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Image charge inclusion in the dielectric sphere revisited

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Abstract
Van Siclen (1988 *Am. J. Phys.* **56** 1142) reported a curious property of a dielectric sphere in the field of an external point charge: the field outside the sphere generated by the combination of the original charge exterior and the Kelvin image charge interior to the sphere is independent of the permittivity of the sphere. In this paper, we simplify and correct the original derivation and give a detailed analysis of the sources of the field. We also present various checks for the theory, providing instructive exercises for advanced undergraduates.

1. Introduction

Consider a perfectly conducting sphere of radius a embedded in a homogeneous, isotropic dielectric of relative permittivity ε_r; a point charge q is set outside the sphere at the distance d from the centre. If the potential of the sphere is zero, then the Legendre series expressing the external potential due to the actual (free plus bound) charge over the sphere can be interpreted in a simple way as the potential due to a single point charge embedded in an infinite dielectric of the same relative permittivity ε_r [1]. The Kelvin image charge, as it is usually termed, is at the distance $d_K \equiv a^2/d$ from the centre of the sphere and it has the charge $q_K \equiv -qa/d$. This discovery which, paraphrasing Maxwell [2], seems to have been reserved for the young William Thomson, later Lord Kelvin, led him in 1845 to the principle of images [3–5], a powerful method for solving boundary-value problems in electrostatics and elsewhere. More recently, Van Siclen made a surprising observation about the classical Kelvin image theory for the conducting sphere: the field outside the grounded conducting sphere remains unchanged if the conducting sphere is replaced by a dielectric sphere with the Kelvin image charge embedded in it [6]. Thus, the exterior field generated by the combination of the original charge exterior and the Kelvin image charge interior to the dielectric sphere is independent of the permittivity of the sphere; conversely, the field inside the dielectric sphere is independent of the permittivity of the surrounding medium. This curious result has been generalized to the spheroidal electrostatic and spherical magnetostatic cases [7–9].

Unfortunately, Van Siclen's original argument is somewhat involved and occasionally erroneous [7]. In this paper, we simplify the argument and show by direct calculation of the sources of the field how the surprising result reported in [6] comes about. Apart from illustrating the fact that finding the various possible sources for a given field can be challenging, the present discussion provides insightful exercises for advanced undergraduates.

2. A point charge outside and the Kelvin image charge inside a dielectric sphere

Consider a dielectric sphere of radius a, whose relative permittivity is ε_r' embedded in an infinite dielectric of relative permittivity ε_r; both media are linear, homogeneous and isotropic. Choosing the origin at the centre of the sphere, we will find the potential of the system, which is obtained by setting a point charge q outside and the Kelvin image charge q_{K} inside the sphere at the respective points with Cartesian coordinates $(0, 0, d)$ and $(0, 0, d_{\mathrm{K}})$ on the positive z axis in the following way. First, we solve separate problems of the point charge q outside (no free charges inside) and a point charge q' at the point $(0, 0, \alpha)$ inside the sphere (no free charges outside it). Then the sum of the respective solutions for the potentials, Ψ_q and $\Psi_{q'}$, is the potential Ψ of the system in the case where both q and q' are present, $\Psi = \Psi_q + \Psi_{q'}$. Eventually, making the Kelvin image charge substitutions $q' = q_{\mathrm{K}}$ and $\alpha = d_{\mathrm{K}}$, we obtain the required potential.

2.1. The dielectric sphere in the field of the external charge q

Taking into account the azimuthal symmetry of the problem, the potential Ψ_q^+ at the point with spherical coordinates (r, θ, ϕ) outside the sphere is given by

$$\Psi_q^+ = \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{q}{R} + \sum_{l=0}^{\infty} B_l r^{-l-1} P_l(\cos\theta), \tag{1}$$

where R is the distance between the point of observation (r, θ, ϕ) and the point on the positive z axis with Cartesian coordinates $(0, 0, d)$ where the charge q is located, $R \equiv |\boldsymbol{r} - d\hat{\mathbf{k}}|$, and B_l are the unknown coefficients. Also, at any point inside the sphere the resultant potential is of the form

$$\Psi_q^- = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta), \tag{2}$$

where A_l are the unknown coefficients. Now using the well-known expansion of $1/R$ in Legendre polynomials [10]:

$$\frac{1}{R} = \frac{1}{\sqrt{r^2 - 2rd\cos\theta + d^2}} = \sum_{l=0}^{\infty} P_l(\cos\theta) \begin{cases} r^{-l-1}d^l, & r \geqslant d, \\ r^l d^{-l-1}, & r \leqslant d, \end{cases} \tag{3}$$

from the continuity of the electrostatic potential as well as from the continuity of normal component of $\boldsymbol{D}$ on the sphere $r = a$ we find

$$B_l = \frac{1}{4\pi\varepsilon_0}\left(1 - \frac{\varepsilon_r'}{\varepsilon_r}\right)\frac{lqad_{\mathrm{K}}^l}{[\varepsilon_r' l + \varepsilon_r(l+1)]d}, \tag{4}$$

$$A_l = \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{q}{d^{l+1}} + B_l a^{-2l-1}. \tag{5}$$

The above results for Ψ_q are well known and are reached in various ways [1, 6, 11] and recast in various forms [12].

For the sake of completeness, we discuss briefly the so-called conducting limit of the above results, recalling that the field of a region whose permittivity tends to infinity remains unchanged if the region is replaced by a perfectly conducting region of the same shape and size, and of the same *total* free charge[3]. Indeed, when $\varepsilon_r' \to \infty$ we have that

$$B_0 = 0, \tag{6}$$

(note that this is so *regardless of* the value of ε_r') and

$$B_l \to \frac{1}{4\pi\varepsilon_0\varepsilon_r} q_{\mathrm{K}} d_{\mathrm{K}}^l, \qquad l > 0, \tag{7}$$

and also that

$$A_0 = \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{q}{d}, \tag{8}$$

$$A_l \to 0, \qquad l > 0. \tag{9}$$

Thus, when $\varepsilon_r' \to \infty$ we infer that

$$\Psi_q^+ \to \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{q}{R} + \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{(-q_{\mathrm{K}})}{r} + \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{q_{\mathrm{K}}}{r}\sum_{l=0}^{\infty}\left(\frac{d_{\mathrm{K}}}{r}\right)^l P_l(\cos\theta), \tag{10}$$

$$\Psi_q^- \to \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{q}{d} \equiv \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{(-q_{\mathrm{K}})}{a}, \tag{11}$$

in agreement with the fact that for the field outside there are two classic point images for an uncharged conducting sphere, the Kelvin image and $-q_{\mathrm{K}}$ at the centre of the sphere, and with the fact that the potential inside the uncharged sphere is given by the right-hand side of limit (11) [1].

(Instead of recognizing that the B_0 term is zero regardless of the value of ε_r', it is sometimes incorrectly stated that $B_l \to (1/4\pi\varepsilon_0\varepsilon_r)q_{\mathrm{K}}d_{\mathrm{K}}^l$ for *all* l, which leads to the wrong conclusion that when $\varepsilon_r' \to \infty$ the dielectric sphere approaches a grounded conducting sphere with the Kelvin image charge for the field outside the sphere [6, 11, 13]. Similarly, some authors state that in the conducting limit a prolate dielectric spheroid in the field of an external point charge q approaches a grounded conducting spheroid in the field of q (i.e. the potential inside the dielectric spheroid approaches zero) [14], which is of course erroneous [15, 16]. It seems that the conducting limit is an active source of errors, especially the zeroth-order terms [17–19].)

As another check, let the radius of the sphere a grow so that the distance $h = d - a$ of the charge q from the surface of the sphere remains constant. Our system thus becomes a point charge q in front of a planar interface, at the distance h from it. It can be verified that in this limit ($a \to \infty$, $h = d - a = \text{const}$), taking into account that $a - a^2/d \to h$ and being careful with the B_0 term,

$$\Psi_q^+ \to \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{q}{R} + \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{(\varepsilon_r - \varepsilon_r')}{(\varepsilon_r + \varepsilon_r')}\frac{q}{R^*},$$

$$\Psi_q^- \to \frac{1}{4\pi\varepsilon_0}\frac{2}{(\varepsilon_r + \varepsilon_r')}\frac{q}{R} = \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{q}{R} + \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{(\varepsilon_r - \varepsilon_r')}{(\varepsilon_r + \varepsilon_r')}\frac{q}{R},$$

where R^* is the distance from the point symmetrical to q relative to the interface to the point of observation, as they should (cf, e.g., [20]). A little reflection reveals that the second terms in

[3] Note that this last condition need not apply in the case of regions extending to infinity.

the resulting formulae for potentials in two dielectric media separated by the planar interface are due to bound charges over the interface [20, 21][4].

2.2. The dielectric sphere in the field of the internal charge q'

Following the same line of argument, introducing *mutatis mutandis* primed quantities whose meaning is analogous to that of the corresponding unprimed quantities, we have that

$$\Psi_{q'}^{+} = \sum_{l=0}^{\infty} B'_l r^{-l-1} P_l(\cos\theta), \tag{12}$$

$$\Psi_{q'}^{-} = \frac{1}{4\pi\varepsilon_0\varepsilon'_r}\frac{q'}{R'} + \sum_{l=0}^{\infty} A'_l r^l P_l(\cos\theta), \tag{13}$$

where R' is the distance from the internal point $(0, 0, \alpha)$ on the positive z axis where the charge q' is located to the point of observation. Using the expansion of $1/R'$ in Legendre polynomials:

$$\frac{1}{R'} = \frac{1}{\sqrt{r^2 - 2r\alpha\cos\theta + \alpha^2}} = \sum_{l=0}^{\infty} P_l(\cos\theta)\begin{cases} r^{-l-1}\alpha^l, & r \geqslant \alpha, \\ r^l\alpha^{-l-1}, & r \leqslant \alpha, \end{cases} \tag{14}$$

and the boundary conditions, we obtain that the expansion coefficients A'_l and B'_l are given by

$$B'_l = \frac{1}{4\pi\varepsilon_0}\frac{q'\alpha^l(2l+1)}{\left[\varepsilon'_r l + \varepsilon_r(l+1)\right]}, \tag{15}$$

$$A'_l = -\frac{1}{4\pi\varepsilon_0\varepsilon'_r}q'\alpha^l a^{-2l-1} + B'_l a^{-2l-1}. \tag{16}$$

As simple checks, for a conducting sphere, $\varepsilon'_r \to \infty$, $B'_0 = q'/4\pi\varepsilon_0\varepsilon_r$ (regardless of the value of ε'_r), $B'_l \to 0$ for $l > 0$, $A'_0 = q'/4\pi\varepsilon_0\varepsilon_r a$ and $A'_l \to 0$ for $l > 0$, as it should be. Also, for the dielectric sphere embedded in an infinite perfect conductor, $\varepsilon_r \to \infty$, $B'_l \to 0$ (thus the potential of the infinite conductor is zero) and $A'_l \to -(q'/4\pi\varepsilon_0\varepsilon'_r)\alpha^l a^{-2l-1}$ consistent with the fact that for the field inside the sphere there is a single image charge $-q'a/\alpha$ at the point $(0, 0, a^2/\alpha)$ on the z axis.

Note that the above expressions for the potentials $\Psi_{q,q'}^{+,-}$ can also be reached as limiting cases from the solutions to the corresponding problems in the prolate [15, 22] and oblate [23] spheroidal geometries.

[4] Another instructive check would be to let the point charge q recede to infinity, $d \to \infty$, and at the same time have $|q|$ grow so that q/d^2 remains finite, which corresponds to the dielectric sphere lying in a finite uniform externally applied electric field, $\boldsymbol{E}_0 = -\hat{\mathbf{k}}q/4\pi\varepsilon_0\varepsilon_r d^2$. In this limit, we obtain

$$\Psi_q^{+} \to -\boldsymbol{E}_0\cdot\boldsymbol{r} + \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3} + \aleph,$$

where

$$\boldsymbol{p} = 4\pi\varepsilon_0\varepsilon_r\frac{\varepsilon'_r - \varepsilon_r}{\varepsilon'_r + 2\varepsilon_r}a^3\boldsymbol{E}_0,$$

is the electric dipole moment of the sphere and $\aleph = \lim_{d,|q|\to\infty}(q/4\pi\varepsilon_0\varepsilon_r d)$ is infinite constant,

$$\Psi_q^{-} \to -\frac{3\varepsilon_r}{\varepsilon'_r + 2\varepsilon_r}\boldsymbol{E}_0\cdot\boldsymbol{r} + \aleph.$$

Interestingly, the above procedure, while not 'legal' (our results for Ψ_q^{+} and Ψ_q^{-} are obtained under the proviso that the sources of the field are localized in a finite region of space) gives the correct results for the potentials of a dielectric sphere in a uniform externally applied field [1], if we ignore the infinite constant $\aleph$, of course. (As can be seen, we can evade $\aleph$ if we start from the dielectric sphere in the field of *two* charges, q at $d\hat{\mathbf{k}}$ and $-q$ at $-d\hat{\mathbf{k}}$, and then take the limit $d \to \infty$, $|q| \to \infty$ so that q/d^2 remains finite.) Note that no infinities arise in the preceding check (the $a \to \infty$ limit), despite bound charges extending to infinity over the planar interface (cf [21]).

2.3. The dielectric sphere in the field of q outside and q_{K} inside the sphere

As pointed out above, when both charges q and q' are present at their respective points outside and inside the dielectric sphere, the potentials outside Ψ^+ and inside Ψ^- are given by

$$\Psi^+ = \Psi_q^+ + \Psi_{q'}^+, \quad \Psi^- = \Psi_q^- + \Psi_{q'}^-. \tag{17}$$

Now making the Kelvin image charge substitutions $q' = q_{\mathrm{K}} \equiv -qa/d$, $\alpha = d_{\mathrm{K}} \equiv a^2/d$, after a simple but somewhat cumbersome calculation, we obtain surprisingly simple expressions for the corresponding potentials

$$\Psi_{\mathrm{K}}^+ = \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{q}{R} + \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{q_{\mathrm{K}}}{r}\sum_{l=0}^{\infty}\left(\frac{d_{\mathrm{K}}}{r}\right)^l P_l(\cos\theta), \tag{18}$$

$$\Psi_{\mathrm{K}}^- = \frac{1}{4\pi\varepsilon_0\varepsilon_r'}\frac{q_{\mathrm{K}}}{R_{\mathrm{K}}'} + \frac{1}{4\pi\varepsilon_0\varepsilon_r'}\frac{q}{d}\sum_{l=0}^{\infty}\left(\frac{r}{d}\right)^l P_l(\cos\theta), \tag{19}$$

where the subscripts K serve as a reminder that Ψ_{K}^+, Ψ_{K}^- and R_{K}' refer to the case when q' is replaced by the Kelvin image charge. Taking into account expansions (3) and (14), Ψ_{K}^+ and Ψ_{K}^- can be recast as

$$\Psi_{\mathrm{K}}^+ = \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{q}{R} + \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{q_{\mathrm{K}}}{R_{\mathrm{K}}'}, \tag{20}$$

$$\Psi_{\mathrm{K}}^- = \frac{1}{4\pi\varepsilon_0\varepsilon_r'}\frac{q_{\mathrm{K}}}{R_{\mathrm{K}}'} + \frac{1}{4\pi\varepsilon_0\varepsilon_r'}\frac{q}{R}. \tag{21}$$

This is Van Siclen's surprising result: the potential outside the sphere is independent of its electric permittivity and remains unchanged when the dielectric sphere is removed (leaving the Kelvin image charge inclusion q_{K} in a spherical cavity), or it and q_{K} are replaced by a grounded conducting sphere, or both q and q_{K} are in an infinite dielectric of relative permittivity ε_r. Conversely, the potential inside the sphere is independent of the electric permittivity of the surrounding medium and remains unchanged when the surroundings are removed (leaving the charge q at $(0, 0, d)$ *in vacuo*), or the surroundings and q are replaced by the perfect conductor, or both q and q_{K} are in an infinite dielectric of relative permittivity ε_r'. (Note that these independence properties also apply in the case where another point charge is embedded at the centre of the sphere for the fields both outside and inside, and the potential outside (but *not* inside) the sphere.)

3. The charge distribution

In what follows we demonstrate how the simple potentials (20) and (21) come about by finding their sources.

First we find the distribution of volume bound charges in the problem considered. Using the law

$$\nabla \cdot \boldsymbol{P} = -\rho_b, \tag{22}$$

where ρ_b is the volume density of bound charge, and the well-known relation

$$\nabla^2\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) = -4\pi\delta(\boldsymbol{r}-\boldsymbol{r}'), \tag{23}$$

and also taking into account that both media are linear, homogeneous and isotropic, from expressions (20) and (21) we find that there are two bound point charges in our problem,

$[-(\varepsilon_r' - 1)/\varepsilon_r']q_{\rm K}$ at $d_{\rm K}\hat{\mathbf{k}}$ and $[-(\varepsilon_r - 1)/\varepsilon_r]q$ at $d\hat{\mathbf{k}}$. The two bound charges reside at the same points as the corresponding free charges $q_{\rm K}$ and q, respectively. Obviously, there are no other volume bound charges in our system.

The total surface density of bound charge is obtained from the boundary condition

$$(\boldsymbol{P}_2 - \boldsymbol{P}_1)\cdot\boldsymbol{n} = -\sigma_{\rm tb}\,, \tag{24}$$

where $\boldsymbol{n}$ is the unit normal vector at a point on the boundary surface pointing from side 1 to side 2 of the surface. A little reflection reveals that contributions to $\sigma_{\rm tb}$ from dielectrics 1 and 2, σ_{1b} and σ_{2b}, respectively, are given by

$$\boldsymbol{P}_1\cdot\boldsymbol{n} = \sigma_{1b}, \tag{25}$$

$$\boldsymbol{P}_2\cdot\boldsymbol{n} = -\sigma_{2b}. \tag{26}$$

Applying conditions (25) and (26) to the spherical boundary $r = a$, using equations (20) and (21) we find the surface densities of bound charges

$$\sigma_b'(r=a) = -\frac{(\varepsilon_r' - 1)}{\varepsilon_r'}q\frac{1}{4\pi ad}\sum_{l=0}^{\infty}(2l+1)\left(\frac{a}{d}\right)^l P_l(\cos\theta), \tag{27}$$

$$\sigma_b(r=a) = \frac{(\varepsilon_r - 1)}{\varepsilon_r}q\frac{1}{4\pi ad}\sum_{l=0}^{\infty}(2l+1)\left(\frac{a}{d}\right)^l P_l(\cos\theta), \tag{28}$$

where σ_b' and σ_b are the contributions from the interior (ε_r') and exterior (ε_r) dielectric, respectively. Using expansion

$$\sum_{l=0}^{\infty}(2l+1)\left(\frac{a}{d}\right)^l P_l(\cos\theta) = \frac{(1-a^2/d^2)}{[1-2(a/d)\cos\theta + a^2/d^2]^{3/2}}, \tag{29}$$

which applies for $a < d$ (cf, e.g., [24])[5], equations (27) and (28) can be recast as

$$\sigma_b'(r=a) = -\frac{(\varepsilon_r' - 1)}{\varepsilon_r'}q\frac{1}{4\pi ad}\frac{(1-a^2/d^2)}{[1-2(a/d)\cos\theta + a^2/d^2]^{3/2}}, \tag{30}$$

$$\sigma_b(r=a) = \frac{(\varepsilon_r - 1)}{\varepsilon_r}q\frac{1}{4\pi ad}\frac{(1-a^2/d^2)}{[1-2(a/d)\cos\theta + a^2/d^2]^{3/2}}. \tag{31}$$

Now recall two identities which are essential for our argument.

Let a charge $-Qa/d$ be distributed over the sphere $r = a$ with surface charge density

$$\sigma(\theta) = -Q\frac{1}{4\pi ad}\frac{(1-a^2/d^2)}{[1-2(a/d)\cos\theta + a^2/d^2]^{3/2}}, \tag{32}$$

[5] A 'physical proof' of expansion (29) would be to calculate the surface density of free charge over a grounded conducting sphere in the field of q from equation $\sigma_f = -\varepsilon_0\varepsilon_r(\partial\Psi_{\rm K}^+/\partial r)_{r=a}$ using expressions (20), (14) (with of course α replaced by $d_{\rm K}$) and (3), which gives

$$\sigma_f(r=a) = -\frac{q}{4\pi ad}\sum_{l=0}^{\infty}(2l+1)\left(\frac{a}{d}\right)^l P_l(\cos\theta),$$

or

$$\sigma_f(r=a) = -\frac{q}{4\pi ad}\frac{(1-a^2/d^2)}{[1-2(a/d)\cos\theta + a^2/d^2]^{3/2}},$$

depending on which expressions for $1/R$ and $1/R_{\rm K}'$ (via Legendre polynomials or via the law of cosines, respectively) are used in the calculation. True, the physical proof appears somewhat cumbersome when compared to its elegant mathematical counterpart (cf, e.g., [24]). Incidentally, it seems to be attractive for students when they assimilate mathematics from live physical problems. For example, the concept of δ-sequence (the weak limit) is vividly illustrated in various physical contexts (cf, e.g., [25, 26]).

where, as before, $a < d$. Then the following identities apply for the potential of the charge distribution (32) outside and inside the sphere *in vacuo*:

$$\frac{1}{4\pi\varepsilon_0}\oint_{r'=a}\frac{\left[-\frac{Q}{4\pi ad}\frac{(1-a^2/d^2)}{[1-2(a/d)\cos\theta'+a^2/d^2]^{3/2}}\right]}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}S' = \frac{1}{4\pi\varepsilon_0}\frac{(-Qa/d)}{|\boldsymbol{r}-(a^2/d)\hat{\mathbf{k}}|}, \tag{33}$$

when the point of observation $\boldsymbol{r}$ lies outside the sphere $r = a$, and

$$\frac{1}{4\pi\varepsilon_0}\oint_{r'=a}\frac{\left[-\frac{Q}{4\pi ad}\frac{(1-a^2/d^2)}{[1-2(a/d)\cos\theta'+a^2/d^2]^{3/2}}\right]}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}S' = \frac{1}{4\pi\varepsilon_0}\frac{(-Q)}{|\boldsymbol{r}-d\hat{\mathbf{k}}|}, \tag{34}$$

when $\boldsymbol{r}$ lies inside the sphere $r = a$; $\mathrm{d}S$ is an infinitesimal element of area of the sphere. The above identities are well known in the context of the image solution to the classical electrostatic problem of a point charge outside a conducting sphere at zero potential *in vacuo* (cf, e.g., [20, 24]). While we introduced them in the language of physics, involving charges and potentials, it should be stressed that they are purely mathematical identities that need not have a physical interpretation. For the sake of completeness, a simple proof of identities (33) and (34) via Green's second identity is sketched in the appendix.

Now we can return to our original task of demonstrating how the potentials (20) and (21) come about. Comparing equations (30) and (31) with equation (32), and making use, *mutatis mutandis*, of identities (33) and (34), it follows that the distribution of bound volume and surface charges determined above, together with free point charges q and q_K, indeed gives rise to simple potentials Ψ_K^+ and Ψ_K^-.

4. Conclusion

In this paper we have presented a thorough analysis of the surprising observation reported in [6]: for a dielectric sphere in the field of a point charge q exterior and the Kelvin image charge q_K interior to the sphere, the potential outside (inside) the sphere is independent of the electric permittivity of the internal (external) medium. By finding the distribution of bound charges, we showed how the curious Van Siclen's result came about. Also, the so-called conducting limit checks for the theory are presented and some familiar results demonstrated in a new way.

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Appendix

Consider two point charges Q and $Q_K \equiv -Qa/d$ *in vacuo*, lying on the positive z axis at the points $d\hat{\mathbf{k}}$ outside and $(a^2/d)\hat{\mathbf{k}}$ inside the sphere of radius a with its centre at the origin, respectively. As is well known, the potential for this system,

$$\Psi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{|\boldsymbol{r}-d\hat{\mathbf{k}}|} + \frac{1}{4\pi\varepsilon_0}\frac{Q_K}{|\boldsymbol{r}-(a^2/d)\hat{\mathbf{k}}|}, \tag{A.1}$$

vanishes on the sphere $r = a$. Thus the potential (A.1) is Q times the corresponding Green's function for the sphere satisfying the Dirichlet boundary condition.

Now recall that Green's second identity implies the following representation formula for a well-behaved scalar function of position Φ:

$$\Phi(\boldsymbol{r}) = -\frac{1}{4\pi}\int_{V^*}\frac{\nabla'^2\Phi(\boldsymbol{r}')}{\Re}\,\mathrm{d}^3r' + \frac{1}{4\pi}\oint_{S^*}\left[\frac{1}{\Re}\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{\Re}\right)\right]\mathrm{d}S', \quad \text{(A.2)}$$

where $\Re \equiv |\boldsymbol{r} - \boldsymbol{r}'|$, which applies in the case where the point of observation $\boldsymbol{r}$ lies within the volume V^* bounded by the closed surface S^* [1, 20]. Substitute now the potential Ψ from equation (A.1) for Φ in equation (A.2) and apply the modified equation to the volume V^* outside the sphere $r = a$. Taking into account that Ψ vanishes on the sphere $r = a$ and also that it is regular at infinity, after a simple calculus we obtain identity (33). Similarly, applying the same procedure to the volume V^* within the sphere $r = a$, identity (34) follows.

Note that instead of the above heuristic method of proving identities (33) and (34), via Green's equivalent stratum [20, 27], more general methods of finding the potential of a spherical shell of given surface density can be used [24, 28][6].

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