. We reject the latter range as being unphysical.

Consider (2.5), which indicates that the shell-focusing singularity occurs at $z = -1/a$. If we chose the range $a > a^{**}$ we would find that the corresponding outgoing RNG occurs after the shell focusing singularity and so is not part of the spacetime.

Remark 2.3.2 We note that this analysis has assumed that the entire spacetime is filled with a dust fluid. A more realistic model would involve introducing a cutoff at some radius $r = r_*$, after which the spacetime would be empty. We would then match the interior matter-filled region to an exterior Schwarszchild spacetime. However, it can be shown that this cutoff spacetime will be globally naked so long as the cutoff radius is chosen to be sufficiently small [29]. We will therefore neglect to introduce such a cutoff.

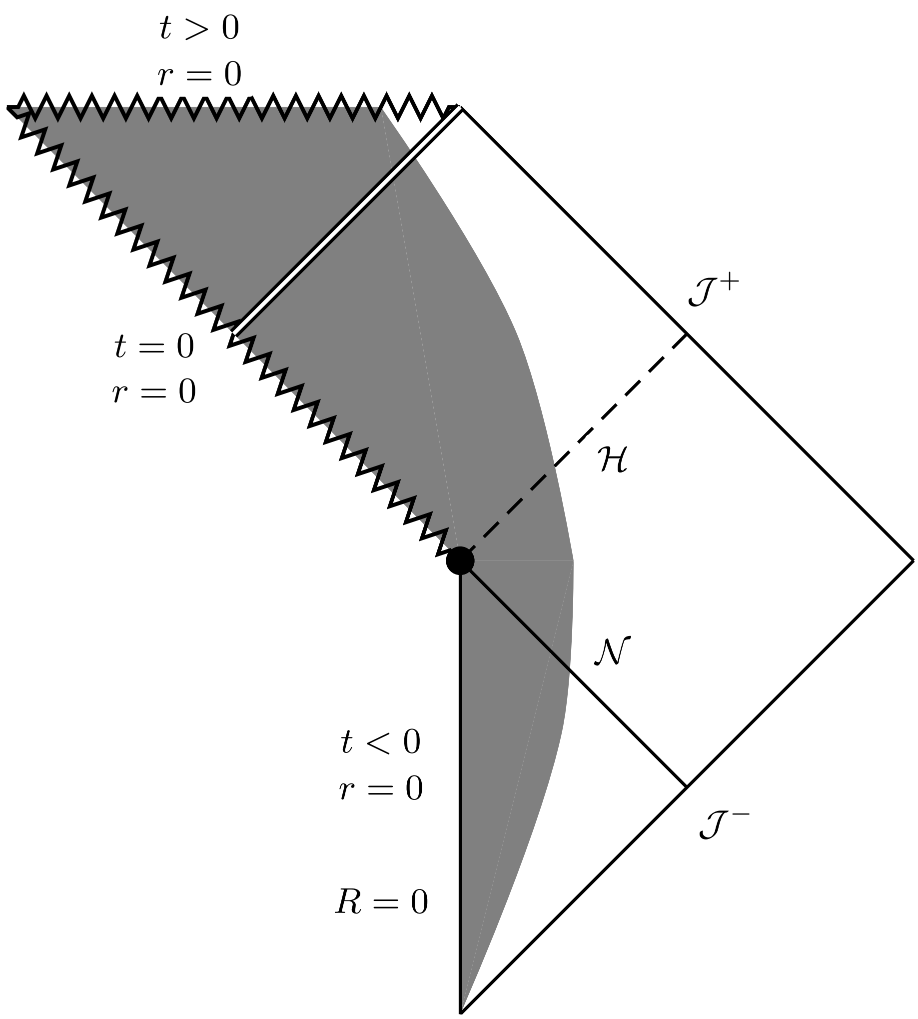

Figure 2.1: Structure of the Self-Similar LTB spacetime. We present here a conformal diagram for the self-similar LTB spacetime. The gray shaded region represents the interior of the collapsing dust cloud. We label the past null cone of the naked singularity by $\mathcal{N}$ and the Cauchy horizon by $\mathcal{H}$. Future and past null infinity are labelled by $\mathcal{J}^+$ and $\mathcal{J}^-$.

Part II
Perturbations

Chapter 3

The Gerlach-Sengupta Method

In this chapter, we present the Gerlach-Sengupta method [19] which provides us with the most general possible linear perturbations of a spherically symmetric spacetime. We shall follow throughout the presentation of [34]. In Section 3.1 we decompose the spherically symmetric background spacetime into two submanifolds (with corresponding metrics) and present the background Einstein equations in terms of this decomposition. In Section 3.2 we expand perturbations of the background in a multipole decomposition and construct gauge invariant combinations of the perturbations. Finally, in Section 3.3 we write the linearised Einstein equations in terms of these perturbations.

3.1 The Background Spacetime

We begin by writing the metric of the entire spacetime $(\mathcal{M}^4, g_{\mu\nu})$ as

$$ds^2 = g_{AB}(x^C)dx^A dx^B + R^2(x^C)\gamma_{ab}dx^a dx^b, \tag{3.1}$$

where g_{AB} is a Lorentzian metric on the 2-dimensional manifold $\mathcal{M}^2$ and γ_{ab} is the metric for the 2-sphere $\mathcal{S}^2$ (and the full manifold is $\mathcal{M}^4 = \mathcal{M}^2 \times \mathcal{S}^2$). The indices $A, B, C...$ indicate coordinates on $\mathcal{M}^2$ and take the values $A, B... = 1, 2$ while the indices $a, b, c...$ indicate coordinates on $\mathcal{S}^2$ and take the values $a, b... = 3, 4$. The covariant derivatives on $\mathcal{M}^4$, $\mathcal{M}^2$ and $\mathcal{S}^2$ are denoted by a semi-colon, a vertical bar and a colon respectively. The stress-energy can be split in a similar fashion,

$$t_{\mu\nu}dx^{\mu}dx^{\nu} = t_{AB}dx^{A}dx^{B} + Q(x^{C})R^{2}\gamma_{ab}dx^{a}dx^{b}, \tag{3.2}$$

where $Q(x^C) = \frac{1}{2}t^a_{\ a}$ is the trace across the stress-energy on $\mathcal{S}^2$. Now if we define

$$v_A = \frac{R_{|A}}{R}, \tag{3.3}$$

$$V_0 = -\frac{1}{R^2} + 2v^A_{\ |A} + 3v^A v_A, \tag{3.4}$$

then the Einstein equations for the background metric and stress-energy read

$$G_{AB} = -2(v_{A|B} + v_A v_B) + V_0 g_{AB} = 8\pi t_{AB}, \tag{3.5}$$

$$\frac{1}{2}G^a_{\ a} = -\mathcal{R} + v^A v_A + v^A_{\ |A} = 8\pi Q(x^C), \tag{3.6}$$

where $G^a_{\ a} = \gamma^{ab}G_{ab}$. $\mathcal{R}$ is the Gaussian curvature of $\mathcal{M}^2$, $\mathcal{R} = \frac{1}{2}R^{(2)A}_{A}$ where $R^{(2)}_{AB}$ indicates the Ricci tensor on $\mathcal{M}^2$.

3.2 Perturbations

We now wish to perturb the metric (3.1), so that $g_{\mu\nu}(x^\alpha) \rightarrow g_{\mu\nu}(x^\alpha) + \delta g_{\mu\nu}(x^\alpha)$. To do this, we use a similar decomposition for $\delta g_{\mu\nu}(x^\alpha)$ and write explicitly the angular dependence using the spherical harmonics. We write the spherical harmonics as $Y_l^m \equiv Y$. $\{Y\}$ forms a basis for scalar harmonics, while $\{Y_a := Y_{:a}, S_a := \epsilon_a^{\ b}Y_b\}$ form a basis for vector harmonics. Finally, $\{Y\gamma_{ab}, Z_{ab} := Y_{a:b} + \frac{l(l+1)}{2}Y\gamma_{ab}, S_{a:b} + S_{b:a}\}$ form a basis for tensor harmonics.

We can classify these harmonics according to their behaviour under spatial inversion $\vec{x} \rightarrow -\vec{x}$: A harmonic with index l is even if it transforms as $(-1)^l$ and odd if it transforms as $(-1)^{l+1}$. According to this classification, Y, Y_a and Z_{ab} are even, while S_a and $S_{(a:b)}$ are odd.

Even and odd perturbations will decouple in what follows. We now expand the metric perturbation in terms of the spherical harmonics. Each perturbation is labelled by (l, m) and the full perturbation is given by a sum over all l and m. However, since each individual perturbation decouples in what follows, we can neglect the labels and summation symbols. The metric perturbation is given by

$$\delta g_{AB} = h_{AB} Y, \qquad \delta g_{Ab} = h^{\mathbf{E}}_A Y_{:b} + h^{\mathbf{O}}_A S_b, \tag{3.7}$$

$$\delta g_{ab} = R^2 K Y \gamma_{ab} + R^2 G Z_{ab} + h(S_{a:b} + S_{b:a}), \tag{3.8}$$

where h_{AB} is a symmetric rank 2 tensor, $h^{\mathbf{E}}_A$ and $h^{\mathbf{O}}_A$ are vectors and K, G and h are scalars, all on $\mathcal{M}^2$. We similarly perturb the stress-energy $t_{\mu\nu} \rightarrow t_{\mu\nu} + \delta t_{\mu\nu}$ and expand the perturbation in terms of the spherical harmonics,

$$\delta t_{AB} = \Delta t_{AB} Y, \qquad \delta t_{Ab} = \Delta t^{\mathbf{E}}_A Y_{:b} + \Delta t^{\mathbf{O}}_A S_b, \tag{3.9}$$

$$\delta t_{ab} = R^2 \Delta t^3 \gamma_{ab} Y + R^2 \Delta t^2 Z_{ab} + 2\Delta t S_{(a:b)}, \tag{3.10}$$

where Δt_{AB} is a symmetric rank 2 tensor, $\Delta t^{\mathbf{E}}_A$ and $\Delta t^{\mathbf{O}}_A$ are vectors and Δt^3, Δt^2 and Δt are scalars, all on $\mathcal{M}^2$.

We wish to work with gauge invariant variables, which can be constructed as follows. Suppose the vector field $\vec{\xi}$ generates an infinitesimal coordinate transformation of our coordinates, $\vec{x} \rightarrow \vec{x'} = \vec{x} + \vec{\xi}$. We wish our variables to be invariant under such a transformation. We can decompose $\vec{\xi}$ into even and odd harmonics and write the one-form fields

$$\underline{\xi}^{\mathbf{E}} = \xi_A(x^C) Y dx^A + \xi^{\mathbf{E}}(x^C) Y_{:a} dx^a, \qquad \underline{\xi}^{\mathbf{O}} = \xi^{\mathbf{O}} S_a dx^a. \tag{3.11}$$

We then construct the transformed perturbations after this coordinate transformation and look for combinations of perturbations which are independent of $\vec{\xi}$ and therefore gauge invariant. The odd parity metric perturbation can be written as a gauge invariant vector field,

$$k_A = h^{\mathbf{O}}_A - h_{|A} + 2h v_A, \tag{3.12}$$

and the odd parity gauge invariant matter perturbation is given by a 2-vector and a scalar,

$$L_A = \Delta t^{\mathbf{O}}_A - Q h^{\mathbf{O}}_A, \tag{3.13}$$

$$L = \Delta t - Q h. \tag{3.14}$$

In the even parity case, the metric perturbation is described by a gauge invariant 2-tensor k_{AB} and a gauge invariant scalar k,

$$k_{AB} = h_{AB} - (p_{A|B} + p_{B|A}), \qquad\qquad k = K - 2v^A p_A, \tag{3.15}$$

where $p_A = h^{\mathbf{E}}_A - \frac{1}{2}R^2 G_{|A}$. The even parity gauge invariant matter perturbation is given by

$$T_{AB} = \Delta t_{AB} - t_{AB}{}^{|C} p_C - 2(t_{CA} p^C{}_{|B} + t_{CB} p^C{}_{|A}), \tag{3.16}$$

$$T_A = \Delta t_A - t_A{}^C p_C - \frac{R^2 Q}{2} G_{|A}, \tag{3.17}$$

$$T^3 = \Delta t^3 - \frac{p^C}{R^2}\left(R^2 Q\right)_{|C}, \qquad\qquad T^2 = \Delta t^2 - R^2 Q G. \tag{3.18}$$

In the Regge-Wheeler gauge, $h^{\mathbf{E}}_A = h = G = 0$, which implies that $p_A = 0$. The gauge invariant matter perturbations then coincide with the bare matter perturbations, which simplifies matters considerably. We will use this gauge in what follows.

3.3 The Linearised Einstein Equations

We now list the linearised Einstein equations, which can be written in terms of the gauge invariant quantities presented above. For the odd parity perturbations, the linearised Einstein equations are

$$k^A_{\ |A} = 16\pi L, \qquad\qquad l \geq 2, \tag{3.19}$$

$$(R^4 D^{AB})_{|B} + \mathcal{L} k^A = 16\pi R^2 L^A, \qquad\qquad l \geq 1, \tag{3.20}$$

where $\mathcal{L} = (l-1)(l+2)$ and D_{AB} is

$$D_{AB} = \left(\frac{k_B}{R^2}\right)_{|A} - \left(\frac{k_A}{R^2}\right)_{|B}. \tag{3.21}$$

By taking a derivative of (3.20), using the fact that D_{AB} is antisymmetric and combining the result with (3.19), one can derive the stress-energy conservation equation,

$$(R^2 L^A)_{|A} = \mathcal{L} L. \tag{3.22}$$

One can show that (3.20) is equivalent to a single scalar equation

$$\left(\frac{1}{R^2}(R^4\Psi)^{|A}\right)_{|A} - \mathcal{L}\Psi = -16\pi\epsilon^{AB} L_{A|B}, \tag{3.23}$$

where the scalar Ψ is defined, for $l \geq 2$, by

$$\Psi = \epsilon^{AB}(R^{-2}k_A)_{|B}, \tag{3.24}$$

The gauge invariant metric perturbation k_A can be recovered from

$$\mathcal{L}k_A = 16\pi R^2 L_A - \epsilon_{AB}(R^4\Psi)^{|B}. \tag{3.25}$$

For the even parity perturbations, the linearised Einstein equations are

$$\begin{aligned}
&(k_{CA|B} + k_{CB|A} - k_{AB|C})v^C - g_{AB}(2k_{CD}{}^{|D} - k_D{}^D{}_{|C})v^C \\
&- (k_{|A}v_B + k_{|B}v_A + k_{|AB}) + \left(V_0 + \frac{l(l+1)}{2R^2}\right)k_{AB} \\
&- g_{AB}\left(k_F{}^F\frac{l(l+1)}{2R^2} + 2k_{DF}v^{D|F} + 3k_{DF}v^Dv^F\right) \\
&+ g_{AB}\left(\frac{(l-1)(l+2)}{2R^2}k - k_{|F}{}^F - 2k_{|F}v^F\right) = 8\pi T_{AB}, \qquad l \geq 0
\end{aligned} \tag{3.26}$$

$$
\begin{aligned}
&- k^{AB}{}_{|AB} + (k_A{}^A)_{|B}^{\ |B} - 2k^{AB}{}_{|A}v_B + k^A{}_{A|B}v^B + R^{AB}(k_{AB} - kg_{AB}) \\
&\qquad - \frac{l(l+1)}{2R^2}k^A_{\ A} + k_{|A}{}^A + 2k_{|A}v^A = 16\pi T^3, \qquad l \geq 0 \quad (3.27) \\
&\qquad k_{AB}{}^{|B} - k^B{}_{B|A} + k^B{}_B v_B - k_{|A} = 16\pi T_A, \qquad l \geq 1 \quad (3.28) \\
&\qquad k^A_{\ A} = 16\pi T^2. \qquad l \geq 2 \quad (3.29)
\end{aligned}
$$

Finally, we present the even parity linearised stress-energy conservation equations,

$$
\begin{aligned}
&\frac{1}{R^2}(R^2 T_{AB})^{|B} - \frac{l(l+1)}{R^2}T_A - 2v_A T^3 = (t_{AB|D} + 2t_{AB}v_D)k^{BD} + \\
&Q(k_{|A} - 2kv_A) - t_{AB}k^{|B} + \frac{1}{2}t^{BC}k_{BC|A} - \frac{1}{2}t_{AB}k_F{}^{F|B} + t_{AB}k^{BF}{}_{|F}, \quad (3.30)
\end{aligned}
$$

$$
\frac{1}{R^2}(R^2 T_B)^{|B} + T^3 - \frac{(l-1)(l+2)}{2}\frac{T^2}{R^2} = \frac{1}{2}k_{AB}t^{AB} + Q(k - \frac{1}{2}k_A{}^A). \quad (3.31)
$$

Chapter 4

The Odd Parity Perturbations

In this chapter, the even parity perturbations are set to zero, and we study the behaviour of odd parity perturbations as they approach the Cauchy horizon. We begin by determining the forms of the matter perturbation and the master equation which describes the evolution of $l \geq 2$ perturbations in Sections 4.1 and 4.2. In Section 4.3 we present a theorem which provides for the existence of unique solutions to this master equation, with a choice of C_0^∞ initial data. In Section 4.4, we show that the perturbation remains finite throughout its evolution up to and on the Cauchy horizon. In Section 4.5 we provide a physical interpretation for this result. Finally, in Section 4.6 we discuss the $l = 1$ perturbation, and in Section 4.7 we briefly discuss the odd parity results generally.

4.1 The Matter Perturbation

We begin by finding a relation between the gauge invariant matter perturbation L^A and the dust density and velocity discussed in Section 2.1. To do this, we write the stress-energy of the full spacetime as a sum of the background stress-energy and the perturbation stress-energy (where a bar indicates a background quantity),

$$T_{\mu\nu} = \overline{T}_{\mu\nu} + \delta T_{\mu\nu}.$$

We will assume that the full stress-energy of the perturbed spacetime also represents dust. We can write the density as $\rho = \overline{\rho} + \delta\rho$ and the fluid velocity as $u_\mu = \overline{u}_\mu + \delta u_\mu$. We can therefore find an expression for the perturbed stress-energy (keeping only first order terms),

$$\delta T_{\mu\nu} = \overline{\rho}(\overline{u}_\mu \delta u_\nu + \overline{u}_\nu \delta u_\mu) + \delta\rho \overline{u}_\mu \overline{u}_\nu. \tag{4.1}$$

The perturbation of the dust velocity can now be expanded in terms of the spherical harmonics as $\delta u_\mu = (\delta u_A Y, \delta u_o S_a) = (0, 0, U(t,r)S_a)$. If we set all even perturbations in (3.9 - 3.10) to zero, then comparison of (3.9) and (3.10) to (4.1) produces the results,

$$\delta\rho = 0, \qquad\qquad \Delta t = 0.$$

Then comparing (3.9) to (4.1) and using (3.13) and (3.14) (remembering that $Q = 0$ in this spacetime) produces

$$L_A = \Delta t_A^{\mathbf{o}} = \overline{\rho} U \bar{u}_A, \qquad\qquad L = 0.$$

If we use these results in conjunction with (3.22) (noting that the relevant perturbation Christoffel symbols all vanish for the case of odd perturbations in the Regge-Wheeler gauge), we find that (3.22) becomes,

$$U_{,t} + U\left(\frac{2R_{,t}}{R} + \frac{\bar{\rho}_{,t}}{\bar{\rho}} + \frac{\nu_{,t}}{2}\right) = 0. \tag{4.2}$$

Conservation of stress-energy on the background spacetime results in

$$\bar{\rho}_{,t} + \bar{\rho}\left(\frac{2R_{,t}}{R} + \frac{\nu_{,t}}{2}\right) = 0. \tag{4.3}$$

Combining (4.2) and (4.3) produces

$$\frac{\partial U}{\partial t} = 0. \tag{4.4}$$

Given this result, the matter perturbation can be completely determined by a choice of initial profile $U(z = z_i, r) = y(r)$ on some suitable initial data surface $z_i \in (z_c, z_p]$, where z_p indicates the past null cone of the scaling origin. We now exploit these results to find a useful form for (3.20).

4.2 The Master Equation

Having specified the matter perturbation in terms of an initial data function, we now consider the remaining odd parity terms. We use the coordinates (z, p) where $z = -t/r$ is the similarity variable introduced in Section 2.2 and $p = \ln r$ is a useful scaling of the radial coordinate. In terms of these coordinates, (3.23) can be written as

$$\beta(z)\frac{\partial^2 A}{\partial z^2} + \gamma(z)\frac{\partial^2 A}{\partial p^2} + \xi(z)\frac{\partial^2 A}{\partial z \partial p} + a(z)\frac{\partial A}{\partial z} + b(z)\frac{\partial A}{\partial p} + c(z)A = e^{\kappa p}\Sigma(z, p), \tag{4.5}$$

where the function $A(z, p)$ is related to the master function by

$$A(z, p) = e^{\kappa p} S^4(z)\Psi(z, p).$$

We introduce a factor $e^{\kappa p} = r^\kappa$, for $\kappa \geq 0$, for reasons which will be explained later. This means that Ψ can be non-zero at the singularity. In what follows, we will find a positive value κ^* such that $\kappa \in [0, \kappa^*]$. The coefficients in (4.5) depend only on z and are given by

$$\beta(z) = 1 - z^2 e^{-\nu}, \tag{4.6}$$

$$\gamma(z) = -e^{-\nu}, \tag{4.7}$$

$$\xi(z) = 2ze^{-\nu}, \tag{4.8}$$

$$a(z) = 2ze^{-\nu}(2 - \kappa) + \frac{\dot{\nu}}{2}(1 + z^2 e^{-\nu}) - \frac{2\dot{S}}{S}\beta(z), \tag{4.9}$$

$$b(z) = e^{-\nu}(2\kappa - 5) - e^{-\nu} z \left(\frac{\dot{\nu}}{2} + \frac{2\dot{S}}{S}\right), \tag{4.10}$$

$$c(z) = -e^{-\nu}(\kappa^2 - 5\kappa + 4) + ze^{-\nu}\left(\frac{\dot{\nu}}{2} + \frac{2\dot{S}}{S}\right)(\kappa - 4) + \mathcal{L}S^{-2}. \qquad (4.11)$$

We note that the three leading coefficients are all metric functions, see (2.8). The source term $\Sigma(z, p)$ is

$$\Sigma(z, p) = -16\pi e^{-\nu/2} S^2 \partial_r(\overline{\rho} U). \qquad (4.12)$$

4.3 Existence and Uniqueness of Solutions

We briefly note that the choice of coordinate $z = -t/r$ means that in the range $z_c < z \leq z_p$, where z_c, z_p are the Cauchy horizon and past null cone respectively, z is a time coordinate. Since the Cauchy horizon $z = z_c$ actually occurs at some negative z-value, we should always integrate from $z = z_c$ up to z. Notice that (2.8) indicates that the Cauchy horizon occurs at $z_c = -e^{\nu(z_c)/2}$, while the past null cone of the scaling origin occurs at $z_p = +e^{\nu(z_p)/2}$.

We wish to prove that there exist unique solutions to the initial value problem comprised of (4.5) with suitable initial conditions. We begin by showing that (4.5) may be written as a first order symmetric hyperbolic system. We define a useful coordinate transformation,

$$\overline{z} := \int_z^{z_i} \frac{ds}{\beta(s)},$$

where z_i labels the initial data surface. By inspection, we can see that $\bar{z}(z_i) = 0$. Also, we can see that $\bar{z}(z_c) = \infty$ if we note that we can write $\beta(z) = z_c^{-2}(z_c + z)(z_c - z)$, so that z_c is a simple root of $\beta(z)$. We now define the vector $\vec{\Phi}$,

$$\vec{\Phi} = \begin{pmatrix} A \\ A_{,\bar{z}} + \xi(z) A_{,p} \\ A_{,p} \end{pmatrix}.$$

Then (4.5) takes the form

$$\vec{\Phi}_{,\bar{z}} = X\vec{\Phi}_{,p} + W\vec{\Phi} + \vec{j}. \tag{4.13}$$

where the matrices X and W, and the vector $\vec{j}$ are given in Appendix A.1. In this appendix, we use standard hyperbolic PDE theory to put the system (4.13) in the form required for the Theorem 4.3.1. We identify the surface $S_i = \{(z_i, p) : z_i = 0, p \in \mathbb{R}\}$ as our initial data surface. Since we have written (4.5) using self-similar coordinates for the region between (z_p, z_c), this is a suitable choice. Here, $C_0^\infty(\mathbb{R}, \mathbb{R})$ is the space of smooth functions with compact support.

We note that the source term (4.12) in (4.5) is separable, and can be written as $\Sigma(z,p) = B(z)C(p)$, where $B(z) = -16\pi e^{\nu/2} S^2(z) q(z)$ and $C(p) = (rU_{,r} - 2U)/r^3$, where $U(r)$ is the initial data function appearing in (4.4). It follows that the source vector $\vec{j}$ appearing in (4.13) is also separable, and can be written as $\vec{j}(z,p) = \vec{h}(z)C(p)$.

Theorem 4.3.1 *Let $\vec{f} \in C_0^\infty(\mathbb{R}, \mathbb{R}^3)$ and $C \in C_0^\infty(\mathbb{R}, \mathbb{R})$. Then there exists a unique solution $\vec{\Phi}(z,p)$, $\vec{\Phi} \in C^\infty(\mathbb{R} \times (z_c, z_i], \mathbb{R}^3)$, to the initial value problem consisting of (4.13) with the initial condition $\vec{\Phi}|_{z_i} = \vec{f}$. For all $z \in (z_c, z_i]$ the vector function $\vec{\Phi}(z, \cdot) : \mathbb{R} \to \mathbb{R}^3$ has compact support.*

Proof See Chapter 12 of [35] for a standard proof of this theorem. □

As a corollary to this theorem, the second order master equation, (4.5), inherits existence and uniqueness.

Corollary 4.3.2 *Let f, g, $C \in C_0^\infty(\mathbb{R}, \mathbb{R})$. Then there exists a unique solution $A \in C^\infty(\mathbb{R} \times (z_c, z_i], \mathbb{R})$, to the initial value problem consisting of (4.5) with the initial conditions*

$$A|_{z_i} = f \qquad\qquad A_{,z}|_{z_i} = g$$

For all $z \in (z_c, z_i]$ the function $A(z, \cdot) : \mathbb{R} \to \mathbb{R}$ has compact support.

This corollary ensures existence and uniqueness for solutions to (4.5) in the region between the initial data surface and the Cauchy horizon. In other words, when $A(z, p)$ has regular initial data, the evolution of $A(z, p)$ remains smooth from the initial data surface up to the Cauchy horizon. However, this does not imply smooth behaviour of the perturbation on the Cauchy horizon.

4.4 Behaviour of Perturbations on the Cauchy Horizon

Having given a theorem which guarantees the existence of solutions to (4.5), we now outline the problem under consideration. We insert an initial perturbation from the set of initial data $C_0^\infty(\mathbb{R}, \mathbb{R})$ on the surface $z = z_i$. We then evolve this perturbation up to the Cauchy horizon. We aim to determine whether or not the perturbation remains finite as it impinges on the Cauchy horizon. See Figure 4.1 for an illustration of this.

We begin by noting that the abovementioned choice of initial data is not ideal. Our choice of initial data surface is dictated by the self-similar nature of the background spacetime, and thus, is a natural choice to make. However, this surface intersects the singular scaling origin $(t = 0, r = 0)$ of the spacetime. We are therefore forced to consider initial data which is compactly supported away from the naked singularity. However, by establishing certain bounds on the behaviour of solutions to (4.5) with this initial data choice, we can then exploit the nature of the space $C_0^\infty(\mathbb{R}, \mathbb{R})$ to extend these bounds to a more satisfactory choice of initial data which can be non-zero at the scaling origin.

Finally, we note that since the leading coefficient in (4.5), $\beta(z)$, vanishes on the Cauchy horizon, the Cauchy horizon is a singular hypersurface for this equation. This means that the question of the behaviour of $A(z, p)$ and its derivatives as we approach the Cauchy horizon is nontrivial. To examine this behaviour, we use energy methods for hyperbolic systems.

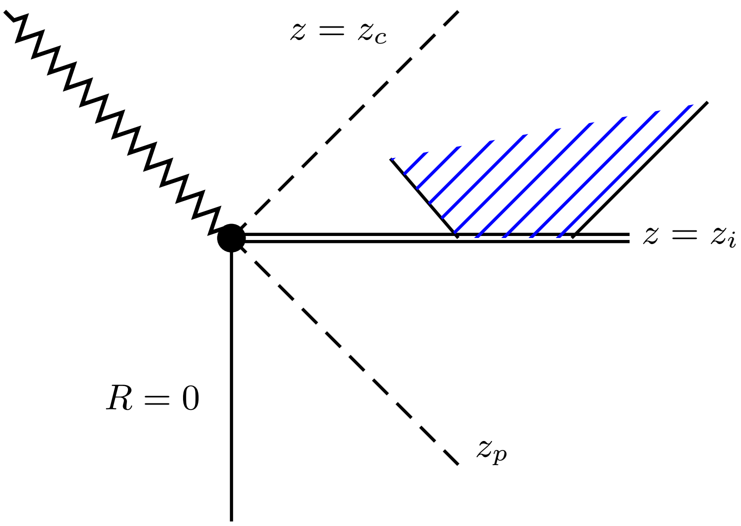

Figure 4.1: The Cauchy Problem. We illustrate here the Cauchy problem associated with the evolution of the perturbation $A(z,p)$ from the initial surface. The support of the initially smooth perturbation (indicated by stripes) spreads causally from the initial surface $z = z_i$ up to the Cauchy horizon.

4.4.1 First Energy Norm

We begin our analysis of the Cauchy horizon behaviour of the perturbation by introducing the energy integral

$$E_1(\bar{z}) = E_1[A](\bar{z}) = \int_{\mathbb{R}} |\vec{\Phi}|^2 dp, \tag{4.14}$$

where $|\cdot|$ indicates the Euclidean norm. The notation

$$\|\vec{f}\|_2^2 = \int_{\mathbb{R}} |\vec{f}|^2 dp$$

indicates the L^2-norm (squared) of the vector function $\vec{f}(z,p)$. We can immediately state a bound on this energy integral, which is a standard result for equations of the form of (4.5).

Corollary 4.4.1 *$E_1[A](\bar{z})$ is differentiable on $[0,\infty)$ and satisfies the bound*

$$E_1[A](\bar{z}) \leq e^{B_0\bar{z}} \left(E_1[A](0) + \int_0^\infty |\vec{j}|^2 dp \right),$$

where $B_0 = \sup_{\bar{z}>0} |I-2W| < \infty$, where W is the matrix appearing in (4.13). As a consequence, the following results also hold,

$$\int_{\mathbb{R}} |A(z,p)|^2 dp \leq e^{B_0\bar{z}} \left(E_1[A](0) + \int_0^\infty |\vec{j}|^2 dp \right), \tag{4.15}$$

$$\int_{\mathbb{R}} |A_{,p}(z,p)|^2 dp \leq e^{B_0\bar{z}} \left(E_1[A](0) + \int_0^\infty |\vec{j}|^2 dp \right), \tag{4.16}$$

$$\int_{\mathbb{R}} |A_{,z}(z,p)|^2 dp \leq C_1 e^{C_0\bar{z}} \left(E_1[A](0) + \int_0^\infty |\vec{j}|^2 dp \right), \tag{4.17}$$

where C_0 and C_1 are constants, not necessarily equal, which depend only on the angular number l and the metric functions $\nu(t,r)$ and $R(t,r)$.

Proof This is a standard result which follows from the definition of $E_1(\bar{z})$, see Chapter 12 of [35]. □

These results indicate that this energy norm is bounded by a divergent term, since $e^{B_0 \bar{z}}$ diverges on the Cauchy horizon. In other words, this theorem only ensures that the growth of the energy norm as we approach the Cauchy horizon is subexponential. In order to proceed, we define a second energy integral, whose behaviour near the Cauchy horizon can be more strongly controlled.

4.4.2 Second Energy Integral

Define

$$E_2[A](z) := \int_{\mathbb{R}} \beta(z) A_{,z}^2 - \gamma(z) A_{,p}^2 + H(z) A^2 + K(z) e^{2\kappa p} \Sigma^2(z,p)\, dp, \quad (4.18)$$

where $H(z) = c(z)$ and $K(z)$ is an arbitrary, non-negative smooth function defined on $(z_c, z_i]$ which will be fixed later. In Corollary 4.4.3, we will find a useful range for κ and in Lemma 4.4.2, we establish that with κ in this range, $H(z) \geq 0$, and thus, $E_2[A](z) \geq 0$. Using these results, we can control the behaviour of dE_2/dz and with this in place, we can finally bound $E_2(z)$.

Lemma 4.4.2 *In the region $\kappa \in [0, \kappa^*]$, where $\kappa^* := \frac{9}{4}$, $H(z) \geq 0$ and $\dot{H}(z) \leq 0$, for all $z \in (z_c, z_i]$.*

Proof We first note that the range $\kappa \in [0, \kappa^*]$ arises from a bound in the next corollary, Corollary 4.4.3. Recall

$$\begin{aligned} H(z) := c(z) = -e^{-\nu}(\kappa^2 - 5\kappa + 4) \\ + z e^{-\nu}\left(\frac{\dot{\nu}}{2} + \frac{2\dot{S}}{S}\right)(\kappa - 4) + \mathcal{L} S^{-2}. \end{aligned} \quad (4.19)$$

From (2.6) and (2.5), we can find explicit forms for each function involved in this definition. We note that $\mathcal{L}$ enters with a coefficient $S^{-2}(z)$, which is always positive. We can therefore safely set $\mathcal{L} = 4$, since if $H(z)$ is positive for $\mathcal{L} = 4$, it will become larger, and therefore more positive, for larger values of $\mathcal{L}$. So, with $\mathcal{L} = 4$, we find that,

$$H(z) = \frac{-m(z) - n(z) + p(z)}{(1 + az)^{4/3}(3 + az)^3},$$

where $m(z) = 9\kappa^2(3 + az)(1 + az)^2$, $n(z) = 8az(36 + 45az + 13a^2z^2)$ and $p(z) = 9\kappa(15 + 39az + 31a^2z^2 + 7a^3z^3)$. The denominator is clearly positive, as can be verified by explicitly checking the allowed ranges of $a \in (0, a^*)$, $z \in (z_c, z_p)$ and $\kappa \in [0, \kappa^*]$. We next consider the numerator. It can easily be confirmed that for a, κ and z in their respective ranges, the numerator above is also positive.

If we consider (4.19), and take a derivative with respect to z, we note that the term containing $\mathcal{L}$ will be $-\frac{4}{3}S^{-3/2}(z)$, where we used the fact that $\dot{S}(z) = \frac{2}{3}aS^{-1/2}(z)$. In other words, $\mathcal{L}$ enters with a coefficient which is always negative. So, if we set $\mathcal{L} = 4$, and can show that in this case, $\dot{H}(z) \leq 0$, then increasing $\mathcal{L}$ will result in $\dot{H}(z)$ becoming more negative. So, calculating the derivative of $H(z)$ and setting $\mathcal{L} = 4$, we find

$$\dot{H}(z) = \frac{4a(o(z) - t(z) + u(z))}{3(1 + az)^{7/3}(3 + az)^4},$$

where $o(z) = azm(z)$, $t(z) = 9k(-9 - 3az + 27a^2z^2 + 28a^3z^3 + 7a^4z^4)$ and $u(z) = 4(-162 - 243az - 63a^2z^2 + 60a63z^3 + 26a^4z^4)$. Again, the denominator is clearly positive, and using the same ranges for a, κ, z and $\mathcal{L}$ we can verify that the numerator is negative. So overall, for $a \in (0, a^*)$, $\kappa \in [0, \kappa^*]$, $z \in (z_c, z_p)$ and $\mathcal{L} \geq 4$ (which corresponds to $l \geq 2$), we have $H(z) \geq 0$ and $\dot{H}(z) \leq 0$. □

We can now move on to examine the behaviour of the derivative of $E_2(z)$.

Corollary 4.4.3 *Let $\kappa \in [0, \kappa^*]$, where $\kappa^* = \frac{9}{4}$. Then there exists some z^* with $z_c < z^* \leq z_i$, a positive constant μ and a choice of function $K(z)$ such that $E_2(z) \geq 0$ and the derivative of the second energy integral obeys the bound*

$$\frac{dE_2}{dz} \geq -\mu E_2(z)$$

in the range $z \in (z_c, z^*]$.

Proof From the definition of $E_2(z)$, (4.18),

$$\frac{dE_2}{dz} = \int_{\mathbb{R}} (\beta_{,z} A_{,z}^2 + 2\beta A_{,z} A_{,zz} - \gamma_{,z} A_{,p}^2 - 2\gamma A_{,p} A_{,pz} + H_{,z} A^2 + 2HA_{,z} A + K_{,z} e^{2\kappa p}\Sigma^2 + 2Ke^{2\kappa p}\Sigma\Sigma_{,z})dp.$$

We now take the following steps. We remove the term containing $A_{,p} A_{,pz}$ by integrating by parts; the resulting surface term will vanish due to the compact support of A. We then replace the term containing $A_{,zz}$ using (4.5). Finally, we remove the term containing $A_{,z} A_{,zp}$ as it is a total derivative. Having followed these steps, we are left with

$$\frac{dE_2}{dz} = \int_{\mathbb{R}} (\beta_{,z} - 2a(z))A_{,z}^2 - \gamma_{,z} A_{,p}^2 + H_{,z} A^2 - 2b(z)A_{,z} A_{,p} + K_{,z} e^{2\kappa p}\Sigma^2 + 2Ke^{2\kappa p}\Sigma\Sigma_{,z} + 2A_{,z} e^{\kappa p}\Sigma \;\; dp.$$

We now use the Cauchy-Schwarz inequality, which states that

$$\int_{\mathbb{R}} 2e^{\kappa p}\Sigma A_{,z}\, dp \geq -\int_{\mathbb{R}} e^{2\kappa p}\Sigma^2 + A_{,z}^2 \;\; dp,$$

to produce

$$\frac{dE_2}{dz} \geq \int_{\mathbb{R}} (\beta_{,z} - 2a(z) - 1)A_{,z}^2 - \gamma_{,z} A_{,p}^2 + H_{,z} A^2 - 2b(z)A_{,z} A_{,p} + (K_{,z} e^{2\kappa p} - e^{2\kappa p})\Sigma^2 + 2Ke^{2\kappa p}\Sigma\Sigma_{,z}\, dp.$$

We now wish to deal with the term containing $\Sigma_{,z}$. To do this, we will need the equation of motion for matter, (4.4). Using this equation, (2.4) and (4.12), it is possible to show that Σ is a separable function of z and r, that is, $\Sigma(z,p) = B(z)C(r)$, where $B(z) = -16\pi e^{-\nu/2}S^2 q(z)$ and $C(r) = (U_{,r} - 2U)/r^3$. We could therefore write $\Sigma_{,z} = (B_{,z} / B(z))\Sigma$. Incorporating this produces

$$\frac{dE_2}{dz} \geq \int_{\mathbb{R}} (\beta,_z - 2a(z) - 1)A,_z^2 - \gamma,_z A,_p^2 + H,_z A^2 \tag{4.20}$$
$$-2b(z)A,_z A,_p + \left(K,_z - 1 + 2K\frac{B,_z}{B(z)} \right) e^{2\kappa p}\Sigma^2 dp.$$

Now set I to equal the integrand on the right hand side of (4.20) and define $I_R = I + \mu I_{E_2}$, where $\mu > 0$ is a positive constant, and I_{E_2} is the integrand such that $E_2(z) = \int_{\mathbb{R}} I_{E_2} dp$. If we can show that $I_R \geq 0$, then this corollary is proven. We have

$$I_R = (\beta,_z - 2a(z) - 1 + \mu\beta)A,_z^2 + (-\gamma,_z - \mu\gamma)A,_p^2 + (H,_z + \mu H)A^2$$
$$+ \left(K,_z - 1 + 2K\frac{B,_z}{B(z)} + \mu K \right) e^{2\kappa p}\Sigma^2 - 2b(z)A,_z A,_p\,.$$

It is possible to pick $K(z)$ so that the Σ^2 coefficient is always positive so we make this choice. Although $H,_z$ is negative, $H(z)$ is positive, and therefore with a choice of large enough μ, the A^2 coefficient will also be positive. This leaves us with

$$\begin{aligned} I_R &\geq (\beta,_z - 2a(z) - 1 + \mu\beta)A,_z^2 + (-\gamma,_z - \mu\gamma)A,_p^2 - 2b(z)A,_z A,_p \\ &:= d(z)A,_z^2 + e(z)A,_p^2 + f(z)A,_z A,_p\,. \end{aligned}$$

We define the quadratic form

$$Q(z,p) := d(z)X^2 + e(z)Y^2 + f(z)XY. \tag{4.21}$$

In order for this form to be positive definite, we will require $d(z) > 0$, $e(z) > 0$ and $D(z) = 4d(z)e(z) - f(z)^2 > 0$. We first investigate the behaviour of $d(z)$, $e(z)$ and $f(z)$ on the Cauchy horizon. Using the fact that the Cauchy horizon occurs at $z_c = -e^{\nu(z_c)/2}$ we can compute $d(z_c)$ (recall $\beta(z_c) = 0$),

$$d(z_c) = -e^{-\nu/2}(-2 + e^{\nu/2}(1 + \dot{\nu}) - 4(2 - \kappa))|_{z=z_c}$$

and so $d(z_c)$ will be positive so long as κ is in the region

$$\kappa < \frac{1}{4}(10 - e^{\nu/2}(1+\dot{\nu})) := \tilde{\kappa}(z).$$

We wish to minimize the function $\tilde{\kappa}(z)$ over the interval $[z_c, z_i]$. To do so, we first find an explicit functional form for this expression, using (2.6). We find

$$\tilde{\kappa}(z) = \frac{1}{4}\left(10 - \frac{9 + 12az + a^2 z(4+3z)}{9(1+az)^{4/3}}\right).$$

We first set $a = a^*$ to minimize this function with respect to a. We can then calculate the derivative of κ^* and find that

$$\frac{d\tilde{\kappa}}{dz} = \frac{-e^{\nu/2}}{4}\left(\frac{2a^2(6 + (3-2a)z + 3az^2)}{9(1+az)^2(3+az)}\right).$$

The term in brackets can easily be shown to be positive. Since the coefficient of this bracket above is negative, it follows that the derivative $\frac{d\tilde{\kappa}}{dz}$ is everywhere negative in the region $(z_c, z_i]$. It follows that the minimal value of $\tilde{\kappa}$ is the value at $z_i = 0$. Inserting this value produces $\kappa^* = \frac{9}{4}$, which was used in the statements of this corollary and Lemma 4.4.2.

Now, $e(z_c)$ is positive, for any choice of positive μ. This follows since by (4.7), $e(z_c) = e^{-\nu(z_c)}(\mu - \dot{\nu}(z_c))$ and using (2.6) we can check that $\dot{\nu}(z)$ is negative at $z = z_c$ for all values of $a \in (0, a^*)$. Finally, we must check that $D(z_c) = 4d(z_c)e(z_c) - b(z_c)^2 > 0$. But $d(z_c) > 0$ (with the above choice for κ) and since $e(z_c)$ can be made arbitrarily large by a choice of large μ, it follows that $D(z_c)$ can always be made positive by a suitable choice of μ. So at the Cauchy horizon, the quadratic form is positive definite. Then for a choice of z^* close enough to z_c, the continuity of the coefficients $d(z)$, $e(z)$ and $f(z)$ ensures that the quadratic form (4.21) is positive definite in the range $(z_c, z^*]$. Therefore, we can conclude that

$$\frac{dE_2}{dz} \geq -\mu E_2(z)$$

for $z \in (z_c, z^*]$. □

Having successfully bound the derivative of $E_2(z)$, we can establish a satisfactory bound on $E_2(z)$ itself, which does not share the defects of (4.14).

Theorem 4.4.4 *Let $A(z,p)$ be a solution to (4.5) which is subject to Theorem 4.3.1 and Lemma 4.4.2. Then the energy $E_2(z)$ of $A(z,p)$ obeys the a priori bound*

$$E_2(z) \leq C_1 E_1[A](0) + C_2 J_\kappa[\Sigma(z_i)],$$

where $J_\kappa[\Sigma(z_i)] = \int_R e^{2\kappa p}\Sigma^2(z_i,p)dp$ and $z \in (z_c, z_i]$.

Proof We can immediately construct a bound on E_2 by considering the results (4.15 - 4.17) of Corollary 4.4.1. Using these results, we can construct the bound

$$E_2(z) \leq h(z)\left(E_1[A](0) + \int_{-\infty}^{\infty} |\vec{j}|^2 dp\right), \tag{4.22}$$

where $h(z) = |C_1\beta(z)e^{C_0\bar{z}}| + |e^{B_0\bar{z}}(H(z) - \gamma(z))|$. The function $h(z)$ clearly diverges on the Cauchy horizon. We now wish to convert the L^2-norm of $\vec{j}$ into an *a priori* bound, that is, a bound which depends on some quantity evaluated on the initial data surface. To do this we note that

$$\int_{-\infty}^{\infty} |\vec{j}|^2 dp = f(z) J_\kappa[\Sigma(z_i)],$$

where $J_\kappa[\Sigma(z_i)] = \int_R e^{2\kappa p}\Sigma^2(z_i,p)dp$ and $f(z) = B^{-2}(z_i)(2B^2(z)\beta^2(z)k^2(z))$ and $k(z) = -\frac{1}{2}(1+z^2e^{-\nu})^{-1/2}e^{\nu/2}$. By inspection, we can see that the function $f(z)$ is finite up to the Cauchy horizon, so we have the bound

$$\int_0^{\infty} |\vec{j}|^2 dp \leq C_0 J_\kappa[\Sigma(z_i)]$$

for some positive and sufficiently large constant C_0 that depends only on the metric functions. Using this in (4.22) produces

$$E_2(z) \leq h(z)\left(E_1[A](0) + C_0 J_\kappa[\Sigma(z_i)]\right).$$

We now integrate the bound on dE_2/dz from Corollary 4.4.3 to find

$$E_2(z) \leq e^{-\mu(z-z^*)} E_2(z^*)$$

in the range $z \in (z_c, z^*]$. Combining these two bounds and noting that $h(z^*)$ is finite results in an *a priori* bound on $E_2(z)$,

$$E_2(z) \leq C_1 E_1[A](0) + C_2 J_\kappa[\Sigma(z_i)],$$

where $C_1 = e^{-\mu(z_c - z^*)} h(z^*)$ and $C_2 = e^{-\mu(z_c - z^*)} h(z^*) C_0$ are finite and $z \in (z_c, z^*]$. $\square$

Having found an *a priori* bound on $E_2(z)$ we can immediately progress to a bound on the function $A(z, p)$. We pause briefly to note that the Sobolev space $\mathbb{H}^{1,2}(\mathbb{R}, \mathbb{R})$ is the set of all functions f with finite $\mathbb{H}^{1,2}$-norm, that is, the set of all functions f such that

$$\int_{\mathbb{R}} |f|^2 + |f,_p|^2 \, dp < \infty.$$

Theorem 4.4.5 *Let $A(z, p)$ be a solution to (4.5) which is subject to Theorem 4.3.1 and Lemma 4.4.2. Then $A(z, p)$ is uniformly bounded on $(z_c, z_i]$. That is, there exists constants $C_1 > 0$, $C_2 > 0$ such that*

$$|A(z, p)| \leq C_1 E_1[A](0) + C_2 J_\kappa[\Sigma(z_i)].$$

Proof From the previous theorem and the fact that (in the definition of $E_2(z)$) the terms $\beta A,_z^2 + K(z) e^{2\kappa p} \Sigma^2$ are positive definite, we can state that

$$\int_{\mathbb{R}} A^2 + A,_p^2 \, dp \leq C_1 E_1(0) + C_2 J_\kappa[\Sigma(z_i)].$$

So we get a bound on the $\mathbb{H}^{1,2}(\mathbb{R}, \mathbb{R})$ norm of $A(z, p)$ directly from Theorem 4.4.4. We now apply Sobolev's inequality,

$$|A| \leq \frac{1}{2} \int_{\mathbb{R}} |A|^2 + |A,_p|^2 dp,$$

to convert this to a bound on $A(z, p)$,

$$|A(z, p)|^2 \leq C_1 E_1(0) + C_2 J_\kappa[\Sigma(z_i)]$$

for all $z \in (z_c, z_i]$. $\square$

Remark 4.4.1 This theorem shows that $A(z,p)$ (and therefore the gauge invariant matter scalar Ψ) is bounded in the approach to the Cauchy horizon. However, this is not itself sufficient to prove that the limit of Ψ (for all $p \in \mathbb{R}$) actually exists in the approach to the Cauchy horizon. The following lemma allows us to control the behaviour of the time derivative of $A(z,p)$ and hence, to prove the existence and finiteness of the limit.

Lemma 4.4.6 *Let $A(z,p)$ be a solution to (4.5) which is subject to Theorem 4.3.1 and Lemma 4.4.2. Then $A_{,z}(z,p)$ is uniformly bounded on $(z_c, z_i]$. That is, there exist constants $\{C_i\},\ i = 0,..,5$ such that*

$$\begin{aligned} |A_{,z}(z,p)| \leq\ & C_0 E_1[A](0) + C_1 E_1[A_{,p}](0) + C_2 E_1[A_{,pp}](0) \\ & + C_3 J_\kappa[\Sigma(z_i)] + C_4 J_\kappa[\Sigma_{,p}(z_i)] + C_5 J_\kappa[\Sigma_{,pp}(z_i)]. \end{aligned} \tag{4.23}$$

Proof We wish to find a bound on the behaviour of $A_{,z}(z,p)$. To achieve this, we first rewrite (4.5) as a first order transport equation for $A_{,z}(z,p)$. If we label $\chi := A_{,z}$ then

$$\beta(z)\chi_{,z} + \xi(z)\chi_{,p} + a\chi = f(z,p), \tag{4.24}$$

where $f(z,p) = e^{\kappa p}\Sigma - c(z)A(z,p) - b(z)A_{,p} - \gamma(z)A_{,pp}$. By inspection, we see that the function $f(z,p)$ is smooth and has compact support on each $z = \textit{constant}$ surface. If we define the differential operator L to be

$$L := \beta(z)\frac{\partial^2}{\partial z^2} + \gamma(z)\frac{\partial^2}{\partial p^2} + \xi(z)\frac{\partial^2}{\partial z \partial p} + a(z)\frac{\partial}{\partial z} + b(z)\frac{\partial}{\partial p} + c(z)$$

then (4.5) would read

$$L[A] = a_0(p)\Sigma(z,p),$$

where $a_0(p) = e^{\kappa p}$. Now since every coefficient in the above differential operator has only z-dependence, we could differentiate (4.5) with respect to

p and write the result as

$$L[A_{,p}] = b_0(p)\Sigma + a_0(p)\Sigma_{,p}\,,$$

where $b_0(p) = da_0/dp = \kappa a_0(p)$. Similarly,

$$L[A_{,pp}] = c_0(p)\Sigma + 2b_0(p)\Sigma_{,p} + a_0(p)\Sigma_{,pp}\,,$$

where $c_0(p) = db_0/dp = \kappa^2 a_0(p)$. So we see that $A_{,p}$ and $A_{,pp}$ satisfy similar differential equations to $A(z, p)$, with different source terms. We can therefore apply Theorem 4.4.5 to $A_{,p}$ and $A_{,pp}$ so long as we modify the bounding terms to take account of the modified source terms,

$$|A_{,p}| \leq C_3 J_\kappa[\Sigma(z_i)] + C_4 J_\kappa[\Sigma_{,p}(z_i)] + C_5 E_1[A_{,p}](0) \tag{4.25}$$

$$\begin{aligned} |A_{,pp}| \leq C_6 J_\kappa[\Sigma(z_i)] + C_7 J_\kappa[\Sigma_{,p}(z_i)] \\ + C_8 J_\kappa[\Sigma_{,pp}(z_i)] + C_9 E_1[A_{,pp}](0) \end{aligned} \tag{4.26}$$

We must now integrate the first order transport equation (4.24) and use the above results to bound $A_{,z}(z, p)$. The charateristics of (4.24) are $dp/dz = \xi(z)/\beta(z)$, which integrates to give

$$p = \alpha + \int_z^{z_i} \frac{\xi(s)}{\beta(s)} ds = \alpha + \omega(z).$$

α labels each characteristic, and at $z = z_i$, it gives the value of p where the characteristic intersects the initial data surface. With this result, the transport equation becomes

$$\beta(z)\frac{d}{dz}\{\chi(z, \alpha + \omega(z))\} + a(z)\chi(z, \alpha + \omega(z)) = f(z, \alpha + \omega(z)), \tag{4.27}$$

where the derivative is taken along characteristics. Now define

$$J(z) := \exp\left[-\int_z^{z_i} \frac{a(s)}{\beta(s)} ds\right].$$

It can easily be verified that a solution of the ordinary differential equation (4.27) (which is (4.24) evaluated along characteristics) can be written as

$$J(z)\chi(z,\alpha+\omega(z)) = \chi(z_i,\alpha) - \int_z^{z_i} \frac{J(s)}{\beta(s)} f(s,\alpha+\omega(s))ds. \tag{4.28}$$

Now recall that $\omega(z) = \int_z^{z_i} \frac{\xi(s)}{\beta(s)}ds$ which tends to infinity as $z \to z_c$. For z close enough to z_c the characteristic at (z,p) will hit $z = z_i$ at some very large negative p value. Therefore, since $A(z,p)$ has compact support, $\chi(z_i,\alpha) = 0$. We now apply the mean value theorem for integrals to find that

$$\int_z^{z_i} \frac{J(s)}{\beta(s)} f(s,\alpha+\omega(s))ds = f(z_*,\alpha+\omega(z_*)) \int_z^{z_i} \frac{J(s)}{\beta(s)}ds,$$

for some z_* in the interval $(z_c, z_i]$. Then from (4.28) we can conclude that

$$\chi(z,\alpha+\omega(z)) = \frac{f(z_*,\alpha+\omega(z_*))}{J(z)} \int_z^{z_i} \frac{J(s)}{\beta(s)}ds.$$

The coefficient of $f(z_*,\alpha+\omega(z_*))$ above is clearly finite away from the Cauchy horizon, and could therefore be bounded by some suitably large constant C_*, so

$$\chi(z,\alpha+\omega(z)) \leq f(z_*,\alpha+\omega(z_*))C_*.$$

Now from (4.24), (4.25) and (4.26), we know that $f(z,p)$ is bounded, so we may finally state the bound on χ,

$$\begin{aligned} |\chi(z,p)| := |A_{,z}(z,p)| \leq\ & C_0E_1[A](0) + C_1E_1[A_{,p}](0) + C_2E_1[A_{,pp}](0) \\ & + C_3J_\kappa[\Sigma(z_i)] + C_4J_\kappa[\Sigma_{,p}(z_i)] + C_5J_\kappa[\Sigma_{,pp}(z_i)] \end{aligned}$$

in the range $z \in (z_c, z_i]$. □

Having bounded the derivative of $A(z,p)$, we are now in a position to bound the perturbation on the Cauchy horizon.

Theorem 4.4.7 *Let $A(z,p)$ be a solution of (4.5) subject to Theorem 4.3.1* *and Lemma 4.4.2. Then* $A_{\mathcal{H}+} := \lim_{z\to z_c} A(z,\cdot) \in C^\infty(\mathbb{R},\mathbb{R})$ *obeys the bound*

$$|A_{\mathcal{H}+}(z,p)| \leq C_1 E_1[A](0) + C_2 J_\kappa[\Sigma(z_i)].$$

Proof We wish to show that $\lim_{z\to z_c} |A(z,p)|$ is bounded. We begin by fixing p and introducing a sequence of z-values that converge to z_c, $\{z^{(n)}\}_{n=0}^\infty \subset (z_c, z_i]$. For all $m, n \geq 1$, we can use the mean value theorem to show that

$$|A(z^{(m)},p) - A(z^{(n)},p)| = |A_{,z}(z_*,p)||z^{(m)} - z^{(n)}|$$

for some $z_* \in (z^{(m)}, z^{(n)})$. Then Lemma 4.4.6 tells us that $A_{,z}$ is bounded, so $|A_{,z}(z_*,p)|$ will be a real number. Then since the sequence $\{z^{(n)}\}_{n=0}^\infty \subset (z_c, z_i]$ tends towards z_c, it follows that for large enough n, m, $|z^{(m)} - z^{(n)}| < \epsilon$ for all $\epsilon > 0$. Therefore $A(z^{(m)},p)$ is a Cauchy sequence of real numbers. Then for each $p \in \mathbb{R}$, $\lim_{z\to z_c} A(z,p)$ exists. Define

$$A_{\mathcal{H}+} := \lim_{z\to z_c} A(z,\cdot).$$

We now wish to take the limit $z \to z_c$ in Theorem 4.4.5, which bounds $|A(z,p)|$. In order to do this, we will need to know that the limits of $A_{,p}$ and $A_{,pp}$ exist. But this follows by a similar argument to the above (recall that we know that all p- derivatives of $A(z,p)$ to arbitrary order can be bounded, using an argument similar to that of Lemma 4.4.6). Finally, we must show that

$$\frac{d}{dp}A_{\mathcal{H}+} = \lim_{z\to z_c} A_{,p}\,.$$

But we know that the sequence $A(z^{(n)},p)$ converges uniformly to $A(z,p)$, so the above result follows. Using these results, we can take the limit $z \to z_c$ in Theorem 4.4.5 to find that

$$|A_{\mathcal{H}+}(z,p)| \leq C_1 E_1[A](0) + C_2 J_\kappa[\Sigma(z_i)].$$

□

We now wish to generalize our choice of initial data. Recall that we chose an initial data surface which intersected the axis at $r = 0$. We therefore had to require that the initial data for the perturbation be supported away from this point, which is an undesirable feature of our analysis so far. We pause briefly to note that the Sobolev spaces $\mathbb{H}^{2,2}(\mathbb{R}, \mathbb{R})$ and $\mathbb{H}^{3,2}(\mathbb{R}, \mathbb{R})$ are the set of all functions f with finite $\mathbb{H}^{2,2}$ and finite $\mathbb{H}^{3,2}$-norms respectively, that is, the set of all functions f such that

$$\int_{\mathbb{R}} |f|^2 + |f_{,p}|^2 + |f_{,pp}|^2 \, dp < \infty,$$

for $f \in \mathbb{H}^{2,2}$ and

$$\int_{\mathbb{R}} |f|^2 + |f_{,p}|^2 + |f_{,pp}|^2 + |f_{,ppp}|^2 \, dp < \infty,$$

for $f \in \mathbb{H}^{3,2}$ respectively.

Theorem 4.4.8 *Let* $\kappa \in [0, \kappa^*)$*.*

(1) Let $f \in \mathbb{H}^{1,2}(\mathbb{R}, \mathbb{R})$*,* $g \in L^2(\mathbb{R}, \mathbb{R})$ *and* $C \in L^2(\mathbb{R}, \mathbb{R})$ *for each fixed* z*. Then there exists a unique solution* $A \in C((z_c, z_i], \mathbb{H}^{1,2}(\mathbb{R}))$ *of the initial value problem consisting of (4.5) with the initial data* $A|_{z_i} = f$*,* $A_{,z}|_{z_i} = g$*. This solution satisfies the a priori bound*

$$|A(z, p)| \leq C_0 E_1[A](0) + C_2 J_\kappa[\Sigma(z_i)]$$

for $z \in (z_c, z_i]$ *and* $p \in \mathbb{R}$*.*

(2) Let $f \in \mathbb{H}^{3,2}(\mathbb{R}, \mathbb{R})$*,* $g \in \mathbb{H}^{2,2}(\mathbb{R}, \mathbb{R})$ *and* $C \in \mathbb{H}^{2,2}(\mathbb{R}, \mathbb{R})$ *for each fixed* z*. Then there exists a unique solution* $A \in C([z_c, z_i], \mathbb{H}^{1,2}(\mathbb{R}))$ *of the initial value problem consisting of (4.5) with the initial data* $A|_{z_i} = f$*,* $A_{,z}|_{z_i} = g$*. This solution satisfies the a priori bound*

$$|A(z, p)| \leq C_0 E_1[A](0) + C_2 J_\kappa[\Sigma(z_i)]$$

for $z \in (z_c, z_i]$ *and* $p \in \mathbb{R}$*, and its time derivative satisfies*

$$|A_{,z}(z,p)| \leq C_0 E_1[A](0) + C_1 E_1[A_{,p}](0) + C_2 E_1[A_{,pp}](0) + C_3 J_\kappa[\Sigma(z_i)] + C_4 J_\kappa[\Sigma_{,p}(z_i)] + C_5 J_\kappa[\Sigma_{,pp}(z_i)]$$

for $z \in (z_c, z_i]$ *and* $p \in \mathbb{R}$*.*

Proof The proof of this theorem is standard, and uses the density of the space $C_0^\infty(\mathbb{R},\mathbb{R})$ in the Banach spaces $\mathbb{H}^{1,2}(\mathbb{R},\mathbb{R})$, $\mathbb{H}^{2,2}(\mathbb{R},\mathbb{R})$, $\mathbb{H}^{3,2}(\mathbb{R},\mathbb{R})$ and $L^2(\mathbb{R},\mathbb{R})$. The proof is essentially identical to that of Theorems 5 and 7 in [41].

Remark 4.4.2 The choice of which Sobolev space to take our initial data functions from in the above proofs is dictated by the nature of the bounds required. For example, to use a bound involving $E_1[A_{,pp}]$, we will require the function f to be in $\mathbb{H}^{3,2}(\mathbb{R})$ so that it and its p-derivatives up to third order are in $L^2(\mathbb{R})$. This is required for the integral involved in $E_1[A_{,pp}]$ to be well defined. All other choices of Sobolev spaces used above can be understood in a similar fashion.

Remark 4.4.3 This theorem successfully generalizes the choice of initial data function for (4.5). This generalisation involves choosing initial data which need not vanish at the scaling centre of the spacetime, which is crucial as it allows for a perturbation which need not vanish at the past endpoint of the naked singularity. Similar finiteness results go through for this general choice of initial data.

4.5 Physical Interpretation of Results

In order to physically interpret the results obtained thus far, we turn to the perturbed Weyl scalars. These scalars are related to the gauge invariant scalar Ψ and can be interpreted in terms of in- and outgoing gravitational radiation. In the case of odd parity perturbations, they are both tetrad and identification gauge invariant. This means that if we make a change of null tetrad, or a change of our background coordinate system, we will find that these terms are invariant under such changes.

Following [39] and [53], we note that $\delta\Psi_0$ and $\delta\Psi_4$ represent transverse gravitational waves propagating radially inwards and outwards, and $\delta\Psi_2$ represents the perturbation of the Coulomb part of the gravitational field[1].

The perturbed Weyl scalars are given by

$$\begin{aligned}
\delta\Psi_0 =& \frac{Q_0}{2R^2}\bar{l}^A\bar{l}^B k_{A|B},\\
\delta\Psi_1 =& \frac{Q_1}{R}\left((R^2\Psi)_{|A}\bar{l}^A - \frac{4}{R^2}k_A\bar{l}^A\right),\\
\delta\Psi_2 =& Q_2\Psi,\\
\delta\Psi_3 =& \frac{Q_1^*}{R}\left((R^2\Psi)_{|A}\bar{n}^A - \frac{4}{R^2}k_A\bar{n}^A\right),\\
\delta\Psi_4 =& \frac{Q_0^*}{2R^2}\bar{n}^A\bar{n}^B k_{A|B},
\end{aligned}$$

where Ψ is the gauge invariant scalar appearing in (4.5), k_A is the gauge invariant vector describing the metric perturbation, (3.12), and $\bar{l}^A$ and $\bar{n}^A$ are the in- and outgoing null vectors given in (2.11). Q_0, Q_1 and Q_2 are angular coefficients depending on the other vectors in the null tetrad, and on the basis constructed from the spherical harmonics. We have made a gauge choice such that the perturbation of the real members of the null tetrad vanishes, that is, $\delta l_\mu = \delta n_\mu = 0$. See [39] for further details.

We note that the quantities δP_{-1}, δP_0 and δP_{+1}, which are defined as follows,

$$\delta P_{-1} = |\delta\Psi_0\delta\Psi_4|^{1/2}, \tag{4.29}$$

$$\delta P_0 = \delta\Psi_2, \tag{4.30}$$

$$\delta P_{+1} = |\delta\Psi_1\delta\Psi_3|^{1/2}, \tag{4.31}$$

are fully gauge invariant (in that they are invariant under a change in the background null tetrad, as well as being invariant under transformations in

[1]We note that [53] refers to $\delta\Psi_1$ and $\delta\Psi_3$ as "longitudinal gravitational waves" propagating radially inwards and outwards.

the perturbed null tetrad and identification gauge transformations) and have physically meaningful magnitudes.

Although we could write these scalars in terms of the coordinates (z, p) used in the previous section, it is advantageous to use null coordinates (u, v) instead, as this simplifies matters considerably. We will therefore consider the master equation in null coordinates, and establish a series of results indicating the boundedness of various of the derivatives of $A(u, v)$ in null coordinates. These results will allow us to show that the perturbed Weyl scalars are bounded as the Cauchy horizon is approached.

4.5.1 Master Equation in Null Coordinates

We first rewrite the master equation (4.5) in terms of the in and out-going null coordinates (2.10). The master equation takes the form

$$\begin{aligned}\alpha_1(u,v)\, A_{,uv} + \alpha_2(u,v)u\, A_{,u} + \alpha_3(u,v)v\, A_{,v} \\ + \alpha_4(u,v)A = e^{\kappa p}\Sigma(u,v),\end{aligned} \tag{4.32}$$

where in terms of the coefficients (4.6 -4.11), the above coefficients are given by

$$\alpha_1(u,v) = 2z\frac{\beta(z)+\xi(z)}{f_+(z)f_-(z)} + 2\gamma(z), \qquad \alpha_2(u,v) = \frac{a(z)}{f_+(z)} + b(z), \tag{4.33}$$
$$\alpha_3(u,v) = \frac{a(z)}{f_-(z)} + b(z), \qquad \alpha_4(u,v) = c(z),$$

where $f_\pm(z)$ are factors coming from (2.10). We can formally solve (4.32) by integrating across the characteristic diamond $\Omega = \{(\bar{u},\bar{v}) : u_0 < \bar{u} \le u, v_o \le \bar{v} \le v\}$ (see Figure 4.2). We find

$$A(u,v) = A(u_0,v) + A(u,v_0) + A(u_0,v_0) + \int_{u_0}^{u}\int_{v_0}^{v} F(\bar{u},\bar{v})d\bar{v}d\bar{u},$$

where $F(u,v) = (\alpha_1)^{-1}\left(-\alpha_2(u,v)u\, A_{,u} - \alpha_3(u,v)v\, A_{,v} - \alpha_4(u,v)A + e^{\kappa p}\Sigma(u,v)\right)$.

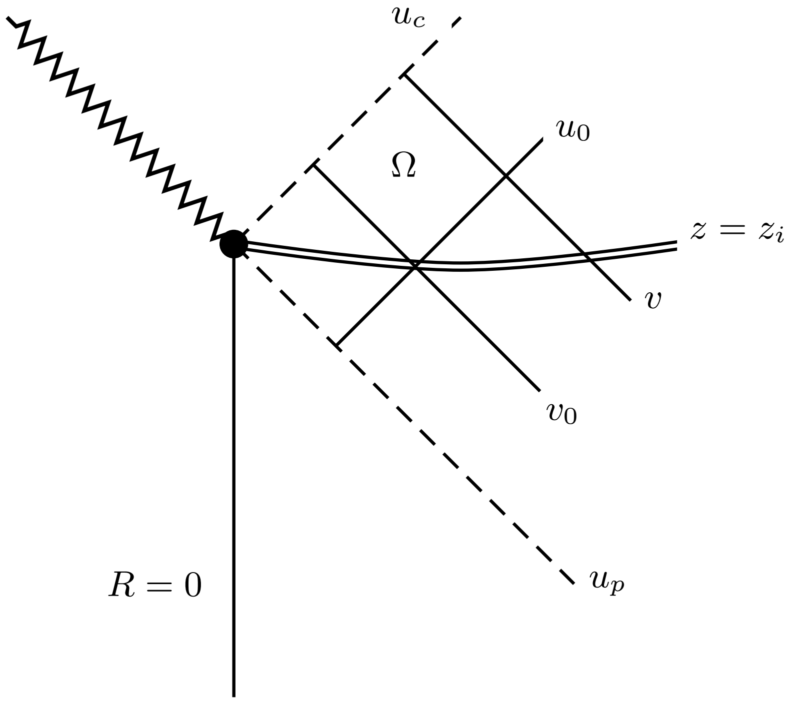

Figure 4.2: The Characteristic Diamond. We integrate over the characteristic diamond labelled Ω, where u and v are the retarded and advanced null coordinates. u_c labels the Cauchy horizon, u_p labels the past null cone of the naked singularity and z_i is the initial data surface.

Now in Section 4.5.2, in order to control the perturbed Weyl scalars, we will need to know that A, $A_{,u}$, $A_{,v}$, $A_{,uu}$ and $A_{,vv}$ are bounded in the approach to the Cauchy horizon.

Lemma 4.5.1 *With a choice of initial data $A(z_i, p) = f(p)$, $A_{,z}(z_i, p) = j(p)$ and $\Sigma(z_i, p) = h(p)$, with $f(p)$, $j(p)$ and $h(p) \in C_0^\infty(\mathbb{R}, \mathbb{R})$, the first order derivatives of $A(u, v)$ with respect to u and v are bounded by a priori terms in the approach to the Cauchy horizon.*

See Appendix A.2 for the proof of this lemma. We can use this result, together with results from Section 4.4.2, to establish that the second order derivatives of A with respect to u and v are also bounded.

Lemma 4.5.2 *With a choice of initial data $A(z_i, p) = f(p)$, $A_{,z}(z_i, p) = j(p)$ and $\Sigma(z_i, p) = h(p)$, with $f(p)$, $j(p)$ and $h(p) \in C_0^\infty(\mathbb{R}, \mathbb{R})$, the second order derivatives $A_{,vv}$, $A_{,uu}$ and $A_{,uv}$ of A are bounded by a priori terms in the approach to the Cauchy horizon.*

See Appendix A.2 for the proof of this lemma. The results so far establish the boundedness of all first and second order derivatives of A with respect to u and v, with a choice of initial data from the space $C_0^\infty(\mathbb{R}, \mathbb{R})$. As discussed in Section 4.4, this choice of initial data does not interact with the past endpoint of the naked singularity. As in Theorem 4.4.8, we can extend this choice of initial data so that the perturbation need not vanish at the Cauchy horizon.

Lemma 4.5.3 *With a choice of initial data $A(z_i, p) = f(p)$, $A_{,z}(z_i, p) = j(p)$ and $\Sigma(z_i, p) = h(p)$, with $f(p) \in \mathbb{H}^{3,2}(\mathbb{R}, \mathbb{R})$, $j(p) \in \mathbb{H}^{1,2}(\mathbb{R}, \mathbb{R})$ and $h(p) \in \mathbb{H}^{3,2}(\mathbb{R}, \mathbb{R})$, the first and second order derivatives $A_{,u}$, $A_{,v}$, $A_{,vv}$, $A_{,uu}$ and $A_{,uv}$ of A are bounded by a priori terms in the approach to the Cauchy horizon.*

See Appendix A.2 for the proof of this lemma. Having bounded the first and second order derivatives of A with a satisfactory choice of initial data, we are now in a position to consider the perturbed Weyl scalars.

4.5.2 Gauge Invariant Curvature Scalars

The in- and outgoing background null vectors $\bar{l}^\mu$ and $\bar{n}^\mu$ are given in (2.11). We note that a factor of $B^{-1}(u,v)$ appears in the definition of $\bar{l}^\mu$, and that this factor involves a power of r^{-2}.

In (u,v) coordinates, the perturbed Weyl scalars take the form

$$\delta\Psi_0 = \frac{Q_0 B^{-2}}{2\mathcal{L}S^2}\left(16\pi((S^2L_0)_{,u} - \gamma_0 S^2 L_0)\right) + \frac{Q_0 B^{-2}}{2\mathcal{L}S^2}\left(\frac{r^2}{B}\left((S^4\Psi)_{,uu} - \gamma_0 (S^4\Psi)_{,u}\right)\right), \tag{4.34}$$

$$\delta\Psi_1 = \frac{Q_1}{SB}\left(r(S^2\Psi)_{,u} - \frac{4}{\mathcal{L}Br}\left(16\pi S^2 L_0 - \frac{r^2(S^4\Psi)_{,u}}{B}\right)\right), \tag{4.35}$$

$$\delta\Psi_2 = Q_2\Psi, \tag{4.36}$$

$$\delta\Psi_3 = \frac{Q_1^*}{S}\left(r(S^2\Psi)_{,v} - \frac{4}{\mathcal{L}r}\left(16\pi S^2 L_1 - \frac{r^2(S^4\Psi)_{,v}}{B}\right)\right), \tag{4.37}$$

$$\delta\Psi_4 = \frac{Q_0^*}{2\mathcal{L}S^2}\left(16\pi((S^2L_0)_{,v} - \gamma_1 S^2 L_1)\right) + \frac{Q_0^*}{2\mathcal{L}S^2}\left(\frac{r^2}{B}\left((S^4\Psi)_{,vv} - \gamma_1 (S^4\Psi)_{,v}\right)\right), \tag{4.38}$$

where we used (3.25) to write $\delta\Psi_0$ and $\delta\Psi_4$ in terms of Ψ. Here, $\gamma_0(u,v)$ and $\gamma_1(u,v)$ are Christoffel symbols, $\mathcal{L} = (l-1)(l+2)$ and $L_A = (L_0, L_1)$ is the gauge invariant matter vector (3.13).

Theorem 4.5.4 *With a choice of initial data $\Psi(z_i,p) = f(p)$, $\Psi_{,z}(z_i,p) = j(p)$ and $\Sigma(z_i,p) = h(p)$, with $f(p) \in \mathbb{H}^{3,2}(\mathbb{R},\mathbb{R})$, $j(p) \in \mathbb{H}^{1,2}(\mathbb{R},\mathbb{R})$ and $h(p) \in \mathbb{H}^{3,2}(\mathbb{R},\mathbb{R})$, the perturbed Weyl scalars, as well as δP_{-1}, δP_0 and δP_{+1}, remain finite on the Cauchy horizon, barring a possible divergence at the past endpoint of the naked singularity, where $r = 0$. They are bounded by a priori terms arising from the bounds on A, $A_{,z}$, $A_{,p}$, $A_{,pp}$, $A_{,zp}$ and Σ.*

Proof If we consider (4.34 - 4.38), we see that the perturbed Weyl scalars depend on the gauge invariant scalar Ψ, its first derivatives $\Psi_{,u}$ and $\Psi_{,v}$, its second derivatives $\Psi_{,uu}$ and $\Psi_{,vv}$, and on the gauge invariant vector L_A. By letting $\kappa = 0$ in Lemma 4.5.3 we can immediately state that Ψ, $\Psi_{,u}$, $\Psi_{,v}$, $\Psi_{,uu}$ and Ψ_{vv} remain finite up to and on the Cauchy horizon. They are bounded by *a priori* terms arising from the bounds on A, $A_{,z}$, $A_{,p}$, $A_{,pp}$, $A_{,zp}$ and Σ.

Thus, the perturbed Weyl scalars remain finite on the Cauchy horizon, and are bounded by the same *a priori* terms, except for a possible divergence at $r = 0$. The terms involving L_A depend on the function $U(r)$, and these may also diverge at $r = 0$, depending on the details of $U(r)$.

From (4.29 - 4.31), δP_{-1}, δP_0 and δP_{+1} are given by products of the perturbed Weyl scalars, and therefore are bounded in the same way, with a similar proviso about a possible divergence at $r = 0$. □

This theorem establishes that the perturbed Weyl scalars remain finite in the approach to the Cauchy horizon, and we can conclude that the various gravitational waves and the perturbation of the Coulomb potential represented by these scalars also remain finite up to and on the Cauchy horizon.

Having studied the behaviour of the perturbed Weyl scalars, it is reasonable to ask whether there are any scalars arising from the perturbed Ricci tensor which we should also consider. We are not aware of any gauge invariant scalars which can be constructed from the perturbed Ricci tensor, but we expect that any such scalars would be related via the Einstein equations to gauge invariant matter scalars. In Section 4.1, we showed that the matter perturbation depends only on an initial data function, and therefore, we expect any such scalars to be trivial in this sense.

4.6 The $l = 1$ Perturbation

We now consider separately the behaviour of the $l = 1$ perturbation. When $l = 1$, k_A is no longer gauge invariant. Instead, we find that under a change of coordinates $\vec{x} \to \vec{x}' = \vec{x} + \vec{\xi}$, where $\vec{\xi} = \xi S_a dx^a$,

$$k_A \to k_A - R^2(R^{-2}\xi)_{,A}\,.$$

Additionally, (3.19) no longer holds. However, Ψ is still gauge invariant, and obeys (3.25). When $l = 1$, $\mathcal{L} = 0$, so that (3.25) reduces to

$$16\pi R^2 L_A - \epsilon_{AB}(R^4\Psi)^{|B} = 0. \tag{4.39}$$

Now, the stress-energy conservation equation, (3.22) reduces to $(R^2 L_A)^{|A} = 0$ when $l = 1$. This indicates that there exists a potential for L_A, which we write as

$$R^2 L_A = \epsilon_A^{\ B}\lambda_{,B}\,. \tag{4.40}$$

As before, in (t,r) coordinates $L^A = (\bar{\rho}(t,r)U(r), 0)$, so (4.40) implies that $\partial\lambda(t,r)/\partial r = R^2\bar{\rho}(t,r)U(r)$. Combining (4.39) and (4.40) produces

$$\epsilon_A^{\ B}(16\pi\lambda - R^4\Psi)_{,B} = 0,$$

which implies that

$$R^4\Psi(t,r) = 16\pi\lambda(t,r) + c,$$

where $c \in \mathbb{R}$ is a constant. This result indicates that Ψ remains finite up to and on the Cauchy horizon, barring a possible divergence at $r = 0$; whether or not this divergence occurs depends on the choice of the initial velocity perturbation $U(r)$.

4.7 Discussion

We have found that the scalar Ψ remains finite as it impinges on the Cauchy horizon of the naked singularity. This scalar describes the odd parity metric perturbation, the matter perturbation being trivial. Finiteness refers to certain natural integral energy measures (as well as pointwise values thereof) which arise in this spacetime, whose value on an initial surface bounds the growth of this scalar.

For the analysis of the Cauchy horizon behaviour, we used a foliation of this spacetime which consists of hypersurfaces that are generated by the homothetic Killing vector field, that is, hypersurfaces of constant z. This is a natural choice to make, as it exploits the self-similarity of the background spacetime. If we use this foliation, we find that the coefficients of the master equation are independent of the radial coordinate. This foliation also dictates our choice of initial data surface for the Cauchy problem.

A disadvantage of this choice of foliation is that these hypersurfaces intersect the singular scaling origin of the spacetime, rather than meeting the regular centre $R = 0$. This forced us to begin our analysis by considering initial data taken from the space $C_0^\infty(\mathbb{R}, \mathbb{R})$ which were compactly supported away from the singular point. We then established Theorems 4.3.1-4.4.7 using this data. We finally extended these results to a more general choice of initial data, taken from various Sobolev spaces, which were capable of having non-zero values at the singular origin. This extension is crucial, as it shows that a perturbation which interacts with the naked singularity still remains bounded at the Cauchy horizon.

Using the perturbed Weyl scalars, one can give a physical interpretation of these results; the gauge invariant scalar Ψ enters into the definition of the perturbed Weyl scalars, which in turn represent ingoing and outgoing gravitational radiation and the perturbation of the Coulomb part of the gravitational field. Now since Ψ remains finite up to and on the Cauchy horizon, this indicates that this radiation will also remain finite on the Cauchy horizon (with the exception of a possible divergence at the past endpoint of the naked singularity).

One deficiency of this work is the choice of initial data surface. The surface $z_i = 0$ intersects the past end point of the naked singularity; it would be preferable to have a surface $t = t_1$ which intersects the regular centre of the spacetime prior to the formation of the naked singularity. Given the results already shown, the challenge here would be to show that regular initial data on such a surface evolves to regular data on the surface $z = z_i$. The results already proven then show that this data remains finite on the Cauchy horizon.

Chapter 5

The Even Parity Perturbations I: The Averaged Perturbation

In this chapter, we begin our study of the even parity perturbations of the self-similar Lemaître-Tolman-Bondi spacetime. In Section 5.1 we determine the form of the even parity matter perturbations, before reducing the linearised Einstein equations for the even parity perturbations into a useful form in Section 5.2. We also state a theorem providing for the existence of unique solutions to these equations, subject to a choice of initial data in either C_0^∞ or L^1. In Section 5.3, we introduce a kind of average of the perturbation. This average is shown to diverge generically on the Cauchy horizon in Section 5.3.1 (where the term generic refers to the open and dense subset of initial data in L^1 which lead to solutions with this behaviour). We use this in Section 5.3.2 to show that the L^p-norm of the perturbation, for $1 \leq p \leq \infty$, diverges generically on the Cauchy horizon.

5.1 The Matter Perturbation

We begin by finding a relation between the gauge invariant matter perturbation terms (3.16 - 3.18) and the dust density and velocity discussed in Section 2.1. This amounts to specifying the matter content of the perturbed spacetime. To do this, we write the stress-energy of the full spacetime as a sum of the background stress-energy and the perturbation stress-energy (where a bar indicates a background quantity),

$$T_{\mu\nu} = \overline{T}_{\mu\nu} + \delta T_{\mu\nu}.$$

We assume that the perturbed spacetime also contains dust and write the density as $\rho = \overline{\rho} + \delta\rho$ and the fluid velocity as $u_\mu = \overline{u}_\mu + \delta u_\mu$. We can now find an expression for the perturbation stress-energy (keeping only first order terms),

$$\delta T_{\mu\nu} = \overline{\rho}(\overline{u}_\mu \delta u_\nu + \overline{u}_\nu \delta u_\mu) + \delta\rho \overline{u}_\mu \overline{u}_\nu. \tag{5.1}$$

The perturbation of the dust velocity can now be expanded in terms of spherical harmonics as $\delta u_\mu = (\delta u_A, \delta u_a) = (\delta u_A Y, \delta u_E Y_{:a})$. By imposing conservation of stress-energy and requiring that the perturbed velocity $u_\mu = \overline{u}_\mu + \delta u_\mu$ obeys $u^\mu u_\mu = -1$, one can show that the perturbed velocity can be written in the form

$$\delta u_\mu = (\partial_A \Gamma(t,r) Y, \gamma(t,r) Y_{:a}), \tag{5.2}$$

where the variable Γ acts as a velocity potential and obeys an equation of motion arising from perturbed stress-energy conservation (specifically, from the acceleration equations arising from stress-energy conservation),

$$\frac{\partial \Gamma}{\partial z} = -\frac{1}{2}\alpha(z,r), \tag{5.3}$$

where we have labelled the first component of k_{AB} (see (3.15)) as $k_{00} = \alpha(z,p)$. In addition, by using (3.31) one can show that

$$\gamma(z,p) = \Gamma(z,p) + g(p), \tag{5.4}$$

where $p = \ln r$ and $g(p)$ is an initial data function for the velocity perturbation. If we compare this form of the perturbed stress-energy to the Gerlach-Sengupta form, we can find the gauge invariant matter perturbations for the LTB spacetime, which we write in terms of (z,p) coordinates. The gauge invariant tensor T_{AB} is given in (z, p) coordinates by

$$
\begin{aligned}
T_{00} &= 2\rho e^{p}\frac{\partial \Gamma}{\partial z} + e^{2p}\delta\rho, \\
T_{01} = T_{10} &= e^{p}\rho\left(\frac{\partial \Gamma}{\partial p} + z\frac{\partial \Gamma}{\partial z}\right) + e^{2p}z\delta\rho, \\
T_{11} &= 2\rho z e^{p}\frac{\partial \Gamma}{\partial p} + e^{2p}z^{2}\delta\rho.
\end{aligned}
$$

The vector T_A is given by $T_A = e^{p}\rho\gamma(z, p)(1, z)$ and the gauge invariant scalars both vanish, $T^2 = 0$, $T^3 = 0$.

5.2 Reduction of the Perturbation Equations and the Main Existence Theorem

In what follows, we omit the exact form of various matrices and vectors if they appear in versions of the system of perturbation equations which we do not use; relevant terms are included in Appendix B.1 as indicated. As we have imposed self-similarity on this spacetime, a natural set of coordinates to present the linearised Einstein equations in is $x^\mu = (z, p, \theta, \phi)$, where $z = -t/r$ is the similarity coordinate and $p = \ln(r)$. The metric in these coordinates takes the form

$$
ds^2 = e^{2p}(-dz^2 + ((S - z\dot{S})^2 - z^2)dp^2 - 2z dz dp + S^2 d\Omega^2).
$$

Our initial full set of perturbation equations consists of both components of (3.28), the $t - p$ and $p - p$ components of (3.26) and the equation of motion (5.3) for $\Gamma(z, p)$. We now discuss the series of simplifications which allows us to make this choice, before stating the initial six dimensional system of mixed evolution and constraint equations.

(1) We note that since $T^2 = 0$ in this spacetime, by (3.29) the metric perturbation tensor k_{AB} is trace-free. We use this to eliminate one component of k_{AB}.

(2) Since T^2 vanishes, (3.29) implies that the metric perturbation k_{AB} is trace-free. Additionally, given that the scalars Q and and T^3 both vanish in the LTB case, one can show that (3.27) is identically satisfied, assuming that the background Einstein equations, (3.28) and (3.31) all hold. We will therefore use (3.31) in preference to (3.27).

(3) We note that the $t-t$ component of (3.26) gives us a relation for the perturbation of the dust velocity $\delta\rho$ in terms of the velocity and metric perturbation. We use this equation to eliminate $\delta\rho$ from the system.

(4) The variable k is the only variable which appears at second order in derivatives in the resulting system. We therefore introduce a first order reduction by letting $u(z,p) = k(z,p)$, $v(z,p) = \partial k/\partial p$ and $w(z,p) = \dot{k}$, where $\cdot = \partial/\partial z$.

(5) The resulting system of equations consists of both components of (3.28), the $t-p$ and $p-p$ components of (3.26) and the equation of motion (5.3) for $\Gamma(z,p)$. To this we append the auxiliary equations $\dot{k} = w$ and $\dot{v} = \partial w/\partial p$, which makes a total of seven equations.

(6) This results in a first order system of seven equations for six variables, which we combine into a vector

$$\vec{X} = (\alpha(z,p), \beta(z,p), u(z,p), v(z,p), w(z,p), \Gamma(z,p))^T \in \mathbb{R}^6.$$

Here $\alpha(z,p)$ and $\beta(z,p)$ are the independent components of the tensor k_{AB} in (z,p) coordinates

$$k_{AB} = \begin{pmatrix} \alpha(z,p) & \beta(z,p) \\ \beta(z,p) & \delta(z,p) \end{pmatrix},$$

and since $T^2 = 0$, (3.29) implies that $\delta = 2z\beta(z,p) + (S(S - 2z\dot{S}) + z^2(-1 + \dot{S}^2))\alpha(z,p)$. Finally $\Gamma(z,p)$ is the velocity potential given in (5.2).

(7) We can write the system of equations in a more compact form as

$$M(z)\frac{\partial \vec{X}}{\partial z} + N(z)\frac{\partial \vec{X}}{\partial p} + O(z)\vec{X} = \vec{S}_7(z,p), \tag{5.5}$$

for 7×6 matrices $M(z)$, $N(z)$ and $O(z)$ and a 7-dimensional source vector $\vec{S}_7$. The dimensions of this system suggest that it may be possible to rewrite it as a 6-dimensional system of evolution equations with a constraint. To identify the constraint, we look for linear combinations of the rows of $M(z)$ which add to give zero. This corresponds to a linear combination of the equations in (5.5) which has no time derivatives, that is, a constraint. We call this the Einstein constraint.

(8) Having identified this constraint, we construct a new system consisting of six of the equations of the original system, with the constraint added to them. This is fully equivalent to the original seven equation system. We can write this system in a similar manner to (5.5),

$$P(z)\frac{\partial \vec{X}}{\partial z} + Q(z)\frac{\partial \vec{X}}{\partial p} + R(z)\vec{X} = \vec{\Sigma}_6(z,p), \tag{5.6}$$

for 6×6 matrices $P(z)$, $Q(z)$ and $R(z)$ and a source vector $\vec{\Sigma}_6$. We note that in addition to the Einstein constraint, we have the trivial constraint $\partial k/\partial p = v(z,p)$.

(9) Having identified the two constraints, we would like to use them to eliminate two variables from the system and thus reduce the number of variables from six to four. To do this, we need to diagonalise the system. We multiply through (5.6) by P^{-1} and find the Jordan canonical form $\tilde{T}$ of the matrix coefficient of $\partial\vec{X}/\partial p$, $T := P^{-1}Q$. We also identify the similarity matrix S such that $\tilde{T} = S^{-1}TS$.

(10) To diagonalise the system, we let $\vec{Y} = S\vec{X}$, where $\det(S) \neq 0$. $\vec{Y}$ obeys the equation

$$\frac{\partial \vec{Y}}{\partial z} + \tilde{T}(z)\frac{\partial \vec{Y}}{\partial p} + \tilde{R}(z)\vec{Y} = \vec{\sigma}_6(z,p), \tag{5.7}$$

where $\tilde{T} = S^{-1}TS$, $\tilde{R} = \dot{S} + S^{-1}P^{-1}RS$ and $\vec{\sigma}_6 = S^{-1}P^{-1}\vec{\Sigma}_6$. The matrix $\tilde{T}$ is in Jordan form but is not diagonal.

(11) In terms of the components of $\vec{X}$, the components of $\vec{Y} \in \mathbb{R}^6$ are

$$\begin{aligned}
Y_1(z,p) &= -\alpha(z,p) - 8\pi z q(z)\Gamma(z,p) + v(z,p) \\
&\qquad - zw(z,p) + \frac{\dot{S}(z)}{S(z)}\beta(z,p), \\
Y_2(z,p) &= 8\pi q(z)\Gamma(z,p), \\
Y_3(z,p) &= u(z,p), \\
Y_4(z,p) &= \frac{-\beta(z,p) + S(8\pi q(z)\Gamma(z,p) + w(z,p))(S - z\dot{S})}{S(S - z\dot{S})} \\
&\qquad + \frac{(z + \dot{S}(S - z\dot{S}))\alpha(z,p)}{S(S - z\dot{S})}, \\
Y_5(z,p) &= \frac{(1 + \dot{S})(\beta(z,p) + (-z - S + z\dot{S})\alpha(z,p))}{2S(S - z\dot{S})}, \\
Y_6(z,p) &= \frac{(-1 + \dot{S})(-\beta(z,p) + (z - S + z\dot{S})\alpha(z,p))}{2S(S - z\dot{S})},
\end{aligned}$$

where $S(z)$ is the radial function and $q(z)$ is the density given by (2.7).

(12) The Einstein constraint can be given in terms of $\vec{Y}$ as

$$\begin{aligned}
&c_1(z)Y_1(z,p) + c_2(z)Y_2(z,p) + c_3(z)Y_3(z,p) + c_4(z)Y_4(z,p) \\
&\qquad + c_5(z)Y_5(z,p) + c_6(z)Y_6(z,p) + c_7(z)\frac{\partial Y_4}{\partial p}(z,p) + c_8(z)g(p) = 0.
\end{aligned} \tag{5.8}$$

where the coefficients are not important. The $g(p)$ term comes from the source term $\vec{\Sigma}_6$ in (5.6), which could be written as $\vec{\Sigma}_6 = \vec{b}(z)g(p)$ where the form of $\vec{b}(z)$ is not needed.

(13) The trivial constraint $\partial k/\partial p = v(z,p)$ can be stated in terms of $\vec{Y}$ as

$$Y_1(z,p) + zY_4(z,p) + \left(z - S + z\dot{S}\right) Y_5(z,p) + \left(z + S - z\dot{S}\right) Y_6(z,p) - \frac{\partial Y_3}{\partial p}(z,p) = 0. \tag{5.9}$$

(14) **Lemma 5.2.1** *Suppose that (5.8) is satisfied on an initial surface $z = z_0$. Then assuming that (5.7) holds, (5.8) will be satisfied on all surfaces $z \in (z_c, z_0]$.*

Proof This is a straightforward but lengthy calculation which was carried out using computer algebra. □

This lemma indicates that the constraint (5.8) is propagated by the system.

(15) The trivial constraint (5.9) is also propagated in the sense of Lemma 5.2.1. The existence of these two constraints suggests that the true number of free variables in this system is four. We therefore aim to reduce this system to a free evolution system of four variables by using the two constraints to eliminate two variables. This is carried out in such a way as to keep the system always first order in all variables.

(16) We carry out this reduction in two steps, by first passing to a five dimensional system (which is still a mixed evolution-constraint system) by solving the trivial constraint for one variable, and then passing to a four dimensional system by solving the non-trivial constraint for another variable. The advantage to carrying out the reduction in this manner is that the five dimensional system is symmetric hyperbolic, which will be useful in what follows.

(17) The five dimensional system is obtained by solving the trivial constraint (5.9) for the variable $Y_1(z,p)$. We eliminate this variable and reduce the system to five variables, with a state vector $\vec{w} \in \mathbb{R}^5$. Again we would like to put the system in a diagonalised form. We do this as before by

calculating the Jordan canonical form of the matrix coefficient $H(z)$ of $\partial\vec{w}/\partial p$ and letting $\vec{u} = S\vec{w} \in \mathbb{R}^5$, where S is the similarity matrix arising from the transformation of $H(z)$ into Jordan canonical form. In terms of $\vec{Y}$, the new variables are given by

$$
\begin{aligned}
u_1(z,p) &= f_1(z)Y_3(z,p), \qquad & u_2(z,p) = Y_2(z,p),\\
u_3(z,p) &= Y_4(z,p) + f_2(z)Y_3(z,p),\\
u_4(z,p) &= Y_5(z,p) + f_3(z)Y_3(z,p),\\
u_5(z,p) &= Y_6(z,p) + f_4(z)Y_3(z,p),
\end{aligned}
$$

where

$$
f_1(z) = \frac{1-\dot{S}(z)}{2S(z)}, \qquad f_2(z) = \frac{-z\dot{S}(z)^2 + S(z)(\dot{S}(z) + z\ddot{S}(z))}{S(z)(S(z) - z\dot{S}(z))},
$$
$$
f_3(z) = \frac{1+\dot{S}(z)}{2S(z)}, \qquad f_4(z) = \frac{-1+\dot{S}(z)}{2S(z)}.
$$

(18) The five dimensional system obeys an equation of motion of the form

$$
\frac{\partial \vec{u}}{\partial z} + \tilde{A}(z)\frac{\partial \vec{u}}{\partial p} + \tilde{C}(z)\vec{u} = \vec{\Sigma}_5(z,p). \tag{5.10}
$$

We shall use this form of the system in this and the next chapter chiefly because it has the useful property that it is symmetric hyperbolic. This means that the matrix $\tilde{A}$,

$$
\tilde{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{z} & 0 & 0 \\ 0 & 0 & 0 & -(z-S+z\dot{S})^{-1} & 0 \\ 0 & 0 & 0 & 0 & -(z+S-z\dot{S})^{-1} \end{pmatrix},
$$

is symmetric. The matrix $\tilde{C}$ and the source term $\vec{\Sigma}_5$ are given in Appendix B.1.

(19) In terms of these new variables, the Einstein constraint becomes

$$g_1(z)u_1(z,p) + g_2(z)u_2(z,p) + g_3(z)u_3(z,p) + g_4(z)u_4(z,p) \\ + g_5(z)u_5(z,p) + g_6(z)\frac{\partial u_3}{\partial p}(z,p) + g_7(z)g(p) = 0, \tag{5.11}$$

where the coefficients $g_i(z)$, $i = 1, \ldots, 7$ are listed in Appendix B.1, and $g(p)$ is an initial data function. We note that $g_5(z)$ vanishes on the Cauchy horizon.

(20) In order to eliminate one more variable, we solve (5.11) for $u_2(z,p)$. As before, we then put the new system in Jordan canonical form by writing it in terms of the vector $\vec{k} \in \mathbb{R}^4$, where in terms of $\vec{u}(z,p)$, the new variables are

$$k_1(z,p) = u_1(z,p), \qquad k_2(z,p) = \frac{u_3(z,p)}{f(z)},$$
$$k_3(z,p) = u_4(z,p), \qquad k_4(z,p) = u_5(z,p),$$

where

$$f(z) = \frac{12(3+2az)(2a-3h(z))^2(2a+3h(z))}{(3+az)(16a^4+108a^2z-81h(z)-48a^3h(z)-27a(-4+3zh(z)))},$$

and $h(z) = (1+az)^{1/3}$.

(21) $\vec{k}$ obeys the differential equation

$$\frac{\partial \vec{k}}{\partial z} + E(z)\frac{\partial \vec{k}}{\partial p} + B(z)\vec{k} = \vec{\Sigma}_4(z,p), \tag{5.12}$$

where $E(z)$ is given in Appendix B.1 and we omit $B(z)$ and $\vec{\Sigma}_4$. This system is a free evolution system in the sense that there are no further constraints which must be obeyed by these variables. The system cannot be reduced to any simpler form than this. However, the matrix $E(z)$ is not symmetrizable, which implies that this system is not symmetric hyperbolic. This is why we choose to work with the five dimensional system (5.10) and the Einstein constraint (5.11).

We will slightly rewrite the five dimensional system (5.10) as

$$t\frac{\partial \vec{u}}{\partial t} + A(t)\frac{\partial \vec{u}}{\partial p} + C(t)\vec{u} = \vec{\Sigma}(t, p) \tag{5.13}$$

where now $t = z - z_c$, so that $t = 0$ is the Cauchy horizon. In terms of the coefficient matrices and source in (5.10), $A(t) = t\tilde{A}(z)$, $C(t) = t\tilde{C}(z)$ and $\vec{\Sigma}(t, p) = t\vec{\Sigma}_5(z, p)$. We briefly list here the most important properties of this system.

- $\vec{u}(t, p)$ is a five dimensional vector, whose components are linear combinations of the components of the gauge invariant metric and matter perturbations.

- $A(t)$ and $C(t)$ are five-by-five matrices. We note that $A(t) = t\tilde{A}(z)$ and the matrix $\tilde{A}(z)$ contains a factor of $h^{-1}(z)$ in the $(5, 5)$ component. Here $h(z) := z + S - z\dot{S} = z - f(z)$, where $S(z)$ is the radial function (see (2.5)) and $f(z) = -S + z\dot{S}$. $f(z_c) = z_c$ so that $h(z)$ vanishes on the Cauchy horizon. If we Taylor expand $h(z) = z - z_c - \dot{f}(z_c)(z - z_c) + O((z - z_c)^2) = t - \dot{f}(z_c)t + O(t^2)$, then we can see that $th^{-1}(z)$ is analytic at the Cauchy horizon where $t = 0$. This in turn implies that $A(t) = t\tilde{A}(z)$ is analytic at the Cauchy horizon. Similar remarks apply to the matrix $C(t) = t\tilde{C}(z)$, since the fifth row of $\tilde{C}(z)$ contains $h^{-1}(z)$ factors. Similarly, the fifth component of $\vec{\Sigma}_5$ contains a factor of $h^{-1}(z)$. So overall, $A(t)$, $C(t)$ and $\vec{\Sigma}(t, p)$ are analytic for $t \geq 0$.

- $A(t)$ is diagonal, whereas $C(t)$ is not. The first four rows of $A(t)$, $C(t)$ and $\vec{\Sigma}(t, p)$ are $O(t)$ as $t \to 0$, while the last row of each is $O(1)$ as $t \to 0$.

- The source $\vec{\Sigma}$ is separable and we can write it as $\vec{\Sigma}(z, p) = \vec{h}(t)g(p)$, where $\vec{h}(t)$ is an analytic vector valued function of t and $g(p)$ is an initial data term.

- $g(p)$ represents the perturbation of the dust velocity and is a free initial data function. The results below follow for $g \in C_0^\infty(\mathbb{R}, \mathbb{R})$ which we assume henceforth.

In what follows, t_1 is the initial data surface and $0 \le t \le t_1$, where $t = 0$ is the Cauchy horizon, so the Cauchy horizon is approached in the direction of decreasing t.

Theorem 5.2.2 *The IVP consisting of the system (5.13) along with the initial data*

$$\vec{u}\Big|_{t_1} = \vec{f}(p)$$

where $\vec{f} \in C_0^\infty(\mathbb{R}, \mathbb{R}^5)$, possesses a unique solution $\vec{u}(t,p)$, $\vec{u} \in \mathbf{C}^\infty(\mathbb{R} \times (0, t_1], \mathbb{R}^5)$. For all $t \in (0, t_1]$, $\vec{u}(t, \cdot) : \mathbb{R} \to \mathbb{R}^5$ has compact support.

Proof This is a standard result from the theory of symmetric hyperbolic systems, see Chapter 12 of [35]. □

We note that since the constraint is propagated by the five dimensional system (see Lemma 5.2.1), a choice of smooth and compactly supported initial data for the components $u_1(z,p)$, $u_3(z,p)$, $u_4(z,p)$ and $u_5(z,p)$ is sufficient to ensure that $u_2(z,p)$ as given by the constraint (5.11) is also smooth and compactly supported. Therefore, this theorem also provides sufficient conditions for the existence of unique solutions to the four dimensional free evolution system.

Remark 5.2.1 In Section 6.1 we will require solutions $\vec{u}(t,p)$ in $L^1(\mathbb{R}, \mathbb{R}^5)$ for a choice of initial data $\vec{f} \in L^1(\mathbb{R}, \mathbb{R}^5)$. It follows immediately from Theorem 5.2.2 by the density of C_0^∞ in L^1 that for $0 < t \le t_1$, $\vec{u}(\cdot, p) \in L^1(\mathbb{R}, \mathbb{R}^5)$. To show that we can extend our choice of initial data to L^1, we require a bound on $\vec{u}$, which is established in the following lemma. For this lemma, we will need Grönwall's inequality [13], which states that for continuous functions $\phi(t)$, $\psi(t)$ and $\chi(t)$, if

$$\phi(t) \le \psi(t) + \int_a^t \chi(s)\phi(s)\, ds,$$

then

$$\phi(t) \leq \psi(t) + \int_a^t \chi(s)\psi(s) \exp\left(\int_s^t \chi(u)\, du\right) ds.$$

Lemma 5.2.3 *The L^1-norm of $\vec{u}(t,p)$ obeys the bound*

$$||\vec{u}(t)||_1 \leq c_1(t)||\vec{u}(t_1)||_1 + c_2(t)||g||_1, \tag{5.14}$$

for $0 < t \leq t_1$, where $c_1(t)$ and $c_2(t)$ are continuous on $(0, t_1]$.

Proof $\vec{u}$ obeys the equation

$$t\frac{\partial \vec{u}}{\partial t} + A(t)\frac{\partial \vec{u}}{\partial p} + C(t)\vec{u} = \vec{\Sigma}(t,p) \tag{5.15}$$

where $\vec{\Sigma} = \vec{h}(t)g(p)$ and $g(p)$ in an initial data term. Each row of (5.15) can be written as

$$t\frac{\partial u_i}{\partial t} + a_i(t)\frac{\partial u_i}{\partial p} + c_i(t)u_i = \Sigma_i(t,p) - \sum_{j=1, j\neq i}^{5} c_{ij}(t)u_j(t,p) = S_i(t,p), \tag{5.16}$$

for $i = 1, \ldots, 5$, where $a_i(t)$ and $c_i(t)$ are the diagonal components of the matrices $A(t)$ and $C(t)$ respectively, and because $C(t)$ is not diagonal, the off-diagonal components $c_{ij}(t)$ are put into the source term $S_i(t,p)$. We can solve (5.16) using the method of characteristics. The characteristics are given by

$$\frac{dp_i}{dt} = \frac{a_i(t)}{t} \qquad \Rightarrow \qquad p_i(t) = \eta_i + \pi_i(t), \tag{5.17}$$

where $\pi_i(t) = -\int_t^{t_1} \frac{a_i(\tau)}{\tau} d\tau$ and $\eta_i = p_i(t_1)$. On characteristics, (5.16) becomes

$$t\frac{du_i}{dt}(t, p_i(t)) + c_i(t)u_i(t, p_i(t)) = S_i(t, p_i(t)). \tag{5.18}$$

The integrating factor for (5.18) is $e^{\xi_i(t)}$ where $\xi_i(t) = -\int_t^{t_1} \frac{c_i(\tau)}{\tau} d\tau$, and the solution to (5.18) is

$$u_i(t, p_i) = e^{-\xi_i(t)} u_i^{(0)}(p_i - \pi_i(t)) - e^{\xi_i(t)} \int_t^{t_1} \frac{e^{\xi_i(\tau)}}{\tau} S_i(\tau, p_i + \pi_i(\tau) - \pi_i(t)) d\tau, \tag{5.19}$$

where $u_i(t_1, p_i) = u_i^{(0)}(p)$ is the initial data. We take the L^1-norm by taking an absolute value and integrating with respect to p; this produces

$$||u_i(t)||_1 \leq e^{-\xi_i(t)}||u_i^{(0)}||_1 \\ + e^{\xi_i(t)} \int_{\mathbb{R}} \left(\int_t^{t_1} \frac{e^{\xi_i(\tau)}}{\tau} |S_i(\tau, p_i + \pi_i(\tau) - \pi_i(t))| d\tau \right) dp.$$

Recall Theorem 5.2.2 which tells us that at each $t \in (0, t_1]$, $u_j \in C_0^\infty(\mathbb{R}, \mathbb{R})$. This allows us to apply Fubini's theorem; that is, to interchange the order of the integrals above. We note that the structure of the characteristics (5.17) indicates that evaluation of the L^1-norm of a function $f(t, p)$ at fixed time t yields the same result as the evaluation of the L^1-norm of f evaluated on characteristics, that is

$$\int_{\mathbb{R}} |f(\tau, p)| \, dp = \int_{\mathbb{R}} |f(\tau, p_i + \pi_i(\tau) - \pi_i(t))| \, dp.$$

So, applying Fubini's theorem and using the form of S_i produces

$$||u_i(t)||_1 \leq e^{-\xi_i(t)}||u_i^{(0)}||_1 + \\ e^{\xi_i(t)} \int_t^{t_1} \frac{e^{\xi_i(\tau)}}{\tau} \left(|h_i(\tau)|||g||_1 + \sum_{j=1, j \neq i}^{5} |c_{ij}(\tau)| \int_{\mathbb{R}} |u_j| \, dp \right) d\tau. \tag{5.20}$$

Then if we note that $||g||_1$ does not depend on t, we can write

$$||\vec{u}(t)||_1 = \sup_i ||u_i(t)||_1 \leq d_1(t)||\vec{u}^{(0)}||_1 + d_2(t)||g||_1 + d_3(t) \int_t^{t_1} d_4(\tau)||\vec{u}(\tau)||_1 d\tau, \tag{5.21}$$

where the $d_i(t)$ functions are the suprema of the various t-dependent functions which appear in (5.20) (and their precise value is not important). Now applying Grönwall's inequality (with $\psi(t) = d_1(t)||\vec{u}^{(0)}||_1 + d_2(t)||g||_1$) produces

$$||\vec{u}(t)||_1 \leq d_1(t)||\vec{u}^{(0)}||_1 + d_2(t)||g||_1$$
$$+ \int_t^{t_1} d_5(\tau)\left(d_1(\tau)||\vec{u}^{(0)}||_1 + d_2(\tau)||g||_1\right)\exp\left(\int_\tau^{t_1} d_5(\tau')d\tau'\right)d\tau,$$

where the exact value of $d_5(t)$ is unimportant. Again, since $||g||_1$ and $||\vec{u}^{(0)}||_1$ do not depend on t, we can summarize this as

$$||\vec{u}(t)||_1 \leq c_1(t)||\vec{u}(t_1)||_1 + c_2(t)||g||_1,$$

for some continuous functions $c_1(t)$ and $c_2(t)$ defined on $(0, t_1]$, whose exact value is not important.

□

Remark 5.2.2 The characteristics $p_i(t)$ provide a C^1 foliation of the region $\Omega = \{(t, p_i) : t \in (0, t_1], p_i \in \mathbb{R}\}$. A typical such foliation is shown in Figure 5.1. For every point $q = (t_2, p_{i,2}) \in \Omega$, there is a unique characteristic C_1 such that $q \in C_1$ which we label by $\eta_{i,2} = p_i(t_2)\Big|_{C_1}$. Define the set $\Omega_t := \{(t', p_i') \in \Omega : t' = t\}$. Then the characteristics provide a natural diffeomorphism of Ω_t,

$$p_{i,2} \in \Omega_{t_2} \rightarrow p_{i,3} \in \Omega_{t_3},$$

with

$$p_{i,2} = p_{i,3} - \int_{t_2}^{t_3} a_i(\tau)d\tau.$$

The fact that $\vec{u}$ obeys a bound of the form (5.14) allows us to extend our initial data to $\vec{f} = \vec{u}(t_1, p) \in L^1(\mathbb{R}, \mathbb{R}^5)$.

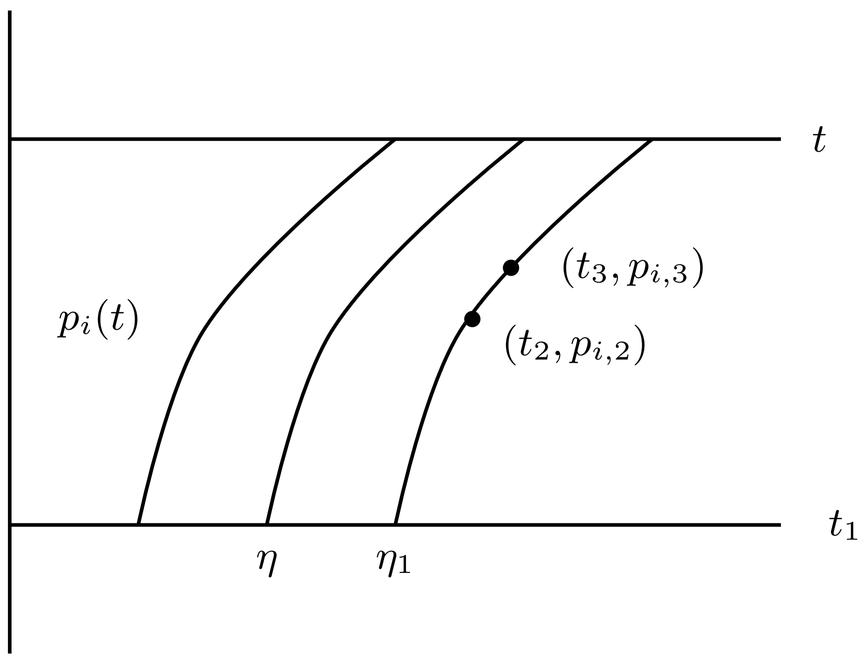

Figure 5.1: Characteristics. We show here typical characteristic curves $p_i = p_i(t)$, along which the solution (5.19) is evaluated.

Theorem 5.2.4 *The IVP consisting of the system (5.13) along with the initial data*

$$\vec{u}\Big|_{t_1} = \vec{f}(p)$$

where $\vec{f} \in L^1(\mathbb{R}, \mathbb{R}^5)$, possesses a unique solution $\vec{u}(t,p)$, $\vec{u} \in C^\infty((0,t_1], L^1(\mathbb{R}, \mathbb{R}^5))$.

Proof The proof of this result relies on the bound (5.14) and follows by a standard argument exploiting the density of $C_0^\infty(\mathbb{R}, \mathbb{R}^5)$ in $L^1(\mathbb{R}, \mathbb{R}^5)$. See Theorems 5 and 7 of [41] for examples of such techniques and Chapter 12 of [35] for background details. □

Although these theorems provide for the existence of smooth or L^1 solutions prior to the Cauchy horizon, it gives us no information about their behaviour as they reach the horizon itself. We must therefore consider this behaviour separately. We note that this problem is rendered nontrivial by the fact that the Cauchy horizon is a singular hypersurface of (5.13).

5.3 Behaviour of the L^q-Norm

In this section, we will use Theorem 5.2.4 to provide for the existence and uniqueness of L^1 solutions to (5.13) with a choice of L^1 initial data. This theorem applies for $t \in (0, t_1]$ only and we must consider the behaviour of $\vec{u}$ on the Cauchy horizon separately. Our strategy in tackling this problem is as follows (see [44]). We expect that any divergence which might arise in the perturbation would be in some sense (to be defined) independent of the radial coordinate, since the Cauchy horizon is a hypersurface of constant t, and since the coefficients of (5.13) are independent of p. Motivated by this observation, we introduce the integral of the perturbation vector with respect to the radial coordinate, which acts as a kind of "average" of the perturbation. This variable obeys a relatively simple system of ODEs, the solutions to which can be determined.

Let $\vec{u}$ be the solution of (5.13) with $\vec{u}(t_1, p) = \vec{u}^{(0)}$. Then define

$$\bar{u}(t) := \int_{\mathbb{R}} \vec{u}(t, p)\, dp, \tag{5.22}$$

which is a kind of "average" of $\vec{u}(t, p)$ (note that the existence of $\bar{u}$ is guaranteed since $|\bar{u}| \leq ||\vec{u}||_1 < \infty$ since $\vec{u} \in L^1$). If we integrate with respect to p through the system (5.13), we find that $\bar{u}$ obeys the ODE

$$t\frac{d\bar{u}}{dt} = -C(t)\bar{u} + \bar{\Sigma}, \tag{5.23}$$

where $\bar{\Sigma}(t) := \int_{\mathbb{R}} \vec{\Sigma}(t, p)\, dp$. This ODE displays a regular singular point at $t = 0$ (see Chapter 2 of [58] and Chapter 4 of [13] for the theory of such points). We now state a theorem which gives the fundamental matrix for this system. Recall that the fundamental matrix for an ODE system is a matrix whose rows are linearly independent solutions to the ODE in question.

Theorem 5.3.1 *The fundamental matrix corresponding to (5.23) is*

$$\begin{aligned} H(t) &= J(t) + K(t) \\ &= P(t)\, t^{-\bar{C}_0} + P(t)\, t^{-\bar{C}_0} \int_t^{t_1} P^{-1}(\tau)\, \tau^{\bar{C}_0 - \mathbb{I}} \bar{\Sigma}\, d\tau, \end{aligned} \tag{5.24}$$

where $P(t) = \mathbb{I} + tP_1 + \ldots$ is a matrix series whose coefficients can be found by a recursion relation from the Taylor expansion of $C(t)$. $\bar{C}_0$ is the Jordan canonical form of the zero order term in the Taylor expansion of $C(t)$. $\bar{C}_0$ takes the form $\bar{C}_0 = diag(0, 0, 0, 0, c)$ where c is a constant given by

$$c = \lim_{t \to 0} t \left(\frac{3 + (S(z_c) - z_c)\ddot{S}(z_c) + \dot{S}(z_c)(-3 + z_c\ddot{S}(z_c))}{(1 - \dot{S}(z_c))(z_c + S(z_c) - z_c\dot{S}(z_c))} \right).$$

c depends only on a, and satisfies $c \in (3, +\infty)$, with

$$\lim_{a \to 0^+} c = 3, \qquad \lim_{a \to a^*} c = +\infty,$$

where a^* is the maximum value of a for which a naked singularity forms.

Proof The proof is a standard result for systems of the form (5.23). See [58] or [13] for details. We have relegated the proof of the results about the behaviour of c to Appendix B.2. [1] $\square$

Our next task is to analyse in more detail the behaviour of this fundamental matrix.

5.3.1 Behaviour of the Fundamental Matrix

We now expand about the Cauchy horizon so that $P(t) = \mathbb{I} + O(t)$ and write the homogeneous part and the particular part of $H(t)$ separately. Given the form of the matrix $\bar{C}_0$ listed in Theorem 5.3.1, the homogeneous part takes the form

$$J(t) = \mathrm{diag}(1 + O(t), 1 + O(t), 1 + O(t), 1 + O(t), t^{-c} + O(t^{-c+1})),$$

and the particular part can be written

$$K(t) = \mathrm{diag}(\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5),$$

where

$$\kappa_i = \int_t^{t_1} \tau^{-1}\bar{\Sigma}_i(\tau)(1 + O(\tau))\, d\tau,$$

for $i = 1, 2, 3, 4$ and

$$\kappa_5 = t^{-c} \int_t^{t_1} \tau^{c-1}\, \bar{\Sigma}_5(\tau)(1 + O(\tau))\, d\tau.$$

We will now examine the particular part. We recall that $\bar{\Sigma}_i$ is separable, so that $\bar{\Sigma} = \vec{h}(t)\, G$, where $\vec{h}(t) = (0, 0, tk_3(t), tk_4(t), k_5(t))$. The $k_i(t)$ functions

[1] We note that there exists a set A of values of a such that $c(a)$ is a natural number, that is $A = \{a \in (0, a^*) : c(a) \in \mathbb{N} \cap (3, \infty)\}$. When $c \in \mathbb{N} \cap (3, \infty)$, the fundamental matrix (5.24) will contain extra log terms. However, since this set has zero measure in the set $a \in (0, a^*)$ we will not consider it further. See [58] and [13] for further details.

are all analytic at $t = 0$ and $G := \int_{\mathbb{R}} g(p)\,dp \in \mathbb{R}$. Now the κ terms become

$$\kappa_1 = \kappa_2 = 0,$$
$$\kappa_j = Gk_j(t^*)(t_1 - t) + O((t_1 - t)^2),$$

for $j = 3, 4$ and

$$\kappa_5 = t^{-c}k_5(t^*)G\left(\frac{t_1^c - t^c}{c}\right) + O(t_1 - t),$$

where we have applied the mean value theorem, and $t^* \in [t_1, t]$. We can see that these integrals have the same order behaviour as the corresponding homogeneous terms, that is, the κ_i are $O(1)$ as $t \to 0$, for $i = 1, \ldots, 4$ and κ_5 is $O(t^{-c})$ as $t \to 0$.

Now since $c > 0$, (5.24) shows that solutions to (5.23) blow up as $t \to 0$. We now examine this divergence.

5.3.2 Blow-up of the L^q-norm

We begin this analysis by determining a way to distinguish between those initial data which lead to diverging solutions to (5.23) and those which do not. If we include both the homogeneous and the inhomogeneous parts, then we can label the five solutions to (5.23) arising from Theorem 5.3.1 as

$$\begin{aligned}
\bar{\phi}_1(z) &= (1 + \kappa_1, 0, 0, 0, 0)^T,\\
\bar{\phi}_2(z) &= (0, 1 + \kappa_2, 0, 0, 0)^T,\\
\bar{\phi}_3(z) &= (0, 0, 1 + \kappa_3, 0, 0)^T,\\
\bar{\phi}_4(z) &= (0, 0, 0, 1 + \kappa_4, 0)^T,\\
\bar{\phi}_5(z) &= (0, 0, 0, 0, t^{-c} + \kappa_5)^T,
\end{aligned}$$

where the $\bar{\phi}_{1,2,3,4}$ are finite as $t \to 0$ and $\bar{\phi}_5$ is divergent. Given that (5.23) has coefficients which are analytic on $(0, t_1]$, it follows that $\bar{\phi}_{1-5}$ are analytic on $(0, t_1]$. Thus these solutions provide a basis for solutions of (5.23) on $(0, t_1]$. Hence given any solution $\bar{u}(t)$ of (5.23), there exist constants $d_i, i = 1, \ldots, 5$

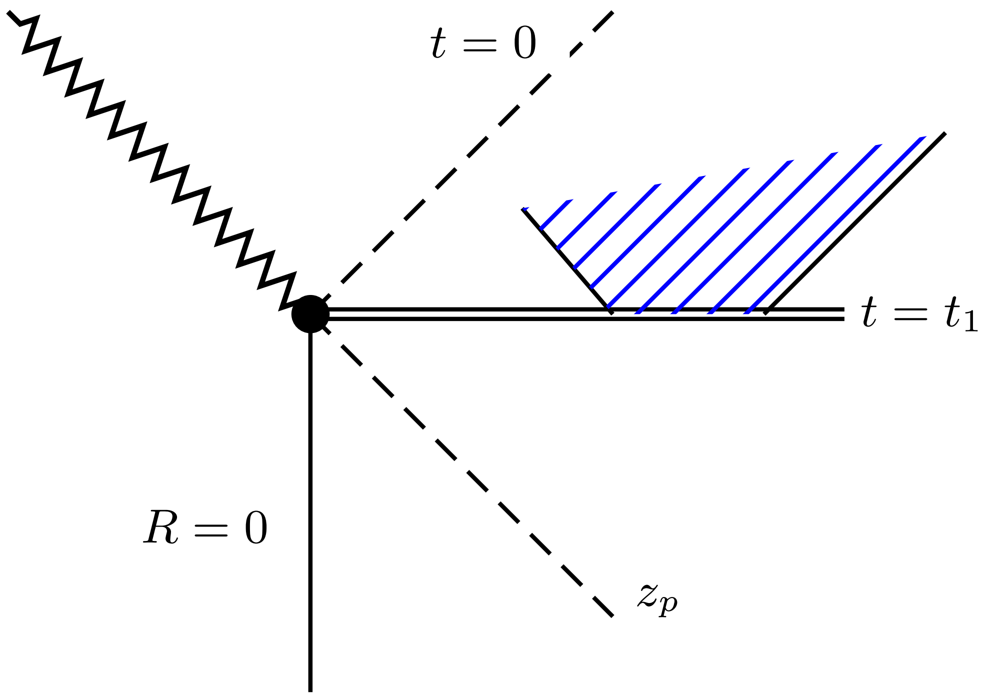

Figure 5.2: The Spread of the Support of $\vec{u}$. We illustrate the spread of the compact support of $\vec{u}$ from the initial data surface t_1 to the Cauchy horizon. The growth of the support is bounded by in- and outgoing null rays starting from the initial data surface.

such that

$$\bar{u}(t) = \sum_{i=1}^{5} d_i \bar{\phi}_i(t).$$

Let $\mathcal{S} = L^1(\mathbb{R}, \mathbb{R}^5)$. We consider solutions of (5.13) with initial data in $\mathcal{S}$. Given $\vec{u}^{(0)} \in \mathcal{S}$, define $\bar{u}_0 = \int_{\mathbb{R}} \vec{u}^{(0)}\, dp$. We can define $d_i(\vec{u}^{(0)})$ via the existence of unique constants $d_i, i = 1, \ldots, 5$ for which

$$\bar{u}_0 = \sum_{i=1}^{5} d_i \bar{\phi}_i(t_1).$$

Define

$$\mathcal{S}' = \{\vec{u}^{(0)} \in \mathcal{S} : d_5(\vec{u}^{(0)}) = 0\}.$$

This set corresponds one-to-one with initial data for (5.13) which give rise to solutions for which the corresponding solutions of (5.23) are finite as $t \to 0$.

We define the set complement of $\mathcal{S}'$ in $\mathcal{S}$ as $\mathcal{S}'' = \mathcal{S} - \mathcal{S}'$.

Lemma 5.3.2 *Given a choice of initial data $\vec{u}^{(0)} \in \mathcal{S}''$, the solution $\vec{u}$ corresponding to this data displays a blow-up of its L^1-norm, that is,*

$$\lim_{t \to 0} ||\vec{u}||_1 = \infty.$$

Proof We define $\bar{u} = \int_{\mathbb{R}} \vec{u}\, dp$ where $\vec{u}$ is the solution of (5.13) corresponding to $\vec{u}^{(0)} \in \mathcal{S}''$. It is immediately clear from (5.24) that

$$\lim_{z \to z_c} |\bar{u}| = \lim_{t \to 0} |\bar{u}| = \infty.$$

Then using the definition of the L^1-norm of $\vec{u}$, $||\vec{u}||_1$, it follows that

$$||\vec{u}||_1 = \int_{\mathbb{R}} |\vec{u}|\, dp \geq \left| \int_{\mathbb{R}} \vec{u}\, dp \right| = |\bar{u}|, \tag{5.25}$$

which implies that $||\vec{u}||_1 \to \infty$ as $t \to 0$. $\square$

This divergent behaviour in the L^1-norm of $\vec{u}$ could be attributed to the divergence of the support of $\vec{u}$ as it spreads from the initial surface t_1, rather than to divergent behaviour in $\vec{u}$ itself (see Figure 5.2 for an illustration of this). We must therefore consider the behaviour of the spread of the support of $\vec{u}$ from the initial surface to the Cauchy horizon. In analysing the growth of the support of $\vec{u}$, it is convenient to briefly return to the self-similar coordinate z.

Lemma 5.3.3 *Let* $vol[\vec{u}](z) = p^+(z) - p^-(z)$, *where*

$$p^+(z) = \sup_{p\in\mathbb{R}}\{p : \vec{u}(z,p) \neq 0\},$$

$$p^-(z) = \inf_{p\in\mathbb{R}}\{p : \vec{u}(z,p) \neq 0\}.$$

Then neglecting terms which remain finite on the Cauchy horizon, $vol[\vec{u}](z)$ *grows as*

$$vol[\vec{u}](z) \sim -\ln|t|, \qquad\qquad t \to 0.$$

Proof Define $\mathrm{vol}[\vec{u}](z) = p^+(z) - p^-(z)$ where

$$p^+(z) = \sup_{p\in\mathbb{R}}\{p : \vec{u}(z,p) \neq 0\},$$

$$p^-(z) = \inf_{p\in\mathbb{R}}\{p : \vec{u}(z,p) \neq 0\}.$$

The support of $\vec{u}$ at some time z, $\mathrm{supp}[\vec{u}](z)$, will obey

$$\mathrm{supp}[\vec{u}](z) \subseteq [p^-(z), p^+(z)].$$

We define the L^q-norm as usual,

$$||\vec{u}||_q = \left[\int_{\mathbb{R}} |\vec{u}(z,p)|^q \, dp\right]^{1/q} = \left[\int_{p^-(z)}^{p^+(z)} |\vec{u}(z,p)|^q \, dp\right]^{1/q},$$

for $1 \leq q < \infty$. $\vec{u}$ has initially compact support which implies that $\mathrm{vol}[\vec{u}](z_0) = p^+(z_0) - p^-(z_0) < \infty$, where z_0 is the initial data surface. This initial support must grow in a causal manner; that is, the growth of $p^{\pm}(z)$ must be bounded by the in- and outgoing null directions. From the metric (2.1), the in- and outgoing null directions are described by the relation $dt/dr = \pm e^{\nu/2}$, which in (z,p) coordinates becomes

$$\frac{dz}{dp} = -(z \pm e^{\nu/2}),$$

which results in

$$p^{\pm}(z) = p^{\pm}(z_0) + \int_z^{z_0} dz (z \pm e^{\nu/2})^{-1}. \tag{5.26}$$

In handling this integral, we first substitute $e^{\nu/2} = S(z) - z\dot{S}(z)$ (see (2.6)), and then make the coordinate change $y = S^{1/2}(z)$. The resulting integral can be performed and results in

$$p^{+}(z) = p^{+}(z_0) + 3\sum_{i=1}^{4} f_{+}(y_i^{+}) \ln |(1+az)^{1/3} - y_i^{+}|,$$

$$p^{-}(z) = p^{-}(z_0) + 3\sum_{i=1}^{4} f_{-}(y_i^{-}) \ln |(1+az)^{1/3} - y_i^{-}|,$$

where

$$f_{\pm}(k) = \frac{k^3}{-1 \pm k^2 + 4k^3},$$

and $y_i^{\pm}$ is the i^{th} root of $\pm 2a - 3k \pm ak^3 + 3k^4 = 0$. Therefore, the volume of the support of $\vec{u}$ grows as

$$\begin{aligned} \mathrm{vol}[\vec{u}](z) = {} & p^{+}(z_0) - p^{-}(z_0) + \\ & 3\sum_{i=1}^{4} \left(f_{+}(y_i^{+}) \ln |(1+az)^{1/3} - y_i^{+}| - f_{-}(y_i^{-}) \ln |(1+az)^{1/3} - y_i^{-}| \right). \end{aligned} \tag{5.27}$$

Now, since in (5.26) there is a divergence at the Cauchy horizon when $z_c = -e^{\nu/2}(z_c)$, the above result must contain a Cauchy horizon divergence. Using the coordinate transformation $y = S(z)^{1/2}$ it is possible to show that in terms of y, the Cauchy horizon is at that y for which $(y^3 - 1)(3y - 2a) + 3ay^3 = 0$, which is precisely where $(1+az)^{1/3} = -z^{-1}(1 + \frac{1}{3}az)$. When this holds, the first log term given above diverges. So in (5.27) we have one finite term describing the initially finite volume, a second term which diverges on the

Cauchy horizon and a third term which is finite everywhere. So overall, if we ignore terms which remain finite as $t \to 0$, then we can describe the behaviour of the volume as

$$\mathrm{vol}[\vec{u}](z) \sim -\ln|t|, \qquad t \to 0.$$

□

We will next need the L^q-embedding theorem [1], which we state as follows:

Theorem 5.3.4 *L^q* ***Embedding Theorem****: Suppose $\Omega \subseteq \mathbb{R}^n$ satisfies $vol(\Omega) = \int_\Omega 1\,dx < \infty$. For $1 \leq p \leq q \leq \infty$, if $u \in L^q(\Omega)$, then $u \in L^p(\Omega)$ and*

$$||u||_p \leq [vol(\Omega)]^{\frac{1}{p}-\frac{1}{q}}||u||_q. \tag{5.28}$$

We are now in a position to show that the L^q-norm of $\vec{u}$ diverges.

Theorem 5.3.5 *Given a choice of initial data $\vec{u}^{(0)} \in \mathcal{S}''$, the solution $\vec{u}$ of (5.13) corresponding to this data displays a blow-up of its L^q-norm for $1 \leq q \leq \infty$, that is,*

$$\lim_{t\to 0}||\vec{u}||_q = \infty.$$

Proof We set $p = 1$ in (5.28) . This produces

$$||\vec{u}||_1 \leq [\mathrm{vol}[\vec{u}](z)]^{1-\frac{1}{q}}||\vec{u}||_q.$$

We know from (5.25) that $||\vec{u}||_1 \geq |\bar{u}(z)| \sim t^{-c}$ so

$$\frac{t^{-c}}{(\mathrm{vol}[\vec{u}](z))^{1-\frac{1}{q}}} \leq ||\vec{u}||_q, \tag{5.29}$$

Now $\lim_{t\to 0} t^c(\ln(t))^{1-\frac{1}{q}} = 0$, since $c > 0$. Therefore, we can conclude that

$$\lim_{t\to 0}||\vec{u}||_q = \infty.$$

So the L^q-norm of the solutions with initial data in $\mathcal{S}''$ blows up as $t \to 0$. □

We note that this behaviour will also hold for the four dimensional free evolution system, as the constraint is propagated (see Lemma 5.2.1). We next prove two theorems which together show that this divergent behaviour is generic with respect to the initial data. In particular, we can show that the set of initial data which corresponds to solutions with divergent behaviour, $\mathcal{S}''$, is open and dense in the set of all initial data, and that it has codimension 1 in $\mathcal{S}$.

Theorem 5.3.6 *The quotient space of $\mathcal{S}'$ in $\mathcal{S}$, $\hat{\mathcal{S}} = \mathcal{S}/\mathcal{S}'$, has codimension one in $\mathcal{S}$.*

Proof A quotient space $\hat{\mathcal{S}} = \mathcal{S}/\mathcal{S}'$ has dimension n if and only if there exist n vectors $\vec{X}_{(1)}, \ldots, \vec{X}_{(n)}$ linearly independent relative to $\mathcal{S}'$ such that for every $\vec{X} \in \mathcal{S}$, there exist unique numbers $c_1, \ldots, c_n$ and a unique $\vec{X}' \in \mathcal{S}'$ such that $\vec{X} = \sum_{i=1}^{n} c_i \vec{X}_i + \vec{X}'$ [28]. So let $\vec{X} \in \mathcal{S}$ and let $\vec{X}_{(1)}$ be any element of $\hat{\mathcal{S}}$. To prove the result, we show that there is a unique value of α for which $\vec{X}^{(\alpha)} = \vec{X} - \alpha \vec{X}_{(1)} \in \mathcal{S}'$. Integrating over the real line and exploiting earlier notation, we have

$$\begin{aligned} \bar{x}^{(\alpha)} &= \bar{x} - \alpha \bar{x}_{(1)} \\ &= \sum_{i=1}^{5} (d_i(\vec{X}) - \alpha d_i(\vec{X}_{(1)})) \bar{\phi}_i(t_1). \end{aligned}$$

We have $\vec{X}^{(\alpha)} \in \mathcal{S}'$ if and only if $d_5(\vec{X}) - \alpha d_5(\vec{X}_{(1)}) = 0$. Since $d_5(\vec{X}_{(1)}) \neq 0$ - as $\vec{X}_{(1)} \in \hat{\mathcal{S}}$ - this occurs for a unique value of α.

□

Theorem 5.3.7 *$\mathcal{S}''$ is dense and open in $\mathcal{S}$ in the topology induced by the L^1-norm.*

Proof To show that $\mathcal{S}''$ is dense in $\mathcal{S}$, we must show that for any $\vec{X} \in \mathcal{S}$ and any $\epsilon > 0$, there exists some $\vec{X}'' \in \mathcal{S}''$ such that the L^1 distance between $\vec{X}$ and $\vec{X}''$ is less than ϵ, that is,

$$||\vec{X} - \vec{X}''||_1 < \epsilon. \tag{5.30}$$

First, suppose that $\vec{X} \in \mathcal{S}''$. Then (5.30) is trivially satisfied by taking $\vec{X}'' = \vec{X} \in \mathcal{S}''$. We therefore assume that $\vec{X} \in \mathcal{S}'$. Now consider some $\psi(p) \in C_0^\infty(\mathbb{R}, \mathbb{R})$, such that $\psi(p) \geq 0$ and $\int \psi(p)\, dp = 1$. We then set

$$\vec{X}'' = \vec{X} + \frac{\epsilon}{2}\psi(p)\frac{\bar{\phi}_5(t_1)}{|\bar{\phi}_5(t_1)|},$$

where $|\cdot|$ indicates the maximum vector norm in $\mathbb{R}^5$, that is $|\bar{\phi}_5(t_1)| = \max_i |(\bar{\phi}_5(t_1))_i|$ [1]. Then

$$||\vec{X} - \vec{X}''||_1 = \int_{\mathbb{R}} \frac{\epsilon}{2}\, \psi(p) \left| \frac{\bar{\phi}_5(t_1)}{|\bar{\phi}_5(t_1)|} \right| dp = \frac{\epsilon}{2} < \epsilon.$$

So we can explicitly construct the $\vec{X}''$ required to satisfy (5.30), and thus $\mathcal{S}''$ is dense in $\mathcal{S}$.

To show that $\mathcal{S}''$ is open in $\mathcal{S}$, we must show that for all $\vec{X}'' \in \mathcal{S}''$, there exists an $\epsilon > 0$ such that $B^1_\epsilon(\vec{X}'') \subset \mathcal{S}''$, where $B^1_\epsilon(\vec{X}'')$ indicates a ball of radius ϵ in the L^1-norm centred at $\vec{X}''$. We fix $\vec{X}'' \in \mathcal{S}''$ and let $\vec{X} \in B^1_\epsilon(\vec{X}'') \subseteq \mathcal{S}$. Then

$$||\vec{X} - \vec{X}''||_1 = \int_{\mathbb{R}} |\vec{X} - \vec{X}''|\, dp < \epsilon. \tag{5.31}$$

There exists unique constants c_i and d_i (for $i = 1, \ldots, 5$) such that

$$\bar{x}'' = d_1\bar{\phi}_1 + \ldots + d_5\bar{\phi}_5,$$
$$\bar{x} = c_1\bar{\phi}_1 + \ldots + c_5\bar{\phi}_5.$$

It follows from (5.31) that $|c_i - d_i| < \alpha_i \epsilon$, for some α_i depending on $\bar{\phi}_{1-5}(t_1)$. Then by making ϵ arbitrarily small, we can make the d_i arbitrarily close to the c_i. We know that $d_5 \neq 0$ since $\vec{X}'' \in \mathcal{S}''$; therefore $c_5 \neq 0$ which implies that $\vec{X} \in \mathcal{S}''$.

□

[1] We note that if $\bar{\phi}_5(t_1) = 0$, we can replace $\bar{\phi}_5$ with $\hat{\bar{\phi}}_5 = \bar{\phi}_5 + \sum_{i=1}^4 c_i\bar{\phi}_i$, for some constants c_i, chosen to guarantee that $\hat{\bar{\phi}}_5(t_1) \neq 0$. This does not affect the definition of $\mathcal{S}'$ or $\mathcal{S}''$.

These two theorems, coupled with Theorem 5.3.5, suffice to show that the averaged form of the solution (5.22) displays a generic divergence of its L^q-norm, where the term generic refers to the open, dense subset of the initial data which lead to this divergence.

Remark 5.3.1 We note that if we define $\vec{x} := t^c \vec{u}$ and let $\bar{x} := \int_{\mathbb{R}} \vec{x}\, dp$, then by multiplying (5.24) by a factor of t^c and taking the limit $t \to 0$, we find

$$\lim_{t \to 0} \bar{x} \neq 0. \tag{5.32}$$

This will be used in the proof of Theorem 6.1.10. The results of Theorems 5.3.6 and 5.3.7 tell us that the set of initial data which gives rise to (5.32) is open and dense in $L^1(\mathbb{R}, \mathbb{R}^5)$.

We note that from Theorem 5.3.5 alone we cannot conclude that the perturbation itself diverges on the Cauchy horizon. The reason for this is that one can easily imagine a function which has finite pointwise behaviour, but a diverging L^q-norm arising from the spatial integration in (5.22). For example, a constant function is clearly pointwise finite, but has a diverging L^q-norm. In the next section, we will determine the pointwise behaviour of the perturbation as the Cauchy horizon is approached and show that it diverges with a characteristic power of t^{-c}.

Chapter 6

The Even Parity Perturbations II: The Pointwise Behaviour

In this chapter, we examine the pointwise behaviour of the even parity perturbation. We work with a scaled version of the perturbation, $\vec{x}$, and aim to show that this scaled version is non-zero on the Cauchy horizon. In Section 6.1.1, we use an extension of odd parity energy methods to show that $\vec{x}$ is bounded in the approach to the Cauchy horizon. In Section 6.1.2, we improve these bounds using the method of characteristics, and by using a series of results about L^p-spaces, we show that $\vec{x}$ is generically non-zero on an open subset of the Cauchy horizon. This in turn is used to show that the perturbation itself generically diverges on the Cauchy horizon. In Section 6.2, we give a physical interpretation of this result in terms of the perturbed Weyl scalars.

6.1 Pointwise Behaviour at the Cauchy horizon

So far we have determined the behaviour of the averaged perturbation $\bar{u}$. In this section, we aim to show that the vector $\vec{u}$ has behaviour similar to that of $\bar{u}$, that is, $O(1)$ behaviour in the first four components, and $O(t^{-c})$ behaviour in the last component. In this section, we will use Theorem 5.2.2 to provide us with smooth, compactly supported solutions $\vec{u}$ to (5.13), with a choice of initial data $\vec{u}(t_1, p) \in C_0^\infty(\mathbb{R}, \mathbb{R}^5)$.

We begin by returning to the five dimensional symmetric hyperbolic system

$$t\frac{\partial \vec{u}}{\partial t} + A(t)\frac{\partial \vec{u}}{\partial p} + C(t)\vec{u} = \vec{\Sigma}(t, p). \tag{6.1}$$

Our strategy is to work with a scaled form of $\vec{u}$, namely $\vec{x} := t^c \vec{u}$. We can write an equation for $\vec{x}$ by using (6.1). We find that

$$t\frac{\partial \vec{x}}{\partial t} + A(t)\frac{\partial \vec{x}}{\partial p} + (C(t) - c\mathbb{I})\vec{x} = t^c \vec{\Sigma}(t, p). \tag{6.2}$$

Before presenting the results which determine the behaviour of $\vec{x}$ at the Cauchy horizon, we present a summary of various steps involved.

- We begin by showing that $\vec{x}$ has a bounded energy throughout its evolution, including on the Cauchy horizon. Initially, we introduce the first energy norm, $E_1[\vec{x}](t)$, which is simply the L^2-norm of $\vec{x}$. In Theorem 6.1.1, we show that this norm is bounded by a term which diverges as the Cauchy horizon is approached.

- We introduce a second energy norm, $E_2[\vec{x}](t)$ and in Theorem 6.1.2 show that it is bounded for $t \in [0, t^*]$, for some t^* sufficiently close to the Cauchy horizon. By combining Theorems 6.1.1 and 6.1.2, we can show that $\vec{x}$ has a bounded energy up to the Cauchy horizon; see Theorem 6.1.3.

- We use this to show that $\vec{x}$ itself is bounded in Corollary 6.1.4. However, there is no guarantee that $\vec{x}$ does not vanish on the Cauchy horizon. If this were to occur, then we would not be able to deduce any information about the behaviour of $\vec{u}$ from that of $\vec{x}$.

- We want to show that $\vec{x}$ is generically non-zero on the Cauchy horizon, for a set of non-zero measure. We can easily show that $\bar{x} := \int_{\mathbb{R}} \vec{x}\, dp$ is non-zero at the Cauchy horizon (see Remark 5.3.1). If we could commute the limit $t \to 0$ with the integral, then we could show that $\int_{\mathbb{R}} \vec{x}(0, p)\, dp \neq 0$, which would be sufficient, since then $\vec{x} \neq 0$ on at least some interval on the Cauchy horizon.

- In order to show that we can commute the limit with the integral, we turn to the Lebesgue dominated convergence theorem, which provides conditions under which one may do this. In order to meet these conditions, we must strengthen the bound on $\vec{x}$ (Lemma 6.1.5), construct a Cauchy sequence of $\vec{x}$ values in L^1 (Lemmas 6.1.6 and 6.1.7) and finally apply the dominated convergence theorem.

- So overall, we can show that $\vec{x}(0,p) \neq 0$ over at least some interval $p \in (a,b)$ on the Cauchy horizon. This applies for a generic choice of initial data. This result in turn shows that $\vec{u}$ generically blows up at the Cauchy horizon in a pointwise fashion (see Theorem 6.1.11)

We begin with our first energy norm for $\vec{x}$.

6.1.1 Energy Bounds for $\vec{x}$

Since we expect $\vec{u}$ to diverge as t^{-c}, if we define $\vec{x} = t^c\vec{u}$, then we expect that $\vec{x}$ should have a bounded energy in the approach to the Cauchy horizon.

Theorem 6.1.1 *Let $\vec{u}(t,p)$ be a solution of (5.13) subject to the hypotheses of Theorem 5.2.2. Then there exists $t^* > 0$ such that for all $t \in (0,t^*]$, the energy norm*

$$E_1(t) := \int_{\mathbb{R}} t^{2c}\,\vec{u}\cdot\vec{u}\,dp = \int_{\mathbb{R}} \vec{x}\cdot\vec{x}\,dp, \tag{6.3}$$

obeys the bound

$$E_1(t) \leq \frac{\mu}{t}, \tag{6.4}$$

for a positive constant μ which depends only on the initial data and the background geometry.

Proof We define $E_1(t)$ as in (6.3) and take its derivative. We substitute (6.2) to find

$$\frac{dE_1}{dt} = \int_{\mathbb{R}} 2t^{2c-1}(c\vec{u}\cdot\vec{u} - \vec{u}\cdot C(t)\vec{u}) - 2t^{2c-1}\vec{u}\cdot A(t)\frac{\partial\vec{u}}{\partial p} + 2t^{2c-1}\vec{u}\cdot\vec{\Sigma}\,dp. \tag{6.5}$$

Integrating by parts shows that the term containing $\partial\vec{u}/\partial p$ vanishes due to the compact support of $\vec{u}$ and the fact that $A(t)$ is symmetric. This leaves

$$\int_{\mathbb{R}} 2t^{2c-1}(c\vec{u}\cdot\vec{u} - \vec{u}\cdot C(t)\vec{u}) + 2t^{2c-1}\vec{u}\cdot\vec{\Sigma}\,dp, \tag{6.6}$$

on the right hand side of (6.5). We focus on the first term in (6.6), and introduce the constant matrix S, which transforms C_0 into its Jordan canonical form and which is listed in Appendix B.1. We use this matrix to show that

$$c\vec{u}\cdot\vec{u} - \vec{u}\cdot C(t)\vec{u} = S^T\vec{u}\cdot(c\mathbb{I} - \bar{C}(t))\vec{v},$$

where $\vec{v} = S^{-1}\vec{u}$ and $\bar{C}(t) = S^{-1}C(t)S$, so that $\bar{C}_0$ is the Jordan canonical form of C_0. Recall that $\bar{C}_0 = \mathrm{diag}(0,0,0,0,c)$. Now since $S^T = S^T\,S\,S^{-1}$, we find that

$$c\vec{u}\cdot\vec{u} - \vec{u}\cdot C(t)\vec{u} = \vec{v}^T S^T S\cdot(c\mathbb{I} - \bar{C}(t))\vec{v}.$$

Now for any matrix A, A^TA is positive definite. We now wish to show that $\langle\vec{v}, S^TS(c\mathbb{I} - \bar{C}(t))\vec{v}\rangle \geq 0$, and that equality holds iff $\vec{v} = 0$. By using the form of S and the matrix $\bar{C}(t=0)$ we find that the matrix $S^TS(c\mathbb{I} - \bar{C}(t))$ has four positive eigenvalues and one zero eigenvalue at $t = 0$. This indicates that it is positive semi-definite, but could still vanish along the direction of one of the eigenvectors. Specifically, it is possible for only the fifth component of $\vec{v}$, v_5 to be non-zero and for this dot product to still vanish. However, by the continuity of the matrices here, it follows that $S^TS(c\mathbb{I} - \bar{C}(t))$ has four positive eigenvalues for $t \in (0, t^*]$ for some $t^* > 0$. Therefore, if $\langle\vec{v}, S^TS(c\mathbb{I} - \bar{C}(t))\vec{v}\rangle = 0$ in this range, it follows that we must have $v_1 = v_2 = v_3 = v_4 = 0$. But if these components vanish, then the only way to have $\langle\vec{v}, S^TS(c\mathbb{I} - \bar{C}(t))\vec{v}\rangle = 0$ is to have $v_5 = 0$ too. We therefore conclude that if $v_1 = v_2 = v_3 = v_4 = 0$, we must also have $v_5 = 0$. Therefore, $\langle\vec{v}, S^TS(c\mathbb{I} - \bar{C}(t))\vec{v}\rangle = 0$ iff $\vec{v} = 0$.

So overall, we conclude that

$$\langle\vec{v}, S^TS(c\mathbb{I} - \bar{C}(t))\vec{v}\rangle \geq 0. \tag{6.7}$$

for $t \in (0, t^*]$ for some positive t^*, with equality holding iff $\vec{v} = 0$. We note that if $\vec{v} = 0$, then $\vec{u} = S\vec{v} = 0$ too.

We now assume that $t \in (0, t^*]$ and using this information about the first two terms of (6.5), we can conclude that

$$\frac{dE_1}{dt} \geq \int_{\mathbb{R}} 2t^{2c-1}\vec{u} \cdot \vec{\Sigma}\, dp.$$

We now apply the Cauchy-Schwarz inequality to show that

$$\frac{dE_1}{dt} \geq \; -\textstyle\int_{\mathbb{R}} t^{2c-1}\vec{u} \cdot \vec{u}\, dp - \int_{\mathbb{R}} t^{2c-1}\vec{\Sigma} \cdot \vec{\Sigma}\, dp \; = -\frac{1}{t}(E_1(t) + J(t)),$$

where $J(t) := \int_{\mathbb{R}} t^{2c}\vec{\Sigma} \cdot \vec{\Sigma}$. Integrating this from some time t up to an initial time t_1 (where $0 < t \leq t_1 \leq t^*$) will produce

$$tE_1(t) \leq t_1 E_1(t_1) + \int_t^{t_1} J(\tau)\, d\tau. \tag{6.8}$$

We now examine the term $J(t)$. We first recall the form of the source vector $\vec{\Sigma}(t, p) = \vec{h}(t)g(p)$, where $\vec{h}(t)$ is an analytic function of t. $J(t)$ can therefore be written as $J(t) = t^{2c}\vec{h} \cdot \vec{h}G$, where $G := \int_{\mathbb{R}} g(p)^2\, dp \in \mathbb{R}$. Then the integral appearing in (6.8) is

$$\int_t^{t_1} J(\tau)\, d\tau = Gt_*^{2c}\vec{h} \cdot \vec{h}\Big|_{t_*} (t_1 - t),$$

where we use the mean value theorem [49] to put the t-dependent terms outside the integral and $t_* \in [t, t_1]$. The right hand side above is clearly bounded as $t \to 0$, and we can therefore conclude that

$$E_1(t) \leq \frac{\mu}{t},$$

for a positive constant μ which depends only on the initial data and the background geometry. □

We note that (6.3) is in fact the L^2-norm of $\vec{x}$. Since $E_1(t)$ is bounded, it follows that $\vec{x}(t, p) \in L^2(\mathbb{R}, \mathbb{R}^5)$ for $t \in (0, t^*]$. We also emphasise that this bound holds only for $t > 0$ and does not hold on the Cauchy horizon. We now introduce a second energy norm $E_2[\vec{x}](t)$ whose bound will extend to the Cauchy horizon.

Theorem 6.1.2 *Let $\vec{u}$ be a solution of (5.13), subject to Theorem 5.2.2. Define*

$$E_2[\vec{x}](t) := \int_{\mathbb{R}} \vec{x} \cdot \vec{x} + (t-1)x_5^2 \, dp. \tag{6.9}$$

Then there exists some $t_2 > 0$ and $\mu > 0$ such that

$$E_2[\vec{x}](t) \leq E_2[\vec{x}](t_2) e^{\mu(t_2 - t)}. \tag{6.10}$$

Here μ is a positive constant that depends only on the components of the background metric tensor.

Proof We begin by noting that the definition (6.9) is equivalent to

$$E_2[\vec{x}](t) := \int_{\mathbb{R}} x_1^2 + x_2^2 + x_3^2 + x_4^2 + t x_5^2 \, dp.$$

The factor of t is intended to control the behaviour of x_5 as the Cauchy horizon is approached. In what follows, we will denote the $(5,5)$ component of the matrix $A(t)$ as $a_5(t)$, the $(5,5)$ component of the matrix $C(t)$ as $c_5(t)$, and the remaining components of the fifth row of the matrix $C(t)$ will be denoted $c_{5j}(t)$, where $j = 1, \ldots, 4$. We also recall that the first four rows of $C(t)$ have $O(t)$ behaviour as $t \to 0$, and the fifth row has $O(1)$ behaviour as $t \to 0$. We begin by taking a t-derivative of (6.9) and substituting from (6.2) to find

$$\begin{aligned} \frac{dE_2}{dt} = \int_{\mathbb{R}} 2\vec{x} \cdot \left(-\frac{A(t)}{t}\frac{\partial \vec{x}}{\partial p} - \frac{C(t) - c\mathbb{I}}{t}\vec{x} + t^{c-1}\vec{\Sigma}(t,p) \right) + x_5^2 \\ + 2(t-1)\frac{x_5}{t}\left(-a_5(t)\frac{\partial x_5}{\partial p} - (c_5(t) - c)x_5 + t^c \Sigma_5 - \sum_{j=1}^{4} c_{5j}(t) x_j \right) dp, \end{aligned} \tag{6.11}$$

where $\sum_{j=1}^4 c_{5j}(t) x_j$ appears in the source term of the x_5 equation due to the fact that the $C(t)$ matrix is not diagonal. Now consider the terms

$$-2\vec{x} \cdot \frac{A(t)}{t}\frac{\partial \vec{x}}{\partial p} - 2(t-1)\frac{x_5}{t}a_5(t)\frac{\partial x_5}{\partial p}.$$

This can be simplified to

$$-2\left(x_1\tilde{a}_1(t)\frac{\partial x_1}{\partial p} + x_2\tilde{a}_2(t)\frac{\partial x_2}{\partial p} + x_3\tilde{a}_3(t)\frac{\partial x_4}{\partial p} + x_4\tilde{a}_4(t)\frac{\partial x_4}{\partial p}\right) - 2x_5a_5(t)\frac{\partial x_5}{\partial p}, \tag{6.12}$$

where we used the fact that $a_i(t) = t\tilde{a}_i(t)$ where $\tilde{a}_i(t)$ is $O(1)$ for $i = 1, \ldots, 4$. After we insert (6.12) into the integral in (6.11), it will vanish after an integration by parts, due to the compact support of $\vec{x}$. Returning to (6.11), we are left with

$$\begin{aligned}\frac{dE_2}{dt} = \int_{\mathbb{R}} 2\vec{x} \cdot \left(-\frac{C(t) - c\mathbb{I}}{t}\vec{x} + t^{c-1}\vec{\Sigma}(t,p)\right) + x_5^2 + \\ 2(t-1)\frac{x_5}{t}\left(-(c_5(t) - c)x_5 + t^c\Sigma_5 - \sum_{j=1}^{4} c_{5j}(t)x_j\right) dp.\end{aligned} \tag{6.13}$$

If we now consider the terms

$$-2\vec{x} \cdot \frac{(C(t) - c\mathbb{I})}{t}\vec{x} - 2(t-1)\frac{x_5^2}{t}(c_5(t) - c) - 2(t-1)\sum_{j=1}^{4}\frac{c_{5j}(t)}{t}x_jx_5,$$

we notice that they can be rewritten as

$$-2\vec{x} \cdot \frac{(\tilde{C}(t) - c\mathbb{I})}{t}\vec{x} - 2(c_5(t) - c)x_5^2 - 2\sum_{j=1}^{4} c_{5j}(t)x_jx_5,$$

where $\tilde{C}(t)$ is a matrix got by replacing the final row of $C(t) - c\mathbb{I}$ with a row of zeroes. In other words, we can write $\tilde{C}(t)$ as

$$\tilde{C}(t) = \begin{pmatrix} & & D & & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where D is a 4×5 $O(t)$ matrix. Equally, we could write $\tilde{C}(t) = t\bar{C}(t)$, where

$\bar{C}(t)$ is an $O(1)$ matrix. The fifth row of $\bar{C}(t)$ contains only zeroes, that is, $\bar{C}_{5i} = 0$ for $i = 1, \ldots, 5$.

We insert this into (6.13) and note that $2c\vec{x} \cdot \vec{x}/t$ is explicitly positive definite. We also use the Cauchy-Schwarz inequality, in the form

$$\int_{\mathbb{R}} 2\vec{x} \cdot \vec{\Sigma}\, dp \geq - \int_{\mathbb{R}} \vec{x} \cdot \vec{x} + \vec{\Sigma} \cdot \vec{\Sigma}\, dp,$$

to find that

$$\frac{dE_2}{dt} \geq \int_{\mathbb{R}} -2\vec{x} \cdot \bar{C}(t)\vec{x} - t^{c-1}(\vec{\Sigma} \cdot \vec{\Sigma} + \vec{x} \cdot \vec{x}) + x_5^2 + \\ - 2x_5^2(c_5(t) - c) + t^{c-1}(t-1)(-\Sigma_5^2 - x_5^2) - 2\sum_{j=1}^{4} c_{5j}(t) x_j x_5\, dp.$$

Now we let I equal the integrand on the right hand side above. We introduce $I_R = I + \mu I_{E_2}$, where $\mu > 0$ is a constant and I_{E_2} indicates the integrand of (6.9). We wish to show that $I_R \geq 0$.

We can write I_R as

$$I_R = \vec{x} \cdot (-2\bar{C}(t) + \mu\mathbb{I} - t^{c-1}\mathbb{I})\vec{x} + (-2(c_5(t) - c) + 1 - (t-1)t^{c-1} \\ +(t-1)\mu)x_5^2 - 2\sum_{j=1}^{4} c_{5j}(t) x_j x_5 - t^{c-1}\vec{\Sigma} \cdot \vec{\Sigma} - (t-1)t^{c-1}\Sigma_5^2.$$

Let $t = 0$ and note that $c > 1$ so that at $t = 0$, $t^{c-1} = 0$. Then I_R simplifies to

$$I_R\bigg|_{t=0} = \vec{x} \cdot (-2\bar{C}(t=0) + \mu\mathbb{I})\vec{x} + (1-\mu)x_5^2 - 2\sum_{j=1}^{4} c_{5j}(t=0) x_j x_5.$$

We can simplify matters by writing $I_R\Big|_{t=0} = \vec{x} \cdot H\vec{x}$, where H can be written as

$$H = \begin{pmatrix} \mu & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + K,$$

where K is a constant matrix, independent of μ, which depends on the components of the matrix $\bar{C}(t)$ evaluated at $t = 0$ and $K_{55} = 1$. For $I_R(0) > 0$, we need H to be positive definite. This implies that all of the principal subdeterminants of H must be non-negative. So we require

$$\mu + K_{11} > 0$$

$$\begin{vmatrix} \mu + K_{11} & K_{12} \\ K_{21} & \mu + K_{22} \end{vmatrix} > 0,$$

$$\begin{vmatrix} \mu + K_{11} & K_{12} & K_{13} \\ K_{21} & \mu + K_{22} & K_{23} \\ K_{31} & K_{32} & \mu + K_{33} \end{vmatrix} > 0,$$

$$\begin{vmatrix} \mu + K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & \mu + K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & \mu + K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & \mu + K_{44} \end{vmatrix} > 0,$$

$$|H| = \begin{vmatrix} \mu + K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\ K_{21} & \mu + K_{22} & K_{23} & K_{24} & K_{25} \\ K_{31} & K_{32} & \mu + K_{33} & K_{34} & K_{35} \\ K_{41} & K_{42} & K_{43} & \mu + K_{44} & K_{45} \\ K_{51} & K_{52} & K_{53} & K_{54} & 1 \end{vmatrix} > 0.$$

These conditions produce a linear equation, a quadratic with leading μ^2, a cubic with leading μ^3 and two quartics with leading μ^4, all of which must be positive. We therefore pick a μ which is large enough to satisfy each of these conditions.

We therefore conclude that at $t = 0$, $I_R > 0$. Now by continuity of the coefficients of $\vec{x}$ in I_R, it follows that there exists some t^* such that $I_R \geq 0$ in the range $t \in [0, t^*]$. We may therefore state that

$$\frac{dE_2}{dt} \geq -\mu E_2, \tag{6.14}$$

in this range. We now integrate (6.14) starting from some initial data surface $t_2 \in (0, t^*)$. This results in

$$E_2[\vec{x}](t) \leq E_2[\vec{x}](t_2) e^{\mu(t_2 - t)}, \tag{6.15}$$

which provides the desired bound for $E_2[\vec{x}](t)$. □

We note that the definition of $E_2[\vec{x}](t)$, (6.9), is a sum of the L^2-norms of x_i for $i = 1, \ldots, 4$ and the L^2-norm of $t^{1/2} x_5$. Since we can bound $E_2[\vec{x}](t)$ in $t \in [0, t^*]$, it follows that in this range, $x_i \in L^2(\mathbb{R}, \mathbb{R})$ for $i = 1, \ldots, 4$ and $t^{1/2} x_5 \in L^2(\mathbb{R}, \mathbb{R})$.

We next combine this result with Theorem 6.1.1 to provide a bound on $\vec{x}$ which holds for the entire range of t.

Theorem 6.1.3 *Let $\vec{u}$ be a solution of (5.13), subject to Theorem 5.2.2. Then $E_2[\vec{x}](t)$ is bounded by an a priori bound for $t \in [0, t_1]$, that is,*

$$E_2[\vec{x}](t) \leq \nu E_1[\vec{x}](t_1), \tag{6.16}$$

for a positive constant ν.

Proof To prove this, we note that by definition, $E_2[\vec{x}](t) = E_1[\vec{x}](t) + (t - 1)E_1[x_5](t)$. Therefore, using the bound on $E_1[\vec{x}]$ from Theorem 5.3.1 produces $E_2[\vec{x}](t_2) \leq E_1[\vec{x}](t_1)$. Inserting this into (6.10) produces (6.16), where $\nu = e^{\mu(t_2 - t)}$.

□

Corollary 6.1.4 *Let $\vec{u}$ be a solution of (5.13), subject to Theorem 5.2.2.*

Then $\vec{x} = t^c \vec{u}$ is uniformly bounded in the range $t \in (0, t^*)$. That is

$$|x_i| \leq \beta_i$$

for $i = 1, \ldots, 4$ and

$$|t^{1/2} x_5| \leq \beta_5$$

where the β_j, $j = 1, \ldots, 5$, are constants depending on the background geometry and on the initial data.

Proof We first note that one of the effects of self-similarity has been to produce a differential operator on the left hand side of (6.2) which has only t-dependent coefficients. This means that the spatial derivative, $\vec{x}_{,p}$ obeys the same differential equation as $\vec{x}$, but with a modified source term. It follows that if we define $E_1[\vec{x}_{,p}](t) := \int t^{2c} \vec{u}_{,p} \cdot \vec{u}_{,p}\, dp$, we can bound this energy in an exactly similar manner to Theorem 5.3.1. Similarly, we can bound the energy $E_2[\vec{x_{,p}}](t)$ using the same argument as that of Theorem 6.1.2.

Now recall Sobolev's inequality which states that

$$|\vec{v}|^2 \leq \frac{1}{2} \int_{\mathbb{R}} |\vec{v}|^2 + |\vec{v}_{,p}|^2\, dp,$$

for $\vec{v} \in C_0^\infty(\mathbb{R}, \mathbb{R}^5)$. Applying this to $\vec{x}$ and using the bound (6.16) (with the corresponding bound for $\vec{x}_{,p}$) produces

$$|x_i| \leq \beta_i$$

for $i = 1, \ldots, 4$ and

$$|t^{1/2} x_5| \leq \beta_5$$

where the β_j, $j = 1, \ldots, 5$, are constants depending on the background geometry and on the initial data. $\square$

6.1.2 Behaviour of $\vec{x}$ at the Cauchy Horizon

Having established a bound on $\vec{x}$ through the use of energy norms, we now wish to determine the behaviour of $\vec{x}$ as $t \to 0$, that is, the behaviour on the Cauchy horizon. In particular, we must establish that $\vec{x} \neq 0$ there. To do so, we must first strengthen the bound on $\vec{x}$ by using the method of characteristics.

Lemma 6.1.5 *Let $\vec{u}$ be a solution of (5.13), subject to Theorem 5.2.2. Then $\vec{x}$ obeys the bounds*

$$|x_i(t,p)| \leq \gamma_i\, t^{1/2}, \qquad\qquad |x_5(t,p)| \leq \gamma_5, \tag{6.17}$$

for $i = 1,\ldots,4$ and constants γ_j, $j = 1,\ldots,5$ which depend only on the initial data and the background geometry of the spacetime.

Proof We begin by considering the first four rows of (6.2), which we write as

$$t\frac{\partial x_i}{\partial t} + a_i(t)\frac{\partial x_i}{\partial p} + (c_i - c)x_i = S_i(t,p), \tag{6.18}$$

where $S_i(t,p) = t^c\Sigma_i - \sum_{j=1,j\neq i}^{5} c_{ij}(t)x_j$. Here $a_i(t)$ and $c_i(t)$ represent each entry on the main diagonal of the matrices $A(t)$ and $C(t)$ respectively (and we note that $a_1(t) = a_2(t) = 0$). Since the matrix $C(t)$ is not diagonal, the term $\sum_{j=1,j\neq i}^{5} c_{ij}(t)x_j$ represents all the off-diagonal terms, which we put into the source.

We solve equations (6.18) along characteristics. The characteristics are given by $p = p_i(t)$ where

$$\frac{dp_i}{dt} = \frac{a_i(t)}{t} = \tilde{a}_i(t) \Rightarrow p_i(t) = \pi_i(t) + \eta_i, \tag{6.19}$$

where we use the fact that $a_i(t)$ is $O(t)$ as $t \to 0$, so that $a_i(t) = t\tilde{a}_i(t)$, where $\tilde{a}_i(t)$ is $O(1)$ as $t \to 0$. $\pi_i(t) = -\int_t^{t_1} \tilde{a}_i(\tau)d\tau$ and $\eta_i = p_i(t_1)$. On characteristics, (6.18) becomes

$$t\frac{dx_i}{dt}(t,p_i(t)) + (c_i - c)x_i(t,p_i(t)) = S_i(t,p_i(t)). \tag{6.20}$$

The integrating factors for these equations are given by $e^{\xi_i(t)}$ where

$$\xi_i(t) = -\int_t^{t_1} \frac{c_i(\tau) - c}{\tau}d\tau,$$

and if we Taylor expand the term inside the integral about $t = 0$, we will find that $e^{-\xi_i} = (t/t_1)^c \alpha_i(t)$, where $\alpha_i(t)$ is an $O(1)$ term containing all terms other than the zero order term from the Taylor expansion. The solution to (6.20) is

$$x_i(t, p_i(t)) = \left(\frac{t}{t_1}\right)^c \frac{\alpha_i(t)}{\alpha_i(t_1)} x_i^{(0)}(\eta_i) - t^c \alpha_i(t) \int_t^{t_1} \frac{\tau^{-c-1}}{\alpha_i(\tau)} S_i(\tau, p_i(\tau)) d\tau, \quad (6.21)$$

where $x_i^{(0)}$ is the initial data at $t = t_1$. Now we fix $t \in [0, t_1]$ and let $p_i \in \mathbb{R}$. Then using (6.19) we can write (6.21) as

$$\begin{aligned} x_i(t, p_i(t)) = &\left(\frac{t}{t_1}\right)^c \frac{\alpha_i(t)}{\alpha_i(t_1)} x_i^{(0)}(p_i(t) - \pi_i(t)) \\ &- t^c \alpha_i(t) \int_t^{t_1} \frac{\tau^{-c-1}}{\alpha_i(\tau)} S_i(\tau, p_i(t) + \pi_i(\tau) - \pi_i(t)) d\tau. \end{aligned} \quad (6.22)$$

Now taking the absolute value of (6.22) will produce two integral terms (coming from the two terms in the source S_i), which we label I_{i1} and I_{i2}. I_{i1} is given by

$$I_{i1} = t^c |\alpha_i(t)| \int_t^{t_1} \frac{\tau^{-c-1}}{|\alpha_i(\tau)|} \tau^{c+1} |h_i(\tau)||g(p_i(t) + \pi_i(\tau) - \pi_i(t))| d\tau, \quad (6.23)$$

where we use the fact that $S_i(t, p) = t^c \Sigma_i - \sum_{j=1, j \neq i}^{5} c_{ij}(t) x_j$ and that $\Sigma_i = t h_i(t) g(p)$ where the $h_i(t)$ terms are $O(1)$ functions as $t \to 0$. We use the mean value theorem to evaluate this, and conclude that

$$I_{i1} = t^c \frac{|\alpha_i(t)|}{|\alpha_i(t^*)|} |h_i(t^*)||g(p_i(t) + \pi_i(t^*) - \pi_i(t))|(t_1 - t),. \quad (6.24)$$

for $t^* \in [t_1, t]$. That is,

$$I_{i1} \leq \mu_i t^c + O(t^{c+1}), \quad (6.25)$$

where $\mu_i = \sup_{t \in [0, t_1]} \frac{|\alpha_i(t)|}{|\alpha_i(t^*)|} |h_i(t^*)||g(p_i(t) + \pi_i(t^*) - \pi_i(t))| t_1$.

The second integral from (6.22) is given by

$$I_{i2} = t^c|\alpha_i(t)| \int_t^{t_1} \frac{\tau^{-c-1}}{|\alpha_i(\tau)|} \left(\sum_{j=1, j\neq i}^{4} |c_{ij}(\tau)||x_j(\tau, p_i(t) + \pi_i(\tau) - \pi_i(t))| \right) + \frac{\tau^{-c-1}}{|\alpha_i(\tau)|} \left(|c_{i5}(\tau)||x_5(\tau, p_i(t) + \pi_i(\tau) - \pi_i(t))|\right) d\tau. \tag{6.26}$$

To handle this integral, we first note that $c_{ij}(t)$ is $O(t)$, so that $c_{ij}(t) = t\tilde{c}_{ij}(t)$, where $\tilde{c}_{ij}(t)$ is $O(1)$ as $t \to 0$. We then use the bounds on $\vec{x}$ coming from Corollary 6.1.4 (as well as using the mean value theorem as before). This produces

$$I_{i2} = t^c \frac{|\alpha_i(t)|}{|\alpha_i(t^*)|} \left(\sum_{j=1, j\neq i}^{4} |\tilde{c}_{ij}(t^*)|\beta_j \frac{(t_1^{-c+1} - t^{-c+1})}{-c+1} + |\tilde{c}_{i5}(t^*)|\beta_5 \frac{(t_1^{-c+1/2} - t^{-c+1/2})}{-c+1/2} \right),$$

for $t^* \in [t_1, t]$. That is,

$$I_{i2} \leq \nu_i t^{1/2} + O(t) \tag{6.27}$$

where $\nu_i = \sup_{t\in[0,t_1]} \frac{|\alpha_i(t)|}{|\alpha_i(t^*)|}|\tilde{c}_{i5}(t^*)|\beta_5(-c+1/2)^{-1}$. Combining (6.25), (6.27) and (6.22) produces

$$|x_i(t,p)| \leq \gamma_i\, t^{1/2}, \tag{6.28}$$

where the γ_i factors are constants depending on the initial data and the background geometry.

Using (6.28), it is also possible to improve our previous bound on $x_5(t,p)$. The equation which x_5 obeys is

$$t\frac{\partial x_5}{\partial t} + a_5(t)\frac{\partial x_5}{\partial p} + (c_5 - c)x_5 = S_5(t,p), \tag{6.29}$$

where $S_5(t,p) = t^c\Sigma_5 - \sum_{j=1}^{4} c_{5j}(t)x_j$. The characteristics for this equation are given by

$$\frac{dp_5}{dt} = \frac{a_5(t)}{t} \Rightarrow p_5(t) = \pi_5(t) + \eta_5,$$

where $\pi_5(t) = -\int_t^{t_1} \frac{a_5(\tau)}{\tau} d\tau$ and $\eta_5 = p_5(t_1)$. On characteristics, (6.29) becomes

$$t\frac{dx_5}{dt}(t, p_5(t)) + (c_5 - c)x_5(t, p_5(t)) = S_5(t, p_5(t)). \tag{6.30}$$

The integrating factor for this equation is given by $e^{\xi_5(t)}$ where

$$\xi_5(t) = -\int_t^{t_1} \frac{c_5(\tau) - c}{\tau} d\tau,$$

and since $c_5(t = 0) = c$, we see that $e^{-\xi_5(t)}$ is an $O(1)$ function as $t \to 0$. The solution to (6.30) is

$$\begin{aligned} x_5(t, p_5(t)) &= e^{-\xi_5} x_5^{(0)}(p_5(t) - \pi(t)) \\ &- e^{-\xi_5} \int_t^{t_1} \frac{e^{\xi_5(\tau)}}{\tau} \left(\tau^c \Sigma_5(\tau, p_5(t) + \pi_5(\tau) - \pi_5(t))\right) \\ &\quad - \frac{e^{\xi_5(\tau)}}{\tau} \left(\sum_{j=1}^{4} c_{5j}(\tau) x_j(\tau, p_5(t) + \pi_5(\tau) - \pi_5(t)) \right) d\tau. \end{aligned} \tag{6.31}$$

The integral above contains two terms,

$$I_1 = e^{-\xi_5} \int_t^{t_1} \frac{e^{\xi_5(\tau)}}{\tau} \tau^c \Sigma_5(\tau, p_5(t) + \pi_5(\tau) - \pi_5(t)) d\tau, \tag{6.32}$$

and

$$I_2 = e^{-\xi_5} \int_t^{t_1} \frac{e^{\xi_5(\tau)}}{\tau} \left(\sum_{j=1}^{4} c_{5j}(\tau) x_j(\tau, p_5(t) + \pi_5(\tau) - \pi_5(t)) \right) d\tau. \tag{6.33}$$

We recall that $\Sigma_5(t, p) = h_5(t)g(p)$ and use the mean value theorem to show that

$$I_1 = e^{-\xi_5} e^{\xi_5(t^*)} h_5(t^*) g(p_5(t) + \pi_5(t^*) - \pi_5(t)) \frac{(t_1^c - t^c)}{c}, \tag{6.34}$$

for $t^* \in [t_1, t]$. For the second integral, we use the mean value theorem and the bound (6.28) which arises from the first part of this theorem. This produces

$$I_2 = \left(2e^{-\xi_5} e^{\xi_5(t^*)} \sum_{j=1}^{4} c_{5j}(t^*)\gamma_j \right) (t_1^{1/2} - t^{1/2}), \tag{6.35}$$

for $t^* \in [t_1, t]$. Combining (6.34), (6.35) and (6.31) produces

$$|x_5(t,p)| \leq \gamma_5,$$

where γ_5 is a constant depending only on the initial data and the background geometry.

□

The next lemma shows that we can bound $t^{1/2}x_{i,t}$. We use this in Lemma 6.1.7 to construct a Cauchy sequence of $\vec{x}$-values in L^1.

Lemma 6.1.6 *Let $\vec{u}$ be a solution to (5.13), subject to Theorem 5.2.2. Define $\vec{\chi} := \partial\vec{x}/\partial t$. Then*

$$|t^{1/2}\chi_i(t,p)| \leq \eta_1, \tag{6.36}$$

for $i = 1, \ldots, 4$, where η_1 is a constant depending only on the background geometry and the initial data.

Proof Define $\vec{\chi} := \partial\vec{x}/\partial t$. By differentiating (6.2), we see that $\vec{\chi}$ obeys

$$t\frac{\partial\vec{\chi}}{\partial t} + A(t)\frac{\partial\vec{\chi}}{\partial p} + (C(t) + (1-c)\mathbb{I})\vec{\chi} = \vec{\sigma}(t,p), \tag{6.37}$$

where $\vec{\sigma} = (t^c\vec{\Sigma})_{,t} - A_{,t}\frac{\partial\vec{x}}{\partial p} - C_{,t}\,\vec{x}$. For $i = 1, \ldots, 4$ this becomes

$$t\frac{\partial\chi_i}{\partial t} + a_i(t)\frac{\partial\chi_i}{\partial p} + (c_i(t) + 1 - c)\chi_i = \sigma_i(t,p) - \sum_{j=1, j\neq i}^{5} c_{ij}(t)\chi_j = S_i(t,p), \tag{6.38}$$

where $a_i(t)$ and $c_i(t)$ label the diagonal elements of the $A(t)$ and $C(t)$ matrices appearing in (6.37) and since $C(t)$ is not diagonal, the $c_{ij}(t)$ label the off-diagonal elements which we put in the source term $S_i(t,p)$. We note that $a_1(t) = a_2(t) = 0$.

As in Lemma 6.1.5, we solve (6.38) on characteristics (and these are the same characteristics which appeared in Lemma 6.1.5). By a similar method to that which lead to (6.22), we find that the solutions to (6.38) can be written as

$$\chi_i(t,p) = \alpha_i(t)\left(\frac{t}{t_1}\right)^{c-1}\chi_i^{(0)}(p_i - \pi_i(t)) - \\ \alpha_i(t)t^{c-1}\int_t^{t_1}\frac{\tau^{-c}}{\alpha_i(\tau)}S_i(\tau, p_i - \pi_i(\tau) + \pi_i(t))d\tau, \tag{6.39}$$

where $\chi_i^{(0)}(p_i - \pi_i(t))$ is the initial data and the integrating factor is $e^{-\xi_i(t)} = t^{c-1}t_1^{1-c}\alpha_i(t)$. As in Lemma 6.1.5, we Taylor expand the integrating factors and put all higher order terms into $O(1)$ functions $\alpha_i(t)$. We note that $c-1>0$. Taking an absolute value of (6.39) produces two integrals (coming from the two terms in $S_i(t,p)$), which we label

$$I_{i1}(t,p_i) = |\alpha_i(t)|t^{c-1}\int_t^{t_1}\frac{\tau^{-c}}{\alpha_i(\tau)}|\sigma_i(\tau, p_i(\tau) + \pi_i(\tau) - \pi_i(t))|d\tau, \tag{6.40}$$

and

$$I_{i2}(t,p_i) = |\alpha_i(t)|t^{c-1}\int_t^{t_1}\frac{\tau^{-c}}{|\alpha_i(\tau)|}\sum_{j=1,j\neq i}^{5}\tau|\tilde{c}_{ij}(\tau)||\chi_j(\tau, p_i(\tau)+\pi_i(\tau)-\pi_i(t))|d\tau, \tag{6.41}$$

where we recall that $c_{ij}(t)$ is $O(t)$, so that $c_{ij}(t) = t\tilde{c}_{ij}(t)$, where $\tilde{c}_{ij}(t)$ is $O(1)$. Now to deal with I_{i1}, we note that $\sigma_i(t,p) = (t^c\Sigma_i)_{,t} - a_{i,t}\frac{\partial x_i}{\partial p} - C_{,t}\,\vec{x}|_i$, where $C_{,t}\,\vec{x}|_i$ indicates the i^{th} row of the matrix $C_{,t}\,\vec{x}$. Using the bounds (6.17) on $\vec{x}$ (and note that $C_{,t}\,\vec{x}|_i$ includes x_5), we can see that overall $\sigma_i(t,p)$ is $O(1)$ in t. It is also C^∞ in t, so we can apply the mean value theorem to find

$$I_{i1}(t,p_i) = t^{c-1}\left|\frac{\alpha_i(t^*)\sigma_i(t^*, p_i(t^*) + \pi_i(t^*) - \pi_i(t))}{\alpha_i(t^*)}\right|\frac{(t_1^{-c+1} - t^{-c+1})}{(-c+1)},$$

for $t^* \in [t_1, t]$. We can abbreviate this by writing

$$I_{i1}(t,p_i) \le \eta_2, \tag{6.42}$$

where η_2 is a constant depending on the initial data and the background geometry (it inherits this dependence from the bounds on $\vec{x}$ entering into σ_i). For I_{i2}, we note that from (6.2), we can deduce that $|t\vec{\chi}|$ is $O(1)$ and it is also C^∞ in t, so as before, we apply the mean value theorem to find

$$I_{i2}(t,p_i) = \sum_{j=1,j\neq i}^{5} \left|\frac{\alpha_i(t)\tilde{c}_{ij}(t^*)}{\alpha_i(t^*)}\right| |t^*\chi_j(t^*,p_i(t*)+\pi_i(t^*)-\pi_i(t))| \frac{(t_1^{-c+1}-t^{-c+1})}{(-c+1)},$$

for $t^* \in [t_1, t]$, which we can summarise as

$$I_{i2}(t,p) \leq \eta_3. \tag{6.43}$$

Combining (6.39), (6.42) and (6.43) produces

$$|x_{i},_t(t,p)| = |\chi_i(t,p)| \leq \eta_1, \tag{6.44}$$

for $i = 1, \ldots, 4$, where we neglect higher order terms.

□

We next use this result to show that we can define a sequence $\vec{x}^{(n)}$ of $\vec{x}$-values which is Cauchy in $L^1(\mathbb{R}, \mathbb{R}^5)$.

Lemma 6.1.7 *Let $\{t^{(n)}\}$ be a sequence of t-values in $(0, t_1]$ with $\lim_{n\to\infty} t^{(n)} = 0$. For each $n \geq 1$, define $\vec{x}^{(n)}(p) = \vec{x}(t^{(n)}, p)$. Then $\{\vec{x}^{(n)}\}$ is a Cauchy sequence in $L^1(\mathbb{R}, \mathbb{R}^5)$.*

Proof We define $\vec{x}^{(n)} := \vec{x}(t^{(n)}, p)$, where the sequence $\{t^{(n)}\}_{n=0}^{\infty}$ tends to zero as $n \to \infty$. The mean value theorem produces

$$|\vec{x}(t^{(m)},p) - \vec{x}(t^{(n)},p)| = |\vec{x},_t(t^*,p)||t^{(m)} - t^{(n)}|, \tag{6.45}$$

for some $t^* \in (t^{(m)}, t^{(n)})$. For $i = 1, \ldots, 4$, we can use the bound from Lemma 6.1.6 and integrate with respect to p to give

$$||x_i(t^{(m)},p) - x_i(t^{(n)},p)||_1 \leq |t^{(m)} - t^{(n)}|\eta_1 \int_{p_1}^{p_2} dp,$$

where $p_1 = \max_{p\in\mathbb{R}}(\mathrm{supp}[x_i(t^{(m)},p)], \mathrm{supp}[x_i(t^{(n)},p)])$ and $p_2 = \min_{p\in\mathbb{R}}(\mathrm{supp}[x_i(t^{(m)},p)], \mathrm{supp}[x_i(t^{(n)},p)])$. From Lemma 5.3.3, we know that $\mathrm{supp}[x_i(t,p)]$ satisfies $\mathrm{supp}[x_i(t,p)] \sim -\ln(t)+\mu$, where μ is a constant which represents terms that remain finite as $t \to 0$. Now since $0 \le t^{(m)} \le t^{(n)} \le t_1$, the largest support is that at $t^{(m)}$, so that

$$||x_i(t^{(m)},p) - x_i(t^{(n)},p)||_1 \le |t^{(m)} - t^{(n)}|\eta_1(-\ln(t^{(m)}) + \mu).$$

We can take the $n \to \infty$ limit above and see that $x_i^{(n)}$, for $i = 1,\ldots,4$, is a Cauchy sequence with respect to the L^1-norm.

To show that $x_5^{(n)}$ is a Cauchy sequence in L^1 we use a different tactic. As in (6.31), we write the solution to the x_5 equation of motion as

$$x_5(t,p_5) = e^{-\xi_5(t)}x_5^{(0)}(p_5 - \pi_5(t)) - F(t,p_5), \tag{6.46}$$

where

$$F(t,p_5) = e^{-\xi_5(t)}\int_t^{t_1} \frac{e^{\xi_5(\tau)}}{\tau} S_5(\tau, p_5 + \pi_5(\tau) - \pi_5(t))d\tau. \tag{6.47}$$

Here $e^{\xi_5(t)}$ is the integrating factor, and $\xi_5(t) = -\int_t^{t_1} \frac{c_5(\tau)-c}{\tau}d\tau$, $\pi_5(t) = -\int_t^{t_1} \frac{a_5(\tau)}{\tau}d\tau$ and $c_5(t)$ and $a_5(t)$ are the $(5,5)$-components of the $C(t)$ and $A(t)$ matrices respectively. $S_5(t,p)$ is the source term, $S_5(t,p) = t^c\Sigma_5(t,p) - \sum_{j=1}^5 c_{5j}(t)x_j(t,p)$, where $c_{5j}(t)$ are the components of the fifth row of the $C(t)$ matrix appearing in (6.2). Using (6.46) we can write

$$\begin{aligned}&|x_5(t^{(n)},p_5) - x_5(t^{(m)},p_5)| \le \\ &|e^{-\xi_5(t^{(n)})}x_5^{(0)}(p_5 - \pi_5(t^{(n)})) - e^{-\xi_5(t^{(m)})}x_5^{(0)}(p_5 - \pi_5(t^{(m)}))| + |F(t^{(n)},p_5) - F(t^{(m)},p_5)|.\end{aligned} \tag{6.48}$$

Now if we suppose that $t^{(n)}$ and $t^{(m)}$ are very close to the Cauchy horizon, $t = 0$, then tracing the characteristic back to the initial data surface, we can see that it will intersect the initial data surface outside the compact support of the solution. That is, there exists some N such that for $n, m \ge N$, $x_5^{(0)}(p_5 - \pi_5(t^{(n)})) = x_5^{(0)}(p_5 - \pi_5(t^{(m)})) = 0$.

Now for the second term in (6.48), we use the mean value theorem to show that for $t^* \in [t^{(n)}, t^{(m)}]$,

$$|F(t^{(n)}, p) - F(t^{(m)}, p)| \leq \left| \frac{\partial F}{\partial t}(t^*, p) \right| |t^{(n)} - t^{(m)}|. \tag{6.49}$$

We can easily calculate

$$\frac{\partial F}{\partial t} = -\frac{d\xi_5}{dt} F(t, p) - \frac{S_5(t, p)}{t}. \tag{6.50}$$

Now using the bounds (6.17) on $\vec{x}$ (and the fact that $\Sigma_5 = h_5(t) g(p)$ where $h_5(t)$ is $O(1)$ as $t \to 0$) we can show that

$$\left| \frac{\partial F}{\partial t}(t, p) \right| \leq \mu \, t^{-1/2},$$

where μ is a constant depending only on the background geometry and the initial data. Combining this with (6.49) produces

$$|F(t^{(n)}, p) - F(t^{(m),p})| \leq \mu (t^*)^{-1/2} |t^{(n)} - t^{(m)}|. \tag{6.51}$$

Returning to (6.48), assuming $n, m \geq N$ and using (6.51) produces

$$|x_5(t^{(n)}, p) - x_5(t^{(m)}, p)| \leq \mu (t^*)^{-1/2} |t^{(n)} - t^{(m)}|.$$

Finally, we take the L^1-norm to find

$$||x_5(t^{(n)}, p) - x_5(t^{(m)}, p)||_1 \leq \mu (t^*)^{-1/2} |t^{(n)} - t^{(m)}| \int_{p_1}^{p_2} dp,$$

where $p_1 = \max_{p \in \mathbb{R}}(\text{supp}[x_5(t^{(m)}, p)], \text{supp}[x_5(t^{(n)}, p)])$ and
$p_2 = \min_{p \in \mathbb{R}}(\text{supp}[x_5(t^{(m)}, p)], \text{supp}[x_5(t^{(n)}, p)])$. From Lemma 5.3.3, we know that $\text{supp}[x_5(t, p)]$ satisfies $\text{supp}[x_i(t, p)] \sim -\ln(t) + \mu$, where μ is a constant which represents terms that remain finite as $t \to 0$. Now since $0 \leq t^{(m)} \leq t^{(n)} \leq t_1$, the largest support is that at $t^{(m)}$, so that

$$||x_5(t^{(n)}, p) - x_5(t^{(m)}, p)||_1 \leq \mu (t^*)^{-1/2} |t^{(n)} - t^{(m)}| (-\ln(t^{(m)}) + \mu).$$

and taking the limit $n \to \infty$, we see that $x_5^{(n)}$ is also a Cauchy sequence in L^1. So we can conclude that $\vec{x}^{(n)}$ is a Cauchy sequence in $L^1(\mathbb{R}, \mathbb{R}^5)$.

□

Now with Lemma 6.1.7 in place, and since we know that $\vec{x} \in L^1(\mathbb{R}, \mathbb{R}^5)$ for $t \in (0, t_1]$, we can show that $\vec{x}$ does not vanish on the Cauchy horizon. To do this, we will make use of two theorems from real analysis, which we state here. The proofs of both theorems are standard; see [33] for details.

Theorem 6.1.8 *Let $1 \leq p \leq \infty$. Suppose Ω is some set, $\Omega \subseteq \mathbb{R}^n$, and let $f^{(i)}$, $i = 1, 2, \ldots$ be a Cauchy sequence in $L^p(\Omega)$. Then there exists a unique function $f \in L^p$ such that $||f^{(i)} - f||_p \to 0$ as $i \to \infty$, that is, $f^{(i)}$ converges strongly in the L^p-norm to f as $i \to \infty$.*

Furthermore, there exists a subsequence $f^{(i_1)}, f^{(i_2)}, \ldots$, $i_1 < i_2 < \ldots$ and a non-negative function $F \in L^p(\Omega)$ such that

- *Domination: $|f^{(i_k)}(x)| \leq F(x)$ for all k and for a dense subset of $x \in \Omega$,*
- *Pointwise Convergence: $\lim_{k\to\infty} f^{(i_k)}(x) = f(x)$ for a dense subset of $x \in \Omega$.*

Theorem 6.1.9 *(Lebesgue's Dominated Convergence Theorem) Let $\{f^{(i)}\}$ be a sequence of summable functions which converges to f pointwise almost everywhere. If there exists a summable $F(x)$ such that $|f^{(i)}(x)| \leq F(x)$ $\forall i$, then $|f(x)| \leq F(x)$ and*

$$\lim_{i\to\infty} \int_\Omega f^{(i)}(x)dx = \int_\Omega f(x)dx,$$

that is, we can commute the taking of the limit with the integration.

Theorem 6.1.10 *Let $\vec{u}$ be a solution to (5.13), subject to Theorem 5.2.2. Then $\vec{x}(t, p) = t^c\vec{u}$ does not vanish as $t \to 0$, for a generic choice of initial data. Here the term generic refers to the open and dense subset of initial data which leads to this result.*

Proof Since $\vec{x} \in L^1$ for $t \in (0, t_1]$, the proof of this theorem follows by an application of the two theorems from analysis quoted above. Consider the sequence $\vec{x}^{(n)} := \vec{x}(t^{(n)}, p)$, where $\{t^{(n)}\}_{n=0}^{\infty}$ tends to zero as $n \to \infty$. In Lemma 6.1.7, we showed that this sequence is Cauchy in L^1. Theorem 6.1.8 therefore provides for the existence of a dominated subsequence of $\vec{x}^{(n)}$. In particular, by applying this theorem we may conclude that there exists a non-negative $H \in L^1(\mathbb{R}, \mathbb{R})$ and a unique $\vec{h} \in L^1(\mathbb{R}, \mathbb{R}^5)$ such that

$$|\vec{x}^{(n_m)}| \leq H(p) \quad \forall m,$$

and $||\vec{x}^{(n_m)} - \vec{h}||_1 \to 0$ as $m \to \infty$. Next we apply the Lebesgue dominated convergence theorem (Theorem 6.1.9) to the dominated subsequence $\vec{x}^{(n_m)}$. This produces

$$\lim_{m\to\infty} \int_{\mathbb{R}} \vec{x}(t^{(n_m)}, p)\, dp = \int_{\mathbb{R}} \vec{x}(0, p) dp, \tag{6.52}$$

where we identify $\vec{h} \in L^1(\mathbb{R}, \mathbb{R}^5)$ with $\vec{x}(0, p)$. [1]

Now if we recall Remark 5.3.1, which indicated that $\lim_{t\to 0} \int_{\mathbb{R}} \vec{x}\, dp \neq 0$, we can conclude that $\lim_{m\to\infty} \int_{\mathbb{R}} \vec{x}(t^{(n_m)}, p)\, dp \neq 0$. Combining this with (6.52) produces

$$\int_{\mathbb{R}} \vec{x}(0, p) dp \neq 0, \tag{6.53}$$

which implies that there exists an open subset (a, b) on the Cauchy horizon such that $\vec{x}(t, p) \neq 0$ for $p \in (a, b)$. We note that (6.53) holds generically since $\lim_{t\to 0} \int_{\mathbb{R}} \vec{x}\, dp \neq 0$ for a generic set of initial data (see Remark 5.3.1).

□

We conclude that $\vec{x} := t^c \vec{u}$ exists and is non-zero on the Cauchy horizon for $p \in (a, b)$, for a general choice of initial data. This in turn tells us that the perturbation $\vec{u}$ diverges in a pointwise manner at the Cauchy horizon, with a characteristic power given by t^{-c}.

[1] That is, $\vec{x}(0, p)$ is defined on a dense subset of $\mathbb{R}$ by the second result of Theorem 6.1.8. It suffices to take any bounded extension to "fill in" the definition of $\vec{x}(0, p)$ on the remaining set of zero measure.

Theorem 6.1.11 *There exists an open and dense subset of initial data* $\vec{u}^{(0)} \in L^1(\mathbb{R}, \mathbb{R}^5)$ *such that the solution* $\vec{u}$ *of (6.1) corresponding to this initial data blows up as* $t \to 0$ *on an open subset* $p \in (a, b)$*, that is*

$$\lim_{t\to 0} \vec{u}(t,p) = \infty, \qquad \forall p \in (a,b). \tag{6.54}$$

Proof It follows immediately from Theorem 6.1.10 that $\vec{u}$ blows up as $t \to 0$ on an open subset $p \in (a, b)$ for a choice of $C_0^\infty(\mathbb{R}, \mathbb{R}^5)$ initial data. Recall Remark 5.3.1 which indicates that $\bar{x} = \int_{\mathbb{R}} \vec{x}\, dp \neq 0$ for a generic choice of initial data from $L^1(\mathbb{R}, \mathbb{R}^5)$. We can therefore extend the results of Theorem 6.1.10 to a choice of initial data taken from an open and dense subset of L^1. We conclude that for such initial data

$$\lim_{t\to 0} \vec{u}(t,p) = \infty, \qquad \forall p \in (a,b).$$

□

6.2 Physical Interpretation of Variables

So far, we have established the behaviour of $\vec{u}$ as it approaches the Cauchy horizon. We now wish to provide an interpretation of these results in terms of the perturbed Weyl scalars, which represent the gravitational radiation produced by the metric and matter perturbations. In this section, we will use the coordinate system (u, v, θ, ϕ), where u and v are the in- and outgoing null coordinates (see (2.10) for their definitions), rather than (z, p, θ, ϕ) coordinates. This is a useful choice of coordinate system to make when considering the perturbed Weyl scalars. We follow throughout the presentation of [39].

For the even parity perturbations, only two of the perturbed Weyl scalars, $\delta\Psi_0$ and $\delta\Psi_4$, are identification and tetrad gauge invariant (see [39] and [53]). This means that if we make a change of null tetrad, or a change of our background coordinate system, we will find that these terms are invariant under such changes. We note that $\delta\Psi_0$ and $\delta\Psi_4$ represent transverse gravitational waves propagating radially inwards and outwards.These terms are given by

$$\delta\Psi_0 = \frac{Q}{2r^2}\bar{l}^A\bar{l}^B k_{AB}, \qquad \delta\Psi_4 = \frac{Q^*}{2r^2}\bar{n}^A\bar{n}^B k_{AB},$$

where Q and Q^* are angular coefficients depending on the other vectors in the null tetrad, and on the basis constructed from the spherical harmonics. The ingoing and outgoing null vectors $\bar{l}^A$ and $\bar{n}^A$ are given in (2.11). The term

$$\delta P_{-1} = |\delta\Psi_0\delta\Psi_4|^{1/2} \tag{6.55}$$

is also invariant under spin-boosts, and therefore has a physically meaningful magnitude [39].

Theorem 6.2.1 *The perturbed Weyl scalars $\delta\Psi_0$ and $\delta\Psi_4$, as well as the scalar δP_{-1}, diverge on the Cauchy horizon.*

Proof We begin by writing the tensor k_{AB} in (u, v) coordinates as

$$k_{AB} = \begin{pmatrix} \eta(u,v) & \nu(u,v) \\ \nu(v,v) & \lambda(v,v) \end{pmatrix}.$$

The condition that k_{AB} be tracefree results in $\nu(u, v) = 0$. In (u, v) coordinates, the perturbed Weyl scalars become

$$\delta\Psi_0 = \frac{Q}{2r^2B^2}\eta(u,v), \qquad \delta\Psi_4 = \frac{Q^*}{2r^2}\lambda(u,v),$$

where the factor of $B(u, v)$ is the same factor which appears in (2.11). Now by performing a coordinate transformation, we can write $\alpha(z, p)$ and $\beta(z, p)$ (the components of k_{AB} in (z, p) coordinates) in terms of $\eta(u, v)$ and $\lambda(u, v)$ and by so doing, we can find $\delta\Psi_0$ and $\delta\Psi_4$ in terms of $\alpha(z, p)$ and $\beta(z, p)$. We find

$$\begin{aligned} \delta\Psi_0 &= F(z,p)(\alpha(z,p) - f_-^{-1}(z)\beta(z,p)), \\ \delta\Psi_4 &= G(z,p)(f_+(z)\alpha(z,p) - \beta(z,p)), \end{aligned} \tag{6.56}$$

where the coefficients F and G are given by

$$F(z,p) = \frac{Q}{2r^2B^2}\frac{f_+^2}{u^2}\left(\frac{f_-}{f_- - f_+}\right), \qquad G(z,p) = \frac{Q^*}{2r^2}\frac{f_-^2}{v^2}\left(\frac{1}{f_+ - f_-}\right),$$

and we note that $F \sim r^{-4}$ and $G \sim r^{-2}$ (recall that $B(u,v) \sim r^2$).

Now, if we retrace our steps through the first order reduction in Section 5.2, we find that $u_5(z,p)$ contributes to $\alpha(z,p)$, $\beta(z,p)$, $k(z,p)$ and its first derivatives. In particular, the pointwise divergence of $u_5(z,p)$ on the Cauchy horizon produces a similar divergence in these terms. We can write α and β as

$$\alpha(z,p) = \frac{S}{1-\dot{S}^2}\left(u_4(-1+\dot{S}) + u_5(1+\dot{S}) + 2u_1(1+\dot{S})\right), \tag{6.57}$$

$$\beta(z,p) = \frac{S}{1+\dot{S}^2}\left(u_4(-1+\dot{S})(z-S+z\dot{S}) + 2z(1+\dot{S})u_1 + u_5(z-z\dot{S}+S(1+\dot{S}))\right),$$

where $S(z) = (1+az)^{2/3}$ is the radial function. Combining (6.56) and (6.57) produces

$$\delta\Psi_0 = F(z,p)(\beta_1(z)u_4(z,p) + \beta_2(z)u_5(z,p) + \beta_3(z)u_1(z,p)),$$
$$\delta\Psi_4 = G(z,p)(\beta_4(z)u_4(z,p) + \beta_5(z)u_1(z,p) + \beta_6(z)u_5(z,p)),$$

where

$$\beta_1(z) = \frac{2(-1+\dot{S})S\dot{S}^2}{1-\dot{S}^4},$$
$$\beta_2(z) = S\left(\frac{1}{1-\dot{S}} - \frac{z-z\dot{S}+S(1+\dot{S})}{(z-S+z\dot{S})(1+\dot{S}^2)}\right),$$
$$\beta_3(z) = 2S\left(\frac{1}{1-\dot{S}} - \frac{z(1+\dot{S})}{(z-S+z\dot{S})(1+\dot{S}^2)}\right),$$
$$\beta_4(z) = \frac{2(-1+\dot{S})}{1-\dot{S}^4}(S+z\dot{S}(-1+\dot{S})),$$

$$\beta_5(z) = \frac{2S^2(1+\dot{S})^2}{1-\dot{S}^4},$$
$$\beta_6(z) = \frac{S}{1-\dot{S}^4}(-\dot{S}(1+\dot{S})(-2\dot{S}S - z + \dot{S}^2 z)).$$

So $\delta\Psi_0$ and $\delta\Psi_4$ depend on u_5 and therefore they diverge as the Cauchy horizon is approached. Similarly, δP_{-1} diverges on the Cauchy horizon, as it depends on $\delta\Psi_0$ and $\delta\Psi_4$. □

To construct a gauge invariant interpretation for the matter term $\Gamma(z,p)$, we note that by comparing the GS terms (3.9) and (3.17) to (5.1), we find that

$$T_A = \bar{\rho}(z,p)\,\bar{u}_A(\Gamma(z,p) + g(p))$$

which we contract with the background dust velocity $\bar{u}^A$ to find

$$\bar{u}^A T_A = -\bar{\rho}(z,p)(\Gamma(z,p) + g(p))$$

where $\bar{u}^A T_A$ is a gauge invariant scalar.

6.3 Discussion

We have determined that even parity perturbations of the naked self-similar Lemaître-Tolman-Bondi spacetimes diverge generically on the Cauchy horizon associated with the naked singularity. Here the term generic refers to the open, dense subset of L^1 initial data which lead to this divergence. We can give a physical interpretation of this result in terms of the perturbed Weyl scalars, and find that these scalars also diverge.

Iguchi, Harada and Nakao [26] studied the behaviour of the quadrupole mode ($l = 2$) of the even parity perturbations of the LTB spacetime. They numerically solved the linearised Einstein equations and found that this perturbation diverged on the Cauchy horizon. In one sense, our results are a generalization of theirs, in that this method allows us to treat all perturbations. However, the extra symmetry of self-similarity is needed in order to apply our methods.

We note a potential issue with this work. Our perturbations are at linear order; it follows that it is somewhat strange to conclude that they diverge on the Cauchy horizon, as they are therefore far too large to remain at linear order. However, this result still indicates that this spacetime is not stable to perturbation.

In Chapter 4, we demonstrated that the odd parity perturbations of this background spacetime are finite for all l, where finiteness was measured with respect to initial values of a natural energy norm for the odd parity system. Taken as a whole, these results supports the hypothesis of cosmic censorship, in that one should expect perturbations on a naked singularity spacetime to diverge as the Cauchy horizon is approached. Overall, we have established the linear instability of the naked singularity in this class of LTB spacetimes.

The background spacetime investigated here is of course not a serious model of gravitational collapse, as at the very least, it ignores the effects of pressure during the collapse. A natural extension of this work would therefore be to consider the self-similar perfect fluid model, which contains a naked singularity for a wide range of the equation of state parameter. The study of the behaviour of perturbations in this spacetime would be a very interesting application of the methods developed here.

Part III
Conclusions

Chapter 7

Conclusions

We summarise here the main results of this thesis. There were two main topics of research in this thesis, the behaviour of odd and even parity perturbations of the self-similar LTB spacetime. For the odd parity perturbations:

- Through the use of energy methods for hyperbolic equations, we established that the odd parity perturbation remains finite on the Cauchy horizon. The term finite refers to certain natural integral energy measures (as well as pointwise values thereof) which arise in this spacetime, whose value bounds the growth of the gauge invariant scalar.

- We used this result to establish the behaviour of the perturbed Weyl scalars, showing that they too remain bounded by similar finite terms.

- We considered separately the behaviour of the $l = 1$ perturbation and found that a similar result pertains in that case.

For the even parity perturbations:

- We considered an averaged form of the perturbation state vector and showed, through the use of methods for systems of ordinary differential equations with regular singular points, that it displays a generic divergence in its L^q-norm as the Cauchy horizon is approached. The term generic refers to the open and dense subset of L^1 initial data which leads to this divergence.

- We next introduced a scaled form of the perturbation and used an extension of odd parity energy methods to bound it in the approach to the Cauchy horizon. Through the use of the method of characteristics, we were able to improve these bounds, and by applying some theorems from real analysis we established that the scaled form of the perturbation is generically non-zero on some interval of the Cauchy horizon. This in turn indicates that the perturbation itself diverges on the Cauchy horizon, for a generic choice of L^1 initial data.

- We provided an interpretation of these results in terms of the perturbed Weyl scalars and established that they also diverge at the Cauchy horizon.

Overall, these results show that linear non-spherical perturbations of this spacetime diverge on the Cauchy horizon, as a general perturbation is a linear combination of odd and even parity perturbations. This establishes the linear instability of the naked singularity in the self-similar LTB spacetime. This result supports the hypothesis of cosmic censorship.

In principle, the methods developed in this work should be applicable to any spacetime which is self-similar and spherically symmetric. We require spherical symmetry since without it, we cannot use the Gerlach-Sengupta formalism at all. Self-similarity is required to prove various results in Chapters 4, 5 and 6, chiefly because it a produces equations of motion with operators whose coefficients depend only on z, rather than on z and p.

A particularly interesting application of these methods would be the self-similar perfect fluid spacetime. This spacetime was considered in detail in [45] and [3]. The naked singularity in this spacetime is one of the strongest counter-examples to cosmic censorship in the literature. In principle, the behaviour of linear non-spherical perturbations in this spacetime should be amenable to the methods developed here. However, there are at least two difficulties which we can anticipate.

Firstly, the self-similar LTB spacetime is exactly solvable, in that one can explicitly solve the background Einstein equations and give the metric functions in terms of z. The same is not true of the self-similar perfect fluid.

An explicit knowledge of background functions was important at various stages throughout this work (for example, in Lemma 4.4.2, in Corollary 4.4.3, in Lemma 4.5.1, in Theorem 5.3.1 and in Theorem 6.1.1). Without this background knowledge, we would need to either modify our methods to avoid this difficulty, or to numerically solve the background Einstein equations so as to access the information we need.

Another general difficultly lies in the additional complexity of the background spacetime due to the non-zero pressure. The first task in a study of this spacetime would be to prove the existence of solutions to the background Einstein equations. This will involve navigating the sonic point and the past null cone of the naked singularity, as well as determining a useful condition for nakedness of the singularity in this spacetime. This adds an extra layer of complexity to the problem. However, despite these additional issues, we expect that these methods should be applicable to general linear perturbations of the self-similar perfect fluid spacetime.

Appendix A

Odd Parity Perturbations

A.1 First Order Reduction

The matrices $X(z)$ and $W(z)$ appearing in (4.13) are given by

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\gamma(z) \\ 0 & 1 & -\xi(z) \end{pmatrix},$$

$$W = \begin{pmatrix} 0 & 1 & -\xi(z) \\ -c(z) & \left(-\frac{\beta_{,\bar{z}}}{\beta(z)} + a(z)\right) & -\xi(z)\left(-\frac{\beta_{,\bar{z}}}{\beta(z)} + a(z)\right) + \left(\xi_{,\bar{z}}(z) - b(z)\right) \\ 0 & 0 & 0 \end{pmatrix}.$$

The source vector $\vec{j}$ is given by $\vec{j} = (0, e^{\kappa p}\Sigma(z,p), 0)^T$.

In order to use the standard theorem which proves existence and uniqueness of solutions to systems sych as (4.13), we require X and W to be smooth, matrix-valued bounded functions of $\bar{z}$ on $[0,\infty)$, such that X is symmetric with real, distinct eigenvalues. The matrix X given above is not symmetric, however, it is easy to check that it is symmetrizable, and therefore a first order symmetric hyperbolic form of (4.5) does exist. The matrix which symmetrizes X is

$$N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\nu}(z - e^{\nu/2}(1+z^2e^{-\nu})^{1/2}) & e^{-\nu}(z + e^{\nu/2}(1+z^2e^{-\nu})^{1/2}) \\ 0 & 1 & 1 \end{pmatrix},$$

so that $\tilde{X} = N^{-1}XN$ is a symmetric matrix. $\tilde{X}(z)$ and $\tilde{W}(z) = N^{-1}WN$ are given by

$$\tilde{X} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & x_+(z) & 0 \\ 0 & 0 & x_-(z) \end{pmatrix},$$

where

$$x_\pm(z) = \frac{\pm 4\gamma(z) \mp \xi(z)(\xi(z) + \sqrt{-4\gamma(z) + \xi^2(z)})}{2\sqrt{-4\gamma(z) + \xi^2(z)}},$$

and

$$\tilde{W} = \begin{pmatrix} 0 & y_1^+(z) & y_1^-(z) \\ \frac{c(z)}{\sqrt{-4\gamma(z)+\xi^2(z)}} & y_2^+(z) & y_2^-(z) \\ -\frac{c(z)}{\sqrt{-4\gamma(z)+\xi^2(z)}} & y_3^+(z) & y_3^-(z) \end{pmatrix}. \tag{A.1}$$

The components of the $\tilde{W}$ matrix are given by

$$y_1^\pm(z) = \frac{1}{2}\left(-\xi \mp \sqrt{-4\gamma(z) + \xi^2(z)}\right),$$

$$y_2^\pm(z) = \frac{\zeta(z)}{2\beta(z)\sqrt{-4\gamma(z) + \xi^2(z)}},$$

$$y_3^\pm(z) = \frac{w(z)}{2\sqrt{-4\gamma(z) + \xi(z)^2}},$$

where

$$\zeta(z) = 3\xi(z) \pm \sqrt{-4\gamma(z) + \xi^2(z)}\dot{\beta}(z) + \beta(z)(2b(z) - 3a(z)\xi(z)) \\ + \beta(z)(a(z)\sqrt{-4\gamma(z) + \xi^2(z)} - 2\dot{\xi}(z)),$$

and

$$w(z) = (\xi(z) \mp \sqrt{-4\gamma(z) + \xi^2(z)}) \left(a(z) - \dot{\beta}(z)/\beta(z)\right)$$
$$-2b(z) - 2\xi(z)\left(-a(z) + \dot{\beta}(z)/\beta(z)\right) + 2\dot{\xi}(z).$$

The symmetric hyperbolic form of (4.5) is given by

$$\vec{\Psi},_{\bar{z}} = \tilde{X}\vec{\Psi},_p + (N_{\bar{z}}^{-1}N + \tilde{Y})\vec{\Psi} + \vec{j}' \tag{A.2}$$

where $\vec{\Psi} := N^{-1}\vec{\Phi}$, and $\vec{j}' = N^{-1}\vec{j}$ is given by

$$\vec{j}' = \begin{pmatrix} 0 \\ -\frac{1}{2}e^{\nu/2}e^{\kappa p}\Sigma \\ \frac{1}{2}e^{\nu/2}e^{\kappa p}\Sigma \end{pmatrix}.$$

A.2 Proof of Lemmas 4.5.1, 4.5.2 and 4.5.3

In this appendix we provide the proofs of Lemmas 4.5.1, 4.5.2 and 4.5.3 which were omitted in the main text.

Proof of Lemma 4.5.1: That $A(u, v)$ is bounded follows immediately from the results in Section 4.4. To bound $A,_u(u, v)$ and $A,_v(u, v)$, we write them in terms of $A,_z$ and $A,_p$. We find that

$$A,_u(u,v) = \frac{f_+(z)}{u}\left(\frac{f_-(z)}{f_-(z) - f_+(z)}\right)\left[\frac{\partial A}{\partial z} - \frac{1}{f_-(z)}\frac{\partial A}{\partial p}\right],$$
$$A,_v(u,v) = \frac{1}{v}\left(\frac{f_-(z)}{f_+(z) - f_-(z)}\right)\left[f_+(z)\frac{\partial A}{\partial z} - \frac{\partial A}{\partial p}\right]. \tag{A.3}$$

We note that by using (2.10), one can show that $\frac{f_+(z)}{u}$ tends to a finite value as $z \to z_c$. Then since $A,_z$ and $A,_p$ can be bounded by *a priori* initial data (see (4.23) and (4.25)), it follows that $A,_u(u, v)$ can be bounded by similar *a priori* terms. By an exactly similar argument, we can show that $A,_v$ is bounded. □

Proof of Lemma 4.5.2: We can write (4.32) as $A_{,uv} = F(u,v)$ and by noting the form of the coefficients (4.33) and that $A_{,u}$ and $A_{,v}$ are bounded, it follows that $A_{,uv}$ is bounded in the approach to the Cauchy horizon.

To deal with $A_{,vv}$, we first write (A.3) as $A_{,v} = \frac{H(z,p)}{v}$, where

$$H(z,p) = \frac{f_-}{f_+ - f_-}(f_+ A_{,z} - A_{,p}).$$

Taking a derivative of $A_{,v}$ with respect to v and converting to (z,p) coordinates produces a set of terms which depend on H, $H_{,p}$ and $H_{,z}$,

$$A_{,vv} = -\frac{1}{v^2}H(z,p) + \frac{1}{v^2}\left(f_-\frac{\partial H}{\partial z} + \frac{\partial H}{\partial p}\right). \tag{A.4}$$

The partial derivatives of $H(z,p)$ with respect to z and p are given by

$$\frac{\partial H}{\partial p} = \frac{f_-}{f_+ - f_-}(f_+ A_{,zp} - A_{,pp}),$$

and

$$\frac{\partial H}{\partial z} = \left(\frac{f_-}{f_+ - f_-}\right)_{,z}(f_+ A_{,z} - A_{,p}) + \frac{f_-}{f_+ - f_-}(f_{+,z} A_{,z} + f_+ A_{,zz} - A_{,zp}). \tag{A.5}$$

Now in (A.4), the terms involving H and $H_{,p}$ remain finite in the approach to the Cauchy horizon. This follows since these terms involve $A_{,z}$, $A_{,p}$, $A_{,pp}$ and $A_{,zp}$, which Theorem 4.4.5 and Lemma 4.4.6 show to be bounded by *a priori* terms (recall that we can take p-derivatives in these results to show that $A_{,p}$, $A_{,pp}$ and $A_{,zp}$ are bounded).

It remains to show that the $H_{,z}$ term remains finite as the Cauchy horizon is approached. If we examine (A.5), we see that we have terms involving $A_{,z}$, $A_{,p}$, $A_{,zp}$ and $f_+ A_{,zz}$. The first three of these remain finite in the approach to the Cauchy horizon as discussed above. Now by solving (4.5) for $A_{,zz}$, we find that we can write $f_+ A_{,zz}$ as

$$f_+ A_{,zz} = -\frac{f_+}{\beta(z)}(\gamma(z)A_{,pp} + \xi(z)A_{,zp} + a(z)A_{,z} + b(z)A_{,p} + c(z)A - e^{\kappa p}\Sigma(z,p)). \tag{A.6}$$

The term $(\gamma(z)A_{,pp} + \xi(z)A_{,zp} + a(z)A_{,z} + b(z)A_{,p} + c(z)A - e^{\kappa p}\Sigma(z,p))$ remains finite in the approach to the Cauchy horizon. This follows immediately from the boundedness of $A_{,pp}$, $A_{,zp}$, $A_{,z}$, $A_{,p}$ and A, and from the boundedness of the coefficients $\gamma(z)$, $\xi(z)$, $a(z)$, $b(z)$ and $c(z)$. The source term $e^{\kappa p}\Sigma(z,p)$ is bounded everywhere, assuming a finite perturbation of the dust velocity, see (4.12).

To deal with the factor of $f_+\beta^{-1}$ in (A.6), we note that $\beta(z)$ can be written as $\beta(z) = e^{-\nu}(e^{\nu/2} - z)(e^{\nu/2} + z)$ (see 4.6), so that $f_+\beta^{-1} = e^{\nu}(e^{\nu/2} - z)^{-1}$. This remains finite as the Cauchy horizon is approached. We can therefore conclude that the term $f_+A_{,zz}$ remains finite, and indeed, is bounded by *a priori* terms. It follows that (A.5) also remains finite, and finally $A_{,vv}$, given by (A.4) also remains finite as the Cauchy horizon is approached, and is bounded by *a priori* terms inherited from the bounds on A, $A_{,z}$, $A_{,p}$, $A_{,pp}$, $A_{,zp}$ and Σ.

Finally, we must show that $A_{,uu}$ is bounded in the approach to the Cauchy horizon. This proof involves two steps. Firstly, by following a procedure similar to that given above, we can establish a bound on $uA_{,uu}$. We write $A_{,u}$ as $A_{,u} = \frac{G(z,p)}{u}$ where

$$G(z,p) = \frac{f_+f_-}{f_- - f_+}\left(A_{,z} - \frac{A_{,p}}{f_-}\right).$$

Taking a u derivative of $A_{,u}$ results in

$$A_{,uu} = -\frac{1}{u^2}G(z,p) + \frac{1}{u^2}\left(f_+\frac{\partial G}{\partial z} + \frac{\partial G}{\partial p}\right). \tag{A.7}$$

$G_{,z}$ and $G_{,p}$ are given by

$$\frac{\partial G}{\partial p} = \frac{f_+f_-}{f_- - f_+}\left(A_{,zp} - \frac{A_{,pp}}{f_-}\right), \tag{A.8}$$

and

$$\frac{\partial G}{\partial z} = \left(\frac{f_+f_-}{f_- - f_+}\right)_{,u}\left(A_{,z} - \frac{A_{,p}}{f_-}\right) + \frac{f_+f_-}{f_- - f_+}\left(A_{,zz} - \frac{A_{,zp}}{f_-} + \frac{f_{-,z}}{f_-^2}A_{,p}\right). \tag{A.9}$$

By the same argument as that given above, we can conclude that $f_+ A_{,zz}$ remains finite as the Cauchy horizon is approached, so that $G_{,z}$ and $G_{,p}$ both remain finite in this limit. If we combine (A.7), (A.8) and (A.9), we find that we can write $A_{,uu}$ as

$$A_{,uu} = \frac{f_+}{u^2}\tilde{G}(z,p),$$

where

$$\tilde{G}(z,p) = \frac{\partial G}{\partial z} + \frac{f_-}{f_- - f_+}\left(A_{,zp} - A_{,z} + \frac{A_{,p} - A_{,pp}}{f_-}\right),$$

and this term is bounded by *a priori* terms as the Cauchy horizon is approached. Now since $\frac{f_+}{u}$ tends to a finite constant as $z \to z_c$, it follows that $uA_{,uu}$ is bounded by *a priori* terms inherited from the bounds on A, $A_{,z}$, $A_{,p}$, $A_{,pp}$, $A_{,zp}$ and Σ . This result is not sufficient for our requirements in Section 4.5.2, but we can use it to establish a stronger bound on $A_{,uu}$.

We return to the wave equation, in the form $A_{,uv} = F(u,v)$. By integrating with respect to v, we find

$$A_{,u}(u,v) - A_{,u}(u,v_0) = \int_{v_0}^{v} F(u,\bar{v})d\bar{v}. \tag{A.10}$$

where $F(u,v) = (\alpha_1)^{-1}(-\alpha_2(u,v)u\,A_{,u} - \alpha_3(u,v)v\,A_{,v} - \alpha_4(u,v)A + e^{\kappa p}\Sigma(u,v))$. Now we can make a choice of v_0 such that v_0 does not intersect the compact support of the perturbation (see Figure 4.2). With this choice, $A_{,u}(u,v_0) = 0$. Then taking a u derivative of (A.10) results in

$$A_{,uu} = \int_{v_0}^{v} \frac{\partial F}{\partial u}(u,\bar{v})d\bar{v}. \tag{A.11}$$

We can write $\frac{\partial F}{\partial u}$ as

$$\frac{\partial F}{\partial u} = \frac{\alpha_{1,u}}{\alpha_1^2}\left(-\alpha_2(u,v)u\,A_{,u} - \alpha_3(u,v)v\,A_{,v} - \alpha_4(u,v)A\right) \qquad \text{(A.12)}$$
$$+ \frac{\alpha_{1,u}}{\alpha_1^2}e^{\kappa p}\Sigma(u,v) - \frac{1}{\alpha_1}\left(\alpha_{2,u}\,uA_{,u} + \alpha_2 A_{,u}\right)$$
$$- \frac{1}{\alpha_1}\left(\alpha_2 uA_{,uu} + \alpha_{3,u}\,vA_{,v} + \alpha_3 vA_{,uv} + \alpha_{4,u}\,A + \alpha_4 A_{,u} - (e^{\kappa p}\Sigma)_{,u}\right).$$

Now, if we consider (4.33), we see that we can write $\alpha_1 = \frac{\tilde{\alpha}_1}{f_+}$, where $\tilde{\alpha}_1 = 2z(\beta+\xi)f_-^{-1} + 2\gamma f_+$. Then we can show that

$$\frac{\alpha_{1,u}}{\alpha_1^2} = \frac{f_+}{u}\frac{1}{\tilde{\alpha}_1^2}(-\tilde{\alpha}_1 f_{+,z} + \tilde{\alpha}_{1,z}\,f_+), \qquad \text{(A.13)}$$

which remains finite as the Cauchy horizon is approached (since $\frac{f_+}{u}$ tends to a finite constant as $z \to z_c$, and by inspection, so do $\tilde{\alpha}_1$, $\tilde{\alpha}_{1,z}$ and $f_{+,z}$). From (A.13), and the results of Lemma 4.5.1, we can conclude that the first term in (A.12) remains finite as the Cauchy horizon is approached.

Similarly, by writing $\alpha_2 = \frac{\tilde{\alpha}_2}{f_+}$, where $\tilde{\alpha}_2 = a(z) + b(z)f_+$, we can show that

$$\frac{\alpha_{2,u}}{\alpha_1}u = \frac{1}{\tilde{\alpha}_1}(-\tilde{\alpha}_2 f_{+,z} + f_+\tilde{\alpha}_{2,z}), \qquad \text{(A.14)}$$

and

$$\frac{\alpha_2}{\alpha_1} = \frac{\tilde{\alpha}_2}{\tilde{\alpha}_1}, \qquad \text{(A.15)}$$

which remain bounded as the Cauchy horizon is approached, since by inspection, $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, $f_{+,z}$ and $\tilde{\alpha}_{2,z}$ tend to finite constants as the Cauchy horizon is approached. Now the terms $\alpha_{3,u}\,vA_{,v} + \alpha_{4,u}\,A + \alpha_4 A_{,u} - (e^{\kappa p}\Sigma)_{,u}$ which appear in the second term in (A.12) are bounded as the Cauchy horizon is approached. This follows from Lemma 4.5.1 and from the boundedness of the coefficients $\alpha_{3,u}$, $\alpha_{4,u}$, α_4 (which can be easily seen by converting to (z,p) coordinates and using 4.33). $(e^{\kappa p}\Sigma)_{,u}$ reduces to z and p derivatives of Σ when we switch to (z,p) coordinates, and these are bounded in the approach to the Cauchy horizon.

The remaining terms from (A.12) which must be dealt with are $\alpha_{2,u}\,uA_{,u}\,\alpha_1^{-1}$,

$\alpha_2 A_{,u}\, \alpha_1^{-1}$, $\alpha_2 u A_{,uu}\, \alpha_1^{-1}$, $\alpha_3 v A_{,uv}\, \alpha_1^{-1}$. Now combining (A.14), (A.15) and the results of Lemma 4.5.1, as well as our result for the boundedness of $uA_{,uu}$, we see that these terms remain bounded as the Cauchy horizon is approached. We can therefore conclude that $\frac{\partial F}{\partial u}$ remains finite, and therefore, from (A.11), $A_{,uu}$ also remains bounded as the Cauchy horizon is approached. In particular, $A_{,uu}$ is bounded by *a priori* terms arising from the bounds on A, $A_{,z}$, $A_{,p}$, $A_{,pp}$, $A_{,zp}$ and Σ. □

Proof of Lemma 4.5.3: That A itself is bounded with this choice of initial data follows immediately from the second part of Theorem 4.4.8. To show the boundedness of the derivatives, we follow a procedure similar to that used to prove Theorem 4.4.8.

The space $C_0^\infty(\mathbb{R},\mathbb{R})$ is dense in each of $\mathbb{H}^{3,2}(\mathbb{R},\mathbb{R})$ and $\mathbb{H}^{1,2}(\mathbb{R},\mathbb{R})$. It follows that there exist sequences $\{f_{(m)}\}_{m=0}^\infty$, $\{j_{(m)}\}_{m=0}^\infty$ and $\{h_{(m)}\}_{m=0}^\infty$, with $f_{(m)}$, $j_{(m)}$ and $h_{(m)} \in C_0^\infty(\mathbb{R},\mathbb{R})$, such that $f_{(m)} \to f$, $j_{(m)} \to j$ and $h_{(m)} \to h$ as $m \to \infty$, with convergence in the $\mathbb{H}^{3,2}(\mathbb{R},\mathbb{R})$ and $\mathbb{H}^{1,2}(\mathbb{R},\mathbb{R})$ norms respectively.

Then for all $m \geq 0$, we take $f_{(m)}$, $j_{(m)}$ and $h_{(m)}$ as initial data for A and $A_{,z}$, and apply Theorems 4.3.1, 4.4.4 and 4.4.5 to find a sequence of solutions $A_{(m)}$ which obeys at each m the *a priori* bounds from Theorem 4.4.5 and Lemma 4.4.6. By taking p-derivatives in these results as required, we can establish similar bounds on $A_{(m),p}$, $A_{(m),pp}$ and $A_{(m),zp}$. We can then take the $m \to \infty$ limit and will find a solution $A \in C([z_c, z], \mathbb{H}^{3,2}(\mathbb{R},\mathbb{R}))$.

Finally, we apply Lemmas 4.5.1 and 4.5.2 to establish bounds on the required u and v derivatives of A. These terms will be bounded by *a priori* terms inherited from the bounds on A, $A_{,z}$, $A_{,p}$, $A_{,pp}$, $A_{,zp}$ and Σ. The order of derivatives of A and Σ required for these bounds dictates the choice of Sobolev spaces for the initial data specified in this lemma. □

Appendix B

Even Parity Perturbations

B.1 Equations of Motion

We list here the various matrix coefficients and source terms omitted in Sections 5.2, 5.3 and 6.1. We neglect to list those terms whose exact form is not important. We note in what follows that $S = S(z) = (1 + az)^{2/3}$ is the radial function, $q = q(z) = \lambda/4\pi(-z\dot{S} + S)S^2$ is the density function and $\cdot = \frac{\partial}{\partial z}$. L is the angular number (see Chapter 3).

The five dimensional system takes the form

$$\frac{\partial \vec{u}}{\partial z} + \tilde{A}(z)\frac{\partial \vec{u}}{\partial p} + \tilde{C}(z)\vec{u} = \vec{\sigma}(z,p)$$

where the matrix coefficient $\tilde{A}(z)$ is given in Section 5.2. $\tilde{C}(z)$ is given by

$$\tilde{C} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{21} & c_{22} & 0 & c_{24} & c_{25} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} \\ c_{41} & 0 & c_{43} & c_{44} & c_{45} \\ c_{51} & 0 & c_{53} & c_{54} & c_{55} \end{pmatrix},$$

where

$$c_{11} = \frac{-3z(-1+\dot{S})\dot{S}^2 - S^2\ddot{S} + S(3\dot{S}^2 - z\ddot{S} + \dot{S}(-3+2z\ddot{S}))}{S(-1+\dot{S})(S-z\dot{S})},$$

$$c_{12} = -\frac{-1+\dot{S}}{2S}, \qquad c_{13} = \frac{-1+\dot{S}}{2S},$$

$$c_{14} = \frac{-1+\dot{S}}{2S}, \qquad c_{15} = \frac{-1+\dot{S}}{2S},$$

$$c_{21} = -\frac{8\pi qS}{-1+\dot{S}}, \qquad c_{22} = -\frac{\dot{q}}{q},$$

$$c_{24} = -\frac{4\pi qS}{1+\dot{S}}, \qquad c_{25} = -\frac{4\pi qS}{-1+\dot{S}},$$

$$c_{31} = \frac{n_1}{zS(-1+\dot{S})(S-z\dot{S})^2},$$

$$n_1 = 2(2z^3\dot{S}^4 - 2z^2S\dot{S}^2(3\dot{S}+z\ddot{S}) + S^3(-2\dot{S}+z^2S^{(3)}) + zS^2(6\dot{S}^2 + z^2\ddot{S}^2 + z\dot{S}(2\ddot{S} - zS^{(3)}))),$$

$$c_{32} = -\frac{2}{z} - \frac{\dot{q}}{q}, \qquad c_{33} = \frac{1}{z} + \frac{2\dot{S}}{S},$$

$$c_{34} = \frac{n_2}{2zS(1+\dot{S})},$$

$$n_2 = z\dot{S}(2 - L - L^2 + 2\dot{S} + 2\dot{S}^2) - 4S^2\ddot{S} + S(-2 + L + L^2 - 2\dot{S}^2 - 2z\ddot{S} + \dot{S}(-2+4z\ddot{S})),$$

$$c_{35} = \frac{n_3}{2zS(-1+\dot{S})},$$

$$n_3 = z\dot{S}(-2 + L + L^2 + 2\dot{S} - 2\dot{S}^2) + 4S^2\ddot{S} - S(-2 + L + L^2 - 2\dot{S}^2 + 2z\ddot{S} + \dot{S}(2+4z\ddot{S})),$$

$$c_{41} = \frac{4(1+\dot{S})(-z\dot{S}^2 + S(\dot{S} + z\ddot{S}))}{S(-1+\dot{S})(z - S + z\dot{S})},$$
$$c_{43} = \frac{(1+\dot{S})(S - z\dot{S})}{S(z - S + z\dot{S})},$$
$$c_{44} = \frac{3 + (z+S)\ddot{S} + \dot{S}(3 + z\ddot{S})}{(1+\dot{S})(z - S + z\dot{S})},$$
$$c_{45} = \frac{(1+\dot{S})(S - z\dot{S})}{S(z - S + z\dot{S})},$$

$$c_{51} = -\frac{4(-z\dot{S}^2 + S(\dot{S} + z\ddot{S}))}{S(z + S - z\dot{S})},$$
$$c_{53} = -\frac{(-1+\dot{S})(S - z\dot{S})}{S(z + S - z\dot{S})},$$
$$c_{54} = -\frac{(-1+\dot{S})(S - z\dot{S})}{S(z + S - z\dot{S})},$$
$$c_{55} = -\frac{3 + (-z+S)\ddot{S} + \dot{S}(-3 + z\ddot{S})}{(-1+\dot{S})(z + S - z\dot{S})}.$$

The five dimensional source term is given by $\vec{\Sigma}_5 = \vec{f}(t)g(p)$ where $\vec{f}(t) = (0, 0, f_1(z), f_2(z), f_3(z))^T$ and

$$f_1(z) = -\frac{16\pi q(-S + z\dot{S})}{zS}, \qquad f_2(z) = -\frac{8\pi q(1+\dot{S})(S - z\dot{S})}{S(z - S + z\dot{S})},$$
$$f_3(z) = \frac{8\pi q(-1+\dot{S})(S - z\dot{S})}{S[z](z + S - z\dot{S})}.$$

In Sections 5.3 and 6.1 we used the system in the form

$$t\frac{\partial \vec{u}}{\partial t} + A(t)\frac{\partial \vec{u}}{\partial p} + C(t)\vec{u} = \vec{\Sigma}(t, p).$$

Here $A(t) = t\tilde{A}(z)$, $C(t) = t\tilde{C}(z)$ and $\vec{\Sigma}(t, p) = t\vec{\Sigma}_5(z, p)$.

The coefficients appearing in (5.11) are given by

$$
\begin{aligned}
g_1(z) &= \frac{m_1(z)}{(-1+\dot{S})(-S+z\dot{S})},\\
g_2(z) &= 2S(2S^2+2z^2\dot{S}^2+zS(-4\dot{S}+z\ddot{S})),\\
g_3(z) &= -2S^2(S+z(-\dot{S}+z\ddot{S})),\\
g_4(z) &= -\frac{m_2(z)}{1+\dot{S}}, & g_5(z) &= \frac{m_3(z)}{1-\dot{S}},\\
g_6(z) &= 2S^2(S-z\dot{S}), & g_7(z) &= 32\pi q S(S-z\dot{S})^2,
\end{aligned}
$$

where

$$
\begin{aligned}
m_1(z) &= 4S(-6z^2S\dot{S}^3+2z^3\dot{S}^4+4\pi zqS^2(S-z\dot{S})^2+S^3(-2\dot{S}+z\ddot{S})\\
&\quad +zS^2(6\dot{S}^2-z\dot{S}\ddot{S}+z^2\ddot{S}^2))\\
m_2(z) &= S(z^2\dot{S}(1+\dot{S})(-2+L+L^2-2\dot{S}-2\dot{S}^2)+S^3(8\pi zq-4\ddot{S})\\
&\quad -zS(-2+L+L^2-4\dot{S}^3-2z\ddot{S}+2\dot{S}(-3+L+L^2+z\ddot{S})+\dot{S}^2(-6+4z\ddot{S}))\\
&\quad +S^2(-2+L+L^2-2\dot{S}^2+4z\ddot{S}+\dot{S}(-2-8\pi z^2q+8z\ddot{S}))),\\
m_3(z) &= S(-z^2(-1+\dot{S})\dot{S}(-2+L+L^2+2\dot{S}-2\dot{S}^2)+4S^3(2\pi zq+\ddot{S})\\
&\quad -zS(-2+L+L^2+4\dot{S}^3+2z\ddot{S}-2\dot{S}(-3+L+L^2-z\ddot{S})-2\dot{S}^2(3+2z\ddot{S}))\\
&\quad -S^2(-2+L+L^2-2\dot{S}^2-4z\ddot{S}+\dot{S}(2+8\pi z^2q+8z\ddot{S}))).
\end{aligned}
$$

The coefficient matrix $E(z)$ appearing in (5.12) is given by

$$
E(z) = \begin{pmatrix}
0 & 1 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & \frac{3(1+az)^{1/3}}{3+az-3z(1+az)^{1/3}} & 0\\
0 & 0 & 0 & -\frac{3(1+az)^{1/3}}{3+az+3z(1+az)^{1/3}}
\end{pmatrix},
$$

so we can explicitly see that this system is not symmetric hyperbolic.

Finally, we present the similarity matrix S (which appears in Section 6.1.1 and is used in the proof of Theorem 6.1.1), a constant matrix which transforms the zero order term of the Taylor expansion of $C(t)$, $C(t=0)$, into $\bar{C}_0$. It is given by

$$S = \begin{pmatrix} s_1 & s_2 & s_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$s_1 = \frac{S(z_c)(3 + (-z_c + S(z_c))\ddot{S}(z_c) + \dot{S}(z_c)(-3 + z_c\ddot{S}(z_c)))}{4(-1 + \dot{S}(z_c))(z_c\dot{S}(z_c)^2 - S(z_c)(\dot{S}(z_c) + z\ddot{S}(z_c)))},$$
$$s_2 = \frac{(-1 + \dot{S}(z_c))(-S(z_c) + z_c\dot{S}(z_c))}{4(-z_c\dot{S}(z_c)^2 + S(z_c)(\dot{S}(z_c) + z_c\ddot{S}(z_c)))}.$$

B.2 The Behaviour of c

In the statement of Theorem 5.3.1, we looked that the Jordan canonical form of the zero order term in the Taylor expansion of the matrix $C(t)$. We claimed that the only non-zero eigenvalue of this matrix was in the range $c \in (3, +\infty)$, for $a \in (0, a^*)$. Here we prove this claim.

In the previous section, we presented the coefficients of the equation of motion for $\vec{u}$. Recall that the coefficient of $\vec{u}$ was given as $\tilde{C}(z)$, where in terms of the notation of (5.13), $C(t) = t\tilde{C}(z)$. After we put $\tilde{C}(z)$ in Jordan canonical form, we find that the only non-zero term is the $(5, 5)$ entry, which is given by

$$\tilde{c}_{55} = \frac{3 + (S - z)\ddot{S} + \dot{S}(-3 + z\ddot{S})}{(1 - \dot{S})(z + S - z\dot{S})}. \tag{B.1}$$

Now define $h(z) := z + S - z\dot{S}$. On the Cauchy horizon, $h(z_c) = 0$ by definition. We can use this to Taylor expand the numerator of (B.1) and to simplify the zero order term. We can also Taylor expand the factor of $h(z)$ which appears in the denominator so that $h(z) = \dot{h}(z_c)(z - z_c) + O((z - z_c)^2) =$

$(1 - z_c\ddot{S}(z_c)) + O((z - z_c)^2)$. Inserting these expansions into (B.1) produces

$$\tilde{c}_{55} = \left(2 + \frac{1}{1 - z_c\ddot{S}(z_c)}\right)\frac{1}{z - z_c} + O(1), \tag{B.2}$$

as $z \to z_c$. So we must determine the value of $\dot{h}(z_c) = 1 - z_c\ddot{S}(z_c) = 1 + \frac{2}{9}a^2 z_c(a)(1 + az_c(a))^{-4/3}$, where we use (2.5) to prove the last equality and we write $z_c = z_c(a)$ to emphasise that the location of the Cauchy horizon depends on the value of a, the nakedness parameter.

We first analyse how z_c depends on a, before using this information to determine the behaviour of $\dot{h}(z_c)$. We define $\lambda(z) = (1 + az)h(z)^3$, so that $\lambda(z) = z^3(1 + az) + (1 + \frac{a}{3}z)^3$. The Cauchy horizon corresponds to the first negative root of the quartic equation $\lambda(z) = 0$. This root must lie in the interval $z \in (-\frac{1}{a}, 0)$, since $z = -\frac{1}{a}$ corresponds to the singularity (see (2.5)). We know that a root exists for $a \in (0, a^*)$, where $a = a^*$ corresponds to a double root of λ, where $\lambda(z_c(a^*)) = \dot{\lambda}(z_c(a^*)) = 0$. Let $z^* := z_c(a^*)$. We can easily show (using $\lambda = 0$) that

$$(1 + \frac{a}{3}z)\dot{\lambda}\Big|_{\lambda=0} = z^2(3 + 4az + \frac{1}{3}a^2z^2),$$

which implies that a^*z^* satisfies the quadratic

$$(a^*)^2(z^*)^2 + 12a^*z^* + 9 = 0. \tag{B.3}$$

By solving this for a^*z^* and picking the larger root, we have

$$z^* = \frac{1}{a^*}(-6 + 3\sqrt{3}).$$

Next, we consider the dependence of z_c on a. If we differentiate the condition $\lambda = 0$ with respect to a, multiply by $(1 + az)$ and use $\lambda = 0$ to simplify, we find that

$$(a^2z_c^2 + 12az_c + 9)\frac{dz_c}{da} = 2az_c^3. \tag{B.4}$$

We note that the coefficient of dz_c/da vanishes at $a = a^*$ (see B.3) and is positive in the interval $a \in (0, a^*)$ (we can see this by calculating its roots and noting that for the range of allowable a, we are always above the larger root). Now since $z_c < 0$, this implies that $dz_c/da < 0$ for all $a \in (0, a^*)$ and

$$\lim_{a \to a^*} \frac{dz_c}{da} = -\infty.$$

So to summarise, we know that $z_c(0) = -1$ (see the definition of λ), $z_c(a^*) = z^*$ and z_c is monotonically decreasing from -1 down to z^* as a increases from 0 to a^*.

We now determine the range of $\dot{h}(z_c)$. Define $u(a) = \dot{h}(z_c(a))$. Then $u(0) = 1$ (since $u(a) = \dot{h}(z_c) = 1 + \frac{2}{9}a^2 z_c(a)(1 + az_c(a))^{-4/3}$), and by definition $u(a^*) = 0$ (recall that $\lambda(a) = (1 + az)h(a)^3$). A straightforward calculation (using (B.4)) shows that

$$\frac{du}{da} = \frac{4}{9} a z_c (1 + az_c)^{-4/3} \frac{(3 + 2az_c)}{3 + 4az_c + \frac{1}{3}a^2 z_c^2}.$$

Since $a > 0$, $z_c < 0$, $1 + az_c > 0$ and $3 + 4az_c + \frac{1}{3}a^2 z_c^2 > 0$ on $a \in (0, a^*)$, it follows that $du/da < 0$ for $a \in (0, a^*)$. So $u(a) = \dot{h}(z_c(a)) \in [0, 1]$ and u decreases monotonically from $u = 1$ at $a = 0$ to $u = 0$ at $a = a^*$. It follows that $\frac{1}{u(a)} \in (1, \infty)$ for $a \in (0, a^*)$ with

$$\lim_{a \to 0^+} \frac{1}{u(a)} = 1, \qquad \lim_{a \to (a^*)^-} \frac{1}{u(a)} = +\infty.$$

We note that (B.2) can be written as

$$\tilde{c}_{55} = \left(2 + \frac{1}{u(a)}\right) \frac{1}{z - z_c} + O(1),$$

and if we note that the $(5, 5)$ entry in the Jordan canonical form of the zero order term in the Taylor expansion of $C(t)$ is $c := c_{55} = t\tilde{c}_{55}$, where $t = z - z_c$, then we can conclude that $c = (2 + \frac{1}{u(a)}) \in (3, +\infty)$ with

$$\lim_{a \to 0^+} c = 3, \qquad \lim_{a \to (a^*)^-} c = +\infty.$$

Bibliography

[1] Adams R.A. and Fournier J.J.F., 2003. *Sobolev Spaces*. Oxford: Elsevier Science Ltd.

[2] Cahill M.E. and Taub A.H. 1971. Spherically symmetric similarity solutions of the Einstein field equations for a perfect fluid. *Commun. Math. Phys.* **21** 1

[3] Carr B.J. and Coley A.A. 2000. Complete classification of spherically symmetric self-similar perfect fluid solutions. *Phys. Rev.* **D62** 044023

[4] Carr B.J. and Coley A.A. 2005. The similarity hypothesis in General Relativity. *Gen. Rel. Grav.* **37** 2165

[5] Carr B.J., Coley A.A., Goliath M., Nilsson U.S. and Uggla C. 2001. The state space and physical interpretation of self-similar spherically symmetric perfect-fluid models. *Class. Quant. Grav.* **18** 303

[6] Chandrasekhar S. and Hartle J. 1982. On crossing the Cauchy horizon of a Reissner-Nordstrom black hole. *Proc. Roy. Soc. Lond.* **A384** 301

[7] Choptuik M.W. 1993. Universality and scaling in gravitational collapse of a massless scalar field. *Phys. Rev. Lett.* **70** 9

[8] Choquet-Bruhat Y. and Geroch R. 1969. Global aspects of the Cauchy problem in General Relativity. *Commun. Math. Phys.* **14** 329

[9] Christodoulou D. 1984. Violation of cosmic censorship in the gravitational collapse of a dust cloud. *Commun. Math. Phys.* **93** 171

[10] Christodoulou D. 1994. Examples of naked singularity formation in the gravitational collapse of a scalar field. *Ann. Math.* **140** 607

[11] Christodoulou D. 1999. The instability of naked singularities in the gravitational collapse of a scalar field. *Ann. Math.* **149** 183

[12] Chruściel P. T. 2005. Recent results in mathematical relativity. In: Florides P, Nolan B and Ottewill A. ed., *General Relativity and Gravitation: Proceedings of GR 17.* Singapore: World Scientific

[13] Coddington E.A. and Levinson N., 1955. *Theory of Ordinary Differential Equations.* Florida: Krieger

[14] Dafermos M. and Rodnianski I. 2008. Lectures on black holes and linear waves. arXiv: 0811.0354v1 [gr-qc]

[15] Duffy E.M. and Nolan B.C. 2011. Odd parity perturbations of the self-similar LTB spacetime. *Class. Quant. Grav.* **28** 105020

[16] Eardley D.M. and Smarr L. 1979. Time functions in numerical relativity: Marginally bound dust collapse. *Phys. Rev.* **D19** 2239

[17] Evans L. C., 1998. *Partial Differential Equations.* Rhode Island: American Mathematical Society

[18] Frolov A.V. 1999. Critical collapse beyond spherical symmetry: General perturbations of the Roberts solution. *Phys. Rev.* **D59** 104011

[19] Gerlach U.H. and Sengupta U.K. 1979. Gauge invariant perturbations on most general spherically symmetric space-times. *Phys. Rev.* **D19** 2268

[20] Gundlach C. 2002. Critical gravitational collapse of a perfect fluid: Non-spherical perturbations. *Phys. Rev.* **D65** 084021

[21] Gundlach C. 2007. Critical phenomena in gravitational collapse. *Living Rev. Relativity* **10**, 5. [Online Article] cited 24/03/2011, http://www.livingreviews.org/lrr-2007-5

[22] Gyulchev G.N. and Yazadjiev S.S. 2008. Gravitational lensing by rotating naked singularities. *Phys. Rev.* **D78** 083004

[23] Harada T. and Maeda H. 2001. Convergence to a self-similar solution in general relativistic gravitational collapse. *Phys. Rev.* **D63** 084022

[24] Horowitz G.T. 2003. Creating naked singularities and negative energy. *Proceedings of Nobel Symposium* **127**

[25] Iguchi H., Nakao K. and Harada T. 1998. Gravitational waves around a naked singularity: Odd parity perturbation of Lemaître-Tolman-Bondi space-time. *Phys. Rev.* **D57** 7262

[26] Iguchi H., Harada T. and Nakao K. 2000. Gravitational radiation from a naked singularity II: Even parity perturbation. *Prog. Theor. Phys.* **103** 53

[27] Harada T., Iguchi H. and Nakao K. 2002. Physical processes in naked singularity formation. *Prog. Theor. Phys* **107** 449

[28] Janich K., 1994. *Linear Algebra.* New York: Springer-Verlag

[29] Joshi P.S. and Dwivedi I.H. 1993. Naked singularities in spherically symmetric inhomogeneous Tolman-Bondi dust cloud collapse. *Phys. Rev.* **D47** 5357

[30] Joshi P.S. 2002. Cosmic censorship: A current perspective. *Mod. Phys. Lett.* **A17** 1067

[31] Kurita Y. and Nakao K. 2006. Naked singularity resolution in cylindrical collapse. *Phys. Rev.* **D73** 064022

[32] Lehner L. and Pretorius P. 2010. Black strings, low viscosity fluids and violation of cosmic censorship. *Phys. Rev. Lett.* **105** 101102

[33] Lieb E.H. and Loss M., 1997. *Analysis.* Rhode Island: American Mathematical Society

[34] Martín-García J.M. and Gundlach C. 1999. All nonspherical perturbations of the Choptuik spacetime decay. *Phys. Rev.* **D59** 064031

[35] McOwen R.C., 2003. *Partial Differential Equations: Methods and Applications*. New Jersey: Prentice Hall

[36] Moncrief V. and Eardley D. M. 1981. The global existence problem and cosmic censorship in General Relativity. *Gen. Rel. Grav.* **13** 887

[37] Nakao K., Kurita Y., Morisawa Y. and Harada T. 2007. Relativistic gravitational collapse of a cylindrical shell of dust. *Prog. Theor. Phys.* **117** 75

[38] Nolan B.C. and Waters T.J. 2002. Cauchy horizon stability in self-similar collapse: Scalar radiation. *Phys. Rev.* **D66** 104012

[39] Nolan B.C. 2004. Physical interpretation of gauge invariant perturbations of spherically symmetric space-times. *Phys. Rev.* **D70** 044004

[40] Nolan B.C. and Waters T.J. 2005. Even perturbations of the self-similar Vaidya spacetime. *Phys. Rev.* **D71** 104030

[41] Nolan B.C. 2006. Bounds for scalar waves on self-similar naked-singularity backgrounds. *Class. Quant. Grav.* **23** 4523

[42] Nolan B.C. 2007. Odd parity perturbations of the self-similar Vaidya spacetime. *Class. Quant. Grav.* **24** 177

[43] Nolan B.C. and Waters T.J. 2009. Gauge invariant perturbations of self-similar Lemaître-Tolman-Bondi spacetime: Even parity modes with $l \geq 2$. *Phys. Rev.* **D79** 084002

[44] Nolan B.C. 2010. Even parity perturbations of the self-similar Vaidya spacetime, Unpublished.

[45] Ori P. and Piran T. 1990. Naked singularities and other features of self-similar general-relativistic gravitational collapse. *Phys. Rev.* **D42** 1068

[46] Penrose R. 1969. Gravitational collapse: the role of general relativity. *Revistas de Nuovo Cimento* **1** 252

[47] Ringström H., 2009. *The Cauchy Problem in General Relativity.* Zürich: EMS Publishing House

[48] Ringström H. 2010. Cosmic censorship for Gowdy spacetimes *Living Rev. Relativity* **13** 2. [Online Article] cited 17/01/2011. http://www.livingreviews.org/lrr-2010-2

[49] Salas S.L. and Hillie E., 1995. *Salas and Hillie's Calculus: One and Several Variables*. New York: Wiley

[50] Schutz B.F., 1985. *A First Course in General Relativity.* Cambridge: Cambridge University Press

[51] Shapiro S.L. and Teukolsky S.A. 1991. Formation of naked singularities: The violation of cosmic censorship. *Phys. Rev. Lett.* **66** 994

[52] Stewart J., 1993. *Advanced General Relativity.* Cambridge: Cambridge University Press

[53] Szekeres P. 1965. The gravitational compass. *J. Math. Phys.* **6** 1387

[54] Thomas G. B., Weir M.D. and Hass J. R., 2010. *Thomas' Calculus.* Boston: Pearson

[55] Wald R.M. 1979. Note on the stability of the Schwarzschild metric. *J. Math. Phys* **20** 1056

[56] Wald R.M., 1984. *General Relativity.* Chicago: Univ. Chicago Press

[57] Wald R.M. 1997. Gravitational collapse and cosmic censorship. arXiv:gr-qc/9710068v3

[58] Wasow W., 1965. *Asymptotic Expansions for Ordinary Differential Equations.* New York: Dover