On the Asymptotic Behaviour of Deterministic and Stochastic Volterra Integro–differential Equations

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Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Doctor of Philosophy in Mathematics is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

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Finally, I would like to dedicate this thesis to the memory of my Dad, Austin.

Contents

Abstract

This thesis examines a question of stability in stochastic and deterministic systems with memory, and involves studying the asymptotic properties of Volterra integro–differential equations. The type of stability that has been established for this class of equations is important in a variety of real–world problems which involve feedback from the past, and are subject to external random forces. These include modelling endemic diseases, and more particularly the modelling of inefficient financial markets.

The theme of the thesis is to subject a dynamical system with memory to increasingly strong and unpredictable external noise. Firstly, a fundamental deterministic Volterra equation is considered. Necessary and sufficient conditions for the solution to approach a nontrivial limit are known. A strengthened version of these conditions is shown to be necessary and sufficient for exponential convergence to a nontrivial limit.

Next, a Volterra equation with a fading stochastic perturbation is studied. Two types of stochastic convergence are considered: mean square and almost sure convergence. Conditions are found which ensure that the solution converges to a non–equilibrium random limit. Moreover, the rate at which this limit is approached is established. In the mean square case, necessary and sufficient conditions on the resolvent, kernel and noise are determined to ensure this rate of convergence. In the almost sure case, the same conditions are found to be sufficient; furthermore, it is shown that the conditions on the resolvent and the kernel are necessary. A corresponding result was also found to hold for a more general class of weakly singular kernels. As in the deterministic case, necessary and sufficient conditions for the solution to converge exponentially fast to its limit are found.

Finally, a stochastic Volterra equation with constant noise intensity is considered. This gives rise to the process analogous to Brownian motion, which has applications to mathematical finance. It can be shown that the increments of the process converge to a stationary statistical distribution, which is Gaussian distributed. The conditions under which such convergence can take place are completely characterised. In fact, a solution of a corresponding Volterra equation with infinite memory is shown to have exactly stationary increments which match the limiting distributions of the increments of solutions.

Chapter 1

Introduction

This thesis examines stochastic and deterministic systems with memory, and involves studying the asymptotic properties of Volterra integro–differential equations. More particularly this thesis examines equations which are poised between stability and instability. Mathematically, the convergence of solutions to a nontrivial and nonequilibrium limit is examined.

The type of stability that has been established for this class of equations is important in a variety of real–world problems which involve feedback from the past, and are subject to external random forces. One of these real–world problems concerns the modelling of population growth or the spread of endemic disease. Various authors have considered these problems (cf. [14, 18]). The advantage of using a Volterra equation as a model is that while the birth rate is a linear function of the population, the convolution term allows death to occur at any time.

More particularly, stochastic Volterra equations may be used to model inefficient financial markets. Surveys of financial markets reveal that a persistently high proportion of traders use past prices as a guide to making investment decisions (cf. [21, 32]). Such feedback trading strategies are thought to be responsible for speculative asset bubbles and crashes: this feedback behaviour is absent from standard non–delay models. It is therefore plausible to postulate that aggregate demand is a functional of past prices: in which case, price dynamics could be modelled by stochastic Volterra equations, (cf. [19]). Further, financial processes such as stock returns and interest rates do not tend to point equilibria but rather to stationary distributions, which are a more general type of equilibrium. The solutions of stochastic Volterra equations exhibit this behaviour under certain conditions.

The theme of the thesis is to subject a dynamical system with memory to increasingly strong and unpredictable external noise. Initially the fundamental Volterra equation known as the resolvent equation is considered. Next a deterministic perturbation is applied to the equation and the resulting stability of the system examined. The equation is then perturbed by a fading noise term before the most unstable type of equation is studied, namely the fundamental equation perturbed by constant noise.

To understand the behaviour of these stochastic Volterra equations, we first study an underlying deterministic equation, namely

$$R'(t) = AR(t) + \int_0^t K(t-s)R(s)\,ds, \quad t > 0,$$
$$R(0) = I,$$

where the solution R is known as the resolvent, A is a constant $n \times n$–dimensional matrix, K is $n \times n$–dimensional function–valued matrix and I is the $n \times n$–dimensional identity matrix.

The behaviour of the resolvent is important in its own right; moreover the solutions of perturbed Volterra equations may be expressed in terms the resolvent using variation of parameters. As such it is important to understand the underlying behaviour of the resolvent equation before considering more complex equations.

The asymptotic behaviour of R has long been a topic of study, and it is well known that a very natural type of asymptotic stability is associated with the solution R being integrable (which implies asymptotic convergence of the solution to zero). In this thesis however we are interested in studying equations where the solution tends to a nontrivial limit. Krisztin and Terjéki [24] studied this case and determined conditions under which the solution R of the resolvent equation converges asymptotically to a limit R_∞ which need not be trivial. In Chapter 2 we extend the result of Krisztin and Terjéki by finding necessary and sufficient conditions for the exponential convergence to R_∞. Much of the inspiration for this work stems from results by Murakami [31, 30]; he studied the speed of convergence of solutions of the resolvent equation to zero.

In the latter part of Chapter 2 we study the behaviour of the resolvent equation which has been influenced by a fading deterministic perturbation,

$$x'(t) = Ax(t) + \int_0^t K(t-s)x(s)\,ds + f(t), \quad t > 0,$$
$$x(0) = x_0,$$

where f is $n \times 1$–dimensional function–valued vector and x_0 is the initial condition. As in the resolvent case we find necessary and sufficient conditions for exponential convergence of the solution to a nontrivial limit. The examination of this perturbed equation provides an insight into the asymptotic behaviour of the more complex stochastic Volterra equations. Moreover, this perturbed equation has direct application in deterministic demographic modelling.

As a next step we perturb the Volterra equation using a fading noise term,

$$dX(t) = \left(AX(t) + \int_0^t K(t-s)X(s)\,ds\right)\,dt + \Sigma(t)dB(t), \quad t > 0,$$
$$X(0) = X_0,$$

where $\Sigma : [0, \infty) \to M_{n\times d}(\mathbb{R})$ and $B(t) = (B_1(t), \dots, B_d(t))$. Here each $B_i(t)$ is a standard Brownian motion for $i = 1\dots, d$. Two types of stochastic convergence, namely mean square and almost sure convergence, are considered. The former considers the 'average' behaviour of all sample paths of the solution, while the latter can give information on the convergence to the solution on a pathwise basis.

In Chapter 3 and 4 the stochastic analogue of results in [24] are proven: Chapter 3 considers the mean square case while Chapter 4 considers the almost sure case. In the mean square case the conditions under which convergence can take place are completely characterised. In the almost sure case sufficient conditions to ensure convergence of the solution may be found. However it has not been possible to show the necessity of each condition. In Chapter 6 we obtain the stochastic analogue to the results in Chapter 2 concerning the asymptotic speed of convergence. The results in Chapters 2, 3 and 4 have been published, see [2], [3] and [4] respectively.

By perturbing the resolvent equation by both a deterministic and stochastic perturbation,

$$dX(t) = \left(AX(t) + \int_0^t K(t-s)X(s)\,ds + f(t)\right)\,dt + \Sigma(t)dB(t), \quad t > 0,$$
$$X(0) = X_0,$$

we obtain an equation which may be used to model the dynamics of a population. In Chapter 7 we prove some theoretical results concerning the long run behaviour of this equation before using them to analyse a demographic model.

Finally we consider the resolvent equation which has been perturbed by a constant noise term,

$$dX(t) = \left(AX(t) + \int_0^t K(t-s)X(s)\,ds\right) dt + \Sigma\,dB(t), \quad t > 0.$$
$$X(t) = X_0.$$

In this case, solutions do not converge to a point equilibria: instead convergence to a stationary process may be observed. As we are interested in the asymptotic convergence of solutions it is important to examine the 'limiting' version of this equation,

$$dX(t) = \left(AX(t) + \int_{-\infty}^t K(t-s)X(s)\,ds\right) dt + \Sigma\,dB(t), \quad t > 0,$$
$$X(t) = \phi(t), \quad t \le 0,$$

where ϕ is a n–dimensional vector–valued function known as the initial function. Authors that have examined the behaviour of a stationary object satisfying a differential equation include Küchler and Mensch [25] and Riedle [36]. Due to the nature of the assumptions on the resolvent it becomes clear that the solution of this equation is not in general stationary. Instead, we investigate whether the increments of the solution might possess stationartiy properties. It is found that for a particular initial condition the increments of the solution are indeed stationary with Gaussian distribution. Furthermore, the increments of the solution of the equation with unbounded but not infinite delay converges to this stationary distribution. This analysis may be found in Chapter 8.

1.1 Mathematical Preliminaries

1.1.1 Deterministic Preliminaries

In this subsection standard deterministic notation used in the sequel is explicitly defined.

Vector Notation. Let $\mathbb{R}$ denote the set of real numbers and let $\mathbb{R}^n$ denote the set of n–dimensional vectors with entries in $\mathbb{R}$. Denote by $\mathbf{e}_i$ the i^{th} *standard basis vector* in $\mathbb{R}^n$. Denote by $\langle A, B\rangle$ the standard inner product of A and $B \in \mathbb{R}^n$ and the standard *Euclidian* *norm*, $\|\cdot\|$, for a vector $A = (a_1, \ldots, a_n)$ is given by

$$\|A\|^2 = \sum_{i=1}^{n} a_i^2 = \mathrm{tr}\ AA^T,$$

where tr denotes the trace of a square matrix. The *Cauchy–Schwarz inequality*,

$$|\langle A, B\rangle| \leq \|A\|\|B\|,$$

proves useful in subsequent proofs.

Matrix Notation. Let $M_{n\times n}(\mathbb{R})$ be the space of $n \times n$ matrices with real entries where I is the *identity matrix*. Let $\mathrm{diag}(a_1, a_2, ..., a_n)$ denote the $n \times n$ matrix with the scalar entries $a_1, a_2, ..., a_n$ on the diagonal and 0 elsewhere. The *transpose* of any matrix A is denoted by A^T. For $A = (a_{ij}) \in M_{n\times d}(\mathbb{R})$ the *norm* denoted by $\|\cdot\|$ is defined by

$$\|A\|^2 = \sum_{i=1}^{n}\left(\sum_{j=1}^{d}|a_{ij}|\right)^2.$$

The *Frobenius norm* is denoted by $\|\cdot\|_F$ and defined by

$$\|A\|_F^2 = \sum_{i=1}^{n}\sum_{j=1}^{d}|a_{ij}|^2.$$

Since $M_{n\times d}(\mathbb{R})$ is a finite–dimensional Banach space the two norms $\|\cdot\|$ and $\|\cdot\|_F$ are equivalent. Thus universal constants $0 \leq c_1(n,d) \leq c_2(n,d)$ can be found such that

$$c_1\|A\| \leq \|A\|_F \leq c_2\|A\|, \quad \text{for all } A \in M_{n\times d}(\mathbb{R}).$$

The following inequality will be used in the sequel:

$$\left(\sum_{i=1}^{n}|x_i|\right)^k \leq n^{k-1}\sum_{i=1}^{n}|x_i|^k \quad n, k \in \mathbb{N}. \tag{1.1.1}$$

It proves useful when manipulating norms.

Little o Notation. In subsequent work it is necessary to characterise the asymptotic behaviour of functions: the function f is $o(t)$ as $t \to \infty$ if

$$\lim_{t\to\infty} f(t)/t = 0.$$

Integrable Functions in the Deterministic Sense. If J is an interval in $\mathbb{R}$ and V a finite–dimensional normed space, with norm $\|\cdot\|$, then $C(J,V)$ denotes the family of *continuous* functions $\phi : J \to V$. The space of Lebesgue integrable functions $\phi : (0,\infty) \to V$ will be denoted by $L^1((0,\infty),V)$ where

$$\int_0^\infty \|\phi(t)\|\,dt < \infty.$$

The space of Lebesgue square–integrable functions $\phi : (0,\infty) \to V$ will be denoted by $L^2((0,\infty),V)$ where

$$\int_0^\infty \|\phi(t)\|^2\,dt < \infty.$$

Where V is clear from the context it is omitted it from the notation. Note that a function of domain J that belongs to $L^1(K,V)$ for every compact subset K of J is known as a locally integrable function and denoted $L^1_{loc}(J,V)$.

Convolutions. The *convolution* of $A : [0,\infty) \to M_{n\times d}(\mathbb{R})$ and $B : [0,\infty) \to M_{d\times r}(\mathbb{R})$ is denoted by $A * B$ and defined to be the function given by

$$(A*B)(t) = \int_0^t A(t-s)B(s)\,ds, \qquad t \geq 0.$$

The following properties of convolutions are utilised in the sequel: $(A*B)*C = A*(B*C) = A*B*C$ and if $A,\ B \in L^1(0,\infty)$,

$$\int_0^\infty (A*B)(t)\,dt = \int_0^\infty A(t)\,dt \int_0^\infty B(t)\,dt. \tag{1.1.2}$$

The following lemma is extracted from [16, Theorem 2.2]:

Lemma 1.1.1. *(i) Let $a \in L^1((0,\infty),M_{n\times n}(\mathbb{R}))$ and $b \in L^p((0,\infty),M_{n\times n}(\mathbb{R}))$ for $p = 1,2$. Then $a*b \in L^p((0,\infty),M_{n\times n}(\mathbb{R}))$.*

(ii) Let $a \in L^1((0,\infty),M_{n\times n}(\mathbb{R}))$ and let the function $b \in L^p((0,\infty),M_{n\times n}(\mathbb{R}))$ be bounded and tending to zero asymptotically. Then $\lim_{t\to\infty}(a*b)(t) = 0$.

(iii) Let $a \in L^p((0,\infty),M_{n\times n}(\mathbb{R}))$ and let the function $b \in L^q((0,\infty),M_{n\times n}(\mathbb{R}))$ where $\frac{1}{p}+\frac{1}{q} = 1$. Then $\lim_{t\to\infty}(a*b)(t) = 0$.

Convolution Equations. There are two results concerning the behaviour of linear convolution equations,

$$x(t) + (k * x)(t) = f(t), \quad t \geq 0, \tag{1.1.3}$$

which are used throughout the thesis:

Theorem 1.1.1. *Let the function k satisfy $k \in L^1_{loc}((0,\infty), M_{n\times n}(\mathbb{R}))$. Then there is a unique solution $r \in L^1_{loc}((0,\infty), M_{n\times n}(\mathbb{R}))$ of the two equations*

$$r(t) + (k * r)(t) = k(t),$$

and

$$r(t) + (r * k)(t) = k(t).$$

Furthermore,

Theorem 1.1.2. *Let $k \in L^1_{loc}((0,\infty), M_{n\times n}(\mathbb{R}))$ and $f \in L^1_{loc}((0,\infty), \mathbb{R}^n)$. Then there is a unique solution $x \in L^1_{loc}((0,\infty), \mathbb{R}^n)$ of (1.1.3). This solution is given by the variation of parameters formula:*

$$x(t) = f(t) - (r * f)(t).$$

The proof of these two results may be found in [16].

Complex Numbers. The set of complex numbers is denoted by $\mathbb{C}$; the real part of z in $\mathbb{C}$ being denoted by $\operatorname{Re} z$ and the imaginary part by $\operatorname{Im} z$. The absolute value of a complex number $z = a + bi$ is defined as $|z| = \sqrt{a^2 + b^2}$. Let $M_{n\times d}(\mathbb{C})$ be the space of $n \times d$ matrices with complex entries. We define the norm of the $n \times d$–dimensional matrix–valued function $A : \mathbb{C} \mapsto M_{n\times d}(\mathbb{C})$ in a similar manner to the norm of a real function:

$$\|A(z)\|^2 = \sum_{i=1}^{n} \left(\sum_{j=1}^{d} |a_{ij}(z)| \right)^2 .$$

The *Riemann–Lebesgue Lemma* may now be stated:

Lemma 1.1.2. *If the function $f : \mathbb{R} \mapsto \mathbb{R}$ is a Riemann integrable function such that $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$ then $\hat{f}(z) \to 0$ as $|z| \to \infty$.*

This lemma is used extensively in Chapter 2. The proof of this result may be found in [23].

Laplace Transforms. The *Laplace transform* of the function $A : [0, \infty) \to M_{n\times n}(\mathbb{R})$ is defined as

$$\hat{A}(z) = \int_0^{\infty} A(t) e^{-zt}\, dt.$$

If $\epsilon \in \mathbb{R}$ and $\int_0^{\infty} \|A(t)\| e^{-\epsilon t}\, dt < \infty$ then $\hat{A}(z)$ exists for $\operatorname{Re} z \geq \epsilon$ and is analytic for $\operatorname{Re} z > \epsilon$. If A is a continuous function which satisfies $\|A(t)\| \leq ce^{\beta t}$ for $t > 0$ then the *Laplace inversion formula,*

$$A(t) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{\epsilon - iT}^{\epsilon + iT} \hat{A}(z) e^{zt}\, dz = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \hat{A}(z) e^{zt}\, dz, \tag{1.1.4}$$

holds for all $\epsilon > \beta$. The following properties of the Laplace transform can easily be shown: if A and $B \in L^1(0, \infty)$ then

$$\widehat{A * B}(z) = \hat{A}(z)\hat{B}(z); \tag{1.1.5}$$

furthermore if $\int_t^{\infty} \|A(s)\|\, ds \in L^1(0, \infty)$ then

$$\hat{A}_1(z) = \frac{1}{z}\hat{A}(0) - \frac{1}{z}\hat{A}(z), \quad \operatorname{Re} z \geq 0,\ z \neq 0, \tag{1.1.6}$$

and

$$\hat{A}_1(0) = -\hat{A}'(0), \tag{1.1.7}$$

where $A_1(t) = \int_t^{\infty} A(s)\, ds$ and $\hat{A}'(0) = \int_0^{\infty} sA(s)\, ds$; also

$$\hat{e}(z) = \frac{1}{z+1}, \quad \operatorname{Re} z > -1, \tag{1.1.8}$$

and

$$\hat{e}(0) = 1, \tag{1.1.9}$$

where the function e is defined by $e(t) = e^{-t}$, $t \geq 0$.

Equilibruim and nonequilibruim limits. If the solution of an integro–differential equation converges to a limit we say that the limit is an *equilibuim limit* if it a solution of the integro–differential equation, we say that it is a *nonequilibuim limit* if it a not solution of the integro–differential equation.

1.1.2 Stochastic Preliminaries

A brief overview of the basic theory concerning stochastic processes is given in this section. For a more details see texts such as Mao [26], Øksendal [33], or Karatzas and Shreve [22].

Probability Spaces. Consider the *probability triple* $(\Omega, \mathcal{F}, \mathbb{P})$. Here Ω denotes the *sample space* where each *outcome* in Ω is denoted by ω. Of particular importance are *σ–algebras*: if C is a family of subsets of Ω then $\sigma(C)$ is the smallest σ–algebra containing the set C which satisfies the following conditions; $\Omega \in \sigma(C)$, $A \in \sigma(C)$ implies $A^c \in \sigma(C)$ and $\{A_i\}_{i\geq 1} \in \sigma(C)$ implies $\cup_{i=1}^{\infty} A_i \in \sigma(C)$. Here, $A^c = \Omega - A$ is the complement of A. The family $\mathcal{F}$ is a σ–algebra; any set which belongs to $\mathcal{F}$ is said to be *$\mathcal{F}$–measurable*, in other words a function $X : \Omega \to \mathbb{R}$ is said to be $\mathcal{F}$-measurable if $\{\omega : X(\omega) \leq a\} \in \mathcal{F}$ for $a \in \mathbb{R}$. A *probability measure* $\mathbb{P}$ on the space $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \mapsto [0, 1]$ which satisfies the following conditions: $\mathbb{P}[\Omega] = 1$; if $A_1, A_2, \ldots$ are disjoint events then $\mathbb{P}[\cup_{i=1}^{\infty} A_i] = \sum_{i=1}^{\infty} \mathbb{P}[A_i]$. In particular, if $\mathbb{P}[A] = 1$ then we say that A is an *almost sure* event where a.s. is often used as shorthand. A *filtration* $\{\mathcal{F}(t)\}_{t\geq 0}$ is an increasing set of *σ–algebras* in $\mathcal{F}$. The filtration at time t represents all the information available at that time. The filtered probability space is denoted by $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$.

Random Variables. A random variable is an $\mathcal{F}$–measurable function $X : \Omega \mapsto \mathbb{R}$. Every random variable X induces a probability measure μ_X on the Borel sets $\mathbb{B}$ of $\mathbb{R}$ where $\mu_X(\mathbb{B}) = \mathbb{P}[\omega : X(\omega) \in \mathbb{B}]$. If X is *integrable* with respect to the probability measure; that is if

$$\int_{\Omega} \|X(\omega)\| \, d\mathbb{P}(\omega) < \infty,$$

then the expectation of X can be expressed as

$$\mathbb{E}[X] = \int_{\Omega} X(\omega)\, d\mathbb{P}(\omega) = \int_{0}^{\infty} x\, d\mu_X(x).$$

Distributions. The *distribution function* of a random variable X is the function $F : \mathbb{R} \mapsto [0,1]$ given by $F(x) = \mathbb{P}(X \leq x)$. The sequence of random variable $X_1, X_2, \ldots$ (with corresponding distribution functions $F_1, F_2 \ldots$) has a *limiting distribution* denoted F if $\lim_{n\to\infty} F_n = F$.

Characteristic function. The characteristic function of the scalar random variable X is the function

$$\Phi : \mathbb{R} \to \mathbb{C} : \quad \lambda \mapsto \Phi(\lambda) = \mathbb{E}\left[e^{i\lambda X}\right],$$

where $i = \sqrt{-1}$. It has a number of important properties: $\Phi(0) = 1$; $|\Phi(\lambda)| \leq 1$ for all λ and Φ is uniformly continuous on $\mathbb{R}$. The joint characteristic function of the vector of random variables $X = (X_1, \ldots, X_n)$ is given by $\Phi_X(\lambda) = \mathbb{E}\left[e^{i\lambda X^T}\right]$ where $\lambda = (\lambda_1, \ldots, \lambda_n)$. Of special interest is the case when X has multivariate normal distribution in which case the joint characteristic function is given by $\Phi_X(\lambda) = e^{-\frac{1}{2}\lambda C \lambda^T}$ where C is the covariance matrix of X.

A useful result is the following: suppose that $F_1, F_2, \ldots$ is a sequence of distribution functions with corresponding characteristic functions $\Phi_1, \Phi_2, \ldots$. If $\lim_{n\to\infty} F_n = F$, where F has the characteristic function Φ, then $\lim_{n\to\infty} \Phi_n(\lambda) = \Phi(\lambda)$ for all λ.

For further detail the reader is referred to Grimmett and Stirzaker [15].

Stochastic Processes. A *stochastic process* is a family $\{X(t)\}_{t\geq 0}$ of $\mathbb{R}^n$–valued random variables. It is *continuous* if for almost all $\omega \in \Omega$ the function $t \mapsto X(t,\omega)$ is continuous. It is $\mathcal{F}(t)$*–adapted* if $X(t)$ is $\mathcal{F}(t)$–measurable for every t. It is said to be *increasing* if $X(t,\omega)$ is nonnegative, nondecreasing and right continuous on $t \geq 0$ for almost all $\omega \in \Omega$. It is a process of *finite variation* if $X(t) = \bar{A}(t) - \hat{A}(t)$ where both $\{\bar{A}(t)\}$ and $\{\hat{A}(t)\}$ are increasing processes.

Stationarity. A process $S = \{S(t)\}_{t\geq 0}$ is called a stationary process if

$$\mathbb{P}[S(t+t_k) \in A_k, k = 1\dots,n] = \mathbb{P}[S(t_k), \in A_k, k = 1\dots,n], \tag{1.1.10}$$

for all $t > 0$, $t_k \in [0,\infty)$ and Borel sets A_k, where $k = 1,\dots,n$.

Standard Brownian Motion. If $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$ is a filtered probability space then a 1–dimensional standard Brownian motion $\{B(t)\}_{t\geq 0}$ is a process which has the following properties: $B(0) = 0$; the increment $B(t) - B(s)$ is normally distributed with mean 0 and variance $t-s$ where $0 \leq s < t < \infty$; the increment $B(t) - B(s)$ is independent of $\mathcal{F}_s$ where $0 \leq s < t < \infty$.

Stochastic Integrability and Convergence. Due to the random nature of stochastic processes various definitions of stochastic integrability exist. A stochastic process X is integrable if $X(t)$ is integrable with respect to the probability measure for each $t \geq 0$, that is if $\mathbb{E}\,\|X(t)\| < \infty$ for each $t \geq 0$; it is square integrable if $\mathbb{E}\,\|X(t)\|^2 < \infty$ for each $t \geq 0$.

The notion of convergence and integrability in p^{th} mean and almost sure senses are now defined: the $\mathbb{R}^n$–valued stochastic process $\{X(t)\}_{t\geq 0}$ *converges in p^{th} mean* to X_∞ if

$$\lim_{t\to\infty} \mathbb{E}\,\|X(t) - X_\infty\|^p = 0;$$

the process is p^{th} mean exponentially convergent to X_∞ if there exists a $\beta_p > 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E}\,\|X(t) - X_\infty\|^p) \leq -\beta_p;$$

we say that the difference between the stochastic process $\{X(t)\}_{t\geq 0}$ and X_∞ is *integrable in the p^{th} mean sense* if

$$\int_0^\infty \mathbb{E}\,\|X(t) - X_\infty\|^p\,dt < \infty.$$

If there exists a $\mathbb{P}$–null set Ω_0 such that for every $\omega \notin \Omega_0$ the following holds:

$$\lim_{t\to\infty} X(t,\omega) = X_\infty(\omega),$$

then X *converges almost surely* to X_∞; we say X is almost surely exponentially convergent to X_∞ if there exists a $\beta_0 > 0$ such that,

$$\limsup_{t\to\infty} \frac{1}{t} \log\|X(t,\omega) - X_\infty(\omega)\| \leq -\beta_0, \quad a.s.$$

Finally, the difference between the stochastic process $\{X(t)\}_{t\geq 0}$ and X_∞ is *square integrable in the almost sure sense* if

$$\int_0^\infty \|X(t,\omega) - X_\infty(\omega)\|^2 \, dt < \infty.$$

Henceforth, $\mathbb{E}[X^p]$ will be denoted by $\mathbb{E}X^p$ except in cases where the meaning may be ambiguous.

Stochastic integrals. The n–dimensional Itô integral is defined as

$$\int_0^t g(s)\, dB(s),$$

where $B(t) = (B_1(t)\ldots, B_d(t))$, each $B_i(t)$ is standard Brownian motion and g is an $n \times d$–dimensional function. If the function g is square integrable then

$$\mathbb{E}\left[\left\|\int_0^t g(s)\, dB(s)\right\|^2\right] = \int_0^t \|g(s)\|_F^2 \, ds.$$

X is an n–dimensional Itô process if there is an adapted n–dimensional vector–valued function f and an adapted $n \times d$ matrix–valued function g such that

$$X(t) = X_0 + \int_0^t f(s)\, ds + \int_0^t g(s)\, dB(s), \quad t \geq 0,$$

where X_0 is deterministic. The stochastic differential used to express this is given by

$$dX(t) = f(t)\, dt + g(s)\, dB(t), \quad t > 0, \tag{1.1.11}$$

$$X(0) = X_0. \tag{1.1.12}$$

One of the most important tools of stochastic calculus is the change of variable formula which is also known as *Itô's formula*:

Lemma 1.1.3. *Let the n–dimensional Itô process X be defined by* (1.1.11) *and assume that the functions f and g are in $L^1(0,\infty)$ and $L^2(0,\infty)$ respectively. If the function $V(\cdot,\cdot) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ then $t \mapsto V(X(t),t)$ is once again an Itô process with stochastic differential given by*

$$dV(X(t),t) = \left[V_t(X(t),t) + V_x(X(t),t)f(t) + \frac{1}{2}tr\ \left(g^T(t)V_{xx}(X(t),t)g(t)\right)\right] dt$$
$$+ V_x(X(t),t)g(t)\, dB(t) \quad a.s.$$

Another important tool used in stochastic calculus is the *integration by parts formula*:

Lemma 1.1.4. *Let the 1-dimensional Itô process X be defined by (1.1.11) with $n = d = 1$ and assume that the functions f and g are in $L^1(0,\infty)$ and $L^2(0,\infty)$ respectively. If Y is a continuous adapted process of finite variation then*

$$d[X(t)Y(t)] = X(t)dY(t) + Y(t)dX(t).$$

Martingales. The stochastic process $M = \{M(t)\}_{t\geq 0}$, defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$, is said to be a martingale with respect to the filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ if $M(t)$ is $\mathcal{F}(t)$–measurable for all $t \geq 0$, $\mathbb{E}[|M(t)|] < \infty$ for all $t \geq 0$ and for all $0 \leq s \leq t$

$$\mathbb{E}[M(t)|\mathcal{F}(s)] = M(s) \quad \text{a.s.}$$

Furthermore, if the process M is a real–valued square integrable martingale then there exists a unique, adapted, increasing, integrable process $\langle M\rangle = \{\langle M\rangle(t)\})_{t\geq 0}$ such that the process $\{M(t)^2 - \langle M\rangle(t)\}_{t\geq 0}$ is a martingale which vanishes at $t = 0$. The process $\langle M\rangle$ is known as the *quadratic variation* of M. The asymptotic behaviour of the quadratic variation characterises the asymptotic behaviour of the martingale, this is seen in the *Martingale Convergence Theorem*, which is stated presicely in Lemma 1.1.5. Before stating this we define a *stopping time* and a *local martingale*. A random variable $\tau : \Omega \mapsto [0,\infty]$ is called a stopping time if $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}(t)$ for any $t \geq 0$. A right continuous adapted process $M = \{M(t)\}_{t\geq 0}$ is a local martingale if there exists a nondecreasing sequence of stopping times $\{\tau_k\}_{k\geq 0}$ with $\tau_k \to \infty$ as $k \to \infty$ almost surely such that every $\{M(\tau_k \wedge t)\}_{t\geq 0}$ is a martingale.

Lemma 1.1.5. *For a continuous local martingale M, the sets $\{\langle M\rangle(\infty) < \infty\}$ and $\{\lim_{t\to\infty} M(t)$ exists$\}$ are almost–surely equal. Furthermore, $\limsup_{t\to\infty} M(t) = \infty$ and $\liminf_{t\to\infty} M(t) = -\infty$ almost surely on the set $\{\langle M\rangle(\infty) = \infty\}$.*

The proof of this lemma may be found in [35]. Another result concerning the asymptotic behaviour of martingales is the *Law of Large Numbers for Martingales*:

Lemma 1.1.6. *Let the process $M = \{M(t)\}_{t\geq 0}$ defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$ be a continuous martingale which vanishes at $t = 0$. If the quadratic variation $\langle M\rangle$ of M satisfies $\lim_{t\to\infty}\langle M\rangle(t) = \infty$ then*

$$\lim_{t\to\infty} \frac{M(t)}{\langle M\rangle(t)} = 0 \quad a.s.$$

The *Burkholder–Davis–Gundy* inequality provides useful bounds on the moments of martingales.

Lemma 1.1.7. *Let M be a continuous local martingale. For every $m > 0$ there exist universal positive constants k_m, K_m (depending only on m) such that*

$$k_m\mathbb{E}\left[\langle M\rangle(T)^m\right] \leq \mathbb{E}\left[\sup_{0\leq t\leq T} M(t)^{2m}\right] \leq K_m\mathbb{E}\left[\langle M\rangle(T)^m\right],$$

holds for every stopping time $T > 0$.

Note that if $g \in L^2(0,\infty)$ then the Itô integral defined by $I(t) = \int_0^t g(s)\,dB(s)$ is a martingale with quadratic variation given by

$$\langle I\rangle(t) = \int_0^t g(s)^2\,ds.$$

Useful Inequalities. *The Hölder* and *Chebyshev* inequalities are used frequently in subsequent chapters. They are stated here for clarity: if the finite–dimensional random variable X satisfies $\mathbb{E}\,\|X\|^p < \infty$ for $p \in (0,\infty)$ and the finite–dimensional random variable Y satisfies $\mathbb{E}\,\|Y\|^q < \infty$ for $q \in (0,\infty)$ then

$$\left\|\mathbb{E}\left[X^TY\right]\right\| \leq \mathbb{E}\left[\|X\|^p\right]^{\frac{1}{p}}\mathbb{E}\left[\|Y\|^q\right]^{\frac{1}{q}}, \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1;$$

furthermore, if c is a positive constant, then

$$\mathbb{P}[\omega : \|X(\omega)\| \geq c] \leq c^{-p}\mathbb{E}\,\|X\|^p.$$

The *Liapunov inequality* is useful when considering the p^{th} mean behaviour of random variables as any exponent $p > 0$ may be considered rather than $p > 1$ in the case of the Hölder inequality:

$$\mathbb{E}\left[\|X\|^p\right]^{\frac{1}{p}} \leq \mathbb{E}\left[\|X\|^q\right]^{\frac{1}{q}}, \quad 0 < p \leq q.$$

Chapter 2

Exponential Convergence to a Nontrivial Deterministic Limit

2.1 Introduction

In this chapter the exponential decay of the solution of

$$R'(t) = AR(t) + \int_0^t K(t-s)R(s)\,ds, \quad t > 0, \tag{2.1.1a}$$
$$R(0) = I, \tag{2.1.1b}$$

to a constant vector is studied. Here the matrix–valued function R is known as the resolvent or fundamental solution and A is a real $n \times n$ matrix. It is assumed throughout that the function $K : [0,\infty) \to M_{n\times n}(\mathbb{R})$ satisfies

$$K \in C([0,\infty), M_{n\times n}(\mathbb{R})) \cap L^1((0,\infty), M_{n\times n}(\mathbb{R})). \tag{2.1.2}$$

Under the hypothesis (2.1.2), it is well–known that (2.1.1) has a unique continuous solution R, which is continuously differentiable.

We also examine the exponential decay of the solution of (2.1.3),

$$x'(t) = Ax(t) + \int_0^t K(t-s)x(s)\,ds + f(t), \quad t > 0, \tag{2.1.3a}$$
$$x(0) = x_0. \tag{2.1.3b}$$

The function $t \mapsto x(t; x_0, f)$ is the unique n–dimensional solution of the initial value problem (2.1.3). A and K are defined as above and the function $f : [0,\infty) \to \mathbb{R}^n$ satisfies

$$f \in C([0,\infty), \mathbb{R}^n) \cap L^1((0,\infty), \mathbb{R}^n). \tag{2.1.4}$$

The representation of the solution of (2.1.3) in terms of R is given by the variation of constants formula

$$x(t; x_0, f) = R(t)x_0 + (R * f)(t), \quad t \geq 0. \tag{2.1.5}$$

Note that where x_0 and f are clear from the context they are omitted from the notation. So it is clear that the behaviour of the perturbed equation is influenced by the behaviour of the underlying resolvent.

It is interesting to understand the relationship between the rate of decay of the kernel, and the rate of decay of solutions of (2.1.1) and (2.1.3). Authors who have shown that some sort of exponential decay in the kernel can be identified with exponential decay of the resolvent include Murakami [30, 31] and Appleby and Reynolds [7]. Murakami shows that the exponential decay of the solution of (2.1.1) is equivalent to an exponential decay property on the kernel K under the restriction that none of the elements of K change sign on $[0, \infty)$. A condition of this type will be employed in this chapter to identify exponential convergence. In a similar spirit, various authors have identified decay conditions on K which give rise to particular decay properties in the resolvent. For example Burton, Huang and Mahfoud [13] have shown that the existence of the 'moments' of the kernel can be identified with the existence of the 'moments' of the solution. Appleby and Reynolds [6] have studied a type of non–exponential decay of solutions (called subexponential decay) which can in certain circumstances be identified with the subexponential decay of the kernel. Jordan and Wheeler [20] and Shea and Wainger [38] have studied the relationship between the existence of the kernel in a certain weighted L^p-space and the existence of the solution in such spaces.

The case where the solutions of (2.1.1) are neither integrable, nor unstable, has also been considered. Krisztin and Terjéki [24] studied this case and determined conditions under which $R(t)$ converges to a limit R_∞ which need not be trivial as $t \to \infty$. In addition to determining a formula for R_∞ in these cases they showed that a condition on the second moment of the kernel is crucial.

Throughout this thesis the case where the resolvent of (2.1.3) is not integrable is considered. In this chapter an equivalence between the exponential decay property of $R - R_\infty$ and an exponential decay property of the kernel is found; it is also shown for solutions of (2.1.3) that the exponential decay of $x - x_\infty$ can be identified with exponential decay in the kernel and the perturbation.

2.2 Discussion of Results

In this section, the connection between the results on exponential decay presented by Murakami in [30, 31] and those in this chapter are explained.

Murakami obtained the following result in the case where the solutions of (2.1.1) are integrable.

Theorem 2.2.1. *Let K satisfy* (2.1.2). *Suppose the resolvent R of* (2.1.1) *satisfies*

$$R \in L^1((0,\infty), M_{n\times n}(\mathbb{R})). \tag{2.2.1}$$

If

$$\textit{each entry of } K \textit{ does not change sign on } [0,\infty), \tag{2.2.2}$$

then the following are equivalent:

(i) there exists a constant $\alpha > 0$ such that

$$\int_0^\infty \|K(s)\| e^{\alpha s} ds < \infty; \tag{2.2.3}$$

(ii) there exist constants $c > 0$ and $\beta > 0$ such that

$$\|R(t)\| \le ce^{-\beta t}, \quad t \ge 0. \tag{2.2.4}$$

A fundamental result on the asymptotic behaviour of the solution of (2.1.3) is the following theorem due to Grossman and Miller [17]:

Theorem 2.2.2. *Let K satisfy* (2.1.2). *The resolvent R of* (2.1.1) *satisfies* (2.2.1) *if and only if*

$$\det[zI - A - \hat{K}(z)] \neq 0, \quad \textit{for } \operatorname{Re} z \ge 0. \tag{2.2.5}$$

Murakami considered the case where the resolvent of (2.1.1) is integrable and consequently was able to make use of (2.2.5) in his proof. In this thesis, the case where the solution of (2.1.1) approaches a constant matrix is considered. This constant matrix need not be trivial, in which case the solution is not integrable, consequently, assumption (2.2.5) no longer holds for all $\operatorname{Re} z \geq 0$. So, it is not possible to apply Murakami's method of proof directly to (2.1.1).

However, a reformulation of (2.1.1) as an integral equation was provided by Krisztin and Terjéki [24], the details of which may be found in Lemma 2.3.1. A fundamental result concerning the integrability of Volterra integral equations is the following theorem by Paley and Wiener [34]:

Theorem 2.2.3. *If the functions F and $G \in L^1(0,\infty)$, then the unique solution Y of the equation $Y(t) + (F * Y)(t) = G(t)$ is integrable if and only if*

$$\det[I + \hat{F}(z)] \neq 0, \quad \operatorname{Re} z \geq 0. \tag{2.2.6}$$

Krisztin and Terjéki were able to show that (2.2.6) of Theorem 2.2.3 holds for their specific integral equation.

Before citing the relevant results from [24] some notation used there and adopted hereinafter is introduced. Let $M = A + \int_0^\infty K(t)\,dt$ and T be an invertible matrix such that $J = T^{-1}MT$ has Jordan canonical form. Let $e_i = 1$ if all the elements of the i^{th} row of J are zero, and $e_i = 0$ otherwise. Let $D_p = \operatorname{diag}(e_1, e_2, ..., e_n)$ and put $P = TD_pT^{-1}$ and $Q = I - P$. The relevant theorem is now stated:

Theorem 2.2.4. *If K satisfies*

$$\int_0^\infty t^2\|K(t)\|dt < \infty, \tag{2.2.7}$$

and the resolvent R of (2.1.1) satisfies

$$R - R_\infty \in L^1((0,\infty), M_{n\times n}(\mathbb{R})), \tag{2.2.8}$$

then

$$\det[zI - A - \hat{K}(z)] \neq 0 \quad \textit{for } \operatorname{Re} z \geq 0 \textit{ and } z \neq 0, \tag{2.2.9}$$

and

$$\det\left[P - M - \int_0^\infty \int_s^\infty PK(u)\,du\,ds\right] \neq 0. \tag{2.2.10}$$

Moreover $R(t) \to R_\infty$ *as* $t \to \infty$ *and*

$$R_\infty = \left[P - M - \int_0^\infty \int_s^\infty PK(u)\,du\,ds\right]^{-1} P. \tag{2.2.11}$$

Here, assumption (2.2.7) corresponds to the integrability condition on F and G in Theorem 2.2.3. In fact, this hypothesis may be improved upon; it is shown in Chapter 3 that a condition on the first moment of the kernel combined with an integrability condition on the solution ensures that assumption (2.2.6) holds. However, in this chapter, this stronger assumption is acceptable as it is required for the existence of Laplace transforms.

It is shown in Lemma 2.4.1 that (2.2.9) and (2.2.10) imply that (2.2.6) holds for the reformulated integral equation. It becomes clear that (2.2.6) is an analogue of (2.2.5) in the case where the solution to (2.1.1) converges a nontrivial limit.

Using the reformulated equation and (2.2.6) we can apply Murakami's method to obtain the main theorem of this chapter:

Theorem 2.2.5. *Let* K *satisfy* (2.1.2) *and* (2.2.7). *Suppose there exists a constant matrix* R_∞ *such that the solution* R *of* (2.1.1) *satisfies* (2.2.8). *If*

$$\text{each entry of } K \text{ does not change sign on } [0,\infty), \tag{2.2.12}$$

then the following are equivalent:

(i) there exists a constant $\alpha > 0$ *such that*

$$\int_0^\infty \|K(s)\| e^{\alpha s}\,ds < \infty; \tag{2.2.13}$$

(ii) there exist constants $\beta > 0$ *and* $c > 0$ *such that*

$$\|R(t) - R_\infty\| \leq c e^{-\beta t}, \quad t \geq 0. \tag{2.2.14}$$

One can readily see the similarities between Theorem 2.2.1 and Theorem 2.2.5; the hypotheses (2.2.1) and (2.2.2) in Theorem 2.2.1 are identical to (2.2.8) and (2.2.12) in Theorem 2.2.5; moreover, the equivalence between (2.2.3) and (2.2.4) in Theorem 2.2.1 is mirrored by the equivalence between (2.2.13) and (2.2.14). The hypothesis in Theorem 2.2.5 which has no counterpart in Theorem 2.2.1 is (2.2.7); however, this hypothesis is natural and sometimes indispensable in the case when the limiting value of R is nontrivial.

Assumption (2.1.2) is stated explicitly in Theorem 2.2.5 to highlight the condition that ensures the existence of a solution and the condition which is needed for convergence.

It is possible to obtain results comparable to Theorem 2.2.5 for the solution of the perturbed equation (2.1.3). More precisely, it is possible to show that the exponential decay of $x - x_\infty$ is equivalent to the exponential decay of the tail of the perturbation and the exponential integrability of the kernel. The following theorem makes this precise:

Theorem 2.2.6. *Let K satisfy* (2.1.2) *and* (2.2.7), *f satisfy* (2.1.4), *and f_1 be defined by*

$$f_1(t) = \int_t^\infty f(s)\,ds, \quad t \geq 0. \tag{2.2.15}$$

Suppose that for all x_0 there is a constant vector $x_\infty(x_0, f)$ such that the solution $t \mapsto x(t; x_0, f)$ of (2.1.3) *satisfies*

$$x(\cdot\,; x_0, f) - x_\infty(x_0, f) \in L^1((0,\infty), \mathbb{R}^n). \tag{2.2.16}$$

If K satisfies (2.2.12), *the following are equivalent:*

(i) there exists $\alpha > 0$ such that statement (i) of Theorem 2.2.5 holds and there exist constants $\gamma > 0$, $c_1 > 0$ independent of x_0 such that

$$\|f_1(t)\| \leq c_1 e^{-\gamma t}, \quad t \geq 0; \tag{2.2.17}$$

(ii) for each x_0 the solution $t \mapsto x(t; x_0, f)$ satisfies

$$\|x(t) - x_\infty\| \leq c_2 e^{-\bar{\beta} t}, \quad t \geq 0, \tag{2.2.18}$$

for some $\tilde{\beta} > 0$ independent of x_0, and $c_2 = c_2(x_0) > 0$.

Note that assumption (2.2.16) may be replaced by (2.2.8).

2.3 Reformulation of Equation (2.1.1)

The resolvent equation (2.1.1) must be reformulated in order to prove Theorem 2.2.5 and Theorem 2.2.6. The proof of these theorems rely on the transformation of the reformulated equation using Laplace transforms. The details of this transformation may be found in the following lemma:

Lemma 2.3.1. *If the kernel K satisfies (2.1.2) and (2.2.7), and there exists a constant matrix R_∞ such that the solution R of (2.1.1) satisfies (2.2.8) then*

$$\hat{Y}(z) + \hat{F}(z)\hat{Y}(z) = \hat{G}(z), \quad \operatorname{Re} z \geq 0, \tag{2.3.1}$$

where $Y(t) = R(t) - R_\infty$, the function $z \mapsto \hat{F}(z)$ is defined for $\operatorname{Re} z \geq 0$ as

$$\hat{F}(z) = \frac{-1}{z+1}Q(I + A + \hat{K}(z)) + \frac{1}{z}P(\hat{K}(0) - \hat{K}(z)) \tag{2.3.2}$$

and

$$\hat{F}(0) = -Q(I + A + \hat{K}(0)) - P\hat{K}'(0), \tag{2.3.3}$$

and the function $z \mapsto \hat{G}(z)$ is defined for $\operatorname{Re} z \geq 0$ as

$$\begin{aligned} \hat{G}(z) = {} & \frac{1}{(z+1)}Q\left(I - (I + A + \hat{K}(z))R_\infty\right) \\ & - \frac{1}{z}\left(P\hat{K}'(0) + Q\hat{K}(0) - Q\hat{K}(z)\right)R_\infty - \frac{1}{z^2}P(\hat{K}(0) - \hat{K}(z))R_\infty \end{aligned} \tag{2.3.4}$$

and

$$\hat{G}(0) = Q - \left(Q(I + A + \hat{K}(0)) - \frac{1}{2}P\hat{K}''(0) - Q\hat{K}'(0)\right)R_\infty. \tag{2.3.5}$$

Proof. As conditions (2.2.7) and (2.2.8) hold it is known from Theorem 2.2.4 that (2.2.11) holds. An idea used in [24, Theorem 2] is now employed. Define the functions Y and Φ, for $t \geq 0$, by $Y(t) = R(t) - R_\infty$ and $\Phi(t) = P + e^{-t}Q$ respectively. Multiply from the left both sides of $R'(s) = AR(s) + (K * R)(s)$ by $\Phi(t-s)$ to get

$$\Phi(t-s)R'(s) = \Phi(t-s)AR(s) + \Phi(t-s)(K * R)(s).$$

Integrating over $[0,t]$ one obtains

$$\int_0^t \Phi(t-s)R'(s)\,ds = \int_0^t \Phi(t-s)AR(s)\,ds + \int_0^t \Phi(t-s)(K * R)(s)\,ds.$$

Using integration by parts on the left hand side of this expression this becomes

$$R(t) - \Phi(t) + (\Phi' * R)(t) = (\Phi A * R)(t) + (\Phi * K * R)(t).$$

On rearrangement the following is obtained:

$$R(t) + (F * R)(t) = \Phi(t), \quad t \geq 0, \tag{2.3.6}$$

where $F = \Phi' - \Phi A - (\Phi * K)$. Expanding the left hand side of this expression we see that

$$F(t) = -e^{-t}(Q + QA) - (e * QK)(t) + P\int_t^\infty K(u)\,du, \quad t \geq 0, \tag{2.3.7}$$

where the function e is defined by $e(t) = e^{-t}$, $t \geq 0$. The details of this expansion may be found in Appendix B. A further calculation yields

$$Y(t) + (F * Y)(t) = G(t), \quad t \geq 0, \tag{2.3.8}$$

where $G(t) = \Phi(t) - R_\infty - (F * R_\infty)(t)$. Again, by expanding the left hand side of this expression one obtains

$$\begin{aligned} G(t) = e^{-t}Q - e^{-t}(QR_\infty + QAR_\infty) + \int_t^\infty \int_s^\infty PK(u)R_\infty\,du\,ds \\ - \int_t^\infty QK(u)R_\infty\,du - (e * QKR_\infty)(t), \quad t \geq 0, \end{aligned} \tag{2.3.9}$$

the details of which may be found in Appendix B.

Since (2.2.8) holds, the Laplace transform of Y denoted by $\hat{Y}$ exists for $\operatorname{Re} z \geq 0$. Since (2.2.7) holds the Laplace transform of F exists for $\operatorname{Re} z \geq 0$. In fact, by applying (1.1.8) to the first term on the right hand side of (2.3.7), (1.1.5) and then (1.1.8) to the second term on the right hand side of (2.3.7) and (1.1.6) to the third term on the right hand side of (2.3.7) we get

$$\hat{F}(z) = \frac{-1}{z+1}Q(I+A) - \frac{1}{z+1}Q\hat{K}(z) + \frac{1}{z}P\hat{K}(0) - \frac{1}{z}P\hat{K}(z), \qquad \operatorname{Re} z \geq 0, z \neq 0,$$

which on rearrangement yields (2.3.2). Now consider the case when $z = 0$. By applying (1.1.9) to the first term on the right hand side of (2.3.7), (1.1.5) to the second term on the right hand side of (2.3.7) and (1.1.7) to the third term on the right hand side of (2.3.7) it is found that

$$\hat{F}(0) = -Q(I+A) - Q\hat{K}(0) - P\hat{K}'(0),$$

which on rearrangement yields (2.3.3). Similarly, as (2.2.7) holds the Laplace transform of G exists for $\operatorname{Re} z \geq 0$. By applying: (1.1.8) to the first and second terms on the right hand side of (2.3.9); (1.1.6) twice to the third term on the right hand side; (1.1.6) to the fourth term on the right hand side; (1.1.5) and then (1.1.8) to the final term on the right hand side of (2.3.9), we get

$$\begin{aligned}\hat{G}(z) = \frac{1}{z+1}Q - \frac{1}{z+1}(QR_\infty + QAR_\infty) + P\left(-\frac{1}{z}\hat{K}'(0) - \frac{1}{z^2}\hat{K}(0) + \frac{1}{z^2}\hat{K}(z)\right)R_\infty \\ - \frac{1}{z}Q\hat{K}(0)R_\infty + \frac{1}{z}Q\hat{K}(z)R_\infty - \frac{1}{z+1}Q\hat{K}(z)R_\infty,\end{aligned}$$

which on rearrangement yields (2.3.4). Now consider the case when $z = 0$. If we apply (1.1.9) to the first and second terms on the right hand side of (2.3.9), calculate the third term on the right hand side explicitly, apply (1.1.7) to the fourth term on the right hand side, before finally applying (1.1.5) then (1.1.9) to the final term on the right hand side of (2.3.9), we arrive at

$$\hat{G}(0) = Q - (QR_\infty + QAR_\infty) + \frac{1}{2}P\hat{K}''(0)R_\infty + Q\hat{K}'(0)R_\infty - Q\hat{K}(0)R_\infty,$$

which on rearrangement yields (2.3.5). This completes the proof.

□

If it is assumed that there exists a constant $\alpha > 0$ such that (2.2.13) of Theorem 2.2.5 holds then the functions $\hat{F}$ and $\hat{G}$ defined by (2.3.2) and (2.3.4) respectively can be extended into the negative half plane. Use is made of this fact when proving that statement (ii) implies (i) in Theorem 2.2.5.

2.4 Proof of Theorem 2.2.5

Theorem 2.2.5 is a consequence of the following results.

Theorem 2.4.1. *Let K satisfy* (2.1.2) *and* (2.2.7), *and R be the solution of* (2.1.1). *Suppose there exists a constant matrix R_∞ such that* (2.2.8) *holds. If there exists a constant $\alpha > 0$ such that K obeys* (2.2.13) *in Theorem 2.2.5, then there exist constants $\beta > 0$ and $c > 0$ such that R obeys* (2.2.14) *of Theorem 2.2.5.*

Theorem 2.4.2. *Let K satisfy* (2.1.2), (2.2.7) *and* (2.2.12), *and let R be the solution of* (2.1.1). *Suppose there exists a constant matrix R_∞ such that* (2.2.8) *holds. If there exist constants $\beta > 0$ and $c > 0$ such that R obeys* (2.2.14) *in Theorem 2.2.5, then there exists a constant $\alpha > 0$ such that K obeys* (2.2.13) *in Theorem 2.2.5.*

The following lemma is used in the proof of Theorem 2.4.1. The proof is deferred until the end of the section.

Lemma 2.4.1. *If the function K satisfies* (2.1.2), (2.2.7) *and there exists a constant matrix R_∞ such that the solution R of* (2.1.1) *satisfies* (2.2.8), *then*

$$\det[I + \hat{F}(z)] \neq 0, \quad \mathrm{Re}\, z \geq 0, \tag{2.4.1}$$

where the function F is defined by (2.3.7).

The following proposition may extracted from [30, 31] and used later in the proof of Theorem 2.4.2.

Proposition 2.4.1. *Let K satisfy* (2.1.2) *and* (2.2.12). *Suppose there exists $\alpha > 0$ and a function B such that $\hat{K}(z) = B(z)$ for $\mathrm{Re}\, z \geq 0$ and B is well defined on $|\mathrm{Re}\, z| \leq \alpha$ and analytic on $|\mathrm{Re}\, z| < \alpha$. Then*

$$\int_0^\infty \|K(s)\| e^{\alpha s}\, ds < \infty.$$

The proof of this proposition is deferred to Appendix A. The proof is identical in all important details to that of Theorem 2 in [31]. The difference is that one must work in finite dimensions.

Proof of Theorem 2.4.1. As (2.2.8) holds the inversion formula for the Laplace Transform of Y is well defined when $\epsilon > 0$:

$$Y(t) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \hat{Y}(z) e^{zt} dz.$$

From Lemma 2.4.1 it is known that $\det[I + \hat{F}(z)] \neq 0$ for $\mathrm{Re}\, z \geq 0$ so one can write

$$\hat{Y}(z) = H_1(z), \quad \mathrm{Re}\, z \geq 0,$$

where

$$H_1(z) = (I + \hat{F}(z))^{-1} \hat{G}(z), \quad \mathrm{Re}\, z \geq 0,$$

by (2.3.8).

Firstly, it is shown that for some $\beta > 0$ one can write

$$Y(t) = \frac{1}{2\pi i}\int_{-\beta-i\infty}^{-\beta+i\infty} H_1(z)e^{zt}\,dz, \quad t > 0.$$

Observe that since $\det[I + \hat{F}(0)] \neq 0$, $H_1(0)$ exists. Using (2.2.13) and the Riemann–Lebesgue Lemma it is seen that $\hat{K}(z) \to 0$ as $|z| \to \infty$ for $\operatorname{Re} z \geq -\alpha$. Thus it can be seen from (2.3.2) that $\hat{F}(z) \to 0$ as $|z| \to \infty$ for $\operatorname{Re} z \geq -\alpha$. Therefore, a constant $T_0 > 0$ exists such that for $|\operatorname{Im} z| > T_0$ the function $z \mapsto \det[I + \hat{F}(z)] \neq 0$ when $\operatorname{Re} z \geq -\alpha$, $|\operatorname{Im} z| > T_0$. Hence, $H_1(z)$ exists when $|\operatorname{Im} z| > T_0$ and $\operatorname{Re} z \geq -\alpha$. Let

$$D = \{z : -\frac{\alpha}{2} \leq \operatorname{Re} z \leq 0, \quad |\operatorname{Im} z| \leq T_0\}$$

and

$$c_0 = \max\{\operatorname{Re} z : z \in D, \quad \det[I + \hat{F}(z)] = 0\}.$$

Since $z \mapsto (I + \hat{F}(z))$ is analytic on the domain $\operatorname{Re} z > -\alpha$, and its determinant is a continuous function of its entries, the function $z \mapsto \det[I + \hat{F}(z)]$ is analytic on the domain $\operatorname{Re} z > -\alpha$. Thus it has at most a finite number of zeros in the set D. As $\det[I + \hat{F}(z)] \neq 0$ for $\operatorname{Re} z = 0$ this means that $c_0 < 0$. Take a constant $\beta > 0$ so that $c_0 < -\beta$ and consider the integration of the function $H_1(z)e^{-zt}$ around the boundary of the region:

$$\{\lambda + i\tau : \ -\beta \leq \lambda \leq \beta, \ -T \leq \tau \leq T\}.$$

Since H_1 exists and is analytic in this region it follows that the integral over the boundary is zero, that is:

$$\left(\int_{\beta-iT}^{\beta+iT} + \int_{\beta+iT}^{-\beta+iT} + \int_{-\beta+iT}^{-\beta-iT} + \int_{-\beta-iT}^{\beta-iT}\right) H_1(z)e^{zt}dz = 0.$$

It is now shown that

$$\lim_{T\to\infty}\int_{-\beta+iT}^{\beta+iT} H_1(z)e^{zt}\,dz = 0 \quad \text{and} \quad \lim_{T\to\infty}\int_{-\beta-iT}^{\beta-iT} H_1(z)e^{zt}\,dz = 0.$$

Consider $\|H_1(z)e^{zt}\|$ where $z = \lambda + iT$ and $-\beta \le \lambda \le \beta$:

$$\|H_1(z)e^{zt}\| = \|(I+\hat{F}(z))^{-1}\hat{G}(z)e^{zt}\| \le e^{\beta t}\|\hat{G}(z)\|\|(I+\hat{F}(z))^{-1}\|.$$

Due to a previous argument, $\det[I+\hat{F}(z)] \ne 0$ on the contour we are considering, so a constant c may be found such that $\|(I+\hat{F}(z))^{-1}\| < c$. Using this and expanding $\hat{G}$ we obtain

$$\begin{aligned}\|H_1(z)e^{zt}\| \le ce^{\beta t}\cdot\Bigg(&\frac{\|Q-(I+A)R_\infty\|}{|z+1|} + \frac{\|Q\hat{K}(z)R_\infty\|}{|z+1|} + \frac{\|P\hat{K}'(0)R_\infty - Q\hat{K}(0)R_\infty\|}{|z|}\\ &+\frac{\|Q\hat{K}(z)R_\infty\|}{|z|} + \frac{\|P\hat{K}(0)R_\infty\|}{|z|^2} + \frac{\|P\hat{K}(z)R_\infty\|}{|z|^2}\Bigg)\\ \le \frac{ce^{\beta t}}{T}\cdot\Bigg(&\|Q-(I+A)R_\infty\| + \|Q\hat{K}(z)R_\infty\| + \|P\hat{K}'(0)R_\infty - Q\hat{K}(0)R_\infty\|\\ &+\|Q\hat{K}(z)R_\infty\| + \frac{\|P\hat{K}(0)R_\infty\|}{|z|} + \frac{\|P\hat{K}(z)R_\infty\|}{|z|}\Bigg)\end{aligned}$$

As $T \to \infty$ this means that $|z| \to \infty$ so from the Riemann–Lebesgue Lemma it is seen that as $|T| \to \infty$, $\hat{K}(z) \to 0$ uniformly, for $\operatorname{Re} z \ge -\alpha$. Now, due to the continuity of K, a positive constant $m < \infty$ can be found such that

$$\|H_1(z)e^{zt}\| \le \frac{m}{T}e^{\beta t},$$

for $|\operatorname{Re} z| \le \beta$, $z = \lambda + iT$, and $t > 0$. Thus,

$$\left\|\int_{-\beta+iT}^{\beta+iT} H_1(z)e^{zt}\,dz\right\| \le \int_{-\beta}^{\beta}\|H_1(\lambda+iT)e^{(\lambda+iT)t}\|\,d\lambda \le \int_{-\beta}^{\beta}\frac{m}{T}e^{\beta t}\,d\lambda \le 2\beta\frac{m}{T}e^{\beta t}.$$

It is obvious that $\int_{-\beta+iT}^{\beta+iT}\|H_1(z)e^{zt}\|dz \to 0$ as $T \to \infty$. A similar argument shows that $\int_{-\beta-iT}^{\beta-iT}\|H_1(z)e^{zt}\|dz \to 0$ as $T \to \infty$. Hence,

$$\int_{\beta-i\infty}^{\beta+i\infty} H_1(z)e^{zt}\,dz = \int_{-\beta-i\infty}^{-\beta+i\infty} H_1(z)e^{zt}\,dz$$

and

$$Y(t) = \frac{1}{2\pi i}\int_{-\beta-i\infty}^{-\beta+i\infty} H_1(z)e^{zt}\,dz,$$

as required.

It is necessary to choose an integrable function H_2 in order to obtain (2.2.14). The function H_2 is defined as follows:

$$\begin{aligned}H_2(z) &= H_1(z) - (z-c_0)^{-1}L \\ &= \frac{1}{z-c_0}\cdot\frac{1}{z+1}\cdot(I+\hat{F}(z))^{-1}\Big[(z-c_0)(z+1)\hat{G}(z) - (z+1)(I+\hat{F}(z))L\Big]\end{aligned}$$

where $L := Q(I-(I+A+\hat{K}(0))R_\infty) - P\hat{K}'(0)R_\infty$. Since H_2 is a continuous function it is clear that H_2 is integrable if $\|z^2H_2(z)\| < \infty$ as $|z|\to\infty$. Obviously,

$$\begin{aligned}\|z^2H_2(z)\| \le{}& \left|\frac{z}{z-c_0}\right|\cdot\left|\frac{z}{z+1}\right|\cdot\|(I+\hat{F}(z))^{-1}\| \\ &\times\Big[\Big\|z(z+1)\hat{G}(z)-zL\Big\| + \Big\|c_0(z+1)\hat{G}(z)\Big\| + \|L\| + \Big\|(z+1)\hat{F}(z)L\Big\|\Big].\end{aligned}$$

Firstly, both $\left|\frac{z}{z-c_0}\right|$ and $\left|\frac{z}{z+1}\right|$ tend to unity as $|z|\to\infty$. Thus, positive constants c_1 and c_2 may be found such that $\left|\frac{z}{z-c_0}\right| < c_1$ and $\left|\frac{z}{z+1}\right| < c_2$. A previous argument showed that $\det[I+\hat{F}(z)] \neq 0$ for $\operatorname{Re} z \ge -\beta$, so we can find a constant $c > 0$ such that $\|(I+\hat{F}(z))\| < c$. As (2.2.13) holds it is known from the Riemann–Lebesgue lemma that $\hat{K}(z) \to 0$ as $|z|\to\infty$ with $\operatorname{Re} z \ge -\alpha$. Thus from (2.3.2) and (2.3.4) it is seen that $\|c_0(z+1)\hat{G}(z)\|$ and $\|(z+1)\hat{F}(z)L\|$ are bounded for $\operatorname{Re} z \ge -\alpha$. Now we examine $z(z+1)\hat{G}(z) - zL$:

$$\begin{aligned}&z(z+1)\hat{G}(z) - zL \\ &\quad= zQ\Big(I-(I+A+\hat{K}(z))R_\infty\Big) - (z+1)\Big(P\hat{K}'(0)+Q\hat{K}(0)-Q\hat{K}(z)\Big)R_\infty \\ &\qquad\qquad -\frac{(z+1)}{z}P(\hat{K}(0)-\hat{K}(z))R_\infty - zL \\ &\quad= z\Big(Q(I-(I+A)R_\infty) - (P\hat{K}'(0)+Q\hat{K}(0))R_\infty - L\Big) \\ &\qquad\qquad + \Big(P\hat{K}'(0)+Q\hat{K}(0)+P\hat{K}(0)\Big)R_\infty + \hat{K}(z)R_\infty \\ &\qquad\qquad -\frac{1}{z}P\hat{K}(0)R_\infty + \frac{1}{z}\hat{K}(z)R_\infty.\end{aligned}$$

Due to the definition of L we see that (2.4) becomes

$$\|z(z+1)\hat{G}(z) - zL\| \le \left\| \left(P\hat{K}'(0) + \hat{K}(0)\right) R_\infty \right\| + \|\hat{K}(z)R_\infty\| + \frac{\|P\hat{K}(0)R_\infty\|}{|z|} + \frac{\|\hat{K}(z)R_\infty\|}{|z|},$$

when we take norms on both sides. The right hand side is clearly tending to a constant as $|z| \to \infty$. So again using the continuity of K we see that $\|z(z+1)\hat{G}(z) - zL\|$ is bounded. Combining the above arguments it is clear that $\|z^2 H_2(z)\| < \infty$ with $\operatorname{Re} z = -\beta$.

Consequently,

$$\int_{-\beta-i\infty}^{-\beta+i\infty} \|H_2(z)\|\, dz < \infty.$$

Therefore,

$$\begin{aligned}
\|Y(t)\| &\le \frac{1}{2\pi}\left\| \int_{-\beta-i\infty}^{-\beta+i\infty} H_1(z)e^{zt}\, dz \right\| \\
&\le \frac{1}{2\pi}\left\| \int_{-\beta-i\infty}^{-\beta+i\infty} H_2(z)e^{zt}\, dz \right\| + \frac{1}{2\pi}\left\| \int_{-\beta-i\infty}^{-\beta+i\infty} \frac{Le^{-zt}}{z-c_0}\, dz \right\| \\
&\le \frac{1}{2\pi}e^{-\beta t} \int_{-\beta-i\infty}^{-\beta+i\infty} \|H_2(z)\|\, dz + \frac{1}{2\pi}\|L\|e^{c_0 t}e^{-\beta t} \\
&\le ce^{-\beta t}.
\end{aligned}$$

This completes the proof. □

Proof of Theorem 2.4.2. Since assumption (2.2.8) holds, the definition of the Laplace Transform of Y is valid. Therefore $\hat{Y}$ exists and is continuous in $\{\operatorname{Re} z \ge -\beta\}$ and is analytic in $\{\operatorname{Re} z > -\beta\}$. As assumptions (2.2.7) and (2.2.8) hold Theorem 2.2.4 can be applied to get (2.2.10). Thus $\det[z\hat{Y}(z) - R_\infty]$ is nonzero at zero. From the continuity of $\hat{Y}$ at zero there exists an open neighbourhood centered at zero on which $\det[z\hat{Y}(z) - R_\infty] \ne 0$. From the definition of P it is clear that its eigenvalues are either 0 or 1. Thus $\det[zI + P]$ is nonzero except at zero in an open neighbourhood centered at zero with radius less than one. Choose $\alpha > 0$ such that $\det[z\hat{Y}(z) - R_\infty]$ and $\det[zI + P]$ are nonzero for

$-\alpha < \operatorname{Re} z < 0$. Define the function B as follows

$$\begin{aligned} B(z) = (P + zI)^{-1}\big[z^2 Q(I - (I + A)(R_\infty - \hat{Y}(z))) \\ - z(z+1)P\hat{K}'(0)R_\infty - (z+1)(P + zQ)\hat{K}(0)R_\infty \\ - z(z+1)P\hat{K}(0)\hat{Y}(z) - z^2(z+1)\hat{Y}(z)\big](z\hat{Y}(z) - R_\infty)^{-1} \end{aligned} \tag{2.4.2}$$

for $-\alpha < \operatorname{Re} z < 0$ and $B(0) := \hat{K}(0)$. Note that $B(z) := \hat{K}(z)$ for $\operatorname{Re} z \geq 0$. Proposition 2.4.1 can be applied to prove Theorem 2.4.2.

The formula for the function B is now derived: the function F can easily be expressed as

$$\hat{F}(z) = \frac{1}{z^2(z+1)}\left[-z^2 Q(I + A) + z(z+1)P\hat{K}(0) + z(P + zI)\hat{K}(z)\right],$$

and the function G can easily be expressed as

$$\begin{aligned} \hat{G}(z) = \frac{1}{z^2(z+1)}\Big[z^2 Q(I - (I + A)R_\infty) - z(z+1)P\hat{K}'(0)R_\infty \\ - (z+1)(P + zQ)\hat{K}(0)R_\infty + (P + zI)\hat{K}(z)R_\infty\Big]. \end{aligned}$$

Using the above expressions we can expand $\hat{Y}(z) + \hat{F}(z)\hat{Y}(z) = \hat{G}(z)$ to obtain

$$\begin{aligned} z^2(z+1)\hat{Y}(z) + \left[-z^2 Q(I + A) + z(z+1)P\hat{K}(0)\right]\hat{Y}(z) + z(P + zI)\hat{K}(z)\hat{Y}(z) \\ = \Big[z^2 Q(I - (I + A)R_\infty) - z(z+1)P\hat{K}'(0)R_\infty \\ - (z+1)(P + zQ)\hat{K}(0)R_\infty\Big] + (P + zI)\hat{K}(z)R_\infty. \end{aligned}$$

Rearranging the equation one obtains

$$\begin{aligned} z(P + zI)\hat{K}(z)\hat{Y}(z) - (P + zI)\hat{K}(z)R_\infty \\ = z^2 Q(I - (I + A)(R_\infty - \hat{Y}(z))) - z(z+1)P\hat{K}'(0)R_\infty \\ - (z+1)(P + zQ)\hat{K}(0)R_\infty - z(z+1)P\hat{K}(0)\hat{Y}(z) - z^2(z+1)\hat{Y}(z). \end{aligned}$$

Both $P + zI$ and $z\hat{Y}(z) - R_\infty$ are invertible in the region considered, so (2.4.2) holds. □

Proof of Lemma 2.4.1. The case where $z = 0$ and $\operatorname{Re} z \geq 0$, $z \neq 0$ are considered separately. If $z = 0$, then using the fact that $QM = M$, it is seen that

$$\det[I + \hat{F}(0)] = \det\left[I - Q(I + A + \hat{K}(0)) - P\hat{K}'(0)\right]$$
$$= \det\left[P - M + \int_0^\infty \int_s^\infty PK(u)\,du\,ds\right].$$

This is nonzero by (2.2.10) of Theorem 2.2.4. Now consider the case when $\operatorname{Re} z \geq 0$, $z \neq 0$;

$$\det[I + \hat{F}(z)] = \det\left[I - \frac{1}{z+1}Q(I+A) - \frac{1}{z+1}Q\hat{K}(z) + P\frac{1}{z}\hat{K}(0) - \frac{1}{z}P\hat{K}(z)\right]$$
$$= \frac{1}{z}\frac{1}{z+1}\det\Big[z(z+1) - zQ(I+A) - zQ\hat{K}(z)$$
$$+(z+1)P\hat{K}(0) - (z+1)P\hat{K}(z)\Big]$$
$$= \frac{1}{z}\frac{1}{z+1}\det\left[z(zI - A - \hat{K}(z)) + P(zI - A - \hat{K}(z))\right]$$
$$= \frac{1}{z}\frac{1}{z+1}\det[zI + P]\det\left[zI - A - \hat{K}(z)\right],$$

using the fact that $PM = 0$. Consider each term individually. Obviously, $\frac{1}{z}\frac{1}{z+1} \neq 0$, since $\operatorname{Re} z \geq 0$ and $z \neq 0$. The function $z \mapsto \det[zI - A - \hat{K}(z)]$ is nonzero due to Theorem 2.2.4. Finally, the function $z \mapsto \det[zI + P]$ is nonzero due to the structure of P: the matrix P has two eigenvalues, 0 and 1. So $\det[zI + P]$ is zero only at $z = 0$ and $z = -1$. From the above it is obvious that (2.4.1) holds. □

2.5 Proof of Theorem 2.2.6

Theorem 2.2.6 is a consequence of the following results:

Theorem 2.5.1. *Let K satisfy* (2.1.2) *and* (2.2.7) *and let f satisfy* (2.1.4). *Suppose that for all x_0 there is a constant vector $x_\infty(x_0, f)$ such that the solution $t \mapsto x(t; x_0, f)$ of* (2.1.3) *satisfies* (2.2.16). *If there exist constants $\alpha > 0$, $\gamma > 0$ and $c_1 > 0$ such that statement (i) of Theorem 2.2.6 holds, then there exist constants $\tilde{\beta} > 0$, independent of x_0, and $c_2 = c_2(x_0) > 0$, such that statement (ii) of Theorem 2.2.6 holds.*

Theorem 2.5.2. *Let K satisfy (2.1.2), (2.2.7) and (2.2.12) and let f satisfy (2.1.4). Suppose that for all x_0 there is a constant vector $x_\infty(x_0, f)$ such that the solution $t \mapsto x(t; x_0, f)$ of (2.1.3) satisfies (2.2.16). If there exist constants $\tilde{\beta} > 0$, independent of x_0, and $c_2 = c_2(x_0) > 0$ such that statement (ii) of Theorem 2.2.6 holds, then there exist constants $\alpha > 0$, $\gamma > 0$ and $c_1 > 0$ such that statement (i) of Theorem 2.2.6 holds.*

If a weaker condition is imposed (that is if (2.2.18) of Theorem 2.2.6(ii) only holds for a basis on initial values) then the same result holds.

Theorem 2.5.1 is proven by examining the variation of parameters representation of the solution while Theorem 2.5.2 is proven by examining the integral version of (2.1.3). The rationale for this is obvious; when showing sufficiency we have information about the underlying resolvent, but when showing necessity we know nothing about the resolvent *a priori*.

Lemma 2.5.1 is required in the proof of Theorem 2.5.1; it is however an interesting result in its own right. It gives conditions which ensure that the constant matrix R_∞ satisfies the differential equation (2.1.1) at ∞.

Lemma 2.5.1. *Let K satisfy (2.2.7). Suppose that there exists a constant matrix R_∞ such that the solution R of (2.1.1) satisfies (2.2.8). Then*

$$\left(A + \int_0^\infty K(s)\, ds\right) R_\infty = 0.$$

Lemma 2.5.2 is the analogue of Lemma 2.5.1. It states conditions that ensure the vector x_∞ satisfies (2.1.3) at ∞. It is required in the proof of Theorem 2.5.2.

Lemma 2.5.2. *Let K satisfy (2.2.7) and let f satisfy (2.1.4). Suppose that there exists a constant matrix x_∞ such that the solution x of (2.1.3) satisfies (2.2.16) and $x(t) \to x_\infty$ as $t \to \infty$. Then*

$$\left(A + \int_0^\infty K(s)\, ds\right) x_\infty = 0.$$

The proof of both Lemma 2.5.1 and 2.5.2 may be found at the end of this section.

Proof of Theorem 2.5.1. Using (2.1.4) and (2.1.5) it is seen that $\lim_{t\to\infty} x(t) = x_\infty$ where x_∞ is finite and

$$x_\infty = R_\infty \left(x_0 + \int_0^\infty f(s)\,ds \right),$$

which implies

$$\begin{aligned} x(t) - x_\infty &= R(t)x_0 + (R * f)(t) - R_\infty \left(x_0 + \int_0^\infty f(s)\,ds \right) \\ &= (R(t) - R_\infty)x_0 + \int_0^t (R(t-s) - R_\infty) f(s)\,ds - R_\infty f_1(t). \end{aligned}$$

Integrating $((R - R_\infty) * f)(t)$ by parts one obtains

$$\begin{aligned} x(t) - x_\infty = (R(t) - R_\infty)x_0 &- (R(0) - R_\infty) f_1(t) \\ &+ (R(t) - R_\infty) f_1(0) - \int_0^t R'(t-s) f_1(s)\,ds - R_\infty f_1(t). \end{aligned} \tag{2.5.1}$$

Due to the fact that K obeys (2.2.13) it follows that $R - R_\infty$ decays exponentially by Theorem 2.2.5. It is proved in the sequel that R' decays exponentially; f_1 also decays exponentially so therefore the convolution of R' and f_1 decays exponentially. By use of the above facts and the hypothesis (2.2.17) on f_1, it is clear that each term on the right hand side of (2.5.1) decays exponentially, which yields (2.2.18).

It is now shown that R' decays exponentially. The resolvent equation (2.1.1) can be rewritten as

$$\begin{aligned} R'(t) = A(R(t) - R_\infty) &+ \int_0^t K(t-s)(R(s) - R_\infty)\,ds \\ &- K_1(t) R_\infty + \left(A + \int_0^\infty K(s) ds \right) R_\infty, \end{aligned} \tag{2.5.2}$$

where

$$K_1(t) = \int_t^\infty K(s)\,ds, \quad t \geq 0. \tag{2.5.3}$$

The first term on the right hand side of (2.5.2) decays exponentially since (2.2.13) holds. An argument is now provided to show that the second term decays exponentially. Since $R - R_\infty$ decays exponentially and (2.2.13) holds it is possible to choose μ such that the functions $t \mapsto e^{\mu t}K(t)$ and $t \mapsto e^{\mu t}(R(t) - R_\infty)$ are both in $L^1((0,\infty), M_{n\times n}(\mathbb{R}))$. The convolution of two integrable functions is itself an integrable function, so

$$e^{\mu t}\left\|\int_0^t K(t-s)(R(s)-R_\infty)\,ds\right\|$$
$$= \left\|\int_0^t e^{\mu(t-s)}K(t-s)e^{\mu s}(R(s)-R_\infty)\,ds\right\| \le c. \quad (2.5.4)$$

Clearly $\int_0^t K(t-s)(R(s)-R_\infty)\,ds$ decays exponentially. It is now shown that the third term on the right hand side of (2.5.2) decays exponentially using the following argument:

$$\bar{K}_\alpha := \int_0^\infty \|K(s)\|e^{\alpha s}ds \ge \int_t^\infty \|K(s)\|e^{\alpha s}ds$$
$$\ge e^{\alpha t}\int_t^\infty \|K(s)\|ds \ge e^{\alpha t}\|K_1(t)\|. \quad (2.5.5)$$

where K_1 is defined by (2.5.3). Finally, it is shown that $(A + \int_0^\infty K(s)\,ds)R_\infty = 0$ in Lemma 2.5.1. Combining the above observations it is seen that R' decays exponentially to 0. □

Proof of Theorem 2.5.2. Firstly assumption (2.2.13) is proved. As (2.2.18) holds for all x_0 the following $n+1$ solutions of (2.1.3) can be considered; $\{x_j(t)\}_{j=1,\cdots,n+1}$ where

$$x_j(0) = \mathbf{e}_j \quad \text{for } j = 1, \cdots, n \quad \text{and} \quad x_{n+1}(0) = 0.$$

It is known that $x_j(t)$ approaches $x_j(\infty)$ exponentially fast. Introduce $s_j(t) = x_j(t) - x_{n+1}(t)$ and notice $s_j(0) = \mathbf{e}_j$. Let $S(t) = [s_1(t), \ldots, s_n(t)] \in M_{n\times n}(\mathbb{R})$. Then

$$S'(t) = AS(t) + (K * S)(t), \; t > 0; \quad S(0) = I.$$

Define $s_j(\infty) = x_j(\infty) - x_{n+1}(\infty)$. Then $S(t) \to S_\infty$ exponentially fast, so Theorem 2.4.2 can be applied to obtain (2.2.13). Note that the rate of convergence of K is independent of x_0.

Using the above, one can now prove (2.2.17). As (2.2.18) holds for all initial conditions we can choose $x_0 = 0$ to simplify calculations. Integrating (2.1.3) the following is obtained:

$$x(t) = \int_0^t Ax(s)\,ds + \int_0^t (K * x)(s)\,ds + \int_0^t f(s)\,ds.$$

The objective now is to write f in terms of K and $x - x_\infty$. Adding and subtracting x_∞ from both sides it is seen that

$$\begin{aligned} x(t) - x_\infty = -x_\infty &+ \int_0^t A(x(s) - x_\infty)\,ds \\ &+ \int_0^t \int_0^s K(s-u)(x(u) - x_\infty)\,du\,ds \\ &+ \int_0^t Ax_\infty\,ds + \int_0^t \int_0^s K(s-u)x_\infty\,du\,ds + \int_0^t f(s)\,ds. \end{aligned}$$

It is shown in the proof of Lemma 2.5.2 that $(A + \int_0^\infty K(u)\,du)x_\infty = 0$, so

$$\begin{aligned} x(t) - x_\infty = -x_\infty &+ \int_0^t A(x(s) - x_\infty)\,ds \\ &+ \int_0^t \int_0^s K(s-u)(x(u) - x_\infty)\,du\,ds \\ &- \int_0^t K_1(s)\,ds\,x_\infty + \int_0^t f(s)\,ds. \qquad (2.5.6) \end{aligned}$$

Changing the order of integration and introducing a change of variable in (2.5.6) the following is obtained:

$$\begin{aligned} x(t) - x_\infty = -x_\infty &+ \int_0^t \left(A + \int_0^{t-u} K(v)\,dv\right)(x(u) - x_\infty)\,du \\ &- \int_0^t K_1(u)\,du\,x_\infty + \int_0^t f(s)\,ds. \end{aligned}$$

A further rearrangement yields

$$x(t) - x_\infty = -x_\infty + \left(A + \int_0^\infty K(v)\,dv\right)\int_0^t (x(u) - x_\infty)\,du$$
$$- \int_0^t K_1(s)\,ds\,x_\infty - \int_0^t K_1(t-u)(x(u) - x_\infty)\,du + \int_0^t f(s)\,ds. \quad (2.5.7)$$

Allowing $t \to \infty$ in (2.5.6) it is seen that

$$- x_\infty = -\int_0^\infty A(x(s) - x_\infty)\,ds$$
$$- \int_0^\infty \int_0^s K(s-u)(x(u) - x_\infty)\,du\,ds$$
$$+ \int_0^\infty K_1(u)\,du\,x_\infty - \int_0^\infty f(s)\,ds. \quad (2.5.8)$$

An application of (1.1.2) yields

$$- x_\infty = -\int_0^\infty A(x(s) - x_\infty)\,ds - \int_0^\infty K(s)\,ds \int_0^\infty (x(s) - x_\infty)\,ds$$
$$+ \int_0^\infty K_1(u)\,du\,x_\infty - \int_0^\infty f(s)\,ds. \quad (2.5.9)$$

Substituting this expression for $-x_\infty$ into (2.5.7) one obtains

$$x(t) - x_\infty = -\left(A + \int_0^\infty K(v)\,dv\right)\int_t^\infty (x(u) - x_\infty)\,du$$
$$+ \int_t^\infty K_1(u)\,du\,x_\infty - f_1(t) - \int_0^t K_1(t-u)(x(u) - x_\infty)\,du.$$

This yields

$$f_1(t) = -\left(A + \int_0^\infty K(u)\,du\right)\int_t^\infty (x(s) - x_\infty)\,ds + \int_t^\infty K_1(u)\,du\,x_\infty$$
$$- (x(t) - x_\infty) - \int_0^t K_1(t-u)(x(u) - x_\infty)\,du. \quad (2.5.10)$$

It is now shown that f_1 decays exponentially. The first and third term on the right hand side of (2.5.10) decay exponentially to zero due to assumption (2.2.18). As K has been shown to obey (2.2.13) it is seen from (2.5.5) that $\int_t^\infty \|K_1(s)\| \leq \bar{K}_\alpha e^{-\alpha t}$, $t \geq 0$. By this estimate it is clear that the second and fourth terms decay exponentially to zero. Note that since K is independent of x_0 and $\|x(t) - x_\infty\| \le c_2(0)e^{-\beta t} = c_2 e^{-\beta t}$ the rate of decay of f_1 is independent of x_0. $\square$

Proof of Lemma 2.5.1. Integrate (2.5.2) over $[0, t]$ and rearrange the terms to obtain:

$$-\left(A + \int_0^\infty K(s)\,ds\right) R_\infty t = \int_0^t A(R(s) - R_\infty)\,ds$$
$$+ \int_0^t \int_0^s K(s-u)(R(u) - R_\infty)\,du\,ds - \int_0^t K_1(s)R_\infty\,ds - (R(t) - R_0).$$

The first term on the right hand side of the equation is bounded as (2.2.8) holds. The second term on the right hand side of the equation is bounded as the convolution of an L^1 function with an L^1 function lies in L^1. The third term is bounded due to (2.2.7). Due to the fact that assumption (2.2.8) holds and R converges to a finite matrix R_∞ the final term is bounded. Combining these arguments it is seen that $(A + \int_0^\infty K(s)\,ds)R_\infty = 0$. $\square$

Proof of Lemma 2.5.2. To prove this consider (2.1.3) and add and subtract x_∞ from the right hand side of the equation to obtain

$$x'(t) = A(x(t) - x_\infty) + \int_0^t K(t-s)(x(s) - x_\infty)\,ds$$
$$+ f(t) - K_1(t)x_\infty + \left(A + \int_0^\infty K(s)\,ds\right) x_\infty.$$

Integrating and rearranging terms the following is obtained

$$\left(A + \int_0^\infty K(s)\,ds\right) x_\infty t = (x(t) - x_\infty) - (x_0 - x_\infty) - \int_0^t A(x(s) - x_\infty)\,ds$$
$$- \int_0^t \int_0^s K(s-u)(x(u) - x_\infty)\,du\,ds - \int_0^t f(s) + \int_0^t K_1(s)x_\infty\,ds. \quad (2.5.11)$$

The first term on the right hand side of (2.5.11) is finite since (2.2.16) holds and $x(t) \to x_\infty$ as $t \to \infty$. The second term is finite since both x_0 and x_∞ are finite. The third term is finite since (2.2.16) holds. The fourth term is finite since an L^1 term convolved with an

L^1 term is in L^1. The fifth term is finite since (2.1.4) holds. Finally, the sixth term is finite since (2.2.7) holds.

The left hand side of (2.5.11) tends to ∞ if $\left(A + \int_0^\infty K(s)\,ds\right) x_\infty \neq 0$, while the right hand side of (2.5.11) is finite. This is contradictory, thus $\left(A + \int_0^\infty K(s)\,ds\right) x_\infty = 0$. $\square$

Chapter 3

Mean square convergence

3.1 Introduction

Here the asymptotic convergence of the solution of

$$dX(t) = \left(AX(t) + \int_0^t K(t-s)X(s)\,ds\right) dt + \Sigma(t)dB(t), \quad t > 0, \tag{3.1.1a}$$

$$X(0) = X_0, \tag{3.1.1b}$$

to a random variable is studied. It is assumed that X_0 is a deterministic n–dimensional vector, that K satisfies (2.1.2) and that the function $\Sigma : [0,\infty) \to M_{n\times d}(\mathbb{R})$ satisfies

$$\Sigma \in C([0,\infty), M_{n\times d}(\mathbb{R})). \tag{3.1.2}$$

Let $\{B(t)\}_{t\geq 0}$ denote d–dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}^B\}_{t\geq 0}, \mathbb{P})$ where the filtration is the natural filtration of B namely, $\mathcal{F}^B(t) = \sigma\{B(s) : 0 \leq s \leq t\}$. The function $t \mapsto X(t; X_0, \Sigma)$ is defined to be the unique continuous adapted process which satisfies the initial value problem (3.1.1). Results concerning the existence and uniqueness of solutions may be found in [9, Theorem 2E] or [37, Chapter 5] for example. As Σ is continuous, for any deterministic initial condition X_0 there exists a unique a.s. continuous and adapted process obeying (3.1.1) given by

$$X(t; X_0, \Sigma) = R(t)X_0 + \int_0^t R(t-s)\Sigma(s)dB(s), \quad t \geq 0. \tag{3.1.3}$$

Again, it is noted that when X_0 and Σ are clear from the context they are omitted from the notation.

In the stochastic case the asymptotic convergence of solutions of (3.1.1) to equilibrium alone has been considered. Authors who have studied this type of convergence include Appleby and Riedle [8], Mao [27] and Mao and Riedle [28]. This work is the first of its type to consider the mean square convergence of solutions of (3.1.1) to non–equilibrium limits. The levels of environmental noise by which the system can be perturbed while still maintaining convergence to an explicit non–equilibrium limit X_∞ is established. These results suggest that when considering the convergence of the system to a non–trivial random variable the square integrability of the noise term is crucial. Conditions for the integrability of $X - X_\infty$ in the mean square case are also considered. Here, the establishment of the necessary and sufficient conditions on the resolvent, kernel and noise is complicated by the fact that X_∞ is not $\mathcal{F}(t)$–adapted. Consequently standard Itô methods cannot be directly applied. The results presented in this chapter have counterparts in the almost sure case, the details of which will be presented in Chapter 4.

3.2 Discussion of Results

The main results of the chapter are presented in this section. The necessary and sufficient conditions for asymptotic convergence of solutions of (3.1.1) to a nontrivial limit and the integrability of these solutions in the mean square sense are considered.

Krisztin and Terjéki considered the case where $R - R_\infty$ is in the space of $L^1(0,\infty)$ functions. However, in the stochastic case it is more natural to consider the case where $R - R_\infty$ lies in $L^2(0,\infty)$. Consequently, Theorem 2.2.4 is converted from the space of integrable functions to the space of square integrable functions. This conversion may be found in Lemma 3.3.1. Furthermore, it is found that an assumption on the existence of the first moment of K—rather than on the second moment as required in (2.2.7) of Theorem 2.2.4—is adequate for the convergence and integrability of solutions of equation (2.1.1). This is due to the fact that $R - R_\infty \in L^2(0,\infty)$ is assumed *a priori* in this case, while Krisztin and Terjéki were interested in the necessary and sufficient conditions to establish $R - R_\infty \in L^1(0,\infty)$.

Initially, sufficient conditions for convergence of solutions to a non–equilibrium limit are considered.

Theorem 3.2.1. *Let K satisfy* (2.1.2) *and*

$$\int_0^\infty t\|K(t)\|\,dt < \infty, \tag{3.2.1}$$

and let Σ satisfy (3.1.2) and

$$\int_0^\infty \|\Sigma(t)\|^2\,dt < \infty. \tag{3.2.2}$$

If the resolvent R of (2.1.1) satisfies

$$R(\cdot) - R_\infty \in L^2((0,\infty), M_{n\times n}(\mathbb{R})), \tag{3.2.3}$$

then the solution X of (3.1.1) satisfies $X(t) \to X_\infty$ as $t \to \infty$ almost surely, where

$$X_\infty = R_\infty \left(X_0 + \int_0^\infty \Sigma(t)\,dB(t) \right) \quad a.s. \tag{3.2.4}$$

and X_∞ is almost surely finite.

The following theorem concerns sufficient conditions for asymptotic convergence in mean square of the solutions of (3.1.1) to a nontrivial limit X_∞ and the integrability of $X - X_\infty$ in the mean square sense. As in the almost sure sense, it is found that conditions (3.2.2) and (3.2.3) are required for convergence in mean square. Additionally, in order to ensure integrability, an assumption on the existence of the first moment of the noise term is required.

Theorem 3.2.2. *Let K satisfy (2.1.2) and (3.2.1) and let Σ satisfy (3.1.2). If Σ satisfies (3.2.2) and there exists a constant matrix R_∞ such that the solution R of (2.1.1) satisfies (3.2.3), then for all initial conditions X_0 there is an almost surely finite $\mathcal{F}^B(\infty)$-measurable random variable $X_\infty(X_0,\Sigma)$ with $\mathbb{E}\,\|X_\infty(X_0,\Sigma)\|^2 < \infty$, such that the unique continuous adapted process $X(\cdot\,; X_0,\Sigma)$ which obeys (3.1.1) satisfies*

$$\lim_{t\to\infty} \mathbb{E}\,\|X(t; X_0,\Sigma) - X_\infty(X_0,\Sigma)\|^2 = 0. \tag{3.2.5}$$

Moreover, if the function Σ also satisfies

$$\int_0^\infty t\|R_\infty\Sigma(t)\|^2\,dt < \infty, \tag{3.2.6}$$

then

$$\mathbb{E}\,\|X(\cdot\,; X_0,\Sigma) - X_\infty(X_0,\Sigma)\|^2 \in L^1((0,\infty),\mathbb{R}). \tag{3.2.7}$$

Furthermore, it is possible to prove that conditions (3.2.2), (3.2.3) and (3.2.6) are necessary for the asymptotic convergence of the solution X of (3.1.1) to the non–equilibrium limit X_∞, and for the integrability of $X - X_\infty$ in the mean square sense.

Theorem 3.2.3. *Let K satisfy (2.1.2) and (3.2.1) and let Σ satisfy (3.1.2). Suppose for all initial conditions X_0 there is an a.s. finite $\mathcal{F}^B(\infty)$-measurable random variable $X_\infty(X_0,\Sigma)$, with $\mathbb{E}\,\|X_\infty(X_0,\Sigma)\|^2 < \infty$, such that the unique continuous adapted process $X(\cdot\,; X_0,\Sigma)$ which obeys (3.1.1) satisfies (3.2.5) and (3.2.7). Then there exists a constant matrix R_∞ such that the solution R of (2.1.1) satisfies (3.2.3) and the function Σ satisfies (3.2.2) and (3.2.6).*

Theorems 3.2.2 and 3.2.3 can be combined to obtain the following equivalence.

Theorem 3.2.4. *Let K satisfy (2.1.2) and (3.2.1) and let Σ satisfy (3.1.2). The following are equivalent.*

(i) There exists a constant matrix R_∞ such that the solution R of (2.1.1) satisfies (3.2.3) and the function Σ satisfies (3.2.2) and (3.2.6).

(ii) For all initial conditions X_0 there is an a.s. finite $\mathcal{F}^B(\infty)$-measurable random variable $X_\infty(X_0,\Sigma)$ with $\mathbb{E}\,\|X_\infty(X_0,\Sigma)\|^2 < \infty$ such that the unique continuous adapted process $X(\cdot\,; X_0,\Sigma)$ which obeys (3.1.1) satisfies (3.2.5) and (3.2.7).

(iii) There is a basis of initial conditions $\{X_j(0)\}_{1\le j\le n}$ and a collection of a.s. finite $\mathcal{F}^B(\infty)$-measurable random variables $\{X_j(\infty)\}_{1\le j\le n}$ with $\mathbb{E}\,\|X_j(\infty)\|^2 < \infty$, such that for each $j = 1,\dots,n$ the unique continuous adapted process $X_j = X(\cdot\,; X_j(0),\Sigma)$ satisfies

$$\lim_{t\to\infty} \mathbb{E}\,\|X_j(t) - X_j(\infty)\|^2 = 0, \tag{3.2.8}$$

and

$$\mathbb{E}\,\|X_j(\cdot) - X_j(\infty)\|^2 \in L^1((0,\infty),\mathbb{R}). \tag{3.2.9}$$

The proof of Theorems 3.2.2, 3.2.3 and 3.2.4 may be found in Section 3.4.

3.3 Convergence of Deterministic and Stochastic Equations to a Nontrivial Limit

In this section, sufficient conditions for convergence of solutions to an explicit non–equilibrium are established. Firstly, the result by Krisztin and Terjéki is converted from the space of integrable functions to the space of square integrable functions. This result is presented in Lemma 3.3.1, the proof of which is technical and deferred to Section 3.6.

Lemma 3.3.1. *Let K satisfy* (2.1.2) *and* (3.2.1). *If there exists a constant matrix C such that the resolvent R of* (2.1.1) *satisfies*

$$R(\cdot) - C \in L^2((0,\infty), M_{n\times n}(\mathbb{R})), \tag{3.3.1}$$

then

$$\det\left[P - M + \int_0^\infty \int_s^\infty PK(u)du\,ds\right] \neq 0. \tag{3.3.2}$$

Moreover, if the constant R_∞ is defined by

$$R_\infty = \left[P - M + \int_0^\infty \int_s^\infty PK(u)\,du\,ds\right]^{-1} P,$$

then $R(t) \to R_\infty$ as $t \to \infty$ and $C = R_\infty$.

The proof of Lemma 3.3.1 relies on proving that, given a particular criterion, the rank of a certain matrix is less than the rank of another. The proof of this is interesting in its own right and so it is stated as the following lemma:

Lemma 3.3.2. *Let K satisfy (2.1.2) and (3.2.1). Suppose that there exists a constant matrix R_∞ such that the resolvent R of (2.1.1) satisfies (3.2.3). If*

$$\det\left[P - M + \int_0^\infty \int_s^\infty PK(u)du\,ds\right] = 0, \tag{3.3.3}$$

then rank (PDN) < rank (N) where $D = I + \int_0^\infty \int_s^\infty K(u)\,du\,ds$ and the matrix N is defined as $N = TD_NT^{-1}$. Here $D_N = diag(f_1, f_2, ..., f_n)$ with $f_i = 1$ if all the elements of the i^{th} column of J are zero, and $f_i = 0$ otherwise.

We prove the result in Section 3.5. Some illustrative examples are included in this section to highlight the complexities of this result.

The following lemma is used in the proof of Theorem 3.2.1. The details of the proof is deferred to Section 3.6.

Lemma 3.3.3. *If K satisfies (2.1.2) and (3.2.1). If there exists a constant matrix R_∞ such that the resolvent R of (2.1.1) satisfies (3.2.3), then*

$$\det[I + \hat{F}(z)] \neq 0, \quad \operatorname{Re} z \geq 0, \tag{3.3.4}$$

where F is defined by (2.3.7).

Proof of Theorem 3.2.1. Begin by writing (3.1.1) in integral form over $[0, s]$, then multiply both sides of the equation by $\Phi'(t-s)$, where $\Phi(t) = P + e^{-t}Q$. Integration of the resulting equation over $[0, t]$ leads to:

$$X(t) + (F * X)(t) = G_S(t), \tag{3.3.5}$$

where the function F is given by (2.3.7), G_S is given by

$$G_S(t) = \Phi(t)X_0 + \mu(t) + \int_0^t \Phi'(t-u)\mu(u)\,du, \tag{3.3.6}$$

and $\mu(t) = \int_0^t \Sigma(s)\,dB(s)$. Now using Itô's Lemma with $dY(t) = \Sigma(s)\,dB(s)$ and $V(y,t) = e^t y$ we see that

$$\int_0^t \Phi'(t-s)\mu(s)\,ds = -Q\mu(t) + Q\int_0^t e^{-(t-s)}\Sigma(s)\,dB(s),$$

and so

$$G_S(t) = \Phi(t)X_0 + \int_0^t \Phi(t-u)\Sigma(u)\,dB(u). \tag{3.3.7}$$

From Theorem 1.1.2, it is known that X can be expressed as

$$X(t) = G_S(t) - \int_0^t r(t-s)G_S(s)\,ds, \tag{3.3.8}$$

where the function r satisfies $r + F * r = F$ and $r + r * F = F$. In order to consider the asymptotic behaviour of X in (3.3.8) we begin by looking at the asymptotic behaviour of G_S in (3.3.6). Clearly, $\Phi(t) \to PX_0$ as $t \to \infty$. Now consider the second term in (3.3.6). Since (3.2.2) holds, we see using the Martingale Convergence Theorem that $\lim_{t\to\infty} \mu(t) = \mu(\infty)$. Now consider the third term in (3.3.6). Since $\lim_{t\to\infty} \mu(t) = \mu(\infty)$ and Φ is integrable, as $t \to \infty$, we have

$$\begin{aligned}
\int_0^t \Phi'(t-s)\mu(s)\,ds &= \int_0^t \Phi'(t-s)(\mu(s) - \mu(\infty))\,ds + \int_0^t \Phi'(t-s)\mu(\infty)\,ds \\
&\to -\int_0^\infty Qe^{-s}\,ds\,\mu(\infty) \\
&= -Q\mu(\infty).
\end{aligned}$$

Combining the above we see that the first term on the right hand side of (3.3.8) has finite limit

$$G_S(\infty) = PX_0 + P\int_0^\infty \Sigma(t)\,dB(t), \qquad \text{a.s.} \tag{3.3.9}$$

Now, consider the second term on the right hand side of (3.3.8) as $t \to \infty$. Due to Theorem 2.2.3 and Lemma 3.3.3 it is seen that r is integrable. Thus one can integrate $r + F * r = F$ over $[0, \infty)$ and rearrange the equation to obtain

$$\int_0^\infty r(s)\,ds = \left(I + \int_0^\infty F(s)\,ds\right)^{-1} \int_0^\infty F(s)\,ds.$$

Now consider the asymptotic behaviour of $\int_0^t r(t-u)G_S(u)\,du$:

$$\lim_{t\to\infty}\int_0^t r(t-s)G_S(s)\,ds = \left(I+\int_0^\infty F(s)\,ds\right)^{-1}\int_0^\infty F(s)\,ds\,G_S(\infty). \tag{3.3.10}$$

It was shown in the proof of Lemma 2.4.1 that $I+\hat{F}(0) = P - M + \int_0^\infty \int_s^\infty PK(u)du\,ds$.

So letting $t\to\infty$ and using the definition of R_∞ given in (2.2.11) it is seen that by taking limits as $t\to\infty$ in (3.3.8) we get

$$\lim_{t\to\infty} X(t) := X_\infty = G_S(\infty) - \int_0^\infty r(s)\,ds\,G_S(\infty) = R_\infty\left(X_0 + \int_0^\infty \Sigma(t)\,dB(t)\right),$$

by combining (3.3.9) and (3.3.10). From assumption (3.2.2) and Lemma 3.3.1 it is seen that X_∞ is finite a.s. This completes the proof. □

3.4 Conditions for Convergence and Integrability in Mean Square

In this section, necessary and sufficient conditions for asymptotic convergence of solutions of (3.1.1) to a nontrivial random variable in the mean square sense are considered. Two technical lemmas used in the proof of Theorem 3.2.3 are presented. Lemma 3.4.1 is the analogue of Lemma 2.5.1 in the stochastic case; it concerns the structure of X_∞. This result enables us to prove Lemma 3.4.2, which concerns the necessity of (3.2.2) for stability of the system. Consequently, one need only assume the continuity of the noise term (which ensures the existence of solutions) at the outset. Lemma 3.4.2 in turn allows us to show the necessity of (3.2.6), the details of which may be found in the proof of Theorem 3.2.3, given below.

Lemma 3.4.1. *Let K satisfy* (2.1.2) *and* (3.2.1). *Suppose that for all initial conditions X_0 there is a $\mathcal{F}^B(\infty)$-measurable and almost surely finite random variable $X_\infty(X_0,\Sigma)$ with $\mathbb{E}\,\|X_\infty\|^2<\infty$ such that the solution $t\mapsto X(t;X_0,\Sigma)$ of* (3.1.1) *satisfies* (3.2.5) *and*

(3.2.7). *Then,* X_∞ *obeys*

$$\left(A+\int_0^\infty K(s)\,ds\right)X_\infty = 0 \quad a.s.$$

Lemma 3.4.2. *Let* K *satisfy* (2.1.2) *and* (3.2.1). *Suppose for all initial conditions* X_0 *there is an almost surely finite random variable* $X_\infty(X_0,\Sigma)$ *which obeys* $\mathbb{E}\,\|X_\infty(X_0,\Sigma)\|^2 < \infty$ *such that the solution* $t\mapsto X(t;X_0,\Sigma)$ *of* (3.1.1) *satisfies* (3.2.5) *and* (3.2.7). *Then* Σ *satisfies* (3.2.2).

The proofs of these Lemmata are deferred until Section 3.6.

Proof of Theorem 3.2.2. Using Theorem 3.2.1, it is shown that $\mathbb{E}\,\|X_\infty\|^2 < \infty$ as follows:

$$\mathbb{E}\,\|X_\infty\|^2 = \mathbb{E}\left[\operatorname{tr}\,X_\infty X_\infty^T\right] = \|R_\infty X_0\|^2 + \int_0^\infty \|R_\infty \Sigma(s)\|^2\,ds. \tag{3.4.1}$$

By applying Lemma 3.3.1, it is seen that the first term on the right hand side of (3.4.1) is finite. Furthermore, the second term is finite by (3.2.2).

It is now shown that (3.2.5) holds. By considering the solution of (3.1.1) expressed in terms of variation of parameters and the expression obtained for X_∞ obtained in Theorem 3.2.1 it is seen that

$$\begin{aligned}\mathbb{E}\,\|X(t)-X_\infty\|^2 &= \mathbb{E}\left[\operatorname{tr}\,(X(t)-X_\infty)(X(t)-X_\infty)^T\right]\\ &= \|(R(t)-R_\infty)X_0\|^2 + \int_0^t \|(R(t-s)-R_\infty)\Sigma(s)\|^2\,ds \\ &\qquad + \int_t^\infty \|R_\infty\Sigma(s)\|^2\,ds.\end{aligned} \tag{3.4.2}$$

From Lemma 3.3.1, it is clear that the first term on the right side of (3.4.2) tends to 0 as $t\to\infty$. Now consider the second term. Using the fact that $R(t)\to R_\infty$ as $t\to\infty$ and (3.2.2) it is seen that $\lim_{t\to\infty}\int_0^t \|(R(t-s)-R_\infty)\Sigma(s)\|^2\,ds = 0$. Finally, the third term is examined. Using the using the continuity of Σ and the existence of R_∞ it is seen that $\lim_{t\to\infty}\int_t^\infty \|R_\infty\Sigma(s)\|^2\,ds = 0$. Combining these observations it is clear that (3.2.5) holds.

In order to show that (3.2.7) holds, the right hand side of (3.4.2) is considered. The first term is in $L^1(0,\infty)$ since (3.2.3) holds. The convolution of two $L^1(0,\infty)$ functions is itself in $L^1(0,\infty)$, so the second term is in $L^1(0,\infty)$. Finally, as (3.2.6) holds it is seen that the third term is in $L^1(0,\infty)$. From this analysis it is evident that (3.2.7) holds. □

Proof of Theorem 3.2.3. Firstly, assumption (3.2.3) is proved. Using Hölder's inequality the following is obtained:

$$\begin{aligned}\|(R(t)-R_\infty)X_0\| &= \|\mathbb{E}\left[X(t)-X_\infty\right]\| \\ &\leq \mathbb{E}\|X(t)-X_\infty\| \leq \mathbb{E}\left[\|(X(t)-X_\infty)\|^2\right]^{\frac{1}{2}}. \quad (3.4.3)\end{aligned}$$

Squaring both sides yields

$$\|(R(t)-R_\infty)X_0\|^2 \leq \mathbb{E}\|(X(t)-X_\infty)\|^2. \quad (3.4.4)$$

Since (3.2.7) holds the required result is proven.

By Lemma 3.4.2, (2.1.2), (3.2.1), (3.2.5) and (3.2.7) imply (3.2.2).

Now using (3.2.7), which was given as a hypothesis to Theorem 3.2.3, together with (3.2.2) and (3.2.3), it is shown that (3.2.6) holds. Rearranging equation (3.4.2) the following is obtained:

$$\begin{aligned}\int_t^\infty \|R_\infty\Sigma(s)\|^2\,ds &= \mathbb{E}\|X(t)-X_\infty\|^2 \\ &\quad - \|(R(t)-R_\infty)X_0\|^2 - \int_0^t \|(R(t-s)-R_\infty)\Sigma(s)\|^2\,ds. \quad (3.4.5)\end{aligned}$$

Each term on the right hand side of (3.4.5) is in $L^1(0,\infty)$ so (3.2.6) must hold. This completes the proof. □

Proof of Theorem 3.2.4. Showing that (i) implies (ii), and that (ii) implies (i) is the subject of Theorems 3.2.2 and 3.2.3 respectively.

It is now shown that (i) implies (iii). For $j = 1, \dots, n$ it can be shown that (3.2.8) and (3.2.9) hold by observing that

$$\mathbb{E}\,\|X_j(t) - X_j(\infty)\|^2 = \|(R(t) - R_\infty)X_j(0)\|^2 + \int_0^t \|(R(t-s) - R_\infty)\Sigma(s)\|^2\,ds + \int_t^\infty \|R_\infty\Sigma(s)\|^2\,ds.$$

Using an identical argument to that used in the proof of Theorem 3.2.2 it can be shown that (3.2.2), (3.2.3) and (3.2.6) imply (3.2.8) and (3.2.9) for $j = 1, \dots, n$.

It is now proven that (iii) implies (i). Using the argument given by (3.4.3) and (3.4.4) for $j = 1, \dots, n$ it is seen that $(R - R_\infty)X_j(0)$ is square integrable. Since $X_1(0), \dots, X_n(0)$ form a basis in $\mathbb{R}^n$ it follows that each column $R_j(t) - R_j(\infty)$ of $R(t) - R_\infty$ is square integrable; thus (3.2.3) must hold. It is clear that Lemma 3.4.2 holds for any initial condition $X_j(0)$, so (3.2.2) holds. Again, for any initial condition $X_j(0)$ the argument given by (3.4.5) can be used to prove that (3.2.6) holds. This completes the proof.

□

3.5 Proof of Lemma 3.3.2 and Explanatory Examples

In this section we prove Lemma 3.3.2. Some illustrative examples are included to highlight the complexities of this result. The proof of Lemma 3.3.2 follows the line of reasoning in [24]; here, the details have been fleshed out.

Proof of Lemma 3.3.2. It is now shown that rank $PDN <$ rank N if (3.3.3) holds. It can easily be seen that the assumption $\det[PD - M] = 0$ excludes the case where M has no zero eigenvalues. Suppose that M has rank r. Consequently, the matrix J has rank r and the matrices D_P and D_N have rank $n - r$. Without loss of generality we can assume that the zero eigenvalues of M occupy the $(r+1)^{th}$ to the n^{th} diagonal position in J, so there exists an invertible $r \times r$–dimensional matrix J_1 and a matrix J_2 in Jordon canonical form such that

$$J = \begin{pmatrix} J_1 & 0_{r\times(n-r)} \\ 0_{(n-r)\times r} & J_2 \end{pmatrix},$$

Now consider $PD - M$:

$$PD - M = TD_PT^{-1}D - TJT^{-1} = T[D_PT^{-1}DT - J]T^{-1} = T[D_P\tilde{D} - J]T^{-1},$$

where $\tilde{D} = T^{-1}DT$ and D_P is defined as in Section 2.2. So we are interested in $D_P\tilde{D} - J$. By definition, the i^{th} diagonal entry, $[D_P]_{ii}$, of D_P is 1 if and only if the i^{th} row of J is zero. Thus, the i^{th} row of $D_P\tilde{D}$ is equal to the i^{th} row of $\tilde{D}$ if $[D_P]_{ii}$ is nonzero, and is a zero row otherwise. Obviously, $D_P\tilde{D}$ has a zero row if and only if the corresponding row in J is nonzero so the ij^{th} entry of $D_P\tilde{D} - J$ is either $\tilde{D}_{ij}$ or $-J_{ij}$. As $\det[D_P\tilde{D} - J] = 0$ it is seen from the structure of J that there must be a linear dependence in the columns of $D_P\tilde{D} - J$ that correspond to the zero columns of J.

When the matrix $D_P\tilde{D} - J$ is postmultiplied by D_N the resulting matrix, $(D_P\tilde{D} - J)D_N$, consists of zero columns and nonzero columns. Clearly, the i^{th} column of $(D_P\tilde{D} - J)D_N$ is zero if $[D_N]_{ii}$ is 0 and is equal to the i^{th} column of $D_P\tilde{D} - J$ if $[D_N]_{ii} = 1$. As $[D_N]_{ii} = 0$ if the i^{th} column of J is nonzero it is clear that the nonzero columns of $(D_P\tilde{D} - J)D_N$ correspond to the zero columns of J. That is the columns consist only of entries of $\tilde{D}$; entries of J are excluded. These nonzero columns are in fact the linearly dependent columns mentioned above.

When the matrix $D_P\tilde{D} - J$ is postmultiplied by D_N, the resulting matrix, $(D_P\tilde{D} - J)D_N$, consists of at most $n - r$ nonzero columns; however the linear dependence in the columns means that the rank of $(D_P\tilde{D} - J)D_N$ is less than $n - r$.

Now observing that $MN = 0$ we see that $(PD - M)N = PDN$ so it is clear that rank $(PDN) <$ rank (N). □

Two illustrative examples are now considered. Example 3.5.1 considers the case where the geometric multiplicity of the zero eigenvalues of M is equal to the algebraic multiplicity. This is the most straightforward type of multiplicity. If the multiplicities do not coincide then the calculations are more complicated; this is because D_P is no longer equal to D_N. Example 3.5.2 considers a multiplicity of this type; that is, the case where the geometric multiplicity of the zero eigenvalues of M is equal to one. Example 3.5.2 can easily be generalised to the case where the geometric multiplicity is greater than one.

Example 3.5.1. Suppose that M has rank r and that the geometric multiplicity of the 0 eigenvalue of M is equal to the algebraic multiplicity. The matrix J is of the form

$$J = \begin{pmatrix} J_1 & 0_{r\times(n-r)} \\ 0_{(n-r)\times r} & 0_{(n-r)\times(n-r)} \end{pmatrix},$$

where J_1 is an invertible r–dimensional matrix and $0_{m\times d}$ is an $m \times d$–dimensional matrix with 0 entries. Clearly,

$$D_P = D_N = \begin{pmatrix} 0_{r\times r} & 0_{r\times(n-r)} \\ 0_{(n-r)\times r} & I_{(n-r)\times(n-r)} \end{pmatrix},$$

where $I_{(n-r)\times(n-r)}$ is the $(n-r)\times(n-r)$-dimensional identity matrix. Now,

$$D_P\tilde{D} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ d_{r+1,1} & \dots & d_{r+1,n} \\ \vdots & \ddots & \vdots \\ d_{n,1} & \dots & d_{n,n} \end{pmatrix},$$

where $\tilde{D} = T^{-1}DT$ as before, and has individual entries $[\tilde{D}]_{ij} = d_{ij}$ for $1 \le i, j \le n$.

Since $\det[D_P\tilde{D} - J] = 0$ and J is an invertible matrix this implies that

$$\det\begin{pmatrix} d_{r+1,r+1} & \cdots & d_{r+1,n} \\ \vdots & \ddots & \vdots \\ d_{n,r+1} & \cdots & d_{n,n} \end{pmatrix} = 0.$$

Now,

$$(D_P\tilde{D} - J)D_N = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & d_{r+1,n-r+1} & \cdots & d_{r+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & d_{n,r+1} & \cdots & d_{n,n} \end{pmatrix}.$$

There are $n-r$ nonzero columns in this matrix; however due to the linear dependence in the columns the rank of the matrix must be less than $n-r$.

Example 3.5.2. Without loss of generality we can consider a matrix M whose eigenvalues are all 0. We consider a 4×4 matrix where the geometric multiplicity of the 0 eigenvalue is 1. The matrix J is of the form:

$$J = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so

$$D_P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } D_N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Obviously,

$$D_P\tilde{D} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{pmatrix},$$

and

$$D_P\tilde{D} - J = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{pmatrix}.$$

Since $\det[D_P\tilde{D} - J] = 0$ this means that the vectors $(0, 0, d_{31}, d_{41})$ and $(0, 0, d_{34}, d_{44})$ are linearly dependent. Postmultiplying $D_P\tilde{D} - J$ by D_N we see that

$$D_P\tilde{D} - J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ d_{31} & 0 & 0 & d_{34} \\ d_{41} & 0 & 0 & d_{44} \end{pmatrix},$$

The rank of $D_N = 2$. However due to the linear dependence in the columns of $(D_P\tilde{D} - J)D_N$ the rank of $(D_P\tilde{D} - J)D_N$ must be less than the rank of D_N.

3.6 Proofs

In this section the proofs of results which were postponed earlier in the chapter are given.

Proof of Lemma 3.3.1. We begin by proving that (3.3.2) holds. A method similar to that used by Krisztin and Terjéki in [24, Theorem 4] is applied. As (3.2.1) holds the matrix $D = I + \int_0^\infty \int_t^\infty K(s)\, ds\, dt$ can be defined. Suppose that (3.3.2) does not hold, that is $\det[PD - M] = 0$.

A simple calculation shows that $MN = 0$:

$$MN = TJT^{-1}TD_NT^{-1} = TJD_NT^{-1} = 0.$$

It is shown in Lemma 3.3.2 that rank $PDN <$ rank N, so there exists a vector $v \in \mathbb{R}^n$ such that v is in the kernel of PDN but not in the kernel of N. That is $Nv \neq 0$ and $PDNv = 0$. Since $PDNv = 0$ it is clear that the vector DNv lies in the kernel of P.

Suppose that the rank of M is r; then the rank of the kernel of P is also r. As $PM = 0$ it is obvious that $Pm_i = 0$ for $i = 1, \dots, n$, where $M = [m_1, \dots, m_n]$. M has a basis $\{b_1, \dots, b_r\}$ and $Pb_i = 0$ for $i = 1, \dots, r$. As the rank of the kernel of P is r this means that $\{b_1, \dots, b_r\}$ is the basis of the kernel of P. Thus, $\{m_1, \dots, m_n\}$ is a spanning set of the kernel of P and so we can choose a vector $w = (w_1, \dots, w_n)$ which is a linear combination of $\{b_1, \dots, b_r\}$ such that such that $Mw = DNv$.

Using the fact that $MNv = 0$ and $Mw = DNv$ it is seen that $Nvt + w$ is a solution of

$$y'(t) = Ay(t) + \int_0^\infty K(s)y(t-s)\,ds$$

since

$$\begin{aligned} A(Nvt+w) + \int_0^\infty K(s)[Nv(t-s)+w]\,ds &= M(Nvt+w) - \int_0^\infty sK(s)\,ds\,Nv \\ &= Mw + Nv - \left(I + \int_0^\infty sK(s)\,ds\right)Nv \\ &= DNv - DNv + Nv = Nv. \end{aligned}$$

Now using variation of parameters the following is obtained:

$$Nvt + w = R(t)w + \int_0^t R(t-s)\int_s^\infty K(u)[Nv(s-u)+w]\,du\,ds. \tag{3.6.1}$$

Adding and subtracting the matrix C from the right hand side of (3.6.1) yields

$$Nvt + w =$$
$$(R(t) - C)w + \int_0^t (R(t-s) - C) \int_s^\infty K(u)[Nv(s-u) + w]\,du\,ds$$
$$+ Cw + C\int_0^t \int_s^\infty K(u)[Nv(s-u) + w]\,du\,ds. \quad (3.6.2)$$

As $Nv \neq 0$, the left hand side of this equation consists of a vector which is growing linearly and a constant vector. The right hand side of (3.6.2) is now examined. The first term lies in $L^2(0,\infty)$. The third term is a constant vector and can therefore be combined with the constant term on the left hand side to create the vector C_1.

Now, examine the fourth term on the left hand side of (3.6.2). As (3.2.1) holds it is clear that $\int_t^\infty K(t)[Nv(t-s) + w]\,ds \to 0$ as $t \to \infty$ since

$$\int_t^\infty \|K(s)\|[\|Nv\|(t+s) + \|w\|]\,ds \leq 2\int_t^\infty \|K(s)\|\|Nv\|s\,ds + \int_t^\infty \|K(s)\|\|w\|\,ds.$$

As $\int_t^\infty \|K(s)\|[\|Nv\|(t+s) + \|w\|]\,ds \to 0$ as $t \to \infty$ for all $\epsilon > 0$ we can choose $T(\epsilon) > 0$ such that for all $t \geq T(\epsilon)$ the function K satisfies $\int_t^\infty \|K(s)\|[\|Nv\|(t-s) + \|w\|]\,ds \leq \epsilon$. Thus,

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \int_s^\infty \|K(u)\|[\|Nv\||s-u| + \|w\|]\,du\,ds \leq$$
$$\limsup_{t\to\infty} \frac{1}{t}\left(\int_0^T \int_s^\infty \|K(u)\|[\|Nv\||s-u| + \|w\|]\,du\,ds + \int_T^t \epsilon\,ds\right) < \epsilon,$$

but since $\epsilon > 0$ is arbitrary we see that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \int_s^\infty K(u)[Nv(s-u) + w]\,du\,ds = 0.$$

So the fourth term on the right hand side of (3.6.2) is $o(t)$ as $t \to \infty$.

Now consider the second term on the right hand side of (3.6.2). It can now be easily shown using the Cauchy–Schwarz inequality that the convolution of a square integrable function with a function which tends to zero is $o(t)$ as $t \to \infty$:

$$\left(\frac{1}{t}\int_0^t \|R(t-s)-C\| \left\| \int_s^\infty K(u)[Nv(s-u)+w]\,du \right\| ds\right)^2$$
$$\leq \frac{1}{t^2}\left(\int_0^t \left\| \int_s^\infty K(u)[Nv(s-u)+w]\,du \right\| ds \times\right.$$
$$\left.\int_0^t \|R(t-s)-C\|^2 \left\| \int_s^\infty K(u)[Nv(s-u)+w]\,du \right\| ds\right), \tag{3.6.3}$$

Since $\frac{1}{t}\int_0^t \int_s^\infty \|K(u)\|[\|Nv\|(s+u)+\|w\|]\,du\,ds \to 0$ as $t \to \infty$ this means that

$$\lim_{t\to\infty} \frac{1}{t^2}\int_0^t \int_s^\infty \|K(u)\|[\|Nv\|(s+u)+\|w\|]\,du\,ds = 0. \tag{3.6.4}$$

Also, the convolution of a L^1 function with a function tending to zero tends to zero. So we see that the second term on the right hand side of (3.6.2) is $o(t)$ as $t \to \infty$.

Thus, the second and final term can be amalgamated with the linear term on the right hand side to obtain $C_2(t)t$ where $C_2(t) \to Nv \neq 0$. Hence, $(R(t)-C)w = C_1 + C_2(t)t$. Rearranging this equation and taking norms one obtains

$$\|(R(t)-C)w\|^2 + \|C_1\|^2 \geq \|C_2(t)\|^2 t^2$$

But the $L^2(0,\infty)$ function $(R(t)-C)w$ cannot satisfy this inequality and so the desired contradiction is obtained.

Next we use the reformulation of (2.1.1) found in Lemma 2.3.1;

$$R(t) + (F*[R-C])(t) = \Phi(t) - (F*C)(t), \tag{3.6.5}$$

where the function F is given by (2.3.7). It can easily be shown that $F \in L^2$. Using the fact that the convolution of an $L^2(0,\infty)$ function with an $L^2(0,\infty)$ function is bounded, continuous and tends to zero at infinity we see that

$$\lim_{t\to\infty} R(t) = P - \int_0^\infty F(s)\,ds\,C := R_\infty.$$

Now, we once again use the reformulation of the resolvent equation given by (2.3.6) to obtain

$$Y(t) + (F * Y)(t) = \Phi(t) - R_\infty - (F * R_\infty)(t),$$

where $Y = R - R_\infty$. Noting that $Y \to 0$ as $t \to \infty$ and $F \in L^1$ we take limits on both sides of this equation

$$0 = P - \left(I + \int_0^\infty F(s)\,ds\right) R_\infty.$$

Now since $\det[I + \hat{F}(0)] \neq 0$ we see that $R_\infty := [I + \hat{F}(0)]^{-1}P$.

A proof by contradiction is used to show that $C = R_\infty$. Suppose that $C \neq R_\infty$. Observe

$$R_\infty - C = (R(t) - C) - (R(t) - R_\infty).$$

Integrating both sides over $[0, t]$ and dividing both sides by t one obtains

$$R_\infty - C = \frac{1}{t}\int_0^t (R(s) - C)\,ds - \frac{1}{t}\int_0^t (R(t) - R_\infty)\,ds. \tag{3.6.6}$$

Using Cauchy–Schwarz we see that the first term on the right hand side of (3.6.6) tends to zero since

$$\left(\frac{1}{t}\int_0^t (R(s) - C)\,ds\right)^2 \leq \frac{1}{t}\int_0^t (R(s) - C)^2\,ds$$

and $R - C \in L^2$ by assumption. Now consider the second term on the right hand side of (3.6.6). Since $R \to R_\infty$ as $t \to \infty$ it is clear that for any $\epsilon > 0$ we can choose $T(\epsilon) > 0$ such that for $t \geq T(\epsilon)$ the following holds: $\|R(t) - R_\infty\| < \epsilon$. Thus

$$\frac{1}{t}\int_0^t \|R(t) - R_\infty\|\,ds = \frac{1}{t}\int_0^{T(\epsilon)} \|R(t) - R_\infty\|\,ds + \frac{1}{t}\int_{T(\epsilon)}^t \|R(t) - R_\infty\|\,ds,$$

taking the limit as $t \to \infty$ on both sides we see that $\frac{1}{t}\int_0^t (R(t) - R_\infty)\,ds \to 0$ as $t \to \infty$ since ϵ is arbitrary. Combining the above arguments we see that $R_\infty = C$. □

Proof of Lemma 3.3.3. From the proof of Lemma 2.4.1 it is known that

$$\det[I+\hat{F}(0)] = \det\left[P - M + \int_0^\infty \int_s^\infty PK(u)\,du\,ds\right],$$

and that

$$\det[I+\hat{F}(z)] = \frac{1}{z}\frac{1}{z+1}\det[zI+P]\det\left[zI - A - \hat{K}(z)\right],$$

when $\operatorname{Re} z \geq 0$, $z \neq 0$.

Using Lemma 3.3.1 we see that $\det[I+\hat{F}(0)] \neq 0$. Now consider the case where $\operatorname{Re} z \geq 0$, $z \neq 0$. Obviously, $\frac{1}{z}\frac{1}{z+1} \neq 0$, since $z \neq 0$. The function $z \mapsto \det[zI+P]$ is nonzero due to the structure of P. The matrix P has two eigenvalues, 0 and 1, thus $\det[zI+P]$ is zero only at 0 and -1. Thus, in order to prove that (3.3.4) holds for $\operatorname{Re} z \geq 0$, $z \neq 0$ it is necessary to show that $\det[zI - A - \hat{K}(z)] \neq 0$. An idea used in [24] is utilised. Suppose that there exists $z_0 \neq 0$, $\operatorname{Re} z_0 \geq 0$ such that $\det[z_0 I - A - \hat{K}(z_0)] = 0$. Thus $v_0 e^{z_0 t}$ is a solution of

$$y'(t) = Ay(t) + \int_0^\infty K(s)y(t-s)\,ds, \quad t > 0,$$

$$y(0) = v_0.$$

Using variation of parameters we see that

$$v_0 e^{z_0 t} = R(t)v_0 + \int_0^t R(t-s)\int_s^\infty K(u)v_0 e^{z_0(s-u)}\,du\,ds. \tag{3.6.7}$$

We consider the cases where $\operatorname{Re} z_0 > 0$ and $\operatorname{Re} z_0 = 0$ separately. When $\operatorname{Re} z_0 > 0$ the real part of the left hand side of (3.6.7) is unbounded as $t \to \infty$. Now consider the right hand side. The first term on the right hand side of (3.6.7) converges to a finite limit $t \to \infty$. Now we consider the second term. Since $t \mapsto \int_t^\infty K(u)e^{z_0(t-u)}\,du$ is integrable and $R(t) \to R_\infty$ as $t \to \infty$ their convolution tends to a finite constant. So the real part of right hand side approaches a finite constant while the real part of the left hand side is unbounded. This yields a contradiction and so $\det[zI - A - \hat{K}(z)] \neq 0$ for $\operatorname{Re} z > 0$.

We now look at the case when $\operatorname{Re} z_0 = 0$. By considering the real part of both sides of (3.6.7), we see that the lefthand side varies sinusoidally while the righthand side tends to a constant. This yields a contradiction and so

$$\det[zI - A - \hat{K}(z)] \neq 0 \text{ for } \operatorname{Re} z \geq 0.$$

□

Proof of Lemma 3.4.1. Define $\Delta(t) = X(t) - X_\infty$, and let Λ be given by

$$\Lambda := \left(A + \int_0^\infty K(s)\,ds\right) X_\infty. \tag{3.6.8}$$

It is shown in the sequel that

$$\mathbb{E}\left\|\frac{1}{t}\int_0^t \Sigma(s)\,dB(s) + \Lambda\right\|^2 \leq \frac{\kappa}{t}, \quad t \geq 1, \tag{3.6.9}$$

where κ is a positive constant. Armed with this estimate, it is now shown that $\Lambda = 0$, a.s. Define for $m \geq 1$ the discrete filtration $\{\mathcal{G}_m\}_{m\geq 1}$ by $\mathcal{G}_m = \mathcal{F}^B(m^2)$, $m \geq 1$. Then, the discrete time process $N = \{N(m) : m \geq 1; \mathcal{G}_m\}$ defined by

$$N(m) = \int_0^{m^2} \Sigma(s)\,dB(s)$$

is a $\mathbb{R}^n$-valued $\mathcal{G}_m$-martingale. Define for $m \geq 1$ the processes N_i and $\tilde{N}_i$ by

$$N_i(m) = \sum_{j=1}^d \int_0^{m^2} \Sigma_{ij}(s)\,dB_j(s)$$

and $\tilde{N}_i(m) = N_i(m)/m^2$ respectively. Consider first the case when σ_i, defined by

$$\sigma_i^2(t) = \sum_{j=1}^d \Sigma_{ij}^2(t), \quad t \geq 0, \tag{3.6.10}$$

obeys $\sigma_i^2 \notin L^1(0,\infty)$. Then $\langle N_i\rangle(m) \to \infty$ as $m \to \infty$, and so

$$\liminf_{m\to\infty} N_i(m) = -\infty, \quad \limsup_{m\to\infty} N_i(m) = \infty, \quad \text{a.s.}$$

This implies

$$\liminf_{m\to\infty} \tilde{N}_i(m) \le 0, \quad \limsup_{m\to\infty} \tilde{N}_i(m) \ge 0, \quad \text{a.s.} \tag{3.6.11}$$

Now, by (3.6.9), for $m \ge 1$,

$$\mathbb{E}[|\tilde{N}_i(m) + \Lambda_i|^2] \le \frac{\kappa}{m^2}.$$

Hence, by Chebyshev's inequality, for every $\varepsilon > 0$,

$$\sum_{m=1}^{\infty} \mathbb{P}\left[|\tilde{N}_i(m) + \Lambda_i| \ge \varepsilon\right] \le \frac{\kappa}{\varepsilon^2} \sum_{m=1}^{\infty} \frac{1}{m^2} < \infty.$$

Therefore, by the first Borel–Cantelli lemma it is seen that

$$\lim_{m\to\infty} \tilde{N}_i(m) = -\Lambda_i, \quad \text{a.s.}$$

Finally, since this limit exists, (3.6.11) may be used to give

$$-\Lambda_i = \lim_{m\to\infty} \tilde{N}_i(m) = \liminf_{m\to\infty} \tilde{N}_i(m) \le 0,$$

so $\Lambda_i \ge 0$, a.s. Similarly, by (3.6.11)

$$-\Lambda_i = \lim_{m\to\infty} \tilde{N}_i(m) = \limsup_{m\to\infty} \tilde{N}_i(m) \ge 0,$$

so $\Lambda_i \le 0$, a.s. Therefore $\Lambda_i = 0$, a.s.

The case when σ_i defined by (3.6.10) obeys $\sigma_i^2 \in L^1(0,\infty)$ is now considered. Clearly,

$$\mathbb{E}[\Lambda_i^2] \le 2\mathbb{E}\left|\Lambda_i + \left(\frac{1}{t}\int_0^t \Sigma(s)\,dB(s)\right)_i\right|^2 + 2\mathbb{E}\left|\left(\frac{1}{t}\int_0^t \Sigma(s)\,dB(s)\right)_i\right|^2.$$

Due to (3.6.9), the first term on the righthand side has zero limit as $t \to \infty$. As for the second term on the righthand side,

$$E\left|\left(\frac{1}{t}\int_0^t \Sigma(s)\,dB(s)\right)_i\right|^2 = \frac{1}{t^2}\int_0^t \sigma_i^2(s)\,ds,$$

and this also tends to zero as $t \to \infty$. Thus $\mathbb{E}[\Lambda_i^2] = 0$, and so $\Lambda_i = 0$, a.s.

Combining the components for which $\sigma_i^2 \in L^1(0,\infty)$ and those for which $\sigma_i^2 \notin L^1(0,\infty)$, it is seen that $\Lambda_i = 0$ a.s. in all cases, and therefore $\Lambda = 0$, a.s., proving the result.

It is now shown that (3.6.9) holds. Rearranging the integral version of (3.1.1), adding and subtracting X_∞ and dividing across by t one obtains:

$$\begin{aligned}
&\frac{1}{t}\int_0^t \Sigma(s)\,dB(s) + \Lambda \\
&\qquad = \frac{1}{t}(X(t) - X_0) - \frac{1}{t}\int_0^t A\Delta(s) + (K*\Delta)(s)\,ds + \frac{1}{t}\int_0^t K_1(s)\,ds\,X_\infty,
\end{aligned}$$

where $\Delta(t) = X(t) - X_\infty$. Taking norms and expectations across both sides of the equation one obtains:

$$\begin{aligned}
\mathbb{E}\left\|\frac{1}{t}\int_0^t \Sigma(s)\,dB(s) + \Lambda\right\|^2 &\le 4^3\Bigg(\frac{1}{t^2}\mathbb{E}\,\|X(t) - X_\infty\|^2 + \frac{1}{t^2}\mathbb{E}\,\|X_\infty - X_0\|^2 \\
&\quad + \frac{1}{t^2}\mathbb{E}\left\|\int_0^t A\Delta(s) + (K*\Delta)(s)\,ds\right\|^2 + \frac{1}{t^2}\mathbb{E}\left\|\int_0^t K_1(s)\,ds\,X_\infty\right\|^2\Bigg). \quad (3.6.12)
\end{aligned}$$

Begin by examining the first term on the right hand side of (3.6.12); it is shown that this tends to zero using the following estimate:

$$\mathbb{E}\,\|X(t)\|^2 \le 2\mathbb{E}\,\|\Delta(t)\|^2 + 2\mathbb{E}\,\|X_\infty\|^2.$$

Therefore, taking the lim sup on both sides of this inequality as $t \to \infty$,

$$\limsup_{t\to\infty} \mathbb{E}\,\|X(t)\|^2 \le \mathbb{E}\,\|X_\infty\|^2. \qquad (3.6.13)$$

Since $t \mapsto \mathbb{E}\,\|X(t)\|^2$ is continuous, it is clear that $\mathbb{E}\,\|X(t)\|^2 \le \kappa_1$ for all $t \ge 0$. Because $\mathbb{E}\,\|\Delta(t)\|^2 \le 2\mathbb{E}\,\|X(t)\|^2 + 2\mathbb{E}\,\|X_\infty\|^2$, there is $\kappa_2 > 0$ such that $\mathbb{E}\,\|\Delta(t)\|^2 \le \kappa_2$ for all $t \ge 0$. Thus,

$$\frac{1}{t^2}\mathbb{E}\,\|\Delta(t)\|^2 \le \frac{\kappa_2}{t}, \quad t \ge 1, \qquad (3.6.14)$$

using the fact that $\frac{1}{t^2} \le \frac{1}{t}$ for $t \ge 1$.

The second term on the right hand side of (3.6.12) satisfies

$$\frac{1}{t^2}\mathbb{E}\,\|X_\infty - X_0\|^2 \le \frac{\kappa_3}{t}, \quad t \ge 1, \tag{3.6.15}$$

as both X_0 and $\mathbb{E}\|X_\infty\|^2$ exist.

Consider the third term on the left hand side of (3.6.12); using the Cauchy–Schwarz inequality, we get

$$\begin{aligned}\frac{1}{t^2}\mathbb{E}\left\|\int_0^t A\Delta(s) + (K*\Delta)(s)\,ds\right\|^2 &\le \frac{1}{t}\int_0^t \mathbb{E}\,\|A\Delta(s) + (K*\Delta)(s)\|^2\,ds\\ &\le \frac{2}{t}\int_0^t \left(\mathbb{E}\,\|A\Delta(s)\|^2 + \mathbb{E}\,\|(K*\Delta)(s)\|^2\right) ds,\end{aligned}$$

and using the Cauchy–Schwarz inequality again, this becomes

$$\begin{aligned}&\frac{1}{t^2}\mathbb{E}\left\|\int_0^t A\Delta(s) + (K*\Delta)(s)\,ds\right\|^2\\ &\qquad\le \frac{2}{t}\int_0^t \left(\mathbb{E}\,\|A\Delta(s)\|^2 + \bar{K}\int_0^s \|K(s-u)\|\mathbb{E}\,\|\Delta(u)\|^2\,du\right) ds,\end{aligned}$$

where $\bar{K} = \int_0^\infty \|K(t)\|\,dt$. Since $\int_0^\infty \mathbb{E}\,\|\Delta(t)\|^2\,dt < \infty$ and $K \in L^1(0,\infty)$, it follows that there is a constant $\kappa_4 > 0$ which depends on A and K such that

$$\frac{1}{t^2}\mathbb{E}\left\|\int_0^t A\Delta(s) + (K*\Delta)(s)\,ds\right\|^2 \le \frac{\kappa_4}{t}. \tag{3.6.16}$$

Next, by (3.2.1), there is a universal constant κ_5 such that

$$\frac{1}{t^2}\mathbb{E}\left\|\int_0^t K_1(s)X_\infty\,ds\right\|^2 \le \frac{1}{t^2}\left(\int_0^t \|K_1(s)\|\,ds\right)^2 \mathbb{E}\,\|X_\infty\|^2 \le \frac{\kappa_5}{t}, \quad t \ge 1. \tag{3.6.17}$$

Thus, by (3.6.14), (3.6.15), (3.6.16) and (3.6.17):

$$\mathbb{E}\left\|\frac{1}{t}\int_0^t \Sigma(s)\,dB(s) + \Lambda\right\|^2 \le \frac{\kappa}{t}, \quad t \ge 1,$$

or in other words, (3.6.9) holds. This completes the result. □

Proof of Lemma 3.4.2. By Itô's rule, we see that

$$\begin{aligned}\|X(t)\|^2 = \|X_0\|^2 + 2\int_0^t \langle X(s), AX(s) + (K*X)(s)\rangle\,ds&\\ + \int_0^t \|\Sigma(s)\|_F^2\,ds + M(t),&\end{aligned} \tag{3.6.18}$$

where

$$M(t) = 2\sum_{j=1}^{d}\sum_{i=1}^{n}\int_0^t X_i(s)\Sigma_{ij}(s)\,dB_j(s).$$

Using the definition $\Delta(t) = X(t) - X_\infty$, and the fact that

$$\int_0^t AX(s) + (K * X)(s)\,ds = X(t) - X_0 - \int_0^t \Sigma(s)\,dB(s)$$

it is clear that

$$\begin{aligned}
&\int_0^t \langle X(s), AX(s) + (K * X)(s)\rangle\,ds \\
&\quad= \int_0^t \langle \Delta(s), AX(s) + (K * X)(s)\rangle\,ds + \int_0^t \langle X_\infty, AX(s) + (K * X)(s)\rangle\,ds \\
&\quad= \int_0^t \langle \Delta(s), A\Delta(s) + (K * \Delta)(s)\rangle\,ds + \int_0^t \langle \Delta(s), AX_\infty + (K * X_\infty)(s)\rangle\,ds \\
&\qquad\qquad + \langle X_\infty, X(t) - X_0 - \int_0^t \Sigma(s)\,dB(s)\rangle.
\end{aligned}$$

Therefore, by Lemma 3.4.1, and the definition of K_1, one obtains

$$\begin{aligned}
\int_0^t \langle X(s), AX(s) + (K * X)(s)\rangle\,ds &= \int_0^t \langle \Delta(s), A\Delta(s) + (K * \Delta)(s)\rangle\,ds \\
&\quad - \int_0^t \langle \Delta(s), K_1(s)X_\infty\rangle\,ds + \left\langle X_\infty, X(t) - X_0 - \int_0^t \Sigma(s)\,dB(s)\right\rangle. \quad (3.6.19)
\end{aligned}$$

Substituting (3.6.19) into (3.6.18), taking expectations and rearranging the equation, one obtains:

$$\begin{aligned}
\int_0^t \|\Sigma(s)\|_F^2\,ds &= \mathbb{E}\,\|X(t)\|^2 - \mathbb{E}\,\|X_0\|^2 - 2\mathbb{E}\int_0^t \langle \Delta(s), A\Delta(s) + (K * \Delta)(s)\rangle\,ds \\
&\quad + 2\mathbb{E}\int_0^t \langle \Delta(s), K_1(s)X_\infty\rangle\,ds - 2\mathbb{E}\,\langle X_\infty, X(t)\rangle + 2\mathbb{E}\,\langle X_\infty, X_0\rangle \\
&\qquad + 2\mathbb{E}\left\langle X_\infty, \int_0^t \Sigma(s)\,dB(s)\right\rangle. \quad (3.6.20)
\end{aligned}$$

Consider each term on the left hand side of (3.6.20). The first term is bounded due the argument given in Lemma 3.4.1. The second term is bounded due to our assumptions.

Consider the third term:

$$
\begin{aligned}
\mathbb{E}\left|\int_0^t \langle\Delta(s), A\Delta(s)+(K*\Delta)(s)\rangle\, ds\right| \\
&\le \mathbb{E}\int_0^t \|\Delta(s)\|\|A\Delta(s)+(K*\Delta)(s)\|\, ds \\
&\le 2\mathbb{E}\int_0^t \|\Delta(s)\|^2 + \|A\Delta(s)+(K*\Delta)(s)\|^2\, ds \\
&\le 2\int_0^t \mathbb{E}\left[\|\Delta(s)\|^2\right] ds + \int_0^t \mathbb{E}\left[\|A\Delta(s)+(K*\Delta)(s)\|^2\right] ds.
\end{aligned}
$$

Since X obeys (3.2.7) it is seen that $\mathbb{E}\left[\int_0^t \|\Delta(s)\|^2\, ds\right] < C_1$. It is easily shown that $\mathbb{E}\left[\int_0^t \|A\Delta(s)+(K*\Delta)(s)\|^2\, ds\right]$ is also bounded:

$$
\begin{aligned}
\mathbb{E}\int_0^t \|A\Delta(s)+(K*\Delta)(s)\|^2\, ds \\
&\le 2\int_0^t \left(\mathbb{E}\,\|A\Delta(s)\|^2 + \mathbb{E}\,\|(K*\Delta)(s)\|^2\right) ds \\
&\le 2\int_0^t \left(\mathbb{E}\,\|A\Delta(s)\|^2 + \bar{K}\int_0^s \|K(s-u)\|\mathbb{E}\,\|\Delta(u)\|^2\, du\right) ds \\
&< C_2,
\end{aligned}
$$

since K is integrable and X obeys (3.2.7). Thus, the third term on the right hand side of (3.6.20) is uniformly bounded.

Now consider the fourth term.

$$
\begin{aligned}
\mathbb{E}\left|\int_0^t \langle\Delta(s), K_1(s)X_\infty\rangle\, ds\right| &\le \mathbb{E}\left[\int_0^t \|\Delta(s)\|\|K_1(s)X_\infty\|\, ds\right] \\
&\le 2\mathbb{E}\left[\int_0^t \|\Delta(s)\|^2 + \|K_1(s)X_\infty\|^2\, ds\right] \\
&\le 2\int_0^t \mathbb{E}\,\|\Delta(s)\|^2\, ds + \int_0^t \|K_1(s)\|^2\, ds\, \mathbb{E}\,\|X_\infty\|^2.
\end{aligned}
$$

Examine the last line on the right hand side of this inequality. The first term is bounded since (3.2.7) holds. The second term is bounded since $\mathbb{E}\,\|X_\infty\|^2 < \infty$ and (3.2.1) holds. Thus the fourth term on the right hand side of (3.6.20) is bounded.

Again, using the argument given in Lemma 3.4.1 we see that the fifth term is uniformly bounded. Finally, the sixth term is bounded due to our assumptions. Consequently, by applying the Cauchy–Schwarz inequality to the last term, there is a constant $C > 0$ such that

$$\int_0^t \|\Sigma(s)\|_F^2 \, ds \leq C + 2\mathbb{E}\left[\|X_\infty\|^2\right]^{1/2} \left(\int_0^t \|\Sigma(s)\|_F^2 \, ds\right)^{1/2}. \tag{3.6.21}$$

Clearly, it must follow that $\Sigma \in L^2(0,\infty)$; for if not, we have

$$\lim_{t\to\infty} \int_0^t \|\Sigma(s)\|_F^2 \, ds = \infty,$$

and so by dividing both sides of (3.6.21) by $\int_0^t \|\Sigma(s)\|_F^2 \, ds$ and letting $t \to \infty$ we obtain a contradiction. This completes the proof. $\square$

Chapter 4

Almost sure convergence

4.1 Introduction

In this chapter we present results analogous to those proved in Chapter 3. As before we consider equation (3.1.1) under the conditions (2.1.2) and (3.1.2) on the kernel K and the noise term Σ respectively. This equation was precisely defined in the previous chapter.

As before, establishing the necessary and sufficient conditions on the resolvent, kernel and noise for convergence is complicated by the fact that X_∞ is not $\mathcal{F}(t)$–adapted. Nonetheless, it is shown that the sufficient conditions for convergence and integrability in the mean square case also suffice in the almost sure case. However, showing that these conditions are necessary is not as straightforward as in the mean square case. This is due to the fact that we can no longer avail of the simplifying effect that taking expectations has on a random variable. Consequently, we cannot show that the condition on the tail of the noise is necessary.

4.2 Discussion of Results

The main results of this chapter are presented in this section. Necessary and sufficient conditions for asymptotic convergence of the solution of (3.1.1) to a nontrivial limit and the integrability of this solution in the almost sure case are considered.

The sufficient conditions for the asymptotic convergence of the solution X of (3.1.1) to a nontrivial limit X_∞, and for the integrability of $X - X_\infty$ in the almost sure sense are considered in Theorem 4.2.1. As in the mean square case, it is found that conditions (3.2.2) and (3.2.3) are required for convergence; in addition, (3.2.6) is required for integrability.

Theorem 4.2.1. *Let K satisfy* (2.1.2) *and* (3.2.1) *and let Σ satisfy* (3.1.2). *If Σ satisfies* (3.2.2) *and if there exists a constant matrix R_∞ such that the solution R of* (2.1.1) *satisfies*

(3.2.3) *then for all initial conditions* X_0 *there is an almost surely finite* $\mathcal{F}^B(\infty)$*-measurable random variable* $X_\infty(X_0, \Sigma)$ *such that the unique continuous adapted process* $X(\cdot\,; X_0, \Sigma)$ *which obeys* (3.1.1) *satisfies*

$$\lim_{t \to \infty} X(t; X_0, \Sigma) = X_\infty(X_0, \Sigma) \quad a.s. \tag{4.2.1}$$

Moreover, if the function Σ *also satisfies* (3.2.6) *then*

$$X(\cdot\,; X_0, \Sigma) - X_\infty(X_0, \Sigma) \in L^2((0,\infty), \mathbb{R}^n) \quad a.s. \tag{4.2.2}$$

The necessary conditions for the asymptotic convergence of the solution X of (3.1.1) to a nontrivial limit X_∞, and for the square integrability of $X - X_\infty$ in the almost sure sense are now stated.

Theorem 4.2.2. *Let* K *satisfy* (2.1.2) *and* (3.2.1) *and let* Σ *satisfy* (3.1.2). *Suppose for all initial conditions* X_0 *there is an almost surely finite* $\mathcal{F}^B(\infty)$*-measurable random variable* $X_\infty(X_0, \Sigma)$ *such that the unique continuous adapted process* $X(\cdot\,; X_0, \Sigma)$ *which obeys* (3.1.1) *satisfies* (4.2.1) *and* (4.2.2). *Then, there exists a constant matrix* R_∞ *such that the solution* R *of* (2.1.1) *satisfies* (3.2.3) *and the function* Σ *satisfies* (3.2.2) *and*

$$\int_0^\infty \left\| \int_t^\infty R_\infty \Sigma(s)\, dB(s) \right\|^2 dt < \infty \quad a.s. \tag{4.2.3}$$

Assumptions (3.2.2) and (3.2.3) are both necessary and sufficient for convergence and square integrability. However, we have not been able to show that (3.2.6) is a necessary condition. By taking expectations it is clear that (3.2.6) implies (4.2.3) but it is not immediate that (4.2.3) implies (3.2.6).

Although much is known about the behaviour of the solution of equation (3.1.1) and its parameters, both by assumption and analysis, we have not been able to conclude that (3.2.6) holds by examining the structure of the equation.

Due to the complicated structure of (4.2.3) it is not obvious how one may obtain (3.2.6) by examining (4.2.3) directly. Analysis of this term is complicated for a number of reasons. Firstly, the expression $\int_t^\infty R_\infty \Sigma(s)\, dB(s)$ is not a martingale, although this may be overcome by examining each term of the vector individually and applying a time change argument. Secondly, although these martingales are normal random variables, the key objects which we encounter is not a normal random variable but rather the integral of the square of a normal random variable (i.e. a Chi–square random variable).

Analogous results may be obtained in the case where the equation is both stochastically and deterministically perturbed. The following theorem places sufficient conditions under which solutions tend to a non–equilibrium limit.

Theorem 4.2.3. *Let K satisfy* (2.1.2) *and* (3.2.1), *let Σ satisfy* (3.1.2) *and* (3.2.2) *and let f satisfy* (2.1.4). *Suppose the resolvent R of* (2.1.1) *satisfies* (3.2.3). *Then the solution $X(t; X_0, \Sigma, f)$ of* (7.1.1) *satisfies $X(\cdot\,; X_0, \Sigma, f) \to X_\infty(X_0, \Sigma, f)$ almost surely, where*

$$X_\infty(X_0, \Sigma, f) = R_\infty\left(X_0 + \int_0^\infty f(t)\,dt + \int_0^\infty \Sigma(t)\,dB(t)\right) \quad a.s. \tag{4.2.4}$$

and X_∞ is almost surely finite. Moreover if Σ satisfies (3.2.6) *and f satisfies*

$$\int_0^\infty t\|R_\infty f(t)\|\,dt < \infty, \tag{4.2.5}$$

then

$$X(\cdot\,; X_0, \Sigma, f) - X_\infty(X_0, \Sigma, f) \in L^2((0,\infty), \mathbb{R}^n) \quad a.s. \tag{4.2.6}$$

This theorem has applications in the study of infinite delay equations. In particular it provides useful insights into the epidemiological model examined in Section 4.3.

4.3 Biological Application

In this section the following epidemiological model is considered:

$$dx(t) = \left(g(x(t)) - \int_{-\infty}^t w(t-s)g(x(s))\,ds\right)dt + \Sigma(t)dB(t), \tag{4.3.1a}$$

$$x(t) = \phi(t), \quad t \le 0. \tag{4.3.1b}$$

Here the solution $x(\cdot\,;\phi,\Sigma)$ is a scalar function on $[0,\infty)$, the function g is a scalar linear function satisfying $g(x) = \alpha x$ for some constant $\alpha > 0$, w is a positive scalar weighting function satisfying

$$\int_0^\infty w(s)\,ds = 1,$$

Σ is a continuous and square integrable scalar function on $[0,\infty)$, $\{B(t)\}_{t\geq 0}$ is a one–dimensional Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}^B(t)_{t\geq 0}, \mathbb{P})$ where the filtration is the natural one $\mathcal{F}^B(t) = \sigma\{B(s) : 0 \leq s \leq t\}$ and the initial function ϕ satisfies

$$\sup_{t\leq 0} |\phi(t)| \leq \bar{\phi}. \tag{4.3.2}$$

Various authors have considered similar models in the deterministic case where $x(t)$ represents the population at time t. Cooke and Yorke [14] proposed the nonlinear delay–differential equation $x'(t) = g(x(t)) - g(x(t-L))$ as a model for the growth of an epidemic; where $g(x(t))$ represents the birth rate when the current population is $x(t)$, while death is certain at an age of L time units. A generalisation of this model was considered by Haddock and Terjéki [18], in which a convolution term was incorporated to allow for deaths at a distribution of ages. Indeed, Burton [12] extended their model and considered

$$x'(t) = \int_{t-L}^{t} p(s-t)g(x(s))\,ds - \int_{-\infty}^{t} q(s-t)g(x(s))\,ds,$$

in which both births and deaths are distributed. Here, death can occur at any time while the number of births is related to the number of conceptions which occurred up to L time units ago. A simple calculation illustrates that this equation is fundamentally the same as the deterministic version of (4.3.1) when appropriate conditions are imposed on the functions p and q.

The following theorem, the proof of which may be found in Section 4.5, considers the conditions under which the solution of this model converges to a nontrivial limit.

Theorem 4.3.1. *Let w satisfy*

$$w \in C([0,\infty), \mathbb{R}) \cap L^1((0,\infty), \mathbb{R}). \tag{4.3.3}$$

and

$$\int_0^\infty t\,|w(t)|\,dt < \infty. \tag{4.3.4}$$

Let Σ satisfy (3.1.2) and (3.2.2) where $n = d = 1$ and let ϕ satisfy (4.3.2). Suppose the resolvent R of (2.1.1) satisfies (3.2.3). Then the solution $x(\cdot\,;\phi,\Sigma)$ of (4.3.1) satisfies $x(\cdot\,;\phi,\Sigma) \to x_\infty(\phi,\Sigma)$ almost surely, where

$$x_\infty(\phi,\Sigma) = R_\infty \left(\phi(0) + \int_0^\infty \int_{-\infty}^0 w(t-s)\phi(s)\,ds\,dt + \int_0^\infty \Sigma(t)\,dB(t)\right) \quad a.s. \tag{4.3.5}$$

and x_∞ is almost surely finite. Moreover if Σ satisfies (3.2.6) and w satisfies

$$\int_0^\infty t^2\,|w(t)|\,dt < \infty, \tag{4.3.6}$$

then

$$x(\cdot\,;\phi,\Sigma) - x_\infty(\phi,\Sigma) \in L^2((0,\infty),\mathbb{R}^n) \quad a.s. \tag{4.3.7}$$

The function w represents the distribution of deaths within a population. It is evident from Theorem 4.3.1 that the growth of a population is influenced by the decay rate of w: if the first moment of w exists then the population will converge to a finite limit.

4.4 Conditions for Asymptotic Convergence and Integrability in the Almost Sure Sense

In this section, sufficient conditions for asymptotic convergence of solutions of (3.1.1) to a nontrivial random variable in the almost sure sense are obtained. The necessity of these conditions is also considered. Two technical lemmas used in the proof of Theorem 4.2.2 are presented. Lemma 4.4.1 concerns the structure of X_∞. This enables us to prove Lemma 4.4.2 which concerns the necessity of (3.2.2) for stability of the system. So, all that is required *a priori* is an assumption on the continuity of the noise intensity Σ; this ensures the existence of solutions at the outset.

Lemma 4.4.2 in turn allows us to show the necessity of (4.2.3); the proof of this inference may be found in the proof of Theorem 4.2.2 below. The proofs of Lemmas 4.4.1 and 4.4.2 are deferred to Section 4.5.

Lemma 4.4.1. *Let K satisfy (2.1.2) and (3.2.1). Suppose that for all initial conditions X_0 there is an almost surely finite random variable $X_\infty(X_0, \Sigma)$ such that the solution $t \mapsto X(t; X_0, \Sigma)$ of (3.1.1) satisfies (4.2.1) and (4.2.2). Then*

$$\left(A + \int_0^\infty K(s)\,ds\right) X_\infty = 0 \quad a.s. \tag{4.4.1}$$

Lemma 4.4.2. *Let K satisfy (2.1.2) and (3.2.1). Suppose for all initial conditions X_0 there is an almost surely finite $\mathcal{F}^B(\infty)$–measurable random variable $X_\infty(X_0, \Sigma)$ such that the solution $t \mapsto X(t; X_0, \Sigma)$ of (3.1.1) satisfies (4.2.1) and (4.2.2). Then Σ satisfies (3.2.2).*

Proof of Theorem 4.2.1. From Theorem 3.2.1 we know that X_∞ is almost surely finite and (4.2.1) holds if (3.2.1), (3.2.2) and (3.2.3) hold.

We know from Theorem 3.2.4 that $\int_0^\infty \mathbb{E}\,\|X(t) - X_\infty\|^2\,dt < \infty$ since (3.2.2), (3.2.3) and (3.2.6) hold. Fubini's theorem allows us to interchange the order of integration in this term; thus $\mathbb{E}\left[\int_0^\infty \|X(t) - X_\infty\|^2\,dt\right] < \infty$. If the expectation of a non–negative random variable is finite then the random variable itself is almost surely finite; applying this fact here means that (4.2.2) holds. □

Proof of Theorem 4.2.2. We begin by proving (3.2.3). Consider the $n+1$ solutions $X_j(t)$ of (3.1.1) with initial conditions $X_j(0) = \mathbf{e}_j$ for $j = 1, \dots, n$ and $X_{n+1}(0) = 0$ where $\mathbf{e}_1, \dots, \mathbf{e}_n$ is the standard basis. Note that $X_j(t) = R(t)\mathbf{e}_j + \rho(t)$ and $X_{n+1}(t) = \rho(t)$ where $\rho(t) = \int_0^t R(t-s)\Sigma(s)\,dB(s)$. Since $X_{n+1}(t) \to X_{n+1}(\infty)$ as $t \to \infty$ this implies that $\rho(t) \to \rho(\infty)$. Now, since $R(t)\mathbf{e}_j = X_j(t) - X_{n+1}(t)$ for $j = 1, \dots, n$ we see that $R(t) \to R_\infty$. Thus for $j = 1, \dots, n$, we can write

$$\begin{aligned}(R(t) - R_\infty)\mathbf{e}_j &= (X_j(t) - X_{n+1}(t)) - (X_j(\infty) - X_{n+1}(\infty)) \\ &= (X_j(t) - X_j(\infty)) - (X_{n+1}(t) - X_{n+1}(\infty)).\end{aligned}$$

Since (4.2.2) holds we see that $(R(\cdot) - R_\infty)\mathbf{e}_j \in L^2(0,\infty)$ for $j = 1,\dots,n$ hence (3.2.3) holds.

In order to show (3.2.2) holds we apply Lemma 4.4.2. Finally, we turn to (4.2.3). Expressing the solution of (3.1.1) using variation of parameters, subtracting X_∞ from both sides and rearranging the equation we obtain

$$\int_t^\infty R_\infty \Sigma(s)\,dB(s) = (R(t) - R_\infty)X_0 + \int_0^t (R(t-s) - R_\infty)\Sigma(s)\,dB(s) - (X(t) - X_\infty). \quad (4.4.2)$$

The first term on the right hand side of (4.4.2) is in $L^2(0,\infty)$ due to the above argument. Using the fact that (3.2.2) and (3.2.3) hold we see that

$$\mathbb{E}\left[\int_0^\infty \left\| \int_0^t (R(t-s) - R_\infty)\Sigma(s)\,dB(s)\right\|^2 dt\right] = \int_0^\infty \int_0^t \|(R(t-s) - R_\infty)\Sigma(s)\|^2\,ds\,dt < \infty. \quad (4.4.3)$$

If the expectation of a random variable is finite then the random variable itself is finite almost surely which means that the second term is in $L^2(0,\infty)$. The third term on the right hand side of (4.4.2) is in $L^2(0,\infty)$ using (4.2.2), and so (4.2.3) holds. This completes our proof. □

4.5 Proofs

In this section the proofs of results which were postponed earlier in this chapter are now given.

Proof of Theorem 4.2.3. The solution $X(t; X_0, \Sigma)$ of (3.1.1) satisfies (3.1.3) and the solution $X(t; X_0, \Sigma, f)$ satisfies

$$X(t; X_0, \Sigma, f) = R(t)X_0 + \int_0^t R(t-s)f(s)\,ds + \int_0^t R(t-s)\Sigma(s)\,dB(s), \quad (4.5.1)$$

thus

$$X(t; X_0, \Sigma, f) = X(t; X_0, \Sigma) + \int_0^t R(t-s)f(s)\,ds, \quad t \geq 0.$$

As $t \to \infty$, we know from Theorem 3.2.1 that $X(t; X_0, \Sigma) \to X_\infty(X_0, \Sigma)$. Also from our assumptions

$$\lim_{t\to\infty} \int_0^t R(t-s)f(s)\,ds = R_\infty \int_0^\infty f(s)\,ds,$$

and so $X(t; X_0, \Sigma, f) \to X_\infty(X_0, \Sigma, f)$ where $X_\infty(X_0, \Sigma, f)$ is given by (4.2.4).

We now prove (4.2.6). Consider

$$\begin{aligned} X(t; X_0, \Sigma, f) - X_\infty(X_0, \Sigma, f) &= (X(t; X_0, \Sigma) - X_\infty(X_0, \Sigma)) \\ &\quad + \int_0^t (R(t-s) - R_\infty)f(s)\,ds - \int_t^\infty R_\infty f(s)\,ds. \end{aligned} \tag{4.5.2}$$

Consider the righthand side of (4.5.2). We know that $X(t; X_0, \Sigma) - X_\infty(X_0, \Sigma) \in L^2(0, \infty)$ by using Theorem 4.2.1. An $L^2(0, \infty)$ term convolved with an $L^1(0, \infty)$ term lies in the space of $L^2(0, \infty)$ functions and so the second term on the right hand side of (4.5.2) must lie in $L^2(0, \infty)$. Next, (4.2.5) and the fact that $\lim_{t\to\infty} \int_t^\infty R_\infty f(s)\,ds = 0$ guarantee that the last term on the righthand side of (4.5.2) is in $L^2(0, \infty)$. Combining the arguments given in this paragraph we see that (4.2.6) must hold. This completes our proof. □

Proof of Theorem 4.3.1. We begin by splitting the convolution term as follows:

$$dx(t) = \alpha \left(x(t) - \int_{-\infty}^0 w(t-s)\phi(s)\,ds - \int_0^t w(t-s)x(s)\,ds \right) dt + \Sigma(t)dB(t).$$

Clearly $-\alpha \int_{-\infty}^0 w(t-s)\phi(s)\,ds$ corresponds to $f(t)$ of (7.1.1) for $t \geq 0$. We see that this term is in $L^1(0, \infty)$ using (4.3.2) and (4.3.4). Thus we can apply Theorem 4.2.3 to show that the solution $x(t; \phi, \Sigma)$ of (4.3.1) satisfies $x(\cdot\,; \phi, \Sigma) \to x_\infty(\phi, \Sigma)$ almost surely, where $x_\infty(\phi, \Sigma)$ is given by (4.3.5).

Furthermore, as (4.3.6) holds, a simple calculation shows that condition (4.2.5) of Theorem 4.2.3 is satisfied and so (4.3.7) must hold. □

Proof of Lemma 4.4.1. Define the random vector Λ as in (3.6.8). Writing (3.1.1) in integral form, adding and subtracting X_∞ from both sides, dividing both sides of the equation by t and then rearranging we obtain

$$\Lambda = \frac{X(t)-X_\infty}{t} - \frac{X_0 - X_\infty}{t} - \frac{\int_0^t A(X(s)-X_\infty)\,ds}{t} - \frac{\int_0^t\int_0^s K(s-u)(X(u)-X_\infty)\,du\,ds}{t} + \frac{\int_0^t\int_s^\infty K(u)\,du\,ds\,X_\infty}{t} - \frac{\int_0^t \Sigma(s)\,dB(s)}{t}. \quad (4.5.3)$$

As $t\to\infty$ we see that the first term on the right hand side of (4.5.3) tends to zero since (4.2.1) holds. The second term tends to zero as $t\to\infty$ since X_0 is a finite deterministic vector and X_∞ is almost surely finite by hypothesis. The third term tends to zero since (4.2.2) holds. Consider the fourth term. Using the Cauchy–Schwarz inequality we see that

$$\begin{aligned}
&\frac{1}{t}\left\|\int_0^t\int_0^s K(s-u)(X(u)-X_\infty)\,du\,ds\right\| \\
&= \left(\frac{1}{t^2}\left\|\int_0^t\int_0^s K(s-u)(X(u)-X_\infty)\,du\,ds\right\|^2\right)^{1/2} \\
&\le \left(\frac{1}{t^2}\left[\int_0^t\int_0^s \|K(s-u)\|\|X(u)-X_\infty\|\,du\,ds\right]^2\right)^{1/2} \\
&\le \left(\frac{1}{t}\int_0^t\left[\int_0^s \|K(s-u)\|\|X(u)-X_\infty\|\,du\right]^2 ds\right)^{1/2} \\
&= \left(\frac{1}{t}\int_0^t\left[\int_0^s \|K(s-u)\|^{1/2}\|K(s-u)\|^{1/2}\|X(u)-X_\infty\|\,du\right]^2 ds\right)^{1/2} \\
&\le \left(\frac{\bar{K}}{t}\int_0^t\int_0^s \|K(s-u)\|\|X(u)-X_\infty\|^2\,du\,ds\right)^{1/2},
\end{aligned}$$

where $\bar{K} = \int_0^\infty \|K(t)\|\,dt$. Using (2.1.2) and (4.2.2) we see that the right hand side of this inequality tends to zero as $t\to\infty$. Thus, the third term on the right hand side of (4.5.3) tends to zero. Since (3.2.1) holds, we see that the fourth term tends to zero as $t\to\infty$.

Therefore if we take limits on both sides of (4.5.3) we obtain

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \Sigma(s)\,dB(s) = -\Lambda. \tag{4.5.4}$$

We now show that $\Lambda = 0$ a.s. Each individual entry of the vector $\frac{1}{t}\int_0^t \Sigma(s)\,dB(s)$ is given by

$$\left[\frac{1}{t}\int_0^t \Sigma(s)\,dB(s)\right]_i = \frac{1}{t}\sum_{j=1}^d \int_0^t \Sigma_{ij}(s)\,dB_j(s).$$

Since Λ is almost surely finite by hypothesis we know that $\mathbb{P}[C_i] = 1$ where $C_i \subset \Omega$ is defined by

$$C_i = \left\{\omega : \left[\lim_{t\to\infty}\frac{1}{t}\int_0^t \Sigma(s)\,dB(s)\right]_i \text{ exists}\right\}, \quad i = 1,\dots,d.$$

For each $i = 1,\dots,d$, define σ_i by (3.6.10) and consider the cases when $\sigma_i^2 \in L^1(0,\infty)$ and $\sigma_i^2 \notin L^1(0,\infty)$ individually. If $\sigma_i^2 \in L^1(0,\infty)$, then $\lim_{t\to\infty}\sum_{j=1}^d \int_0^t \Sigma_{ij}(s)\,dB_j(s)$ exists and is a.s. finite, and so

$$\lim_{t\to\infty}\frac{1}{t}\sum_{j=1}^d \int_0^t \Sigma_{ij}(s)\,dB_j(s) = 0, \quad \text{a.s.}$$

Thus, if $\sigma_i^2 \in L^1(0,\infty)$, then $\Lambda_i = 0$, a.s.

In the case when $\sigma_i^2 \notin L^1(0,\infty)$, we have that

$$\liminf_{t\to\infty}\sum_{j=1}^d \int_0^t \Sigma_{ij}(s)\,dB_j(s) = -\infty, \quad \limsup_{t\to\infty}\sum_{j=1}^d \int_0^t \Sigma_{ij}(s)\,dB_j(s) = \infty, \quad \text{a.s.}$$

Therefore

$$\liminf_{t\to\infty}\frac{1}{t}\sum_{j=1}^d \int_0^t \Sigma_{ij}(s)\,dB_j(s) \le 0, \quad \limsup_{t\to\infty}\frac{1}{t}\sum_{j=1}^d \int_0^t \Sigma_{ij}(s)\,dB_j(s) \ge 0, \quad \text{a.s.}$$

Since $\lim_{t\to\infty}\frac{1}{t}\sum_{j=0}^d \int_0^t \Sigma_{ij}(s)\,dB_j(s) = \Lambda_i$ a.s., and Λ_i is almost surely finite, we have

$$\Lambda_i = \lim_{t\to\infty}\frac{1}{t}\sum_{j=1}^d \int_0^t \Sigma_{ij}(s)\,dB_j(s) = \liminf_{t\to\infty}\frac{1}{t}\sum_{j=1}^d \int_0^t \Sigma_{ij}(s)\,dB_j(s) \le 0,$$

so $\Lambda_i \leq 0$, a.s. Similarly,

$$\Lambda_i = \lim_{t\to\infty} \frac{1}{t}\sum_{j=1}^{d}\int_0^t \Sigma_{ij}(s)\,dB_j(s) = \limsup_{t\to\infty} \frac{1}{t}\sum_{j=1}^{d}\int_0^t \Sigma_{ij}(s)\,dB_j(s) \geq 0,$$

so $\Lambda_i \geq 0$, a.s. Therefore, in the case when $\sigma_i^2 \notin L^2(0,\infty)$, we have that $\Lambda_i = 0$, a.s. Hence $\Lambda_i = 0$ for all $i = 1,\dots,d$, a.s., and so $\Lambda = 0$, a.s. That is (4.4.1) must hold. □

Proof of Lemma 4.4.2. We suppose that

$$\lim_{t\to\infty}\int_0^t \|\Sigma(s)\|_F^2\,ds = \infty, \tag{4.5.5}$$

and prove that this is false by contradiction. In order to do this we analyse two different formulations of $\int_0^t \langle X(s), AX(s) + (K * X)(s)\rangle\,ds$.

We obtain the first formulation by rearranging (3.6.18):

$$2\int_0^t \langle X(s), AX(s) + (K * X)(s)\rangle\,ds = \|X(t)\|^2 - \|X_0\|^2 - \int_0^t \|\Sigma(s)\|_F^2\,ds - M(t), \tag{4.5.6}$$

where

$$M(t) = 2\sum_{j=1}^{d}\sum_{i=1}^{n}\int_0^t X_i(s)\Sigma_{ij}(s)\,dB_j(s).$$

The quadratic variation of M is given by

$$\langle M\rangle(t) = 4\sum_{i=1}^{n}\sum_{j=1}^{d}\int_0^t (X_i(s)\Sigma_{ij}(s))^2\,ds.$$

Therefore,

$$\begin{aligned}\langle M\rangle(t) &= 4\sum_{j=1}^{d}\int_0^t\sum_{i=1}^{n}(X_i(s)\Sigma_{ij}(s))^2\,ds\\ &\leq 4\sum_{j=1}^{d}\int_0^t\sum_{i=1}^{n}X_i(s)^2\sum_{i=1}^{n}\Sigma_{ij}(s)^2\,ds\\ &\leq 4\int_0^t \|X(s)\|^2\|\Sigma(s)\|_F^2\,ds.\end{aligned}$$

If we define the event C by

$$C = \{\omega : \lim_{t\to\infty} \langle M\rangle(t,\omega) = \infty\},$$

then by L'Hôpital's rule, (4.5.5) and (4.2.1) we get

$$\limsup_{t\to\infty} \frac{\langle M\rangle(t)}{\int_0^t \|\Sigma(s)\|_F^2\, ds} \le 4\|X_\infty\|^2, \quad \text{a.s. on } C.$$

Therefore, by the law of large numbers for martingales (see Lemma 1.1.6) we get

$$\lim_{t\to\infty} \frac{|M(t)|}{\int_0^t \|\Sigma(s)\|_F^2\, ds} = \lim_{t\to\infty} \frac{|M(t)|}{\langle M\rangle(t)} \cdot \frac{\langle M\rangle(t)}{\int_0^t \|\Sigma(s)\|_F^2\, ds} = 0, \quad \text{a.s. on } C.$$

On $\overline{C}$, we have that $\lim_{t\to\infty} M(t)$ exists a.s. and is a.s. finite. Therefore, on account of (4.5.5), we have

$$\lim_{t\to\infty} \frac{|M(t)|}{\int_0^t \|\Sigma(s)\|_F^2\, ds} = 0, \quad \text{a.s. on } \overline{C}.$$

Hence,

$$\lim_{t\to\infty} \frac{M(t)}{\int_0^t \|\Sigma(s)\|_F^2\, ds} = 0, \quad \text{a.s.}$$

By applying this result and using (4.2.1) in (4.5.6) we now may conclude that

$$\lim_{t\to\infty} \frac{\int_0^t \langle X(s), AX(s) + (K * X)(s)\rangle\, ds}{\int_0^t \|\Sigma(s)\|_F^2\, ds} = -\frac{1}{2}, \quad \text{a.s.} \tag{4.5.7}$$

We now analyse the second reformulation of $\int_0^t \langle X(s), AX(s) + (K*X)(s)\rangle\, ds$. We obtain this reformulation as follows: introduce the function Δ defined by $\Delta(t) = X(t) - X_\infty$, apply Lemma 4.4.1, use the definition of K_1 and the fact that

$$\int_0^t AX(s) + (K * X)(s)\, ds = X(t) - X_0 - \int_0^t \Sigma(s)\, dB(s)$$

to obtain

$$\begin{aligned}\int_0^t \langle X(s), AX(s) + (K * X)(s)\rangle\, ds &= \int_0^t \langle \Delta(s), A\Delta(s) + (K * \Delta)(s)\rangle\, ds \\ &\quad - \int_0^t \langle \Delta(s), K_1(s)X_\infty\rangle\, ds + \left\langle X_\infty, X(t) - X_0 - \int_0^t \Sigma(s)\, dB(s)\right\rangle.\end{aligned} \tag{4.5.8}$$

We now analyse the right hand side of (4.5.8) and show that its limit must be zero, thereby inducing a contradiction to the hypothesis that (4.5.5) holds. Dividing (4.5.8) by $\int_0^t \|\Sigma(s)\|_F^2\,ds$ we get

$$\begin{aligned}
&\frac{1}{\int_0^t \|\Sigma(s)\|_F^2\,ds}\int_0^t \langle X(s), AX(s) + (K * X)(s)\rangle\,ds \\
&\qquad = \frac{1}{\int_0^t \|\Sigma(s)\|_F^2\,ds}\int_0^t \langle \Delta(s), A\Delta(s) + (K * \Delta)(s)\rangle\,ds \\
&\qquad\quad - \frac{\int_0^t \langle \Delta(s), K_1(s)X_\infty\rangle\,ds}{\int_0^t \|\Sigma(s)\|_F^2\,ds} + \frac{\langle X_\infty, X(t) - X_0\rangle}{\int_0^t \|\Sigma(s)\|_F^2\,ds} \\
&\qquad\quad - \frac{1}{\int_0^t \|\Sigma(s)\|_F^2\,ds}\langle X_\infty, \int_0^t \Sigma(s)\,dB(s)\rangle. \qquad (4.5.9)
\end{aligned}$$

Now consider each term on the right hand side of (4.5.9). Applying the Cauchy–Schwarz inequality twice to the numerator of the first term one obtains:

$$\begin{aligned}
&\left|\int_0^t \langle \Delta(s), A\Delta(s) + (K * \Delta)(s)\rangle\,ds\right| \\
&\quad \le \int_0^t \|\Delta(s)\|\|A\Delta(s) + (K * \Delta)(s)\|\,ds \\
&\quad \le \int_0^t \|\Delta(s)\|^2\,ds \int_0^t \|A\Delta(s) + (K * \Delta)(s)\|^2\,ds \\
&\quad \le 2\int_0^t \|\Delta(s)\|^2\,ds \int_0^t \|A\|^2\,\|\Delta(s)\|^2 + \|(K * \Delta)(s)\|^2\,ds \\
&\quad \le 2\int_0^t \|\Delta(s)\|^2\,ds \times \\
&\qquad \left(\int_0^t \|A\|^2\,\|\Delta(s)\|^2\,ds + \int_0^t \left[\int_0^s \|K(u)\|\,du \int_0^s \|K(s-u)\|\,\|\Delta(u)\|^2\,du\right]\,ds\right),
\end{aligned}$$

This term is finite as (4.2.2) and (2.1.2) hold. Consequently, the first term on the right hand side of (4.5.9) has zero limit as $t \to \infty$, a.s. Now consider the second term on the right hand side of (4.5.9). Again, by applying the Cauchy–Schwarz inequality to the numerator one obtains:

$$\begin{aligned}\left|\int_0^t \langle \Delta(s), K_1(s)X_\infty\rangle\, ds\right| &\le \int_0^t \|\Delta(s)\|^2\|K_1(s)\|^2\|X_\infty\|^2\, ds \\ &\le \bar{K}^2\|X_\infty\|^2 \int_0^t \|\Delta(s)\|^2\, ds\end{aligned}$$

where $\bar{K} = \int_0^\infty \|K(s)\|\, ds$ as before. As X_∞ is finite almost surely due to our assumptions, and assumptions (4.2.2) and (3.2.1) hold, it is seen that the numerator of the second term tends to a finite limit, consequently the second term on the right hand side of (4.5.9) tends to 0. Now consider the third term. Again, by applying the Cauchy–Schwarz inequality to the numerator one obtains:

$$\langle X_\infty, X(t) - X_0\rangle \le 2^2\|X_\infty\|^2(\|\Delta(t)\|^2 + \|X_0 - X_\infty\|^2),$$

By (4.2.1) and (4.2.2), it follows that for each ω in an almost sure event $t \mapsto \|\Delta(t,\omega)\|$ is uniformly bounded. Hence, the third term has zero limit as $t \to \infty$, a.s. Thus, by considering the final term on the righthand side of (4.5.9), it is evident that

$$\lim_{t\to\infty} \frac{1}{\int_0^t \|\Sigma(s)\|_F^2\, ds} \int_0^t \langle X(s), AX(s) + (K * X)(s)\rangle\, ds = 0, \quad \text{a.s.} \tag{4.5.10}$$

if it can be shown that (4.5.5) implies

$$\lim_{t\to\infty} \frac{\int_0^t \Sigma(s)\, dB(s)}{\int_0^t \|\Sigma(s)\|_F^2\, ds} = 0, \quad \text{a.s.} \tag{4.5.11}$$

Hence proving (4.5.11) provides the desired contradiction to (4.5.7) in the shape of (4.5.10).

The proof of (4.5.11) is quite straightforward. Define $N(t) = \int_0^t \Sigma(s)\, dB(s)$, for $t \ge 0$ and

$$N_i(t) = \sum_{j=1}^d \int_0^t \Sigma_{ij}(s)\, dB_j(s), \quad t \ge 0,$$

so that $N_i(t) = \langle N(t), \mathrm{e}_i\rangle$. Then each N_i is a local martingale with square variation

$$\langle N_i\rangle(t) = \int_0^t \sigma_i^2(s)\, ds, \quad t \ge 0,$$

where σ_i is defined by (3.6.10). It is easily seen that

$$\langle N_i\rangle(t) \le \int_0^t \|\Sigma(s)\|_F^2 \, ds. \tag{4.5.12}$$

In the case when $\lim_{t\to\infty}\langle N_i\rangle(t) = \infty$, the law of large numbers for martingales and (4.5.12) give

$$\lim_{t\to\infty} \frac{|N_i(t)|}{\int_0^t \|\Sigma(s)\|_F^2 \, ds} = \lim_{t\to\infty} \frac{|N_i(t)|}{\langle N_i\rangle(t)} \cdot \frac{\langle N_i\rangle(t)}{\int_0^t \|\Sigma(s)\|_F^2 \, ds} = 0, \quad \text{a.s.}$$

On the other hand, due to the martingale convergence theorem (see Lemma 1.1.5), if $\lim_{t\to\infty}\langle N_i\rangle(t) < \infty$, then $\lim_{t\to\infty} N_i(t)$ exists a.s. and is a.s. finite. Since Σ obeys (4.5.5), it is immediate that

$$\lim_{t\to\infty} \frac{|N_i(t)|}{\int_0^t \|\Sigma(s)\|_F^2 \, ds} = 0, \quad \text{a.s.}$$

Therefore,

$$\lim_{t\to\infty} \frac{|N_i(t)|}{\int_0^t \|\Sigma(s)\|_F^2 \, ds} = 0, \quad \text{for all } i = 1, \dots, d, \text{ a.s.},$$

from which (4.5.11) follows immediately. □

Chapter 5

Equations with Weakly Singular Kernels

5.1 Introduction

The behaviour of Volterra equations with weakly singular kernels has been studied by several authors including Miller and Feldstein [29] and Brunner *et al.* [10, 11]. We briefly examine the effect of a weakly singular kernel of algebraic or logarithmic type on the convergence and integrability of the solution of (3.1.1). It is found that singularities of this type have no effect on the convergence of the solution. The type of singularity used in the sequel is precisely defined below.

5.2 Discussion of Results

In this chapter, the behaviour of the solution of equation (3.1.1) when the kernel is weakly singular is considered. While Miller and Feldstein considered a general definition for weak singularities in the kernel, Brunner *et al.* [10, 11] considered Volterra equations with weakly singular kernels of algebraic or logarithmic type. In these papers singularities not only in the kernel itself but also in its derivatives were considered. In keeping with earlier assumptions made in this paper no new assumptions concerning the existence of the derivatives of the kernel are made in this section. Instead, the investigation is restricted to the study of singularities in the kernel alone. Consequently, an abridged version of the definition of a weakly singular function used in [11] may be adopted; the function $K : (0, \infty) \to M_{n \times n}(\mathbb{R})$ satisfies

$$K \in \mathcal{W}^{\nu}((0, \infty), M_{n \times n}(\mathbb{R})), \tag{5.2.1}$$

if K is continuous on $(0, \infty)$ and

$$\|K(t)\| \leq c(K) \begin{cases} 1 + |\log(t)| & \nu = 0, \quad 0 < t < \infty, \\ t^{-\nu} & 0 < \nu < 1, \quad 0 < t < \infty. \end{cases}$$

Before considering the effect of a weakly singular kernel on the results discussed in the previous chapter, it is necessary to prove that a solution exists under assumption (5.2.1).

Theorem 5.2.1. *Let K satisfy* (5.2.1) *and let Σ satisfy* (3.1.2) *then there is a unique adapted process $X(\cdot, X_0, \Sigma) \in C([0,\infty), \mathbb{R}^n)$ obeying* (3.1.1).

The proof of this theorem breaks into several steps. Firstly, it is necessary to show that there exists a process X which satisfies a reformulated version of (3.1.1) for $t \in [0, T]$:

$$X(t) = X_0 + \int_0^t \left[A + \int_0^{t-s} K(u)\,du\right] X(s)\,ds + \mu(t), \quad 0 < t \leq T, \tag{5.2.2a}$$

$$X(0) = X_0, \tag{5.2.2b}$$

where $\mu(t) = \int_0^t \Sigma(s)\,dB(s)$. The weak singularity in the kernel introduces added complexity into the analysis of the equation. However, if the reformulated equation is considered this analysis is simplified. Secondly, by applying Fubini's Theorem to (5.2.2) we can show the process X is actually a solution of (3.1.1). Next, a Gronwall–type argument can be implemented to show that this is in fact a unique solution. Finally, standard arguments can be applied to show that X is in fact a unique, continuous, adapted process obeying (3.1.1) on $[0, \infty)$.

A consequence of Theorem 5.2.1 is that assumption (2.1.2) may be replaced by

$$K \in \mathcal{W}^{\nu}((0,\infty), M_{n\times n}(\mathbb{R})) \cap L^1((0,\infty), M_{n\times n}(\mathbb{R})), \quad 0 \leq \nu < 1, \tag{5.2.3}$$

in Theorem 3.2.1. In order to show that the conclusion of this proof is unaltered a number of results must be reanalysed very carefully: firstly, it must be shown that (3.1.1) may still be reformulated as (3.3.5); secondly, in order use expression (3.3.8) for the solution X of (3.1.1) the hypotheses of Theorem 1.1.1 and 1.1.2 must be shown to be valid under (5.2.3); finally, Lemma 3.3.3 needs to be proved afresh under the weaker hypothesis. Each of the above points are stated precisely and proved in Section 5.4. Using them it is possible to prove the analogue of Theorem 3.2.1:

Theorem 5.2.2. *Let K satisfy* (5.2.3) *and* (3.2.1) *and let Σ satisfy* (3.1.2) *and* (3.2.2). *If the resolvent R of* (2.1.1) *satisfies* (3.2.3) *then the solution X of* (3.1.1) *satisfies $X(t) \to X_\infty$ as $t \to \infty$ almost surely, where X_∞ is given by* (3.2.4) *and is almost surely finite.*

The proof of this theorem may be found in Section 5.4 below. The primary reason that the analogue of Theorem 3.2.1 holds is that the reformulation of equation (3.1.1) still holds. In fact, the structure of the reformulated equation ensures that the type of singularity considered in the kernel has no influence on the convergence of the solutions.

The question of integrability of solutions is more delicate and requires careful analysis. The proof of this result requires the use of the variation of parameters representation of the solution. It will be necessary to prove the validity of this formula, which will involve a close examination of stochastic Fubini theorems (see [9], for example), before the integrability of the solution can be tackled. This work lies outside the scope of the thesis and will be examined elsewhere.

In [11, 29] the extent to which the regularity in the kernel influences the regularity of the solution of the deterministic equation was investigated. However, the presence of the non–differentiable Brownian motion in the stochastic equation prohibits the existence of a derivative in the solution. Indeed it is known that the solution to the stochastic equation will be Hölder continuous with exponent $1/2-\epsilon$ for every $\epsilon > 0$. Consequently, we cannot expect to obtain the same amount of regularity in the solution of the stochastic equation as obtained in the deterministic case regardless of the regularity of the kernel.

5.3 Proof of Theorem 5.2.1

In the proof of Theorem 5.2.1 use is made of Fubini's Theorem. Due to the presence of the weakly singular kernel it is crucial that the hypotheses of this theorem are examined closely to ensure that they are satisfied before the conclusion of the theorem is applied.

Proof of Theorem 5.2.1. We begin by proving the existence of a continuous adapted process X which satisfies (5.2.2). Since the dispersion coefficient Σ is dependent on the parameter t alone we can fix an arbitrary $\omega \in \Omega$ and regard equation (5.2.2) as a deterministic equation with forcing function $\{\mu(t,\omega) : 0 \le t \le T\}$. Define the Picard iterations as

$$X_k(t,\omega) = X_0 + \int_0^t \left[A + \int_0^{t-s} K(u)\,du\right] X_{k-1}(s,\omega)\,ds + \mu(t,\omega), \quad 0 < t \le T,$$
$$X_k(0,\omega) = X_0,$$

when $k \ge 1$ and

$$X_0(t,\omega) = X_0,$$

otherwise. By mathematical induction it can easily be shown that X_k is continuous and adapted on $[0,T]$ for each $k \ge 0$.

On the interval $[0,T]$ observe that

$$\begin{aligned}
\|X_1(t,\omega) - X_0(t,\omega)\| &= \|X_1(t,\omega) - X_0\| \\
&= \left\| \int_0^t \left[A + \int_0^{t-s} K(u)\,du\right] X_0\,ds + \mu(t,\omega)\right\| \\
&\le \int_0^t \left[\|A\| + \int_0^{t-s} \|K(u)\|\,du\right] \|X_0\|\,ds + \|\mu(t,\omega)\| \\
&\le \bar{M}\|X_0\|t + \bar{\mu} \\
&\le \bar{M}\|X_0\|T + \bar{\mu} := C,
\end{aligned}$$

where $\bar{M} = \|A\| + \int_0^\infty \|K(u)\|\,du$ and $\bar{\mu} = \sup_{0\le t\le T} \|\mu(t,\omega)\|$. So it is clear that

$$\sup_{0\le t\le T} \|X_1(t,\omega) - X_0(t,\omega)\| \le C. \tag{5.3.1}$$

Again on the interval $[0,T]$ and for $k > 0$ we have that

$$X_{k+1}(t,\omega) - X_k(t,\omega) = \int_0^t \left[A + \int_0^{t-s} K(u)\,du\right] (X_k(s,\omega) - X_{k-1}(s,\omega))\,ds.$$

Taking norms across this equation we see that

$$\begin{aligned}
\|X_{k+1}(t,\omega) - X_k(t,\omega)\| &\le \int_0^t \left[\|A\| + \int_0^{t-s} \|K(u)\|\,du\right] \|X_k(s,\omega) - X_{k-1}(s,\omega)\|\,ds \\
&\le \bar{M} \int_0^t \|X_k(s,\omega) - X_{k-1}(s,\omega)\|\,ds.
\end{aligned} \tag{5.3.2}$$

Using (5.3.1) and (5.3.2) we now prove by mathematical induction that

$$\|X_{k+1}(t,\omega) - X_k(t,\omega)\| \leq \frac{1}{k!}C(\bar{M}t)^k, \quad 0 \leq k, \quad 0 \leq t \leq T. \tag{5.3.3}$$

As a consequence of (5.3.1) it is clear that (5.3.3) holds for $k = 0$. Now assume that (5.3.3) holds for $k = l$. Using this assumption we show that (5.3.3) holds for $k = l+1$ using the following argument:

$$\begin{aligned}
\|X_{l+2}(t,\omega) - X_{l+1}(t,\omega)\| &\leq \bar{M}\int_0^t \|X_{l+1}(s,\omega) - X_l(s,\omega)\|\,ds \\
&\leq \bar{M}\int_0^t \frac{1}{l!}C(\bar{M}s)^l\,ds \\
&= \frac{1}{(l+1)!}C(\bar{M}t)^{l+1}.
\end{aligned}$$

Clearly,

$$\sup_{0\leq t\leq T}\|X_{k+1}(t,\omega) - X_k(t,\omega)\| \leq \frac{1}{k!}C(\bar{M}T)^k, \quad 0 \leq k, \quad 0 \leq t \leq T. \tag{5.3.4}$$

Let $X_k^i(t,\omega)$ denote the i^{th} component of the vector $X_k(t,\omega)$ where $0 \leq i \leq n$. Notice that due to the definition of the norm and (5.3.4) it is seen that

$$\sup_{0\leq t\leq T}\left|X_{k+1}^i(t,\omega) - X_k^i(t,\omega)\right| \leq \frac{1}{k!}C(\bar{M}T)^k, \quad 0 \leq k, \quad 0 \leq t \leq T.$$

The expression $\frac{1}{k!}C(\bar{M}T)^k$ is the general term of the Taylor expansion of $Ce^{\bar{M}T}$; so $\sum_{k=0}^{\infty}\frac{1}{k!}C(\bar{M}T)^k$ is a convergent sequence. Now, due to the Weierstrauss M–test it is clear that $X_k^i(t,\omega)$, which may be expressed as a partial sum

$$X_k^i(t,\omega) = X_0^i(t,\omega) + \sum_{j=0}^{k-1}(X_{j+1}^i(t,\omega) - X_j^i(t,\omega)),$$

must converge to a limit, which we call $X^i(t,\omega)$. Thus,

$$\lim_{k\to\infty} X_k(t,\omega) = X(t,\omega) = (X^1(t,\omega),\ldots,X^n(t,\omega)).$$

We now show that X is a solution of (5.2.2). Consider the expression

$$X(t,\omega) - X_0 - \int_0^t \left[A + \int_0^{t-s} K(u)\,du\right] X(s,\omega)\,ds - \mu(t,\omega).$$

Using the definition of the Picard iteration we see that

$$\begin{aligned}
X(t,\omega) - X_0 &- \int_0^t \left[A + \int_0^{t-s} K(u)\,du\right] X(s,\omega)\,ds - \mu(t,\omega)\\
&= (X(t,\omega) - X_k(t,\omega)) - X_0 - \int_0^t \left[A + \int_0^{t-s} K(u)\,du\right] (X(s,\omega) - X_{k-1}(s,\omega))\,ds\\
&\qquad - \mu(t,\omega) + X_k(t,\omega) - \int_0^t \left[A + \int_0^{t-s} K(u)\,du\right] X_{k-1}(s,\omega)\,ds.
\end{aligned}$$

Rearranging this and then taking norms across both sides the above equation becomes

$$\begin{aligned}
&\left\| X(t,\omega) - X_0 - \int_0^t \left[A + \int_0^{t-s} K(u)\,du\right] X(s,\omega)\,ds - \mu(t,\omega)\right\|\\
&\qquad \le \|X(t,\omega) - X_k(t,\omega)\| + \bar{M} \int_0^t \|X(s,\omega) - X_{k-1}(s,\omega)\|\,ds\\
&\qquad + \left\| X_k(t,\omega) - \left(X_0 + \int_0^t \left[A + \int_0^{t-s} K(u)\,du\right] X_{k-1}(s,\omega)\,ds + \mu(t,\omega)\right)\right\|.
\end{aligned}$$

Using the definition of the Picard iteration for k we see that the last term on the right hand side of this inequality is equal to 0. Now, taking the supremum over $[0,T]$ on both sides of the equation we see that

$$\begin{aligned}
&\sup_{0\le t\le T} \left\| X(t,\omega) - X_0 - \int_0^t \left[A + \int_0^{t-s} K(u)\,du\right] X(s,\omega)\,ds - \mu(t,\omega)\right\|\\
&\qquad \le \sup_{0\le t\le T} \|X(t,\omega) - X_k(t,\omega)\| + \bar{M} \int_0^T \|X(s,\omega) - X_{k-1}(s,\omega)\|\,ds.
\end{aligned}$$

Considering the limit as $k \to \infty$ we see that

$$\lim_{k\to\infty} \sup_{0\le t\le T} \left\| X(t,\omega) - X_0 - \int_0^t \left[A + \int_0^{t-s} K(u)\,du\right] X(s,\omega)\,ds - \mu(t,\omega)\right\| = 0,$$

that is, X is a solution of (5.2.2). Hence, the continuous adapted process $t \mapsto X(t,\Sigma)$ also satisfies (3.1.1) for $0 \le t \le T$.

The following argument is provided to show that $t \mapsto X(t,\Sigma)$ is the unique continuous adapted process satisfying (3.1.1) on $0 \le t \le T$. By applying Fubini's Theorem, we see that a continuous solution X of (5.2.2) must also be a solution of (3.1.1). Suppose that there are two continuous adapted processes X and Y satisfying (3.1.1). Subtracting Y

from X, changing the order of integration in the double integral once again and taking norms across the equation we obtain

$$\|X(t) - Y(t)\| \leq \bar{M} \int_0^t \|X(s) - Y(s)\| \, ds,$$

where $\bar{M} = \|A\| + \int_0^\infty \|K(t)\| \, dt$. Using Gronwall's inequality we see that $\|X(t) - Y(t)\| = 0$, in other words $t \mapsto X(t, \Sigma)$ is unique process obeying (3.1.1) on $0 \leq t \leq T$. However, since T is arbitrary we see that X is the unique, continuous, adapted process which satisfies (3.1.1) on the positive real line.

□

5.4 Proof of Theorem 5.2.2

In order to prove Theorem 5.2.2 the following lemmata are required.

Lemma 5.4.1. *Let K satisfy* (5.2.3) *and let Σ satisfy* (3.1.2). *Then* (3.1.1) *may be reformulated as* (3.3.5) *where the function $t \mapsto F(t)$ can be written as* (2.3.7) *and is well defined for all $t \geq 0$.*

Lemma 5.4.2. *Let K satisfy* (5.2.3) *and* (3.2.1). *Then the function F satisfies*

$$\int_0^\infty \|F(t)\| \, dt < \infty, \tag{5.4.1}$$

and the Laplace transform of F, $z \mapsto \hat{F}(z)$, is given by (2.3.2) *for $\operatorname{Re} z \geq 0$ and $z \neq 0$ and* (2.3.3) *for $z = 0$.*

Lemma 5.4.3. *Let K satisfy* (5.2.3) *and let f satisfy* (2.1.4). *Then the solution x of* (2.1.3) *expressed using the variation of parameters representation given by* (2.1.5).

Lemma 5.4.4. *Let K satisfy* (5.2.3). *Suppose that the kernel K satisfies* (3.2.1) *and the resolvent R of* (2.1.1) *satisfies* (3.3.1). *If* (3.3.3) *holds, then rank (PDN) < rank (N)*

where $D = I + \int_0^\infty \int_s^\infty K(u)\,du\,ds$ *and the matrix* N *is defined as* $N = TD_NT^{-1}$. *Here* $D_N = diag(f_1, f_2, ..., f_n)$ *with* $f_i = 1$ *if all the elements of the* i^{th} *column of* J *are zero, and* $f_i = 0$ *otherwise.*

Lemma 5.4.5. *Let* K *satisfy* (5.2.3). *Suppose that the kernel* K *satisfies* (3.2.1) *and there exists a constant* C *such that the resolvent* R *of* (2.1.1) *satisfies* (3.3.1). *Then* (3.3.2) *holds. Moreover, if the constant* R_∞ *is defined by* (2.2.11) *then* $R(t) \to R_\infty$ *as* $t \to \infty$ *and* $C = R_\infty$.

Lemma 5.4.6. *If* K *satisfies* (5.2.3) *and* (3.2.1) *and the resolvent* R *of* (2.1.1) *satisfies* (3.2.3), *then* (3.3.4) *holds.*

Proof of Lemma 5.4.1. We begin by considering (5.2.2). Similarly to Lemma 3.2.1 both sides of (5.2.2) are premultiplied by $\Phi'(t-s)$ to obtain

$$\Phi'(t-s)X(s) = \Phi'(t-s)X_0 + \Phi'(t-s)\rho(s) + \Phi'(t-s)\mu(s), \quad 0 \le s \le t \le T,$$

where $\rho(t) = \int_0^t \left[A + \int_0^{t-s} K(u)\,du\right] X(s)\,ds$ and $\Phi(t) = P + Qe^{-t}$. Integrating this over $[0,T]$ one obtains

$$\begin{aligned}\int_0^t \Phi'(t-s)X(s)\,ds \\ = \int_0^t \Phi'(t-s)X_0\,ds + \int_0^t \Phi'(t-s)\rho(s)\,ds + \int_0^t \Phi'(t-s)\mu(s)\,ds, \quad 0 \le s \le t \le T.\end{aligned} \tag{5.4.2}$$

Now examine each term in (5.4.2). As Theorem 5.2.1 holds it is clear that the integral on the left hand side (5.4.2) exists on $[0,T]$ and may be written notationally as $\Phi' * X$. The first term on the right hand side of (5.4.2) can be evaluated in a straightforward manner due to the continuity of the integrand: $\int_0^t \Phi'(t-s)X_0\,ds = \Phi(t)X_0 - X_0$. Now examine the second term. Due to the fact that

$$\int_0^t \left[\int_0^{t-s} \|K(u)\|\,du\right] \|X(s)\|\,ds < \infty, \tag{5.4.3}$$

we see that ρ is continuous on $[0,t]$ and $\rho(0) = 0$. So the integral $\int_0^t \Phi'(t-s)\rho(s)\,ds$ is well–defined for $t \in [0,T]$. We can reformulate this integral using integration by parts:

$$\int_0^t \Phi'(t-s)\rho(s)\,ds = -\rho(t) + (\Phi A * X)(t) + (\Phi * K * X)(t),$$

where we have rewritten ρ according to $\rho(t) = \int_0^t \left[AX(s) + \int_0^s K(s-u)X(u)\,du\right] ds$.

Using the above and the fact that $\mu(t) = X(t) - X_0 - \rho(t)$ we see that (5.2.2) becomes $X(t) + ([\Phi' - \Phi A - \Phi * K] * X)(t) = G_S(t)$ where G_S is given by (3.3.7). Letting $F = \Phi' - \Phi A - \Phi * K$ as before we now check that the reformulated version of F given by (2.3.7) is still valid:

$$\begin{aligned} F(t) &= e^{-t}Q - PA - e^{-t}QA - \int_0^t PK(s)\,ds - \int_0^t e^{-(t-s)}QK(s)\,ds \\ &= -e^{-t}(Q+QA) - P\left(A + \int_0^\infty K(s)\,ds\right) + P\int_t^\infty K(u)\,du - \int_0^t e^{-(t-s)}QK(s)\,ds. \end{aligned}$$

As the function K is integrable this means that the matrix M which is defined as $M = A + \int_0^\infty K(s)\,ds$ exists and so $PM = 0$ as before (where P is defined as in Section 2.2). Now consider the second and third terms: both terms are bounded by $\int_0^\infty \|K(t)\|\,dt$ so they are well–defined. Thus, F is a well–defined function on $[0,T]$. Again, since T is arbitrary the reformulated equation is valid for $t \geq 0$. □

Proof of Lemma 5.4.2. We use (2.3.7) to obtain

$$\begin{aligned} \int_0^\infty \|F(t)\|\,dt \leq \|Q + QA\| + \|Q\| \int_0^\infty \int_0^t e^{-(t-s)} \|K(s)\|\,ds\,dt \\ + \|P\| \int_0^\infty \int_t^\infty \|K(s)\|\,ds\,dt, \quad (5.4.4) \end{aligned}$$

and use this to show that (5.4.1) holds. Obviously the first term on the right hand side of (5.4.4) is bounded. We now show that the second term on the right hand side of (5.4.4) is bounded. Note that for an arbitrary constant T, with $0 \leq T < \infty$,

$$\begin{aligned}\int_0^T \int_s^T e^{-t}e^s\|K(s)\|\,dt\,ds &= \int_0^T (e^{-s} - e^{-T})e^s\|K(s)\|\,ds \\ &= \int_0^T \|K(s)\|\,ds - \int_0^T e^{-(T-s)}\|K(s)\|\,ds < \infty.\end{aligned} \tag{5.4.5}$$

Thus, we can apply Fubini's Theorem to obtain

$$\begin{aligned}\int_0^\infty \int_0^t e^{-(t-s)}\|K(s)\|\,ds\,dt &= \lim_{T\to\infty} \int_0^T \int_0^t e^{-(t-s)}\|K(s)\|\,ds\,dt \\ &= \lim_{T\to\infty} \int_0^T \int_s^T e^{-(t-s)}\|K(s)\|\,dt\,ds \\ &= \int_0^\infty \|K(s)\|\,ds - \lim_{T\to\infty} \int_0^T e^{-(T-s)}\|K(s)\|\,ds.\end{aligned}$$

Now consider the final line of this expression. Clearly the first term is finite due to our assumptions. We see that the second term on the right hand side tends to zero by using the Dominated Convergence Theorem. (Define a sequence $f_T(t) = e^{-(T-t)}\|K(t)\|$ for $0 \leq t \leq T$ and zero otherwise, clearly $\lim_{T\to\infty} f_T(t) = 0$. As $|f_T(t)| \leq \|K(t)\|$ and K is integrable we can apply the Dominated Convergence Theorem.) Thus the second term on the right hand side of (5.4.4) tends to zero.

Now consider the third term on the right hand side of (5.4.4). Note that

$$\int_0^T s\|K(s)\|\,ds < \infty.$$

So, using Fubini's Theorem and the Dominated Convergence Theorem,

$$\begin{aligned}\int_0^\infty s\|K(s)\|\,ds &= \lim_{T\to\infty} \int_0^T s\|K(s)\|\,ds \\ &= \lim_{T\to\infty} \int_0^T \int_0^s \|K(s)\|\,dt\,ds \\ &= \lim_{T\to\infty} \int_0^T \int_t^T \|K(s)\|\,ds\,dt \\ &= \int_0^\infty \int_t^\infty \|K(s)\|\,ds\,dt.\end{aligned}$$

So we see that the third term on the right hand side of (5.4.4) is bounded. Combining the above arguments, we see that the left hand side of (5.4.4) is bounded; that is (5.4.1) must hold.

As assumption (5.4.1) holds it is known that the Laplace Transform of F, denoted $\hat{F}(z)$, exists for $\operatorname{Re} z \geq 0$. We now show that the expression for the Laplace Transform of $\hat{F}(z)$ given by (2.3.2) for $\operatorname{Re} z \geq 0$ and $z \neq 0$ and (2.3.3) for $z = 0$ is valid.

We begin by considering the case when $\operatorname{Re} z \geq 0$ and $z \neq 0$. The first term of F given in (2.3.7) in not influenced by K and so can be calculated in a straightforward manner as before. Now consider the second term. Before calculating the Laplace transform of $(e * QK)(t)$ we note for $\operatorname{Re} z \geq 0$ and $z \neq 0$ that

$$\int_0^T \int_s^T e^{-(z+1)t} e^s K(s)\, dt\, ds = \frac{1}{z+1} \int_0^T (e^{-(z+1)s} - e^{-(z+1)T}) e^s K(s)\, ds,$$

so

$$\begin{aligned}\int_0^T \int_s^T & e^{-(z+1)t} e^s \|K(s)\|\, dt\, ds \\ &\leq \frac{1}{|z+1|}\left(\int_0^T e^{-zs}\|K(s)\|\, ds + e^{-zT} \int_0^T e^{-(T-s)}\|K(s)\|\, ds \right) \\ &\leq \frac{2}{|z+1|} \int_0^T \|K(s)\|\, ds < \infty.\end{aligned}$$

Using this we can apply Fubini's Theorem and the Dominated Convergence Theorem to obtain

$$\begin{aligned}\int_0^\infty \int_0^t & e^{-zt} e^{-(t-s)} QK(s)\, ds\, dt \\ &= \lim_{T\to\infty} \int_0^T \int_0^t e^{-(z+1)t} e^s QK(s)\, ds\, dt \\ &= \lim_{T\to\infty} \int_0^T \int_s^T e^{-(z+1)t} e^s K(s)\, dt\, ds \\ &= \lim_{T\to\infty} \frac{1}{z+1}\left(\int_0^T e^{-zs} K(s)\, ds - e^{-zT} \int_0^T e^{-(T-s)} K(s)\, ds \right) \\ &= \frac{1}{z+1}\hat{K}(z).\end{aligned}$$

Now consider the third term on the right hand side of (2.3.7). It is necessary to simplify the Laplace Transform of the function $t \mapsto \int_t^\infty PK(s)\, ds$. Note that

$$\int_0^T \int_0^s e^{-zt} K(s)\, dt\, ds = \frac{1}{z} \int_0^T (1 - e^{-zs}) K(s)\, ds,$$

so

$$\int_0^T \int_0^s e^{-zt} \|K(s)\| \, dt \, ds \leq \frac{1}{|z|} \int_0^T (1 - e^{-zs}) \|K(s)\| \, ds < \infty.$$

So applying Fubini's theorem and the Dominated Convergence Theorem we see that

$$\begin{aligned}
\frac{1}{z} \int_0^\infty (1 - e^{-zs}) K(s) \, ds &= \lim_{T \to \infty} \int_0^T \int_0^s e^{-zt} K(s) \, dt \, ds \\
&= \lim_{T \to \infty} \int_0^T \int_t^T e^{-zt} K(s) \, ds \, dt \\
&= \int_0^\infty \int_t^\infty e^{-zt} K(s) \, ds \, dt.
\end{aligned}$$

Thus, $\int_0^\infty \int_t^\infty e^{-zt} PK(s) \, ds \, dt = \frac{1}{z} P\hat{K}(0) - \frac{1}{z} P\hat{K}(z)$. Combining the above arguments we see that $\hat{F}(z)$, given by (2.3.2), holds as before.

Now consider the case when $z = 0$. The first term in of F in (2.3.7) in not influenced by K and so can be calculated in a straightforward manner as before. Now consider the second term on the right hand side of (2.3.7). Since (5.4.5) holds we can apply Fubini's Theorem and the Dominated Convergence Theorem to obtain $\int_0^\infty \int_0^t e^{-(t-s)} QK(s) \, ds = Q\hat{K}(0)$. Also, since (3.2.1) holds we can change the order of integration in the third term on the right hand side of (2.3.7) to obtain $\int_0^\infty \int_t^\infty PK(s) \, ds \, dt = -P\hat{K}'(0)$. Combining the above arguments we see that $\hat{F}(0)$, given by (2.3.2), holds as before. $\square$

Proof of Lemma 5.4.3. We want to show that x given by (2.1.5) satisfies equation (2.1.3). Initially we show that this representation holds for $0 \leq t \leq T$. Consider the left hand side of (2.1.3) and substitute the variation of parameters representation of the solution into the integral version:

$$\begin{aligned}
x_0 &+ \int_0^t \left(Ax(s) + \int_0^s K(t-s)x(s)\,ds\right)\,dt + \int_0^t f(s)\,ds \\
&= x_0 + \int_0^t \left(A[R(s)x_0 + (R*f)(s)] + \int_0^s K(s-u)[R(u)x_0 + (R*f)(u)]\,du\right)\,ds \\
&\qquad + \int_0^t f(s)\,ds \\
&= x_0 + \int_0^t AR(s)\,ds\,x_0 + \int_0^t\int_0^s K(s-u)R(u)\,du\,ds\,x_0 + \int_0^t\int_0^s AR(s-u)f(u)\,du\,ds \\
&\qquad + \int_0^t\int_0^s\int_0^u K(s-u)R(u-v)f(v)\,dv\,du\,ds + \int_0^t f(s)\,ds.
\end{aligned}$$

Consider the final line on the right hand side. Using the continuity of f and R on finite intervals it is clear that we can interchange the order of integration of the fourth term. Since $\int_0^t\int_0^s\int_0^{s-v} \|K(s-v-z)\|\,\|R(u-v)\|\,\|f(v)\|\,dz\,dv\,ds < \infty$ for $0 \le t \le T$ we can change the order of integration twice in the fifth term. So we obtain

$$\begin{aligned}
x_0 &+ \int_0^t \left(Ax(s) + \int_0^s K(s-u)x(u)\,du\right)\,ds + \int_0^t f(s)\,ds \\
&= x_0 + \int_0^t [AR(s) + (K*R)(s)]\,ds\,x_0 + \int_0^t \left(\int_0^{t-v} AR(w)\,dw\right) f(v)\,dv \\
&\qquad + \int_0^t \left(\int_0^{t-v} (K*R)(w)\,dw\right) f(v)\,dv + \int_0^t f(s)\,ds \\
&= x_0 + \int_0^t R'(s)\,ds\,x_0 + \int_0^t\int_0^{t-v} R'(w)\,dw\,f(v)\,dv + \int_0^t f(s)\,ds \\
&= R(t)x_0 + (R*f)(t),
\end{aligned}$$

as required. Now since T is arbitrary this representation holds for all $0 \le t < \infty$. □

Proof of Lemma 5.4.3. As the kernel is integrable, the matrix M exists. The proof of this lemma now follows the line of reasoning given in the proof of Lemma 3.3.2. □

Proof of Lemma 5.4.5. Due to (i) the existence of the variation of parameters representation of the solution; (ii) the assumption that the kernel is integrable; (iii) the existence of the function F which allows the differential equation (5.2.2) to be reformulated as an integral equation, we may follow the line of reasoning in Lemma 3.3.1 to show that Lemma 5.4.5 holds. □

Proof of Lemma 5.4.6. Using the definition of $\hat{F}(0)$ obtained in Lemma 5.4.2 we can show that

$$\det[I + \hat{F}(0)] = \det\left[P - M + \int_0^\infty \int_s^\infty PK(u)\,du\,ds\right].$$

Now using Lemma 5.4.4 we see that this expression is nonzero.

Using the definition of $\hat{F}(z)$ obtained in Lemma 5.4.2 we can show that

$$\det[I + \hat{F}(z)] = \frac{1}{z}\frac{1}{z+1}\det[zI + P]\det\left[zI - A - \hat{K}(z)\right],$$

when $\operatorname{Re} z \geq 0$, $z \neq 0$. Now, using the fact that the variation of parameters representation of the solution holds we can apply the line of reasoning found in Lemma 3.3.3 to complete the proof. □

Proof of Lemma 5.2.2. By applying Lemma 5.4.1 we see that equation (3.1.1) may be reformulated as the integral equation defined as (3.3.5). Since the function F defined by (2.3.7) does not contain a singularity we can apply Theorem 1.1.1 and Lemma 5.4.6 to see that there exists a unique solution $r \in L^1((0,\infty), M_{n\times n}(\mathbb{R}))$ of the equations $r + F * r = F$ and $r + r * F = F$. We can integrate $r + F * r = F$ over $[0,\infty)$ and rearrange the equation to obtain

$$\int_0^\infty r(s)\,ds = \left(I + \int_0^\infty F(s)\,ds\right)^{-1}\int_0^\infty F(s)\,ds.$$

Using Theorem 1.1.2 we see that the solution X of (3.1.1) can be expressed using variation of parameters: $X(t) = G_S(t) - (r * G_S)(t)$. Letting $t \to \infty$ we obtain an expression for X_∞ as before. □

Chapter 6

Exponential Convergence of Solutions of (3.1.1) to a Nontrivial Stochastic Random Variable

6.1 Introduction

In this chapter we study the exponential decay of the solution of X (3.1.1) to a non–equilibrium limit under conditions (2.1.2) and (3.1.2) on the kernel K and the noise term Σ respectively.

A paper by Appleby and Freeman [5] considered the speed of convergence of solutions of (3.1.1) to a trivial equilibrium. It was shown that under the condition that the kernel does not change sign on $[0, \infty)$ then (i) the almost sure exponential convergence of the solution to zero, (ii) the p-th mean exponential convergence of the solution to zero and (iii) the exponential integrability of the kernel and the exponential square integrability of the noise are equivalent.

This chapter extends these results. Conditions are determined on the resolvent, kernel and noise terms which ensure exponential convergence of solutions to a non–equilibrium limit in the p-th mean and almost sure senses. Similarly to [5] it is shown that the exponential integrability of the kernel and the exponential square integrability of the noise are crucial.

6.2 Discussion of Results

In this section, the necessary and sufficient conditions for the solution of (3.1.1) to converge exponentially to a nontrivial limit are stated. Appleby and Freeman [5] considered the exponential convergence of solutions of (3.1.1) to zero. The connections between their result and the main result presented in this chapter are discussed, as are the connections between Theorem 2.2.5 and the main result presented in this chapter.

Appleby and Freeman obtained the following result in the case where the solution of (3.1.1) is integrable.

Theorem 6.2.1. *Let K satisfy* (2.1.2) *and let Σ satisfy* (3.1.2) *and* (3.2.2)*. If* (2.2.12) *holds, then the following are equivalent.*

(i) *There exist constants $\alpha > 0$ and $\gamma > 0$ such that* (2.2.13) *holds, Σ satisfies*

$$\int_0^\infty \|\Sigma(s)\|^2 e^{2\gamma s}\, ds < \infty, \tag{6.2.1}$$

and the solution R of (2.1.1) *satisfies* (2.2.1)

(ii) *For all initial conditions X_0 there exists $\lambda_p > 0$, such that for every $p > 0$ there exists a constant $M_p(X_0) > 0$ such that the solution $t \to X(t; X_0, \Sigma)$ of* (3.1.1) *satisfies*

$$\mathbb{E}\left[\|X(t)\|^p\right] \leq M_p(X_0) e^{-\lambda_p t}, \quad t \geq 0. \tag{6.2.2}$$

(iii) *For all initial conditions X_0 there exists a constant $\beta > 0$ such that the solution $t \to X(t; X_0, \Sigma)$ of* (3.1.1) *satisfies*

$$\limsup_{t\to\infty} \frac{1}{t} \log \|X(t)\| \leq -\beta \quad a.s. \tag{6.2.3}$$

In this chapter, a result in the case where the solution is not integrable but approaches a nontrivial limit is considered. In Chapter 2 the exponential convergence of the solution of (2.1.1) to a non–equilibrium limit was considered. By writing the solution of (3.1.1) in terms of variation of parameters the theory contained in Chapter 2 may be applied in the analysis of the solutions of (3.1.1). The main theorem of this chapter is now stated.

Theorem 6.2.2. *Let K satisfy* (2.1.2) *and* (2.2.7) *and let Σ satisfy* (3.1.2)*. If K satisfies* (2.2.12) *then the following are equivalent.*

(i) There exists a constant R_∞ such that the solution R of (2.1.1) satisfies (3.2.3) and the function Σ satisfies (3.2.2) and (3.2.6), and there exist constants $\alpha > 0$ and $\gamma > 0$ such that K and Σ satisfy (2.2.13) and (6.2.1) respectively.

(ii) For all initial conditions X_0 and constants $p > 0$ there exists an almost surely finite $\mathcal{F}^B(\infty)$-measurable random variable $X_\infty(X_0, \Sigma)$ with $\mathbb{E}\,\|X_\infty\|^p < \infty$ such that the unique continuous adapted process $X(\cdot\,; X_0, \Sigma)$ which obeys (3.1.1) satisfies

$$\mathbb{E}\left[\|X(t) - X_\infty\|^p\right] \leq m_p e^{-\beta_p t}, \quad t \geq 0, \tag{6.2.4}$$

where β_p and $m_p = m_p(X_0)$ are positive constants.

(iii) For all initial conditions X_0 there exists an a.s. finite $\mathcal{F}^B(\infty)$-measurable random variable $X_\infty(X_0, \Sigma)$ such that the unique continuous adapted process $X(\cdot\,; X_0, \Sigma)$ which obeys (3.1.1) satisfies

$$\limsup_{t\to\infty} \frac{1}{t} \log \|X(t) - X_\infty\| \leq -\beta_0 \quad a.s. \tag{6.2.5}$$

where β_0 is a positive constant.

It is clear the assumption (6.2.1) guarantees that assumptions (3.2.2) and (3.2.6) hold. In this theorem they are explicitly assumed in order to highlight the conditions that guarantee the existence of a limit and the integrability of the solution minus its limit in contrast to the conditions that guarantee the speed of convergence.

Whether considering the exponential convergence of solutions to a non-trivial limit or the exponential convergence to zero it is seen that the conditions on the exponential integrability of the kernel and noise term remain the same. The additional assumptions in Theorem 6.2.2 ensure the convergence to an explicit almost surely finite random variable. In addition, it is required that the $R - R_\infty$ lies in the space of square integrable functions. This is due to the fact that in general it is more natural to consider Itô integrals with square integrable integrands rather than integrable integrands. Here, $R - R_\infty \in L^2(0, \infty)$

rather than in the space of $L^1(0,\infty)$ functions is considered. Nonetheless it is possible to make use of Theorem 2.2.6 in the proof of Theorem 6.2.2. This is due to the fact that $R - R_\infty \in L^2(0,\infty)$ together with an assumption on the second moment of the kernel implies that $R - R_\infty \in L^1(0,\infty)$. This statement is made precise in Lemma 6.3.2.

6.3 Sufficient Conditions for Exponential Convergence of Solutions of (3.1.1)

In this section, sufficient conditions for exponential convergence of solutions of (3.1.1) to a non–equilibrium limit are obtained. Proposition 6.3.1 concerns convergence in the p^{th} mean sense while Proposition 6.3.2 deals with the almost sure case.

Proposition 6.3.1. *Let K satisfy* (2.1.2) *and* (2.2.7)*, Σ satisfy* (3.1.2) *and* (3.2.2) *and let R_∞ be a constant matrix such that the solution R of* (2.1.1) *satisfies* (3.2.3). *If there exist constants $\alpha > 0$ and $\gamma > 0$ such that such that* (2.2.13) *and* (6.2.1) *of Theorem 6.2.2(i) hold, then there exist constants $\beta_p > 0$, independent of X_0, and $m_p = m_p(X_0) > 0$, such that statement (ii) of Theorem 6.2.2 holds.*

Proposition 6.3.2. *Let K satisfy* (2.1.2) *and* (2.2.7) *and let Σ satisfy* (3.1.2), (3.2.2) *and* (3.2.6) *where R_∞ is a constant matrix such that the solution R of* (2.1.1) *satisfies* (3.2.3). *If there exist constants $\alpha > 0$ and $\gamma > 0$ such that such that* (2.2.13) *and* (6.2.1) *of Theorem 6.2.2(i) hold, then there exists a constant $\beta_0 > 0$, independent of X_0 such that statement (iii) of Theorem 6.2.2 holds.*

Proposition 6.3.1 may be proved by examining the variation of parameters representation of the solution. Proposition 6.3.2 may be proved by considering the integral representation of the equation.

Extensive use is made of Liapunov's inequality in the proof of Proposition 6.3.1 and Proposition 6.3.2. This inequality is more appropriate in this instance than Hölder's inequality: using Hölder's inequality, the exponential p^{th} mean convergence is obtained for $p > 1$ while exponential convergence is obtained for $p > 0$ using Liaponov's inequality, which is a stronger result.

The proof of Proposition 6.3.1 is divided into two cases: $0 < p \leq 2$ and $p > 2$. In the first case, Lemma 6.3.1 and Liapunov's inequality may by applied initially to show the existence of the p^{th} moment of the limit and then to show that the speed of convergence to the limit is exponential. In the second case, the result is proved for the $2m^{th}$ moment, where $2(m-1) < p \leq 2m$. Liapunov's inequality may then be applied to prove the result for p. Here the even exponent simplifies calculations.

In Chapter 2, the conditions which give mean square convergence to a nontrivial limit were considered. So a natural progression in this chapter is the examination of the speed of convergence in the mean square case. Lemma 6.3.1 examines the case when $p = 2$ in order to highlight this important case. This lemma may be then used when generalising the result to all $p > 0$.

Lemma 6.3.1. *Let K satisfy (2.1.2) and (2.2.7), let Σ satisfy (3.1.2) and (3.2.2) and let R_∞ be a constant matrix such that the solution R of (2.1.1) satisfies (3.2.3). If there exist constants $\alpha > 0$ and $\gamma > 0$ such that (2.2.13) and (6.2.1) of Theorem 6.2.2(i) hold, then there exist constants $\lambda > 0$, independent of X_0, and $m = m(X_0) > 0$, such that*

$$\mathbb{E}\,\|X(t) - X_\infty\|^2 \leq m(X_0)e^{-2\lambda t}, \quad t \geq 0. \tag{6.3.1}$$

From Chapters 3 and 4 it is evident that $R - R_\infty \in L^2((0,\infty), M_{n\times n}(\mathbb{R}))$ is a more natural condition on the resolvent than $R - R_\infty \in L^1((0,\infty), M_{n\times n}(\mathbb{R}))$ when studying convergence of solutions of (3.1.1). However, the deterministic results obtained in Chapters 2 are based on the assumption that $R - R_\infty \in L^1((0,\infty), M_{n\times n}(\mathbb{R}))$. In order to make use of Theorem 2.2.5 in this chapter Lemma 6.3.2 is required; this result isolates conditions that ensure the integrability of $R - R_\infty$ once $R - R_\infty$ is square integrable.

Lemma 6.3.2. *Let K satisfy (2.1.2) and (2.2.7) and let R_∞ be a constant matrix such that the solution R of (2.1.1) satisfies (3.2.3). Then the solution R of (2.1.1) satisfies*

(2.2.8).

Two technical lemmata are now stated. Lemma 6.3.3 concerns the moments of a normally distributed random variable. It can be extracted from [5, Theorem 3.3] and it is used in Proposition 6.3.1. Its proof may be found at the end of this section.

Lemma 6.3.3. *Suppose the function $\sigma \in C((0,\infty)\times(0,\infty), M_{p\times r}(\mathbb{R}))$ then*

$$\mathbb{E}\left\|\int_a^b \sigma(s,t)\,dB(s)\right\|^{2m} \leq d_m(p,r)\left(\int_a^b \|\sigma(s,t)\|^2\,ds\right)^m$$

where $d_m(p,r) = p^{m+1}r^{2m+1}(2m)!(m!2^m)^{-1}c_2(p,r)^m$.

The following lemma is used in Proposition 6.3.2. A similar result is proved in [5]; the proof is included in Appendix A for completeness.

Lemma 6.3.4. *Suppose that $\tilde{K} \in C([0,\infty), M_{n\times n}(\mathbb{R})) \cap L^1((0,\infty), M_{n\times n}(\mathbb{R}))$ and*

$$\int_0^\infty \|\tilde{K}(s)\|e^{\tilde{\alpha}s}ds < \infty.$$

If $\tilde{\lambda} > 0$ and $\tilde{\eta} = 2\tilde{\lambda}\wedge\tilde{\alpha}$ then

$$\int_0^t e^{-2\tilde{\lambda}(t-s)}e^{-\tilde{\alpha}s}\|\tilde{K}(s)\|\,ds \leq ce^{-\tilde{\eta}t},$$

where c is a positive constant.

The proofs of Proposition 6.3.1 and 6.3.2, Lemma 6.3.1 and 6.3.3 are now given.

Proof of Lemma 6.3.1. From Theorem 3.2.1 we see that $X(t) \to X_\infty$ where X_∞ is given by (3.2.4). From Lemma 3.3.1, the definition of the norm and the expression for X_∞ we see that

$$\mathbb{E}\,\|X_\infty\|^2 = \mathbb{E}\left[\text{tr}\,(X_\infty X_\infty^T)\right] = \|R_\infty X_0\|^2 + \int_0^\infty \|R_\infty\Sigma(s)\|^2\,ds < \infty. \tag{6.3.2}$$

Since

$$\mathbb{E}\left[\|X(t)-X_\infty\|^2\right] = \mathbb{E}\left[\text{tr}\,(X(t)-X_\infty)(X(t)-X_\infty)^T\right], \tag{6.3.3}$$

we use (3.1.3) and (3.2.4) to expand the right hand side of (6.3.3) to obtain

$$\mathbb{E}\left[\|X(t)-X_\infty\|^2\right] = \|(R(t)-R_\infty)X_0\|^2 + \int_0^t \|(R(t-s)-R_\infty)\Sigma(s)\|^2\,ds + \int_t^\infty \|R_\infty\Sigma(s)\|^2\,ds. \quad (6.3.4)$$

In order to obtain an exponential upper bound on (6.3.4) each term must be considered individually. We begin by considering the first term on the right hand side of (6.3.4). Using (2.2.7) and (3.2.3) we can apply Lemma 6.3.2 to obtain (2.2.8). Then using (2.2.7), (2.2.8) and (2.2.13) we see from Theorem 2.2.5 that

$$\|(R(t)-R_\infty)X_0\|^2 \le \|R(t)-R_\infty\|^2\|X_0\|^2 \le c_1\|X_0\|^2 e^{-2\beta t}. \quad (6.3.5)$$

We provide an argument to show that the second term decays exponentially. Using (6.2.1) and the fact that R decays exponentially quickly to R_∞ we can choose $0<\lambda<\min(\beta,\gamma)$ such that $e_\lambda\Sigma$ and $e_\lambda(R-R_\infty)\in L^2(0,\infty)$ where the function e_λ is defined by $e_\lambda(t)=e^{\lambda t}$. The convolution of an $L^2(0,\infty)$ function with an $L^2(0,\infty)$ function is itself an $L^2(0,\infty)$ function, thus,

$$e^{2\lambda t}\int_0^t \|(R(t-s)-R_\infty)\Sigma(s)\|^2\,ds \le \int_0^t e^{2\lambda(t-s)}\|R(t-s)-R_\infty\|^2 e^{2\lambda s}\|\Sigma(s)\|^2\,ds \le c_2. \quad (6.3.6)$$

Using the argument given by (6.3.6) we see that the second term of (6.3.4) decays exponentially quickly.

We can show that the third term on the right hand side of (6.3.3) decays exponentially using the following argument:

$$\bar{\Sigma} := \int_0^\infty \|\Sigma(s)\|^2 e^{2\gamma s}ds \ge \int_t^\infty \|\Sigma(s)\|^2 e^{2\gamma s}ds \ge e^{2\gamma t}\int_t^\infty \|\Sigma(s)\|^2 ds. \quad (6.3.7)$$

Combining these facts we see that

$$\mathbb{E}\left[\|X(t)-X_\infty\|^2\right] \le m(X_0)e^{-2\lambda t},$$

where $m(X_0) = c_1\|X_0\|^2 + c_2 + \bar{\Sigma}\|R_\infty\|^2$ and $\lambda < \min(\beta, \gamma)$. $\square$

Proof of Proposition 6.3.1. Consider the case where $0 < p \leq 2$ and $p > 2$ separately. We begin with the case where $0 < p \leq 2$. Using the argument given by (6.3.2) shows that $\mathbb{E}\left[\|X_\infty\|^2\right] < \infty$. Now applying Liapunov's inequality we see that

$$\mathbb{E}\,\|X_\infty\|^p \leq \mathbb{E}\left[\|X_\infty\|^2\right]^{\frac{p}{2}} < \infty.$$

We now show that (6.2.4) holds for $0 \leq p \leq 2$. Liapunov's inequality and Lemma 6.3.1 can be applied as follows:

$$\mathbb{E}\left[\|X(t) - X_\infty\|^p\right] \leq \mathbb{E}\left[\|X(t) - X_\infty\|^2\right]^{\frac{p}{2}} \leq m_p(X_0)e^{-\beta_p t}, \quad t \geq 0,$$

where $m_p(X_0) = m(X_0)^{\frac{p}{2}}$and $\beta_p = \lambda p$.

Now consider the case where $p > 2$. In this case, there exists a constant $m \in \mathbb{N}$ such that $2(m-1) < p \leq 2m$. We now seek an upper bound on $\mathbb{E}\,\|X_\infty\|^{2m}$ and $\mathbb{E}\left[\|X(t) - X_\infty\|^{2m}\right]$, which will in turn give an upper bound on $\mathbb{E}\,\|X_\infty\|^p$ and $\mathbb{E}\left[\|X(t) - X_\infty\|^p\right]$ by using Liapunov's inequality. Clearly, we see that

$$\begin{aligned}
\mathbb{E}\,\|X_\infty\|^{2m} &= \mathbb{E}\left\|R_\infty X_0 + \int_0^\infty R_\infty \Sigma(s)\,dB(s)\right\|^{2m} \\
&\leq 2^{2m-1}\mathbb{E}\left[\|R_\infty X_0\|^{2m} + \left\|\int_0^\infty R_\infty \Sigma(s)\,dB(s)\right\|^{2m}\right] \\
&\leq c\left(\|R_\infty X_0\|^{2m} + \int_0^\infty \|R_\infty \Sigma(s)\|^{2m}\,ds\right) < \infty,
\end{aligned}$$

by applying Lemmas 3.3.1 and 6.3.3.

Now consider $\mathbb{E}\left[\|X(t) - X_\infty\|^{2m}\right]$. Using the variation of parameters representation of the solution and the expression obtained for X_∞ we see that

$$X(t) - X_\infty = (R(t) - R_\infty)X_0 + \int_0^t (R(t-s) - R_\infty)\Sigma(s)\,dB(s) - \int_t^\infty R_\infty \Sigma(s)\,dB(s).$$

Now taking norms,

$$\begin{aligned}&\|X(t) - X_\infty\| \\ &\quad \leq \|(R(t) - R_\infty)X_0\| + \left\|\int_0^t (R(t-s) - R_\infty)\Sigma(s)\,dB(s)\right\| + \left\|\int_t^\infty R_\infty\Sigma(s)\,dB(s)\right\|,\end{aligned}$$

raising both sides of the equation to the $2m^{th}$ power,

$$\begin{aligned}&\|X(t) - X_\infty\|^{2m} \\ &\quad \leq 3^{2m-1}\Bigg(\|(R(t) - R_\infty)X_0\|^{2m} + \left\|\int_0^t (R(t-s) - R_\infty)\Sigma(s)\,dB(s)\right\|^{2m} \\ &\qquad\qquad + \left\|\int_t^\infty R_\infty\Sigma(s)\,dB(s)\right\|^{2m}\Bigg),\end{aligned}$$

then taking expectations across the inequality, we arrive at

$$\begin{aligned}&\mathbb{E}\left[\|X(t) - X_\infty\|^{2m}\right] \\ &\quad \leq 3^{2m-1}\Bigg(\|(R(t) - R_\infty)X_0\|^{2m} + \mathbb{E}\left[\left\|\int_0^t (R(t-s) - R_\infty)\Sigma(s)\,dB(s)\right\|^{2m}\right] \\ &\qquad\qquad + \mathbb{E}\left[\left\|\int_t^\infty R_\infty\Sigma(s)\,dB(s)\right\|^{2m}\right]\Bigg). \qquad (6.3.8)\end{aligned}$$

We consider each term on the right hand side of (6.3.8). Using (2.2.7) and (3.2.3) we can apply Lemma 6.3.2 to obtain (2.2.8). Then using (2.2.7), (2.2.8) and (2.2.13) we see from Theorem 2.2.5 that

$$\|(R(t) - R_\infty)X_0\|^{2m} \leq c_1\|X_0\|^{2m}e^{-2m\beta t}. \qquad (6.3.9)$$

Now, consider the second term on the right hand side of (6.3.8). Using the argument given by (6.3.6) we see that $\int_0^t \|(R(t-s) - R_\infty)\Sigma(s)\|^2\,ds \leq c_2 e^{-2\lambda t}$ where $\lambda < \min(\beta, \gamma)$. Now using Lemma 6.3.3 we see that

$$\begin{aligned}
&\mathbb{E}\left[\left\| \int_0^t (R(t-s) - R_\infty)\Sigma(s)\, dB(s)\right\|^{2m}\right] \\
&\qquad \leq d_m(n,d) \left(\int_0^t \|(R(t-s) - R_\infty)\|^2 \|\Sigma(s)\|^2 \, ds \right)^m \\
&\qquad \leq d_m(n,d) \left(c_2 e^{-2\lambda t} \right)^m \\
&\qquad \leq d_m(n,d) c_2^m e^{-2m\lambda t}.
\end{aligned} \tag{6.3.10}$$

Using (6.3.7) combined with Lemma 6.3.3 and the Dominated Convergence Theorem, we show that the third term decays exponentially quickly:

$$\begin{aligned}
\mathbb{E}\left[\left\| \int_t^\infty \Sigma(s)\, dB(s)\right\|^{2m}\right] &= \mathbb{E}\left[\lim_{T\to\infty} \left\| \int_t^T \Sigma(s)\, dB(s)\right\|^{2m}\right] \\
&= \lim_{T\to\infty} \mathbb{E}\left[\left\| \int_t^T \Sigma(s)\, dB(s)\right\|^{2m}\right] \\
&\leq \lim_{T\to\infty} d_m(n,d) \left(\int_t^T \|\Sigma(s)\|^2 \, ds \right)^m \\
&= d_m(n,d) \left(\int_t^\infty \|\Sigma(s)\|^2 \, ds \right)^m \\
&\leq d_m(n,d) \bar{\Sigma}^m e^{-2m\gamma t}.
\end{aligned} \tag{6.3.11}$$

Combining (6.3.9), (6.3.10) and (6.3.11) the inequality (6.3.8) becomes

$$\begin{aligned}
&\mathbb{E}\left[\|X(t) - X_\infty\|^{2m}\right] \leq \\
&3^{2m-1} \left(c_1 \|X_0\|^{2m} e^{-2m\beta t} + d_m(n,d) c_2^m e^{-2m\lambda t} + d_m(n,d) \|R_\infty\|^{2m} \bar{\Sigma}^m e^{-2m\gamma t} \right).
\end{aligned} \tag{6.3.12}$$

Using Liapunov's inequality, the inequality (6.3.12) implies

$$\mathbb{E}\left[\|X(t) - X_\infty\|^p\right] \leq m_p(X_0) e^{-\beta_p t},$$

where $m_p(X_0) = 3^{p\frac{2m-1}{2m}} (c_1 \|X_0\|^{2m} + d_m(n,d) c_2^m + d_m(n,d) \|R_\infty\|^{2m} \bar{\Sigma}^m)^{\frac{p}{2m}}$ and $\beta_p = \lambda p$.

$\square$

Proof of Proposition 6.3.2. In order to prove this proposition we must show that

$$\mathbb{E}\left[\sup_{n-1\le t\le n}\|X(t)-X_\infty\|^2\right]\le \tilde{m}(X_0)e^{-2\eta(n-1)},\quad \eta>0. \tag{6.3.13}$$

For each $t>0$ there exists $n\in\mathbb{N}$ such that $n-1\le t<n$. Define $\Delta(t)=X(t)-X_\infty$. Integrating (3.1.1) over $[n-1,t]$, then adding and subtracting X_∞ on both sides we obtain

$$\begin{aligned}X(t)-X_\infty&=(X(n-1)-X_\infty)+\int_{n-1}^t A(X(s)-X_\infty)\,ds\\&\quad+\int_{n-1}^t\int_0^s K(s-u)(X(u)-X_\infty)\,du\,ds+\int_{n-1}^t\Sigma(s)\,dB(s)\\&\qquad+\int_{n-1}^t\left(A+\int_0^\infty K(u)\,du\right)X_\infty\,ds-\int_{n-1}^t\int_s^\infty K(v)X_\infty\,dv\,ds,\end{aligned}$$

applying Lemma 4.4.1 this becomes

$$\begin{aligned}\Delta(t)=\Delta(n-1)+\int_{n-1}^t(A\Delta(s)+(K*\Delta)(s))\,ds+\int_{n-1}^t\Sigma(s)\,dB(s)\\-\int_{n-1}^t K_1(s)\,ds\,X_\infty.\end{aligned} \tag{6.3.14}$$

Taking norms on both sides of (6.3.14),

$$\begin{aligned}\|\Delta(t)\|\le\|\Delta(n-1)\|+\left\|\int_{n-1}^t\left(A\Delta(s)+\int_0^s K(s-u)\Delta(u)\,du\right)\,ds\right\|\\+\left\|\int_{n-1}^t\Sigma(s)\,dB(s)\right\|+\left\|\int_{n-1}^t\int_s^\infty K(v)X_\infty\,dv\,ds\right\|,\end{aligned}$$

and squaring both sides this becomes,

$$\begin{aligned}\|\Delta(t)\|^2\le4\Bigg[\|\Delta(n-1)\|^2+\left\|\int_{n-1}^t\left(A\Delta(s)+\int_0^s K(s-u)\Delta(u)\,du\right)\,ds\right\|^2\\+\left\|\int_{n-1}^t\Sigma(s)\,dB(s)\right\|^2+\left\|\int_{n-1}^t\int_s^\infty K(v)X_\infty\,dv\,ds\right\|^2\Bigg],\end{aligned}$$

This then becomes

$$\begin{aligned}\|\Delta(t)\|^2\le4\Bigg[\|\Delta(n-1)\|^2+\left(\int_{n-1}^t\left[\|A\|\,\|\Delta(s)\|+\int_0^s\|K(s-u)\|\,\|\Delta(u)\|\,du\right]\,ds\right)^2\\+\left\|\int_{n-1}^t\Sigma(s)\,dB(s)\right\|^2+\left(\int_{n-1}^t\|K_1(s)\|\,\|X_\infty\|\,ds\right)^2\Bigg],\end{aligned}$$

taking suprema,

$$\sup_{n-1\le t\le n} \|\Delta(t)\|^2 \le 4\left[\|\Delta(n-1)\|^2 + \left(\int_{n-1}^{n}\left[\|A\|\,\|\Delta(s)\|\,ds + \int_0^s \|K(s-u)\|\,\|\Delta(u)\|\,du\right]ds\right)^2 + \sup_{n-1\le t\le n}\left\|\int_{n-1}^{t}\Sigma(s)\,dB(s)\right\|^2 + \left(\int_{n-1}^{n}\|K_1(s)\|\,\|X_\infty\|\,ds\right)^2\right].$$

before finally taking expectations yields:

$$\mathbb{E}\left[\sup_{n-1\le t\le n}\|\Delta(t)\|^2\right] \le 4\Bigg\{\mathbb{E}\,\|\Delta(n-1)\|^2 + \mathbb{E}\left[\left(\int_{n-1}^{n}\|A\|\,\|\Delta(s)\| + (\|K\|*\|\Delta\|)(s)\,ds\right)^2\right] + \mathbb{E}\left[\sup_{n-1\le t\le n}\left\|\int_{n-1}^{t}\Sigma(s)\,dB(s)\right\|^2\right] + \left(\int_{n-1}^{n}\|K_1(s)\|\,ds\right)^2\mathbb{E}\,\|X_\infty\|^2\Bigg\}. \quad (6.3.15)$$

We now consider each term on the right hand side of (6.3.15). From Lemma 6.3.1 we see that the first term satisfies

$$\mathbb{E}\left[\|\Delta(n-1)\|^2\right] \le m(X_0)e^{-2\lambda(n-1)}. \quad (6.3.16)$$

In order to obtain an exponential bound on the second term on the right hand side of (6.3.8) we make use of the Cauchy–Schwarz inequality as follows:

$$\begin{aligned}
&\left(\int_{n-1}^{n}\|A\|\,\|\Delta(s)\| + (\|K\|*\|\Delta\|)(s)\,ds\right)^2 \\
&\le \int_{n-1}^{n}\left(\|A\|\,\|\Delta(s)\| + (\|K\|*\|\Delta\|)(s)\right)^2 ds \\
&\le 2\int_{n-1}^{n}\left(\|A\|^2\|\Delta(s)\|^2 + (\|K\|*\|\Delta\|)^2(s)\right) ds \\
&\le 2\int_{n-1}^{n}\Bigg[\|A\|^2\|\Delta(s)\|^2 + \left(\int_0^s e^{\alpha(s-u)/2}\|K(s-u)\|^{1/2}e^{-\alpha(s-u)/2}\|K(s-u)\|^{1/2}\|\Delta(s)\|\,du\right)^2\Bigg] ds \\
&\le 2\int_{n-1}^{n}\left[\|A\|^2\|\Delta(s)\|^2 + \bar{K}_\alpha\int_0^s e^{-\alpha(s-u)}\|K(s-u)\|\,\|\Delta(s)\|^2\,du\right] ds
\end{aligned}$$

where $\bar{K}_\alpha = \int_0^\infty e^{\alpha t}\|K(t)\|\,dt$. Take expectations and examine the two terms within the integral. Using Lemma 6.3.1 we obtain

$$\begin{aligned}\mathbb{E}\left[\int_{n-1}^{n} \|A\|^2\|\Delta(s)\|^2\,ds\right] &\leq \int_{n-1}^{n} \|A\|^2\mathbb{E}\left[\|\Delta(s)\|^2\right]\,ds\\ &\leq \|A\|^2 m(X_0)\int_{n-1}^{n} e^{-2\lambda s}\,ds \\ &\leq c_1(X_0)e^{-2\lambda(n-1)}.\end{aligned} \tag{6.3.17}$$

In order to obtain an exponential upper bound for the second term within the integral we apply Lemma 6.3.4 with $K=\tilde{K}$, $\alpha=\tilde{\alpha}$, $\lambda=\tilde{\lambda}$ and $\eta=\tilde{\eta}$.

$$\begin{aligned}&\mathbb{E}\left[\int_{n-1}^{n} \bar{K}_\alpha\int_0^s e^{-\alpha(s-u)}\|K(s-u)\|\,\|\Delta(s)\|^2\,du\,ds\right]\\ &\qquad\leq \bar{K}_\alpha\int_{n-1}^{n}\int_0^s e^{-\alpha(s-u)}\|K(s-u)\|\,\mathbb{E}\left[\|\Delta(s)\|^2\right]\,du\,ds\\ &\qquad\leq m(X_0)\bar{K}_\alpha\int_{n-1}^{n}\int_0^s e^{-\alpha u}\|K(u)\|e^{-2\lambda(s-u)}\,du\,ds\\ &\qquad\leq m(X_0)c\bar{K}_\alpha\int_{n-1}^{n} e^{-\eta s}\,ds\\ &\qquad\leq c_2(X_0)e^{-\eta(n-1)}.\end{aligned} \tag{6.3.18}$$

Next, we obtain an exponential upper bound on the third term. By the Burkholder–Davis–Gundy inequality, there exists a constant $c_3>0$ such that

$$\mathbb{E}\left[\sup_{n-1\leq t\leq n}\left\|\int_{n-1}^{t}\Sigma(s)\,dB(s)\right\|^2\right] \leq c_3\int_{n-1}^{n}\|\Sigma(s)\|^2\,ds \tag{6.3.19}$$

Applying (6.3.7) we see that

$$\mathbb{E}\left[\sup_{n-1\leq t\leq n}\left\|\int_{n-1}^{t}\Sigma(s)\,dB(s)\right\|^2\right] \leq c_3\bar{\Sigma}e^{-2\gamma(n-1)}. \tag{6.3.20}$$

Now consider the last term on the right hand side of (6.3.15). Using (2.2.13) we see that (2.5.5) holds. Using this and the fact that $\mathbb{E}\|X_\infty\|^2<\infty$ (see (6.3.2)) we obtain

$$\left(\int_{n-1}^{n}\|K_1(s)\|\,ds\right)^2\mathbb{E}\|X_\infty\|^2 \leq \mathbb{E}\|X_\infty\|^2\left(\int_{n-1}^{n}\bar{K}e^{-\alpha s}\,ds\right)^2 \leq c_4e^{-2\alpha(n-1)}. \tag{6.3.21}$$

Combining (6.3.16), (6.3.17), (6.3.18), (6.3.20) and (6.3.21) we obtain

$$\mathbb{E}\left[\sup_{n-1\leq t\leq n}\|X(t)-X_\infty\|^2\right] \leq \bar{m}(X_0)e^{-2\eta(n-1)},$$

where $\tilde{m}(X_0) = 4\left(m(X_0) + c_1(X_0) + c_2(X_0) + c_3\bar{\Sigma} + c_4\right)$.

We can now apply the line of reasoning used in [26, Theorem 4.4.2] to obtain (6.2.5). Let $\epsilon \in (0, \eta)$ be arbitrary. Since (6.3.13) holds we can apply Chebyshev's inequality:

$$\begin{aligned}\mathbb{P}\left[\omega : \sup_{n-1\leq t\leq n} \|X(t,\omega) - X_\infty(\omega)\|^2 > e^{-2(\eta-\epsilon)(n-1)}\right] \\ &\leq e^{2(\eta-\epsilon)(n-1)}\mathbb{E}\left[\sup_{n-1\leq t\leq n} \|X(t,\omega) - X_\infty(\omega)\|^2\right] \\ &\leq \tilde{m}(X_0)e^{-2\epsilon(n-1)}.\end{aligned}$$

Since $\sum_{n=1}^{\infty} e^{-\epsilon(n-1)} < \infty$ we can apply the first Borel–Cantelli lemma to see that for $\omega \in \Omega_0$, with $\mathbb{P}[\Omega_0] = 1$,

$$\sup_{n-1\leq t\leq n} \|X(t,\omega) - X_\infty(\omega)\|^2 \leq e^{-2(\eta-\epsilon)(n-1)}, \quad n \geq n_0(\omega).$$

Consequently, for $n - 1 \leq t \leq n$,

$$\frac{1}{t}\log\|X(t,\omega) - X_\infty(\omega)\| \leq \frac{1}{2t}\log\|X(t,\omega) - X_\infty(\omega)\|^2 \leq -(\eta-\epsilon)\frac{(n-1)}{t}, \quad n \geq n_0(\omega).$$

But since ϵ is arbitrary this means that

$$\limsup_{t\to\infty} \frac{1}{t}\log\|X(t) - X_\infty\| \leq -\eta, \quad \text{a.s.}$$

□

Proof of Lemma 6.3.2. Consider the reformulation of (2.1.1) given by (2.3.8). From Theorem 1.1.2 we know that Y can be expressed as

$$Y(t) = G(t) - \int_0^t r(t-s)G(s)\,ds, \tag{6.3.22}$$

where the function r satisfies $r + F * r = F$ and $r + r * F = F$. Consider the first term on the right hand side of (6.3.22). As (2.2.7) holds it is clear that the function G is integrable. Now consider the second term. Since (2.2.8) and (3.2.1) hold we may apply Lemma 3.3.3

to obtain (3.3.4). Now we may apply Theorem 2.2.3 to see that r is integrable. The convolution of an integrable function with an integrable function is itself integrable. Now combining the arguments for the first and second terms we see that (2.2.8) must hold. $\square$

Proof of Lemma 6.3.3. We see that

$$\left\|\int_a^b \sigma(s,t)\,dB(s)\right\|^{2m} = \left(\sum_{i=1}^{p}\left(\sum_{j=1}^{r}\int_a^b \sigma_{ij}(s,t)\,dB_j(s)\right)^2\right)^m.$$

Applying (1.1.1) twice we obtain

$$\begin{aligned}\left\|\int_a^b \sigma(s,t)\,dB(s)\right\|^{2m} &\le p^{m-1}\sum_{i=1}^{p}\left(\sum_{j=1}^{r}\int_a^b \sigma_{ij}(s,t)\,dB_j(s)\right)^{2m}\\ &\le p^{m-1}r^{2m-1}\sum_{i=1}^{p}\sum_{j=1}^{r}\left(\int_a^b \sigma_{ij}(s,t)\,dB_j(s)\right)^{2m}.\end{aligned}$$

As

$$\int_a^b \sigma_{ij}(s,t)\,dB_j(s) \sim \mathcal{N}\left(0, \int_a^b \sigma_{ij}(s,t)^2\,ds\right)$$

we can use the fact that for a normally distributed random variable X ($X \sim \mathcal{N}(0,\sigma^2)$) obeys $\mathbb{E}\left[X^{2m}\right] = \frac{(2m)!\sigma^{2m}}{m!2^m}$ to obtain

$$\mathbb{E}\left[\left\|\int_a^b \sigma(s,t)\,dB(s)\right\|^{2m}\right] \le p^m r^{2m}(2m)!(m!2^m)^{-1}\sum_{i=1}^{p}\sum_{j=1}^{r}\left(\int_a^b \sigma_{ij}(s,t)^2\,ds\right)^m.$$

But $\sigma_{ij}(s,t)^2 \le \|\sigma(s,t)\|_F^2 \le c_2(p,r)\|\sigma(s,t)\|^2$, hence,

$$\mathbb{E}\left[\left\|\int_a^b \sigma(s,t)\,dB(s)\right\|^{2m}\right] \le p^{m+1}r^{2m+1}(2m)!(m!2^m)^{-1}c_2(p,r)^m\left(\int_a^b \|\sigma(s,t)\|^2\,ds\right)^m.$$

$\square$

6.4 Necessary Conditions for Exponential Convergence of Solutions of (3.1.1)

In this section, the necessity of condition (6.2.1) for exponential convergence in the almost sure and p^{th} mean senses is shown. Proposition 6.4.1 concerns the necessity of the condition in the almost sure case while Proposition 6.4.2 deals with the p^{th} mean case.

Proposition 6.4.1. *Let K satisfy (2.1.2) and (3.2.1) and let Σ satisfy (3.1.2). If there exists a constant $\alpha > 0$ such that (2.2.13) holds, and if for all X_0 there is a constant vector $X_\infty(X_0, \Sigma)$ such that the solution $t \mapsto X(t; X_0, \Sigma)$ of (3.1.1) satisfies statement (iii) of Theorem 6.2.2, then there exists a constant $\gamma > 0$, independent of X_0, such that (6.2.1) of Theorem 6.2.2(i) holds.*

Proposition 6.4.2. *Let K satisfy (2.1.2) and (3.2.1) and let Σ satisfy (3.1.2). If there exists a constant $\alpha > 0$ such that (2.2.13) of Theorem 6.2.2(i) holds, and if for all X_0 there is a constant vector $X_\infty(X_0, \Sigma)$ such that the solution $t \mapsto X(t; X_0, \Sigma)$ of (3.1.1) satisfies statement (ii) of Theorem 6.2.2, then there exists a constant $\gamma > 0$, independent of X_0, such that (6.2.1) of Theorem 6.2.2(i) holds.*

In order to prove these propositions the integral version of (3.1.1) is considered. By reformulating this version of the equation an expression for a term related to the exponential integrability of the perturbation is found. Using various arguments, including the Martingale Convergence Theorem in the almost sure case, this term is used to show that (6.2.1) holds.

A supporting result is now stated. This was proved in [5], but for completeness the proof is given in Appendix A.

Lemma 6.4.1. *Let $N = (N_1, \ldots, N_n)$ where $N_i \sim \mathcal{N}(0, v_i^2)$ for $i = 1, \ldots, n$, then there exists a v_i-independent constant $d_1 > 0$ such that*

$$\mathbb{E}\left[\|N\|^2\right] \leq d_1 \mathbb{E}\left[\|N\|\right]^2.$$

Proof of Proposition 6.4.1. In order to prove this result we follow the argument used in [5, Theorem 4.1]. Let $0 < \gamma < \alpha \wedge \beta_0$. By defining the process $Z(t) = e^{\gamma t} X(t)$ and the matrix $\kappa(t) = e^{\gamma t} K(t)$ we can rewrite (3.1.1) as

$$d(e^{-\gamma t} Z(t)) = \left(A(e^{-\gamma t} Z(t)) + \int_0^t e^{-\gamma(t-s)} \kappa(t-s) e^{-\gamma s} Z(s)\, ds \right) dt + \Sigma(t) dB(t).$$

Using integration by parts this becomes:

$$-\gamma e^{-\gamma t}Z(t) + e^{-\gamma t}dZ(t) = \left(A(e^{-\gamma t}Z(t)) + \int_0^t e^{-\gamma(t-s)}\kappa(t-s)e^{-\gamma s}Z(s)\,ds\right)dt + \Sigma(t)dB(t).$$

Rearranging the equation one obtains:

$$e^{-\gamma t}dZ(t) = \left((\gamma I + A)e^{-\gamma t}Z(t) + \int_0^t e^{-\gamma(t-s)}\kappa(t-s)e^{-\gamma s}Z(s)\,ds\right)dt + \Sigma(t)dB(t).$$

Multiplying both sides by $e^{\gamma t}$:

$$dZ(t) = \left((\gamma I + A)Z(t) + \int_0^t \kappa(t-s)Z(s)\,ds\right)dt + e^{\gamma t}\Sigma(t)dB(t),$$

the integral form of which is

$$Z(t) - Z(0) = (\gamma I + A)\int_0^t Z(s)\,ds + \int_0^t\int_0^s \kappa(s-u)Z(u)\,du\,ds + \int_0^t e^{\gamma s}\Sigma(s)dB(s).$$

Using $Z(t) = e^{\gamma t}X(t)$ and rearranging the equation this becomes

$$\int_0^t e^{\gamma s}\Sigma(s)\,dB(s) = e^{\gamma t}X(t) - X_0 - (\gamma I + A)\int_0^t e^{\gamma s}X(s)\,ds - \int_0^t e^{\gamma s}\int_0^s K(s-u)X(u)\,du\,ds.$$

Adding and subtracting X_∞ from the right hand side

$$\begin{aligned}\int_0^t e^{\gamma s}\Sigma(s)\,dB(s) &= e^{\gamma t}(X(t)-X_\infty) - X_0 - (\gamma I + A)\int_0^t e^{\gamma s}(X(s)-X_\infty)\,ds \\ &\quad - \int_0^t e^{\gamma s}\int_0^s K(s-u)(X(u)-X_\infty)\,du\,ds + e^{\gamma t}X_\infty \\ &\quad - (\gamma I + A)\int_0^t e^{\gamma s}X_\infty\,ds - \int_0^t e^{\gamma s}\int_0^s K(s-u)X_\infty\,du\,ds \\ &= e^{\gamma t}(X(t)-X_\infty) - (X_0 - X_\infty) - (\gamma I + A)\int_0^t e^{\gamma s}(X(s)-X_\infty)\,ds \\ &\quad - \int_0^t e^{\gamma s}\int_0^s K(s-u)(X(u)-X_\infty)\,du\,ds \\ &\quad - \int_0^t e^{\gamma s}\left(A + \int_0^\infty K(u)\,du\right)X_\infty\,ds \\ &\quad + \int_0^t e^{\gamma s}\int_s^\infty K(u)X_\infty\,du\,ds.\end{aligned}$$

Applying Lemma 4.4.1 we obtain:

$$\int_0^t e^{\gamma s}\Sigma(s)\,dB(s) = e^{\gamma t}(X(t)-X_\infty) - (X_0 - X_\infty) - (\gamma I + A)\int_0^t e^{\gamma s}(X(s)-X_\infty)\,ds$$
$$-\int_0^t e^{\gamma s}\int_0^s K(s-u)(X(u)-X_\infty)\,du\,ds + \int_0^t e^{\gamma s}K_1(u)\,du\,ds\,X_\infty. \quad (6.4.1)$$

Consider each term on the right hand side of (6.4.1). We see that the first term tends to zero as (6.2.5) holds and $\gamma < \beta_0$. The second term is finite by hypothesis. Again, using the fact that $\gamma < \beta_0$ and that assumption (6.2.5) holds we see that $e_\gamma(X - X_\infty) \in L^2(0,\infty)$, so the third term tends to a limit as $t \to \infty$. Now consider the fourth term. Since $0 < \gamma < \alpha \wedge \beta_0$, we can choose $\gamma_1 > 0$ such that $\gamma < \gamma_1 < \alpha \wedge \beta_0$. Now, we can apply an argument similar to (2.5.4) to show that

$$\left\| \int_0^s K(s-u)(X(u)-X_\infty)\,du \right\| < ce^{-\gamma_1 s}.$$

Thus, it is clear that the fourth term has a finite limit as $t \to \infty$. Finally, the fifth term on the right hand side of (6.4.1) has a finite limit at infinity, using the argument given by (2.5.5). Each term on the right hand side of the inequality has a finite limit as $t \to \infty$, so therefore

$$\lim_{t\to\infty} \int_0^t e^{\gamma s}\Sigma(s)\,dB(s) \quad \text{exists and is almost surely finite.}$$

The Martingale Convergence Theorem may now be applied component by component to obtain (6.2.1). $\square$

Proof of Proposition 6.4.2. By Lemma 3.4.1, (6.4.1) still holds. Define $\gamma < \alpha \wedge \beta_1$, take norms and expectations across (6.4.1) to obtain

$$\begin{aligned}\mathbb{E}\left[\left\|\int_0^t e^{\gamma s}\Sigma(s)\,dB(s)\right\|\right] &\leq \mathbb{E}\left[e^{\gamma t}\|X(t)-X_\infty\|\right] + \mathbb{E}\left[\|X_0 - X_\infty\|\right] \\ &\quad + \|\gamma I + A\| \int_0^t \mathbb{E}\left[e^{\gamma s}\|X(s)-X_\infty\|\right] ds \\ &\quad + \int_0^t e^{\gamma s}\int_0^s \|K(u)\|\,\mathbb{E}\left[\|X(s-u)-X_\infty\|\right] du\, ds \\ &\quad + \int_0^t e^{\gamma s}\|K_1(s)\|\,ds\,\mathbb{E}\,\|X_\infty\|. \end{aligned} \quad (6.4.2)$$

By Theorem 6.3.1 there exists m_1 such that

$$\mathbb{E}\left[e^{\gamma t}\|X(t)-X_\infty\|\right] \leq m_1 e^{-(\beta_1-\gamma)t}, \quad (6.4.3)$$

thus the first, second and third terms on the right hand side of (6.4.2) are uniformly bounded on $[0,\infty)$. Now consider the fourth term. Since $0<\gamma<\alpha\wedge\beta_1$, we can choose $\gamma_1>0$ such that $\gamma<\gamma_1<\alpha\wedge\beta_1$. Now we can apply an argument similar to (2.5.4) to show that

$$\int_0^s \|K(s-u)\|\mathbb{E}\,\|X(u)-X_\infty\|\,du < ce^{-\gamma_1 s},$$

so it is clear that the fourth term is uniformly bounded on $[0,\infty)$. Finally, we consider the final term on the right hand side of (6.4.2). Using (2.5.5) we obtain

$$\int_0^t e^{\gamma s}\|K_1(s)\|\,ds\,\mathbb{E}\,\|X_\infty\| \leq \bar{K}\mathbb{E}\,\|X_\infty\| \int_0^t e^{-(\alpha-\gamma)s}\,ds < \infty,$$

since $\gamma<\alpha$. Thus there is a constant $c>0$ such that

$$\mathbb{E}\left[\left\|\int_0^t e^{\gamma s}\Sigma(s)\,dB(s)\right\|\right] \leq c. \quad (6.4.4)$$

The proof now follows the line of reasoning found in [5, Theorem 4.3]: observe that

$$\left\|\int_0^t e^{\gamma s}\Sigma(s)\,dB(s)\right\|^2 = \sum_{i=1}^n N_i(t)^2,$$

where

$$N_i(t) = \sum_{j=1}^{d} \int_0^t e^{\gamma s} \Sigma_{ij}(s)\, dB_j(s).$$

It is clear that $N_i(t)$ is normally distributed with zero mean and variance given by

$$v_i(t)^2 = \sum_{j=1}^{d} \int_0^t e^{2\gamma s} \Sigma_{ij}(s)^2\, ds.$$

Lemma 6.4.1 and (6.4.4) may now by applied to obtain:

$$\begin{aligned}\int_0^t e^{2\gamma s} \|\Sigma(s)\|_F^2\, ds &= \sum_{i=1}^{n}\sum_{j=1}^{d} \int_0^t e^{2\gamma s} \left|\Sigma_{ij}(s)\right|^2\, ds = \sum_{i=1}^{n} v_i(t)^2 \\ &= \mathbb{E}\left[\left\|\int_0^t e^{\gamma s}\Sigma(s)\, dB(s)\right\|^2\right] \leq d_1 \mathbb{E}\left[\left\|\int_0^t e^{\gamma s}\Sigma(s)\, dB(s)\right\|\right]^2 \leq d_1 c^2.\end{aligned}$$

Allowing $t \to \infty$ on both sides of this inequality yields the desired result. $\square$

6.5 On the Necessary and Sufficient Conditions for Exponential Convergence of Solutions of (3.1.1)

We now combine the results from Sections 6.3 and 6.4 to prove Theorem 6.2.2.

Proof of Theorem 6.2.2. We begin by proving the equivalence between (i) and (ii). The implication (i) implies (ii) is the subject of Theorem 6.3.1. We can demonstrate that (ii) implies (i) as follows: we begin by proving that (6.2.4) implies (2.2.13). We consider the following n solutions of (3.1.1); $X_j(t)_{j=1,\cdots,n}$ where $X_j(0) = \mathbf{e}_j$. Since (6.2.4) holds we obtain

$$\begin{aligned} m_1(\mathbf{e}_j)e^{-\beta_1 t} &\geq \mathbb{E}\left\|X_j(t) - X_j(\infty)\right\| \\ &\geq \left\|\mathbb{E}\left[X_j(t) - X_j(\infty)\right]\right\| = \left\|(R(t) - R(\infty))\mathbf{e}_j\right\|. \end{aligned} \tag{6.5.1}$$

for each $j = 1 \ldots, n$. Thus, the resolvent R of (2.1.1) decays exponentially to R_∞. We can apply Theorem 2.2.5 to obtain (2.2.13) after which Proposition 6.4.2 can be applied to obtain (6.2.1). As (6.5.1) holds it is clear that (3.2.3) holds and as (6.2.1) and as R_∞ is finite it is clear that (3.2.6) holds.

We now prove the equivalence between (i) and (iii). The implication (i) implies (iii) is the subject of Theorem 6.3.2. We now demonstrate that (iii) implies (i). We begin by proving that (6.2.5) implies (2.2.13). As (6.2.5) holds for all X_0 we can consider the following $n+1$ solutions of (3.1.1); $X_j(t)_{j=1,\cdots,n+1}$ where

$$X_j(0) = \mathbf{e}_j \quad \text{for } j = 1, \cdots, n \quad \text{and} \quad X_{n+1}(0) = 0.$$

We know that $X_j(t)$ approaches $X_j(\infty)$ exponentially quickly in the almost sure sense. Introduce

$$S_j(t) = X_j(t) - X_{n+1}(t),$$

and notice $S_j(0) = \mathbf{e}_j$. Let $S = [S_1, \ldots, S_n] \in M_{n\times n}(0,\infty)$. Then

$$S'(t) = AS(t) + (K * S)(t), \quad t > 0,$$

$$S(0) = I.$$

If we define $S_j(\infty) = X_j(\infty) - X_{n+1}(\infty)$ then $S(t) \to S_\infty$ exponentially quickly so we can apply Theorem 2.2.5 to obtain (2.2.13). As (2.2.13) and (6.2.5) hold we can apply Proposition 6.4.1 to obtain (6.2.1). Again, as (6.5.1) holds it is clear that (3.2.3) holds and as (6.2.1) holds and R_∞ is finite it is clear that (3.2.6) holds. This proves that (iii) implies (i). $\square$

Chapter 7

Exponential Convergence of Solutions of (7.1.1) to a Nontrivial Random Variable

7.1 Introduction

In this chapter we study the exponential convergence of solutions of

$$dX(t) = \left(AX(t) + \int_0^t K(t-s)X(s)\,ds + f(t)\right) dt + \Sigma(t)dB(t), \quad t > 0, \tag{7.1.1a}$$

$$X(0) = X_0, \tag{7.1.1b}$$

to a nontrivial random variable. Note that A, B are defined as before and K, f and Σ satisfy (2.1.2), (2.1.4) and (3.1.2) respectively.

Results analogous to those obtained in Chapter 6 are proven: we find the necessary and sufficient conditions for the exponential convergence of solutions of (7.1.1). Here, the analysis is complicated, particularly in the almost sure case, due to presence of both a deterministic and stochastic perturbation.

These theoretical results are then used to interpret the equation as an epidemiological model. Conditions under which a disease becomes endemic, which is the interpretation when solutions settle down to a non–trivial are studied. The theoretical results are exploited to highlight the speed at which this can occur within a population.

7.2 Discussion of Results

In this section, the necessary and sufficient conditions for the solution of (7.1.1) to converge exponentially to a nontrivial limit are stated.

Theorem 7.2.1, stated below, is an analogue of Theorem 6.2.2. In Theorem 7.2.1 the behaviour of the solution to equation (7.1.1) is considered rather than the behaviour of the solution to equation (3.1.1). The proof of this theorem is complicated by the presence not one but two perturbations.

Theorem 7.2.1. *Let K satisfy (2.1.2) and (2.2.7), let Σ satisfy (3.1.2) and let f satisfy (2.1.4). If K satisfies (2.2.12) the following are equivalent.*

(i) There exists a constant R_∞ such that the solution R of (2.1.1) satisfies (3.2.3) and there exist constants $\alpha > 0$, $\gamma > 0$, $\rho > 0$ and $c_1 > 0$ such that K and Σ and the tail of the perturbation f_1 defined by (2.2.15) satisfy (2.2.13), (6.2.1) and

$$\|f_1(t)\| \leq c_1 e^{-\rho t} \tag{7.2.1}$$

respectively.

(ii) For all initial conditions X_0 and constants $p > 0$ there exists an a.s. finite $\mathcal{F}^B(\infty)$-measurable random variable $X_\infty(X_0, \Sigma, f)$ with $\mathbb{E}\,\|X_\infty\|^p < \infty$ such that the unique continuous adapted process $X(\cdot\,; X_0, \Sigma, f)$ which obeys (7.1.1) satisfies

$$\mathbb{E}\left[\|X(t) - X_\infty\|^p\right] \leq m_p^* e^{-\beta_p^* t}, \quad t \geq 0, \tag{7.2.2}$$

where β_p^* and $m_p^* = m_p^*(X_0)$ are positive constants.

(iii) For all initial conditions X_0 there exists an a.s. finite $\mathcal{F}^B(\infty)$-measurable random variable $X_\infty(X_0, \Sigma, f)$ such that the unique continuous adapted process $X(\cdot\,; X_0, \Sigma, f)$ which obeys (7.1.1) satisfies

$$\limsup_{t\to\infty} \frac{1}{t} \log \|X(t) - X_\infty\| \leq -\beta_0^* \quad a.s. \tag{7.2.3}$$

where β_0^* is a positive constant.

It is interesting to note that the condition on the deterministic perturbation f which ensures that the solution of (2.1.1) converges exponentially to a nontrivial random variable is the same condition which ensures that the solution of (7.1.1) converges exponentially to a nontrivial random variable.

7.3 Biological Application

In this section, the biological application considered in Section 4.3 is once again examined. Theorem 7.3.1 builds on Theorem 4.3.1. In Theorem 4.3.1 sufficient conditions for convergence to a nontrivial random variable are considered; Theorem 7.3.1 considers necessary and sufficient conditions for exponential convergence to a nontrivial random variable.

Theorem 7.3.1. *Let $n = d = 1$. Let w satisfy (4.3.3) and (4.3.6), let Σ satisfy (3.1.2) and let ϕ satisfy (4.3.2). If w satisfies*

$$w \text{ does not change sign on } [0,\infty), \tag{7.3.1}$$

the following are equivalent.

(i) There exists a constant R_∞ such that the solution R of (2.1.1) satisfies (3.2.3) and there exist constants $\alpha > 0$ and $\gamma > 0$ such that w satisfies

$$\int_0^\infty |w(s)|\, e^{\alpha s} ds < \infty, \tag{7.3.2}$$

and Σ satisfies (6.2.1).

(ii) For all initial functions ϕ satisfying (4.3.2) and constants $p > 0$ there exists an almost surely finite $\mathcal{F}^B(\infty)$-measurable random variable $X_\infty(\phi,\Sigma)$ with $\mathbb{E}\,|X_\infty|^p < \infty$ such that the unique continuous adapted process $X(\cdot\,;\phi,\Sigma)$ which obeys (4.3.1) satisfies

$$\mathbb{E}\left[|X(t) - X_\infty|^p\right] \le m_p^* e^{-\beta_p^* t}, \quad t \ge 0, \tag{7.3.3}$$

where β_p^* and $m_p^* = m_p^*(X_0)$ are positive constants.

(iii) For all initial functions ϕ satisfying (4.3.2) there exists an a.s. finite $\mathcal{F}^B(\infty)$-measurable random variable $X_\infty(\phi,\Sigma)$ such that the unique continuous adapted process $X(\cdot\,;\phi,\Sigma)$ which obeys (4.3.1) satisfies

$$\limsup_{t\to\infty} \frac{1}{t} \log|X(t) - X_\infty| \le -\beta_0^* \quad a.s. \tag{7.3.4}$$

where β_0^* is a positive constant.

It is evident that the exponential decay of the population is reliant not only on the deterministic factors but also on the external random influences.

7.4 Sufficient Conditions for Exponential Convergence of Solutions of (7.1.1)

In this section, sufficient conditions for exponential convergence of solutions of (7.1.1) to a non–equilibrium limit are found. Proposition 7.4.1 concerns the p^{th} mean sense while Proposition 7.4.2 deals with the almost sure case.

Proposition 7.4.1. *Let K satisfy* (2.1.2) *and* (2.2.7), *Σ satisfy* (3.1.2), *f satisfy* (2.1.4) *and let R_∞ be a constant matrix such that the solution R of* (2.1.1) *satisfies* (3.2.3). *If there exist constants $\alpha > 0$, $\gamma > 0$ and $\rho > 0$ such that such that* (2.2.13), (6.2.1) *and* (7.2.1) *hold, then there exist constants $\beta_p^* > 0$, independent of X_0, and $m_p^* = m_p^*(X_0) > 0$, such that statement (ii) of Theorem 7.2.1 holds.*

Proposition 7.4.2. *Let K satisfy* (2.1.2) *and* (2.2.7), *Σ satisfy* (3.1.2), *f satisfy* (2.1.4) *and let R_∞ be a constant matrix such that the solution R of* (2.1.1) *satisfies* (3.2.3). *If there exist constants $\alpha > 0$, $\gamma > 0$ and $\rho > 0$ such that such that* (2.2.13), (6.2.1) *and* (7.2.1) *hold, then there exists a constant $\beta_0^* > 0$, independent of X_0 such that statement (iii) of Theorem 7.2.1 holds.*

Lemma 7.4.1 is an analogue of Lemma 6.3.1; it is used in the proof of Proposition 7.4.1.

Lemma 7.4.1. *Let K satisfy* (2.1.2) *and* (2.2.7), *Σ satisfy* (3.1.2), *f satisfy* (2.1.4) *and let R_∞ be a constant matrix such that the solution R of* (2.1.1) *satisfies* (3.2.3). *If there exist constants $\alpha > 0$, $\gamma > 0$ and $\rho > 0$ such that such that* (2.2.13), (6.2.1) *and* (7.2.1) *hold, then there exist constants $\lambda > 0$, independent of X_0, and $m^* = m^*(X_0) > 0$, such that*

$$\mathbb{E}\,\|X(t) - X_\infty\|^2 \le m^*(X_0)e^{-2\lambda^* t}, \quad t \ge 0. \tag{7.4.1}$$

In order to prove the above propositions and lemma the following result is required.

Lemma 7.4.2. *Let K satisfy (2.1.2) and (3.2.1) and let R_∞ be a constant matrix such that the solution R of (2.1.1) satisfies (3.2.3) then*

$$\left(A + \int_0^\infty K(s)\,ds\right) R_\infty = 0.$$

Proof of Lemma 7.4.1. We begin by considering the difference between the solution $X(\cdot\,; X_0, \Sigma, f)$ of (7.1.1) and its limit $X_\infty(X_0, \Sigma, f)$ given by (4.2.4):

$$\begin{aligned} &X(t; X_0, \Sigma, f) - X_\infty(X_0, \Sigma, f) \\ &\quad = \big(X(t; X_0, \Sigma) - X_\infty(X_0, \Sigma)\big) + \int_0^t (R(t-s) - R_\infty) f(s)\,ds - \int_t^\infty R_\infty f(s)\,ds. \end{aligned}$$

Using integration by parts this expression becomes

$$\begin{aligned} X(t; X_0, \Sigma, f) - X_\infty(X_0, \Sigma, f) &= \big(X(t; X_0, \Sigma) - X_\infty(X_0, \Sigma)\big) \\ &\quad - f_1(t) + (R(t) - R_\infty) f_1(0) - \int_0^t R'(t-s) f_1(s)\,ds. \end{aligned} \tag{7.4.2}$$

Taking norms on both sides of the equation, squaring both sides, and taking expectations across we obtain

$$\begin{aligned} \mathbb{E}\,\|X(t; X_0, \Sigma, f) - X_\infty(X_0, \Sigma, f)\|^2 &\le 4\Bigg(\mathbb{E}\,\|X(t; X_0, \Sigma) - X_\infty(X_0, \Sigma)\|^2 \\ &\quad + \|f_1(t)\|^2 + \|R(t) - R_\infty\|^2 \|f_1(0)\|^2 + \left(\int_0^t \|R'(t-s)\|\|f_1(s)\|\,ds\right)^2\Bigg). \end{aligned} \tag{7.4.3}$$

Now consider the right hand side of (7.4.3). The first term decays exponentially quickly due to Proposition 6.3.1. The second term decays exponentially quickly due to assumption

(7.2.1). We can apply Theorem 2.2.5 to show that the third term must decay exponentially. In the sequel, an argument is provided to show that R' decays exponentially. Thus the final term must decay exponentially. Combining the above arguments we see that (7.4.1) holds where $\lambda^* < \min(\lambda, \rho)$.

It is now shown that R' decays exponentially. It is clear from the resolvent equation (2.1.1) that

$$R'(t) = A(R(t) - R_\infty) + \int_0^t K(t-s)(R(s) - R_\infty)\, ds \\ - K_1(t)R_\infty + \left(A + \int_0^\infty K(s)\, ds\right) R_\infty. \quad (7.4.4)$$

Consider each term on the right hand side of (7.4.4). We can apply Theorem 2.2.5 to obtain that R decays exponentially quickly to R_∞. Using an argument similar to (2.5.4) and (2.5.5) we see that the second and third term respectively on the right hand side decay exponentially. Finally, using Lemma 7.4.2 we see that the fourth term equals zero. Thus R' decays exponentially quickly to 0. □

Proof of Proposition 7.4.1. The case where $0 < p \leq 2$ and $p > 2$ are considered separately. We begin with the case where $0 < p \leq 2$. Using the expression for $X_\infty(X_0, \Sigma)$ given in Theorem 4.2.3 and (7.2.1) we obtain

$$\mathbb{E}\, \|X_\infty(X_0, \Sigma, f)\|^2 \leq 2\left(\mathbb{E}\, \|X_\infty(X_0, \Sigma)\|^2 + \left\|\int_0^\infty R_\infty f(s)\, ds\right\|^2\right) < \infty. \quad (7.4.5)$$

Applying Liapunov's inequality we see that

$$\mathbb{E}\, \|X_\infty(X_0, \Sigma, f)\|^p \leq \mathbb{E}\left[\|X_\infty(X_0, \Sigma, f)\|^2\right]^{\frac{p}{2}} < \infty.$$

We now show that (6.2.4) holds for $0 \leq p \leq 2$. Liapunov's inequality and Lemma 7.4.1 can be applied as follows

$$\mathbb{E}\left[\|X(t) - X_\infty\|^p\right] \leq \mathbb{E}\left[\|X(t) - X_\infty\|^2\right]^{\frac{p}{2}} \leq m_p^*(X_0)e^{-\beta_p^* t}, \quad t \geq 0,$$

where $m_p^*(X_0) = m^*(X_0)^{\frac{p}{2}}$and $\beta_p^* = \lambda^* p$.

Now consider the case where $p > 2$. In this case there exists a constant $m \in \mathbb{N}$ such that $2(m-1) < p \leq 2m$. We now seek and upper bound on $\mathbb{E}\,\|X_\infty\|^{2m}$ and $\mathbb{E}\left[\|X(t) - X_\infty\|^{2m}\right]$, which will in turn give an upper bound on $\mathbb{E}\,\|X_\infty\|^p$ and $\mathbb{E}\left[\|X(t) - X_\infty\|^p\right]$. Clearly, we see that

$$\mathbb{E}\,\|X_\infty(X_0,\Sigma,f)\|^{2m} \leq 2^{2m-1}\left(\mathbb{E}\,\|X_\infty(X_0,\Sigma)\|^{2m} + \left\|\int_0^\infty R_\infty f(s)\,ds\right\|^{2m}\right) < \infty.$$

Now consider $\mathbb{E}\left[\|X(t) - X_\infty\|^{2m}\right]$. Taking norms on both sides of equation (7.4.2), raising the power to $2m$ on both sides, and taking expectations across we obtain

$$\begin{aligned}&\mathbb{E}\,\|X(t;X_0,\Sigma,f) - X_\infty(X_0,\Sigma,f)\|^{2m}\\ &\quad\leq 4^{2m-1}\Big(\mathbb{E}\,\|X(t;X_0,\Sigma) - X_\infty(X_0,\Sigma)\|^{2m} + \|f_1(t)\|^{2m}\\ &\qquad + \|R(t) - R_\infty\|^{2m}\|f_1(0)\|^{2m} + \left(\int_0^t \|R'(t-s)\|\|f_1(s)\|\,ds\right)^{2m}\Big)\end{aligned} \tag{7.4.6}$$

Now consider the right hand side of (7.4.6). The first term decays exponentially quickly due to Proposition 6.3.1. The second term decays exponentially quickly due to assumption (7.2.1). We can apply Theorem 2.2.5 to show that the third term must decay exponentially. In Lemma 7.4.1 we provided an argument to show that R' decays exponentially. Thus the final term must decay exponentially. Combining the above arguments we see that (7.4.1) holds. □

Proof of Proposition 7.4.2. Take norms across (7.4.2) to obtain

$$\begin{aligned}\|X(t;X_0,\Sigma,f) - X_\infty(X_0,\Sigma,f)\| &\leq \|X(t;X_0,\Sigma) - X_\infty(X_0,\Sigma)\|\\ &\quad + \|f_1(t)\| + \|R(t) - R_\infty)\|\|f_1(0)\| + \left\|\int_0^t R'(t-s)f_1(s)\,ds\right\|.\end{aligned} \tag{7.4.7}$$

Using Proposition 6.3.2 we see that the first term on the right hand side of (7.4.7) decays exponentially. The second term on the right hand side decay exponentially as (7.2.1) holds.

We can apply Theorem 2.2.5 to show that the third term must decay exponentially. An argument was provided in Lemma 7.4.1 to show that R' decays exponentially. Combining this with (7.2.1) enables us to show that the fourth term decays exponentially. Using the above arguments we obtain (7.2.3). □

Proof of Lemma 7.4.2. We now show that $\left(A + \int_0^\infty K(s)\,ds\right) R_\infty = 0$. Integrating the resolvent equation (2.1.1), adding and subtracting R_∞ from both sides, dividing by t and rearranging the equation we obtain

$$\begin{aligned}\left(A + \int_0^\infty K(s)\,ds\right) R_\infty &= \frac{R(t)-I}{t} \\ &\quad - \frac{1}{t}\int_0^t \left[A(R(s)-R_\infty) + \int_0^s K(s-u)(R(u)-R_\infty)\,du\right] ds \\ &\quad + \frac{1}{t}\int_0^t K_1(s) R_\infty. \qquad (7.4.8)\end{aligned}$$

Using the fact that (3.2.3) holds and R_∞ is a finite constant we see that the first term on the right hand side of (7.4.8) tends to 0 as $t \to \infty$. We know that K_1 decays exponentially due to (2.5.5): thus, the third term on the right hand side of (7.4.8) tends to zero. Now consider the second term.

$$\begin{aligned}&\frac{1}{t}\left\| \int_0^t \left[A(R(s)-R_\infty) + \int_0^s K(s-u)(R(u)-R_\infty)\,du\right] ds \right\| \\ &\quad \le \left[\left(\frac{1}{t}\int_0^t \left\|A(R(s)-R_\infty) + \int_0^s K(s-u)(R(u)-R_\infty)\,du\right\| ds\right)^2\right]^{\frac{1}{2}} \\ &\quad \le \left[\frac{1}{t}\int_0^t \left\|A(R(s)-R_\infty) + \int_0^s K(s-u)(R(u)-R_\infty)\,du\right\|^2 ds\right]^{\frac{1}{2}} \\ &\quad \le \left[\frac{1}{t}\left(\int_0^t 2\|A\|^2\|R(s)-R_\infty\|^2 \right.\right. \\ &\qquad \left.\left. +2\left[\int_0^s \|K(s-u)\|^{1/2}\|K(s-u)\|^{1/2}\,\|R(u)-R_\infty\|\,du\right]^2 ds\right)\right]^{\frac{1}{2}} \\ &\quad \le \left[\frac{1}{t}\left(2\int_0^t \|A\|^2\|R(s)-R_\infty\|^2 \right.\right. \\ &\qquad \left.\left. +2\bar{K}\int_0^s \|K(s-u)\|\,\|R(u)-R_\infty\|^2\,du\,ds\right)\right]^{\frac{1}{2}},\end{aligned}$$

where $\bar{K} = \int_0^\infty \|K(s)\|\,ds$. Using the fact that (3.2.3) holds we see that $\int_0^t \|A\|^2\|R(s) - R_\infty\|^2\,ds < C$. As the convolution of an L^1 function with a L^1 function is in L^1 we see that $\int_0^t\int_0^s \|K(s-u)\|\,\|R(u) - R_\infty\|^2\,du\,ds < C$. Thus, the second term on the right hand side of (7.4.8) tends to zero. The right hand side of (7.4.8) tends to zero while the left hand side is a finite constant. This implies that $\left(A + \int_0^\infty K(s)\,ds\right) R_\infty = 0$ as required. □

7.5 Necessary Conditions for Exponential Convergence of Solutions of (7.1.1)

In this section, the necessity of (6.2.1) and (7.2.1) for exponential convergence of solutions of (7.1.1) in the almost sure and p^{th} mean senses is shown. Proposition 7.5.1 concerns the necessity of the conditions in the p^{th} mean case while Proposition 7.5.2 deals with the almost sure case.

Proposition 7.5.1. *Let K satisfy* (2.1.2) *and* (3.2.1), *let Σ satisfy* (3.1.2) *and let f satisfy* (2.1.4). *If there exists a constant $\alpha > 0$ such that* (2.2.13) *holds, and if for all X_0 there is a constant vector $X_\infty(X_0, \Sigma, f)$ such that the solution $t \mapsto X(t; X_0, \Sigma, f)$ of* (7.1.1) *satisfies statement (ii) of Theorem 7.2.1, then there exists constants $\gamma > 0$ and $\rho > 0$, independent of X_0, such that* (6.2.1) *and* (7.2.1) *hold.*

Proposition 7.5.2. *Let K satisfy* (2.1.2) *and* (3.2.1), *let Σ satisfy* (3.1.2) *and let f satisfy* (2.1.4). *If there exists a constant $\alpha > 0$ such that* (2.2.13) *holds, and if for all X_0 there is a constant vector $X_\infty(X_0, \Sigma, f)$ such that the solution $t \mapsto X(t; X_0, \Sigma, f)$ of* (7.1.1) *satisfies statement (iii) of Theorem 7.2.1, then there exists constants $\gamma > 0$ and $\rho > 0$, independent of X_0, such that* (6.2.1) *and* (7.2.1) *hold.*

The following lemma is used in the proof of Proposition 7.5.2.

Lemma 7.5.1. *Suppose $c \geq 0$ is an almost surely finite random variable and*

$$\|f_1(t) + \mu_1(t,\omega)\| \leq c(\omega)e^{-\lambda t},$$

where $\lambda > 0$, $\omega \in \Omega^*$, $\mathbb{P}[\Omega^*] = 1$ and the functions f_1 and μ_1 are defined by (2.2.15) *and*

$$\mu_1(t) = \int_t^\infty \Sigma(s)\,dB(s), \quad t \geq 0, \tag{7.5.1}$$

respectively. Then (6.2.1) *and* (7.2.1) *hold.*

In Lemma 7.5.1 the exponential decay of f, (7.2.1), and the exponential square integrability of Σ, (6.2.1), can be proven given an assumption of the tail on f and the tail of Σ. In order to prove this result the object defined by (6.2.1) is not considered directly, instead a variant of this condition given by (7.5.13) is examined. The reason for this is that (7.5.13) may be reformulated in terms of the tail of Σ.

The following lemma is proved in [1] and used in the proof of Lemma 7.5.1.

Lemma 7.5.2. *If there is a $\gamma > 0$ such that $\sigma \in C([0,\infty),\mathbb{R})$ and*

$$\int_0^\infty \sigma(s)^2 e^{2\gamma s}\,ds < \infty, \tag{7.5.2}$$

then

$$\limsup_{t\to\infty} \frac{1}{t}\log\left|\int_t^\infty \sigma(s)\,dB(s)\right| < -\gamma \quad a.s.$$

where $\{B(t)\}_{t\geq 0}$ is a one-dimensional standard Brownian motion.

Lemmas 7.5.3 and 7.5.4 are used in the proofs of Propositions 7.5.1 and 7.5.2 respectively and are the analogues of Lemma 3.4.1 and 4.4.1. Their proofs are identical to the proofs of Lemma 3.4.1 and 4.4.1 in all important aspects and so are omitted.

Lemma 7.5.3. *Let K satisfy* (2.1.2) *and* (3.2.1). *Suppose that for all initial conditions X_0 there is a $\mathcal{F}^B(\infty)$-measurable and almost surely finite random variable $X_\infty(X_0,\Sigma)$ with $\mathbb{E}\|X_\infty\|^2 < \infty$ such that the solution $t \mapsto X(t;X_0,\Sigma)$ of* (7.1.1) *satisfies*

$$\lim_{t\to\infty} \mathbb{E}\|X(t;X_0,\Sigma,f) - X_\infty(X_0,\Sigma,f)\|^2 = 0$$

and

$$\mathbb{E}\,\|X(\cdot\,;X_0,\Sigma,f)-X_\infty(X_0,\Sigma,f)\|^2\in L^1((0,\infty),\mathbb{R}).$$

Then, X_∞ obeys

$$\left(A+\int_0^\infty K(s)\,ds\right)X_\infty=0\quad a.s. \tag{7.5.3}$$

Lemma 7.5.4. *Let K satisfy* (2.1.2) *and* (3.2.1). *Suppose that for all initial conditions X_0 there is a $\mathcal{F}^B(\infty)$-measurable and almost surely finite random variable $X_\infty(X_0,\Sigma,f)$ such that the solution $t\mapsto X(t;X_0,\Sigma,f)$ of* (7.1.1) *satisfies*

$$\lim_{t\to\infty}X(t;X_0,\Sigma,f)=X_\infty(X_0,\Sigma,f)\quad a.s.$$

and

$$X(\cdot\,;X_0,\Sigma,f)-X_\infty(X_0,\Sigma,f)\in L^2((0,\infty),\mathbb{R}^n)\quad a.s.$$

Then, X_∞ obeys

$$\left(A+\int_0^\infty K(s)\,ds\right)X_\infty=0\quad a.s.$$

Proof of Proposition 7.5.1. Since (7.2.2) holds for all initial conditions we can choose $X_0=0$: this simplifies calculations. Moreover using (7.2.2) in Lemma 7.5.4 it is clear that assumption (7.5.3) holds. Consider the integral form of (7.1.1). Adding and subtracting X_∞ from both sides and applying Lemma 7.5.3 we obtain

$$\Delta(t)=-X_\infty+\int_0^t\kappa(s)\,ds+\int_0^t f(s)\,ds+\mu(t)-\int_0^t K_1(s)\,ds\,X_\infty, \tag{7.5.4}$$

where $\Delta(t)=X(t)-X_\infty$, the function κ is defined by

$$\kappa(t)=A\Delta(t)+(K*\Delta)(t), \tag{7.5.5}$$

and $\mu(t) = \int_0^t \Sigma(s)\,dB(s)$. Taking expectations across (7.5.4) and allowing $t \to \infty$ we obtain

$$-\mathbb{E}[X_\infty] = -\int_0^\infty \kappa_{\mathbb{E}}(s)\,ds - \int_0^\infty f(s)\,ds + \int_0^\infty K_1(s)\,ds\,\mathbb{E}[X_\infty], \tag{7.5.6}$$

where $\kappa_{\mathbb{E}}(t) = A\mathbb{E}[\Delta(t)] + (K * \mathbb{E}[\Delta])(t)$. Using this expression for $\mathbb{E}[X_\infty]$ we obtain

$$f_1(t) = -\mathbb{E}[\Delta(t)] - \int_t^\infty \kappa_{\mathbb{E}}(s)\,ds + \int_t^\infty K_1(s)\,ds\,\mathbb{E}[X_\infty], \tag{7.5.7}$$

The first term on the right hand side of (7.5.7) decays exponentially due to (7.2.1). Assumptions (2.2.13) and (7.2.2) imply that $\kappa_{\mathbb{E}}$ decays exponentially so the second term decays exponentially. The third term on the right hand side of (7.5.7) decays exponentially due to the argument given by (2.5.5). Hence, f_1 decays exponentially.

Proving that (6.2.1) holds breaks into two steps. We begin by showing that

$$\left\| \int_0^\infty e^{\rho_1 t} f(t)\,dt \right\| < \infty, \tag{7.5.8}$$

where $\rho_1 > 0$. By choosing $\rho_1 < \alpha \wedge \beta_1$ we can obtain the following reformulation of (7.1.1) using methods applied in Proposition 6.4.2:

$$\begin{aligned} e^{\rho_1 t}\Delta(t) &= \Delta(0) + (\rho_1 I + A)\int_0^t e^{\rho_1 s}\Delta(s)\,ds \\ &\quad + \int_0^t e^{\rho_1 s}\int_0^s K(s-u)\Delta(u)\,du\,ds - \int_0^t e^{\rho_1 s}K_1(s)\,du\,ds\,X_\infty \\ &\quad + \int_0^t e^{\rho_1 s} f(s)\,ds + \int_0^t e^{\rho_1 s}\Sigma(s)\,dB(s). \end{aligned} \tag{7.5.9}$$

Rearranging (7.5.9), taking expectations then norms across (7.5.9) we can obtain

$$\begin{aligned} \left\| \int_0^t e^{\rho_1 s} f(s)\,ds \right\| &\le \mathbb{E}\left\|e^{\rho_1 t}\Delta(t)\right\| + \mathbb{E}\left\|\Delta(0)\right\| \\ &\quad + \|\rho_1 I + A\| \int_0^t e^{\rho_1 s}\mathbb{E}\|\Delta(s)\|\,ds \\ &\quad + \int_0^t e^{\rho_1 s}\int_0^s \|K(s-u)\|\mathbb{E}\|\Delta(u)\|\,du\,ds + \int_0^t e^{\rho_1 s}\|K_1(s)\|\,ds\,\mathbb{E}\|X_\infty\|. \end{aligned} \tag{7.5.10}$$

Since (7.2.2) holds this implies that both the first and third terms on the right hand side of (7.5.10) are bounded. The second term is bounded due to our assumptions. Since $0 < \rho_1 < \alpha \wedge \beta_1$, we can choose $\rho_2 > 0$ such that $\rho_1 < \rho_2 < \alpha \wedge \beta_1$. Now we can apply an argument similar to (2.5.4) to show that

$$\int_0^s \|K(s-u)\| \mathbb{E}\left[\|\Delta(u)\|\right] du < ce^{-\rho_2 s}.$$

Finally, we see that the fifth term is bounded using (2.5.5). So, (7.5.8) holds.

We now return to (7.5.9). Again rearranging the equation, taking norms and then expectations across both sides we obtain

$$\begin{aligned}\mathbb{E}\left\|\int_0^t e^{\rho_1 s}\Sigma(s)\,dB(s)\right\| &\leq \mathbb{E}\left\|e^{\rho_1 t}\Delta(t)\right\| + \mathbb{E}\left\|\Delta(0)\right\| \\ &\quad + \|\rho_1 I + A\| \int_0^t e^{\rho_1 s}\mathbb{E}\left\|\Delta(s)\right\| ds + \int_0^t e^{\rho_1 s}\int_0^s \|K(s-u)\|\,\mathbb{E}\left\|\Delta(u)\right\| du\, ds \\ &\quad + \int_0^t e^{\rho_1 s}\|K_1(u)\|\, du\, ds\, \mathbb{E}\left\|X_\infty\right\| + \left\|\int_0^t e^{\rho_1 s} f(s)\, ds\right\|.\end{aligned}$$

We already provided an argument above to show that the first five terms on the right hand side of this expression are bounded. Also, we know that (7.5.8) holds. Thus,

$$\mathbb{E}\left\|\int_0^t e^{\gamma s}\Sigma(s)\,dB(s)\right\| \leq C, \quad a.s.$$

The proof is now identical to Proposition 6.4.1. □

Proof of Proposition 7.5.2. Consider equation (7.5.4). Since Lemma 7.5.3 holds we can obtain (7.5.4). Thus, as $t \to \infty$ we obtain

$$-X_\infty = -\int_0^\infty \kappa(s)\,ds - \int_0^\infty f(s)\,ds - \mu(\infty) + \int_0^\infty K_1(s)\,ds\, X_\infty,$$

where κ is defined by (7.5.5). Using this expression for X_∞, equation (7.5.4) becomes

$$\Delta(t) = -\int_t^\infty \kappa(s)\,ds - f_1(t) - \mu_1(t) + \int_t^\infty K_1(s)\,ds\, X_\infty,$$

where $\mu_1(t) = \int_t^\infty \Sigma(s)\,dB(s)$. Rearranging the equation and taking norms yields

$$\|f_1(t) + \mu_1(t)\| = \|\Delta(t)\| + \left\|\int_t^\infty \kappa(s)\,ds\right\| + \left\|\int_t^\infty K_1(s)\,ds\,X_\infty\right\|, \tag{7.5.11}$$

The first term on the right hand side of (7.5.11) decays exponentially due to (7.2.3). Using the argument given in (2.5.5) we see that the third term on the right hand side of (7.5.11) decays exponentially. Finally, we consider the second term. Clearly $\int_t^\infty A\Delta(s)\,ds$ decays exponentially due to (7.2.3). In order to show that $\int_t^\infty (K * \Delta)(s)\,ds$ decays exponentially we use an argument similar to that applied in (2.5.4). Thus

$$\|f_1(t) + \mu_1(t)\| \le ce^{-\lambda t} \quad \text{a.s.} \tag{7.5.12}$$

We can now apply Lemma 7.5.1 to obtain (7.2.1) and (6.2.1). □

Proof of Lemma 7.5.1. Choose $0 < \gamma < \lambda$. We begin by supposing that (6.2.1) holds. Using the equivalence of norms we see that for all $1 \le i \le n$ and $1 \le j \le d$ assumption (6.2.1) implies that

$$\int_0^\infty \Sigma_{ij}(s)^2 e^{2\gamma s}\,ds < \infty.$$

Applying Lemma 7.5.2 we obtain

$$\limsup_{t\to\infty} \frac{1}{t} \log \left|\int_t^\infty \Sigma_{ij}(s)\,dB(s)\right| < -\gamma \quad \text{a.s.}$$

Thus, we can choose $c_{ij}(\omega) > 0$ such that

$$|\mu_{ij}(t)| = \left|\int_t^\infty \Sigma_{ij}(s)\,dB_j(s)\right| \le c_{ij}(\omega)e^{-\gamma t}, \quad \omega \in \Omega_{ij}, \quad \mathbb{P}[\Omega_{ij}] = 1.$$

Now, summing over j we see that

$$|\mu_1^i(t)| \le c_i(\omega)e^{-\gamma t},$$

where $\omega \in \Omega_i = \cap_{j=1}^d \Omega_{ij}$, $c_i = \sum_{j=1}^d c_{ij}$ and $\mu_1^i(t) = \sum_{j=1}^n |\mu_{ij}(t)|$. Now, since

$$\left|f_1^i(t) + \mu_1^i(t)\right|^2 \le \sum_{i=1}^n \left|f_1^i(t) + \mu_1^i(t)\right|^2 = \|f_1(t) + \mu_1(t)\|^2$$

we see that

$$\left|f_1^i(t) + \mu_1^i(t)\right| \le c(\omega)e^{-\lambda t}, \quad \omega \in \Omega^*.$$

Thus $|f_1^i(t)| \le |f_1^i(t) + \mu_1^i(t)| + |\mu_1^i(t)| \le c(\omega)e^{-\lambda t} + c_i(\omega)e^{-\gamma t}$. This gives

$$\left|f_1^i(t)\right| \le \bar{c}_i(\omega)e^{-\rho t},$$

where $\bar{c}_i > 0$ is finite and $\rho \le \gamma$. Now summing over i we obtain (7.2.1), by picking out any $\omega \in \Omega^*$.

Now define the martingale M as

$$M(t) = \int_0^t \frac{1}{\gamma}(e^{\gamma s} - 1)\Sigma(s)\, dB(s), \tag{7.5.13}$$

and the function d as

$$d(t) = \int_0^t \frac{1}{\gamma}(e^{\gamma s} - 1)f(s)\, ds. \tag{7.5.14}$$

We show in the sequel that

$$\|d(t) + M(t)\| < c(\omega), \quad \omega \in \Omega^*, \tag{7.5.15}$$

and therefore assume it for the time being.

Now, consider the case where assumption (6.2.1) fails to hold. Clearly, the quadratic variation of M is given by

$$\langle M\rangle(t) = \int_0^t \frac{1}{\gamma^2}(e^{\gamma s} - 1)^2\Sigma(s)^2\, ds.$$

Since (6.2.1) fails to hold there exists at least one entry i, $1 \le i \le n$ such that

$$\langle M\rangle_i(t) = \sum_{j=1}^d \int_0^t \frac{1}{\gamma^2}(e^{\gamma s} - 1)^2\Sigma_{ij}(s)^2\, ds \to \infty, \quad \text{as } t \to \infty, \tag{7.5.16}$$

where $\langle M\rangle_i$ is the i^{th} component of the vector $\langle M\rangle$. Thus,

$$\liminf_{t\to\infty} M_i(t) = -\infty \text{ and } \limsup_{t\to\infty} M_i(t) = \infty.$$

Consider the function d_i; it is either bounded or unbounded. If d_i is bounded then M is bounded and so by the Martingale Convergence Theorem this means that $\langle M\rangle(t)$ is bounded: a contradiction. So, we suppose the latter, that is d is unbounded, and proceed to show this is also contradictory. Since $|d_i(t)+M_i(t)|<c_1(\omega)$, for $\omega\in\Omega^*$ it is clear that $-c_1-M_i(t)<d_i(t)$. Taking lim sup's on both sides of the inequality yields

$$\infty=-c_1-\liminf_{t\to\infty}M_i(t)\leq\limsup_{t\to\infty}d_i(t).$$

Thus, as d is deterministic, there exists a sequence of deterministic times $\{t_n\}_{n=1}^{\infty}$ such that $d_i(t_n)\to\infty$ as $n\to\infty$. This in turn implies $M_i(t_n)\to-\infty$ as $n\to\infty$.

A simple proof now shows that this is inconsistent. Define $N(n)=M_i(t_n)$ for $\{t_n\}_{n=0}^{\infty}$, where M_i is the i^{th} component of M. Clearly N is a martingale such that $\lim_{n\to\infty}N(n)=-\infty$. Also $\lim_{n\to\infty}\langle N\rangle(n)=\infty$ due to assumption (7.5.16). By the Martingale Convergence Theorem, this implies that $\limsup_{n\to\infty}N(n)=\infty$ and $\liminf_{n\to\infty}N(n)=-\infty$. However, as $\lim_{n\to\infty}N(n)=-\infty$ this provides the desired contradiction: that is, there cannot exist a sequence of times $\{t_n\}_{n=1}^{\infty}$ such that $M_i(t_n)\to-\infty$ as $n\to\infty$. This excludes (7.5.16), and hence (6.2.1) holds.

We now show that assumption (7.5.15) holds. By changing the order of integration we can show that

$$d(t)=\int_0^t e^{\gamma s}(f_1(s)-f_1(t))\,ds \text{ and } M(t)=\int_0^t e^{\gamma s}(\mu_1(s)-\mu_1(t))\,ds.$$

Thus, as $0<\gamma<\lambda$,

$$\begin{aligned}\|d(t)+M(t)\|&\leq\int_0^t e^{\gamma s}(\|f_1(t)+\mu_1(t)\|+\|f_1(s)+\mu_1(s)\|)\,ds\\&\leq c(\omega)\int_0^t e^{\gamma s}(e^{-\lambda t}+e^{-\lambda s})\,ds<c(\omega),\quad\omega\in\Omega^*.\end{aligned}$$

□

7.6 On the Necessary and Sufficient Conditions for Exponential Convergence of Solutions of (7.1.1)

We now combine the results from Sections 7.4 and 7.5 to prove Theorem 7.2.1.

Proof of Theorem 7.2.1. We begin by proving the equivalence between (i) and (ii). The implication (i) implies (ii) is the subject of Proposition 7.4.1. Now consider the implication (ii) implies (i) Using (7.4.1) we see that

$$\|\mathbb{E}\left[X(t) - X_\infty\right]\| \le \mathbb{E}\,\|X(t) - X_\infty\| \le e^{-\beta_1 t}.$$

Consider the $n+1$ solutions $X_j(t)$ of (3.1.1) with initial conditions $X_j(0) = \mathbf{e}_j$ for $j = 1, \dots, n$ and $X_{n+1}(0) = 0$. Since $R(t)\mathbf{e}_j = X_j(t) - X_{n+1}(t)$ we see that

$$\mathbb{E}\left[X_j(t) - X_j(\infty)\right] + \mathbb{E}\left[X_{n+1}(t) - X_{n+1}(\infty)\right] = R(t)\mathbf{e}_j - \mathbb{E}\left[c_j\right],$$

where $c_j = X_j(\infty) - X_{n+1}(\infty)$ is an almost surely finite constant. As both terms on the right hand side of this expression are decaying exponentially to zero, $t \mapsto R(t)\mathbf{e}_j$ must decay exponentially to $\mathbb{E}\left[c_j\right]$ as $t \to \infty$. Thus R must satisfy (2.2.14). Now, apply Theorem 2.2.5 to obtain (2.2.13) and Proposition 7.5.1 to obtain (7.2.1) and (6.2.1).

We now prove the equivalence between (i) and (iii). The implication (i) implies (iii) is the subject of Proposition 7.4.2. Consider the $n+1$ solutions $X_j(t)$ of (3.1.1) with initial conditions $X_j(0) = \mathbf{e}_j$ for $j = 1, \dots, n$ and $X_{n+1}(0) = 0$. Since $R(t)\mathbf{e}_j = X_j(t) - X_{n+1}(t)$ for $j = 1, \dots, n$, we can write

$$(X_j(t) - X_j(\infty)) - (X_{n+1}(t) - X_{n+1}(\infty)) = R(t)\mathbf{e}_j - c_j,$$

where $c_j = X_j(\infty) - X_{n+1}(\infty)$ is an almost surely finite random variable. From (7.2.3) we know that X_j decays exponentially quickly to $X_j(\infty)$, similarly X_{n+1} decays exponentially quickly to $X_{n+1}(\infty)$. Thus, R decays exponentially to a limit R_∞. As a result (3.2.3) must hold. Now apply Theorem 2.2.5 to obtain (2.2.13) and Proposition 7.5.2 to obtain (7.2.1) and (6.2.1). □

7.7 Proof of Theorem 7.3.1

We begin by proving that (i) implies (ii) and (iii). Rewrite (4.3.1) as

$$dx(t) = \alpha\left(x(t) - \int_{-\infty}^{0} w(t-s)\phi(s)\,ds - \int_{0}^{t} w(t-s)x(s)\,ds\right)dt + \Sigma(t)dB(t). \quad (7.7.1)$$

Let

$$f(t) = -\alpha\int_{-\infty}^{0} w(t-s)\phi(s)\,ds, \quad (7.7.2)$$

then

$$f_1(t) = -\alpha\int_{t}^{\infty}\int_{s}^{\infty} w(v)\phi(s-v)\,dv\,ds.$$

Using assumptions (4.3.2) and (7.3.2) we see that

$$|f_1(t)| \le c_1 e^{-\alpha t}.$$

Using this fact combined with (3.2.3) and (7.3.2) we can apply Theorem 7.2.1 to see that both (7.3.3) and (7.3.4) hold.

In order to prove the implication that (ii) implies (i) and (iii) implies (i) we once again rewrite the equation as in (7.7.1) and define f as in (7.7.2). Again applying Theorem 7.2.1 yields the desired result.

Chapter 8

Behaviour of Equations with Constant Noise

8.1 Introduction

In this chapter we consider the equation

$$dX(t) = \left(AX(t) + \int_0^t K(t-s)X(s)\,ds\right) dt + \Sigma\, dB(t), \quad t > 0, \tag{8.1.1a}$$

$$X(t) = X_0. \tag{8.1.1b}$$

As always we are interested in the case when $R - R_\infty$ is square integrable. It is evident that the solution of (8.1.1) is not in general stationary, nor does the solution converge to a stationary process. However, we pose the following question: do the increments of the solution of (8.1.1) converge in distribution to a stationary process?

Before tackling this question it is natural to examine the infinite delay differential equation:

$$dX(t) = \left(AX(t) + \int_{-\infty}^t K(t-s)X(s)\,ds\right) dt + \Sigma\, dB(t), \quad t > 0, \tag{8.1.2a}$$

$$X(t) = \phi(t), \quad t \le 0, \tag{8.1.2b}$$

where ϕ is a $n \times 1$–dimensional vector–valued function is the initial function. We define standard Brownian motion B on the entire real line as follows

$$B(t) = \begin{cases} B_1(t), & t \ge 0 \\ B_2(-t), & t < 0, \end{cases}$$

where B_1 and B_2 are independent Brownian motions.

Riedle [36] considered an equation of this type for a class of measures and proved that the integrability of the resolvent was necessary and sufficient for a stationary solution to exist. In this chapter we explore the case where $R - R_\infty$ is square integrable and examine the existence of a solution which gives rise to stationary increments.

In the sequel a number of interesting results concerning the increments of the solutions of (8.1.1) and (8.1.2) are proved under assumptions on the integrability of moments of the kernel and the behaviour of the initial function. It is proved that for a particular initial function the corresponding solution of (8.1.2) has stationary increments. Furthermore, it is proven that for any initial function the increments of the corresponding solution converge to a stationary process. Finally, it is shown that the solution of (8.1.1) also converges to a stationary process.

These results are stated and discussed in Section 8.2. Note that all of these results are proven in the scalar case, although it is thought that they also hold in the n-dimensional case.

8.2 Discussion of Results

It is possible to show that for a particular initial function that there exists a solution to the infinite delay equation with stationary increments. Indeed, if the initial history satisfies a particular constraint then it is possible to show that the corresponding solution converges in distribution to the stationary solution.

We begin by considering the existence of a unique continuous $\mathcal{F}^B$-adapted process satisfying (8.1.2).

Theorem 8.2.1. *Suppose there is $\epsilon > 0$ such that the kernel K satisfies*

$$\int_0^\infty (1+t)^{1+\epsilon}|K(t)|\,dt < \infty, \tag{8.2.1}$$

and the initial function ϕ satisfies

$$\mathbb{E}\left[\sup_{-\infty<t\leq 0}(1+|t|)^{-(1+\epsilon)}\phi(t)^2\right] < \infty. \tag{8.2.2}$$

Then there exists a unique continuous $\mathcal{F}^B$-adapted process X which satisfies (8.1.2).

This theorem gives a sufficient condition on the initial function which will ensure the existence of a solution under (8.2.1). It should be noted that as the initial function becomes less well behaved a stronger condition is required on the integrability of the kernel in order to counteract this.

Although most of the proof of Theorem 8.2.1 follows the corresponding proofs for equations without infinite delay, there are a number of differences which should be highlighted. A non–standard norm must by used to ensure the existence of the solution on the entire real line; many of these differences arise due to the use of this norm. Consequently we supply the proof of this theorem in full in Section 8.6.

We now consider the solution X of (8.1.2) for a particular initial function. The key hypothesis which ensures the relevant stationarity property of the solution of (8.1.2) is that there exists R_∞ such that $R - R_\infty$ is square–integrable where R is the resolvent of (2.1.1). In this case X can be written for all $t \in \mathbb{R}$ as the sum of a standard Brownian motion and stationary Gaussian process.

Theorem 8.2.2. *For $\epsilon > 0$, let the kernel K satisfy*

$$\int_0^\infty (1+t)^{\frac{5+\epsilon}{2}} |K(t)|\, dt < \infty. \tag{8.2.3}$$

Suppose that there exists a constant R_∞ such that the resolvent R of (2.1.1) satisfies

$$R - R_\infty \in L^2([0,\infty),\mathbb{R}). \tag{8.2.4}$$

If the initial function ϕ is given by

$$\phi(t) = R_\infty \Sigma B(t) + \int_{-\infty}^t (R(t-s) - R_\infty)\Sigma\, dB(s), \quad t \le 0, \tag{8.2.5}$$

then the process S defined by

$$S(t) = R_\infty \Sigma B(t) + \int_{-\infty}^t (R(t-s) - R_\infty)\Sigma\, dB(s), \quad t \in \mathbb{R}, \tag{8.2.6}$$

satisfies (8.1.2).

Since (8.2.4) holds we see that the variance of $S(t)$ defined by (8.2.6) is given by

$$\mathbb{E}\left[S(t)^2\right] = R_\infty^2 \Sigma^2 t + 2R_\infty \Sigma \int_0^t (R(s) - R_\infty)\, ds + \int_0^\infty (R(s) - R_\infty)^2\, ds.$$

It is clear that the process S is not stationary unless $R_\infty = 0$. However, the following theorem asserts that the increments of S are stationary.

Theorem 8.2.3. *Suppose that there exists a constant R_∞ such that the resolvent R of* (2.1.1) *satisfies* (8.2.4) *and let the process S be defined by* (8.2.6). *Then for every $\Delta > 0$ the increment $S_\Delta = \{S_\Delta(t)\}_{t\geq 0}$ defined by*

$$S_\Delta(t) = S(t) - S(t-\Delta),$$

is a stationary process.

In Theorem 8.2.4 it is shown that (8.2.4) is in fact necessary for the existence of a solution with stationary increments.

Theorem 8.2.4. *For $\epsilon > 0$, let K satisfy* (8.2.3) *and let ϕ satisfy* (8.2.2) *where $\{\phi(t)\}_{t\leq 0}$ is independent of the Brownian motion $\{B(t)\}_{t\geq 0}$. The following are equivalent:*

(i) There exists a constant R_∞ such that the resolvent R of (2.1.1) *satisfies* (8.2.4).

(ii) There exists a function $\phi \in C((-\infty, 0], \mathbb{R})$ such that the corresponding process X satisfying (8.1.2) *has stationary increments.*

Theorem 8.2.3 and Theorem 8.2.4 are proved in Section 8.4.

If a general initial function is considered then the increments of the resulting process converge to the increments of the process given by (8.2.6).

Theorem 8.2.5. *For $\epsilon > 0$, let K satisfy* (2.2.7) *and let ϕ satisfy* (8.2.2) *where $\{\phi(t)\}_{t\leq 0}$ is independent of the Brownian motion $\{B(t)\}_{t>0}$. The following are equivalent:*

(i) There exists a constant R_∞ such that the resolvent R of (2.1.1) *satisfies* (8.2.4).

(ii) The increment of the process X satisfying (8.1.2) *has a limitimg distribution.*

Finally we consider the solution of (8.1.1).

Theorem 8.2.6. *For $\epsilon > 0$, let K satisfy*

$$\int_0^\infty (1+t)^{2+\epsilon}|K(t)|\,dt < \infty, \tag{8.2.7}$$

The following are equivalent:

(i) There exists a constant R_∞ such that the resolvent R of (2.1.1) *satisfies* (8.2.4).

(ii) The increment X_Δ of the process X satisfying (8.1.1) *has a limitimg distribution.*

The condition on the kernel K given by (8.2.7) is weaker than (8.2.3) because it is no longer necessary to allow integrability in order to control the impact of the possibility of an unbounded process.

Theorem 8.2.6 may be proved in a manner identical to Theorem 8.2.5 and so the proof is omitted.

8.3 Proof of Theorem 8.2.2

Before proving Theorem 8.2.2 we state some following supporting results. We begin by stating that the initial function defined by (8.2.5) satisfies (8.2.2) in Lemma 8.3.1.

Lemma 8.3.1. *Suppose that there exists a constant R_∞ such that the resolvent R of* (2.1.1) *satisfies* (8.2.4). *Then the initial function ϕ defined by* (8.2.5) *satisfies* (8.2.2).

Proving that Lemma 8.3.1 holds requires the stochastic integral in the initial function (8.2.5) to be rewritten. This is in order to avoid the need to consider Fubini–type theorems for iterated stochastic integrals over infinite domains. The reformulation enables all stochastic integrals to be replaced by Riemann integrals with stochastic integrands.

In Lemma 8.3.2 a general square integrable function is considered and a reformulation obtained under certain growth conditions.

Lemma 8.3.2. *Let $\eta \in L^2((0,\infty),\mathbb{R})$, and satisfy*

$$\int_0^\infty (1+t)^{\frac{1+\epsilon}{2}} |\eta(t)|\, dt < \infty,$$

and

$$\lim_{t\to\infty} (1+t)^{\frac{1+\epsilon}{2}} |\eta(t)| = 0,$$

and let η' satisfy

$$\int_0^\infty (1+t)^{\frac{1+\epsilon}{2}} |\eta'(t)|\, dt < \infty,$$

then

$$\int_{-\infty}^t \eta(t-s)\, dB(s) = \eta(0)B(t) + \int_{-\infty}^t \eta'(t-s)B(s)\, ds, \quad t \in \mathbb{R}. \tag{8.3.1}$$

In Lemma 8.3.3 we state a condition of the kernel which ensures that the growth conditions hold when considering $R - R_\infty$ in the role of η in Lemma 8.3.2 above.

Lemma 8.3.3. *Let K satisfy (8.2.3) and suppose that there exists a constant R_∞ such that the resolvent R of (2.1.1) satisfies (8.2.4), then R satisfies*

$$\int_0^\infty (1+t)^{\frac{1+\epsilon}{2}} |R(t) - R_\infty|\, dt < \infty, \tag{8.3.2}$$

and

$$\lim_{t\to\infty} (1+t)^{\frac{1+\epsilon}{2}} |R(t) - R_\infty| = 0, \tag{8.3.3}$$

and R' satisfies

$$\int_0^\infty (1+t)^{\frac{1+\epsilon}{2}} |R'(t)|\, dt < \infty. \tag{8.3.4}$$

We require growth conditions such as these on the solution of the resolvent equation and on its derivative to show that S defined by (8.2.6) satisfies (8.1.2). They are needed to show that an infinite integral containing an integrand involving Brownian motion exists.

Before proving these lemmata we state a version of a result by Shea and Wainger, [38], which is needed to prove Lemma 8.3.3. In this result it is shown that if the kernel of a resolvent integral equation is in a certain weighted space then so is its solution.

Theorem 8.3.1. *Let A be a continuous function which satisfies*

$$\int_0^\infty \rho(t)|A(t)|\, dt < \infty,$$

where $\rho \in C[0,\infty)$ is positive and

$$\rho(t_1+t_2) \leq \rho(t_1)\rho(t_2), \quad 0 \leq t_1, t_2 < \infty. \tag{8.3.5}$$

Define

$$\rho_* = -\lim_{t\to\infty} \frac{\log \rho(t)}{t} \tag{8.3.6}$$

and suppose that

$$1 + \hat{A}(z) \neq 0, \quad \mathrm{Re}\, z \geq \rho_*. \tag{8.3.7}$$

Then the resolvent $r \in C([0,\infty),\mathbb{R})$ satisfying

$$r(t) + (A * r)(t) = A(t), \quad t \geq 0,$$

exists and satisfies

$$\int_0^\infty \rho(t)|r(t)|\,dt < \infty.$$

In [38], Shea and Wainger state and prove the above result for an integro–differential equation and comment that the result also holds for an integral equation. The statement of Theorem 8.3.1 may not be found in [38], however a statement may be found in [16].

Proof of Lemma 8.3.1. We want to prove that

$$\mathbb{E}\left[\sup_{-\infty<t\leq 0}(1+|t|)^{-(1+\epsilon)}\left(R_\infty \Sigma B(t) + \int_{-\infty}^t (R(t-s)-R_\infty)\Sigma\,dB(s)\right)^2\right] < \infty.$$

Defining $Y = R - R_\infty$ as before and applying Lemma 8.3.2 and 8.3.3 we see that

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{-\infty<t\leq 0}(1+|t|)^{-(1+\epsilon)}\left(R_\infty \Sigma B(t) + \int_{-\infty}^t Y(t-s)\Sigma\,dB(s)\right)^2\Big] \\
&\quad= \mathbb{E}\Big[\sup_{-\infty<t\leq 0}(1+|t|)^{-(1+\epsilon)}\left(\Sigma B(t) + \int_{-\infty}^t Y'(t-s)\Sigma B(s)\,ds\right)^2\Big] \\
&\quad\leq 2\Sigma^2\mathbb{E}\Big[\sup_{-\infty<t\leq 0}(1+|t|)^{-(1+\epsilon)}B(t)^2\Big] \\
&\qquad+ 2\Sigma^2\mathbb{E}\Big[\sup_{-\infty<t\leq 0}(1+|t|)^{-(1+\epsilon)}\left(\int_{-\infty}^t Y'(t-s)B(s)\,ds\right)^2\Big]
\end{aligned} \tag{8.3.8}$$

In order to obtain a bound we consider the terms on the last line of (8.3.8). We begin with the first term; using the fact that $\bar{B}(t) = B(-t)$ is a standard Brownian motion we see that

$$\mathbb{E}\Big[\sup_{-\infty<t\le 0}(1+|t|)^{-(1+\epsilon)}B(t)^2\Big] = \mathbb{E}\Big[\sup_{0\le t<\infty}(1+t)^{-(1+\epsilon)}B(t)^2\Big].$$

We now apply Itô's Lemma to $(1+t)^{-(1+\epsilon)}B(t)^2$ to obtain

$$\begin{aligned}
&(1+t)^{-(1+\epsilon)}B(t)^2\\
&= -(1+\epsilon)\int_0^t \frac{B(s)^2}{(1+s)^{2+\epsilon}}\,ds + \int_0^t \frac{1}{(1+s)^{1+\epsilon}}\,ds + 2\int_0^t \frac{B(s)}{(1+s)^{1+\epsilon}}\,dB(s)\\
&= -(1+\epsilon)\int_0^t \frac{B(s)^2}{(1+s)^{2+\epsilon}}\,ds + \frac{1}{\epsilon}\left[1-\frac{1}{(t+1)^\epsilon}\right] + 2\int_0^t \frac{B(s)}{(1+s)^{1+\epsilon}}\,dB(s)\\
&\le \frac{1}{\epsilon} + 2\left|\int_0^t (1+s)^{-(1+\epsilon)}B(s)\,dB(s)\right|.
\end{aligned}$$

Taking the supremum over $[0,T]$, then taking expectations and using the Burkholder–Davis–Gundy inequality one obtains

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le T}(1+t)^{-(1+\epsilon)}B(t)^2\Big]\\
&\le \frac{1}{\epsilon} + 2\mathbb{E}\left[\sup_{0\le t<T}\left|\int_0^t (1+s)^{-(1+\epsilon)}B(s)\,dB(s)\right|\right]\\
&\le \frac{1}{\epsilon} + 2C_0\left(\mathbb{E}\left[\int_0^T (1+s)^{-2(1+\epsilon)}B(s)^2\,ds\right]\right)^{1/2} \qquad (8.3.9)\\
&\le \frac{1}{\epsilon} + 2C_0\left[\int_0^T (1+s)^{-2(1+\epsilon)}s\,ds\right]^{1/2}\\
&\le C(\epsilon),
\end{aligned}$$

where C_0 is a constant independent of T, and $C(\epsilon) > 0$ is also independent of T.

Obviously, $\sup_{0\le t<T}(1+t)^{-(1+\epsilon)}B(t)^2$ is a random variable increasing in T. So we can use the monotone convergence theorem to obtain

$$\begin{aligned}
\mathbb{E}\Big[\sup_{0\le t<\infty}(1+t)^{-(1+\epsilon)}B(t)^2\Big] &= \mathbb{E}\Big[\lim_{T\to\infty}\sup_{0\le t<T}(1+t)^{-(1+\epsilon)}B(t)^2\Big]\\
&= \lim_{T\to\infty}\mathbb{E}\Big[\sup_{0\le t<T}(1+t)^{-(1+\epsilon)}B(t)^2\Big]\\
&\le C(\epsilon).
\end{aligned}$$

We now turn to the second term on the last line of (8.3.8). Using the fact that R' is integrable and the fact that $\mathbb{E}\left[\sup_{-\infty<t\le 0}(1+|t|)^{-(1+\epsilon)}B(t)^2\right]<\infty$ it is clear that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{-\infty<t\le 0}(1+|t|)^{-(1+\epsilon)}\left(\int_{-\infty}^{t}R'(t-s)B(s)\,ds\right)^2\right]\\
&\le \mathbb{E}\left[\sup_{-\infty<t\le 0}(1+|t|)^{-(1+\epsilon)}\int_{-\infty}^{t}|R'(t-s)|\,ds\int_{-\infty}^{t}|R'(t-s)|B(s)^2\,ds\right]\\
&= C\mathbb{E}\left[\sup_{-\infty<t\le 0}(1+|t|)^{-(1+\epsilon)}\int_{-\infty}^{t}|R'(t-s)|B(s)^2\,ds\right]\\
&\le C\mathbb{E}\left[\sup_{-\infty<t\le 0}\sup_{-\infty<s\le t}(1+|s|)^{-(1+\epsilon)}B(s)^2\int_{-\infty}^{t}\left(\frac{1-s}{1-t}\right)^{(1+\epsilon)}|R'(t-s)|\,ds\right]\\
&\le C_1\sup_{-\infty<t\le 0}\int_0^\infty\left(\frac{1-u+t}{1-t}\right)^{(1+\epsilon)}|R'(u)|\,du\\
&\le C_2\sup_{-\infty<t\le 0}\left[\left(\frac{1+t}{1-t}\right)^{(1+\epsilon)}\int_0^\infty|R'(u)|\,du+\int_0^\infty\left(\frac{u}{1-t}\right)^{(1+\epsilon)}|R'(u)|\,du\right]\\
&<\infty.
\end{aligned}$$

So, (8.2.5) satisfies (8.2.2). □

Proof of Theorem 8.2.2. We consider (8.2.6). By Lemma 8.3.2 and Lemma 8.3.3 we can rewrite this as

$$S(t)=\Sigma B(t)+\int_{-\infty}^{t}Y'(t-s)B(s)\Sigma\,ds,\quad t\in\mathbb{R}. \tag{8.3.10}$$

We now aim to prove that

$$\begin{aligned}S(t)=\phi(0)+\int_0^t AS(s)\,ds+\int_0^t\int_0^s K(s-u)S(u)\,du\,ds+\\
\int_0^t\int_{-\infty}^{0}K(s-u)\phi(u)\,du\,ds+\Sigma B(t),\end{aligned} \tag{8.3.11}$$

which establishes that S is the unique continuous adapted solution of (8.1.2).

We examine each term on the right hand side of (8.3.11). Obviously by (8.3.10),

$$\phi(0)=\int_{-\infty}^{0}Y'(-v)B(v)\Sigma\,dv. \tag{8.3.12}$$

Now consider the second term on the right hand side of (8.3.11). Clearly,

$$\int_0^t AS(s)\,ds = \Sigma\int_0^t AB(s)\,ds + \int_0^t\int_0^s AY'(s-v)B(v)\Sigma\,dv\,ds + \int_0^t\int_{-\infty}^0 AY'(s-v)B(v)\Sigma\,dv\,ds. \tag{8.3.13}$$

Now consider each term on the right hand side of (8.3.13). Due to the continuity of R' and B and the compactness of intervals, we can use Fubini's Theorem to exchange the order of the integration in the second term to obtain

$$\begin{aligned}\int_0^t\int_0^s AY'(s-v)B(v)\Sigma\,dv\,ds &= \int_0^t\int_v^t AY'(s-v)B(v)\Sigma\,ds\,dv\\ &= \int_0^t AY(t-v)B(v)\Sigma\,dv - \int_0^t AY(0)B(v)\Sigma\,dv.\end{aligned} \tag{8.3.14}$$

Now consider the third term on the right hand side of (8.3.13). We define the function f_T by

$$f_T(s) = \int_{-T}^0 AY'(s-v)B(v)\Sigma\,dv.$$

Due to the continuity of Y' and B we see that

$$\lim_{T\to\infty} f_T(s) = \int_{-\infty}^0 AY'(s-v)B(v)\Sigma\,dv.$$

Note that, by the Law of the Iterated Logarithm, Lemma 8.3.3 and the fact that $s>0$,

$$\begin{aligned}\sup_{T\geq 0}\left|\int_{-T}^0 Y'(s-v)B(v)\,dv\right| &\leq \int_{-\infty}^0 |Y'(s-v)|\,|B(v)|\,dv\\ &\leq \int_{-\infty}^0 (1+|s-v|)^{\frac{1+\epsilon}{2}}\,|Y'(s-v)|\frac{|B(v)|}{(1+|v|)^{\frac{1+\epsilon}{2}}}\,dv\\ &\leq I(s),\end{aligned}$$

where I is a continuous function integrable over compact intervals. So using the Dominated

Convergence Theorem and then Fubini's Theorem we see that

$$
\begin{aligned}
\int_0^t &\left[\int_{-\infty}^0 AY'(s-v)B(v)\Sigma\,dv\right] ds \\
&= \int_0^t \left[\lim_{T\to\infty}\int_{-T}^0 AY'(s-v)B(v)\Sigma\,dv\right] ds \\
&= \lim_{T\to\infty}\int_0^t \left[\int_{-T}^0 AY'(s-v)B(v)\Sigma\,dv\right] ds \\
&= \lim_{T\to\infty}\int_{-T}^0 \int_0^t AY'(s-v)B(v)\Sigma\,ds\,dv \\
&= \int_{-\infty}^0 A\left[\int_{-v}^{t-v} Y'(w)\,dw\right] B(v)\Sigma\,dv \\
&= \int_{-\infty}^0 AY(t-v)B(v)\Sigma\,dv - \int_{-\infty}^0 AY(-v)B(v)\Sigma\,dv.
\end{aligned}
\tag{8.3.15}
$$

So using the fact that $Y(0) = 1 - R_\infty$, (8.3.13) may be rewritten according to

$$
\begin{aligned}
\int_0^t &AS(s)\,ds \\
&= \int_0^t AB(v)R_\infty\Sigma\,dv + \int_0^t AY(t-v)B(v)\Sigma\,dv \\
&\quad + \int_{-\infty}^0 AY(t-v)B(v)\Sigma\,dv - \int_{-\infty}^0 AY(-v)B(v)\Sigma\,dv, \\
&= \int_0^t AB(v)R_\infty\Sigma\,dv + \int_{-\infty}^t AY(t-v)B(v)\Sigma\,dv - \int_{-\infty}^0 AY(-v)B(v)\Sigma\,dv.
\end{aligned}
\tag{8.3.16}
$$

using (8.3.14) and (8.3.15).

Now consider the third term on the right hand side of (8.3.11). Obviously

$$
\begin{aligned}
\int_0^t &\int_0^s K(s-u)S(u)\,du\,ds \\
&= \int_0^t \int_0^s K(s-u)\Sigma B(u)\,du\,ds \\
&\quad + \int_0^t \int_0^s \int_0^u K(s-u)Y'(u-v)B(v)\Sigma\,dv\,du\,ds \\
&\quad \int_0^t \int_0^s \int_{-\infty}^0 K(s-u)Y'(u-v)B(v)\Sigma\,dv\,du\,ds
\end{aligned}
\tag{8.3.17}
$$

Now consider the first term on the righthand side of (8.3.17). Due to the compactness of intervals and the continuity of K and B we can exchange the order of integration to obtain

$$\int_0^t \int_0^s K(s-u)\Sigma B(u)\,du\,ds = \int_0^t \left[\int_0^{t-u} K(w)\,dw\right] B(u)\Sigma\,du. \tag{8.3.18}$$

Now consider the second term on the right side of (8.3.17). Due the compactness of the intervals and the continuity of the integrand we can exchange the order of integration three times to obtain

$$\begin{aligned}
&\int_0^t \int_0^s \int_0^u K(s-u)Y'(u-v)B(v)\Sigma\,dv\,du\,ds \\
&\quad = \int_0^t \int_0^s \int_v^s K(s-u)Y'(u-v)B(v)\Sigma\,du\,dv\,ds \\
&\quad = \int_0^t \left[\int_v^t \int_v^s K(s-u)Y'(u-v)\,du\,ds\right] B(v)\Sigma\,dv \\
&\quad = \int_0^t \left[\int_v^t \int_u^t K(s-u)Y'(u-v)\,ds\,du\right] B(v)\Sigma\,dv \\
&\quad = \int_0^t \left[\int_v^t \left(\int_0^{t-u} K(w)\,dw\right) Y'(u-v)\,du\right] B(v)\Sigma\,dv.
\end{aligned}$$

Now using integration by parts on the inner integral one obtains

$$\int_v^t \left(\int_0^{t-u} K(w)\,dw\right) Y'(u-v)\,du = -Y(0)\int_0^{t-v} K(w)\,dw + (K*Y)(t-v),$$

so

$$\begin{aligned}
&\int_0^t \int_0^s \int_0^u K(s-u)Y'(u-v)B(v)\Sigma\,dv\,du\,ds \\
&\quad = -\int_0^t Y(0)\left(\int_0^{t-v} K(w)\,dw\right) B(v)\Sigma\,dv + \int_0^t (K*Y)(t-v)B(v)\Sigma\,dv. \quad (8.3.19)
\end{aligned}$$

Now consider the third term on the right hand side of (8.3.17). Observe that

$$\begin{aligned}
&\int_{-\infty}^0 |K(s-u)|\,|Y'(u-v)|\,|B(v)|\,dv \\
&\quad \le \int_{-\infty}^0 |K(s-u)|(1+|u-v|)^{\frac{1+\epsilon}{2}}\,|Y'(u-v)|\frac{|B(v)|}{(1+|v|)^{\frac{1+\epsilon}{2}}}\,dv \le I(s,u),
\end{aligned}$$

where $u \mapsto |I(s,u)|$ is a continuous function, integrable on compact intervals for every $s \ge 0$. So we can use the Dominated Convergence Theorem and Fubini's Theorem to change the order of integration as follows

$$\begin{aligned}
&\int_0^t \int_0^s \int_{-\infty}^0 K(s-u)Y'(u-v)B(v)\Sigma\,dv\,du\,ds \\
&\qquad = \int_0^t \int_0^s \lim_{T\to\infty} \int_{-T}^0 K(s-u)Y'(u-v)B(v)\Sigma\,dv\,du\,ds \\
&\qquad = \int_0^t \lim_{T\to\infty} \int_0^s \int_{-T}^0 K(s-u)Y'(u-v)B(v)\Sigma\,dv\,du\,ds \\
&\qquad = \int_0^t \lim_{T\to\infty} \int_{-T}^0 \int_0^s K(s-u)Y'(u-v)B(v)\Sigma\,du\,dv\,ds.
\end{aligned}$$

Now since

$$\begin{aligned}
&\int_{-\infty}^0 \int_0^s |K(s-u)|\,|Y'(u-v)|\,du\,|B(v)|\,dv \\
&\qquad \le \int_{-\infty}^0 \int_0^s |K(s-u)|(1+|u-v|)^{\frac{1+\epsilon}{2}}|Y'(u-v)|\,du\,\frac{|B(v)|}{(1+|v|)^{\frac{1+\epsilon}{2}}}\,dv \\
&\qquad \le I(s),
\end{aligned}$$

we can obtain

$$\begin{aligned}
&\int_0^t \int_0^s \int_{-\infty}^0 K(s-u)Y'(u-v)B(v)\Sigma\,dv\,du\,ds \\
&\qquad = \int_0^t \lim_{T\to\infty} \int_{-T}^0 \int_0^s K(s-u)Y'(u-v)B(v)\Sigma\,du\,dv\,ds \\
&\qquad = \lim_{T\to\infty} \int_0^t \left[\int_{-T}^0 \int_0^s K(s-u)Y'(u-v)B(v)\Sigma\,du\,dv\right] ds \\
&\qquad = \lim_{T\to\infty} \int_{-T}^0 \left[\int_0^t \left(\int_0^{t-u} K(w)\,dw\right) Y'(u-v)\,du\right] B(v)\Sigma\,dv,
\end{aligned}$$

by changing the order of integration twice. Using integration by parts we see that

$$\begin{aligned}
&\int_0^t \left(\int_0^{t-u} K(w)\,dw\right) Y'(u-v)\,du \\
&\qquad = -\int_0^t K(w)\,dw\,Y(-v) + \int_v^t K(t-u)Y(u-v)\,du + \int_0^v K(t-u)Y(u-v)\,du \\
&\qquad = -\int_0^t K(w)\,dw\,Y(-v) + (K*Y)(t-v) + \int_0^v K(t-u)Y(u-v)\,du.
\end{aligned}$$

So

$$\begin{aligned}
&\int_0^t \int_0^s \int_{-\infty}^0 K(s-u)Y'(u-v)B(v)\Sigma\,dv\,du\,ds \\
&\qquad = -\int_{-\infty}^0 \left(\int_0^t K(w)\,dw\right) Y(-v)B(v)\Sigma\,dv + \int_{-\infty}^0 (K*Y)(t-v)B(v)\Sigma\,dv \\
&\qquad\qquad + \int_{-\infty}^0 \left(\int_0^v K(t-u)Y(u-v)\,du\right) B(v)\Sigma\,dv. \qquad (8.3.20)
\end{aligned}$$

Hence by combining (8.3.18), (8.3.19) and (8.3.20) we see that (8.3.17) becomes

$$
\begin{aligned}
&\int_0^t \int_0^s K(s-u)S(u)\,du\,ds \\
&\quad= \int_0^t \left[\int_0^{t-v} K(w)\,dw\right] B(v)\Sigma\,dv - \int_0^t Y(0)\int_0^{t-v} K(w)\,dw\,B(v)\Sigma\,dv \\
&\qquad + \int_0^t (K*Y)(t-v)B(v)\Sigma\,dv - \int_{-\infty}^0 \left(\int_0^t K(w)\,dw\right) Y(-v)B(v)\Sigma\,dv \\
&\qquad\quad + \int_{-\infty}^0 (K*Y)(t-v)B(v)\Sigma\,dv \\
&\qquad\qquad + \int_{-\infty}^0 \left(\int_0^v K(t-u)Y(u-v)\,du\right) B(v)\Sigma\,dv \\
&\quad= \int_0^t \left[\int_0^{t-v} K(w)\,dw\right] B(v)R_\infty\Sigma\,dv + \int_{-\infty}^t (K*Y)(t-v)B(v)\Sigma\,dv \\
&\qquad - \int_{-\infty}^0 \left(\int_0^t K(w)\,dw\right) Y(-v)B(v)\Sigma\,dv \\
&\qquad\quad + \int_{-\infty}^0 \left(\int_0^v K(t-u)Y(u-v)\,du\right) B(v)\Sigma\,dv
\end{aligned}
\tag{8.3.21}
$$

We may now combine (8.3.16) and (8.3.21) to obtain

$$
\begin{aligned}
&\int_0^t AS(s)\,ds + \int_0^t \int_0^s K(s-u)S(u)\,du\,ds \\
&\quad= \int_0^t AB(v)R_\infty \Sigma\,dv + \int_{-\infty}^t AY(t-v)B(v)\Sigma\,dv \\
&\qquad - \int_{-\infty}^0 AY(-v)B(v)\Sigma\,dv + \int_0^t \left[\int_0^{t-v} K(w)\,dw\right] R_\infty B(v)\Sigma\,dv \\
&\qquad\quad + \int_{-\infty}^t (K*Y)(t-v)B(v)\Sigma\,dv \\
&\qquad\qquad - \int_{-\infty}^0 \left(\int_0^t K(w)\,dw\right) Y(-v)B(v)\Sigma\,dv \\
&\qquad\qquad\quad + \int_{-\infty}^0 \left(\int_0^v K(t-u)Y(u-v)\,du\right) B(v)\Sigma\,dv \\
&\quad= \int_0^t \left[A + \int_0^{t-v} K(w)\,dw\right] R_\infty B(v)\Sigma\,dv \\
&\qquad + \int_{-\infty}^t \left[AY(t-v) + (K*Y)(t-v)\right] B(v)\Sigma\,dv \\
&\qquad\quad - \int_{-\infty}^0 \left[AY(-v) + (K*Y)(-v)\right] B(v)\Sigma\,dv \\
&\qquad\qquad + \int_{-\infty}^0 \left(\int_0^v K(t-u)Y(u-v)\,du\right) B(v)\Sigma\,dv \\
&\qquad\qquad\quad + \int_{-\infty}^0 (K*Y)(-v)B(v)\Sigma\,dv \\
&\qquad\qquad\qquad - \int_{-\infty}^0 \left(\int_0^t K(w)\,dw\right) Y(-v)B(v)\Sigma\,dv.
\end{aligned}
\tag{8.3.22}
$$

We can simplify this equation by considering each of the terms. From Lemma 7.4.2 it is known that $(A + \int_0^\infty K(s)\,ds)R_\infty = 0$. Using this we see that

$$
\int_0^t \left[A + \int_0^{t-v} K(w)\,dw\right] R_\infty B(v)\Sigma\,dv = -\int_0^t K_1(t-v)R_\infty B(v)\Sigma\,dv. \tag{8.3.23}
$$

Now consider the second term on the right hand side of (8.3.22). Using the fact the

$Y = R - R_\infty$ we see that

$$\begin{aligned}
&\int_{-\infty}^{t} \left[AY(t-v) + (K*Y)(t-v)\right] B(v)\Sigma\, dv \\
&\quad = \int_{-\infty}^{t} \left[AR(t-v) + (K*R)(t-v)\right] B(v)\Sigma\, dv \\
&\qquad - \int_{-\infty}^{t} \left[A + \int_{0}^{t-v} K(w)\, dw\right] R_\infty B(v)\Sigma\, dv \\
&\quad = \int_{-\infty}^{t} Y'(t-v)B(v)\Sigma\, dv + \int_{-\infty}^{t} K_1(t-v)R_\infty B(v)\Sigma\, dv.
\end{aligned} \tag{8.3.24}$$

Similarly, the third term on the right hand side of (8.3.22) becomes

$$\begin{aligned}
&\int_{-\infty}^{0} \left[AY(-v) + (K*Y)(-v)\right] B(v)\Sigma\, dv \\
&\qquad = \int_{-\infty}^{0} Y'(-v)B(v)\Sigma\, dv + \int_{-\infty}^{0} K_1(-v)R_\infty B(v)\Sigma\, dv.
\end{aligned} \tag{8.3.25}$$

We now use (8.3.23), (8.3.24) and (8.3.25) to reformulate (8.3.22) as

$$\begin{aligned}
&\int_{0}^{t} AS(s)\, ds + \int_{0}^{t}\int_{0}^{s} K(s-u)S(u)\, du\, ds \\
&\quad = -\int_{0}^{t} K_1(t-v)R_\infty B(v)\Sigma\, dv + \int_{-\infty}^{t} Y'(t-v)B(v)\Sigma\, dv \\
&\qquad + \int_{-\infty}^{t} K_1(t-v)R_\infty B(v)\Sigma\, dv - \int_{-\infty}^{0} Y'(-v)B(v)\Sigma\, dv \\
&\qquad - \int_{-\infty}^{0} K_1(-v)R_\infty B(v)\Sigma\, dv \\
&\qquad + \int_{-\infty}^{0} \left(\int_{0}^{v} K(t-u)Y(u-v)\, du\right) B(v)\Sigma\, dv \\
&\qquad + \int_{-\infty}^{0} (K*Y)(-v)B(v)\Sigma\, dv \\
&\qquad - \int_{-\infty}^{0} \left(\int_{0}^{t} K(w)\, dw\right) Y(-v)B(v)\Sigma\, dv.
\end{aligned} \tag{8.3.26}$$

But the first, third and fifth terms of this equation may be combined to obtain

$$\begin{aligned}
&-\int_{0}^{t} K_1(t-v)R_\infty B(v)\Sigma\, dv + \int_{-\infty}^{t} K_1(t-v)R_\infty B(v)\Sigma\, dv - \int_{-\infty}^{0} K_1(-v)R_\infty B(v)\Sigma\, dv \\
&\qquad = \int_{-\infty}^{0} \left[K_1(t-v) - K_1(-v)\right] R_\infty B(v)\Sigma\, dv \\
&\qquad = -\int_{-\infty}^{0} \left(\int_{-v}^{t-v} K(u)\, du\right) R_\infty B(v)\Sigma\, dv,
\end{aligned}$$

and so (8.3.26) becomes

$$\begin{aligned}
&\int_0^t AS(s)\,ds + \int_0^t \int_0^s K(s-u)S(u)\,du\,ds \\
&\quad = -\int_{-\infty}^0 \left(\int_{-v}^{t-v} K(u)\,du\right) R_\infty B(v)\Sigma\,dv + \int_{-\infty}^t Y'(t-v)B(v)\Sigma\,dv \\
&\qquad - \int_{-\infty}^0 Y'(-v)B(v)\Sigma\,dv + \int_{-\infty}^0 (K*Y)(-v)B(v)\Sigma\,dv \\
&\qquad + \int_{-\infty}^0 \left(\int_0^v K(t-u)Y(u-v)\,du\right) B(v)\Sigma\,dv \\
&\qquad - \int_{-\infty}^0 \left(\int_0^t K(w)\,dw\right) Y(-v)B(v)\Sigma\,dv.
\end{aligned} \tag{8.3.27}$$

We now consider the fourth term on the right hand side of (8.3.11). Since

$$\int_{-\infty}^0 |K(s-u)|\,|B(u)|\,du = \int_{-\infty}^0 (1+|s-u|)^{\frac{1+\epsilon}{2}}|K(s-u)|\frac{|B(u)|}{(1+|u|)^{\frac{1+\epsilon}{2}}}\,du < I(s),$$

where $s \mapsto I(s)$ is continuous and so integrable on compact intervals. So, we obtain

$$\begin{aligned}
&\int_0^t \int_{-\infty}^0 K(s-u)\phi(u)\,du\,ds \\
&\quad = \int_0^t \int_{-\infty}^0 K(s-u)\Sigma B(u)\,du\,ds + \int_0^t \int_{-\infty}^0 \int_{-\infty}^u K(s-u)Y'(u-v)B(v)\Sigma\,dv\,du\,ds \\
&\quad = \int_{-\infty}^0 \left(\int_{-v}^{t-v} K(w)\,dw\right) \Sigma B(v)\,dv + \int_0^t \int_{-\infty}^0 \int_{-\infty}^u K(s-u)Y'(u-v)B(v)\Sigma\,dv\,du\,ds.
\end{aligned} \tag{8.3.28}$$

Now combining (8.3.12), (8.3.27) and (8.3.28) we see that

$$
\begin{aligned}
\phi(0) &+ \int_0^t [AS(s)\,ds + (K*S)(s)]\,ds + \int_0^t \int_{-\infty}^0 K(s-u)\phi(u)\,du\,ds + \Sigma B(t) \\
&= \int_{-\infty}^0 Y'(-v)B(v)\Sigma\,dv - \int_{-\infty}^0 \left(\int_{-v}^{t-v} K(u)\,du\right) R_\infty B(v)\Sigma\,dv \\
&\quad + \int_{-\infty}^t Y'(t-v)B(v)\Sigma\,dv - \int_{-\infty}^0 Y'(-v)B(v)\Sigma\,dv \\
&\quad + \int_{-\infty}^0 (K*Y)(-v)B(v)\Sigma\,dv \\
&\quad - \left(\int_0^t K(w)\,dw\right) \int_{-\infty}^0 Y(-v)B(v)\Sigma\,dv \\
&\quad + \int_{-\infty}^0 \left(\int_0^v K(t-u)Y(u-v)\,du\right) B(v)\Sigma\,dv \\
&\quad + \int_{-\infty}^0 \left(\int_{-v}^{t-v} K(w)\,dw\right) \Sigma B(u)\,ds \\
&\quad + \int_0^t \int_{-\infty}^0 \int_{-\infty}^u K(s-u)Y'(u-v)B(v)\Sigma\,dv\,du\,ds \\
&\quad + \Sigma B(t),
\end{aligned}
$$

which becomes

$$
\begin{aligned}
\phi(0) &+ \int_0^t [AS(s)\,ds + (K*S)(s)]\,ds + \int_0^t \int_{-\infty}^0 K(s-u)\phi(u)\,du\,ds + \Sigma B(t) \\
&= S(t) - \left(\int_0^t K(w)\,dw\right) \int_{-\infty}^0 Y(-v)B(v)\Sigma\,dv \\
&\quad + \int_{-\infty}^0 (K*Y)(-v)B(v)\Sigma\,dv \\
&\quad + \int_{-\infty}^0 \left(\int_{-v}^{t-v} K(w)\,dw\right) Y(0)\Sigma B(u)\,ds \\
&\quad + \int_{-\infty}^0 \left(\int_0^v K(t-u)Y(u-v)\,du\right) B(v)\Sigma\,dv \\
&\quad + \int_0^t \int_{-\infty}^0 \int_{-\infty}^u K(s-u)Y'(u-v)B(v)\Sigma\,dv\,du\,ds,
\end{aligned}
$$

since $Y(0) = 1 - R_\infty$. So we will have completed the proof if

$$
\begin{aligned}
&- \int_0^t \int_{-\infty}^0 \int_{-\infty}^u K(s-u)Y'(u-v)B(v)\Sigma\,dv\,du\,ds \\
&= - \left(\int_0^t K(w)\,dw\right) \int_{-\infty}^0 Y(-v)B(v)\Sigma\,dv + \int_{-\infty}^0 \left(\int_{-v}^{t-v} K(w)\,dw\right) Y(0)\Sigma B(u)\,ds \\
&\quad + \int_{-\infty}^0 (K*Y)(-v)B(v)\Sigma\,dv + \int_{-\infty}^0 \left(\int_0^v K(t-u)Y(u-v)\,du\right) B(v)\Sigma\,dv.
\end{aligned}
$$

Rearranging the right hand side of this equation we obtain

$$\int_{-\infty}^{0}\Big[\Big(\int_{-v}^{t-v} K(w)\,dw\Big)Y(0) - \Big(\int_{0}^{t} K(w)\,dw\Big)Y(-v) + \int_{0}^{v}[K(t-u)-K(-u)]Y(u-v)\,du\Big]B(v)\Sigma\,dv$$
$$= \int_{-\infty}^{0}\Big[\int_{0}^{v}\Big(\int_{-u}^{t-u} K(w)\,dw\Big)Y'(u-v)\,du\Big]\,B(v)\Sigma\,dv$$
$$= \int_{-\infty}^{0}\Big[\int_{0}^{v}\Big(\int_{0}^{t} K(s-u)\,ds\Big)Y'(u-v)\,du\Big]\,B(v)\Sigma\,dv$$
$$= \int_{-\infty}^{0}\Big[\int_{0}^{t}\int_{0}^{v} K(s-u)Y'(u-v)\,du\,ds\Big]\,B(v)\Sigma\,dv,$$

using Fubini's Theorem and integration by parts. Since

$$\int_{-\infty}^{0}\Big[\int_{0}^{v}|K(s-u)||Y'(u-v)|\,du\Big]\,|B(v)|\,|\Sigma|\,dv$$
$$= \int_{-\infty}^{0}\Big[\int_{0}^{v}(1+|v|)^{\frac{1+\epsilon}{2}}|K(s-u)||Y'(u-v)|\,du\Big]\,\frac{|B(v)|}{(1+|v|)^{\frac{1+\epsilon}{2}}}\,|\Sigma|\,dv$$
$$\le C(\omega)\int_{-\infty}^{0}\Big[\int_{0}^{v}(1+|u-v|+|u|)^{\frac{1+\epsilon}{2}}|K(s-u)||Y'(u-v)|\,du\Big]\,dv$$
$$\le C(\omega)\int_{-\infty}^{0}\Big[\int_{0}^{v}(1+|u-v|)^{\frac{1+\epsilon}{2}}|K(s-u)||Y'(u-v)|\,du\Big]\,dv + C(\omega)\int_{-\infty}^{0}\Big[\int_{0}^{v}(1+|u|)^{\frac{1+\epsilon}{2}}|K(s-u)||Y'(u-v)|\,du\Big]\,dv$$
$$\le C_1(\omega)\int_{-\infty}^{0}\Big[\int_{0}^{-v}|K(s-v-w)|(1+|w|)^{\frac{1+\epsilon}{2}}|Y'(w)|\,dw\Big]\,dv + C(\omega)\int_{-\infty}^{0}\Big[\int_{0}^{v}(1+|s-u|)^{\frac{1+\epsilon}{2}}|K(s-u)||Y'(u-v)|\,du\Big]\,dv$$
$$\le C_1(\omega)\int_{-\infty}^{0}\Big[\int_{0}^{-v}|K(s-v-w)|(1+|w|)^{\frac{1+\epsilon}{2}}|Y'(w)|\,dw\Big]\,dv + C_1(\omega)\int_{-\infty}^{0}\Big[\int_{0}^{-v}(1+|s-v-w|)^{\frac{1+\epsilon}{2}}|K(s-v-w)||Y'(w)|\,dw\Big]\,dv$$
$$\le C_1(\omega)\int_{-\infty}^{0}\Big[\int_{0}^{s-v}|K(s-v-w)|(1+|w|)^{\frac{1+\epsilon}{2}}|Y'(w)|\,dw\Big]\,dv + C_1(\omega)\int_{-\infty}^{0}\Big[\int_{0}^{s-v}(1+|s-v-w|)^{\frac{1+\epsilon}{2}}|K(s-v-w)||Y'(w)|\,dw\Big]\,dv$$
$$\le C_1(\omega)\int_{s}^{\infty}\Big[\int_{0}^{u}|K(u-w)|(1+|w|)^{\frac{1+\epsilon}{2}}|Y'(w)|\,dw\Big]\,dv + C_1(\omega)\int_{s}^{\infty}\Big[\int_{0}^{u}(1+|u-w|)^{\frac{1+\epsilon}{2}}|K(u-w)||Y'(w)|\,dw\Big]\,dv < \infty,$$

we can exchange the order of integration to obtain

$$\int_{-\infty}^{0}\int_{0}^{t}\int_{0}^{v} K(s-u)Y'(u-v)\,B(v)\Sigma\,du\,ds\,dv$$
$$= \int_{0}^{t}\int_{-\infty}^{0}\int_{0}^{v} K(s-u)Y'(u-v)\,B(v)\Sigma\,du\,dv\,ds.$$

We now wish to change to order of integration of the inner integrals.

$$\begin{aligned}
&\int_{-\infty}^{0}\int_{0}^{v} K(s-u)Y'(u-v)\,B(v)\Sigma\,du\,dv \\
&\quad= \lim_{T\to\infty} -\int_{-T}^{0}\int_{v}^{0} K(s-u)Y'(u-v)\,B(v)\Sigma\,du\,dv \\
&\quad= -\lim_{T\to\infty}\int_{-T}^{0}\int_{-T}^{u} K(s-u)Y'(u-v)\,B(v)\Sigma\,dv\,du \\
&\quad= -\lim_{T\to\infty}\Big[\int_{-\infty}^{0}\int_{-T}^{u} K(s-u)Y'(u-v)\,B(v)\Sigma\,dv\,du \\
&\qquad - \int_{-\infty}^{-T}\int_{-T}^{u} K(s-u)Y'(u-v)\,B(v)\Sigma\,dv\,du\Big].
\end{aligned}$$

Now

$$\begin{aligned}
&\sup_{T\geq 0}\left|\int_{-T}^{u} K(s-u)Y'(u-v)\,B(v)\,dv\right| \\
&\quad\leq |K(s-u)|\int_{-\infty}^{u}(1+|v|)^{\frac{1+\epsilon}{2}}|Y'(u-v)|\,\frac{|B(v)|}{(1+|v|)^{\frac{1+\epsilon}{2}}}\,dv \\
&\quad\leq C(\omega)|K(s-u)|\int_{-\infty}^{u}(1+|(u-v)-u|)^{\frac{1+\epsilon}{2}}|Y'(u-v)|\,dv \\
&\quad\leq C_1(\omega)|K(s-u)|\int_{-\infty}^{u}(1+|u-v|)^{\frac{1+\epsilon}{2}}|Y'(u-v)|\,dv \\
&\qquad + C_1(\omega)|u|^{\frac{1+\epsilon}{2}}|K(s-u)|\int_{-\infty}^{u}|Y'(u-v)|\,dv \\
&\quad\leq C_1(\omega)|K(s-u)|\int_{0}^{\infty}(1+|w|)^{\frac{1+\epsilon}{2}}|Y'(w)|\,dw \\
&\qquad + C_1(\omega)|s-u|^{\frac{1+\epsilon}{2}}|K(s-u)|\int_{0}^{\infty}|Y'(w)|\,dw \\
&\quad\leq C_3(\omega)\left(1+|s-u|^{\frac{1+\epsilon}{2}}\right)|K(s-u)|.
\end{aligned}$$

Since (8.2.1) holds we have

$$\lim_{T\to\infty}\int_{-\infty}^{-T}\int_{-T}^{u} K(s-u)Y'(u-v)\,B(v)\Sigma\,dv\,du = 0.$$

and

$$\lim_{T\to\infty}\int_{-\infty}^{0}\int_{-T}^{u} K(s-u)Y'(u-v)\,B(v)\Sigma\,dv\,du = \int_{-\infty}^{0}\int_{0}^{v} K(s-u)Y'(u-v)\,B(v)\Sigma\,du\,dv$$

so

$$\int_{-\infty}^{0}\int_{0}^{v} K(s-u)Y'(u-v)\,B(v)\Sigma\,du\,dv = \int_{-\infty}^{0}\int_{-\infty}^{u} K(s-u)Y'(u-v)\,B(v)\Sigma\,dv\,du.$$

This completes our proof. □

8.4 Proof of Theorem 8.2.3 and Theorem 8.2.4

The following theorem highlights the necessity of (8.2.4) for the existence of a stationary solution.

Theorem 8.4.1. *Let K satisfy* (2.2.7) *and let the initial function ϕ satisfy* (8.2.2) *where ϕ is independent of the Brownian motion $\{B(t)\}_{t\geq 0}$. If the increment X_Δ defined by*

$$X_\Delta(t) = X(t) - X(t-\Delta), \quad \Delta > 0, \tag{8.4.1}$$

is stationary, then there exists a constant R_∞ such that the resolvent R of (2.1.1) *satisfies* (8.2.4).

The following lemma is required in the proof of Theorem 8.4.1.

Lemma 8.4.1. *Let K satisfy* (2.2.7). *If the Δ-increment of the solution R of the resolvent equation* (2.1.1) *defined by*

$$R_\Delta(t) = R(t) - R(t-\Delta), \quad \Delta > 0, \tag{8.4.2}$$

satisfies

$$\int_0^\infty R_\Delta(t)^2\,dt < \infty, \tag{8.4.3}$$

then there exists a constant R_∞ such that the resolvent R of (2.1.1) *satisfies* (8.2.4).

Proof of Theorem 8.2.3. We choose ϕ to be the process in (8.2.5). Then the process S in (8.2.6) obeys (8.1.2) by Theorem 8.2.2. Using the fact that $R(t) := 0$ for $t < 0$ we see that

$$\begin{aligned}
S_\Delta(t) &= S(t) - S(t-\Delta) \\
&= R_\infty \Sigma\, B(t) + \int_{-\infty}^{t} (R(t-s) - R_\infty)\Sigma\, dB(s) \\
&\quad - R_\infty \Sigma\, B(t-\Delta) - \int_{-\infty}^{t-\Delta} (R(t-\Delta-s) - R_\infty)\Sigma\, dB(s) \\
&= R_\infty \Sigma\,(B(t) - B(t-\Delta)) + \int_{-\infty}^{t} (R(t-s) - R(t-\Delta-s))\Sigma\, dB(s) \\
&\quad + \int_{t-\Delta}^{t} (R(t-\Delta-s) - R_\infty)\Sigma\, dB(s) \\
&= \int_{-\infty}^{t} R_\Delta(t-s)\Sigma\, dB(s),
\end{aligned}$$

where R_Δ is defined by (8.4.2).

Consider the vector

$$\mathbf{S}_{\boldsymbol{\Delta}}(\tau) = (S_\Delta(t+t_1), \dots, S_\Delta(t+t_n)),$$

where $\tau = (t_1, \dots, t_n)$ and $0 \le t_1 < t_2 \cdots < t_n$. Clearly $\mathbf{S}_{\boldsymbol{\Delta}}(\tau)$ has multivariate normal distribution

$$\mathbf{S}_{\boldsymbol{\Delta}}(\tau) \sim \mathcal{N}(0, C),$$

where the ij^{th} entry of the matrix C is given by

$$\begin{aligned}
C_{ij} &= \mathrm{Cov}[S_\Delta(t+t_i), S_\Delta(t+t_j)] \\
&= \mathbb{E}\left[\int_{-\infty}^{t+t_i} R_\Delta(t+t_i-s)\Sigma\, dB(s) \int_{-\infty}^{t+t_j} R_\Delta(t+t_j-s)\Sigma\, dB(s)\right] \\
&= \int_{-\infty}^{t+t_i \wedge t_j} R_\Delta(t+t_i-s) R_\Delta(t+t_j-s)\Sigma^2\, ds \\
&= \int_{0}^{\infty} R_\Delta(u) R_\Delta(|t_i - t_j| + u)\Sigma^2\, du.
\end{aligned}$$

It is clear that C is independent of t.

The characteristic function of a distribution completely characterises the distribution, so in order to show that the process S_Δ is stationary we calculate the characteristic function of the vector $\mathbf{S}_\Delta(\tau)$:

$$\Phi_{\mathbf{S}_\Delta(\tau)}(\lambda) = \mathbb{E}\left[\exp\{i\lambda \mathbf{S}_\Delta(\tau)^T\}\right]$$

where $\lambda = (\lambda_1, \dots, \lambda_n)$. Since $\mathbf{S}_\Delta(\tau)$ is multivariate normal we see that

$$\Phi_{\mathbf{S}_\Delta(\tau)}(\lambda) = \exp\left\{-\frac{1}{2}\lambda C \lambda^T\right\},$$

which is independent of t. This means that the process is stationary. □

Proof of Theorem 8.4.1. Using variation of parameters the process X which satisfies (8.1.2) may be represented as

$$X(t) = R(t)\phi(0) + \int_0^t R(t-s) \int_{-\infty}^0 K(s-u)\phi(u)\,du\,ds + \int_0^t R(t-s)\Sigma\,dB(s), \quad t \geq 0,$$

so

$$X_\Delta(t) = R_\Delta(t)\phi(0) + \int_0^t R_\Delta(t-s)f(s)\,ds + \int_0^t R_\Delta(t-s)\Sigma\,dB(s), \quad t \geq 0,$$

where

$$f(t) = \int_{-\infty}^0 K(t-u)\phi(u)\,du.$$

Now consider the characteristic function of $X_\Delta(t)$ given by

$$\Phi_{X_\Delta(t)}(\lambda) = \mathbb{E}\left[\exp\{i\lambda X_\Delta(t)\}\right].$$

For any characteristic function $\Phi(0) = 1$ and Φ is continuous in λ, so we can choose $\tilde{\lambda}$ close to zero so that $\Phi_{X_\Delta(t)}(\tilde{\lambda}) = C \neq 0$.

So

$$
\begin{aligned}
&C \\
&= \mathbb{E}\left[\exp\left\{i\tilde{\lambda}\left(R_\Delta(t)\phi(0) + \int_0^t R_\Delta(t-s)f(s)\,ds + \int_0^t R_\Delta(t-s)\Sigma\,dB(s)\right)\right\}\right] \\
&= \mathbb{E}\left[\exp\left\{i\tilde{\lambda}\left(R_\Delta(t)\phi(0) + \int_0^t R_\Delta(t-s)f(s)\,ds\right)\right\}\right]\exp\left\{-\frac{\tilde{\lambda}^2}{2}\int_0^t R_\Delta(s)^2\Sigma^2\,ds\right\}
\end{aligned}
$$

using the independence of the increments of $\{B(t)\}_{t>0}$ from $\{\phi(t)\}_{t\leq 0}$, and the fact that $\int_0^t R_\Delta(t-s)\Sigma\,dB(s)$ is a normally distributed random variable. Multiplying both sides of the equation by $e^{\frac{\tilde{\lambda}^2}{2}\int_0^t R_\Delta(s)^2\Sigma^2\,ds}$, taking absolute values, using the fact that any characteristic function is bounded by 1 and rearranging the equation one obtains

$$
\exp\left\{\frac{\tilde{\lambda}^2}{2}\int_0^t R_\Delta(s)^2\Sigma^2\,ds\right\} \leq \frac{1}{|C|}.
$$

So it follows that $\int_0^\infty R_\Delta(s)^2\,ds < \infty$. We now apply Lemma 8.4.1 to obtain (8.2.4) as required. □

Proof of Theorem 8.2.4. By choosing the initial function ϕ as in (8.2.5) we see that implication (i) implies (ii) is the subject of Theorem 8.2.3. The implication (ii) implies (i) is the subject of Theorem 8.4.1. □

8.5 Proof of Theorem 8.2.5

We use the following two theorems to prove Theorem 8.2.5.

Theorem 8.5.1. *Suppose that K satisfies (2.2.7) and that the initial function ϕ satisfies (8.2.2) where $\{\phi(t)\}_{t\leq 0}$ is independent of the Brownian motion $\{B(t)\}_{t>0}$. Suppose there exists a constant R_∞ such that the resolvent R of (2.1.1) satisfies (8.2.4). Then the increment X_Δ defined by (8.4.1) has a limitimg distribution. Moreover this limiting distribution is the same as the distribution of the increment S_Δ defined in Theorem 8.2.3.*

Theorem 8.5.2. *Let K satisfy* (2.2.7) *and let the initial function ϕ satisfy* (8.2.2) *where $\{\phi(t)\}_{t\leq 0}$ is independent of the Brownian motion $\{B(t)\}_{t>0}$. If the increment X_Δ has a limitimg distribution then there exists a constant R_∞ such that the resolvent R of* (2.1.1) *satisfies* (8.2.4).

Proof of Theorem 8.5.1. We know that

$$X_\Delta(t) = R_\Delta(t)\phi(0) + \int_0^t R_\Delta(t-s)f(s)\,ds + \int_0^t R_\Delta(t-s)\Sigma\,dB(s), \quad t \geq 0.$$

where

$$f(t) = \int_{-\infty}^0 K(t-u)\phi(u)\,du.$$

Now consider the vector

$$\mathbf{X}_\Delta(\tau) = \{X_\Delta(t+t_1), \dots, X_\Delta(t+t_n)\},$$

where $\tau = (t_1, \dots, t_n)$ as before and $0 \leq t_1 < t_2 \cdots < t_n$. With $\lambda = (\lambda_1, \dots, \lambda_n)$, the characteristic function of this vector is given by

$$\begin{aligned}
\Phi_{X_\Delta}(\lambda) &= \mathbb{E}\left[\exp\{i\lambda X_\Delta(\tau)^T\}\right] \\
&= \mathbb{E}\biggl[\exp\biggl\{i\sum_{j=1}^n \lambda_j\Bigl(R_\Delta(t+t_j)\phi(0) + \int_0^{t+t_j} R_\Delta(t+t_j-s)f(s)\,ds \\
&\qquad + \int_0^{t+t_j} R_\Delta(t+t_j-s)\Sigma\,dB(s)\Bigr)\biggr\}\biggr] \\
&= \mathbb{E}\biggl[\exp\biggl\{i\sum_{j=1}^n \lambda_j\Bigl(R_\Delta(t+t_j)\phi(0) + \int_0^{t+t_j} R_\Delta(t+t_j-s)f(s)\,ds\Bigr)\biggr\}\biggr] \times \\
&\qquad \exp\Bigl\{\lambda C(t)\lambda^T\Bigr\},
\end{aligned}$$

where $C(t)$ is an $n \times n$ matrix with ij^{th} entry given by

$$C_{ij}(t) = \int_0^{t+t_i\wedge t_j} R_\Delta(s)R_\Delta(s+|t_i-t_j|)\,ds.$$

Since (8.2.4) and (2.2.7) hold it is known that

$$\lim_{t\to\infty} R_\Delta(t) = 0,$$

also we have that

$$\begin{aligned}
\mathbb{E}\left[f(t)^2\right] &= \mathbb{E}\left[\left(\int_{-\infty}^{0} K(t-u)\phi(u)\,du\right)^2\right] \\
&\leq \tilde{K}\mathbb{E}\left[\int_{-\infty}^{0} (1+|u|)^{1+\epsilon}|K(t-u)|(1+|u|)^{-(1+\epsilon)}\phi(u)^2\,du\right] \\
&\leq \tilde{K}\mathbb{E}\left[\sup_{-\infty<t\leq 0}(1+|t|)^{-(1+\epsilon)}\phi(t)^2\right]\int_{-\infty}^{0}(1+|u|)^{1+\epsilon}|K(t-u)|\,du \\
&\leq \tilde{K}\mathbb{E}\left[\sup_{-\infty<t\leq 0}(1+|t|)^{-(1+\epsilon)}\phi(t)^2\right]\int_{t}^{\infty}(1-t+u)^{1+\epsilon}|K(u)|\,du
\end{aligned}$$

so

$$\begin{aligned}
\mathbb{E}\left[\int_0^\infty f(t)^2\,dt\right] &\leq C\int_0^\infty\int_t^\infty (1-t+u)^{1+\epsilon}|K(u)|\,du\,dt \\
&= C\int_0^\infty\int_1^{1+u} z^{1+\epsilon}|K(u)|\,dz\,du \\
&\leq C_1\int_0^\infty (1+u)^{2+\epsilon}|K(u)|\,du < \infty.
\end{aligned}$$

So $f \in L^2$ almost surely. As the convolution of an L^2 function with an L^2 function tends to zero asymptotically we see that

$$\lim_{t\to\infty}\int_0^t R_\Delta(t-s)f(s)\,ds = 0 \qquad \text{a.s.}$$

Now since

$$\left|\exp\left\{i\sum_{j=1}^{n}\lambda_j\left(R_\Delta(t+t_j)\phi(0)+\int_0^{t+t_j}R_\Delta(t+t_j-s)f(s)\,ds\right)\right\}\right| = 1,$$

we can use the bounded convergence theorem to obtain

$$\begin{aligned}
&\lim_{t\to\infty}\mathbb{E}\left[\exp\left\{i\sum_{j=1}^{n}\lambda_j\left(R_\Delta(t+t_j)\phi(0)+\int_0^{t+t_j}R_\Delta(t+t_j-s)f(s)\,ds\right)\right\}\right] \\
&\quad= \mathbb{E}\left[\lim_{t\to\infty}\exp\left\{i\sum_{j=1}^{n}\lambda_j\left(R_\Delta(t+t_j)\phi(0)+\int_0^{t+t_j}R_\Delta(t+t_j-s)f(s)\,ds\right)\right\}\right] = 1,
\end{aligned}$$

so

$$\begin{aligned}
&\lim_{t\to\infty} \Phi_{X_\Delta}(\lambda) \\
&\quad = \lim_{t\to\infty} \mathbb{E}\Big[\exp\Big\{i\sum_{j=1}^{n}\lambda_j\Big(R_\Delta(t+t_j)\phi(0)+\int_0^{t+t_j} R_\Delta(t+t_j-s)f(s)\,ds\Big)\Big\}\Big]\times \\
&\qquad\qquad \exp\Big\{\lambda C(t)\lambda^T\Big\} \\
&\quad = \exp\Big\{\lambda C(\infty)\lambda^T\Big\},
\end{aligned}$$

where $C(\infty)$ is an $n \times n$ matrix with ij^{th} entry given by

$$C_{ij}(\infty) = \int_0^\infty R_\Delta(s)R_\Delta(s+|t_i-t_j|)\,ds.$$

As this is independent of t we have the desired result. □

Proof of Theorem 8.5.2. We know that convergence in distribution implies

$$\lim_{t\to\infty} \mathbb{E}\left[\exp\{i\lambda X_\Delta(t)\}\right] = \Phi_\Delta(\lambda),$$

where Φ_Δ is a characteristic function. Due to the continuity of Φ_Δ and the fact that $\Phi_\Delta(0) = 1$ we can choose a $\tilde{\lambda}$ such that $\Phi_\Delta(\tilde{\lambda}) \neq 0$. Now as before we have that

$$\begin{aligned}
&\mathbb{E}\left[\exp\{i\tilde{\lambda}X_\Delta(t)\}\right] \\
&= \mathbb{E}\left[\exp\left\{i\tilde{\lambda}\left(R_\Delta(t)\phi(0)+\int_0^t R_\Delta(t-s)\int_{-\infty}^0 K(s-u)\phi(u)\,du\,ds\right)\right\}\right]\times \\
&\qquad \exp\left\{-\frac{\tilde{\lambda}^2}{2}\int_0^t R_\Delta(s)^2\Sigma^2\,ds\right\},
\end{aligned}$$

so

$$\left|\mathbb{E}\left[\exp\{i\tilde{\lambda}X_\Delta(t)\}\right]\right|\exp\left\{\frac{\tilde{\lambda}^2}{2}\int_0^t R_\Delta(s)^2\Sigma^2\,ds\right\} \le 1.$$

Taking limits on both sides we obtain

$$|\Phi_\Delta(\tilde{\lambda})|\exp\left\{\frac{\tilde{\lambda}^2}{2}\int_0^\infty R_\Delta(s)^2\Sigma^2\,ds\right\} \le 1.$$

So it follows that $\int_0^\infty R_\Delta(s)^2\,ds < \infty$. We now apply Lemma 8.4.1 to obtain (8.2.4) as required. □

8.6 Proof of Technical Results

Proof of Theorem 8.2.1. Let $\epsilon > 0$ and T be constant. Before proceeding we define for convenience the norm $\|\cdot\|_T$ on a subspace of continuous $\mathcal{F}^B$-adapted processes by

$$\|X\|_T^2 = \mathbb{E}\left[\sup_{-\infty<t\le T} \frac{X(t)^2}{(1+|t|)^{1+\epsilon}}\right].$$

We now prove the existence of a continuous $\mathcal{F}^B$-adapted process X which satisfies (8.1.2) on the interval $(-\infty, T]$. Define the Picard iterations

$$X_k(t,\omega) = \phi(0) + \int_0^t AX_{k-1}(s,\omega)\,ds + \int_0^t \int_0^s K(s-u)X_{k-1}(u,\omega)\,du\,ds +$$
$$\int_0^t \int_{-\infty}^0 K(s-u)\phi(u,\omega)\,du\,ds + \Sigma B(t,\omega), \quad 0 < t \le T,$$
$$X_k(t,\omega) = \phi(t,\omega) \quad t \le 0,$$

when $k \ge 1$ and

$$X_0(t,\omega) = \phi(0,\omega), \quad 0 < t \le T,$$
$$X_0(t,\omega) = \phi(t,\omega), \quad t \le 0.$$

otherwise. By mathematical induction it can easily be shown that X_k is continuous and adapted on $[0,T]$ for each $k \ge 0$.

Obviously,

$$X_1(t,\omega) - X_0(t,\omega) = \phi(0,\omega)\int_0^t \left[A + \int_0^{t-s} K(u)\,du\right] ds$$
$$+ \int_0^t \int_{-\infty}^0 K(s-u)\phi(u,\omega)\,du\,ds + \Sigma B(t,\omega), \quad 0 < t < T, \tag{8.6.1}$$

and

$$X_1(t,\omega) - X_0(t,\omega) = 0, \quad t \le 0.$$

Once again we consider the interval $[0,T]$ and square both sides of (8.6.1) to obtain

$$\begin{aligned}
(X_1(t,\omega)&-X_0(t,\omega))^2 \\
&\le 3^2\Big[\phi(0,\omega)^2\left(\int_0^t\left[A+\int_0^{t-s}K(u)\,du\right]ds\right)^2 \\
&\quad+\left(\int_0^t\int_{-\infty}^0 K(s-u)\phi(u,\omega)\,du\,ds\right)^2+\Sigma^2B(t,\omega)^2\Big] \\
&\le 3^2\Big[\phi(0,\omega)^2\bar{M}^2t \\
&\quad+\bar{K}T\sup_{-\infty<s\le 0}\frac{\phi(s)^2}{(1+|s|)^{1+\epsilon}}\int_0^t\int_{-\infty}^0(1+|u|)^{1+\epsilon}K(s-u)\,du\,ds \\
&\quad+\Sigma^2(1+|t|)^{1+\epsilon}\frac{B(t,\omega)^2}{(1+|t|)^{1+\epsilon}}\Big] \\
&\le 3^2\Big[\phi(0,\omega)^2\bar{M}^2T+\bar{K}\tilde{K}(T)\sup_{-\infty<t\le 0}\frac{\phi(t)^2}{(1+|t|)^{1+\epsilon}} \\
&\quad+(1+|T|)^{1+\epsilon}\sup_{0<t\le T}\frac{B(t,\omega)^2}{(1+|t|)^{1+\epsilon}}\Big] \\
&:=C(T,\omega).
\end{aligned}$$

where $\bar{M}=|A|+\int_0^\infty|K(t)|\,dt$. Note that

$$\begin{aligned}
&\mathbb{E}\left[C(T)\right]=3^2\bar{M}^2T\mathbb{E}\left[\phi(0)^2\right] \\
&+3^2\bar{K}\tilde{K}(T)\mathbb{E}\left[\sup_{-\infty<t\le 0}\frac{\phi(t)^2}{(1+|t|)^{1+\epsilon}}\right]+3^2\Sigma^2(1+|T|)^{1+\epsilon}\mathbb{E}\left[\sup_{0<t\le T}\frac{B(t)^2}{(1+|t|)^{1+\epsilon}}\right]<\infty.
\end{aligned}$$

It is clear that

$$\begin{aligned}
&X_{k+1}(t,\omega)-X_k(t,\omega) \\
&\qquad=\begin{cases}\int_0^t\left[A+\int_0^{t-s}K(u)\,du\right](X_k(s,\omega)-X_{k-1}(s,\omega))\,ds, & 0\le t<T, \\ 0, \quad t\le 0.\end{cases}
\end{aligned}$$

Consider the case when $t>0$, squaring both sides of the above, one obtains

$$\begin{aligned}
(X_{k+1}(t,\omega)-X_k(t,\omega))^2&=\left(\int_0^t\left[A+\int_0^{t-s}K(u)\,du\right](X_k(s,\omega)-X_{k-1}(s,\omega))\,ds\right)^2 \\
&\le T\bar{M}^2\int_0^t(X_k(s,\omega)-X_{k-1}(s,\omega))^2\,ds.
\end{aligned}$$

We now prove by mathematical induction that

$$(X_{k+1}(t,\omega) - X_k(t,\omega))^2 \leq \frac{1}{k!}C(T,\omega)[T\bar{M}^2 t]^k. \tag{8.6.2}$$

Obviously (8.6.2) holds when $k = 0$. When $k = 1$ we obtain

$$\begin{aligned}(X_2(t,\omega) - X_1(t,\omega))^2 &\leq T\bar{M}^2 \int_0^t (X_1(s,\omega) - X_0(s,\omega))^2\, ds \\ &= C(T,\omega)T\bar{M}^2 t.\end{aligned}$$

When $k = 2$ we obtain

$$\begin{aligned}(X_3(t,\omega) - X_2(t,\omega))^2 &\leq T\bar{M}^2 \int_0^t (X_2(s,\omega) - X_1(s,\omega))^2\, ds \\ &= \frac{1}{2}C(T,\omega)T^2\bar{M}^4 t^2.\end{aligned}$$

So we assume that (8.6.2) holds for $k = l$, that is

$$(X_{l+1}(t,\omega) - X_l(t,\omega))^2 \leq \frac{1}{l!}C(T,\omega)[T\bar{M}^2 t]^l.$$

and prove that it holds for $k = l + 1$:

$$\begin{aligned}(X_{l+2}(t,\omega) - X_{l+1}(t,\omega))^2 &\leq T\bar{M}^2 \int_0^t (X_{l+1}(s,\omega) - X_l(s,\omega))^2\, ds \\ &\leq T\bar{M}^2 \int_0^t \frac{1}{l!}C(T,\omega)[T\bar{M}^2 s]^l\, ds \\ &= \frac{1}{(l+1)!}C(T,\omega)[T\bar{M}^2 t]^{l+1},\end{aligned}$$

as required.

Now

$$\begin{aligned}\|X_{k+1}(t,\omega) - X_k(t,\omega)\|_T^2 &= \mathbb{E}\left[\sup_{-\infty<t\leq T} \frac{(X_{k+1}(t,\omega) - X_k(t,\omega))^2}{(1+|t|)^{1+\epsilon}}\right] \\ &\leq \mathbb{E}\left[\sup_{0<t\leq T} \frac{\frac{1}{k!}C(T,\omega)[T\bar{M}^2 t]^k}{(1+|t|)^{1+\epsilon}}\right] \\ &\leq \frac{1}{k!}[T^2\bar{M}^2]^k \mathbb{E}\left[C(T,\omega)\right] \\ &< C_1(T)\frac{1}{k!}[T^2\bar{M}^2]^k.\end{aligned} \tag{8.6.3}$$

We can now use Chebyshev's inequality and (8.6.3) to obtain

$$\mathbb{P}\left[\sup_{0\leq t\leq T}|X_{k+1}(t,\omega)-X_k(t,\omega)|>\frac{1}{2^{k+1}}\right]\leq 4C_1(T)\frac{1}{k!}[4T^2\bar{M}^2]^k.$$

This upper bound is the general term in a convergent sequence so we may apply the Borel–Cantelli Lemma to conclude that for each $\omega\in\Omega^*$ there exists $N(\omega)\in\mathbb{N}$ such that

$$\sup_{0\leq t\leq T}|X_{k+1}(t,\omega)-X_k(t,\omega)|\leq\frac{1}{2^{k+1}},\quad k\geq N(\omega),$$

with $\mathbb{P}[\Omega^*]=1$. Consequently the sequence $\{X_k(t,\omega)\}_{k=0}^{\infty}$ converges in the supremum topology to a continuous limit which we call $X(t,\omega)$ on $0\leq t\leq T$.

We now show that X is a solution of (8.1.2) for $t\in[0,T]$. Consider the expression

$$\begin{aligned}X(t,\omega)-\phi(0,\omega)-\int_0^t\left[A+\int_0^{t-s}K(u)\,du\right]X(s,\omega)\,ds\\-\int_0^t\int_{-\infty}^0 K(s-u)\phi(u,\omega)\,du\,ds-\Sigma B(t,\omega).\end{aligned}$$

Using the definition of the Picard iteration we see that

$$\begin{aligned}&X(t,\omega)-\phi(0,\omega)-\int_0^t\left[A+\int_0^{t-s}K(u)\,du\right]X(s,\omega)\,ds-\\&\qquad\int_0^t\int_{-\infty}^0 K(s-u)\phi(u,\omega)\,du\,ds-\Sigma B(t,\omega)\\&=(X(t,\omega)-X_k(t,\omega))-\int_0^t\left[A+\int_0^{t-s}K(u)\,du\right](X(s,\omega)-X_{k-1}(s,\omega))\,ds\quad 0\leq t\leq T.\end{aligned}$$

Taking absolute values across both sides

$$\begin{aligned}&\Bigg|X(t,\omega)-\phi(0,\omega)-\int_0^t\left[A+\int_0^{t-s}K(u)\,du\right]X(s,\omega)\,ds\\&\qquad-\int_0^t\int_{-\infty}^0 K(s-u)\phi(u,\omega)\,du\,ds-\Sigma B(t,\omega)\Bigg|\\&\qquad\leq|X(t,\omega)-X_k(t,\omega)|+\bar{M}\int_0^t|X(s,\omega)-X_{k-1}(s,\omega)|\,ds,\quad 0\leq t\leq T.\end{aligned}$$

Hence,

$$\sup_{0\le t\le T}\left|X(t,\omega)-\phi(0,\omega)-\int_0^t\left[A+\int_0^{t-s}K(u)\,du\right]X(s,\omega)\,ds\right.$$
$$\left.-\int_0^t\int_{-\infty}^0 K(s-u)\phi(u,\omega)\,du\,ds-\Sigma B(t,\omega)\right|$$
$$\le(1+\bar{M}T)\sup_{0\le t\le T}|X(t,\omega)-X_k(t,\omega)|.$$

Now taking the limit as $k\to\infty$ we see that and using the fact that $X_k\to X$ as $k\to\infty$ we see that

$$\sup_{0\le t\le T}\left|X(t,\omega)-\phi(0,\omega)-\int_0^t\left[A+\int_0^{t-s}K(u)\,du\right]X(s,\omega)\,ds\right.$$
$$\left.-\int_0^t\int_{-\infty}^0 K(s-u)\phi(u,\omega)\,du\,ds-\Sigma B(t,\omega)\right|=0.$$

that is, X is a solution of (8.1.2) for $0\le t\le T$.

The following argument is provided to show that $t\mapsto X(t,\Sigma)$ is the unique continuous adapted process satisfying (8.1.2) on $0\le t\le T$. Suppose both X and Y are continuous adapted processes satisfying (8.1.2). Subtracting Y from X and taking absolute values across the equation we obtain

$$|X(t)-Y(t)|\le\bar{M}\int_0^t|X(s)-Y(s)|\,ds,$$

where $\bar{M}=|A|+\int_0^\infty|K(t)|\,dt$. By construction $X(t)=Y(t)$ for $t\le 0$ and using Gronwall's inequality we see that $|X(t)-Y(t)|=0$ for all $t\in[0,T]$, in other words $t\mapsto X(t,\Sigma)$ is unique on $-\infty\le t\le T$. However, since T is arbitrary we see that X is the unique, continuous, adapted process which satisfies (8.1.2) on the whole real line. □

Proof of Lemma 8.3.2. For $T < 0$, we obtain

$$\begin{aligned}
\int_T^t \eta'(t-s)B(s)\,ds &= \int_T^t \eta'(t-s)\left(\int_T^t dB(u)\right) ds + \int_T^t \eta'(t-s)B(T)\,ds \\
&= \int_T^t \int_T^t \eta'(t-s)\,dB(u)\,ds + B(T)\int_0^{t-T} \eta'(u)\,ds \\
&= \int_T^t \int_u^t \eta'(t-s)\,ds\,dB(u) + B(T)\eta(t-T) - B(T)\eta(0) \\
&= \int_T^t \left(\int_0^{t-u} \eta'(z)\,dz\right) dB(u) + B(T)\eta(t-T) - B(T)\eta(0) \\
&= \int_T^t \eta(t-u)\,dB(u) - \eta(0)\int_T^t dB(u) + B(T)\eta(t-T) - B(T)\eta(0) \\
&= \int_T^t \eta(t-u)\,dB(u) - \eta(0)B(t) + B(T)\eta(t-T)
\end{aligned}$$

which on rearrangement becomes

$$\int_T^t \eta(t-u)\,dB(u) = \eta(0)B(t) - B(T)\eta(t-T) + \int_T^t \eta'(t-s)B(s)\,ds. \tag{8.6.4}$$

Since $\eta \in L^2$, the limit as $T \to -\infty$ on the left hand side of (8.6.4) exists almost surely. The limit as $T \to -\infty$ of the last term on the right hand side of (8.6.4) exists since

$$\begin{aligned}
\int_{-\infty}^t |\eta'(t-s)|\,|B(s)|\,ds &= \int_{-\infty}^t |\eta'(t-s)|(1+|s|)^{\frac{1+\epsilon}{2}} \frac{|B(s)|}{(1+|s|)^{\frac{1+\epsilon}{2}}}\,ds \\
&\le C(\omega)\int_{-\infty}^t |\eta'(t-s)|(1+|t-s|)^{\frac{1+\epsilon}{2}}\,ds < \infty.
\end{aligned} \tag{8.6.5}$$

So taking the limit as $T \to -\infty$ on both sides of (8.6.4) we obtain

$$\int_{-\infty}^t \eta(t-u)\,dB(u) = \eta(0)B(t) + \int_{-\infty}^t \eta'(t-s)B(s)\,ds,$$

since the limit as $T \to -\infty$ of $B(T)\eta(t-T)$ exists (since the limit of each of the other terms in (8.6.4) exist) and equals zero due to

$$\limsup_{T\to-\infty} |B(T)|\,|\eta(t-T)| \le \limsup_{T\to-\infty} \frac{|B(T)|}{(1+|T|)^{(1+\epsilon)}}(1+|t-T|)^{(1+\epsilon)}|\eta(t-T)| = 0.$$

□

Proof of Lemma 8.3.3. The integro–differential equation (2.1.1) may be reformulated as

$$Y(t) + (K_1 * Y)(t) = K_2(t), \quad t \ge 0, \tag{8.6.6}$$

where $Y = R - R_\infty$ and

$$K_2(t) = \int_t^\infty K_1(s)\,ds.$$

If r is the solution of

$$r(t) + (K_1 * r)(t) = K_1(t), \quad t \geq 0,$$

then the solution Y of (8.6.6) may be expressed as

$$Y(t) = K_2(t) - \int_0^t r(t-s)K_2(s)\,ds, \quad t \geq 0.$$

Thus,

$$\begin{aligned}(1+t)^{\frac{1+\epsilon}{2}}|Y(t)| \leq (1+t)^{\frac{1+\epsilon}{2}}|K_2(t)| &+ c\int_0^t (1+t-s)^{\frac{1+\epsilon}{2}}|r(t-s)|\,|K_2(s)|\,ds \\ &+ c\int_0^t s^{\frac{1+\epsilon}{2}}|r(t-s)|\,|K_2(s)|\,ds, \quad (8.6.7)\end{aligned}$$

where c is a positive constant.

We begin by considering (8.3.2). Integrating (8.6.7) over $[0,\infty)$ one obtains

$$\begin{aligned}&\int_0^\infty (1+t)^{\frac{1+\epsilon}{2}}|Y(t)|\,dt \\ &\quad\leq \int_0^\infty (1+t)^{\frac{1+\epsilon}{2}}|K_2(t)|\,dt + c\int_0^\infty\int_0^t (1+t-s)^{\frac{1+\epsilon}{2}}|r(t-s)|\,|K_2(s)|\,ds\,dt \\ &\qquad + c\int_0^\infty\int_0^t s^{\frac{1+\epsilon}{2}}|r(t-s)|\,|K_2(s)|\,ds\,dt. \quad (8.6.8)\end{aligned}$$

Consider the first term on the right hand side of (8.6.8). Clearly,

$$\begin{aligned}\int_0^\infty (1+t)^{\frac{1+\epsilon}{2}}|K_2(t)|\,dt &\leq \int_0^\infty\int_t^\infty (1+t)^{\frac{1+\epsilon}{2}}|K_1(s)|\,ds\,dt \\ &\leq C\int_0^\infty (1+s)^{\frac{3+\epsilon}{2}}|K_1(s)|\,ds \qquad (8.6.9) \\ &\leq C_1\int_0^\infty (1+u)^{\frac{5+\epsilon}{2}}|K(u)|\,du < \infty.\end{aligned}$$

Now consider the second term and third term in the right hand side of (8.6.8). Let $\rho(t) = (1+t)^{\frac{1+\epsilon}{2}}$. It can easily be shown that (8.3.5) holds and that (8.3.6) holds for

$\rho_* = 0$. Also since (8.2.4) holds and the second moment of the kernel exists the scalar version of Theorem 2.2.4 may be applied to obtain condition (8.3.7) with K_1 in the role of A. So, we can apply Theorem 8.3.1 to obtain

$$\int_0^\infty (1+t)^{\frac{1+\epsilon}{2}} |r(t)|\, dt < \infty. \tag{8.6.10}$$

Using this and (8.6.9) we see that the second and third terms in (8.6.8) are finite so (8.3.2) holds.

We now consider (8.3.3). Taking the limit as $t \to \infty$ on both sides of (8.6.7) we obtain

$$\begin{aligned}\limsup_{t\to\infty}(1+t)^{\frac{1+\epsilon}{2}}|Y(t)| &\leq \limsup_{t\to\infty}(1+t)^{\frac{1+\epsilon}{2}}|K_2(t)| \\ &+ \limsup_{t\to\infty} c\int_0^t (1+t-s)^{\frac{1+\epsilon}{2}}|r(t-s)|\,|K_2(s)|\,ds + \limsup_{t\to\infty} c\int_0^t s^{\frac{1+\epsilon}{2}}|r(t-s)|\,|K_2(s)|\,ds.\end{aligned} \tag{8.6.11}$$

The first term on the right hand side of (8.6.11) is zero since

$$(1+t)^{\frac{1+\epsilon}{2}}|K_2(t)| \leq C\int_t^\infty s^{\frac{5+\epsilon}{2}}|K(s)|\,ds \tag{8.6.12}$$

Now consider the second term. Since (8.6.10) holds and $K_2 \to 0$ as $t \to \infty$ we see that second term tends to zero since the convolution of an L^1 term with one which tends to zero is tending to zero. Finally we consider the third term. Since r is integrable and K_2 tends to zero due to (8.6.12) we see that the third term on the right hand side of (8.6.11) tends to zero. Combining the above arguments we see that

$$\lim_{t\to\infty}(1+t)^{\frac{1+\epsilon}{2}}|Y(t)| = 0,$$

as required.

Now consider (8.3.4). Using equation (2.1.1) and the fact that $(A+\int_0^\infty K(s)\,ds)R_\infty = 0$ we see that

$$Y'(t) = AY(t) + (K*Y)(t) - K_1(t)R_\infty.$$

So,

$$\int_0^\infty (1+t)^{\frac{(1+\epsilon)}{2}} |Y'(t)|\,dt$$
$$\leq \int_0^\infty (1+t)^{\frac{(1+\epsilon)}{2}} |A||Y(t)|\,dt + \int_0^\infty (1+t)^{\frac{(1+\epsilon)}{2}} (|K| * |Y|)(t)\,dt$$
$$+ \int_0^\infty (1+t)^{\frac{(1+\epsilon)}{2}} |K_1(t)|\,dt\,|R_\infty|. \quad (8.6.13)$$

The first term on the right hand side of (8.6.13) is finite due to (8.3.2). Consider the second term:

$$\int_0^\infty (1+t)^{\frac{(1+\epsilon)}{2}} (|K| * |Y|)(t)\,dt$$
$$\leq c\int_0^\infty \int_0^t (1+t-s)^{\frac{(1+\epsilon)}{2}} |K(t-s)||Y(s)|\,ds\,dt$$
$$+ c\int_0^\infty \int_0^t |K(t-s)| s^{\frac{(1+\epsilon)}{2}} |Y(s)|\,ds\,dt < \infty,$$

due to (8.2.3) and (8.3.2). We may show that the third term is integrable in a manner similar to (8.6.9). So (8.3.4) holds. □

Proof of Lemma 8.4.1. In order to prove this result we must prove that (i)

$$z - A - \hat{K}(z) \neq 0, \quad \mathrm{Re}\, z \geq 0,$$

and apply a result of Grossman and Miller, Theorem 2.2.2, to obtain that the resolvent R satisfying (2.1.1) is integrable, from which it may easily be shown that (8.2.4) holds with $R_\infty = 0$ or (ii) show that

$$1 + \hat{K}_1(z) \neq 0, \quad \mathrm{Re}\, z \geq 0. \quad (8.6.14)$$

and apply a result of Paley and Weiner, Theorem 2.2.3, from which it may easily be shown that (8.2.4) holds.

We begin by considering the case when $z \neq 0$ and prove

$$z - A - \hat{K}(z) \neq 0, \quad \mathrm{Re}\, z \geq 0, z \neq 0. \quad (8.6.15)$$

We begin by assuming the opposite: that is there exists a non zero constant $\tilde{z} = \tilde{z}_R + i\tilde{z}_I$ with $\tilde{z}_R \geq 0$ such that

$$\tilde{z} - A - \hat{K}(\tilde{z}) = 0, \quad \operatorname{Re}\tilde{z} \geq 0, \tilde{z} \neq 0.$$

Consequently $y_0 e^{\tilde{z}t}$ is a solution of

$$y'(t) = Ay(t) + \int_0^\infty K(s)y(t-s)\,ds, \quad t > 0, \tag{8.6.16}$$

$$y(0) = y_0 \neq 0, \tag{8.6.17}$$

so using variation of parameters

$$e^{\tilde{z}t} = R(t) + \int_0^t R(t-s)\int_s^\infty K(u)e^{\tilde{z}(s-u)}\,du\,ds,$$

since $y_0 \neq 0$ and

$$e^{\tilde{z}(t-\Delta)} = R(t-\Delta) + \int_0^{t-\Delta} R(t-\Delta-s)\int_s^\infty K(u)e^{\tilde{z}(s-u)}\,du\,ds.$$

So

$$ce^{\tilde{z}t} = R_\Delta(t) + \int_0^t R_\Delta(t-s)\int_s^\infty K(u)e^{\tilde{z}(s-u)}\,du\,ds, \tag{8.6.18}$$

where $c = \left(1 - e^{-\tilde{z}\Delta}\right)$. We consider the cases when $\operatorname{Re}\tilde{z} > 0$ and $\operatorname{Re}\tilde{z} = 0$ separately.

We begin by considering $\operatorname{Re}\tilde{z} > 0$. Examine the right hand side of (8.6.18). The first term on the right hand side is square integrable due to assumption (8.4.3). Since

$$\begin{aligned}\int_0^\infty \left|\int_s^\infty K(u)e^{\tilde{z}(s-u)}\,du\right| ds &\leq \int_0^\infty \int_s^\infty |K(u)|e^{\tilde{z}_R(s-u)}\,du\,ds \\ &< c_1 \int_0^\infty |K(u)|e^{-\tilde{z}_R u}\,du < \infty,\end{aligned}$$

we see that the second term on the right hand side of (8.6.18) is square integrable as the convolution of a square integrable function with an integrable function is square integrable. So, the right hand side of (8.6.18) is square integrable while the left hand side is unbounded as t tends to ∞. This produces a contradiction and so $z - A - \hat{K}(z) \neq 0$ for $\operatorname{Re}\tilde{z} > 0$.

Now consider the case when $\operatorname{Re}\tilde{z} = 0$. Equation (8.6.18) becomes

$$c(\cos(\tilde{z}_I t) + i\sin(\tilde{z}_I t))$$
$$= R_\Delta(t) + \int_0^t R_\Delta(t-s)\int_s^\infty K(u)(\cos(\tilde{z}_I(s-u)) + i\sin(\tilde{z}_I(s-u)))\,du\,ds,$$

The left hand side of this equation varies sinusoidally while the left hand side in L^2. This produces a contradiction and so $z - A - \hat{K}(z) \neq 0$ for $\operatorname{Re}\tilde{z} = 0$.

So combining the above arguments we see that (8.6.15) holds. Now if $A + \int_0^\infty K(t)\,dt \neq 0$ we may apply Theorem 2.2.2 to obtain (8.2.4) with $R_\infty = 0$.

If $A + \int_0^\infty K(t)\,dt = 0$, we must prove that (8.4.2) and (2.2.7) imply that

$$1 + \hat{K}_1(z) \neq 0, \quad \operatorname{Re} z \geq 0.$$

We begin by considering the case when $z = 0$. We assume that

$$1 + \int_0^\infty sK(s)\,ds = 0,$$

and prove by contradiction that this cannot hold. If this holds, we can choose any nonzero constant c so that $ct + y_0$ is a solution of (8.6.16). So by using variation of parameters it is seen that

$$ct + y_0 = R(t)y_0 + \int_0^t R(t-s)\int_s^\infty K(u)[c(s-u) + y_0]\,du\,ds,$$

and

$$c[t-\Delta] + y_0 = R(t-\Delta)y_0 + \int_0^{t-\Delta} R(t-\Delta-s)\int_s^\infty K(u)[c(s-u) + y_0]\,du\,ds.$$

Subtracting these two equations one obtains

$$c\Delta = R_\Delta(t)y_0 + \int_0^t R_\Delta(t-s)f(s)\,ds, \tag{8.6.19}$$

where f is given by

$$f(t) = \int_t^\infty K(u)[c(t-u) + y_0]\,du.$$

We see that f is integrable since

$$\int_0^\infty \int_s^\infty |K(u)|[c(s+u)+y_0]\,du \le C\int_0^\infty \int_s^\infty u|K(u)|\,du$$
$$\le C_1 \int_0^\infty u^2|K(u)|\,du < \infty.$$

Moreover $f(t) \to 0$ as $t \to \infty$ so f is square integrable. Consider equation (8.6.19). The first term on the right hand side is square integrable due to our assumptions. The second term is also square integrable since a square integrable function convolved with a square integrable function is square integrable. So the right hand side of (8.6.19) is square integrable while the left hand side is a non zero constant. These facts are incompatible which implies that $1+\hat{K_1}(0) \neq 0$.

Now consider the case when $z \neq 0$. It is easy to show that

$$1+\hat{K_1}(z) = \frac{1}{z}(z - A - \hat{K}(z)), \quad \operatorname{Re} z \geq 0,$$

As we have shown that (8.6.15) holds we see that

$$1+\hat{K_1}(z) \neq 0, \quad \operatorname{Re} z \geq 0, z \neq 0,$$

combining this with the fact that $1+\hat{K_1}(0) \neq 0$ we obtain (8.6.14). So using Theorem 2.2.3 we see that the solution r of $r + r * K_1 = K_1$ is integrable. Now using the representation of the solution of (8.6.6) given in Theorem 1.1.2:

$$Y(t) = K_2(t) - \int_0^t r(t-s)K_2(s)\,ds, \quad t \geq 0,$$

and using the fact that K_2 and r are integrable we see that (8.2.4) holds. □

Chapter 9

Conclusion

We end by summarising the main results of this thesis and highlighting the links and relationships between them. Further to this some avenues for the extension and application of this thesis are discussed.

The paper by Krisztin and Terjéki [24] lies at the foundation of this thesis. Here the most fundamental equation considered in the preceding chapters, namely the resolvent equation, is analysed. The conditions which completely characterise asymptotic convergence of the resolvent to a nontrivial and nonequilibrium limit are found.

Building on this foundation, the analysis in this thesis initially consists of three separate strands before becoming interlinked as more complex and involved questions are considered.

The first strand consists of the analysis of deterministic Volterra equations. Under the condition that the sign of the kernel does not change sign on $[0, \infty)$ and that the second moment of the kernel exists, we can use methods applied in [30, 31] to find that the exponential integrability of the kernel is necessary and sufficient for exponential convergence of the resolvent to a nontrivial limit.

The above result is then used as a building block to develop this strand. A perturbed Volterra equation is considered and it is found that the exponential decay of the tail of the perturbation is the only additional assumption required for exponential convergence to a nontrivial limit. This condition is in fact necessary.

The second strand considers a stochastically perturbed Volterra equation with asymptotically fading noise. As the study of stochastic Volterra equations is newer than deterministic Volterra equations, this strand begins at a less advanced stage: initially a stochastic analogue of Krisztin and Terjéki result is found. Under the condition that the first moment of the kernel exists, necessary and sufficient conditions for the convergence of the solution to a nontrivial random variable and for the integrability of the solution minus its limit in the mean square case is found. These conditions are the square integrability of the resolvent minus its limit and the existence of the tail of the noise term. In fact the same conditions are sufficient in the almost sure case but the proof of the necessity of the condition on the noise term has so far been elusive.

This strand is developed in a way analogous to the deterministic case: necessary and sufficient conditions for exponential convergence to a nontrivial random variable are found—the kernel must be exponentially integrable and the noise term exponentially square integrable. Obvious comparisons can be drawn with the corresponding result in the deterministic case. Indeed the proof of the stochastic result draws much of its inspiration from the deterministic case.

The final strand considers a stochastic Volterra equation with constant noise. Once again the resolvent minus its limit being square integrable manifests itself as the necessary and sufficient condition for convergence, in this case to a stationary distribution.

This strand has the most scope for further development. Perhaps more interesting is the applicability of the equation as a model of a financial market.

The solution of this equation is a continuous–time stochastic process which, like Brownian motion, has Gaussian and stationary increments, but in contrast to Brownian motion, has correlated increments. In a financial model, this would mean that past returns of asset prices can influence current returns. It also enjoys a semimartingale property which enables it to be used in the pricing of derivative securities in inefficient financial markets, which is not shared by existing long–memory processes. The process also has potential application in time series modelling, as it is a continuous–time and long memory analogue of popular discrete–time and short memory econometric models.

Proofs

Proof of Proposition 2.4.1. We begin by defining the norm:

$$\|A\|_N = \sum_{i=1}^{n}\sum_{j=1}^{d} |a_{ij}|\,, \quad A \in M_{n\times d}(\mathbb{R}).$$

This proof follows by a sequence of contradictions. We begin by showing that the first moment of the kernel is integrable, that is

$$\int_0^\infty t\|K(t)\|_N\,dt < \infty. \tag{A.0.1}$$

This is not the case if there exists a constant $T > 1$ such that

$$\int_0^T \|K(t)\|_N(t-1)\,dt > \bar{B}, \tag{A.0.2}$$

where $\bar{B} := \|B'(0)\|_N$.

We begin by assuming (A.0.2). By choosing $0 < h < \delta_T := \min(1,\ e^{-T})$,

$$\begin{aligned}
\sup_{0\le t\le T}\left|\frac{1-e^{-ht}}{h} - t\right| &= \sup_{0\le t\le T}\left|\frac{1-(1-hat+\frac{(hat)^2}{2!}-\ldots)}{h} - t\right| \\
&= \sup_{0\le t\le T}\left|\frac{ht^2}{2!} - \frac{h^2t^3}{3!} + \ldots\right| \\
&\le \sup_{0\le t\le T}|h|\left|\frac{t^2}{2!} - \frac{ht^3}{3!} + \ldots\right| \\
&\le |h|\left|\frac{T^2}{2!} + \frac{T^3}{3!} + \ldots\right| \\
&\le |h|\,e^T \\
&\le 1, \quad t \in [0,T].
\end{aligned}$$

Manipulating this inequality one obtains:

$$\frac{1-e^{-ht}}{h} \ge t - 1.$$

Consequently,

$$\begin{aligned}
\left\|\frac{B(h)-B(0)}{h}\right\|_N &= \left\|\frac{\hat{K}(h)-\hat{K}(0)}{h}\right\|_N \\
&= \left\|\int_0^\infty K(t)\frac{1-e^{-ht}}{h}\,dt\right\|_N \\
&= \sum_{i=1}^n\sum_{j=1}^n \left|\int_0^\infty K_{ij}(t)\frac{1-e^{-ht}}{h}\,dt\right| \\
&= \sum_{i=1}^n\sum_{j=1}^n \int_0^\infty |K_{ij}(t)|\,\frac{1-e^{-ht}}{h}\,dt,
\end{aligned}$$

using the fact that $K_{ij}(t)$ has the same sign on $[0,\infty)$ at the last step. So for $0<h<\delta_T$ we obtain

$$\bar{B} = \|B'(0)\|_N = \lim_{h\to 0^+}\left\|\frac{\hat{K}(h)-\hat{K}(0)}{h}\right\|_N \geq \int_0^T \|K(t)\|_N(t-1)\,dt > \bar{B},$$

a contradiction. Hence (A.0.1) holds.

We now proceed to show that $B'(z) := -\int_0^\infty te^{-zt}K(z)\,dt$ for $z\geq 0$. Using l'Hôpital's rule,

$$\begin{aligned}
B_{ij}'(z) &= \lim_{h\to 0^+}\frac{B_{ij}(z+h)-B_{ij}(z)}{h} \\
&= \lim_{h\to 0^+}\frac{\hat{K}_{ij}(z+h)-\hat{K}_{ij}(z)}{h} \\
&= \int_0^\infty \lim_{h\to 0^+}\frac{e^{-ht}-1}{h}e^{-zt}K_{ij}(t)\,dt \\
&= -\int_0^\infty tK_{ij}(t)e^{-zt}dt,
\end{aligned}$$

where $z>0$ and $h>0$. Therefore $B'(z) = -\int_0^\infty te^{-zt}K(t)dt$.

These procedures can be repeated to obtain the following:

$$\int_0^\infty t^n\|K(t)\|_N\,dt < \infty,$$

and $B^n(z) = (-1)^n\int_0^\infty t^n e^{-zt}K(t)\,dt$ for $z\geq 0$ and $n\in\mathbb{N}$.

Since B is analytic for $|\operatorname{Re} z|\leq\alpha$ the Maclaurin's series, $\sum_{n=0}^\infty \frac{B^n(0)}{n!}s^n$ is absolutely convergent on the closed disk of radius α centered at zero. Hence

$$\sum_{n=0}^{\infty} \frac{B_{ij}^n(0)}{n!}\alpha^n < \infty \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \int_0^{\infty} t^n \left|K_{ij}(t)\right| dt < \infty$$

and so using Fatou's Lemma

$$\begin{aligned}
\int_0^{\infty} e^{\alpha t}\|K(t)\|_N \, dt &= \int_0^{\infty} e^{\alpha t} \sum_{i=1}^{n}\sum_{j=1}^{n} \left|K_{ij}(t)\right| dt \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \int_0^{\infty} \sum_{m=0}^{\infty} \frac{(\alpha t)^m}{m!} \left|K_{ij}(t)\right| dt \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \left[\sum_{m=0}^{\infty} \frac{(\alpha)^m}{m!} \int_0^{\infty} t^m \left|K_{ij}(t)\right| dt \right] < \infty,
\end{aligned}$$

Now due to the equivalence of norms on a finite Banach space we see that the proof is complete. $\square$

Proof of Lemma 6.3.4. We consider two cases: $\tilde{\eta} = 2\tilde{\lambda}$ and $\tilde{\eta} = \tilde{\alpha}$. We begin by looking at the case where $\eta = 2\lambda$:

$$\begin{aligned}
\int_0^t e^{-2\tilde{\lambda}(t-s)} e^{-\tilde{\alpha}s} \|\tilde{K}(s)\| \, ds &= e^{-2\tilde{\lambda}t} \int_0^t e^{2\tilde{\lambda}s} e^{-\tilde{\alpha}s} \|\tilde{K}(s)\| \, ds \\
&\le e^{-2\tilde{\lambda}t} \int_0^t e^{2\tilde{\alpha}s} e^{-\tilde{\alpha}s} \|\tilde{K}(s)\| \, ds \\
&= e^{-2\tilde{\eta}t} \int_0^t e^{\tilde{\alpha}s} \|\tilde{K}(s)\| \, ds \\
&\le ce^{-2\tilde{\eta}t}.
\end{aligned}$$

Now consider the case when $\tilde{\eta} = \tilde{\alpha}$:

$$\begin{aligned}
\int_0^t e^{-2\tilde{\lambda}(t-s)} e^{-\tilde{\alpha}s} \|\tilde{K}(s)\| \, ds &= e^{-\tilde{\alpha}t} \int_0^t e^{-2\tilde{\lambda}(t-s)} e^{\tilde{\alpha}(t-s)} \|\tilde{K}(s)\| \, ds \\
&= e^{-\tilde{\alpha}t} \int_0^t e^{-(2\tilde{\lambda}-\tilde{\alpha})(t-s)} \|\tilde{K}(s)\| \, ds \\
&= e^{-\tilde{\alpha}t} \int_0^t \|\tilde{K}(s)\| \, ds \\
&\le ce^{-2\tilde{\eta}},
\end{aligned}$$

since $2\tilde{\lambda} - \tilde{\alpha} \ge 0$. $\square$

Proof of Lemma 6.4.1. By the equivalence of norms, there exist n-dependent positive constants $c_1(n) \le c_2(n)$ such that

$$c_1 \sum_{i=1}^{n} |N_i| \le \left(\sum_{i=1}^{n} |N_i|^2 \right)^{1/2} \le c_2 \sum_{n=1}^{n} |N_i| \,.$$

Clearly, $\|N\| \ge c_1 \sum_{i=1}^{n} |N_i|$. Taking expectations across this expression one obtains

$$\mathbb{E}\left[\|N\|\right] \ge c_1 \sum_{i=1}^{n} \mathbb{E}\left[|N_i|\right],$$

now squaring this inequality we obtain

$$\mathbb{E}\left[\|N\|\right]^2 \ge c_1^2 \left(\sum_{i=1}^{n} \mathbb{E}\left[|N_i|\right] \right)^2 .$$

Using the equivalence of norms again,

$$\mathbb{E}\left[\|N\|\right]^2 \ge c_1^2 \left(\sum_{i=1}^{n} \mathbb{E}\left[|N_i|\right] \right)^2 \ge \frac{c_1^2}{c_2^2} \sum_{i=1}^{n} \mathbb{E}\left[|N_i|\right]^2 . \tag{A.0.3}$$

Now, the following simple calculation shows that there exists a v_i-independent constant $c_3 > 0$ such that $\mathbb{E}\left[|N_i|\right] = c_3 v_i$:

$$\mathbb{E}\left[|N_i|\right] = \frac{1}{v_i\sqrt{2\pi}} \int_{-\infty}^{\infty} |x|\, e^{-\frac{1}{2}\left(\frac{x}{v_i}\right)^2} \, dx = \frac{2v_i}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}u} \, du = c_3 v_i.$$

Thus (A.0.3) becomes

$$\mathbb{E}\left[\|N\|\right]^2 \ge \frac{c_1^2 c_3^2}{c_2^2} \sum_{i=1}^{n} v_i^2.$$

Due to the independence of the normal random variables we see that $\mathbb{E}\left[\|N\|^2\right] = \sum_{i=1}^{n} v_i^2$ so

$$\mathbb{E}\left[\|N\|\right]^2 \ge d\mathbb{E}\left[\|N\|^2\right],$$

where $d = \frac{c_1^2 c_3^2}{c^2}$. □

Appendix B

Derivation of functions

Lemma B.0.1. *Let K satisfy* (2.1.2). *Then the function F, given by*

$$F(t) = \Phi'(t) - \Phi(t)A - (\Phi * K)(t),$$

may be reformulated as

$$F(t) = -e^{-t}(Q + QA) + P \int_t^\infty K(s)\, ds - (e * QK)(t),$$

where Φ, P, Q and e are defined as before.

Proof of Lemma B.0.1. Since K is integrable the matrix M is well defined so $PM = 0$, where $M = A + \int_0^\infty K(s)\, ds$ as before, it is seen that

$$\begin{aligned} F(t) &= -Qe^{-t} - (P + Qe^{-t})A - \int_0^t PK(s)\, ds - \int_0^t Qe^{-(t-s)}K(s)\, ds \\ &= -e^{-t}(Q + QA) - P\left(A + \int_0^\infty K(s)\, ds\right) + P\int_t^\infty K(s)\, ds - (e * QK)(t) \\ &= -e^{-t}(Q + QA) + P\int_t^\infty K(s)\, ds - (e * QK)(t), \end{aligned}$$

as required. □

Lemma B.0.2. *Let K satisfy* (2.1.2). *Then the function G, given by*

$$G(t) = \Phi(t) - R_\infty - (F * R_\infty)(t),$$

may be reformulated as

$$\begin{aligned} G(t) = e^{-t}Q - e^{-t}Q\,(I + A)\,R_\infty + \int_t^\infty \int_s^\infty PK(u)\, du\, ds\, R_\infty - (e * QK)(t)R_\infty \\ - \int_t^\infty QK(u)\, du\, R_\infty. \end{aligned}$$

Proof of Lemma B.0.2. It is known that

$$G(t) = \Phi(t) - R_\infty - (F * R_\infty)(t).$$

By expanding the final term on the right hand side of this expression it is seen that

$$(F * R_\infty)(t) = -(Q + QA)(e * R_\infty)(t) + P(K_1 * R_\infty)(t) - (e * QK * R_\infty)(t). \quad \text{(B.0.1)}$$

On evaluation of the first term on the left hand side of (B.0.1) the following is obtained:

$$(Q + QA)(e * R_\infty)(t) = (1 - e^{-t})(Q + QA)R_\infty. \quad \text{(B.0.2)}$$

Now consider the final term on the left hand side of (B.0.1):

$$\begin{aligned}(e * QK * R_\infty)(t) &= \int_0^t e^{-(t-s)} \left(\int_0^s QK(u)\, du \right) ds\, R_\infty \\ &= \int_0^t e^{-(t-s)} \left(\int_0^\infty QK(u)\, du \right) ds\, R_\infty - \int_0^t e^{-(t-s)} \left(\int_s^\infty QK(u)\, du \right) ds\, R_\infty. \quad \text{(B.0.3)}\end{aligned}$$

Evaluating the first term on the left hand side of (B.0.3) one obtains

$$\int_0^t e^{-(t-s)} \left(\int_0^\infty QK(u)\, du \right) ds\, R_\infty = (1 - e^{-t}) \int_0^\infty QK(s)\, ds\, R_\infty. \quad \text{(B.0.4)}$$

In order to evaluate the second term on the left hand side of (B.0.3) one splits the integral in two, changes the order of integration and recombines the terms as follows:

$$\begin{aligned}&\int_0^t e^{-(t-s)} \left(\int_s^\infty QK(u)\, du \right) ds\, R_\infty \\ &\quad = \int_0^t e^{-(t-s)} \left(\int_s^t QK(u)\, du \right) ds\, R_\infty + \int_0^t e^{-(t-s)} \left(\int_t^\infty QK(u)\, du \right) ds\, R_\infty \\ &\quad = e^{-t} \int_0^t \left(\int_0^u e^s\, ds \right) QK(u)\, du\, R_\infty + e^{-t} \int_t^\infty \left(\int_0^t e^s\, ds \right) QK(u)\, du\, R_\infty \\ &\quad = e^{-t} \int_0^t (e^u - 1) QK(u)\, du\, R_\infty + (1 - e^{-t}) \int_t^\infty QK(u)\, du\, R_\infty \\ &\quad = (e * QK)(t) - e^{-t} \int_0^\infty QK(u)\, du\, R_\infty + \int_t^\infty QK(u)\, du\, R_\infty.\end{aligned}$$

(B.0.5)

Combining (B.0.4) and (B.0.5) one obtains

$$(e * QK * R_\infty)(t) = (1-e^{-t})\int_0^\infty QK(s)\,ds - (e*QK)(t)R_\infty$$
$$+ e^{-t}\int_0^\infty QK(s)\,ds\,R_\infty - \int_t^\infty QK(u)\,du\,R_\infty. \quad \text{(B.0.6)}$$

Combining (B.0.2) with (B.0.6) one obtains

$$\begin{aligned}(F * R_\infty)(t) &= -(1-e^{-t})(Q+QA)R_\infty + \int_0^t\int_s^\infty PK(u)\,du\,ds\,R_\infty \\ &\quad -(1-e^{-t})\int_0^\infty QK(s)\,ds\,R_\infty + (e*QK)(t)R_\infty \\ &\quad -e^{-t}\int_0^\infty QK(u)\,du\,R_\infty + \int_t^\infty QK(u)\,du\,R_\infty \\ &= (e^{-t}-1)Q\,(I+M)\,R_\infty + \int_0^t\int_s^\infty PK(u)\,du\,ds\,R_\infty + (e*QK)(t)R_\infty \\ &\quad -e^{-t}\int_0^\infty QK(u)\,du\,R_\infty + \int_t^\infty QK(u)\,du\,R_\infty.\end{aligned}$$

Now using the fact that $QM = M$ and $Q = I - P$ is seen that

$$\begin{aligned}(F * R_\infty)(t) &= -R_\infty + \left(P - M + \int_0^\infty\int_s^\infty PK(u)\,du\,ds\right)R_\infty + e^{-t}Q\,(I+M)\,R_\infty \\ &\quad -\int_t^\infty\int_s^\infty PK(u)\,du\,ds\,R_\infty + (e*QK)(t)R_\infty - e^{-t}\int_0^\infty QK(u)\,du\,R_\infty \\ &\quad + \int_t^\infty QK(u)\,du\,R_\infty.\end{aligned}$$

Now using the definition of R_∞ given by (2.2.11) this becomes

$$\begin{aligned}(F*R_\infty)(t) &= -R_\infty + P + e^{-t}Q\,(I+A)\,R_\infty - \int_t^\infty\int_s^\infty PK(u)\,du\,ds\,R_\infty + (e*QK)(t)R_\infty \\ &\quad + \int_t^\infty QK(u)\,du\,R_\infty.\end{aligned}$$

Thus

$$\begin{aligned}G(t) &= \Phi(t) - R_\infty - (F*R_\infty)(t) \\ &= e^{-t}Q - e^{-t}Q\,(I+A)\,R_\infty + \int_t^\infty\int_s^\infty PK(u)\,du\,ds\,R_\infty - (e*QK)(t)R_\infty \\ &\quad - \int_t^\infty QK(u)\,du\,R_\infty.\end{aligned}$$

□

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