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SUBEXPONENTIAL SOLUTIONS OF SCALAR LINEAR INTEGRO– DIFFERENTIAL EQUATIONS WITH DELAY

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Abstract. This paper considers the asymptotic behaviour of solutions of the scalar linear convolution integro-differential equation with delay

$$\begin{aligned} x'(t) &= -\sum_{i=1}^{n} a_i x(t-\tau_i) + \int_0^t k(t-s)x(s)\,\mathrm{d}s, \quad t > 0, \\ x(t) &= \phi(t), \quad -\tau \le t \le 0, \end{aligned}$$

where $\tau = \max_{1\le i\le n} \tau_i$. In this problem, k is a non-negative function in $L^1(0,\infty)\cap C[0,\infty)$, $\tau_i \ge 0$, $a_i > 0$ and ϕ is a continuous function on $[-\tau, 0]$. The kernel k is subexponential in the sense that $\lim_{t\to\infty} k(t)\alpha(t)^{-1} > 0$ where α is a positive subexponential function. A consequence of this is that $k(t)e^{\epsilon t} \to \infty$ as $t \to \infty$ for every $\epsilon > 0$.

Key Words. Volterra integro–differential equations, subexponential function, exponential asymptotic stability.

AMS(MOS) subject classification. 34K20, 34K25, 34K06, 45D05, 45J05

1. Introduction and Results. This paper examines the asymptotic behaviour of solutions of the scalar linear integrodifferential equation with delay

$$x'(t) = -\sum_{i=1}^{n} a_i x(t-\tau_i) + \int_0^t k(t-s)x(s)\,ds, \quad t > 0, \tag{1}$$

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subject to the initial condition

$$x(t) = \phi(t), \quad -\tau \leq t \leq 0. \tag{2}$$

The following hypotheses are postulated.

(H_1) $\tau_i \geq 0$, $a_i \geq 0$, $\sum_{i=1}^n a_i > 0$ and $\tau = \max_{1\leq i\leq n} \tau_i$.

(H_2) The characteristic equation

$$p = \sum_{i=1}^{n} a_i e^{\tau_i p},$$

has a real root.

(H_3) The kernel k is a non-trivial function in $L^1(0,\infty)\cap C[0,\infty)$ with $k(t) \geq 0$ for all $t \geq 0$.

(H_4) $\int_0^\infty k(s)\,ds < \sum_{i=1}^n a_i$.

(H_5) $\lim_{t\to\infty} k(t)/\alpha(t) > 0$ for some positive subexponential function α.

(H_6) The initial function ϕ is in $C[-\tau, 0]$.

The significance of (H_2), and sufficient conditions for it to hold, are discussed in Section 2. The definition of positive subexponential functions and some of their important properties are also reviewed there. The main result of this paper is the following theorem.

THEOREM 1. *Suppose that* (H_1)–(H_6) *hold. Then the solution of (1) and (2) satisfies*

$$\lim_{t\to\infty} \frac{x(t)}{k(t)} = \frac{\phi(0) - \sum_{i=1}^n a_i \int_{-\tau_i}^0 \phi(s)\,ds}{\left(\sum_{i=1}^n a_i - \int_0^\infty k(s)\,ds\right)^2}, \tag{3}$$

$$\lim_{t\to\infty} \frac{x'(t)}{k(t)} = 0. \tag{4}$$

The decay rate given in (3) can also be written as

$$\lim_{t\to\infty} \frac{x(t)}{k(t)} = \frac{\int_0^\infty x(s)\,ds}{\sum_{i=1}^n a_i - \int_0^\infty k(s)\,ds},$$

It is shown in [1, 2] that the decay rate of two classes of stochastic Volterra integrodifferential equations with subexponential kernels, can also be expressed in this form.

In order to prove Theorem 1 we introduce the resolvent for (1), which is the solution of the equation

$$r'(t) = -\sum_{i=1}^{n} a_i r(t-\tau_i) + \int_0^t k(t-s) r(s)\,ds, \quad t > 0, \tag{5}$$

which satisfies the initial condition

$$r(t) = \begin{cases} 1, & t = 0, \\ 0, & -\tau \le t < 0. \end{cases} \tag{6}$$

The significance of r is that the solution of (1) which obeys (2) is given by the variation of parameters formula

$$x(t) = \phi(0)r(t) + \int_0^t r(t-s)\tilde{\phi}(s)\,ds, \quad t \ge 0, \tag{7}$$

where

$$\tilde{\phi}(t) = -\sum_{i=1}^{n} a_i \phi(t-\tau_i)\chi_{[0,\tau_i]}(t), \quad t \ge 0, \tag{8}$$

and χ_J denotes the indicator function of a set J. The asymptotic behaviour of the resolvent is described in the next theorem.

Theorem 2. *Suppose that* (H_1)–(H_4) *hold. Then the resolvent* r*, defined by (5) and (6), is in* $L^1(0,\infty)$, $r(t) > 0$ *for all* $t \ge 0$ *and* $r(t) \to 0$ *as* $t \to \infty$. *If, in addition,* (H_5) *holds,*

$$\lim_{t\to\infty} \frac{r(t)}{k(t)} = \left(\sum_{i=1}^{n} a_i - \int_0^\infty k(s)\,ds\right)^{-2}, \quad \lim_{t\to\infty} \frac{r'(t)}{k(t)} = 0. \tag{9}$$

Moreover

$$\lim_{t\to\infty} \frac{\int_0^t r(t-s)r(s)\,ds}{r(t)} = 2\int_0^\infty r(s)\,ds. \tag{10}$$

Theorems 1 and 2 are generalisations of [3, Theorem 6.2], which concerns linear scalar convolution integro-differential equations with subexponential kernels but without delays. Theorem 2 has the following converse, which is an extension of [3, Theorem 6.4].

Theorem 3. *Suppose that* (H_1)–(H_4) *hold, and that* $k(t) > 0$ *for all* $t \ge 0$. *If the resolvent* r *satisfies (9), then* k *is a positive subexponential function and (10) is true.*

2. Mathematical Preliminaries. The convolution of two appropriate functions f and g defined on $[0,\infty)$ is denoted, as usual, by

$$(f * g)(t) = \int_0^t f(t-s)g(s)\,ds, \quad t \ge 0.$$

We recall a definition from [3], based on the hypotheses of Theorem 3 of [5].

DEFINITION 1. *A* positive subexponential function *is a continuous integrable function* $\alpha : [0,\infty) \to (0,\infty)$ *satisfying*

$$\lim_{t\to\infty} \frac{(\alpha * \alpha)(t)}{\alpha(t)} = 2\int_0^\infty \alpha(s)\,ds, \tag{11}$$

$$\lim_{t\to\infty} \sup_{0\le s\le A} \left| \frac{\alpha(t-s)}{\alpha(t)} - 1 \right| = 0, \quad \textit{for all } A > 0. \tag{12}$$

It is noted in [3] that the class of positive subexponential functions includes all positive, continuous, integrable functions which are regularly varying at infinity. It is known that (12) implies for every $\epsilon > 0$ that

$$\alpha(t)e^{\epsilon t} \to \infty \quad \text{as } t \to \infty, \tag{13}$$

(cf., e.g., [4, Lemma 2.2]), and hence by (H_5) that $k(t)e^{\epsilon t} \to \infty$ as $t \to \infty$ for every $\epsilon > 0$.

If α is a positive subexponential function and f is a function on $(0,\infty)$ such that $\lim_{t\to\infty} f(t)/\alpha(t)$ exists, we define

$$L_\alpha f = \lim_{t\to\infty} \frac{f(t)}{\alpha(t)}.$$

An important result is the following lemma. It is essentially Theorem 4.1 of [3]. Perusal of the proof of this theorem shows that the hypotheses that f/α and g/α be bounded continuous functions on $[0,\infty)$ are redundant, and are therefore omitted here.

LEMMA 1. *Suppose that* α *is a positive subexponential function. Let* f *and* g *be integrable functions on* $(0,\infty)$ *for which* $L_\alpha f$ *and* $L_\alpha g$ *both exist. Then* $L_\alpha(f*g)$ *exists and is given by*

$$L_\alpha(f*g) = L_\alpha f \int_0^\infty g(s)\,ds + L_\alpha g \int_0^\infty f(s)\,ds. \tag{14}$$

Next we introduce the resolvent z associated with the purely point delay part of (1). It satisfies the equation

$$z'(t) = -\sum_{i=1}^n a_i z(t-\tau_i), \quad t > 0, \tag{15}$$

and the initial condition

$$z(t) = \begin{cases} 1, & t = 0, \\ 0, & -\tau \le t < 0. \end{cases} \tag{16}$$

We collect in a lemma some salient properties of z.

LEMMA 2. *Suppose that* (H_1) *and* (H_2) *hold. Then* $z(t) > 0$ *for* $t \ge 0$ *and*

$$\int_0^\infty z(t)\,dt = \frac{1}{\sum_{i=1}^n a_i}, \tag{17}$$

$$z(t) \to 0 \quad \textit{exponentially as } t \to \infty. \tag{18}$$

Thus if α *is a subexponential function*

$$L_\alpha z = 0. \tag{19}$$

The positivity of z and (17) are part of Proposition 2.1 of [7]; (18) follows from the same proposition and Lemma 6.5.3 of [8]; (19) is a consequence of (13) and (18).

It is shown in [9] that a necessary condition for (H_2) to be true is $\sum_{i=1}^n a_i\tau_i \le \mathrm{e}^{-1}$, and that $\tau \sum_{i=1}^n a_i \le e^{-1}$ is a sufficient condition. In the case of a single delay with $n = 1$, $a_1 = a > 0$, $\tau_1 = \tau$, a necessary and sufficient condition for (H_2) to hold is $a\tau \le e^{-1}$.

The following yields a representation of the solution of

$$y'(t) = -\sum_{i=1}^n a_i y(t - \tau_i) + f(t), \quad t > 0, \tag{20}$$

$$y(t) = \begin{cases} 1, & t = 0, \\ 0, & -\tau \le t < 0. \end{cases} \tag{21}$$

LEMMA 3. *Let* f *be in* $C[0,\infty)$. *Then the solution of (20) and (21) can be represented as* $y(t) = z(t) + (z * f)(t)$, $t \ge 0$.

3. Proofs. *Proof. (Theorem 2)* The resolvent r of (1) satisfies (5) and (6). It is a consequence of Lemma 3 that r satisfies

$$r = z + z * (k * r) = z + h * r, \tag{22}$$

where $h = z * k$. Due to (H_3) and Lemma 2, $h(t) \ge 0$ for all $t \ge 0$. A standard argument shows that $r(t) > 0$ for all $t \ge 0$. By taking the convolution of each term in (22) with k, we see that

$$\rho = h + h * \rho,$$

where $\rho = k * r$. Since by (17) and (H_4),

$$\int_0^\infty h(s)\,ds = \int_0^\infty z(s)\,ds \int_0^\infty k(s)\,ds = \frac{1}{\sum_{i=1}^n a_i} \int_0^\infty k(s)\,ds < 1,$$

it can be deduced from a theorem of Paley and Wiener (cf., e.g, [6, Theorem 2.4.1]) that ρ is in $L^1(0,\infty)$. It is then an immediate consequence of Lemma 2 and

$$r = z + \rho * z,$$

that r is in $L^1(0,\infty)$. It then follows from this equation that $r(t) \to 0$ as $t \to \infty$, since z is a bounded continuous function obeying (18).

Integration of (5) gives

$$-1 = -\sum_{i=1}^n a_i \int_0^\infty r(t-\tau_i)\,dt + \int_0^\infty r(s)\,ds \int_0^\infty k(s)\,ds,$$

which the aid of (6) leads to

$$\int_0^\infty r(s)\,ds = \frac{1}{\sum_{i=1}^n a_i - \int_0^\infty k(s)\,ds}.$$

Also we can deduce from Lemma 1, (17) and (19) that

$$L_\alpha h = L_\alpha(z * k) = L_\alpha z \int_0^\infty k(s)\,ds + L_\alpha k \int_0^\infty z(s)\,ds = \frac{L_\alpha k}{\sum_{i=1}^n a_i}.$$

Suppose now that $L_\alpha r$ exists. Then we can infer from Lemma 1, (19), (22) and the above formulae that

$$\begin{aligned} L_\alpha r &= L_\alpha z + L_\alpha h \int_0^\infty r(s)\,ds + L_\alpha r \int_0^\infty h(s)\,ds \\ &= \frac{L_\alpha k}{\sum_{i=1}^n a_i(\sum_{i=1}^n a_i - \int_0^\infty k(s)\,ds)} + L_\alpha r \frac{1}{\sum_{i=1}^n a_i} \int_0^\infty k(s)\,ds. \end{aligned}$$

Rearranging yields the first formula in (9). To obtain the second, note that (12) implies $L_\alpha r(\cdot - \tau_i) = L_\alpha r$, so then, by applying L_α to (5) and using Lemma 1, we get that

$$L_\alpha r' = -L_\alpha r\Big(\sum_{i=1}^n a_i - \int_0^\infty k(s)\,ds\Big) + L_\alpha k \int_0^\infty r(s)\,ds = 0.$$

We note that by Lemma 1

$$L_\alpha(r * r) = 2L_\alpha r \int_0^\infty r(s)\,ds,$$

which immediately implies (10).

To complete the proof, it only remains to show that $L_\alpha r$ exists. For the sake of brevity a proof is indicated here under the additional (and unnecessary) assumption that $k(t) > 0$ for all $t \geq 0$. It follows then that $h(t) > 0$ for all $t > 0$. Then by Lemma 4.3 of [3], h is a subexponential function. By applying Theorem 5.2 of [3] to (22), we conclude that $L_h r$ exists and hence $L_\alpha r$. □

Proof. (Theorem 1) First, we observe from (8) that $L_\alpha \tilde{\phi} = 0$ and

$$\int_0^\infty \tilde{\phi}(t)\,dt = -\sum_{i=1}^n a_i \int_{-\tau_i}^0 \phi(s)\,ds.$$

Then, by applying L_α to (7) and using Lemma 1, we obtain

$$L_\alpha x = \phi(0) L_\alpha r + L_\alpha r \int_0^\infty \tilde{\phi}(s)\,ds + L_\alpha \tilde{\phi} \int_0^\infty r(s)\,ds.$$

Therefore we can conclude that

$$L_\alpha x = L_\alpha k \frac{\phi(0) - \sum_{i=1}^n a_i \int_{-\tau_i}^0 \phi(s)\,ds}{\left(\sum_{i=1}^n a_i - \int_0^\infty k(s)\,ds\right)^2},$$

and hence that (3) holds.

It also follows easily from (7) that

$$\int_0^\infty x(t)\,dt = \frac{\phi(0) - \sum_{i=1}^n a_i \int_{-\tau_i}^0 \phi(s)\,ds}{\sum_{i=1}^n a_i - \int_0^\infty k(s)\,ds}.$$

By applying Lemma 1 to (1), we then see that

$$\begin{aligned}
L_\alpha x' &= -\sum_{i=1}^n a_i L_\alpha x(\cdot - \tau_i) + L_\alpha x \int_0^\infty k(s)\,ds + L_\alpha k \int_0^\infty x(s)\,ds \\
&= \left(\int_0^\infty k(s)\,ds - \sum_{i=1}^n a_i\right) L_\alpha x + L_\alpha k \int_0^\infty x(s)\,ds \\
&= -L_\alpha k \frac{\phi(0) - \sum_{i=1}^n a_i \int_{-\tau_i}^0 \phi(s)\,ds}{\sum_{i=1}^n a_i - \int_0^\infty k(s)\,ds} + L_\alpha k \int_0^\infty x(s)\,ds = 0.
\end{aligned}$$

Therefore (4) is true. □

Proof. (Theorem 3) For convenience we introduce

$$\eta = \sum_{i=1}^n a_i - \int_0^\infty k(s)\,ds.$$

Dividing (5) by $k(t)$, we see that

$$\frac{r'(t)}{k(t)} = -\sum_{i=1}^{n} a_i \frac{r(t-\tau_i)}{k(t)} + \frac{(k*r)(t)}{k(t)}.$$

By letting $t \to \infty$ and using (9),

$$\lim_{t\to\infty} \frac{(k*r)(t)}{k(t)} = \frac{1}{\eta^2}\sum_{i=1}^{n} a_i.$$

Hence $(k*r)(t)/r(t) \to \sum_{i=1}^{n} a_i$ as $t \to \infty$. Since r and k are positive and $\lim_{t\to\infty} k(t)/r(t) > 0$, Lemma 3.8 of [3] applies. We can conclude from it that k satisfies (12) and

$$\lim_{t\to\infty} \frac{(k*k)(t)}{k(t)} = \sum_{i=1}^{n} a_i + \int_0^\infty k(s)\,ds - \eta^2 \int_0^\infty r(s)\,ds = 2\int_0^\infty k(s)\,ds.$$

Thus k satisfies (11), finishing the proof. □

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