The Valuation of Collateralised Debt Obligations – multi-period modelling in a risk-neutral framework


Degree: Ph.D.

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Statement of Originality

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Ph.D. is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

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To Mom and Dad
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Abstract

For over fifty years, mortgages have been securitised by selling the rights to the mortgage cash flows to third party investors. Over the past ten years or so, a similar securitisation process has been undertaken with corporate debt. The claims on the cash flowing from the corporate debt portfolio are called collateralised debt obligations (CDO).

CDO cash flows are dependent on the interaction of a portfolio of debt securities over many time periods. They are particularly sensitive to the correlation among the underlying securities and to the terms of the indenture. While much progress has been made in modelling debt portfolios over a single period, there has been a lot less published about the interaction of debt securities in a portfolio over many periods.

This thesis develops a model for valuing CDOs using a risk-neutral approach in a multi-period setting. A model is also developed which reproduces Moody’s CDO rating. The Moody’s rating is compared to that which is implied from applying the risk-neutral model, the differences analysed and the implications for regulatory capital for CDOs explored.
Executive Summary

In recent years, as liquidity in the credit default swap market has increased for the largest borrowing firms in the market, many debt portfolio-based structured products have been created. For example, single-tranche collateralised debt obligations and \( n^{th} \)-to-default swaps are traded in huge volume. The theoretical framework necessary to underpin an analysis of these products has been developed and has gained widespread acceptance. The emphasis has now moved onto the analysis of more complex products referencing the same underlying names for which there is a liquid credit default swap (CDS) available.

By comparison, the market for structured credit products referencing names for which there is no liquid credit default swap market has grown a lot more slowly. This section of the market includes approximately 34,000 publicly quoted firms worldwide compared to the 500 or so for which an active CDS market exists. It is estimated that the debt issuance by these 34,000 firms exceeds that of the top 500 issuers. It comprises, in the main, privately issued debt, provided primarily by banks. Relatively little of this debt is traded; hence price information is lacking.

However, since the firms in this latter category have publicly-quoted equity, it is possible to infer the credit quality of the issuer’s debt from the characteristics of its equity and the correlation between the firms’ asset returns, as one firm, Moody’s KMV (KMV), has done. With this information, the probability distribution of debt portfolio values at a future date can be derived, giving the portfolio owner the information necessary to manage the credit risks presented.

KMV have never publicly disclosed the methodology that they employ in developing their credit portfolio model. Thus, the academic community has largely ignored their approach. But the market has embraced their approach, and their portfolio management product is the clear market leader. This gap between the academic and market approaches is a puzzle.

Credit Portfolio Model Development

This first innovation presented in this thesis is the development of a credit portfolio model in the spirit of KMV. In so doing, it shows for the very first time how the KMV approach sits within current academic thinking. Using only their data and their very
limited public references to their methodology, their portfolio modelling approach is re-created and their results replicated. It thus provides confirmation that their approach is founded on traditional portfolio management principles identified within the academic literature. It is hoped that this will enable a debate to begin in academic circles regarding the merits of their approach, a debate that has not occurred to date because of the lack of understanding of the approach that KMV adopted.

**Assessing the Marginal Impact of a Loan on a Portfolio**

A major disadvantage of current portfolio models is their failure to assess the marginal impact of a proposed new facility on a pre-existing portfolio of credits. They are primarily directed at analysing the performance of a portfolio of credit exposures. In short, they determine the impact of a facility on the portfolio after the fact. While this is clearly important information, it is being delivered too late to give effect to portfolio management action. Within most banks, exposures are being written by a large number of credit officers dispersed throughout the organisation while the portfolio is being modelled at periodic intervals by the credit portfolio function.

This model enables putative new facilities to be added to the current portfolio and the capital required to support the new facility to be determined immediately. Thus, the second innovation of this thesis is the creation of a model which is capable of giving effect to portfolio decisions in real-time since the portfolio impact of potential new facilities can be determined *ex ante* compared to current models which deliver this information *ex post*.

**Developing a Coherent Measure of Credit Concentration**

A primary concern of banks and bank regulators has been with credit risk concentration. However, the approaches adopted hitherto have been largely intuitive with very little by way of theoretical underpinning. Most banks use simplistic rules of thumb to place limits on acceptable maximum exposures to individual obligors, industries and countries. Even under the Basel II proposals, the Bank for International Settlements (BIS) has not attempted to quantify the impact of concentrations on the capital required to support credit portfolios, withdrawing their original proposal for a granularity adjustment in calculating credit portfolio capital.
The third innovation presented in this thesis is the development of an approach to measuring the impact of concentrations on credit portfolios. In particular, a key issue of concern to the managers of credit portfolios—namely, the \textit{ex ante} assessment of the maximum economic holding of a syndicated loan—is answered. The framework provides portfolio managers the basis for measuring the cost of concentrations to favoured relationship clients in a theoretically rigorous manner. It also provides a basis for setting limits for clients—a single capital number can become the basis for limits to all customers replacing the qualitative limit framework currently employed in the market. Finally, it provides regulators a basis on which to set capital requirements.

The analysis also questions the use of contribution to the volatility of portfolio value—the market standard method of allocating portfolio capital among the component securities—for allocating capital in debt portfolios. This approach is well suited to allocating capital in portfolios of traded securities whose returns are near-normal over the holding period. However, illiquid credit portfolios do not meet these requirements. An alternative framework, contribution to Expected Tail Loss, is proposed which is found to give results that accord more closely with intuition. In particular, the proposed alternative is shown to be much more sensitive to credit concentrations than the contribution to portfolio volatility framework.

*Extending the Credit Portfolio Model to a Multi-Period Setting*

The credit portfolio models most commonly used in the marketplace are based on a single time period. They are ill equipped to assess structured securities such as CDOs that derive their value from cash flows from a credit portfolio over many time periods. While the reduced form approach has been applied to value structured securities which reference names for which liquid CDS exist, it is not well suited to modeling portfolios of names which lack the pricing transparency that CDS provide.

The fourth innovation presented in this thesis is the development of a multi-period credit portfolio model. This model extends current modeling approaches along two dimensions:

- It takes the structural model from its single-period frame of reference to the multi-period frame necessary to deal with the complexities of portfolio-based securities.

The reduced form paradigm is the preferred approach when modeling credit...
exposures over more than one time period. The author is unaware of any published research that uses the structural model in a multi-period setting.

- It incorporates the complexities of the waterfall, which are central to the structuring of portfolios of cash flow securitisations. By comparison, the market standard for modelling credit portfolios, a copula approach, is primarily geared to modelling credit exposures in synthetic form.

Furthermore, since the model tracks the portfolio of underlying securities over time, it can be adapted to deal with a CDO where the underlying collateral comprises tranches of other CDOs. These CDO-squared securities, as they are called, have never before been analysed using a structural approach to the author’s knowledge.

**Comparing the Rating Assessments**

The rating agencies serve a critical role in assessing the credit risk of firms and securities. As products have become more structured, and hence more complex, market participants have come to rely to an ever greater extent on the rating agency assessment of credit risk. Their credit assessment of collateralised debt obligations will become still more important in the future given the special position granted to them in the revised Basel Accord.

The fifth innovation of this thesis is to compare the rating agency assessment of CDO risk with that of the structural model based on KMV data. This required that the model developed by Moody's for evaluating CDOs - their so-called Binomial Expansion Model - be re-created based on the publications in which they outline their approach. The model is then tested to confirm that it successfully replicates the Moody’s rating for a sample of deals in the marketplace. The expected loss under the Binomial Expansion Model, from which Moody’s infer their rating, is compared with the expected loss predicted by the multi-period model. This provides a basis for assessing the validity of the binomial approach.

**Assessing the Moody's Rating Assessment Methodology**

Despite making the rating agencies the sole arbiters of the creditworthiness of CDOs purchased by investors, regulatory doubts about the validity of the rating agency approach remain. In particular, their decision to apply different risk weightings to CDOs and corporate debt of the same rating confirms their unease with the meaning of ratings
given to structured securities. Equally, the market demands higher spreads for structured products than for corporate debt with the same rating. It is unclear whether this additional premium is a charge for the reduced liquidity of CDOs, or compensation for the extra effort in coming to understand the complexity of the product, or a charge for risks not adequately captured in the agency rating process.

The sixth innovation is to examine the validity of the agency credit rating approach to structured debt. It highlights the shortcoming of expected loss as a measure of CDO risk, ignoring as it does the variability around the average. In particular, by ignoring the systematic risk that these products bring to a credit portfolio, it fails to provide the regulator with a coherent basis for setting capital to be assigned to CDOs. It confirms that the regulators have good reason for demanding that more capital is held against subordinated CDO tranches than similarly rated corporate securities.

**Conclusion**

This thesis extends current knowledge as outlined above. The approach is new and the results have important implications for investors and regulators. In particular, the new model demonstrates that the rating agency approach to grading CDOs under-estimates the embedded risk. It suggests that the more onerous risk-weighting of mezzanine CDOs compared with similarly-rated corporate debt under Basel II is justified. It also supports the market requirement for wider spreads for these subordinated tranches compared to similarly-rated corporate debt.
Chapter 1. Introduction

1.1 Chapter Overview

This chapter
❖ examines the background to the development of the collateralised debt obligations (CDO) market, explores the reasons for its phenomenal growth and discusses the unique valuation challenges which CDOs present,
❖ details the objective of this thesis, namely, to create a model which values CDO tranches in a rigorous manner,
❖ describes the key research issues which are addressed, and
❖ gives a chapter-by-chapter overview of the research undertaken

1.2 Background

The CDO product will be examined in detail in Chapter 2. However, a brief introduction will be provided here in order to give context to the discussion.

A CDO is an asset-backed security (ABS) where the underlying securities are debt instruments. The CDO market has grown at a tremendous pace since first introduced ten years ago. Many factors have contributed to this growth. Some banks have embraced securitisation as a way to manage their regulatory capital requirements. Lower quality financial institutions – with a rating of single-A or less – which cannot fund themselves in the inter-bank market at Libor, or which lack a deposit base, have securitised their debt assets as a means to achieving funding at a lower rate. Various financial organisations – including investment companies and banks – have seen debt securitisation as an asset management opportunity enabling them to earn fee income managing other people’s capital rather than margin income through investing their own capital.

However, the new securities that are created through debt securitisation present many new and challenging valuation issues that remain unresolved.
❖ The resulting securities – CDO – are the product of the interaction of a portfolio of debt securities, most of which are unquoted or are highly illiquid.
The portfolio theory that exists was developed to cater for equities rather than debt. There is very little research published in the academic literature in the area of debt portfolio management. Most of the debate is occurring within the trade literature or at practitioner conferences.

The rating agencies have assumed a central role in creating the framework within which these securities are structured and graded. Without a rating agency grade, a CDO tranche becomes almost unmarketable and, therefore, much of the structuring which takes place and most of the underlying collateral purchase decisions are driven by rating agency requirements.

The practice of modelling debt portfolios is still at an embryonic stage and regulators remain unconvinced by rating agency assessments of the credit risk of structured debt. (BIS 1999a). This is confirmed by the more onerous treatment of lower-rated CDO securities compared to equally rated single name corporate debt securities (BIS 2001a).

The CDO rating methodologies employed by the rating agencies were subject to particular criticism in the 2001-2 period due to the significantly higher level of re-rating compared to similarly-rated corporate debt securities.

1.3 Objective

There is an obvious gap between academic scientific research and market practice in relation to CDOs. The rigorous academic approach to the subject fails to address the many critical structural issues. Likewise, the practitioners, while addressing these structural issues, settle for extremely heuristic approaches to many other aspects of the structure.

This thesis attempts to embed academic rigour in a model that incorporates the many complex structural features typical of the CDO product. It develops a new model that transforms the current state-of-the-art portfolio modelling approach from a single time period framework to a multi-period setting and values the CDO tranches created from the credit portfolio in a risk-neutral framework. The results of this model will then be compared to those obtained by one of the rating agencies, Moody’s Investors Service (Moody’s). In order to do this, it will be necessary to replicate the state-of-the-art credit portfolio model currently employed by KMV (KMV), the leading credit portfolio risk software vendors, in their product, Portfolio Manager and the CDO tranche rating methodology employed by Moody’s.
It is hoped that the main contribution of this thesis to the literature will be the development of a new model for valuing CDO tranches based on the structural approach, and the comparison of this alternative approach with the rating process currently employed by the rating agencies.

The research proceeds as follows:

A one-time period structural model is developed using KMV's assessment of default probabilities and asset correlations. This enables the value of the individual facilities to be determined, the capital required to support the loan portfolio estimated, and the risk contribution of each facility within the portfolio gauged.

This one time period model is extended to a multi-period setting in order to value CDO tranches. This model comprises two modules: (i) A multi-time step Monte Carlo simulation module to ascertain the behaviour of the credit portfolio between CDO coupon payment dates, and (ii) a cash flow model to disburse the cash flows to the tranches in accordance with the cash flow waterfall.

The Moody's CDO tranche rating model is re-created and the grade of each of the tranches is inferred from this model. This grade is compared to that suggested by application of the Moody's rating process.

In view of the enormous size of the CDO market – in excess of $250 billion are extant - and the controversy which surrounds the rating process, it is suggested that this research is timely and it is hoped it will make a valuable contribution to the literature and help in furthering current understanding of the issues.

1.4 Key Research Issues Addressed

The thesis addresses many significant research issues:

❖ It examines the nature of the interaction among debt securities in portfolios, a subject about which little has been written in the academic literature.
❖ It measures the impact of facilities – both new and existing - on a credit portfolio.
❖ It develops a framework for deciding on the maximum amount of a syndicated loan a bank should purchase.
❖ It presents a methodology for setting credit limits.
❖ It takes the structural approach to credit risk from its standard single period framework into a multi-period setting.
❖ It draws on the two main types of rating data – the rating agency letter grade and the probability of default (PD) metric, expected default frequency (EDF), employed by KMV

❖ It examines actual CDO structures under both approaches

1.5 Overview of Thesis Approach

The thesis proceeds as follows

Chapter 2 introduces the CDO product, discusses the size of the market and its development over the years, and the regulatory attitude to the product

Chapter 3 undertakes a review of the academic literature in the areas of portfolio management, credit default probability assessment and credit portfolio management. It examines the two principal competing approaches – the contingent claims approach and the reduced form – and summarises the main strands in the literature. It then proceeds to discuss the practical implementation of these approaches. It draws on research published in practitioner journals and trade literature published by systems vendors.

The state-of-the-art single time period credit portfolio model in use in the market, that employed by KMV and delivered in their Portfolio Manager software offering, is re-created in Chapter 4. A new method for determining the impact of a new facility on an existing loan portfolio is also developed in this chapter as well as a framework for determining the cost of portfolio concentration and borrower limit setting.

Chapter 5 describes the development of a new model for valuing CDO tranches that is the centrepiece of this thesis. This new model takes the state-of-the-art single time period credit portfolio model and converts it into a multi-period model that incorporates the CDO indenture. In so doing, it is, to the author’s knowledge, the first academic research to take the current market standard credit portfolio paradigm into the structured securities arena.

The principal measure of CDO tranche quality quoted in the market is the rating agency-assigned rating. Chapter 6 replicates the tranche rating methodology employed by Moody’s so that their rating can be attributed to tranches of any deal.

Chapter 7 compares the results of the newly developed model with those from Moody’s model for a variety of CDO structures. The differences between the model-implied
rating and the Moody's rating are explored for each of the tranches. The reasons for these differences are examined and the validity of the competing approaches is assessed.

Chapter 8 summarises the contribution of this thesis to the literature. It critically evaluates the model's assumptions and makes suggestions for further research.

The Appendix develops the CreditMetrics approach to credit portfolio modelling and confirms their published results.

1.6 Conclusion

This chapter provided a brief overview of the CDO market and a summary of the research agenda. The next chapter examines the CDO product in much greater detail and the regulatory approach thereto.
Chapter 2. The CDO Market

2.1 Chapter Overview

In this chapter, the market for CDOs is examined and the many product variations are introduced. The nature of the CDO product gives regulators particular cause for concern; the regulatory attitude and response are detailed.

2.2 CDO Market

While the CDO market began in the late 1980s, the market really only became significant in 1996 as reported by Tavakoli (2003) citing Bank of America and Moody’s and shown in Figure 2.1 below:

![Rated CDO Volume](image)

Figure 2.1 CDO Market Size

However, Tavakoli (2003) distinguishes between what she terms ‘the old paradigm’ of cash flow CDOs which accounted for most of the volume up to 1999 and ‘the new paradigm’ of synthetic CDOs which accounts for most of the growth since then. “Synthetics facilitate more efficient portfolio ramp-up, synthetics facilitate getting a higher average credit rating, and synthetics facilitate more efficient portfolio
diversification." (p8) The synthetic arbitrage is facilitated by the feasibility of a smaller equity tranche, which creates more leverage. The synthetic arbitrage gets a further huge boost from the large, inexpensive super senior tranche that makes up the bulk of the synthetic deal.

The synthetic market accounted for $187.5 billion of the CDOs extant in 2002 compared to $62.5 billion of cash flow CDOs, Tavakoh notes. But she emphasises that the absolute size of synthetic CDO issuance is exaggerated by these figures. "Assuming the super senior tranche makes up 90 per cent on average of the synthetic CDO, only about $18.75 billion of synthetic CDO product is available to traditional investors." (p12) She further notes that the super senior tranche is held in the trading book and is 'marked to market' in theory, but not in practice.

The Moody's review of 2001 was the last one to have been reported on a global basis. It reported 277 transactions covering $101bn of tranche issuance in 2001 compared to 189 transactions in 2000 and $121bn of issuance. Their report on 2003 U.S. CDO activity reflects similar trends but, since it focuses only on U.S. tranches which they rated, presents somewhat different numbers. Figure 2.2 below shows the number of new transactions and associated volume of Moody's-rated CDO tranches year-by-year since the CDO market took off in the mid-1990s as reported by Gluck (2004).
The paradigm shift that Tavakoli noted may well have run its course. Commenting on activity in the fourth quarter of 2003, Gluck notes the notable reversal of the long-term trend towards synthetics. Only 12 of the 58 fourth quarter CDOs took synthetic form, or just over 20%. “The pattern shift was entirely due to the lack of arbitrage opportunity in the investment-grade corporate sector.” Gluck (2004, p2)

The product mix underlying the CDO market has shifted dramatically in the past few years. The product which was the primary driver of the market in its early years, the high-yield collateralised bond obligation (CBO), was “nearly dormant” in 2003 (p6). High-yield CBOs accounted for 30% of Moody’s-rated U.S. CDOs in 2001; by 2003, that had fallen to a mere 2% as shown in Figure 2.3 below. All acronyms are explained in the List of Acronyms below:
Likewise, there has been a dramatic shift from bonds to structured debt in the collateral underlying the CDO product as evidenced in Table 2-1 below based on Gluck (2004, 2003)

<table>
<thead>
<tr>
<th></th>
<th>2003</th>
<th>2002</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corporate Bonds</td>
<td>36%</td>
<td>54%</td>
</tr>
<tr>
<td>Loans</td>
<td>24%</td>
<td>17%</td>
</tr>
<tr>
<td>Structured Debt</td>
<td>40%</td>
<td>29%</td>
</tr>
</tbody>
</table>

Table 2-1 Collateral Underlying Moody's-rated U.S. CDO Deals

2.3 **Overview of the CDO Product**

A CDO is an asset-backed security backed by a diversified pool of one or more classes of debt. In a CDO structure, there is an asset manager responsible for managing the portfolio of debt obligations. There are restrictive covenants imposed on what the manager may do and certain tests that must be satisfied for the debt obligations to maintain the credit rating assigned at the time of issuance.
Collateralised debt obligations are structured debt products. They are liabilities of a special purpose vehicle whose only assets are debt securities – either loans or bonds - issued by corporates. These assets by definition, present credit risk. Therefore, the starting point for this research is to gain a thorough understanding of the credit risk of individual firms.

“A traditional securitisation is a structure where the cash flow from an underlying pool of exposures is used to service at least two different stratified risk positions or tranches reflecting different degrees of credit risk. Payments to the investors depend upon the performance of the specified underlying exposures, as opposed to being derived from an obligation of the entity originating those exposures.” BIS (2003, p100) They suggest the primary difference between the stratified/tranched structures that characterise securitisations and ordinary senior/subordinated debt instruments relates to the cash flow diversion mechanism: junior securitisation tranches can absorb losses without interrupting contractual payments to more senior tranches, whereas subordination in a senior/subordinated debt structure is a matter of priority of rights to the proceeds of a liquidation.

An understanding of the credit risk of the issuing firms individually will not suffice. The value of a CDO is determined by the interaction of many debt securities. Any attempt to value a CDO must seek to gain an in-depth appreciation of the manner in which those individual credit risks behave as a group because the cash flows from the asset pool are channelled through the cash flow waterfall to the individual CDO tranches in order of priority.

2.3.1 CDO Sponsor Motivation

BIS (2001a) notes that banks that securitise assets are able to accomplish several objectives. By securitising rather than holding the originated assets, they suggest banks attain a number of objectives: (i) they can secure a reduction in regulatory capital requirements; (ii) they can tap an additional source of funding, generally at a lower cost; (iii) they can enhance their financial ratios; and (iv) they can manage their portfolio risk by reducing large exposures or sectoral concentrations. By investing in tranches of other banks’ securitisations, they suggest banks are able to diversify their portfolios by acquiring different asset types from different geographic areas.

Bluhm (2003) suggests four possible motives for CDO creation:
Spread arbitrage: This occurs where the total spread collected on single credit risky instruments at the asset side of the transaction exceeds the total ‘diversified’ spread to be paid to investors on the tranched liability side of the structure. Such a mismatch typically creates a significant arbitrage potential which offers an attractive excess spread to the equity or subordinated notes investor.

Regulatory capital relief: The Basel I regulatory capital requirements often exceed the economic capital required given the risks that many loan assets embed. In such circumstances, it is possible for a bank to obtain credit protection for a relatively modest cost once it retains a tranche that absorbs most of the loss which is likely to be experienced. The capital associated with the first-loss piece combined with the 20% risk-weighting of the super-senior credit default swap will necessitate significantly less – 50% or less - regulatory capital than would otherwise be required to hold low-risk assets on balance sheet. As ‘opportunity costs’ for capital relief, the originating bank has to pay interest to notes investors, a super senior swap premium, upfront costs (rating agencies, lawyers, structuring and underwriting costs) ongoing administration costs and possibly some other expenses. “A full calculation of costs compared to the decline of regulatory capital costs is required to judge about the economics of such transactions.” (p7)

Funding: For banks with sub-AA ratings, funding can become too expensive to allow them put high-quality assets on balance sheet. Equally, even highly rated institutions like to have a range of funding sources available should they ever need it. For non-bank institutions, accessing relatively inexpensive funding through securitisation is a key consideration. The advantage of refinancing by means of securitisations is that resulting funding costs are mainly related to the “credit quality of the transferred assets and not so much to the rating of the originator.” (p8) He notes, however, that some linkage remains to the originator’s rating, if the SPV also enters into a servicer agreement with the originating bank. In such cases, investors and rating agencies will evaluate the servicer risk inherent in the transaction.

Economic risk transfer: The final motivation Bluhm suggests is economic risk reduction. This is a key reason for many regional banks undertaking securitisations. Where they have a strong local franchise but no global presence, concentrations to key customers develop. These concentrations, if held on balance sheet, require significant capital; however, they will not present any concentration risk to investors outside the
region and can be supported by substantially less capital. Securitising its excess exposures to its best customers is a way of deriving portfolio benefits. However, Bluhm, commenting on some transactions undertaken primarily for regulatory or funding reasons, notes that the capitalisation rate for the remaining portfolio can be higher as a result of poor sub-portfolio selection. "[S]ecuritising a subportfolio can cause some negative effect on the economic capital of the residual source portfolio" (p9) due to the diversification turn-down caused by taking away a pool of diversifying assets.

2.3.2 CDO Market Practice

The 2002 Survey of Credit Portfolio Management Practices undertaken by the International Association of Credit Portfolio Managers asked financial institutions about their use of securitisation. The questions and responses are given below.

In order to transfer loans from the institution, has your institution issued a CLO – either cash or synthetic?

- No 27%
- Yes, traditional CLOs 20%
- Yes, synthetic CLOs 24%
- Yes – both cash and synthetic CLOs 29%

If your institution has issued a CLO, rank these motivations by order of importance (Use 1 to denote the most important and 3 to denote the least important).

- Regulatory capital 1 68
- Economic capital 2 21
- Exposure management (freeing lines) 2 07

Has your institution used a CLO structure as a way of transferring loan exposures into the institution? That is, have you purchased the equity or subordinated tranches of someone else’s CLO or have you set up a CLO structure using assets from other originators as a way of importing credit risk?

- No 59%
- Yes, traditional CLOs 10%
- Yes, synthetic CLOs 13%
- Yes – both cash and synthetic CLOs 18%
It is clear from these survey results that financial institutions have embraced CLOs largely as a way of circumventing the Basel 1 rules which militate against holding high-quality assets on balance sheet. The preferred vehicle for achieving this result is the synthetic CLO.

### 2.4 CDO Categories

The two main categories of CDO transaction can be distinguished based on sponsor motivation described in 2.3.1 above.

**Arbitrage Transaction** This is the name given to a transaction where the primary motivation of the sponsor is to earn a spread between the yield offered on the collateral assets and the payments made to the various tranches.

**Balance Sheet Transaction** When the sponsor's main concern is to remove debt instruments from its balance sheet, it is classed as a balance sheet transaction. This type of structure is often adopted by a financial institution seeking to reduce its capital requirements where the regulatory capital necessary to support the debt exceeds the economic capital.

Merritt *et al* (2001) further sub-divide these categories as shown in Figure 2.4 below.

![Figure 2.4 Categorisation of CDOs](image)

There is a clear gap between the treatment of synthetic structures and those based on cash flows. These differences are neatly summarised by Cifuentes *et al* (2004, p101).
<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Typical Cash CDO</th>
<th>Typical Synthetic CDO</th>
</tr>
</thead>
<tbody>
<tr>
<td>Collateral</td>
<td>Leveraged loans</td>
<td>CDS referencing balance sheet assets</td>
</tr>
<tr>
<td></td>
<td>High-yield bonds</td>
<td></td>
</tr>
<tr>
<td>Size</td>
<td>$200m - $600m</td>
<td>$1bn plus</td>
</tr>
<tr>
<td>Collateral Quality</td>
<td>Sub-investment grade</td>
<td></td>
</tr>
<tr>
<td>Diversity</td>
<td>60</td>
<td>80</td>
</tr>
<tr>
<td>Payment Frequency</td>
<td>Semi-annually</td>
<td>Quarterly</td>
</tr>
<tr>
<td>Maturity</td>
<td>7-12 years</td>
<td>3-5 years</td>
</tr>
<tr>
<td>Prepayment Risk</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Equity Leverage</td>
<td>8-12 times</td>
<td>30-100 times</td>
</tr>
<tr>
<td>Interest Rate Risk</td>
<td>Managed with swaps</td>
<td>None</td>
</tr>
<tr>
<td>Management</td>
<td>Typically managed</td>
<td>Typically static</td>
</tr>
</tbody>
</table>

Table 2-2 Comparison of Cash and Synthetic CDOs

This thesis will focus on cash CDOs with the characteristics summarised in Table 2-2

2.4.1 Arbitrage Transactions

Arbitrage CDO transactions may be further categorised based on the primary source of funds to repay the tranches. 

*Market value transactions* rely heavily on the total return generated from the active management of the collateral assets. Funds used to repay liability principal are derived primarily from collateral liquidation.

In contrast, *cash flow transactions* are those in which the interest and principal from maturing assets are the primary source of cash with which to repay the tranches.

2.4.1.1 Market Value Transactions

Market value CDOs are transactions "in which the credit enhancement is reflected in a cushion between the current market value of the collateral and the face value of the structure's obligations" (Falcone and Gluck, 1998, p1). Whereas cash flow transactions normally provide for the diversion of cash flows from junior to senior classes if certain tests that relate to the structure's soundness are not met, in a market value transaction, "the collateral must normally be liquidated, either in whole or in part, if the ratio of the market value of the collateral to the obligations falls below some threshold" (p1). The liquidated collateral is used to pay down obligations, bringing the structure back into balance.
Market value transactions depend upon the ability of the fund manager to maintain and improve the market value of the collateral. “Ratings are based on collateral price volatility, liquidity, and market value.” Goodman and Fabozzi (2001, p174). The manager focuses on maximising total return subject to an acceptable volatility level. The market value of the collateral assets, multiplied by rating agency-specified advance rates, must exceed the value of debt outstanding. Failure to meet these over-collateralisation tests requires the manager to undertake collateral sales and liability redemption to bring the test back into compliance.

Market value transactions give more flexibility to the manager in choosing collateral. Distressed debt and debt which matures beyond the life of the transaction can be accommodated within a market value structure whereas they would prove wholly unsuited to a cash flow structure. The liquidity premium, which has made high-yield debt attractive to the buy-and-hold investor, will be relinquished if the manager is obliged to sell the asset.

Cash flow CDOs exhibit minimal trading. On the other hand, market value CDOs may be expected to trade frequently. “A market value CDO is one for which the CDO tranches receive payments based essentially on the mark-to-market returns of the collateral pool, which depends on the trading performance of the CDO asset manager.” Duffie and Singleton (2003, p250). Analysis of market value CDOs is primarily an analysis of the trading behaviour of the CDO manager they suggest. Thus, the portfolio manager has a much bigger influence on the performance of a market value transaction. Potential investors must carefully examine the manager’s investment style and philosophy and the investment criteria adopted.

Market value deals face risks that are distinctly different from those faced by cash flow deals despite the fact that the underlying assets are largely similar. The biggest risk in a market value transaction is a sudden decline in the value of the collateral pool, according to Goodman and Fabozzi (2001, p174). Thus, the rating agency focus is on the price volatility and liquidity of the assets and this is reflected in a set of advance rates designed to provide a cushion against market risk.

Market value deals represent a minority of CDOs. Indeed, Moody’s rated only a single market value deal in the whole of 2003 though they were projecting an increase in activity for 2004. Current market value proposals tend to focus on more liquid, and thus more easily marked, asset classes, Gluck (2004, p6) suggests.
This thesis addresses the risk of credit assets in a primarily static portfolio. It does not attempt to model the market risk created by short-term price changes and the requirement to sell assets to return to compliance. While the risk driver in market value CDOs is credit, the time frame is short and credit risk presents itself as market risk. The framework that is developed is incapable of addressing this market risk. For this reason, but also because of the fact that this sector of the market is particularly small, market value deals will not be examined further.

2.4.1.2 Cash Flow Transactions

According to Duffie and Singleton (2003, p250), a cash flow CDO is one for which the collateral portfolio is not subject to active trading by the CDO manager, implying that the uncertainty regarding interest and principal payments to the CDO tranches is determined mainly by the number and timing of defaults of the collateral securities.

The objective of the asset manager in a cash flow transaction is to generate cash flow for the senior and mezzanine tranches without active trading of bonds. Because the cash flows from the structure are designed to accomplish the objective for each tranche, restrictions are imposed on the asset manager. Goodman and Fabozzi (2001, p15) note that the asset manager is very limited in his or her authority to buy and sell bonds. The conditions for disposing of issues held are specified and are usually driven by credit risk management. Also, in assembling the portfolio, the asset manager must meet certain requirements set forth by the rating agencies that rate the deal. They conclude that the most important of these requirements are embedded in the, so-called, cash flow waterfall, described in 2.5.2 below.

Two further tests are imposed by the rating agencies to ensure that the asset manager does not adversely affect the quality of the collateral:

- A maximum weighted-average rating factor (WARF) is set for the collateral pool.
- A minimum diversity score is set for the asset pool.

More complete details of the Moody’s papers describing the calculation of WARF and diversity score are given in 3.9.1.1 below.

2.4.1.3 Source and Sustainability of the Arbitrage

Tavakoli cautions: “There is no such thing as a CDO arbitrage.” (2003, p15) Cifuentes (2004, p4) et al concur, choosing instead to characterise the ‘arbitrage’ as a ‘funding
gap'. This is, of course, true since profit, if it is made, will not be riskless. However, the 
market persists in using the term loosely.

Deal economics are determined by the extent of the arbitrage that exists between the 
assets and liabilities and issuance volume rises or falls with the ‘arb’. The most common 
CDO asset class is the high-yield bond or leveraged loan, with an average rating 
between B1 and B3. Typically, 70% of the liabilities will earn an AAA rating; a further 
15% could earn a BBB rating with the 15% balance supported by equity. Clearly, the 
mezzanine tranche could be further tranched to achieve higher and lower ratings, or 
attempting to achieve higher diversity could reduce the equity.

Cifuentes et al (p3) describe the market conditions which must exist for an arbitrage 
CDO to be created. The portfolio weighted average yield less the weighted average cost 
of debt less expenses associated with arranging the CDO must leave sufficient residual 
cash flows to make the equity position attractive, they conclude.

A number of researchers have addressed the reason for the existence of the arbitrage 
that is fundamental to the existence of the market. Common themes running through 
this literature are market imperfection and adverse selection.

Part of the reason for the arbitrage Cifuentes et al suggest is the low funding cost locked 
in at the outset of the transaction. They suggest that the senior noteholders are the ones 
providing the funding subsidy: If the investors were to borrow money from a bank to 
fund the purchase of the portfolio, the “costs would clearly be higher, as bank funding 
costs, up-front fees and bank profit margins are factored in.” (2004, p4)

The CDO product was created to address problems arising from market imperfections: 
Duffie and Singleton (2003, p252) contend that in perfect capital markets, CDOs would 
serve no purpose as the costs of constructing and marketing a CDO would inhibit its 
creation. They cite two imperfections which could support a CDO market: first, banks 
and certain other financial institutions have regulatory capital requirements that make it 
valuable for them to securitise and sell some portion of their assets, reducing the amount 
of (expensive) regulatory capital they must hold. They also note that as individual bonds 
or loans may be illiquid, this may lead to a reduction in their market values. 
Securitisation and prioritisation may improve liquidity, they suggest, and thereby raise 
the total market value of the CDO structure relative to the sum of the market values of 
its collateral components.
Falcone and Gluck (1998, p1) contend that, in large part, cash flow CDOs have succeeded because they exploit the illiquidity of the high-yield markets. The spreads on high-yield debt have historically more than compensated for the default risk associated with such debt, according to Bencivenga (1997). The gap between the yield on the high-yielding assets and the cost of the lower-yielding liabilities offers the equity the opportunity to earn a return. Clearly, the extent to which that potential return is realised depends on the ability of the asset manager to show good selection skills. In most deals, the asset manager also holds between 20% and 49.9% of the equity of the deal and also earns a performance-based bonus. In most cases, 2% of the liabilities issued are used to meet upfront expenses and a further 70bp p.a. of ongoing expenses is typical. When the projected internal rate of return (IRR) on the equity tranche rises above 15% p.a., activity in the CDO market is known to pick up. Since equity is approximately a six-times leveraged position in the underlying assets, any improvement in the ‘arb’ makes the equity decidedly more attractive.

A substantial part of the arbitrage CDO market is based on sub-investment grade debt, so-called ‘junk’ bonds and loans. There is a suspicion that significant amounts of private information exist regarding the credit quality of this debt and outside investors may find themselves ‘picked off’ when trading this debt. The reduction in price owing to adverse selection was called a lemon’s premium by Akerlof (1970).

DeMarzo (1998) suggests that the CDO structure helps mitigate this lemon’s premium, drawing investors to the CDO market who would be unwilling to invest in the underlying debt directly. Duffie and Singleton (2003, p253), surmise that the seller achieves a higher total valuation (for what is sold and what is retained) by designing the CDO structure so as to concentrate the majority of the risk about which there may be fear of adverse selection into small subordinate tranches. They conclude that this allows a large senior tranche, relatively immune to the effects of adverse selection, to be sold at a small lemon’s premium.

The extent of the arbitrage varies widely over the credit cycle and issuance volumes are highly correlated with its size. The narrowing of the arbitrage in recent years has had a marked impact on the market. Gluck (2004, p3) comments that the single most striking development in the U.S. CDO market during 2003 was the narrowing of corporate, and, to a lesser extent, structured instrument credit spreads. Because the impact of this spread narrowing was far more dramatic on the collateral side than on the CDO liability
tranche side, the opportunity to earn arbitrage gains from the gap between asset- and liability-side spreads contracted sharply. He concludes that the collapse in the arbitrage opportunity was most notable for transactions backed by investment-grade corporates. *I.e.* for conventional synthetics.

Some are questioning the future of the arbitrage CDO market. Gluck (2004, p6) projected near-zero growth in the U.S. market during 2004 and opined that it was difficult to envision a widening of credit spreads to the point where the corporate CDO arbitrage opportunity improves dramatically. He wonders if the increasing ability to hedge corporate credit risk through synthetics had resulted in a permanent contraction of the liquidity component within corporate credit spreads. He suggests that hedging costs have at least theoretically been cut through the increase in the number of liquid names traded in the credit default swap market. He surmises that a more likely source of a restored arbitrage opportunity would be a narrowing of CDO liability costs.

### 2.4.2 Balance Sheet Transactions

The benefit of securitisation to issuers is in the off balance sheet treatment achieved, as well as the capital relief gained to the extent that the underlying assets attract regulatory capital charges. According to Duffie and Singleton (2003, 252), the balance sheet CDO, typically in the form of a CLO, is usually designed to remove loans from the balance sheet of banks, achieving capital relief and perhaps also increasing the valuation of the assets through an increase in liquidity. Another essential benefit is the diversification of funding sources. Funding through securitisation is sometimes cheaper than raising unsecured debt for banks with ratings less than AA, they suggest.

When balance sheet transactions were first undertaken in the early 1990s, a bank typically sold a pool of high-quality loans to a special purpose vehicle (SPV) and took back the first loss piece. The bank benefited to the extent of the difference between the mandated 8% regulatory capital requirement and the lower economic capital requirement.

However, while a balance sheet CLO solved the capital problem, two problems remained: (i) funding cost and (ii) confidentiality.

**Funding Cost** banks have a lower cost of funds than the typical purchaser of AAA-rated debt does. Banks give up this funding advantage by issuing AAA-rated debt to fund its loans.
Confidentiality If a loan is transferred into an SPV, borrower notification and, sometimes, borrower consent are required. Banks are loath to make their clients aware that they are selling their loans. Duffie and Singleton (2003, p252) claim that the direct sale of loans to SPVs may sometimes compromise client relationships or secrecy, or can be costly because of contractual restrictions on transferring the underlying loans.

For these reasons, balance sheet securitisations have migrated from being fully funded CLOs involving asset transfer to fully funded synthetic structures where the assets stayed on the institution's balance sheet but their credit risk was hedged using credit derivatives. Still later, partially funded synthetic CDOs replaced fully funded where a guarantee from an OECD bank replaced the collateral as the source of reimbursement in case of default. More recently still, the most senior piece – typically the top 85% of the structure – is treated as 20% risk-weighted for regulatory purposes regardless as to whether there is a credit default swap referencing it. This most senior tranche is often referred to as the 'super senior piece' because it ranks ahead of other debt that is rated AAA.

This thesis focuses on cash flow structures rather than synthetic structures that absorb credit risk through the use of credit default swaps. There have been no new cash flow-based balance sheet CDOs issued in recent years. Thus, what follows may be seen to refer only to arbitrage structures.

2.5 **CDO Structure**

A typical CDO structure is shown in Figure 2.5 above.

![Figure 2.5 Typical CDO Structure](image-url)
The funds to purchase the collateral assets are obtained from the issuance of debt obligations, known as tranches. Typically, three tranches of debt are issued: senior, mezzanine and subordinate/ equity and these tranches may be further sub-divided. There will be a rating sought for all but the subordinate/ equity tranche. An A-rating at least will usually be sought for the senior tranche while a rating of BBB but no less than B will be sought for the mezzanine tranche. The subordinate/ equity tranche receives the residual cash flow and is invariably unrated.

The fund manager decides on the composition of the collateral portfolio and earns a fee. Should there be any mismatch between the liabilities and the assets, either in currency or interest rate, hedges need to be put in place to avoid penal treatment from the rating agencies. Amortising interest rate swaps are put in place if there is an interest rate mismatch. Since prepayments and defaults cannot be anticipated exactly, swaptions are taken out on a portion of the mismatch.

The order of priority of the payments of interest and principal to the CDO tranches is specified in the prospectus. Payments are made in such a way as to provide the highest level of protection to senior tranches in the structure. This is achieved by providing certain tests that must be satisfied before any distribution of interest and principal may be made to other tranches in the structure. If these tests are failed, the senior tranches are paid down until the tests are passed.

The ability of the asset manager to make the interest and principal payments to the debt holders depends on the performance of the collateral assets. The proceeds to meet the tranche obligations come from (i) coupon interest payments from the collateral assets, (ii) maturity of collateral assets, and (iii) recovery on defaulted assets.

2.5.1 CDO Life Cycle

There are three relevant periods in the life of a CDO:

Ramp-up period: The first period, known as the ramp-up period, usually lasts less than a year; during this period, the asset manager begins investing the proceeds from the sale of the debt obligations. Frequently, when a financial institution is the asset manager, many of the assets will be pre-purchased by the manager and held on the institution’s own balance sheet so that the ramp-up period may be shortened and the negative carry associated with having investors’ funds only earning Libor may be minimised.
Reinvestment period The next period, known as the reinvestment period is that in which the manager may reinvest principal proceeds from maturing or pre-paying assets, subject to compliance with the relevant tests. This period is typically, five years or more.

Final period The final period sees the cash flow from maturing assets paid to the investors. However, early termination may be triggered by failure to comply with certain covenants or failure to meet payments to the senior tranches. The equity-holders may also trigger the collapse of the structure by calling the deal if they perceive that there is greater value for them in doing so.

2.5.2 CDO Cash Flow Waterfall

One of the most important details of a CDO structure is the specified priority of payments to the tranches. This payment priority is usually called the cash flow waterfall, getting its name from the fact that cash flows down the structure based on a set of tests described below. A typical cash flow waterfall is shown in Figure 2.7.

![Cash Flow Waterfall Diagram](image-url)

Figure 2.6 Cash Flow Waterfall
Interest payments are allocated to the tranches in sequence in accordance with the priority shown. Key to the channelling of cash through the waterfall is the passing of coverage tests. Two types of coverage tests exist: overcollateralisation (O/C) tests and interest coverage (I/C) tests.

The O/C ratio for a tranche is defined as follows:

\[
Tranche \ O/C = \frac{\text{Principal par value of the Collateral Assets}}{\text{Principal for Tranche} + \text{All Tranches senior to it}}
\]  
Equation 2-1

The O/C test for a tranche involves a comparison of the tranche’s O/C ratio with the required minimum ratio, the O/C trigger. The lower the seniority, the lower the trigger, not surprisingly given that the denominator is larger the more junior the tranche.

The other test is an I/C test. The I/C ratio, is defined analogously:

\[
Tranche \ I/C = \frac{\text{Interest Due to the Collateral Assets}}{\text{Interest due to Tranche} + \text{All tranches senior to it}}
\]  
Equation 2-2

The I/C test is passed if the I/C ratio exceeds the minimum ratio specified in the prospectus.

2.5.3 Problems in Structuring CDOs

Other researchers have argued that, while the CDO structure helps mitigate the lemon’s premium, it creates problems of moral hazard in its place. Investors fear the manager may engage in cherry-picking the worst assets from its own portfolio for inclusion in the CDO. Also of concern to investors is that the manager may engage in front-running the CDO.

Thus, the issuer has an incentive to indicate to the market that it will not engage in such activities. Many managers retain significant portions – typically between 20% and 49.9% - of one or more subordinate tranches. Likewise, Schorin and Weinrich (1998) point out that in many deals, more than half of the management fees may be subordinated to the issued tranches.

2.6 Synthetic Credit Products

Traditional CDOs enable the transfer of credit portfolio risk on a fully funded basis. The requirement to fund the purchase of the underlying credit assets necessitates the issuance of CDO tranches. This requirement to pay cash for CDO tranches introduces
two complications: (i) the purchaser of a CDO tranche needs to be able to fund itself in an efficient manner and (ii) the fully funded nature of the assets requires that a trustee be retained to hold the assets and collect and disburse the associated cash flows.

Traditional CDOs, with all their attendant cash flow complications, remained the only form of portfolio credit risk transfer throughout most of the 1990s. Their high-cost nature substantially limited their scope for application. Modelling, structuring and placing the tranches in the market required that an investment bank be retained. The need to fund the entire structure demanded that all the tranches apart from the first loss piece be rated. Furthermore, a trustee was required to keep the assets separate from the sponsor who typically managed the assets.

Funding the CDO tranches significantly reduces the potential investor universe. Insurance companies and hedge funds are loath to fund such investment though they may be quite happy with corporate credit as an asset class. Likewise, the handling of cash flows adds significantly to the administration costs.

Interest rate risk further complicates the issue. While many investors may feel comfortable assuming pure credit risk, they inadvertently assume an element of interest rate risk when investing in CDOs. Swaps alone will fail to hedge interest rate mismatches between assets and liabilities because of the uncertain principal repayment profile caused by prepayment, reinvestment and default. The residual interest risk needs to be hedged using options, an additional expense that erodes the return that would otherwise be available to those willing to bear credit risk.

Participants sought ways of circumventing these problems thereby creating a corporate credit portfolio asset class that was less expensive and could appeal to a wider investor base.

2.6.1 Credit Default Swaps

Credit derivatives provided the answer. Credit derivatives – in particular, credit default swaps (CDS) – have become the investment medium of choice in recent years for those wishing to take on pure credit risk. Their unfunded nature makes them the ideal instrument for those without easy access to funding at a competitive rate. This has greatly increased market liquidity that in turn has narrowed bid-ask spreads, improving efficiency. It enables those with a credit risk appetite to take on the credit risk of
companies with a higher credit rating than their own since they can do so on an unfunded basis.

Figure 2.7 Credit Derivative Market: Breakdown by Derivative Type

Figure 2.7 above and Figure 2.8 below show the composition of the credit derivative market by derivative type and market participant, respectively, based on a poll of twelve dealers at the end of 2002. The total notional market outstanding across all credit derivative products was estimated at $2.3 trillion, a 50% increase on the previous year. O’Kane et al (2003, p1) remark: “While not directly comparable, it is worth noting that the total notional outstanding of global investment-grade corporate bond issuance currently stands at $3.1 trillion.”
Banks are the biggest users of credit derivatives, using them both to take on and to hedge credit risk. Credit default swaps are the dominant product type but portfolio correlation products have increased in significance compared to previous years.

The credit default swap is the basic building block for most exotic credit derivatives, transferring as it does the credit risk of a reference entity from one party to another. Following a credit event, the protection buyer typically delivers the cheapest reference asset in return for par. Approximately 500 names world-wide have liquid CDS. These firms are large household names, and predominantly investment-grade.

2.6.2 Credit Correlation Products

The portfolio correlation product category is comprised of synthetic CDOs and default baskets with a total notional value of $449 billion. The market for portfolio correlation products has grown in line with the credit derivatives market itself. This is to be expected because there is a symbiotic relationship between the single-name CDS market and the synthetic CDO market: synthetic tranche products are hedged using CDS.
2621 Basket Default Products

The simplest correlation product is the basket default swap. It is similar to a CDS except that the trigger is the n-th credit event in a specified basket of reference entities. The contingent payment typically involves physical delivery of the defaulted asset in return for a payment of the par amount in cash. A first-to-default (FTD) basket is a way of leveraging the credit risk by increasing the probability of loss without increasing the size of the potential loss.

2622 Traded CDS Portfolio Products

Traded CDS portfolio products have developed significant liquidity in recent years. They enable the investor to go long or short a portfolio of CDS in one transaction. TRAC-X is one such index.

2623 Synthetic CDOs

Synthetic CDOs were first used in 1997 as a flexible, low-cost mechanism for transferring credit risk off bank balance sheets. Their primary motivation was regulatory capital arbitrage. They provided banks with a mechanism for transferring the credit risk of loans without the need to sell these loans, which could otherwise have required informing the borrower or possibly even seeking borrower consent. They also provided a way of managing the credit risk of revolving credits, something to which fully-funded CDOs were unsuited.

Under Basel II, synthetic CDOs will no longer be created for regulatory capital arbitrage because the risk-weighting of high-quality assets will be reduced to reflect their lower risk. Furthermore, as discussed in 274, Basel II ensures that the regulatory capital required to be held against a portfolio of loans on a bank's balance sheet will be less than the capital to be held against the CDO tranches of synthetic CDO referencing those same assets.

In more recent years, synthetic CDOs have found a whole new role. Gluck (2003, p6) notes that, beginning in 2000, the synthetic structure began to be adopted for arbitrage CDOs and became close substitutes for cash flow, investment-grade CDOs. The so-called customised CDO has been created which enables investors assume credit risk that exactly matches their appetite. The Risk 2003 survey shows the total market size to be
approximately $500 billion. Investors can specify the amount of tranche subordination – the attachment point – and the tranche thickness, matching their exact requirements.

**Full Capital Structure Synthetic CDOs**

In a typical full capital structure synthetic, 10% or less of the credit protection is funded through the issuance of notes. These proceeds are typically invested in high-quality securities. The remaining 90%, or more, of the credit protection is distributed in an unfunded format via a senior swap. This substantially reduces the cost of obtaining credit protection compared to the fully funded traditional CDO. Instead of paying 45bp over Libor for funding, which in many cases could have been achieved by a bank at close to Libor, a CDS premium of less than 10bp was required, significantly reducing the cost. AAA-rated reinsurers who were keen to provide the CDS would be incapable of funding a senior tranche.

Another major advantage that the synthetic CDO enjoys over its traditional cousin is simplicity. The reference portfolio, typically, is static, all the referenced credits are for a single maturity, there is no interest rate risk, and there are no cash flows to be managed. Little wonder, therefore, that synthetic CDO volumes have far out-weighed traditional CDO volumes in recent years for investment-grade names, which, in total issuance terms, dominate the market.

**Single-Tranche CDOs**

As the name implies, single-tranche and full capital structure CDOs differ as regards the extent of liability tranches which are created. While the full capital structure CDO issued equity, mezzanine and senior parts of the capital structure, customised synthetics, more frequently called ‘single-tranche’, may issue just one tranche.

Gluck (2004, p5) comments that single-tranche synthetics allow investors to take on exposures to credit baskets of the investors’ choosing. The investors, he surmises, may be motivated by views on default/recovery rates that differ from those of the market, or by different views regarding default correlation. Dealers can absorb the risks that arise from these reverse-mquiry deals by delta-hedging in the single-name CDS market.

They are customised in that the investor can specify the credits in the collateral, the trade maturity, the attachment point, and the tranche width. The tranches frequently carry an agency rating, which avoids regulatory risk-weighting issues that frequently
accompany FTD baskets. The resulting tranches often achieve investment-grade ratings despite their subordination within the capital structure and offer higher spreads than comparably rated single-name corporate debt. It is a matter of debate whether the rating is comparable though it may take many years for the truth to emerge.

**Other Synthetics**

The standard synthetic CDO product is a relatively simple product to structure and price. Closed-form solutions exist for pricing and for delta-hedging the various underlying credits. Inevitably, variations on the basic scheme were developed in order to make some of the tranches appeal to new investor categories.

Some introduced structural features such as reserve account funding which divert excess spread into a reserve account which is then available to absorb losses. Others reintroduced the over-collateralisation triggers commonly found in traditional structures. Still others introduced principal-protected versions in which only the size of coupon is impacted by losses. The recently introduced managed synthetic gives the asset manager the flexibility to trade names in the portfolio.

Finally, the CDO of CDOs, more commonly known as ‘CDO squared’, has grown in popularity. The underlying collateral, typically, comprises mezzanine tranches of other CDOs. Thus, an additional layer of leverage is applied to the traditional CDO. This product clearly compounds the complexity of the traditional CDO where than the basic synthetic CDO simplifies it.

### 2.7 Regulatory Approach

The Basel Committee on Banking Supervision, known as the Committee, of BIS has expressed its views regarding credit risk modelling and securitisation on a number of occasions in the past few years. In particular, it has considered the potential for portfolio credit risk models in setting regulatory capital requirements. "The Committee commends the use and continued development of such models." BIS (1999, p41) It acknowledges that credit risk models may enable better risk management within banks. It also suggests that such models would have the potential to be used by regulators in their bank supervisory role.
271 Regulatory Attitude to Credit Risk Modelling

However, it expresses reservations about the use of such models in setting regulatory capital for credit risk. Among the concerns it cites, data availability, model validation and the need for banks to prove that they are actively managing risk based on model outputs, were paramount. The Committee has stated that it will monitor developments. It is clear, however, that it views as significant the hurdles that remain to be cleared. It is equally clear that the adoption of a credit risk model-based approach to the setting of regulatory capital is many years away.

Credit portfolio risk modelling is not sufficiently well-developed in the Committee’s view to trust model outputs for tranche capital. “[T]he Group has ruled out the possibility of allowing banks to rely on their own assessments of the credit risk of securitisation exposures for regulatory capital purposes” (2001, p2) The reason they suggest is that this would require banks to use credit risk models for assessing correlation effects within the underlying pool. They contend that “credit risk models are not yet at the stage where they can play an explicit part in setting regulatory capital requirements” (1999, p14) Five years later, they concede that the final Basel II document “stops short of allowing the results of such credit risk models to be used for regulatory capital purposes” BIS (2004, p5)

272 Regulatory Attitude to Asset Securitisation

BIS has expressed misgivings about the role of asset securitisation for many years. For example, in discussing the merits and weaknesses of the 1988 Capital Accord in its 1999 consultative paper, it speaks of the ability of banks to arbitrage their regulatory capital requirement and exploit divergences between true economic risk and risk measured under the Accord. BIS (1999, p9) They note that securitisation facilitates regulatory capital arbitrage and can lead to a shift in banks’ portfolio concentrations to lower quality assets. They note that through such techniques, a bank may be able to achieve an overall risk-based capital ratio that is nominally high but which may hide capital weakness in relation to the actual economic risks inherent in the bank’s portfolio. (p36) CDO structures are explicitly targeted when they say that their proposal is primarily addressing transactions that result in a special purpose vehicle (SPV) issuing paper secured on a pool of assets.
They include securitisation tranches in the higher risk category and propose to establish a 150% risk weighting category to include securitisation tranches that are rated between BB+ and BB- (p32). They also propose that securitisation tranches "rated B+ or below or unrated would be deducted from capital" (p36). In so doing, they were, for the first time, suggesting that the capital required to support a securitisation tranche should exceed that required to support similarly rated corporate debt. In a later document, BIS (2003), they revised the risk weighting to be applied to the tranches in the BB+ to BB- category from 150% to 350%. However, this treatment was limited to investing banks only. Originating banks are obliged to treat as a capital deduction all retained securitisation exposures rated below investment grade (p106).

They elaborate on their view of the risks involved in securitisations in their first working paper on asset securitisation (2001): "[A] well-diversified portfolio of ABS tranches (each backed by a diversified pool of corporate loans) can be expected to exhibit higher default correlations among underlying tranches than a well-diversified portfolio of similarly-rated corporate loans" (p7). They suggest that this reflects the fact that much of the credit risk inherent in a single corporate loan is idiosyncratic risk that can be diversified away within a larger portfolio. They state that the IRB risk weight formula for corporate exposures assumes that bank portfolios are well diversified and, hence, presumes substantial risk reductions through diversification when individual corporate loans are combined within a bank’s portfolio. "In contrast, the automatic pooling of loans within a securitisation implies that an ABS tranche already is purged of much of the idiosyncratic risks of the underlying assets" (p7).

In their Working Paper on the Treatment of Asset Securitisations, they comment that in developing an IRB treatment for securitisations, "the Group has ruled out the possibility of basing the capital requirement on banks' internal assessments of the credit risk of individual securitisation tranches." BIS (2001, p2).

BIS (2003, p8) makes its reservations even more explicit: "One noteworthy point is the difference in treatment of lower and unrated securitisations vis-a-vis comparable corporate exposures. In a securitisation, such exposures are designed to absorb all losses on the underlying pool of exposures up to a certain level." Therefore, the Committee decided this concentration of risk warranted higher capital requirements. In particular, for banks using the standardised approach, unrated securitisation positions must be deducted from capital.
273 Regulatory Attitude to Agency CDO Ratings

The Committee emphasises the difference between rating agency measures of credit risk and the dimension of risk that is of concern to the regulator. They suggest agency ratings are linked to default probabilities or expected losses (EL) on the tranche and do not directly reflect unexpected losses (UL). This expression of standalone risk is not what concerns the regulator; capital charges are intended to capture an asset’s marginal contribution to portfolio risk (defined as EL + UL) under the assumption that the bank’s overall credit portfolio is well diversified and highly granular. (In the final Basel II document, published in June, 2004, capital was defined as covering UL only, EL was explicitly excluded in the final agreed formula.)

They suggest that the link between EL and UL can be expressed in a fairly straightforward fashion for whole loans with “only a single additional regulatory parameter (representing the correlation of the borrower’s performance with systematic risk)” required BIS (2002, p5). However, they suggest that for tranches of a securitisation, the relationship between EL and UL is much more complex and is sensitive to the composition of the underlying pool (p6).

Not only does the economic capital for a securitisation tranche depend on the risk characteristics (e.g., PD and LGD) of the individual underlying exposures securitised, they suggest it also depends on the average asset correlation among the exposures, the number of exposures in the pool, the credit enhancement level of the tranche in question, and the tranche’s thickness. They state that the current Ratings-Based Approach (RBA) risk-weights attempt to take account of these variables in a way that ensures prudential capital levels for a wide variety of possible securitisation structures (p6).

The Committee cites tranche thickness, systematic risk and pool granularity as reasons for requiring much more capital to support a CDO than a similarly rated corporate bond.

Tranche Thickness Apart from the very senior positions, tranches are very thin, accounting for only a small portion of the pool. This will cause the tranches to exhibit loss-rates in the event of default that exceed those for corporate bonds having the same rating (p6).

Systematic Risk Structured securities backed by a granular pool likely embody more systematic risk than a similarly rated corporate loan whose risk is largely idiosyncratic.
The diversification that occurs within the securitisation structure creates systematic risk within securitisation tranches. They point out that the stand-alone credit risk of a securitisation tranche backed by an infinitely granular pool will be effectively all systematic. Thus, they conclude “the marginal contribution to portfolio risk of such a tranche will be larger than a corporate bond with a similar rating” (p6).

**Pool Granularity** They suggest that pool granularity is a key parameter in determining the risk weighting of senior tranches in non-granular pools. As the pool of exposures underlying a securitisation becomes less diversified, the volatility of payoffs on the pool increase. The marginal value-at-risk measures for tranches with different levels of protection, they suggest, become increasingly similar and hence appropriate capital charges for more senior tranches increase (2002, p7).

They comment that senior and higher mezzanine tranches backed by less diversified pools are accompanied by lower external ratings than those backed by diversified pools. They surmise that this seems to reflect the impact of the implied increase in volatility on the expected loss or default probability of these tranches. They wonder if the adjustments made are intended to allow for the increase in unexpected loss on tranches that occurs. They conclude that within the ratings-based approach, the higher capital requirement that a tranche attracts when its pool is less diversified simply because of the lower rating may still not be sufficient to reflect the greater unexpected loss (p7).

The industry reaction to the higher risk weightings was, rather unsurprisingly, one of dismay. In the second Working Paper on Securitisation (WP2), however, the Committee conclude that many within the risk management community “now seem to accept the view that securitisation tranches and loans having identical ratings may warrant different capital charges” BIS (2002, p6).

### 2.7.4 CDO Treatment under Basel II

The Basel II document (BIS, 2004) was finalised in May 2004. Banks opting for the standardised approach to credit risk under Basel II must apply the risk weightings to CDO tranches outlined in Table 2-3 below.
<table>
<thead>
<tr>
<th>Credit Rating</th>
<th>AAA to AA-</th>
<th>A+ to A-</th>
<th>BBB+ to BBB-</th>
<th>BB+ to BB</th>
<th>B- and below or unrated</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk Weight</td>
<td>20%</td>
<td>50%</td>
<td>100%</td>
<td>350%</td>
<td>Deduction</td>
</tr>
</tbody>
</table>

**Table 2-3 CDO Tranche Risk Weightings by Rating Agency Rating**

Furthermore, originators must deduct from capital below investment-grade exposures which they retain.

Banks adopting the RBA must apply the following risk weights:

<table>
<thead>
<tr>
<th>External Rating (Illustrative)</th>
<th>Risk weights for senior positions and eligible senior IAA exposures</th>
<th>Base risk weights</th>
<th>Risk weights for tranches backed by non-granular pools</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>7%</td>
<td>12%</td>
<td>20%</td>
</tr>
<tr>
<td>AA</td>
<td>8%</td>
<td>15%</td>
<td>25%</td>
</tr>
<tr>
<td>A+</td>
<td>10%</td>
<td>18%</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>12%</td>
<td>20%</td>
<td></td>
</tr>
<tr>
<td>A-</td>
<td>20%</td>
<td>35%</td>
<td></td>
</tr>
<tr>
<td>BBB+</td>
<td>35%</td>
<td>50%</td>
<td></td>
</tr>
<tr>
<td>BBB-</td>
<td>60%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>BBB+</td>
<td>100%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>BB+</td>
<td>650%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>BB-</td>
<td>425%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Below BB- and unrated</td>
<td>Deduction</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Figure 2-9 RBA risk weights**

The effect of applying the Supervisory Formula under the Internal Ratings-Based (IRB) approach for a CDO portfolio with an IRB-determined capitalisation rate of 5% is shown in Figure 2-10.
Thus, if the bank chooses to hold on its own balance sheet a first-loss tranche up to 5% in thickness, the bank would suffer a full capital deduction equal to the amount retained. However, if their tranche exceeded the $K_{IRB}$ capital requirement, they would be obliged to hold still more capital. For example, if they chose to retain the lowest 10% tranche, they would be obliged to hold 6.66%, exceeding the 5% $K_{IRB}$ level. This will effectively put an end to the arbitraging of regulatory capital requirements by ensuring that the capital required to support a loan portfolio will increase on securitisation compared to holding the same portfolio on balance sheet. This can also be seen by plotting the capital required to support a €1m tranche of a €100m portfolio with an IRB capital requirement of 5% as shown in Figure 2.11:
This demonstrates that €1m tranches within the $K_{IRB}$ layer bear a €1m regulatory capital burden but, for example, a €1m tranche with 10% subordination will still require €175,000 of capital; the regulators clearly are uncomfortable with tranched credit portfolio risk.

However, following discussions with industry, BIS (2002, p4) relented and agreed to cap the total capital allocated to all securitisation tranches retained by an originator at $K_{IRB}$. Thus, while securitisation is not penalised, banks will no longer securitise their loans for the purpose of regulatory capital arbitrage.

### 2.8 Conclusion

This chapter presented a detailed analysis of the CDO product, market and regulatory treatment. It highlighted the important role of the CDO in the financial markets, it identified the complexities that an evaluation of the cash flow CDO presents, and it discussed the regulators’ concerns.

The next chapter summarises the academic research that has been undertaken in the areas of relevance to structured securities. It examines the development of methodologies for assessing borrower default probability and it summarises the
literature regarding portfolio theory as applied to equities and the extension of that theory to credit portfolios
Chapter 3. Literature Review

3.1 Chapter Overview

This chapter explores the current state of research as it relates to CDO valuation. It begins by establishing the areas of the literature which are relevant to structured debt valuation and proceeds to summarise the seminal contributions to the literature. These contributions lie in the areas of default probability estimation, portfolio theory as originally developed in an equity context, and the adaptation of portfolio concepts to the credit context.

The performance of the theory when subjected to empirical testing is then examined and the shortcomings identified by the empirical academic research come centre-stage. The test results reflect rather poorly on the theory suggesting that the theory can provide no more than a framework for thinking about the issues.

Next, industry efforts to make the academic research discussed in this chapter work in practice are addressed.

Financial products that have assumed a central role in credit portfolio valuation are explored. In particular, the role of credit default swaps and credit correlation products are explored.

The CDO indenture contains many details that the academic literature ignores. These details are central to the rating agency modelling approach. Furthermore, these are central to the valuation issue but are rarely addressed in the academic journals. The chapter concludes with an overview of a rating agency approach to CDO tranche grading.

The theoretical framework that the academic researchers developed is central to the solution. However, in many cases, it has to be supplemented by empirical research. Chapter 4 examines the market-leading solution to the credit portfolio modelling challenge in greater detail.

3.2 Risk Dimensions of Debt Portfolios

Before attempting to build a model for the valuation of CDO tranches, the literature relevant to the various aspects of the issues to be encountered will be reviewed.
First, the literature relating to the estimation of the default risk of individual firms will be reviewed followed by the literature dealing with the valuation of debt securities issued by corporate entities.

Then, the focus switches to the behaviour of securities in portfolios. The literature relating to portfolio theory as applied to equities will be reviewed briefly before the emerging theory on the behaviour of portfolios of debt securities is examined.

3.3 Individual Borrower Credit Risk

A fundamental concern of all involved in the extension of credit is the development of robust methodologies for the evaluation of the credit risk that a borrower presents. This credit risk measurement challenge is sometimes disaggregated into the separate calculation of PD and loss given default (LGD).

Two principal schools of thought have emerged about how best to address this credit risk measurement issue: the contingent claims and the reduced form approaches. The following sections explore the literature on assessing the credit risk of the individual borrower using these two approaches.

3.3.1 The Contingent Claims Approach

The fundamental concept underlying the contingent claims approach is that default is a structural issue: when a borrowing firm’s assets falls below the level of its outstanding liabilities, the firm will avail of the right which limited liability confers on it, the right to renege on debt repayment. To the extent that default is a logical outcome of the decline in a firm’s fortunes as reflected in the market value of its assets, default is considered to be ‘structural’. It is this structural feature of default that characterises the approach that distinguishes it from the main competing approach, the reduced form approach, in which default is characterised by a Poisson arrival time, which, by definition, is incapable of being anticipated.

Jarrow and Turnbull (1995, p55) state that the Merton model is called a ‘structural’ model of credit risk because the assumptions underlying the model are imposed on the firm’s balance sheet, the firm’s structure. The structural approach is frequently called the contingent claims approach because it regards all corporate securities as claims on company cash flows. The paragraphs that follow summarise the contributions of the principal proponents of this approach.
Mason and Merton (1985, p25) proclaim the benefits of this approach. They bemoan the fact that the traditional approach to the pricing of corporate liabilities is disjointed, as exemplified by the structure of a typical, vintage corporate finance textbook, with separate chapters on the pricing of equity, and on the pricing of long-term debt, each employing a different valuation technique. Rarely, if ever, they comment, are any attempts made to integrate the various components of a firm’s capital structure as even a check on the internal consistency of these diverse valuation methodologies. In contrast, they claim that the contingent claims approach to the pricing of corporate liabilities begins with the firm’s total capital structure and uses a single evaluation technique to simultaneously price each of the individual components of that structure.

3.3.1.1 Black and Scholes

Black and Scholes (1973) put research in the area of credit risk on a sound theoretical basis for the first time. As almost all corporate liabilities can be viewed as combinations of options, they suggest, the option-pricing formula and the analysis that led to it are also applicable to corporate liabilities such as common stock, corporate bonds, and warrants. In particular, they note, the formula can be used to derive the discount that should be applied to a corporate bond because of the possibility of default (p637). They note further that corporate liabilities other than warrants may be viewed as options. The bondholders own the company’s assets, but they have given options to the stockholders to buy the assets back, they comment “By subtracting the value of the bonds given by this formula from the value they would have if there were no default risk, we can figure the discount that should be applied to the bonds due to the existence of default risk” (p 649-650). They further note that if a company has coupon bonds rather than pure discount bonds outstanding, then the common stock can be viewed as a ‘compound option’.

Jones et al (1984) argue that Black and Scholes’ contingent claims insight is of more academic and practical value than their option pricing model.

Option Pricing Formulae

The stock price is assumed to follow geometric Brownian motion, namely,

\[
\frac{dS}{S} = \mu dt + \sigma dz
\]

Equation 3-1
where $S$ is the stock price, $\mu$ the expected stock return, $\sigma$ the standard deviation of stock returns, $t$ is time and $dz$ a drawing from a standard normal distribution. Using Itô's Lemma, Black and Scholes (1973) showed that a function, $F$, a derivative of $S$, would satisfy

$$dF = \left( \frac{\partial F}{\partial S} \mu S + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial F}{\partial S} \sigma S dz \quad \text{Equation 3-2}$$

where $r$ is the continuously compounded risk-free spot rate over period $t$.

By letting $F = \ln(S)$, we get

$$dF = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dz \quad \text{Equation 3-3}$$

Black and Scholes demonstrated that it is possible to create a hedged position, consisting of a long position in the stock and a short position in the option, whose value will not depend on the price of the stock, but will depend only on time and the values of known constants (p 641). The long position in the stock is set equal to the partial derivative of the option price with respect to the stock price. If the hedge is maintained continuously, they note, the return on the hedged position becomes certain (p 641).

They showed that any derivative of $S$ would satisfy the partial differential equation

$$\frac{\partial F}{\partial t} + rS \frac{\partial F}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} = rF \quad \text{Equation 3-4}$$

subject to appropriate boundary conditions. For a non-dividend-paying stock, the solution for a European call, $c$, on $S$ with a strike price, $X$, expiring at $T$ is

$$c = SN \left[ \ln \left( \frac{S}{X} \right) + \frac{r + \sigma^2}{2} T \right] - X e^{-rT} N \left[ \ln \left( \frac{S}{X} \right) + \frac{r - \sigma^2}{2} T \right] \quad \text{Equation 3-5}$$

where $N(\bullet)$ is the standard normal cumulative probability distribution function.

### 3.3.1.2 Merton

Merton (1974) formalised these insights in the context of a company financed by zero-coupon debt and non-dividend-paying stock. Assuming the value of the firm's assets
follows geometric Brownian motion, the value of the debt, \( F \), must satisfy a similar partial differential equation to that developed already by Black and Scholes

\[
\frac{\partial F}{\partial t} + rV \frac{\partial F}{\partial V} + \frac{1}{2} \sigma^2 V^2 \frac{\partial^2 F}{\partial V^2} = rF \quad \text{Equation 3-6}
\]

where \( V \), the value of the firm, now takes the place of \( S \), the value of the shares The boundary condition at maturity is

\[
F = \min[ V, B ] \quad \text{Equation 3-7}
\]

where \( B \) is the face value of the debt Equity, \( f \), is the exact equivalent of a call on \( V \) with a strike price, \( B \) (Equation 9, p 454)

\[
f = V N \left( \frac{\ln \left( \frac{V}{B} \right) + \left( r + \sigma^2/2 \right) T}{\sigma \sqrt{T}} \right) - B e^{-rT} N \left( \frac{\ln \left( \frac{V}{B} \right) + \left( r - \sigma^2/2 \right) T}{\sigma \sqrt{T}} \right) \quad \text{Equation 3-8}
\]

The difference between \( V \) and \( f \) is \( F \), the value of risky debt (Equation 13)

\[
F = B e^{-rT} N \left[ \frac{-\sigma^2 T}{2} - \ln \left( \frac{B e^{-rT}}{V} \right) \right] + V N \left[ \frac{-\sigma^2 T}{2} + \ln \left( \frac{B e^{-rT}}{V} \right) \right] \quad \text{Equation 3-9}
\]

The credit spread, \( R(T) - r \), is the difference in yield between the riskless and the risky bonds (Equation 14) where the bond matures at \( T \)

\[
R(T) - r = -\frac{1}{T} \ln \left[ V \left( \frac{\ln \left( \frac{V}{2} \right) + \ln(d)}{\sigma \sqrt{T}} \right) + \frac{\sigma^2 T}{2} e^{-rT} N \left( \frac{\ln \left( \frac{V}{2} \right) - \ln(d)}{\sigma \sqrt{T}} \right) \right] \quad \text{Equation 3-10}
\]
Thus, for a given asset volatility and debt level, the value of the debt and equity can be uniquely determined, as shown in Figure 3.1:

![Figure 3.1 Value of Debt and Equity Contingent on Value of Firm](image)

Figure 3.2 below examines the effect of $\frac{V}{B_0 e^{-rt}}$, which Merton calls the leverage ratio, $d$, on the term structure of credit spreads. This is one of the more controversial results of the Merton model. While it indicates a rising term structure of credit spreads for low-leverage firms, it suggests a declining term structure for highly leveraged firms. Sarig and Warga (1989), Franks and Torous (1989) and Wei and Guo (1997) discuss the issue at length. A declining term structure is rarely observed in practice although this may be because many issues contain a prepayment option, a factor that Merton did not consider. Furthermore, it shows the credit spread goes to zero for low leverage firms as they approach maturity. “Empirically we do not observe this to be the case.” Jarrow and Turnbull, (1998, p16) These issues will be re-examined in 3.3.3 below.
Figure 3.2 The Term Structure of Credit Spreads

Given the unique relationship that both debt and equity have with the asset value for a given asset volatility, they also have a unique relationship with one another, as shown in Figure 3.3:
The value of the default put option, $p$, or equivalently, the cost of the credit derivative, may be expressed as:

$$
p = -V N \left( \ln \left( \frac{V}{Be^{-rt}} \right) + \frac{1}{2} \sigma^2 T \right) + Be^{-rt} N \left( \ln \left( \frac{V}{Be^{-rt}} \right) - \frac{1}{2} \sigma^2 T \right) \quad \text{Equation 3-11}
$$

This cost is a homogeneous function of the leverage ratio, $d$.

The PD in a risk-neutral world is

$$
1 - N \left( \ln \left( \frac{V}{Be^{-rt}} \right) - \frac{1}{2} \sigma^2 T \right) \quad \text{Equation 3-12}
$$

and this enables the value of the credit derivative to be written as:

$$
\left[ \frac{-N(-d_1)}{N(-d_2)} V + Be^{-rt} \right] N(-d_2) \quad \text{Equation 3-13}
$$

where
In Equation 3-14
\[
d_1 = \frac{\ln \left( \frac{V}{Be^{-rT}} \right) + rT}{\sigma \sqrt{T}}
\]

and
\[
d_2 = \frac{\ln \left( \frac{V}{Be^{-rT}} \right) - rT}{\sigma \sqrt{T}}
\]

Crouhy et al (2001, p 364) point out

The absolute value of the first term inside the brackets in Equation 3-13 for the credit derivative is the expected discounted recovery value of the loan, conditional on default. It represents the risk-neutral expected payment to the bank in the case where the firm defaults. The second term inside the brackets is the value of a risk-free bond. Thus, the sum of the two terms inside the brackets is the expected shortfall in present value terms, conditional on the firm being in default at \( T \). Multiplying this present value of the expected shortfall by the PD gives the premium for insurance against default. The price of a credit derivative (CD) and its component parts – PD, also known as expected default frequency (EDF), and the present value of the LGD – are shown in Figure 3.4 for a loan of 100 to a firm.
However, practitioners do not think of LGD in present value terms, and rating agencies quote recovery rates as a percentage of par value. Hence, it is common practice to express LGD in actual monetary terms. Re-arranging the previous equation and expressing it in future value terms, we get:

$$EL_t = B\left(1 - N(d_2) - \frac{N(-d_1)}{d}\right)$$  \hspace{1cm} \text{Equation 3-16}

Hence, the expected payoff from the debt at maturity is

$$B - EL_t = B\left(N(d_2) + \frac{N(-d_1)}{d}\right)$$  \hspace{1cm} \text{Equation 3-17}

so the expected cost of default, in yield terms, is:

$$-\frac{1}{T} \ln \left(\frac{B \left(N(d_2) + N(-d_1) \frac{V}{Be^{-rT}}\right)}{B}\right)$$  \hspace{1cm} \text{Equation 3-18}

which is the same as the expression previously derived for the credit spread.
Merton showed how the cost of eliminating credit risk can be derived from the value of the firm’s assets, \( V \). However, it is not possible to observe \( V \) in most instances – usually only the equity is traded and there are no liquid prices quoted for the other liabilities in the firm’s capital structure. As a practical matter, therefore, it is necessary to be able to express a relationship between debt and equity rather than debt and assets as heretofore. If debt is to be hedged and priced, it must be done via the equity.

The value of equity according to Merton (1974) is:

\[
f = V N(d_1) - F e^{-rT} N(d_2)
\]

Equation 3-19

As Crouhy et al (2001, p367) note, the equity value is a function of the same parameters as the default put option. They note that a put can be created synthetically by selling short \( N(-d_1) \) units of the firm’s assets, and buying \( B e^{-rT} N(-d_2) \) units of government bonds maturing at \( T \), with face value of \( B \). They conclude that by selling short \( \frac{N(-d_1)}{N(d_1)} \) units of the stock, \( f \), a short position in the firm’s assets of \( N(-d_1) \) units is created. Thus, even if \( V \) is not directly traded or observed, a put option can be created dynamically by selling short the appropriate number of shares.

Equity is a leveraged position in the asset and its volatility relationship with the assets reflects this leverage:

\[
\sigma_f = \eta_{fV} \sigma = \frac{N(d_1)V}{f} \sigma
\]

Equation 3-20

where \( \eta_{fV} \) is the instantaneous elasticity of equity with respect to the firm’s value, \( \frac{\partial f}{\partial V} \). Bensoussan et al (1994, 1995) showed that hedge ratio and the formula linking firm volatility and equity volatility above can both be used despite the fact that firm volatility is stochastic, changing with \( V \), which is not theoretically correct if the Black and Scholes model is to be applicable.

### 3.3.1.3 Vasicek

Vasicek (1997, p1) summarises the contingent claims approach to the measurement of borrower PD. Traditional credit analysis, he notes, involves detailed examination of the company’s operations, projection of cash flows, and assessment of the future earning
power of the firm. But, he contends, such analysis is not necessary, not because future prospects of the firm are not of primary importance—clearly they are—but because an assessment, based on all currently available information on the company’s future, has already been made by the aggregate of the market participants, and reflected in the firm’s current market value. He proceeds to emphasise that this assessment is accurate not in the sense that its implicit forecasts of future prospects will be realised, only that any one person or institution is unlikely to arrive at a superior valuation. The challenge, he says, is properly to interpret the changing share prices.

He proceeds to extend the contingent claims approach to the valuation of subordinate debt within a capital structure that contains senior debt and equity also. He derives the expected loss in a risk-neutral environment, $Q$, as

$$
Q = (D_T + C_T) N \left[ \frac{\ln(D_T + C_T) - \ln(A - F) - rT + \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}} \right] - (A - F) e^{-rT} N \left[ \frac{\ln(D_T + C_T) - \ln(A - F) - rT - \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}} \right] - C_T N \left[ \frac{\ln(C_T) - \ln(A - F) - rT + \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}} \right] + (A - F) e^{-rT} N \left[ \frac{\ln(C_T) - \ln(A - F) - rT - \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}} \right]
$$

Equation 3-21

where $D$ is the market value of the subordinated debt, $D_T$ is the face value of the subordinated debt, $C$ is the market value of the senior debt, $C_T$ is the face value of the senior debt, $A$ is the market value of the assets, $F$ is the present value of dividends and interest paid over the term of the loan, $T$, and where $r$ is the risk-free interest rate.

The value of the subordinated debt is

$$
D = (D_T - Q) e^{-rT}
$$

Equation 3-22

The impact of firm value and volatility on the value of the subordinated debt with a risk-free present value of $50$ is shown in Figure 3.5 for a range of asset values.
3.3.1.4 Other Contributions to the Contingent Claims Literature

Black and Cox (1976, p351) focus on the assumptions made by Black and Scholes and Merton who had assumed that the bond contract renders the firm’s investment, payout, and further financing policies determinate and that the fortunes of the firm may cause its value to dwindle to nearly nothing without any sort of reorganisation occurring in the firm’s financial arrangements. (p352) As they point out, in reality, the firm may be reorganised if the asset value reaches upper or lower boundaries. These boundaries, they suggest, may be given exogenously by the contract specifications or determined endogenously as part of the optimal decision problem. (p352)

Black and Cox allow for coupon-paying debt and for default prior to maturity by introducing an exogenously determined lower boundary, which, when crossed, triggers default. In the Merton framework, they note, the time of receipt of each potential payment was known but not the amount which would actually be received. They contrast this with their new approach in which the amount to be received at each boundary is a known function specified by the contract, but the time of receipt is a random variable. (p353) The closed form solution that they develop confirms the
benefits to debtholders of covenants that trigger debt repayment in the face of deteriorating company fortunes

Shimko, Tejima and van Deventer (1993) allowed for stochastic interest rates as per Vasicek (1977) in the Merton (1974) framework. They conclude that the correlation between interest rate movements and the returns on the underlying asset is an important variable in determining the credit spread on risky debt having shown that for reasonable parameter values, as the correlation increases, the credit spread increases (p64).

Longstaff and Schwartz (1995) address the “clearly unrealistic” assumption in the standard contingent claims approach that default will only occur “when the firm exhausts its assets” (p789). They attempt to extend the Black and Cox (1976) model by incorporating both default and interest rate risk and by explicitly allowing for deviations from strict absolute priority which they do by exogenously imposing a recovery rate for different securities. They conclude that “credit spreads are strongly negatively related to the level of interest rates” (p791) and that for investment-grade bonds, “changes in interest rates account for more of the variation in credit spreads than changes in the value of the assets of the firm” (p815).

3.3.5 Other Contingent Claims Models

Models based on this contingent-claims approach compare the value of an issuer’s assets with the level of debt in the issuer’s capital structure to determine the PD. Duffie and Singleton (1998) define this modelling framework as the “structural” approach to risky debt valuation. Bohn (2000) says that in the Black and Scholes and Merton version of this model, default is assumed to occur when the market value of assets has fallen to a sufficiently low level relative to the issuer’s total liabilities. Bohn notes that the key characteristic shared by structural models is their reliance on economic arguments for why firms default (p54).

Lando (1997) demonstrates that the Merton formula for the value of risky debt, \( F \), can be re-cast as the value of a default-free loan of the same amount plus a short position in a put option on the firm’s assets with a strike price equal to the debt’s face value.
Thus, the value of risky debt is the value of otherwise similar, default risk-free debt less the present value of the expected loss, $EL$, given that the company defaults. Bohn notes that this expected loss term can be divided into two components. The first term, the expected loss on the debt in the case of no recovery, equals the face value of the debt, $B$, multiplied by the risk-neutral PD. The second term represents the expected recovery in the event of default.

However, Bohn comments the main difficulty with the formulation is empirically finding all the necessary inputs. He proposes a simpler characterisation of default as a binary option in which the lender incurs a loss of a fixed amount, $L$, when the borrower defaults, where $L$ is fixed as a percentage of face value, and nothing if the borrower does not default. The formulation for $EL$ above can then be seen as the expected loss with no recovery (i.e., risk-neutral PD times the face value of the debt) less the expected recovery in the event of default.

The expected payoff at maturity in a risk-neutral world is the sum of the payoff in the case of no default times the probability of no default plus the payoff in the case of default times the probability of default. He calls the risk-neutral PD, $Q$, and derives the following equation:

$$F = e^{-rT} \left( B - EL \right)$$

$$EL = B N \left( -\ln(V) + rT - \frac{1}{2} \sigma^2 T - \ln(B) \right) - \frac{V}{e^{-rT}} N \left( -\ln(V) + rT + \frac{1}{2} \sigma^2 T - \ln(B) \right)$$

Equation 3-23

$$F = e^{-rT} \left[ (1 - Q)(1 + Q(1 - L)) \right] = Be^{-rT} [1 - QL]$$

Equation 3-24

Bohn proposes an approach to calculating $Q$ by adjusting the actual PD by the market price of risk and a function of time. The actual PD, $p$, that the value of the firm’s assets will be less than the face value of debt at maturity is:

$$p = N \left[ \ln(B) - \ln(V) - \mu T + \frac{1}{2} \sigma^2 T \right]$$

Equation 3-25

The risk-neutral PD is given by the same formula with the expected asset return, $\mu$, replaced by the default risk-free rate, $r$. 

58
Bohn proposes a factor-pricing framework to formulate a relationship between the expected return on the firm’s assets and the overall expected return on the market. Using the Capital Asset Pricing Model to describe this relationship

\[ \mu - r = \frac{\text{cov}(r, r_e)}{\sigma_m} \lambda \]

Equation 3-27

where \( \mu_m \) is the expected return on the market, \( \sigma_m \) is the volatility of market returns and \( \lambda \) is the overall market Sharpe ratio. Substituting into the previous equation for \( \rho \), a formula for \( Q \) can be derived

\[ Q = N \left( N^{-1}(\rho) + \rho \lambda \sqrt{T} \right) \]

Equation 3-28

In this formulation, \( \rho \) is the correlation of the return on the firm’s assets, \( r \), with the return on the market, \( r_m \). Although suggesting the CAPM framework to describe the concept, he proposes the use of a more sophisticated factor model to determine the amount of variation in the return explained by the firm’s sensitivity to certain market factors. He suggests that the sensitivity parameter, \( \rho \), be set equal to \( \sqrt{R^2} \), where \( R^2 \) is the coefficient of determination resulting from estimation of a suitable multi-factor model.

In this framework, the firm defaults with a probability, \( Q \), and in which case the firm pays \((1 - L)B\) or it does not default with a probability \((1 - Q)\) and pays back \( B \)

\[ F = e^{-rT}(B - LBQ) \]

Equation 3-29

Hence, the term structure of credit spreads is then given by

\[ R(T) - r = -\frac{\ln(1 - LQ)}{T} \]

Equation 3-30
This valuation framework has been adopted by KMV. They use a 120-factor model in their credit portfolio modelling product, Portfolio Manager, described in detail in 4.3.2.1 below.

### 3.3.2 The Reduced Form Approach

Structural models begin with an economic argument about why a firm defaults. In this framework, default can never occur by surprise. As time to maturity goes to zero, credit spreads also approach zero. However, in practice, non-zero credit spreads are observable in the market regardless of maturity.

By comparison, in the reduced form framework, default is always an unpredictable event governed by an intensity-based or hazard-rate process according to Duffie and Singleton (1998). Reduced form models eliminate the need for an economic explanation of default, comments Bohn (2000, p54). According to Jarrow and van Deventer (1999), the approach is ‘reduced form’, because the assumptions underlying the model are imposed on the prices of the firm’s traded liabilities that can be deduced from the structural models.

Iovino (1999) characterised the difference between the structural and reduced form approaches. She stated that both approaches attempted to model the time a firm defaults. However, while the structural approach addressed the problem by “modelling the time a firm defaults”, that is, attaching meaning to each of the underlined words, the reduced form approach instead models the time a firm defaults. (p15)

Bohn (2000) formulates the value of a zero-coupon bond issued by a firm with one class of equity as follows:

\[
F = B e^{-rT} \left(1 - L Q(\tau^* < T)\right) \quad \text{Equation 3-31}
\]

where, as previously, \(F\) is the market value of the zero-coupon debt, \(B\) is the face value of the debt, \(L\) is the LGD expressed as a fraction of face value, \(r\) is the risk-free rate and \(T\) is the time to maturity.

He states that the difference between the structural characterisation and reduced form characterisation of the model lies in the specification of \(Q\). In the reduced form model, \(Q\) indicates the risk-neutral probability the unpredictable event of default occurred at time \(\tau^*\), which happened to precede the maturity of the debt. The time of default is assumed to follow a stochastic process governed by its own distribution that must be
parameterised by an intensity or hazard rate process. “The default or ‘stopping’ time is inaccessible i.e. it jumps out at you (from nowhere).” p.63

Jarrow and Turnbull (1995) developed one of the first reduced form models in which they assumed a constant LGD and an exponentially distributed default-time. They modelled risky bonds as foreign currency bonds denominated in “promised” dollars. The exchange rate is 1 in the absence of default and (1-LGD) if default has occurred. Default is a Poisson arrival. This assumption of constant default intensity is unrealistic, however. For example, in reality, strong firms become weaker over time indicating the necessity for time varying default intensity.

Jarrow, Lando and Turnbull (1997) addressed this weakness. They modelled default as the first time a continuous-time Markov chain with \( K \) states hits the absorbing \( K\)-th state. States 1 to \( K \) are associated with credit ratings where 1 is the strongest rating and the \( K\)-th state is default. However, in order to implement such a model, one needs to estimate an entire generator matrix to arrive at transition probabilities for each possible change in state. As a first approximation, they suggest using historical rating agency transition probability matrices.

Duffie and Singleton use reduced form models to value risky debt as if it were default risk-free by replacing the usual short-term default risk-free rate with the default-adjusted short-rate process. They show how to specify a reduced form model in the context of popular default risk-free term structure models such as Heath, Jarrow and Morton (1992):

\[
F = E^Q \left[ e^{-\int_0^t Rdt} B \right] \\
R = r_t + h_t L_t + l_t
\]

Equation 3-32

where \( r_t \) is the default risk-free rate, \( h_t \) is the arrival intensity at time \( t \) (under \( Q \)) of a Poisson process whose first jump occurs at default, \( L_t \) is the fractional LGD, and \( l_t \) is a variable that is intended to capture liquidity effects. Credit spread data can be used to imply the risk-neutral mean expected loss rate, \( h_t L_T \).

Duffie and Lando (1997) show how to formulate a structural model such that it can be represented as a reduced form model in the Duffie and Singleton framework. They begin with a diffusion process for the firm’s asset value and a default barrier that marks
the asset value at which the firm defaults. They then derive a formula for the hazard rate, $h_t$, in terms of the asset value volatility, the default barrier, and the conditional distribution of asset value given the history of information available to investors. The mechanism creating the inaccessible default stopping time is imperfect accounting information. With imperfect accounting data, credit spreads remain bounded away from zero even as maturity approaches zero. Thus, this version of a structural model is recast in a reduced form framework.

### 3.3.3 Assessment of the Different Modelling Approaches

Next, the success, or otherwise, of the structural and reduced form approaches to evaluating credit risk is examined.

#### 3.3.3.1 Evaluating Structural Models

Jones, Mason and Rosenfeld (1983, 1984) found that contingent claims models produced credit spreads which were significantly lower than actual credit spreads. They concluded that they produced results which were no more accurate than those obtained by discounting at the risk-free rate in the case of investment-grade debt. The Contingent Claims Analysis (CCA) model, they assert, is not an improvement over a naive, riskless model for investment-grade bonds. However, the CCA model "does appear to have incremental explanatory power over the naive model for non-investment-grade bonds" (1984, p624).

The naive model prices are obtained by discounting the promised cash flows at the risk-free rate. Based on a sign test, the CCA model outperforms the naive model for 139 of the 176 investment-grade debt and for 117 of the 129 non-investment grade debt. All these results are significant at the 95% level. However, for pricing, the results were less impressive.
Table 3-1 Pricing comparisons CCA and Naive Models vs Market Prices

Table 3 of their paper, reproduced as Table 3-1 above, shows the difference between the model price and market price expressed as a percentage of the market price. The mean pricing error for investment grade debt was small (0.47% of market value), but the mean error for sub-investment grade debt was large and positive (10.05%). Likewise, fluctuations around the mean errors were large in both sub-samples – the standard deviations were 7.27% and 10.63%, respectively, and the mean absolute errors were 5.87% and 11.97%, respectively.

They surmise that the key assumptions which give rise to these negative conclusions include constant asset volatility, the absolute priority rule (APR), perfect asset liquidity enabling firms to sell assets as necessary, Itô dynamics, the frequent requirement to retire bonds via periodic sinking fund provisions, and a non-stochastic term structure and suggest that “introducing stochastic interest rates, as well as taxes, would improve the model’s performance.” Franks and Torous (1989) concurred with their findings.

Sang and Warga (1989) estimated the term structure of credit spreads using a small number of zero-coupon corporate bonds and zero-coupon treasury bonds. They demonstrated curve shapes consistent with contingent claims model predictions, namely, upward sloping for investment grade debt, humped shaped for lower grade debt, and downward sloping for very low grade debt.

Delianedis and Geske (1998) used the Black Scholes Merton framework to estimate risk-neutral probabilities of default. They found that rating migrations and defaults are detected months before in the equity markets, lending support to the modelling of default as a diffusion process rather than as a Poisson event.
Fons (1987) used a risk-neutral model to examine low-grade bonds and concluded either that there is systematic mispricing by investors of low-rated corporate bonds or that the risk-neutral model he derived could not fully capture the market's assessment of the PD of those securities. In a 1994 article, Fons again found his risk-neutral model seriously underestimated the spreads he obtained from fitting linear regressions through data within different classes, but particularly investment grade bonds.

A fundamental assumption of the structural model is that APR, which requires that senior claimants be paid in full before more junior claimants get anything, holds. This is seldom the case in practice. Franks and Torous (1989) found that 21 out of 27 recapitalisations exhibited a violation of APR. In 21 of 30 cases examined by LoPucki and Whitford (1990) in which the total value to be distributed was less than that due to creditors, stockholders received value, averaging 5.6% of the total value of all distributions. Garbade (2001, p104) concludes that “[v]iolation of the absolute priority rule does not come as a surprise to market participants” and that “[s]enior debt is not priced on the assumption that it will be paid in full before subordinated creditors and stockholders get anything.”

The structural model assumes that bankruptcy is instantaneous and costless. In practice, it is neither. Warner (1977) found that bankruptcy costs for eleven large railroads averaged 5.3%, ranging between 1.7% and 9.1%. Altman (1984) found bankruptcy costs averaged 6.0% of the debtor's value for a sample of seven industrial firms and twelve retailers.

While acknowledging the APR violation and bankruptcy cost issues, Garbade (2001) suggests that the problem is deeper, as the CCA methodology demands that an analyst recognise not only the existing operating characteristics, capital structure, and contractual obligations of the firm but also the prospect of change attributable to managerial discretion and decision making. (p387) He concludes: “The challenge is to extend the analytical framework to include the role of discretion in the exercise of implicit management options.” (p387)

Structural models have been used to value callable debt, with mixed results. Some of the pricing errors can be attributed to a failure to allow for fluctuations in yields on default-proof fixed income securities, Garbade suggests (2001, p114). Vu (1986) found evidence of 'unexpectedly late' redemption of callable bonds that he suggested might be optimal because of the costs associated with calling and refinancing. Likewise, he found
evidence of ‘unexpectedly early’ redemption conjecturing that firms chose to do so in order to eliminate restrictive covenants. Asquith and Wizman (1990) found that firms commonly called bonds for early redemption to eliminate restrictive covenants that would otherwise impede a planned buyout. However, this type of ‘event risk’ associated with apparently suboptimal calling of debt cannot be modelled correctly in the structural framework.

Ingersoll (1977a) and Brennan and Schwartz (1977, 1980) were among the first to analyse the contingent value of convertible debt. However, their models failed to explain corporate behaviour satisfactorily. It was Ingersoll (1977b, 463) himself who pointed out that, compared to the conversion behaviour suggested by his structural form model, companies commonly waited “too long” to call their convertible debt. Jaffee and Shleifer (1990) suggested that taking proper account of the notice period between announcement of a call and actual redemption could rectify the most egregious failings of the Ingersoll model.

Kim, Ramaswamy and Sundaresan (1993) valued coupon-paying debt using the Cox, Ingersoll and Ross (1985) interest rate model and an exogenously defined recovery rate. They conclude that their approach is plausible as it generates yield spreads on corporate bonds consistent with those observed in the marketplace. They further conclude that stochastic interest rates seem to play an important role in determining the yield differentials between a callable corporate bond and an equivalent government bond “due to the interactions between call provisions and default risk” (p127).

Jarrow and van Deventer (1999) criticise the implicit assumptions embedded in the structural approach regarding corporate capital structure policy. The structural approach, they claim, assumes that the corporate capital structure policy is static, with the liability structure fixed and unchanging. This assumes that management puts a debt structure in place and leaves it unchanged even if the value of corporate assets has doubled, they note “This is too simplistic to realistically capture management behaviour and the dynamics of bankruptcy” (p302).

Conventional wisdom has it that structural models provide good insights into the cause of default but fail the test of providing good models for valuation. However, Bohn (2000) suggests that small sample sizes, doubts about the quality of bond pricing data, and the lack of focus on the appropriate default risk-free rate leave us without
conclusive evidence regarding the power of structural models. "The resolution of these empirical issues awaits further research", Bohn comments (p66).

Gemmill (2002) examined the credit spreads on zero-dividend preference shares (ZDP) issued by split-capital trusts in the U.K. He confirms that two of the biases consistently reported are present in his analysis, namely, that model credit spreads are too small for bonds that are near maturity and also for companies with low leverage and volatility. His study once again casts doubt on the appropriateness of the Merton model that assumes the assets follow geometric Brownian motion and, hence, that the spread should reduce to zero as time to maturity and/or volatility goes to zero. It is clear the market has not excluded the possibility of a jump in asset value.

However, of greater significance is Gemmill's finding that market and model spreads are of similar magnitude and, in line with market practice, the Merton model consistently produces an upward-sloping term-structure (p1).

The standard Merton model suggests a downward-sloping term structure of credit spreads for risky bonds as shown in Figure 3.2 above whereas the observed term structures in the market are invariably rising. Gemmill comments: "The solution to this conundrum appears to be that the leverage of the companies changes over time, being higher when bonds are issued than when they mature" (p3). The Gemmill model allows for this leverage effect and leads to model term structures of credit risk which are upward-sloping, as empirically observed (p3). He notes his results reinforce the conclusion of Collin-Dufresne and Goldstein (2001) "that it is important to take account of the expected trajectory of leverage when computing credit spreads" (p3). He suggests that firms have a target leverage ratio which they try to maintain by issuing or retiring debt (p4).

### 3.3.2 Evaluating Reduced Form Models

Many reduced form models are parameterised on rating agency ratings and transition data. Kealhofer et al. (1998) take issue with the rating agency approach to grading securities in discrete rating classes. In particular they reject the notion that all assets within the rating grade have a single default rate, and the default rate is equal to the historical average default rate (p40).

They assert that even when all loans within a grade have the same default rate, the historical average default rate can deviate significantly from the actual default rate.
Similarly, they state that the historical transition probabilities can deviate significantly from the actual transition probabilities.

There are substantial differences of default rate within bond rating grade, with some bonds in a higher grade having greater default rates than some bonds in a lower grade, indicating an overlap in default probability ranges. The overlap, they suggest, appears to be caused by lack of timeliness in upgrade and downgrade decisions. They claim the range of default rates within a rating grade can cause the mean default rate to significantly exceed the median default rate within a grade. They suggest the mean may be almost twice as large as the median, and as many as 75% of the borrowers within a rating grade may have default rates that are less than the mean. They also state that historical default rates are statistics for the mean default rate, and thus may be biased upwards by as much as double from the typical default rate within the grade.

They claim that the lack of timeliness in rating changes causes a significant bias in transition probabilities. In consequence, the “probability of remaining at the same quality is overstated by about double for most grades, whereas the probabilities for changing to other non-default grades are significantly understated” (p40).

They also take issue with the assumption that ratings and default rates are synonymous. They find that the highest EDF within a given grade is in excess of four times the lowest whereas the ratio of mean EDFs from one grade to the next is approximately two to one. They conclude that grades overlap significantly. They further conclude that the equity markets are faster than the rating agencies in reflecting information on the condition of firms. This is borne out by their research, which concludes that “about 70% of apparent spread variation is actually due to EDF variation within a rating grade over time” (p50).

Using non-overlapping ranges of default probabilities, they calculate the one-year transition matrix for 6,000 publicly rated U.S. firms reproduced in Table 3-2.
<table>
<thead>
<tr>
<th>Initial Rating</th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
<th>Default</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>66.26%</td>
<td>22.22%</td>
<td>7.37%</td>
<td>2.45%</td>
<td>0.86%</td>
<td>0.67%</td>
<td>0.14%</td>
<td>0.02%</td>
</tr>
<tr>
<td>AA</td>
<td>21.66%</td>
<td>43.04%</td>
<td>25.83%</td>
<td>6.56%</td>
<td>1.99%</td>
<td>0.68%</td>
<td>0.20%</td>
<td>0.04%</td>
</tr>
<tr>
<td>A</td>
<td>2.76%</td>
<td>20.34%</td>
<td>44.19%</td>
<td>22.94%</td>
<td>7.42%</td>
<td>1.97%</td>
<td>0.28%</td>
<td>0.10%</td>
</tr>
<tr>
<td>BBB</td>
<td>0.30%</td>
<td>2.80%</td>
<td>22.63%</td>
<td>42.54%</td>
<td>23.52%</td>
<td>6.95%</td>
<td>1.00%</td>
<td>0.26%</td>
</tr>
<tr>
<td>BB</td>
<td>0.08%</td>
<td>0.24%</td>
<td>3.69%</td>
<td>22.93%</td>
<td>44.41%</td>
<td>24.53%</td>
<td>3.41%</td>
<td>0.71%</td>
</tr>
<tr>
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<td>0.01%</td>
<td>0.05%</td>
<td>0.39%</td>
<td>3.48%</td>
<td>20.47%</td>
<td>53.00%</td>
<td>20.58%</td>
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</tr>
<tr>
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<td>0.01%</td>
<td>0.09%</td>
<td>0.26%</td>
<td>1.79%</td>
<td>17.77%</td>
<td>69.94%</td>
<td>10.13%</td>
</tr>
</tbody>
</table>

Table 3-2 Implied KMV Transition Matrix

The corresponding table, drawn from Standard & Poor’s CreditWeek, April 15, 1996, and reproduced in Table 3-3, notes the transition of rated firms

<table>
<thead>
<tr>
<th>Initial Rating</th>
<th>AAA</th>
<th>AA</th>
<th>A</th>
<th>BBB</th>
<th>BB</th>
<th>B</th>
<th>CCC</th>
<th>Default</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>90.81%</td>
<td>8.33%</td>
<td>0.68%</td>
<td>0.06%</td>
<td>0.12%</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>AA</td>
<td>0.70%</td>
<td>90.65%</td>
<td>7.79%</td>
<td>0.64%</td>
<td>0.06%</td>
<td>0.14%</td>
<td>0.02%</td>
<td>0.00%</td>
</tr>
<tr>
<td>A</td>
<td>0.09%</td>
<td>2.27%</td>
<td>91.05%</td>
<td>5.52%</td>
<td>0.74%</td>
<td>0.26%</td>
<td>0.01%</td>
<td>0.06%</td>
</tr>
<tr>
<td>BBB</td>
<td>0.02%</td>
<td>0.33%</td>
<td>5.95%</td>
<td>86.93%</td>
<td>5.30%</td>
<td>1.17%</td>
<td>0.12%</td>
<td>0.18%</td>
</tr>
<tr>
<td>BB</td>
<td>0.03%</td>
<td>0.14%</td>
<td>0.67%</td>
<td>7.73%</td>
<td>80.53%</td>
<td>8.84%</td>
<td>1.00%</td>
<td>1.06%</td>
</tr>
<tr>
<td>B</td>
<td>0.00%</td>
<td>0.11%</td>
<td>0.24%</td>
<td>0.43%</td>
<td>6.48%</td>
<td>83.46%</td>
<td>4.07%</td>
<td>5.20%</td>
</tr>
<tr>
<td>CCC</td>
<td>0.22%</td>
<td>0.00%</td>
<td>0.22%</td>
<td>1.30%</td>
<td>2.38%</td>
<td>11.24%</td>
<td>64.86%</td>
<td>19.79%</td>
</tr>
</tbody>
</table>

Table 3-3 S&P Transition Matrix

The probability of staying in an S&P grade is approximately 90% for most ratings, which is about twice that recorded by KMV. Furthermore, the default probability for the lower S&P grades is about twice that in the KMV matrix.

In sum, they conclude that ratings-based probabilities will tend to overstate the risk for maturities near to the measurement horizon, due to the overstated default rate, but will tend to understate the risk for longer maturities, due to the overstated probability of credit quality remaining the same (p53).

Duffee (1996) used monthly prices from 1985 to 1994 for the corporate bonds in the Lehman Brothers Bond indexes and the Jarrow, Lando and Turnbull model. He finds strong evidence of mis-specification as the model fails to produce the term structures of...
credit spreads most commonly experienced in the market. Although the model fits investment-grade corporate bond prices reasonably well, he concludes that single-factor models of instantaneous default probabilities "face a substantial challenge in matching the dynamic behaviour of corporate bond term structures" (p 26).

As discussed above, the structural model is often criticised for producing a declining term structure for low-grade bonds. A similar criticism can be levelled at reduced form models such as that of Jarrow, Lando and Turnbull (1995) and Markov-chain models which rely on rating agency transition matrices. If a low-grade firm does not default in the first year, its annual default probability declines in subsequent years.

### 3.4 Portfolio Risk

All research in the area of portfolio theory invariably begins with references to the work of Markowitz (1952, 1959). His work attempts to characterise the interaction of a portfolio of equities and proceeds to calculate the efficient portfolio, that combination of available equities which maximises the mean-variance trade-off.

Markowitz assumes that equity returns are normally distributed and that investors' utility functions are quadratic. At the end of an investment period, an investor's portfolio has a value \( W = W_0 (1 + \tilde{r}_p) \). Using Taylor's theorem, the utility derived from that portfolio is given by

\[
U(\tilde{r}_p) = U(E(\tilde{r}_p)) + \left( r_p - E(\tilde{r}_p) \right) U' + \frac{1}{2} \left( r_p - E(\tilde{r}_p) \right)^2 U'' + \text{Equation 3-33}
\]

Taking the expectation of this expression and assuming third- and higher-order terms are of minor importance, Markowitz get the expression

\[
E[U(\tilde{r}_p)] = U(E(\tilde{r}_p)) + \frac{1}{2} V(\tilde{r}_p) U'' E(\tilde{r}_p) \quad \text{Equation 3-34}
\]

where \( V(\tilde{r}_p) \) is the variance of the rate of return. Under these assumptions, the investor's utility is a function only of the mean and variance of the rate of return on the portfolio.

The exclusive focus on the mean and the variance of returns is inappropriate in a credit context for two reasons: (i) Credit returns are far from normal, with limited upside potential - the highest price that can be achieved is a risk-free price that will never be much higher than par - whereas the minimum price is zero. (ii) Credit portfolios are...
invariably leveraged. Thus, the focus cannot solely rest on the standard deviation. The downside tail is of critical importance as the institution that holds the portfolio becomes insolvent when losses exceed its capital. This is not a concern for equity portfolios that are mostly unlevered.

Thus, credit portfolio research has had to adapt traditional equity portfolio concepts quite significantly to make them applicable to the issues they face.

3.4.1 Credit Portfolio Risk

When the asset return distributions for two firms, $r_A$ and $r_B$, are normally distributed with an instantaneous correlation coefficient of $\rho$ between their returns, then their joint returns will be described by the bivariate normal distribution

$$f(r_A, r_B, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(r_A^2 - 2\rho r_A r_B + r_B^2)}$$

Equation 3-35

The bivariate normal distribution can be used to calculate the joint probability of both borrowers defaulting over a period

$$\Pr(\text{Def}_A, \text{Def}_B) = \Pr(V_A \leq V_{\text{Def}_A}, V_B \leq V_{\text{Def}_B}) = \Pr(r_A \leq d_A^*, r_B \leq d_B^*) = N_2(-d_A^*, d_B^*, \rho)$$

Equation 3-36

where $d_A^*$ and $d_B^*$ are the distances to default for $A$ and $B$, respectively.

The probability of joint default is, therefore, the volume under the cumulative standard bivariate normal distribution.

3.4.1.1 The Normal Inverse Distribution

Vasicek (1997b) explored the situation in which a portfolio of debt securities is financed by equity and notes. He developed a methodology for determining the capital necessary to support the desired rating of a lender's notes. The credit quality of the lender's notes will depend on the probability that the loss on the portfolio exceeds the equity capital, he suggests. The equity capital allocated to the portfolio "must be equal to the ordinate of the distribution of the portfolio loss that corresponds to the desired probability" (p1).

He also examined the probability distribution of portfolio losses. He assumed that the portfolio consisted of $n$ loans of equal dollar amounts, the PD of any one loan is $p$, the
asset returns of the borrowing companies are correlated with a coefficient $\rho$ for any two companies, and all loans had the same term, $T$.

He defines a default indicator $L_i$ to be one if the $i$-th borrower defaults and zero otherwise. Thus, the variable $L_i$ is the gross loss (before recoveries) on the $i$-th loan and

$$
P[L_i = 1] = p, \quad P[L_i = 0] = 1 - p \quad \text{ Equation 3-37}
$$

and the expected value and variance of the loan loss is

$$
E[L_i] = p, \quad Var(L_i) = p(1 - p) \quad \text{ Equation 3-38}
$$

He defines $L = \frac{1}{n} \sum_{i=1}^{n} L_i$ to be the portfolio percentage gross loss and he proceeds to calculate the probability distribution of $L$, that is, the probabilities

$$
P_k = P[L = k/n], \quad k = 0, 1, \ldots, n \quad \text{ Equation 3-39}
$$

Assuming that all borrowers’ assets follow the process

$$
dA_i = \mu_i A_i dt + \sigma A_i dx_i \quad \text{ Equation 3-40}
$$

where $x_i$ are correlated joint Wiener processes with correlation $\rho$

$$
E(dx_i)^2 = dt \\
E(dx_i)(dx_j) = \rho dt, i \neq j \quad \text{ Equation 3-41}
$$

The PD is

$$
p = P[L = 1] = P[A_i(T) < F_i] = P[X_i < c] = N \left( \frac{\ln(F_i) - \ln(A_i(0)) - \mu_i T + \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}} \right) \quad \text{ Equation 3-42}
$$

where $A_i$ is the value of the $i$-th borrower’s assets, $F_i$ is the value of obligations payable, and $X_i$ is defined as

$$
X_i = \frac{(x_i(T) - x_i(0))}{\sqrt{T}} \quad \text{ Equation 3-43}
$$
The variables $X_t$ are jointly normal with equal pairwise correlations $\rho$ and he represents them as

$$X_t = \sqrt{\rho} Y + \sqrt{1 - \rho} Z_t \quad \text{Equation 3-44}$$

where $Y, Z_1, Z_2, \ldots, Z_n$ are mutually independent standard normal variables He interprets the $Y$ variable as a common factor, such as the state of the economy, over the interval $(0, T)$, the term $\sqrt{\rho} Y$ as the company exposure to the common factor (the systematic risk) and the term $\sqrt{1 - \rho} Z_t$ as the company-specific risk.

Conditional on $Y$, the variables $L_t$ are independent equally distributed zero-one variables with the conditional probability

$$p(Y) = P[L_t = 1|Y] = P[X_t < c|Y] = P[Z_t < \frac{1}{\sqrt{1 - \rho}}(c - \sqrt{\rho} Y)] = N\left(\frac{1}{\sqrt{1 - \rho}}(c - \sqrt{\rho} Y)\right) \quad \text{Equation 3-45}$$

He calculates the portfolio loss distribution as

$$P_k = P\left[L = \frac{k}{n}\right] = \binom{n}{k} \int_0^\infty (p(Y))^k (1 - p(Y))^{n-k} dN(Y) \quad \text{Equation 3-46}$$

which can be evaluated numerically.

He then proceeds to examine the behaviour of this integral as the number of loans in the portfolio increases Because the defaults are not independent, the conditions of the central limit theorem are not satisfied, he argues, and $L$ is not asymptotically normal It turns out, he notes, that the distribution of the portfolio loss does converge to a limiting form, which “can be then conveniently used for large portfolios instead of the integral form” (p5).

He calls the cumulative distribution function

$$Q(x, p, \rho) = N\left(\frac{1}{\sqrt{\rho}} N^{-1}(x) - N^{-1}(p)\right) \quad \text{Equation 3-47}$$

the normal inverse distribution, being the distribution of the variable
\[ L = N\left( \frac{1}{\sqrt{\rho}} \left( N^{-1}(p) - \sqrt{\rho} X \right) \right) \]  \hspace{1cm} \text{Equation 3-48} \]

where \( X \) is a standard normal distribution. He derives the density function for this distribution as

\[ Q'(x; p, \rho) = \sqrt{\frac{1-\rho}{\rho}} \exp\left( -\frac{1}{2\rho} \left[ (1-2\rho)(N^{-1}(x))^2 - 2\sqrt{1-\rho} N^{-1}(x)N^{-1}(p) + (N^{-1}(p))^2 \right] \right) \]

\hspace{1cm} \text{Equation 3-49} \]

The mean of the distribution is \( p \) and the variance is \( N^2 \left( N^{-1}(p), N^{-1}(p), \rho \right) - p^2 \) while the \( \alpha \)-percentile value of \( L \) is

\[ L_\alpha = Q^{-1}(\alpha; p, \rho) = Q(\alpha; 1 - p, 1 - p) \]  \hspace{1cm} \text{Equation 3-50} \]

"The normal inverse distribution is highly skewed and leptokurtic." (p8) This is observed in Figure 3.6 below in which the loan loss distribution is plotted for a range of asset correlation values:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{loan_loss_distribution.png}
\caption{Normal Inverse Loan Loss Distribution for Varying Asset Correlation}
\end{figure}

This family of distributions varies between two extremes: a normal distribution centred on the loan loss probability when the asset correlation is zero, and a binomial
distribution exhibiting 0% loss with a probability \((1 - p)\) and 100% loss with a probability \(p\)

This, albeit simplistic, characterisation of the loan loss distribution points to the challenge of modelling loan loss portfolios and, consequently, the valuation of derivative products such as CDOs. The number of standard deviations from the mean, 
\[
\frac{L_a - p}{\sigma}
\]
for combinations of asset correlation and \(\alpha\)-percentile for a portfolio with a portfolio PD of 1% is shown in Table 3-4 below.

<table>
<thead>
<tr>
<th>Alpha-percentile</th>
<th>10%</th>
<th>1</th>
<th>0.1</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset 0.1</td>
<td>1.19</td>
<td>3.82</td>
<td>7.01</td>
<td>10.67</td>
</tr>
<tr>
<td>Correlation 0.2</td>
<td>0.97</td>
<td>4.22</td>
<td>8.77</td>
<td>14.19</td>
</tr>
<tr>
<td>Normal 0.3</td>
<td>0.75</td>
<td>4.41</td>
<td>10.04</td>
<td>16.61</td>
</tr>
<tr>
<td>Normal 0.4</td>
<td>0.55</td>
<td>4.51</td>
<td>11.04</td>
<td>18.19</td>
</tr>
</tbody>
</table>

Table 3-4 Standard Deviations Corresponding to Percentiles of the Inverse Normal PD = 1%

However, this situation is further exacerbated if there is low default probability. Table 3-5 shows the corresponding table where the borrower default probability is 10bp, typical of an investment-grade portfolio. It shows that an institution with a portfolio of loans with average asset correlation of 40% will have to hold capital equal to 13 standard deviations if it wishes to secure an AA- rating – a PD of 10bp - for itself.

<table>
<thead>
<tr>
<th>Alpha-percentile</th>
<th>10%</th>
<th>1</th>
<th>0.1</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset 0.1</td>
<td>0.98</td>
<td>4.09</td>
<td>8.83</td>
<td>15.37</td>
</tr>
<tr>
<td>Correlation 0.2</td>
<td>0.60</td>
<td>4.10</td>
<td>11.16</td>
<td>22.39</td>
</tr>
<tr>
<td>Normal 0.3</td>
<td>0.31</td>
<td>3.75</td>
<td>12.45</td>
<td>27.65</td>
</tr>
<tr>
<td>Normal 0.4</td>
<td>0.12</td>
<td>3.25</td>
<td>13.18</td>
<td>31.76</td>
</tr>
</tbody>
</table>

Table 3-5 Standard Deviations Corresponding to Percentiles of the Inverse Normal PD = 0.1%

74
3.4.1.2 Conditional Approaches to Credit Portfolio Loss Distributions

Finger (1999) addressed the issue of credit portfolio loss distributions conditioned on the outcome of a market variable. His interest in the issue stemmed from the slow convergence exhibited by Monte Carlo simulation of credit portfolios. He noted that reliance on Monte Carlo simulation was unnecessary once account was taken of the assumed correlation structure. "[O]nce we condition on the industry factors that drive the model, all defaults and rating changes are independent" (p14).

He adopted the same approach as Vasicek (1997b) outlined in 3.4.1.1 above whereby each normalised asset value change can be expressed as

\[ Z_i = wZ + \sqrt{1 - w^2} \varepsilon_i \]  \hspace{1cm} \text{Equation 3-51}

and each obligor has the same default probability, \( p \), and default threshold, \( \alpha \), where

\[ \alpha = \Phi^{-1}(p) \]  \hspace{1cm} \text{Equation 3-52}

and all obligors have the same pairwise asset correlation, \( w \). Once the market factor, \( Z \), is fixed, he notes, everything else that happens to the obligors is independent, the obligors are conditionally independent given \( Z \) (p15). The conditional independence, he contends, proves crucial, as it transforms the complex problem of aggregating correlated exposures into the well understood problem of convolution, or the aggregation of independent exposures.

Conditioned on \( Z \), an obligor defaults if

\[ \varepsilon_i < \frac{\alpha - wZ}{\sqrt{1 - w^2}} \]  \hspace{1cm} \text{Equation 3-53}

and since \( \varepsilon_i \) is a standard normal deviate, the PD conditioned on \( Z \) is

\[ p(Z) = \Phi \left( \frac{\alpha - wZ}{\sqrt{1 - w^2}} \right) \]  \hspace{1cm} \text{Equation 3-54}

He deduces that the portfolio variance is

\[ \left[ \Phi(\alpha, \alpha, w^2) - p^2 \right] + \frac{p - \Phi_2(\alpha, \alpha, w^2)}{N} \]  \hspace{1cm} \text{Equation 3-55}
where the first term is the same as Vasicek's and the last term is due to idiosyncratic variance which is not perfectly diversified due to the finite number of obligors.

He derives the portfolio distribution as

\[
Pr[V < v] = \int_{-\infty}^{\infty} \phi(z) \Phi \left( \frac{v - (1 - p(z))}{p(z)(1 - p(z))} \right) dz \quad \text{Equation 3-56}
\]

While this does not have an analytical solution, a numerical solution can be found substantially more quickly than would the Monte Carlo solution. He notes that "the real dimensionality of the problem is not the number of obligors, but the number of market factors" (p33) and suggests that employing this approach will enable the same accuracy as standard Monte Carlo to be achieved in a fraction of the time.

### 3.4.1.3 Other Credit Portfolio Models

Koyluoglu and Hickman (1998) showed that despite the apparent differences in approach adopted by the main models used in the marketplace – CreditMetrics, Credit Suisse Financial Products' CreditRisk+, McKinsey’s CreditPortfolioView and KMV’s Portfolio Manager – the models "in fact represent a remarkable consensus in the underlying framework, differing primarily in calculation procedures and parameters rather than financial intuition" (p29).

All the models, they suggest, fit within a general framework consisting of three components (p32):

- **Joint default behaviour**: portfolio correlation is reflected by borrowers’ conditional default rates varying together in different states.

- **Conditional distribution of portfolio default rate**: for a given 'state of the world', the conditional distribution can be calculated, they note, as if borrowers are independent because the joint default behaviour is accounted for in generating conditional default rates.

- **Convolution/Aggregation**: They comment that the unconditional distribution of portfolio defaults is obtained by combining conditional default rate distributions in each state.
They decompose the change in asset value into a set of normally distributed orthogonal systemic factors, \( x_k \), and a normally distributed idiosyncratic component, \( \varepsilon_i \),

\[
\Delta A_i = b_{i1} x_i + b_{i2} x_2 + \sqrt{1 - \sum_k b_{ik}^2} \varepsilon_i \tag{Equation 3-57}
\]

where \( b_{ik} \) are the factor loadings, and \( x_k \) and \( \varepsilon_i \) are \( \sim \text{N}[0,1] \)

For a given set of values for the systemic factors, the portfolio default rate can be expressed as

\[
p_{|x} = \Phi \left[ \frac{c - \sum_k b_{ik} x_k}{\sqrt{1 - \sum_k b_{ik}^2}} \right] \tag{Equation 3-58}
\]

where \( c \) is the threshold value of the standard normal variable at which default occurs.

For a homogeneous portfolio, they summarise the systemic factors by a single variable, \( m \), simplifying the expression for the portfolio default rate to

\[
p_{|m} = \Phi \left[ \frac{c - \sqrt{\rho m}}{\sqrt{1 - \rho}} \right] \tag{Equation 3-59}
\]

where \( m \sim \text{N}[0,1] \) and \( \rho = \sum_k b_{ik}^2 \) is the asset correlation.

They proceed to derive the probability density function for the default rate, \( f(p) \)

\[
f(p) = \sqrt{1 - \rho} \frac{\varphi \left( \frac{c - \sqrt{1 - \rho} \Phi^{-1}(p)}{\sqrt{\rho}} \right)}{\sqrt{\rho \varphi(\Phi^{-1}(p))}} \tag{Equation 3-60}
\]

For a homogeneous portfolio of loans with a PD of 1.16%, \( c = \Phi^{-1}[1.16\%] = -2.27 \), the probability distribution of the default rate is plotted in Figure 3.7.
Figure 3.7 Probability Distribution of Portfolio Default Rate

The unconditional probability distribution of portfolio defaults is obtained by combining the conditional distributions across all ‘states of the world’. While they point out the similarities in approach to developing the unconditional default probability, they identify differences in the way joint default behaviour is modelled. Whereas the Merton-based models of CreditMetrics and KMV use pairwise asset correlation, the actuarial model employed by CreditRisk+ uses sector weightings and default rate volatilities. They emphasise, however, that despite the parameter differences, “they contain equivalent information to characterise default behaviour.” (p35)

They present the generalised framework reproduced as Figure 3.8 below. While their comments are specifically directed at CreditMetrics, they suggest they apply ‘reasonably well to Portfolio Manager’ also:
They conclude "Any significant model differences can be attributed to parameter value estimates that have inconsistent implications for the observable default rate behaviour" (p35)

### 3.5 Modelling Correlation

Duffie and Singleton (2003, p230), summarise the most popular approaches to modelling correlated defaults

*CreditMetrics*, they characterise as a method by which ratings transitions for multiple entities can be simulated with the correlation induced by underlying correlated *drivers*, such as asset returns

*Doubly stochastic correlated default-intensity processes* is, they state, an approach to modelling multi-entity default risk in which “correlation is captured through correlated changes in the default intensities of the entities”

*Copulas*, they describe as devices that allow entity-by-entity default models “to be linked with auxiliary correlating variables”

*Intensity-based models of default with joint credit events* can, they suggest, cause multiple issuers to default simultaneously “The simplest example is the multivariate exponential model of default times, which has constant default intensities”
3.5.1 The CreditMetrics Approach

CreditMetrics uses the counterparty’s asset returns, $X_i$, as the driving variable. Each counterparty’s asset returns are assumed to be normally distributed, and the asset returns for multiple counterparties, $X_1, X_2, \ldots, X_n$, are assumed to be multivariate normally distributed, with a covariance matrix, $\Sigma$. The Cholesky decomposition of $\Sigma$ is $C$ such that $CC^T = \Sigma$. By simulating independent standard normal variables, $Z_1, Z_2, \ldots, Z_n$, the drivers with the appropriate means and covariances can be simulated by letting

$$X_i = E(X_i) + C_{i1}Z_1 + C_{i2}Z_2 + \ldots + C_{in}Z_n$$

CreditMetrics suggest that the necessary covariance information can be obtained from the volatilities and correlations of equity returns for the $n$ firms in the case of publicly traded firms. Duffie and Singleton (2003, p232), suggest that one could also “take the drivers to be the KMV measures of distance to default, firm by firm, which is more in the spirit of the asset-return foundation of the CreditMetrics model.” They add that time series data on distances to default for pairs of firms could be used to estimate the covariance matrix.

Lando (1998) considered correlation within the framework of finite-state continuous-time Markov chains for each entity’s rating. However, the approach proves to be rather intractable because the state space is $n^b$ where there are $b$ borrowers and $n$ rating states.

3.5.2 Copula-Based Correlation Modelling

It was Li (2000) who first applied the copula-based approach to simulating correlated defaults. He began by letting $T$ represent a security’s time-until-default, and $F(t)$ denote the distribution function of $T$,

$$F(t) = \Pr(T \leq t), \quad t \geq 0 \quad \text{Equation 3-61}$$

He defines the probability density function

$$f(t) = F'(t) = \lim_{\Delta \to 0^+} \frac{\Pr[t \leq T < t + \Delta]}{\Delta} \quad \text{Equation 3-62}$$

and he defines

$$q_x = \Pr[T - x \leq 1 | T > x] \quad \text{Equation 3-63}$$
as the marginal probability of default in the next year conditional on survival until the
beginning of the year. "A credit curve is then simply defined as the sequence of
$q_0, q_1, \ldots, q_n$ in discrete models" (p43)

Next, he introduces the hazard rate function, $h(x)$, as the instantaneous default
probability for a security that has attained age $x$

$$
\Pr[x < T \leq x + \Delta x | T > x] = \frac{F(x + \Delta x) - F(x)}{1 - F(x)}
$$

$$
= \frac{f(x) \Delta x}{1 - F(x)} \quad \text{Equation 3-64}
$$

$$
= h(x) \Delta x
$$

where

$$
h(x) = \frac{f(x)}{1 - F(x)} \quad \text{Equation 3-65}
$$

The survival function can be defined as

$$
S(t) = e^{-\int_0^t h(x) \, dx} \quad \text{Equation 3-66}
$$

so that

$$
f(t) = S(t) h(t) \quad \text{Equation 3-67}
$$

Li points out that if the typical assumption of a constant hazard rate is made, the density
function is

$$
f(t) = h e^{-ht} \quad \text{Equation 3-68}
$$

He states "Modelling a default process is equivalent to modelling a hazard function" (p47) He also notes “[T]here are a lot of similarities between the hazard rate function and the short rate. Many modelling techniques for the short rate processes can be readily borrowed to model the hazard rate” (p48) Indeed this is exactly how much credit risk modelling has progressed over the past ten years, and the hazard rate function is called the credit curve because of its similarity to a yield curve.

He proceeds to define the joint distributional function for two entities, $A$ and $B$, as
\[ F(s, t) = \Pr[T_A \leq s, T_B \leq t] \quad \text{Equation 3-69} \]

and defines the survival time correlation as
\[ \rho_{AB} = \frac{\text{Cov}(T_A, T_B)}{\sqrt{\text{Var}(T_A)\text{Var}(T_B)}} = \frac{\text{E}(T_A T_B) - \text{E}(T_A)\text{E}(T_B)}{\sqrt{\text{Var}(T_A)\text{Var}(T_B)}} \quad \text{Equation 3-70} \]

He suggests three methods could be used to extract the term structure of default rates (i) using historical default information from rating agencies, (ii) applying the Merton option theoretical approach, and (iii) taking an implied approach using market prices of defaultable bonds or asset swap spreads.

The last approach is the “one used by most credit derivative desks” (p53) Li (1998), demonstrated how to build the credit curve for individual credits from market information based on the Duffie and Singleton (1996) default treatment. The challenge, he says, is to create “a joint distribution function with given marginal distributions and a correlation structure” (p9) While it is straightforward to derive the marginal distributions and the correlation structure if the joint distribution is known, creating a joint distribution from a given set of marginals and a correlation structure, he comments, is rather more difficult. The copula function is the mechanism he chooses to accomplish this.

A copula function links univariate marginal distributions with a joint distribution. For given univariate marginal distribution functions, \( F_1(x_1), F_2(x_2), \ldots, F_m(x_m), \) the function
\[ C(F_1(x_1), F_2(x_2), \ldots, F_m(x_m)) = F(x_1, x_2, \ldots, x_m) \quad \text{Equation 3-71} \]

which is defined using a copula function, \( C, \) results in a multivariate distribution with univariate marginal distributions, \( F_1(x_1), F_2(x_2), \ldots, F_m(x_m) \)

Sklar (1959) proved the converse. He showed that any multivariate distribution, \( F, \) could be written in the form of a copula function. He proved that if \( F(x_1, x_2, \ldots, x_m) \) is a joint multivariate distribution function with univariate marginal distribution functions \( F_1(x_1), F_2(x_2), \ldots, F_m(x_m), \) then there exists a copula function \( C(u_1, u_2, \ldots, u_m) \) such that
\[ F(x_1, x_2, \ldots, x_m) = C(F_1(x_1), F_2(x_2), \ldots, F_m(x_m)) \quad \text{Equation 3-72} \]

If each \( F_i \) is continuous then \( C \) is unique.
Li showed that the copula function embedded in CreditMetrics is the bivariate normal

\[ C(u, v) = \Phi_2(\Phi^{-1}(u), \Phi^{-1}(v), \rho) \quad \text{Equation 3-73} \]

where \( \rho \) is defined as the correlation between the default times of \( u \) and \( v \). For example, if the one-year default probabilities for two credits, \( A \) and \( B \), are \( q_A \) and \( q_B \), respectively, the default thresholds are given by

\[
q_A = \Pr[Z < Z_A] \quad \text{Equation 3-74}
\]
\[
q_B = \Pr[Z < Z_B]
\]

where \( Z \) is a standard normal random variable. If \( \rho \) is the asset correlation, the joint default probability for credit \( A \) and \( B \) is

\[
\Pr[Z < Z_A \land Z < Z_B] = \int_{-\infty}^{Z_A} \int_{-\infty}^{Z_B} \phi(x, y | \rho) \, dx \, dy = \Phi_2(Z_A, Z_B, \rho) \quad \text{Equation 3-75}
\]

This is graphed in Figure 3.9 for an asset correlation of 0.3. This probability is most easily visualised as the volume under the surface in Figure 3.10 in the bottom left hand corner below the specified \( y \)-value and to the left of the specified \( x \)-value.

He comments that CreditMetrics uses a bivariate normal copula function with the asset correlation as the correlation parameter in the copula function. Thus, to generate survival times of two credit risks, a bivariate normal copula function is used with a correlation parameter equal to the CreditMetrics asset correlation (p13).
He further notes: "Conveniently, the marginal distribution of any subset of an $n$ dimensional normal distribution is still a normal distribution. Using asset correlations,
we can construct high dimensional normal copula functions to model the credit portfolio of any size” (p13) This has become the standard modelling methodology in the market O’Kane et al (2003) comment that although both the structural and the reduced form approaches can in principle be extended to the multivariate case, structural models calibrated to market-implied default probabilities (often called ‘hybrid’ models) have gained favour among practitioners because of their tractability in high dimensions “Hybrid models use the dependence among asset returns to generate joint defaults, therefore avoiding the need for a direct estimation of joint default probabilities” (p34) These hybrid models are Monte Carlo models which generate default paths, where each path is a list of default times for each of the credits in the reference portfolio drawn at random from the joint default distribution Once the time and identity of each default event is known, any credit portfolio product may be valued The choice of copula is non-trivial as pointed out by Marshal and Naldi (2002) They demonstrate that the Gaussian copula is unable to explain the extreme co-movements that are observed in the market The assumption of normality of asset returns, however, “is certainly not innocuous, since a multivariate normal distribution does not allow for extreme joint events to happen with the frequency that the data suggests” (p41) The multivariate normal distribution exhibits a tail dependence of zero for all correlations less than 1 They propose instead a $t$ copula that they suggest is consistent with asset returns being multivariate $t$ distributed Their analysis suggests nine as the maximum likelihood degrees of freedom They demonstrate that the impact on a first-to-default credit derivative is to reduce the price by between 5% and 10% compared to the Gaussian copula However, the impact farther out in the tail is quite dramatic they calculate that second-to-default protection will be under-estimated by as much as 58% by using a Gaussian copula While the copula-based approach to credit correlation modelling has become the market standard for portfolios of debt extended to firms for which there is a liquid CDS market, the approach has limited applicability to the vast majority of firms in order to obtain risk-neutral probabilities of default for a particular company, a precise yield curve specific to Company X debt (or a precise yield curve for debt of other companies that are deemed to be of similar credit risk) is required “Thus it will be difficult to apply reduced form models to middle market companies or illiquid markets” Smithson (2003, p215) concludes
3.6 Pricing Credit Risk

Merton's version of the structural approach automatically prices credit risk but it assumes that LGD is endogenous. Other variations on this approach assume that default probability only is determined by the asset value process and that recovery is exogenously specified thereby taking account of the fact that the absolute priority rule is frequently violated in practice.

Smithson (2003, p209), suggests a 'family tree' of models for pricing default-risky claims as shown in Figure 3.11

![Figure 3.11 A Family Tree of Pricing Models for Default-Risky Claims (after Smithson)](image)

Smithson remarks that at the risk of oversimplifying, credit derivatives and traditional derivatives can all be valued as the present value of their risk-adjusted expected future cash flows. He qualifies this remark, however "The bad news is that credit models are much more difficult to implement." He lists three separate areas that cause difficulty:

Default definition: "Default is an imprecise concept subject to various legal and economic definitions." (p209) He adds that a pricing model will necessarily have to simplify the economics of default or very carefully define the precise conditions being modelled.

LGD: Pricing models for credit must address the uncertainty in LGD or assume that loss given default is known, he comments.

Data on which to parameterise models: Data on credit losses are "notoriously limited" and credit spread data are available only for the largest and most liquid markets, he opines (p209).
3.6.1 Pricing Credit Derivatives

This imprecision regarding default definition is evident in the lack of standardisation as regards credit events in credit derivative documentation. Despite the improvements in the 2003 ISDA master agreement over its 1999 predecessor, the market remains split between an US and a non-US standard. In the US, the market has adopted the Modified Restructuring (Mod-Re) standard while the European market has adopted the Modified-Modified Restructuring (Mod-Mod-re) convention. While Duffie (1999) comments that credit default swaps involve some risk of disagreement about whether the event has, in fact, occurred, in his discussion of valuing the credit swap, he ignores issues surrounding documentation and enforceability.

Structural models are seldom used to price credit default swaps; instead, structural models are generally used to say at what spread corporate bonds should trade based on the internal structure of the company, according to O’Kane et al. (2003, p5) They state that they require information about the balance sheet of the company and can be used to establish a link between pricing in the equity and debt markets. They add that they are limited in a number of ways including the fact that they generally lack the flexibility to fit exactly a given term structure of spreads, and “they cannot be easily extended to price complex credit derivatives” (p5).

The reduced form approach of Jarrow and Turnbull (1995) is the basis for what has become the market standard method of valuing CDS. They characterise a credit event as the first event of a Poisson counting process that occurs at time $t$ with a probability defined as

$$\Pr[\tau < t + dt | \tau > t] = \lambda(t) dt \quad \text{Equation 3-76}$$

Thus, the probability of defaulting in the time interval $[t, t + dt]$ conditional on surviving to time $t$ is proportional to $\lambda(t)$, which they call the hazard rate and the length of the time interval, $dt$. Thus, the survival probability is

$$Q(0, T) = E^Q \left[ e^{-\int_0^T \lambda(s) ds} \right] \quad \text{Equation 3-77}$$
where the expectation is taken under the risk-neutral measure, $Q$. The standard market assumption is that the hazard rate process is deterministic, and, therefore, independent of interest rates and recovery rates.

In this framework, the spread is set so that the present values of the premium and protection legs of the CDS contract are equal. If the hazard rate and risk-free rate term structures are flat, and $R$ is the recovery rate, the present value of the protection leg is:

$$ (1-R) \int_0^T \lambda e^{-(r+\lambda)s} ds = \frac{\lambda (1-R)(1-e^{-(r+\lambda)T})}{r+\lambda} $$  \hspace{1cm} \text{Equation 3-78}$$

If the spread, $S$, on the premium leg is paid continuously, the present value of the premium leg is:

$$ S \int_0^T e^{-(r+\lambda)s} ds = \frac{S(1-e^{-(r+\lambda)T})}{r+\lambda} $$  \hspace{1cm} \text{Equation 3-79}$$

Setting the value of the premium leg equal to that of the protection leg, the value of the spread is extracted:

$$ S = \lambda (1-R) $$  \hspace{1cm} \text{Equation 3-80}$$

and the risky PV01, or RPV01 as it is known in the market, is:

$$ RPV01(t) = \frac{(1-e^{-(r+\lambda)(T-t)})}{r+\lambda} $$  \hspace{1cm} \text{Equation 3-81}$$

In reality, the interest rate and hazard rate term structures are not flat and it becomes necessary to build a full term structure using bootstrapping techniques.

Duffie (1999) remarks that the term ‘swap’ applies to credit swaps because they can be viewed under certain ideal conditions as a swap of a default-free note for a defaultable floating-rate note. He proceeds to calculate the at-market annuity premium rate for which the market value of the credit swap is zero at the outset.

### 3.7 Measuring Probability of Default

There are numerous approaches to the measurement of PD in use in the marketplace. These range from the multivariate approach pioneered by Altman thirty years ago to neural networks. For our purposes, it suffices to examine in greater detail the two methodologies which are most widely used in the market and which are also
representative of the two approaches which underpin the models which will be compared later in this thesis. The KMV approach is the most widely used default probability measure commercially available while the Moody's rating is the standard on which many market participants still rely. These two approaches will be examined in turn.

### 3.7.1 KMV

According to the Merton approach, the PD is a function of (i) the value of the firm's assets, (ii) the volatility of this value, and (iii) the amount of debt in the firm's capital structure. KMV have adopted this approach but have adapted it to empirical data to convert the outputs into practical results by developing a measure of risk they call 'distance-to-default'.

Crosbie (2002) discusses KMV's understanding of 'default point'. He notes that in KMV's studies of defaults, they have found that in general firms do not default when their asset value reaches the book value of their total liabilities. He acknowledges that while some firms certainly default at this point, many continue to trade and service their debts. The long-term nature of some of their liabilities provides these firms with some breathing space, he surmises. KMV found that the default point, the asset value at which the firm will default, lies "somewhere between total liabilities and current, or short-term, liabilities". The relevant net worth of the firm is the market value of the firm's assets minus the firm's default point. A firm will default "when its market net worth reaches zero", he notes (p3).

He explains that asset value, business risk, and leverage can be combined into a single measure of default risk which compares the market net worth to the size of a one standard deviation move in the asset value. KMV refer to this ratio as the distance-to-default and it is calculated as

\[
\text{Distance to Default} = \frac{\text{Market Value of Assets} - \text{Default Point}}{\text{Market Value of Assets} \times \text{Asset Volatility}}
\]

"The default probability can be computed directly from the distance-to-default if the default rate for a given level of distance-to-default is known" (p6).

KMV have compiled a very large database of defaults and have computed the distance-to-default metric for these firms for the years prior to their defaulting. A default database is used to derive an empirical distribution relating the distance-to-default to a
default probability. “In this way, the relationship between asset value and liabilities can be captured without resorting to a substantially more complex model characterising a firm’s liability process.” (p7) They call the resulting PD, the expected default frequency, EDF. They have implemented this process in their software product, Credit Edge, and deliver annualised EDF values over 1 to 5 years on a daily basis over the internet on over 35,000 companies. A schematic representation of EDF measurement is shown in Figure 3.12 below:

![Figure 3.12 Schematic for Estimation of Expected Default Frequency](image)

If the future distribution of the distance-to-default were known, the default probability, EDF, would be the likelihood that the final asset value was below the default point. However, in practice, Crosbie says that the distribution of the distance-to-default is difficult to measure. He states that the usual assumptions of normal or lognormal distributions cannot be used. For default measurement, the likelihood of large adverse changes in the relationship of asset value to the firm’s default point is critical to the accurate determination of the default probability, he suggests. These changes may come about from changes in asset value or changes in the firm’s leverage. In fact, changes in asset value and changes in leverage may be highly correlated. Consequently, KMV first measures the distance-to-default as the number of standard deviations the asset value is away from default and then uses empirical data to determine the corresponding default
They obtain the relationship between distance-to-default and default probability from data on historical default and bankruptcy frequencies. Their database includes over 250,000 company-years and over 4,700 incidents of default or bankruptcy. From this data, a lookup or frequency table is created which generates the likelihood of default to various levels of distance-to-default. Thus, if they are interested in determining the default probability over the next year for a firm that is seven standard deviations away from default, they query the default history for the proportion of the firms, seven standard deviations away from default that defaulted over the next year. The answer, he says, is about 5bp, 0.05%, or an equivalent rating of AA.

The PD by time $T$, $p_T$, is the probability that the market value of the firm’s assets, $V_A^T$, will be less than the book value of the firm’s liabilities due by time $T$, $X_T$, where the asset volatility is $\sigma_A$. In the Merton framework, the PD is

$$p_T = \Pr \left[ \ln \left( \frac{V_A}{X_T} \right) + \left( \mu - \frac{1}{2} \sigma_A^2 \right) T + \sigma_A \sqrt{T} \, \varepsilon \right] \leq \ln(X_T) \quad \text{Equation 3-82}$$

Rearranging,

$$p_T = \Pr \left[ - \frac{\ln \left( \frac{V_A}{X_T} \right) + \left( \mu - \frac{1}{2} \sigma_A^2 \right) T}{\sigma_A \sqrt{T}} \geq \varepsilon \right] \quad \text{Equation 3-83}$$

Assuming the asset returns are normally distributed, this probability is

$$p_T = N \left[ - \frac{\ln \left( \frac{V_A}{X_T} \right) + \left( \mu - \frac{1}{2} \sigma_A^2 \right) T}{\sigma_A \sqrt{T}} \right] = N(-DD) \quad \text{Equation 3-84}$$

In order to calculate these EDF values, KMV must calculate the asset value and asset volatility of the firm from the market value and volatility of equity and the book value of the liabilities. They have extended the Merton model into what they call the Vasicek-Kealhofer model to incorporate more complex capital structures including long- and short-term debt, convertible debt and the perpetuity nature of equity. Thus, they solve the following two relationships simultaneously.
Equity Value = \( f(\text{Asset Value, Asset Volatility, Capital Structure, Interest Rate}) \)
Equity Volatility = \( f(\text{Asset Value, Asset Volatility, Capital Structure, Interest Rate}) \)

Asset value and volatility are the only unknown quantities in these relationships and thus the two equations can be solved to determine the values implied by the current equity value, volatility and capital structure.

KMV start with the Merton relationship

\[
V_E = V_A N \left( \ln \left( \frac{V_A}{X e^{-rT}} \right) + \frac{1}{2} \sigma^2_A T \right) - X e^{-rT} \left( \frac{\ln \left( \frac{V_A}{X e^{-rT}} \right) - \frac{1}{2} \sigma^2_A T}{\sigma \sqrt{T}} \right)
\]

Equation 3-85

They then recognise that equity is a leveraged interest in the underlying assets and that its volatility is higher than that of the underlying assets as follows

\[
\sigma_A = \frac{V_A}{V_E} \Delta N(d_i) \text{where}
\]

\[
d_i = -\frac{\ln \left( \frac{V_A}{X e^{-rT}} \right) + \frac{1}{2} \sigma^2_A T}{\sigma \sqrt{T}}
\]

Equation 3-86

\[
\sigma_A = \frac{V_A}{V_E} \Delta N(d_i)
\]

where \( \Delta \) is the delta of the equity option on the assets.

Once again, they need to modify the Merton model to derive meaningful results. They adjust the distance-to-default to include not only the increases in asset value given by the rate but also adjust for any cash outflows to service debt, dividends, and so on. In addition, they state that the Normal distribution is a very poor choice to define the PD. The most important reason they give is the fact that the default point is in reality also a random variable. Thus, while they have assumed that the firm’s default point is described by the firm’s liabilities and amortisation schedule, they acknowledge that this is not true and recognise that firms will often adjust their liabilities as they near default. They note that it is common to observe the liabilities of commercial and industrial firms increase as they near default while the liabilities of financial institutions often decrease as they approach default. The difference, they suggest, is usually just a reflection of the liquidity in the firm’s assets and thus their ability to adjust their leverage as they encounter difficulties. However, KMV have found themselves unable to specify \textit{ex ante}
the behaviour of the liabilities and thus they must capture the uncertainty in the adjustments in the liabilities elsewhere. They choose to include this uncertainty in the mapping of distance-to-default to the EDF credit measure. They observe that the resulting empirical distribution of default rates has much wider tails than the Normal distribution. "For example, a distance to default of four, four standard deviations, maps to a default rate of 100bp. The equivalent probability from the Normal distribution is essentially zero." Crosbie (2000, p17)

KMV determine the PD of a firm by reference to its distance-to-default. All the firms in its database over many years are categorised based on their distance-to-default and the subsequently realised default rates over one-, two-, three-, four- and five-year periods thereafter. Thus, KMV can extract not just a one-year PD for a firm but the term structure of a firm’s PD out to five years.

3.7.2 Rating Agencies

The rating agencies, and indeed, most financial institutions, assign ratings to borrowers based on a variety of financial and non-financial measures. These ratings are merely ordinal rankings and the descriptions of what the ratings mean are extremely general.

In recent years, the rating agencies have responded to market demand for more quantitative risk measures by publishing historical default statistics. Two types of statistics in particular are quoted: (i) historical default rates over various time periods, and (ii) transition matrices which report not just on the frequency of transition from a given grade to default but also on the frequency of transition from a given grade to other non-default grades.

However, two criticisms are frequently levelled at these statistics. The first criticism is that default rates are simply historical and these rates vary significantly from one period to the next. The second is that transition matrices are unconditional averages whereas what the market really needs is a transition matrix that is conditioned to the current state of the credit market.

Despite the criticisms and the many rating errors that the agencies have made – especially notable is their slow reaction to credit deterioration – their rating is still the market benchmark for many companies and debt issues.
3.8 Modelling Credit Portfolios

Crouhy et al (2000), cite two additional difficulties in calculating credit VaR compared to market VaR. The first difficulty cited is that the portfolio distribution is far from being normal, and the second is that measuring the portfolio effect due to credit diversification is much more complex than for market risk. While it was legitimate to assume normality of the portfolio changes due to market risk, they suggest, it is not feasible for credit returns which are by nature highly skewed and fat-tailed. There is limited upside to be expected from any improvement in credit quality, they state, while there is substantial downside consecutive to downgrading and default. "The percentile levels of the distribution cannot be any longer estimated from the mean and variance only. The calculation of VaR for credit risk requires simulating the full distribution of the changes in portfolio value." (p320)

Gupton et al (1997) concur with this view. Modelling portfolio risk in credit portfolios is neither analytically nor practically easy, they note. Fundamental differences between credit risks and equity price risks "make equity portfolio theory problematic when applied to credit portfolios." (p7) They expand on this point by identifying two fundamental problems which credit portfolio modelling presents which are absent when modelling equities. The first problem to which they refer is that oft-cited problem that equity returns are relatively symmetric and are well approximated by normal or Gaussian distributions whereas credit returns are highly skewed and fat-tailed. Thus, more than just the mean and standard deviation is required fully to understand a credit portfolio’s distribution. The second problem they cite is the difficulty of modelling correlations. For equities, they note, the correlations can be directly estimated by observing high-frequency liquid market prices. "For credit quality, the lack of data makes it difficult to estimate any type of credit correlation from history." (p8)

Another important difference between equity portfolio management and credit portfolio management to which they refer is in relation to firm-specific risk. Whereas market risks can be diversified with a relatively small portfolio or hedged using liquid instruments, credit risks, they suggest, are more problematic. For credit portfolios, simply having many obligors’ names represented within a portfolio does not assure good diversification (e.g., they may all be large banks in one country) they state. They conclude that when diversification is possible, it is typically achieved by "much larger numbers of exposures than for market portfolios." (p81)
3 8 1 Credit Portfolio Models used in the Market

According to the 2002 Survey of Credit Portfolio Management Practices undertaken by the International Association of Credit Portfolio Managers (IACPM), the International Swaps and Derivatives Association (ISDA) and the Risk Management Association (RMA) as reported by Rutter (2003, p i61), 85% of the 41 large financial institutions which responded stated they used a portfolio management model. Of these, 69% stated they used KMV’s Portfolio Manager while the CreditMetrics product, CreditManager, was employed by 20%. In addition, 17% used an internally developed model (Rutter notes that the responses sum to more than 100% as some respondents used more than one model.) Neither McKinsey’s CreditPortfolioView nor CSFB’s CreditRisk+ was used by any of the respondents. However, the author is aware that many German mortgage banks use CreditRisk+ so the results may reflect the survey emphasis on commercial and investment banks rather than mortgage/retail banks.

Therefore, it is considered appropriate to examine the two credit portfolio models that are most frequently used in industry, namely, CreditMetrics’ CreditManager and KMV’s Portfolio Manager

3 8 1 1 CreditMetrics

In April 1997, J P Morgan, in conjunction with six bank sponsors and KMV, launched CreditMetrics. It followed the successful launch four years earlier of RiskMetrics, a product for modelling market risk.

While the products were similar, the challenge they faced in creating CreditMetrics was substantially greater. As they acknowledged in their technical document, (Gupton et al., 1994), one major difference in the models was driven by the difference in available data. In RiskMetrics, there was an abundance of available daily liquid pricing data on which to construct a model of conditional volatility. “In CreditMetrics, we have relatively sparse and infrequently priced data on which to construct a model of unconditional volatility.” (p 111)

Nor was there any market agreement on the correct modelling approach. Unlike market risks where daily liquid price observations allow a direct calculation of VaR, they state that CreditMetrics seeks to construct what it cannot directly observe, the volatility of value due to credit quality changes. This constructive approach, they claim, makes
CreditMetrics less an exercise in fitting distributions to observed price data, and more an exercise in proposing models which explain the changes in credit related instruments. Furthermore, the nature of credit returns presented special challenges the models that best describe credit risk “do not rely on the assumption that returns are normally distributed, marking a significant departure from the RiskMetrics framework” (p iv)

CreditMetrics’ goal, they state, was to estimate the volatility of value due to changes in credit quality, not just the expected loss. In their view, as important as default likelihood estimation is, it is only one link in the long chain of modelling and estimation that is necessary fully to assess credit risk within a portfolio “CreditMetrics is a tool for assessing portfolio risk due to changes in debt value caused by changes in obligor credit quality” (p 5) They assess risk within the full context of a portfolio, addressing the correlation of credit quality moves across obligors.

Their outline of their valuation framework is shown in Figure 3.13

Figure 3.13 CreditMetrics Schematic

Figure 3.13 shows the two main building blocks, namely ‘value at risk due to credit’ for a single financial instrument and ‘portfolio value at risk due to credit’ which accounts for portfolio diversification effects. ‘Correlations’ derive the asset return correlations that are used to generate the joint migration probabilities and ‘Exposures’ produce future exposures of derivatives.

CreditMetrics, they state, looks to a horizon and constructs a distribution of historically estimated credit outcomes including rating migrations and, potentially, default. Each
credit quality migration is weighted by its likelihood using transition matrix analysis. Each outcome has an estimate of change in value, given by either credit spreads or studies of recovery rates in default. They then aggregate volatilities across the portfolio, applying estimates of correlation. CreditMetrics accepts any discrete classification of credit quality and an associated credit migration probability matrix.

Gupton et al. (p6) suggest that the primary reason to have a quantitative portfolio approach to credit risk management is so that "we can more systematically address concentration risk." In particular, they suggest that intuitive - but arbitrary - exposure-based credit limits fail to recognize the relationship between risk and return. They suggest that their approach allows a portfolio manager to state credit lines and limits in units of marginal portfolio volatility.

The decision to take on ever higher exposure to an obligor will meet ever higher marginal risk as risk that grows geometrically with the concentration on that name, they claim. They also note how their approach differs markedly from that mandated by the BIS. "The BIS risk-based capital guidelines do not distinguish high quality and well-diversified portfolios from low quality and concentrated portfolios." (p6) However, they acknowledge the difficulty in estimating credit correlation because of a lack of data.

They state their goal is to estimate portfolio risk due to credit events, the uncertainty in the forward value of the portfolio at the risk horizon caused by the possibility of obligor credit quality changes. (p8)

**The CreditMetrics Credit Modelling Approach**

The starting point for the CreditMetrics model is the transition matrix. The CreditMetrics approach is "based on credit migration analysis i.e. estimating the probability of moving from one credit quality to another, including default, within a given time horizon, which is often taken arbitrarily as one year." Crouhy et al. (2000, p316) CreditMetrics models the full distribution of the values of any bond or loan portfolio, say one year forward, with the changes in values being related to credit migration only, while interest rates are assumed to evolve in a deterministic fashion. Credit-VaR of a portfolio is then derived in a similar fashion as for market risk. It is simply the percentile of the distribution corresponding to the desired confidence level.

CreditMetrics assumes that future migration probabilities are equal to historical rating transition experience. They further assume that the value of the facility at the modelling
horizon may be obtained by discounting the cash flows using the borrower’s forward zero curve, a “curve that is different for each rating category.” (p10) Additionally, their framework accommodates variable loss rates assuming a beta distribution based on a mean loss rate and a standard deviation specified by the user based on the seniority of the debt issue.

To measure the effect of portfolio diversification, CreditMetrics need to estimate the correlations in credit quality changes for all pairs of obligors. But these correlations are not directly observable. They base their evaluation on the joint probability of asset returns, “which itself results from strong simplifying assumptions on the capital structure of the obligor, and on the generating process for equity returns.” Crouhy et al (2000, p321)

CreditMetrics suggest a model of firm value that has the log of each firm’s value at horizon described by a normal distribution. The distribution is divided into discrete areas such that the probability of being in a given area corresponds with the probabilities in the transition matrix. Furthermore, all firms may be modelled jointly as multivariate normal based.

Further discussion of the CreditMetrics approach is presented in the Appendix.

**Critique of the CreditMetrics Credit Modelling Approach**

CreditMetrics make many assumptions that may invalidate their results. They assume (i) that each obligor will migrate to a credit rating at the horizon date based on its senior, unsecured credit rating and the transition matrix of historical migrations, (ii) that all obligors in a given rating category will face the same forward zero curve at the horizon date – in other words, that interest rates will evolve to the forward rates in a deterministic fashion and that the forward credit spreads will be realised at the horizon, (iii) that credit rating distributions are multivariate normal and (iv) that asset correlation may be approximated by equity correlation.

Crouhy et al (2000) state that the CreditMetrics approach to measuring credit risk is rather appealing as a methodology, but that “unfortunately it has a major weakness: reliance on ratings transition probabilities that are based on average historical frequencies of default and credit migration.” (p357) As a result, they suggest, the accuracy of CreditMetrics calculation depends upon two critical assumptions, namely, that all firms within the same rating class have the same default rate and the same
spread curve even when recovery rates differ among obligors, and that the actual default rate is equal to the historical average default rate. But this cannot be true because default rates evolve continuously whereas ratings are adjusted in a discrete fashion.

3.8.1.2 KMV Portfolio Manager

KMV’s approach to modelling debt portfolios is embedded in their software offering, *Portfolio Manager*. This product is based on the same structural model underlying their default probability software, *Credit Edge*, and the default probability, their so-called expected default frequency (EDF), is a key input to the portfolio model. Furthermore, the asset value time series imputed for each company forms the basis for creating asset value indices by country and industry, from which they extract their factors for the model underlying their correlation estimates.

KMV are less transparent about their methodology and their data than CreditMetrics. They make their EDF data and their factor sensitivity data available to clients. They also explain in high-level terms the approach they adopt. However, they avoid explaining their methodology in detail and most of the key steps in the simulation and revaluation elements of the program are not transparent.

The CDO model that is developed in this thesis relies on KMV data for its default probabilities and its correlation data. In building the model, therefore, the first task was to check that the KMV results could be reproduced to confirm that the data was being interpreted in the appropriate manner. A complete description of the tasks undertaken in achieving this goal is given in Chapter 4 and will not, therefore, be repeated here.

3.8.1.3 Comparison of the CreditMetrics and KMV Approaches

The most fundamental difference between the CreditMetrics and KMV approaches relates to their default probability estimation methodology. Whereas KMV develop an estimate of expected default frequency for each borrower, CreditMetrics rely upon the average historical transition frequencies produced by the rating agencies for each credit class.

Another key difference relates to the manner in which they measure correlation. CreditMetrics relies on equity values and equity indices for various industry-country combinations in order to imply asset correlation. KMV, on the other hand, create asset
return series based on all the firms in their database and use principal component analysis to extract orthogonal factors which are then used to drive future asset values.

Both approaches rely on the asset value model originally proposed by Merton (1974), but they differ quite substantially in the simplifying assumptions they require in order to facilitate its implementation. "How damaging are, in practice, these compromises to a satisfactory capture of the complexity of credit measurement stays an open issue." Crouhy et al, (2000, p357)

3.8.1.4 Other Portfolio Models

Credit Suisse Financial Products released CreditRisk+ in 1997. It is a 'default only' model that assumes default for individual loans or bonds follows a Poisson process.

McKinsey also has a portfolio model offering which is focused on default risk only. It is a discrete time multi-period model, where default probabilities are a function of macrovariables like unemployment, the level of interest rates, and the growth rate in the economy, government expenses and foreign exchange rates, which drive credit cycles.

3.9 Moody’s Binomial Expansion Technique

Cifuentes and O’Connor (1996) outline Moody’s Binomial Expansion Technique (BET) as applied to CDO analysis. “Moody’s ratings of CBOs and CLOs are ultimately based on the expected loss concept” (p1). They suggest that a number of methods can be used to estimate the expected loss, ranging from Monte Carlo simulation techniques to rather simple single-event models. However, they propose an alternative to simulation or single-event models, the so-called BET. They suggest it captures the effects of “tail events”, by accounting for all possible default scenarios (p1).

3.9.1 Overview of the BET

The BET is based on the diversity score concept. “The idea is to use the diversity score to build a hypothetical pool of uncorrelated and homogeneous assets that will mimic the default behaviour of the original pool.” (p2) If $D$ is the diversity score of the collateral portfolio, they suggest the behaviour of the original pool can be modelled using a fictitious portfolio consisting of $D$ bonds, each of which has the same par value (total collateral par value divided by $D$). They assume that these bonds have the same probability of default (determined by the weighted average probability of default of the
original pool) They further assume that the behaviour of this homogeneous pool of $D$
assets can be fully described in terms of $D$ possible scenarios: one default, two defaults
and so on up to $D$ defaults. The probability $P_j$ that scenario $j$ ($j$ defaults) could happen
can be computed using the binomial formula where $P_j$ represents the weighted average
probability of default of the pool $E_j$, is the term they assign to the loss for the note to
be rated under scenario $j$. They calculate the total expected loss, considering all possible
loss scenarios, as

$$\text{Expected Loss} = \sum_{i}^{D} P_j E_j$$ \hspace{1cm} \text{Equation 3-87}

3.9.1.1 Weighted Average Rating Factor

Each rating is mapped to a rating factor and the credit quality of the asset pool is
determined by the weighted-average rating factor, WARF. This is calculated as the par
value-weighted average. The WARF score measures, in basis points, the Moody’s
idealised cumulative default rate over ten years. The rating factor equivalents from that
paper are reproduced in Figure 3.14

<table>
<thead>
<tr>
<th>Rating of Debt Security</th>
<th>Rating Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaa</td>
<td>1</td>
</tr>
<tr>
<td>Aa1</td>
<td>10</td>
</tr>
<tr>
<td>Aa2</td>
<td>20</td>
</tr>
<tr>
<td>Aa3</td>
<td>40</td>
</tr>
<tr>
<td>A1</td>
<td>70</td>
</tr>
<tr>
<td>A2</td>
<td>120</td>
</tr>
<tr>
<td>A3</td>
<td>180</td>
</tr>
<tr>
<td>Baa1</td>
<td>260</td>
</tr>
<tr>
<td>Baa2</td>
<td>360</td>
</tr>
<tr>
<td>Baa3</td>
<td>610</td>
</tr>
<tr>
<td>Ba1</td>
<td>940</td>
</tr>
<tr>
<td>Ba2</td>
<td>1,350</td>
</tr>
<tr>
<td>Ba3</td>
<td>1,780</td>
</tr>
<tr>
<td>B1</td>
<td>2,220</td>
</tr>
<tr>
<td>B2</td>
<td>2,720</td>
</tr>
<tr>
<td>B3</td>
<td>3,490</td>
</tr>
<tr>
<td>Caa</td>
<td>6,500</td>
</tr>
<tr>
<td>Ca</td>
<td>10,000</td>
</tr>
<tr>
<td>C</td>
<td>10,000</td>
</tr>
</tbody>
</table>

Figure 3.14 Moody’s Rating Factor Equivalents
3912 Diversity Score

The other key attribute of the asset pool is the level of diversification that exists within it. To measure this, Moody’s have developed the concept of diversity score. They look at both the number of firms in the collateral pool and their distribution among industry groups.

Diversity Score penalises the structure for having issuers in the same industry. For example, Table 4 of their paper, reproduced below as Figure 3.15, shows that the first firm in a particular industry earns the transaction a Diversity Score of 1. The second name in the same industry increases the Diversity Score in that particular industry to 1.5. Subsequent additions from the same industry earn a still lesser addition to diversity score. The transaction’s total Diversity Score is computed by summing the Diversity Scores of all industries represented in the portfolio.

<table>
<thead>
<tr>
<th>Number of Firms in Same Industry</th>
<th>Diversity Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00</td>
</tr>
<tr>
<td>2</td>
<td>1.50</td>
</tr>
<tr>
<td>3</td>
<td>2.00</td>
</tr>
<tr>
<td>4</td>
<td>2.33</td>
</tr>
<tr>
<td>5</td>
<td>2.67</td>
</tr>
<tr>
<td>6</td>
<td>3.00</td>
</tr>
<tr>
<td>7</td>
<td>3.25</td>
</tr>
<tr>
<td>8</td>
<td>3.50</td>
</tr>
<tr>
<td>9</td>
<td>3.75</td>
</tr>
<tr>
<td>10</td>
<td>4.00</td>
</tr>
<tr>
<td>&gt;10</td>
<td>Evaluated on a case-by-case basis</td>
</tr>
</tbody>
</table>

Figure 3.15 Moody’s Diversity Score Measurement Methodology

Moody’s 32-industry classification system used to measure intra-industry correlation is described by Backman and O’Connor (1995, p11). Appendix B of that paper is reproduced as Figure 3.16. Since the paper was published, an additional industry, “Broadcasting and Entertainment”, has been added to the list.
### Table 6

<table>
<thead>
<tr>
<th>Number</th>
<th>Industry Classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Aerospace and Defense: Major Contractor, Subsystems Research Aircraft Manufacturing, Arms Ammunition</td>
</tr>
<tr>
<td>2</td>
<td>Automotive Automotive Equipment, Auto-Part Manufacturing, Personal-Use Trailers, Motor Homes Dealers</td>
</tr>
<tr>
<td>3</td>
<td>Banking Bank Holding, Savings and Loans, Consumer Credit, Small Loan Agency, Factoring, Receivables</td>
</tr>
</tbody>
</table>
3.9.2.1 The BET Approach to Rating Senior Notes

As discussed previously, the BET rating approach was first outlined in Cifuentes and O’Connor (1996). The portfolio they use to demonstrate the approach is shown in Figure 3.17.

![Chart 1: Hypothetical CBO Structure](image)

Figure 3.17 Moody’s Hypothetical CBO Structure

They consider the simple two-tier structure depicted in Figure 3.17. They assume that the collateral pool has a diversity score of 20, an average probability of default of 25% (after factoring in the stressing factor), a recovery rate of 30%, a six-year time to maturity, and pays an average coupon of 11%. They also assume that all bonds have bullet repayments, that there are no overcollateralisation or interest rate triggers, and that the excess cash is reinvested at 11% per year.

The author built a model to confirm Moody’s results. An example of the output is shown in Table 3-6 below when the number of defaults is set at 10. In these circumstances, the senior note receives \{2.4, 2.4, 2.4, 2.4, 2.4, 2.4, 2.4, 2.4, 2.4, 2.4, 78.86\}. The final payment is 78.86 instead of 82.4, yielding a loss of 3.1%.
Collateral 100
Coupon 11%
p 25%
D 20
Number of Defaults 10
Recovery 30%
Maturity 6 yrs
Senior 80
Senior coupon 6%
Equity 20
Equity coupon 12%

<table>
<thead>
<tr>
<th>Year</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
<th>45</th>
<th>50</th>
<th>55</th>
<th>60</th>
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<tbody>
<tr>
<td>0%</td>
<td>0</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Default Distribution</td>
<td>0</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Cumulative Defaults</td>
<td>0</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Defaulted Collateral</td>
<td>0</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Cum Def Collateral</td>
<td>0</td>
<td>5</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Recoveries</td>
<td>0</td>
<td>7.5</td>
<td>12.5</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
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<tr>
<td>Collateral Outstanding</td>
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<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>Collateral Coupon</td>
<td>55</td>
<td>45</td>
<td>45</td>
<td>43</td>
<td>43</td>
<td>42</td>
<td>42</td>
<td>40</td>
<td>40</td>
<td>38</td>
<td>38</td>
<td>36</td>
<td>36</td>
</tr>
<tr>
<td>Surplus Account</td>
<td>0</td>
<td>20</td>
<td>30</td>
<td>40</td>
<td>50</td>
<td>60</td>
<td>70</td>
<td>80</td>
<td>90</td>
<td>100</td>
<td>110</td>
<td>120</td>
<td>130</td>
</tr>
<tr>
<td>Cash Available</td>
<td>55</td>
<td>65</td>
<td>75</td>
<td>85</td>
<td>95</td>
<td>105</td>
<td>115</td>
<td>125</td>
<td>135</td>
<td>145</td>
<td>155</td>
<td>165</td>
<td>175</td>
</tr>
<tr>
<td>Payment to Senior</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
<td>24</td>
</tr>
<tr>
<td>Interest to Equity</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>Principal to Equity</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Surplus in year</td>
<td>19</td>
<td>22</td>
<td>25</td>
<td>28</td>
<td>31</td>
<td>34</td>
<td>37</td>
<td>40</td>
<td>43</td>
<td>46</td>
<td>49</td>
<td>52</td>
<td>55</td>
</tr>
</tbody>
</table>

Table 3-6 Moody’s Cash Flow Model

Repeating the analysis for varying numbers of defaults and collating confirms the output in Table 1 of their paper Moody’s assume that the portfolio comprises a number of assets equal to the diversity score that default independently. Hence the distribution of losses is given by the binomial probability distribution. The results are reproduced as Table 3-7 below.
<table>
<thead>
<tr>
<th>Probability</th>
<th># Defaults</th>
<th>Loss %</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3171%</td>
<td>0</td>
<td>0.0000%</td>
</tr>
<tr>
<td>2.1141%</td>
<td>1</td>
<td>0.0000%</td>
</tr>
<tr>
<td>6.6948%</td>
<td>2</td>
<td>0.0000%</td>
</tr>
<tr>
<td>13.3896%</td>
<td>3</td>
<td>0.0000%</td>
</tr>
<tr>
<td>18.9685%</td>
<td>4</td>
<td>0.0000%</td>
</tr>
<tr>
<td>20.2331%</td>
<td>5</td>
<td>0.0000%</td>
</tr>
<tr>
<td>16.8609%</td>
<td>6</td>
<td>0.0000%</td>
</tr>
<tr>
<td>11.2406%</td>
<td>7</td>
<td>0.0000%</td>
</tr>
<tr>
<td>6.0887%</td>
<td>8</td>
<td>0.0000%</td>
</tr>
<tr>
<td>2.7061%</td>
<td>9</td>
<td>0.0000%</td>
</tr>
<tr>
<td>0.9922%</td>
<td>10</td>
<td>3.1026%</td>
</tr>
<tr>
<td>0.3007%</td>
<td>11</td>
<td>7.8958%</td>
</tr>
<tr>
<td>0.0752%</td>
<td>12</td>
<td>12.6890%</td>
</tr>
<tr>
<td>0.0154%</td>
<td>13</td>
<td>17.4822%</td>
</tr>
<tr>
<td>0.0026%</td>
<td>14</td>
<td>22.2754%</td>
</tr>
<tr>
<td>0.0003%</td>
<td>15</td>
<td>27.0686%</td>
</tr>
<tr>
<td>0.0000%</td>
<td>16</td>
<td>31.5621%</td>
</tr>
<tr>
<td>0.0000%</td>
<td>17</td>
<td>34.7819%</td>
</tr>
<tr>
<td>0.0000%</td>
<td>18</td>
<td>38.0531%</td>
</tr>
<tr>
<td>0.0000%</td>
<td>19</td>
<td>41.6080%</td>
</tr>
<tr>
<td>0.0000%</td>
<td>20</td>
<td>45.1629%</td>
</tr>
</tbody>
</table>

**Table 3-7 Scenario Default Probability and Senior Tranche Loss Percentage**

Repeating the analysis for varying diversity scores confirms the results Cifuentes and O’Connor have graphed in Chart 2 of their paper. Chart 2 is re-created here as Figure 3.18:

![Expected Loss versus Diversity](chart.png)

**Figure 3.18 Expected Loss for Senior Tranche as a Function of Diversity**
<table>
<thead>
<tr>
<th>Rating</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaa</td>
<td>0.000028</td>
<td>0.000011</td>
<td>0.000039</td>
<td>0.000099</td>
<td>0.000160</td>
<td>0.000220</td>
<td>0.000286</td>
<td>0.000363</td>
<td>0.000451</td>
<td>0.000550</td>
</tr>
<tr>
<td>Aa1</td>
<td>0.0000314</td>
<td>0.0000165</td>
<td>0.0000550</td>
<td>0.0001155</td>
<td>0.0001705</td>
<td>0.0002310</td>
<td>0.0002970</td>
<td>0.0003685</td>
<td>0.0004510</td>
<td>0.0005500</td>
</tr>
<tr>
<td>Aa2</td>
<td>0.0000748</td>
<td>0.0000440</td>
<td>0.0001430</td>
<td>0.0002585</td>
<td>0.0003740</td>
<td>0.0004895</td>
<td>0.0006105</td>
<td>0.0007425</td>
<td>0.0009020</td>
<td>0.011000</td>
</tr>
<tr>
<td>Aa3</td>
<td>0.0001661</td>
<td>0.0001045</td>
<td>0.0003245</td>
<td>0.0005555</td>
<td>0.0007810</td>
<td>0.0010065</td>
<td>0.0012485</td>
<td>0.0014960</td>
<td>0.0179850</td>
<td>0.220000</td>
</tr>
<tr>
<td>A1</td>
<td>0.0003196</td>
<td>0.0002035</td>
<td>0.0006435</td>
<td>0.0103950</td>
<td>0.0143550</td>
<td>0.0181500</td>
<td>0.0223300</td>
<td>0.0264000</td>
<td>0.0315150</td>
<td>0.385000</td>
</tr>
<tr>
<td>A2</td>
<td>0.0005979</td>
<td>0.0003850</td>
<td>0.0122100</td>
<td>0.0189750</td>
<td>0.0256850</td>
<td>0.0320650</td>
<td>0.0390500</td>
<td>0.0455950</td>
<td>0.0540100</td>
<td>0.660000</td>
</tr>
<tr>
<td>A3</td>
<td>0.0013686</td>
<td>0.0008250</td>
<td>0.0198000</td>
<td>0.0297000</td>
<td>0.0401500</td>
<td>0.0500500</td>
<td>0.0610500</td>
<td>0.0715000</td>
<td>0.0836000</td>
<td>0.990000</td>
</tr>
<tr>
<td>Baa1</td>
<td>0.0049500</td>
<td>0.0154000</td>
<td>0.0308000</td>
<td>0.0456500</td>
<td>0.0605000</td>
<td>0.0753500</td>
<td>0.0918500</td>
<td>1.0835000</td>
<td>1.2485000</td>
<td>1.430000</td>
</tr>
<tr>
<td>Baa2</td>
<td>0.0093500</td>
<td>0.0258500</td>
<td>0.0456500</td>
<td>0.0660000</td>
<td>0.0860000</td>
<td>0.1083500</td>
<td>0.1325500</td>
<td>0.1567500</td>
<td>0.1782000</td>
<td>0.198000</td>
</tr>
<tr>
<td>Baa3</td>
<td>0.0231000</td>
<td>0.0577500</td>
<td>0.0940500</td>
<td>0.1309000</td>
<td>0.1677500</td>
<td>0.2035000</td>
<td>0.2381500</td>
<td>0.2733500</td>
<td>0.3063500</td>
<td>0.355000</td>
</tr>
<tr>
<td>Ba1</td>
<td>0.0478500</td>
<td>1.111000</td>
<td>1.721500</td>
<td>2.310000</td>
<td>2.904000</td>
<td>3.437500</td>
<td>3.883000</td>
<td>4.339500</td>
<td>4.779500</td>
<td>5.170000</td>
</tr>
<tr>
<td>Ba2</td>
<td>0.0858000</td>
<td>1.908500</td>
<td>2.849000</td>
<td>3.740000</td>
<td>4.625500</td>
<td>5.373500</td>
<td>5.885000</td>
<td>6.413000</td>
<td>6.957500</td>
<td>7.425000</td>
</tr>
</tbody>
</table>
The expected loss suffered by the senior note is 0.067%. Referring to Moody’s Idealised Cumulative Expected Loss Rates in Table 2 of that paper, reproduced above as Table 3-8, the senior note would be rated Aa3 (the cut-off value is 0.10065% for the Aa3).

### 3.9.2.2 The BET Approach to Rating Senior/Subordinate Structures

Anderson (1997) describes how diversification of a securitised pool affects Moody’s ratings of senior/subordinated structures. The author built a model to replicate the structure he described. The model results are shown in Table 3-9. These concur exactly with the results he reports in this paper.

<table>
<thead>
<tr>
<th>Diversity</th>
<th>30 Pool</th>
<th>100 Pool</th>
<th>Probability of Loss</th>
<th>Pool Loss</th>
<th>Senior Loss</th>
<th>Mezz. Loss</th>
<th>Junior Loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>PD 10% Senior</td>
<td>60</td>
<td>30.00</td>
<td>30.00</td>
<td>10.00</td>
<td>95.8%</td>
<td>0.0%</td>
<td>17.5%</td>
</tr>
<tr>
<td>LGD 70% Mezzanine</td>
<td>30</td>
<td>90.00</td>
<td>30.00</td>
<td>5.00</td>
<td>7.00%</td>
<td>0.00%</td>
<td>1.83%</td>
</tr>
</tbody>
</table>

Table 3-9 Probability of Loss, LGD and EL for Pool and CDO Tranches

The effectiveness of diversification and subordination in protecting the senior notes at the expense of the junior notes is observed by varying the diversity score. Anderson’s results, as summarised in Table 1 of his paper, agree exactly with Table 3-10 below. The impact of diversity is best appreciated by graphing these results. Figure 3.19 below
plots the model results. Again, these agree exactly with the results depicted in Chart 1 of Anderson’s paper: it shows how the expected loss of the pool is increasingly concentrated in the subordinated tranches as diversity increases. Similarly, Figure 3.20 shows how the probability of the senior tranche suffering a loss declines to zero as diversity increases while the probability of the junior notes incurring a loss goes to 100%. Finally, Figure 3.21 plots the model results which agree exactly with Anderson’s results shown in Chart 3 of the paper: it shows, as might be expected, that the probability distribution of losses narrows as diversity increases.

<table>
<thead>
<tr>
<th>Diversity</th>
<th>Pool</th>
<th>Expected Loss</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Senior</td>
<td>Mezz.</td>
</tr>
<tr>
<td>1</td>
<td>7.000%</td>
<td>5.000%</td>
</tr>
<tr>
<td>2</td>
<td>7.000%</td>
<td>0.500%</td>
</tr>
<tr>
<td>3</td>
<td>7.000%</td>
<td>0.350%</td>
</tr>
<tr>
<td>5</td>
<td>7.000%</td>
<td>0.039%</td>
</tr>
<tr>
<td>10</td>
<td>7.000%</td>
<td>0.001%</td>
</tr>
<tr>
<td>20</td>
<td>7.000%</td>
<td>0.000%</td>
</tr>
<tr>
<td>30</td>
<td>7.000%</td>
<td>0.000%</td>
</tr>
<tr>
<td>50</td>
<td>7.000%</td>
<td>0.000%</td>
</tr>
<tr>
<td>100</td>
<td>7.000%</td>
<td>0.000%</td>
</tr>
</tbody>
</table>

Table 3-10 Expected Loss and Probability of Loss as a Function of Diversity

Figure 3.19 Expected Loss vs Diversity for Pool and CDO Tranches
3.9.3 Critique of the Moody's Approach

The Moody's approach is extremely heuristic and some would suggest it doesn’t merit citing as a portfolio model. The simplistic manner in which the diversity score is
calculated is scorned. Likewise, the distillation of a range of credit ratings into a weighted average has been criticised. This latter criticism has prompted them to develop a Double Binomial Method to accommodate portfolios where two groups of assets have distinctly different default probabilities.

Among other criticisms, the role played by the diversity score is often questioned. Greater diversification will always lead to a higher quality senior tranche or will enable the sponsor to achieve a higher percentage of AAA-rated debt. However, in practice, greater diversity has been blamed for some of the problems that have been encountered in the CDO market.

Typical of these comments is that of Goodman and Fabozzi (2001, p34). They claim that a very high diversity score can limit flexibility by requiring an asset manager with broad expertise to invest in an industry he does not like. They suggest that too much diversification is even more of a problem for a smaller asset manager, where the portfolio may have selective strengths in fewer industries. They conclude: “Investors should certainly be wary of deals in which very high diversity scores are achieved by managers straying from their fields of expertise.” (p34)

### 3.10 Modelling Credit Portfolios over Multiple Time Periods

The KMV portfolio model developed in Chapter 4, in common with other major models in the market – CreditMetrics, CreditRisk+ and CreditPortfolioView – are single time period models. The models describe for a specific risk horizon, whether each asset of interest defaults within the horizon. The timing of defaults within the risk horizon is not considered, nor is the possibility of defaults beyond the horizon. “This is not a flaw of the current models, but rather an indication of their genesis as approaches to risk management and capital allocation for a fixed portfolio.” Finger (2000, p49)

However, this framework is incapable of dealing with the modelling of CDOs. Finger comments that the performance of a CDO structure depends on the default behaviour of a pool of assets. He notes that the dependence of is not just on whether the assets default over the life of the structure, but also on when the defaults occur. Thus, he concludes that while an “application of the existing models can give a cursory view of the structure, a more rigorous analysis requires a model of the timing of defaults.” (p49)

In the paper, he compares the performance of four models – a discrete CreditMetrics extension, a diffusion-driven CreditMetrics extension, a copula approach and the
stochastic default intensity approach of Duffie and Garleanu (2001) Each model is calibrated to the same one-year default rate and single period correlation parameter He modelled a portfolio of 100 obligors under low and high correlation assumptions on the assumption of 40% recovery He then tranched the liabilities into a 10% first loss, a 20% second loss and a senior piece

The discrepancies between the models were not too large for the first loss tranche – the cost of first loss protection was approximately 20% higher for the most expensive compared to the least expensive Likewise, under the high correlation assumption, the variation in the cost of second loss protection was of a similar magnitude However, under the low correlation assumption, the most expensive second loss protection was almost twice that of the least expensive and, for the senior tranche, the subordination necessary to achieve a target rating varied by a factor greater than two At the 30% subordination level, the senior notes were rated Aaa in the most benign model compared to A3 in the most severe

He notes that in the single period case, a number of studies have concluded that the various models do not produce vastly different conclusions when calibrated to the same first and second order information However, in the case of CDOs, “the issue of model choice is much more important, and any analysis of structures over multiple horizons should heed this potential model error” (p64)

However, the actual problem is even greater than Fnger suggests, the disparities he has identified remain even after the models have been calibrated to the same input data In fact, the input data is, in most cases, unavailable “Currently the weakest link in the chain of CDO analysis is the limited availability of empirical data bearing on the correlation of default risk ” Duffie and Singleton (2003, p252)

3.11 Choosing a Modelling Paradigm

The two market-leading portfolio models – those of CreditMetrics and KMV - have been examined in this chapter Despite the fact that they are both structural models, they adopt very different approaches to the solution of the credit portfolio modelling problem

CreditMetrics relies on credit ratings and rating transition matrices Its most obvious application is to portfolios of publicly rated names for which there is a substantial volume of data on which to build transition matrices While it is possible for any
financial institution to build its own transition matrices based on its own internal rating system, few would have a sufficient number of names in the various ratings to enable the extraction of reliable transition probabilities.

In contrast, the approach adopted by KMV enables them to generate probabilities of default for all publicly quoted firms – some 35,000 at the time of writing – and update these estimates on a daily basis. The agency-rated universe, by comparison, is substantially less, numbering less than 5,000. This is a particular problem in Europe where most debt is privately issued and only the largest companies aiming to sell their debt worldwide seek a rating from (predominantly US) rating agencies. Adopting a framework which naturally provides the key input, namely, default probability, greatly expands the universe of firms which are amenable to analysis. Many of the so-called ‘arbitrage’ CDOs purchase the debt of firms which would not normally seek an agency rating because of their smaller size and their lesser creditworthiness, having a ready measure of default probability is critical in these circumstances.

The rating agencies’ own research has identified the issue of autocorrelation in downgrades. CreditMetrics, if it were expanded to a multi-period framework, would ignore this autocorrelation in its Markov modelling framework that assumes grading transition is a Markov process. This would systematically under-estimate the tendency for serial downgrades, an issue that is critical to the performance of CDO tranches. In contrast, KMV offer a term structure of default probability for each borrower that can inform the evolution of borrower creditworthiness over successive time periods.

The transition matrix most frequently used is an average of transitions over a long time period. However, the realised transition and default probabilities vary quite substantially over the years depending on whether the economy is in recession or expanding. “When implementing a model that relies on transition probabilities, one may have to adjust the average historical values to be consistent with one’s assessment of the current economic environment.” Crouhy et al (2001, p325) There is little published in the literature to help inform the user how to condition the transition matrix on the state of the economy. Not having a mechanism for conditioning the transition matrix on the stage in the economic cycle is a critical issue militating against using CreditMetrics in a multi-period setting.

The grading systems developed by rating agencies and by most financial institutions are long-term ratings. That is, they take a ‘through-the-cycle’ view of credit compared to
the KMV view that is characterised as ‘point-in-time’. "A strong assumption made by CreditMetrics is that all issuers are credit-homogeneous within the same rating class, with the same transition probabilities and the same default probabilities." Crouhy et al (2000, p27) In KMV’s approach, in contrast, each issuer is specific and is characterised by its own returns distribution, its own capital structure, and, consequently, its own default probability term structure.

While some agency ratings apply to the issuer, most agency ratings attach to the debt issue. “Bond-rating systems are supposed to rate an individual loan (including its covenants and collateral backing), whereas loan-rating systems are more oriented to rating the overall borrower.” Saunders et al. (2002, p18) In Moody’s words, a rating is, “an opinion on the future ability and legal obligation of an issuer to make timely payments of principal and interest on a specific fixed-income security.” Moody’s (1998, p4). The following year, they elaborate as follows: “Moody’s ratings of industrial and financial companies have primarily reflected default probability, while expected severity of loss in the event of default has played an important secondary role. In the speculative-grade portion of the market, which has been developing into a distinct sector, Moody’s ratings place more emphasis on expected loss than on relative default risk.”

For these reasons, it was decided to choose the KMV modelling paradigm instead of that adopted by CreditMetrics. Throughout what follows, the default probability estimates used will be KMV’s expected default frequencies. Likewise, KMV’s asset correlation estimates will be embedded in the portfolio model through the use of the factor sensitivities in their factor model framework. Furthermore, it was decided to adopt Moody’s BET approach to CDO tranche rating in preference to those of other rating agencies because of its more widespread usage and acceptance.

### 3.12 Conclusion

This chapter presented an overview of the principal research strands that are relevant to an analysis of a portfolio of debt securities.

The most fundamental issue in debt valuation is an assessment of the PD of the individual obligors so the chapter began by assessing various approaches to the measurement of default probability proposed in the literature.
The next section undertakes an examination of the literature concerning the interaction of equity securities in a portfolio. While portfolios of equities behave in a fundamentally different way from portfolios of debt securities, it is in the area of equity portfolio modelling that the research is most advanced and it is to these approaches those who are faced with modelling debt securities have turned in the first instance.

The unique challenges that confront researchers in the area of debt portfolio modelling are explored in the next section. Few closed form solutions are available because of the particularly complex nature of the interaction among the securities. One formulaic solution to the portfolio value distribution problem for a portfolio of infinite granularity allows us to examine the main attributes of debt portfolios and gives guidance on the challenges facing those who need to model actual portfolios in a more realistic manner.

The approach to valuing credit default swaps was then introduced. The valuation method of choice for structured products, the copula approach, was also summarised. Structured products referencing credits for which a liquid CDS exist are successfully accommodated within this framework.

The credit portfolio models which are most used in the market – those of CreditMetrics and KMV – are examined next. These models were contrasted and critically assessed. The KMV model proved to be particularly suited to the modelling of exposures to sub-investment grade names which are not traded in the market. Also, the approach adopted by Moody’s to rate CDO tranches, BET, was summarised and critiqued.

There is an obvious disconnect between (i) the rating agency approach to CDO tranche grading which takes explicit account of the CDO indenture and the multi-period dimension to the tranche-rating issue but largely ignores the work of academic researchers, and (ii) the more theoretically rigorous academic approaches which have been adopted by the software vendors but which are fundamentally single period models and ignore the CDO indenture which is central to tranche rating and valuation.

The challenge that is being taken up in this thesis is to apply the intellectual rigour of the academic approach embedded in one vendor offering to a multi-period framework which takes explicit account of the CDO indenture. In undertaking that challenge, the decision was made to adopt the KMV approach in preference to the competing market offerings because of its more robust theoretical foundations and its greater data coverage.
As a first step to achieving this goal, the current state-of-the-art – as represented by KMV’s credit portfolio modelling approach – will be re-created next in Chapter 4. On completion of the single period model, we will then be ready to take on the challenge of developing a multi-period version. This model will incorporate the best features of the single period credit portfolio model and the rating agency model. It will model the asset migration over time in an academically rigorous risk-neutral pricing framework and incorporate all the features specified in the CDO indenture in channelling the cash flows to the liabilities. The development of this model is the subject of Chapter 5.
Chapter 4. Developing a Single Period Credit Portfolio Model

4.1 Chapter Overview

In Chapter 3, the market-leading models were overviewed and the fundamentally different approaches taken by the rating agencies and the portfolio model vendors were noted. The differences between the two approaches are such that it is not possible to make meaningful comparison between them.

The portfolio models are framed on a single period and are incapable of analysing CDO tranches whose values are fundamentally dependent on the performance of the portfolio over multiple time periods. It is only by modelling the underlying debt portfolio over the life of the transaction that the cash flows available to the tranches can be calculated and hence valued. Single period models can give the value of the debt portfolio at the period end but are incapable of saying how that value is divided among the different tranches.

The rating agency model takes full account of the CDO indenture and explicitly models the cash flow waterfall. However, it adopts a heuristic approach to the modelling of correlation and default probability. Ignoring all the modelling advances of the past decade is clearly undesirable.

The goal of this thesis is to develop a new model that incorporates the best features of the rating agency approach into a multi-period version of the market-leading credit portfolio model. A further goal is to compare the rating implied by this new model - which is potentially more rigorous than the current rating agency model - with that assigned by the rating agency model to the various different CDO tranches.

In order to achieve these goals, three tasks must be undertaken:

1. The KMV approach to modelling credit portfolios in a single period time frame is re-created from public data sources.

2. This single period model is expanded into a multi-period setting in order to be capable of valuing CDOs.

3. Moody’s BET is replicated, once again by reference to published articles.

The completion of the first of these tasks is the subject of this chapter.
4.2 The KMV Modelling Paradigm

Traditionally, credit risk in a portfolio context was managed qualitatively with stringent underwriting standards, limit enforcement and counterparty monitoring. However, such an approach fails because of its inability to measure the correlated nature of credit defaults. Furthermore, it fails to inform a bank’s efforts to build portfolios with superior return-to-risk characteristics.

KMV attempts to replace this qualitative approach with a quantitative one. Rather than measure portfolio risk as an exposure-based amount, they focus instead on the amount of economic capital needed to maintain a particular level of risk in the debt issued by the institution holding the portfolio. Portfolio performance is based on a comparison between the portfolio’s promised return and the capital required to support the portfolio. Individual facilities are allocated capital on the basis of their contribution to portfolio variance and individual facility performance can then be measured based on the ratio of the facility’s return to this capital.

KMV’s portfolio management software product, Portfolio Manager, is designed to:

- Produce a mark-to-model price for credit-risky exposures,
- Characterise the return and risk of exposures in the context of a credit portfolio, and the return and risk of the portfolio as a whole,
- Compute the distribution of portfolio values at a specified horizon date and use this distribution to calculate required economic capital today,
- Determine optimal transactions—buy or sell—for a given set of trading or origination opportunities, and
- Calculate optimal portfolios by rearranging the weights of existing holdings.

Whereas performance was traditionally measured in terms of earnings per share or return on equity, the focus within the KMV modelling paradigm is on return on risk-adjusted capital (RORAC) whereby the portfolio return is measured against the capital required to support the portfolio and return per unit of portfolio unexpected loss, the portfolio Sharpe ratio.

The key building blocks of the KMV approach are a methodology for measuring default probability and default correlation.
4.2.1.1 Measuring Default Probability

The methodology employed by KMV to calculate the term structure of default probabilities has been described in detail in 3.7.1 above. The value of the firm’s assets is imputed from the value of the firm’s equity, the volatility of equity and the amount of the firm’s debt.

4.2.1.2 Measuring Correlation

KMV apply the Markowitz variance-covariance approach to determine portfolio value variance. They have constructed a factor model to explain the correlation in the underlying asset values of the obligors. The asset returns are assumed to be multivariate normally distributed. By isolating the systematic variation in asset returns, they create a framework within which future asset values, and, hence, future credit exposure values can be modelled.

4.3 Re-creating KMV’s Portfolio Modelling Approach

Before building a multi-period model to value a CDO using KMV EDF and correlation data, it is necessary to build a single time period model using these data. Developing a model which re-creates the results which KMV produce in their current portfolio modelling product, Portfolio Manager, will serve the further purpose of confirming the theoretical validity of the KMV modelling paradigm. Such independent validation has not previously been available and KMV, presumably for commercial reasons, have been unwilling to disclose their methodology.

While KMV have not published their methodology in detail, they have outlined their modelling approach in various articles in academic and practitioner journals. See for example, Kealhofer (2003, 2003a), Bohn (2000) and Crosbie (2002). Furthermore, their approach has been analysed by various commentators. Using these outlines and commentaries and an understanding of the simulation process, the Portfolio Manager modelling approach is successfully re-created – for the first time, to the author’s knowledge - as confirmed by the model outputs that replicate almost exactly those from Portfolio Manager.

The task of replicating comprised the following stages:

- Each asset in the portfolio is valued at the outset using a risk-neutral valuation approach.
The asset correlation structure is derived using the KMV factor model.

The asset migration process is modelled.

The portfolio value at the horizon date is simulated repeatedly.

The portfolio distribution is derived, and the portfolio parameters — standard deviation, called unexpected loss by KMV, and capital — are calculated.

The capital and unexpected loss are allocated to the component assets and individual asset performance — both Sharpe Ratio and RORAC — are calculated.

Each of these stages will now be described in detail. A schematic outlining the various tasks undertaken is shown in Figure 4.1 below.

4.3.1 Asset Valuation at the Outset

The method that KMV employ for asset valuation was outlined in Bohn (2000). Technical details regarding the manner in which they undertake interpolation of interest rates and cumulative default probabilities are further described in KMV’s *Portfolio Manager* product manual, *Modelling Portfolio Risk* (2003). These latter details are of no theoretical importance but it was necessary to account for them correctly if results matching KMV’s were to be obtained.

As Bohn describes, the KMV so-called risk-comparable valuation method begins by calculating the risk-adjusted PD by adjusting the actual PD to account for the market price of risk in the standard risk-neutral approach outlined in Hull (2003, p203).

4.3.1.1 Calculating Cash Flows

The expected cash flows for an exposure are derived from the reference rate term structure. It is assumed that the future spot rate will equal the forward rate. Hence, all the expected cash flows are assumed known at the outset. There is no modelling of interest rate volatility.
4.3.1.2 Cumulative Quasi-Default Probability

The actual cumulative probability of default from 0 to time \( t \), \( CEDF_t \), is given by the Merton formula

\[
CEDF_t = N \left( -\frac{\ln(A_0) + \left(\mu - \frac{1}{2}\sigma^2\right)t - \ln(DPT)}{\sigma\sqrt{t}} \right)
\]

Equation 4-1

where
\( A_0 \) = market value of the firm’s assets at time 0

\( DPT \) = default point

\( \sigma \) = volatility of return on the firm’s assets, and

\( \mu \) = drift rate, or expected return, on the firm’s assets

However, for valuation purposes, it is the risk-adjusted PD - quasi-probability of default in KMV terminology - that is required. The actual drift rate is replaced by \( r \), the continuously compounded risk-free rate to obtain the cumulative quasi-probability of default, \( CQDF_t \).

\[
CQDF_t = N \left\{ -\frac{\ln(A_0) + \left( r - \frac{1}{2} \sigma^2 \right) t - \ln(DPT)}{\sigma \sqrt{t}} \right\}
\]

Equation 4-2

Re-arranging this equation, we get

\[
CQDF_t = N \left\{ N^{-1}(CEDF_t) + \left( \frac{\mu - r}{\sigma} \right) \sqrt{t} \right\}
\]

Equation 4-3

which specifies the cumulative quasi-default probability, \( CQDF_t \), as a function of the actual cumulative probability of default, \( CEDF_t \), and the Sharpe ratio of the asset. KMV (2003, p151) suggest that the expected excess return on the asset is a function of its sensitivity to systematic market risk factors

\[
\beta \pi = \mu - r \quad \text{Equation 4-4}
\]

where \( \beta \) is the asset’s beta to the market and \( \pi \) is the market risk premium, \( (\mu_m - r) \)

The asset’s beta to the market, \( \beta \), may be re-stated as

\[
\beta = \frac{\text{cov(asset, market)}}{\sigma_m^2} = R \frac{\sigma}{\sigma_m}
\]

Equation 4-5

where

\( R \) = the correlation between the asset return and the market return, and

\( \sigma_m \) = the volatility of the market return

This enables the individual asset’s Sharpe ratio to be expressed as a function of the market’s Sharpe ratio, \( \lambda \). The market’s Sharpe ratio is the market price of risk,
\[ \lambda = \frac{(\mu_m - r)}{\sigma_m} \quad \text{Equation 4-6} \]

linking the amount of one year's excess return to annual return volatility

KMV (2003, p51) suggest that the market price of risk will vary with the square root of time since the excess return will be a linear function of time while volatility will increase with the square root of time. Furthermore,

\[ \frac{(\mu - r)}{\sigma} = R \lambda \quad \text{Equation 4-7} \]

Hence,

\[ CQDF_t = N \left\{ N^{-1}(CEDF_t) + R \lambda \sqrt{t} \right\} \quad \text{Equation 4-8} \]

KMV suggest an appropriate proxy for the market is the custom index based on the country and industry (or industries) within which the firm operates. (Using a multifactor model, described in detail later, they create asset value indices for industries and countries.) They regress the firm's asset return series on the custom index to determine the percentage of the asset return variability that is explained by the custom index, \( R^2 \). The square root of \( R^2 \) is a measure of the asset correlation with the market, they suggest based on the assumption that the custom index proxies for the market.

The market risk premium, called the market Sharpe ratio, measures the required return over and above the risk-free rate for holding a unit of risk at the aggregate level. Research performed at KMV on the risk premium implicit in credit spreads for U.S. corporate bonds "reveals that the market Sharpe Ratio parameter is relatively stable over time and typically around 0.4" KMV (2003, p19). They continue "Note that this is the risk premium associated with the market value of firm assets and not the value of the firm's equity. The risk premium for equities fluctuates more than that for assets because of the dynamic nature of firm leverage." (p19) The value of 0.4 will be used throughout the rest of this study.

### 4.3.1.3 Valuation

A risky exposure will pay \((1-LGD)\) if it defaults and 1 otherwise. KMV decompose this exposure into two separate cash flows. The first will pay the recovery amount \((1-LGD)\) whether it defaults or not. The second pays 0 in the event of default and LGD otherwise.
The first cash flow has no default risk and can be discounted at the risk-free rate. The second cash flow contains credit risk, valuation of this component must include a discount for risk.

KMV’s risk-comparable valuation (RCV) methodology embeds their risk-neutral pricing technique. The RCV value at the modelling date, time 0, is

\[ V_0^{RCV} = (1 - LGD)RFV_0 + LGD RYV_0 \]  \hspace{1cm} \text{Equation 4-9}

where

\[ V_0^{RCV} = \text{Risk-comparable value at time 0} \]

\[ RFV_0 = \text{Risk-free value at time 0 and} \]

\[ RYV_0 = \text{Risk comparable risky value at time 0} \]

The risk-free value at \( t_0 \) discounts each cash flow at the risk-free rate

\[ RFV_0 = \sum_{t=0}^{M} C_t DF_t^{R_f} \]  \hspace{1cm} \text{Equation 4-10}

where

\[ M = \text{time to maturity} \]

\( t = \text{time to payment of a given cash flow} \)

\[ C_t = \text{amount of cash flow at time } t \]

\[ DF_t^{R_f} = \text{risk-free discount factor to time } t \]

The risky value calculation adjusts each cash flow by the quasi-probability and then uses the risk-free discount factor for discounting

\[ RYV_0^{RCV} = \sum_{t=0}^{M} \left(1 - CQDF_t\right) C_t DF_t^{R_f} \]  \hspace{1cm} \text{Equation 4-11}

KMV give the term structure of EDF values annually from one to five years. Using these values, the cumulative probabilities of default are calculated as follows

\[ CEDF_t = 1 - (1 - EDF_t)^t \]  \hspace{1cm} \text{Equation 4-12}

Interpolation and extrapolation are based on the following formulae.

124
\[
CEDF_t = 1 - \left[ \left( 1 - CEDF_{t_1} \right)^\frac{t - t_1}{T_2 - T_1} \right]\text{ for } t \leq 1
\]
\[
CEDF_t = 1 - \left[ \left( 1 - CEDF_{t_1} \right) \left( \frac{1 - CEDF_{t_2}}{1 - CEDF_{t_1}} \right)^\frac{t - t_1}{T_2 - T_1} \right]\text{ for } 1 \leq T_1 \leq t \leq T_2 \leq 5
\]
\[
CEDF_t = 1 - \left( 1 - CEDF_{t_1} \right)^\frac{t}{5}\text{ for } t > 5
\]

**Equation 4-13**

As shown earlier, the \(CQDF\) value may be written in terms of the actual probability of default to time \(t\), the market’s Sharpe ratio, \(\lambda\), and the asset’s correlation with the market, \(R\)

\[
CQDF_t = N \left\{ N^{-1}\left( CEDF_t \right) + \lambda \sqrt{t} \right\}
\]

**Equation 4-14**

This enables the asset to be valued at time 0

### 4.3.2 Asset Correlation

The key reason for adopting the KMV-based approach to modelling is the opportunity this affords to use their correlation framework. KMV have a substantial database of public companies – approximately 35,000 at the time of writing – for which they have equity price time series over many years and from which they have calculated asset price time series.

They aggregate these individual time series to create 61 industry time series and 45 country time series. They then use principal components analysis to extract orthogonal factors – two global, five regional and seven sectoral – which may be used to calculate asset correlation between all pairs of obligors. The resulting factor sensitivities may be used to model asset migration and hence value a debt portfolio at the modelling horizon. KMV update these factor sensitivities on an annual basis and include the file with their monthly updates to obligor EDF values.

#### 4.3.2.1 The KMV Factor Structure

The correlation between the market value of a firm’s assets and the market embodies the extent to which systematic risk factors in the economy drive the value of the firm’s assets. While the market, in theory, comprises all available assets, KMV make the assumption that the market becomes “what we can observe that reasonably proxies for
the theoretical market” KMV (2003, p61) They define the market as the custom index comprised of the country and industry indexes within which the firm operates
KMV’s factor model imposes a structure on the correlation of asset returns, which implies that the correlation between the asset returns of any pair of firms can be explained by the firms’ relationships to a set of common factors. There are three levels used in KMV’s factor structure, KMV (2003, p117) (i) A composite company-specific factor, (ii) country and industry factors, and (iii) global, regional and industrial sector factors
The first level of the structure differentiates between firm-specific and systematic risk. Systematic risk is captured by a single common factor. This factor is unique to each firm and is a weighted sum of country and industry factors to which the firm has exposure.
The country and industry factors at the second level of the factor structure are correlated with each other. Therefore, their risk can also be decomposed into systematic and idiosyncratic components. The systematic component of the risk is captured by the basic economic factors in the third level of the structure. The idiosyncratic risk components of countries and industries are retained as country- and industry-specific factors.
KMV emphasise (2003, p118) that the third level of factors is only needed for interpreting the drivers of correlation. The actual correlation estimate depends only on the division between the systematic and idiosyncratic parts of the country and industry risks. This is shown in schematic format below, adapted from KMV (2003, p118)

![Schematic Diagram](image-url)

**Figure 4.2 KMV’s Factor Model Structure**

The global, regional and sector factors capture all the common risk between countries and industries. That is, they capture all of the correlation between the country and
industry factors. Likewise, these basic factors also explain all of the common risk between firms in different countries and industries. Firms with exposure to the same country or industry also share country- or industry-specific risks.

4.3.2.2 Estimating the KMV Factor Model

KMV (2003, pp. 119-122) describe the process whereby they estimate the relevant parameters for their factor model. They construct 14 orthogonal factors from the 106 - 45 countries and 61 industries - indices. They regress the 106 country and industry return indices on the 14 orthogonal factors to obtain the country and industry betas on these 14 factors. They also obtain the country- and industry-specific risks. They regress each firm's return series on its composite index returns to obtain the firm-specific beta and the $R^2$.

Each index is regressed in sequence on the residual of the previous regression so that, by construction, each of the factors is orthogonal. This means that the Interest Sensitivity factor, for example, is not the total effect of interest rates but only the portion of that effect that cannot be explained by the global and regional factors.

![Figure 4.3 KMV's Correlation Schematic](image)

Country or industry risk is decomposed into systematic risk arising from either global, regional or sector effects and specific, or idiosyncratic, risk.

The regression model can be written as

```
Country Return = Global Economic Effect + Regional Factor Effect + Sector Factor Effect + Country-Specific Effect
Industry Return = Global Economic Effect + Regional Factor Effect + Sector Factor Effect + Industry-Specific Effect
```
\[ r_c = \sum_{G=1}^{2} \beta_{cG} r_G + \sum_{R=1}^{5} \beta_{cR} r_R + \sum_{S=1}^{7} \beta_{cS} r_S + \epsilon_c \]  
\[ r_i = \sum_{G=1}^{2} \beta_{iG} r_G + \sum_{R=1}^{5} \beta_{iR} r_R + \sum_{S=1}^{7} \beta_{iS} r_S + \epsilon_i \]  

where

\( r_c = \) return for country \( c \)

\( r_i = \) return for industry \( i \)

\( r_G = \) return for global market \( G \)

\( r_R = \) return for region \( R \)

\( r_S = \) return for sector \( S \)

\( \beta_{cG} = \) effect of global market \( G \) on country \( c \)

\( \beta_{iG} = \) effect of global market \( G \) on industry \( i \)

\( \beta_{cR} = \) effect of region \( R \) on country \( c \)

\( \beta_{iR} = \) effect of region \( R \) on industry \( i \)

\( \beta_{cS} = \) effect of sector \( S \) on country \( c \)

\( \beta_{iS} = \) effect of sector \( S \) on industry \( I \)

\( \epsilon_c = \) country-specific effect for country \( c \)

\( \epsilon_i = \) industry-specific effect for industry \( i \)

The variance of the industry and sector returns is, therefore

\[ \sigma_c^2 = \sum_{G=1}^{2} \beta_{cG}^2 \sigma_G^2 + \sum_{R=1}^{5} \beta_{cR}^2 \sigma_R^2 + \sum_{S=1}^{7} \beta_{cS}^2 \sigma_S^2 + \sigma_{\epsilon_c}^2 \]  
\[ \sigma_i^2 = \sum_{G=1}^{2} \beta_{iG}^2 \sigma_G^2 + \sum_{R=1}^{5} \beta_{iR}^2 \sigma_R^2 + \sum_{S=1}^{7} \beta_{iS}^2 \sigma_S^2 + \sigma_{\epsilon_i}^2 \]  

where

\( \sigma_G^2 = \) variance of global market factor \( G \)'s return

\( \sigma_R^2 = \) variance of region factor \( R \)'s return

\( \sigma_S^2 = \) variance of sector factor \( S \)'s return
The firm risk is decomposed into systematic and idiosyncratic components. The composite factor is constructed individually for each firm based on the countries and industries to which it is exposed. These country and industry classifications are determined from the firm’s reported sales and asset levels in a particular country or industry.

The composite factor can be written as

$$\phi_k = \sum_{c=1}^{45} w_{kc} r_c + \sum_{i=1}^{61} w_{ki} r_i \quad \text{Equation 4-17}$$

where

- $\phi_k$ = composite factor for firm $k$
- $w_{kc}$ = weight of firm $k$ in country $c$
- $w_{ki}$ = weight of firm $k$ in industry $I$
- $r_c$ = return for country $c$
- $r_i$ = return for industry $I$

and

$$\sum_{c=1}^{45} w_{kc} = \sum_{i=1}^{61} w_{ki} = 1 \quad \text{Equation 4-18}$$

KMV run a regression of each firm’s weekly returns against the returns of its composite factor

$$r_k = \beta_k \phi_k + \varepsilon_k \quad \text{Equation 4-19}$$

where

- $r_k$ = return for firm $k$
- $\beta_k$ = beta for firm $k$
- $\varepsilon_k$ = firm-specific effect for firm $k$

KMV (2003, p126) call the composite factor coefficient $\beta_k$, “the firm’s beta”, noting that it differs from the firm’s stock beta which is against the market index whereas in their model, each firm has its own unique index. KMV publish the $R^2$ for the regression.
for each firm in their database. These values range from 0.1 to 0.65 with the lower values typical of smaller firms and higher values for larger firms.

For private firms, the appropriate industry and country weights are used as for public companies and an estimate is made of the firm’s beta. This latter value is calculated using an estimate of \( R^2 \) for the firm that is based on the value of \( R^2 \) for public firms of similar size in the same country and industries. With this estimate, the firm’s beta is calculated using

\[
\beta_k = \sqrt{\frac{R^2}{\sigma_{\hat{\epsilon}_k}}} \quad \text{Equation 4-20}
\]

We can re-state the firm’s returns as follows

\[
r_k = \beta_k \phi_k + \epsilon_k
\]

\[
= \beta_k \left( \sum_{c=1}^{45} w_{kc} r_c + \sum_{i=1}^{61} w_{ki} r_i \right) + \epsilon_k
\]

\[
= \beta_k \left( \sum_{c=1}^{45} w_{kc} \left( \sum_{G=1}^{2} \beta_{cG} r_G + \sum_{R=1}^{5} \beta_{cR} r_R + \sum_{S=1}^{7} \beta_{cS} r_S \right) \right) + \epsilon_k
\]

\[
= \beta_k \left( \sum_{i=1}^{61} w_{ki} \left( \sum_{G=1}^{2} \beta_{iG} r_G + \sum_{R=1}^{5} \beta_{iR} r_R + \sum_{S=1}^{7} \beta_{iS} r_S \right) \right) + \epsilon_k
\]

\[
= \beta_k \left[ \sum_{G=1}^{2} \left( \sum_{c=1}^{45} \beta_{cG} w_{kc} \right) r_G + \sum_{R=1}^{5} \left( \sum_{c=1}^{45} \beta_{cR} w_{kc} \right) r_R + \sum_{S=1}^{7} \left( \sum_{c=1}^{45} \beta_{cS} w_{kc} \right) r_S + \sum_{i=1}^{61} w_{ki} \epsilon_i \right] + \epsilon_k
\]

\[
= \beta_k \left[ \sum_{G=1}^{2} \left( \sum_{i=1}^{61} \beta_{iG} w_{ki} \right) r_G + \sum_{R=1}^{5} \left( \sum_{i=1}^{61} \beta_{iR} w_{ki} \right) r_R + \sum_{S=1}^{7} \left( \sum_{i=1}^{61} \beta_{iS} w_{ki} \right) r_S + \left( \sum_{i=1}^{61} w_{ki} \epsilon_i \right) \right] + \epsilon_k
\]

Equation 4-21
\[ \begin{align*}
= & \sum_{G=1}^{2} \beta_{k} \left( \sum_{c=1}^{45} \beta_{cG} w_{kc} + \sum_{i=1}^{61} \beta_{AG} w_{ki} \right) r_{G} + \sum_{R=1}^{5} \beta_{k} \left( \sum_{c=1}^{45} \beta_{cR} w_{kc} + \sum_{i=1}^{61} \beta_{kR} w_{ki} \right) r_{R} \\
+ & \sum_{S=1}^{7} \beta_{k} \left( \sum_{c=1}^{45} \beta_{cS} w_{kc} + \sum_{i=1}^{61} \beta_{kS} w_{ki} \right) r_{S} + \sum_{c=1}^{45} \beta_{k} w_{kc} \varepsilon_{c} + \sum_{i=1}^{61} \beta_{k} w_{ki} \varepsilon_{i} + \varepsilon_{k} \\
= & \sum_{G=1}^{2} \beta_{kG} r_{G} + \sum_{R=1}^{5} \beta_{kR} r_{R} + \sum_{S=1}^{7} \beta_{kS} r_{S} + \sum_{c=1}^{45} \beta_{kc} \varepsilon_{c} + \sum_{i=1}^{61} \beta_{ki} \varepsilon_{i} + \varepsilon_{k} \\
= & \sum_{G=1}^{2} \beta_{kG} r_{G} + \sum_{R=1}^{5} \beta_{kR} r_{R} + \sum_{S=1}^{7} \beta_{kS} r_{S} + \sum_{c=1}^{45} \beta_{kc} \varepsilon_{c} + \sum_{i=1}^{61} \beta_{ki} \varepsilon_{i} + \varepsilon_{k}
\end{align*} \]

\textbf{Equation 4-22}

where

\[ \begin{align*}
\beta_{kc} &= \beta_{k} w_{kc} \\
\beta_{ki} &= \beta_{k} w_{ki}
\end{align*} \textbf{Equation 4-23} \]

The total risk of the firm can thus be expressed as follows

\[ \sigma_{k}^{2} = \sum_{G=1}^{2} \beta_{kG}^{2} \sigma_{G}^{2} + \sum_{R=1}^{5} \beta_{kR}^{2} \sigma_{R}^{2} + \sum_{S=1}^{7} \beta_{kS}^{2} \sigma_{S}^{2} + \sum_{c=1}^{45} \beta_{kc}^{2} \sigma_{c}^{2} + \sum_{i=1}^{61} \beta_{ki}^{2} \sigma_{i}^{2} + \varepsilon_{k}^{2} \textbf{Equation 4-24} \]

The covariance in the asset returns of two firms can be calculated by summing their joint sensitivity to the common factors

\[ \sigma_{jk} = \sum_{G=1}^{2} \beta_{jG} \beta_{kG} \sigma_{G}^{2} + \sum_{R=1}^{5} \beta_{jR} \beta_{kR} \sigma_{R}^{2} + \sum_{S=1}^{7} \beta_{jS} \beta_{kS} \sigma_{S}^{2} + \sum_{c=1}^{45} \beta_{jc} \beta_{kc} \sigma_{c}^{2} + \sum_{i=1}^{61} \beta_{ji} \beta_{ki} \sigma_{i}^{2} \textbf{Equation 4-25} \]

and the return correlation is obtained using the standard formula

\[ \rho_{jk} = \frac{\sigma_{jk}}{\sigma_{j} \sigma_{k}} \textbf{Equation 4-26} \]

**4.3.3 Simulating Asset Migration**

The asset migration process is modelled using a Monte Carlo simulation approach. The approach proceeds as follows

1. Draw a set of factor realisations, one for each factor. The factors are independent and identically distributed (i.i.d.) standard normal variables.
2. Draw a specific risk random variable for each firm. These draws are also standard normal random variables.

3. Compute the random component of each firm's asset return as the weighted sum of the specific and systematic risks.

4. If the random value drawn is below the default threshold set by the firm's default probability to the modelling horizon, $CEDF_H$, the firm has defaulted. The loss incurred on defaulted securities is obtained by making a random drawing from a beta distribution, characterized by the average loss rate expected for the facility as well as its standard deviation.

5. Compute the value of each exposure at the horizon from its asset value realization. This is calculated using the RCV methodology previously described. The value will be a function of each exposure's LGD, EDF value, $R^2$ and the random realization of asset value at the horizon.

6. The value of the portfolio at the horizon is obtained by summing the values of the individual exposures in the portfolio.

7. Repeat steps 1 to 6 sufficiently often to achieve the requisite resolution in the extreme tail of the distribution. The number of iterations typically required for this is of the order of 100,000.

In the KMV framework, sampling in the Monte Carlo simulation takes place over the asset values of the individual obligors. The asset value at the horizon for an obligor, $A_H$, is calculated as

$$\ln(A_H) = \ln(A_0) + \left(\mu - \frac{\sigma^2}{2}\right)t_H + \sigma \sqrt{t_H} \tilde{e}_H$$

Equation 4-27

where

$A_0$ = the borrower's underlying asset value at $t_0$

$\mu$ = expected return on the underlying assets

$\sigma$ = volatility of the return on the underlying assets

$\tilde{e}_H$ = the random component of the asset return

This allows the process of simulating asset migration to be completed.
The $\varepsilon_H$ are assumed to be drawn from a multi-variate normal distribution. Thus, the simulation must draw a value of $\varepsilon_H$ for each obligor to embed this correlation structure. The factor model, described already, provides the structure necessary to embed the matrix of correlations among asset returns. The independent draws for the 120 factors - two global macroeconomic, five regional, seven sectoral, 45 country-specific and 61 industry-specific effects - are combined as follows

$$\varepsilon_H = \frac{1}{\sqrt{\sum_{j=1}^{120} \beta_j^2 \sigma_j^2 + \sigma_u^2}} \left[ \sum_{j=1}^{120} \beta_j \lambda_j + \sigma_u \tilde{u} \right]$$

Equation 4-28

The $\lambda_j$ are the 120 systematic risk factors, while $\tilde{u}$ is the firm-specific factor, and, as mentioned already, all are independent draws of standard normally distributed random variables. The random component of asset return, $\varepsilon_H$, is obtained by first calculating the weighted sum of the firm-specific return, $\tilde{u}$, and the 120 systematic risk factors, $\lambda_j$, the weights being the coefficients in the last equation, and then scaling the sum by dividing by the standard deviation of the firm’s asset return. Thus, by construction, the random component of the firm’s asset return, $\varepsilon_H$, is standard normally distributed and has a correlation structure consistent with the factor model of correlation.

4.3.4 LGD

KMV, in common with many others in the industry - most notably, CreditMetrics - use the beta distribution to model the recovery rate. The beta distribution has desirable characteristics for a recovery function. First, it can be bounded at whatever level is chosen clearly, the desired upper and lower bounds for the recovery rate are 0 and 1. Furthermore, it can accommodate many different distributional shapes, which offers the flexibility to represent the lender’s view on recovery uncertainty.

The Beta function is characterised by two parameters

$$B(a,b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a + b)}$$

Equation 4-29

where $\Gamma$ is the gamma function, defined as
\[ \Gamma(a) = \int_0^\infty e^{-y} y^{a-1} dy \]  \hspace{1cm} \text{Equation 4-30}

A Beta distribution density function is given by

\[ f(x,a,b) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} \]  \hspace{1cm} \text{Equation 4-31}

The mean of the beta distribution is given by

\[ \mu(a,b) = \frac{a}{a+b} \]  \hspace{1cm} \text{Equation 4-32}

and its variance is given by

\[ \sigma^2(a,b) = \frac{ab}{a+b+1} \]  \hspace{1cm} \text{Equation 4-33}

Clearly, the mean of the distribution must be $LGD$, so, by definition

\[ LGD = \frac{a}{a+b} \]  \hspace{1cm} \text{Equation 4-34}

The relationship between $a$ and $b$, which determines the shape of the Beta distribution for a given $LGD$ is determined by their relationship in the variance of the distribution

\[ \sigma^2 = \frac{LGD(1-LGD)}{a+b+1} \]  \hspace{1cm} \text{Equation 4-35}

KMV (2003, p 103) suggest that this shape be controlled through the use of a single parameter, $k$, which they define as

\[ k = a + b + 1 \]  \hspace{1cm} \text{Equation 4-36}

Hence, the shape parameters, $a$ and $b$, can be determined from $LGD$ and $k$

\[ a = (k-1)LGD \]
\[ b = (k-1)(1-LGD) \]  \hspace{1cm} \text{Equation 4-37}

and the variance can be expressed as

\[ \sigma^2 = \frac{LGD(1-LGD)}{k} \]  \hspace{1cm} \text{Equation 4-38}
The reason the beta distribution has become the distribution of choice for \( LGD \) in the market is the flexibility it offers. If nothing is known about the distributional characteristics of \( LGD \), a \( k \)-value of 3 might be chosen; this will result in a uniform distribution. Higher \( k \) values will result in a distribution resembling the normal. If, however, as is sometimes observed, \( LGD \) is either 0% or 100% with very low probabilities of intermediate outcomes, a \( k \)-parameter of 2 or less will give a U-shaped distribution which captures these features. These distributions are shown in Figure 4.4:

![Beta Distribution: LGD](image)

**Figure 4.4 Beta Distribution with Average LGD = 50%**

Thus, as \( k \) becomes large, the variance of the distribution goes to 0, implying little uncertainty about the estimate of mean \( LGD \), while lesser values of \( k \) are appropriate where the lender has less confidence in the \( LGD \) estimate. KMV suggest a value of 4 as being appropriate for many lenders.

The beta distribution is plotted in Figure 4.5 below for various values of \( k \) when the average \( LGD \) is 45%:
The fractional loss of the exposure, $LGD$, is a random variable drawn from a Beta distribution with mean $LGD$ and variance $\frac{LGD(1-LGD)}{k}$. $LGD$ will be modelled using a $k$ value of 4 throughout this thesis. This sees $LGD$ values close to the average much of the time but allows full recovery and complete loss occur also.

### 4.3.5 Asset Revaluation at the Horizon

Once the borrower’s asset value realisation at horizon is determined, exposures to the borrower at the horizon can be evaluated. The value of a given exposure is determined as the sum of two exposures, a riskless portion, which pays $(1 - LGD)(RFV_H)$ whether the exposure defaults or not, and a risky portion, which pays $LGD \cdot RFY_H$ when the exposure is in default and zero otherwise:

$$V_H = (1 - LGD)RFV_H + LGD \cdot RFY_H \quad \text{Equation 4-39}$$

where $RFV_H$ is the risk-free value of the exposure from horizon to maturity which includes the risk-free value at $H$ of any cash flows received from 0 to $H$. $RFY_H$ is the value of the risky portion of the exposure from horizon to maturity, and $LGD$ is the
expected LGD. $RYV_H$ also includes coupons from time 0 to $H$. According to KMV (2003, p161), the coupons or cash flows received before horizon are assumed to be risk-free and enter into the LGD calculation.

If the exposure defaults before the horizon, then the value of the risky portion of the exposure, $RYV_H$, is equal to zero and the obligor is assumed to lose an amount based on the parameterised LGD distribution. Since LGD is assumed to be random, a random draw is made to determine the fractional loss for each defaulted exposure, as discussed next.

4.3.5.1 Exposure Value in the Default State

If the asset value at the modelling horizon, $\ln(A_H)$, falls below the default point, $\ln(DPT_H)$, then the obligor will default. In other words, the obligor will default if the realisation of the random component of the obligor’s asset return, $\tilde{\epsilon}_H$, is larger (in absolute terms) than the $H$-period distance to default (DD) at time 0, which is the normal inverse of the $H$-period EDF:

$$\tilde{\epsilon}_H \leq - \left[ \frac{\ln \left( \frac{A_0}{DPT_H} \right) + \left( \mu - \frac{\sigma^2}{2} \right) H}{\sigma \sqrt{H}} \right] = N^{-1}(CEDF_H) \quad \text{Equation 4-40}$$

where

$DPT_H = \text{standardised default point at the horizon}$

$CEDF_H = \text{cumulative probability of default to the horizon}$

In the event of default, the risky value of the exposure, $RYV_H$, is set to zero. A random draw from the beta distribution determines the loss incurred as explained previously.

4.3.5.2 Exposure Value in the Non-Default State

In order to determine the risk which a portfolio of loans presents, the portfolio value distribution at horizon must be calculated. As described above, the portfolio must be simulated under the true risk measure to determine whether default has occurred for each individual loan. This will depend on whether the realised $\tilde{\epsilon}_H$ for an obligor causes the borrower to fall below its default threshold.
If the obligor does not default, the facility must be revalued at the modelling horizon.
The facility value will be a function of the realised EDF term structure. The modelling
requires that the realised value distribution for each facility should be consistent with
the EDF term structure as observed at the outset. This, in turn, requires that the
distribution of distance to default for an obligor at the modelling horizon be explicitly
linked to the term structure of DD observed at the outset.

In order to undertake valuation in the non-default state at the modelling horizon, the
risk-comparable approach is adopted once again. The cumulative quasi-EDF values
from the modelling horizon to the cash flow date must be calculated as a function of the
asset return at horizon, $\widehat{E}_H$. Thus, the challenge is to calculate the value for $CQDF_{H,M}$,
the cumulative quasi-EDF from the modelling horizon to maturity, in a manner that
incorporates the information about the borrower’s DD at the outset:

$$CQDF_{H,M} = N(-DD_{H,M}|ND) \quad \text{Equation 4-41}$$

where $DD_{H,M}$ is the risk-neutral DD from horizon, $H$, to maturity, $M$, in the non-default
(ND) state at the modelling horizon, $H$.

This DD can be expressed as a function of the realised asset value at horizon, $A_H$, and
the default point at maturity, $DP_M$:

$$DD_{H,M} = -\frac{\ln \left( \frac{A_H}{DP_M} \right) + \left( r - \frac{\sigma_A^2}{2} \right)(M - H)}{\sigma_A \sqrt{M - H}} \quad \text{Equation 4-42}$$

By adding and subtracting $(\mu - r)(M - H)$ in the numerator, this simplifies as follows.
The Brownian processes at the modelling horizon, \( B_H \), and at maturity, \( B_M \), are correlated. \( B_M \) may be re-written as

\[
B_M = B_H + B_{M-H} \quad \text{Equation 4-44}
\]

The correlation between \( B_H \) and \( B_M \) may be expressed as follows

\[
\rho_{B_{H,M}} = \frac{\text{Cov}(B_H, B_M)}{\sqrt{\text{Cov}(B_H, B_H)} \cdot \sqrt{\text{Cov}(B_M, B_M)}}
\]

\[
= \frac{\text{Cov}(B_H, B_H) + \text{Cov}(B_H, B_{M-H})}{\sqrt{\text{Cov}(B_H, B_H)} \cdot \sqrt{\text{Cov}(B_M, B_M)}}
\]

\[
= \frac{H}{\sqrt{H} \sqrt{M}} = \frac{H}{M}
\]

Since \( \rho = \frac{H}{\sqrt{M}} \), we get \( \sqrt{\frac{M-H}{M}} = \sqrt{1 - \rho^2} \)

Furthermore,

\[
DD_{0,M} = \frac{\ln \left( \frac{A_0}{DP_m} \right) + \left( \mu - \frac{\sigma^2}{2} \right) M}{\sigma_A \sqrt{M}} \quad \text{Equation 4-46}
\]

and
\[ \frac{(\mu - r)}{\sigma_A} = R \lambda \quad \text{Equation 4-47} \]

Hence, we can re-write \( DD_{H,M} \) as follows

\[
DD_{H,M} = \frac{DD_{0,M} \sqrt{M}}{\sqrt{M-H}} + \frac{\sqrt{H}}{\sqrt{M-H}} \hat{\varepsilon}_H - \frac{(\mu - r)\sqrt{M-H}}{\sigma_A} \quad \text{Equation 4-48}
\]

This gives us an expression that connects the DD at maturity to the realised return to the modelling horizon.

Default can occur at any time However, we can approximate this by replacing the continuous barrier with a situation in which default can occur only at the horizon modelling date or at facility maturity The DD from the outset to the modelling horizon is given by the obligor’s default probability

\[ DD_{0,H} = -N^{-1}(CDF_{H}) \quad \text{Equation 4-49} \]

In these circumstances, the modelled DD from the outset to maturity, \( DD_{0,M} \), must be consistent with the known probability of default from the outset to facility maturity, \( CDF_M \) Given our assumption that default can occur only at the modelling horizon or at maturity, the probability of default may be calculated as the combined probability of default occurring before maturity or horizon.

The two Brownian motion processes - to the modelling horizon and maturity, respectively - are bivariate normally distributed with a correlation coefficient of \( \rho \) as described above Thus, we know that the probability of defaulting between the start of the modelling period and facility maturity is comprised of the volume under the bivariate surface to the left of \( -DD_{0,H} \) and below \( -DD_{0,M} \) As shown in Figure 4 6, the probability of surviving is the volume under the bivariate above and to the right of the default thresholds given by \( -DD_{0,H} \) and \( -DD_{0,M} \)
Calculating Distance to Default from the Outset to Maturity

By symmetry, this can be calculated as the volume to the left and below \( DD_{0,H} \) and \( DD_{0,M} \). Hence,

\[
N_2(-N^{-1}(CEDF_H), DD_{0,M}, \rho) = 1 - CEDF_M \quad \text{Equation 4-50}
\]

If the borrower does not default at the horizon, then its new cumulative quasi-EDF term structure must be derived conditional on its realised distance-to-default at the horizon. Only when this is calculated can all the cash flows from \( H \) to \( M \) be valued.

We interpolate to calculate all values \( CQDF(H,t) \), the cumulative quasi-EDF to date \( t \) between \( H \) and \( M \):

\[
CQDF_{H,t} = 1 - \left(1 - CQDF_{H,M} \right)^{\frac{t-H}{M-H}} \quad \text{Equation 4-51}
\]

All values \( CQDF(H, t) \), the cumulative quasi-EDF to date \( t \) between \( H \) and \( M \) may be obtained using this interpolation. The facility values at the horizon date can then be calculated and hence the portfolio value. Repeated simulation yields the portfolio value distribution.
4.4 Checking the Successful Replication of the KMV Modelling Framework

The model described above was implemented in a Matlab environment using a portfolio of ninety loans of $5m each to ninety different B-rated obligors with maturities varying between five and ten years each offering a 1.5% spread. The modelling horizon was set to one year and the portfolio value distribution at the horizon date was obtained by using Monte Carlo simulation with 100,000 trials. The same portfolio was modelled in Portfolio Manager and, once again, the portfolio value distribution at the horizon date was derived.

The summary output from Portfolio Manager is presented in Figure 4.7 below. It shows that the portfolio of loans with a par value of $450m has a market value of $447.9m. This value is obtained from the analytic calculation shown earlier. This exact same value is obtained within the newly constructed model.

The portfolio yields a total spread of 203bp which is eroded by an expected loss of 98bp to yield an expected spread of 105bp over the one-year horizon. The standard deviation of loss, the unexpected loss, is 239bp and, hence, the Sharpe ratio, which equals the expected spread divided by unexpected loss, is $\frac{105}{239} = 0.438$.

An item of particular importance to portfolio managers and regulators alike is the economic capital required to support the portfolio. Economic capital is typically set at a level such that the debt which ranks ahead of the economic capital can achieve a particular rating, or, equivalently, a probability of default at the horizon of a given level. In this instance, a default probability of 10bp at the one-year horizon is chosen as the target default probability for the debt. Thus, the 99.9th percentile on the portfolio loss distribution is measured. The difference between the mean portfolio value and the 99.9th percentile is the loss from which the debt must be protected at the horizon. The present value of this amount represents the economic capital, which is the amount which we must set aside now to absorb portfolio losses over the modelling horizon.
<table>
<thead>
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</tr>
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<td>Book Value Drawn</td>
<td>450,000,000</td>
</tr>
<tr>
<td>Market Value Drawn</td>
<td>447,902,425</td>
</tr>
<tr>
<td>Total Spread Revenue, Annualized</td>
<td>9,090,968</td>
</tr>
<tr>
<td>Expected Loss, Annualized</td>
<td>4,407,690</td>
</tr>
<tr>
<td>Expected Spread Revenue, Annualized</td>
<td>4,683,278</td>
</tr>
<tr>
<td>Unexpected Loss (Simulated)</td>
<td>10,685,779</td>
</tr>
<tr>
<td>Capital (10.00 bp in excess of Expected Loss)</td>
<td>47,689,875</td>
</tr>
<tr>
<td>Current Non-Zero Records</td>
<td>90</td>
</tr>
<tr>
<td>Total Spread, Annualized</td>
<td>0.02030</td>
</tr>
<tr>
<td>Expected Loss, Annualized</td>
<td>0.00984</td>
</tr>
<tr>
<td>Expected Spread, Annualized</td>
<td>0.01046</td>
</tr>
<tr>
<td>Unexpected Loss (Simulated)</td>
<td>0.02366</td>
</tr>
<tr>
<td>Capital (10.00 bp in excess of Expected Loss)</td>
<td>0.10647</td>
</tr>
<tr>
<td>Sharpe Ratio</td>
<td>0.43827</td>
</tr>
<tr>
<td>RORAC, Annualized</td>
<td>0.12092</td>
</tr>
</tbody>
</table>

**Figure 4.7 Portfolio Manager Summary Output**

The capital required to support the portfolio equals 10.65% of the market value of the portfolio. Furthermore, the return on risk-adjusted capital (RORAC) which equals expected spread divided by the capital plus the risk-free rate equals 1.05%/10.65% + 2.27% = 12.09%.

The two portfolio distributions are shown in Figure 4.8 below:
It is clear from the above that the model captures the portfolio dynamic embedded in the KMV modelling methodology and that the differences between the distributions are attributable to simulation error.

This portfolio capital is, in turn, allocated to the individual facilities that comprise the portfolio. The capital allocation methodology adopted by KMV is the standard contribution to variance methodology applied in most VaR contexts.

\[
RC_i = \frac{\partial UL_p}{\partial w_i} = \sum_{j=1}^{n} w_j \rho_{ij} UL_i UL_j = \frac{\rho_{ip} UL_i UL_p}{UL_p} = \rho_{ip} UL_i \quad \text{Equation 4-52}
\]

where

\( RC_i \) is the risk contribution of facility \( i \), the fraction of portfolio capital which is attributed to facility \( i \),

\( w_i \) is the weight of the facility in the portfolio,

\( UL_i, UL_j \) and \( UL_p \) are the standard deviations of the facility values \( i, j \) and the portfolio, respectively,

and \( \rho_{ip} \) is the correlation between the value of facility \( i \) and the value of the portfolio.
The sum of all the capital attributed to the facilities equals the capital for the portfolio:

\[ UL_p = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \rho_{ij} U_i U_j \]

\[ = \sum_{i=1}^{n} w_i \sum_{j=1}^{n} w_j \rho_{ij} U_i U_j \]

\[ = \sum_{i=1}^{n} w_i \rho_i U_i = \sum_{i=1}^{n} w_i R C_i \]

Equation 4-53

Risk contribution can be interpreted as the portion of the individual facility’s risk that remains after diversification. This is the key focus of managerial attention.

Therefore, in addition to checking that the portfolio distribution is calculated in a manner consistent with that employed by KMV, a check on capital attribution is necessary to ensure that individual facilities are being modelled with similar consistency. To that end, the capital attributed to individual facilities under the model and that attributed by KMV in Portfolio Manager are compared. The results of that comparison are plotted in Figure 4.9:

![Figure 4.9 Comparison of Capital Allocation under New Model and Portfolio Manager](image)

Figure 4.9 Comparison of Capital Allocation under New Model and Portfolio Manager

Again, the results confirm consistency between the two models at the facility level. This is a necessary pre-requisite to extending the modelling approach into a multi-period framework. This confirms the successful replication of the KMV modelling framework.
4.5 **Using the Model to Address Key Credit Portfolio Issues**

The primary concern of this thesis is CDO tranche valuation. This requires that a single period credit portfolio model be created as a first step towards the building of the multi-period model necessary for CDO tranche valuation.

However, the availability of this single period model affords insights into other areas of interest in the credit portfolio management field. In particular, having such a portfolio model enables us to answer four questions that have long challenged academe and industry alike:

- How can a new facility be assessed in the context of the portfolio to which it will be added?
- What capital attribution methodology is most appropriate in the context of a bank credit portfolio which is financed largely with debt?
- What is the optimal amount of a new syndicated loan for which a bank should subscribe given its portfolio composition? Or, equivalently, and of more relevance to the bank’s biggest customers, what penalty is being incurred by holding a facility which is larger than the optimal size?
- What framework should a bank use to set limits on the amount of exposure it should be willing to accept to different borrowers, sectors and geographies consistent with its risk appetite?

**4.5.1 Determining the Marginal Impact of a Loan on a Debt Portfolio**

Standard portfolio management software is run centrally and in isolation from the business line. Those who compete for business have little appreciation of the impact of the new facilities which they consider writing on the portfolio. The best that can be achieved in such circumstances is to give the business line general instructions about the concentrations which exist in the portfolio to guide their market positioning and loan pricing decisions.

However, with the model that has just been built, all the inputs necessary to guide the pricing decision are available. The value of the new loan at the outset can be calculated using the formula outlined at 4.3.1 above. The new loan value distribution at the modelling horizon can be obtained by using the stored realisations for the 120 random variables for the 100,000 iterations.
Thus, the value distribution for the new loan and for the portfolio including the proposed new loan may be calculated using the stored portfolio values. This provides all the inputs required for the calculation of Risk Contribution and facility capital as described in 4.4 above. This, in turn, enables the facility Sharpe ratio, 

\[
\frac{\text{Expected Spread}}{\text{Risk Contribution}}, \quad \text{and facility RORAC, } \frac{\text{Expected Spread}}{\text{Capitalisation Rate}} + \text{Risk - free rate},
\]

to be calculated. Finally, having facility capital allows us calculate the facility Economic Value Added (EVA).

None of these measures is available to the business line from Portfolio Manager or other credit portfolio software offerings at present. Having these values available before the decision to provide the loan has to be made allows all new loan-granting decisions to be made in the full knowledge of their portfolio impact. This can turn the portfolio management function from its current reactive stance of trying to mitigate the worst effects of low RORAC facilities after they have been written to identifying them in advance and avoiding taking them into the portfolio.

In summary, portfolio management is primarily about identifying the capital required to support individual facilities. According as the concentration of a facility in a portfolio increases, the capitalisation rate for that facility increases and the facility becomes less desirable. Knowing the capital required to support a putative new facility \textit{ex ante} has the potential to transform the loan portfolio management business.

4.5.2 An Alternative Capital Attribution Framework

Capital is fundamentally a portfolio concept. Allocating capital to a facility requires an attribution methodology. The challenge of devising a logical attribution methodology has been faced in a trading room environment for many years. The methodology of choice for distributing limits across desks, for example, has been to use the covariance of the desk returns with the trading room returns.

KMV and many others in the credit marketplace have adopted this methodology. But, is this appropriate? The instruments dealt in a trading room context are all liquid and the trading horizon is short. In these circumstances, the distribution of returns is approximately normal. This differs significantly from the situation with bank loan portfolios. A typical bank portfolio comprises very illiquid loans - indeed, some would venture to suggest that the role of banks is liquidity intermediation – so the modelling...
horizon is much longer than that considered by trading desks. In fact, the primary justification offered for the one-year modelling horizon which is the standard used in the marketplace is that one year is a period long enough to allow the bank to be recapitalised if its loan portfolio declines significantly in value.

Loan portfolio distributions are skewed and fat-tailed over the one year modelling horizon. A bank is a highly leveraged institution. It is concerned not only about standard deviation, it is also concerned about becoming decapitalised. Capital is fundamentally about insulating the bank from extreme losses. But the risk contribution methodology penalises facilities based on their contribution to variance and not on their contribution to those scenarios for which capital is actually required.

A further consideration relates to the nature of capital, while capital is set by reference to an extreme loss percentile, such as 99.9% or 99.95% depending on the bank’s target debt rating, the bank will be economically bankrupt at much lower loss levels. Thus, while the amount of capital is set by reference to an extreme percentile, a less extreme percentile is a more relevant threshold for a bank which concerned about its economic independence.

4.5.2.1 Capital Allocation Based on Contribution to ETL

An alternative is suggested: adopt an expected tail loss (ETL) approach which allocates capital to facilities based on their contribution to portfolio outcomes below a chosen portfolio threshold.

\[ \text{ETL}_P = \mathbb{E}[L \mid L > \text{Threshold}] \]  \text{Equation 4-54}

In other words, the expected tail loss is the average of the portfolio losses, \( L \), for those scenarios where the portfolio loss is beyond the chosen threshold.

Portfolio capital, \( C_p \), is then allocated to facilities, \( C_i \), based on the extent to which the average facility value in the scenarios which give rise to a portfolio loss greater than the portfolio loss threshold is below the facility’s unconditional average.

\[ C_i = \frac{\mathbb{E}[L_i \mid L_p > \text{Threshold}]}{\text{ETL}_P} C_p \]  \text{Equation 4-55}

Once again, this measure satisfies the requirement that \( \sum_i C_i = C_p \).
This proposed capital allocation methodology has two desirable properties. Firstly, it satisfies the sub-additivity, homogeneity and monotonicity requirements for a coherent risk measure specified by Artzner et al (1997, 1999). Secondly, it is more sensitive to exposure concentrations. A criticism that is often levelled at the contribution to variance methodology is that concentrations have to increase significantly before a noticeable increase in allocated capital is observed.

There is no published work of which the author is aware which suggests how sensitive a portfolio should be to exposure size. However, industry practice is that single-name exposures greater than four times the average exposure size are seldom held voluntarily. This rule of thumb is based on the view that exposures of this size will wipe out the net income contribution of over 300 exposures, assuming a 40% LGD and 0.4% net income. However, this rule assumes a portfolio with thousands of exposures; in a portfolio of just 90 exposures, the four times multiplier would certainly be reduced.

This point is borne out in Figure 4.10 below:

![Figure 4.10 Capital Allocation: ETL and Risk Contribution methodologies compared](image)

When a €5m facility is added to a €450m portfolio of 90 loans of €5m each, the capitalisation rate for the facility is 6.9% under the ETL allocation approach and 8.3%
under the Risk Contribution approach. However, as the facility size increases, the ETL-based capitalisation rate increases faster and by the time the facility has increased to €50m representing 10% of the total portfolio, the capitalisation rate has increased to 37.9% compared to 21.7% under the Risk Contribution approach. While the example is stylised in the sense that corporate banking portfolios are much more diversified with, typically, thousands or tens of thousands of facilities, the same pattern is repeated when tested on a typical bank portfolio of corporate loans. This alternative allocation methodology ensures that taking on exposures significantly larger than the average size within the portfolio will exhibit low RORAC and consequently will not be undertaken. This accords with typical bank policy whereby larger exposures are only extended to high-grade borrowers or to low-grade borrowers on a secured basis.

### 4.5.3 Determining the Optimal Hold Level and the Concentration Penalty

The model outlined at 4.5.2.1 above determines the capital required to support a putative new facility to be added to a loan portfolio. However, in many instances, it is the amount of the loan that is at issue, and not whether or not to grant the loan. For example, when participating in a syndicated loan, we need to determine the optimal level of participation. Likewise, when extending facilities to a relationship customer, we need to understand what costs are incurred by making credit available which exceeds the optimal hold level. What is required is a framework for determining both these quantities.

The credit portfolio model provides just such a framework. Typically, loans will be issued at a spread in excess of their true value given by the formula at 4.3.1 above. However, according as the facility size increases, it becomes a source of concentration in the portfolio and the capital increases at an increasing rate as shown in Figure 4.11 below for the same €450m portfolio described above.

The average capitalisation rate increases and the marginal capitalisation rate increases at an even faster rate, as demonstrated in Figure 4.12 below. The increased capitalisation rate translates into an increased cost of writing incremental exposure. Figure 4.13 demonstrates that the increased capitalisation requirements translate into higher costs for the bank as exposure increases, the marginal cost of assuming additional exposure increases until it exceeds the spread, and, at still higher exposure levels, the average cost exceeds the spread.
Figure 4.11 Capital as a function of Exposure

Figure 4.12 Marginal and Average Capitalisation Rates as a function of Exposure
EVA is the primary measure of value in most banks. It is calculated by subtracting the cost of capital times the amount of capital from the expected spread. It is a measure of annualised value added. The EVA in Figure 4.14 rises at first before peaking where the expected spread equals the cost of capital times the marginal capitalisation rate. Beyond that exposure level, the net revenue turns negative as the marginal cost of capital exceeds the expected spread. Clearly, the optimal exposure is that at which the EVA per unit of regulatory capital consumed is above the threshold set by management. If Tier 1 equity— and hence, regulatory capital - is not constrained, then the optimal exposure is that which maximises EVA.

In the €450m portfolio under consideration, EVA is at a maximum for a particular facility at an exposure size of €5.9m. But, the maximum desirable exposure size scales with portfolio size and with the number and diversity of exposures. Thus, larger institutions can comfortably accommodate much larger exposures before detracting from EVA. Likewise, better diversified portfolios will reduce the correlation between the value of a new facility and that of the portfolio.
However, according as the facility becomes a larger fraction of the total portfolio, the benefits of diversification are eroded and that facility’s marginal EVA will decline and eventually turn negative. This approach enables a bank to determine two important facts:

- The optimal amount of a syndicated loan to assume is that which maximises EVA or which ensures the marginal EVA per unit of exposure just exceeds the threshold set by management.
- The cost of exceeding the optimal hold level for a relationship customer is the EVA destroyed on the exposure beyond the optimal.

By implementing a loan pricing model linked to the marginal capitalisation rates, the business line will have an incentive to buy credit protection against the names where the exposure exceeds the optimal. It also creates an alignment of interest between the portfolio management function and the business line.

### 4.5.4 Limit-Setting in Debt Portfolios

The measure of risk of relevance to a bank is economic capital. Thus, economic capital, it is suggested, provides an obvious basis for setting limits to borrowers, sectors and
geographies. No further exposure may be assumed if the economic capital required to
support the exposure to the obligor exceeds the agreed capital threshold. Such a
framework would provide a far more coherent basis for limit setting than the exposure-
based limits currently in use which are linked to customer grade based on rather vague
qualitative factors.

4.6 Implications for Bank Regulation

The framework created in this chapter demonstrates the impact of portfolio composition
on capital. Under Basel II, regulatory capital depends on the obligor PD and on the
LGD, exposure at default (EAD) and maturity (M) of the facility. This chapter
highlights that facility capital is dependent not just on these variables but, crucially, on
the composition of the portfolio also. Poorly diversified portfolios will require
significantly more capital than well diversified ones. While this has been understood in
regulatory circles for a long time, this framework enables the debate to move from
being purely qualitative in nature to quantitative. The regulators, understandably, are
reluctant to commit to a quantitative approach until they are satisfied about data quality,
modelling and backtesting. However, it is suggested that this approach could be
employed as the basis for setting capital in the future.

4.7 Conclusion

In this chapter, a single period credit portfolio model was developed which replicated
the current state-of-the-art model.

The KMV approach to the modelling of a portfolio of debt securities over a single time
period was replicated based on outline descriptions of their approach in the literature.
The success of this modelling effort is confirmed by comparing the loan value
distributions at maturity and the value, expected loss, unexpected loss and risk
contribution of each loan in the portfolio produced by running the portfolio through
KMV's Portfolio Manager software.

This model was then used to propose a modification to current market practice and two
significant extensions to the state-of-the-art. The capital attribution methodology
adopted by industry and embedded in KMV's portfolio model allocates capital in
accordance with contribution to variance. This model, the standard in market risk
environments, implicitly assumes that variability of value around the mean is the investor's main concern.

However, this author argues that for leveraged portfolios, contribution to extreme outcomes is of much more concern and proposes an allocation scheme based on expected tail loss. This has the effect of penalising concentrations more severely than the market standard framework and offers a concentration sensitivity that is more consistent with intuition and market practice.

The single period model which KMV offers models portfolios \textit{ex post}. Unfortunately, this tells the portfolio manager the effect of individual facilities on the portfolio after they have already been added to the portfolio and the manager has no opportunity to influence the decision. This thesis creates a framework in which the impact of a facility on the portfolio can be measured \textit{ex ante} giving the portfolio manager the ability to influence the decision on whether to add the facility before the fact.

In reality, however, the decision to extend a facility to a customer is seldom a Yes/No. Particularly in syndicated lending, the key concern is "How much?" not "Whether or not"; the issue is one of deciding what size of facility should be made available. Likewise, a bank's best relationship customers will require facilities that exceed those which can be economically accommodated within the bank's portfolio. The bank's appetite will vary depending on the size and composition of its book.

This thesis proposes a model that enables the portfolio manager to determine the impact of a loan increment on the portfolio. As the capitalisation rate increases, the cost of the incremental capital drives the total cost of holding the facility on balance sheet higher than the spread on offer.

This provides the line with a signal as to what the optimal hold amount is. This accords with intuition that suggests that limits not be set as absolute amounts: rather, they should ensure that the return per unit of risk of the last increment of exposure is greater than the minimum threshold. A larger hold amount is warranted if the spread on offer is wider.

Likewise, it measures the cost of holding an excessive exposure to a given obligor. This 'concentration penalty' can be levied on the relationship manager. The penalty provides the portfolio manager with adequate income to spend on the purchase of protection on that name should a CDS market exist.
Finally, it provides the basis for setting borrower limits, an issue with which the market has long struggled. Typically such limits have been set as absolute amounts of exposure. This thesis suggests that a single capital amount can be used to set limits regardless of the borrower grade.

Having replicated the KMV modelling approach in a single time period framework, the next step is to extend this approach in two ways: (i) First, model the assets that comprise the portfolio. To achieve this, the portfolio model must be converted from a single- to a multi-period framework. This will enable the cash flowing from the portfolio over time to be determined. (ii) Then, model the liabilities. Build a cash flow model to disburse the cash from the assets to the CDO tranches in accordance with the cash flow waterfall specified in the CDO indenture.

The development of this new model is the subject of the next chapter.
Chapter 5. Developing a Multi-Period Credit Portfolio Model to Value CDO Tranches

5.1 Chapter Overview

In this chapter, a new model is developed which will enable the valuation of CDO tranches using a multi-time step Monte Carlo modelling approach in a risk-neutral framework.

5.2 Motivation for Building a New Model

This thesis grew from dissatisfaction with the process that the rating agencies applied to rating CDO tranches:

- The models they employ rely heavily on the rating of the assets contained in the SPV. That asset rating process was already seen to be overly heuristic and had failed to identify on a timely basis many assets whose quality had materially changed.
- The assumptions regarding asset correlation, the manner in which the portfolio characteristics are distilled into their binomial framework – described more fully in Chapter 6 – and the use of the expected loss measure to assign grade seemed to the author to be overly simplistic.

It seemed unlikely that a process that relied on such asset ratings and a rather arbitrary approach to the assessment of portfolio interactions could successfully grade as complex a structured debt product as a CDO tranche.

Nor does the market seem much more enlightened. The analysis undertaken by many of the systems most commonly used in the market – Intex, ICDO and CDO Vantage, for example - amounts to nothing more than scenario analysis supported by a cash flow model. Most purchasers of senior tranches settle for an assessment that suggests full repayment as long as the number of defaults in the first three years is less than a given multiple of historical loss levels for collateral of that quality. No attempt is made to analyse the portfolio interactions, it is hoped that the rating agency requirements regarding industry limits will avoid losses due to concentration that will penetrate the senior tranche.
The author was attracted to the structural approach that potentially offered a more rigorous alternative to the rating agency approach. The adoption of this approach required appropriate data and it was decided to use KMV data because of its widespread acceptance in the market and its potential to offer a more rigorous approach to modelling the correlation of credit risks.

5.3 Outline of Task of Extending the Single Period Model to a Multi-Period Framework

The single period credit portfolio model developed in Chapter 4 used KMV estimates of probability of default – its EDF measure – and of asset correlation. The steps taken in developing that model can be summarised as follows:

- The assets are valued at the outset using the risk-comparable approach. This involved the discounting of expected cash flows using a risk-neutral approach that increased the probability of default to account for the systematic risk that they contained.

- The evolution of each obligor’s credit term structure was simulated under the true risk measure – that is under the Cumulative Expected Default Frequency measure, $CEDF_h$, to the modelling horizon.
  - If the obligor defaulted – in other words, if the random obligor return was more negative than the negative of the obligor’s DD - recovery was modelled as a drawing from a beta distribution with the mean and standard deviation specified.
  - If the obligor survived, the term structure of the obligor’s credit risk was recomputed based on the realised random return over the modelling period. With this new credit term structure, the value of the debt instrument is recomputed using the same risk-neutral approach as was applied at the outset.

The simulation applied a factor model approach to embed the KMV asset correlation measures.

- The frequency distribution of facility values and portfolio values at the modelling horizon is calculated. An analysis of these distributions produces a number of key performance measures.
Calculating the portfolio average and the average for each of the individual facilities enables the expected portfolio return and expected return for each facility to be calculated in a mark-to-model framework.

The standard deviation of the facility and portfolio values — called unexpected loss — may be computed. The expected return and unexpected loss measures can be combined to obtain the Sharpe ratio. The Sharpe ratio is a measure of portfolio performance and of facility performance. This is a key metric for investors in unleveraged debt portfolios.

Where investment in debt portfolios is financed largely by debt — as is the case in the banking sector — the Sharpe ratio alone is insufficient to inform the investor about the desirability of a given investment since it is focused on value volatility. In these circumstances, portfolio capital is set by reference to the present value of the difference between the expected portfolio value and some extreme percentile of the distribution. Investment performance is then measured by RORAC.

Logic would suggest that a similar approach could be applied to the valuation of a CDO tranche and to the measurement of its risk characteristics. However, a number of issues complicate this approach:

- The value of a tranche at the start of the modelling period can no longer be calculated in the formulaic manner applied to the valuation of loans.
- It is no longer possible to determine the amount and the timing of cash flows ex ante. The cash flows will derive from the evolution of the portfolio over many time periods in the future. Thus, rather than modelling over one period as previously, the portfolio must be modelled over every period at which cash is disbursed to the tranches until all the underlying portfolio assets have matured or defaulted.
- The modelling must occur under the risk-neutral measure and not the true measure as previously since the focus for now is on valuation and not on risk.
- Measuring the risk of a tranche is more complicated still.
- As for the single period model, the probability distribution of security values at the horizon must be calculated.
♦ This requires that the portfolio of underlying securities be simulated under the true measure out to the investor's risk horizon.

❖ At the horizon, each tranche must be revalued. However, this task is a repeat of that undertaken at the start of the modelling period described above. Hence, valuation at horizon demands that a Monte Carlo simulation be undertaken for each point on the probability distribution. Thus, the calculation of a tranche Sharpe ratio is a 'Monte Carlo squared' problem. Obtaining a Sharpe ratio estimate for a CDO tranche with the same precision as was obtained for an individual facility in the single period model would require $100,000^2 = 10,000$ million multi-period simulations. If the underlying portfolio contains ten-year maturity assets and the CDO tranches receive quarterly payments, a total of 400 billion simulations are necessary. It is clear that lesser accuracy must be accepted or a good approximation methodology developed.

This thesis is focused on the valuation of CDO tranches and therefore concentrates on the first of these challenges. The latter will be addressed in subsequent research.

5.4 Modelling to Horizon in a Risk-Neutral Framework

In the single period credit portfolio model, simulation took place in actual default space. Such modelling is described in the literature as occurring under the true measure, to distinguish it from modelling under the risk-neutral which is undertaken when value, and not risk, is the focus of concern. The focus of this model is the valuation of CDO tranches and thus, all the modelling in this chapter will occur under the risk-neutral measure. The first task, therefore, is to adapt the modelling paradigm of Chapter 4 for use in a risk-neutral framework.

As was done previously in the single period model, the portfolio value distribution at horizon must be calculated. However, unlike previously, the portfolio must be simulated under the risk-neutral measure to determine whether default has occurred for each individual loan since our interest is in portfolio valuation, and later, tranche valuation. This will depend on whether the realised $\tilde{\varepsilon}_m$ for an obligor causes the borrower to fall below its risk-neutral default threshold. Clearly, the probability of default will be higher in the risk-neutral framework since $CQDF$ will be greater than $CEDF$.
5.4.1 Exposure Value in the Default State

The value of the facility in default is obtained in exactly the same way as described previously in 4.3.5.1. Making a drawing from a beta distribution simulates the LGD.

5.4.2 Forward CQDF in the Non-Default State

Once again, the value of the facility at the modelling horizon is a function of the shape of the credit curve and the simulated asset return, $\widetilde{\epsilon}_H$, over the period.

However, in Equation 4-50 above, modelling occurred under the true measure, so we had

$$N_2\left(-N^{-1}(CEDF_H), DD_{0,M}, \rho\right) = 1 - CEDF_M$$  \hspace{1cm} \text{Equation 5-1}$$

where $CEDF_H$ and $CEDF_M$ are the true cumulative default probability to the horizon and facility maturity, respectively, $DD_{0,M}$ is the DD over the period to facility maturity, and $\rho$ is the correlation between the Brownian processes at horizon and maturity, respectively.

Now, when modelling under the risk-neutral measure, we have

$$N_2\left(-N^{-1}(CQDF_H), DD_{0,M}, \rho\right) = 1 - CQDF_M$$  \hspace{1cm} \text{Equation 5-2}$$

where the variables are as in Equation 5-1 except they are now measured under the risk-neutral measure. The relationship between the risk-neutral and true cumulative default probability measures was given in Equation 4-8 above as

$$CQDF = N\left(N^{-1}(CEDF), + \lambda R \sqrt{t}\right)$$  \hspace{1cm} \text{Equation 5-3}$$

By the same token, under the true measure, the impact of $\widetilde{\epsilon}_H$ on the DD was captured in Equation 4-48 above as

$$DD_{H,M} = \frac{DD_{0,M}}{\sqrt{1 - \rho^2}} + \frac{\rho}{\sqrt{1 - \rho^2}} \widetilde{\epsilon}_H - R \lambda \sqrt{M - H}$$  \hspace{1cm} \text{Equation 5-4}$$

Under the risk-neutral measure, this formula is modified to

$$DD_{H,M}^0 = \frac{DD_{0,M}^0}{\sqrt{1 - \rho^2}} + \frac{\rho}{\sqrt{1 - \rho^2}} \widetilde{\epsilon}_H$$  \hspace{1cm} \text{Equation 5-5}$$

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since no additional adjustment needs to be made for systematic risk

Hence,

\[ CQDF_{H,M}^0 = N(-DD_{H,M}^0) \quad \text{Equation 5-6} \]

5.5 Modelling to Subsequent Horizons

From 5.4.2, we have the cumulative quasi-default probability from the first modelling horizon to maturity. However, we are no longer interested in valuing the potential future cash flows at the first horizon; instead, we wish to model the portfolio over the next time period. Thus, we wish to determine the probability of default under the risk-neutral measure from one horizon to the next. For CDO valuation purposes, the modelling horizons correspond to the dates on which cash is paid to the tranches, this typically occurs quarterly or semi-annually.

We determine the cumulative quasi probability of default from \( H \) to \( 2H \) by interpolating between \( H \) and \( M \) using the same interpolation scheme as we used previously in 4.3.5.2

\[ CQDF_{H,2H}^0 = 1 - \left(1 - CQDF_{H,M}^0 \right)^{\frac{2H-H}{M-H}} \quad \text{Equation 5-7} \]

This allows us to repeat the modelling procedure at the second and subsequent modelling horizons.

5.6 Modelling the Liabilities

The procedure outlined above allows the cash flows from the assets to be modelled. The next stage is to distribute the available cash to the various tranches in accordance with the CDO indenture.

The most important factors to incorporate are the O/C and the I/C tests. Breaching these tests will cause the cash available from the assets to be diverted to more senior tranches. Breaching the mezzanine O/C test will see the residual cash flow used to repay the most senior outstanding principal or outstanding interest should there be any. Likewise, breaching the senior O/C test causes all cash beyond that required to pay senior interest to be used to repay senior principal.

The overall result is that all the cash flowing from the assets is disbursed to the liabilities.
5.7 Implementing the Model

This new model is now applied to the valuation of the portfolio modelled above in 4.4. This portfolio comprises ninety loans of $5m each, rated B1 to B3 by Moody's, paying Libor + 250bp. The loans have maturities between 4.5 and 10 years. The portfolio is financed by $360m of Aaa-rated senior notes paying Libor + 50bp and $40m of mezzanine paying Libor + 150bp. The senior O/C test is 1.2 while the corresponding mezzanine test is 1.05. The senior I/C test is 1.8 and mezzanine I/C is 1.2.

The model which has been developed can cater for any combination of asset characteristics and securitisation structures. However, the portfolio chosen has the characteristics of a typical arbitrage cash flow CDO. These comprise 80-100 loans to sub-investment grade names with maturities in the five- to ten-year bracket. Likewise the O/C and I/C tests are set at levels typically seen in the market. The results which follow should therefore be representative of those for the securitisation class.

5.7.1 Tranche Cash Flows

An example of the cash flow waterfall is shown in Figure 5.1 and Figure 5.2 below. The two figures give the key cash flows, balance sheet values and O/C and I/C test values from five simulations semi-annually over the ten year period:

The cash inflow to the structure comes from three sources: (i) Interest from the assets, (ii) principal from maturing assets; and (iii) principal recovered from defaulting assets.

On the cash outflow side of the structure, the cash is distributed to the tranches in priority: (i) Senior tranche interest is paid first. (ii) Subject to passing all relevant tests, mezzanine tranche interest is paid. (iii) Outstanding interest, if any, is paid. (iv) Principal – both recovered and maturing - is channelled to the tranches in priority order, and (v) should all relevant tests be passed and there not be any interest outstanding on any tranche, equity is entitled to the residual.

Interest from Assets gives the amount of interest flowing from the underlying loans. It declines over time as assets default or mature. No assets mature during the first five years. When assets default, a random draw is made to determine the recovery amount.

If the senior tranche O/C or I/C tests are breached, no cash is paid to the mezzanine or equity tranches. Instead, that cash is diverted to pay down the senior tranche principal. The amount of interest due to the mezzanine tranche accrues in the outstanding interest
account which is repaid as soon as interest payments can be resumed. Only if all tests are in compliance will any cash be paid to the equity tranche.

All recoveries and maturing principal amounts are applied to the senior tranche first. Only when the senior is fully repaid is any cash paid to the mezzanine tranche. Finally, if all the notes are repaid, all remaining interest and principal flowing from the assets is applied to equity.

**5.7.2 Tranche Valuation**

Since all the cash flows are modelled in risk-neutral space, their present value may be obtained by discounting at the risk-free interest rate.

The average present value of the cash flow from the assets equals the value of the assets calculated using the method outlined in 4.3.1.3 above. The sum of the tranche values will clearly equal the value of the assets since all the cash flows from the assets are distributed to the tranches.

Thus, we are able to value each of the tranches across all the simulations.
Figure 5.1 Key Variable Values for Five Random Asset Paths
The best-selling CDO valuation software in the market is RiskMetrics. This model applies a copula approach to determine the time of default for CDOs.

<table>
<thead>
<tr>
<th>Period</th>
<th>5 Yrs</th>
<th>6 Yrs</th>
<th>7 Yrs</th>
<th>8 Yrs</th>
<th>9 Yrs</th>
<th>10 Yrs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interest from Assets</td>
<td>9,925,03</td>
<td>8,734,004</td>
<td>7,535,100</td>
<td>6,073,407</td>
<td>4,936,467</td>
<td>4,213,410</td>
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<tr>
<td>Maturing Principal</td>
<td>10,780,352</td>
<td>9,270,059</td>
<td>7,867,179</td>
<td>6,362,710</td>
<td>5,069,260</td>
<td>3,953,406</td>
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<tr>
<td>Recovered Principal</td>
<td>11,791,000</td>
<td>10,209,925</td>
<td>8,826,709</td>
<td>7,028,960</td>
<td>5,938,235</td>
<td>4,036,970</td>
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<tr>
<td></td>
<td>12,689,000</td>
<td>11,664,739</td>
<td>8,706,113</td>
<td>6,207,657</td>
<td>5,272,230</td>
<td>4,478,350</td>
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<td>8,918,014</td>
<td>7,407,288</td>
<td>6,542,246</td>
<td>5,387,962</td>
<td>4,285,793</td>
<td>3,550,932</td>
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<table>
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<th>6 Yrs</th>
<th>7 Yrs</th>
<th>8 Yrs</th>
<th>9 Yrs</th>
<th>10 Yrs</th>
</tr>
</thead>
<tbody>
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<td>Maturing Principal</td>
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<td>6,060,000</td>
<td>5,460,000</td>
<td>4,500,000</td>
<td>3,500,000</td>
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<td>1,500,000</td>
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<td>3,940,051</td>
<td>1,533,330</td>
<td>2,063,424</td>
<td>2,163,287</td>
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<td>2,346,684</td>
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<tr>
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<td>6,289,945</td>
<td>3,769,433</td>
<td>4,870,349</td>
<td>3,674,556</td>
<td>2,866,753</td>
<td>2,254,831</td>
<td>1,622,180</td>
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<td>2,379,747</td>
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<td>5,422,956</td>
<td>4,037,479</td>
<td>3,167,812</td>
<td>2,406,833</td>
<td>1,645,894</td>
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<tr>
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<td>7,391,391</td>
<td>6,336,691</td>
<td>5,032,737</td>
<td>3,728,236</td>
<td>2,967,297</td>
<td>2,315,027</td>
<td>1,780,193</td>
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<tr>
<td></td>
<td>5,695,753</td>
<td>4,911,806</td>
<td>4,205,355</td>
<td>3,172,271</td>
<td>2,411,312</td>
<td>1,797,491</td>
<td>1,255,532</td>
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</table>

<table>
<thead>
<tr>
<th>Mezzanine Tranche</th>
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<th>1,009,667</th>
<th>1,009,667</th>
<th>1,009,667</th>
<th>1,009,667</th>
<th>977,107</th>
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</thead>
<tbody>
<tr>
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<td>2,322,261</td>
<td>2,495,682</td>
<td>2,282,891</td>
<td>2,115,691</td>
<td>1,903,451</td>
<td>1,709,667</td>
<td>1,503,920</td>
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<tr>
<td></td>
<td>3,401,628</td>
<td>2,306,458</td>
<td>2,394,694</td>
<td>1,981,814</td>
<td>1,760,736</td>
<td>1,620,450</td>
<td>1,500,294</td>
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<tr>
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<td>5,386,944</td>
<td>2,332,331</td>
<td>1,764,349</td>
<td>1,467,754</td>
<td>1,295,386</td>
<td>1,133,556</td>
<td>1,053,853</td>
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</table>

<table>
<thead>
<tr>
<th>Cash to Equity</th>
<th>2,452,991</th>
<th>1,954,987</th>
<th>1,655,283</th>
<th>1,388,184</th>
<th>1,028,085</th>
<th>948,912</th>
<th>886,908</th>
</tr>
</thead>
</table>

The comparison between the CDO valuation software RiskMetrics and its copula approach is a key method for determining the time of default for CDOs. The model uses a copula approach to handle the interdependence of default times among different tranches of CDOs. The table above shows the key variable values for five random asset paths, which are crucial for understanding the dynamics of CDO valuation.
each asset in the portfolio. This approach is decidedly easier to implement since each simulation produces a time at which each asset defaults. This obviates the need for multi-period simulation. If the asset defaults beyond its maturity date, it repays the full principal amount. Knowing when the assets default or mature, all the cash flows can be derived. Equity correlation is used in place of asset correlation to generate the correlated random variables which simulate the default time. RiskMetrics then apply a cash flow waterfall in a similar manner to that described above. More frequently, however, *CDO Manager* is used to value synthetic CDOs — this simplifies matters still further since only the premium payments and payments on default need to be recorded.

This method has become the market standard. Its simplicity is its attraction. However, it needs CDS spreads in order to provide the risk-neutral default probability term structure measure necessary to inform the model. For the names which are included in synthetic securitisation structures, these are spreads readily available. But for the sub-investment grade names which comprise arbitrage securitisation structures, this information is absent. Banks are obliged to rely on their internal ratings to determine the default probability and to make some — presumably heuristic — adjustments to these to make them risk-neutral.

The author is unaware of any other CDO valuation model, details of which have been published in the literature. It is clear that many banks, particularly investment banks, have bespoke models, these are presumably deemed to be proprietary and have not been published.

### 5.9 Model Results

The graphs that follow show the distributions of the relevant variables.
5.9.1 Asset and Tranche Value Distributions

Figure 5.3 Asset Value Probability Distribution

Figure 5.3 above shows the probability distribution of asset values. This variability in asset values translates into variability of tranche values as shown in Figure 5.4, Figure 5.6 and Figure 5.7 below:

Figure 5.4 Senior Tranche Value Probability Distribution
The variability in senior tranche value arises from the variability in the timing of principal repayment: the higher the incidence of asset default early in the life of the structure, the earlier the senior repayments and the lower the value of the asset as the senior tranche earns the spread for a shorter period. This may be seen clearly by plotting the value of the senior tranche against average life: the value increases monotonically with average life in Figure 5.5 below:

![Figure 5.5 Average Life vs Value of Senior Tranche](image)

The mezzanine tranche value distribution is shown in Figure 5.6:
Figure 5.6 Mezzanine Tranche Value Probability Distribution

The equity tranche value is extremely volatile as evidenced in Figure 5.7:

Figure 5.7 Equity Tranche Value Probability Distribution

The variability in the value of the underlying assets is transmitted to the tranches but not in a uniform manner as shown in Table 5-1:
The senior tranche is insulated from the volatility in the value of the underlying assets by the subordinated tranches. The equity and mezzanine tranches that are providing this protection to the senior tranche experience significantly higher value volatility than the underlying assets.

This is seen more clearly in Figure 5.8 below. When the underlying assets are arranged in value order, the average senior tranche value in the lowest five percent range is 99.57% of the average senior tranche value in the highest five percent range. In contrast, the corresponding figure for the mezzanine is 76.63% and for equity, it's a mere 7.53%. The structure has functioned as intended the senior tranche remains immune to the losses in the underlying collateral.

By the same token, the equity tranche absorbs the losses until the collateral starts to default at a very high rate. The mezzanine value in the second lowest 5% bucket has a value of 99.22% of the highest 5% bucket losses only penetrate the mezzanine tranche when the equity has effectively been decimated.

<table>
<thead>
<tr>
<th></th>
<th>Assets</th>
<th>Senior</th>
<th>Mezzanine</th>
<th>Equity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>453,162,648</td>
<td>370,252,103</td>
<td>43,288,550</td>
<td>39,621,940</td>
</tr>
<tr>
<td>St Dev</td>
<td>18,098,251</td>
<td>476,154</td>
<td>2,876,867</td>
<td>16,384,908</td>
</tr>
<tr>
<td>Coefficient of Variation</td>
<td>3.99%</td>
<td>0.13%</td>
<td>6.65%</td>
<td>41.35%</td>
</tr>
</tbody>
</table>
This value redistribution from the junior tranches to the senior tranche is highlighted again in Figure 5.9 below: whereas the senior receives only 77% of the value in the highest 5% of asset value outcomes, it receives 90.5% in the lowest 5%. The equity share declines from 13.9% to 1.2% over the same range.
5.9.2 The Nature of the Risk in CDO Tranches

The senior debt never experienced loss of principal in any of the simulations. However, as alluded to already, there is some uncertainty regarding timing of receipt of principal. The average principal profile is shown in Figure 5.10 below:
However, there is some variation around this average that causes the variation in average life shown in Figure 5.11 below:

Figure 5.11 Probability Distribution of Senior Tranche Average Life
The average cash received by the senior tranche varies widely dictated by the pattern of defaults among the assets. The cash flow pattern is that of an amortising loan with an uncertain amortisation schedule.

Figure 5.12 Senior Tranche Cash Flow Profile

Figure 5.6 shows that the mezzanine tranche trades in a tight value range much of the time. However, when losses occur, they can be substantial as evidenced in Figure 5.13 below:
While the average LGD is only 36.9%, in some cases the entire principal is lost. It is this latter feature of mezzanine tranches that so concerns the regulator.

The cash flows to the mezzanine tranche resemble those of a bullet maturity bond as shown in Figure 5.14 below. However, mezzanine debt can be PIKed, that is, paid in kind: this occurs when interest is capitalised and the cash that was available to pay the interest is used to pay down senior tranche principal. This means there is potential for significant variation around this average cash flow, a major source of worry for mezzanine debt purchasers.
The uncertainty regarding timing of receipt of cash flows is a risk dimension unlike that associated with typical corporate loans. This is shown in Figure 5.15 below which graphs average interest outstanding over time:

An appreciation of the variation around this average can be gained by graphing the distribution of the number of periods in which the mezzanine tranche was PIKed.
Figure 5.16 Frequency Distribution of Mezzanine Tranche PIKings

Over 62% of the time, the mezzanine tranche will experience at least one occasion when it does not receive interest due. In some cases, it becomes a zero-coupon bond for a period of up to five years. This type of behaviour renders it unsuitable as an investment product for many investors despite its relatively low loss experience.

5.10 The Expected Loss on Rated Tranches

Moody’s assign ratings to CDO tranches based on the expected loss of the tranche over its life. This new model produces an expected loss value for the tranches also. We can use the output to compare the Moody’s rating, derived in Chapter 6, with the comparable rating implied by this model.

The expected loss for the senior debt across all simulations is 0%. This is not unusual and it suggests a Moody’s Aaa rating.

The expected loss for the mezzanine debt is 1.28%. This can be seen to be the product of the frequency of ‘default’ – though the term is nebulous for tranches – and LGD. In 6.05% of the simulations, there was a shortfall to some extent on the repayment of principal and/or interest. The average loss in the event of default was 36.9%, though that varied widely as indicated in Figure 5.13. This is the key output for comparing this model with Moody’s.
5.11 Conclusion

This chapter presents a newly created multi-period CDO model in a structural framework. The model takes the existing state-of-the-art and adapts it to incorporate an analysis of CDO tranches. In so doing, it presents an alternative perspective on the CDO rating question:

- It enables the Moody's CDO tranche rating methodology to be compared with a new alternative.
- It also provides a richer framework for thinking about the risk which CDO tranches present.

It concludes that the uni-dimensional view of risk – that of expected loss only – which guides the agency rating process fails to meet the needs of the regulator for a basis on which to assign capital. In fact, expected loss in banking is simply regarded as a cost of doing business and risk is measured by loss variability. It is suggested that Moody's CDO ratings framework is flawed since it does not measure risk at all. In particular, the rating methodology fails to meet the regulator's need for a measure on which to base its requirement for bank capital.

In Chapter 6, the Moody's rating framework will be developed and in Chapter 7, the results of the two modelling approaches will be compared.
Chapter 6. Re-constructing Moody’s CDO Tranche Rating Approach

6.1 Chapter Overview

Moody’s assign ratings to CDO tranches using their BET. This approach was outlined by Cifuentes and O’Connor (1996) and re-created in 3.9 above. But, the outline, while giving the basic Moody’s rating philosophy, is insufficient to enable their tranche ratings to be replicated. Neither have they issued a model that embeds their CDO tranche rating methodology.

However, a model that is capable of replicating Moody’s approach is a requirement in order to enable comparison with the alternative methodology that has been developed in this thesis. In this chapter, the author builds a fully-functional BET model which can accommodate the complete details which their papers do not specifically address.

Moody’s further requirements - not specified in the literature but gleaned by the author from conversations with Moody’s personnel, investment banks and other market participants - were noted. These requirements, together with Moody’s published guidelines, were incorporated into a new cash flow model designed to replicate the Moody’s rating process.

The success of the model in replicating their rating approach is then confirmed by testing the model on newly issued deals. The model-derived ratings are very closely aligned to the actual ratings granted by Moody’s to the rated tranches of the deals.

6.2 Creating a Moody’s BET Model

The results shown in 3.9.1.1 prove that the models the author constructed produce the same results as those quoted by Moody’s in their publications. However, the structures addressed by Moody’s in those three papers is very stylised and it ignores many of the details that Moody’s take into account in practice.

The tasks to be undertaken in constructing a full-blown BET model are sketched in Figure 6.1 below.
In developing the model, details of the idealised portfolio – the number of binomial bonds, the default rate, the recovery rate on defaulted assets, the weighted average maturity and the weighted average coupon – as discussed already must be combined with details of the actual portfolio, including the value of the portfolio, the price paid and the timing of the acquisition, the so-called ‘ramp-up’ period. Other factors that are also considered include (i) interest rate stresses, (ii) expenses – management, rating agency, accounting and trustee fees, (iii) hedging (if any), (iv) liquidity requirements, and (v) tax.

The model that was built tracks the various cash flows from the idealised portfolio as described below.

6.2.1 Principal Cash Flows

Principal derives from three separate sources (i) redemption of collateral, (ii) recoveries, and (iii) excess spread. On the other hand, principal amounts may be applied to pay down note principal or to reinvest in new assets during the reinvestment period.
6.2.2 Revenue Cash Flows

In the BET framework, there are five sources of revenue (i) collateral interest income, (ii) reinvestment income, (iii) revenue from hedging, (iv) drawings on the liquidity reserve, and (v) release of the liquidity reserve at maturity.

Revenue is used to (i) pay senior expenses, (ii) pay note interest liabilities, (iii) replenish the liquidity reserve, and (iv) pay surplus to the holder of the equity. Excess spread may be diverted to the principal ledger in particular circumstances.

6.2.3 Other Cash Flow Modelling Stresses

Moody's require that the default timing be varied. The standard test has 50% of total defaults occurring in the first year with 10% occurring annually thereafter for the next five years. While this proves to be the most severe timing for most tranches, Moody's oblige that back-end loaded defaults and mid-loaded defaults be undertaken in addition to the standard front-loading and the most conservative rating outcome is applied to the tranche. The complete list of default scenarios which must be examined are outlined in Table 6-1 below.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Year 1</th>
<th>Year 2</th>
<th>Year 3</th>
<th>Year 4</th>
<th>Year 5</th>
<th>Year 6</th>
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<tr>
<td>6</td>
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<td>10%</td>
<td>10%</td>
<td>10%</td>
<td>10%</td>
<td>50%</td>
</tr>
</tbody>
</table>

Table 6-1 Default Timing Scenarios

6.3 Replicating Moody's Rating Results

The model tracks the cash flows from the assets to the various tranches over time. Each scenario assumes a different number of defaults. The cash flows received by each tranche are discounted at Libor plus the promised tranche spread, if no losses occur, the tranche value will equal par. The loss incurred by each tranche weighted by the binomial probability of the loss occurring will equal the tranche expected loss. This expected loss is compared to the average loss incurred by debt of similar maturity in
Moody's loss database The rating whose loss matches the expected loss of the tranche is the rating that is assigned to the tranche

A sample of the intermediate calculations follows to indicate how these calculations were undertaken

6.4 Discussion of Modelling Details

The figures which follow summarise the results for a single scenario The diversity score is 47 based on the industry and country composition of the 90-loan portfolio, the weighted average rating is B2 and the weighted average life is 7.25 years, or 29 quarters The scenario shown involves front-loaded defaults – 50% in the first year and 10% per annum in the five subsequent years – and six of the 47 idealised loans defaulting

Figure 6.2 shows a high default rate over the first four quarters followed by a lower default rate over the remaining quarters until the end of year 6 The assets which have not defaulted are assumed to be redeemed as bullet payments at the end of the weighted average life The LGD is assumed to be 45% and occurs after a one-year delay

The Class A notes are the senior notes in the structure and the Class B are the mezzanine If the senior O/C or I/C test is breached – as happens in Quarters 4, 5 and 6 – the mezzanine tranche is not paid interest and any cash beyond that required to pay the senior interest is diverted to pay down senior principal until the test is corrected If the senior O/C and I/C tests are passed but the mezzanine tests are not, no cash is paid to equity and the residual is diverted to pay down the senior principal The senior expenses include rating agency costs, trustee fees and management fees

The last rows of Figure 6.6 and Figure 6.7 give the total cash – comprising interest and principal – received quarterly over the 29 quarters of the structure’s life The present value of each cash flow stream, discounted at the promised yield, gives the value of the senior and mezzanine notes The shortfall from par, expressed as a percentage of par gives the loss rate incurred
### 6.4.1 Account Ledgers

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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

| **Class A Notes** | | | | | | | | | | | | | | | |
| Opening Balance | 360,000,000 | 360,000,000 | 360,000,000 | 360,000,000 | 358,004,648 | 352,181,510 | 346,284,534 | 342,335,066 | 338,385,598 | 337,595,705 | 336,805,811 | 336,015,917 | 335,226,024 | 334,436,130 | 333,646,236 |
| Redemptions | 0 | 0 | 0 | 1,995,352 | 5,823,137 | 3,949,468 | 789,894 | 789,894 | 789,894 | 789,894 | 789,894 | 789,894 | 789,894 | 789,894 | 789,894 |
| Closing Balance | 360,000,000 | 360,000,000 | 360,000,000 | 360,000,000 | 358,004,648 | 352,181,510 | 346,284,534 | 342,335,066 | 338,385,598 | 337,595,705 | 336,805,811 | 336,015,917 | 335,226,024 | 334,436,130 | 333,646,236 |

| **Class B Notes** | | | | | | | | | | | | | | | |
| Opening Balance | 40,000,000 | 40,000,000 | 40,000,000 | 40,000,000 | 40,000,000 | 40,000,000 | 40,000,000 | 40,000,000 | 40,000,000 | 40,000,000 | 40,000,000 | 40,000,000 | 40,000,000 | 40,000,000 | 40,000,000 |
| Redemptions | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Closing Balance | 40,000,000 | 40,000,000 | 40,000,000 | 40,000,000 | 40,000,000 | 40,000,000 | 40,000,000 | 40,000,000 | 40,000,000 | 40,000,000 | 40,000,000 | 40,000,000 | 40,000,000 | 40,000,000 | 40,000,000 |

| **Seller Equity** | | | | | | | | | | | | | | | |
| Opening Balance | 50,000,000 | 50,000,000 | 50,000,000 | 50,000,000 | 50,000,000 | 50,000,000 | 50,000,000 | 50,000,000 | 50,000,000 | 50,000,000 | 50,000,000 | 50,000,000 | 50,000,000 | 50,000,000 | 50,000,000 |
| Redemptions | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Closing Balance | 50,000,000 | 50,000,000 | 50,000,000 | 50,000,000 | 50,000,000 | 50,000,000 | 50,000,000 | 50,000,000 | 50,000,000 | 50,000,000 | 50,000,000 | 50,000,000 | 50,000,000 | 50,000,000 | 50,000,000 |

*Figure 6.2: The Securitisation Balance Sheet*
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Figure 6.3 The Securitisation Balance Sheet (contd)
## 6.4.2 Cash Flow Waterfall

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**Figure 6.4 Cash Flow Waterfall**
### Cash Flow Waterfall (contd)

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<td>789,894</td>
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#### Application Source

| Class A Notes | 789,894 | 789,894 | 789,894 | 789,894 | 789,894 | 789,894 | 789,894 | 789,894 | 789,894 | 789,894 | 789,894 | 789,894 | 789,894 | 789,894 | 789,894 | 789,894 | 789,894 | 0 |
| Class B Notes | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Seller Equity | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

#### Interest Source

| Portfolio Yield | 9,399,734 | 9,366,559 | 9,333,383 | 9,300,207 | 9,267,032 | 9,233,856 | 9,200,681 | 9,167,505 | 9,134,330 | 9,101,154 | 9,067,979 | 9,067,979 | 9,067,979 | 9,067,979 | 9,067,979 |
| Liquidity Drawings | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Closing Balance | 9,399,734 | 9,366,559 | 9,333,383 | 9,300,207 | 9,267,032 | 9,233,856 | 9,200,681 | 9,167,505 | 9,134,330 | 9,101,154 | 9,067,979 | 9,067,979 | 9,067,979 | 9,067,979 | 9,067,979 | 9,067,979 |

#### Application Source

| Senior Expenses | 507,779 | 506,164 | 504,548 | 502,932 | 501,316 | 499,701 | 498,085 | 496,469 | 494,854 | 493,238 | 491,622 | 491,622 | 491,622 | 491,622 |
| Class A Notes | 6,038,997 | 6,024,700 | 6,010,403 | 5,996,106 | 5,981,809 | 5,967,512 | 5,953,214 | 5,938,917 | 5,924,620 | 5,910,323 | 5,896,026 | 5,881,729 | 5,867,432 | 5,853,135 | 5,838,838 |
| Class B Notes | 854,000 | 854,000 | 854,000 | 854,000 | 854,000 | 854,000 | 854,000 | 854,000 | 854,000 | 854,000 | 854,000 | 854,000 | 854,000 | 854,000 |
| Diversion of Seller Revenue | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Seller Equity | 1,998,958 | 1,981,695 | 1,964,432 | 1,947,170 | 1,929,907 | 1,912,644 | 1,895,381 | 1,878,119 | 1,860,856 | 1,843,593 | -1,826,330 | 1,840,627 | 1,854,924 | 1,869,221 | 1,883,519 |

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**Figure 6.5 Cash Flow Waterfall (contd)**
## 6.4.3 Cash Flow and Coverage Tests

### Quarter 1 - 15

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<tbody>
<tr>
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<td>125%</td>
<td>124%</td>
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<td>128%</td>
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### Overcollateralisation

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<th>121%</th>
<th>119%</th>
<th>118%</th>
<th>119%</th>
<th>121%</th>
<th>122%</th>
<th>123%</th>
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<th>122%</th>
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<th>122%</th>
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</tr>
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<tbody>
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<td>109%</td>
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<td>106%</td>
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</tr>
</tbody>
</table>

### Liquidity Drawings

| Note Interest   | -7,370,000 | -7,370,000 | -7,370,000 | -7,370,000 | -7,333,884 | -7,228,485 | -7,121,750 | -7,050,265 | -6,978,779 | -6,964,482 | -6,950,185 | -6,935,888 | -6,921,591 | -6,897,294 | -6,892,997 |
| Available Revenue | 10,395,000 | 10,229,122 | 10,063,245 | 9,897,367 | 9,731,469 | 9,598,314 | 9,665,138 | 9,631,963 | 9,598,787 | 9,565,612 | 9,532,436 | 9,499,261 | 9,466,085 | 9,432,910 | 9,399,734 |

### RESULTS

| Principal & Interest | Class A Notes | 6,516,000 | 6,516,000 | 6,516,000 | 8,511,352 | 12,303,021 | 12,271,461 | 10,217,218 | 10,145,733 | 6,914,673 | 6,900,376 | 6,886,079 | 6,871,782 | 6,857,485 | 6,843,188 | 6,828,890 |
| Class B Notes       | 854,000 | 854,000 | 854,000 | 854,000 | 854,000 | 854,000 | 854,000 | 854,000 | 854,000 | 854,000 | 854,000 | 854,000 | 854,000 | 854,000 | 854,000 |
| Seller Equity       | 2,468,750 | 2,310,951 | 2,153,152 | 0       | 0       | 2,022,683 | 2,062,609 | 2,102,535 | 2,085,272 | 2,068,099 | 2,050,746 | 2,033,483 | 2,016,221 | 1,998,958 |       |

Figure 6.6 Cash Flow and Coverage Tests

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<th>17</th>
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</tr>
<tr>
<td>Senior Expenses</td>
<td>507,779</td>
<td>506,164</td>
<td>504,548</td>
<td>502,932</td>
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<td>491,622</td>
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<td>5,953,214</td>
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<td>5,910,323</td>
<td>5,896,026</td>
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<tr>
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<tr>
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<td>504,548</td>
<td>502,932</td>
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<td>499,701</td>
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Figure 6.7 Cash Flow and Coverage Tests (contd.)
### 6.4.4 Tranche Losses under the Various Scenarios

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<table>
<thead>
<tr>
<th>Loss %</th>
<th>Implied Rating</th>
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<tr>
<td>0.00023%</td>
<td>Aaa</td>
</tr>
<tr>
<td>2.14226%</td>
<td>Baa3</td>
</tr>
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Table 6-2 The Binomial Probability Loss Weighting Scheme

190
The expected loss on the senior and mezzanine tranches are 0.00023% and 2.14226% implying ratings of Aaa and Baa3, respectively.

6.5 Checking the Replication of Moody’s BET

The model described above attempts to replicate Moody’s approach to the rating of CDO tranches. In order to gauge the success of the replication effort, the model was used to rate the tranches of four securitisations

<table>
<thead>
<tr>
<th>CDO</th>
<th>Tranche</th>
<th>Moody's Rating</th>
<th>Model Rating</th>
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</thead>
<tbody>
<tr>
<td>Galway Bay</td>
<td>Class I</td>
<td>Aaa</td>
<td>Aaa</td>
</tr>
<tr>
<td></td>
<td>Class II</td>
<td>A2</td>
<td>A3</td>
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<tr>
<td></td>
<td>Class III</td>
<td>Baa2</td>
<td>Baa3</td>
</tr>
<tr>
<td></td>
<td>Class IV</td>
<td>Ba3</td>
<td>Ba3</td>
</tr>
<tr>
<td>Clare Island</td>
<td>Class I</td>
<td>Aaa</td>
<td>Aaa</td>
</tr>
<tr>
<td></td>
<td>Class II</td>
<td>Aa2</td>
<td>Aa1</td>
</tr>
<tr>
<td></td>
<td>Class III</td>
<td>Baa2</td>
<td>Baa2</td>
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<td></td>
<td>Class IV</td>
<td>Ba3</td>
<td>Ba2</td>
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<tr>
<td>Cashel Rock</td>
<td>A1</td>
<td>Aaa</td>
<td>Aaa</td>
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<td></td>
<td>A2</td>
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<td></td>
<td>A3</td>
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<td>Baa3</td>
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<td>Tara Hill</td>
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<tr>
<td></td>
<td>Class III</td>
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</tr>
<tr>
<td></td>
<td>Class IV</td>
<td>Ba3</td>
<td>B1</td>
</tr>
</tbody>
</table>

Table 6-3 Comparison between Model Ratings to Moody’s Ratings

Comparing the model tranche ratings to those which Moody’s assigned, it is observed that the ratings assigned by Moody’s were replicated exactly for 10 of the 15 tranches while in each of the five remaining tranches, the difference in ratings was one notch.

In order to determine whether this represents a good replication performance, S&P and Moody’s ratings were compared for 104 CDO tranches. Of these tranches, 75 received the same rating from both agencies, 25 differed by one notch and four differed by two notches.

Set against this background, the replication appears successful. It may be concluded that the model is capable of inferring Moody’s ratings to CDO tranches.
6.6 Conclusion

This chapter described how Moody’s BET was replicated. Such a model will be necessary if we are to be able to examine the source of any difference in tranche quality assessment between the multi-period structural model developed in Chapter 5 and Moody’s.

The comparison between the assessment of the new multi-period structural model developed in this thesis and that of Moody’s is the subject of Chapter 7.
Chapter 7. Comparing the CDO Model Results with Moody's Rating and Market Prices

7.1 Chapter Overview

In Chapter 5, a new model was developed which extended the existing state-of-the-art credit portfolio modelling paradigm to a multi-period framework and incorporated the details of the indenture in disbursing the cash flows to the multiple CDO tranches. This enabled CDO tranches to be valued and the ratings implied to the tranches.

In Chapter 6, Moody's BET was successfully replicated. This allows a Moody's rating to be attributed to tranches of any proposed securitisation.

The aim of this chapter is to compare the tranche ratings implied from the multi-period credit portfolio model with Moody's ratings. The differences between the two ratings will be examined and the reasons for these differences explored.

7.2 Comparing Model Outputs

The Moody's ratings for the senior and mezzanine tranches are Aaa and Baa3 based on loss rates of 0.00023% and 2.14%, respectively. The corresponding loss rates under the new CDO valuation model are 0% and 1.28% suggesting ratings of Aaa and Baa2 for the senior and mezzanine tranches, respectively.

At first glance, the differences appear small – differences of one notch for the same tranche between the rating agencies are commonplace – suggesting that the two models are capturing a similar dynamic, albeit in completely different ways. However, this may not be the result of a close alignment of methodologies.

7.2.1 Comparing Default Probability Assumptions

Moody's default rates are derived from their expected loss table in Table 3-8 above. The assumption underlying these expected loss rates is of a LGD of 55%. Based on these figures, the probability of default attributed to B-rated assets over a seven-year period are calculated in Table 7-1.
Table 7-1 Moody’s Default Probabilities

The distribution of Moody’s and KMV’s default probabilities of the assets in the portfolio is plotted in Figure 7.1:

![Histogram of 7-Year Cumulative Default Probabilities](image)

**Figure 7.1 Distribution of KMV and Moody’s Default Probabilities in the Portfolio**

The Moody’s average cumulative default probability over the seven-year period is 23.91% compared to KMV’s average of 10.67%. This probably reflects the fact that the date of the analysis is September 2004, a benign point in the credit cycle.

KMV have observed that the average default probability within agency rating categories moves with the stage in the credit cycle. Figure 7.2 below shows the evolution of the Moody’s B-rated universe over the five-year period from November 1999 to November 2004. The median one-year probability of default is 0.76% compared to the long-term
average of approximately 3%. KMV’s analysis suggests that Moody’s will systematically over-estimate default probability in the benign phase of the cycle and under-estimate it during the stressed stage. Moody’s stated ‘through the cycle’ approach to creditworthiness estimation lends further support to this view.

In view of the fundamental difference between Moody’s and KMV’s assessment of default probability, it is unlikely that there will ever be a close alignment between the ratings of structured debt based on the two different modelling approaches. Equally, Moody’s assumption that the portfolio’s default characteristics can be summarised in one single number differs significantly from the KMV approach where each borrower is modelled individually. Given the PD range in Figure 7.1 which KMV estimates exists, the assumption appears untenable.

![Figure 7.2 EDF Percentiles for Moody’s B-rated debt over time](image)

Given this disparity in the fundamental input to the portfolio model, it is unlikely that it would ever be possible to reconcile the ratings under the two approaches.

### 7.2.2 Comparing Correlation Assumptions

A section of asset correlation matrix embedded in the KMV factor model implementation is shown in Figure 7.3 below. The average asset correlation among all...
obligor pairs is 0.148. Likewise, the average default correlation among all obligors equals 0.019. A section of the default correlation matrix is given in Figure 7.4.

![Asset Correlation Matrix](image)

![Default Correlation Matrix](image)

To the author’s knowledge, there is no obvious way to compare Moody’s Diversity Score with asset correlation or default correlation. Moody’s distill the portfolio of ninety
names down to a portfolio of forty-seven independent entities. Equally, Moody’s would suggest the same Diversity Score regardless of the firms within the industry. On the other hand, KMV’s correlation value will be heavily dependent on company size: larger companies are found to have higher R-squared.

7.3 Comparing Implied Spreads to Market Spreads

As shown in Table 5-1 above, the senior tranche, which has a par value of €360m and pays a 50bp spread over Libor, was valued at €370.25m while the mezzanine tranche has a par value of €40m, pays a 150bp spread and is valued at €43.29m.

The senior and mezzanine tranches would trade at par if they were to be paid spreads of 24.9bp and 98.4bp, respectively. The spreads chosen for the tranches were typical of spreads available for similarly-rated tranches in 2003. Spreads in the CDO market have tightened considerably since then, though not nearly as much as spreads on individual corporate names. Current spreads on Aaa- and Baa2-rated CDO tranches based on an asset pool with an average maturity of seven years are approximately 30bp and 110bp, respectively.

The model spreads are narrower than the market is demanding. However, the spread difference is small especially when considered in the light of spreads demanded in the market only two years ago. Furthermore, there is likely to be some additional spread required as compensation for the illiquidity of the tranches vis-à-vis equally-rated corporate debt.

7.4 Conclusion

In this chapter, the tranche ratings implied by the newly developed CDO valuation model are compared with the ratings attributed to these tranches by Moody’s and the model-implied spreads are compared to those available in the market:

- It is suggested that the differences in the default probability assessment of the underlying assets will make it extremely unlikely that the rating agency rating will align with the model-implied rating based on the model expected loss.
- The factor model framework makes the correlation measurement in the new model explicit. The rating agency approach is heuristic and it is impossible to make any comparison between the two methodologies.
• The model-implied spreads are less than, but nevertheless quite close to, those available in the market for similarly-rated CDO tranches. This is reassuring the extra margin demanded in the market is no more than could be explained by liquidity differences.

In summary, the market prices seem to provide reassurance that the model outputs are realistic.
Chapter 8. Conclusions and Suggestions for Further Research

8.1 Chapter Overview

This thesis developed a new model that is capable of valuing CDO tranches using the current state-of-the-art framework. The model-implied tranche ratings were compared to those assigned to the tranches by Moody’s using their BET approach. The results from the two approaches differed, the reasons for these differences were explored and the implications for the rating agency approach were assessed.

Likewise, the spreads suggested by the model were compared to those demanded in the marketplace. The differences were, once again analysed and reasons were suggested which could explain these differences.

This final chapter

- summarises the contribution of this thesis to the literature,
- examines other applications of the modelling framework developed in this thesis,
- looks back on the research approach which was adopted and critically examines the weaknesses of the approach, and
- suggests how the modelling framework that has been built can be extended to evaluate other securities.

8.2 Contribution of the Thesis to the Literature

The primary aims of this thesis were twofold

- Extend the market-leading structural approach to credit portfolio modelling from its single period framework into a multi-period risk-neutral framework capable of pricing structured debt
- Use this multi-period model to gauge whether the rating agency assessment of CDO tranche credit quality accurately captures the risks that such investments present
8 2.1 Model Construction

The approach that was adopted involved the development of three major models. The first model required that the single-period model developed by KMV be re-constructed. Such a re-construction has never previously been published. This stems from the fact that KMV have divulged very little regarding their actual approach, their limited public utterances were couched in very generic terms. The model results aligned almost completely with those produced by KMV in their Portfolio Manager software offering. The labelling of their approach as 'black box' - a criticism which has been frequently been levelled at them because of their unwillingness to publish their portfolio modelling approach - now seems inappropriate. These results confirm that KMV's methodology is firmly rooted in the standard factor model implementation of the Markowitz framework.

The second major model and the key extension to the current literature is the conversion of the single-period model into a risk-neutral multi-period model capable of valuing structured debt. While the model addresses CDOs specifically, the framework is sufficiently generic to accommodate any credit product whose cash flows depend on portfolio interactions over many time periods.

The final model was a re-creation of Moody’s BET that they employ to rate CDOs. This re-construction was necessary in order to allow a Moody’s rating to be assigned to the various tranches of the CDOs that were modelled. The successful replication was confirmed by rating structures previously rated by Moody’s.

8 2.2 Assessment of Agency Rating for CDO Tranches

The completion of the multi-period portfolio model enabled the calculation of the expected loss for the CDO tranche. This same measure is also an output from the Moody’s BET model and is the basis on which they assign their rating. A comparison of these expected loss measures provides the basis for comparing the two assessments of CDO tranche quality. This offers an alternative perspective on tranche quality to that provided by the agencies. Potentially, the expected loss measure from the new model ought to be more theoretically-sound than Moody’s measure which is based on an heuristic approach. Reference to market prices offers an independent assessment of the same issue.
8.2.3 Assessment of Validity of Agency Approach to the Rating of CDO Tranches

The nature of the cash flows from the various tranches points to the multi-dimensional nature of CDO tranche risk. The multi-period model shows that the risks presented by CDOs are significantly more complex than those presented by the loans which comprise the portfolio. The potential for mezzanine debt, for example, not to receive the interest due to it in the current period – to receive ‘payment-in-kind’ – substantially complicates risk assessment for the investor. This points to inadequacies in the expected loss-based measure to summarise the risks that the tranche presents.

More fundamentally, the agency view that expected loss is a measure of risk is questioned. The view is expressed that expected loss is merely a cost of doing business. As such, the Moody’s rating tells the portfolio manager very little of relevance about the risk which the tranche presents. This problem is worse still for the regulator for whom expected loss is a matter of indifference since expected loss is a cost borne by bank shareholders. The only real concern to regulators is systemic risk, the potential for loss substantially higher than the expected to occur which arises when multiple obligors default during the period of interest.

The results of this research confirm the view expressed in the recent Basel II publication that the risks of CDO tranches differ significantly from those of similarly rated corporate debt. Subordinated tranches of CDOs embed significantly higher unexpected loss than equally rated corporate debt. Furthermore, AAA-rated senior tranches built on poorly diversified loan portfolios – ‘non-granular’ portfolios in Basel-speak – contain more systematic risk than AAA-rated corporate loans. The very notion of rating is compromised.

If the agency rating concept is to be redeemed, it will need to be enriched. While the current rating could continue to be used as a measure of expected loss, the rating should be qualified to indicate the extent of variation around the expected loss. A separate qualifier may be of interest to those managing the institution’s liquidity for whom interruption of cash flows may represent unacceptable risk. However, it is hard to see how any meaningful measure of risk can be obtained which would satisfy the regulatory need for an assessment of systemic risk embedded in the tranche. The agency methodology is too contrived to be capable of adaptation to measure tranche risk in the context of bank portfolio.
8.2.4 Additional Insights for the Credit Portfolio Manager

The single-period model offers two significant additional insights to the credit portfolio manager which are not available from the current models available in the market. Furthermore, the analysis which the model accommodates supports two recommendations for changes to the way in which credit portfolios are managed.

Marginal impact of a new facility on a debt portfolio. The model developed enables the effect of adding a new facility to a debt portfolio to be measured and the facility Sharpe ratio to be evaluated having taken the portfolio characteristics into account. This addresses the key concern of portfolio managers, namely, to measure the effect of adding a new facility to the existing portfolio.

This new approach is developed in a structural framework using the KMV measure of obligor quality, and their measure of asset correlation. Heretofore, most research has relied on the reduced form approach.

The further benefit of this approach is that it greatly expands the universe of obligors that can be accommodated. The reduced form approach requires that a liquid credit default swap market exist. However, such a market exists only for well-known names that have access to the bond market and typically to not avail of bank loans. The framework developed addresses the needs of the typical bank since the portfolio impact can be determined for all companies whether or not they have quoted debt securities.

Optimal Hold Level. Almost all banks impose arbitrary limits on facility size. They have no mechanism for determining the exposure size at which it becomes uneconomical to assume further exposure.

Using the model developed in this thesis and the approach advocated, the appropriate limit on exposure amount to a given counterparty may be set by comparing the available market spread to the cost of writing new business at the marginal capital rate. The exposure threshold can be set at some minimum EVA spread taking explicit account of the portfolio composition.

Relationship managers in banks constantly argue for further limits to secure other income from clients. This framework provides a basis for determining the cost of excessive exposure to a relationship client and the basis for levying a concentration penalty. While most banks that do not implement ‘hard’ limits levy a ‘hog tax’ on these.
customers, such penalties are usually based on intuition rather than quantitative approaches.

New Capital Attribution Framework: The model enables a new capital attribution methodology – namely, contribution to ETL - to be tested. The proposed new methodology is far more sensitive to concentrations and gives results which are much more aligned with market intuition.

Limit-Setting in Portfolios: Finally, it is suggested that this new measure of capital forms an appropriate basis on which to set limits on exposure to customers, sectors and geographies which is consistent with the bank’s risk appetite.

8.3 Assessment of the Modelling Assumptions

The complexity of the credit portfolio interactions addressed in this thesis and the paucity of data necessitated that certain assumptions and approximations be made. Among the most significant of these are the following:

Deterministic Interest Rates: Interest rates were assumed to evolve in a deterministic fashion. The potential mismatch between fixed- and floating-rate assets and liabilities is not addressed. It is naïvely assumed that the interest rate mismatch that exists between the assets and liabilities is perfectly hedged. In reality, this will never occur because the swap can never be structured to cater for all potential defaults and prepayments. The interest rate mis-match would require modelling the joint movement of interest rate and credit quality.

Prepayment due to Credit Quality Improvement: The option for the underlying assets to prepay was ignored. Thus, the prepayment of loans and calling of bonds that accompany credit quality improvement were not addressed. This would cause the actual portfolio losses to be under-estimated as the portfolio manager is obliged to re-invest in new loans during the re-investment period. Furthermore, the effect beyond the reinvestment period will be to cause earlier de-leveraging and paydown of the senior debt with consequent higher risk for the subordinate tranches.

Prepayment due to Declines in Interest Rates: Fixed-rate bonds will often be redeemed if interest rates decline. An interest rate model would need to be integrated into the current model to capture this aspect of prepayment risk.
Credit Quality Migration Assumptions: The KMV EDF measure is the key assessment of default probability over various time periods. Unfortunately, this measure assumes that the obligor’s liability structure is constant while it models the impact of asset value. This ignores the potential for a firm to take on significant new debt if its fortunes improve, as many companies are wont to do. Thus, an improvement in a company’s fortunes will not necessarily convert into an improvement in the value of its debt if the equity-holders seize the opportunity to re-leverage. Unfortunately, there is no way around this problem without access to KMV’s database of EDF migration histories.

Manager Gamesmanship: The model has not attempted to model manager gamesmanship. In practice, some managers may attempt to game the O/C test by selling assets which are trading above par and buying assets which are trading below par. Effectively, they are capitalising on circumstances where the rating agencies are slow to downgrade or upgrade; since the O/C test is based on par values, all similarly-rated bonds count equally, regardless of market price. In this way, the manager can keep the cash flowing to equity in circumstances where the collateral quality has deteriorated significantly. Since most managers are themselves equity-holders, they will find it in their interest to do so. Moody’s have recognised this and now identify in their research those managers who most egregiously engage in such activity. However, it would be very difficult to capture this type of behaviour in the current model.

8.4 Suggestions for Further Research

The model that was developed was applied to the valuation of tranches of a CDO structure. However, the framework is sufficiently general to allow the model to be applied to many different types of structured debt securities.

- A financial institution’s equity interest in its portfolio of debt securities is the most obvious candidate for evaluation using this approach. This could provide a novel approach to the valuation of bank portfolios.
- Many investment funds – for example, split-capital investment trusts – are structured to take leveraged positions in portfolios of debt securities. The model developed here is capable of being adapted to incorporate the market value triggers instead of the over-collateralisation triggers of the CDO structure. This would enable the valuation of the equity, senior debt and zero-dividend preference shares that comprise the fund’s liabilities.
• When the credit quality of an obligor with a floating-rate loan improves, the obligor will often choose to prepay the loan. This feature could be readily incorporated into the model once data was acquired to parameterise this behaviour.

• It is well documented in the literature – see, for example, Altman et al. (2002) – that there is a strong link between default rate and recovery rates. In other words, recovery rates decline when default rates in the economy are higher than average. This is intuitive – the value of assets realised in the event of default is less in a more stressed credit environment as many firms find themselves as forced sellers. Making the beta distribution from which the LGD is drawn correlated with the portfolio default frequency easily incorporates this correlation between portfolio default rates and LGD.

• CDOs of CDOs, or CDO-squared as they have become known in the marketplace, are a particularly difficult debt instrument to value. The methodologies that the rating agencies employ are merely extensions of their current, heuristic CDO rating methodologies. The current model could easily accommodate the additional complexity presented by the multiple layering. A second waterfall would need to be overlaid on the individual CDO waterfalls to determine what cash would flow to the individual CDO-squared tranches. Clearly, the challenge of collating all the underlying data would be substantial, but the additional modelling effort should not be overly onerous.

• The use of expected loss as a measure of the risk of CDO tranches has been questioned. A more theoretically correct approach would be to measure the variability of tranche loss. A more relevant measure for a portfolio manager would be a measure of tranche loss contribution to the portfolio. Similarly, regulators would want to measure tranche loss contribution to credit portfolios in stressed credit environments. The task of producing such measures – a huge number of simulations is required – is challenging and requires new insights.

8.5 Final Comment

The 1974 Merton approach – subsequently called the structural approach – was framed around a single obligor. Subsequent work, most notably by KMV and CreditMetrics,
moved forward from that single obligor view to a multi-obligor portfolio view in a
single-period time-frame which could assess the behaviour of corporate debt portfolios.
This thesis builds on this portfolio view by extending it to a multi-period time frame
capable of valuing structured debt. As such, it is a natural extension of previous research
efforts. However, it takes previous research down a path which has been somewhat
ignored of late because of the spectacular growth which has occurred in the CDS market
and the associated price transparency. Unfortunately, the lure of plentiful data and
robust pricing methodologies has drawn researchers to focus on the liquid sector of the
market – the investment-grade market primarily – ignoring the bulk of bank obligors,
those sub-investment-grade names whose debt is seldom, if ever, traded. It is hoped that
this thesis goes some way towards redressing that imbalance.
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**List of Acronyms**

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
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<tbody>
<tr>
<td>BET</td>
<td>Binomial Expansion Technique</td>
</tr>
<tr>
<td>BIS</td>
<td>Bank for International Settlements</td>
</tr>
<tr>
<td>CBO</td>
<td>Collateralised bond obligation</td>
</tr>
<tr>
<td>CCA</td>
<td>Contingent claims analysis</td>
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<tr>
<td>CD</td>
<td>Credit derivative</td>
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<tr>
<td>CDO</td>
<td>Collateralised debt obligation</td>
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<tr>
<td>CLO</td>
<td>Collateralised loan obligation</td>
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<tr>
<td>DD</td>
<td>Distance to default</td>
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<tr>
<td>EAD</td>
<td>Exposure at default</td>
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<tr>
<td>EL</td>
<td>Expected loss</td>
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<tr>
<td>EVA</td>
<td>Economic value added</td>
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<tr>
<td>IAA</td>
<td>Internal assessment approach</td>
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<tr>
<td>IRR</td>
<td>Internal rate of return</td>
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<tr>
<td>M</td>
<td>Maturity</td>
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<tr>
<td>PIK</td>
<td>Paid in kind</td>
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<tr>
<td>RBA</td>
<td>Ratings-based approach</td>
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<tr>
<td>SPV</td>
<td>Special purpose vehicle</td>
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<tr>
<td>TRUPS</td>
<td>Trust preferred securities</td>
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<tr>
<td>UL</td>
<td>Unexpected loss</td>
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<tr>
<td>ZDP</td>
<td>Zero-dividend preference shares</td>
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Appendix The CreditMetrics Credit Portfolio Modelling Approach

CreditMetrics rely on a model which connects rating changes and defaults to movements in an obligor's asset value. "This allows us to model joint rating changes across multiple obligors without relying on historical rating change or bond spread data" (Gupton et al, p81). They do so because of the problems with alternative approaches such as non-parametric methods using direct estimation of joint credit moves and estimates based on bond spreads. They summarise their two-step asset value model for joint probabilities of credit rating changes:

- They propose an underlying process which drives credit rating changes. They attempt to establish a connection between the events that they ultimately want to describe (rating changes), but which are not readily observable, and a process that they understand and can observe.

- They estimate the parameters from the process above. "If we have been successful in the first part, this should be easier than estimating the joint rating change probabilities directly" (p85).
They propose that a firm’s asset value be the process that drives its credit rating changes and defaults, a model they claim is essentially Merton’s option theoretic model. However, since their focus is on portfolio value changes resulting from changes in credit rating as well as default, they do not concentrate solely on the default threshold but identify all the rating boundaries. They assume that there are asset levels such that they can construct a mapping from asset value in one year’s time to rating in one year’s time. Knowing the asset thresholds that correspond to rating boundaries, “we only need to model the company’s change in asset value in order to describe its credit rating evolution.” (p86)

They then state a fundamental premise on which their model’s validity rests: “To do this [modelling], we assert that the percentage changes in asset value (that is, asset “returns”, which we will denote by $R$) are normally distributed, and parameterised by a mean $\mu$ and standard deviation (or volatility) $\sigma$.” (p86) They can then define the rating thresholds as corresponding to a cumulative probability of the standard normal distribution. Using S&P’s transition matrix, they establish the rating thresholds corresponding to asset return values. An example of these thresholds is shown in Figure 8.1. They comment that for one obligor, they only
need the transition probabilities to describe the evolution of credit rating changes, and the asset value process is not necessary. The benefit of the asset value process, they claim, is only in the consideration of multiple obligors.

They assume that the asset returns for a two-asset portfolio are bivariate normally distributed though they note that any multivariate distribution (including those incorporating fat tails or skewness effects) where the joint movements of asset values can be characterised fully by one correlation parameter would be applicable. By extension, they assume that the joint distribution of the asset returns of any collection of firms is multivariate normal. By way of example, they assume a two-asset portfolio comprising a BB- and an A-rated obligor with a correlation of 0.3 between their asset returns. The distribution is sketched in Figure 8.2:

![Bivariate Normal Distribution of Asset Returns](image)

**Figure 8.2 Bivariate Normal Distribution of Asset Returns**

They calculate the probabilities that the two assets will be in the various combinations of credit states by integrating under the probability density surface. The results are shown in Table 8-1:
Table 8-1 Joint Rating Change Probabilities for BB- and A-rated Obligors

The results shown in Table 8-1 are fundamentally dependent on the asset correlation estimate. CreditMetrics suggest a variety of approaches could be taken to the estimation of asset correlation, from a simple average correlation approach to one that uses equity correlations. "One fundamental—and typically very observable—source of firm-specific correlation information is equity returns." (p93) They use the correlation between equity returns as a proxy for the correlation of asset returns. While they acknowledge that this method has the drawback of overlooking the differences between equity and asset correlations, they assert that it is more accurate than using a fixed correlation and is based on more readily available data than credit spreads or actual joint rating changes. They accept that it would be desirable if they could produce correlations for any pair of obligors, but assert that scarcity of data for many obligors as well as the impossibility of storing a correlation matrix of the size that would be necessary, would make this approach untenable. Therefore, they "resort to a methodology which relies on correlations within a set of indices and a mapping scheme to build the obligor-by-obligor correlations from the index correlations." (p93)

Thus, to produce individual obligor correlations, the correlation between industry indices in particular countries is calculated. Then they map individual obligors by industry participation. They also calculate the volatility of each index and the correlation between each index pair. In these calculations, they use the last 190 weekly returns and weight each of these equally. Their motivation for using this approach, they say, is that they are interested in computing correlations which are valid over the longer horizons for which CreditMetrics will be used. The statistics tend to be more stable over time, they claim, and reflect longer term trends, whereas the statistics in RiskMetrics vary more from day to day, and capture shorter term behaviour.
They summarise their simulation approach as follows

- Assign weights to each obligor according to its participation in countries and industries, and specify how much of the obligor's equity movements are not explained by the relevant indices
- Express the standardised returns for each obligor as a weighted sum of the returns on the indices and a company-specific component
- Use the weights along with the index correlations to compute the correlations between obligors

"By specifying the amount of an obligor's equity price movements not explained by the relevant indices, we are describing the obligor's firm-specific, or idiosyncratic, risk" (p98)