THE SLOW-MOTION APPROXIMATION FOR THE GOR'KOV-ÉLIASHBERG EQUATIONS

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March 25, 2003

Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Doctor of Philosophy in Applied Mathematical Sciences is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

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Acknowledgements

It gives me great pleasure to thank my research supervisor, Jurgen Burzlaff, for his attention, advice, guidance and, non last, the interesting conversations, especially those regarding topics in Particle Physics. Moreover I wish to thank Jurgen for his patience and the reassurance he gave me throughout the last two years.

A particular acknowledgement is also addressed to Dr. Laurent Plantie for the more than precious discussions and suggestions that helped me to find rigorous proofs of several theorems.

I wish to thank the staff of the Mathematics Department on whom I have called for technical advice from time to time. In this respect, I particularly thank John Appleby, Michael Clancy and Eugene O'Riordan for their suggestions and explanations.

Finally, but in a special way, I am delighted to thank Ronan and Sol to have contributed to make my stay in Dublin so pleasant.

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Abstract

We want to describe the dynamics of magnetic vortices in type-II superconductors using the Gor'kov-Éliashberg equations. To solve this system is very difficult so we want to use an approximation, called Slow-Motion Approximation. This approximation is used quite a lot in physics and for our system of nonlinear partial differential equations we want to show, using rigorous mathematical arguments, that it is in fact an approximation to the exact solution. For the Abelian-Higgs model which shares the same time independent equations with the Gor'kov- Éliashberg system, such a mathematically rigorous proof was given by Stuart (1994).

The mathematical discussion starts with an *ansatz* for the solution that involves the exact solution of the static problem and a small correction. It is well-known that the static solution is a $2N$ real-parameter family. Let us denote the parameters by q. In the Slow-Motion Approximation we assume that the parameters are time dependent. In our case we want to find the trajectory in the space of static solutions which is the closest, in some sense, to the exact solution.

As in many approximation techniques we need a small parameter such that the approximation gets better and better the smaller the parameter becomes. The small parameter, denoted by ε, is given by the Higgs self-coupling constant $k^2 = (1+\varepsilon)/2$.
Guided by Stuart's proof we assume that the time derivative of the parameters q is $\mathcal{O}(\varepsilon)$.

So the problem of proving the validity of the approximation is now turned into proving the existence and smallness of the corrections, which are the solutions of a parabolic linear partial differential equations system on $\mathbb{R}^2$. In order to prove this we try to imitate the techniques for finding solutions of the same class of equations in a bounded domain. We need also an iterative method that provide us with certain estimates in suitable Sobolev spaces. We get a system of equations for the parameters $q(t)$ that is a Cauchy problem as soon as we fix initial conditions for q. Imposing initial conditions as well on the corrections of the static solution we simplify the equations for q and solve them.
Substituting these $q(t)$'s into the static solution we obtain a good approximation for the exact solution of Gor'kov-Éliashberg equations.

Notation

$$\begin{aligned} detA &= \text{determinant of } A. \\ \mathbb{R}^n &= \text{n-dimensional real Euclidean space.} \\ e_i &= (0,\ldots,0,1,0,\ldots,0) = i^{th} \text{ standard coordinate vector.} \\ \partial U &= \text{boundary of } U. \\ B(x,r) &= \text{closed ball with centre } x\text{, radius } r > 0. \end{aligned}$$

For numbers $a, b \in \mathbb{C}$ we use the usual inner product

$$(a,b) = \frac{\bar{a}b + a\bar{b}}{2}.$$

In particular

$$|a| = \sqrt{(a,a)}.$$

Definition 1 *Let* $\boldsymbol{a} = (a_1, a_2)$ *a vector potential for the magnetic field* $\vec{H}$*. We define the covariant derivative with respect to the background field* $\boldsymbol{a}$ *of a complex field* $\boldsymbol{f}$ *as*

$$D_j^{(\boldsymbol{a})}\boldsymbol{f} := \partial_j \boldsymbol{f} - a_j \boldsymbol{f}$$

for $j = 1, 2$.

A vector of the form $\beta = (\beta_1, \ldots, \beta_n)$, where each component β_i is a nonnegative integer, is called a *multiindex* of *order*

$$|\beta| = \beta_1 + \cdots + \beta_n.$$

Given a scalar function $u : U \longrightarrow \mathbb{R}$ and a multiindex β, we define

$$\frac{\partial^{|\beta|}u}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}}(x) := \partial_1^{\beta_1} \dots \partial_n^{\beta_n} u(x).$$

We will use the same definition for the covariant derivative instead of the normal derivative.

Definition 2 *A function $u : U \longrightarrow \mathbb{R}$ is called Holderian if a multiindex β and $\gamma > 0$ exist such that*

$$|\partial_1^{\beta_1} \dots \partial_n^{\beta_n} u(x) - \partial_1^{\beta_1} \dots \partial_n^{\beta_n} u(y)| \leq C|x - y|^\gamma$$

for some constant $C > 0$ and all $x, y \in U$.

The Laplacian of a scalar function u is defined as

$$\Delta u(x) = \sum_{l=1}^{n} \partial_{ll}^2 u(x).$$

The gradient of a scalar function u is defined as

$$\nabla u(x) = \big(\partial_1 u(x), \dots, \partial_n u(x)\big).$$

If now $m = 2, 3$ and $U \subseteq \mathbb{R}^m$, the divergence of a vector-valued function $\vec{u} : U \longrightarrow \mathbb{R}^m$, $\vec{u}(x) = \big(u_1(x), \dots, u_m(x)\big)$ is defined as

$$\vec{\nabla} \cdot \vec{u}(x) = \sum_{l=1}^{m} \partial_l u_l(x).$$

$$C^k(U) = \big\{u : U \longrightarrow \mathbb{R} \mid u \text{ is } k\text{-times continuously differentiable}\big\}.$$

$$C^\infty(U) = \{u : U \longrightarrow \mathbb{R} \mid u \text{ is infinitely differentiable}\}.$$

$$C_c^\infty(U) = \{u \in C^\infty(U) \mid u \equiv 0 \text{ outside a compact subset of } U\}.$$

$$L^p(U) = \{u : U \longrightarrow \mathbb{R} \mid u \text{ is Lebesgue measurable, } |u|_{L^p(U)} < +\infty\},$$

where

$$|u|_{L^p(U)} = \Big(\int_U |u(x)|^p dx\Big)^{\frac{1}{p}} \qquad (1 \leq p < +\infty),$$

$$L^\infty(U) = \{u : U \longrightarrow \mathbb{R} \mid u \text{ is Lebesgue measurable, } |u|_{L^\infty(U)} < +\infty\},$$

where

$$|u|_{L^\infty(U)} = \sup_{x \in U} |u(x)|.$$

Fix $1 \leq p \leq +\infty$ and let k be a nonnegative integer. We define now certain function spaces, whose elements have weak derivatives of various orders lying in various L^p spaces.

Definition 3 *The Sobolev space*

$$W^{k,p}(U)$$

consists of all locally summable functions $u : U \longrightarrow \mathbb{R}$ such that for each multiindex β with $|\beta| \leq k$, $\partial_1^{\beta_1} \dots \partial_n^{\beta_n} u$ exists in the distributions sense and belongs to $L^p(U)$.

Remark. If $p = 2$, we usually write

$$H^k(U) = W^{k,2}(U) \qquad (k = 0, 1, \dots).$$

Definition 4 *If* $u \in W^{k,p}(U)$, *we define its norm to be*

$$|u|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\beta|\le k} \int_U |\partial_1^{\beta_1} \dots \partial_n^{\beta_n} u(x)|^p dx\right)^{\frac{1}{p}} & (1 \le p < +\infty) \\ \sum_{|\beta|\le k} \sup_{x\in U} |\partial_1^{\beta_1} \dots \partial_n^{\beta_n} u(x)| & (p = +\infty). \end{cases}$$

Definition 5 *Let* α *a multiindex. If* $\psi(x,t) := \begin{pmatrix} \widetilde{a}(x,t) \\ \widetilde{\phi}(x,t) \end{pmatrix} \in \mathbb{R}^2 \times \mathbb{C}$, *then we call*

$$H^{r,\alpha(\cdot,q(0))}(\mathbb{R}^2) = \{\psi : \mathbb{R}^2 \times [0,T] \longrightarrow \mathbb{R}^2 \times \mathbb{C} \mid \forall t \in [0,T] \\ |\psi(\cdot,t)|_{r,\alpha(\cdot,q(0))} < +\infty\},$$

where

$$|\psi(\cdot,t)|_{r,\alpha(\cdot,q(0))} := \sum_{|\beta|=0}^{r} \sum_{l=1}^{2} \int_{\mathbb{R}^2} \Big(|\partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} \widetilde{a}_l(x,t)|^2 \\ + |(D_1^{(\alpha(\cdot,q(0)))})^{\beta_1} \dots (D_n^{(\alpha(\cdot,q(0)))})^{\beta_n} \widetilde{\phi}(x,t)|^2\Big) d^2x.$$

Cauchy's inequality. Let $a, b > 0$ and $\eta > 0$. Then

$$ab \le \eta a^2 + \frac{b^2}{4\eta}.$$

Hölder's inequality. Assume $1 \le p, q \le +\infty, \frac{1}{p} + \frac{1}{q} = 1$. Then if $u \in L^p(U), v \in L^q(U)$, we have

$$|uv|_{L^1(U)} \le |u|_{L^p(U)} |v|_{L^q(U)}.$$

Minkowski's inequality. Assume $1 \le p \le +\infty$ and $u, v \in L^p(U)$.

$$|u + v|_{L^p(U)} \le |u|_{L^p(U)} + |v|_{L^p(U)}.$$

Cauchy-Schwartz's inequality. For every $x, y \in \mathbb{R}^n$

$$|(x, y)| \leq |x||y|.$$

Definition 6 *If ∂U is C^1, then along ∂U the outward pointing unit normal vector field*

$$\nu(x) = \big(\nu_1(x), \ldots, \nu_n(x)\big)$$

is defined at any point $x \in \partial U$.
Let $u \in C^1(\overline{U})$. We call

$$\frac{\partial u}{\partial \nu}(x) := \nu(x) \cdot \nabla u(x)$$

the outward normal derivative of u.

Theorem (Green's formulas). Let $u, v \in C^2(\overline{U})$. Then

- $\displaystyle\int_U \triangle u(x)dx = \int_{\partial U} \frac{\partial u}{\partial \nu}(x)d\sigma(x),$
- $\displaystyle\int_U \nabla v(x) \cdot \nabla u(x)dx = -\int_U u(x)\triangle v(x)dx + \int_{\partial U} u(x)\frac{\partial v}{\partial \nu}(x)d\sigma(x),$
- $\displaystyle\int_U \big(u(x)\triangle v(x) - v(x)\triangle u(x)\big)dx = \int_{\partial U} \big(u(x)\frac{\partial v}{\partial \nu}(x) - v(x)\frac{\partial u}{\partial \nu}(x)\big)d\sigma(x).$

Definition 7 *We define the inner product for functions exponentially decaying at infinity as*

$$\langle \begin{pmatrix} \widetilde{a}(\cdot,t) \\ \widetilde{\phi}(\cdot,t) \end{pmatrix}, \begin{pmatrix} \widetilde{a}'(\cdot,t) \\ \widetilde{\phi}'(\cdot,t) \end{pmatrix} \rangle := \int_{\mathbb{R}^2} \Big(\sum_{l=1}^{2} \widetilde{a}_l(x,t)\widetilde{a}'_l(x,t) + \big(\widetilde{\phi}(x,t), \widetilde{\phi}'(x,t)\big)\Big) d^2x.$$

Throughout this work we use an arbitrary positive constant Γ. Also a constant k is used. It is related to the material in which the equations describe the vortex motion. The parameters $q_1, \ldots, q_{2N}$ are the real and imaginary parts of the zeros of the static Higgs field. They are made time dependent. A short time scale $\tau := \varepsilon t$ is introduced. The derivative with respect to τ is indicated with $\dot{q}$. Finally the functions $\widetilde{a}$, $\widetilde{\phi}$ and $\widetilde{\widetilde{a}}_0$ represent the deviation of the exact solution from the solution to the static Ginzburg-Landau equations.

Chapter 1

Introduction

1.1 Phenomenology of Superconductivity

An explanation of the microscopical understanding of superconductivity has been given by Bardeen, Cooper and Schrieffer (B-C-S) in [BCS57]. In their theory, the onset of superconductivity is due to the formation of bound electron pairs, the so called Cooper pairs. With respect to small applied forces, the electron pairs interact as single entities, particles with twice the charge of a single electron. Furthermore this effective particle is a boson of spin zero, since the spins of the single particles of the pairs are supposed antiparallel. Large external forces and the temperature of the materials, usually near 0°K for metals and about 100°K for ceramics, disrupt the pairing and force the superconductor to change to the normal state.

The supeconducting state is characterized by an ensemble of Cooper pairs and these in turn are described by a scalar field Φ for charged, spinless particles. As usual, such a field is a complex-valued function of position. The density $|\Phi(\vec{x})|^2$ is proportional to the number density of Cooper pairs in $\vec{x}$. The field Φ is called the order parameter. The state where $\Phi \sim 0$ has a few pairs and behaves as a normal conductor. The case in which $|\Phi|$ is bounded away from zero describes the state of condensed pairs and superconductivity.

The order parameter provides a macroscopic description of the system described microscopically by the B-C-S theory. While this macroscopic theory can be obtained as an approximate consequence of the more general B-C-S theory, it was proposed well before the justification of a microscopic theory existed. A system of equations for Φ and its interaction with the electromagnetic potential $\vec{A}$ was proposed by Ginzburg and Landau [GL50].

In this system a parameter λ appears, called Higgs self-coupling constant, which is related to the material and describes different physical properties. When λ is smaller than 1 in a certain *normalization*, the material, called type-I superconductor, once embedded in an external magnetic field $\vec{H}$, expels utterly this field. If sufficient intensity has been achieved, the external field will penetrate uniformly the material and the superconducting properties vanish. When λ is greater than 1 the material, now called type-II superconductor, once subjected to an external magnetic field $\vec{H}$, can exist in one of three phases:

- If $|\vec{H}|$ is smaller than a certain value, say $|\vec{H}_0|$, the external field is pushed out of the sample. The material is in a fully superconducting phase.

- If $|\vec{H}|$ is between $|\vec{H}_0|$ and another value, say $|\vec{H}_1|$, the external field penetrates the sample only partially. What one detects physically is that the total flux of the magnetic field through the sample is quantized and every single quantum of flux, called fluxon, can be regarded as a lump of magnetic field lines or vortex. In the centre of the vortex $\Phi \sim 0$.
- If $|\vec{H}|$ is greater than $|\vec{H}_1|$, the external magnetic field gets through the material uniformly with consequent total break-down of superconductivity.

1.2 The Ginzburg-Landau Model

The macroscopic model for superconductivity presented by Ginzburg and Landau in [GL50] is based on thermodynamic considerations and is an extension of Landau's theory of second order phase transitions. The time-independent Ginzburg-Landau equations are given by the following second-order system:

$$\begin{aligned}&\triangle A_h(\vec{x}) - \partial_h \vec{\nabla} \cdot \vec{A}(\vec{x}) \\ &\quad = \frac{1}{2}\Big(i\bar{\Phi}(\vec{x})\big(\partial_h \Phi(\vec{x}) - iA_h(\vec{x})\Phi(\vec{x})\big) + i\Phi(\vec{x})\big(\partial_h \bar{\Phi}(\vec{x}) + iA_h(\vec{x})\bar{\Phi}(\vec{x})\big)\Big),\end{aligned}$$

$$-\sum_{l=1}^{2}\big(\partial_l - iA_l(\vec{x})\big)^2 \Phi(\vec{x}) = \frac{\lambda}{2}\Phi(\vec{x})\big(1 - |\Phi(\vec{x})|^2\big) \tag{1.1}$$

where Φ is the Higgs field or the order parameter, $\vec{A}$ is the vector potential and λ is the so-called Higgs self-coupling constant. The term $\vec{\nabla} - i\vec{A}$ is called covariant derivative and makes the equations invariant under a particular class of transformations, the so-called gauge transformations. The invariance of the solutions of the Ginzburg-Landau equations under such transformations allows us to consider several mathematical solutions without altering the physical measurable quantities which are the electrical and magnetic fields.

A generalization of the Ginzburg-Landau system to the time-dependent case was given by Gor'kov and Éliashberg in [GE68] for the case of type-II superconductors. They found out, using the Feynman technique for the Green functions at finite temperatures, that the generalization to the non-stationary case of the Ginzburg-Landau equations is still gauge invariant and is given by

$$\partial_t \bar{\Phi}' + \frac{\tau_s}{3}\Big[\Big(-\pi^2(T_c^2 - T) + \frac{|\Phi'|^2}{2}\Big)|\bar{\Phi}'| - \frac{v^2\tau_1}{\tau_s}\Big(\vec{\nabla} + \frac{2ie}{c}\vec{A}\Big)^2\bar{\Phi}'\Big] + 2ieA_0\bar{\Phi}' = 0,$$

$$\vec{j} = \sigma\Big(\frac{1}{c}\partial_t\vec{A} - \vec{\nabla}A_0\Big) + \frac{N\tau_s\tau_1}{2m}\Big[ie(\Phi\vec{\nabla}\bar{\Phi} - \bar{\Phi}\vec{\nabla}\Phi) - \frac{4e^2}{c}|\Phi|^2\vec{A}\Big],$$

$$\vec{j} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}). \tag{1.2}$$

Here Φ' is the Higgs field, $\vec{A}$ is the vector potential, A_0 is the scalar potential, $\vec{j}$ is the current, $v^2\tau_1$ is a constant proportional to the diffusion coefficient, τ_s is a parameter which has the meaning of a free flight time between collisions associated with the electron spin flip in a material with paramagnetic impurities, T_c is the critical temperature at which the superconducting behaviour starts, T is the actual temperature of the material, σ is the conductivity of the electrons in the material, e is the electron charge, c is the speed of light and N and m are other coefficients.

The motion of the vortex structure was investigated on the basis of the B-C-S microscopy theory of superconductivity only in the vicinity of the upper critical field $\vec{H}_1$. The model proposed by Gor'kov and Éliashberg enabled Gor'kov and Kopnin [GK71] to find an approximate solution for any value, of $|\vec{H}|$ between $|\vec{H}_0|$ and $|\vec{H}_1|$, provided that the parameter of the Ginzburg-Landau theory λ is much bigger than 1. Their reasoning was based on the idea that the solution is formed by the solution of the corresponding static Ginzburg-Landau system and a correction which must solve another set of equations. We point out that, as many experiments have shown, superconductors of the second type (alloys, ceramics, etc.) have a finite resistance in a magnetic field exceeding the lower critical value $|\vec{H}_0|$. The nature of these dissipative effects is due to the motion of the Abrikosov vortex structure under the influence of the Lorentz force resulting from the flow of current [Abr57]. For $|\vec{H}| << |\vec{H}_1|$, however, the resistance is very small.

1.3 The Multivortex Static Solution

A feature of the Ginzburg-Landau model is the existence of a Lagrangian from which the Ginzburg-Landau equations are derived as its equations of motion. In [Bog76] it was shown that, for the case $\lambda = 1$ in (1.1), a lower bound on the action associated to the lagrangian of Ginzburg-Landau exists, and minimizing such an action means solving the so-called Bogomol'nyi equations

$$\partial_1 \mathcal{R}e\, \Phi(\vec{x}) + A_1(\vec{x}) \mathcal{I}m\, \Phi(\vec{x}) - \partial_2 \mathcal{I}m\, \Phi(\vec{x}) + A_2(\vec{x}) \mathcal{R}e\, \Phi(\vec{x}) = 0,$$

$$\partial_2 \mathcal{R}e\, \Phi(\vec{x}) + A_2(\vec{x}) \mathcal{I}m\, \Phi(\vec{x}) + \partial_1 \mathcal{I}m\, \Phi(\vec{x}) - A_1(\vec{x}) \mathcal{R}e\, \Phi(\vec{x}) = 0,$$

$$\partial_1 A_2(\vec{x}) - \partial_2 A_1(\vec{x}) + \frac{1}{2}\left(|\Phi(\vec{x})|^2 - 1\right) = 0$$

with Φ and $\vec{A} = (A_1, A_2, 0)$ as above.

In [Wei79] it is shown that, assuming the existence of an arbitrary solution of minimal energy, for every integer N there is a $2N$-real parameter family of N-vortex solutions satisfying the Bogomol'nyi bound, the lowest static energy state. It can be proven that the number N has the meaning of counting how many times the Higgs field at infinity rotates in the complex plane, hence it is called winding number. Furthermore, every solution satisfying this bound must belong to such a family. Weinberg conjectured that these parameters may be chosen to be the $2N$ co-ordinates specifying the positions of the vortices.

In 1980 Jaffe and Taubes made great improvements to the understanding of the static Ginzburg-Landau model. They proved that a finite action solution to the Bogomol'nyi equations exists, called multi-vortex solution, such that:

- The solution is globally smooth.
- A finite number of zeros for the Higgs field exists.

Identifying these zeros of Φ with the parameters discussed by Weinberg, one can refer to them as the centres of the vortices. According to the interpretation of the modulus of Φ as the density of the Cooper pairs, the centre of the vortices represent the points of the material where the superconducting properties vanish.

To prove the existence theorem, Taubes and Jaffe defined

$$\Phi =: e^{\frac{1}{2}(u+i\theta)}, \qquad A := \frac{1}{2}(A_1 - iA_2), \qquad \partial_z := \frac{1}{2}(\partial_1 - i\partial_2)$$

with $\{Z_1, \ldots, Z_N\}$ the zeros of Φ. They noticed that away from the zeros of Φ the Bogomol'nyi equations turn into

$$A = i\partial_z \ln \bar{\Phi},$$

$$-\Delta u + e^u - 1 = -4\pi \sum_{k=1}^{N} \delta(z - Z_k)$$

with the condition

$$\lim_{|z|\to+\infty} u(z) = 0$$

arising from

$$\lim_{|z|\to+\infty} |\Phi(z)| = 1.$$

A more explicit static solution of the Ginzburg-Landau equations for a general λ is given by the radially symmetric functions

$$\vec{A}(\vec{x}) = Na(r)(\sin\theta, -\cos\theta, 0), \qquad \Phi(\vec{x}) = f(r)e^{iN\theta}$$

where $a(r)$, $f(r) \in C^\infty(]0, +\infty[)$ satisfy an O.D.E. system and r, θ are polar co-ordinates in $\mathbb{R}^2$. Such a solution represents N vortices sitting on top of each other.

1.4 The Idea of the Slow-Motion Approximation.

Another generalization of the static Ginzburg-Landau model to the time-dependent case but from the Gor'kov-Éliashberg equation is given by the gauge invariant Abelian-Higgs model, i.e.

$$-\triangle A_0(\vec{x},t) + \partial_h \vec{\nabla}\cdot\vec{A}(\vec{x},t) = \frac{i}{2}\Big(- \bar{\Phi}(\vec{x},t)\big(\partial_t\Phi(\vec{x},t) - iA_0(\vec{x},t)\Phi(\vec{x},t)\big) + \Phi(\vec{x},t)\big(\partial_t\bar{\Phi}(\vec{x},t) + iA_0(\vec{x},t)\bar{\Phi}(\vec{x},t)\big)\Big),$$

$$\partial^2_{tt}A_h(\vec{x},t) - \triangle A_h(\vec{x},t) - \partial_t\partial_h A_0(\vec{x},t) + \partial_h\vec{\nabla}\cdot\vec{A}(\vec{x},t) = \frac{i}{2}\Big(- \bar{\Phi}(\vec{x},t)\big(\partial_h\Phi(\vec{x},t) - iA_h(\vec{x},t)\Phi(\vec{x},t)\big) + \Phi(\vec{x},t)\big(\partial_h\bar{\Phi}(\vec{x},t) + iA_h(\vec{x},t)\bar{\Phi}(\vec{x},t)\big)\Big),$$

$$\big(\partial_t - iA_0(\vec{x},t)\big)^2\Phi(\vec{x},t) - \sum_{l=1}^{2}\big(\partial_l - iA_l(\vec{x},t)\big)^2\Phi(\vec{x},t) = \frac{\lambda}{2}\Phi(\vec{x},t)\big(1-|\Phi(\vec{x},t)|^2\big)$$

with the notation as above. Even if this model does not describe the dynamics of the vortices in the superconductors, it will be useful to cite some its mathematical characteristics in order to develop our method applied to the present work.

As for the static Ginzburg-Landau equations, this system admits a Lagrangian, which allows to define an action or energy. A global existence of time-dependent solutions of this system has been proven in [BM85]. The next step would be to find an explicit expression for these solutions. Since this aim appears over ambitious, for some applications it might be enough to exhibit an approximate solution. A fruitful idea emerged in 1982 from a remark by Manton on the scattering of monopoles evolving according to the second-order field equations of the Yang-Mills-Higgs theory, [Man82]. At large separations monopoles move freely in straight lines, because there are no forces. But what happens in a collision? Manton pointed out that it should be possible to study the scattering of B-P-S (Bogomol'nyi-Prasad-Sommerfield) monopoles, i.e. the static solutions of the B-P-S equations, moving at slow velocities. At best his idea would give approximate results, becoming exact as the velocities tended to zero.

Like in the case of vortices, Taubes and Jaffe proved the existence of the static multimonopole solution dependent on a family of parameters denoted collectively by γ, say $\left(\vec{A}(\vec{x},\gamma),\Phi(\vec{x},\gamma)\right)$. Consider now a smooth trajectory $\gamma(t)$ in the parameter space, [Man82]. Then $\left(\vec{A}(\vec{x},\gamma(t)),\Phi(\vec{x},\gamma(t))\right)$ does not in general satisfy the field equations, but it might provide a good approximation if the velocities are small. Since some of the parameters of the multimonopole solutions represent positions, at least for well-separated monopoles, such motion would describe well-separated monopoles in relative motion. Although the evolution of the fields does not exactly follow a trajectory in the set of exact static solutions, it was shown that it does to a good approximation for a specific $\gamma(t)$. As $\frac{d\gamma}{dt}(t_0)$ tends to zero, the approximation becomes better and better.

A rigorous proof of Manton's idea was given by Stuart for the Abelian-Higgs model [Stu94,I] as well as for the three-dimensional Yang-Mills-Higgs model [Stu94,II]. Since the Slow-Motion Approximation idea depends on slowly moving vortices or monopoles given by static solutions, which only exist for $\lambda = 1$, one would expect that the perturbative forces should be small, in other words, the system should be near the Bogomol'nyi regime. The small parameter ε introduced to describe the order of deviation from $\lambda = 1$ allowed Stuart to estimate also the time scale for which the Slow-Motion Approximation furnishes a good approximation of the exact solution of the Abelian-Higgs model. He found analogous results for the solution of the Yang-Mills-Higgs model. In both models it turns out that the approximation is valid for a $\mathcal{O}(\frac{1}{\varepsilon})$ time scale.

1.5 Solutions of the Time-Dependent Ginzburg-Landau System

The existence of a continuous solution of the time-dependent Ginzburg-Landau system for a short time defined on the whole of $\mathbb{R}^2$ is the first result arising from this work.

The existence of global weak solutions was already proven in the more physical case of a bounded domain, [WZ97]. In particular, when the spatial dimension equals 3, the existence of global (in time) weak solutions of the time-dependent Ginzburg-Landau equations for arbitrary L^p ($p \geq 4$) initial data has been proven by these authors. When the space dimension $n =$

2, first existence of global weak solution was obtained in [TW95]. Other properties as uniqueness and asymptotic behaviour of the solution for large time were found in [LQ93] and [RWW99].

Secondly we seek a solution which is a $\mathcal{O}(\varepsilon)$ perturbation of the static configuration when we perturb the Bogomol'nyi regime to the same order in ε. Our aim is to follow Stuart's reasoning, since both the Gor'kov-Éliashberg equations and the Abelian-Higgs equations reduce to the Ginzburg-Landau equations once all the fields are time-independent. While the space derivatives of all the fields are supposed to be $\mathcal{O}(1)$, we will define a slow-time variable such that the velocities of the centres of the vortices and the velocities of the perturbative components of the static fields are $\mathcal{O}(\varepsilon)$.

In [CDGP95] the case with Higgs self-coupling constant $\lambda \to +\infty$ and critical external magnetic field $\vec{H}_1 = \mathcal{O}(\lambda)$ is examined. Making a formal expansion in powers of λ^{-2} in the Ginzburg-Landau equations, the authors derive a sequence of recursive linear equations. The main result show that, in a suitable sense when $\lambda \to +\infty$, the energy-minimizing solutions of the full Ginzburg-Landau equations approach the solutions of the leading order equations in the recursive system both in a bounded-domain setting and in a periodic-lattice setting.

1.6 Summary

In chapter 2 we introduce the Time Dependent Ginzburg-Landau model along with an *ansatz* for the candidate solution that brings a small parameter into the equations. This parameter represent the intensity of perturbation of the static solution of the time independent Ginzburg-Landau equations. Since the system of equations is gauge invariant a gauge condition will be imposed. Then an additional equation for the electric scalar potential is derived. In section 2.4 the idea of the Slow-Motion Approximation is made explicit: a $2N$-parameter family associated with the static solution is made time dependent and moves in a manifold, called the moduli space, provided with a metric. The invertibility of such metric is discussed in section 2.5. From here it follows that the $2N$-parameter family is the solution of a particular O.D.E. system.

In chapter 3 we start describing some decay properties at spatial infinity for the static solution of the Ginzburg-Landau equations, and a recursive method to prove the existence of solution to the Gor'kov-Éliashberg equations is applied. A new system of linear elliptic and parabolic equations is obtained and general results are exhibited, such as existence in particular Sobolev spaces and behaviour at infinity of their solutions. This method produces then iterated functions and they are shown to satisfy firstly certain estimates using the Sobolev norms.

In chapter 4 these estimates are improved in order to show that, under certain initial conditions, the iterated functions remain bounded for a small time. Using these fact and defining a new complete space, in chapters 5 and 6 we prove that the sequences of the iterated functions are Cauchy and then convergent in a Sobolev norm. The way to show this is based on a contraction of certain functions. We firstly use the mean value theorem to bound the difference between two successive iterates through the partial derivatives of those functions. Then we establish that the bounds on their partial derivatives are dependent on time and a parameter ε. By choosing a short interval of time and small ε it can then be said that the iterates do form a contracting sequence.

Finally, in section 6.2 we show that the sums of these sequences define the solution of the Gor'kov-Éliashberg system for a short time and for small perturbation of the static configuration. Within the short time this solution is continuous in t along with the $2N$ parameters. Notice that this result is only local in time and it is valid when the static configuration of the vortices is only slightly perturbed.

Chapter 2

The Gor'kov-Éliashberg Equations

2.1 The Model

The model describing the time dependent physical quantities is displayed. The cylindrical symmetry of the fields enables us to represent the vector potential as a 2-dimensional field defined on the complex plane. Also the Higgs field is defined on the complex plane. This amounts to considering the cross section of the vortices on a plane perpendicular to the magnetic field within the material. A further equation, which we will use later on to study the behaviour of the scalar potential, is derived from the time-dependent Ginzburg-Landau model. All the equations are gauge invariant. Considering cylindrical symmetry here amounts to imposing the conditions

$$\vec{A}(\vec{x},t) = (A_1(x,y,t), A_2(x,y,t)), \quad \Phi(\vec{x},t) = \Phi(x,y,t).$$

The Gor'kov-Éliashberg system describing the dynamics of the magnetic vortices in type-II superconductors given in section 1.2 can be rewritten in the dimensionless form

$$\begin{aligned}&\big(\partial_t - ikA_0(\vec{x},t)\big)\Phi(\vec{x},t)\\&\qquad + k^2\Phi(\vec{x},t)\big(|\Phi(\vec{x},t)|^2-1\big) - \big(\vec{\nabla} - ik\vec{A}(\vec{x},t)\big)^2\Phi(\vec{x},t) = 0,\end{aligned} \tag{2.1}$$

$$\begin{aligned}&-\vec{\nabla}\times\big(\vec{\nabla}\times\vec{A}(\vec{x},t)\big) = \sigma\big(\partial_t\vec{A}(\vec{x},t) - \vec{\nabla}A_0(\vec{x},t)\big)\\&\qquad + \frac{i}{2k}\big(\bar{\Phi}(\vec{x},t)\vec{\nabla}\Phi(\vec{x},t) - \Phi(\vec{x},t)\vec{\nabla}\bar{\Phi}(\vec{x},t)\big) + |\Phi(\vec{x},t)|^2\vec{A}(\vec{x},t).\end{aligned} \tag{2.2}$$

Here Φ is the complex Higgs field, $\vec{A}$ is the vector potential, A_0 is the electric scalar potential, and k and σ are constants depending on the material.

The equations (2.1) and (2.2) are gauge invariant. Gauge invariance means that when a solution of certain equations is transformed by a gauge transformation, it is still a solution of the same equations. Here the gauge transformation is

$$\begin{aligned}A_\mu &\longmapsto A_\mu + \partial_\mu\chi, \qquad \mu = 0,1,2\\ \Phi &\longmapsto \Phi e^{i\chi}\end{aligned}$$

where $\chi = \chi(\vec{x},t)$ is a function sufficiently regular in the spatial variable. The equations (2.1) and (2.2) are 4 real equations for 5 real functions. Due to the fact that the solution is gauge invariant, an additional condition, a so called gauge condition, may be imposed, e.g. $\sum_{\mu=0}^{2}\partial_\mu A_\mu = 0$, so that a system of 5 real equations for 5 real functions is obtained. It turns out that to obtain certain estimates later on, a different gauge condition has to be imposed.

If we let

$$\partial_0 := \partial_t, \quad A'_\mu := kA_\mu, \quad D_\mu := \partial_\mu - iA'_\mu$$

for $\mu = 0, 1, 2$, then (2.1) and (2.2) can be written in the following way:

$$D_0\Phi(\vec{x}, t) - \sum_{l=1}^{2} D_l^2\Phi(\vec{x}, t) = k^2\big(1 - |\Phi(\vec{x}, t)|^2\big)\Phi(\vec{x}, t), \tag{2.3}$$

$$\begin{aligned} &\sigma\partial_t A'_h(\vec{x}, t) - \Delta A'_h(\vec{x}, t) - \sigma\partial_h A'_0(\vec{x}, t) \\ &\quad + \partial_h\big(\vec{\nabla} \cdot \vec{A}'(\vec{x}, t)\big) = \big(i\Phi(\vec{x}, t), D_h\Phi(\vec{x}, t)\big) \end{aligned} \tag{2.4}$$

for $h = 1, 2$. Here

$$\big(i\Phi, D_h\Phi\big) = \frac{i\bar{\Phi}D_h\Phi + i\Phi\overline{D_h\Phi}}{2}.$$

To obtain certain estimates we also need an equation for A'_0 which follows from (2.2). We obtain

$$\begin{aligned} 0 &= \vec{\nabla} \cdot \big(\vec{\nabla} \times \vec{\nabla} \times \vec{A}\big) = -\sigma\vec{\nabla} \cdot \vec{E} - \vec{\nabla} \cdot \vec{J}_s \\ &= -\vec{\nabla} \cdot (\sigma\vec{E}) - \vec{\nabla} \cdot \vec{J}_s = -\vec{\nabla} \cdot \vec{J}_c - \vec{\nabla} \cdot \vec{J}_s, \end{aligned}$$

with

$$\vec{E} := -\partial_t\vec{A} + \vec{\nabla}A_0, \quad \vec{J}_s := \frac{1}{k}\big(i\Phi, \vec{\nabla}\Phi - ik\vec{A}\Phi\big), \quad \vec{J}_c := \sigma\vec{E}.$$

Therefore

$$\vec{\nabla} \cdot \vec{J_c} = -\vec{\nabla} \cdot \vec{J_s}.$$

Since

$$\begin{aligned} -\vec{\nabla} \cdot \vec{J_s} &= \vec{\nabla} \cdot \vec{J_c} = \vec{\nabla} \cdot \sigma \vec{E} \\ &= \sigma \vec{\nabla} \cdot (-\partial_t \vec{A} + \vec{\nabla} A_0) = -\sigma \partial_t (\vec{\nabla} \cdot \vec{A}) + \sigma \Delta A_0, \end{aligned}$$

one obtains

$$-\sigma \Delta A_0' + \sigma \partial_t (\vec{\nabla} \cdot \vec{A}') = k \sum_{l=1}^{2} \partial_l \big(i\Phi, D_l \Phi \big). \tag{2.5}$$

Remark 2.1.1 *Notice that, supposing* $A_0'(\vec{x}, 0) = 0$ *and considering the time independent solution of* (2.3) *and* (2.4), *the equation* (2.5) *turns into*

$$\sum_{l=1}^{2} \partial_l \big(i\Phi, (\partial_l - iA_l')\Phi \big) = 0$$

for any static configuration $\big(\vec{A}'(\vec{x}), \Phi(\vec{x})\big)$.

From here on, A_0' and $\vec{A}'$ will be replaced by A_0 and $\vec{A}$, respectively, in (2.3), (2.4) and (2.5).

2.2 The Ansatz and the Gauge Conditions

The *ansatz* for the candidate solution of the Gor'kov-Éliashberg system is presented here. As is the case in a gauge theory, the solution is not defined uniquely since it can be transformed by a gauge transformation. At different orders in a small parameters, two conditions are imposed for such a solution and it is shown that they are equivalent to fixing the gauge, i.e., these two conditions eliminate the gauge freedom.

Let us use the *ansatz*:

$$A_h(\vec{x},t) = a_h(x,y,q(t)) + \varepsilon a_h^{'}(x,y,t),$$

$$\Phi(\vec{x},t) = \phi(x,y,q(t)) + \varepsilon\phi^{'}(x,y,t) \tag{2.6}$$

for $h = 1, 2$, with $\varepsilon > 0$,

$$k^2 = (1+\varepsilon)/2,$$

(a, ϕ) a static solution of (2.3) and (2.4) and q a family of $2N$ real parameters.

The integer number N is the winding number. If $|\phi|$ approaches a constant, γ, at infinity, then a winding number can be defined in the following way: if

$$\phi \longrightarrow \phi_\infty(\theta) = \gamma e^{i\sigma(\theta)} \quad \text{as } r \to +\infty$$

then

$$\begin{aligned} \phi_\infty/\gamma : \ S^1 &\longrightarrow U(1) \\ \theta &\longmapsto e^{i\sigma}(\theta) \end{aligned}$$

i.e. ϕ_∞/γ maps the circle at infinity, S^1, to the unit circle in the complex plane, $U(1)$. To each continuous ϕ_∞ we can therefore associate a winding number

$$N = \frac{i}{2\pi}\int_0^{2\pi} \phi_\infty(\theta)\frac{d\bar{\phi}_\infty}{d\theta}(\theta)d\theta.$$

Two gauge transformations are now defined as follows: the first one is

$$\begin{aligned} A_0 &\longmapsto \widetilde{A}_0 := A_0 + \varepsilon\partial_t\xi \\ A_h &\longmapsto \widetilde{A}_h := A_h + \partial_h(\varepsilon\xi) = a_h + \varepsilon(a_h^{'} + \partial_h\xi) \\ \Phi &\longmapsto \widetilde{\Phi} := \Phi e^{\imath\varepsilon\xi} = (\phi + \varepsilon\phi^{'})e^{\imath\varepsilon\xi} \end{aligned}$$

in terms of $\xi = \xi(x, y, t)$. The second, in terms of $\Xi = \Xi(x, y, q(t))$, reads

$$\begin{aligned} \widetilde{A}_0 &\longmapsto \widetilde{\widetilde{A}}_0 := \widetilde{A}_0 + \partial_t\Xi \\ \widetilde{A}_h &\longmapsto \widetilde{\widetilde{A}}_h := \widetilde{A}_h + \partial_h\Xi = \alpha_h + \varepsilon\widetilde{a}_h \\ \widetilde{\Phi} &\longmapsto \widetilde{\widetilde{\Phi}} := \widetilde{\Phi}e^{\imath\Xi} = (\phi + \varepsilon\phi^{'})e^{\imath\varepsilon\xi}e^{\imath\Xi} \\ &\qquad = \phi e^{\imath\Xi}e^{\imath\varepsilon\xi} + \varepsilon\phi^{'}e^{\imath\Xi}e^{\imath\varepsilon\xi} \\ &\qquad = \phi e^{\imath\Xi} + \varepsilon\Big(\frac{e^{\imath\varepsilon\xi} - 1}{\varepsilon}\phi e^{\imath\Xi} + \phi^{'}e^{\imath\Xi}e^{\imath\varepsilon\xi}\Big), \end{aligned}$$

with

$$\begin{aligned} \alpha_h &:= a_h + \partial_h\Xi, \qquad \varphi := \phi e^{\imath\Xi}, \\ \widetilde{a}_h &:= a_h^{'} + \partial_h\xi, \qquad \widetilde{\phi} := \frac{e^{\imath\varepsilon\xi} - 1}{\varepsilon}\phi e^{\imath\Xi} + \phi^{'}e^{\imath\Xi}e^{\imath\varepsilon\xi}. \end{aligned}$$

The functions ξ and Ξ are chosen such that the following conditions are satisfied:

$$\nabla \cdot \widetilde{a}(x, y, t) - \varepsilon\big(i\varphi(x, y, q(t)), \widetilde{\phi}(x, y, t)\big) = 0 \tag{2.7}$$

and

$$\sigma\nabla \cdot \partial_t\alpha(x, y, q(t)) - \big(i\varphi(x, y, q(t)), \partial_t\varphi(x, y, q(t))\big) = 0. \tag{2.8}$$

To show that ξ and Ξ can be found to satisfy the conditions (2.7) and (2.8) we rewrite (2.7) and (2.8) as follows:

$$\sum_{l=1}^{2} \partial_l(a_l^{'} + \partial_l \xi) - \varepsilon\big(i\phi e^{i\Xi}, \frac{e^{i\varepsilon\xi} - 1}{\varepsilon}\phi e^{i\Xi} + \phi^{'} e^{i\Xi} e^{i\varepsilon\xi}\big) = 0,$$

$$\sigma\partial_t \sum_{l=1}^{2} \partial_l(a_l + \partial_l \Xi) - \big(i\phi e^{i\Xi}, \partial_t(\phi e^{i\Xi})\big) = 0. \tag{2.9}$$

The above system admits solutions. In fact, (2.9) is equivalent to

$$\Delta\xi + \nabla\cdot a^{'} - \varepsilon\big(i\phi, \frac{e^{i\varepsilon\xi} - 1}{\varepsilon}\phi\big) - \varepsilon\big(i\phi, \phi^{'} e^{i\varepsilon\xi}\big) = 0,$$

$$\sigma\partial_t(\nabla\cdot a) + \sigma\Delta\partial_t\Xi - \big(i\phi e^{i\Xi}, e^{i\Xi}\partial_t\phi + ie^{i\Xi}\phi\partial_t\Xi\big) = 0,$$

which in turn is equivalent to

$$\Delta\xi = -\nabla\cdot a^{'} + \varepsilon\frac{\cos(\varepsilon\xi) - 1}{\varepsilon}\big(i\phi, \phi\big) + \varepsilon\frac{\sin(\varepsilon\xi)}{\varepsilon}\big(i\phi, i\phi\big) + \varepsilon\big(i\phi, \phi^{'} e^{i\varepsilon\xi}\big),$$

$$\sigma\Delta\partial_t\Xi - |\phi|^2\partial_t\Xi = -\sigma\partial_t(\nabla\cdot a) + \big(i\phi, \partial_t\phi\big),$$

or,

$$\Delta\xi = -\nabla\cdot a^{'} + |\phi|^2\sin(\varepsilon\xi) + \varepsilon\big(i\phi, \phi^{'} e^{i\varepsilon\xi}\big), \tag{2.10}$$

$$\Delta\partial_t\Xi - \frac{|\phi|^2}{\sigma}\partial_t\Xi = -\partial_t(\nabla\cdot a) + \frac{1}{\sigma}\big(i\phi, \partial_t\phi\big). \tag{2.11}$$

The equation (2.10) is a nonlinear Poisson equation. The existence of solution is proven using the method of variations, [Eva98, chapter 8].

To solve the equation (2.11) it is necessary to examine the properties of the static solution of the Ginzburg-Landau equations. Let

$$Z_\phi := \{z \in \mathbb{C} \mid \phi(z) = 0\} = \{Z_1, \ldots, Z_N\}$$

where $Z_1, \ldots, Z_N$ have $q_1, \ldots, q_{2N}$ as the real and the imaginary parts. According to [JT80], let the fields be represented using u and θ, two real functions of x, y and $q_1, \ldots, q_{2N}$ such that

$$\phi := e^{\frac{u+i\theta}{2}}.$$

From the Bogomol'nyi equations it follows that

$$a_1 = \frac{1}{2}(\partial_2 u + \partial_1 \theta), \quad a_2 = -\frac{1}{2}(\partial_1 u - \partial_2 \theta),$$

in $\mathbb{R}^2 \setminus Z_\phi$. For any $\mu \in \{1, \ldots, 2N\}$, the equations

$$\Delta \frac{\partial \Xi}{\partial q_\mu} = -\nabla \cdot \frac{\partial a}{\partial q_\mu} \quad \text{and} \quad |\phi|^2 \frac{\partial \Xi}{\partial q_\mu} = -\big(i\phi, \frac{\partial \phi}{\partial q_\mu}\big)$$

are satisfied in $\mathbb{R}^2 \setminus Z_\phi$ by $\Xi = -\theta/2$. This amounts to satisfying

$$\nabla \cdot \frac{\partial \alpha}{\partial q_\mu} = 0 \quad \text{and} \quad \big(i\varphi, \frac{\partial \varphi}{\partial q_\mu}\big) = 0 \tag{2.12}$$

in $\mathbb{R}^2 \setminus Z_\phi$ for any $\mu \in \{1, \ldots, 2N\}$. Knowing that the static solution is globally C^∞ in the variables and in the parameters, [JT80] and [Stu94,I],

$$\nabla \cdot \frac{\partial \alpha}{\partial q_\mu} = 0 \quad \text{and} \quad \big(i\varphi, \frac{\partial \varphi}{\partial q_\mu}\big) = 0$$

hold on the whole of $\mathbb{R}^2$. Then (2.11) admits a solution on $\mathbb{R}^2$.

N.B. The same arguments as those used to prove the validity of condition (2.8) enable us to prove the condition

$$\nabla \cdot \frac{\partial \alpha}{\partial q_\mu}(x, y, q(t)) - \big(i\varphi(x, y, q(t)), \frac{\partial \varphi}{\partial q_\mu}(x, y, q(t))\big) = 0. \tag{2.13}$$

We will need (2.13) to justify certain results later on.

After fixing the gauge transformations in the way just presented, the initial *ansatz* turns into:

$$A_h(\vec{x},t) := \alpha_h(x,y,q(t)) + \varepsilon\widetilde{a}_h(x,y,t),$$

$$\Phi(\vec{x},t) := \varphi(x,y,q(t)) + \varepsilon\widetilde{\phi}(x,y,t),$$

with α, φ, $\widetilde{a}$ and $\widetilde{\phi}$ satisfying the conditions (2.7), (2.8) and (2.13).

2.3 The Equations for $\widetilde{\widetilde{A}}_0$, $\widetilde{a}$ and $\widetilde{\phi}$

The *ansatz* is substituted into the equation for the scalar potential and into the Gor'kov-Éliashberg equations. Also the conditions (2.7) and (2.8) are taken into account in order to rewrite the original equations as a new system. Using that new system, an operator is defined and some of its properties are pointed out. A slow-time variable $\tau := \varepsilon t$ is introduced. This procedure will allow us to apply the Slow-Motion Approximation to the new system in the next section.

In the following we will adopt the notation $\dot{q} := \frac{dq}{d\tau} = \frac{1}{\varepsilon}\frac{dq}{dt}$. Putting the *ansatz* into the equation (2.5), one obtains

$$\begin{aligned} -\Delta\widetilde{\widetilde{A}}_0 = &-\partial_t \sum_{l=1}^{2} \partial_l(\alpha_l + \varepsilon\widetilde{a}_l) \\ &+ \frac{k}{\sigma}\sum_{l=1}^{2} \partial_l\Big(i(\varphi+\varepsilon\widetilde{\phi}), \big(\partial_l - i(\alpha_l+\varepsilon\widetilde{a}_l)\big)(\varphi+\varepsilon\widetilde{\phi})\Big). \end{aligned}$$

Using Remark 2.1.1 and the conditions (2.7) and (2.8), this is equivalent to

$$
\begin{aligned}
-\Delta\widetilde{\widetilde{A}}_0 = \varepsilon\Big(&-\frac{1}{\sigma}\sum_{\mu=1}^{2N}(i\varphi,\frac{\partial\varphi}{\partial q_\mu})\dot{q}_\mu + \frac{k}{\sigma}\sum_{l=1}^{2}(i\partial_l\varphi, D_l^{(0)}\widetilde{\phi}) \\
&+\frac{k}{\sigma}\sum_{l=1}^{2}(i\widetilde{\phi},\partial_l D_l^{(0)}\varphi) + \frac{k}{\sigma}\sum_{l=1}^{2}(i\varphi,\partial_l D_l^{(0)}\widetilde{\phi}) \\
&-\frac{2k}{\sigma}\sum_{l=1}^{2}\widetilde{a}_l(\varphi,\partial_l\varphi) + \frac{k}{\sigma}\sum_{l=1}^{2}(i\partial_l\widetilde{\phi}, D_l^{(0)}\varphi)\Big) \\
&+\varepsilon^2\Big((i\varphi,\partial_t\widetilde{\phi}) + \frac{k}{\sigma}|\phi|^2(i\varphi,\widetilde{\phi}) - \frac{2k}{\sigma}\sum_{l=1}^{2}\widetilde{a}_l(\widetilde{\phi},\partial_l\varphi) \\
&-\frac{2k}{\sigma}\sum_{l=1}^{2}\widetilde{a}_l(\varphi,\partial_l\widetilde{\phi}) + \frac{k}{\sigma}\sum_{l=1}^{2}(i\partial_l\widetilde{\phi}, D_l^{(0)}\widetilde{\phi}) + \frac{k}{\sigma}\sum_{l=1}^{2}(i\widetilde{\phi},\partial_l D_l^{(0)}\widetilde{\phi})\Big) \\
&+\varepsilon^3\Big(\sum_{\mu=1}^{2N}(\widetilde{\phi}, i\frac{\partial\varphi}{\partial q_\mu})\dot{q}_\mu - \frac{2k}{\sigma}(\varphi,\widetilde{\phi})(i\varphi,\widetilde{\phi}) - \frac{2k}{\sigma}\sum_{l=1}^{2}\widetilde{a}_l(\widetilde{\phi},\partial_l\widetilde{\phi})\Big) \\
&-\varepsilon^4\frac{k}{\sigma}|\widetilde{\phi}|^2(i\varphi,\widetilde{\phi}),
\end{aligned}
$$

with

$$D_l^{(0)} := \partial_l - i\alpha_l$$

for $l = 1, 2$. Therefore, without loss of generality, $\widetilde{\widetilde{A}}_0$ can be $\mathcal{O}(\varepsilon)$, i.e., one can write $\widetilde{\widetilde{A}}_0(\vec{x},t) := \varepsilon\widetilde{\widetilde{a}}_0(x,y,t)$.

Using this definition for $\widetilde{\widetilde{A}}_0$, one obtains

$$-\Delta\widetilde{\widetilde{a}}_0 = f_0''(\widetilde{a},\widetilde{\phi},\partial_t\widetilde{\phi},q,\dot{q}) \tag{2.14}$$

with

$$
\begin{aligned}
f_0''(\widetilde{a},\widetilde{\phi},\partial_t\widetilde{\phi},q,\dot{q}) := &-\frac{1}{\sigma}\sum_{\mu=1}^{2N}\big(i\varphi(\cdot,q),\frac{\partial\varphi}{\partial q_\mu}(\cdot,q)\big)\dot{q}_\mu \\
&+\frac{k}{\sigma}\sum_{l=1}^{2}\big(i\partial_l\varphi(\cdot,q),D_l^{(0)}\widetilde{\phi}\big)+\frac{k}{\sigma}\sum_{l=1}^{2}\big(i\widetilde{\phi},\partial_l D_l^{(0)}\varphi(\cdot,q)\big) \\
&+\frac{k}{\sigma}\sum_{l=1}^{2}\big(i\varphi(\cdot,q),\partial_l D_l^{(0)}\widetilde{\phi}\big)-\frac{2k}{\sigma}\sum_{l=1}^{2}\widetilde{a}_l\big(\varphi(\cdot,q),\partial_l\varphi(\cdot,q)\big) \\
&+\frac{k}{\sigma}\sum_{l=1}^{2}\big(i\partial_l\widetilde{\phi},D_l^{(0)}\varphi(\cdot,q)\big)+\varepsilon\Big(\big(i\varphi(\cdot,q),\partial_t\widetilde{\phi}\big) \\
&+\frac{k}{\sigma}|\phi|^2\big(i\varphi(\cdot,q),\widetilde{\phi}\big)-\frac{2k}{\sigma}\sum_{l=1}^{2}\widetilde{a}_l\big(\widetilde{\phi},\partial_l\varphi(\cdot,q)\big) \\
&-\frac{2k}{\sigma}\sum_{l=1}^{2}\widetilde{a}_l\big(\varphi(\cdot,q),\partial_l\widetilde{\phi}\big)+\frac{k}{\sigma}\sum_{l=1}^{2}\big(i\partial_l\widetilde{\phi},D_l^{(0)}\widetilde{\phi}\big)+\frac{k}{\sigma}\sum_{l=1}^{2}\big(i\widetilde{\phi},\partial_l D_l^{(0)}\widetilde{\phi}\big)\Big) \\
&+\varepsilon^2\Big(\sum_{\mu=1}^{2N}\big(\widetilde{\phi},i\frac{\partial\varphi}{\partial q_\mu}(\cdot,q)\big)\dot{q}_\mu-\frac{2k}{\sigma}\big(\varphi(\cdot,q),\widetilde{\phi}\big)\big(i\varphi(\cdot,q),\widetilde{\phi}\big) \\
&-\frac{2k}{\sigma}\sum_{l=1}^{2}\widetilde{a}_l\big(\widetilde{\phi},\partial_l\widetilde{\phi}\big)\Big)-\varepsilon^3\frac{k}{\sigma}|\widetilde{\phi}|^2\big(i\varphi(\cdot,q),\widetilde{\phi}\big).
\end{aligned}
\tag{2.15}
$$

Putting the *ansatz* into (2.3), one obtains

$$
\begin{aligned}
&\partial_t(\varphi+\varepsilon\widetilde{\phi})-i\varepsilon\widetilde{\widetilde{a}}_0(\varphi+\varepsilon\widetilde{\phi}) \\
&\quad-\sum_{l=1}^{2}\big(\partial_l-i(\alpha_l+\varepsilon\widetilde{a}_l)\big)^2(\varphi+\varepsilon\widetilde{\phi})=\frac{1+\varepsilon}{2}(1-|\varphi+\varepsilon\widetilde{\phi}|^2)(\varphi+\varepsilon\widetilde{\phi}).
\end{aligned}
\tag{2.16}
$$

After eliminating the terms involved in the static Ginzburg-Landau equation for the Higgs field, (2.16) becomes

$$
\begin{aligned}
&\partial_t\widetilde{\phi} - \sum_{l=1}^{2}\big(D_l^{(0)}\big)^2\widetilde{\phi} + i2\sum_{l=1}^{2}\widetilde{a}_l D_l^{(0)}\varphi \\
&\quad + |\phi|^2\widetilde{\phi} - \frac{1}{2}\widetilde{\phi}(1-|\phi|^2) - \frac{1}{2}|\phi|^2\widetilde{\phi} + \frac{1}{2}\varphi^2\overline{\widetilde{\phi}} \\
&\quad - i\varphi\sum_{l=1}^{2}\partial_l\widetilde{a}_l = -\partial_\tau\varphi + \frac{1}{2}\varphi(1-|\phi|^2) + i\widetilde{\widetilde{a}}_0\varphi + \varepsilon j_3
\end{aligned}
\tag{2.17}
$$

with

$$
\begin{aligned}
j_3(\widetilde{a},\widetilde{\phi},\widetilde{\widetilde{a}}_0,q) := &-i\widetilde{\phi}\sum_{l=1}^{2}\partial_l\widetilde{a}_l - i2\sum_{l=1}^{2}\widetilde{a}_l D_l^{(0)}\widetilde{\phi} - i\varphi(\cdot,q)\sum_{l=1}^{2}\widetilde{a}_l^2 \\
&- i\varepsilon\widetilde{\phi}\sum_{l=1}^{2}\widetilde{a}_l^2 + \frac{1}{2}\widetilde{\phi}(1-|\phi(\cdot,q)|^2) - (1+\varepsilon)\widetilde{\phi}\big(\varphi(\cdot,q),\widetilde{\phi}\big) \\
&- \varphi(\cdot,q)\big(\varphi(\cdot,q),\widetilde{\phi}\big) - \frac{1+\varepsilon}{2}|\widetilde{\phi}|^2\varphi(\cdot,q) - \frac{\varepsilon}{2}(1+\varepsilon)|\widetilde{\phi}|^2\widetilde{\phi} \\
&+ i\widetilde{\widetilde{a}}_0\widetilde{\phi}.
\end{aligned}
\tag{2.18}
$$

Putting the *ansatz* into (2.4), one obtains

$$
\begin{aligned}
&\sigma\partial_t(\alpha_h + \varepsilon\widetilde{a}_h) - \Delta(\alpha_h + \varepsilon\widetilde{a}_h) - \varepsilon\sigma\partial_h\widetilde{\widetilde{a}}_0 \\
&\quad + \sum_{l=1}^{2}\partial_{hl}^2(\alpha_l + \varepsilon\widetilde{a}_l) = \big(i(\varphi+\varepsilon\widetilde{\phi}), (\partial_h - i\alpha_h - i\varepsilon\widetilde{a}_h)(\varphi+\varepsilon\widetilde{\phi})\big).
\end{aligned}
\tag{2.19}
$$

Again eliminating the terms involved in the static Ginzburg-Landau equation for the vector potential and using condition (2.7), (2.19) becomes

$$
\begin{aligned}
&\sigma\partial_t\widetilde{a}_h - \Delta\widetilde{a}_h + |\phi|^2\widetilde{a}_h - 2\big(i\widetilde{\phi}, D_h^{(0)}\varphi\big) \\
&\quad - \big(i\partial_h\varphi,\widetilde{\phi}\big) - \big(i\varphi,\partial_h\widetilde{\phi}\big) = -\sigma\partial_\tau\alpha_h + \sigma\partial_h\widetilde{\widetilde{a}}_0 + \varepsilon j_h
\end{aligned}
\tag{2.20}
$$

with

$$
\begin{aligned}
j_h(\widetilde{a},\widetilde{\phi},q) :=& -2\widetilde{a}_h\big(\varphi(\cdot,q),\widetilde{\phi}\big) + \big(i\widetilde{\phi}, D_h^{(0)}\widetilde{\phi}\big) \\
& - \varepsilon|\widetilde{\phi}|^2\widetilde{a}_h - \big(i\partial_h\varphi(\cdot,q),\widetilde{\phi}\big) - \big(i\varphi(\cdot,q),\partial_h\widetilde{\phi}\big).
\end{aligned} \tag{2.21}
$$

The equations (2.17) and (2.20) can be rewritten in the form

$$
\begin{aligned}
\partial_t \begin{pmatrix} \sigma\widetilde{a} \\ \widetilde{\phi} \end{pmatrix} + L^{(\alpha,\varphi)} \begin{pmatrix} \widetilde{a} \\ \widetilde{\phi} \end{pmatrix} =& - \begin{pmatrix} \sigma\partial_\tau\alpha \\ \partial_\tau\varphi \end{pmatrix} \\
& + \begin{pmatrix} 0 \\ \frac{1}{2}\varphi(1-|\phi|^2) \end{pmatrix} + \begin{pmatrix} \sigma\nabla\widetilde{\widetilde{a}}_0 \\ i\varphi\widetilde{\widetilde{a}}_0 \end{pmatrix} + \varepsilon j
\end{aligned} \tag{2.22}
$$

with

$$
\widetilde{a} := \begin{pmatrix} \widetilde{a}_1 \\ \widetilde{a}_2 \end{pmatrix}, \quad \alpha := \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \quad j := \begin{pmatrix} j_1 \\ j_2 \\ j_3 \end{pmatrix}
$$

and

$$
\begin{aligned}
L^{(\alpha,\varphi)} \begin{pmatrix} \widetilde{a} \\ \widetilde{\phi} \end{pmatrix} :=& \begin{pmatrix} -\Delta\widetilde{a}_1 + |\phi|^2\widetilde{a}_1 \\ \\ -\Delta\widetilde{a}_2 + |\phi|^2\widetilde{a}_2 \\ \\ -\sum_{l=1}^2 (D_l^{(0)})^2\widetilde{\phi} + |\phi|^2\widetilde{\phi} \end{pmatrix} \\
& + \begin{pmatrix} -2\big(i\widetilde{\phi}, D_1^{(0)}\varphi\big) - \partial_1\big(i\varphi,\widetilde{\phi}\big) \\ \\ -2\big(i\widetilde{\phi}, D_2^{(0)}\varphi\big) - \partial_2\big(i\varphi,\widetilde{\phi}\big) \\ \\ i2\sum_{l=1}^2 \widetilde{a}_l D_l^{(0)}\varphi - \frac{1}{2}\widetilde{\phi}(1-|\phi|^2) - i\varphi\big(i\varphi,\widetilde{\phi}\big) \end{pmatrix}.
\end{aligned}
$$

Now two important properties of the operator $L^{(\alpha,\varphi)}$ are pointed out.

Property 2.3.1 *The operator $L^{(\alpha,\varphi)}$ is self-adjoint on a domain $\mathcal{D}_1 \cap \mathcal{D}_2$, where*

$$\mathcal{D}_1 := \left\{(\widetilde{a}, \widetilde{\phi}) \in H^{3,\alpha(\cdot,q(0))}(\mathbb{R}^2) \mid \nabla \cdot \widetilde{a} - \varepsilon\big(i\varphi, \widetilde{\phi}\big) = 0\right\}$$

and

$$\mathcal{D}_2 := \left\{(\widetilde{a}, \widetilde{\phi}) \in H^{3,\alpha(\cdot,q(0))}(\mathbb{R}^2) \mid \quad |\widetilde{a}(x,\cdot)|, |\widetilde{\phi}(x,\cdot)| \leq Ce^{-m|x|} \quad \exists C, m > 0\right\}.$$

In particular

$$\langle L^{(\alpha,\varphi)} \begin{pmatrix} \widetilde{a} \\ \widetilde{\phi} \end{pmatrix}, \begin{pmatrix} n_\mu^1 \\ n_\mu^2 \end{pmatrix} \rangle = \langle \begin{pmatrix} \widetilde{a} \\ \widetilde{\phi} \end{pmatrix}, L^{(\alpha,\varphi)} \begin{pmatrix} n_\mu^1 \\ n_\mu^2 \end{pmatrix} \rangle \tag{2.23}$$

holds, where

$$n_\mu^1 := \begin{pmatrix} \frac{\partial \alpha_1}{\partial q_\mu} \\ \frac{\partial \alpha_2}{\partial q_\mu} \end{pmatrix}, \qquad n_\mu^2 := \frac{\partial \varphi}{\partial q_\mu}$$

are the $2N$ so-called zero modes.

Since $n_\mu := \begin{pmatrix} n_\mu^1 \\ n_\mu^2 \end{pmatrix} \in \mathcal{D}_1 \cap \mathcal{D}_2$, once we prove (2.23), the self-adjointness of $L^{(\alpha,\varphi)}$ is proven on $\mathcal{D}_1 \cap \mathcal{D}_2$. In fact

$$\langle L^{(\alpha,\varphi)} \begin{pmatrix} \widetilde{a} \\ \widetilde{\phi} \end{pmatrix}, \begin{pmatrix} n_\mu^1 \\ n_\mu^2 \end{pmatrix} \rangle = \sum_{l=1}^{2} \int_{\mathbb{R}^2} \big(-\Delta \widetilde{a}_l(x,t) + |\phi|^2 \widetilde{a}_l(x,t)\big) \frac{\partial \alpha_l}{\partial q_\mu}(x,q) d^2x$$
$$-2\sum_{l=1}^{2} \int_{\mathbb{R}^2} \big(i\widetilde{\phi}(x,t), D_l^{(0)}\varphi(x,q)\big) \frac{\partial \alpha_l}{\partial q_\mu}(x,q) d^2x$$

$$
\begin{aligned}
&- \sum_{l=1}^{2} \int_{\mathbb{R}^2} \frac{\partial \alpha_l}{\partial q_\mu}(x,q) \partial_l \big(i\varphi(x,q), \widetilde{\phi}(x,t) \big) d^2x \\
&- \sum_{l=1}^{2} \int_{\mathbb{R}^2} \big((D_l^{(0)})^2 \widetilde{\phi}(x,t), \frac{\partial \varphi}{\partial q_\mu}(x,q) \big) d^2x \\
&+ \int_{\mathbb{R}^2} |\phi(x,q)|^2 \big(\widetilde{\phi}(x,t), \frac{\partial \varphi}{\partial q_\mu}(x,q) \big) d^2x \\
&+ 2 \sum_{l=1}^{2} \int_{\mathbb{R}^2} \widetilde{a}_l(x,t) \big(iD_l^{(0)} \varphi(x,q), \frac{\partial \varphi}{\partial q_\mu}(x,q) \big) d^2x \\
&- \int_{\mathbb{R}^2} \frac{1}{2}(1 - |\phi(x,q)|^2) \big(\widetilde{\phi}(x,t), \frac{\partial \varphi}{\partial q_\mu}(x,q) \big) d^2x \\
&- \int_{\mathbb{R}^2} \big(i\varphi(x,q), \widetilde{\phi}(x,t) \big) \big(i\varphi(x,q), \frac{\partial \varphi}{\partial q_\mu}(x,q) \big) d^2x \\
&= \sum_{l=1}^{2} \int_{\mathbb{R}^2} \widetilde{a}_l(x,t) \big(-\Delta \frac{\partial \alpha_l}{\partial q_\mu}(x,q) + |\phi(x,q)|^2 \frac{\partial \alpha_l}{\partial q_\mu}(x,q) \big) d^2x \\
&+ 2 \sum_{l=1}^{2} \int_{\mathbb{R}^2} \frac{\partial \alpha_l}{\partial q_\mu}(x,q) \big(iD_l^{(0)} \varphi(x,q), \widetilde{\phi}(x,t) \big) d^2x \\
&+ \sum_{l=1}^{2} \int_{\mathbb{R}^2} \big(i\varphi(x,q), \widetilde{\phi}(x,t) \big) \partial_l \frac{\partial \alpha_l}{\partial q_\mu}(x,q) d^2x \\
&- \sum_{l=1}^{2} \int_{\mathbb{R}^2} \partial_{ll}^2 \big(\widetilde{\phi}(x,t), \frac{\partial \varphi}{\partial q_\mu}(x,q) \big) d^2x \\
&+ 2 \sum_{l=1}^{2} \int_{\mathbb{R}^2} \partial_l \big(D_l^{(0)} \frac{\partial \varphi}{\partial q_\mu}(x,q), \widetilde{\phi}(x,t) \big) d^2x \\
&- \sum_{l=1}^{2} \int_{\mathbb{R}^2} \big((D_l^{(0)})^2 \frac{\partial \varphi}{\partial q_\mu}(x,q), \widetilde{\phi}(x,t) \big) d^2x \\
&+ \int_{\mathbb{R}^2} |\phi(x,q)|^2 \big(\frac{\partial \varphi}{\partial q_\mu}(x,q), \widetilde{\phi}(x,t) \big) d^2x \\
&- 2 \sum_{l=1}^{2} \int_{\mathbb{R}^2} \widetilde{a}_l(x,t) \big(i \frac{\partial \varphi}{\partial q_\mu}(x,q), D_l^{(0)} \varphi(x,q) \big) d^2x \\
&- \int_{\mathbb{R}^2} \frac{1}{2}(1 - |\phi(x,q)|^2) \big(\frac{\partial \varphi}{\partial q_\mu}(x,q), \widetilde{\phi}(x,t) \big) d^2x
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{\mathbb{R}^2} \big(i\varphi(x,q), \frac{\partial \varphi}{\partial q_\mu}(x,q)\big)\big(i\varphi(x,q), \widetilde{\phi}(x,t)\big) d^2x \\
&= \langle \begin{pmatrix} \widetilde{a} \\ \widetilde{\phi} \end{pmatrix}, L^{(\alpha,\varphi)} \begin{pmatrix} n^1_\mu \\ n^2_\mu \end{pmatrix} \rangle,
\end{aligned}
$$

where the last equality is obtained by noticing that from (2.12) it follows

$$
(i\varphi, \widetilde{\phi}) \sum_{l=1}^{2} \partial_l \frac{\partial \alpha_l}{\partial q_\mu} = \sum_{l=1}^{2} \widetilde{a}_l \partial_l (i\varphi, \frac{\partial \varphi}{\partial q_\mu}) = 0
$$

and by using the exponential decay at infinity of the zero modes and their spatial partial derivatives.

Property 2.3.2 *The operator $L^{(\alpha,\varphi)}$ vanishes on the zero modes.*

In fact, differentiating the Ginzburg-Landau equations with respect to q_μ and knowing that

$$
\nabla \cdot \frac{\partial \alpha}{\partial q_\mu} = 0
$$

for any $\mu \in \{1, \ldots, 2N\}$, one obtains

$$
\begin{aligned}
&- \sum_{l=1}^{2} (D_l^{(0)})^2 \frac{\partial \varphi}{\partial q_\mu} + |\phi|^2 \frac{\partial \varphi}{\partial q_\mu} - \frac{1}{2} \frac{\partial \varphi}{\partial q_\mu}(1 - |\phi|^2) \\
&\qquad + i2 \sum_{l=1}^{2} \frac{\partial \alpha_l}{\partial q_\mu} D_l^{(0)} \varphi - i\varphi(i\varphi, \frac{\partial \varphi}{\partial q_\mu}) = 0,
\end{aligned}
$$

$$
-\Delta \frac{\partial \alpha_h}{\partial q_\mu} + |\phi|^2 \frac{\partial \alpha_h}{\partial q_\mu} - 2(i\frac{\partial \varphi}{\partial q_\mu}, D_h^{(0)} \varphi) - \partial_h (i\varphi, \frac{\partial \varphi}{\partial q_\mu}) = 0
$$

for $h = 1, 2$, namely

$$
L^{(\alpha,\varphi)} \begin{pmatrix} \frac{\partial \alpha}{\partial q_\mu} \\ \frac{\partial \varphi}{\partial q_\mu} \end{pmatrix} = 0 \tag{2.24}
$$

for any $\mu \in \{1, \ldots, 2N\}$. Also note the following:

Remark 2.3.1 *The operator $L^{(\alpha,\varphi)}$ can be written as follows*

$$L^{(\alpha,\varphi)}\begin{pmatrix}\widetilde{a}\\ \widetilde{\phi}\end{pmatrix} = L_0^{(\alpha,\varphi)}\begin{pmatrix}\sigma\widetilde{a}\\ \widetilde{\phi}\end{pmatrix} + V^{(\alpha,\varphi)}\begin{pmatrix}\sigma\widetilde{a}\\ \widetilde{\phi}\end{pmatrix}, \tag{2.25}$$

with

$$L_0^{(\alpha,\varphi)}\begin{pmatrix}\sigma\widetilde{a}\\ \widetilde{\phi}\end{pmatrix} := \begin{pmatrix} -\frac{1}{\sigma}\Delta(\sigma\widetilde{a}_1) + \frac{1}{\sigma}|\phi|^2(\sigma\widetilde{a}_1)\\ -\frac{1}{\sigma}\Delta(\sigma\widetilde{a}_2) + \frac{1}{\sigma}|\phi|^2(\sigma\widetilde{a}_2)\\ -\sum_{l=1}^{2}(D_l^{(0)})^2\widetilde{\phi} + |\phi|^2\widetilde{\phi}\end{pmatrix} \tag{2.26}$$

and

$$V^{(\alpha,\varphi)}\begin{pmatrix}\sigma\widetilde{a}\\ \widetilde{\phi}\end{pmatrix} := \begin{pmatrix} -2\big(i\widetilde{\phi}, D_1^{(0)}\varphi\big) - \partial_1\big(i\varphi, \widetilde{\phi}\big)\\ -2\big(i\widetilde{\phi}, D_2^{(0)}\varphi\big) - \partial_2\big(i\varphi, \widetilde{\phi}\big)\\ i\frac{2}{\sigma}\sum_{l=1}^{2}(\sigma\widetilde{a}_l)D_l^{(0)}\varphi - \frac{1}{2}\widetilde{\phi}(1-|\phi|^2) - i\varphi\big(i\varphi, \widetilde{\phi}\big)\end{pmatrix}. \tag{2.27}$$

2.4 The Slow-Motion Approximation

The idea of the Slow-Motion Approximation is presented by using a particular distance in the space of the solutions. The idea is to minimize the distance between the solution of the Gor'kov-Éliashberg equations and a certain $2N$-parameter family of static solutions of the Ginzburg-Landau equations. It is shown that this is equivalent to imposing another condition on $(\widetilde{a}, \widetilde{\phi})$ in the *ansatz*. Also a condition for $\widetilde{\widetilde{a}}_0$ is derived. Finally, a system of O.D.E.s for the parameters is obtained.

Given the static solution of the Ginzburg-Landau equations (α, φ), the Slow-Motion Approximation amounts to finding the solution of the Gor'kov-

Éliashberg equations $(\vec{A}, \Phi)$ which minimizes the distance

$$\int_{\mathbb{R}^2} \big(\sigma |\vec{A}(x,t) - \alpha(x,q)|^2 + |\Phi(x,t) - \varphi(x,q)|^2\big) d^2x$$

in the moduli space of the parameters q. The choice of this distance is inspired by the Abelian-Higgs case, [Stu94,I]. Since a constant σ appears in (2.1) and (2.2), we need to give this definition of distance in order to be able to find an approximate O.D.E. system for q. At the minimum

$$\frac{\partial}{\partial q_\mu} \int_{\mathbb{R}^2} \big(\sigma |\vec{A}(x,t) - \alpha(x,q)|^2 + |\Phi(x,t) - \varphi(x,q)|^2\big) d^2x = 0$$

must hold for any $\mu \in \{1, \ldots, 2N\}$, which is equivalent to

$$\langle \begin{pmatrix} \sigma \widetilde{a}(\cdot, t) \\ \widetilde{\phi}(\cdot, t) \end{pmatrix}, \begin{pmatrix} n_\mu^1(\cdot, q(t)) \\ n_\mu^2(\cdot, q(t)) \end{pmatrix} \rangle = 0 \tag{2.28}$$

for any $\mu \in \{1, \ldots, 2N\}$.

Moreover, for any $\mu \in \{1, \ldots, 2N\}$, we have

$$\langle \begin{pmatrix} \sigma \nabla \widetilde{\widetilde{a}}_0(\cdot, t) \\ i\widetilde{\widetilde{a}}_0(\cdot, t)\varphi(\cdot, q(t)) \end{pmatrix}, \begin{pmatrix} n_\mu^1(\cdot, q(t)) \\ n_\mu^2(\cdot, q(t)) \end{pmatrix} \rangle = \int_{\mathbb{R}^2} \Big(\sigma \sum_{l=1}^{2} \partial_l \widetilde{\widetilde{a}}_0(x,t) \frac{\partial \alpha_l}{\partial q_\mu}(x, q(t))$$
$$+ \big(i\widetilde{\widetilde{a}}_0(x,t)\varphi(x,q(t)), \frac{\partial \varphi}{\partial q_\mu}(x, q(t))\big)\Big) d^2x$$
$$= - \int_{\mathbb{R}^2} \widetilde{\widetilde{a}}_0(x,t) \Big(\sigma \sum_{l=1}^{2} \partial_l \frac{\partial \alpha_l}{\partial q_\mu}(x, q(t)) - \big(i\varphi(x,q(t)), \frac{\partial \varphi}{\partial q_\mu}(x, q(t))\big)\Big) d^2x.$$

Then, using condition (2.8), one obtains

$$\langle \begin{pmatrix} \sigma \nabla \widetilde{\widetilde{a}}_0 \\ i\widetilde{\widetilde{a}}_0 \varphi \end{pmatrix}, \begin{pmatrix} n_\mu^1 \\ n_\mu^2 \end{pmatrix} \rangle = 0 \tag{2.29}$$

for any $\mu \in \{1, \dots, 2N\}$. Also, differentiating (2.28) with respect to t, one obtains

$$\langle \partial_t \begin{pmatrix} \sigma\widetilde{a} \\ \widetilde{\phi} \end{pmatrix}, \begin{pmatrix} n_\mu^1 \\ n_\mu^2 \end{pmatrix} \rangle = -\varepsilon \langle \begin{pmatrix} \sigma\widetilde{a} \\ \widetilde{\phi} \end{pmatrix}, \partial_\tau \begin{pmatrix} n_\mu^1 \\ n_\mu^2 \end{pmatrix} \rangle. \tag{2.30}$$

Now a system of O.D.E.s for the parameter q is derived from the equations, using the conditions mentioned so far. In fact, from (2.22)

$$\begin{aligned} &\langle \partial_t \begin{pmatrix} \sigma\widetilde{a} \\ \widetilde{\phi} \end{pmatrix}, \begin{pmatrix} n_\mu^1 \\ n_\mu^2 \end{pmatrix} \rangle + \langle L^{(\alpha,\varphi)} \begin{pmatrix} \widetilde{a} \\ \widetilde{\phi} \end{pmatrix}, \begin{pmatrix} n_\mu^1 \\ n_\mu^2 \end{pmatrix} \rangle \\ &= -\langle \begin{pmatrix} \sigma\partial_\tau\alpha \\ \partial_\tau\varphi \end{pmatrix}, \begin{pmatrix} n_\mu^1 \\ n_\mu^2 \end{pmatrix} \rangle + \langle \begin{pmatrix} 0 \\ \frac{1}{2}\varphi(1-|\phi|^2) \end{pmatrix}, \begin{pmatrix} n_\mu^1 \\ n_\mu^2 \end{pmatrix} \rangle \\ &+ \langle \begin{pmatrix} \sigma\nabla\widetilde{\widetilde{a}}_0 \\ i\widetilde{\widetilde{a}}_0\varphi \end{pmatrix}, \begin{pmatrix} n_\mu^1 \\ n_\mu^2 \end{pmatrix} \rangle + \varepsilon\langle j, n_\mu \rangle \end{aligned}$$

follows. Then, using (2.23), (2.24), (2.29) and (2.30), one obtains

$$\begin{aligned} &\sum_{\nu=1}^{2N} \Big[\langle \begin{pmatrix} \sigma\frac{\partial\alpha}{\partial q_\nu}(\cdot, q(t)) \\ \frac{\partial\varphi}{\partial q_\nu}(\cdot, q(t)) \end{pmatrix}, \begin{pmatrix} n_\mu^1(\cdot, q(t)) \\ n_\mu^2(\cdot, q(t)) \end{pmatrix} \rangle \\ &\quad - \varepsilon \langle \begin{pmatrix} \sigma\widetilde{a}(\cdot, t) \\ \widetilde{\phi}(\cdot, t) \end{pmatrix}, \begin{pmatrix} \frac{\partial n_\mu^1}{\partial q_\nu}(\cdot, q(t)) \\ \frac{\partial n_\mu^2}{\partial q_\nu}(\cdot, q(t)) \end{pmatrix} \rangle \Big] \dot{q}_\nu(t) \\ &\quad = \frac{1}{2}\int_{\mathbb{R}^2} \big(\varphi(x, q(t))(1 - |\phi(x, q(t))|^2), n_\mu^2(x, q(t))\big) d^2x \\ &\quad + \varepsilon \langle j(\widetilde{a}(\cdot, t), \widetilde{\phi}(\cdot, t), \widetilde{\widetilde{a}}_0(\cdot, t), q(t)), n_\mu(\cdot, q(t)) \rangle \end{aligned} \tag{2.31}$$

for any $\mu \in \{1, \dots, 2N\}$. To leading order,

$$\begin{aligned} &\sum_{\nu=1}^{2N} \langle \begin{pmatrix} \sigma\frac{\partial\alpha}{\partial q_\nu}(\cdot, q(t)) \\ \frac{\partial\varphi}{\partial q_\nu}(\cdot, q(t)) \end{pmatrix}, \begin{pmatrix} n_\mu^1(\cdot, q(t)) \\ n_\mu^2(\cdot, q(t)) \end{pmatrix} \rangle \dot{q}_\nu(t) \\ &\quad = \frac{1}{2}\int_{\mathbb{R}^2} \big(\varphi(x, q(t))(1 - |\phi(x, q(t))|^2), n_\mu^2(x, q(t))\big) d^2x \end{aligned}$$

for any $\mu \in \{1, \ldots, 2N\}$.

2.5 The Metric

The matrix in the left-hand side of the O.D.E. system (2.31) for q is the sum of a part depending only on the static solution and another part of order ε. The first part is associated with the metric in the moduli space of the parameters q and it is shown to be invertible. The metric is examined away from the zeros of the Higgs field. All the results will extend by continuity to all the points of the moduli space. The entire procedure in studying this metric is based on work by Samols, [Sam92]. The invertibility of the first part and the small correction given by the second part to the determinant of the first matrix guarantee the invertibility of the complete matrix.

The invertibility of the matrix

$$A(\widetilde{a}, \widetilde{\phi}, q) = \left(a_{\mu\nu}(\widetilde{a}, \widetilde{\phi}, q)\right)_{\mu,\nu \in \{1,\ldots,2N\}},$$

with

$$\begin{aligned} a_{\mu\nu}(\widetilde{a}, \widetilde{\phi}, q) := & \langle \begin{pmatrix} \sigma \frac{\partial \alpha}{\partial q_\nu}(\cdot, q) \\ \frac{\partial \varphi}{\partial q_\nu}(\cdot, q) \end{pmatrix}, \begin{pmatrix} n_\mu^1(\cdot, q) \\ n_\mu^2(\cdot, q) \end{pmatrix} \rangle \\ & - \varepsilon \langle \begin{pmatrix} \sigma \widetilde{a} \\ \widetilde{\phi} \end{pmatrix}, \begin{pmatrix} \frac{\partial n_\mu^1}{\partial q_\nu}(\cdot, q) \\ \frac{\partial n_\mu^2}{\partial q_\nu}(\cdot, q) \end{pmatrix} \rangle, \end{aligned}$$

must be proven. First of all the metric associated with

$$A'(q) = \left(\langle \begin{pmatrix} \sigma \frac{\partial \alpha}{\partial q_\nu}(\cdot, q) \\ \frac{\partial \varphi}{\partial q_\nu}(\cdot, q) \end{pmatrix}, \begin{pmatrix} n_\mu^1(\cdot, q) \\ n_\mu^2(\cdot, q) \end{pmatrix} \rangle \right)_{\mu,\nu \in \{1,\ldots,2N\}}$$

is examined for $q \in \mathbb{R}^{2N} \setminus Z_\phi$.

After defining

$$\alpha^* := \frac{1}{2}(\alpha_1 - i\alpha_2) \quad \text{and} \quad \eta := \frac{\bar{\varphi}}{|\varphi|^2}\partial_t\varphi,$$

the first Bogomol'nyi equation can be rewritten as

$$\alpha^* = i\partial_z \ln\bar{\varphi},$$

with $\partial_z := \frac{1}{2}(\partial_1 - i\partial_2)$ [JT80, p. 61], and then

$$\partial_t\alpha^* = i\partial_z\bar{\eta}.$$

From $u = \ln|\varphi|^2$ one obtains

$$\partial_t u = 2\mathcal{R}e\,\eta.$$

Moreover

$$\partial_t\bar{\alpha}^* = \frac{1}{2}\big(\partial_1\mathcal{I}m\,\eta + \partial_2\mathcal{R}e\,\eta + i(\partial_2\mathcal{I}m\,\eta - \partial_1\mathcal{R}e\,\eta)\big)$$

and then, because of

$$\partial_t\alpha_1 = \partial_1\mathcal{I}m\,\eta + \partial_2\mathcal{R}e\,\eta \quad \text{and} \quad \partial_t\alpha_2 = \partial_2\mathcal{I}m\,\eta - \partial_1\mathcal{R}e\,\eta,$$

we have

$$\nabla\cdot\partial_t\alpha = \Delta\mathcal{I}m\,\eta.$$

Therefore equation (2.8) can be written

$$\sigma\Delta\mathcal{I}m\,\eta - e^u\mathcal{I}m\,\eta = 0. \tag{2.32}$$

After differentiating with respect to t the second Bogomol'nyi equation for u on $\mathbb{R}^{2N} \setminus Z_\phi$

$$-\Delta u + e^u - 1 = 0 \tag{2.33}$$

yields

$$\Delta \mathcal{R}e\,\eta - e^u \mathcal{R}e\,\eta = 0. \tag{2.34}$$

We now define S_ϱ as a set of nonoverlapping discs of radius ϱ centred at $z \in Z_\phi$ and use it in the following calculation:

$$\begin{aligned}
T &:= \sum_{\mu,\nu=1}^{2N} \int_{\mathbb{R}^2} \Big(\sigma \sum_{l=1}^{2} \frac{\partial \alpha_l}{\partial q_\mu}(x,q) \frac{\partial \alpha_l}{\partial q_\nu}(x,q) + \big(\frac{\partial \varphi}{\partial q_\mu}(x,q), \frac{\partial \varphi}{\partial q_\nu}(x,q)\big)\Big) d^2x \frac{dq_\mu}{dt}\frac{dq_\nu}{dt} \\
&= \int_{\mathbb{R}^2} \big(4\sigma \partial_t \alpha^*(x,q) \partial_t \bar{\alpha}^*(x,q) + |\partial_t \varphi(x,q)|^2\big) d^2x \\
&= \int_{\mathbb{R}^2 \setminus S_\varrho} \big(4\sigma \partial_z \bar{\eta}(x,q) \partial_{\bar{z}} \eta(x,q) + |\varphi(x,q)|^2 \eta(x,q) \bar{\eta}(x,q)\big) d^2x \\
&+ \int_{S_\varrho} \big(4\sigma \partial_t \alpha^*(x,q) \partial_t \bar{\alpha}^*(x,q) + |\partial_t \varphi(x,q)|^2\big) d^2x \\
&= 4\sigma \mathcal{R}e \int_{\mathbb{R}^2 \setminus S_\varrho} \partial_z \big(\bar{\eta}(x,q) \partial_{\bar{z}} \eta(x,q)\big) d^2x \\
&+ \mathcal{R}e \int_{\mathbb{R}^2 \setminus S_\varrho} \big(-\sigma \Delta \eta(x,q) + |\varphi(x,q)|^2 \eta(x,q)\big) \bar{\eta}(x,q) d^2x, \\
&+ \mathcal{R}e \int_{S_\varrho} \big(4\sigma \partial_t \alpha^*(x,q) \partial_t \bar{\alpha}^*(x,q) + |\partial_t \varphi(x,q)|^2\big) d^2x
\end{aligned} \tag{2.35}$$

with ∂_z defined above in this section and

$$\partial_{\bar{z}} := \frac{1}{2}(\partial_1 + i\partial_2).$$

Since the integrand of T is smooth the last part of (2.35) is $\mathcal{O}(N\varrho^2)$, which vanishes as $\varrho \to 0$. For

$$\mathcal{R}e \int_{\mathbb{R}^2\setminus S_\varrho} \partial_z\big(\bar{\eta}(x,q)\partial_{\bar{z}}\eta(x,q)\big)d^2x$$

the same arguments as those in [Sam92, section 4.2] are applied. There it was shown that, by using the divergence theorem only the contributions from the neighbourhoods of the zeros S_ϱ are left so that

$$\int_{\mathbb{R}^2\setminus S_\varrho} \partial_z\big(\bar{\eta}(x,q)\partial_{\bar{z}}\eta(x,q)\big)d^2x = -\frac{i}{2}\int_{\partial S_\varrho} \bar{\eta}(x,q)\partial_{\bar{z}}\eta(x,q)d\bar{z}. \tag{2.36}$$

We assume that the zeros of the Higgs field $\{Z_1,\ldots,Z_N\}$ are distinct, so all have multiplicity one. In a neighbourhood of Z_r from $\varphi(z) = (z - Z_r)h_r(z)$, [JT80], it follows that

$$u(z) = \ln|\varphi(z)|^2 = \ln|z - Z_r|^2 + \text{smooth} \tag{2.37}$$

and, using the definition of η, that

$$\eta(z) = -\frac{1}{z - Z_r}\frac{dZ_r}{dt}. \tag{2.38}$$

Writing $u = \ln|\varphi|^2$ (2.13) becomes

$$\Delta\mathcal{I}m\,\eta - e^u\mathcal{I}m\,\eta = 0. \tag{2.39}$$

Adding (2.34) to (2.39) it turns out that, for $z \neq Z_r$,

$$\Delta\eta - e^u\eta = 0$$

which, extended to all points of $\mathbb{R}^2$, becomes

$$\Delta\eta - e^u\eta = -4\pi\sum_{r=1}^{N}\frac{dZ_r}{dt}\partial_z\delta(z - Z_r). \tag{2.40}$$

Equation (2.33) extended to all points of $\mathbb{R}^2$ becomes

$$-\Delta u + e^u - 1 = 4\pi \sum_{r=1}^{N} \delta(z - Z_r). \tag{2.41}$$

Note that (2.41) is the equation analysed by Taubes in his proof of the existence of vortex solutions.

The solutions of (2.40) and (2.41) may be related in a simple way. Differentiating (2.41) with respect to Z_r gives

$$\Delta \frac{\partial u}{\partial Z_r} - e^u \frac{\partial u}{\partial Z_r} = -4\pi \partial_z \delta(z - Z_r). \tag{2.42}$$

Turning now to the boundary conditions on u and η, we note that the finiteness of the static energy requires

$$\lim_{|z| \to +\infty} u(z) = 0$$

and the fact that a metric on the moduli space of the parameters $Z_1, \ldots, Z_N$ has to be well-defined everywhere, [Wei79], requires

$$\lim_{|z| \to +\infty} \eta(z) = 0.$$

Thus by the linearity of (2.40), and for the boundary conditions on u and η just mentioned,

$$\eta = \sum_{r=1}^{N} \frac{dZ_r}{dt} \frac{\partial u}{\partial Z_r}. \tag{2.43}$$

Using successively (2.38) and (2.43), and neglecting terms of $\mathcal{O}(\varrho)$, the right-hand side of (2.36) can be rewritten as

$$\sum_{s=1}^{N} \int_0^{2\pi} \frac{d\bar{Z}_s}{dt} \partial_{\bar{z}} \eta d\theta_s = \sum_{r,s=1}^{N} \int_0^{2\pi} \partial_{\bar{z}} \frac{\partial u}{\partial Z_r} \frac{dZ_r}{dt} \frac{d\bar{Z}_s}{dt} d\theta_s, \tag{2.44}$$

where for each s the integration is around the circle $|z - Z_s| = \varrho$. Now, near Z_s, a Taylor expansion of the smooth part of u in (2.37) gives

$$u(z) = \ln|z - Z_s|^2 + a_s + \frac{1}{2}\{b_s(z - Z_s) + \bar{b}_s(\bar{z} - \bar{Z}_s)\} \\ + c_s(z - Z_s)^2 + d_s(z - Z_s)(\bar{z} - \bar{Z}_s) + \bar{c}_s(\bar{z} - \bar{Z}_s)^2 + \mathcal{O}(\varrho^3), \tag{2.45}$$

where to satisfy (2.41) we require

$$d_s = -\frac{1}{4}.$$

Hence for z near but not equal to Z_s, it follows that

$$\partial_{\bar{z}} \frac{\partial u}{\partial Z_r} = \frac{1}{2}\frac{\partial \bar{b}_s}{\partial Z_r} + \frac{1}{4}\delta_{rs} + \mathcal{O}(\varrho).$$

Substituting into (2.44) we obtain finally

$$\int_{\mathbb{R}^2 \setminus S_\varrho} \partial_z \big(\bar{\eta}(x, q)\partial_{\bar{z}}\eta(x, q)\big) d^2x = \frac{\pi}{4}\sum_{r,s=1}^{N} \left(\delta_{rs} + 2\frac{\partial \bar{b}_s}{\partial Z_r}\right)\frac{dZ_r}{dt}\frac{d\bar{Z}_s}{dt}.$$

For a single vortex ($N = 1$), u is rotationally symmetric and the coefficient b_1 of the linear term in (2.45) vanishes. When there is more than one vortex, the b_r in the Taylor expansion are non-zero; they describe the leading local change in the fields at each vortex due to the presence of the rest. Since (see [Sam92, Appendix A]) a system of well-separated vortices is approximated by the superposition of 1-vortex solutions with an error exponentially small in the separation, the b_r will then be small of the same order and then

$$\int_{\mathbb{R}^2 \setminus S_\varrho} \partial_z \big(\bar{\eta}(x, q)\partial_{\bar{z}}\eta(x, q)\big) d^2x \approx \frac{\pi}{4}\sum_{r=1}^{N} \frac{dZ_r}{dt}\frac{d\bar{Z}_r}{dt}. \tag{2.46}$$

For

$$\mathcal{R}e \int_{\mathbb{R}^2 \setminus S_\varrho} \big(- \sigma \Delta \eta(x,q) + |\varphi(x,q)|^2 \eta(x,q) \big) \bar{\eta}(x,q) d^2x$$

one notices that, for (2.32,)

$$\mathcal{R}e \left\{ (-\sigma \Delta \eta + |\varphi|^2 \eta) \bar{\eta} \right\} = (-\sigma \Delta \mathcal{R}e\, \eta + |\varphi|^2 \mathcal{R}e\, \eta) \mathcal{R}e\, \eta \tag{2.47}$$

and, by invoking (2.34), one obtains

$$-\sigma \Delta \mathcal{R}e\, \eta + |\varphi|^2 \mathcal{R}e\, \eta = (1 - \sigma) \Delta \mathcal{R}e\, \eta. \tag{2.48}$$

As discussed in the introduction, we restrict our attention to $\sigma = 1 + \mathcal{O}(\varepsilon)$. Therefore, using (2.48) in (2.47), one obtains

$$\mathcal{R}e \int_{\mathbb{R}^2 \setminus S_\varrho} \big(- \sigma \Delta \eta(x,q) + |\varphi(x,q)|^2 \eta(x,q) \big) \bar{\eta}(x,q) d^2x \;\; = \;\; \mathcal{O}(\varepsilon). \tag{2.49}$$

Using now (2.46) and (2.49) in (2.35) it turns out that at leading order T is

$$\frac{\pi}{4} \sum_{r=1}^{N} \frac{dZ_r}{dt} \frac{d\bar{Z}_r}{dt}$$

and then the matrix associated with the kinetic metric T is proportional to the identity matrix, which is trivially invertible. Then $detA'(q) \neq 0$ is invertible and consequently, for $\varepsilon > 0$ small enough, even $detA(q) \neq 0$ for $q \in \mathbb{R}^{2N} \setminus Z_\phi$. Extending this result by continuity to Z_ϕ, the invertibility of $A(q)$ is proven everywhere in $\mathbb{R}^{2N}$.

Chapter 3

An Iterative Scheme

In this chapter the original coupled P.D.E.s associated with the Gor'kov-Éliashberg system are used to define a system of elliptic equations, parabolic equations and a set of O.D.E.s. The existence of a solution to this new system is shown in sections 3.3 and 3.4. The exponential decay of the solutions at infinity in space is discussed as well. The iterative method allows us to find certain estimates for the solutions in terms of the initial data and the functions on the right-hand side of the system. The first section aims to describe the exponential decay at infinity of the static solutions of the Ginzburg-Landau model.

3.1 Higher Derivatives of the Static Solution

As given in section 2.5, the second Bogomol'nyi equation can be written

$$-\Delta u + e^u - 1 = 0 \tag{3.1}$$

on $\mathbb{R}^2 \setminus Z_\phi$. Differentiating (3.1) with respect to x_j yields

$$-\Delta\partial_j u + e^u \partial_j u = 0 \tag{3.2}$$

on $\mathbb{R}^2 \setminus B(0,R)$, where $B(0,R)$ is a closed ball with regular boundary containing all the zeros of φ, which are the singularities of u. Given the definition of u, $u = \ln|\varphi|^2$, it turns out that

$$e^u \partial_j u = \partial_j(|\varphi|^2) = 2\big(\varphi, D_j^{(0)}\varphi\big) \tag{3.3}$$

holds. Then, using (3.3) and the fact that the covariant derivative of the static Higgs field is exponentially decaying at infinity and that u vanishes at infinity, [JT80], one obtains that

$$\lim_{|x|\to+\infty} |\partial_j u(x,q)| \le 2 \lim_{|x|\to+\infty} |D_j^{(0)}\varphi(x,q)| = 0$$

uniformly in x.

We now invoke the following,

Proposition 3.1.1 *Let $u \in C^\infty(\mathbb{R}^d)$, $v \in C^0(\mathbb{R}^d)$. Suppose that for $|x| \ge R$,*

$$\Delta u(x) - 2b(x)\cdot\nabla u(x) - m^2\big(1 - v(x)\big)u(x) = q(x)$$

and

$$\lim_{|x|\to+\infty} |u(x)| = 0, \qquad \lim_{|x|\to+\infty} |v(x)| = 0$$

uniformly in $arg(x)$. Suppose that b is bounded and

$$|q(x)| \le M_1 e^{-\sigma|x|}$$

for $M_1 > 0$ *and* $0 < \sigma < +\infty$, *and define* $b_0 := \sup_{|x|>R} |b(x)|$. *Then given* $0 < \varepsilon < 1$, *there exists* $M(\varepsilon) < +\infty$ *such that*

$$|u(x)| \leq M(\varepsilon) e^{-(1-\varepsilon)\overline{m}|x|}$$

with $\overline{m} := \min\{(m^2 + b_0)^{1/2} - b_0, \sigma\}$.

Proof. See [JT80, *III*, Proposition 7.4].

From Proposition 3.1.1 it follows that a solution of (3.2) is such that

$$|\partial_j u(x,q)| \leq C_1 e^{-m_1|x|} \tag{3.4}$$

with constants $C_1, m_1 > 0$ depending only on q.

Now the definition of u is used to define a new smooth function $\widetilde{u}$ on the whole plane as follows:

$$\widetilde{u}(x,q) := \begin{cases} u(x,q) & \text{if } x \in \mathbb{R}^2 \setminus B(0,R_1) \\ 0 & \text{if } x \in B(0,R_2) \\ \text{smooth interpolation between } u \text{ and } 0 & \\ & \text{if } x \in B(0,R_1) \setminus B(0,R_2) \end{cases}$$

with $R_2 < R_1$. Then, it follows from (3.2) that $\widetilde{u}$ solves on $\mathbb{R}^2$ the equation

$$-\Delta \partial_j \widetilde{u} + e^{\widetilde{u}} \partial_j \widetilde{u} = f \tag{3.5}$$

with f smooth and different from zero only in $B(0,R_1) \setminus B(0,R_2)$. Since $\partial_j \widetilde{u}$ has the same behaviour at infinity as $\partial_j u \in L^2(\mathbb{R}^2)$, and since $e^{\widetilde{u}}$ is bounded, the Fourier transform may be taken. Thus $\widehat{\partial_j \widetilde{u}} \in L^2(\mathbb{R}^2)$ and

$$\widehat{\Delta \partial_j \widetilde{u}} = \widehat{e^{\widetilde{u}} \partial_j \widetilde{u}} - \hat{f} \in L^2(\mathbb{R}^2).$$

One of the properties of the Fourier transform on $L^2(\mathbb{R}^2)$ [Eva98, IV, theorem 2] tells us that, since $\Delta\partial_j\widetilde{u} \in L^2(\mathbb{R}^2)$,

$$\widehat{\Delta\partial_j\widetilde{u}}(\xi) = -|\xi|^2\widehat{\partial_j\widetilde{u}} \in L^2(\mathbb{R}^2).$$

Thus $(1+|\xi|^2)\widehat{\partial_j\widetilde{u}}$ is in $L^2(\mathbb{R}^2)$, which is equivalent to

$$\partial_j\widetilde{u} \in H^2(\mathbb{R}^2)$$

as shown in [Kes98, page 52]. Differentiating (3.5) with respect to x_k one obtains

$$-\Delta\partial^2_{jk}\widetilde{u} + e^{\widetilde{u}}\partial^2_{jk}\widetilde{u} = \partial_k f - e^{\widetilde{u}}\partial_j\widetilde{u}\partial_k\widetilde{u}. \tag{3.6}$$

Since $\partial_j\widetilde{u}$, $\partial_k\widetilde{u}$ and $\partial_k f$ are exponentially decaying at infinity and $e^{\widetilde{u}}$ is bounded, the right-hand side of (3.6) is exponentially decaying at infinity. Moreover it has just been proven that $\partial^2_{jk}\widetilde{u}$ is in $L^2(\mathbb{R}^2)$. Iterating the previous procedure one obtains

$$\partial^2_{jk}\widetilde{u} \in H^2(\mathbb{R}^2).$$

Now we recall two results:

Proposition 3.1.2 (Sobolev Imbeddings) *There exists a constant* $c = c(m,p,q,d)$ *such that for all* $f \in W^{j+m,q}(\mathbb{R}^d)$ *and for* $pd/(d+pm) < q \leq p$,

$$|f|_{W^{j,p}(\mathbb{R}^d)} \leq c|f|_{W^{j+m,q}(\mathbb{R}^d)},$$

i.e.,

$$W^{j+m,q}(\mathbb{R}^d) \subseteq W^{j,p}(\mathbb{R}^d).$$

Proof. See [JT80, VI, Proposition 2.5].

Proposition 3.1.3 *Suppose $u \in W^{1,p}(\mathbb{R}^d)$ for some $p > d$. Then u satisfies*

$$\lim_{R\to+\infty} \sup_{|x|=R} |u(x)| = 0.$$

Proof. See [JT80, III, Proposition 7.5].

In particular, from Proposition 3.1.2 it follows that

$$H^2(\mathbb{R}^2) := W^{2,2}(\mathbb{R}^2) \subseteq W^{1,3}(\mathbb{R}^2)$$

and from Proposition 3.1.3 it turns out that

$$\lim_{R\to+\infty} \sup_{|x|=R} |\partial^2_{jk}\widetilde{u}(x,q)| = 0$$

holds. Thus, all the hypotheses of Proposition 3.1.1 are satisfied and a solution of (3.6) exists for which

$$|\partial^2_{jk}u(x,q)| \leq C_2 e^{-m_2|x|} \tag{3.7}$$

with constants $C_2, m_2 > 0$ depending only on q. Differentiating (3.6) with respect to x_l and repeating the previous arguments one obtains

$$|\partial^3_{jkl}u(x,q)| \leq C_3 e^{-m_3|x|} \tag{3.8}$$

and so on for the higher derivatives.

As a consequence of our choice of gauge in section 2.2, in $\mathbb{R}^2 \setminus B(0,R)$ the fields may be written

$$\varphi = e^{\frac{u}{2}}, \quad \alpha_1 = \frac{1}{2}\partial_2 u, \quad \alpha_2 = -\frac{1}{2}\partial_1 u.$$

Now all these results are used to prove,

Remark 3.1.1 *The following quantities:*

$$\sup_{x\in\mathbb{R}^2} |\underbrace{\partial_j\partial_k\partial_l\ldots}_{r}\varphi(x,q)|$$

and

$$\sup_{x\in\mathbb{R}^2} |\underbrace{\partial_j\partial_k\partial_l\ldots}_{r}\alpha_h(x,q)| \quad \textit{with } h=1,2$$

are bounded by a constant dependent on q and r. Moreover, if q is within a compact set of $\mathbb{R}^{2N}$, say Π, there exists a constant $M>0$ independent of x and q and dependent only on Π, r and s such that

$$\sup_{(x,q)\in\mathbb{R}^2\times\Pi} |\underbrace{\partial_j\partial_k\partial_l\ldots}_{r}\underbrace{\frac{\partial}{\partial q_\mu}\frac{\partial}{\partial q_\nu}\frac{\partial}{\partial q_\tau}\ldots}_{s}\varphi(x,q)| \leq M$$

and

$$\sup_{(x,q)\in\mathbb{R}^2\times\Pi} |\underbrace{\partial_j\partial_k\partial_l\ldots}_{r}\underbrace{\frac{\partial}{\partial q_\mu}\frac{\partial}{\partial q_\nu}\frac{\partial}{\partial q_\tau}\ldots}_{s}\alpha_h(x,q)| \leq M \quad \textit{with } h=1,2$$

for $r,s=0,1,2,\ldots$.

In fact

$$\sup_{x\in\mathbb{R}^2}|\partial_j\varphi(x,q)| = \max\Big\{\max_{x\in B(0,R)}|\partial_j\varphi(x,q)|, \sup_{x\in\mathbb{R}^2\setminus B(0,R)}|\partial_j\varphi(x,q)|\Big\}.$$

This supremum is shown to be bounded because $\max_{x\in B(0,R)} |\partial_j\varphi(x,q)|$ is finite since $\partial_j\varphi$ is a continuous function on a compact domain and because, from the definition of the static Higgs field and from what has been proven at the beginning of this section, we know that

$$\sup_{x\in\mathbb{R}^2\setminus B(0,R)} |\partial_j\varphi(x,q)| = \sup_{x\in\mathbb{R}^2\setminus B(0,R)} |\partial_j(e^{\frac{1}{2}u(x,q)})|$$
$$\leq \frac{1}{2}e^{\frac{1}{2}u(\overline{x},q)} \sup_{x\in\mathbb{R}^2\setminus B(0,R)} |\partial_j u(x,q)| < +\infty,$$

where $\overline{x}$ yields a maximum of u. Analogously

$$\sup_{x\in\mathbb{R}^2} |\partial^2_{jk}\varphi(x,q)| = \max\big\{ \max_{x\in B(0,R)} |\partial^2_{jk}\varphi(x,q)|, \sup_{x\in\mathbb{R}^2\setminus B(0,R)} |\partial^2_{jk}\varphi(x,q)|\big\}$$

is shown to be bounded because $\max_{x\in B(0,R)} |\partial^2_{jk}\varphi(x,q)|$ is finite since $\partial^2_{jk}\varphi$ is a continuous function on a compact domain and because, from the definition of the static Higgs field and from what has been proven at the beginning of this section, we have

$$\sup_{x\in\mathbb{R}^2\setminus B(0,R)} |\partial^2_{jk}\varphi(x,q)| = \sup_{x\in\mathbb{R}^2\setminus B(0,R)} |\partial^2_{jk}(e^{\frac{1}{2}u(x,q)})|$$
$$\leq \frac{1}{2}e^{\frac{1}{2}u(\overline{x},q)} \sup_{x\in\mathbb{R}^2\setminus B(0,R)} |\partial^2_{jk}u(x,q)|$$
$$+ \frac{1}{4}e^{\frac{1}{2}u(\overline{x},q)} \sup_{x\in\mathbb{R}^2\setminus B(0,R)} |\partial_j u(x,q)||\partial_k u(x,q)| < +\infty,$$

with $\overline{x}$ as above. The same method can be used for the higher derivatives.

One proceeds analogously for α_h. For example,

$$\sup_{x\in\mathbb{R}^2} |\alpha_1(x,q)| = \max\big\{ \max_{x\in B(0,R)} |\alpha_1(x,q)|, \sup_{x\in\mathbb{R}^2\setminus B(0,R)} |\alpha_1(x,q)|\big\}.$$

This supremum is shown to be bounded because $\max_{x\in B(0,R)} |\alpha_1(x,q)|$ is finite since α_1 is a continuous function on a compact domain and because, from the definition of the static vector potential and from what has been proven at the beginning of this section, we have

$$\sup_{x\in\mathbb{R}^2\setminus B(0,R)} |\alpha_1(x,q)| = \frac{1}{2} \sup_{x\in\mathbb{R}^2\setminus B(0,R)} |\partial_2 u(x,q)| < +\infty.$$

Analogously

$$\sup_{x\in\mathbb{R}^2} |\partial_j\alpha_1(x,q)| = \max\Big\{ \max_{x\in B(0,R)} |\partial_j\alpha_1(x,q)|, \sup_{x\in\mathbb{R}^2\setminus B(0,R)} |\partial_j\alpha_1(x,q)|\Big\}$$

is shown to be bounded. The same method can be used for the higher derivatives. For the derivatives with respect to q we proceed in the same way as for the derivatives with respect to x. We notice that from (3.1) an equation like (3.2) for the $\frac{\partial u}{\partial q_\mu}$ holds. Since

$$\lim_{|x|\to+\infty} |\frac{\partial u}{\partial q_\mu}(x,q)| = 0,$$

from proposition 3.1.1 it follows that they are exponentially decaying. Then we iterate this method for the higher derivatives and the exponential fall-off is proven for all the derivatives with respect to x as well as q of the static solution (α, φ).

3.2 The Iteration System

The new system for the fields is still coupled. An iterative system is defined in order to uncouple it. This system consists of an elliptic equation, two parabolic equations and a set of O.D.E.s.

Putting together (2.14), (2.22) and (2.31), the aim becomes to prove that the system

$$-\Delta\tilde{\tilde{a}}_0 + \varepsilon\tilde{\tilde{a}}_0 = f_0(\tilde{a}, \tilde{\phi}, \tilde{\tilde{a}}_0, \partial_t\tilde{\phi}, q, \dot{q}),$$

$$\partial_t \begin{pmatrix} \sigma\widetilde{a} \\ \widetilde{\phi} \end{pmatrix} + L^{(\alpha,\varphi)} \begin{pmatrix} \widetilde{a} \\ \widetilde{\phi} \end{pmatrix} = f_1'(\widetilde{a}, \widetilde{\phi}, \widetilde{\widetilde{a}}_0, q, \dot{q}),$$

$$\dot{q} = f_2(\widetilde{a}, \widetilde{\phi}, \widetilde{\widetilde{a}}_0, q) \tag{3.9}$$

with

$$f_0(\widetilde{a}, \widetilde{\phi}, \widetilde{\widetilde{a}}_0, \partial_t\widetilde{\phi}, q, \dot{q}) := f_0''(\widetilde{a}, \widetilde{\phi}, \partial_t\widetilde{\phi}, q, \dot{q}) + \varepsilon\widetilde{\widetilde{a}}_0, \tag{3.10}$$

for f_0'' like in (2.15),

$$\begin{aligned} f_1'(\widetilde{a}, \widetilde{\phi}, \widetilde{\widetilde{a}}_0, q, \dot{q}) := -\sum_{\nu=1}^{2N} \begin{pmatrix} \sigma\frac{\partial\alpha}{\partial q_\nu}(\cdot, q) \\ \frac{\partial\varphi}{\partial q_\nu}(\cdot, q) \end{pmatrix} \dot{q}_\nu \\ + \begin{pmatrix} 0 \\ \frac{1}{2}\varphi(1 - |\phi(\cdot, q)|^2) \end{pmatrix} + \begin{pmatrix} \sigma\nabla\widetilde{\widetilde{a}}_0 \\ i\widetilde{\widetilde{a}}_0\varphi(\cdot, q) \end{pmatrix} + \varepsilon j(\widetilde{a}, \widetilde{\phi}, \widetilde{\widetilde{a}}_0, q), \end{aligned} \tag{3.11}$$

$$\begin{aligned} f_2(\widetilde{a}, \widetilde{\phi}, \widetilde{\widetilde{a}}_0, q) \quad &:= \quad A^{-1}(\widetilde{a}, \widetilde{\phi}, q)\Big(\frac{1}{2}\int_{\mathbb{R}^2} \big(\varphi(x, q)(1 - |\phi(x, q)|^2), n^2(x, q)\big) d^2x \\ &\quad + \varepsilon\langle j(\widetilde{a}, \widetilde{\phi}, \widetilde{\widetilde{a}}_0, q), n(\cdot, q)\rangle\Big), \end{aligned} \tag{3.12}$$

has solutions bounded in some normed space. An iterative method will be used to show this.

The following system of linear partial differential equations is defined:

$$-\Delta\widetilde{\widetilde{a}}_0^{(i+1)} + \varepsilon\widetilde{\widetilde{a}}_0^{(i+1)} \quad = \quad f_0(\widetilde{a}^{(i)}, \widetilde{\phi}^{(i)}, \widetilde{\widetilde{a}}_0^{(i)}, \partial_t\widetilde{\phi}^{(i)}, q^{(i)}, \dot{q}^{(i)}), \tag{3.13}$$

$$\partial_t \begin{pmatrix} \sigma\widetilde{a}^{(i+1)} \\ \widetilde{\phi}^{(i+1)} \end{pmatrix} + L_0^{(\alpha^{(i)},\varphi^{(i)})} \begin{pmatrix} \sigma\widetilde{a}^{(i+1)} \\ \widetilde{\phi}^{(i+1)} \end{pmatrix} = f_1(\widetilde{a}^{(i)}, \widetilde{\phi}^{(i)}, \widetilde{\widetilde{a}}_0^{(i+1)}, q^{(i)}, \dot{q}^{(i)}), \tag{3.14}$$

$$\dot{q}^{(i+1)} = f_2(\widetilde{a}^{(i)}, \widetilde{\phi}^{(i)}, \widetilde{\widetilde{a}}_0^{(i+1)}, q^{(i)}), \tag{3.15}$$

with

$$\alpha^{(i)}(\cdot,t) := \alpha(\cdot, q^{(i)}(t)), \quad \varphi^{(i)}(\cdot,t) := \varphi(\cdot, q^{(i)}(t)),$$

$$f_1(\widetilde{a}^{(i)}, \widetilde{\phi}^{(i)}, \widetilde{\widetilde{a}}_0^{(i+1)}, q^{(i)}, \dot{q}^{(i)}) := -V^{(\alpha^{(i)},\varphi^{(i)})} \begin{pmatrix} \sigma\widetilde{a}^{(i)} \\ \widetilde{\phi}^{(i)} \end{pmatrix} + f_1'(\widetilde{a}^{(i)}, \widetilde{\phi}^{(i)}, \widetilde{\widetilde{a}}_0^{(i+1)}, q^{(i)}, \dot{q}^{(i)}).$$

Define

$$\psi^{(i)}(x,t) := \begin{pmatrix} \sigma\widetilde{a}^{(i)}(x,t) \\ \widetilde{\phi}^{(i)}(x,t) \end{pmatrix}$$

and, for the initial data, choose

$$\widetilde{\widetilde{a}}_0^{(0)}(x,t) := \widetilde{\widetilde{a}}_0(x,0) \equiv 0, \quad \psi^{(0)}(x,t) := \psi(x,0), \quad q^{(0)}(t) := q(0).$$

Below we will use the following notation:

$$f_0(i,x,t) := f_0(\widetilde{a}^{(i)}(x,t), \widetilde{\phi}^{(i)}(x,t), \widetilde{\widetilde{a}}_0^{(i)}(x,t), \partial_t\widetilde{\phi}^{(i)}(x,t), q^{(i)}(t), \dot{q}^{(i)}(t)),$$

$$f_1(i,x,t) := f_1(\widetilde{a}^{(i)}(x,t), \widetilde{\phi}^{(i)}(x,t), \widetilde{\widetilde{a}}_0^{(i+1)}(x,t), q^{(i)}(t), \dot{q}^{(i)}(t))$$

and

$$f_2(i,x,t) := f_2(\widetilde{a}^{(i)}(x,t), \widetilde{\phi}^{(i)}(x,t), \widetilde{\widetilde{a}}_0^{(i+1)}(x,t), q^{(i)}(t)).$$

3.3 The Elliptic Equation

The existence of a solution to (3.13) is now shown. We outline a possible proof based on the knowledge of a *fundamental solution* of the equation. For a complete proof two theorems about the equation (3.13) are displayed. These theorems also establish the regularity of the solution.

Theorem 3.3.1 *A solution of* (3.13) *exists on the whole of* $\mathbb{R}^2$ *and has the following form:*

$$\widetilde{\widetilde{a}}_0^{(i+1)}(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}^2} f_0(i,y,t)K_0(\sqrt{\varepsilon}|x-y|)d^2y \tag{3.16}$$

where K_0 *is the first* K*-Bessel function.*

Proof. It is known that $u(x) := \frac{1}{2\pi}K_0(\sqrt{\varepsilon}|x|)$ is a solution of

$$-\Delta u(x) + \varepsilon u(x) = 0$$

for $x \in \mathbb{R}^2 \setminus \{0\}$, where K_0 is the first K-Bessel function [ES92, 2.9, example 2.116]. (For a complete discussion of the K-Bessel functions see [Abr57, 9.6]). Now we prove that (3.16) solves (3.13):

$$\begin{aligned}
-\Delta\widetilde{\widetilde{a}}_0^{(i+1)}&(x,t) + \varepsilon\widetilde{\widetilde{a}}_0^{(i+1)}(x,t) \\
&= -\frac{1}{2\pi}\Delta_x\int_{\mathbb{R}^2} f_0(i,x-y,t)K_0(\sqrt{\varepsilon}|y|)d^2y \\
&+ \frac{\varepsilon}{2\pi}\int_{\mathbb{R}^2} f_0(i,x-y,t)K_0(\sqrt{\varepsilon}|y|)d^2y \\
&= -\frac{1}{2\pi}\int_{\mathbb{R}^2} \Delta_y f_0(i,x-y,t)K_0(\sqrt{\varepsilon}|y|)d^2y \\
&+ \frac{\varepsilon}{2\pi}\int_{\mathbb{R}^2} f_0(i,x-y,t)K_0(\sqrt{\varepsilon}|y|)d^2y
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2\pi}\lim_{\eta\to 0}\int_{\mathbb{R}^2\setminus B(0,\eta)} \Delta_y f_0(i,x-y,t)K_0(\sqrt{\varepsilon}|y|)d^2y \\
&+ \frac{\varepsilon}{2\pi}\lim_{\eta\to 0}\int_{\mathbb{R}^2\setminus B(0,\eta)} f_0(i,x-y,t)K_0(\sqrt{\varepsilon}|y|)d^2y \\
&= -\frac{1}{2\pi}\lim_{\eta\to 0}\int_{\mathbb{R}^2\setminus B(0,\eta)} f_0(i,x-y,t)\Delta_y K_0(\sqrt{\varepsilon}|y|)d^2y \\
&+ \frac{1}{2\pi}\lim_{\eta\to 0}\int_{\partial B(0,\eta)} f_0(i,x-y,t)\frac{\partial K_0}{\partial n}(\sqrt{\varepsilon}|y|)d^2y \\
&- \frac{1}{2\pi}\lim_{\eta\to 0}\int_{\partial B(0,\eta)} \frac{\partial f_0}{\partial n}(i,x-y,t)K_0(\sqrt{\varepsilon}|y|)d^2y \\
&+ \frac{\varepsilon}{2\pi}\lim_{\eta\to 0}\int_{\mathbb{R}^2\setminus B(0,\eta)} f_0(i,x-y,t)K_0(\sqrt{\varepsilon}|y|)d^2y \\
&= \frac{1}{2\pi}\lim_{\eta\to 0}\int_{\mathbb{R}^2\setminus B(0,\eta)} f_0(i,x-y,t)\big(-\Delta_y K_0(\sqrt{\varepsilon}|y|)+\varepsilon K_0(\sqrt{\varepsilon}|y|)\big)d^2y \\
&+ \frac{1}{2\pi}\lim_{\eta\to 0}\int_{\partial B(0,\eta)} f_0(i,x-y,t)\frac{\partial K_0}{\partial n}(\sqrt{\varepsilon}|y|)d^2y \\
&- \frac{1}{2\pi}\lim_{\eta\to 0}\int_{\partial B(0,\eta)} \frac{\partial f_0}{\partial n}(i,x-y,t)K_0(\sqrt{\varepsilon}|y|)d^2y \\
&= -\frac{1}{2\pi}\lim_{\eta\to 0}\int_0^{2\pi} \eta f_0(i,x-\eta\vec{u},t)\Big(\frac{\partial K_0}{\partial r}(\sqrt{\varepsilon}r)\Big)_{|r=\eta}d\theta \\
&+ \frac{1}{2\pi}\lim_{\eta\to 0}\int_0^{2\pi} \eta\Big(\frac{\partial f_0}{\partial r}(i,x-r\vec{u},t)\Big)_{|r=\eta}K_0(\sqrt{\varepsilon}\eta)d\theta \\
&= -\lim_{\eta\to 0}\eta f_0(i,x-\eta\vec{u},t)\Big(\frac{\partial K_0}{\partial r}(\sqrt{\varepsilon}r)\Big)_{|r=\eta} \\
&+ \lim_{\eta\to 0}\eta\Big(\frac{\partial f_0}{\partial r}(i,x-r\vec{u},t)\Big)_{|r=\eta}K_0(\sqrt{\varepsilon}\eta) \\
&= -f_0(i,x,t)\lim_{\eta\to 0}\eta\Big(\frac{\partial K_0}{\partial r}(\sqrt{\varepsilon}r)\Big)_{|r=\eta} + \frac{\partial f_0}{\partial r}(i,x,t)\lim_{\eta\to 0}\eta K_0(\sqrt{\varepsilon}\eta).
\end{aligned}
$$

According to [Abr57, 9.6.13, 9.6.12] a constant γ exists such that

$$
\begin{aligned}
K_0(\sqrt{\varepsilon}\eta) = &-\ln(\sqrt{\varepsilon}\eta/2)\Big(1+\frac{\varepsilon\eta^2/4}{(1!)^2}+\frac{(\varepsilon\eta^2/4)^2}{(2!)^2}+\dots\Big) \\
&-\gamma+(-\gamma+1)\frac{\varepsilon\eta^2/4}{(1!)^2}+(-\gamma+3/2)\frac{(\varepsilon\eta^2/4)^2}{(2!)^2}+\dots
\end{aligned}
$$

and

$$\Big(\frac{\partial K_0}{\partial r}(\sqrt{\varepsilon}r)\Big)_{|r=\eta} = -\frac{1}{\eta} - \frac{\varepsilon/4}{(1!)^2}\eta - \frac{(\varepsilon/4)^2}{(2!)^2}\eta^3 - \dots$$
$$- \frac{\varepsilon/2}{(1!)^2}\eta\ln(\sqrt{\varepsilon}\eta/2) - \frac{\varepsilon}{(2!)^2}\eta^3\ln(\sqrt{\varepsilon}\eta/2)$$
$$+ \frac{-\gamma+1}{(1!)^2}(\varepsilon\eta/2) + \frac{-\gamma+3/2}{(2!)^2}(\varepsilon^2\eta^3/4) + \dots.$$

Therefore

$$\lim_{\eta\to 0}\eta K_0(\sqrt{\varepsilon}\eta) = 0 \quad \text{and} \quad \lim_{\eta\to 0}\eta\Big(\frac{\partial K_0}{\partial r}(\sqrt{\varepsilon}r)\Big)_{|r=\eta} = -1,$$

which allows us to conclude that

$$-\Delta\widetilde{\widetilde{a}}_0^{(i+1)} + \varepsilon\widetilde{\widetilde{a}}_0^{(i+1)} = f_0(\widetilde{a}^{(i)}, \widetilde{\phi}^{(i)}, \widetilde{\widetilde{a}}_0^{(i)}, \partial_t\widetilde{\phi}^{(i)}, q^{(i)}, \dot{q}^{(i)}).$$

The aim is now to discuss the regularity of a solution of (3.13). We recall the following results:

Proposition 3.3.1 *Let $f \in L^2(\mathbb{R}^n)$ and let ω be a positive real constant. The problem*

$$-\Delta u(x) + \omega u(x) = f(x), \qquad x \in \mathbb{R}^n$$

$$\lim_{|x|\to+\infty} u(x) = 0 \tag{3.17}$$

has a solution $u \in H^1(\mathbb{R}^n)$ in the distributional sense. This solution is unique.

Proof. See [Kes98, 3.2.6].

Proposition 3.3.2 *Let* $u \in H^1(\mathbb{R}^n)$ *such that*

$$-\Delta u(x) + \omega u(x) = f(x), \qquad x \in \mathbb{R}^n$$

in the distributional sense, with $f \in H^m(\mathbb{R}^n)$ *and where* $m \geq 0$ *is an integer. Then* $u \in H^{m+2}(\mathbb{R}^n)$ *and*

$$|u|_{H^{m+2}(\mathbb{R}^n)} \leq C|f|_{H^m(\mathbb{R}^n)}, \tag{3.18}$$

where $C > 0$ *is a constant independent of* f*.*

Proof. See [Kes98, 3.3, theorem 3.3.1.].

Finally a result about the regularity of the solution of (3.13) will be used later on:

Proposition 3.3.3 *Let* $u \in H^1(\mathbb{R}^2)$ *be a solution in the distributional sense of the elliptic equation* $-\Delta u + \omega u = f$ *in* $\mathbb{R}^2$ *and suppose that the function* f *is in* $H^k(\mathbb{R}^2)$ *for any* $k \in \mathbb{N}$ *with* $k \geq \overline{k}$*. Then* $u \in C^\infty(\mathbb{R}^2)$*.*

Proof. From Proposition 3.3.2 it follows that u is in $H^{k+2}(\mathbb{R}^2)$ for any $k \in \mathbb{N}$ with $k \geq \overline{k}$. Moreover from Proposition 3.1.2 it turns out that

$$H^{k+2}(\mathbb{R}^2) \subseteq W^{1,p}(\mathbb{R}^2)$$

for $p \geq 3$. Knowing that, if $p > 2$ then every $u \in W^{1,p}(\mathbb{R}^2)$ is Holderian. In fact, from [JT80, 6, corollary 2.7, (b)],

$$|u(x) - u(x_0)| \leq C|\partial_j u|_{W^{0,p}(\mathbb{R}^2)}|x - x_0|^\beta$$

with $\beta := 1 - 2/p$, for any $x, x_0 \in \mathbb{R}^2$. Since $u \in W^{1,p}(\mathbb{R}^2)$

$$|u(x) - u(x_0)| \leq C' |x - x_0|^\beta \tag{3.19}$$

yields. Thus, for every $\varepsilon > 0$ exists $\delta(\varepsilon) := \frac{\varepsilon^{-\beta}}{C'} > 0$ such that, if $|x - x_0| \leq \delta(\varepsilon)$, then, using (3.19),

$$|u(x) - u(x_0)| \leq \varepsilon.$$

So the continuity of u on $\mathbb{R}^2$ follows. Iterate the same method for the higher derivatives of u with respect to x in order to prove the smoothness of u.

3.4 The Parabolic Equation

The existence of a solution of (3.14) is proven by induction. Such an equation can be split into two parabolic equations, one for $\widetilde{a}$ and the other for $\widetilde{\phi}$. The general theory about the parabolic equations is used to prove the existence of solutions and to study their behaviour as $|x|$ tends to infinity.

Consider a second-order parabolic operator:

$$\begin{aligned} L_{x,t}u(x,t) &:= -\partial_t u(x,t) + \sum_{i,j=1}^{n} a_{i,j}(x,t)\partial^2_{ij}u(x,t) \\ &+ \sum_{i=1}^{n} b_i(x,t)\partial_i u(x,t) + c(x,t)u(x,t) = f(x,t). \end{aligned} \tag{3.20}$$

Assume that the functions $a_{i,j}$, b_i and c are continuous on $\mathbb{R}^n \times [0,T]$ and that $\lambda > 0$ exists such that

$$\sum_{i,j=1}^{n} a_{i,j}(x,t)\xi_i\xi_j \geq \lambda|\xi|^2 \tag{3.21}$$

for any $\xi \in \mathbb{R}^n$ and for any $(x,t) \in \mathbb{R}^n \times [0,T]$. Condition (3.21) is called the *ellipticity condition.*

It is possible to find a solution of the Cauchy problem (3.20) using Green's functions. A *fundamental solution* of (3.20) is a function $\Gamma(x,\xi,t,\tau)$ defined for $t > \tau$ and $x, \xi \in \mathbb{R}^n$ such that for each continuous bounded function ϱ on $\mathbb{R}^n$ the integral

$$u(x,t) = \int_{\mathbb{R}^n} \varrho(\xi)\Gamma(x,\xi,t,0)d^n\xi$$

converges, $L_{x,t}u(x,t) = 0$ for $t > \tau$ and

$$\lim_{t\to 0^+} u(x,t) = \varrho(x).$$

The existence of such a *fundamental solution* has been shown in [Fri64, chapter 9].

Now we want to discuss the existence of a solution of (3.20) and to analyze the behaviour of that solution and its derivatives with respect to x, as $|x|$ tends to infinity.

Proposition 3.4.1 *Let the function ϱ be continuous on $\mathbb{R}^n$ and, given $H := \mathbb{R}^n \times]0,T]$, let f be continuous on $\overline{H}$ and satisfy a Hölder condition on x. Let*

$$|\varrho(x)| + |f(x,t)| \le Ce^{C_1|x|^{2-\eta}}, \quad \eta > 0, \quad 0 \le t \le T \tag{3.22}$$

for some constants C and C_1. Then for $0 < t \le T$ the function

$$u(x,t) = \int_{\mathbb{R}^n} \varrho(\xi)\Gamma(x,\xi,t,0)d^n\xi + \int_0^t \int_{\mathbb{R}^n} f(\xi,\tau)\Gamma(x,\xi,t,\tau)d^n\xi d\tau \tag{3.23}$$

satisfies (3.20) *and*

$$\lim_{t\to 0^+} u(x,t) = \varrho(x). \tag{3.24}$$

Here $|u(x,t)| \le C_2 e^{C_1|x|^{2-\eta}}$, *for some constant* C_2 *and for* $0 \le t \le T$.

Proof. See [ES92, 2.5.7, theorem 2.82] for references.

A regularity result for the solution of (3.20) is given by

Proposition 3.4.2 *Given* $H := \mathbb{R}^n \times]0,T]$, *if the coefficients of* $L_{x,t}$ *and* f *are infinitely often differentiable with respect to* x *in* $\overline{H}$, *then every solution* u *of* (3.20) *is infinitely often differentiable with respect to* x *in* $\overline{H}$.

Proof. See [Fri64, 9.6, theorem 10].

The uniqueness of the solution of (3.14) is established in the following

Theorem 3.4.1 *Suppose that* $H := \mathbb{R}^n \times]0,T]$ *and that the coefficients of the operator* $L_{x,t}$ *are bounded in* $\overline{H}$. *Then the Cauchy problem for* (3.20) *can have at most one solution in* H *in the class of functions of* $C(\overline{H})$ *satisfying condition*

$$|u(x,t)| \le Ce^{\sigma|x|^2} \quad \textit{in } H, \tag{3.25}$$

for positive constants C *and* σ.

Proof. See [ES92, 2, corollary 2.75].

In particular, since $L_0^{(\alpha^{(i)},\varphi^{(i)})}$ satisfies the *ellipticity condition* and all its coefficients are continuous and bounded functions in $\overline{H}$, the solutions $\sigma\widetilde{a}^{(i+1)}$ and $\widetilde{\phi}^{(i+1)}$ exist satisfying (3.25).

It is possible to extend the outcome about the behaviour at infinity of u in Proposition 3.4.1 to its derivatives with respect to x:

Proposition 3.4.3 *Let all the hypotheses of Proposition 3.4.1 and Proposition 3.4.2 hold, with $a_{i,j}$ and b_i independent of x and $\frac{\partial^{|\beta|}c}{\partial x_1^{\beta_1}...\partial x_n^{\beta_n}}(x,t)$ bounded in x on $\overline{H}$ for all $0 \leq |\beta| \leq r$, with r an arbitrary positive integer. Furthermore, let us suppose that the functions $\frac{\partial^{|\beta|}\varrho}{\partial x_1^{\beta_1}...\partial x_n^{\beta_n}}(x)$ are continuous on $\mathbb{R}^n$, $\frac{\partial^{|\beta|}f}{\partial x_1^{\beta_1}...\partial x_n^{\beta_n}}(x,t)$ are continuous on $\overline{H}$, satisfy a Hölder condition on x and*

$$|\frac{\partial^{|\beta|}\varrho}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}(x)| + |\frac{\partial^{|\beta|}f}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}(x,t)| \leq Ce^{C_1|x|^{2-\eta}} \tag{3.26}$$

in the interval $0 \leq t \leq T$, for $\eta > 0$ and for all $0 \leq |\beta| \leq r$. Then

$$|\frac{\partial^{|\beta|}u}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}(x,t)| \leq C_3 e^{C_1|x|} \tag{3.27}$$

for some constant C_3 and for $0 \leq t \leq T$.

Proof. From Proposition 3.4.2 and using the hypothesis on the differentiability of ϱ, f and the coefficients of $L_{x,t}$, by differentiating (3.20) with respect to x_j, it turns out that $\partial_j u$ is the only solution of

$$L_{x,t}v(x,t) = \partial_j f(x,t) - \partial_j c(x,t)u(x,t) \tag{3.28}$$

satisfying the condition

$$v(x,0) = \partial_j u(x,0) \tag{3.29}$$

on $\overline{H}$. By using the conditions on ϱ, f and applying Proposition 3.4.1, it follows that $|\partial_j u(x,t)| \leq C_3 e^{C_1|x|^{2-\eta}}$, for some constant C_3 and for $0 \leq t \leq T$. An iteration of the previous method for the higher derivatives with respect to x allows us to prove (3.27) for all $0 \leq |\beta| \leq r$.

3.5 Exponential Decay of $\widetilde{\widetilde{a}}_0^{(i)}$ and $\widetilde{\psi}^{(i)}$

In the sections 3.3 and 3.4 the existence of solutions to (3.13) and for (3.14) has been shown. Now we want to discuss the behaviour at infinity of the functions $\widetilde{\widetilde{a}}_0^{(i)}$ and $\psi^{(i)}$ for all $i \in \mathbb{N}$. Let us suppose that the first iterates, i.e. the initial data $\widetilde{\widetilde{a}}_0(\cdot, 0)$, $\widetilde{a}(\cdot, 0)$ and $\widetilde{\phi}(\cdot, 0)$ are C_c^∞ in x. During the proof we will need the following:

Lemma 3.5.1 *Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that $\partial_j f \in L^\infty(\mathbb{R}^2)$ exists, for $j = 1, 2$. Then f is Hölderian.*

Proof. Let us start with

$$\begin{aligned} |f(x,y) - f(x',y')| &= |f(x,y) - f(x',y) + f(x',y) - f(x',y')| \\ &\leq |f(x,y) - f(x',y)| + |f(x',y) - f(x',y')|. \end{aligned}$$

From the mean value theorem

$$\frac{|f(x,y) - f(x',y)|}{|x - x'|} = |\partial_1 f(\eta, y)|,$$

follows for some $\eta \in]x', x[$ and also

$$\frac{|f(x',y) - f(x',y')|}{|y - y'|} = |\partial_1 f(x', \xi)|$$

for some $\xi \in]y', y[$. Therefore

$$\frac{|f(x,y) - f(x',y)|}{|x - x'|} \leq \sup_{(v,w)\in\mathbb{R}^2} |\partial_1 f(v,w)| \leq C$$

and

$$\frac{|f(x',y) - f(x',y')|}{|y - y'|} \leq \sup_{(v,w)\in\mathbb{R}^2} |\partial_2 f(v,w)| \leq C,$$

where such a constant $C > 0$ exists by hypothesis.

Under the previous hypothesis we can show the following

Theorem 3.5.1 *Let us suppose that the initial data* $\widetilde{\widetilde{a}}_0(\cdot, 0) = 0$, $\widetilde{a}(\cdot, 0)$ *and* $\widetilde{\phi}(\cdot, 0)$ *are in* C_c^∞ *in* $\mathbb{R}^2$. *Then* $\widetilde{\widetilde{a}}_0^{(i+1)}$ *and* $\psi^{(i+1)}$, *which are the solutions of* (3.13) *and* (3.14) *respectively, are in* C^∞ *in* $\mathbb{R}^2$ *for any* $i \in \mathbb{N}$ *and their partial derivatives with respect to* x *are exponentially decaying as* x *tends to infinity.*

Proof. The results will be proven by induction on i. For $i = 0$, using the hypothesis on the initial data, the definition of $\widetilde{\phi}^{(0)}$ given at the end of section 3.2 and the fact that the zero modes decay exponentially at infinity with respect to x, it follows that

$$|f_0(0, x, t)| \leq C^{(0)} e^{-m^{(0)}|x|}$$

for some constants $C^{(0)}, m^{(0)} > 0$. Then, since $f_0(0, \cdot, t) \in L^2(\mathbb{R}^2)$, from Proposition 3.3.1 it turns out that $\widetilde{\widetilde{a}}_0^{(1)}(\cdot, t)$ is the unique solution in the sense of distributions of (3.13) such that

$$\lim_{|x|\to+\infty} \widetilde{\widetilde{a}}_0^{(1)}(x, t) = 0$$

and $\widetilde{\widetilde{a}}_0^{(1)}(\cdot,t) \in H^1(\mathbb{R}^2)$. Actually, since $\widetilde{a}(\cdot,0)$, $\widetilde{\phi}(\cdot,0)$, $\widetilde{\widetilde{a}}_0(\cdot,0)$ and $\partial_t\widetilde{\phi}(\cdot,0)$ are C^∞ functions in x identically zero outside a certain compact subset of $\mathbb{R}^2$, since the zero modes are also C^∞ in x and all their partial derivatives with respect to x are exponentially decaying when x tends to infinity, it follows that $f_0(0,\cdot,t) \in C^\infty(\mathbb{R}^2)$ for any $0 \le t \le T$ and, for any positive integer r,

$$|\frac{\partial^{|\beta|} f_0}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}(0,x,t)| \le C^{(\beta)} e^{-m^{(\beta)}|x|} \tag{3.30}$$

for some constants $C^{(\beta)}, m^{(\beta)} > 0$, for all $0 \le |\beta| \le r$, and $0 \le t \le T$. Therefore $f_0(0,\cdot,t) \in H^m(\mathbb{R}^2)$, with an arbitrary large integer $m \ge 0$. Thus, applying Proposition 3.3.2, one obtains $\widetilde{\widetilde{a}}_0^{(1)}(\cdot,t) \in H^{m+2}(\mathbb{R}^2)$. Now, using Proposition 3.3.3, one obtains that $\widetilde{\widetilde{a}}_0^{(1)}(\cdot,t) \in C^\infty(\mathbb{R}^2)$ when $0 \le t \le T$. From Proposition 3.1.2 it follows that, by choosing $d = 2$, $j = 1$, $p = 3$ and $q = 2$, $H^{m+2}(\mathbb{R}^2) \subseteq W^{1,3}(\mathbb{R}^2)$. Then, for any positive integer r, another positive integer m big enough can be chosen such that

$$\frac{\partial^{|\beta|}\widetilde{\widetilde{a}}_0^{(1)}}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}(\cdot,t) \in W^{1,3}(\mathbb{R}^2)$$

for all $0 \le |\beta| \le r$, when $0 \le t \le T$. Thus, Proposition 3.1.3 allows us to conclude that

$$\lim_{|x|\to+\infty} \frac{\partial^{|\beta|}\widetilde{\widetilde{a}}_0^{(1)}}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}(x,t) = 0 \tag{3.31}$$

uniformly in x for all $0 \le |\beta| \le r$, and $0 \le t \le T$. By differentiating (3.13) with respect to x on $\mathbb{R}^2$ and recalling (3.30) and (3.31) one can apply Proposition 3.1.1 so that

$$|\frac{\partial^{|\beta|}\widetilde{\widetilde{a}}_0^{(1)}}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}(x,t)| \le C_1^{(\beta)} e^{-m_1^{(\beta)}|x|}$$

with constants $C_1^{(\beta)}, m_1^{(\beta)} > 0$, for all $0 \leq |\beta| \leq r$, when $0 \leq t \leq T$.

Now, since $\widetilde{a}(\cdot,0)$, $\widetilde{\phi}(\cdot,0)$, $\widetilde{\widetilde{a}}_0^{(1)}(\cdot,t)$ and the zero modes are C^∞ on $\mathbb{R}^2$, it turns out that also $f_1(0,\cdot,t) \in C^\infty(\mathbb{R}^2)$ for any $t \in [0,T]$. Now, using Proposition 3.4.2, one finds that $\begin{pmatrix} \sigma\widetilde{a}^{(1)}(\cdot,t) \\ \widetilde{\phi}^{(1)}(\cdot,t) \end{pmatrix}$, the solution of (3.14), is C^∞ on $\mathbb{R}^2$ for any $t \in [0,T]$. Furthermore, since the derivatives with respect to x of $\widetilde{a}(\cdot,0)$, $\widetilde{\phi}(\cdot,0)$, $\widetilde{\widetilde{a}}_0^{(1)}(\cdot,t)$ and of the zero modes up to any order are exponentially decaying as $|x|$ tends to infinity, one obtains that the derivatives of $f_1(0,\cdot,t)$ with respect to x up to any order are exponentially decaying as $|x|$ tends to infinity, for any $t \in [0,T]$. In particular, from

$$|\frac{\partial^{|\beta|} f_1}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}(0,x,t)| \leq C_2^{(\beta)} e^{-m_2^{(\beta)}|x|}$$

with constants $C_2^{(\beta)}, m_2^{(\beta)} > 0$, for all $0 \leq |\beta| \leq r$, when $0 \leq t \leq T$, it follows immediately that

$$\frac{\partial^{|\beta|} f_1}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}(0,\cdot,t) \in L^\infty(\mathbb{R}^2)$$

for all $0 \leq |\beta| \leq r$, when $0 \leq t \leq T$. Thus, using Lemma 3.5.1, one obtains that $\frac{\partial^{|\beta|} f_1}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}(0,\cdot,t)$ is Hölderian on x, for all $0 \leq |\beta| \leq r-1$ and $0 \leq t \leq T$. Now, by using Proposition 3.4.3, we find that two constants $C_3^{(\beta)}, m_3^{(\beta)} > 0$ exist such that

$$|\frac{\partial^{|\beta|} \widetilde{a}^{(1)}}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}(x,t)| + |\frac{\partial^{|\beta|} \widetilde{\phi}^{(1)}}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}(x,t)| \leq C_3^{(\beta)} e^{-m_3^{(\beta)}|x|}$$

for all $0 \leq |\beta| \leq r-1$, when $0 \leq t \leq T$. We now assume that $\widetilde{\widetilde{a}}_0^{(l)}$, $\widetilde{a}^{(l)}$ and $\widetilde{\phi}^{(l)}$ are C^∞ on $\mathbb{R}^2$ and that, for any positive integer r, constants $B_1^{(\beta)}, B_2^{(\beta)}, B_3^{(\beta)}, k_1^{(\beta)}, k_2^{(\beta)}, k_3^{(\beta)} > 0$ exist for which

$$|\frac{\partial^{|\beta|} \widetilde{\widetilde{a}}_0^{(l)}}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}(x,t)| \leq B_1^{(\beta)} e^{-k_1^{(\beta)}|x|},$$

$$|\frac{\partial^{|\beta|}\widetilde{a}^{(l)}}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}(x,t)| \leq B_2^{(\beta)} e^{-k_2^{(\beta)}|x|}$$

and

$$|\frac{\partial^{|\beta|}\widetilde{\phi}^{(l)}}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}(x,t)| \leq B_3^{(\beta)} e^{-k_3^{(\beta)}|x|}$$

for all $0 \leq |\beta| \leq r$, when $0 \leq t \leq T$, for $l \leq i$. The aim is to prove that the same inequalities hold for $\widetilde{\widetilde{a}}_0^{(i+1)}$, $\widetilde{a}^{(i+1)}$ and $\widetilde{\phi}^{(i+1)}$.

Using (3.14) and the induction hypothesis one obtains that $\partial_t\widetilde{\phi}^{(i)}$ and all its derivatives with respect to x are exponentially decaying, as x tends to infinity. Using this result, again the induction hypothesis on $\widetilde{\widetilde{a}}_0^{(i)}$, $\widetilde{a}^{(i)}$ and $\widetilde{\phi}^{(i)}$ and the fact that the zero modes decay exponentially at infinity with respect to x, it follows that

$$|f_0(i,x,t)| \leq \overline{C}^{(0)} e^{-\overline{m}^{(0)}|x|}$$

for some constants $\overline{C}^{(0)}, \overline{m}^{(0)} > 0$. Then, since $f_0(i,\cdot,t) \in L^2(\mathbb{R}^2)$, from Proposition 3.3.1 it turns out that $\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)$ is the unique solution in the distributional sense of (3.13) such that

$$\lim_{|x|\to+\infty} \widetilde{\widetilde{a}}_0^{(i+1)}(x,t) = 0$$

and $\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t) \in H^1(\mathbb{R}^2)$. Actually, since $\widetilde{a}^{(i)}(\cdot,t)$, $\widetilde{\phi}^{(i)}(\cdot,t)$, $\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)$, $\partial_t\widetilde{\phi}^{(i)}(\cdot,t)$ and the zero modes are C^∞ functions in x and all their derivatives with respect to x are exponentially decaying when x tends to infinity, it follows that $f_0(i,\cdot,t) \in C^\infty(\mathbb{R}^2)$ for any $0 \leq t \leq T$, and for any positive integer r,

$$|\frac{\partial^{|\beta|} f_0}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}(i,x,t)| \leq \overline{C}^{(\beta)} e^{-\overline{m}^{(\beta)}|x|} \tag{3.32}$$

for some constants $\overline{C}^{(\beta)}, \overline{m}^{(\beta)} > 0$, for all $0 \le |\beta| \le r$, when $0 \le t \le T$. Therefore $f_0(i, \cdot, t) \in H^m(\mathbb{R}^2)$, with an arbitrary large integer $m \ge 0$. Thus, applying Proposition 3.3.2, one obtains $\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot, t) \in H^{m+2}(\mathbb{R}^2)$. Now, using Proposition 3.3.3, one obtains that $\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot, t) \in C^\infty(\mathbb{R}^2)$ when $0 \le t \le T$. From Proposition 3.1.2 it follows that, by choosing $d = 2$, $j = 1$, $p = 3$ and $q = 2$, $H^{m+2}(\mathbb{R}^2) \subseteq W^{1,3}(\mathbb{R}^2)$ and then, for any positive integer r, another positive integer m big enough can be chosen such that

$$\frac{\partial^{|\beta|}\widetilde{\widetilde{a}}_0^{(i+1)}}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}}(\cdot, t) \in W^{1,3}(\mathbb{R}^2)$$

for all $0 \le |\beta| \le r$, when $0 \le t \le T$. Thus, Proposition 3.1.3 allows us to conclude that

$$\lim_{|x| \to +\infty} \frac{\partial^{|\beta|}\widetilde{\widetilde{a}}_0^{(i+1)}}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}}(x, t) = 0 \tag{3.33}$$

uniformly in x for all $0 \le |\beta| \le r$, when $0 \le t \le T$. By differentiating (3.13) with respect to x on $\mathbb{R}^2$ and recalling (3.32) and (3.33) one can apply Proposition 3.1.1 so that

$$\left|\frac{\partial^{|\beta|}\widetilde{\widetilde{a}}_0^{(i+1)}}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}}(x, t)\right| \le \overline{C}_1^{(\beta)} e^{-\overline{m}_1^{(\beta)}|x|}$$

with constants $\overline{C}_1^{(\beta)}, \overline{m}_1^{(\beta)} > 0$, for all $0 \le |\beta| \le r$, when $0 \le t \le T$.
Since $\widetilde{a}^{(i)}(\cdot, t)$, $\widetilde{\phi}^{(i)}(\cdot, t)$, $\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot, t)$ and the zero modes are C^∞ on $\mathbb{R}^2$, it turns out that also $f_1(i, \cdot, t) \in C^\infty(\mathbb{R}^2)$ for any $t \in [0, T]$. Now, using Proposition 3.4.2, one finds that $\begin{pmatrix} \sigma\widetilde{a}^{(i+1)}(\cdot, t) \\ \widetilde{\phi}^{(i+1)}(\cdot, t) \end{pmatrix}$, the solution of (3.14), is C^∞ on $\mathbb{R}^2$ for any $t \in [0, T]$. Furthermore, since every derivative with respect to x of $\widetilde{a}^{(i)}(\cdot, t)$, $\widetilde{\phi}^{(i)}(\cdot, t)$, $\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot, t)$ and the zero modes are exponentially decaying

as $|x|$ tends to infinity, one obtains that also any derivative with respect to x of $f_1(i,\cdot,t)$ is exponentially decaying as x tends to infinity, for any $t \in [0,T]$. In particular, from

$$|\frac{\partial^{|\beta|} f_1}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}}(i,x,t)| \leq \overline{C}_2^{(\beta)} e^{-\overline{m}_2^{(\beta)}|x|}$$

with constants $\overline{C}_2^{(\beta)}, \overline{m}_2^{(\beta)} > 0$, for all $0 \leq |\beta| \leq r$, when $0 \leq t \leq T$, it follows immediately that

$$\frac{\partial^{|\beta|} f_1}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}}(i,\cdot,t) \in L^\infty(\mathbb{R}^2)$$

for all $0 \leq |\beta| \leq r$, when $0 \leq t \leq T$. Thus, using Lemma 3.5.1, one obtains that $\frac{\partial^{|\beta|} f_1}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}}(i,\cdot,t)$ is Hölderian on x, for all $0 \leq |\beta| \leq r-1$ and when $0 \leq t \leq T$. Now, by using Proposition 3.4.3, we see that two constants $\overline{C}_3^{(\beta)}, \overline{m}_3^{(\beta)} > 0$ exist such that

$$|\frac{\partial^{|\beta|} \widetilde{a}^{(i+1)}}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}}(x,t)| + |\frac{\partial^{|\beta|} \widetilde{\phi}^{(i+1)}}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}}(x,t)| \leq \overline{C}_3^{(\beta)} e^{-\overline{m}_3^{(\beta)}|x|}$$

for all $0 \leq |\beta| \leq r-1$, when $0 \leq t \leq T$. So we conclude that all the derivatives with respect to x of $\widetilde{a}^{(i+1)}$ and $\widetilde{\phi}^{(i+1)}$ are exponentially decaying at infinity.

3.6 Estimates for $\widetilde{\widetilde{a}}_0^{(i)}$ and $\psi^{(i)}$

Estimates in certain Sobolev spaces are derived from the equations of the iterative system. First of all the norms for those spaces are defined. Then estimates for the fields $\widetilde{\widetilde{a}}_0^{(i)}$ and $\psi^{(i)}$ are given in terms of the right-hand sides of (3.13) and (3.14).

From Theorem 3.5.1 one knows that all the iterates $\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)$ are in $H^3(\mathbb{R}^2)$, and $\psi^{(i)}(\cdot,t)$ are in $H^{3,\alpha(\cdot,q(0))}(\mathbb{R}^2)$. This allows us to define

$$||\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)|| := \max_{0\leq s\leq t} |\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)|^2_{H^3(\mathbb{R}^2)}$$

and

$$||\psi^{(i)}(\cdot,t)|| := \max_{0\leq s\leq t} |\psi^{(i)}(\cdot,s)|_{3,\alpha(\cdot,q(0))}.$$

For the $2N$-real parameter family the following norm is defined:

$$|q^{(i)}(t)| := \sum_{\mu=1}^{2N} |q_\mu^{(i)}(t)|.$$

Firstly we state

Lemma 3.6.1 *Let the initial data* $\widetilde{\widetilde{a}}_0(\cdot,0) = 0$, $\widetilde{a}(\cdot,0)$ *and* $\widetilde{\phi}(\cdot,0)$ *be in* $C_c^\infty(\mathbb{R}^2)$. *Then, for any* $\widetilde{\widetilde{a}}_0^{(i+1)}$, *the solution of* (3.13), *a constant* $C > 0$ *independent of* i *exists such that*

$$|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)|^2_{H^3(\mathbb{R}^2)} \leq C|f_0(i,\cdot,t)|^2_{H^1(\mathbb{R}^2)}. \tag{3.34}$$

Proof. From the hypothesis, using Theorem 3.5.1, it turns out that the quantities in (3.34) are all well-defined. Thus the inequality in (3.34) follows directly from (3.18) for $m=1$ and with the constant $C>0$ independent of $f(i,\cdot,t)$, i.e. independent of i.

Furthermore, an estimate for $\psi^{(i)}$ is given by the following

Lemma 3.6.2 *Let the initial data* $\widetilde{\widetilde{a}}_0(\cdot,0) = 0$, $\widetilde{a}(\cdot,0)$ *and* $\widetilde{\phi}(\cdot,0)$ *be in* $C_c^\infty(\mathbb{R}^2)$. *Then, for any* $\psi^{(i+1)}$, *the solution of* (3.14), *constants* $C, K > 0$

independent of i exist, where $K := 2\big(1 + R_1 + R_2 + R_3\big)^2 > 2$ *with constants* $R_1, R_2, R_3 > 0$ *independent of i, such that*

$$\max_{0\le s\le t} |\psi^{(i+1)}(\cdot,s)|_{3,\alpha(\cdot,q(0))}$$
$$\le K|\psi^{(i+1)}(\cdot,0)|_{3,\alpha(\cdot,q(0))} + C\int_0^t |f_1(i,\cdot,s)|_{2,\alpha(\cdot,q(0))}ds. \tag{3.35}$$

Proof. The inequality (3.35) will be proven for any component of $\psi^{(i+1)}$. Multiplying the real component of (3.15) by $\sigma\widetilde{a}_h^{(i+1)}$, noticing that

$$\sigma\widetilde{a}_h^{(i+1)}\partial_t\sigma\widetilde{a}_h^{(i+1)} = \frac{1}{2}\partial_t(|\sigma\widetilde{a}_h^{(i+1)}|^2)$$

and integrating on $\mathbb{R}^2$, one obtains

$$\frac{1}{2}\partial_t|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + \big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t), (-\Delta + |\phi^{(i)}(\cdot,t)|^2)\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)$$
$$\le |\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)f^1_{1h}(i,\cdot,t)|_{L^1(\mathbb{R}^2)}. \tag{3.36}$$

From [Stu94,I, lemma 9.1] it is known that a constant $c > 1$ independent of i exists such that

$$c^{-1}|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|^2_{H^1(\mathbb{R}^2)} \le \big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t), (-\Delta + |\phi^{(i)}(\cdot,t)|^2)\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big). \tag{3.37}$$

By means of the Hölder inequality,

$$|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)f^1_{1h}(i,\cdot,t)|_{L^1(\mathbb{R}^2)} \le |\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|_{L^2(\mathbb{R}^2)}|f^1_{1h}(i,\cdot,t)|_{L^2(\mathbb{R}^2)} \tag{3.38}$$

holds, and using the Cauchy inequality,

$$|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|_{L^2(\mathbb{R}^2)}|f^1_{1h}(i,\cdot,t)|_{L^2(\mathbb{R}^2)}$$
$$\le \eta|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + \frac{1}{4\eta}|f^1_{1h}(i,\cdot,t)|^2_{L^2(\mathbb{R}^2)} \tag{3.39}$$

with $\eta > 0$ small. From (3.36), (3.37), (3.38), (3.39) one obtains

$$\partial_t|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + (c^{-1}-2\eta)|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|^2_{H^1(\mathbb{R}^2)} \le \frac{1}{2\eta}|f^1_{1h}(i,\cdot,t)|^2_{H^2(\mathbb{R}^2)}$$

and then, as $c^{-1}-2\eta > 0$,

$$\partial_t|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} \le \frac{1}{2\eta}|f^1_{1h}(i,\cdot,t)|^2_{H^2(\mathbb{R}^2)}.$$

Integrating the last inequality with respect to t, one obtains

$$|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} \le |\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)|^2_{L^2(\mathbb{R}^2)} + \frac{1}{2\eta}\int_0^t |f^1_{1h}(i,\cdot,s)|^2_{H^2(\mathbb{R}^2)}ds. \tag{3.40}$$

We now use exactly the same arguments for the higher derivatives. Differentiating with respect to x_j the real component of (3.14), multiplying it by $\partial_j(\sigma\widetilde{a}_h^{(i+1)})$, noticing that

$$\partial_j(\sigma\widetilde{a}_h^{(i+1)})\partial_t\partial_j(\sigma\widetilde{a}_h^{(i+1)}) = \frac{1}{2}\partial_t\partial_j(|\sigma\widetilde{a}_h^{(i+1)}|^2)$$

and integrating on $\mathbb{R}^2$, we obtain

$$\begin{aligned}
&\frac{1}{2}\partial_t|\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \sigma^2\big(\partial_j\widetilde{a}_h^{(i+1)}(\cdot,t), (-\Delta + |\phi^{(i)}(\cdot,t)|^2)\partial_j\widetilde{a}_h^{(i+1)}(\cdot,t)\big) \\
&\quad \le |\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)\partial_j f^1_{1h}(i,\cdot,t)|_{L^1(\mathbb{R}^2)} \\
&\quad + |\partial_j(|\phi^{(i)}(\cdot,t)|^2)\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^1(\mathbb{R}^2)}.
\end{aligned} \tag{3.41}$$

From [Stu94,I, lemma 9.1] it is known that a constant $c > 1$ independent of i exists such that

$$c^{-1}|\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{H^1(\mathbb{R}^2)} \le \sigma^2\big(\partial_j\widetilde{a}^{(i+1)}(\cdot,t), (-\Delta + |\phi^{(i)}(\cdot,t)|^2)\partial_j\widetilde{a}_h^{(i+1)}(\cdot,t)\big). \tag{3.42}$$

By means of the Hölder inequality and Remark 3.1.1 we get

$$
\begin{aligned}
&|\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)\partial_j f_{1h}^1(i,\cdot,t)|_{L^1(\mathbb{R}^2)} \\
&\quad + |\partial_j(|\phi^{(i)}(\cdot,t)|^2)\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^1(\mathbb{R}^2)} \\
&\quad \le |\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^2(\mathbb{R}^2)}|\partial_j f_{1h}^1(i,\cdot,t)|_{L^2(\mathbb{R}^2)} \\
&\quad + M|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|_{L^2(\mathbb{R}^2)}|\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^2(\mathbb{R}^2)}
\end{aligned}
\tag{3.43}
$$

and, using the Cauchy inequality, one obtains

$$
\begin{aligned}
&|\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^2(\mathbb{R}^2)}|\partial_j f_{1h}^1(i,\cdot,t)|_{L^2(\mathbb{R}^2)} \\
&\quad + M|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|_{L^2(\mathbb{R}^2)}|\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^2(\mathbb{R}^2)} \\
&\quad \le \eta|\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} + \frac{1}{4\eta}|\partial_j f_{1h}^1(i,\cdot,t)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \frac{M}{4\eta}|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + M\eta|\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)}
\end{aligned}
\tag{3.44}
$$

with $\eta > 0$ small. From (3.41), (3.42), (3.43), (3.44), (3.45)

$$
\begin{aligned}
&\partial_t|\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \big(2c^{-1} - 2\eta(M+1)\big)|\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{H^1(\mathbb{R}^2)} \\
&\quad \le \frac{1}{2\eta}|\partial_j f_{1h}^1(i,\cdot,t)|^2_{L^2(\mathbb{R}^2)} + \frac{M}{2\eta}|\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \frac{M}{4\eta^2}\int_0^t |f_{1h}^1(i,\cdot,s)|^2_{H^2(\mathbb{R}^2)}ds
\end{aligned}
\tag{3.45}
$$

follows and then, as $2c^{-1} - 2\eta(M+1) > 0$,

$$
\begin{aligned}
\partial_t|\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} &\le \frac{1}{2\eta}|\partial_j f_{1h}^1(i,\cdot,t)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \frac{M}{2\eta}|\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)|^2_{L^2(\mathbb{R}^2)} + \frac{M}{4\eta^2}\int_0^t |f_{1h}^1(i,\cdot,s)|^2_{H^2(\mathbb{R}^2)}ds.
\end{aligned}
$$

Integrating the last inequality with respect to t, one obtains

$$
\begin{aligned}
&|\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} \le |\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)\big)|^2_{L^2(\mathbb{R}^2)}\\
&\quad + \frac{1}{2\eta}\int_0^t |\partial_j f^1_{1h}(i,\cdot,s)|^2_{L^2(\mathbb{R}^2)}ds + \frac{M}{2\eta}\int_0^t |\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)|^2_{L^2(\mathbb{R}^2)}ds\\
&\quad + \frac{M}{4\eta^2}\int_0^t\int_0^s |f^1_{1h}(i,\cdot,\tau)|^2_{H^2(\mathbb{R}^2)}d\tau ds\\
&\quad \le |\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)\big)|^2_{L^2(\mathbb{R}^2)} + \frac{1}{2\eta}\int_0^t |\partial_j f^1_{1h}(i,\cdot,s)|^2_{L^2(\mathbb{R}^2)}ds\\
&\quad + \frac{Mt}{2\eta}|\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)|^2_{L^2(\mathbb{R}^2)} + \frac{Mt}{4\eta^2}\int_0^t |f^1_{1h}(i,\cdot,s)|^2_{H^2(\mathbb{R}^2)}ds,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\sum_{j=1}^2 |\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} \le \frac{Mt}{\eta}|\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)|^2_{L^2(\mathbb{R}^2)}\\
&\quad + \sum_{j=1}^2 |\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)\big)|^2_{L^2(\mathbb{R}^2)} + \Big(\frac{1}{\eta} + \frac{Mt}{2\eta^2}\Big)\int_0^t |f^1_{1h}(i,\cdot,s)|^2_{H^2(\mathbb{R}^2)}ds.
\end{aligned} \tag{3.46}
$$

Differentiating with respect to x_j, x_k the real component of (3.14), multiplying it by $\partial^2_{jk}(\sigma\widetilde{a}_h^{(i+1)})$, noticing that

$$
\partial^2_{jk}(\sigma\widetilde{a}_h^{(i+1)})\partial_t\partial^2_{jk}(\sigma\widetilde{a}_h^{(i+1)}) = \frac{1}{2}\partial_t\partial^2_{jk}(|\sigma\widetilde{a}_h^{(i+1)}|^2)
$$

and integrating on $\mathbb{R}^2$,

$$
\begin{aligned}
&\frac{1}{2}\partial_t|\partial^2_{jk}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)}\\
&\quad + \sigma^2\big(\partial^2_{jk}\widetilde{a}_h^{(i+1)}(\cdot,t), (-\Delta + |\phi^{(i)}(\cdot,t)|^2)\partial^2_{jk}\widetilde{a}_h^{(i+1)}(\cdot,t)\big)\\
&\quad \le |\partial^2_{jk}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)\partial^2_{jk}f^1_{1h}(i,\cdot,t)|_{L^1(\mathbb{R}^2)}\\
&\quad + |\partial^2_{jk}(|\phi^{(i)}(\cdot,t)|^2)\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\partial^2_{jk}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^1(\mathbb{R}^2)}\\
&\quad + 2|\partial_k(|\phi^{(i)}(\cdot,t)|^2)\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)\partial^2_{jk}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^1(\mathbb{R}^2)}
\end{aligned} \tag{3.47}
$$

follows. From [Stu94,I, lemma 9.1] it is known that a constant $c > 1$ independent of i exists such that

$$c^{-1}|\partial^2_{jk}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{H^1(\mathbb{R}^2)} \\ \leq \sigma^2\big(\partial^2_{jk}\widetilde{a}_h^{(i+1)}(\cdot,t),(-\Delta+|\phi^{(i)}(\cdot,t)|^2)\partial^2_{jk}\widetilde{a}_h^{(i+1)}(\cdot,t)\big). \tag{3.48}$$

By means of the Hölder inequality and Remark 3.1.1 one obtains

$$|\partial^2_{jk}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)\partial^2_{jk}f^1_{1h}(i,\cdot,t)|_{L^1(\mathbb{R}^2)} \\ + |\partial^2_{jk}(|\phi^{(i)}(\cdot,t)|^2)\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\partial^2_{jk}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^1(\mathbb{R}^2)} \\ + 2|\partial_k(|\phi^{(i)}(\cdot,t)|^2)\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)\partial^2_{jk}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^1(\mathbb{R}^2)} \\ \leq |\partial^2_{jk}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^2(\mathbb{R}^2)}|\partial^2_{jk}f^1_{1h}(i,\cdot,t)|_{L^2(\mathbb{R}^2)} \\ + M|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|_{L^2(\mathbb{R}^2)}|\partial^2_{jk}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^2(\mathbb{R}^2)} \\ +2M|\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^2(\mathbb{R}^2)}|\partial^2_{jk}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^2(\mathbb{R}^2)} \tag{3.49}$$

and, using the Cauchy inequality,

$$|\partial^2_{jk}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^2(\mathbb{R}^2)}|\partial^2_{jk}f^1_{1h}(i,\cdot,t)|_{L^2(\mathbb{R}^2)} \\ + M|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|_{L^2(\mathbb{R}^2)}|\partial^2_{jk}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^2(\mathbb{R}^2)} \\ + 2M|\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^2(\mathbb{R}^2)}|\partial^2_{jk}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^2(\mathbb{R}^2)} \\ \leq |\partial^2_{jk}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} + \frac{1}{4\eta}|\partial^2_{jk}f^1_{1h}(i,\cdot,t)|^2_{L^2(\mathbb{R}^2)} \\ + \frac{M}{4\eta}|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + \eta M|\partial^2_{jk}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} \\ + \frac{M}{2\eta}|\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} + 2\eta M|\partial^2_{jk}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} \tag{3.50}$$

follows with $\eta > 0$ small. From (3.47), (3.48), (3.49), (3.50), (3.40), (3.46), one obtains

$$\begin{aligned}&\frac{1}{2}\partial_t|\partial_{jk}^2\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} + \big(c^{-1} - \eta(3M+1)\big)|\partial_{jk}^2\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{H^1(\mathbb{R}^2)}\\&\quad\le \frac{1}{4\eta}|\partial_{jk}^2 f_{1h}^1(i,\cdot,t)|^2_{L^2(\mathbb{R}^2)} + \Big(\frac{M}{4\eta} + \frac{M^2t}{8\eta^2}\Big)|\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)|^2_{L^2(\mathbb{R}^2)}\\&\quad+ \frac{M}{4\eta}|\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)\big)|^2_{L^2(\mathbb{R}^2)} + \Big(\frac{M}{8\eta^2} + \frac{M^2t}{16\eta^3}\Big)\int_0^t |f_{1h}^1(i,\cdot,s)|^2_{H^2(\mathbb{R}^2)}ds\end{aligned} \tag{3.51}$$

and then, as $c^{-1} - \eta(3M+1) > 0$,

$$\begin{aligned}\partial_t|\partial_{jk}^2\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} &\le \frac{1}{2\eta}|\partial_{jk}^2 f_{1h}^1(i,\cdot,t)|^2_{L^2(\mathbb{R}^2)}\\&+ \Big(\frac{M}{2\eta} + \frac{M^2t}{4\eta^2}\Big)|\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)|^2_{L^2(\mathbb{R}^2)} + \frac{M}{2\eta}|\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)\big)|^2_{L^2(\mathbb{R}^2)}\\&+ \Big(\frac{M}{4\eta^2} + \frac{M^2t}{8\eta^3}\Big)\int_0^t |f_{1h}^1(i,\cdot,s)|^2_{H^2(\mathbb{R}^2)}ds.\end{aligned}$$

Integrating the last inequality with respect to t, yields

$$\begin{aligned}&|\partial_{jk}^2\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} \le |\partial_{jk}^2\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)\big)|^2_{L^2(\mathbb{R}^2)}\\&\quad+ \frac{1}{2\eta}\int_0^t |\partial_{jk}^2 f_{1h}^1(i,\cdot,s)|^2_{L^2(\mathbb{R}^2)}ds + \int_0^t \Big(\frac{M}{2\eta} + \frac{M^2s}{4\eta^2}\Big)|\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)|^2_{L^2(\mathbb{R}^2)}ds\\&\quad+ \int_0^t\int_0^s \Big(\frac{M}{2\eta^2} + \frac{M^2s}{8\eta^3}\Big)|f_{1h}^1(i,\cdot,\tau)|^2_{H^2(\mathbb{R}^2)}d\tau ds\\&\quad+ \int_0^t \frac{M}{2\eta}|\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)\big)|^2_{L^2(\mathbb{R}^2)}ds\\&\quad\le |\partial_{jk}^2\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)\big)|^2_{L^2(\mathbb{R}^2)} + \frac{1}{2\eta}\int_0^t |\partial_{jk}^2 f_{1h}^1(i,\cdot,s)|^2_{L^2(\mathbb{R}^2)}ds\\&\quad+ \Big(\frac{Mt}{2\eta} + \frac{M^2t^2}{4\eta^2}\Big)|\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)|^2_{L^2(\mathbb{R}^2)}\\&\quad+ \Big(\frac{Mt}{2\eta^2} + \frac{M^2t^2}{8\eta^3}\Big)\int_0^t |f_{1h}^1(i,\cdot,s)|^2_{H^2(\mathbb{R}^2)}ds + \frac{Mt}{2\eta}|\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)\big)|^2_{L^2(\mathbb{R}^2)},\end{aligned}$$

which implies

$$\sum_{j,k=1}^{2} |\partial^2_{jk}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} \le \Big(\frac{2Mt}{\eta}+\frac{M^2t^2}{\eta^2}\Big)|\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)|^2_{L^2(\mathbb{R}^2)}$$
$$+\frac{Mt}{\eta}\sum_{j=1}^{2}|\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)\big)|^2_{L^2(\mathbb{R}^2)}+\sum_{j,k=1}^{2}|\partial^2_{jk}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)\big)|^2_{L^2(\mathbb{R}^2)}$$
$$+\Big(\frac{2}{\eta}+\frac{2Mt}{\eta^2}+\frac{M^2t^2}{2\eta^3}\Big)\int_0^t |f^1_{1h}(i,\cdot,s)|^2_{H^2(\mathbb{R}^2)}ds. \tag{3.52}$$

Differentiating with respect to x_j, x_k, x_l the real component of (3.14), multiplying it by $\partial^3_{jkl}(\sigma\widetilde{a}_h^{(i+1)})$, noticing that

$$\partial^3_{jkl}(\sigma\widetilde{a}_h^{(i+1)})\partial_t\partial^3_{jkl}(\sigma\widetilde{a}_h^{(i+1)})=\frac{1}{2}\partial_t\partial^3_{jkl}(|\sigma\widetilde{a}_h^{(i+1)}|^2),$$

integrating on $\mathbb{R}^2$ and using Remark 3.1.1, we get

$$\partial_t|\partial^3_{jkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)}$$
$$+\sigma^2\big(\partial^3_{jkl}\widetilde{a}^{(i+1)}(\cdot,t),(-\Delta+|\phi^{(i)}(\cdot,t)|^2)\partial^3_{jkl}\widetilde{a}_h^{(i+1)}(\cdot,t)\big)$$
$$\le |\partial^4_{jjkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)\partial^2_{kl}f^1_{1h}(i,\cdot,t)|_{L^1(\mathbb{R}^2)}$$
$$+M|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\partial^3_{jkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^1(\mathbb{R}^2)}$$
$$+3M|\partial_l\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)\partial^3_{jkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^1(\mathbb{R}^2)}$$
$$+3M|\partial^2_{jl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)\partial^3_{jkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^1(\mathbb{R}^2)}. \tag{3.53}$$

From [Stu94,I, lemma 9.1] it is known that a constant $c>1$ independent of i exists such that

$$c^{-1}|\partial^3_{jkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{H^1(\mathbb{R}^2)}$$
$$\le\sigma^2\big(\partial^3_{jkl}\widetilde{a}^{(i+1)}(\cdot,t),(-\Delta+|\phi^{(i)}(\cdot,t)|^2)\partial^3_{jkl}\widetilde{a}_h^{(i+1)}(\cdot,t)\big). \tag{3.54}$$

By means of the Hölder inequality the following holds:

$$
\begin{aligned}
&|\partial^4_{jjkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)\partial^2_{kl}f^1_{1h}(i,\cdot,t)|_{L^1(\mathbb{R}^2)} \\
&\quad + M|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\partial^3_{jkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^1(\mathbb{R}^2)} \\
&\quad + 3M|\partial_l\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)\partial^3_{jkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^1(\mathbb{R}^2)} \\
&\quad + 3M|\partial^2_{jl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)\partial^3_{jkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^1(\mathbb{R}^2)} \\
&\quad \leq |\partial^4_{jjkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^2(\mathbb{R}^2)}|\partial^2_{kl}f^1_{1h}(i,\cdot,t)|_{L^2(\mathbb{R}^2)} \\
&\quad + M|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|_{L^2(\mathbb{R}^2)}|\partial^3_{jkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^2(\mathbb{R}^2)} \\
&\quad + 3M|\partial_l\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^2(\mathbb{R}^2)}|\partial^3_{jkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^2(\mathbb{R}^2)} \\
&\quad + 3M|\partial^2_{jl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^2(\mathbb{R}^2)}|\partial^3_{jkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^2(\mathbb{R}^2)}
\end{aligned}
\tag{3.55}
$$

and, using Cauchy's inequality, one obtains

$$
\begin{aligned}
&|\partial^4_{jjlk}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^2(\mathbb{R}^2)}|\partial^2_{kl}f^1_{1h}(i,\cdot,t)|_{L^2(\mathbb{R}^2)} \\
&\quad + M|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|_{L^2(\mathbb{R}^2)}|\partial^3_{jkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^2(\mathbb{R}^2)} \\
&\quad + 3M|\partial_l\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^2(\mathbb{R}^2)}|\partial^3_{jkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^2(\mathbb{R}^2)} \\
&\quad + 3M|\partial^2_{jl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^2(\mathbb{R}^2)}|\partial^3_{jkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|_{L^2(\mathbb{R}^2)} \\
&\quad \leq \eta|\partial^4_{jjkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} + \frac{1}{4\eta}|\partial^2_{jk}f^1_{1h}(i,\cdot,t)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \frac{M}{4\eta}|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + M\eta|\partial^3_{jkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \frac{3M}{4\eta}|\partial_l\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} + 3M\eta|\partial^3_{jkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \frac{3M}{4\eta}|\partial^2_{jl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} + 3M\eta|\partial^3_{jkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)}
\end{aligned}
\tag{3.56}
$$

with $\eta > 0$ small. It should be noticed that

$$
\begin{aligned}
&|\partial^4_{jjkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} \\
&\quad = \sigma^2\big(\partial^3_{jkl}\widetilde{a}_h^{(i+1)}(\cdot,t), (-\Delta + |\phi^{(i)}(\cdot,t)|^2)\partial^3_{jkl}\widetilde{a}_h^{(i+1)}(\cdot,t)\big) \\
&\quad - \sigma^2\big(\partial^3_{jkl}\widetilde{a}_h^{(i+1)}(\cdot,t), |\phi^{(i)}(\cdot,t)|^2\partial^3_{jkl}\widetilde{a}_h^{(i+1)}(\cdot,t)\big).
\end{aligned}
\tag{3.57}
$$

Now, from (3.53), (3.54), (3.55), (3.56), (3.57), (3.40), (3.46), (3.52),

$$
\begin{aligned}
&\frac{1}{2}\partial_t|\partial^3_{jkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} + \sigma^2\big((1-\eta)c^{-1} - 7M\eta\big)|\partial^3_{jkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{H^1(\mathbb{R}^2)} \\
&\quad + \eta\sigma^2\big(\partial^3_{jkl}\widetilde{a}_h^{(i+1)}(\cdot,t), |\phi^{(i)}(\cdot,t)|^2\partial^3_{jkl}\widetilde{a}_h^{(i+1)}(\cdot,t)\big) \\
&\quad \le \frac{1}{4\eta}|\partial^2_{jk}f^1_{1h}(i,\cdot,t)|^2_{L^2(\mathbb{R}^2)} + \Big(\frac{M}{4\eta} + \frac{3M^2t}{4\eta^2} + \frac{3M^3t^2}{16\eta^3}\Big)|\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \Big(\frac{7M}{8\eta^2} + \frac{9M^2t}{16\eta^3} + \frac{3M^3t^2}{32\eta^4}\Big)\int_0^t |f^1_{1h}(i,\cdot,s)|^2_{H^2(\mathbb{R}^2)}ds \\
&\quad + \Big(\frac{3M}{4\eta} + \frac{3M^2t}{8\eta^2}\Big)|\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)\big)|^2_{L^2(\mathbb{R}^2)} + \frac{3M}{4\eta}|\partial^2_{jk}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)\big)|^2_{L^2(\mathbb{R}^2)}
\end{aligned}
\tag{3.58}
$$

follows and then, as $(1-\eta)c^{-1} - 7M\eta > 0$,

$$
\begin{aligned}
\partial_t|\partial^3_{jkl}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} &\le \frac{1}{2\eta}|\partial^2_{jk}f^1_{1h}(i,\cdot,t)|^2_{L^2(\mathbb{R}^2)} \\
&+ \Big(\frac{M}{2\eta} + \frac{3M^2t}{2\eta^2} + \frac{3M^3t^2}{8\eta^3}\Big)|\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)|^2_{L^2(\mathbb{R}^2)} \\
&+ \Big(\frac{7M}{4\eta^2} + \frac{9M^2t}{8\eta^3} + \frac{3M^3t^2}{16\eta^4}\Big)\int_0^t |f^1_{1h}(i,\cdot,s)|^2_{H^2(\mathbb{R}^2)}ds \\
&+ \Big(\frac{3M}{2\eta} + \frac{3M^2t}{4\eta^2}\Big)|\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)\big)|^2_{L^2(\mathbb{R}^2)} \\
&+ \frac{3M}{2\eta}|\partial^2_{jk}\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)\big)|^2_{L^2(\mathbb{R}^2)}.
\end{aligned}
$$

Integrating the last inequality with respect to t, one obtains

$$
\begin{aligned}
&|\partial^3_{jkl}(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t))|^2_{L^2(\mathbb{R}^2)} \le |\partial^3_{jkl}(\sigma\widetilde{a}_h^{(i+1)}(\cdot,0))|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \frac{1}{2\eta}\int_0^t |\partial^2_{jk} f^1_{1h}(i,\cdot,s)|^2_{L^2(\mathbb{R}^2)} ds \\
&\quad + \int_0^t \Big(\frac{M}{2\eta} + \frac{3M^2 s}{2\eta^2} + \frac{3M^3 s^2}{8\eta^3}\Big)|\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)|^2_{L^2(\mathbb{R}^2)} ds \\
&\quad + \int_0^t\int_0^s \Big(\frac{7M}{4\eta^2} + \frac{3M^2 s}{8\eta^3} + \frac{3M^3 s^2}{16\eta^4}\Big)|f^1_{1h}(i,\cdot,\tau)|^2_{H^2(\mathbb{R}^2)} d\tau ds \\
&\quad + \int_0^t \Big(\frac{3M^2 s}{4\eta^2} + \frac{3M}{2\eta}\Big)|\partial_j(\sigma\widetilde{a}_h^{(i+1)}(\cdot,0))|^2_{L^2(\mathbb{R}^2)} ds \\
&\quad + \frac{3Mt}{2\eta}|\partial^2_{jk}(\sigma\widetilde{a}_h^{(i+1)}(\cdot,0))|^2_{L^2(\mathbb{R}^2)} \\
&\quad \le |\partial^3_{jkl}(\sigma\widetilde{a}_h^{(i+1)}(\cdot,0))|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \Big(\frac{1}{2\eta} + \frac{7Mt}{4\eta^2} + \frac{9M^2t^2}{8\eta^3} + \frac{3M^3t^3}{16\eta^4}\Big)\int_0^t |f^1_{1h}(i,\cdot,s)|^2_{H^2(\mathbb{R}^2)} ds \\
&\quad + \Big(\frac{Mt}{2\eta} + \frac{3M^2t^2}{2\eta^2} + \frac{3M^3t^3}{8\eta^3}\Big)|\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \Big(\frac{3M^2t^2}{4\eta^2} + \frac{3Mt}{2\eta}\Big)|\partial_j(\sigma\widetilde{a}_h^{(i+1)}(\cdot,0))|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \frac{3Mt}{2\eta}|\partial^2_{jk}(\sigma\widetilde{a}_h^{(i+1)}(\cdot,0))|^2_{L^2(\mathbb{R}^2)},
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\sum_{j,k,l=1}^{2} |\partial^3_{jkl}(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t))|^2_{L^2(\mathbb{R}^2)} \\
&\quad \le \Big(\frac{4Mt}{\eta} + \frac{12M^2t^2}{\eta^2} + \frac{3M^3t^3}{\eta^3}\Big)|\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \Big(\frac{3M^2t^2}{\eta^2} + \frac{6Mt}{\eta}\Big)\sum_{j=1}^{2}|\partial_j(\sigma\widetilde{a}_h^{(i+1)}(\cdot,0))|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \frac{3Mt}{\eta}\sum_{j,k=1}^{2}|\partial^2_{jk}(\sigma\widetilde{a}_h^{(i+1)}(\cdot,0))|^2_{L^2(\mathbb{R}^2)} + \sum_{j,k,l=1}^{2}|\partial^3_{jkl}(\sigma\widetilde{a}_h^{(i+1)}(\cdot,0))|^2_{L^2(\mathbb{R}^2)}
\end{aligned}
$$

$$+\Big(\frac{1}{2\eta}+\frac{7Mt}{4\eta^2}+\frac{9M^2t^2}{8\eta^3}+\frac{3M^3t^3}{16\eta^4}\Big)\int_0^t |f^1_{1h}(i,\cdot,s)|^2_{H^2(\mathbb{R}^2)}ds. \tag{3.59}$$

Now, adding (3.40), (3.46), (3.52), (3.59), it turns out that there exist a constant t small enough and a constant $C > 0$ independent of i, for which

$$|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|^2_{H^3(\mathbb{R}^2)} \leq 2|\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)|^2_{H^3(\mathbb{R}^2)} + C\int_0^t |f^1_{1h}(i,\cdot,s)|^2_{H^2(\mathbb{R}^2)}ds$$

and then

$$\max_{0\leq s\leq t}|\sigma\widetilde{a}_h^{(i+1)}(\cdot,s)|^2_{H^3(\mathbb{R}^2)} \;\leq\; 2|\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)|^2_{H^3(\mathbb{R}^2)} + C\int_0^t |f^1_{1h}(i,\cdot,s)|^2_{H^2(\mathbb{R}^2)}ds. \tag{3.60}$$

For the complex component of (3.15) one proceeds in the same way as for the real component. Unlike in the previous case, however now the covariant derivatives $D^{(0)}$ must be used instead of the usual derivatives and therefore one needs to find a constant $c > 1$ independent of i such that

$$\begin{aligned}&c^{-1}|\widetilde{\phi}^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + c^{-1}\sum_{l=1}^{2}|D_l^{(0)}\widetilde{\phi}^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\\ &\quad\leq \big(\widetilde{\phi}^{(i+1)}(\cdot,t), -\sum_{l=1}^{2}(D_l^{(0)})^2\widetilde{\phi}^{(i+1)}(\cdot,t) + |\phi^{(i)}(\cdot,t)|^2\widetilde{\phi}^{(i+1)}(\cdot,t)\big).\end{aligned}$$

In fact, from [Stu94,I, lemma 9.1], it is known that such a constant exists, since

$$\begin{aligned}&c^{-1}|\widetilde{\phi}^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + c^{-1}\sum_{l=1}^{2}|D_l^{(0)}\widetilde{\phi}^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\\ &= c^{-1}|\widetilde{\phi}^{(i+1)}(\cdot,t)|^2_{H^1(\mathbb{R}^2)}\\ &+ c^{-1}\int_{\mathbb{R}^2}\big(\widetilde{\phi}^{(i+1)}(x,t), i2\alpha^{(i)}(x,t)\cdot\nabla\widetilde{\phi}^{(i+1)}(x,t)\big)d^2x\end{aligned}$$

$$
\begin{aligned}
&+ c^{-1} \int_{\mathbb{R}^2} |\alpha^{(i)}(x,t)|^2 |\widetilde{\phi}^{(i+1)}(x,t)|^2 d^2x \\
&\le \big(\widetilde{\phi}^{(i+1)}(\cdot,t), (-\Delta + |\phi^{(i)}(\cdot,t)|^2)\widetilde{\phi}^{(i+1)}(\cdot,t)\big) \\
&+ \int_{\mathbb{R}^2} \big(\widetilde{\phi}^{(i+1)}(x,t), i2\alpha^{(i)}(x,t)\cdot\nabla\widetilde{\phi}^{(i+1)}(x,t)\big) d^2x \\
&+ \int_{\mathbb{R}^2} |\alpha^{(i)}(x,t)|^2 |\widetilde{\phi}^{(i+1)}(x,t)|^2 d^2x \\
&= \big(\widetilde{\phi}^{(i+1)}(\cdot,t), -\sum_{l=1}^{2} (D_l^{(0)})^2 \widetilde{\phi}^{(i+1)}(\cdot,t) + |\phi^{(i)}(\cdot,t)|^2 \widetilde{\phi}^{(i+1)}(\cdot,t)\big).
\end{aligned}
$$

Moreover, now the partial covariant derivatives $D_j^{(0)}$ are used where, in the previous case, the usual partial derivatives were used, and, exactly in the same way as before, one obtains

$$
\begin{aligned}
&\max_{0\le s\le t} \big(|\widetilde{\phi}^{(i+1)}(\cdot,s)|^2_{L^2(\mathbb{R}^2)} + \sum_{j=1}^{2} |D_j^{(0)}\widetilde{\phi}^{(i+1)}(\cdot,s)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \sum_{j,k=1}^{2} |D_j^{(0)}D_k^{(0)}\widetilde{\phi}^{(i+1)}(\cdot,s)|^2_{L^2(\mathbb{R}^2)} + \sum_{j,k,l=1}^{2} |D_j^{(0)}D_k^{(0)}D_l^{(0)}\widetilde{\phi}^{(i+1)}(\cdot,s)|^2_{L^2(\mathbb{R}^2)}\big) \\
&\quad \le 2\big(|\widetilde{\phi}^{(i+1)}(\cdot,0)|^2_{L^2(\mathbb{R}^2)} + \sum_{j=1}^{2} |D_j^{(0)}\widetilde{\phi}^{(i+1)}(\cdot,0)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \sum_{j,k=1}^{2} |D_j^{(0)}D_k^{(0)}\widetilde{\phi}^{(i+1)}(\cdot,0)|^2_{L^2(\mathbb{R}^2)} + \sum_{j,k,l=1}^{2} |D_j^{(0)}D_k^{(0)}D_l^{(0)}\widetilde{\phi}^{(i+1)}(\cdot,0)|^2_{L^2(\mathbb{R}^2)}\big) \\
&\quad + C\int_0^t \big(|f_1^2(i,\cdot,s)|^2_{L^2(\mathbb{R}^2)} + \sum_{j=1}^{2} |D_j^{(0)} f_1^2(i,\cdot,s)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \sum_{j,k=1}^{2} |D_j^{(0)}D_k^{(0)} f_1^2(i,\cdot,s)|^2_{L^2(\mathbb{R}^2)}\big) ds. \qquad (3.61)
\end{aligned}
$$

Since the inequality (3.35) involves the norms $|\cdot|_{3,\alpha(\cdot,q(0))}$ and $|\cdot|_{2,\alpha(\cdot,q(0))}$ and the estimate just obtained involves the covariant derivative $D^{(0)}$, we search for a relationship between $D^{(\alpha(\cdot,q(0)))}$ and $D^{(0)}$. Using Cauchy's inequality and Remark 3.1.1, we get the result

$$
\begin{aligned}
|D_j^{(\alpha(\cdot,q(0)))}u(\cdot,t)|_{L^2(\mathbb{R}^2)}^2 &= |\partial_j u(\cdot,t) - i\alpha_j(\cdot,q(0))u(\cdot,t)|_{L^2(\mathbb{R}^2)}^2 \\
&= |D_j^{(0)}u(\cdot,t) + i\big(\alpha_j(\cdot,q(t)) - \alpha_j(\cdot,q(0))\big)u(\cdot,t)|_{L^2(\mathbb{R}^2)}^2 \\
&\leq (1+2\eta)|D_j^{(0)}u(\cdot,t)|_{L^2(\mathbb{R}^2)}^2 + \Big(1+\frac{1}{2\eta}\Big)M^2|u(\cdot,t)|_{L^2(\mathbb{R}^2)}^2 \\
&\leq R_1\big(|u(\cdot,t)|_{L^2(\mathbb{R}^2)}^2 + |D_j^{(0)}u(\cdot,t)|_{L^2(\mathbb{R}^2)}^2\big)
\end{aligned}
$$

with $R_1 > 0$ constant and independent of i. Then

$$
|D_j^{(\alpha(\cdot,q(0)))}u(\cdot,t)|_{L^2(\mathbb{R}^2)}^2 \leq R_1\big(|u(\cdot,t)|_{L^2(\mathbb{R}^2)}^2 + |D_j^{(0)}u(\cdot,t)|_{L^2(\mathbb{R}^2)}^2\big). \qquad (3.62)
$$

In an analogous way

$$
|D_j^{(0)}u(\cdot,t)|_{L^2(\mathbb{R}^2)}^2 \leq R_1\big(|u(\cdot,t)|_{L^2(\mathbb{R}^2)}^2 + |D_j^{(\alpha(\cdot,q(0)))}u(\cdot,t)|_{L^2(\mathbb{R}^2)}^2\big). \qquad (3.63)
$$

We now proceed analogously for the higher covariant derivatives. Using Cauchy's inequality and Remark 3.1.1, one obtains

$$
\begin{aligned}
&|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}u(\cdot,t)|_{L^2(\mathbb{R}^2)}^2 \\
&= |D_j^{(\alpha(\cdot,q(0)))}\big(\partial_k u(\cdot,t) - i\alpha_k(\cdot,q(0))u(\cdot,t)\big)|_{L^2(\mathbb{R}^2)}^2 \\
&= |D_j^{(\alpha(\cdot,q(0)))}D_k^{(0)}u(\cdot,t) \\
&+ iD_j^{(\alpha(\cdot,q(0)))}\Big(\big(\alpha_k(\cdot,q(t)) - \alpha_k(\cdot,q(0))\big)u(\cdot,t)\Big)|_{L^2(\mathbb{R}^2)}^2 \\
&= |D_j^{(0)}D_k^{(0)}u(\cdot,t) + i\big(\alpha_j(\cdot,q(t)) - \alpha_j(\cdot,q(0))\big)D_k^{(0)}u(\cdot,t) \\
&+ i\partial_j\big(\alpha_k(\cdot,q(t)) - \alpha_k(\cdot,q(0))\big)u(\cdot,t) \\
&+ i\big(\alpha_k(\cdot,q(t)) - \alpha_k(\cdot,q(0))\big)D_j^{(0)}u(\cdot,t) \\
&- \alpha_j(\cdot,q(t)))\big(\alpha_k(\cdot,q(t)) - \alpha_k(\cdot,q(0))\big)u(\cdot,t) \\
&+ \alpha_j(\cdot,q(0))\big(\alpha_k(\cdot,q(t))) - \alpha_k(\cdot,q(0))\big)u(\cdot,t)|_{L^2(\mathbb{R}^2)}^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \big(1+12M\eta+8M^2\eta\big)|D_j^{(0)}D_k^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\\
&+\Big(2M^2+24M^2\eta+16M^3\eta+\frac{2M}{\eta}\Big)|D_k^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\\
&+\Big(2M^2+16M^3+12M^4+\frac{2M}{\eta}+\frac{12M^2}{\eta}+\frac{16M^3}{\eta}\Big)|u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\\
&+\Big(2M^2+16M^3\eta+\frac{2M}{\eta}+\frac{4M^2}{\eta}\Big)|D_j^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\\
&\leq R_2\big(|u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}+|D_j^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}+|D_j^{(0)}D_k^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\big)
\end{aligned}
$$

with $R_2>0$ constant independent of i and then

$$
\begin{aligned}
&|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\\
&\quad\leq R_2\big(|u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}+|D_j^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}+|D_j^{(0)}D_k^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\big).
\end{aligned}
\tag{3.64}
$$

Analogously,

$$
\begin{aligned}
&|D_j^{(0)}D_k^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\leq R_2\big(|u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\\
&\quad+|D_j^{(\alpha(\cdot,q(0)))}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}+|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\big).
\end{aligned}
\tag{3.65}
$$

Using Cauchy's inequality and Remark 3.1.1, the following holds

$$
\begin{aligned}
&|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}D_l^{(\alpha(\cdot,q(0)))}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\\
&=|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}\Big(D_l^{(0)}u(\cdot,t)-i\big(\alpha_l(\cdot,q(t))-\alpha_l(\cdot,q(0))\big)u(\cdot,t)\Big)|^2_{L^2(\mathbb{R}^2)}\\
&=|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}D_l^{(0)}u(\cdot,t)\\
&-iD_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}\Big(\big(\alpha_l(\cdot,q(t))-\alpha_l(\cdot,q(0))\big)u(\cdot,t)\Big)|^2_{L^2(\mathbb{R}^2)}\\
&=|D_j^{(\alpha(\cdot,q(0)))}\Big(D_k^{(0)}D_l^{(0)}u(\cdot,t)+i\big(\alpha_k(\cdot,q(t))-\alpha_k(\cdot,q(0))\big)D_l^{(0)}u(\cdot,t)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&- iD_j^{(\alpha(\cdot,q(0)))}\Big(\partial_k\big(\alpha_l(\cdot,q(t)) - \alpha_l(\cdot,q(0))\big)u(\cdot,t)\\
&+ \big(\alpha_l(\cdot,q(t)) - \alpha_l(\cdot,q(0))\big)D_k^{(0)}u(\cdot,t)\\
&+ i\alpha_l(\cdot,q(t))\big(\alpha_k(\cdot,q(t)) - \alpha_k(\cdot,q(0))\big)u(\cdot,t)\Big)|_{L^2(\mathbb{R}^2)}^2\\
&= |D_j^{(\alpha(\cdot,q(0)))}D_k^{(0)}D_l^{(0)}u(\cdot,t) + iD_j^{(\alpha(\cdot,q(0)))}\big(\alpha_k(\cdot,q(t))D_l^{(0)}u(\cdot,t)\big)\\
&- iD_j^{(\alpha(\cdot,q(0)))}\big(\alpha_k(\cdot,q(0))D_l^{(0)}u(\cdot,t)\big)\\
&- iD_j^{(\alpha(\cdot,q(0)))}\Big(\partial_k\big(\alpha_l(\cdot,q(t)) - \alpha_l(\cdot,q(0))\big)u(\cdot,t)\Big)\\
&- iD_j^{(\alpha(\cdot,q(0)))}\big(\alpha_l(\cdot,q(t))D_k^{(0)}u(\cdot,t)\big)\\
&+ D_j^{(\alpha(\cdot,q(0)))}\Big(\alpha_l(\cdot,q(t))\big(\alpha_k(\cdot,q(t)) - \alpha_k(\cdot,q(0))\big)u(\cdot,t)\Big)\\
&+ iD_j^{(\alpha(\cdot,q(0)))}\big(\alpha_l(\cdot,q(0))D_k^{(0)}u(\cdot,t)\big)|_{L^2(\mathbb{R}^2)}^2\\
&\le |D_j^{(0)}D_k^{(0)}D_l^{(0)}u(\cdot,t) + i\alpha_j(\cdot,q(t))\alpha_j(\cdot,q(0))D_k^{(0)}D_l^{(0)}u(\cdot,t)\\
&+ \partial_j\alpha_k(\cdot,q(t))D_l^{(0)}u(\cdot,t) + i\alpha_k(\cdot,q(t))D_j^{(0)}D_l^{(0)}u(\cdot,t)\\
&- \alpha_j(\cdot,q(t))\alpha_k(\cdot,q(t))D_l^{(0)}u(\cdot,t) + \alpha_j(\cdot,q(0))\alpha_k(\cdot,q(t))D_l^{(0)}u(\cdot,t)\\
&- i\partial_j\alpha_k(\cdot,q(0))D_l^{(0)}u(\cdot,t) - i\alpha_k(\cdot,q(0))D_j^{(0)}D_l^{(0)}u(\cdot,t)\\
&+ \alpha_j(\cdot,q(t))\alpha_k(\cdot,q(0))D_l^{(0)}u(\cdot,t) + \alpha_j(\cdot,q(0))\alpha_k(\cdot,q(0))D_l^{(0)}u(\cdot,t)\\
&- i\partial_{jk}^2\big(\alpha_l(\cdot,q(t)) - \alpha_l(\cdot,q(0))\big)u(\cdot,t) - i\partial_k\big(\alpha_l(\cdot,q(t)) - \alpha_l(\cdot,q(0))\big)D_j^{(0)}u(\cdot,t)\\
&+ \alpha_j(\cdot,q(t))\partial_k\big(\alpha_l(\cdot,q(t)) - \alpha_l(\cdot,q(0))\big)u(\cdot,t)\\
&+ \alpha_j(\cdot,q(0))\partial_k\big(\alpha_l(\cdot,q(t)) - \alpha_l(\cdot,q(0))\big)u(\cdot,t) - i\partial_j\alpha_l(\cdot,q(t))D_k^{(0)}u(\cdot,t)\\
&- i\alpha_l(\cdot,q(t))D_j^{(0)}D_k^{(0)}u(\cdot,t) + \alpha_j(\cdot,q(t))\alpha_l(\cdot,q(t))D_k^{(0)}u(\cdot,t)\\
&+ \alpha_j(\cdot,q(0))\alpha_l(\cdot,q(t))D_k^{(0)}u(\cdot,t) + \partial_j\alpha_l(\cdot,q(t))\big(\alpha_k(\cdot,q(t)) - \alpha_k(\cdot,q(0))\big)u(\cdot,t)\\
&+ \alpha_l(\cdot,q(t))\partial_j\big(\alpha_k(\cdot,q(t)) - \alpha_k(\cdot,q(0))\big)u(\cdot,t)\\
&+ \alpha_l(\cdot,q(t))\big(\alpha_k(\cdot,q(t)) - \alpha_k(\cdot,q(0))\big)D_j^{(\alpha(\cdot,q(0)))}u(\cdot,t)\\
&+ i\alpha_j(\cdot,q(t))\alpha_l(\cdot,q(t))\big(\alpha_k(\cdot,q(t)) - \alpha_k(\cdot,q(0))\big)u(\cdot,t)
\end{aligned}
$$

$$
\begin{aligned}
&- i\alpha_j(\cdot,q(0))\alpha_l(\cdot,q(t))\big(\alpha_k(\cdot,q(t)) - \alpha_k(\cdot,q(0))\big)u(\cdot,t)\\
&+ i\partial_j\alpha_l(\cdot,q(0))D_k^{(0)}u(\cdot,t) + i\alpha_l(\cdot,q(0))D_j^{(0)}D_k^{(0)}u(\cdot,t)\\
&- \alpha_j(\cdot,q(t))\alpha_l(\cdot,q(0))D_k^{(0)}u(\cdot,t) - \alpha_j(\cdot,q(0))\alpha_l(\cdot,q(0))D_k^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\\
&\le (2CM + 8CM^2 + 4CM^3)|u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\\
&+ (2CM + 2CM^2)|D_j^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + (2CM + 4CM^2)|D_k^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\\
&+ (2CM + 4CM^2)|D_l^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + 2CM|D_j^{(0)}D_k^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\\
&+ 2CM|D_j^{(0)}D_l^{(0)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + CM^2|D_k^{(0)}D_l^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\\
&+ C|D_j^{(0)}D_k^{(0)}D_l^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\\
&\le R_3\big(|u(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + |D_j^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\\
&+ |D_j^{(0)}D_k^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + |D_j^{(0)}D_k^{(0)}D_l^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\big),
\end{aligned}
$$

with $C, R_3 > 0$ constants independent of i and then

$$
\begin{aligned}
|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}D_l^{(\alpha(\cdot,q(0)))}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)} &\le R_3\big(|u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\\
+ |D_j^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + |D_j^{(0)}D_k^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)} &+ |D_j^{(0)}D_k^{(0)}D_l^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\big).
\end{aligned}
\tag{3.66}
$$

In an analogous way

$$
\begin{aligned}
|D_j^{(0)}D_k^{(0)}D_l^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)} &\le R_3\big(|u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\\
&+ |D_j^{(\alpha(\cdot,q(0)))}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\\
&+ |D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\\
&+ |D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}D_l^{(\alpha(\cdot,q(0)))}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\big).
\end{aligned}
\tag{3.67}
$$

Now, summing (3.62), (3.64), (3.66) and (3.63), (3.65), (3.67) one obtains

$$
\begin{aligned}
&(1+R_1+R_2+R_3)^{-1}\big(|u(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + |D_j^{(\alpha(\cdot,q(0)))}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + |D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + |D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}D_l^{(\alpha(\cdot,q(0)))}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\big) \\
&\quad \le |u(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + |D_j^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + |D_j^{(0)}D_k^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + |D_j^{(0)}D_k^{(0)}D_l^{(0)}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)} \\
&\quad \le (1+R_1+R_2+R_3)\big(|u(\cdot,t)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + |D_j^{(\alpha(\cdot,q(0)))}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + |D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + |D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}D_l^{(\alpha(\cdot,q(0)))}u(\cdot,t)|^2_{L^2(\mathbb{R}^2)}\big).
\end{aligned}
$$

The norm involving $D^{(0)}$ and the norms involving $D^{(\alpha(\cdot,q(0)))}$ are equivalent and then for (3.61), (3.35) holds.

Finally an estimate for the $2N$-parameter family $q^{(i+1)}$ is given by the following

Lemma 3.6.3 *Let $q^{(i+1)}$ be a solution of* (3.15). *Then*

$$|q^{(i+1)}(t)| \le |q^{(i+1)}(0)| + \varepsilon \int_0^t |f_2(i,\cdot,s)|ds. \tag{3.68}$$

Proof. From (3.15) one obtains

$$\frac{dq^{(i+1)}}{dt}(t) = \varepsilon \dot{q}^{(i+1)}(t) = \varepsilon f_2(i,\cdot,t).$$

Integrating with respect to t, we obviously have

$$q^{(i+1)}(t) = q^{(i+1)}(0) + \varepsilon \int_0^t f_2(i,\cdot,s)ds,$$

from which (3.68) follows immediately.

Chapter 4

Upper Bounds for the Iterated Functions

Under certain initial condition it is shown that all the iterated functions from the previous chapter are bounded for a suitably short time. The estimates proven in chapter 3 serve as the first step of this proof. An induction technique is used to prove the boundedness of $\widetilde{\widetilde{a}}_0^{(i+1)}$, $\psi^{(i+1)}$ and $q^{(i+1)}$.

4.1 A New Estimate for $\widetilde{\widetilde{a}}_0^{(i+1)}$

The aim of this section is to find a new estimate for $\|\widetilde{\widetilde{a}}_0^{(i+1)}\|$ in terms of $\|\widetilde{\widetilde{a}}_0^{(i)}\|$, $|\dot{q}^{(i)}|$ and $|\partial_t \psi^{(i)}|$. We start with (3.34) and study the form of the function $f_0(i, \cdot, t)$.

Let $K > 2$ in Lemma 3.6.2 and $\delta \leq (2 - 1/K)\Gamma$. Let us suppose also that for any $i \in \mathbb{N}$

$$\|\psi^{(i)}(\cdot, 0)\|, \quad |q^{(i)}(0)| \leq \frac{\Gamma}{2K}$$

and

$$|\psi^{(i)}(\cdot, 0) - \psi(\cdot, 0)|_{3,\alpha(\cdot,q(0))}, \quad |q^{(i)}(0) - q(0)| \leq \frac{\delta}{2^i}.$$

Firstly the case with $i = 0$ is examined. Using the definition of the first iterated function one obtains

$$\begin{aligned}\|\psi^{(0)}(\cdot)\| &= \|\psi(\cdot, 0)\| = |\psi(\cdot, 0)|_{3,\alpha(\cdot,q(0))} \\ &\leq |\psi(\cdot, 0) - \psi^{(i)}(\cdot, 0)|_{3,\alpha(\cdot,q(0))} + |\psi^{(i)}(\cdot, 0)|_{3,\alpha(\cdot,q(0))} \\ &\leq \frac{\delta}{2^i} + \|\psi^{(i)}(\cdot, 0)\| \leq \frac{\delta}{2^i} + \frac{\Gamma}{2K} \leq \Gamma\end{aligned}$$

and

$$|q^{(0)}(t)| = |q(0)| \leq |q^{(i)}(0) - q(0)| + |q^{(i)}(0)| \leq \frac{\delta}{2^i} + \frac{\Gamma}{2K} \leq \Gamma.$$

Now, let us suppose that for any $l \leq i$ the following inequalities hold:

$$\|\psi^{(l)}(\cdot, t)\|, \quad |q^{(l)}(t)| \leq \Gamma.$$

Then the aim is to show that

$$\|\psi^{(i+1)}(\cdot, t)\|, \quad |q^{(i+1)}(t)| \leq \Gamma.$$

for some t. From Lemma 3.6.1 one knows that

$$\|\approx{a}_0^{(i+1)}(\cdot, t)\| \leq C \max_{0 \leq s \leq t} |f_0(i, \cdot, s)|^2_{H^1(\mathbb{R}^2)}$$

with $C > 0$ constant independent of i. To have a more precise estimate of $\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\|$ one looks at the form of $f_0(i,\cdot,s)$ in (3.10). Using the Schwartz inequality and the Cauchy inequality one obtains

$$\max_{0\le s\le t}|f_0(i,\cdot,s)|^2_{H^1(\mathbb{R}^2)} \le \max_{0\le s\le t}\Big(\varepsilon^2|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)|^2_{H^1(\mathbb{R}^2)}$$
$$+\frac{C}{\sigma^2}\sum_{\mu=1}^{2N}|\big(i\varphi(\cdot,q^{(i)}(s)),\frac{\partial\varphi}{\partial q_\mu}(\cdot,q^{(i)}(s))\big)\dot{q}_\mu^{(i)}(s)|^2_{H^1(\mathbb{R}^2)}$$
$$+\frac{C}{\sigma^2}\sum_{l=1}^{2}|\big(i\partial_l\varphi(\cdot,q^{(i)}(s)),D_l^{(0)}\widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^1(\mathbb{R}^2)}$$
$$+\frac{C}{\sigma^2}\sum_{l=1}^{2}|\big(i\widetilde{\phi}^{(i)}(\cdot,s),\partial_l D_l^{(0)}\varphi(\cdot,q^{(i)}(s))\big)|^2_{H^1(\mathbb{R}^2)}$$
$$+\frac{C}{\sigma^2}\sum_{l=1}^{2}|\big(i\varphi(\cdot,q^{(i)}(s)),\partial_l D_l^{(0)}\widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^1(\mathbb{R}^2)}$$
$$+\frac{4C}{\sigma^2}\sum_{l=1}^{2}|\widetilde{a}_l^{(i)}(\cdot,s)\big(\varphi(\cdot,q^{(i)}(s)),\partial_l\varphi(\cdot,q^{(i)}(s))\big)|^2_{H^1(\mathbb{R}^2)}$$
$$+\frac{C}{\sigma^2}\sum_{l=1}^{2}|\big(i\partial_l\widetilde{\phi}^{(i)}(\cdot,s),D_l^{(0)}\varphi(\cdot,q^{(i)}(s))\big)|^2_{H^1(\mathbb{R}^2)}$$
$$+\varepsilon^2|\big(i\varphi(\cdot,q^{(i)}(s)),\partial_t\widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^1(\mathbb{R}^2)}$$
$$+\frac{C\varepsilon^2}{\sigma^2}||\phi(\cdot,q^{(i)}(s))|^2\big(i\varphi(\cdot,q^{(i)}(s)),\widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^1(\mathbb{R}^2)}$$
$$+\frac{4C\varepsilon^2}{\sigma^2}\sum_{l=1}^{2}|\widetilde{a}_l^{(i)}(\cdot,s)\big(\widetilde{\phi}^{(i)}(\cdot,s),\partial_l\varphi^{(i)}(\cdot,q^{(i)}(s))\big)|^2_{H^1(\mathbb{R}^2)}$$
$$+\frac{4C\varepsilon^2}{\sigma^2}\sum_{l=1}^{2}|\widetilde{a}_l^{(i)}(\cdot,s)\big(\varphi(\cdot,q^{(i)}(s)),\partial_l\widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^1(\mathbb{R}^2)}$$
$$+\frac{C\varepsilon^2}{\sigma^2}\sum_{l=1}^{2}|\big(i\partial_l\widetilde{\phi}^{(i)}(\cdot,s),D_l^{(0)}\widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^1(\mathbb{R}^2)}$$

$$
\begin{aligned}
&+\frac{C\varepsilon^2}{\sigma^2}\sum_{l=1}^{2}|\big(i\widetilde{\phi}^{(i)}(\cdot,s),\partial_l D_l^{(0)}\widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^1(\mathbb{R}^2)}\\
&+C\varepsilon^4\sum_{\mu=1}^{2N}|\big(\widetilde{\phi}^{(i)}(\cdot,s),i\frac{\partial\varphi}{\partial q_\mu}(\cdot,q^{(i)}(s))\big)\dot{q}_\mu^{(i)}(s)|^2_{H^1(\mathbb{R}^2)}\\
&+\frac{4C\varepsilon^4}{\sigma^2}|\big(\varphi(\cdot,q^{(i)}(s)),\widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^1(\mathbb{R}^2)}\\
&+\frac{4C\varepsilon^4}{\sigma^2}\sum_{l=1}^{2}|\widetilde{a}_l^{(i)}(\cdot,s)\big(\widetilde{\phi}^{(i)}(\cdot,s),\partial_l\widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^1(\mathbb{R}^2)}\\
&+\frac{C\varepsilon^6}{\sigma^2}||\widetilde{\phi}^{(i)}(\cdot,s)|^2\big(i\varphi(\cdot,q^{(i)}(s)),\widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^1(\mathbb{R}^2)}\Big)\\
&\le\max_{0\le s\le t}\Big(\varepsilon^2|\widetilde{a}_0^{(i)}(\cdot,s)|^2_{H^1(\mathbb{R}^2)}+C\sum_{\mu=1}^{2N}|\dot{q}_\mu^{(i)}(s)|^2\\
&+C\sum_{l=1}^{2}|D_l^{(0)}\widetilde{\phi}^{(i)}(\cdot,s)|^2_{H^1(\mathbb{R}^2)}+C|\widetilde{\phi}^{(i)}(\cdot,s)|^2_{H^1(\mathbb{R}^2)}\\
&+C\sum_{l=1}^{2}|\partial_l D_l^{(0)}\widetilde{\phi}^{(i)}(\cdot,s)|^2_{H^1(\mathbb{R}^2)}+C\sum_{l=1}^{2}|\widetilde{a}_l^{(i)}(\cdot,s)|^2_{H^1(\mathbb{R}^2)}\\
&+C\sum_{l=1}^{2}|\partial_l\widetilde{\phi}^{(i)}(\cdot,s)|^2_{H^1(\mathbb{R}^2)}+C\varepsilon^2|\partial_t\widetilde{\phi}^{(i)}(\cdot,s)|^2_{H^1(\mathbb{R}^2)}+C|\widetilde{\phi}^{(i)}(\cdot,s)|^2_{H^1(\mathbb{R}^2)}\\
&+\frac{4C\varepsilon^2}{\sigma^2}\sum_{l=1}^{2}|\widetilde{a}_l^{(i)}(\cdot,s)\big(\widetilde{\phi}^{(i)}(\cdot,s),\partial_l\varphi(\cdot,q^{(i)}(s))\big)|^2_{H^1(\mathbb{R}^2)}\\
&+\frac{4C\varepsilon^2}{\sigma^2}\sum_{l=1}^{2}|\widetilde{a}_l^{(i)}(\cdot,s)\big(\varphi(\cdot,q^{(i)}(s)),\partial_l\widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^1(\mathbb{R}^2)}\\
&+C\sum_{l=1}^{2}|\big(\partial_l\widetilde{\phi}^{(i)}(\cdot,s),D_l^{(0)}\widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^1(\mathbb{R}^2)}\\
&+\frac{C\varepsilon^2}{\sigma^2}\sum_{l=1}^{2}|\big(i\widetilde{\phi}^{(i)}(\cdot,s),\partial_l D_l^{(0)}\widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^1(\mathbb{R}^2)}\\
&+C|\widetilde{\phi}^{(i)}(\cdot,s)|^2_{H^1(\mathbb{R}^2)}\sum_{\mu=1}^{2N}|\dot{q}_\mu^{(i)}(s)|^2
\end{aligned}
$$

$$+ \frac{4C\varepsilon^4}{\sigma^2} |\big(\varphi(\cdot, q^{(i)}(s)), \widetilde{\phi}^{(i)}(\cdot, s)\big)\big(i\varphi(\cdot, q^{(i)}(s)), \widetilde{\phi}^{(i)}(\cdot, s)\big)|^2_{H^1(\mathbb{R}^2)}$$

$$+ \frac{4C\varepsilon^4}{\sigma^2} \sum_{l=1}^{2} |\widetilde{a}_l^{(i)}(\cdot, s)\big(\widetilde{\phi}^{(i)}(\cdot, s), \partial_l \widetilde{\phi}^{(i)}(\cdot, s)\big)|^2_{H^1(\mathbb{R}^2)}$$

$$+ \frac{C\varepsilon^6}{\sigma^2} ||\widetilde{\phi}^{(i)}(\cdot, s)|^2 \big(i\varphi(\cdot, q^{(i)}(s)), \widetilde{\phi}^{(i)}(\cdot, s)\big)|^2_{H^1(\mathbb{R}^2)}\Big). \tag{4.1}$$

Here $C > 0$ is a suitable constant independent of i.

Now every component of the right hand-side of (4.1) is estimated. For example, using the Schwartz inequality and the Cauchy inequality, one finds a constant $C > 0$ independent of i such that

$$|\widetilde{a}_l^{(i)}(\cdot, s)\big(\widetilde{\phi}^{(i)}(\cdot, s), \partial_l \varphi(\cdot, q^{(i)}(s))\big)|^2_{H^1(\mathbb{R}^2)}$$
$$\leq C|\widetilde{a}_l^{(i)}(\cdot, s)\big(\widetilde{\phi}^{(i)}(\cdot, s), \partial_l \varphi(\cdot, q^{(i)}(s))\big)|^2_{L^2(\mathbb{R}^2)}$$
$$+ C\sum_{j=1}^{2} |\widetilde{a}_l^{(i)}(\cdot, s)\big(\partial_j \widetilde{\phi}^{(i)}(\cdot, s), \partial_l \varphi(\cdot, q^{(i)}(s))\big)|^2_{L^2(\mathbb{R}^2)}$$
$$+ C\sum_{j=1}^{2} |\widetilde{a}_l^{(i)}(\cdot, s)\big(\widetilde{\phi}^{(i)}(\cdot, s), \partial^2_{jl} \varphi(\cdot, q^{(i)}(s))\big)|^2_{L^2(\mathbb{R}^2)}$$
$$+ C\sum_{j=1}^{2} |\partial_j \widetilde{a}_l^{(i)}(\cdot, s)\big(\widetilde{\phi}^{(i)}(\cdot, s), \partial_l \varphi(\cdot, q^{(i)}(s))\big)|^2_{L^2(\mathbb{R}^2)}.$$

Then, because of the Cauchy-Schwartz inequality and Remark 3.1.1, the terms on the right-hand side are smaller than

$$CM \int_{\mathbb{R}^2} |\widetilde{a}_l^{(i)}(x, s)|^2 |\widetilde{\phi}^{(i)}(x, s)|^2 d^2x$$
$$+ CM \sum_{j=1}^{2} \int_{\mathbb{R}^2} |\widetilde{a}_l^{(i)}(x, s)|^2 |\partial_j \widetilde{\phi}^{(i)}(x, s)|^2 d^2x$$
$$+ CM \int_{\mathbb{R}^2} |\widetilde{a}_l^{(i)}(x, s)|^2 |\widetilde{\phi}^{(i)}(x, s)|^2 d^2x$$

$$+CM\sum_{j=1}^{2}\int_{\mathbb{R}^2}|\partial_j\widetilde{a}_l^{(i)}(x,s)|^2|\widetilde{\phi}^{(i)}(x,s)|^2d^2x.$$

Then, by definition of $|\cdot|_{L^\infty(\mathbb{R}^2)}$, one obtains that the last expression is smaller than

$$\begin{aligned}&CM|\widetilde{\phi}^{(i)}(\cdot,s)|^2_{L^\infty(\mathbb{R}^2)}|\widetilde{a}_l^{(i)}(\cdot,s)|^2_{L^2(\mathbb{R}^2)}\\&\quad+CM|\widetilde{a}_l^{(i)}(\cdot,s)|^2_{L^\infty(\mathbb{R}^2)}\sum_{j=1}^{2}|\partial_j\widetilde{\phi}^{(i)}(\cdot,s)|^2_{L^2(\mathbb{R}^2)}\\&\quad+CM|\widetilde{\phi}^{(i)}(\cdot,s)|^2_{L^\infty(\mathbb{R}^2)}|\widetilde{a}_l^{(i)}(\cdot,s)|^2_{L^2(\mathbb{R}^2)}\\&\quad+CM|\widetilde{\phi}^{(i)}(\cdot,s)|^2_{L^\infty(\mathbb{R}^2)}\sum_{j=1}^{2}|\partial_j\widetilde{a}_l^{(i)}(\cdot,s)|^2_{L^2(\mathbb{R}^2)}.\end{aligned}$$

From [Eva98, page 292, ex. 18] it follows that $|\cdot|_{L^\infty(\mathbb{R}^2)}\leq|\cdot|_{H^2(\mathbb{R}^2)}$, so that the last terms become smaller than

$$\begin{aligned}&CM|\widetilde{\phi}^{(i)}(\cdot,s)|^2_{H^2(\mathbb{R}^2)}|\widetilde{a}_l^{(i)}(\cdot,s)|^2_{H^3(\mathbb{R}^2)}\\&\quad+CM|\widetilde{a}_l^{(i)}(\cdot,s)|^2_{H^3(\mathbb{R}^2)}|\widetilde{\phi}^{(i)}(\cdot,s)|^2_{H^3(\mathbb{R}^2)}\\&\quad+CM|\widetilde{\phi}^{(i)}(\cdot,s)|^2_{H^2(\mathbb{R}^2)}|\widetilde{a}_l^{(i)}(\cdot,s)|^2_{H^3(\mathbb{R}^2)}\\&\quad+CM|\widetilde{\phi}^{(i)}(\cdot,s)|^2_{H^2(\mathbb{R}^2)}|\widetilde{a}_l^{(i)}(\cdot,s)|^2_{H^3(\mathbb{R}^2)}.\end{aligned}$$

Now, since $|\cdot|_{H^k(\mathbb{R}^2)}$ with $k\in\mathbb{N}$ is equivalent to the norm involving $D^{(\alpha(\cdot,q(0)))}$ and, using the fact that $\left\|\psi^{(l)}(\cdot,\overline{T})\right\|\leq\Gamma$ for any $l\leq i$, it turns out that the last sum is smaller than a constant $C(\Gamma)>0$ independent of i.

In the same way one proves:

$$|\widetilde{a}_l^{(i)}(\cdot,s)\big(\varphi(\cdot,q^{(i)}(s)),\partial_l\widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^1(\mathbb{R}^2)}\leq C(\Gamma),$$

$$|\big(\partial_l\widetilde{\phi}^{(i)}(\cdot,s),D_l^{(0)}\widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^1(\mathbb{R}^2)}\leq C(\Gamma),$$

$$|\big(\partial^2_{jl}\widetilde{\phi}^{(i)}(\cdot,s), D^{(0)}_l\widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^1(\mathbb{R}^2)} \le C(\Gamma),$$

$$|\big(\partial_l\widetilde{\phi}^{(i)}(\cdot,s), \partial_j D^{(0)}_l\widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^1(\mathbb{R}^2)} \le C(\Gamma),$$

$$|\big(i\widetilde{\phi}^{(i)}(\cdot,s), \partial_l D^{(0)}_l\widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^1(\mathbb{R}^2)} \le C(\Gamma),$$

$$|\big(\varphi(\cdot,q^{(i)}(s)), \widetilde{\phi}^{(i)}(\cdot,s)\big)\big(i\varphi(\cdot,q^{(i)}(s)), \widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^1(\mathbb{R}^2)} \le C(\Gamma),$$

$$|\widetilde{a}^{(i)}_l(\cdot,s)\big(\widetilde{\phi}^{(i)}(\cdot,s), \partial_l\widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^1(\mathbb{R}^2)} \le C(\Gamma),$$

and

$$||\widetilde{\phi}^{(i)}(\cdot,s)|^2\big(i\varphi(\cdot,q^{(i)}(s)), \widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^1(\mathbb{R}^2)} \le C(\Gamma).$$

Now, using all these estimates for the right-hand side of (4.1), one obtains

$$\begin{aligned}
\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\| &\le \varepsilon^2 C^2(\Gamma)\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\| \\
&+ 2CC^2(\Gamma)\Big(\max_{0\le s\le t}\sum_{\mu=1}^{2N}|\dot{q}^{(i)}_\mu(s)|\Big)^2 + 8CC^3(\Gamma) \\
&+ \varepsilon^2 CC(\Gamma)\max_{0\le s\le t}|\partial_t\widetilde{\phi}^{(i)}(\cdot,s)|^2_{H^1(\mathbb{R}^2)} + \frac{4\varepsilon^2 CC^4(\Gamma)}{\sigma^2} \\
&+ \frac{4\varepsilon^2 CC^3(\Gamma)}{\sigma^2} + CC^4(\Gamma) + \frac{\varepsilon^2 CC^3(\Gamma)}{\sigma^2} \\
&+ \frac{8\varepsilon^4 CC^3(\Gamma)}{\sigma^2} + \frac{\varepsilon^6 CC^3(\Gamma)}{\sigma^2}
\end{aligned} \tag{4.2}$$

and, possibly with a redefined $C(\Gamma) > 0$ independent of i, (4.2) can be written as follows:

$$\begin{aligned}
\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\| &\le \varepsilon C(\Gamma)\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\| \\
&+ C(\Gamma)\max_{0\le s\le t}|\dot{q}^{(i)}(s)|^2 + \varepsilon C(\Gamma)\max_{0\le s\le t}|\partial_t\psi^{(i)}(\cdot,s)|_{1,\alpha(\cdot,q(0))}.
\end{aligned} \tag{4.3}$$

4.2 An Estimate for $\dot{q}^{(i+1)}$ and $q^{(i+1)}$

The equation (3.15) is the starting point to improve the estimate for $\widetilde{\widetilde{a}}_0^{(i+1)}$. Here the aim is to find an estimate for $\dot{q}^{(i+1)}$ in terms of $\|\widetilde{\widetilde{a}}_0^{(i+1)}\|$. We then use such an estimate to evaluate $\widetilde{\widetilde{a}}_0^{(i+1)}$ in terms of $\|\widetilde{\widetilde{a}}_0^{(i)}\|$ and $|\partial_t\psi^{(i)}|$ as well as to evaluate $|q^{(i+1)}|$ by means of $\|\widetilde{\widetilde{a}}_0^{(i+1)}\|$.

From here on we use the notation $A_{(i)}(t)$ for $A\big(\psi^{(i)}(t), q^{(i)}(t)\big)$ and $a_{rs}(i,t)$ for $a_{rs}\big(\psi^{(i)}(t), q^{(i)}(t)\big)$. In (4.3) we need an estimate for $\dot{q}^{(i)}(s)$. It should be noticed that, since

$$A_{(i)}^{-1}(t) = \big(a_{rs}^{-1}(i,t)\big)_{r,s\in\{1,\dots,2N\}}$$

exists, then for any $r,s \in \{1,\dots,2N\}$ the functions

$$a_{rs}^{-1} : \Lambda \longrightarrow \mathbb{R}$$

are continuous in $\psi^{(i)}, q^{(i)}$. Here Λ is a compact subset of $H^{3,\alpha(\cdot,q(0))}(\mathbb{R}^2)\times\mathbb{R}^{2N}$ as, by induction hypothesis, $\|\psi^{(i)}\|$, $|q^{(i)}| \le \Gamma$. Then a_{rs}^{-1} is bounded by a constant $C(\Gamma) > 0$ independent of i, for any $r,s \in \{1,\dots,2N\}$. From (3.15) we obtain

$$\begin{aligned}
|\dot{q}^{(i+1)}(t)| &= \sum_{\nu=1}^{2N} |\dot{q}_\nu^{(i+1)}(t)| = \sum_{\nu=1}^{2N} |f_{2\nu}(i,\cdot,t)| \\
&= \sum_{\nu=1}^{2N} \Big|\frac{1}{2}\sum_{\mu=1}^{2N} a_{\nu\mu}^{-1}(i,t)\int_{\mathbb{R}^2}\big(\varphi(x,q^{(i)}(t))(1-|\phi(x,q^{(i)}(t))|^2), n_\mu^2(x,q^{(i)}(t))\big)d^2x \\
&+ \varepsilon\sum_{\mu=1}^{2N} a_{\nu\mu}^{-1}(i,t)\langle j(\psi^{(i)}(\cdot,t),\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t),q^{(i)}(t)), n_\mu(\cdot,q^{(i)}(t))\rangle\Big| \\
&\le \frac{1}{2}\sum_{\mu,\nu=1}^{2N} \Big|a_{\nu\mu}^{-1}(i,t)\int_{\mathbb{R}^2}\big(\varphi(x,q^{(i)}(t))(1-|\phi(x,q^{(i)}(t))|^2), n_\mu^2(x,q^{(i)}(t))\big)d^2x\Big|
\end{aligned}$$

$$+\varepsilon \sum_{\mu,\nu=1}^{2N} |a_{\nu\mu}^{-1}(i,t)\langle j(\psi^{(i)}(\cdot,t),\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t),q^{(i)}(t)),n_\mu(\cdot,q^{(i)}(t))\rangle|$$

$$\leq \frac{1}{2}C(\Gamma)\sum_{\mu=1}^{2N} |\int_{\mathbb{R}^2} \big(\varphi(x,q^{(i)}(t))(1-|\phi(x,q^{(i)}(t))|^2),n_\mu^2(x,q^{(i)}(t))\big)d^2x|$$

$$+\varepsilon C(\Gamma)\sum_{\mu=1}^{2N} |\langle j(\psi^{(i)}(\cdot,t),\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t),q^{(i)}(t)),n_\mu(\cdot,q^{(i)}(t))\rangle|.$$

Hence, up to redefining $C(\Gamma)$,

$$|\dot{q}^{(i+1)}(t)| \leq C(\Gamma)\sum_{\mu=1}^{2N} |\int_{\mathbb{R}^2} \big(\varphi(x,q^{(i)}(t))(1-|\phi(x,q^{(i)}(t))|^2),n_\mu^2(x,q^{(i)}(t))\big)d^2x|$$

$$+\varepsilon C(\Gamma)\sum_{\mu=1}^{2N} |\langle j(\psi^{(i)}(\cdot,t),\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t),q^{(i)}(t)),n_\mu(\cdot,q^{(i)}(t))\rangle|. \tag{4.4}$$

The first term on the right-hand side of (4.4) is estimated as follows:

$$|\int_{\mathbb{R}^2} \big(\varphi(x,q^{(i)}(t))(1-|\phi(x,q^{(i)}(t))|^2),n_\mu^2(x,q^{(i)}(t))\big)d^2x|$$

$$\leq \int_{\mathbb{R}^2} \big|(1-|\phi(x,q^{(i)}(t))|^2)\big(\varphi(x,q^{(i)}(t)),n_\mu^2(x,q^{(i)}(t))\big)\big|d^2x.$$

We use the Hölder inequality and the fact that $1-|\phi(\cdot,q^{(i)}(t))|^2$ is exponentially decaying as x tends to infinity [JT80, 3.8, theorem 8.5], to show that

$$|\int_{\mathbb{R}^2} \big(\varphi(x,q^{(i)}(t))(1-|\phi(x,q^{(i)}(t))|^2),n_\mu^2(x,q^{(i)}(t))\big)d^2x| \leq C(\Gamma) \tag{4.5}$$

for some constant $C(\Gamma)>0$ independent of i.

In a similar way one proves an estimate for the second term in the right-hand side of (4.4):

$$
\begin{aligned}
&|\langle j(\psi^{(i)}(\cdot,t),\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t),q^{(i)}(t)),n_\mu(\cdot,q^{(i)}(t))\rangle| \\
&\quad\le \sum_{l=1}^{2}\int_{\mathbb{R}^2}|j_l(\psi^{(i)}(x,t),\widetilde{\widetilde{a}}_0^{(i+1)}(x,t),q^{(i)}(t))\frac{\partial\alpha_l}{\partial q_\mu}(x,q^{(i)}(t))|d^2x \\
&\quad+\int_{\mathbb{R}^2}|\big(j_3(\psi^{(i)}(x,t),\widetilde{\widetilde{a}}_0^{(i+1)}(x,t),q^{(i)}(t)),\frac{\partial\varphi}{\partial q_\mu}(x,q^{(i)}(t))\big)|d^2x. \quad (4.6)
\end{aligned}
$$

Now, considering the form of $j(\psi^{(i)}(\cdot,t),\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t),q^{(i)}(t))$ in (2.21) and (2.18) and using the Cauchy-Schwartz inequality, the right-hand side of (4.6) is shown to be smaller than

$$
\begin{aligned}
&C\sum_{l=1}^{2}\int_{\mathbb{R}^2}|\widetilde{a}_l^{(i)}(x,t)||\big(\varphi(x,q^{(i)}(t)),\widetilde{\phi}^{(i)}(x,t)\big)||\frac{\partial\alpha_l}{\partial q_\mu}(x,q^{(i)}(t))|d^2x \\
&\quad+C\sum_{l=1}^{2}\int_{\mathbb{R}^2}|\big(i\widetilde{\phi}^{(i)}(x,t),D_l^{(0)}\widetilde{\phi}^{(i)}(x,t)\big)||\frac{\partial\alpha_l}{\partial q_\mu}(x,q^{(i)}(t))|d^2x \\
&\quad+C\sum_{l=1}^{2}\int_{\mathbb{R}^2}|\widetilde{a}_l^{(i)}(x,t)||\widetilde{\phi}^{(i)}(x,t)|^2|\frac{\partial\alpha_l}{\partial q_\mu}(x,q^{(i)}(t))|d^2x \\
&\quad+C\sum_{l=1}^{2}\int_{\mathbb{R}^2}|\big(i\partial_l\varphi(x,q^{(i)}(t)),\widetilde{\phi}^{(i)}(x,t)\big)||\frac{\partial\alpha_l}{\partial q_\mu}(x,q^{(i)}(t))|d^2x \\
&\quad+C\sum_{l=1}^{2}\int_{\mathbb{R}^2}|\big(i\varphi(x,q^{(i)}(t)),\partial_l\widetilde{\phi}^{(i)}(x,t)\big)||\frac{\partial\alpha_l}{\partial q_\mu}(x,q^{(i)}(t))|d^2x \\
&\quad+C\sum_{l=1}^{2}\int_{\mathbb{R}^2}|\partial_l\widetilde{a}_l^{(i)}(x,t)||\big(i\widetilde{\phi}^{(i)}(x,t),\frac{\partial\varphi}{\partial q_\mu}(x,q^{(i)}(t))\big)|d^2x \\
&\quad+C\sum_{l=1}^{2}\int_{\mathbb{R}^2}|\widetilde{a}_l^{(i)}(x,t)|^2|\big(D_l^{(0)}\widetilde{\phi}^{(i)}(x,t),\frac{\partial\varphi}{\partial q_\mu}(x,q^{(i)}(t))\big)|d^2x \\
&\quad+C\sum_{l=1}^{2}\int_{\mathbb{R}^2}|\widetilde{a}_l^{(i)}(x,t)|^2|\big(i\varphi(x,q^{(i)}(t)),\frac{\partial\varphi}{\partial q_\mu}(x,q^{(i)}(t))\big)|d^2x
\end{aligned}
$$

$$
\begin{aligned}
&+C\sum_{l=1}^{2}\int_{\mathbb{R}^2}|\widetilde{a}_l^{(i)}(x,t)|^2\big|\big(i\widetilde{\phi}^{(i)}(x,t),\frac{\partial\varphi}{\partial q_\mu}(x,q^{(i)}(t))\big)\big|d^2x\\
&+C\int_{\mathbb{R}^2}|1-|\phi(x,q^{(i)}(t))|^2|\big|\big(\widetilde{\phi}^{(i)}(x,t),\frac{\partial\varphi}{\partial q_\mu}(x,q^{(i)}(t))\big)\big|d^2x\\
&+C\int_{\mathbb{R}^2}\big|\big(\varphi(x,q^{(i)}(t)),\widetilde{\phi}^{(i)}(x,t)\big)\big|\big|\big(\widetilde{\phi}^{(i)}(x,t),\frac{\partial\varphi}{\partial q_\mu}(x,q^{(i)}(t))\big)\big|d^2x\\
&+C\int_{\mathbb{R}^2}\big|\big(\varphi(x,q^{(i)}(t)),\widetilde{\phi}^{(i)}(x,t)\big)\big|\big|\big(\varphi(x,q^{(i)}(t)),\frac{\partial\varphi}{\partial q_\mu}(x,q^{(i)}(t))\big)\big|d^2x\\
&+C\int_{\mathbb{R}^2}|\widetilde{\phi}^{(i)}(x,t)|^2\big(\varphi(x,q^{(i)}(t)),\frac{\partial\varphi}{\partial q_\mu}(x,q^{(i)}(t))\big)\big|d^2x\\
&+C\int_{\mathbb{R}^2}|\widetilde{\phi}^{(i)}(x,t)|^2\big|\big(\widetilde{\phi}^{(i)}(x,t),\frac{\partial\varphi}{\partial q_\mu}(x,q^{(i)}(t))\big)\big|d^2x\\
&+C\int_{\mathbb{R}^2}|\widetilde{\widetilde{a}}_0^{(i+1)}(x,t)|\big|\big(\widetilde{\phi}^{(i)}(x,t),\frac{\partial\varphi}{\partial q_\mu}(x,q^{(i)}(t))\big)\big|d^2x,
\end{aligned}
\tag{4.7}
$$

with $C>0$ constant and independent of i. Now every single component of (4.7) is estimated by using Remark 3.1.1, the Hölder inequality, the exponential behaviour of the zero modes as x tends to infinity and the fact that $|\cdot|_{L^\infty(\mathbb{R}^2)}\leq|\cdot|_{H^2(\mathbb{R}^2)}$, in order to find that a constant $C(\Gamma)>0$ exists, independent of i, for which (4.7) becomes smaller than

$$
\begin{aligned}
&C(\Gamma)|\widetilde{\phi}^{(i)}(\cdot,t)|_{L^2(\mathbb{R}^2)}\sum_{l=1}^{2}|\widetilde{a}_l^{(i)}(\cdot,t)|_{H^3(\mathbb{R}^2)}\\
&\quad+C(\Gamma)|\widetilde{\phi}^{(i)}(\cdot,t)|_{H^3(\mathbb{R}^2)}|\widetilde{\phi}^{(i)}(\cdot,t)|_{L^2(\mathbb{R}^2)}\\
&\quad+C(\Gamma)|\widetilde{\phi}^{(i)}(\cdot,t)|^2_{H^3(\mathbb{R}^2)}\sum_{l=1}^{2}|\widetilde{a}_l^{(i)}(\cdot,t)|_{L^2(\mathbb{R}^2)}\\
&\quad+C(\Gamma)|\widetilde{\phi}^{(i)}(\cdot,t)|_{L^2(\mathbb{R}^2)}+C(\Gamma)\sum_{l=1}^{2}|\partial_l\widetilde{\phi}^{(i)}(\cdot,t)|_{L^2(\mathbb{R}^2)}\\
&\quad+|\widetilde{\phi}^{(i)}(\cdot,t)|_{L^2(\mathbb{R}^2)}\sum_{l=1}^{2}|\widetilde{a}_l^{(i)}(\cdot,t)|_{H^3(\mathbb{R}^2)}
\end{aligned}
$$

$$+ C(\Gamma)|\widetilde{\phi}^{(i)}(\cdot,t)|_{H^3(\mathbb{R}^2)}\sum_{l=1}^{2}|\widetilde{a}_l^{(i)}(\cdot,t)|_{H^3(\mathbb{R}^2)}|\widetilde{a}_l^{(i)}(\cdot,t)|_{L^2(\mathbb{R}^2)}$$

$$+ C(\Gamma)\sum_{l=1}^{2}|\widetilde{a}_l^{(i)}(\cdot,t)|_{H^3(\mathbb{R}^2)}|\widetilde{a}_l^{(i)}(\cdot,t)|_{L^2(\mathbb{R}^2)}$$

$$+ C(\Gamma)|\widetilde{\phi}^{(i)}(\cdot,t)|_{L^2(\mathbb{R}^2)}\sum_{l=1}^{2}|\widetilde{a}_l^{(i)}(\cdot,t)|^2_{H^3(\mathbb{R}^2)} + C(\Gamma)|\widetilde{\phi}^{(i)}(\cdot,t)|_{L^2(\mathbb{R}^2)}$$

$$+ C(\Gamma)|\widetilde{\phi}^{(i)}(\cdot,t)|_{H^3(\mathbb{R}^2)}|\widetilde{\phi}^{(i)}(\cdot,t)|_{L^2(\mathbb{R}^2)} + C(\Gamma)|\widetilde{\phi}^{(i)}(\cdot,t)|_{L^2(\mathbb{R}^2)}$$

$$+ C(\Gamma)|\widetilde{\phi}^{(i)}(\cdot,t)|_{H^3(\mathbb{R}^2)}|\widetilde{\phi}^{(i)}(\cdot,t)|_{L^2(\mathbb{R}^2)} + C(\Gamma)|\widetilde{\phi}^{(i)}(\cdot,t)|^2_{H^3(\mathbb{R}^2)}|\widetilde{\phi}^{(i)}(\cdot,t)|_{L^2(\mathbb{R}^2)}$$

$$+ C(\Gamma)|\widetilde{\phi}^{(i)}(\cdot,t)|_{H^3(\mathbb{R}^2)}|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)|_{L^2(\mathbb{R}^2)}$$

$$\leq C(\Gamma) + C(\Gamma)\left\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\right\|^{1/2}. \tag{4.8}$$

Putting together (4.6), (4.7) and (4.8) it turns out that

$$\begin{aligned}&|\langle j(\psi^{(i)}(\cdot,t),\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t),q^{(i)}(t)),n_\mu(\cdot,q^{(i)}(t))\rangle|\\&\qquad\leq C(\Gamma) + C(\Gamma)\left\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\right\|^{1/2}.\end{aligned} \tag{4.9}$$

Now, substituting (4.5) and (4.9) into (4.4), one obtains that two constants $E(\Gamma), F(\Gamma) > 0$ exist, independent of i, for which

$$|\dot{q}^{(i+1)}(t)| \leq E(\Gamma) + \varepsilon F(\Gamma)\left\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\right\|^{1/2}. \tag{4.10}$$

By the same arguments,

$$|\dot{q}^{(i)}(s)| \leq E(\Gamma) + \varepsilon F(\Gamma)\left\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\right\|^{1/2}. \tag{4.11}$$

Now we want to use (4.11) to improve the estimate for $\widetilde{\widetilde{a}}_0^{(i)}$. Substituting (4.11) into (4.3) leads to the result

$$\left\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\right\| \leq \varepsilon C(\Gamma)\left\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\right\|$$

$$+ C(\Gamma) \max_{0\le s\le t} \left(E(\Gamma) + \varepsilon F(\Gamma) \|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\|^{1/2}\right)^2$$
$$+ \varepsilon C(\Gamma) \max_{0\le s\le t} |\partial_t \psi^{(i)}(\cdot,s)|_{1,\alpha(\cdot,q(0))}.$$

Then, using the Cauchy inequality and redefining the constant $C(\Gamma)$, one obtains

$$\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\| \le C(\Gamma) + \varepsilon C(\Gamma) \|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\|$$
$$+ \varepsilon C(\Gamma) \max_{0\le s\le t} |\partial_t \psi^{(i)}(\cdot,s)|_{1,\alpha(\cdot,q(0))}. \tag{4.12}$$

Moreover we use (4.10) to find an estimate more suitable to our purposes. Apply (4.10) to (3.68) and it follows that

$$|q^{(i+1)}(t)| \le \frac{\Gamma}{2} + \varepsilon C(\Gamma) \int_0^t \left(1 + \varepsilon \|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\|^{1/2}\right) ds \tag{4.13}$$

for some constant $C(\Gamma) > 0$ independent of i.

4.3 A New Estimate for $\psi^{(i+1)}$

The aim of this section is to find a new estimate for $\|\psi^{(i+1)}\|$ in terms of $\|\widetilde{\widetilde{a}}_0^{(i)}\|$ and $\|\widetilde{\widetilde{a}}_0^{(i+1)}\|$. We start with (3.35) and study next the form of the function $f_1(i,\cdot,t)$.

Using the definition of $\|\cdot\|$ given at the start of section 3.6, (3.35) can be written as

$$\|\psi^{(i+1)}(\cdot,t)\| \le K \|\psi^{(i+1)}(\cdot,0)\| + C \int_0^t |f_1(i,\cdot,s)|_{2,\alpha(\cdot,q(0))} ds \tag{4.14}$$

where $K > 0$ and $C > 0$ are suitable constants independent of i. The next step is to estimate $f_1(i,\cdot,s)$ in order to improve (4.14). It follows from the definition of $|\cdot|_{2,\alpha(\cdot,q(0))}$ and the Cauchy inequality that a constant $C > 0$ exists independent of i, for which

$$
\begin{aligned}
|f_1(i,\cdot,s)|_{2,\alpha(\cdot,q(0))} &= \sum_{l=1}^{2} |f_{1l}^1(i,\cdot,s)|^2_{H^2(\mathbb{R}^2)} \\
&+ |f_1^2(i,\cdot,s)|^2_{L^2(\mathbb{R}^2)} + \sum_{j=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} f_1^2(i,\cdot,s)|^2_{L^2(\mathbb{R}^2)} \\
&+ \sum_{j,k=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} D_k^{(\alpha(\cdot,q(0)))} f_1^2(i,\cdot,s)|^2_{L^2(\mathbb{R}^2)} \\
&\leq C \sum_{l=1}^{2} |\big(i\widetilde{\phi}^{(i)}(\cdot,s), D_l^{(0)}\varphi(\cdot,q^{(i)}(s))\big)|^2_{H^2(\mathbb{R}^2)} \\
&+ C \sum_{l=1}^{2} \sum_{\mu=1}^{2N} |\frac{\partial \alpha_l}{\partial q_\mu}(\cdot,q^{(i)}(s))|^2_{H^2(\mathbb{R}^2)} |\dot{q}_\mu^{(i)}(s)|^2 \\
&+ C \sum_{l=1}^{2} |\partial_l \widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)|^2_{H^2(\mathbb{R}^2)} + C \sum_{l=1}^{2} |\big(i\partial_l\varphi(\cdot,q^{(i)}(s)), \widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^2(\mathbb{R}^2)} \\
&+ C \sum_{l=1}^{2} |\big(i\varphi(\cdot,q^{(i)}(s)), \partial_l\widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^2(\mathbb{R}^2)} \\
&+ C \sum_{l=1}^{2} |\widetilde{a}_l^{(i)}(\cdot,s)\big(\varphi(\cdot,q^{(i)}(s)), \widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^2(\mathbb{R}^2)} \\
&+ C \sum_{l=1}^{2} |\big(i\widetilde{\phi}^{(i)}(\cdot,s), D_l^{(0)}\widetilde{\phi}^{(i)}(\cdot,s)\big)|^2_{H^2(\mathbb{R}^2)} \\
&+ C \sum_{l=1}^{2} ||\widetilde{\phi}^{(i)}(\cdot,s)|^2 \widetilde{a}_l^{(i)}(\cdot,s)|^2_{H^2(\mathbb{R}^2)} + C \sum_{l=1}^{2} |\widetilde{a}_l^{(i)}(\cdot,s) D_l^{(0)}\varphi(\cdot,q^{(i)}(s))|^2_{L^2(\mathbb{R}^2)} \\
&+ C|\widetilde{\phi}^{(i)}(\cdot,s)(1 - |\phi(\cdot,q^{(i)}(s))|^2)|^2_{L^2(\mathbb{R}^2)} \\
&+ C \sum_{\mu=1}^{2N} |\frac{\partial \varphi}{\partial q_\mu}(\cdot,q^{(i)}(s))|^2_{L^2(\mathbb{R}^2)} |\dot{q}_\mu^{(i)}(s)|^2
\end{aligned}
$$

$$+ C|\varphi(\cdot, q^{(i)}(s))(1 - |\phi(\cdot, q^{(i)}(s))|^2)|^2_{L^2(\mathbb{R}^2)} + C|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot, s)\varphi(\cdot, q^{(i)}(s))|^2_{L^2(\mathbb{R}^2)}$$

$$+ C|\varphi(\cdot, q^{(i)}(s))\big(i\varphi(\cdot, q^{(i)}(s)), \widetilde{\phi}^{(i)}(\cdot, s)\big)|^2_{L^2(\mathbb{R}^2)}$$

$$+ C\sum_{l=1}^{2} |\partial_l \widetilde{a}_l^{(i)}(\cdot, s)\widetilde{\phi}^{(i)}(\cdot, s)|^2_{L^2(\mathbb{R}^2)} + C\sum_{l=1}^{2} |\widetilde{a}_l^{(i)}(\cdot, s)D_l^{(0)}\widetilde{\phi}^{(i)}(\cdot, s)|^2_{L^2(\mathbb{R}^2)}$$

$$+ C\sum_{l=1}^{2} ||\widetilde{a}_l^{(i)}(\cdot, s)|^2\varphi(\cdot, q^{(i)}(s))|^2_{L^2(\mathbb{R}^2)} + C\sum_{l=1}^{2} ||\widetilde{a}_l^{(i)}(\cdot, s)|^2\widetilde{\phi}^{(i)}(\cdot, s)|^2_{L^2(\mathbb{R}^2)}$$

$$+ C|\widetilde{\phi}^{(i)}(\cdot, s)(1 - |\phi(\cdot, q^{(i)}(s))|^2)|^2_{L^2(\mathbb{R}^2)}$$

$$+ C|\widetilde{\phi}^{(i)}(\cdot, s)\big(\varphi(\cdot, q^{(i)}(s)), \widetilde{\phi}^{(i)}(\cdot, s)\big)|^2_{L^2(\mathbb{R}^2)}$$

$$+ C|\varphi(\cdot, q^{(i)}(s))\big(\varphi(\cdot, q^{(i)}(s)), \widetilde{\phi}^{(i)}(\cdot, s)\big)|^2_{L^2(\mathbb{R}^2)}$$

$$+ C||\widetilde{\phi}^{(i)}(\cdot, s)|^2\varphi(\cdot, q^{(i)}(s))|^2_{L^2(\mathbb{R}^2)} + C||\widetilde{\phi}^{(i)}(\cdot, s)|^2\widetilde{\phi}^{(i)}(\cdot, s)|^2_{L^2(\mathbb{R}^2)}$$

$$+ \varepsilon C|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot, s)\widetilde{\phi}^{(i)}(\cdot, s)|^2_{L^2(\mathbb{R}^2)}$$

$$+ C\sum_{j,l=1}^{2} |D_j^{(\alpha(\cdot, q(0)))}\big(\widetilde{a}_l^{(i)}(\cdot, s)D_l^{(0)}\varphi(\cdot, q^{(i)}(s))\big)|^2_{L^2(\mathbb{R}^2)}$$

$$+ C\sum_{j=1}^{2} |D_j^{(\alpha(\cdot, q(0)))}\big(\widetilde{\phi}^{(i)}(\cdot, s)(1 - |\phi(\cdot, q^{(i)}(s))|^2)\big)|^2_{L^2(\mathbb{R}^2)}$$

$$+ C\sum_{j=1}^{2}\sum_{\mu=1}^{2N} |D_j^{(\alpha(\cdot, q(0)))}\frac{\partial\varphi}{\partial q_\mu}(\cdot, q^{(i)}(s))|^2_{L^2(\mathbb{R}^2)}|\dot{q}_\mu^{(i)}(s)|^2$$

$$+ C\sum_{j=1}^{2} |D_j^{(\alpha(\cdot, q(0)))}\big(\varphi(\cdot, q^{(i)}(s))(1 - |\phi(\cdot, q^{(i)}(s))|^2)\big)|^2_{L^2(\mathbb{R}^2)}$$

$$+ C\sum_{j=1}^{2} |D_j^{(\alpha(\cdot, q(0)))}\big(\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot, s)\varphi(\cdot, q^{(i)}(s))\big)|^2_{L^2(\mathbb{R}^2)}$$

$$+ C\sum_{j=1}^{2} |D_j^{(\alpha(\cdot, q(0)))}\Big(\varphi(\cdot, q^{(i)}(s))\big(i\varphi(\cdot, q^{(i)}(s)), \widetilde{\phi}^{(i)}(\cdot, s)\big)\Big)|^2_{L^2(\mathbb{R}^2)}$$

$$+ C\sum_{j,l=1}^{2} |D_j^{(\alpha(\cdot, q(0)))}\big(\partial_l \widetilde{a}_l^{(i)}(\cdot, s)\widetilde{\phi}^{(i)}(\cdot, s)\big)|^2_{L^2(\mathbb{R}^2)}$$

$$
\begin{aligned}
&+ C \sum_{j,l=1}^{2} |D_j^{(\alpha(\cdot,q(0)))}\big(\widetilde{a}_l^{(i)}(\cdot,s) D_l^{(0)} \widetilde{\phi}^{(i)}(\cdot,s)\big)|_{L^2(\mathbb{R}^2)}^2 \\
&+ C \sum_{j,l=1}^{2} |D_j^{(\alpha(\cdot,q(0)))}\big(|\widetilde{a}_l^{(i)}(\cdot,s)|^2 \varphi(\cdot,q^{(i)}(s))\big)|_{L^2(\mathbb{R}^2)}^2 \\
&+ C \sum_{j,l=1}^{2} |D_j^{(\alpha(\cdot,q(0)))}\big(|\widetilde{a}_l^{(i)}(\cdot,s)|^2 \widetilde{\phi}^{(i)}(\cdot,s)\big)|_{L^2(\mathbb{R}^2)}^2 \\
&+ C \sum_{j=1}^{2} |D_j^{(\alpha(\cdot,q(0)))}\big(\widetilde{\phi}^{(i)}(\cdot,s)(1 - |\phi(\cdot,q^{(i)}(s))|^2)\big)|_{L^2(\mathbb{R}^2)}^2 \\
&+ C \sum_{j=1}^{2} |D_j^{(\alpha(\cdot,q(0)))}\Big(\widetilde{\phi}^{(i)}(\cdot,s)\big(\varphi(\cdot,q^{(i)}(s)), \widetilde{\phi}^{(i)}(\cdot,s)\big)\Big)|_{L^2(\mathbb{R}^2)}^2 \\
&+ C \sum_{j=1}^{2} |D_j^{(\alpha(\cdot,q(0)))}\Big(\varphi(\cdot,q^{(i)}(s))\big(\varphi(\cdot,q^{(i)}(s)), \widetilde{\phi}^{(i)}(\cdot,s)\big)\Big)|_{L^2(\mathbb{R}^2)}^2 \\
&+ C \sum_{j=1}^{2} |D_j^{(\alpha(\cdot,q(0)))}\big(|\widetilde{\phi}^{(i)}(\cdot,s)|^2 \varphi(\cdot,q^{(i)}(s))\big)|_{L^2(\mathbb{R}^2)}^2 \\
&+ C \sum_{j=1}^{2} |D_j^{(\alpha(\cdot,q(0)))}\big(|\widetilde{\phi}^{(i)}(\cdot,s)|^2 \widetilde{\phi}^{(i)}(\cdot,s)\big)|_{L^2(\mathbb{R}^2)}^2 \\
&+ \varepsilon C \sum_{j=1}^{2} |\widetilde{a}_0^{(i+1)}(\cdot,s) D_j^{(\alpha(\cdot,q(0)))} \widetilde{\phi}^{(i)}(\cdot,s)|_{L^2(\mathbb{R}^2)}^2 \\
&+ C \sum_{j,k,l=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} D_k^{(\alpha(\cdot,q(0)))}\big(\widetilde{a}_l^{(i)}(\cdot,s) D_l^{(0)} \varphi(\cdot,q^{(i)}(s))\big)|_{L^2(\mathbb{R}^2)}^2 \\
&+ C \sum_{j,k=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} D_k^{(\alpha(\cdot,q(0)))}\big(\widetilde{\phi}^{(i)}(\cdot,s)(1 - |\phi(\cdot,q^{(i)}(s))|^2)\big)|_{L^2(\mathbb{R}^2)}^2 \\
&+ C \sum_{j,k=1}^{2} \sum_{\mu=1}^{2N} |D_j^{(\alpha(\cdot,q(0)))} D_k^{(\alpha(\cdot,q(0)))} \frac{\partial \varphi}{\partial q_\mu}(\cdot,q^{(i)}(s))|_{L^2(\mathbb{R}^2)}^2 |\dot{q}_\mu^{(i)}(s)|^2 \\
&+ C \sum_{j,k=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} D_k^{(\alpha(\cdot,q(0)))}\big(\varphi(\cdot,q^{(i)}(s))(1 - |\phi(\cdot,q^{(i)}(s))|^2)\big)|_{L^2(\mathbb{R}^2)}^2
\end{aligned}
$$

$$
\begin{aligned}
&+ C \sum_{j,k=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} D_k^{(\alpha(\cdot,q(0)))} \Big(\widetilde{\phi}^{(i)}(\cdot,s) \big(\varphi(\cdot,q^{(i)}(s)), \widetilde{\phi}^{(i)}(\cdot,s) \big) \Big) |_{L^2(\mathbb{R}^2)}^2 \\
&+ C \sum_{j,k=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} D_k^{(\alpha(\cdot,q(0)))} \Big(\varphi(\cdot,q^{(i)}(s)) \big(\varphi(\cdot,q^{(i)}(s)), \widetilde{\phi}^{(i)}(\cdot,s) \big) \Big) |_{L^2(\mathbb{R}^2)}^2 \\
&+ C \sum_{j,k=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} D_k^{(\alpha(\cdot,q(0)))} \big(|\widetilde{\phi}^{(i)}(\cdot,s)|^2 \varphi(\cdot,q^{(i)}(s)) \big) |_{L^2(\mathbb{R}^2)}^2 \\
&+ C \sum_{j,k=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} D_k^{(\alpha(\cdot,q(0)))} \big(|\widetilde{\phi}^{(i)}(\cdot,s)|^2 \widetilde{\phi}^{(i)}(\cdot,s) \big) |_{L^2(\mathbb{R}^2)}^2 \\
&+ \varepsilon C \sum_{j,k=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} D_k^{(\alpha(\cdot,q(0)))} \big(\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s) \widetilde{\phi}^{(i)}(\cdot,s) \big) |_{L^2(\mathbb{R}^2)}^2 . \qquad (4.15)
\end{aligned}
$$

$$
\begin{aligned}
&+ C \sum_{j,k=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} D_k^{(\alpha(\cdot,q(0)))} \big(\widetilde{a}_0^{(i+1)}(\cdot,s)\varphi(\cdot,q^{(i)}(s))\big)|_{L^2(\mathbb{R}^2)}^2 \\
&+ C \sum_{j,k=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} D_k^{(\alpha(\cdot,q(0)))} \Big(\varphi(\cdot,q^{(i)}(s))\big(i\varphi(\cdot,q^{(i)}(s)),\widetilde{\phi}^{(i)}(\cdot,s)\big)\Big)|_{L^2(\mathbb{R}^2)}^2 \\
&+ C \sum_{j,k,l=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} D_k^{(\alpha(\cdot,q(0)))} \big(\partial_l \widetilde{a}_l^{(i)}(\cdot,s)\widetilde{\phi}^{(i)}(\cdot,s)\big)|_{L^2(\mathbb{R}^2)}^2 \\
&+ C \sum_{j,k,l=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} D_k^{(\alpha(\cdot,q(0)))} \big(\widetilde{a}_l^{(i)}(\cdot,s) D_l^{(0)} \widetilde{\phi}^{(i)}(\cdot,s)\big)|_{L^2(\mathbb{R}^2)}^2 \\
&+ C \sum_{j,k,l=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} D_k^{(\alpha(\cdot,q(0)))} \big(|\widetilde{a}_l^{(i)}(\cdot,s)|^2 \varphi(\cdot,q^{(i)}(s))\big)|_{L^2(\mathbb{R}^2)}^2 \\
&+ C \sum_{j,k,l=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} D_k^{(\alpha(\cdot,q(0)))} \big(|\widetilde{a}_l^{(i)}(\cdot,s)|^2 \widetilde{\phi}^{(i)}(\cdot,s)\big)|_{L^2(\mathbb{R}^2)}^2 \\
&+ C \sum_{j,k=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} D_k^{(\alpha(\cdot,q(0)))} \big(\widetilde{\phi}^{(i)}(\cdot,s)(1-|\phi(\cdot,q^{(i)}(s))|^2)\big)|_{L^2(\mathbb{R}^2)}^2
\end{aligned}
$$

Now, use a technique similar to that employed to prove (4.9): every single component of the right-hand side of (4.15) is estimated invoking Remark 3.1.1, the Hölder inequality, the exponential behaviour of the zero modes as x tends to infinity, the fact that $|\cdot|_{L^\infty(\mathbb{R}^2)} \leq |\cdot|_{H^2(\mathbb{R}^2)}$ and the induction hypothesis. Thus one finds that a constant $C(\Gamma) > 0$ exists, independent of i, for which the right-hand side of (4.15) is smaller than

$$\begin{aligned}
&C(\Gamma) + C(\Gamma)|\dot{q}^{(i)}(s)|^2 + \varepsilon C(\Gamma)|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)|^2_{H^3(\mathbb{R}^2)} \\
&\quad + C(\Gamma)|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)|^2_{H^3(\mathbb{R}^2)} + C(\Gamma)\sum_{l=1}^{2}|\widetilde{a}_l^{(i)}(\cdot,s)|^2_{H^3(\mathbb{R}^2)} \\
&\quad + C(\Gamma)|\widetilde{\phi}^{(i)}(\cdot,s)|^2_{L^2(\mathbb{R}^2)} + C(\Gamma)\sum_{j=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}\widetilde{\phi}^{(i)}(\cdot,s)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + C(\Gamma)\sum_{j,k=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}\widetilde{\phi}^{(i)}(\cdot,s)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + C(\Gamma)\sum_{j,k,l=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}D_l^{(\alpha(\cdot,q(0)))}\widetilde{\phi}^{(i)}(\cdot,s)|^2_{L^2(\mathbb{R}^2)}.
\end{aligned} \tag{4.16}$$

Again, invoking the induction hypothesis, one obtains that another constant $C(\Gamma) > 0$ exists, independent of i, such that (4.16) is smaller than

$$C(\Gamma)\big(1 + |\dot{q}^{(i)}(s)|^2 + \|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\| + \varepsilon\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\|\big). \tag{4.17}$$

Hence, using (4.17), (4.11), the Cauchy inequality and the hypotheses on the initial data in (4.14) it turns out that

$$\begin{aligned}
\|\psi^{(i+1)}(\cdot,t)\| \leq \frac{\Gamma}{2} + C(\Gamma)\int_0^t \big(1 + \|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\| \\
+ \varepsilon\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\| + \|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\| + \varepsilon\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\|\big)ds
\end{aligned} \tag{4.18}$$

for some constant $C(\Gamma) > 0$ independent of i.

4.4 An Estimate for $\partial_t \psi^{(i+1)}$

An estimate for $\partial_t \psi^{(i+1)}$ is discussed in this section. Such an estimate will be coupled with (4.12) in order to make $\|\widetilde{\widetilde{a}}_0^{(i+1)}\|$ bounded for a nontrivial time interval.

We start our discussion for the real component of $\partial_t \psi^{(i+1)}$. From (3.14), by using the Cauchy inequality, it is known that a constant $C > 0$ exists, independent of i, such that

$$\begin{aligned}
|\partial_t\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{H^1(\mathbb{R}^2)} &= |\Delta\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big) - |\phi^{(i)}(\cdot,t)|^2\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big) \\
&\quad + f_{1h}^1(i,\cdot,t)|^2_{H^1(\mathbb{R}^2)} \\
&\leq C|\Delta\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{H^1(\mathbb{R}^2)} + C||\phi^{(i)}(\cdot,t)|^2\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|^2_{H^1(\mathbb{R}^2)} \\
&\quad + C|f_{1h}^1(i,\cdot,t)|^2_{H^1(\mathbb{R}^2)} \\
&= C|\Delta\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} + C\sum_{j=1}^{2}|\partial_j\Delta\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + C||\phi^{(i)}(\cdot,t)|^2\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + C\sum_{j=1}^{2}|\partial_j\big(|\phi^{(i)}(\cdot,t)|^2\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + C|f_{1h}^1(i,\cdot,t)|^2_{H^1(\mathbb{R}^2)}. \qquad (4.19)
\end{aligned}$$

Again, using the Cauchy inequality and Remark 3.1.1, the right-hand side of (4.19) can be bounded by

$$C\sum_{k=1}^{2}|\partial_{kk}^2\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} + C\sum_{j,k=1}^{2}|\partial_{jkk}^3\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)}$$

$$+ CM|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + CM|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)}$$

$$+ CM\sum_{j=1}^{2}|\partial_j\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} + C|f_{1h}^1(i,\cdot,t)|^2_{H^1(\mathbb{R}^2)} \tag{4.20}$$

which, by defining a new $C(\Gamma) > 0$, is smaller than

$$C(\Gamma)|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|^2_{H^3(\mathbb{R}^2)} + C(\Gamma)|f_{1h}^1(i,\cdot,t)|^2_{H^1(\mathbb{R}^2)}. \tag{4.21}$$

Putting together (4.19) and (4.21), it yields

$$|\partial_t\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big)|^2_{H^1(\mathbb{R}^2)} \leq C(\Gamma)|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)|^2_{H^3(\mathbb{R}^2)} + C(\Gamma)|f_{1h}^1(i,\cdot,t)|^2_{H^1(\mathbb{R}^2)} \tag{4.22}$$

with $C(\Gamma) > 0$ constant and independent of i.

The complex component of $\partial_t\psi^{(i+1)}$ is estimated proceeding in the same way as for the real component, except for using the covariant derivative $D^{(0)}$ instead of the classical derivative. Then

$$\begin{aligned}
&|\partial_t\widetilde{\phi}^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + \sum_{j=1}^{2}|D_j^{(0)}\partial_t\widetilde{\phi}^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} \\
&\leq C|\widetilde{\phi}^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + C\sum_{j=1}^{2}|D_j^{(0)}\widetilde{\phi}^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} \\
&+ C\sum_{j,k=1}^{2}|D_j^{(0)}D_k^{(0)}\widetilde{\phi}^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} \\
&+ C\sum_{j,k,l=1}^{2}|D_j^{(0)}D_k^{(0)}D_l^{(0)}\widetilde{\phi}^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} \\
&+ C|f_1^2(i,\cdot,t)|^2_{L^2(\mathbb{R}^2)} + C\sum_{j=1}^{2}|D_j^{(0)}f_1^2(i,\cdot,t)|^2_{L^2(\mathbb{R}^2)}
\end{aligned} \tag{4.23}$$

follows. However, an estimate with norms involving $D^{(\alpha(\cdot,q(0)))}$ instead of $D^{(0)}$ is required due to the definition of $|\cdot|_{1,\alpha(\cdot,q(0))}$. Because the norms defined with the two different covariant derivative are equivalent, from (4.23)

$$
\begin{aligned}
&|\partial_t\widetilde{\phi}^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + \sum_{j=1}^{2} |D_j^{(\alpha(\cdot,q(0)))}\partial_t\widetilde{\phi}^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} \\
&\le C|\widetilde{\phi}^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + C\sum_{j=1}^{2} |D_j^{(\alpha(\cdot,q(0)))}\widetilde{\phi}^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} \\
&+ C\sum_{j,k=1}^{2} |D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}\widetilde{\phi}^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} \\
&+ C\sum_{j,k,l=1}^{2} |D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}D_l^{(\alpha(\cdot,q(0)))}\widetilde{\phi}^{(i+1)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} \\
&+ C|f_1^2(i,\cdot,t)|^2_{L^2(\mathbb{R}^2)} + C\sum_{j=1}^{2} |D_j^{(\alpha(\cdot,q(0)))}f_1^2(i,\cdot,t)|^2_{L^2(\mathbb{R}^2)}
\end{aligned}
\tag{4.24}
$$

follows, for a constant $C > 0$ independent of i. Now, adding term by term (4.22) to (4.24), it turns out that

$$
|\partial_t\psi^{(i+1)}(\cdot,t)|_{1,\alpha(\cdot,q(0))} \le C|\psi^{(i+1)}(\cdot,t)|_{3,\alpha(\cdot,q(0))} + C|f_1(i,\cdot,t)|_{1,\alpha(\cdot,q(0))} \tag{4.25}
$$

for $C > 0$ constant and independent of i. In section 4.3 it was proven that

$$
\begin{aligned}
|f_1(i,\cdot,t)|_{1,\alpha(\cdot,q(0))} &\le C(\Gamma)\big(1 + \|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\| \\
&+ \varepsilon\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\| + \|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\| + \varepsilon\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\|\big)
\end{aligned}
\tag{4.26}
$$

for some constant $C(\Gamma) > 0$, independent of i.

Using (4.18) and (4.26) in (4.25), one obtains

$$
\begin{aligned}
|\partial_t\psi^{(i+1)}(\cdot,t)|_{1,\alpha(\cdot,q(0))} &\leq C\Big(\frac{\Gamma}{2}+C(\Gamma)\int_0^t\big(1+\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\| \\
&\quad+\varepsilon\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\|\big)ds\Big)+C\Big(C(\Gamma)\big(1+\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\|+\varepsilon\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\| \\
&\quad+\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\|+\varepsilon\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\|\big)\Big) \\
&\leq C(\Gamma)+C(\Gamma)\int_0^t\big(1+\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\|+\varepsilon\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\|\big)ds \\
&\quad+C(\Gamma)\big(1+\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\|+\varepsilon\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\|+\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\|+\varepsilon\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\|\big)
\end{aligned}
$$

from which, possibly with a redefinition of $C(\Gamma)$,

$$
\begin{aligned}
|\partial_t\psi^{(i+1)}(\cdot,t)|_{1,\alpha(\cdot,q(0))} &\leq C(\Gamma)+tC(\Gamma) \\
&\quad+tC(\Gamma)\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\|+\varepsilon tC(\Gamma)\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\| \\
&\quad+C(\Gamma)\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\|+\varepsilon C(\Gamma)\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\| \\
&\quad+C(\Gamma)\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\|+\varepsilon C(\Gamma)\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\| \qquad (4.27)
\end{aligned}
$$

follows.

4.5 Convergent Sequences and Upper Bounds for $\psi^{(i+1)}$ and $q^{(i+1)}$

The last estimates for $\widetilde{\widetilde{a}}_0^{(i+1)}$, $\psi^{(i+1)}$, $q^{(i+1)}$ and $\partial_t\psi^{(i+1)}$ derived in sections 4.1, 4.2, 4.3 and 4.4 are now used to prove the convergence of a particular series. It is shown that this is equivalent to proving that, after choosing the parameter ε small enough, the functions $\psi^{(i+1)}$ and $q^{(i+1)}$ are bounded for a sufficiently short time.

Replacing i with $i-1$ in (4.27) one obtains

$$\begin{aligned}|\partial_t\psi^{(i)}(\cdot,t)|_{1,\alpha(\cdot,q(0))} &\le C(\Gamma)+tC(\Gamma)\\ &+tC(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\big\|+\varepsilon tC(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\big\|\\ &+C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)\big\|+\varepsilon C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)\big\|\\ &+C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\big\|+\varepsilon C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\big\| \qquad (4.28)\end{aligned}$$

for some $C(\Gamma)>0$, independent of i. Now, using (4.28) in (4.12) yields

$$\begin{aligned}\big\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\big\| &\le C(\Gamma)\big(1+2\varepsilon C(\Gamma)+\varepsilon tC(\Gamma)\big)\\ &+\varepsilon C(\Gamma)\big(2+tC(\Gamma)+\varepsilon tC(\Gamma)+\varepsilon tC(\Gamma)\big)\big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\big\|\\ &+\varepsilon C^2(\Gamma)(1+\varepsilon)\big\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)\big\|\\ &=P_1(\varepsilon,\Gamma)+\varepsilon P_2(\varepsilon,t,\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\big\|+\varepsilon P_3(\varepsilon,\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)\big\| \quad (4.29)\end{aligned}$$

where

$$P_1(\varepsilon,\Gamma):=C(\Gamma)\big(1+2\varepsilon C(\Gamma)\big),$$

$$P_2(\varepsilon,t,\Gamma):=C(\Gamma)\big(2+tC(\Gamma)+\varepsilon tC(\Gamma)+\varepsilon tC(\Gamma)\big),$$

$$P_3(\varepsilon,\Gamma):=C^2(\Gamma)(1+\varepsilon).$$

Using (4.29) for i and $i-1$ it follows that

$$\begin{aligned}\big\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\big\| &\le P_1(\varepsilon,\Gamma)+\varepsilon P_1(\varepsilon,\Gamma)P_2(\varepsilon,t,\Gamma)+\varepsilon P_1(\varepsilon,\Gamma)P_3(\varepsilon,\Gamma)\\ &+\varepsilon^2P_1(\varepsilon,\Gamma)P_2^2(\varepsilon,t,\Gamma)+\big(\varepsilon^3P_2^3(\varepsilon,t,\Gamma)+\varepsilon^2 2P_2(\varepsilon,t,\Gamma)P_3(\varepsilon,\Gamma)\big)\big\|\widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,t)\big\|\\ &+\big(\varepsilon^3P_2^2(\varepsilon,t,\Gamma)P_3(\varepsilon,\Gamma)+\varepsilon^2P_3^2(\varepsilon,\Gamma)\big)\big\|\widetilde{\widetilde{a}}_0^{(i-3)}(\cdot,t)\big\|. \qquad (4.30)\end{aligned}$$

Again, estimating $\|\widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,t)\|$ and $\|\widetilde{\widetilde{a}}_0^{(i-3)}(\cdot,t)\|$ according to (4.29) it follows that

$$\begin{aligned}
\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\| &\le P_1(\varepsilon,\Gamma) + \varepsilon P_1(\varepsilon,\Gamma)\big(P_2(\varepsilon,t,\Gamma) + P_3(\varepsilon,\Gamma)\big) \\
&+ \varepsilon^2 P_1(\varepsilon,\Gamma)\big(P_2(\varepsilon,t,\Gamma) + P_3(\varepsilon,\Gamma)\big)^2 + \varepsilon^3 P_1(\varepsilon,\Gamma)\big(P_2(\varepsilon,t,\Gamma) + P_3(\varepsilon,\Gamma)\big)^3 \\
&+ \varepsilon^4\big(2P_2^2(\varepsilon,t,\Gamma)P_3^2(\varepsilon,\Gamma)\|\widetilde{\widetilde{a}}_0^{(i-3)}(\cdot,t)\| + 3P_2^2(\varepsilon,t,\Gamma)P_3(\varepsilon,\Gamma)\|\widetilde{\widetilde{a}}_0^{(i-4)}(\cdot,t)\| \\
&+ 2P_2(\varepsilon,t,\Gamma)P_3^3(\varepsilon,\Gamma)\|\widetilde{\widetilde{a}}_0^{(i-4)}(\cdot,t)\| + P_2^3(\varepsilon,t,\Gamma)P_3(\varepsilon,\Gamma)\|\widetilde{\widetilde{a}}_0^{(i-4)}(\cdot,t)\| \\
&+ 4P_2^2(\varepsilon,t,\Gamma)P_3^2(\varepsilon,\Gamma)\|\widetilde{\widetilde{a}}_0^{(i-5)}(\cdot,t)\| + 2P_2(\varepsilon,t\Gamma)P_3^2(\varepsilon,\Gamma)\|\widetilde{\widetilde{a}}_0^{(i-6)}(\cdot,t)\| \\
&+ P_3^4(\varepsilon,\Gamma)\|\widetilde{\widetilde{a}}_0^{(i-7)}(\cdot,t)\|\big).
\end{aligned} \tag{4.31}$$

By iteration of (4.29), one obtains that

$$\begin{aligned}
\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\| &\le P_1(\varepsilon,\Gamma) + \varepsilon P_1(\varepsilon,\Gamma)\big(P_2(\varepsilon,t,\Gamma) + P_3(\varepsilon,\Gamma)\big) \\
&+ \varepsilon^2 P_1(\varepsilon,\Gamma)\big(P_2(\varepsilon,t,\Gamma) + P_3(\varepsilon,\Gamma)\big)^2 + \varepsilon^3 P_1(\varepsilon,\Gamma)\big(P_2(\varepsilon,t,\Gamma) + P_3(\varepsilon,\Gamma)\big)^3 \\
&+ \varepsilon^4 P_1(\varepsilon,\Gamma)\big(P_2(\varepsilon,t,\Gamma) + P_3(\varepsilon,\Gamma)\big)^4 + \dots.
\end{aligned}$$

Now, if ε is chosen such that

$$P_1(\varepsilon,\Gamma) < P_1'(\Gamma) := 2C(\Gamma),$$

$$P_2(\varepsilon,t,\Gamma) < P_2'(t,\Gamma) := C(\Gamma)\big(3 + tC(\Gamma)\big),$$

$$P_3(\varepsilon,\Gamma) < P_3'(\Gamma) := 2C^2(\Gamma),$$

then

$$\begin{aligned}
\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\| &\le P_1'(\Gamma)\Big(1 + \varepsilon\big(P_2'(t,\Gamma) + P_3'(\Gamma)\big) \\
&+ \varepsilon^2\big(P_2'(t,\Gamma) + P_3'(\Gamma)\big)^2 + \varepsilon^3\big(P_2'(t,\Gamma) + P_3'(\Gamma)\big)^3 \\
&+ \varepsilon^4\big(P_2'(t,\Gamma) + P_3'(\Gamma)\big)^4 + \dots\Big).
\end{aligned} \tag{4.32}$$

Since the series on the right-hand side of (4.32) is a geometric series, it is convergent, when

$$\varepsilon < \left(P_2'(t,\Gamma) + P_3'(\Gamma)\right)^{-1}.$$

In that case, there exists a constant $A(t,\Gamma) > 0$, independent of i, namely the sum of the geometric series in (4.32), such that

$$\left\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\right\| \leq A(t,\Gamma). \tag{4.33}$$

Now, using (4.33) in (4.18), one obtains

$$\begin{aligned}\left\|\psi^{(i+1)}(\cdot,t)\right\| &\leq \frac{\Gamma}{2} + C(\Gamma)\int_0^t \Big(1 + \left\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\right\| \\ &\quad + \varepsilon\left\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\right\| + \left\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\right\| + \varepsilon\left\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\right\|\Big)ds \\ &\leq \frac{\Gamma}{2} + C(\Gamma)\int_0^t \big(1 + 2A(t,\Gamma) + \varepsilon 2A(t,\Gamma)\big)ds \\ &= \frac{\Gamma}{2} + tC(\Gamma)\big(1 + 2A(t,\Gamma) + \varepsilon 2A(t,\Gamma)\big).\end{aligned}$$

By defining

$$T_1 = T_1(\Gamma) := \frac{\Gamma}{2\big(1 + 2A(t,\Gamma) + \varepsilon 2A(t,\Gamma)\big)}, \tag{4.34}$$

it turns out that

$$\left\|\psi^{(i+1)}(\cdot,t)\right\| \leq \Gamma \tag{4.35}$$

for any $t \in [0, T_1]$.

Furthermore, using (4.33) in (4.27), with the same choice of T_1, one obtains

$$|\partial_t\psi^{(i+1)}(\cdot,t)|_{1,\alpha(\cdot,q(0))} \leq 2C(\Gamma) + tC(\Gamma)$$

$$+ tC(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\big\| + \varepsilon tC(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\big\| + C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\big\|$$
$$+ \varepsilon C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\big\| + C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\big\| + \varepsilon C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)\big\|$$
$$\leq 2C(\Gamma) + C(\Gamma)T_1(\Gamma) + A(T_1,\Gamma)C(\Gamma)T_1(\Gamma) + \varepsilon A(T_1,\Gamma)C(\Gamma)T_1(\Gamma)$$
$$+ 2A(T_1,\Gamma)C(\Gamma) + \varepsilon 2A(T_1,\Gamma)C(\Gamma)$$

for any $t \in [0, T_1]$, and then

$$|\partial_t\psi^{(i+1)}(\cdot,t)|_{1,\alpha(\cdot,q(0))} \leq B(T_1,\Gamma). \tag{4.36}$$

The constant $B(T_1, \Gamma)$, independent of i, is given by

$$B(T_1,\Gamma) := 2C(\Gamma) + C(\Gamma)T_1(\Gamma) + A(T_1,\Gamma)C(\Gamma)T_1(\Gamma)$$
$$+ \varepsilon A(T_1,\Gamma)C(\Gamma)T_1(\Gamma) + 2A(T_1,\Gamma)C(\Gamma) + \varepsilon 2A(T_1,\Gamma)C(\Gamma).$$

Finally, using (4.33) in (4.13), one obtains

$$\begin{aligned}|q^{(i+1)}(t)| &\leq \frac{\Gamma}{2} + \varepsilon C(\Gamma)\int_0^t \left(1 + \varepsilon\big\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\big\|^{1/2}\right)ds\\ &\leq \frac{\Gamma}{2} + \varepsilon C(\Gamma)\int_0^t \left(1 + \varepsilon\sqrt{2A(T_1,\Gamma)}\right)ds\\ &= \frac{\Gamma}{2} + \varepsilon tC(\Gamma)\left(1 + \varepsilon 2A(T_1,\Gamma)\right).\end{aligned}$$

Then, defining

$$T_2 = T_2(\Gamma) := \frac{\Gamma}{\varepsilon 2C(\Gamma)\left(1 + \varepsilon 2A(T_1,\Gamma)\right)}, \tag{4.37}$$

we have

$$|q^{(i+1)}(t)| \leq \Gamma \tag{4.38}$$

for any $t \in [0, T_2]$.

We can collect all the results of this chapter in the following

Theorem 4.5.1 *Let $K > 2$ as given in Lemma 3.6.2 and $\delta \leq (2 - 1/K)\Gamma$. Let us suppose also that for any $i \in \mathbb{N}$*

$$\left\|\psi^{(i)}(\cdot, 0)\right\|, \quad |q^{(i)}(0)| \leq \frac{\Gamma}{2K}$$

and

$$|\psi^{(i)}(\cdot, 0) - \psi(\cdot, 0)|_{3,\alpha(\cdot, q(0))}, \quad |q^{(i)}(0) - q(0)| \leq \frac{\delta}{2^i}.$$

Then there exists $\overline{T} := \min\{T_1, T_2\} > 0$ with T_1 and T_2 given respectively by (4.34) and (4.37) such that for any $i \in \mathbb{N}$

$$\left\|\psi^{(i)}(\cdot, t)\right\|, \quad |q^{(i)}(t)| \leq \Gamma \quad \forall t \in [0, \overline{T}].$$

Chapter 5

Contraction Estimates

5.1 A Mean Value Theorem

A new Banach space is defined. The choice of its norm is based on the norms previously discussed. It is straightforward to see that the sequence of the quantities $\psi^{(i)}$ and $q^{(i)}$ generated by (3.13), (3.14) and (3.13) are in such a space. Moreover a mean value theorem is presented in a general case.

The following space is defined:

$$\begin{aligned} X(\overline{T}) \;:=\; & \big\{(\tilde{\tilde{a}}_0(\cdot,\overline{T}), \psi(\cdot,\overline{T}), q(\overline{T})) \in H^3(\mathbb{R}^2) \times H^{3,\alpha(\cdot,q(0))}(\mathbb{R}^2) \times \mathbb{R}^{2N} \big| \\ & \|\tilde{\tilde{a}}_0(\cdot,\overline{T})\| \leq A(T_1,\Gamma), \|\psi(\cdot,\overline{T})\| \leq \Gamma, \max_{0\leq t\leq \overline{T}} |q(t)| \leq \Gamma, \\ & \max_{0\leq t\leq \overline{T}} |\partial_t \psi(\cdot,t)|_{1,\alpha(\cdot,q(0))} \leq B(T_1,\Gamma)\big\}, \end{aligned}$$

with the norm

$$\begin{aligned} |(\tilde{\tilde{a}}_0(\cdot,\overline{T}), \psi(\cdot,\overline{T}), q(\overline{T}))|_{X(\overline{T})} := & \|\tilde{\tilde{a}}_0(\cdot,\overline{T})\| + \|\psi(\cdot,\overline{T})\| \\ & + \max_{0\leq t\leq \overline{T}} |\partial_t \psi(\cdot,t)|_{1,\alpha(\cdot,q(0))} + \max_{0\leq t\leq \overline{T}} |q(t)| \end{aligned}$$

and $\overline{T}$ as in Theorem 4.5.1. From Theorem 4.5.1, from (4.36) and from (4.33) one derives that $\varepsilon_2 > 0$ exists such that, for any $\varepsilon < \varepsilon_2$, the sequence $\left((\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t), \psi^{(i)}(\cdot,t), q^{(i)}(t))\right)_{i\in\mathbb{N}}$ is in $X(t)$, for any $t \in [0, \overline{T}]$.

It is known that the cross product of Banach spaces is complete. In particular, since $H^3(\mathbb{R}^2)$, $H^{3,\alpha(\cdot,q(0))}(\mathbb{R}^2)$ and $\mathbb{R}^{2N}$ are Banach spaces, [Eva98, theorem 2, p. 249], it must follow that also $X(\overline{T})$ is complete as it is a closed subset of a complete metric space. Then, if one proves that $T = T(\Gamma) > 0$ exists, small enough such that $\left((\widetilde{\widetilde{a}}_0^{(i)}(x,t), \psi^{(i)}(x,t), q^{(i)}(t))\right)_{i\in\mathbb{N}}$ is a Cauchy sequence, it follows that such a sequence is convergent in $\Big(X(t), |\cdot|_{X(t)}\Big)$ for any $t \in [0, T]$.

In the remainder the following theorem will be used to find the contraction estimates we need:

Theorem 5.1.1 (Mean Value Theorem) *Let* $(X_1, |\cdot|_{X_1}), \ldots, (X_p, |\cdot|_{X_p})$ *and* $(Y, |\cdot|_Y)$ *be normed spaces and let*

$$f : X_1 \times \cdots \times X_p \longrightarrow Y$$

such that $\partial_j f \in L^\infty(\mathbb{R}^2)$ *exists for* $j = 1, \ldots, p$. *Then*

$$|f(x_1, \ldots, x_p) - f(x_1', \ldots, x_p')|_Y \le \sup_{\overline{x}\in X_1\times\cdots\times X_p} |\partial_1 f(\overline{x})|_Y |x_1 - x_1'|_{X_1} + \cdots + \sup_{\overline{x}\in X_1\times\cdots\times X_p} |\partial_p f(\overline{x})|_Y |x_p - x_p'|_{X_p}.$$

Proof. The proof is similar to that of Lemma 3.5.1 with the usual Euclidean norm replaced by the norms in $X_1, \ldots, X_p, Y$. For details see proof of Lemma 3.5.1.

In our case the partial derivatives are shown to be smaller than 1 for a suitable choice of ε and t. Then theorem 5.1.1 provides a contraction.

5.2 An Estimate for $\psi^{(i+1)} - \psi^{(i)}$

An estimate of $\psi^{(i+1)} - \psi^{(i)}$ in terms of the initial data, $|q^{(i)} - q^{(i-1)}|$ and $|f_1(\widetilde{a}^{(i)}, \widetilde{\phi}^{(i)}, \widetilde{\widetilde{a}}_0^{(i+1)}, q^{(i)}, \dot{q}^{(i)}) - f_1(\widetilde{a}^{(i-1)}, \widetilde{\phi}^{(i-1)}, \widetilde{\widetilde{a}}_0^{(i)}, q^{(i-1)}, \dot{q}^{(i-1)})|^2_{H^{2,\alpha(\cdot,q(0))}(\mathbb{R}^2)}$ is given. The procedure is similar to that followed in section 4.3. The real component and the complex component of $\psi^{(i+1)} - \psi^{(i)}$ are considered separately.

We start the discussion with the real component. Write the real part of (3.14) in the form

$$\begin{aligned}&\partial_t\big(\sigma\widetilde{a}_h^{(i)}(x,t)\big) - \Delta\big(\sigma\widetilde{a}_h^{(i)}(x,t)\big)\\&\quad + |\phi(x,q^{(i-1)}(t))|^2\sigma\widetilde{a}_h^{(i)}(x,t) = f_{1h}^1(i-1,x,t)\end{aligned}$$

and

$$\begin{aligned}&\partial_t\big(\sigma\widetilde{a}_h^{(i+1)}(x,t)\big) - \Delta\big(\sigma\widetilde{a}_h^{(i+1)}(x,t)\big)\\&\quad + |\phi(x,q^{(i)}(t))|^2\sigma\widetilde{a}_h^{(i+1)}(x,t) = f_{1h}^1(i,x,t).\end{aligned}$$

Then, subtracting one from the other, we obtain

$$\begin{aligned}&\partial_t\big(\sigma\widetilde{a}_h^{(i+1)}(x,t) - \sigma\widetilde{a}_h^{(i)}(x,t)\big) - \Delta\big(\sigma\widetilde{a}_h^{(i+1)}(x,t) - \sigma\widetilde{a}_h^{(i)}(x,t)\big)\\&\quad + |\phi(x,q^{(i)}(t))|^2\big(\sigma\widetilde{a}_h^{(i+1)}(x,t) - \sigma\widetilde{a}_h^{(i)}(x,t)\big)\\&\quad = \sigma\widetilde{a}_h^{(i)}(x,t)(|\phi(x,q^{(i-1)}(t))|^2 - |\phi(x,q^{(i)}(t))|^2)\\&\quad + f_{1h}^1(i,x,t) - f_{1h}^1(i-1,x,t). \qquad (5.1)\end{aligned}$$

Now, applying to (5.1) the same arguments as in the proof of Lemma 3.6.2, one obtains that two constants $K, C(\Gamma) > 0$ independent of i exist such that, for $0 \le t \le \overline{T}$,

$$|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t) - \sigma\widetilde{a}_h^{(i)}(\cdot,t)|^2_{H^3(\mathbb{R}^2)} \le K|\sigma\widetilde{a}_h^{(i+1)}(\cdot,0) - \sigma\widetilde{a}_h^{(i)}(\cdot,0)|^2_{H^3(\mathbb{R}^2)}$$
$$+ C(\Gamma)\int_0^t |\sigma\widetilde{a}_h^{(i)}(\cdot,s)(|\phi(\cdot,q^{(i)}(s))|^2 - |\phi(\cdot,q^{(i-1)}(s))|^2)$$
$$+ f^1_{1h}(i,\cdot,s) - f^1_{1h}(i-1,\cdot,s)|^2_{H^2(\mathbb{R}^2)}ds$$

holds. Then

$$\begin{aligned}
|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t) - \sigma\widetilde{a}_h^{(i)}(\cdot,t)|^2_{H^3(\mathbb{R}^2)} &\le K|\sigma\widetilde{a}_h^{(i+1)}(\cdot,0) - \sigma\widetilde{a}_h^{(i)}(\cdot,0)|^2_{H^3(\mathbb{R}^2)} \\
&+ C(\Gamma)\int_0^t |\sigma\widetilde{a}_h^{(i)}(\cdot,s)\big(\varphi(\cdot,q^{(i)}(s)) - \varphi(\cdot,q^{(i-1)}(s)),\varphi(\cdot,q^{(i)}(s))\big)|^2_{H^2(\mathbb{R}^2)}ds \\
&+ C(\Gamma)\int_0^t |\sigma\widetilde{a}_h^{(i)}(\cdot,s)\big(\varphi(\cdot,q^{(i)}(s)) - \varphi(\cdot,q^{(i-1)}(s)),\varphi(\cdot,q^{(i-1)}(s))\big)|^2_{H^2(\mathbb{R}^2)}ds \\
&+ C(\Gamma)\int_0^t |f^1_{1h}(i,\cdot,s) - f^1_{1h}(i-1,\cdot,s)|^2_{H^2(\mathbb{R}^2)}ds \qquad (5.2)
\end{aligned}$$

for some constants $K, C(\Gamma) > 0$, independent of i, for $0 \le t \le \overline{T}$.

Now every component of the right-hand side of (5.2) will be estimated. For example, because of Cauchy's inequality, a constant $C > 0$ exists, independent of i, such that

$$\begin{aligned}
&|\sigma\widetilde{a}_h^{(i)}(\cdot,s)\big(\varphi(\cdot,q^{(i)}(s)) - \varphi(\cdot,q^{(i-1)}(s)),\varphi(\cdot,q^{(i-1)}(s))\big)|^2_{H^2(\mathbb{R}^2)} \\
&\le C|\sigma\widetilde{a}_h^{(i)}(\cdot,s)\big(\varphi(\cdot,q^{(i)}(s)) - \varphi(\cdot,q^{(i-1)}(s)),\varphi(\cdot,q^{(i-1)}(s))\big)|^2_{L^2(\mathbb{R}^2)} \\
&+ C\sum_{j=1}^2 |\sigma\widetilde{a}_h^{(i)}(\cdot,s)\big(\partial_j\varphi(\cdot,q^{(i)}(s)) - \partial_j\varphi(\cdot,q^{(i-1)}(s)),\varphi(\cdot,q^{(i-1)}(s))\big)|^2_{L^2(\mathbb{R}^2)} \\
&+ C\sum_{j=1}^2 |\sigma\widetilde{a}_h^{(i)}(\cdot,s)\big(\varphi(\cdot,q^{(i)}(s)) - \varphi(\cdot,q^{(i-1)}(s)),\partial_j\varphi(\cdot,q^{(i-1)}(s))\big)|^2_{L^2(\mathbb{R}^2)}
\end{aligned}$$

$$+C\sum_{j=1}^{2}|\partial_j\big(\sigma\widetilde{a}_h^{(i)}(\cdot,s)\big)\big(\varphi(\cdot,q^{(i)}(s))-\varphi(\cdot,q^{(i-1)}(s)),\varphi(\cdot,q^{(i-1)}(s))\big)|^2_{L^2(\mathbb{R}^2)}$$
$$+C\sum_{j,k=1}^{2}|\sigma\widetilde{a}_h^{(i)}(\cdot,s)\big(\partial^2_{jk}\varphi(\cdot,q^{(i)}(s))-\partial^2_{jk}\varphi(\cdot,q^{(i-1)}(s)),\varphi(\cdot,q^{(i-1)}(s))\big)|^2_{L^2(\mathbb{R}^2)}$$
$$+C\sum_{j,k=1}^{2}|\sigma\widetilde{a}_h^{(i)}(\cdot,s)\big(\partial_j\varphi(\cdot,q^{(i)}(s))-\partial_j\varphi(\cdot,q^{(i-1)}(s)),\partial_k\varphi(\cdot,q^{(i-1)}(s))\big)|^2_{L^2(\mathbb{R}^2)}$$
$$+C\sum_{j,k=1}^{2}|\partial_k\big(\sigma\widetilde{a}_h^{(i)}(\cdot,s)\big)\big(\partial_j\varphi(\cdot,q^{(i)}(s))-\partial_j\varphi(\cdot,q^{(i-1)}s),\varphi(\cdot,q^{(i-1)}(s))\big)|^2_{L^2(\mathbb{R}^2)}$$
$$+C\sum_{j,k=1}^{2}|\sigma\widetilde{a}_h^{(i)}(\cdot,s)\big(\varphi(\cdot,q^{(i)}(s))-\varphi(\cdot,q^{(i-1)}(s)),\partial^2_{jk}\varphi(\cdot,q^{(i-1)}(s))\big)|^2_{L^2(\mathbb{R}^2)}$$
$$+C\sum_{j,k=1}^{2}|\partial_k\big(\sigma\widetilde{a}_h^{(i)}(\cdot,s)\big)\big(\varphi(\cdot,q^{(i)}(s))-\varphi(\cdot,q^{(i-1)}(s)),\partial_j\varphi(\cdot,q^{(i-1)}(s))\big)|^2_{L^2(\mathbb{R}^2)}$$
$$+C\sum_{j,k=1}^{2}|\partial^2_{jk}\big(\sigma\widetilde{a}_h^{(i)}(\cdot,s)\big)\big(\varphi(\cdot,q^{(i)}(s))-\varphi(\cdot,q^{(i-1)}(s)),\varphi(\cdot,q^{(i-1)}(s))\big)|^2_{L^2(\mathbb{R}^2)}. \tag{5.3}$$

Now, for the first term of the right-hand side of (5.3), using Remark 3.1.1 and knowing that $\widetilde{a}_h^{(i)}\in X(\overline{T})$, we find that a constant $C(\Gamma)>0$ exists, independent of i, for which

$$\begin{aligned}
&|\sigma\widetilde{a}_h^{(i)}(\cdot,s)\big(\varphi(\cdot,q^{(i)}(s))-\varphi(\cdot,q^{(i-1)}(s)),\varphi(\cdot,q^{(i-1)}(s))\big)|^2_{L^2(\mathbb{R}^2)}\\
&\quad=\int_{\mathbb{R}^2}|\sigma\widetilde{a}_h^{(i)}(x,s)\big(\varphi(x,q^{(i)}(s))-\varphi(x,q^{(i-1)}(s)),\varphi(x,q^{(i-1)}(s))\big)|^2d^2x\\
&\quad\le\int_{\mathbb{R}^2}|\sigma\widetilde{a}_h^{(i)}(x,s)|^2|\varphi(x,q^{(i-1)}(s))|^2|\varphi(x,q^{(i)}(s))-\varphi(x,q^{(i-1)}(s))|^2d^2x\\
&\quad\le C(\Gamma)\int_{\mathbb{R}^2}|\varphi(x,q^{(i)}(s))-\varphi(x,q^{(i-1)}(s))|^2d^2x
\end{aligned}\tag{5.4}$$

for $0\le t\le\overline{T}$. Using Lemma 3.5.1 and the fact that the zero modes are exponentially decaying as x tends to infinity, it turns out that the right-hand side of (5.4) is smaller than

$$C(\Gamma)\int_{\mathbb{R}^2}|\frac{\partial\varphi}{\partial q_\mu}(x,q')|^2|q^{(i)}(s)-q^{(i-1)}(s)|^2d^2x \leq C(\Gamma)|q^{(i)}(s)-q^{(i-1)}(s)|,$$

for some $C(\Gamma)>0$ independent of i, for $0\leq t\leq \overline{T}$. Applying the same arguments to the other terms of the right-hand side of (5.3) one obtains that

$$\begin{aligned}&|\sigma\widetilde{a}_h^{(i)}(\cdot,s)\big(\varphi(\cdot,q^{(i)}(s))-\varphi(\cdot,q^{(i-1)}(s)),\varphi(\cdot,q^{(i-1)}(s))\big)|^2_{H^2(\mathbb{R}^2)}\\ &\leq C(\Gamma)|q^{(i)}(s)-q^{(i-1)}(s)| \qquad (5.5)\end{aligned}$$

for some constant $C(\Gamma)>0$ independent of i, for $0\leq t\leq \overline{T}$.

In the same way, for the second term of the right-hand side of (5.2) one proves that

$$\begin{aligned}&|\sigma\widetilde{a}_h^{(i)}(\cdot,s)\big(\varphi(\cdot,q^{(i)}(s))-\varphi(\cdot,q^{(i-1)}(s)),\varphi(\cdot,q^{(i)}(s))\big)|^2_{H^2(\mathbb{R}^2)}\\ &\leq C(\Gamma)|q^{(i)}(s)-q^{(i-1)}(s)| \qquad (5.6)\end{aligned}$$

for some constant $C(\Gamma)>0$ independent of i, for $0\leq t\leq \overline{T}$. Thus, (5.2) becomes

$$\begin{aligned}|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)-\sigma\widetilde{a}_h^{(i)}(\cdot,t)|^2_{H^3(\mathbb{R}^2)} &\leq K|\sigma\widetilde{a}_h^{(i+1)}(\cdot,0)-\sigma\widetilde{a}_h^{(i)}(\cdot,0)|^2_{H^3(\mathbb{R}^2)}\\ &+C(\Gamma)\int_0^t \max_{0\leq\tau\leq s}|q^{(i)}(\tau)-q^{(i-1)}(\tau)|ds\\ &+C(\Gamma)\int_0^t |f^1_{1h}(i,\cdot,s)-f^1_{1h}(i-1,\cdot,s)|^2_{H^2(\mathbb{R}^2)}ds, \qquad (5.7)\end{aligned}$$

for constants $K, C(\Gamma)>0$ independent of i, for $0\leq t\leq \overline{T}$.

A similar line of reasoning is followed for the complex component of $\psi^{(i+1)} - \psi^{(i)}$. From the complex part of (3.14) one obtains

$$\begin{aligned}
&\partial_t\big(\widetilde{\phi}^{(i+1)}(x,t) - \widetilde{\phi}^{(i)}(x,t)\big) - \sum_{l=1}^{2}\big(D_l^{(0)}\big)^2\big(\widetilde{\phi}^{(i+1)}(x,t) - \widetilde{\phi}^{(i)}(x,t)\big) \\
&\quad + |\phi(x,q^{(i)}(t))|^2\big(\widetilde{\phi}^{(i+1)}(x,t) - \widetilde{\phi}^{(i)}(x,t)\big) = f_1^2(i,x,t) - f_1^2(i-1,x,t) \\
&\quad - i\sum_{l=1}^{2}\widetilde{\phi}^{(i)}(x,t)\big(\partial_l\alpha_l(x,q^{(i)}(t)) - \partial_l\alpha_l(x,q^{(i-1)}(t))\big) \\
&\quad - i2\sum_{l=1}^{2}D_l^{(0)}\widetilde{\phi}^{(i)}(x,t)\big(\alpha_l(x,q^{(i)}(t)) - \alpha_l(x,q^{(i-1)}(t))\big) \\
&\quad + \sum_{l=1}^{2}\widetilde{\phi}^{(i)}(x,t)\big(\alpha_l(x,q^{(i)}(t)) - \alpha_l(x,q^{(i-1)}(t))\big)^2 \\
&\quad - \widetilde{\phi}^{(i)}(x,t)\big(|\phi(x,q^{(i)}(t))|^2 - |\phi(x,q^{(i-1)}(t))|^2\big). \qquad (5.8)
\end{aligned}$$

Now, applying to (5.8) the same arguments as in the proof of Lemma 3.6.2, it turns out that the following inequality holds for $0 \le t \le \overline{T}$:

$$\begin{aligned}
&|\widetilde{\phi}^{(i+1)}(\cdot,t) - \widetilde{\phi}^{(i)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} + \sum_{j=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}\big(\widetilde{\phi}^{(i+1)}(\cdot,t) - \widetilde{\phi}^{(i)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \sum_{j,k=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}\big(\widetilde{\phi}^{(i+1)}(\cdot,t) - \widetilde{\phi}^{(i)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \sum_{j,k,l=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}D_l^{(\alpha(\cdot,q(0)))}\big(\widetilde{\phi}^{(i+1)}(\cdot,t) - \widetilde{\phi}^{(i)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} \\
&\quad \le K|\widetilde{\phi}^{(i+1)}(\cdot,0) - \widetilde{\phi}^{(i)}(\cdot,0)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + K\sum_{j=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}\big(\widetilde{\phi}^{(i+1)}(\cdot,0) - \widetilde{\phi}^{(i)}(\cdot,0)\big)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + K\sum_{j,k=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}\big(\widetilde{\phi}^{(i+1)}(\cdot,0) - \widetilde{\phi}^{(i)}(\cdot,0)\big)|^2_{L^2(\mathbb{R}^2)}
\end{aligned}$$

$$
\begin{aligned}
&+ K \sum_{j,k,l=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} D_k^{(\alpha(\cdot,q(0)))} D_l^{(\alpha(\cdot,q(0)))} \big(\widetilde{\phi}^{(i+1)}(\cdot,0) - \widetilde{\phi}^{(i)}(\cdot,0)\big)|^2_{L^2(\mathbb{R}^2)} \\
&+ C(\Gamma) \int_0^t \Big(|f_1^2(i,\cdot,s) - f_1^2(i-1,\cdot,s)|^2_{L^2(\mathbb{R}^2)} \\
&+ \sum_{j=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} \big(f_1^2(i,\cdot,s) - f_1^2(i-1,\cdot,s)|^2_{L^2(\mathbb{R}^2)} \\
&+ \sum_{j,k=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} D_k^{(\alpha(\cdot,q(0)))} \big(f_1^2(i,\cdot,s) - f_1^2(i-1,\cdot,s)\big)|^2_{L^2(\mathbb{R}^2)} \\
&+ \sum_{l=1}^{2} |\widetilde{\phi}^{(i)}(\cdot,s)\big(\partial_l\alpha_l(\cdot,q^{(i)}(s)) - \partial_l\alpha_l(\cdot,q^{(i-1)}(s))\big)|^2_{L^2(\mathbb{R}^2)} \\
&+ \sum_{j,l=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} \Big(\widetilde{\phi}^{(i)}(\cdot,s)\big(\partial_l\alpha_l(\cdot,q^{(i)}(s)) - \partial_l\alpha_l(\cdot,q^{(i-1)}(s))\big)\Big)|^2_{L^2(\mathbb{R}^2)} \\
&+ \sum_{j,k,l=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} D_k^{(\alpha(\cdot,q(0)))} \Big(\widetilde{\phi}^{(i)}(\cdot,s)\big(\partial_l\alpha_l(\cdot,q^{(i)}(s)) \\
&- \partial_l\alpha_l(\cdot,q^{(i-1)}(s))\big)\Big)|^2_{L^2(\mathbb{R}^2)} \\
&+ \sum_{l=1}^{2} |D_l^{(0)}\widetilde{\phi}^{(i)}(\cdot,s)\big(\alpha_l(\cdot,q^{(i)}(s)) - \alpha_l(\cdot,q^{(i-1)}(s))\big)|^2_{L^2(\mathbb{R}^2)} \\
&+ \sum_{j,l=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} \Big(D_l^{(0)}\widetilde{\phi}^{(i)}(\cdot,s)\big(\alpha_l(\cdot,q^{(i)}(s)) - \alpha_l(\cdot,q^{(i-1)}(s))\big)\Big)|^2_{L^2(\mathbb{R}^2)} \\
&+ \sum_{j,k,l=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} D_k^{(\alpha(\cdot,q(0)))} \Big(D_l^{(0)}\widetilde{\phi}^{(i)}(\cdot,s)\big(\alpha_l(\cdot,q^{(i)}(s)) \\
&- \alpha_l(\cdot,q^{(i-1)}(s))\big)\Big)|^2_{L^2(\mathbb{R}^2)} \\
&+ \sum_{l=1}^{2} |\widetilde{\phi}^{(i)}(\cdot,s)\big(\alpha_l(\cdot,q^{(i)}(s)) - \alpha_l(\cdot,q^{(i-1)}(s))\big)^2|^2_{L^2(\mathbb{R}^2)} \\
&+ \sum_{j,l=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} \Big(\widetilde{\phi}^{(i)}(\cdot,s)\big(\alpha_l(\cdot,q^{(i)}(s)) - \alpha_l(\cdot,q^{(i-1)}(s))\big)^2\Big)|^2_{L^2(\mathbb{R}^2)}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j,k,l=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}\Big(\widetilde{\phi}^{(i)}(\cdot,s)\big(\alpha_l(\cdot,q^{(i)}(s))\\
&-\alpha_l(\cdot,q^{(i-1)}(s))\big)^2\Big)|^2_{L^2(\mathbb{R}^2)}+|\widetilde{\phi}^{(i)}(\cdot,s)\big(|\phi(\cdot,q^{(i)}(s))|^2\\
&-|\phi(\cdot,q^{(i-1)}(s))|^2\big)|^2_{L^2(\mathbb{R}^2)}+\sum_{j=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}\Big(\widetilde{\phi}^{(i)}(\cdot,s)\big(|\phi(\cdot,q^{(i)}(s))|^2\\
&-|\phi(\cdot,q^{(i-1)}(s))|^2\big)\Big)|^2_{L^2(\mathbb{R}^2)}+\sum_{j,k=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}\Big(\widetilde{\phi}^{(i)}(\cdot,s)\big(|\phi(\cdot,q^{(i)}(s))|^2\\
&-|\phi(\cdot,q^{(i-1)}(s))|^2\big)\Big)|^2_{L^2(\mathbb{R}^2)}\Big)ds.
\end{aligned}
\tag{5.9}
$$

Now, by estimating every term of the right-hand side of (5.9), as in the case of (5.2), one obtains the equivalent of (5.7) for the complex component:

$$
\begin{aligned}
&|\widetilde{\phi}^{(i+1)}(\cdot,t)-\widetilde{\phi}^{(i)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)}+\sum_{j=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}\big(\widetilde{\phi}^{(i+1)}(\cdot,t)-\widetilde{\phi}^{(i)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)}\\
&\quad+\sum_{j,k=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}\big(\widetilde{\phi}^{(i+1)}(\cdot,t)-\widetilde{\phi}^{(i)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)}\\
&\quad+\sum_{j,k,l=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}D_l^{(\alpha(\cdot,q(0)))}\big(\widetilde{\phi}^{(i+1)}(\cdot,t)-\widetilde{\phi}^{(i)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)}\\
&\leq K|\widetilde{\phi}^{(i+1)}(\cdot,0)-\widetilde{\phi}^{(i)}(\cdot,0)|^2_{L^2(\mathbb{R}^2)}\\
&\quad+K\sum_{j=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}\big(\widetilde{\phi}^{(i+1)}(\cdot,0)-\widetilde{\phi}^{(i)}(\cdot,0)\big)|^2_{L^2(\mathbb{R}^2)}\\
&\quad+K\sum_{j,k=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}\big(\widetilde{\phi}^{(i+1)}(\cdot,0)-\widetilde{\phi}^{(i)}(\cdot,0)\big)|^2_{L^2(\mathbb{R}^2)}\\
&\quad+K\sum_{j,k,l=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}D_l^{(\alpha(\cdot,q(0)))}\big(\widetilde{\phi}^{(i+1)}(\cdot,0)-\widetilde{\phi}^{(i)}(\cdot,0)\big)|^2_{L^2(\mathbb{R}^2)}
\end{aligned}
$$

$$+ C(\Gamma)\int_0^t \max_{0\le\tau\le s} |q^{(i)}(\tau) - q^{(i-1)}(\tau)|ds$$
$$+ C(\Gamma)\int_0^t \Big(|f_1^2(i,\cdot,s) - f_1^2(i-1,\cdot,s)|^2_{L^2(\mathbb{R}^2)}$$
$$+ \sum_{j=1}^2 |D_j^{(\alpha(\cdot,q(0)))}\big(f_1^2(i,\cdot,s) - f_1^2(i-1,\cdot,s)\big)|^2_{L^2(\mathbb{R}^2)}$$
$$+ \sum_{j,k=1}^2 |D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}\big(f_1^2(i,\cdot,s) - f_1^2(i-1,\cdot,s)\big)|^2_{L^2(\mathbb{R}^2)}\Big)ds \tag{5.10}$$

for constants $K, C(\Gamma) > 0$ independent of i, for $0 \le t \le \overline{T}$.

5.3 An Improved Estimate for $\psi^{(i+1)} - \psi^{(i)}$

In this section an improvement of the estimates (5.7) and (5.10) are presented. The mean value theorem is the main tool used here which helps to estimate $\psi^{(i+1)}-\psi^{(i)}$ by using the initial data, $\|\psi^{(i)}-\psi^{(i-1)}\|$, $\|\widetilde{\widetilde{a}}_0^{(i+1)} - \widetilde{\widetilde{a}}_0^{(i)}\|$, $|q^{(i)}-q^{(i-1)}|$ and $|\dot{q}^{(i)} - \dot{q}^{(i-1)}|$.

Firstly, we focus on improving (5.7). The mean value theorem is applied to prove an estimate for $f_{1h}(\widetilde{a}^{(i)}, \widetilde{\phi}^{(i)}, \widetilde{\widetilde{a}}_0^{(i+1)}, q^{(i)}, \dot{q}^{(i)})$ $-f_{1h}(\widetilde{a}^{(i-1)}, \widetilde{\phi}^{(i-1)}, \widetilde{\widetilde{a}}_0^{(i)}, q^{(i-1)}, \dot{q}^{(i-1)})$. Following the notation of Theorem 5.1.1, we define

$$X_1 \times X_2 \times X_3 \times X_4 := X(\overline{T}), \qquad X_5 := \mathbb{R}^{2N}, \qquad Y := H^2(\mathbb{R}^2), \qquad f := f_{1h}^1.$$

Now, provided that all the partial derivatives of f_{1h}^1 with respect to $\widetilde{a}$, $\widetilde{\phi}$, $\widetilde{\widetilde{a}}_0$, q and $\dot{q}$ are bounded in $L^2(\mathbb{R}^2)$, applying the mean value theorem, an estimate for $f_{1h}(\widetilde{a}^{(i)}, \widetilde{\phi}^{(i)}, \widetilde{\widetilde{a}}_0^{(i+1)}, q^{(i)}, \dot{q}^{(i)}) - f_{1h}(\widetilde{a}^{(i-1)}, \widetilde{\phi}^{(i-1)}, \widetilde{\widetilde{a}}_0^{(i)}, q^{(i-1)}, \dot{q}^{(i-1)})$

is straightforwardly obtained in terms of the quantities $\|\psi^{(i)} - \psi^{(i-1)}\|$, $\|\widetilde{\widetilde{a}}_0^{(i+1)} - \widetilde{\widetilde{a}}_0^{(i)}\|$, $|q^{(i)} - q^{(i-1)}|$ and $|\dot{q}^{(i)} - \dot{q}^{(i-1)}|$. It is known that

$$f_1(\psi, \widetilde{\widetilde{a}}_0, q, \dot{q}) := \begin{pmatrix} -\sum_{\mu=1}^{2N} \sigma \frac{\partial \alpha_1}{\partial q_\mu}(\cdot, q)\dot{q}_\mu + \sigma\partial_1 \widetilde{\widetilde{a}}_0 - \frac{2\varepsilon}{\sigma}\sigma\widetilde{a}_1\big(\varphi(\cdot, q), \widetilde{\phi}\big) \\ -\sum_{\mu=1}^{2N} \sigma \frac{\partial \alpha_2}{\partial q_\mu}(\cdot, q)\dot{q}_\mu + \sigma\partial_2 \widetilde{\widetilde{a}}_0 - \frac{2\varepsilon}{\sigma}\sigma\widetilde{a}_2\big(\varphi(\cdot, q), \widetilde{\phi}\big) \\ -\sum_{\mu=1}^{2N} \frac{\partial \varphi}{\partial q_\mu}(\cdot, q)\dot{q}_\mu + \frac{1}{2}\varphi(\cdot, q)(1 - |\phi(\cdot, q)|^2) + i\widetilde{\widetilde{a}}_0\varphi(\cdot, q) \end{pmatrix}$$

$$- \begin{pmatrix} -\varepsilon\big(i\widetilde{\phi}, D_1^{(0)}\widetilde{\phi}\big) + \frac{\varepsilon^2}{\sigma}\sigma\widetilde{a}_1|\widetilde{\phi}|^2 \\ -\varepsilon\big(i\widetilde{\phi}, D_2^{(0)}\widetilde{\phi}\big) + \frac{\varepsilon^2}{\sigma}\sigma\widetilde{a}_2|\widetilde{\phi}|^2 \\ i\frac{\varepsilon}{\sigma}\widetilde{\phi}\sum_{l=1}^2 \partial_l(\sigma\widetilde{a}_l) + i\frac{2\varepsilon}{\sigma}\sum_{l=1}^2 \sigma\widetilde{a}_l D_l^{(0)}\widetilde{\phi} + i\frac{\varepsilon}{\sigma^2}\varphi(\cdot, q)\sum_{l=1}^2(\sigma\widetilde{a}_l)^2 \end{pmatrix}$$

$$- \begin{pmatrix} \frac{1}{\sigma}\partial_1 \sum_{l=1}^2 \partial_l(\sigma\widetilde{a}_l) \\ \frac{1}{\sigma}\partial_2 \sum_{l=1}^2 \partial_l(\sigma\widetilde{a}_l) \\ i\frac{\varepsilon^2}{\sigma^2}\widetilde{\phi}\sum_{l=1}^2(\sigma\widetilde{a}_l)^2 - \frac{\varepsilon}{2}\widetilde{\phi}(1 - |\phi(\cdot, q)|^2) + \varepsilon(1+\varepsilon)\widetilde{\phi}\big(\varphi(\cdot, q), \widetilde{\phi}\big) \end{pmatrix}$$

$$- \begin{pmatrix} -2\big(i\widetilde{\phi}, D_1^{(0)}\varphi(\cdot, q)\big) \\ -2\big(i\widetilde{\phi}, D_2^{(0)}\varphi(\cdot, q)\big) \\ \varepsilon\varphi(\cdot, q)\big(\varphi(\cdot, q), \widetilde{\phi}\big) + \varepsilon\Big(\frac{1}{2} + \frac{\varepsilon}{2}\Big)|\widetilde{\phi}|^2\varphi(\cdot, q) + \varepsilon^2\Big(\frac{1}{2} + \frac{\varepsilon}{2}\Big)|\widetilde{\phi}|^2\widetilde{\phi} \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{1}{\sigma\varepsilon}\partial_1 \sum_{l=1}^2 \partial_l(\sigma\widetilde{a}_l) \\ \frac{1}{\sigma\varepsilon}\partial_2 \sum_{l=1}^2 \partial_l(\sigma\widetilde{a}_l) \\ \frac{1}{2}\widetilde{\phi}(1 - |\phi(\cdot, q)|^2) + i\frac{1}{\varepsilon\sigma}\varphi(\cdot, q)\sum_{l=1}^2 \partial_l(\sigma\widetilde{a}_l) + i\varepsilon\widetilde{\widetilde{a}}_0\widetilde{\phi} \end{pmatrix}.$$

Differentiating f_{1h}^1 with respect to ψ, $\widetilde{\widetilde{a}}_0$, q and $\dot{q}$, we obtain

$$\frac{\partial f_{1h}^1}{\partial(\sigma\widetilde{a}_k)}(\psi, \widetilde{\widetilde{a}}_0, q, \dot{q}) = -\frac{2\varepsilon}{\sigma}\delta_{hk}\big(\varphi(\cdot, q), \widetilde{\phi}\big) - \frac{\varepsilon^2}{\sigma}\delta_{hk}|\widetilde{\phi}|^2,$$

$$\begin{aligned} \frac{\partial f_{1h}^1}{\partial\widetilde{\phi}}(\psi, \widetilde{\widetilde{a}}_0, q, \dot{q}) = &-\frac{2\varepsilon}{\sigma}\sigma\widetilde{a}_h\big(\varphi(\cdot, q), 1\big) + \varepsilon\big(i, D_h^{(0)}\widetilde{\phi}\big) \\ &- \varepsilon\alpha_h(\cdot, q)\big(\widetilde{\phi}, 1\big) - \frac{2\varepsilon^2}{\sigma}\sigma\widetilde{a}_h\big(i, \widetilde{\phi}\big) + 2\big(i, D_h^{(0)}\varphi(\cdot, q)\big), \end{aligned}$$

$$\frac{\partial f^1_{1h}}{\partial \widetilde{\widetilde{a}}_0}(\psi, \widetilde{\widetilde{a}}_0, q, \dot{q}) = 0,$$

$$\begin{aligned}\frac{\partial f^1_{1h}}{\partial q_\nu}(\psi, \widetilde{\widetilde{a}}_0, q, \dot{q}) &= -\sigma \sum_{\mu=1}^{2N} \frac{\partial^2 \alpha_h}{\partial q_\mu \partial q_\nu}(\cdot, q)\dot{q}_\mu \\ &- \frac{2\varepsilon}{\sigma}\sigma\widetilde{a}_h\big(\widetilde{\phi}, \frac{\partial \varphi}{\partial q_\nu}(\cdot, q)\big) - \varepsilon\frac{\partial \alpha_h}{\partial q_\mu}(\cdot, q)|\widetilde{\phi}|^2 + 2\big(i\widetilde{\phi}, \partial_h \frac{\partial \varphi}{\partial q_\nu}(\cdot, q)\big) \\ &- \frac{\partial \alpha_h}{\partial q_\nu}(\cdot, q)\big(\varphi(\cdot, q), \widetilde{\phi}\big) - \alpha_h(\cdot, q)\big(\widetilde{\phi}, \frac{\partial \varphi}{\partial q_\nu}(\cdot, q)\big)\end{aligned}$$

and

$$\frac{\partial f^1_{1h}}{\partial \dot{q}_\nu}(\psi, \widetilde{\widetilde{a}}_0, q, \dot{q}) = -\sigma\frac{\partial \alpha_h}{\partial q_\nu}(\cdot, q).$$

We notice $\frac{\partial f^1_{1h}}{\partial(\sigma\widetilde{a}_k)}$, $\frac{\partial f^1_{1h}}{\partial \widetilde{\phi}}$, $\frac{\partial f^1_{1h}}{\partial \widetilde{\widetilde{a}}_0}$, $\frac{\partial f^1_{1h}}{\partial q_\nu}$ and $\frac{\partial f^1_{1h}}{\partial \dot{q}_\nu}$ are bounded in $H^2(\mathbb{R}^2)$ for $0 \leq t \leq \overline{T}$. Because of Cauchy's inequality a constant $C > 0$ exists, independent of i, such that

$$|\frac{\partial f^1_{1h}}{\partial(\sigma\widetilde{a}_k)}(\psi, \widetilde{\widetilde{a}}_0, q, \dot{q})|^2_{H^2(\mathbb{R}^2)} \leq C|\big(\varphi(\cdot, q), \widetilde{\phi}\big)|^2_{H^2(\mathbb{R}^2)} + C|\big(\widetilde{\phi}, \widetilde{\phi}\big)|^2_{H^2(\mathbb{R}^2)}. \tag{5.11}$$

Because of Remark 3.1.1 and since $(\psi, \widetilde{\widetilde{a}}_0, q) \in X(\overline{T})$, the right-hand side of (5.11) is bounded for $0 \leq t \leq \overline{T}$. Analogously, we find that a constant $C > 0$ exists, independent of i, such that

$$\begin{aligned}|\frac{\partial f^1_{1h}}{\partial \widetilde{\phi}}(\psi, \widetilde{\widetilde{a}}_0, q, \dot{q})|^2_{H^2(\mathbb{R}^2)} &\leq C|\sigma\widetilde{a}_h\big(\varphi(\cdot, q), 1\big)|^2_{H^2(\mathbb{R}^2)} \\ &+ C|\big(i, D_h^{(0)}\widetilde{\phi}\big)|^2_{H^2(\mathbb{R}^2)} + C|\alpha_h(\cdot, q)\big(\widetilde{\phi}, 1\big)|^2_{H^2(\mathbb{R}^2)} \\ &+ C|\sigma\widetilde{a}_h\big(i, \widetilde{\phi}\big)|^2_{H^2(\mathbb{R}^2)} + C|\big(i, D_h^{(0)}\varphi(\cdot, q)\big)|^2_{H^2(\mathbb{R}^2)}.\end{aligned} \tag{5.12}$$

We use Remark 3.1.1, $(\psi, \widetilde{\widetilde{a}}_0, q) \in X(\overline{T})$ and the fact that the covariant derivative of the static Higgs field and any of its derivatives with respect to

x is exponentially decaying as x tends to infinity, [Stu94,I, corollary 2.4], we find that the right-hand side of (5.12) is bounded for $0 \leq t \leq \overline{T}$. Obviously

$$|\frac{\partial f_{1h}^1}{\partial \widetilde{\widetilde{a}}_0}(\psi, \widetilde{\widetilde{a}}_0, q, \dot{q})|_{H^2(\mathbb{R}^2)}^2 = 0$$

is bounded for $0 \leq t \leq \overline{T}$. Moreover, because of Cauchy's inequality, a constant $C > 0$ exists, independent of i, such that

$$\begin{aligned}
|\frac{\partial f_{1h}^1}{\partial q_\nu}(\psi, \widetilde{\widetilde{a}}_0, q, \dot{q})|_{H^2(\mathbb{R}^2)}^2 &\leq C \sum_{\mu=1}^{2N} |\frac{\partial^2 \alpha_h}{\partial q_\mu \partial q_\nu}(\cdot, q)\dot{q}_\mu|_{H^2(\mathbb{R}^2)}^2 \\
&+ C|\sigma \widetilde{a}_h\big(\widetilde{\phi}, \frac{\partial \varphi}{\partial q_\nu}(\cdot, q)\big)|_{H^2(\mathbb{R}^2)}^2 + C|\frac{\partial \alpha_h}{\partial q_\nu}(\cdot, q)\big(\widetilde{\phi}, \widetilde{\phi}\big)|_{H^2(\mathbb{R}^2)}^2 \\
&+ C|\big(\partial_h \widetilde{\phi}, i\frac{\partial \varphi}{\partial q_\nu}(\cdot, q)\big)|_{H^2(\mathbb{R}^2)}^2 + C|\big(i\widetilde{\phi}, D_h^{(0)}\frac{\partial \varphi}{\partial q_\nu}(\cdot, q)\big)|_{H^2(\mathbb{R}^2)}^2 \\
&+ C|\frac{\partial \alpha_h}{\partial q_\nu}(\cdot, q)\big(i\widetilde{\phi}, \varphi(\cdot, q)\big)|_{H^2(\mathbb{R}^2)}^2 + C|\big(\partial_h \widetilde{\phi}, i\frac{\partial \varphi}{\partial q_\nu}(\cdot, q)\big)|_{H^2(\mathbb{R}^2)}^2.
\end{aligned} \tag{5.13}$$

Using again Remark 3.1.1, $(\psi, \widetilde{\widetilde{a}}_0, q) \in X(\overline{T})$ and the fact that all the derivatives with respect to x and q of the zero modes are exponentially decaying as x tends to infinity, the right-hand side of (5.13) is seen to be bounded for $0 \leq t \leq \overline{T}$. In an analogous way, a constant $C > 0$ exists, independent of i, such that

$$|\frac{\partial f_{1h}^1}{\partial \dot{q}_\nu}(\psi, \widetilde{\widetilde{a}}_0, q, \dot{q})|_{H^2(\mathbb{R}^2)}^2 \leq C|\frac{\partial \alpha_h}{\partial q_\nu}(\cdot, q)|_{H^2(\mathbb{R}^2)}^2 \tag{5.14}$$

As in the previous case, the right-hand side of (5.14) is bounded for $0 \leq t \leq \overline{T}$.

Then, applying the mean value theorem to the function $f^1_{1h}(\psi, \widetilde{\widetilde{a}}_0, q, \dot{q})$ one obtains

$$|f^1_{1h}(i,\cdot,s) - f^1_{1h}(i-1,\cdot,s)|^2_{H^2(\mathbb{R}^2)} \leq C(\Gamma)\big\|\psi^{(i)}(\cdot,s) - \psi^{(i-1)}(\cdot,s)\big\|$$
$$+ C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s) - \widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\big\| + C(\Gamma)\max_{0\leq\tau\leq s}|q^{(i)}(\tau) - q^{(i-1)}(\tau)|$$
$$+ C(\Gamma)|\dot{q}^{(i)}(s) - \dot{q}^{(i-1)}(s)|. \tag{5.15}$$

Finally, using (5.15) in (5.7), it turns out that, for $0 \leq t \leq \overline{T}$

$$|\sigma\widetilde{a}_h^{(i+1)}(\cdot,t) - \sigma\widetilde{a}_h^{(i)}(\cdot,t)|^2_{H^3(\mathbb{R}^2)} \leq K|\sigma\widetilde{a}_h^{(i+1)}(\cdot,0) - \sigma\widetilde{a}_h^{(i)}(\cdot,0)|^2_{H^3(\mathbb{R}^2)}$$
$$+ C(\Gamma)\int_0^t \Big(\big\|\psi^{(i)}(\cdot,s) - \psi^{(i-1)}(\cdot,s)\big\| + \big\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s) - \widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\big\|$$
$$+ \max_{0\leq\tau\leq s}|q^{(i)}(\tau) - q^{(i-1)}(\tau)| + |\dot{q}^{(i)}(s) - \dot{q}^{(i-1)}(s)|\Big)ds \tag{5.16}$$

for constants $K, C(\Gamma) > 0$ independent of i.

Now the mean value theorem is applied to the complex part of $f_1(\widetilde{a}^{(i)}, \widetilde{\phi}^{(i)}, \widetilde{\widetilde{a}}_0^{(i+1)}, q^{(i)}, \dot{q}^{(i)}) - f_1(\widetilde{a}^{(i-1)}, \widetilde{\phi}^{(i-1)}, \widetilde{\widetilde{a}}_0^{(i)}, q^{(i-1)}, \dot{q}^{(i-1)})$. In this case, with the same notation as in Theorem 5.1.1, we define

$$X_1 \times X_2 \times X_3 \times X_4 := X(\overline{T}), \quad X_5 := \mathbb{R}^{2N}, \quad Y := H^2(\mathbb{R}^2), \quad f := f^1_2,$$

where $H^2(\mathbb{R}^2)$ is equipped with the covariant derivative $D^{(\alpha(\cdot,q(0)))}$ instead of the normal derivative. After differentiating f^2_1 with respect to ψ, $\widetilde{\widetilde{a}}_0$, q and $\dot{q}$, we obtain

$$\frac{\partial f^2_1}{\partial(\sigma\widetilde{a}_k)}(\psi, \widetilde{\widetilde{a}}_0, q, \dot{q}) = -i\frac{2\varepsilon}{\sigma}D^{(0)}_k\widetilde{\phi} - i\frac{2\varepsilon}{\sigma^2}\sigma\widetilde{a}_k\varphi(\cdot,q) - i\frac{2\varepsilon^2}{\sigma^2}\sigma\widetilde{a}_k\widetilde{\phi},$$

$$\frac{\partial f_1^2}{\partial \widetilde{\phi}}(\psi, \widetilde{\widetilde{a}}_0, q, \dot{q}) = -i\frac{2\varepsilon}{\sigma}\sum_{l=1}^{2}\partial_l(\sigma\widetilde{a}_l) - \frac{2\varepsilon}{\sigma}\sum_{l=1}^{2}\sigma\widetilde{a}_l\alpha_l(\cdot, q)$$
$$- i\frac{\varepsilon^2}{\sigma^2}\sum_{l=1}^{2}(\sigma\widetilde{a}_l)^2 - \varepsilon(1+\varepsilon)\varphi(\cdot, q)\big(\widetilde{\phi}, 1\big) - \varepsilon^2(1+\varepsilon)\widetilde{\phi}\big(\widetilde{\phi}, 1\big)$$
$$- \varepsilon^2\frac{1+\varepsilon}{2}|\widetilde{\phi}|^2 + \frac{1}{2}(1 - |\phi(\cdot, q)|^2) + i\varepsilon\widetilde{\widetilde{a}}_0,$$

$$\frac{\partial f_1^2}{\partial \widetilde{\widetilde{a}}_0}(\psi, \widetilde{\widetilde{a}}_0, q, \dot{q}) = i\varphi(\cdot, q) + i\varepsilon\widetilde{\phi},$$

$$\frac{\partial f_1^2}{\partial q_\nu}(\psi, \widetilde{\widetilde{a}}_0, q, \dot{q}) = -\sigma\sum_{\mu=1}^{2N}\frac{\partial^2\varphi}{\partial q_\mu\partial q_\nu}(\cdot, q)\dot{q}_\mu$$
$$+ \frac{1}{2}(1 - |\phi(\cdot, q)|^2)\frac{\partial\varphi}{\partial q_\nu}(\cdot, q) - \varphi(\cdot, q)\big(\varphi(\cdot, q), \frac{\partial\varphi}{\partial q_\nu}(\cdot, q)\big) + i\widetilde{\widetilde{a}}_0\frac{\partial\varphi}{\partial q_\nu}(\cdot, q)$$
$$- \frac{2\varepsilon}{\sigma}\widetilde{\phi}\sum_{l=1}^{2}\sigma\widetilde{a}_l\frac{\partial\alpha_l}{\partial q_\nu}(\cdot, q) - i\frac{\varepsilon}{\sigma^2}\frac{\partial\varphi}{\partial q_\nu}(\cdot, q)\sum_{l=1}^{2}(\sigma\widetilde{a}_l)^2$$
$$- (1+\varepsilon)\widetilde{\phi}\big(\varphi(\cdot, q), \frac{\partial\varphi}{\partial q_\nu}(\cdot, q)\big) - \varepsilon(1+\varepsilon)\widetilde{\phi}\big(\widetilde{\phi}, \frac{\partial\varphi}{\partial q_\nu}(\cdot, q)\big)$$
$$- \varepsilon\frac{\partial\varphi}{\partial q_\nu}(\cdot, q)\big(\varphi(\cdot, q), \widetilde{\phi}\big) - \varepsilon\varphi(\cdot, q)\big(\widetilde{\phi}, \frac{\partial\varphi}{\partial q_\nu}(\cdot, q)\big)$$
$$- \varepsilon\frac{1+\varepsilon}{2}|\widetilde{\phi}|^2\frac{\partial\varphi}{\partial q_\nu}(\cdot, q) + i\frac{1}{\varepsilon\sigma}\frac{\partial\varphi}{\partial q_\nu}(\cdot, q)\sum_{l=1}^{2}\partial_l(\sigma\widetilde{a}_l)$$

and

$$\frac{\partial f_1^2}{\partial \dot{q}_\nu}(\psi, \widetilde{\widetilde{a}}_0, q, \dot{q}) = -\frac{\partial\varphi}{\partial q_\nu}(\cdot, q).$$

We notice that, while $\frac{\partial f_1^2}{\partial(\sigma\widetilde{a}_k)}$, $\frac{\partial f_1^2}{\partial\widetilde{\phi}}$, $\frac{\partial f_1^2}{\partial q_\nu}$ and $\frac{\partial f_1^2}{\partial\dot{q}_\nu}$ are bounded in $H^2(\mathbb{R}^2)$ for $0 \le t \le \overline{T}$, $\frac{\partial f_1^2}{\partial\widetilde{\widetilde{a}}_0}$ is unbounded since $\varphi(\cdot, q) \notin L^2(\mathbb{R}^2)$. In fact, because of the

Cauchy inequality, a constant $C > 0$ independent of i exists such that

$$|\frac{\partial f_1^2}{\partial(\sigma\widetilde{a}_k)}(\psi, \widetilde{\widetilde{a}}_0, q, \dot{q})|^2_{H^2(\mathbb{R}^2)} \leq C|D_k^{(0)}\widetilde{\phi}|^2_{H^2(\mathbb{R}^2)} + C|\sigma\widetilde{a}_k\varphi(\cdot, q)|^2_{H^2(\mathbb{R}^2)} + C|\sigma\widetilde{a}_k\widetilde{\phi}|^2_{H^2(\mathbb{R}^2)}. \tag{5.17}$$

From Remark 3.1.1 and since $(\psi, \widetilde{\widetilde{a}}_0, q) \in X(\overline{T})$ the right-hand side of (5.17) is bounded for $0 \leq t \leq \overline{T}$. Moreover, because of Cauchy inequality, a constant $C > 0$ independent of i exists such that

$$\begin{aligned}|\frac{\partial f_1^2}{\partial\widetilde{\phi}}(\psi, \widetilde{\widetilde{a}}_0, q, \dot{q})|^2_{H^2(\mathbb{R}^2)} &\leq C|\partial_l(\sigma\widetilde{a}_l)|^2_{H^2(\mathbb{R}^2)} + C|\sigma\widetilde{a}_l\alpha_l(\cdot, q)|^2_{H^2(\mathbb{R}^2)} \\ &+ C|(\sigma\widetilde{a}_l)^2|^2_{H^2(\mathbb{R}^2)} + C|\varphi(\cdot, q)(\widetilde{\phi}, 1)|^2_{H^2(\mathbb{R}^2)} + C|\widetilde{\phi}(\widetilde{\phi}, 1)|^2_{H^2(\mathbb{R}^2)} \\ &+ C|(\widetilde{\phi}, \widetilde{\phi})|^2_{H^2(\mathbb{R}^2)} + C|1 - |\phi(\cdot, q)|^2|^2_{H^2(\mathbb{R}^2)} + C|\widetilde{\widetilde{a}}_0|^2_{H^2(\mathbb{R}^2)}.\end{aligned} \tag{5.18}$$

Because of Remark 3.1.1, since $(\psi, \widetilde{\widetilde{a}}_0, q) \in X(\overline{T})$ and since $1 - |\phi(\cdot, q)|^2$ and its derivatives with respect to x are exponentially decaying as x tends to infinity, the right-hand side of (5.18) is bounded for $0 \leq t \leq \overline{T}$. On the other hand

$$|\frac{\partial f_1^2}{\partial\widetilde{\widetilde{a}}_0}(\psi, \widetilde{\widetilde{a}}_0, q, \dot{q})|^2_{H^2(\mathbb{R}^2)} = C|\varphi(\cdot, q)|^2_{H^2(\mathbb{R}^2)} + C|\widetilde{\phi}|^2_{H^2(\mathbb{R}^2)}$$

is unbounded since $\varphi(\cdot, q) \notin L^2(\mathbb{R}^2)$. Again, using the Cauchy inequality, a constant $C > 0$ independent of i exists such that

$$\begin{aligned}|\frac{\partial f_1^2}{\partial q_\nu}(\psi, \widetilde{\widetilde{a}}_0, q, \dot{q})|^2_{H^2(\mathbb{R}^2)} &\leq C\sum_{\mu=1}^{2N}|\frac{\partial^2\varphi}{\partial q_\mu\partial q_\nu}(\cdot, q)\dot{q}_\nu|^2_{H^2(\mathbb{R}^2)} \\ &+ C|\frac{\partial\varphi}{\partial q_\nu}(\cdot, q)(1 - |\phi(\cdot, q)|^2)|^2_{H^2(\mathbb{R}^2)} + C|\varphi(\cdot, q)(\varphi(\cdot, q), \frac{\partial\varphi}{\partial q_\nu}(\cdot, q))|^2_{H^2(\mathbb{R}^2)} \\ &+ C|\widetilde{\widetilde{a}}_0\frac{\partial\varphi}{\partial q_\nu}(\cdot, q)|^2_{H^2(\mathbb{R}^2)} + C\sum_{l=1}^{2}|\sigma\widetilde{a}_l\widetilde{\phi}\frac{\partial\alpha_l}{\partial q_\nu}(\cdot, q)|^2_{H^2(\mathbb{R}^2)}\end{aligned}$$

$$+C\sum_{l=1}^{2}|(\sigma\widetilde{a}_l)^2\frac{\partial\varphi}{\partial q_\nu}(\cdot,q)|^2_{H^2(\mathbb{R}^2)}+C|\widetilde{\phi}(\varphi(\cdot,q),\frac{\partial\varphi}{\partial q_\nu}(\cdot,q))|^2_{H^2(\mathbb{R}^2)}$$
$$+C|\widetilde{\phi}(\widetilde{\phi},\frac{\partial\varphi}{\partial q_\nu}(\cdot,q))|^2_{H^2(\mathbb{R}^2)}+C|\frac{\partial\varphi}{\partial q_\nu}(\cdot,q)(\varphi(\cdot,q),\widetilde{\phi})|^2_{H^2(\mathbb{R}^2)}$$
$$+C|\varphi(\cdot,q)(\widetilde{\phi},\frac{\partial\varphi}{\partial q_\nu}(\cdot,q))|^2_{H^2(\mathbb{R}^2)}+C|\frac{\partial\varphi}{\partial q_\nu}(\cdot,q)(\widetilde{\phi},\widetilde{\phi})|^2_{H^2(\mathbb{R}^2)}$$
$$+C\sum_{l=1}^{2}|\partial_l(\sigma\widetilde{a}_l)\frac{\partial\varphi}{\partial q_\nu}(\cdot,q)|^2_{H^2(\mathbb{R}^2)}. \tag{5.19}$$

Because of Remark 3.1.1, since $(\psi,\widetilde{\widetilde{a}}_0,q)\in X(\overline{T})$ and all the derivatives with respect to x and q of the zero modes are exponentially decaying as x tends to infinity, the right-hand side of (5.19) is bounded for $0\le t\le\overline{T}$. Finally, in an analogous way, we see that a constant $C>0$, independent of i, exists such that

$$|\frac{\partial f_1^2}{\partial\dot{q}_\nu}(\psi,\widetilde{\widetilde{a}}_0,q,\dot{q})|^2_{H^2(\mathbb{R}^2)}\le C|\frac{\partial\varphi}{\partial q_\nu}(\cdot,q)|^2_{H^2(\mathbb{R}^2)}. \tag{5.20}$$

Since all the derivatives with respect to x of the zero modes are exponentially decaying as x tends to infinity, the right-hand side of (5.20) is bounded for $0\le t\le\overline{T}$.

Subtracting from $f_1^2(\psi,\widetilde{\widetilde{a}}_0,q,\dot{q})$ the component producing this unboundedness, namely $i\widetilde{\widetilde{a}}_0\varphi(\cdot,q)$, the mean value theorem can be applied. Then, because of the Cauchy inequality, a constant $C>0$ independent of i exists such that

$$|f_1^2(i,\cdot,s)-f_1^2(i-1,\cdot,s)|^2_{L^2(\mathbb{R}^2)}$$
$$+\sum_{j=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}\big(f_1^2(i,\cdot,s)-f_1^2(i-1,\cdot,s)\big)|^2_{L^2(\mathbb{R}^2)}$$

$$
\begin{aligned}
&+\sum_{j,k=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}\big(f_1^2(i,\cdot,s)-f_1^2(i-1,\cdot,s)\big)|^2_{L^2(\mathbb{R}^2)}\\
&=|-\sum_{\mu=1}^{2N}\frac{\partial\varphi}{\partial q_\mu}(\cdot,q^{(i)}(s))\dot q_\mu^{(i)}(s)+\frac{1}{2}\varphi(\cdot,q^{(i)}(s))(1-|\phi(\cdot,q^{(i)}(s))|^2)\\
&+i\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\varphi(\cdot,q^{(i)}(s))-i\frac{\varepsilon}{\sigma}\widetilde{\phi}^{(i)}(\cdot,s)\sum_{l=1}^{2}\partial_l\big(\sigma\widetilde{a}_l^{(i)}(\cdot,s)\big)\\
&-i\frac{2\varepsilon}{\sigma}\sum_{l=1}^{2}\sigma\widetilde{a}_l^{(i)}(\cdot,s)D_l^{(0)}\widetilde{\phi}^{(i)}(\cdot,s)-i\frac{\varepsilon}{\sigma^2}\varphi(\cdot,q^{(i)}(s))\sum_{l=1}^{2}\big(\sigma\widetilde{a}_l^{(i)}(\cdot,s)\big)^2\\
&-i\frac{\varepsilon^2}{\sigma^2}\widetilde{\phi}^{(i)}(\cdot,s)\sum_{l=1}^{2}\big(\sigma\widetilde{a}_l^{(i)}(\cdot,s)\big)^2+\frac{\varepsilon}{2}\widetilde{\phi}^{(i)}(\cdot,s)(1-|\phi(\cdot,q^{(i)}(s))|^2)\\
&-\varepsilon(1+\varepsilon)\widetilde{\phi}^{(i)}(\cdot,s)\big(\varphi(\cdot,q^{(i)}(s)),\widetilde{\phi}^{(i)}(\cdot,s)\big)\\
&-\varepsilon\varphi(\cdot,q^{(i)}(s))\big(\varphi(\cdot,q^{(i)}(s)),\widetilde{\phi}^{(i)}(\cdot,s)\big)\\
&-\varepsilon\frac{1+\varepsilon}{2}|\widetilde{\phi}^{(i)}(\cdot,s)|^2\varphi(\cdot,q^{(i)}(s))-\varepsilon^2\frac{1+\varepsilon}{2}|\widetilde{\phi}^{(i)}(\cdot,s)|^2\widetilde{\phi}^{(i)}(\cdot,s)\\
&+\frac{1}{2}\widetilde{\phi}^{(i)}(\cdot,s)(1-|\phi(\cdot,q^{(i)}(s))|^2)+i\frac{1}{\varepsilon\sigma}\varphi(\cdot,q^{(i)}(s))\sum_{l=1}^{2}\partial_l\big(\sigma\widetilde{a}_l^{(i)}(\cdot,s)\big)\\
&+i\varepsilon\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\widetilde{\phi}^{(i)}(\cdot,s)+\sum_{\mu=1}^{2N}\frac{\partial\varphi}{\partial q_\mu}(\cdot,q^{(i-1)}(s))\dot q_\mu^{(i-1)}(s)\\
&-\frac{1}{2}\varphi(\cdot,q^{(i-1)}(s))(1-|\phi(\cdot,q^{(i-1)}(s))|^2)-i\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\varphi(\cdot,q^{(i-1)}(s)\\
&+i\frac{\varepsilon}{\sigma}\widetilde{\phi}^{(i-1)}(\cdot,s)\sum_{l=1}^{2}\partial_l\big(\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)\big)+i\frac{2\varepsilon}{\sigma}\sum_{l=1}^{2}\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)D_l^{(0)}\widetilde{\phi}^{(i-1)}(\cdot,s)\\
&+i\frac{\varepsilon}{\sigma^2}\varphi(\cdot,q^{(i-1)}(s))\sum_{l=1}^{2}\big(\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)\big)^2\\
&+i\frac{\varepsilon^2}{\sigma^2}\widetilde{\phi}^{(i-1)}(\cdot,s)\sum_{l=1}^{2}\big(\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)\big)^2-\frac{\varepsilon}{2}\widetilde{\phi}^{(i-1)}(\cdot,s)(1-|\phi(\cdot,q^{(i-1)}(s))|^2)\\
&+\varepsilon(1+\varepsilon)\widetilde{\phi}^{(i-1)}(\cdot,s)\big(\varphi(\cdot,q^{(i-1)}(s)),\widetilde{\phi}^{(i-1)}(\cdot,s)\big)\\
&+\varepsilon\varphi(\cdot,q^{(i-1)}(s))\big(\varphi(\cdot,q^{(i-1)}(s)),\widetilde{\phi}^{(i-1)}(\cdot,s)\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\varepsilon\frac{1+\varepsilon}{2}|\widetilde{\phi}^{(i-1)}(\cdot,s)|^2\varphi(\cdot,q^{(i-1)}(s))+\varepsilon^2\frac{1+\varepsilon}{2}|\widetilde{\phi}^{(i-1)}(\cdot,s)|^2\widetilde{\phi}^{(i-1)}(\cdot,s)\\
&-\frac{1}{2}\widetilde{\phi}^{(i-1)}(\cdot,s)(1-|\phi(\cdot,q^{(i-1)}(s))|^2)\\
&-i\frac{1}{\varepsilon\sigma}\varphi(\cdot,q^{(i-1)}(s))\sum_{l=1}^{2}\partial_l\big(\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)\big)-i\varepsilon\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\widetilde{\phi}^{(i-1)}(\cdot,s)|^2_{L^2(\mathbb{R}^2)}\\
&+\sum_{j=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}\Big(-\sum_{\mu=1}^{2N}\frac{\partial\varphi}{\partial q_\mu}(\cdot,q^{(i)}(s))\dot{q}_\mu^{(i)}(s)\\
&+\frac{1}{2}\varphi(\cdot,q^{(i)}(s))(1-|\phi(\cdot,q^{(i)}(s))|^2)+i\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\varphi(\cdot,q^{(i)}(s))\\
&-i\frac{\varepsilon}{\sigma}\widetilde{\phi}^{(i)}(\cdot,s)\sum_{l=1}^{2}\partial_l\big(\sigma\widetilde{a}_l^{(i)}(\cdot,s)\big)-i\frac{2\varepsilon}{\sigma}\sum_{l=1}^{2}\sigma\widetilde{a}_l^{(i)}(\cdot,s)D_l^{(0)}\widetilde{\phi}^{(i)}(\cdot,s)\\
&-i\frac{\varepsilon}{\sigma^2}\varphi(\cdot,q^{(i)}(s))\sum_{l=1}^{2}\big(\sigma\widetilde{a}_l^{(i)}(\cdot,s)\big)^2-i\frac{\varepsilon^2}{\sigma^2}\widetilde{\phi}^{(i)}(\cdot,s)\sum_{l=1}^{2}\big(\sigma\widetilde{a}_l^{(i)}(\cdot,s)\big)^2\\
&+\frac{\varepsilon}{2}\widetilde{\phi}^{(i)}(\cdot,s)(1-|\phi(\cdot,q^{(i)}(s))|^2)-\varepsilon(1+\varepsilon)\widetilde{\phi}^{(i)}(\cdot,s)\big(\varphi(\cdot,q^{(i)}(s)),\widetilde{\phi}^{(i)}(\cdot,s)\big)\\
&-\varepsilon\varphi(\cdot,q^{(i)}(s))\big(\varphi(\cdot,q^{(i)}(s)),\widetilde{\phi}^{(i)}(\cdot,s)\big)-\varepsilon\frac{1+\varepsilon}{2}|\widetilde{\phi}^{(i)}(\cdot,s)|^2\varphi(\cdot,q^{(i)}(s))\\
&-\varepsilon^2\frac{1+\varepsilon}{2}|\widetilde{\phi}^{(i)}(\cdot,s)|^2\widetilde{\phi}^{(i)}(\cdot,s)+\frac{1}{2}\widetilde{\phi}^{(i)}(\cdot,s)(1-|\phi(\cdot,q^{(i)}(s))|^2)\\
&+i\frac{1}{\varepsilon\sigma}\varphi(\cdot,q^{(i)}(s))\sum_{l=1}^{2}\partial_l\big(\sigma\widetilde{a}_l^{(i)}(\cdot,s)\big)+i\varepsilon\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\widetilde{\phi}^{(i)}(\cdot,s)\\
&+\sum_{\mu=1}^{2N}\frac{\partial\varphi}{\partial q_\mu}(\cdot,q^{(i-1)}(s))\dot{q}_\mu^{(i-1)}(s)-\frac{1}{2}\varphi(\cdot,q^{(i-1)}(s))(1-|\phi(\cdot,q^{(i-1)}(s))|^2)\\
&-i\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\varphi(\cdot,q^{(i-1)}(s))+i\frac{\varepsilon}{\sigma}\widetilde{\phi}^{(i-1)}(\cdot,s)\sum_{l=1}^{2}\partial_l\big(\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)\big)\\
&+i\frac{2\varepsilon}{\sigma}\sum_{l=1}^{2}\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)D_l^{(0)}\widetilde{\phi}^{(i-1)}(\cdot,s)\\
&+i\frac{\varepsilon}{\sigma^2}\varphi(\cdot,q^{(i-1)}(s))\sum_{l=1}^{2}\big(\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)\big)^2+i\frac{\varepsilon^2}{\sigma^2}\widetilde{\phi}^{(i-1)}(\cdot,s)\sum_{l=1}^{2}\big(\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)\big)^2
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\varepsilon}{2}\widetilde{\phi}^{(i-1)}(\cdot,s)(1-|\phi(\cdot,q^{(i-1)}(s))|^2)\\
&+ \varepsilon(1+\varepsilon)\widetilde{\phi}^{(i-1)}(\cdot,s)\big(\varphi(\cdot,q^{(i-1)}(s)),\widetilde{\phi}^{(i-1)}(\cdot,s)\big)\\
&+ \varepsilon\varphi(\cdot,q^{(i-1)}(s))\big(\varphi(\cdot,q^{(i-1)}(s)),\widetilde{\phi}^{(i-1)}(\cdot,s)\big)\\
&+ \varepsilon\frac{1+\varepsilon}{2}|\widetilde{\phi}^{(i-1)}(\cdot,s)|^2\varphi(\cdot,q^{(i-1)}(s)) + \varepsilon^2\frac{1+\varepsilon}{2}|\widetilde{\phi}^{(i-1)}(\cdot,s)|^2\widetilde{\phi}^{(i-1)}(\cdot,s)\\
&-\frac{1}{2}\widetilde{\phi}^{(i-1)}(\cdot,s)(1-|\phi(\cdot,q^{(i-1)}(s))|^2)\\
&- i\frac{1}{\varepsilon\sigma}\varphi(\cdot,q^{(i-1)}(s))\sum_{l=1}^{2}\partial_l\big(\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)\big) - i\varepsilon\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\widetilde{\phi}^{(i-1)}(\cdot,s)\Big)|^2_{L^2(\mathbb{R}^2)}\\
&+ \sum_{j,k=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}\Big(-\sum_{\mu=1}^{2N}\frac{\partial\varphi}{\partial q_\mu}(\cdot,q^{(i)}(s))\dot{q}_\mu^{(i)}(s)\\
&+ \frac{1}{2}\varphi(\cdot,q^{(i)}(s))(1-|\phi(\cdot,q^{(i)}(s))|^2) + i\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\varphi(\cdot,q^{(i)}(s))\\
&- i\frac{\varepsilon}{\sigma}\widetilde{\phi}^{(i)}(\cdot,s)\sum_{l=1}^{2}\partial_l\big(\sigma\widetilde{a}_l^{(i)}(\cdot,s)\big) - i\frac{2\varepsilon}{\sigma}\sum_{l=1}^{2}\sigma\widetilde{a}_l^{(i)}(\cdot,s)D_l^{(0)}\widetilde{\phi}^{(i)}(\cdot,s)\\
&- i\frac{\varepsilon}{\sigma^2}\varphi(\cdot,q^{(i)}(s))\sum_{l=1}^{2}\big(\sigma\widetilde{a}_l^{(i)}(\cdot,s)\big)^2 - i\frac{\varepsilon^2}{\sigma^2}\widetilde{\phi}^{(i)}(\cdot,s)\sum_{l=1}^{2}\big(\sigma\widetilde{a}_l^{(i)}(\cdot,s)\big)^2\\
&+ \frac{\varepsilon}{2}\widetilde{\phi}^{(i)}(\cdot,s)(1-|\phi(\cdot,q^{(i)}(s))|^2) - \varepsilon(1+\varepsilon)\widetilde{\phi}^{(i)}(\cdot,s)\big(\varphi(\cdot,q^{(i)}(s)),\widetilde{\phi}^{(i)}(\cdot,s)\big)\\
&- \varepsilon\varphi(\cdot,q^{(i)}(s))\big(\varphi(\cdot,q^{(i)}(s)),\widetilde{\phi}^{(i)}(\cdot,s)\big) - \varepsilon\frac{1+\varepsilon}{2}|\widetilde{\phi}^{(i)}(\cdot,s)|^2\varphi(\cdot,q^{(i)}(s))\\
&- \varepsilon^2\frac{1+\varepsilon}{2}|\widetilde{\phi}^{(i)}(\cdot,s)|^2\widetilde{\phi}^{(i)}(\cdot,s) + \frac{1}{2}\widetilde{\phi}^{(i)}(\cdot,s)(1-|\phi(\cdot,q^{(i)}(s))|^2)\\
&+ i\frac{1}{\varepsilon\sigma}\varphi(\cdot,q^{(i)}(s))\sum_{l=1}^{2}\partial_l\big(\sigma\widetilde{a}_l^{(i)}(\cdot,s)\big) + i\varepsilon\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\widetilde{\phi}^{(i)}(\cdot,s)\\
&+ \sum_{\mu=1}^{2N}\frac{\partial\varphi}{\partial q_\mu}(\cdot,q^{(i-1)}(s))\dot{q}_\mu^{(i-1)}(s) - \frac{1}{2}\varphi(\cdot,q^{(i-1)}(s))(1-|\phi(\cdot,q^{(i-1)}(s))|^2)\\
&- i\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\varphi(\cdot,q^{(i-1)}(s)) + i\frac{\varepsilon}{\sigma}\widetilde{\phi}^{(i-1)}(\cdot,s)\sum_{l=1}^{2}\partial_l\big(\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ i\frac{2\varepsilon}{\sigma}\sum_{l=1}^{2}\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)D_l^{(0)}\widetilde{\phi}^{(i-1)}(\cdot,s) + i\frac{\varepsilon}{\sigma^2}\varphi(\cdot,q^{(i-1)}(s))\sum_{l=1}^{2}\big(\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)\big)^2 \\
&+ i\frac{\varepsilon^2}{\sigma^2}\widetilde{\phi}^{(i-1)}(\cdot,s)\sum_{l=1}^{2}\big(\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)\big)^2 - \frac{\varepsilon}{2}\widetilde{\phi}^{(i-1)}(\cdot,s)(1-|\phi(\cdot,q^{(i-1)}(s))|^2) \\
&+ \varepsilon(1+\varepsilon)\widetilde{\phi}^{(i-1)}(\cdot,s)\big(\varphi(\cdot,q^{(i-1)}(s)),\widetilde{\phi}^{(i-1)}(\cdot,s)\big) \\
&+ \varepsilon\varphi(\cdot,q^{(i-1)}(s))\big(\varphi(\cdot,q^{(i-1)}(s)),\widetilde{\phi}^{(i-1)}(\cdot,s)\big) \\
&+ \varepsilon\frac{1+\varepsilon}{2}|\widetilde{\phi}^{(i-1)}(\cdot,s)|^2\varphi(\cdot,q^{(i-1)}(s)) + \varepsilon^2\frac{1+\varepsilon}{2}|\widetilde{\phi}^{(i-1)}(\cdot,s)|^2\widetilde{\phi}^{(i-1)}(\cdot,s) \\
&- \frac{1}{2}\widetilde{\phi}^{(i-1)}(\cdot,s)(1-|\phi(\cdot,q^{(i-1)}(s))|^2) \\
&- i\frac{1}{\varepsilon\sigma}\varphi(\cdot,q^{(i-1)}(s))\sum_{l=1}^{2}\partial_l\big(\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)\big) - i\varepsilon\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\widetilde{\phi}^{(i-1)}(\cdot,s)\Big)|^2_{L^2(\mathbb{R}^2)} \\
&\le C|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\varphi(\cdot,q^{(i)}(s)) - \widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\varphi(\cdot,q^{(i-1)}(s))|^2_{L^2(\mathbb{R}^2)} \\
&+ C\sum_{j=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}\big(\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\varphi(\cdot,q^{(i)}(s)) - \widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\varphi(\cdot,q^{(i-1)}(s))\big)|^2_{L^2(\mathbb{R}^2)} \\
&+ C\sum_{j,k=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}\big(\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\varphi(\cdot,q^{(i)}(s)) \\
&- \widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\varphi(\cdot,q^{(i-1)}(s))\big)|^2_{L^2(\mathbb{R}^2)} + C| - \sum_{\mu=1}^{2N}\frac{\partial\varphi}{\partial q_\mu}(\cdot,q^{(i)}(s))\dot{q}_\mu^{(i)}(s) \\
&+ \frac{1}{2}\varphi(\cdot,q^{(i)}(s))(1-|\phi(\cdot,q^{(i)}(s))|^2) - i\frac{\varepsilon}{\sigma}\widetilde{\phi}^{(i)}(\cdot,s)\sum_{l=1}^{2}\partial_l\big(\sigma\widetilde{a}_l^{(i)}(\cdot,s)\big) \\
&- i\frac{2\varepsilon}{\sigma}\sum_{l=1}^{2}\sigma\widetilde{a}_l^{(i)}(\cdot,s)D_l^{(0)}\widetilde{\phi}^{(i)}(\cdot,s) - i\frac{\varepsilon}{\sigma^2}\varphi(\cdot,q^{(i)}(s))\sum_{l=1}^{2}\big(\sigma\widetilde{a}_l^{(i)}(\cdot,s)\big)^2 \\
&- i\frac{\varepsilon^2}{\sigma^2}\widetilde{\phi}^{(i)}(\cdot,s)\sum_{l=1}^{2}\big(\sigma\widetilde{a}_l^{(i)}(\cdot,s)\big)^2 - \varepsilon(1+\varepsilon)\widetilde{\phi}^{(i)}(\cdot,s)\big(\varphi(\cdot,q^{(i)}(s)),\widetilde{\phi}^{(i)}(\cdot,s)\big) \\
&+ \frac{1+\varepsilon}{2}\widetilde{\phi}^{(i)}(\cdot,s)(1-|\phi(\cdot,q^{(i)}(s))|^2) - \varepsilon\varphi(\cdot,q^{(i)}(s))\big(\varphi(\cdot,q^{(i)}(s)),\widetilde{\phi}^{(i)}(\cdot,s)\big)
\end{aligned}
$$

$$
\begin{aligned}
&- \varepsilon\frac{1+\varepsilon}{2}|\widetilde{\phi}^{(i)}(\cdot,s)|^2\varphi(\cdot,q^{(i)}(s)) - \varepsilon\frac{1+\varepsilon}{2}|\widetilde{\phi}^{(i)}(\cdot,s)|^2\widetilde{\phi}^{(i)}(\cdot,s)\\
&+ i\frac{1}{\varepsilon\sigma}\varphi(\cdot,q^{(i)}(s))\sum_{l=1}^{2}\partial_l\big(\sigma\widetilde{a}_l^{(i)}(\cdot,s)\big) + i\varepsilon\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\widetilde{\phi}^{(i)}(\cdot,s)\\
&+ \sum_{\mu=1}^{2N}\frac{\partial\varphi}{\partial q_\mu}(\cdot,q^{(i-1)}(s))\dot{q}_\mu^{(i-1)}(s) - \frac{1}{2}\varphi(\cdot,q^{(i-1)}(s))(1-|\phi(\cdot,q^{(i-1)}(s))|^2)\\
&+ i\frac{\varepsilon}{\sigma}\widetilde{\phi}^{(i-1)}(\cdot,s)\sum_{l=1}^{2}\partial_l\big(\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)\big) + i\frac{2\varepsilon}{\sigma}\sum_{l=1}^{2}\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)D_l^{(0)}\widetilde{\phi}^{(i-1)}(\cdot,s)\\
&+ i\frac{\varepsilon}{\sigma^2}\varphi(\cdot,q^{(i-1)}(s))\sum_{l=1}^{2}\big(\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)\big)^2 + i\frac{\varepsilon^2}{\sigma^2}\widetilde{\phi}^{(i-1)}(\cdot,s)\sum_{l=1}^{2}\big(\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)\big)^2\\
&+ \varepsilon(1+\varepsilon)\widetilde{\phi}^{(i-1)}(\cdot,s)\big(\varphi(\cdot,q^{(i-1)}(s)),\widetilde{\phi}^{(i-1)}(\cdot,s)\big)\\
&- \frac{1+\varepsilon}{2}\widetilde{\phi}^{(i-1)}(\cdot,s)(1-|\phi(\cdot,q^{(i-1)}(s))|^2)\\
&+ \varepsilon\varphi(\cdot,q^{(i-1)}(s))\big(\varphi(\cdot,q^{(i-1)}(s)),\widetilde{\phi}^{(i-1)}(\cdot,s)\big)\\
&+ \varepsilon\frac{1+\varepsilon}{2}|\widetilde{\phi}^{(i-1)}(\cdot,s)|^2\varphi(\cdot,q^{(i-1)}(s)) + \varepsilon^2\frac{1+\varepsilon}{2}|\widetilde{\phi}^{(i-1)}(\cdot,s)|^2\widetilde{\phi}^{(i-1)}(\cdot,s)\\
&- i\frac{1}{\varepsilon\sigma}\varphi(\cdot,q^{(i-1)}(s))\sum_{l=1}^{2}\partial_l\big(\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)\big) - i\varepsilon\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\widetilde{\phi}^{(i-1)}(\cdot,s)|^2_{L^2(\mathbb{R}^2)}\\
&+ C\sum_{j=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}\Big(-\sum_{\mu=1}^{2N}\frac{\partial\varphi}{\partial q_\mu}(\cdot,q^{(i)}(s))\dot{q}_\mu^{(i)}(s)\\
&+ \frac{1}{2}\varphi(\cdot,q^{(i)}(s))(1-|\phi(\cdot,q^{(i)}(s))|^2) - i\frac{\varepsilon}{\sigma}\widetilde{\phi}^{(i)}(\cdot,s)\sum_{l=1}^{2}\partial_l\big(\sigma\widetilde{a}_l^{(i)}(\cdot,s)\big)\\
&- i\frac{2\varepsilon}{\sigma}\sum_{l=1}^{2}\sigma\widetilde{a}_l^{(i)}(\cdot,s)D_l^{(0)}\widetilde{\phi}^{(i)}(\cdot,s) - i\frac{\varepsilon}{\sigma^2}\varphi(\cdot,q^{(i)}(s))\sum_{l=1}^{2}\big(\sigma\widetilde{a}_l^{(i)}(\cdot,s)\big)^2\\
&- i\frac{\varepsilon^2}{\sigma^2}\widetilde{\phi}^{(i)}(\cdot,s)\sum_{l=1}^{2}\big(\sigma\widetilde{a}_l^{(i)}(\cdot,s)\big)^2 - \varepsilon(1+\varepsilon)\widetilde{\phi}^{(i)}(\cdot,s)\big(\varphi(\cdot,q^{(i)}(s)),\widetilde{\phi}^{(i)}(\cdot,s)\big)\\
&+ \frac{1+\varepsilon}{2}\widetilde{\phi}^{(i)}(\cdot,s)(1-|\phi(\cdot,q^{(i)}(s))|^2) - \varepsilon\varphi(\cdot,q^{(i)}(s))\big(\varphi(\cdot,q^{(i)}(s)),\widetilde{\phi}^{(i)}(\cdot,s)\big)\\
&- \varepsilon\frac{1+\varepsilon}{2}|\widetilde{\phi}^{(i)}(\cdot,s)|^2\varphi(\cdot,q^{(i)}(s)) - \varepsilon^2\frac{1+\varepsilon}{2}|\widetilde{\phi}^{(i)}(\cdot,s)|^2\widetilde{\phi}^{(i)}(\cdot,s)
\end{aligned}
$$

$$
\begin{aligned}
&+ i\frac{1}{\varepsilon\sigma}\varphi(\cdot, q^{(i)}(s)) \sum_{l=1}^{2} \partial_l\big(\sigma\widetilde{a}_l^{(i)}(\cdot, s)\big) + i\varepsilon\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot, s)\widetilde{\phi}^{(i)}(\cdot, s) \\
&+ \sum_{\mu=1}^{2N} \frac{\partial\varphi}{\partial q_\mu}(\cdot, q^{(i-1)}(s))\dot{q}_\mu^{(i-1)}(s) - \frac{1}{2}\varphi(\cdot, q^{(i-1)}(s))(1 - |\phi(\cdot, q^{(i-1)}(s))|^2) \\
&+ i\frac{\varepsilon}{\sigma}\widetilde{\phi}^{(i-1)}(\cdot, s) \sum_{l=1}^{2} \partial_l\big(\sigma\widetilde{a}_l^{(i-1)}(\cdot, s)\big) + i\frac{2\varepsilon}{\sigma} \sum_{l=1}^{2} \sigma\widetilde{a}_l^{(i-1)}(\cdot, s) D_l^{(0)}\widetilde{\phi}^{(i-1)}(\cdot, s) \\
&+ i\frac{\varepsilon}{\sigma^2}\varphi(\cdot, q^{(i-1)}(s)) \sum_{l=1}^{2} \big(\sigma\widetilde{a}_l^{(i-1)}(\cdot, s)\big)^2 + i\frac{\varepsilon^2}{\sigma^2}\widetilde{\phi}^{(i-1)}(\cdot, s) \sum_{l=1}^{2} \big(\sigma\widetilde{a}_l^{(i-1)}(\cdot, s)\big)^2 \\
&+ \varepsilon(1+\varepsilon)\widetilde{\phi}^{(i-1)}(\cdot, s)\big(\varphi(\cdot, q^{(i-1)}(s)), \widetilde{\phi}^{(i-1)}(\cdot, s)\big) \\
&- \frac{1+\varepsilon}{2}\widetilde{\phi}^{(i-1)}(\cdot, s)(1 - |\phi(\cdot, q^{(i-1)}(s))|^2) \\
&+ \varepsilon\varphi(\cdot, q^{(i-1)}(s))\big(\varphi(\cdot, q^{(i-1)}(s)), \widetilde{\phi}^{(i-1)}(\cdot, s)\big) \\
&+ \varepsilon\frac{1+\varepsilon}{2}|\widetilde{\phi}^{(i-1)}(\cdot, s)|^2\varphi(\cdot, q^{(i-1)}(s)) + \varepsilon^2\frac{1+\varepsilon}{2}|\widetilde{\phi}^{(i-1)}(\cdot, s)|^2\widetilde{\phi}^{(i-1)}(\cdot, s) \\
&- i\frac{1}{\varepsilon\sigma}\varphi(\cdot, q^{(i-1)}(s)) \sum_{l=1}^{2} \partial_l\big(\sigma\widetilde{a}_l^{(i-1)}(\cdot, s)\big) - i\varepsilon\widetilde{\widetilde{a}}_0^{(i)}(\cdot, s)\widetilde{\phi}^{(i-1)}(\cdot, s)\Big)|^2_{L^2(\mathbb{R}^2)} \\
&+ C \sum_{j,k=1}^{2} |D_j^{(\alpha(\cdot, q(0)))} D_k^{(\alpha(\cdot, q(0)))}\Big(- \sum_{\mu=1}^{2N} \frac{\partial\varphi}{\partial q_\mu}(\cdot, q^{(i)}(s))\dot{q}_\mu^{(i)}(s) \\
&+ \frac{1}{2}\varphi(\cdot, q^{(i)}(s))(1 - |\phi(\cdot, q^{(i)}(s))|^2) - i\frac{\varepsilon}{\sigma}\widetilde{\phi}^{(i)}(\cdot, s) \sum_{l=1}^{2} \partial_l\big(\sigma\widetilde{a}_l^{(i)}(\cdot, s)\big) \\
&- i\frac{2\varepsilon}{\sigma} \sum_{l=1}^{2} \sigma\widetilde{a}_l^{(i)}(\cdot, s) D_l^{(0)}\widetilde{\phi}^{(i)}(\cdot, s) - i\frac{\varepsilon}{\sigma^2}\varphi(\cdot, q^{(i)}(s)) \sum_{l=1}^{2} \big(\sigma\widetilde{a}_l^{(i)}(\cdot, s)\big)^2 \\
&- i\frac{\varepsilon^2}{\sigma^2}\widetilde{\phi}^{(i)}(\cdot, s) \sum_{l=1}^{2} \big(\sigma\widetilde{a}_l^{(i)}(\cdot, s)\big)^2 - \varepsilon(1+\varepsilon)\widetilde{\phi}^{(i)}(\cdot, s)\big(\varphi(\cdot, q^{(i)}(s)), \widetilde{\phi}^{(i)}(\cdot, s)\big) \\
&+ \frac{1+\varepsilon}{2} D_j^{(\alpha(\cdot, q(0)))} D_k^{(\alpha(\cdot, q(0)))}\big(\widetilde{\phi}^{(i)}(\cdot, s)(1 - |\phi(\cdot, q^{(i)}(s))|^2\big) \\
&- \varepsilon\varphi(\cdot, q^{(i)}(s))\big(\varphi(\cdot, q^{(i)}(s)), \widetilde{\phi}^{(i)}(\cdot, s)\big) - \varepsilon\frac{1+\varepsilon}{2}|\widetilde{\phi}^{(i)}(\cdot, s)|^2\varphi(\cdot, q^{(i)}(s))
\end{aligned}
$$

$$
\begin{aligned}
&- \varepsilon^2 \frac{1+\varepsilon}{2} |\widetilde{\phi}^{(i)}(\cdot,s)|^2 \widetilde{\phi}^{(i)}(\cdot,s) + i\frac{1}{\varepsilon\sigma}\varphi(\cdot,q^{(i)}(s)) \sum_{l=1}^{2} \partial_l \big(\sigma \widetilde{a}_l^{(i)}(\cdot,s)\big) \\
&+ i\varepsilon \widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\widetilde{\phi}^{(i)}(\cdot,s) + \sum_{\mu=1}^{2N} \frac{\partial\varphi}{\partial q_\mu}(\cdot,q^{(i-1)}(s))\dot{q}_\mu^{(i-1)}(s) \\
&- \frac{1}{2}\varphi(\cdot,q^{(i-1)}(s))(1-|\phi(\cdot,q^{(i-1)}(s))|^2) + i\frac{\varepsilon}{\sigma}\widetilde{\phi}^{(i-1)}(\cdot,s)\sum_{l=1}^{2}\partial_l\big(\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)\big) \\
&+ i\frac{2\varepsilon}{\sigma}\sum_{l=1}^{2}\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)D_l^{(0)}\widetilde{\phi}^{(i-1)}(\cdot,s) + i\frac{\varepsilon}{\sigma^2}\varphi(\cdot,q^{(i-1)}(s))\sum_{l=1}^{2}\big(\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)\big)^2 \\
&+ i\frac{\varepsilon^2}{\sigma^2}\widetilde{\phi}^{(i-1)}(\cdot,s)\sum_{l=1}^{2}\big(\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)\big)^2 \\
&+ \varepsilon(1+\varepsilon)\widetilde{\phi}^{(i-1)}(\cdot,s)\big(\varphi(\cdot,q^{(i-1)}(s)),\widetilde{\phi}^{(i-1)}(\cdot,s)\big) \\
&- \frac{1+\varepsilon}{2}\widetilde{\phi}^{(i-1)}(\cdot,s)(1-|\phi(\cdot,q^{(i-1)}(s))|^2) \\
&+ \varepsilon\varphi(\cdot,q^{(i-1)}(s))\big(\varphi(\cdot,q^{(i-1)}(s)),\widetilde{\phi}^{(i-1)}(\cdot,s)\big) \\
&+ \varepsilon\frac{1+\varepsilon}{2}|\widetilde{\phi}^{(i-1)}(\cdot,s)|^2\varphi(\cdot,q^{(i-1)}(s)) + \varepsilon^2\frac{1+\varepsilon}{2}|\widetilde{\phi}^{(i-1)}(\cdot,s)|^2\widetilde{\phi}^{(i-1)}(\cdot,s) \\
&- i\frac{1}{\varepsilon\sigma}\varphi(\cdot,q^{(i-1)}(s))\sum_{l=1}^{2}\partial_l\big(\sigma\widetilde{a}_l^{(i-1)}(\cdot,s)\big) - i\varepsilon\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\widetilde{\phi}^{(i-1)}(\cdot,s)\Big)|^2_{L^2(\mathbb{R}^2)}.
\end{aligned}
\tag{5.21}
$$

The first three norms in the right-hand side of (5.21) are now examined. Because of the Cauchy-Schwartz inequality a constant $C > 0$ exists, independent of i, such that

$$
\begin{aligned}
&|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\varphi(\cdot,q^{(i)}(s)) - \widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\varphi(\cdot,q^{(i-1)}(s))|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \sum_{j=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}\big(\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\varphi(\cdot,q^{(i)}(s)) - \widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\varphi(\cdot,q^{(i-1)}(s))\big)|^2_{L^2(\mathbb{R}^2)}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j,k=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}\big(\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\varphi(\cdot,q^{(i)}(s))\\
&-\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\varphi(\cdot,q^{(i-1)}(s))\big)|_{L^2(\mathbb{R}^2)}^2\\
&\leq C|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\big(\varphi(\cdot,q^{(i)}(s))-\varphi(\cdot,q^{(i-1)}(s))\big)|_{L^2(\mathbb{R}^2)}^2\\
&+C|\varphi(\cdot,q^{(i-1)}(s))\big(\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)-\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\big)|_{L^2(\mathbb{R}^2)}^2\\
&+C\sum_{j=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}\Big(\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\big(\varphi(\cdot,q^{(i)}(s))-\varphi(\cdot,q^{(i-1)}(s))\big)\Big)|_{L^2(\mathbb{R}^2)}^2\\
&+C\sum_{j=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}\Big(\varphi(\cdot,q^{(i-1)}(s))\big(\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)-\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\big)\Big)|_{L^2(\mathbb{R}^2)}^2\\
&+C\sum_{j,k=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}\Big(\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\big(\varphi(\cdot,q^{(i)}(s))\\
&-\varphi(\cdot,q^{(i-1)}(s))\big)\Big)|_{L^2(\mathbb{R}^2)}^2\\
&+C\sum_{j,k=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}\Big(\varphi(\cdot,q^{(i-1)}(s))\big(\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)-\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\big)\Big)|_{L^2(\mathbb{R}^2)}^2.
\end{aligned}
\tag{5.22}
$$

Using Remark 3.1.1, Lemma 3.5.1, the fact that $(\psi^{(i)},\widetilde{\widetilde{a}}_0^{(i+1)},q^{(i)})\in X(\overline{T})$ and that all the derivatives with respect to x of the zero modes are exponentially decaying as x tends to infinity, one obtains from (5.22)

$$
\begin{aligned}
&|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\varphi(\cdot,q^{(i)}(s))-\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\varphi(\cdot,q^{(i-1)}(s))|_{L^2(\mathbb{R}^2)}^2\\
&\quad+\sum_{j=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}\big(\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\varphi(\cdot,q^{(i)}(s))-\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\varphi(\cdot,q^{(i-1)}(s))\big)|_{L^2(\mathbb{R}^2)}^2\\
&\quad+\sum_{j,k=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}\big(\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)\varphi(\cdot,q^{(i)}(s))
\end{aligned}
$$

$$
\begin{aligned}
&- \tilde{\tilde{a}}_0^{(i)}(\cdot, s)\varphi(\cdot, q^{(i-1)}(s))\big)|^2_{L^2(\mathbb{R}^2)} \\
&\leq C(\Gamma)\big\|\tilde{\tilde{a}}_0^{(i+1)}(\cdot, s) - \tilde{\tilde{a}}_0^{(i)}(\cdot, s)\big\| + C(\Gamma)\max_{0\leq\tau\leq s}|q^{(i)}(\tau) - q^{(i-1)}(\tau)|
\end{aligned}
\tag{5.23}
$$

for some constant $C(\Gamma) > 0$, independent of i, for $0 \leq s \leq \overline{T}$.

The norm in the remainder of the right-hand side of (5.21) satisfies the conditions of the mean value theorem. Therefore, it is bounded by

$$
\begin{aligned}
&C(\Gamma)\big\|\psi^{(i)}(\cdot, s) - \psi^{(i-1)}(\cdot, s)\big\| + C(\Gamma)\big\|\tilde{\tilde{a}}_0^{(i+1)}(\cdot, s) - \tilde{\tilde{a}}_0^{(i)}(\cdot, s)\big\| \\
&\quad + C(\Gamma)\max_{0\leq\tau\leq s}|q^{(i)}(\tau) - q^{(i-1)}(\tau)| + C(\Gamma)|\dot{q}^{(i)}(s) - \dot{q}^{(i-1)}(s)|
\end{aligned}
\tag{5.24}
$$

for some constant $C(\Gamma) > 0$ independent of i, for $0 \leq s \leq \overline{T}$. Then, using (5.23) and (5.24) to estimate the right-hand side of (5.21), yields

$$
\begin{aligned}
&|f_1^2(i, \cdot, s) - f_1^2(i-1, \cdot, s)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \sum_{j=1}^{2} |D_j^{(\alpha(\cdot,q(0)))}\big(f_1^2(i, \cdot, s) - f_1^2(i-1, \cdot, s)\big)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \sum_{j,k=1}^{2} |D_j^{(\alpha(\cdot,q(0)))} D_k^{(\alpha(\cdot,q(0)))}\big(f_1^2(i, \cdot, s) - f_1^2(i-1, \cdot, s)\big)|^2_{L^2(\mathbb{R}^2)} \\
&\quad \leq C(\Gamma)\big\|\psi^{(i)}(\cdot, s) - \psi^{(i-1)}(\cdot, s)\big\| + C(\Gamma)\big\|\tilde{\tilde{a}}_0^{(i+1)}(\cdot, s) - \tilde{\tilde{a}}_0^{(i)}(\cdot, s)\big\| \\
&\quad + C(\Gamma)\max_{0\leq\tau\leq s}|q^{(i)}(\tau) - q^{(i-1)}(\tau)| + C(\Gamma)|\dot{q}^{(i)}(s) - \dot{q}^{(i-1)}(s)|
\end{aligned}
\tag{5.25}
$$

for $0 \leq s \leq \overline{T}$.

Now, using (5.25) in (5.10), one obtains

$$
\begin{aligned}
&|\widetilde{\phi}^{(i+1)}(\cdot,t)-\widetilde{\phi}^{(i)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} \\
&\quad+\sum_{j=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}\big(\widetilde{\phi}^{(i+1)}(\cdot,t)-\widetilde{\phi}^{(i)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} \\
&\quad+\sum_{j,k=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}\big(\widetilde{\phi}^{(i+1)}(\cdot,t)-\widetilde{\phi}^{(i)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} \\
&\quad+\sum_{j,k,l=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}D_l^{(\alpha(\cdot,q(0)))}\big(\widetilde{\phi}^{(i+1)}(\cdot,t)-\widetilde{\phi}^{(i)}(\cdot,t)\big)|^2_{L^2(\mathbb{R}^2)} \\
&\quad\le K|\widetilde{\phi}^{(i+1)}(\cdot,0)-\widetilde{\phi}^{(i)}(\cdot,0)|^2_{L^2(\mathbb{R}^2)} \\
&\quad+K\sum_{j=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}\big(\widetilde{\phi}^{(i+1)}(\cdot,0)-\widetilde{\phi}^{(i)}(\cdot,0)\big)|^2_{L^2(\mathbb{R}^2)} \\
&\quad+K\sum_{j,k=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}\big(\widetilde{\phi}^{(i+1)}(\cdot,0)-\widetilde{\phi}^{(i)}(\cdot,0)\big)|^2_{L^2(\mathbb{R}^2)} \\
&\quad+K\sum_{j,k,l=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}D_k^{(\alpha(\cdot,q(0)))}D_l^{(\alpha(\cdot,q(0)))}\big(\widetilde{\phi}^{(i+1)}(\cdot,0)-\widetilde{\phi}^{(i)}(\cdot,0)\big)|^2_{L^2(\mathbb{R}^2)} \\
&\quad+C(\Gamma)\int_0^t\Big(\big\|\psi^{(i)}(\cdot,s)-\psi^{(i-1)}(\cdot,s)\big\|+\big\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)-\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\big\| \\
&\quad+\max_{0\le\tau\le s}|q^{(i)}(\tau)-q^{(i-1)}(\tau)|+|\dot{q}^{(i)}(s)-\dot{q}^{(i-1)}(s)|\Big)ds \qquad (5.26)
\end{aligned}
$$

with $K, C(\Gamma) > 0$ constants independent of i, for $0 \le t \le \overline{T}$.

Adding (5.16) to (5.26) term by term, we finally obtain

$$
\begin{aligned}
|\psi^{(i+1)}(\cdot,t)-\psi^{(i)}(\cdot,t)|_{3,\alpha(\cdot,q(0))} &\le K|\psi^{(i+1)}(\cdot,0)-\psi^{(i)}(\cdot,0)|_{3,\alpha(\cdot,q(0))} \\
&+C(\Gamma)\int_0^t\Big(\big\|\psi^{(i)}(\cdot,s)-\psi^{(i-1)}(\cdot,s)\big\|+\big\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s)-\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\big\| \\
&+\max_{0\le\tau\le s}|q^{(i)}(\tau)-q^{(i-1)}(\tau)|+|\dot{q}^{(i)}(s)-\dot{q}^{(i-1)}(s)|\Big)ds \qquad (5.27)
\end{aligned}
$$

with constants $K, C(\Gamma) > 0$ independent of i, for $0 \le t \le \overline{T}$.

5.4 An Estimate for $\dot{q}^{(i)} - \dot{q}^{(i-1)}$ and $q^{(i+1)} - q^{(i)}$

Here an estimate for $|\dot{q}^{(i)} - \dot{q}^{(i-1)}|$ in terms of $\|\psi^{(i-1)} - \psi^{(i-2)}\|$, $\|\widetilde{\widetilde{a}}_0^{(i)} - \widetilde{\widetilde{a}}_0^{(i-1)}\|$ and $|q^{(i-1)} - q^{(i-2)}|$ is presented. Then this estimate is substituted into (5.27) in order to make it smaller than a linear combination of $\|\psi^{(i)} - \psi^{(i-1)}\|$, $\|\psi^{(i-1)} - \psi^{(i-2)}\|$, $\|\widetilde{\widetilde{a}}_0^{(i+1)} - \widetilde{\widetilde{a}}_0^{(i)}\|$, $\|\widetilde{\widetilde{a}}_0^{(i)} - \widetilde{\widetilde{a}}_0^{(i-1)}\|$, $|q^{(i)} - q^{(i-1)}|$ and $|q^{(i-1)} - q^{(i-2)}|$. In the same way also an estimate for $|q^{(i+1)} - q^{(i)}|$ is derived in terms of $\|\psi^{(i)} - \psi^{(i-1)}\|$, $\|\widetilde{\widetilde{a}}_0^{(i+1)} - \widetilde{\widetilde{a}}_0^{(i)}\|$ and $|q^{(i)} - q^{(i-1)}|$.

From (3.15) it is known that

$$\sum_{\nu=1}^{2N} a_{\mu\nu}(i-1,s)\dot{q}_\nu^{(i)}(s)$$
$$= \frac{1}{2}\int_{\mathbb{R}^2} \big(\varphi(x, q^{(i-1)}(s))(1 - |\phi(x, q^{(i-1)}(s))|^2), n_\mu^2(x, q^{(i-1)}(s))\big) d^2x$$
$$+ \varepsilon\langle j(\widetilde{a}^{(i-1)}(\cdot, s), \widetilde{\phi}^{(i-1)}(\cdot, s), \widetilde{\widetilde{a}}_0^{(i)}(\cdot, s), q^{(i-1)}(s)), n_\mu(\cdot, q^{(i-1)}(s))\rangle$$

and

$$\sum_{\nu=1}^{2N} a_{\mu\nu}(i-2,s)\dot{q}_\nu^{(i-1)}(s)$$
$$= \frac{1}{2}\int_{\mathbb{R}^2} \big(\varphi(x, q^{(i-2)}(s))(1 - |\phi(x, q^{(i-2)}(s))|^2), n_\mu^2(x, q^{(i-2)}(s))\big) d^2x$$
$$+ \varepsilon\langle j(\widetilde{a}^{(i-2)}(\cdot, s), \widetilde{\phi}^{(i-2)}(\cdot, s), \widetilde{\widetilde{a}}_0^{(i-1)}(\cdot, s), q^{(i-2)}(t)), n_\mu(\cdot, q^{(i-2)}(s))\rangle.$$

Subtracting from each other, inverting the matrix $A_{(i-1)}(s)$ and $A_{(i-2)}(s)$ and taking the norm, one obtains

$$|\dot{q}^{(i)}(s) - \dot{q}^{(i-1)}(s)| = \sum_{\tau=1}^{2N} |\dot{q}_\tau^{(i)}(s) - \dot{q}_\tau^{(i-1)}(s)|$$

$$= \sum_{\tau=1}^{2N} \Big| \sum_{\mu,\nu=1}^{2N} a^{-1}_{\tau\mu}(i-1,s)\big(a_{\mu\nu}(i-1,s) - a_{\mu\nu}(i-2,s)\big)\dot{q}^{(i-1)}_{\nu}(s)$$

$$+ \sum_{\mu=1}^{2N} a^{-1}_{\tau\mu}(i-1,s)$$

$$\Big(\frac{1}{2}\int_{\mathbb{R}^2} \big((1-|\phi(x,q^{(i-1)}(s))|^2)\varphi(x,q^{(i-1)}(s)), n^2_{\mu}(x,q^{(i-1)}(s))\big)d^2x$$

$$+ \varepsilon\langle j(\widetilde{a}^{(i-1)}(\cdot,s), \widetilde{\phi}^{(i-1)}(\cdot,s), \widetilde{\widetilde{a}}{}^{(i)}_0(\cdot,s), q^{(i-1)}(s)), n_{\mu}(\cdot,q^{(i-1)}(s))\rangle$$

$$- \frac{1}{2}\int_{\mathbb{R}^2} \big((1-|\phi(x,q^{(i-2)}(s))|^2)\varphi(x,q^{(i-2)}(s)), n^2_{\mu}(x,q^{(i-2)}(s))\big)d^2x$$

$$- \varepsilon\langle j(\widetilde{a}^{(i-2)}(\cdot,s), \widetilde{\phi}^{(i-2)}(\cdot,s), \widetilde{\widetilde{a}}{}^{(i-1)}_0(\cdot,s), q^{(i-2)}(s)), n_{\mu}(\cdot,q^{(i-2)}(s))\rangle\Big)\Big|$$

$$\leq \sum_{\tau,\mu,\nu=1}^{2N} |a^{-1}_{\tau\mu}(i-1,s)|\big|\big(a_{\mu\nu}(i-1,s) - a_{\mu\nu}(i-2,s)\big)\big||\dot{q}^{(i-1)}_{\nu}(s)|$$

$$+ \sum_{\tau,\mu=1}^{2N} |a^{-1}_{\tau\mu}(i-1,s)|$$

$$\Big|\frac{1}{2}\int_{\mathbb{R}^2} \big((1-|\phi(x,q^{(i-1)}(s))|^2)\varphi(x,q^{(i-1)}(s)), n^2_{\mu}(x,q^{(i-1)}(s))\big)d^2x$$

$$+ \varepsilon\langle j(\widetilde{a}^{(i-1)}(\cdot,s), \widetilde{\phi}^{(i-1)}(\cdot,s), \widetilde{\widetilde{a}}{}^{(i)}_0(\cdot,s), q^{(i-1)}(s)), n_{\mu}(\cdot,q^{(i-1)}(s))\rangle$$

$$- \frac{1}{2}\int_{\mathbb{R}^2} \big((1-|\phi(x,q^{(i-2)}(s))|^2)\varphi(x,q^{(i-2)}(s)), n^2_{\mu}(x,q^{(i-2)}(s))\big)d^2x$$

$$- \varepsilon\langle j(\widetilde{a}^{(i-2)}(\cdot,s), \widetilde{\phi}^{(i-2)}(\cdot,s), \widetilde{\widetilde{a}}{}^{(i-1)}_0(\cdot,s), q^{(i-2)}(s)), n_{\mu}(\cdot,q^{(i-2)}(s))\rangle\Big|. \tag{5.28}$$

Using the fact that for every $r, s \in \{1,\ldots,2N\}$ the entry $a^{-1}_{rs}(i-1,s)$ is bounded, as was shown in the proof of Lemma 3.6.3, and using the definition of the norm in $\mathbb{R}^{2N}$, from (5.28)

$$|\dot{q}^{(i)}(s) - \dot{q}^{(i-1)}(s)| \leq C(\Gamma)\sum_{\mu,\nu=1}^{2N} |a_{\mu\nu}(i-1,s) - a_{\mu\nu}(i-2,s)||\dot{q}^{(i-1)}_{\nu}(s)|$$

$$+C(\Gamma)\sum_{\mu=1}^{2N}|\frac{1}{2}\int_{\mathbb{R}^2}\big((1-|\phi(x,q^{(i-1)}(s))|^2)\varphi(x,q^{(i-1)}(s)),n_\mu^2(x,q^{(i-1)}(s))\big)d^2x$$
$$+\varepsilon\langle j(\widetilde{a}^{(i-1)}(\cdot,s),\widetilde{\phi}^{(i-1)}(\cdot,s),\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s),q^{(i-1)}(s)),n_\mu(\cdot,q^{(i-1)}(s))\rangle$$
$$-\frac{1}{2}\int_{\mathbb{R}^2}\big((1-|\phi(x,q^{(i-2)}(s))|^2)\varphi(x,q^{(i-2)}(s)),n_\mu^2(x,q^{(i-2)}(s))\big)d^2x$$
$$-\varepsilon\langle j(\widetilde{a}^{(i-2)}(\cdot,s),\widetilde{\phi}^{(i-2)}(\cdot,s),\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,s),q^{(i-2)}(s)),n_\mu(\cdot,q^{(i-2)}(s))\rangle| \tag{5.29}$$

follows for some constant $C(\Gamma)>0$ independent of i, for $0\leq s\leq\overline{T}$.

Now every single term on the right-hand side of (5.29) will be estimated. By applying (4.11) to the first sum on the right-hand side of (5.29) and using the definition of $a_{\mu\nu}$, one obtains

$$\sum_{\mu,\nu=1}^{2N}|a_{\mu\nu}(i-1,s)-a_{\mu\nu}(i-2,s)||\dot{q}_\nu^{(i-1)}(s)|$$
$$\leq C(\Gamma)\sum_{\mu,\nu=1}^{2N}|\langle\begin{pmatrix}\sigma n_\nu^1(\cdot,q^{(i-1)}(s))\\ n_\nu^2(\cdot,q^{(i-1)}(s))\end{pmatrix},\begin{pmatrix}n_\mu^1(\cdot,q^{(i-1)}(s))\\ n_\mu^2(\cdot,q^{(i-1)}(s))\end{pmatrix}\rangle$$
$$-\varepsilon\langle\begin{pmatrix}\sigma\widetilde{a}^{(i-1)}(\cdot,s)\\ \widetilde{\phi}^{(i-1)}(\cdot,s)\end{pmatrix},\begin{pmatrix}\frac{\partial n_\mu^1}{\partial q_\nu}(\cdot,q^{(i-1)}(s))\\ \frac{\partial n_\mu^2}{\partial q_\nu}(\cdot,q^{(i-1)}(s))\end{pmatrix}\rangle$$
$$-\langle\begin{pmatrix}\sigma n_\nu^1(\cdot,q^{(i-2)}(s))\\ n_\nu^2(\cdot,q^{(i-2)}(s))\end{pmatrix},\begin{pmatrix}n_\mu^1(\cdot,q^{(i-2)}(s))\\ n_\mu^2(\cdot,q^{(i-2)}(s))\end{pmatrix}\rangle$$
$$+\varepsilon\langle\begin{pmatrix}\sigma\widetilde{a}^{(i-2)}(\cdot,s)\\ \widetilde{\phi}^{(i-2)}(\cdot,s)\end{pmatrix},\begin{pmatrix}\frac{\partial n_\mu^1}{\partial q_\nu}(\cdot,q^{(i-2)}(s))\\ \frac{\partial n_\mu^2}{\partial q_\nu}(\cdot,q^{(i-2)}(s))\end{pmatrix}\rangle|$$

$$
\begin{aligned}
&= C(\Gamma) \\
&\sum_{\mu,\nu=1}^{2N} |\langle \begin{pmatrix} \sigma\big(n_\nu^1(\cdot, q^{(i-1)}(s)) - n_\nu^2(\cdot, q^{(i-2)}(s))\big) \\ n_\nu^2(\cdot, q^{(i-1)}(s)) - n_\nu^2(\cdot, q^{(i-2)}(s)) \end{pmatrix}, \begin{pmatrix} n_\mu^1(\cdot, q^{(i-1)}(s)) \\ n_\mu^2(\cdot, q^{(i-1)}(s)) \end{pmatrix} \rangle \\
&+ \langle \begin{pmatrix} n_\mu^1(\cdot, q^{(i-1)}(s)) - n_\mu^1(\cdot, q^{(i-2)}(s)) \\ n_\mu^2(\cdot, q^{(i-1)}(s)) - n_\mu^2(\cdot, q^{(i-2)}(s)) \end{pmatrix}, \begin{pmatrix} \sigma n_\nu^1(\cdot, q^{(i-2)}(s)) \\ n_\nu^2(\cdot, q^{(i-2)}(s)) \end{pmatrix} \rangle \\
&- \varepsilon \langle \begin{pmatrix} \sigma \widetilde{a}^{(i-1)}(\cdot, s) - \sigma \widetilde{a}^{(i-2)}(\cdot, s) \\ \widetilde{\phi}^{(i-1)}(\cdot, s) - \widetilde{\phi}^{(i-2)}(\cdot, s) \end{pmatrix}, \begin{pmatrix} \frac{\partial n_\mu^1}{\partial q_\nu}(\cdot, q^{(i-1)}(s)) \\ \frac{\partial n_\mu^2}{\partial q_\nu}(\cdot, q^{(i-1)}(s)) \end{pmatrix} \rangle \\
&- \varepsilon \langle \begin{pmatrix} \sigma \widetilde{a}^{(i-2)}(\cdot, s) \\ \widetilde{\phi}^{(i-2)}(\cdot, s) \end{pmatrix}, \begin{pmatrix} \frac{\partial n_\mu^1}{\partial q_\nu}(\cdot, q^{(i-1)}(s)) - \frac{\partial n_\mu^1}{\partial q_\nu}(\cdot, q^{(i-2)}(s)) \\ \frac{\partial n_\mu^2}{\partial q_\nu}(\cdot, q^{(i-1)}(s)) - \frac{\partial n_\mu^2}{\partial q_\nu}(\cdot, q^{(i-2)}(s)) \end{pmatrix} \rangle |
\end{aligned}
\tag{5.30}
$$

for some constant $C(\Gamma) > 0$ independent of i, for $0 \leq s \leq \overline{T}$.

Now for the first term on the right-hand side of (5.30) the following inequality holds:

$$
\begin{aligned}
&|\langle \begin{pmatrix} \sigma\big(n_\nu^1(\cdot, q^{(i-1)}(s)) - n_\nu^1(\cdot, q^{(i-2)}(s))\big) \\ n_\nu^2(\cdot, q^{(i-1)}(s)) - n_\nu^2(\cdot, q^{(i-2)}(s)) \end{pmatrix}, \begin{pmatrix} n_\mu^1(\cdot, q^{(i-1)}(s)) \\ n_\mu^2(\cdot, q^{(i-1)}(s)) \end{pmatrix} \rangle | \\
&\leq \sigma \sum_{l=1}^{2} |\frac{\partial \alpha_l}{\partial q_\nu}(\cdot, q^{(i-1)}(s)) - \frac{\partial \alpha_l}{\partial q_\nu}(\cdot, q^{(i-2)}(s))|_{L^2(\mathbb{R}^2)} |\frac{\partial \alpha_l}{\partial q_\mu}(\cdot, q^{(i-1)}(s))|_{L^2(\mathbb{R}^2)} \\
&+ |\frac{\partial \varphi}{\partial q_\nu}(\cdot, q^{(i-1)}(s)) - \frac{\partial \varphi}{\partial q_\nu}(\cdot, q^{(i-2)}(s))|_{L^2(\mathbb{R}^2)} |\frac{\partial \varphi}{\partial q_\mu}(\cdot, q^{(i-1)}(s))|_{L^2(\mathbb{R}^2)}.
\end{aligned}
\tag{5.31}
$$

Because of Remark 3.1.1, Lemma 3.5.1 and the fact that the zero modes are exponentially decaying as x tends to infinity, (5.31) yields

$$|\langle \begin{pmatrix} \sigma\big(n_\nu^1(\cdot, q^{(i-1)}(s)) - n_\nu^1(\cdot, q^{(i-2)}(s))\big) \\ n_\nu^2(\cdot, q^{(i-1)}(s)) - n_\nu^2(\cdot, q^{(i-2)}(s)) \end{pmatrix}, \begin{pmatrix} n_\mu^1(\cdot, q^{(i-1)}(s)) \\ n_\mu^2(\cdot, q^{(i-1)}(s)) \end{pmatrix} \rangle| \leq C(\Gamma) \max_{0\leq\tau\leq s} |q^{(i-1)}(\tau) - q^{(i-2)}(\tau)| \qquad (5.32)$$

for some constant $C(\Gamma) > 0$ independent of i, for $0 \leq s \leq \overline{T}$.

Exactly in the same way it is shown that

$$|\langle \begin{pmatrix} n_\mu^1(\cdot, q^{(i-1)}(s)) - n_\mu^1(\cdot, q^{(i-2)}(s)) \\ n_\mu^2(\cdot, q^{(i-1)}(s)) - n_\mu^2(\cdot, q^{(i-2)}(s)) \end{pmatrix}, \begin{pmatrix} \sigma n_\nu^1(\cdot, q^{(i-2)}(s)) \\ n_\nu^2(\cdot, q^{(i-2)}(s)) \end{pmatrix} \rangle| \leq C(\Gamma) \max_{0\leq\tau\leq s} |q^{(i-1)}(\tau) - q^{(i-2)}(\tau)| \qquad (5.33)$$

and

$$|\langle \begin{pmatrix} \sigma\widetilde{a}^{(i-2)}(\cdot, s) \\ \widetilde{\phi}^{(i-2)}(\cdot, s) \end{pmatrix}, \begin{pmatrix} \frac{\partial n_\mu^1}{\partial q_\nu}(\cdot, q^{(i-1)}(s)) - \frac{\partial n_\mu^1}{\partial q_\nu}(\cdot, q^{(i-2)}(s)) \\ \frac{\partial n_\mu^2}{\partial q_\nu}(\cdot, q^{(i-1)}(s)) - \frac{\partial n_\mu^2}{\partial q_\nu}(\cdot, q^{(i-2)}(s)) \end{pmatrix} \rangle| \leq C(\Gamma) \max_{0\leq\tau\leq s} |q^{(i-1)}(\tau) - q^{(i-2)}(\tau)| \qquad (5.34)$$

for a constant $C(\Gamma) > 0$ independent of i, for $0 \leq s \leq \overline{T}$.

Now we want to estimate

$$\langle \begin{pmatrix} \sigma\widetilde{a}^{(i-1)}(\cdot, s) - \sigma\widetilde{a}^{(i-2)}(\cdot, s) \\ \widetilde{\phi}^{(i-1)}(\cdot, s) - \widetilde{\phi}^{(i-2)}(\cdot, s) \end{pmatrix}, \begin{pmatrix} \frac{\partial n_\mu^1}{\partial q_\nu}(\cdot, q^{(i-1)}(s)) \\ \frac{\partial n_\mu^2}{\partial q_\nu}(\cdot, q^{(i-1)}(s)) \end{pmatrix} \rangle.$$

We try applying the mean value theorem to the function

$$u_i : H^{3,\alpha(\cdot, q(0))}(\mathbb{R}^2) \longrightarrow \mathbb{R}$$

defined as follows:

$$u_i(\sigma\widetilde{a},\widetilde{\phi}) := \langle \begin{pmatrix} \sigma\widetilde{a} \\ \widetilde{\phi} \end{pmatrix}, \begin{pmatrix} \frac{\partial n_\mu^1}{\partial q_\nu}(\cdot, q^{(i-1)}) \\ \frac{\partial n_\mu^2}{\partial q_\nu}(\cdot, q^{(i-1)}) \end{pmatrix} \rangle.$$

First of all we have to check if the conditions of the mean value theorem are satisfied in this case. It should be noticed that

$$\frac{\partial u_i}{\partial(\sigma\widetilde{a}_k)}(\sigma\widetilde{a},\widetilde{\phi}) = \langle \begin{pmatrix} \delta_{1k} \\ \delta_{2k} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\partial^2\alpha_1}{\partial q_\mu \partial q_\nu}(\cdot, q^{(i-1)}) \\ \frac{\partial^2\alpha_2}{\partial q_\mu \partial q_\nu}(\cdot, q^{(i-1)}) \\ \frac{\partial^2\varphi}{\partial q_\mu \partial q_\nu}(\cdot, q^{(i-1)}) \end{pmatrix} \rangle$$

and then

$$|\frac{\partial u_i}{\partial(\sigma\widetilde{a}_k)}(\sigma\widetilde{a},\widetilde{\phi})| \leq |\frac{\partial^2\alpha_k}{\partial q_\mu \partial q_\nu}(\cdot, q^{(i-1)})|_{L^1(\mathbb{R}^2)}. \tag{5.35}$$

The right-hand side of (5.35) is bounded by a constant independent of i, for $0 \leq s \leq \overline{T}$, since all the derivatives with respect to q are exponentially decaying as x tends to infinity, [Stu94,I, corollary 2.4]. For the same reason, given

$$\frac{\partial u}{\partial\widetilde{\phi}}(\sigma\widetilde{a},\widetilde{\phi}) = \langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{\partial^2\alpha_1}{\partial q_\mu \partial q_\nu}(\cdot, q^{(i-1)}) \\ \frac{\partial^2\alpha_2}{\partial q_\mu \partial q_\nu}(\cdot, q^{(i-1)}) \\ \frac{\partial^2\varphi}{\partial q_\mu \partial q_\nu}(\cdot, q^{(i-1)}) \end{pmatrix} \rangle,$$

one obtains that

$$|\frac{\partial u}{\partial\widetilde{\phi}}(\sigma\widetilde{a},\widetilde{\phi})| \leq |\frac{\partial^2\varphi}{\partial q_\mu \partial q_\nu}(\cdot, q^{(i-1)})|_{L^1(\mathbb{R}^2)}, \tag{5.36}$$

with the right-hand side of (5.36) bounded by a constant independent of i, for $0 \leq s \leq \overline{T}$. Then, since the mean value theorem can be applied, one obtains

$$|\langle \begin{pmatrix} \sigma\widetilde{a}^{(i-1)}(\cdot,s) - \sigma\widetilde{a}^{(i-2)}(\cdot,s) \\ \widetilde{\phi}^{(i-1)}(\cdot,s) - \widetilde{\phi}^{(i-2)}(\cdot,s) \end{pmatrix}, \begin{pmatrix} \frac{\partial n_\mu^1}{\partial q_\nu}(\cdot, q^{(i-1)}(s)) \\ \frac{\partial n_\mu^2}{\partial q_\nu}(\cdot, q^{(i-1)}(s)) \end{pmatrix} \rangle|$$
$$\leq C(\Gamma)\big\|\psi^{(i-1)}(\cdot,s) - \psi^{(i-2)}(\cdot,s)\big\| \tag{5.37}$$

for some constant $C(\Gamma) > 0$ independent of i, for $0 \leq s \leq \overline{T}$. Now, using (5.30), (5.32), (5.33), (5.34) and (5.37), one obtains

$$\sum_{\mu=1}^{2N} |\sum_{\nu=1}^{2N} \big(a_{\mu\nu}(i-1,s) - a_{\mu\nu}(i-2,s)\big)\dot{q}_\nu^{(i-1)}(s)|$$
$$\leq C(\Gamma)\big\|\psi^{(i-1)}(\cdot,s) - \psi^{(i-2)}(\cdot,s)\big\| + C(\Gamma)\max_{0\leq\tau\leq s}|q^{(i-1)}(\tau) - q^{(i-2)}(\tau)| \tag{5.38}$$

for a constant $C(\Gamma) > 0$ independent of i, for $0 \leq s \leq \overline{T}$.

Now we estimate the second term in (5.29). The aim is to apply the mean value theorem to the function

$$v_\mu : X(T) \longrightarrow \mathbb{R}$$

defined as follows:

$$v_\mu(\widetilde{a}, \widetilde{\phi}, \widetilde{\widetilde{a}}_0, q) := \frac{1}{2}\int_{\mathbb{R}^2} \big(\varphi(x,q)(1 - |\phi(x,q)|^2), n_\mu^2(x,q)\big)d^2x$$
$$+ \varepsilon\langle j(\widetilde{a}, \widetilde{\phi}, \widetilde{\widetilde{a}}_0, q), n_\mu(\cdot,q)\rangle.$$

All the partial derivatives of v_μ must be shown to be bounded in $\mathbb{R}$ in order to apply the mean value theorem. From

$$\frac{\partial v_\mu}{\partial(\sigma\widetilde{a}_k)}(\widetilde{a},\widetilde{\phi},\widetilde{\widetilde{a}}_0,q)=\varepsilon\langle\frac{\partial j}{\partial(\sigma\widetilde{a}_k)}(\widetilde{a},\widetilde{\phi},\widetilde{\widetilde{a}}_0,q),n_\mu(\cdot,q)\rangle$$

$$=-\varepsilon\langle\begin{pmatrix}\frac{2}{\sigma}\delta_{1k}\big(\varphi(\cdot,q),\widetilde{\phi}\big)+\frac{\varepsilon}{\sigma}\delta_{1k}|\widetilde{\phi}|^2\\ \frac{2}{\sigma}\delta_{2k}\big(\varphi(\cdot,q),\widetilde{\phi}\big)+\frac{\varepsilon}{\sigma}\delta_{2k}|\widetilde{\phi}|^2\\ i\frac{2}{\sigma}D_k^{(0)}\widetilde{\phi}+i\frac{2}{\sigma^2}\sigma\widetilde{a}_k\varphi(\cdot,q)+i\frac{2\varepsilon}{\sigma^2}\sigma\widetilde{a}_k\widetilde{\phi}\end{pmatrix},\begin{pmatrix}\frac{\partial\alpha_1}{\partial q_\mu}(\cdot,q)\\ \frac{\partial\alpha_2}{\partial q_\mu}(\cdot,q)\\ \frac{\partial\varphi}{\partial q_\mu}(\cdot,q)\end{pmatrix}\rangle$$

it follows that

$$\begin{aligned}|\frac{\partial v_\mu}{\partial(\sigma\widetilde{a}_k)}(\widetilde{a},\widetilde{\phi},\widetilde{\widetilde{a}}_0,q)|\le{}&\frac{2\varepsilon}{\sigma}\sum_{l=1}^{2}\int_{\mathbb{R}^2}|\varphi(x,q(s))||\widetilde{\phi}(x,s)||\frac{\partial\alpha_l}{\partial q_\mu}(x,q(s))|d^2x\\ &+\frac{\varepsilon^2}{\sigma}\sum_{l=1}^{2}\int_{\mathbb{R}^2}|\widetilde{\phi}(x,s)|^2|\frac{\partial\alpha_l}{\partial q_\mu}(x,q(s))|d^2x\\ &+\frac{2}{\sigma}\int_{\mathbb{R}^2}|D_k^{(0)}\widetilde{\phi}(x,s)||\frac{\partial\varphi}{\partial q_\mu}(x,q(s))|d^2x\\ &+\frac{2}{\sigma^2}\int_{\mathbb{R}^2}|\sigma\widetilde{a}_k(x,s)||\varphi(x,q(s))||\frac{\partial\varphi}{\partial q_\mu}(x,q(s))|d^2x\\ &+\frac{2\varepsilon}{\sigma^2}\int_{\mathbb{R}^2}|\sigma\widetilde{a}_k(x,s)||\widetilde{\phi}(x,s)||\frac{\partial\varphi}{\partial q_\mu}(x,q(s))|d^2x.\end{aligned}\tag{5.39}$$

Using the Cauchy inequality, Remark 3.1.1 and the fact that $(\widetilde{\psi},\widetilde{\widetilde{a}}_0,q)\in X(\overline{T})$, it turns out that the right-hand side of (5.39) is bounded for $0\le t\le\overline{T}$. From

$$\frac{\partial v_\mu}{\partial\widetilde{\phi}}(\widetilde{a},\widetilde{\phi},\widetilde{\widetilde{a}}_0,q)=\varepsilon\langle\frac{\partial j}{\partial\widetilde{\phi}}(\widetilde{a},\widetilde{\phi},\widetilde{\widetilde{a}}_0,q),n_\mu(\cdot,q)\rangle$$

$$=-\varepsilon\langle\begin{pmatrix}\frac{2}{\sigma}\sigma\widetilde{a}_1\big(\varphi(\cdot,q),1\big)\\ \frac{2}{\sigma}\sigma\widetilde{a}_2\big(\varphi(\cdot,q),1\big)\\ i\frac{1}{\sigma^2}\sum_{l=1}^{2}\partial_l(\sigma\widetilde{a}_l)+\frac{2}{\sigma}\sum_{l=1}^{2}\sigma\widetilde{a}_l\alpha_l(\cdot,q)\end{pmatrix},\begin{pmatrix}\frac{\partial\alpha_1}{\partial q_\mu}(\cdot,q)\\ \frac{\partial\alpha_2}{\partial q_\mu}(\cdot,q)\\ \frac{\partial\varphi}{\partial q_\mu}(\cdot,q)\end{pmatrix}\rangle$$

$$-\varepsilon\langle\begin{pmatrix} -\big(i, D_1^{(0)}\widetilde{\phi}\big) \\ -\big(i, D_2^{(0)}\widetilde{\phi}\big) \\ i\frac{\varepsilon}{\sigma^2}\sum_{l=1}^2(\sigma\widetilde{a}_l)^2 - \frac{1}{2}(1-|\phi(\cdot,q)|^2) \end{pmatrix}, \begin{pmatrix} \frac{\partial\alpha_1}{\partial q_\mu}(\cdot,q) \\ \frac{\partial\alpha_2}{\partial q_\mu}(\cdot,q) \\ \frac{\partial\varphi}{\partial q_\mu}(\cdot,q) \end{pmatrix}\rangle$$

$$-\varepsilon\langle\begin{pmatrix} \alpha_1(\cdot,q)\big(\widetilde{\phi},1\big) \\ \alpha_2(\cdot,q)\big(\widetilde{\phi},1\big) \\ (1+\varepsilon)\widetilde{\phi}\big(\varphi(\cdot,q),1\big) + (1+\varepsilon)\big(\varphi(\cdot,q),\widetilde{\phi}\big) \end{pmatrix}, \begin{pmatrix} \frac{\partial\alpha_1}{\partial q_\mu}(\cdot,q) \\ \frac{\partial\alpha_2}{\partial q_\mu}(\cdot,q) \\ \frac{\partial\varphi}{\partial q_\mu}(\cdot,q) \end{pmatrix}\rangle$$

$$-\varepsilon\langle\begin{pmatrix} \frac{2\varepsilon}{\sigma}\big(\widetilde{\phi},\sigma\widetilde{a}_1\big) \\ \frac{2\varepsilon}{\sigma}\big(\widetilde{\phi},\sigma\widetilde{a}_1\big) \\ \varphi(\cdot,q)\big(\varphi(\cdot,q),1\big) + (1+\varepsilon)\varphi(\cdot,q)\big(\widetilde{\phi},1\big) \end{pmatrix}, \begin{pmatrix} \frac{\partial\alpha_1}{\partial q_\mu}(\cdot,q) \\ \frac{\partial\alpha_2}{\partial q_\mu}(\cdot,q) \\ \frac{\partial\varphi}{\partial q_\mu}(\cdot,q) \end{pmatrix}\rangle$$

$$-\varepsilon\langle\begin{pmatrix} \big(i\partial_1\varphi(\cdot,q),1\big) \\ \big(i\partial_2\varphi(\cdot,q),1\big) \\ \varepsilon(1+\varepsilon)\widetilde{\phi}\big(\widetilde{\phi},1\big) + \frac{\varepsilon}{2}(1+\varepsilon)|\widetilde{\phi}|^2 - i\widetilde{\widetilde{a}}_0 \end{pmatrix}, \begin{pmatrix} \frac{\partial\alpha_1}{\partial q_\mu}(\cdot,q) \\ \frac{\partial\alpha_2}{\partial q_\mu}(\cdot,q) \\ \frac{\partial\varphi}{\partial q_\mu}(\cdot,q) \end{pmatrix}\rangle$$

it follows that

$$\begin{aligned}
|\frac{\partial v_\mu}{\partial\widetilde{\phi}}(\widetilde{a},\widetilde{\phi},\widetilde{\widetilde{a}}_0,q)| \leq{}& \frac{2\varepsilon}{\sigma}\sum_{l=1}^2\int_{\mathbb{R}^2}|\sigma\widetilde{a}_l(x,s)||\varphi(x,q(s))||\frac{\partial\alpha_l}{\partial q_\mu}(x,q(s))|d^2x \\
&+\varepsilon\sum_{l=1}^2\int_{\mathbb{R}^2}|D_l^{(0)}\widetilde{\phi}(x,s)||\frac{\partial\alpha_l}{\partial q_\mu}(x,q(s))|d^2x \\
&+\varepsilon\sum_{l=1}^2\int_{\mathbb{R}^2}|\alpha_l(x,q(s))||\widetilde{\phi}(x,s)||\frac{\partial\alpha_l}{\partial q_\mu}(x,q(s))|d^2x \\
&+\frac{2\varepsilon^2}{\sigma}\sum_{l=1}^2\int_{\mathbb{R}^2}|\sigma\widetilde{a}_l(x,s)||\widetilde{\phi}(x,s)||\frac{\partial\alpha_l}{\partial q_\mu}(x,q(s))|d^2x \\
&+\varepsilon\sum_{l=1}^2\int_{\mathbb{R}^2}|\partial_l\varphi(x,q(s))||\frac{\partial\alpha_l}{\partial q_\mu}(x,q(s))|d^2x \\
&+\frac{\varepsilon}{\sigma}\sum_{l=1}^2\int_{\mathbb{R}^2}|\partial_l(\sigma\widetilde{a}_l)(x,s)||\frac{\partial\varphi}{\partial q_\mu}(x,q(s))|d^2x
\end{aligned}$$

$$+\frac{2\varepsilon}{\sigma}\sum_{l=1}^{2}\int_{\mathbb{R}^2}|\sigma\widetilde{a}_l(x,s)||\alpha_l(x,q(s))||\frac{\partial\varphi}{\partial q_\mu}(x,q(s))|d^2x$$
$$+\frac{\varepsilon^2}{\sigma^2}\sum_{l=1}^{2}\int_{\mathbb{R}^2}|\sigma\widetilde{a}_l(x,s)|^2|\frac{\partial\varphi}{\partial q_\mu}(x,q(s))|d^2x$$
$$+\frac{\varepsilon}{2}\int_{\mathbb{R}^2}|1-|\phi(x,q(s))|^2||\frac{\partial\varphi}{\partial q_\mu}(x,q(s))|d^2x$$
$$-3\varepsilon(1+\varepsilon)\int_{\mathbb{R}^2}|\varphi(x,q(s))||\widetilde{\phi}(x,s)||\frac{\partial\varphi}{\partial q_\mu}(x,q(s))|d^2x$$
$$-\varepsilon\int_{\mathbb{R}^2}|\varphi(x,q(s))|^2|\frac{\partial\varphi}{\partial q_\mu}(x,q(s))|d^2x$$
$$-\frac{3\varepsilon^2}{2}(1+\varepsilon)\int_{\mathbb{R}^2}|\widetilde{\phi}(x,s)||\frac{\partial\varphi}{\partial q_\mu}(x,q(s))|d^2x$$
$$+\varepsilon\int_{\mathbb{R}^2}|\widetilde{\widetilde{a}}_0(x,s)||\frac{\partial\varphi}{\partial q_\mu}(x,q(s))|d^2x. \quad (5.40)$$

Using the Cauchy inequality, Remark 3.1.1 and the fact that $(\widetilde{\psi},\widetilde{\widetilde{a}}_0,q)\in X(\overline{T})$, one finds that the right-hand side of (5.40) is bounded for $0\le t\le\overline{T}$. From

$$\frac{\partial v_\mu}{\partial\widetilde{\widetilde{a}}_0}(\widetilde{a},\widetilde{\phi},\widetilde{\widetilde{a}}_0,q)$$

$$=\varepsilon\langle\frac{\partial j}{\partial\widetilde{\widetilde{a}}_0}(\widetilde{a},\widetilde{\phi},\widetilde{\widetilde{a}}_0,q),n_\mu(\cdot,q)\rangle$$
$$=-\varepsilon\langle\begin{pmatrix}0\\0\\i\widetilde{\phi}\end{pmatrix},\begin{pmatrix}\frac{\partial\alpha_1}{\partial q_\mu}(\cdot,q)\\\frac{\partial\alpha_2}{\partial q_\mu}(\cdot,q)\\\frac{\partial\varphi}{\partial q_\mu}(\cdot,q)\end{pmatrix}\rangle$$

it follows that

$$|\frac{\partial v_\mu}{\partial\widetilde{\widetilde{a}}_0}(\widetilde{a},\widetilde{\phi},\widetilde{\widetilde{a}}_0,q)|\le\varepsilon\int_{\mathbb{R}^2}|\widetilde{\phi}(x,s)||\frac{\partial\varphi}{\partial q_\mu}(x,q(s))|d^2x. \quad (5.41)$$

Using the Cauchy inequality and the fact that $(\widetilde{\psi}, \widetilde{\widetilde{a}}_0, q) \in X(\overline{T})$, it turns out that the right-hand side of (5.41) is bounded for $0 \le t \le \overline{T}$. Finally, from

$$\frac{\partial v_\mu}{\partial q_\nu}(\widetilde{a}, \widetilde{\phi}, \widetilde{\widetilde{a}}_0, q) = -\int_{\mathbb{R}^2} \big(\varphi(x,q), \frac{\partial \varphi}{\partial q_\mu}(x,q)\big)\big(\varphi(x,q), \frac{\partial \varphi}{\partial q_\nu}(x,q)\big) d^2x$$
$$+ \frac{1}{2}\int_{\mathbb{R}^2} \big(1 - |\phi(x,q)|^2\big)\big(\frac{\partial \varphi}{\partial q_\mu}(x,q), \frac{\partial \varphi}{\partial q_\nu}(x,q)\big) d^2x$$
$$+ \frac{1}{2}\int_{\mathbb{R}^2} \big(1 - |\phi(x,q)|^2\big)\big(\varphi(x,q), \frac{\partial^2 \varphi}{\partial q_\mu \partial q_\nu}(x,q)\big) d^2x$$
$$+ \varepsilon\langle \frac{\partial j}{\partial q_\nu}(\widetilde{a}, \widetilde{\phi}, \widetilde{\widetilde{a}}_0, q), n_\mu(\cdot, q)\rangle + \varepsilon\langle j(\widetilde{a}, \widetilde{\phi}, \widetilde{\widetilde{a}}_0, q), \frac{\partial n_\mu}{\partial q_\nu}(\cdot, q)\rangle$$
$$= -\varepsilon\langle \begin{pmatrix} \frac{2}{\sigma}\sigma\widetilde{a}_1\big(\widetilde{\phi}, \frac{\partial \varphi}{\partial q_\nu}(\cdot,q)\big) \\ \frac{2}{\sigma}\sigma\widetilde{a}_2\big(\widetilde{\phi}, \frac{\partial \varphi}{\partial q_\nu}(\cdot,q)\big) \\ \frac{2}{\sigma}\widetilde{\phi}\sum_{l=1}^2 \sigma\widetilde{a}_l \frac{\partial \alpha_l}{\partial q_\nu}(\cdot,q) + i\frac{1}{\sigma^2}\frac{\partial \varphi}{\partial q_\nu}(\cdot,q)\sum_{l=1}^2(\sigma\widetilde{a}_l)^2 \end{pmatrix}, \begin{pmatrix} \frac{\partial \alpha_1}{\partial q_\mu}(\cdot,q) \\ \frac{\partial \alpha_2}{\partial q_\mu}(\cdot,q) \\ \frac{\partial \varphi}{\partial q_\mu}(\cdot,q) \end{pmatrix}\rangle$$
$$- \varepsilon\langle \begin{pmatrix} -|\widetilde{\phi}|^2 \frac{\partial \alpha_1}{\partial q_\nu}(\cdot,q) \\ -|\widetilde{\phi}|^2 \frac{\partial \alpha_2}{\partial q_\nu}(\cdot,q) \\ \widetilde{\phi}\big(\varphi(\cdot,q), \frac{\partial \varphi}{\partial q_\nu}(\cdot,q)\big) + (1+\varepsilon)\widetilde{\phi}\big(\widetilde{\phi}, \frac{\partial \varphi}{\partial q_\nu}(\cdot,q)\big) \end{pmatrix}, \begin{pmatrix} \frac{\partial \alpha_1}{\partial q_\mu}(\cdot,q) \\ \frac{\partial \alpha_2}{\partial q_\mu}(\cdot,q) \\ \frac{\partial \varphi}{\partial q_\mu}(\cdot,q) \end{pmatrix}\rangle$$
$$- \varepsilon\langle \begin{pmatrix} \big(\widetilde{\phi}, i\partial_1 \frac{\partial \varphi}{\partial q_\nu}(\cdot,q)\big) \\ \big(\widetilde{\phi}, i\partial_2 \frac{\partial \varphi}{\partial q_\nu}(\cdot,q)\big) \\ \frac{\partial \varphi}{\partial q_\nu}(\cdot,q)\big(\varphi(\cdot,q), \widetilde{\phi}\big) + \varphi(\cdot,q)\big(\widetilde{\phi}, \frac{\partial \varphi}{\partial q_\nu}(\cdot,q)\big) \end{pmatrix}, \begin{pmatrix} \frac{\partial \alpha_1}{\partial q_\mu}(\cdot,q) \\ \frac{\partial \alpha_2}{\partial q_\mu}(\cdot,q) \\ \frac{\partial \varphi}{\partial q_\mu}(\cdot,q) \end{pmatrix}\rangle$$
$$- \varepsilon\langle \begin{pmatrix} \big(\partial_1\widetilde{\phi}, i\frac{\partial \varphi}{\partial q_\nu}(\cdot,q)\big) \\ \big(\partial_2\widetilde{\phi}, i\frac{\partial \varphi}{\partial q_\nu}(\cdot,q)\big) \\ \frac{1+\varepsilon}{2}|\widetilde{\phi}|^2 \frac{\partial \varphi}{\partial q_\nu}(\cdot,q) \end{pmatrix}, \begin{pmatrix} \frac{\partial \alpha_1}{\partial q_\mu}(\cdot,q) \\ \frac{\partial \alpha_2}{\partial q_\mu}(\cdot,q) \\ \frac{\partial \varphi}{\partial q_\mu}(\cdot,q) \end{pmatrix}\rangle$$
$$- \varepsilon\langle \begin{pmatrix} \frac{2}{\sigma}\sigma\widetilde{a}_1\big(\varphi(\cdot,q), \widetilde{\phi}\big) \\ \frac{2}{\sigma}\sigma\widetilde{a}_2\big(\varphi(\cdot,q), \widetilde{\phi}\big) \\ i\frac{1}{\sigma}\widetilde{\phi}\sum_{l=1}^2 \partial_l(\sigma\widetilde{a}_l) + i\frac{2}{\sigma}\sum_{l=1}^2 \sigma\widetilde{a}_l D_l^{(0)}\widetilde{\phi} \end{pmatrix}, \begin{pmatrix} \frac{\partial^2 \alpha_1}{\partial q_\mu \partial q_\nu}(\cdot,q) \\ \frac{\partial^2 \alpha_2}{\partial q_\mu \partial q_\nu}(\cdot,q) \\ \frac{\partial^2 \varphi}{\partial q_\mu \partial q_\nu}(\cdot,q) \end{pmatrix}\rangle$$

$$- \varepsilon\langle \begin{pmatrix} -(i\widetilde{\phi}, D_1^{(0)}\widetilde{\phi}) \\ -(i\widetilde{\phi}, D_2^{(0)}\widetilde{\phi}) \\ i\frac{1}{\sigma^2}\varphi\sum_{l=1}^2(\sigma\widetilde{a}_l)^2 + i\frac{\varepsilon}{\sigma^2}\widetilde{\phi}\sum_{l=1}^2(\sigma\widetilde{a}_l)^2 \end{pmatrix}, \begin{pmatrix} \frac{\partial^2\alpha_1}{\partial q_\mu\partial q_\nu}(\cdot,q) \\ \frac{\partial^2\alpha_2}{\partial q_\mu\partial q_\nu}(\cdot,q) \\ \frac{\partial^2\varphi}{\partial q_\mu\partial q_\nu}(\cdot,q) \end{pmatrix}\rangle$$

$$+ \varepsilon\langle \begin{pmatrix} -\frac{\varepsilon}{\sigma}\sigma\widetilde{a}_1|\widetilde{\phi}|^2 \\ -\frac{\varepsilon}{\sigma}\sigma\widetilde{a}_2|\widetilde{\phi}|^2 \\ \frac{1}{2}\widetilde{\phi}(1-|\phi(\cdot,q)|^2) - (1+\varepsilon)\widetilde{\phi}(\varphi(\cdot,q),\widetilde{\phi}) \end{pmatrix}, \begin{pmatrix} \frac{\partial^2\alpha_1}{\partial q_\mu\partial q_\nu}(\cdot,q) \\ \frac{\partial^2\alpha_2}{\partial q_\mu\partial q_\nu}(\cdot,q) \\ \frac{\partial^2\varphi}{\partial q_\mu\partial q_\nu}(\cdot,q) \end{pmatrix}\rangle$$

$$+ \varepsilon\langle \begin{pmatrix} (i\partial_1\varphi(\cdot,q),\widetilde{\phi}) \\ (i\partial_2\varphi(\cdot,q),\widetilde{\phi}) \\ \varphi(\cdot,q)(\varphi(\cdot,q),\widetilde{\phi}) + \frac{1+\varepsilon}{2}\varphi(\cdot,q)|\widetilde{\phi}|^2 \end{pmatrix}, \begin{pmatrix} \frac{\partial^2\alpha_1}{\partial q_\mu\partial q_\nu}(\cdot,q) \\ \frac{\partial^2\alpha_2}{\partial q_\mu\partial q_\nu}(\cdot,q) \\ \frac{\partial^2\varphi}{\partial q_\mu\partial q_\nu}(\cdot,q) \end{pmatrix}\rangle$$

$$- \varepsilon\langle \begin{pmatrix} (i\varphi(\cdot,q),\partial_1\widetilde{\phi}) \\ (i\varphi(\cdot,q),\partial_2\widetilde{\phi}) \\ -\varepsilon\frac{1+\varepsilon}{2}\widetilde{\phi}|\widetilde{\phi}|^2 + i\widetilde{\widetilde{a}}_0\widetilde{\phi} \end{pmatrix}, \begin{pmatrix} \frac{\partial^2\alpha_1}{\partial q_\mu\partial q_\nu}(\cdot,q) \\ \frac{\partial^2\alpha_2}{\partial q_\mu\partial q_\nu}(\cdot,q) \\ \frac{\partial^2\varphi}{\partial q_\mu\partial q_\nu}(\cdot,q) \end{pmatrix}\rangle \tag{5.42}$$

it follows that

$$\begin{aligned}
|\frac{\partial v_\mu}{\partial q_\nu}(\widetilde{a},\widetilde{\phi},\widetilde{\widetilde{a}}_0,q)| \leq & \int_{\mathbb{R}^2} |\varphi(x,q(s))|^2|\frac{\partial\varphi}{\partial q_\mu}(x,q(s))||\frac{\partial\varphi}{\partial q_\nu}(x,q(s))|d^2x \\
& + \frac{1}{2}\int_{\mathbb{R}^2} |1-|\phi(x,q(s))|^2||\frac{\partial\varphi}{\partial q_\mu}(x,q(s))||\frac{\partial\varphi}{\partial q_\nu}(x,q(s))|d^2x \\
& + \frac{1}{2}\int_{\mathbb{R}^2} |1-|\phi(x,q(s))|^2|\varphi(x,q(s))||\frac{\partial^2\varphi}{\partial q_\mu\partial q_\nu}(x,q(s))|d^2x \\
& \frac{4\varepsilon}{\sigma}\sum_{l=1}^2\int_{\mathbb{R}^2} |\sigma\widetilde{a}_l(x,s)||\widetilde{\phi}(x,s)||\frac{\partial\alpha_l}{\partial q_\mu}(x,q(s))||\frac{\partial\varphi}{\partial q_\nu}(x,q(s))|d^2x \\
& + \varepsilon\sum_{l=1}^2\int_{\mathbb{R}^2} |\widetilde{\phi}(x,s)|^2|\frac{\partial\alpha_l}{\partial q_\mu}(x,q(s))|^2 d^2x \\
& + \varepsilon\sum_{l=1}^2\int_{\mathbb{R}^2} |\widetilde{\phi}(x,s)||\frac{\partial\alpha_l}{\partial q_\mu}(x,q(s))||\partial_l\frac{\partial\varphi}{\partial q_\nu}(x,q(s))|d^2x
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\varepsilon^2}{\sigma^2} \sum_{l=1}^{2} \int_{\mathbb{R}^2} |\sigma \widetilde{a}_l(x,s)| |\widetilde{\phi}(x,s)|^2 | \frac{\partial^2 \alpha_l}{\partial q_\mu \partial q_\nu}(x, q(s))| d^2x \\
&+ \varepsilon \sum_{l=1}^{2} \int_{\mathbb{R}^2} |\widetilde{\phi}(x,s)| |\partial_l \varphi(x, q(s))| | \frac{\partial^2 \alpha_l}{\partial q_\mu \partial q_\nu}(x, q(s))| d^2x \\
&+ \varepsilon \sum_{l=1}^{2} \int_{\mathbb{R}^2} |\partial_l \widetilde{\phi}(x,s)| |\varphi(x, q(s))| | \frac{\partial^2 \alpha_l}{\partial q_\mu \partial q_\nu}(x, q(s))| d^2x \\
&+ \frac{\varepsilon}{\sigma} \sum_{l=1}^{2} \int_{\mathbb{R}^2} |\widetilde{\phi}(x,s)|^2 |\partial_l \big(\sigma \widetilde{a}_l(x,s)\big)| | \frac{\partial^2 \varphi}{\partial q_\mu \partial q_\nu}(x, q(s))| d^2x \\
&+ \frac{2\varepsilon}{\sigma} \sum_{l=1}^{2} \int_{\mathbb{R}^2} |\sigma \widetilde{a}_l(x,s)| |D_l^{(0)} \widetilde{\phi}(x,s)| | \frac{\partial^2 \varphi}{\partial q_\mu \partial q_\nu}(x, q(s))| d^2x \\
&+ \frac{\varepsilon^2}{\sigma^2} \sum_{l=1}^{2} \int_{\mathbb{R}^2} |\sigma \widetilde{a}_l(x,s)|^2 |\widetilde{\phi}(x,s)|^2 | \frac{\partial^2 \varphi}{\partial q_\mu \partial q_\nu}(x, q(s))| d^2x
\end{aligned}
$$

$$+\varepsilon\sum_{l=1}^{2}\int_{\mathbb{R}^2}|\partial_l\widetilde{\phi}(x,s)||\frac{\partial\alpha_l}{\partial q_\mu}(x,q(s))||\frac{\partial\varphi}{\partial q_\nu}(x,q(s))|d^2x$$

$$+2\varepsilon\sum_{l=1}^{2}\int_{\mathbb{R}^2}|\varphi(x,q(s))||\widetilde{\phi}(x,s)||\frac{\partial\alpha_l}{\partial q_\mu}(x,q(s))||\frac{\partial\varphi}{\partial q_\nu}(x,q(s))|d^2x$$

$$+\frac{\varepsilon}{2}(1+\varepsilon)\sum_{l=1}^{2}\int_{\mathbb{R}^2}|\widetilde{\phi}(x,s)|^2|\frac{\partial\alpha_l}{\partial q_\mu}(x,q(s))||\frac{\partial\varphi}{\partial q_\nu}(x,q(s))|d^2x$$

$$+\frac{\varepsilon}{\sigma^2}\sum_{l=1}^{2}\int_{\mathbb{R}^2}|\sigma\widetilde{a}_l(x,s)|^2|\frac{\partial\varphi}{\partial q_\mu}(x,q(s))||\frac{\partial\varphi}{\partial q_\nu}(x,q(s))|d^2x$$

$$+3\varepsilon\int_{\mathbb{R}^2}|\widetilde{\phi}(x,s)||\varphi(x,q(s))||\frac{\partial\varphi}{\partial q_\mu}(x,q(s))||\frac{\partial\varphi}{\partial q_\nu}(x,q(s))|d^2x$$

$$+\frac{3\varepsilon}{2}\int_{\mathbb{R}^2}|\widetilde{\phi}(x,s)|^2|\frac{\partial\varphi}{\partial q_\mu}(x,q(s))||\frac{\partial\varphi}{\partial q_\nu}(x,q(s))|d^2x$$

$$+\frac{2\varepsilon}{\sigma}\sum_{l=1}^{2}\int_{\mathbb{R}^2}|\varphi(x,q(s))||\widetilde{\phi}(x,s)||\sigma\widetilde{a}_l(x,s)||\frac{\partial^2\alpha_l}{\partial q_\mu\partial q_\nu}(x,q(s))|d^2x$$

$$+\varepsilon\sum_{l=1}^{2}\int_{\mathbb{R}^2}|\widetilde{\phi}(x,s)||D_l^{(0)}\widetilde{\phi}(x,s)||\frac{\partial^2\alpha_l}{\partial q_\mu\partial q_\nu}(x,q(s))|d^2x$$

$$
\begin{aligned}
&+\frac{\varepsilon}{2}\int_{\mathbb{R}^2}|\widetilde{\phi}(x,s)||1-|\phi(x,q(s))|^2||\frac{\partial^2\varphi}{\partial q_\mu \partial q_\nu}(x,q(s))|d^2x \\
&+\frac{3\varepsilon}{2}(1+\varepsilon)\int_{\mathbb{R}^2}|\varphi(x,q(s))||\widetilde{\phi}(x,s)|^2|\frac{\partial^2\varphi}{\partial q_\mu \partial q_\nu}(x,q(s))|d^2x \\
&+\varepsilon\int_{\mathbb{R}^2}|\widetilde{\phi}(x,s)||\varphi(x,q(s))|^2|\frac{\partial^2\varphi}{\partial q_\mu \partial q_\nu}(x,q(s))|d^2x \\
&+\frac{\varepsilon^2}{2}(1+\varepsilon)\int_{\mathbb{R}^2}|\widetilde{\phi}(x,s)|^3|\frac{\partial^2\varphi}{\partial q_\mu \partial q_\nu}(x,q(s))|d^2x \\
&+\varepsilon\int_{\mathbb{R}^2}|\widetilde{a}_0(x,s)||\widetilde{\phi}(x,s)||\frac{\partial^2\varphi}{\partial q_\mu \partial q_\nu}(x,q(s))|d^2x. \qquad (5.43)
\end{aligned}
$$

Using the Cauchy inequality, Remark 3.1.1 and the fact that $(\widetilde{\psi},\widetilde{\widetilde{a}}_0,q)\in X(\overline{T})$, it turns out that the right-hand side of (5.43) is bounded for $0\le t\le\overline{T}$. Then, applying the mean value theorem to the function v_μ, one obtains

$$
\begin{aligned}
&\sum_{\mu=1}^{2N}|\frac{1}{2}\int_{\mathbb{R}^2}\big(\varphi(x,q^{(i-1)}(s))(1-|\phi(x,q^{(i-1)}(s))|^2),n_\mu^2(x,q^{(i-1)}(s))\big)d^2x \\
&\quad+\varepsilon\langle j(\widetilde{a}^{(i-1)}(\cdot,s),\widetilde{\phi}^{(i-1)}(\cdot,s),\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s),q^{(i-1)}(s)),n_\mu(\cdot,q^{(i-1)}(s))\rangle \\
&\quad-\frac{1}{2}\int_{\mathbb{R}^2}\big(\varphi(x,q^{(i-2)}(s))(1-|\phi(x,q^{(i-2)}(s))|^2),n_\mu^2(x,q^{(i-2)}(s))\big)d^2x \\
&\quad-\varepsilon\langle j(\widetilde{a}^{(i-2)}(\cdot,s),\widetilde{\phi}^{(i-2)}(\cdot,s),\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,s),q^{(i-2)}(s)),n_\mu(\cdot,q^{(i-2)}(s))\rangle| \\
&\quad\le C(\Gamma)\|\psi^{(i-1)}(\cdot,s)-\psi^{(i-2)}(\cdot,s)\|+C(\Gamma)\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)-\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,s)\| \\
&\quad+C(\Gamma)\max_{0\le\tau\le s}|q^{(i-1)}(\tau)-q^{(i-2)}(\tau)| \qquad (5.44)
\end{aligned}
$$

for some constant $C(\Gamma)>0$ independent of i, for $0\le t\le\overline{T}$. Now, using (5.28), (5.29), (5.38) and (5.44) it turns out that

$$
\begin{aligned}
&|\dot{q}^{(i)}(s)-\dot{q}^{(i-1)}(s)|\le C(\Gamma)\|\psi^{(i-1)}(\cdot,s)-\psi^{(i-2)}(\cdot,s)\| \\
&\quad+C(\Gamma)\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)-\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,s)\|+C(\Gamma)\max_{0\le\tau\le s}|q^{(i-1)}(\tau)-q^{(i-2)}(\tau)| \\
&\qquad (5.45)
\end{aligned}
$$

for $C(\Gamma)>0$ constant independent of i, for $0\le t\le\overline{T}$.

Using (5.45) in (5.27) and the hypotheses on the initial data as in Theorem 4.5.1, one finds

$$\begin{aligned}
&\|\psi^{(i+1)}(\cdot,t) - \psi^{(i)}(\cdot,t)\| \le C(\Gamma)\frac{\delta}{2^i} \\
&+ C(\Gamma)\int_0^t \Big(\|\psi^{(i)}(\cdot,s) - \psi^{(i-1)}(\cdot,s)\| + \|\psi^{(i-1)}(\cdot,s) - \psi^{(i-2)}(\cdot,s)\| \\
&+ \|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s) - \widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\| + \|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s) - \widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,s)\| \\
&+ \max_{0\le\tau\le s}|q^{(i)}(\tau) - q^{(i-1)}(\tau)| + \max_{0\le\tau\le s}|q^{(i-1)}(\tau) - q^{(i-2)}(\tau)|\Big)ds
\end{aligned} \tag{5.46}$$

for some constant $C(\Gamma) > 0$ independent of i, for $0 \le t \le \overline{T}$.

Furthermore, an estimate for $|q^{(i+1)} - q^{(i)}|$ can be easily derived from the previous inequalities. From (3.15),

$$q^{(i+1)}(t) = q^{(i+1)}(0) + \varepsilon\int_0^t f_2(i,\cdot,s)ds$$

and

$$q^{(i)}(t) = q^{(i)}(0) + \varepsilon\int_0^t f_2(i-1,\cdot,s)ds$$

follow. Then, subtracting the second equation from the first and taking the norm of $\mathbb{R}^{2N}$, one obtains

$$\begin{aligned}
|q^{(i+1)}(t) - q^{(i)}(t)| &\le |q^{(i+1)}(0) - q^{(i)}(0)| \\
&+ \varepsilon\int_0^t |f_2(i,\cdot,s) - f_2(i-1,\cdot,s)|ds.
\end{aligned} \tag{5.47}$$

With the same reasoning that lead us to prove (5.45), by replacing i with $i+1$, one obtains

$$|\dot{q}^{(i+1)}(t) - \dot{q}^{(i)}(t)| \leq C(\Gamma)\|\psi^{(i)}(\cdot,t) - \psi^{(i-1)}(\cdot,t)\|$$
$$+ C(\Gamma)\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\| + C(\Gamma)\max_{0\leq s\leq t}|q^{(i)}(s) - q^{(i-1)}(s)| \tag{5.48}$$

for a constant $C(\Gamma) > 0$ independent of i, for $0 \leq t \leq \overline{T}$. We notice that the integrand in (5.47) coincides with the left-hand side of (5.48). Using the hypotheses on the initial data, from (5.47) one obtains

$$|q^{(i+1)}(t) - q^{(i)}(t)| \leq C(\Gamma)\frac{\delta}{2^{i+1}} + \varepsilon C(\Gamma)\int_0^t \Big(\|\psi^{(i)}(\cdot,s) - \psi^{(i-1)}(\cdot,s)\|$$
$$+ \|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s) - \widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\| + \max_{0\leq\tau\leq s}|q^{(i)}(\tau) - q^{(i-1)}(\tau)|\Big)ds \tag{5.49}$$

for some constant $C(\Gamma) > 0$ constant independent of i, for $0 \leq t \leq \overline{T}$.

5.5 An Estimate for $\widetilde{\widetilde{a}}_0^{(i+1)} - \widetilde{\widetilde{a}}_0^{(i)}$

The equation (3.13) is the key to finding an estimate for $\|\widetilde{\widetilde{a}}_0^{(i+1)} - \widetilde{\widetilde{a}}_0^{(i)}\|$ in terms of $\|\psi^{(i-1)} - \psi^{(i-2)}\|$, $\|\psi^{(i-2)} - \psi^{(i-3)}\|$, $\|\widetilde{\widetilde{a}}_0^{(i)} - \widetilde{\widetilde{a}}_0^{(i-1)}\|$, $\|\widetilde{\widetilde{a}}_0^{(i-1)} - \widetilde{\widetilde{a}}_0^{(i-2)}\|$, $|q^{(i-1)} - q^{(i-2)}|$, $|q^{(i-2)} - q^{(i-3)}|$ and $|\partial_t\psi^{(i)} - \partial_t\psi^{(i-1)}|_{1,\alpha(\cdot,q(0))}$. Again the mean value theorem is the main tool used in the proof. From (3.13) it is known that

$$-\Delta\widetilde{\widetilde{a}}_0^{(i+1)}(x,t) + \varepsilon\widetilde{\widetilde{a}}_0^{(i+1)}(x,t) = f_0(i,x,t)$$

and

$$-\Delta\widetilde{\widetilde{a}}_0^{(i)}(x,t) + \varepsilon\widetilde{\widetilde{a}}_0^{(i)}(x,t) = f_0(i-1,x,t)$$

and, subtracting one from the other, one obtains

$$-\Delta\big(\widetilde{\widetilde{a}}_0^{(i+1)}(x,t)-\widetilde{\widetilde{a}}_0^{(i)}(x,t)\big)+|\phi(x,q^{(i)}(t))|^2\big(\widetilde{\widetilde{a}}_0^{(i+1)}(x,t)-\widetilde{\widetilde{a}}_0^{(i)}(x,t)\big)$$
$$=f_0(i,x,t)-f_0(i-1,x,t)-\widetilde{\widetilde{a}}_0^{(i)}(x,t)\big(|\phi(x,q^{(i)}(t))|^2-|\phi(x,q^{(i-1)}(t))|^2\big). \tag{5.50}$$

Now, applying the same arguments as in the proof of Lemma 3.6.1 to (5.50), it turns out that

$$|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)|^2_{H^3(\mathbb{R}^2)}\le C(\Gamma)|f_0(i,\cdot,t)-f_0(i-1,\cdot,t)|^2_{H^1(\mathbb{R}^2)}$$
$$+C(\Gamma)|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\big(|\phi(\cdot,q^{(i)}(t))|^2-|\phi(\cdot,q^{(i-1)}(t))|^2\big)|^2_{H^1(\mathbb{R}^2)}$$

for some constant $C(\Gamma)>0$ independent of i for $0\le t\le\overline{T}$, and then

$$|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)|^2_{H^3(\mathbb{R}^2)}\le C(\Gamma)|f_0(i,\cdot,t)-f_0(i-1,\cdot,t)|^2_{H^1(\mathbb{R}^2)}$$
$$+C(\Gamma)|\big(\varphi(\cdot,q^{(i)}(t)),\varphi(\cdot,q^{(i)}(t))-\varphi(\cdot,q^{(i-1)}(t))\big)|^2_{H^1(\mathbb{R}^2)}$$
$$+C(\Gamma)|\big(\varphi(\cdot,q^{(i)}(t))-\varphi(\cdot,q^{(i-1)}(t)),\varphi(\cdot,q^{(i-1)}(t))\big)|^2_{H^1(\mathbb{R}^2)} \tag{5.51}$$

for some constant $C(\Gamma)>0$ independent of i, for $0\le t\le\overline{T}$.

Now every component on the right-hand side of (5.51) will be estimated. For example, for the last two components of the right-hand side of (5.51), proceeding as for (5.5), one obtains

$$|\big(\varphi(\cdot,q^{(i)}(t)),\varphi(\cdot,q^{(i)}(t))-\varphi(\cdot,q^{(i-1)}(t))\big)|^2_{H^1(\mathbb{R}^2)}$$
$$\le C(\Gamma)\max_{0\le s\le t}|q^{(i)}(s)-q^{(i-1)}(s)| \tag{5.52}$$

and

$$\begin{aligned}&|\big(\varphi(\cdot,q^{(i)}(t))-\varphi(\cdot,q^{(i-1)}(t)),\varphi(\cdot,q^{(i-1)}(t))\big)|^2_{H^1(\mathbb{R}^2)}\\&\qquad\le C(\Gamma)\max_{0\le s\le t}|q^{(i)}(s)-q^{(i-1)}(s)| \qquad (5.53)\end{aligned}$$

for some constant $C(\Gamma)>0$ independent of i, for $0\le t\le \overline{T}$.

Now, on the right-hand side of (5.51), also the term $f_0(i,\cdot,t)-f_0(i-1,\cdot,t)$ must be estimated. We try to apply the mean value theorem to find such an estimate. Following the notation of this theorem,

$$\begin{aligned}&X_1\times X_2\times X_3\times X_4\times X_5 := X(\overline{T})\times H^{1,\alpha(\cdot,q(0))}(\mathbb{R}^2),\\&X_6:=\mathbb{R}^{2N},\quad Y:=H^1(\mathbb{R}^2),\quad f:=f_0.\end{aligned}$$

Then it must be checked if the partial derivatives of $f_0(\psi,\tilde{\tilde{a}}_0,\partial_t\psi,q,\dot q)$ are bounded in $H^1(\mathbb{R}^2)$. From the form of f_0 it turns out that

$$\begin{aligned}\frac{\partial f_0}{\partial(\sigma\tilde a_k)}(\psi,\tilde{\tilde{a}}_0,\partial_t\psi,q,\dot q) &= -\frac{2}{\sigma^2}\big(\varphi(\cdot,q),\partial_k\varphi(\cdot,q)\big)\\&-\frac{2\varepsilon}{\sigma^2}\big(\tilde\phi,\partial_k\varphi(\cdot,q)\big)-\frac{2\varepsilon}{\sigma^2}\big(\varphi(\cdot,q),\partial_k\tilde\phi\big)-\frac{2\varepsilon^2}{\sigma^2}\big(\tilde\phi,\partial_k\tilde\phi\big),\end{aligned}$$

$$\begin{aligned}\frac{\partial f_0}{\partial\tilde\phi}(\psi,\tilde{\tilde{a}}_0,\partial_t\psi,q,\dot q) &= -\frac{1}{\sigma}\sum_{l=1}^{2}\alpha_l(\cdot,q)\big(\partial_l\varphi(\cdot,q),1\big)\\&+\frac{1}{\sigma}\sum_{l=1}^{2}\big(i,\partial_l D_l^{(0)}\varphi(\cdot,q)\big)-\frac{1}{\sigma}\sum_{l=1}^{2}\partial_l\alpha_l(\cdot,q)\big(\varphi(\cdot,q),1\big)\\&-\frac{2\varepsilon}{\sigma^2}\sum_{l=1}^{2}\sigma\tilde a_l\big(i,\partial_l\varphi(\cdot,q)\big)-\frac{\varepsilon}{\sigma}\sum_{l=1}^{2}\alpha_l(\cdot,q)\big(\partial_l\tilde\phi,1\big)\\&+\frac{\varepsilon}{\sigma}\sum_{l=1}^{2}\big(i,\partial_l D_l^{(0)}\tilde\phi\big)-\frac{\varepsilon}{\sigma}\sum_{l=1}^{2}\partial_l\alpha_l(\cdot,q)\big(\tilde\phi,1\big)\end{aligned}$$

$$
\begin{aligned}
&- \frac{2\varepsilon^2}{\sigma}\big(\varphi(\cdot,q),1\big)\big(i\varphi(\cdot,q),\widetilde{\phi}\big) - \frac{2\varepsilon^2}{\sigma}\big(i\varphi(\cdot,q),1\big)\big(\varphi(\cdot,q),\widetilde{\phi}\big) \\
&- \frac{2\varepsilon^2}{\sigma^2}\sum_{l=1}^{2}\sigma\widetilde{a}_l\big(\partial_l\widetilde{\phi},1\big) - \frac{2\varepsilon^3}{\sigma}\big(\widetilde{\phi},1\big)\big(i\varphi(\cdot,q),\widetilde{\phi}\big) - \frac{\varepsilon^3}{\sigma}|\widetilde{\phi}|^2\big(i\varphi(\cdot,q),1\big),
\end{aligned}
$$

$$
\frac{\partial f_0}{\partial\widetilde{\widetilde{a}}_0}(\psi,\widetilde{\widetilde{a}}_0,\partial_t\psi,q,\dot{q}) = \varepsilon,
$$

$$
\frac{\partial f_0}{\partial(\partial_t\psi)}(\psi,\widetilde{\widetilde{a}}_0,\partial_t\psi,q,\dot{q}) = 0,
$$

$$
\begin{aligned}
\frac{\partial f_0}{\partial q_\nu}(\psi,\widetilde{\widetilde{a}}_0,\partial_t\psi,q,\dot{q}) &= -\frac{1}{\sigma}\sum_{\mu=1}^{2N}\big(i\frac{\partial\varphi}{\partial q_\nu}(\cdot,q),\frac{\partial\varphi}{\partial q_\mu}(\cdot,q)\big)\dot{q}_\mu \\
&- \frac{1}{\sigma}\sum_{\mu=1}^{2N}\big(i\varphi(\cdot,q),\frac{\partial^2\varphi}{\partial q_\mu\partial q_\nu}(\cdot,q)\big)\dot{q}_\mu - \frac{1}{\sigma}\sum_{l=1}^{2}\frac{\partial\alpha_l}{\partial q_\nu}(\cdot,q)\big(\widetilde{\phi},\partial_l\frac{\partial\varphi}{\partial q_\nu}(\cdot,q)\big) \\
&+ \frac{1}{\sigma}\sum_{l=1}^{2}\big(i\widetilde{\phi},\partial_{ll}^2\frac{\partial\varphi}{\partial q_\nu}(\cdot,q)\big) - \frac{1}{\sigma}\big(\varphi(\cdot,q),\widetilde{\phi}\big)\sum_{l=1}^{2}\partial_l\frac{\partial\alpha_l}{\partial q_\nu}(\cdot,q) \\
&- \frac{1}{\sigma}\big(\widetilde{\phi},\frac{\partial\varphi}{\partial q_\nu}(\cdot,q)\big)\sum_{l=1}^{2}\partial_l\alpha_l(\cdot,q) - \frac{1}{\sigma}\sum_{l=1}^{2}\frac{\partial\alpha_l}{\partial q_\nu}(\cdot,q)\big(\partial_l\varphi(\cdot,q),\widetilde{\phi}\big) \\
&- \frac{1}{\sigma}\sum_{l=1}^{2}\alpha_l(\cdot,q)\big(\widetilde{\phi},\partial_l\frac{\partial\varphi}{\partial q_\nu}(\cdot,q),\big) + \frac{1}{\sigma}\sum_{l=1}^{2}\big(\partial_l D_l^{(0)}\widetilde{\phi},i\frac{\partial\varphi}{\partial q_\nu}(\cdot,q)\big) \\
&- \frac{2}{\sigma^2}\sum_{l=1}^{2}(\sigma\widetilde{a}_l)\big(\partial_l\varphi(\cdot,q),\frac{\partial\varphi}{\partial q_\nu}(\cdot,q)\big) - \frac{2}{\sigma^2}\sum_{l=1}^{2}(\sigma\widetilde{a}_l)\big(\varphi(\cdot,q),\partial_l\frac{\partial\varphi}{\partial q_\nu}(\cdot,q)\big) \\
&+ \frac{1}{\sigma}\sum_{l=1}^{2}\big(i\partial_l\widetilde{\phi},\partial_l\frac{\partial\varphi}{\partial q_\nu}(\cdot,q)\big) - \frac{1}{\sigma}\sum_{l=1}^{2}\frac{\partial\alpha_l}{\partial q_\nu}(\cdot,q)\big(\varphi(\cdot,q),\partial_l\widetilde{\phi}\big) \\
&- \frac{1}{\sigma}\sum_{l=1}^{2}\alpha_l(\cdot,q)\big(\partial_l\widetilde{\phi},\frac{\partial\varphi}{\partial q_\nu}(\cdot,q)\big) + \frac{2}{\sigma^2}\big(\varphi(\cdot,q),\frac{\partial\varphi}{\partial q_\nu}(\cdot,q)\big)\sum_{l=1}^{2}\partial_l(\sigma\widetilde{a}_l)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2\varepsilon}{\sigma^2}\sum_{l=1}^{2}(\sigma\widetilde{a}_l)\big(\widetilde{\phi},\partial_l\frac{\partial\varphi}{\partial q_\nu}(\cdot,q)\big)-\frac{2\varepsilon}{\sigma^2}\sum_{l=1}^{2}(\sigma\widetilde{a}_l)\big(\partial_l\widetilde{\phi},\frac{\partial\varphi}{\partial q_\nu}(\cdot,q)\big)\\
&-\frac{2\varepsilon}{\sigma}\sum_{l=1}^{2}\frac{\partial\alpha_l}{\partial q_\nu}(\cdot,q)\big(\widetilde{\phi},\partial_l\widetilde{\phi}\big)-\frac{\varepsilon}{\sigma}|\widetilde{\phi}|^2\sum_{l=1}^{2}\partial_l\frac{\partial\alpha_l}{\partial q_\nu}(\cdot,q)\\
&-\frac{2\varepsilon^2}{\sigma}\big(i\varphi(\cdot,q),\widetilde{\phi}\big)\big(\widetilde{\phi},\frac{\partial\varphi}{\partial q_\nu}(\cdot,q)\big)-\frac{2\varepsilon^2}{\sigma}\big(\varphi(\cdot,q),\widetilde{\phi}\big)\big(\widetilde{\phi},i\frac{\partial\varphi}{\partial q_\nu}(\cdot,q)\big)\\
&-\frac{\varepsilon^3}{\sigma}|\widetilde{\phi}|^2\big(\widetilde{\phi},i\frac{\partial\varphi}{\partial q_\nu}(\cdot,q)\big)
\end{aligned}
$$

and

$$
\frac{\partial f_0}{\partial\dot{q}_\nu}(\psi,\widetilde{\widetilde{a}}_0,\partial_t\psi,q,\dot{q})=-\frac{1}{\sigma}\big(i\varphi(\cdot,q),\frac{\partial\varphi}{\partial q_\nu}(\cdot,q)\big).
$$

While $\frac{\partial f_0}{\partial(\sigma\widetilde{a}_k)}$, $\frac{\partial f_0}{\partial\widetilde{\phi}}$, $\frac{\partial f_0}{\partial(\partial_t\psi)}$, $\frac{\partial f_0}{\partial q_\nu}$ and $\frac{\partial f_0}{\partial\dot{q}_\nu}$ are bounded in $H^1(\mathbb{R}^2)$ for $0\le t\le\overline{T}$, $\frac{\partial f_0}{\partial\widetilde{\widetilde{a}}_0}$ is unbounded since a constant function not zero is unbounded in $H^1(\mathbb{R}^2)$. Because of Cauchy's inequality a constant $C>0$ exists, independent of i, such that

$$
\begin{aligned}
&|\frac{\partial f_0}{\partial(\sigma\widetilde{a}_k)}(\psi,\widetilde{\widetilde{a}}_0,\partial_t\psi,q,\dot{q})|^2_{H^1(\mathbb{R}^2)}\le C|\big(\varphi(\cdot,q),\partial_k\varphi(\cdot,q)\big)|^2_{H^1(\mathbb{R}^2)}\\
&\quad+C|\big(\widetilde{\phi},\partial_k\varphi(\cdot,q)\big)|^2_{H^1(\mathbb{R}^2)}+C|\big(\varphi(\cdot,q),\partial_k\widetilde{\phi}\big)|^2_{H^1(\mathbb{R}^2)}+C|\big(\widetilde{\phi},\partial_k\widetilde{\phi}\big)|^2_{H^1(\mathbb{R}^2)}.
\end{aligned}
\tag{5.54}
$$

Because of Remark 3.1.1, since all the derivatives with respect to x of the static Higgs field are exponentially decaying as x tends to infinity, as it has been shown in section 3.1, and since $(\psi,\widetilde{\widetilde{a}}_0,q)\in X(\overline{T})$, the right-hand side of (5.54) is bounded for $0\le t\le\overline{T}$. Moreover, because of Cauchy's inequality, a constant $C>0$ exists, independent of i, such that

$$
|\frac{\partial f_0}{\partial\widetilde{\phi}}(\psi,\widetilde{\widetilde{a}}_0,\partial_t\psi,q,\dot{q})|^2_{H^1(\mathbb{R}^2)}\le C\sum_{l=1}^{2}|\alpha_l(\cdot,q)\big(\partial_l\varphi(\cdot,q),1\big)|^2_{H^1(\mathbb{R}^2)}
$$

$$+ C\sum_{l=1}^{2}|\big(i,\partial_l D_l^{(0)}\varphi(\cdot,q)\big)|^2_{H^1(\mathbb{R}^2)} + C\sum_{l=1}^{2}|\partial_l\alpha_l(\cdot,q)\big(\varphi(\cdot,q),1\big)|^2_{H^1(\mathbb{R}^2)}$$
$$+ C\sum_{l=1}^{2}|\sigma\widetilde{a}_l\big(i,\partial_l\varphi(\cdot,q)\big)|^2_{H^1(\mathbb{R}^2)} + C\sum_{l=1}^{2}|\alpha_l(\cdot,q)\big(\partial_l\widetilde{\phi},1\big)|^2_{H^1(\mathbb{R}^2)}$$
$$+ C\sum_{l=1}^{2}|\big(i,\partial_l D_l^{(0)}\widetilde{\phi}\big)|^2_{H^1(\mathbb{R}^2)} + C\sum_{l=1}^{2}|\partial_l\alpha_l(\cdot,q)\big(\widetilde{\phi},1\big)|^2_{H^1(\mathbb{R}^2)}$$
$$+ C|\big(\varphi(\cdot,q),1\big)\big(i\varphi(\cdot,q),\widetilde{\phi}\big)|^2_{H^1(\mathbb{R}^2)} + C|\big(i\varphi(\cdot,q),1\big)\big(\varphi(\cdot,q),\widetilde{\phi}\big)|^2_{H^1(\mathbb{R}^2)}$$
$$+ C\sum_{l=1}^{2}|\sigma\widetilde{a}_l\big(\partial_l\widetilde{\phi},1\big)|^2_{H^1(\mathbb{R}^2)} + C|\big(\widetilde{\phi},1\big)\big(i\varphi(\cdot,q),\widetilde{\phi}\big)|^2_{H^1(\mathbb{R}^2)}$$
$$+ C||\widetilde{\phi}|^2\big(i\varphi(\cdot,q),1\big)|^2_{H^1(\mathbb{R}^2)}. \tag{5.55}$$

Again because of Remark 3.1.1, since all the derivatives with respect to x of the static Higgs field and the static vector potential, all the derivatives with respect to x of the covariant derivative of the static Higgs field are exponentially decaying as x tends to infinity, as it has been shown in section 3.1, and since $(\psi,\widetilde{\widetilde{a}}_0,q)\in X(\overline{T})$, the right-hand side of (5.55) is bounded for $0\le t\le\overline{T}$. Unlike the previous partial derivatives of f_0,

$$|\frac{\partial f_0}{\partial\widetilde{\widetilde{a}}_0}(\psi,\widetilde{\widetilde{a}}_0,\partial_t\psi,q,\dot{q})|^2_{H^1(\mathbb{R}^2)} = |\varepsilon|^2_{H^1(\mathbb{R}^2)},$$

which is unbounded since a constant function not zero is not even square integrable. On the other hand, using Cauchy's inequality, one obtains that

$$|\frac{\partial f_0}{\partial(\partial_t\psi)}(\psi,\widetilde{\widetilde{a}}_0,\partial_t\psi,q,\dot{q})|^2_{H^1(\mathbb{R}^2)} = 0,$$

which is trivially bounded. Moreover, because of Cauchy's inequality, a constant $C>0$ exists, independent of i, such that

$$|\frac{\partial f_0}{\partial q_\nu}(\psi,\widetilde{\widetilde{a}}_0,\partial_t\psi,q,\dot{q})|^2_{H^1(\mathbb{R}^2)} \le C\sum_{\mu=1}^{2N}|\big(\frac{\partial\varphi}{\partial q_\mu}(\cdot,q),i\frac{\partial\varphi}{\partial q_\nu}(\cdot,q)\big)|^2_{H^1(\mathbb{R}^2)}$$

$$
\begin{aligned}
&+ C \sum_{\mu=1}^{2N} |\big(i\varphi(\cdot,q), \frac{\partial^2 \varphi}{\partial q_\mu \partial q_\nu}(\cdot,q)\big)|^2_{H^1(\mathbb{R}^2)} \\
&+ C \sum_{l=1}^{2} |\frac{\partial \alpha_l}{\partial q_\nu}(\cdot,q)\big(\widetilde{\phi}, \partial_l \frac{\partial \varphi}{\partial q_\nu}(\cdot,q)\big)|^2_{H^1(\mathbb{R}^2)} \\
&+ C \sum_{l=1}^{2} |\big(i\widetilde{\phi}, \partial^2_{ll} \frac{\partial \varphi}{\partial q_\nu}(\cdot,q)\big)|^2_{H^1(\mathbb{R}^2)} + C \sum_{l=1}^{2} |\partial_l \frac{\partial \alpha_l}{\partial q_\nu}(\cdot,q)\big(\varphi(\cdot,q), \widetilde{\phi}\big)|^2_{H^1(\mathbb{R}^2)} \\
&+ C \sum_{l=1}^{2} |\partial_l \alpha_l(\cdot,q)\big(\widetilde{\phi}, \frac{\partial \varphi}{\partial q_\nu}(\cdot,q)\big)|^2_{H^1(\mathbb{R}^2)} \\
&+ C \sum_{l=1}^{2} |\frac{\partial \alpha_l}{\partial q_\nu}(\cdot,q)\big(\partial_l \varphi(\cdot,q), \widetilde{\phi}\big)|^2_{H^1(\mathbb{R}^2)} \\
&+ C \sum_{l=1}^{2} |\alpha_l(\cdot,q)\big(\widetilde{\phi}, \partial_l \frac{\partial \varphi}{\partial q_\nu}(\cdot,q)\big)|^2_{H^1(\mathbb{R}^2)} \\
&+ C \sum_{l=1}^{2} |\big(\partial_l D_l^{(0)} \widetilde{\phi}, i \frac{\partial \varphi}{\partial q_\nu}(\cdot,q)\big)|^2_{H^1(\mathbb{R}^2)} \\
&+ C \sum_{l=1}^{2} |\sigma \widetilde{a}_l \big(\partial_l \varphi(\cdot,q), \frac{\partial \varphi}{\partial q_\nu}(\cdot,q)\big)|^2_{H^1(\mathbb{R}^2)} \\
&+ C \sum_{l=1}^{2} |\sigma \widetilde{a}_l \big(\varphi(\cdot,q), \partial_l \frac{\partial \varphi}{\partial q_\nu}(\cdot,q)\big)|^2_{H^1(\mathbb{R}^2)} \\
&+ C \sum_{l=1}^{2} |\big(i\partial_l \widetilde{\phi}, \partial_l \frac{\partial \varphi}{\partial q_\nu}(\cdot,q)\big)|^2_{H^1(\mathbb{R}^2)} \\
&+ C \sum_{l=1}^{2} |\frac{\partial \alpha_l}{\partial q_\nu}(\cdot,q)\big(\varphi(\cdot,q), \partial_l \widetilde{\phi}\big)|^2_{H^1(\mathbb{R}^2)} \\
&+ C \sum_{l=1}^{2} |\alpha_l(\cdot,q)\big(\partial_l \widetilde{\phi}, \frac{\partial \varphi}{\partial q_\nu}(\cdot,q)\big)|^2_{H^1(\mathbb{R}^2)} \\
&+ C \sum_{l=1}^{2} |\partial_l(\sigma \widetilde{a}_l)\big(\varphi(\cdot,q), \frac{\partial \varphi}{\partial q_\nu}(\cdot,q)\big)|^2_{H^1(\mathbb{R}^2)}
\end{aligned}
$$

$$+ C \sum_{l=1}^{2} |\sigma \widetilde{a}_l (\widetilde{\phi}, \partial_l \frac{\partial \varphi}{\partial q_\nu}(\cdot, q))|^2_{H^1(\mathbb{R}^2)} + C \sum_{l=1}^{2} |\sigma \widetilde{a}_l (\partial_l \widetilde{\phi}, \frac{\partial \varphi}{\partial q_\nu}(\cdot, q))|^2_{H^1(\mathbb{R}^2)}$$

$$+ C \sum_{l=1}^{2} |\frac{\partial \alpha_l}{\partial q_\nu}(\cdot, q)(\widetilde{\phi}, \partial_l \widetilde{\phi})|^2_{H^1(\mathbb{R}^2)} + C \sum_{l=1}^{2} ||\widetilde{\phi}|^2 \partial_l \frac{\partial \alpha_l}{\partial q_\nu}(\cdot, q)|^2_{H^1(\mathbb{R}^2)}$$

$$+ C|(\widetilde{\phi}, \frac{\partial \varphi}{\partial q_\nu}(\cdot, q))(i\varphi(\cdot, q), \widetilde{\phi})|^2_{H^1(\mathbb{R}^2)} + C|(\varphi(\cdot, q), \widetilde{\phi})(\widetilde{\phi}, i\frac{\partial \varphi}{\partial q_\nu}(\cdot, q))|^2_{H^1(\mathbb{R}^2)}$$

$$+ C||\widetilde{\phi}|^2(\widetilde{\phi}, i\frac{\partial \varphi}{\partial q_\nu}(\cdot, q))|^2_{H^1(\mathbb{R}^2)}. \quad (5.56)$$

We use again Remark 3.1.1, the fact that all the partial derivatives of the zero modes with respect to q and x are exponentially decaying as x tends to infinity and $(\psi, \widetilde{\widetilde{a}}_0, q) \in X(\overline{T})$. It follows that the right-hand side of (5.56) is bounded for $0 \leq t \leq \overline{T}$. Finally, because of Cauchy's inequality, a constant $C > 0$ exists, independent of i, such that

$$|\frac{\partial f_0}{\partial \dot{q}_\nu}(\psi, \widetilde{\widetilde{a}}_0, \partial_t \psi, q, \dot{q})|^2_{H^1(\mathbb{R}^2)} \leq C|(i\varphi(\cdot, q), \frac{\partial \varphi}{\partial q_\nu}(\cdot, q))|^2_{H^1(\mathbb{R}^2)}. \quad (5.57)$$

Because of Remark 3.1.1 and since the zero modes are exponentially decaying as x tends to infinity, the right-hand side of (5.57) is bounded for $0 \leq t \leq \overline{T}$.

Subtracting from $f_0(\psi, \widetilde{\widetilde{a}}_0, \partial_t \psi, q, \dot{q})$ the component producing the unboundedness, it is possible to apply the mean value theorem to the remaining function. This component is $\varepsilon \widetilde{\widetilde{a}}_0$. Then, because of Cauchy's inequality, a constant $C > 0$ exists, independent of i, such that

$$|f_0(i, \cdot, t) - f_0(i-1, \cdot, t)|^2_{H^1(\mathbb{R}^2)} \leq C|\frac{1}{\sigma} \sum_{\mu=1}^{2N} (i\varphi(\cdot, q^{(i)}(t)), \frac{\partial \varphi}{\partial q_\mu}(\cdot, q^{(i)}(t)))\dot{q}^{(i)}_\mu(t)$$

$$- \frac{1}{\sigma} \sum_{l=1}^{2} (i\partial_l \varphi(\cdot, q^{(i)}(t)), D^{(0)}_l \widetilde{\phi}^{(i)}(\cdot, t))$$

$$
\begin{aligned}
&- \frac{1}{\sigma}\sum_{l=1}^{2}\big(i\widetilde{\phi}^{(i)}(\cdot,t), \partial_l D_l^{(0)}\varphi(\cdot,q^{(i)}(t))\big)\\
&- \frac{1}{\sigma}\sum_{l=1}^{2}\big(i\varphi(\cdot,q^{(i)}(t)), \partial_l D_l^{(0)}\widetilde{\phi}^{(i)}(\cdot,t)\big)\\
&+ \frac{2}{\sigma^2}\sum_{l=1}^{2}\sigma\widetilde{a}_l^{(i)}(\cdot,t)\big(\varphi(\cdot,q^{(i)}(t)), \partial_l\varphi(\cdot,q^{(i)}(t))\big)\\
&- \frac{1}{\sigma}\sum_{l=1}^{2}\big(i\partial_l\widetilde{\phi}^{(i)}(\cdot,t), D_l^{(0)}\varphi(\cdot,q^{(i)}(t))\big) - \frac{1}{\sigma}\sum_{l=1}^{2}\partial_l\partial_t\big(\sigma\widetilde{a}_l^{(i)}(\cdot,t)\big)\\
&- \frac{1}{\sigma^2}|\phi(\cdot,q^{(i)}(t))|^2\sum_{l=1}^{2}\partial_l\big(\sigma\widetilde{a}_l^{(i)}(\cdot,t)\big)\\
&- \varepsilon\Big(\frac{2}{\sigma^2}\sum_{l=1}^{2}\sigma\widetilde{a}_l^{(i)}(\cdot,t)\big(\widetilde{\phi}^{(i)}(\cdot,t), \partial_l\varphi(\cdot,q^{(i)}(t))\big)\\
&+ \frac{2}{\sigma^2}\sum_{l=1}^{2}\sigma\widetilde{a}_l^{(i)}(\cdot,t)\big(\varphi(\cdot,q^{(i)}(t)), \partial_l\widetilde{\phi}^{(i)}(\cdot,t)\big)\\
&- \frac{1}{\sigma}\sum_{l=1}^{2}\big(i\partial_l\widetilde{\phi}^{(i)}(\cdot,t), D_l^{(0)}\widetilde{\phi}^{(i)}(\cdot,t)\big) - \frac{1}{\sigma}\sum_{l=1}^{2}\big(i\widetilde{\phi}^{(i)}(\cdot,t), \partial_l D_l^{(0)}\widetilde{\phi}^{(i)}(\cdot,t)\big)\Big)\\
&- \Big(\frac{2\varepsilon^2}{\sigma}\big(\varphi(\cdot,q^{(i)}(t)), \widetilde{\phi}^{(i)}(\cdot,t)\big)\big(i\varphi(\cdot,q^{(i)}(t)), \widetilde{\phi}^{(i)}(\cdot,t)\big)\\
&+ \frac{2}{\sigma^2}\sum_{l=1}^{2}\sigma\widetilde{a}_l^{(i)}(\cdot,t)\big(\widetilde{\phi}^{(i)}(\cdot,t), \partial_l\widetilde{\phi}^{(i)}(\cdot,t)\big)\Big)\\
&+ \frac{\varepsilon^3}{\sigma}|\widetilde{\phi}^{(i)}(\cdot,t)|^2\big(i\varphi(\cdot,q^{(i)}(t)), \widetilde{\phi}^{(i)}(\cdot,t)\big)\\
&- \frac{1}{\sigma}\sum_{\mu=1}^{2N}\big(i\varphi(\cdot,q^{(i-1)}(t)), \frac{\partial\varphi}{\partial q_\mu}(\cdot,q^{(i-1)}(t))\big)\dot{q}_\mu^{(i-1)}(t)\\
&+ \frac{1}{\sigma}\big(i\partial_l\varphi(\cdot,q^{(i-1)}(t)), D_l^{(0)}\widetilde{\phi}^{(i-1)}(\cdot,t)\big) + \frac{1}{\sigma}\big(i\widetilde{\phi}^{(i)}(\cdot,t), \partial_l D_l^{(0)}\varphi(\cdot,q^{(i)}(t))\big)\\
&+ \frac{1}{\sigma}\big(i\varphi(\cdot,q^{(i-1)}(t)), \partial_l D_l^{(0)}\widetilde{\phi}^{(i-1)}(\cdot,t)\big)|\\
&- \frac{2}{\sigma^2}\sigma\widetilde{a}_l^{(i-1)}(\cdot,t)\big(\varphi(\cdot,q^{(i-1)}(t)), \partial_l\varphi(\cdot,q^{(i-1)}(t))\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\sigma}\sum_{l=1}^{2}\big(i\partial_l\widetilde{\phi}^{(i-1)}(\cdot,t),D_l^{(0)}\varphi(\cdot,q^{(i-1)}(t))\big)\\
&+\frac{1}{\sigma}\sum_{l=1}^{2}\partial_l\partial_t\big(\sigma\widetilde{a}_l^{(i-1)}(\cdot,t)\big)+\frac{1}{\sigma^2}|\phi(\cdot,q^{(i-1)}(t))|^2\sum_{l=1}^{2}\partial_l\big(\sigma\widetilde{a}_l^{(i-1)}(\cdot,t)\big)\\
&+\varepsilon\Big(\frac{2}{\sigma^2}\sum_{l=1}^{2}\sigma\widetilde{a}_l^{(i-1)}(\cdot,t)\big(\widetilde{\phi}^{(i-1)}(\cdot,t),\partial_l\varphi(\cdot,q^{(i-1)}(t))\big)\\
&+\frac{2}{\sigma^2}\sum_{l=1}^{2}\sigma\widetilde{a}_l^{(i-1)}(\cdot,t)\big(\varphi(\cdot,q^{(i-1)}(t)),\partial_l\widetilde{\phi}^{(i-1)}(\cdot,t)\big)\\
&-\frac{1}{\sigma}\sum_{l=1}^{2}\big(i\partial_l\widetilde{\phi}^{(i-1)}(\cdot,t),D_l^{(0)}\widetilde{\phi}^{(i-1)}(\cdot,t)\big)\\
&-\frac{1}{\sigma}\sum_{l=1}^{2}\big(i\widetilde{\phi}^{(i-1)}(\cdot,t),\partial_l D_l^{(0)}\widetilde{\phi}^{(i-1)}(\cdot,t)\big)\Big)\\
&+\varepsilon^2\Big(\frac{2}{\sigma}\big(\varphi(\cdot,q^{(i-1)}(t)),\widetilde{\phi}^{(i-1)}(\cdot,t)\big)\big(i\varphi(\cdot,q^{(i-1)}(t)),\widetilde{\phi}^{(i-1)}(\cdot,t)\big)\\
&+\frac{2}{\sigma^2}\sum_{l=1}^{2}\sigma\widetilde{a}_l^{(i-1)}(\cdot,t)\big(\widetilde{\phi}^{(i-1)}(\cdot,t),\partial_l\widetilde{\phi}^{(i-1)}(\cdot,t)\big)\Big)\\
&-\frac{\varepsilon^3}{\sigma}|\widetilde{\phi}^{(i-1)}(\cdot,t)|^2\big(i\varphi(\cdot,q^{(i-1)}(t)),\widetilde{\phi}^{(i-1)}(\cdot,t)\big)|^2_{H^1(\mathbb{R}^2)}\\
&+C|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)|^2_{H^1(\mathbb{R}^2)}. \qquad (5.58)
\end{aligned}
$$

Applying the mean value theorem to all the terms bar the last one on the right-hand side of (5.58) it turns out that their norms are smaller than

$$
\begin{aligned}
&C(\Gamma)\big\|\psi^{(i)}(\cdot,t)-\psi^{(i-1)}(\cdot,t)\big\|\\
&\quad+\varepsilon C(\Gamma)\max_{0\le s\le t}|\partial_t\psi^{(i)}(\cdot,s)-\partial_t\psi^{(i-1)}(\cdot,s)|_{1,\alpha(\cdot,q(0))}\\
&\quad+C(\Gamma)\max_{0\le s\le t}|q^{(i)}(s)-q^{(i-1)}(s)|+C(\Gamma)|\dot{q}^{(i)}(t)-\dot{q}^{(i-1)}(t)|
\end{aligned}
\qquad (5.59)
$$

for some constant $C(\Gamma) > 0$ independent of i, for $0 \leq t \leq \overline{T}$. Furthermore, for the last term on the right-hand side of (5.58) it follows easily that

$$|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)|^2_{H^1(\mathbb{R}^2)} \leq C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)\big\| \tag{5.60}$$

for some constant $C(\Gamma) > 0$ independent of i.

Using (5.59) and (5.60) to estimate the right-hand side of (5.58), one obtains

$$\begin{aligned}
|f_0(i,\cdot,t) - f_0(i-1,\cdot,t)|^2_{H^1(\mathbb{R}^2)} &\leq C(\Gamma)\big\|\psi^{(i)}(\cdot,t) - \psi^{(i-1)}(\cdot,t)\big\| \\
&+ \varepsilon C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)\big\| \\
&+ \varepsilon C(\Gamma)\max_{0\leq s\leq t}|\partial_t\psi^{(i)}(\cdot,s) - \partial_t\psi^{(i-1)}(\cdot,s)|_{1,\alpha(\cdot,q(0))} \\
&+ C(\Gamma)\max_{0\leq s\leq t}|q^{(i)}(s) - q^{(i-1)}(s)| + C(\Gamma)\max_{0\leq s\leq t}|q^{(i-1)}(s) - q^{(i-2)}(s)|
\end{aligned} \tag{5.61}$$

for some constant $C(\Gamma) > 0$ independent of i, for $0 \leq t \leq \overline{T}$.

Now, substituting (5.52), (5.53) and (5.61) into (5.51), it turns out that

$$\begin{aligned}
\big\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\big\| &\leq C(\Gamma)\frac{\delta}{2^{i-1}} \\
&+ C(\Gamma)\int_0^t \Big(\big\|\psi^{(i-1)}(\cdot,s) - \psi^{(i-2)}(\cdot,s)\big\| + \big\|\psi^{(i-2)}(\cdot,s) - \psi^{(i-3)}(\cdot,s)\big\| \\
&+ \big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s) - \widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,s)\big\| + \big\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,s) - \widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,s)\big\| \\
&+ \max_{0\leq\tau\leq s}|q^{(i-1)}(\tau) - q^{(i-2)}(\tau)| + \max_{0\leq\tau\leq s}|q^{(i-2)}(\tau) - q^{(i-3)}(\tau)|\Big)ds \\
&+ \varepsilon C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)\big\| \\
&+ \varepsilon C(\Gamma)\max_{0\leq s\leq t}|\partial_t\psi^{(i)}(\cdot,s) - \partial_t\psi^{(i-1)}(\cdot,s)|_{1,\alpha(\cdot,q(0))}
\end{aligned}$$

$$
\begin{aligned}
&+ C(\Gamma)\frac{\delta}{2^i} + \varepsilon t C(\Gamma)\big\|\psi^{(i-1)}(\cdot,t) - \psi^{(i-2)}(\cdot,t)\big\| \\
&+ \varepsilon t C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)\big\| \\
&+ \varepsilon t C(\Gamma)\max_{0\le s\le t}|q^{(i-1)}(s) - q^{(i-2)}(s)| + C(\Gamma)\frac{\delta}{2^{i-1}} \\
&+ \varepsilon t C(\Gamma)\big\|\psi^{(i-2)}(\cdot,t) - \psi^{(i-3)}(\cdot,t)\big\| \\
&+ \varepsilon t C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,t)\big\| \\
&+ \varepsilon t C(\Gamma)\max_{0\le s\le t}|q^{(i-2)}(s) - q^{(i-3)}(s)| \qquad (5.62)
\end{aligned}
$$

for some constant $C(\Gamma) > 0$ independent of i, for $0 \le t \le \overline{T}$.

5.6 An Estimate for $\partial_t\psi^{(i+1)} - \partial_t\psi^{(i)}$

An estimate for $|\partial_t\psi^{(i+1)} - \partial_t\psi^{(i)}|$ in terms of $\big\|\psi^{(i+1)} - \psi^{(i)}\big\|$, $\big\|\psi^{(i)} - \psi^{(i-1)}\big\|$, $\big\|\widetilde{\widetilde{a}}_0^{(i+1)} - \widetilde{\widetilde{a}}_0^{(i)}\big\|$, $|q^{(i)} - q^{(i-1)}|$, $\big\|\psi^{(i-1)} - \psi^{(i-2)}\big\|$, $\big\|\widetilde{\widetilde{a}}_0^{(i)} - \widetilde{\widetilde{a}}_0^{(i-1)}\big\|$ and $|q^{(i-1)} - q^{(i-2)}|$ is presented here. Again the mean value theorem is the main tool of the proof. As usual, the real part of $\partial_t\psi^{(i+1)} - \partial_t\psi^{(i)}$ is treated separately from its complex part.

For the real part of (5.50), by considering the norm in $H^1(\mathbb{R}^2)$ and using Cauchy's inequality, one obtains

$$
\begin{aligned}
|\partial_t\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)\big) - \partial_t\big(\sigma\widetilde{a}_h^{(i)}(\cdot,t)\big)|^2_{H^1(\mathbb{R}^2)} &= |\Delta\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t) - \sigma\widetilde{a}_h^{(i)}(\cdot,t)\big) \\
&- |\phi(\cdot,q^{(i)}(t))|^2\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t) - \sigma\widetilde{a}_h^{(i)}(\cdot,t)\big) \\
&- \sigma\widetilde{a}_h^{(i)}(\cdot,t)\big(|\phi(\cdot,q^{(i)}(t))|^2 - |\phi(\cdot,q^{(i-1)}(t))|^2\big) \\
&+ f_{1h}^1(i,\cdot,t) - f_{1h}^1(i-1,\cdot,t)|^2_{H^1(\mathbb{R}^2)}
\end{aligned}
$$

$$\leq C|\Delta\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t) - \sigma\widetilde{a}_h^{(i)}(\cdot,t)\big)|^2_{H^1(\mathbb{R}^2)}$$
$$+ C||\phi(\cdot,q^{(i)}(t))|^2\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t) - \sigma\widetilde{a}_h^{(i)}(\cdot,t)\big)|^2_{H^1(\mathbb{R}^2)}$$
$$+ C|\sigma\widetilde{a}_h^{(i)}(\cdot,t)\big(\varphi(\cdot,q^{(i)}(t)) - \varphi(\cdot,q^{(i-1)}(t)),\varphi(\cdot,q^{(i)}(t))\big)|^2_{H^1(\mathbb{R}^2)}$$
$$+ C|\sigma\widetilde{a}_h^{(i)}(\cdot,t)\big(\varphi(\cdot,q^{(i)}(t)) - \varphi(\cdot,q^{(i-1)}(t)),\varphi(\cdot,q^{(i-1)}(t))\big)|^2_{H^1(\mathbb{R}^2)}$$
$$+ C|f^1_{1h}(i,\cdot,t) - f^1_{1h}(i-1,\cdot,t)|^2_{H^1(\mathbb{R}^2)} \tag{5.63}$$

for some constant $C > 0$, independent of i. Using Remark 3.1.1, from (5.63) we obtain

$$|\Delta\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t) - \sigma\widetilde{a}_h^{(i)}(\cdot,t)\big)|^2_{H^1(\mathbb{R}^2)}$$
$$+ ||\phi(\cdot,q^{(i)}(t))|^2\big(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t) - \sigma\widetilde{a}_h^{(i)}(\cdot,t)\big)|^2_{H^1(\mathbb{R}^2)}$$
$$\leq C\big\|\psi^{(i+1)}(\cdot,t) - \psi^{(i)}(\cdot,t)\big\|. \tag{5.64}$$

As we saw in (5.5), (5.6), (5.15) and (5.45), for the remaining norms on the right-hand side of (5.63) the following estimate holds for $0 \leq t \leq \overline{T}$:

$$C|\sigma\widetilde{a}_h^{(i)}(\cdot,t)\big(\varphi(\cdot,q^{(i)}(t)) - \varphi(\cdot,q^{(i-1)}(t)),\varphi(\cdot,q^{(i)}(t))\big)|^2_{H^1(\mathbb{R}^2)}$$
$$+ C|\sigma\widetilde{a}_h^{(i)}(\cdot,t)\big(\varphi(\cdot,q^{(i)}(t)) - \varphi(\cdot,q^{(i-1)}(t)),\varphi(\cdot,q^{(i-1)}(t))\big)|^2_{H^1(\mathbb{R}^2)}$$
$$+ C|f^1_{1h}(i,\cdot,t) - f^1_{1h}(i-1,\cdot,t)|^2_{H^1(\mathbb{R}^2)}$$

$$\leq C(\Gamma)\big\|\psi^{(i)}(\cdot,t) - \psi^{(i-1)}(\cdot,t)\big\| + C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\big\|$$
$$+ C(\Gamma)\max_{0\leq s\leq t}|q^{(i)}(s) - q^{(i-1)}(s)| + C(\Gamma)\big\|\psi^{(i-1)}(\cdot,t) - \psi^{(i-2)}(\cdot,t)\big\|$$
$$+ C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)\big\| + C(\Gamma)\max_{0\leq s\leq t}|q^{(i-1)}(s) - q^{(i-2)}(s)| \tag{5.65}$$

for some constant $C(\Gamma) > 0$, independent of i, and then, from (5.63), using (5.64) and (5.65), one obtains

$$\begin{aligned}
&|\partial_t(\sigma\widetilde{a}_h^{(i+1)}(\cdot,t)) - \partial_t(\sigma\widetilde{a}_h^{(i)}(\cdot,t))|^2_{H^1(\mathbb{R}^2)} \\
&\quad \leq C(\Gamma)\|\psi^{(i+1)}(\cdot,t) - \psi^{(i)}(\cdot,t)\| + C(\Gamma)\|\psi^{(i)}(\cdot,t) - \psi^{(i-1)}(\cdot,t)\| \\
&\quad + C(\Gamma)\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\| + C(\Gamma)\max_{0\leq s\leq t}|q^{(i)}(s) - q^{(i-1)}(s)| \\
&\quad + C(\Gamma)\|\psi^{(i-1)}(\cdot,t) - \psi^{(i-2)}(\cdot,t)\| + C(\Gamma)\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)\| \\
&\quad + C(\Gamma)\max_{0\leq s\leq t}|q^{(i-1)}(s) - q^{(i-2)}(s)| \qquad (5.66)
\end{aligned}$$

for some constant $C(\Gamma) > 0$ independent of i, for $0 \leq t \leq \overline{T}$.

For the complex part, proceeding in a similar way, one finds a constant $C(\Gamma) > 0$, independent of i, such that

$$\begin{aligned}
&|\partial_t\widetilde{\phi}^{(i+1)}(\cdot,t) - \partial_t\widetilde{\phi}^{(i)}(\cdot,t)|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \sum_{j=1}^{2}|D_j^{(\alpha(\cdot,q(0)))}(\partial_t\widetilde{\phi}^{(i+1)}(\cdot,t) - \partial_t\widetilde{\phi}^{(i)}(\cdot,t))|^2_{L^2(\mathbb{R}^2)} \\
&\quad \leq C(\Gamma)\|\psi^{(i+1)}(\cdot,t) - \psi^{(i)}(\cdot,t)\| + C(\Gamma)\|\psi^{(i)}(\cdot,t) - \psi^{(i-1)}(\cdot,t)\| \\
&\quad + C(\Gamma)\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\| + C(\Gamma)\max_{0\leq s\leq t}|q^{(i)}(s) - q^{(i-1)}(s)| \\
&\quad + C(\Gamma)\|\psi^{(i-1)}(\cdot,t) - \psi^{(i-2)}(\cdot,t)\| + C(\Gamma)\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)\| \\
&\quad + C(\Gamma)\max_{0\leq s\leq t}|q^{(i-1)}(s) - q^{(i-2)}(s)| \qquad (5.67)
\end{aligned}$$

for $0 \leq t \leq \overline{T}$. Thus, adding (5.66) to (5.67) it turns out that

$$\max_{0\leq s\leq t}|\partial_t\psi^{(i+1)}(\cdot,s) - \partial_t\psi^{(i)}(\cdot,s)|_{1,\alpha(\cdot,q(0))} \leq C(\Gamma)\|\psi^{(i+1)}(\cdot,t) - \psi^{(i)}(\cdot,t)\|$$

$$+ C(\Gamma)\big\|\psi^{(i)}(\cdot,t) - \psi^{(i-1)}(\cdot,t)\big\| + C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\big\|$$
$$+ C(\Gamma)\max_{0\le s\le t}|q^{(i)}(s) - q^{(i-1)}(s)| + C(\Gamma)\big\|\psi^{(i-1)}(\cdot,t) - \psi^{(i-2)}(\cdot,t)\big\|$$
$$+ C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)\big\| + C(\Gamma)\max_{0\le s\le t}|q^{(i-1)}(s) - q^{(i-2)}(s)| \tag{5.68}$$

for $0 \le t \le \overline{T}$.

By replacing i with $i-1$ and $i-2$ in (5.46) we get, respectively,

$$\big\|\psi^{(i)}(\cdot,t) - \psi^{(i-1)}(\cdot,t)\big\| \le C(\Gamma)\frac{\delta}{2^{i-1}}$$
$$+ C(\Gamma)\int_0^t \Big(\big\|\psi^{(i-1)}(\cdot,s) - \psi^{(i-2)}(\cdot,s)\big\| + \big\|\psi^{(i-2)}(\cdot,s) - \psi^{(i-3)}(\cdot,s)\big\|$$
$$+ \big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s) - \widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,s)\big\| + \big\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,s) - \widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,s)\big\|$$
$$+ \max_{0\le\tau\le s}|q^{(i-1)}(\tau) - q^{(i-2)}(\tau)| + \max_{0\le\tau\le s}|q^{(i-2)}(\tau) - q^{(i-3)}(\tau)|\Big)ds \tag{5.69}$$

and

$$\big\|\psi^{(i-1)}(\cdot,t) - \psi^{(i-2)}(\cdot,t)\big\| \le C(\Gamma)\frac{\delta}{2^{i-2}}$$
$$+ C(\Gamma)\int_0^t \Big(\big\|\psi^{(i-2)}(\cdot,s) - \psi^{(i-2)}(\cdot,s)\big\| + \big\|\psi^{(i-3)}(\cdot,s) - \psi^{(i-4)}(\cdot,s)\big\|$$
$$+ \big\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,s) - \widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,s)\big\| + \big\|\widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,s) - \widetilde{\widetilde{a}}_0^{(i-3)}(\cdot,s)\big\|$$
$$+ \max_{0\le\tau\le s}|q^{(i-2)}(\tau) - q^{(i-3)}(\tau)| + \max_{0\le\tau\le s}|q^{(i-3)}(\tau) - q^{(i-4)}(\tau)|\Big)ds \tag{5.70}$$

for some constant $C(\Gamma) > 0$ independent of i, for $0 \le t \le \overline{T}$.

By replacing i with $i-1$ and $i-2$ in (5.62), one obtains

$$
\begin{aligned}
&\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)\| \le C(\Gamma)\frac{\delta}{2^{i-2}} \\
&\quad + C(\Gamma)\int_0^t \big(\|\psi^{(i-2)}(\cdot,s)-\psi^{(i-3)}(\cdot,s)\| + \|\psi^{(i-3)}(\cdot,s)-\psi^{(i-4)}(\cdot,s)\| \\
&\quad + \|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,s)-\widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,s)\| + \|\widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,s)-\widetilde{\widetilde{a}}_0^{(i-3)}(\cdot,s)\| \\
&\quad + \max_{0\le\tau\le s}|q^{(i-2)}(\tau)-q^{(i-3)}(\tau)| + \max_{0\le\tau\le s}|q^{(i-3)}(\tau)-q^{(i-4)}(\tau)|\big)ds \\
&\quad + \varepsilon C(\Gamma)\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,t)\| \\
&\quad + \varepsilon C(\Gamma)\max_{0\le s\le t}|\partial_t\psi^{(i-1)}(\cdot,s)-\partial_t\psi^{(i-2)}(\cdot,s)|_{1,\alpha(\cdot,q(0))} \\
&\quad + C(\Gamma)\frac{\delta}{2^{i-1}} + \varepsilon t C(\Gamma)\|\psi^{(i-2)}(\cdot,t)-\psi^{(i-3)}(\cdot,t)\| \\
&\quad + \varepsilon t C(\Gamma)\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,t)\| \\
&\quad + \varepsilon t C(\Gamma)\max_{0\le s\le t}|q^{(i-2)}(s)-q^{(i-3)}(s)| + C(\Gamma)\frac{\delta}{2^{i-2}} \\
&\quad + \varepsilon t C(\Gamma)\|\psi^{(i-3)}(\cdot,t)-\psi^{(i-4)}(\cdot,t)\| \\
&\quad + \varepsilon t C(\Gamma)\|\widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i-3)}(\cdot,t)\| \\
&\quad + \varepsilon t C(\Gamma)\max_{0\le s\le t}|q^{(i-3)}(s)-q^{(i-4)}(s)| \qquad (5.71)
\end{aligned}
$$

for some constant $C(\Gamma)>0$ independent of i, for $0\le t\le \overline{T}$.

Finally, by replacing i with $i-1$ and $i-2$ in (5.49) one obtains respectively,

$$
\begin{aligned}
|q^{(i)}(t)-q^{(i-1)}(t)| &\le C(\Gamma)\frac{\delta}{2^i} + \varepsilon C(\Gamma)\int_0^t \big(\|\psi^{(i-1)}(\cdot,s)-\psi^{(i-2)}(\cdot,s)\| \\
&\quad + \|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)-\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,s)\| + \max_{0\le\tau\le s}|q^{(i-1)}(\tau)-q^{(i-2)}(\tau)|\big)ds
\end{aligned}
\qquad (5.72)
$$

and

$$|q^{(i-1)}(t) - q^{(i-2)}(t)| \leq C(\Gamma)\frac{\delta}{2^{i-1}} + \varepsilon C(\Gamma)\int_0^t \Big(\big\|\psi^{(i-2)}(\cdot,s) - \psi^{(i-3)}(\cdot,s)\big\|$$
$$+ \big\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,s) - \widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,s)\big\| + \max_{0\leq\tau\leq s}|q^{(i-2)}(\tau) - q^{(i-3)}(\tau)|\Big)ds \tag{5.73}$$

for some constant $C(\Gamma) > 0$ independent of i, for $0 \leq t \leq \overline{T}$.

Now, using (5.46), (5.69), (5.70), (5.62), (5.71), (5.72) and (5.73) in (5.68) one obtains

$$\max_{0\leq s\leq t}|\partial_t\psi^{(i+1)}(\cdot,s) - \partial_t\psi^{(i)}(\cdot,s)|_{1,\alpha(\cdot,q(0))} \leq C(\Gamma)\frac{\delta}{2^i}$$
$$+ C(\Gamma)\int_0^t \Big(\big\|\psi^{(i)}(\cdot,s) - \psi^{(i-1)}(\cdot,s)\big\| + \big\|\psi^{(i-1)}(\cdot,s) - \psi^{(i-2)}(\cdot,s)\big\|$$
$$+ \big\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,s) - \widetilde{\widetilde{a}}_0^{(i)}(\cdot,s)\big\| + \big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s) - \widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,s)\big\|$$
$$+ \max_{0\leq\tau\leq s}|q^{(i)}(\tau) - q^{(i-1)}(\tau)| + \max_{0\leq\tau\leq s}|q^{(i-1)}(\tau) - q^{(i-2)}(\tau)|\Big)ds$$
$$+ C(\Gamma)\frac{\delta}{2^{i-1}} + C(\Gamma)\int_0^t \Big(\big\|\psi^{(i-1)}(\cdot,s) - \psi^{(i-2)}(\cdot,s)\big\|$$
$$+ \big\|\psi^{(i-2)}(\cdot,s) - \psi^{(i-3)}(\cdot,s)\big\| + \big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s) - \widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,s)\big\|$$
$$+ \big\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,s) - \widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,s)\big\| + \max_{0\leq\tau\leq s}|q^{(i-1)}(\tau) - q^{(i-2)}(\tau)|$$
$$+ \max_{0\leq\tau\leq s}|q^{(i-2)}(\tau) - q^{(i-3)}(\tau)|\Big)ds + C(\Gamma)\frac{\delta}{2^{i-1}}$$
$$+ C(\Gamma)\int_0^t \Big(\big\|\psi^{(i-1)}(\cdot,s) - \psi^{(i-2)}(\cdot,s)\big\| + \big\|\psi^{(i-2)}(\cdot,s) - \psi^{(i-3)}(\cdot,s)\big\|$$
$$+ \big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s) - \widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,s)\big\| + \big\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,s) - \widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,s)\big\|$$
$$+ \max_{0\leq\tau\leq s}|q^{(i-1)}(\tau) - q^{(i-2)}(\tau)| + \max_{0\leq\tau\leq s}|q^{(i-2)}(\tau) - q^{(i-3)}(\tau)|\Big)ds$$

$$
\begin{aligned}
&+ \varepsilon C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)\big\| \\
&+ \varepsilon C(\Gamma) \max_{0\le s\le t} |\partial_t\psi^{(i)}(\cdot,s) - \partial_t\psi^{(i-1)}(\cdot,s)|_{1,\alpha(\cdot,q(0))} \\
&+ C(\Gamma)\frac{\delta}{2^i} + \varepsilon t C(\Gamma)\big\|\psi^{(i-1)}(\cdot,t) - \psi^{(i-2)}(\cdot,t)\big\| \\
&+ \varepsilon t C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)\big\| + \varepsilon t C(\Gamma) \max_{0\le s\le t} |q^{(i-1)}(s) - q^{(i-2)}(s)| \\
&+ C(\Gamma)\frac{\delta}{2^{i-1}} + \varepsilon t C(\Gamma)\big\|\psi^{(i-2)}(\cdot,t) - \psi^{(i-3)}(\cdot,t)\big\| \\
&+ \varepsilon t C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,t)\big\| \\
&+ \varepsilon t C(\Gamma) \max_{0\le s\le t} |q^{(i-2)}(s) - q^{(i-3)}(s)| \\
&+ C(\Gamma)\frac{\delta}{2^i} + \varepsilon C(\Gamma)\int_0^t \Big(\big\|\psi^{(i-1)}(\cdot,s) - \psi^{(i-2)}(\cdot,s)\big\| \\
&+ \big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,s) - \widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,s)\big\| + \max_{0\le\tau\le s} |q^{(i-1)}(\tau) - q^{(i-2)}(\tau)|\Big) ds \\
&+ C(\Gamma)\frac{\delta}{2^{i-2}} + C(\Gamma)\int_0^t \Big(\big\|\psi^{(i-1)}(\cdot,s) - \psi^{(i-2)}(\cdot,s)\big\| \\
&+ \big\|\psi^{(i-3)}(\cdot,s) - \psi^{(i-4)}(\cdot,s)\big\| + \big\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,s) - \widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,s)\big\| \\
&+ \big\|\widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,s) - \widetilde{\widetilde{a}}_0^{(i-3)}(\cdot,s)\big\| + \max_{0\le\tau\le s} |q^{(i-2)}(\tau) - q^{(i-3)}(\tau)| \\
&+ \max_{0\le\tau\le s} |q^{(i-3)}(\tau) - q^{(i-4)}(\tau)|\Big) ds + C(\Gamma)\frac{\delta}{2^{i-2}} \\
&+ C(\Gamma)\int_0^t \Big(\big\|\psi^{(i-2)}(\cdot,s) - \psi^{(i-3)}(\cdot,s)\big\| + \big\|\psi^{(i-3)}(\cdot,s) - \psi^{(i-4)}(\cdot,s)\big\| \\
&+ \big\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,s) - \widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,s)\big\| + \big\|\widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,s) - \widetilde{\widetilde{a}}_0^{(i-3)}(\cdot,s)\big\| \\
&+ \max_{0\le\tau\le s} |q^{(i-2)}(\tau) - q^{(i-3)}(\tau)| + \max_{0\le\tau\le s} |q^{(i-3)}(\tau) - q^{(i-4)}(\tau)|\Big) ds \\
&+ \varepsilon C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,t)\big\| \\
&+ \varepsilon C(\Gamma) \max_{0\le s\le t} |\partial_t\psi^{(i-1)}(\cdot,s) - \partial_t\psi^{(i-2)}(\cdot,s)|_{1,\alpha(\cdot,q(0))} \\
&+ C(\Gamma)\frac{\delta}{2^{i-1}} + \varepsilon t C(\Gamma)\big\|\psi^{(i-2)}(\cdot,t) - \psi^{(i-3)}(\cdot,t)\big\|
\end{aligned}
$$

$$
\begin{aligned}
&+\varepsilon t C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,t)\big\| \\
&+\varepsilon t C(\Gamma)\max_{0\le s\le t}|q^{(i-2)}(s)-q^{(i-3)}(s)|+C(\Gamma)\frac{\delta}{2^{i-2}} \\
&+\varepsilon t C(\Gamma)\big\|\psi^{(i-3)}(\cdot,t)-\psi^{(i-4)}(\cdot,t)\big\| \\
&+\varepsilon t C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i-3)}(\cdot,t)\big\| \\
&+\varepsilon t C(\Gamma)\max_{0\le s\le t}|q^{(i-3)}(s)-q^{(i-4)}(s)| \\
&+C(\Gamma)\frac{\delta}{2^{i-1}}+\varepsilon C(\Gamma)\int_0^t\Big(\big\|\psi^{(i-2)}(\cdot,s)-\psi^{(i-3)}(\cdot,s)\big\| \\
&+\big\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,s)-\widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,s)\big\|+\max_{0\le\tau\le s}|q^{(i-2)}(\tau)-q^{(i-3)}(\tau)|\Big)ds
\end{aligned}
\tag{5.74}
$$

for $0 \le t \le \overline{T}$.

To sum up, we collect all the main results of the present chapter in the following

Theorem 5.6.1 *Let* $\big((\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t),\psi^{(i)}(\cdot,t),q^{(i)}(t))\big)_{i\in\mathbb{N}}$ *be a sequence in* $X(\overline{T})$. *If the initial conditions of Theorem 4.5.1 are satisfied, then exists a constant* $C(\Gamma)>0$ *independent of* i *such that* (5.46), (5.49), (5.62) *and* (5.74) *hold for* $0 \le t \le \overline{T}$.

Chapter 6

Existence of a Solution

6.1 Contractions for the Iterates

The estimates derived in chapter 5 are shown to produce certain contractions in order to prove the existence of a solution of (3.9). What we show here is a straightforward sequel to (5.46), (5.49), (5.62) and (5.74).

From (5.46) follows

$$\begin{aligned}
\left\|\psi^{(i+1)}(\cdot,t)-\psi^{(i)}(\cdot,t)\right\| &\leq C(\Gamma)\frac{\delta}{2^i}+C(\Gamma)t\left\|\psi^{(i)}(\cdot,t)-\psi^{(i-1)}(\cdot,t)\right\| \\
&+C(\Gamma)t\left\|\psi^{(i-1)}(\cdot,t)-\psi^{(i-2)}(\cdot,t)\right\|+C(\Gamma)t\left\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\right\| \\
&+C(\Gamma)t\left\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)\right\|+C(\Gamma)t\max_{0\leq s\leq t}|q^{(i)}(s)-q^{(i-1)}(s)| \\
&+C(\Gamma)t\max_{0\leq s\leq t}|q^{(i-1)}(s)-q^{(i-2)}(s)|
\end{aligned}$$

and then

$$\left\|\psi^{(i+1)}(\cdot,t)-\psi^{(i)}(\cdot,t)\right\|-C(\Gamma)t\left\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\right\|$$

$$
\begin{aligned}
&\leq C(\Gamma)\frac{\delta}{2^i} + C(\Gamma)t\big\|\psi^{(i)}(\cdot,t) - \psi^{(i-1)}(\cdot,t)\big\| \\
&+ C(\Gamma)t\big\|\psi^{(i-1)}(\cdot,t) - \psi^{(i-2)}(\cdot,t)\big\| + C(\Gamma)t\big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)\big\| \\
&+ C(\Gamma)t \max_{0\leq s\leq t} |q^{(i)}(s) - q^{(i-1)}(s)| + C(\Gamma)t \max_{0\leq s\leq t} |q^{(i-1)}(s) - q^{(i-2)}(s)|
\end{aligned} \tag{6.1}
$$

for a constant $C(\Gamma) > 0$, independent of i.

From (5.62) it is known that there exists a constant $C(\Gamma) > 0$, independent of i, such that

$$
\begin{aligned}
\big\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\big\| &\leq C(\Gamma)\frac{\delta}{2^{i-1}} + C(\Gamma)\frac{\delta}{2^i} \\
&+ (1+\varepsilon)tC(\Gamma)\big\|\psi^{(i-1)}(\cdot,t) - \psi^{(i-2)}(\cdot,t)\big\| \\
&+ (1+\varepsilon)tC(\Gamma)\big\|\psi^{(i-2)}(\cdot,t) - \psi^{(i-3)}(\cdot,t)\big\| \\
&+ C(\Gamma)(\varepsilon + t + \varepsilon t)\big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)\big\| \\
&+ (1+\varepsilon)tC(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,t)\big\| \\
&+ (1+\varepsilon)tC(\Gamma) \max_{0\leq s\leq t} |q^{(i-1)}(s) - q^{(i-2)}(s)| \\
&+ (1+\varepsilon)tC(\Gamma) \max_{0\leq s\leq t} |q^{(i-2)}(s) - q^{(i-3)}(s)| \\
&+ \varepsilon C(\Gamma) \max_{0\leq s\leq t} |\partial_t\psi^{(i)}(\cdot,s) - \partial_t\psi^{(i-1)}(\cdot,s)|_{1,\alpha(\cdot,q(0))}.
\end{aligned} \tag{6.2}
$$

From (5.49) it is known that

$$
\begin{aligned}
\max_{0\leq s\leq t} |q^{(i+1)}(s) - q^{(i)}(s)| &\leq C(\Gamma)\frac{\delta}{2^{i+1}} + \varepsilon t C(\Gamma)\big\|\psi^{(i)}(\cdot,t) - \psi^{(i-1)}(\cdot,t)\big\| \\
&+ \varepsilon t C(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\big\| + \varepsilon t C(\Gamma) \max_{0\leq s\leq t} |q^{(i)}(s) - q^{(i-1)}(s)|
\end{aligned}
$$

and then

$$\max_{0\le s\le t}|q^{(i+1)}(s)-q^{(i)}(s)|-\varepsilon tC(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\big\|$$
$$\le C(\Gamma)\frac{\delta}{2^{i+1}}+\varepsilon tC(\Gamma)\big\|\psi^{(i)}(\cdot,t)-\psi^{(i-1)}(\cdot,t)\big\|$$
$$+\varepsilon tC(\Gamma)\max_{0\le s\le t}|q^{(i)}(s)-q^{(i-1)}(s)| \tag{6.3}$$

for some constant $C(\Gamma)>0$, independent of i.

With the same reasoning that lead us to (6.1), except for replacing i with $i-1$ and $i-2$ respectively, it turns out that

$$\big\|\psi^{(i)}(\cdot,t)-\psi^{(i-1)}(\cdot,t)\big\|-C(\Gamma)t\big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)\big\|$$
$$\le C(\Gamma)\frac{\delta}{2^{i-1}}+C(\Gamma)t\big\|\psi^{(i-1)}(\cdot,t)-\psi^{(i-2)}(\cdot,t)\big\|$$
$$+C(\Gamma)t\big\|\psi^{(i-2)}(\cdot,t)-\psi^{(i-3)}(\cdot,t)\big\|+C(\Gamma)t\big\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,t)\big\|$$
$$+C(\Gamma)t\max_{0\le s\le t}|q^{(i-1)}(s)-q^{(i-2)}(s)|+C(\Gamma)t\max_{0\le s\le t}|q^{(i-2)}(s)-q^{(i-3)}(s)| \tag{6.4}$$

and

$$\big\|\psi^{(i-1)}(\cdot,t)-\psi^{(i-2)}(\cdot,t)\big\|-C(\Gamma)t\big\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,t)\big\|$$
$$\le C(\Gamma)\frac{\delta}{2^{i-2}}+C(\Gamma)t\big\|\psi^{(i-2)}(\cdot,t)-\psi^{(i-3)}(\cdot,t)\big\|$$
$$+C(\Gamma)t\big\|\psi^{(i-3)}(\cdot,t)-\psi^{(i-4)}(\cdot,t)\big\|+C(\Gamma)t\big\|\widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i-3)}(\cdot,t)\big\|$$
$$+C(\Gamma)t\max_{0\le s\le t}|q^{(i-2)}(s)-q^{(i-3)}(s)|+C(\Gamma)t\max_{0\le s\le t}|q^{(i-3)}(s)-q^{(i-4)}(s)| \tag{6.5}$$

for some $C(\Gamma)>0$, independent of i.

With the same reasoning that lead us to (6.2), except for replacing i with $i-1$, it turns out that

$$
\begin{aligned}
\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)\| &\le C(\Gamma)\frac{\delta}{2^{i-2}}+C(\Gamma)\frac{\delta}{2^{i-1}}\\
&+(1+\varepsilon)tC(\Gamma)\|\psi^{(i-2)}(\cdot,t)-\psi^{(i-3)}(\cdot,t)\|\\
&+(1+\varepsilon)tC(\Gamma)\|\psi^{(i-3)}(\cdot,t)-\psi^{(i-4)}(\cdot,t)\|\\
&+C(\Gamma)(\varepsilon+t+\varepsilon t)\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,t)\|\\
&+(1+\varepsilon)tC(\Gamma)\|\widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i-3)}(\cdot,t)\|\\
&+(1+\varepsilon)tC(\Gamma)\max_{0\le s\le t}|q^{(i-2)}(s)-q^{(i-3)}(s)|\\
&+(1+\varepsilon)tC(\Gamma)\max_{0\le s\le t}|q^{(i-3)}(s)-q^{(i-4)}(s)|\\
&+\varepsilon C(\Gamma)\max_{0\le s\le t}|\partial_t\psi^{(i-1)}(\cdot,s)-\partial_t\psi^{(i-2)}(\cdot,s)|_{1,\alpha(\cdot,q(0))} \qquad (6.6)
\end{aligned}
$$

for some $C(\Gamma)>0$ independent of i.

Again with the same reasoning that lead us to (6.3) except for replacing i with $i-1$ and $i-2$ respectively, it turns out that

$$
\begin{aligned}
&\max_{0\le s\le t}|q^{(i)}(s)-q^{(i-1)}(s)|-\varepsilon tC(\Gamma)\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)\|\\
&\quad\le C(\Gamma)\frac{\delta}{2^{i}}+\varepsilon tC(\Gamma)\|\psi^{(i-1)}(\cdot,t)-\psi^{(i-2)}(\cdot,t)\|\\
&\quad+\varepsilon tC(\Gamma)\max_{0\le s\le t}|q^{(i-1)}(s)-q^{(i-2)}(s)| \qquad (6.7)
\end{aligned}
$$

and

$$
\begin{aligned}
&\max_{0\le s\le t}|q^{(i-1)}(s)-q^{(i-2)}(s)|-\varepsilon tC(\Gamma)\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,t)\|\\
&\quad\le C(\Gamma)\frac{\delta}{2^{i-1}}+\varepsilon tC(\Gamma)\|\psi^{(i-2)}(\cdot,t)-\psi^{(i-3)}(\cdot,t)\|\\
&\quad+\varepsilon tC(\Gamma)\max_{0\le s\le t}|q^{(i-2)}(s)-q^{(i-3)}(s)| \qquad (6.8)
\end{aligned}
$$

for some constant $C(\Gamma) > 0$, independent of i. Using (6.1), (6.4), (6.5), (6.3), (6.7) and (6.8), from (5.68) it turns out that

$$
\begin{aligned}
\max_{0\le s\le t} |\partial_t\psi^{(i+1)}(\cdot,s) - \partial_t\psi^{(i)}(\cdot,s)|_{1,\alpha(\cdot,q(0))} &\le C(\Gamma)\frac{\delta}{2^i} + C(\Gamma)\frac{\delta}{2^{i-1}} \\
&+ C(\Gamma)\frac{\delta}{2^{i-2}} + C(\Gamma)\frac{\delta}{2^{i-3}} + tC(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\big\| \\
&+ tC(\Gamma)\big\|\psi^{(i)}(\cdot,t) - \psi^{(i-1)}(\cdot,t)\big\| \\
&+ \big(\varepsilon + tC(\Gamma) + \varepsilon tC(\Gamma)\big)\big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)\big\| \\
&+ tC(\Gamma)\max_{0\le s\le t}|q^{(i)}(s) - q^{(i-1)}(s)| \\
&+ \varepsilon C(\Gamma)\max_{0\le s\le t}|\partial_t\psi^{(i)}(\cdot,s) - \partial_t\psi^{(i-1)}(\cdot,s)|_{1,\alpha(\cdot,q(0))} \\
&+ (1+\varepsilon)tC(\Gamma)\big\|\psi^{(i-1)}(\cdot,t) - \psi^{(i-2)}(\cdot,t)\big\| \\
&+ (1+\varepsilon)tC(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,t)\big\| \\
&+ (1+\varepsilon)tC(\Gamma)\max_{0\le s\le t}|q^{(i-1)}(s) - q^{(i-2)}(s)| \\
&+ (1+\varepsilon)tC(\Gamma)\big\|\psi^{(i-2)}(\cdot,s) - \psi^{(i-3)}(\cdot,s)\big\| \\
&+ \big(\varepsilon + tC(\Gamma) + \varepsilon tC(\Gamma)\big)\big\|\widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-3)}(\cdot,t)\big\| \\
&+ (1+\varepsilon)tC(\Gamma)\max_{0\le s\le t}|q^{(i-2)}(s) - q^{(i-3)}(s)| \\
&+ \varepsilon C(\Gamma)\max_{0\le s\le t}|\partial_t\psi^{(i-2)}(\cdot,s) - \partial_t\psi^{(i-3)}(\cdot,s)|_{1,\alpha(\cdot,q(0))} \\
&+ (1+\varepsilon)tC(\Gamma)\big\|\psi^{(i-3)}(\cdot,t) - \psi^{(i-4)}(\cdot,t)\big\| \\
&+ (1+\varepsilon)tC(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i-3)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-4)}(\cdot,t)\big\| \\
&+ (1+\varepsilon)tC(\Gamma)\max_{0\le s\le t}|q^{(i-3)}(s) - q^{(i-4)}(s)| \\
&+ (1+\varepsilon)tC(\Gamma)\big\|\psi^{(i-4)}(\cdot,t) - \psi^{(i-5)}(\cdot,t)\big\| \\
&+ (1+\varepsilon)tC(\Gamma)\max_{0\le s\le t}|q^{(i-4)}(s) - q^{(i-5)}(s)|
\end{aligned}
$$

and finally

$$
\begin{aligned}
&\max_{0\le s\le t} |\partial_t\psi^{(i+1)}(\cdot,s) - \partial_t\psi^{(i)}(\cdot,s)|_{1,\alpha(\cdot,q(0))} \\
&\quad - tC(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\big\| \\
&\quad \le C(\Gamma)\frac{\delta}{2^i} + C(\Gamma)\frac{\delta}{2^{i-1}} + C(\Gamma)\frac{\delta}{2^{i-2}} + C(\Gamma)\frac{\delta}{2^{i-3}} \\
&\quad + tC(\Gamma)\big\|\psi^{(i)}(\cdot,t) - \psi^{(i-1)}(\cdot,t)\big\| \\
&\quad + \big(\varepsilon + tC(\Gamma) + \varepsilon tC(\Gamma)\big)\big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)\big\| \\
&\quad + tC(\Gamma)\max_{0\le s\le t}|q^{(i)}(s) - q^{(i-1)}(s)| \\
&\quad + \varepsilon C(\Gamma)\max_{0\le s\le t}|\partial_t\psi^{(i)}(\cdot,s) - \partial_t\psi^{(i-1)}(\cdot,s)|_{1,\alpha(\cdot,q(0))} \\
&\quad + (1+\varepsilon)tC(\Gamma)\big\|\psi^{(i-1)}(\cdot,t) - \psi^{(i-2)}(\cdot,t)\big\| \\
&\quad + (1+\varepsilon)tC(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,t)\big\| \\
&\quad + (1+\varepsilon)tC(\Gamma)\max_{0\le s\le t}|q^{(i-1)}(s) - q^{(i-2)}(s)| \\
&\quad + (1+\varepsilon)tC(\Gamma)\big\|\psi^{(i-2)}(\cdot,t) - \psi^{(i-3)}(\cdot,t)\big\| \\
&\quad + \big(\varepsilon + tC(\Gamma) + \varepsilon tC(\Gamma)\big)\big\|\widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-3)}(\cdot,t)\big\| \\
&\quad + (1+\varepsilon)tC(\Gamma)\max_{0\le s\le t}|q^{(i-2)}(s) - q^{(i-3)}(s)| \\
&\quad + \varepsilon C(\Gamma)\max_{0\le s\le t}|\partial_t\psi^{(i-2)}(\cdot,s) - \partial_t\psi^{(i-3)}(\cdot,s)|_{1,\alpha(\cdot,q(0))} \\
&\quad + (1+\varepsilon)tC(\Gamma)\big\|\psi^{(i-3)}(\cdot,s) - \psi^{(i-4)}(\cdot,s)\big\| \\
&\quad + (1+\varepsilon)tC(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i-3)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i-4)}(\cdot,t)\big\| \\
&\quad + (1+\varepsilon)tC(\Gamma)\max_{0\le s\le t}|q^{(i-3)}(s) - q^{(i-4)}(s)| \\
&\quad + (1+\varepsilon)tC(\Gamma)\big\|\psi^{(i-4)}(\cdot,t) - \psi^{(i-5)}(\cdot,t)\big\| \\
&\quad + (1+\varepsilon)tC(\Gamma)\max_{0\le s\le t}|q^{(i-4)}(s) - q^{(i-5)}(s)|. \qquad (6.9)
\end{aligned}
$$

6.2 Cauchy Sequence and Existence of a Solution

A new sequence is defined. It is shown to be a Cauchy sequence in the complete space $X(T)$ for a time T small enough. The convergence of such a sequence is the last step to prove the existence of a solution for the system (3.9) and equivalently for the system given by (2.5), (2.3) and (2.4).

A new sequence in $\mathbb{R}$ is now defined by

$$Q^{(i)}(t) := \left\|\psi^{(i+1)}(\cdot,t) - \psi^{(i)}(\cdot,t)\right\| + \left\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t) - \widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\right\| + \max_{0\le s\le t} |\partial_t\psi^{(i+1)}(\cdot,s) - \partial_t\psi^{(i)}(\cdot,s)|_{1,\alpha(\cdot,q(0))} + \max_{0\le s\le t} |q^{(i+1)}(s) - q^{(i)}(s)|$$

for any $t > 0$. Then, the next aim is to prove that $T = T(\Gamma) > 0$ exists such that

$$\sum_{i=0}^{+\infty} Q^{(i)}(T) < +\infty.$$

In order to show this, we state the following

Theorem 6.2.1 *Given $r \in]0,1[$, $T = T(\Gamma) > 0$ and $\varepsilon_2 > 0$ exist such that*

$$Q^{(i)}(T) \le C(\Gamma)\frac{\delta}{2^{i+1}} + C(\Gamma)\frac{\delta}{2^{i}} + C(\Gamma)\frac{\delta}{2^{i-1}} + C(\Gamma)\frac{\delta}{2^{i-2}} + C(\Gamma)\frac{\delta}{2^{i-3}} + r\big(Q^{(i-1)}(T) + Q^{(i-2)}(T) + Q^{(i-3)}(T) + Q^{(i-4)}(T) + Q^{(i-5)}(T)\big) \tag{6.10}$$

for some constant $C(\Gamma) > 0$ independent of i.

Proof. From (6.1), (6.2), (6.3) and (6.9) one obtains

$$\begin{aligned}
&\big\|\psi^{(i+1)}(\cdot,t)-\psi^{(i)}(\cdot,t)\big\| + \big(1-tC(\Gamma)-\varepsilon tC(\Gamma)\big)\big\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\big\| \\
&\quad + \max_{0\le s\le t}|q^{(i+1)}(s)-q^{(i)}(s)| + \max_{0\le s\le t}|\partial_t\psi^{(i+1)}(\cdot,s)-\partial_t\psi^{(i)}(\cdot,s)|_{1,\alpha(\cdot,q(0))} \\
&\quad \le C(\Gamma)\frac{\delta}{2^{i+1}} + C(\Gamma)\frac{\delta}{2^{i}} + C(\Gamma)\frac{\delta}{2^{i-1}} + C(\Gamma)\frac{\delta}{2^{i-2}} + C(\Gamma)\frac{\delta}{2^{i-3}} \\
&\quad + (1+\varepsilon)tC(\Gamma)\big\|\psi^{(i)}(\cdot,t)-\psi^{(i-1)}(\cdot,t)\big\| \\
&\quad + \big(\varepsilon+\varepsilon C(\Gamma)+tC(\Gamma)+\varepsilon tC(\Gamma)\big)\big\|\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)\big\| \\
&\quad + (1+\varepsilon)tC(\Gamma)\max_{0\le s\le t}|q^{(i)}(s)-q^{(i-1)}(s)| \\
&\quad + \varepsilon C(\Gamma)\max_{0\le s\le t}|\partial_t\psi^{(i)}(\cdot,s)-\partial_t\psi^{(i-1)}(\cdot,s)|_{1,\alpha(\cdot,q(0))} \\
&\quad + (1+\varepsilon)tC(\Gamma)\big\|\psi^{(i-1)}(\cdot,t)-\psi^{(i-2)}(\cdot,t)\big\| \\
&\quad + (1+\varepsilon)tC(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i-1)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,t)\big\| \\
&\quad + (1+\varepsilon)tC(\Gamma)\max_{0\le s\le t}|q^{(i-1)}(s)-q^{(i-2)}(s)| \\
&\quad + (1+\varepsilon)tC(\Gamma)\big\|\psi^{(i-2)}(\cdot,t)-\psi^{(i-3)}(\cdot,t)\big\| \\
&\quad + \big(\varepsilon+tC(\Gamma)+\varepsilon tC(\Gamma)\big)\big\|\widetilde{\widetilde{a}}_0^{(i-2)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i-3)}(\cdot,t)\big\| \\
&\quad + (1+\varepsilon)tC(\Gamma)\max_{0\le s\le t}|q^{(i-2)}(s)-q^{(i-3)}(s)| \\
&\quad + \varepsilon C(\Gamma)\max_{0\le s\le t}|\partial_t\psi^{(i-2)}(\cdot,s)-\partial_t\psi^{(i-3)}(\cdot,s)|_{1,\alpha(\cdot,q(0))} \\
&\quad + (1+\varepsilon)tC(\Gamma)\big\|\psi^{(i-3)}(\cdot,t)-\psi^{(i-4)}(\cdot,t)\big\| \\
&\quad + (1+\varepsilon)tC(\Gamma)\big\|\widetilde{\widetilde{a}}_0^{(i-3)}(\cdot,t)-\widetilde{\widetilde{a}}_0^{(i-4)}(\cdot,t)\big\| \\
&\quad + (1+\varepsilon)tC(\Gamma)\max_{0\le s\le t}|q^{(i-3)}(s)-q^{(i-4)}(s)| \\
&\quad + (1+\varepsilon)tC(\Gamma)\big\|\psi^{(i-4)}(\cdot,t)-\psi^{(i-5)}(\cdot,t)\big\| \\
&\quad + (1+\varepsilon)tC(\Gamma)\max_{0\le s\le t}|q^{(i-4)}(s)-q^{(i-5)}(s)| \qquad (6.11)
\end{aligned}$$

for $0 \leq t \leq \overline{T}$. From (6.11) it follows immediately that

$$\big(1 - tC(\Gamma) - \varepsilon tC(\Gamma)\big)Q^{(i)}(t) \leq C(\Gamma)\frac{\delta}{2^{i+1}} + C(\Gamma)\frac{\delta}{2^{i}} + C(\Gamma)\frac{\delta}{2^{i-1}} + C(\Gamma)\frac{\delta}{2^{i-2}} + C(\Gamma)\frac{\delta}{2^{i-3}} + \big(\varepsilon + \varepsilon C(\Gamma) + tC(\Gamma) + \varepsilon tC(\Gamma)\big)\big(Q^{(i-1)}(t) + Q^{(i-2)}(t) + Q^{(i-3)}(t) + Q^{(i-4)}(t) + Q^{(i-5)}(t)\big)$$

for $0 \leq t \leq \overline{T}$. Then

$$Q^{(i)}(\overline{T}) \leq \frac{C(\Gamma)}{1 - \overline{T}C(\Gamma) - \varepsilon\overline{T}C(\Gamma)}\Big(\frac{\delta}{2^{i+1}} + \frac{\delta}{2^{i}} + \frac{\delta}{2^{i-1}} + \frac{\delta}{2^{i-2}} + \frac{\delta}{2^{i-3}}\Big) + \frac{\varepsilon + \varepsilon C(\Gamma) + \overline{T}C(\Gamma) + \varepsilon\overline{T}C(\Gamma)}{1 - \overline{T}C(\Gamma) - \varepsilon\overline{T}C(\Gamma)}\big(Q^{(i-1)}(\overline{T}) + Q^{(i-2)}(\overline{T}) + Q^{(i-3)}(\overline{T}) + Q^{(i-4)}(\overline{T}) + Q^{(i-5)}(\overline{T})\big). \tag{6.12}$$

For $\varepsilon_2 > 0$ sufficiently small

$$T := \frac{r - \varepsilon - \varepsilon C(\Gamma)}{C(\Gamma) + \varepsilon C(\Gamma) + rC(\Gamma) + r\varepsilon C(\Gamma)} > 0$$

exists such that

$$\frac{\varepsilon + \varepsilon C(\Gamma) + TC(\Gamma) + \varepsilon TC(\Gamma)}{1 - TC(\Gamma) - \varepsilon TC(\Gamma)} \leq r$$

for any $0 < \varepsilon \leq \varepsilon_2$. It is convenient to notice that such a time is $\mathcal{O}(1)$. Therefore

$$Q^{(i)}(T) \leq \frac{C(\Gamma)}{1 - TC(\Gamma) - \varepsilon TC(\Gamma)}\Big(\frac{\delta}{2^{i+1}} + \frac{\delta}{2^{i}} + \frac{\delta}{2^{i-1}} + \frac{\delta}{2^{i-2}} + \frac{\delta}{2^{i-3}}\Big) + r\big(Q^{(i-1)}(T) + Q^{(i-2)}(T) + Q^{(i-3)}(T) + Q^{(i-4)}(T) + Q^{(i-5)}(T)\big)$$

and (6.10) is proven.

From (6.10) it follows that

$$\sum_{i=5}^{+\infty} Q^{(i)}(T) - r\sum_{i=5}^{+\infty} Q^{(i-1)}(T) - r\sum_{i=5}^{+\infty} Q^{(i-2)}(T) - r\sum_{i=5}^{+\infty} Q^{(i-3)}(T)$$
$$- r\sum_{i=5}^{+\infty} Q^{(i-4)}(T) - r\sum_{i=5}^{+\infty} Q^{(i-5)}(T)$$
$$\leq \frac{C(\Gamma)}{1 - TC(\Gamma) - \varepsilon TC(\Gamma)}\Big(\sum_{i=5}^{+\infty}\frac{\delta}{2^{i+1}} + \sum_{i=5}^{+\infty}\frac{\delta}{2^{i}} + \sum_{i=5}^{+\infty}\frac{\delta}{2^{i-1}} + \sum_{i=5}^{+\infty}\frac{\delta}{2^{i-2}}$$
$$+ \sum_{i=5}^{+\infty}\frac{\delta}{2^{i-3}}\Big)$$

and then

$$\sum_{i=5}^{+\infty} Q^{(i)}(T) - r\sum_{i=4}^{+\infty} Q^{(i)}(T) - r\sum_{i=3}^{+\infty} Q^{(i)}(T) - r\sum_{i=2}^{+\infty} Q^{(i)}(T)$$
$$- r\sum_{i=1}^{+\infty} Q^{(i)}(T) - r\sum_{i=0}^{+\infty} Q^{(i)}(T) \leq \frac{5C(\Gamma)}{1 - TC(\Gamma) - \varepsilon TC(\Gamma)}\sum_{i=0}^{+\infty}\frac{\delta}{2^{i}}.$$

From the last inequality we find

$$(1 - 5r)\sum_{i=5}^{+\infty} Q^{(i)}(T) \leq \frac{5C(\Gamma)}{1 - TC(\Gamma) - \varepsilon TC(\Gamma)}\sum_{i=0}^{+\infty}\frac{\delta}{2^{i}} + 5r\sum_{i=0}^{4} Q^{(i)}(T)$$

which, for $r < 1/5$, implies

$$\sum_{i=0}^{+\infty} Q^{(i)}(T) < +\infty. \tag{6.13}$$

From (6.13) it follows that

$$\sum_{i=0}^{+\infty} \left\|\psi^{(i+1)}(\cdot, t) - \psi^{(i)}(\cdot, t)\right\| < +\infty, \tag{6.14}$$

$$\sum_{i=0}^{+\infty} \left\|\widetilde{\widetilde{a}}_0^{(i+1)}(\cdot, t) - \widetilde{\widetilde{a}}_0^{(i)}(\cdot, t)\right\| < +\infty, \tag{6.15}$$

$$\sum_{i=0}^{+\infty} \max_{0\le s\le t} |\partial_t \psi^{(i+1)}(\cdot,s) - \partial_t \psi^{(i)}(\cdot,s)|_{1,\alpha(\cdot,q(0))} < +\infty \tag{6.16}$$

and

$$\sum_{i=0}^{+\infty} \max_{0\le s\le t} |q^{(i+1)}(s) - q^{(i)}(s)| < +\infty \tag{6.17}$$

for any $0 \le t \le T$.

The convergence of these series is equivalent to the Cauchy condition. For example, from (6.14) it follows that S exists such that

$$\forall \varepsilon > 0 \quad \exists N \quad \text{such that} \quad \forall h \ge 0 \quad |\sum_{i=0}^{N+h} \|\psi^{(i+1)}(\cdot,t) - \psi^{(i)}(\cdot,t)\| - S| \le \varepsilon$$

which is equivalent to the Cauchy condition:

$$\forall \varepsilon > 0 \quad \exists \overline{N} \quad \text{such that} \quad \forall h > k \ge \overline{N}$$
$$|\sum_{i=0}^{\overline{N}+h} \|\psi^{(i+1)}(\cdot,t) - \psi^{(i)}(\cdot,t)\| - \sum_{i=0}^{\overline{N}+k} \|\psi^{(i+1)}(\cdot,t) - \psi^{(i)}(\cdot,t)\|| \le \varepsilon.$$

From the Cauchy condition it follows that

$$\forall \varepsilon > 0 \quad \exists \overline{M} \quad \text{such that} \quad \forall h > k \ge \overline{M}$$
$$\|\psi^{(M+h)}(\cdot,t) - \psi^{(M+k)}(\cdot,t)\| \le \varepsilon$$

which means, by definition, that $\left(\psi^{(i)}(\cdot,t)\right)_{i\in\mathbb{N}}$ is a Cauchy sequence in the complete space $X(t)$ for any $0 \le t \le T$. Then $\left(\psi^{(i)}(\cdot,t)\right)_{i\in\mathbb{N}}$ is convergent in $X(t)$ for any $0 \le t \le T$. In particular, since $\|\cdot\|$ is a sup norm with respect to t, the series of functions

$$t \longmapsto \psi^{(i)}(\cdot,t)$$

is *uniformly convergent* on $[0,T]$. In an analogous way one shows that also

$$\left(\psi^{(i)}(\cdot,t)\right)_{i\in\mathbb{N}}, \quad \left(\widetilde{\widetilde{a}}_0^{(i)}(\cdot,t)\right)_{i\in\mathbb{N}}, \quad \left(\partial_t\psi^{(i)}(\cdot,t)\right)_{i\in\mathbb{N}}, \quad \left(q^{(i)}(t)\right)_{i\in\mathbb{N}}$$

are convergent in $X(t)$ for any $0 \leq t \leq T$ and

$$t \longmapsto \widetilde{\widetilde{a}}_0^{(i)}(\cdot,t),$$

$$t \longmapsto \partial_t\psi^{(i)}(\cdot,t)$$

and

$$t \longmapsto q^{(i)}(t)$$

are uniformly convergent on $[0,T]$.

Moreover, since

$$t \longmapsto \left(\psi^{(i)}(\cdot,t), \widetilde{\widetilde{a}}_0^{(i)}(\cdot,t), q^{(i)}(t)\right)$$

are continuous functions in t uniformly convergent on $[0,T]$, also

$$t \longmapsto f_2(\psi^{(i-1)}(\cdot,t), \widetilde{\widetilde{a}}_0^{(i)}(\cdot,t), q^{(i-1)}(t))$$

are continuous functions in t uniformly convergent on $[0,T]$ and then so are

$$t \longmapsto \dot{q}^{(i)}(t).$$

Thus, if $q^{(i)}$ converges pointwise to q, $\dot{q}^{(i)}$ converges pointwise to $\dot{q}$. Then, the existence of the solution of (3.9) has been proven and this is given by $(\widetilde{\widetilde{a}}_0, \psi, q)$.

Chapter 7

Conclusion

In the Abelian-Higgs model, Stuart extended the local proof to a longer period of time by means of two almost conserved quantities. Our outcome is still only local because the Gor'kov-Éliashberg equations, unlike the Abelian-Higgs model, do not admit a Lagrangian from which one can derive certain energies. Perhaps it would be possible to find quasi-conserved quantities also for the time-dependent Ginzburg-Landau equations and then use them to give an estimate of the time-scale for which the Slow-Motion Approximation can be applied in that case. We hope that the local result is the first step towards a more complete description of the dynamics of vortices through the Slow-Motion Approximation.

Nevertheless we see some immediate application of our result. An expansion in the separation parameters for two vortices close together was given in [BK01] up to the third order in the distance parameter. This expansion could be used to study some aspects of the scattering of vortices in type-II superconductors. In the Abelian-Higgs model vortex scattering was studied using the geodesic equations for the parameters and 90°-scattering was explained analytically.

Since the Gor'kov-Éliashberg equations are first-order in time, the O.D.E.s for the parameters are first-order as well. If we want to enhance the understanding of the multivortex dynamics, we have to choose initial conditions for the exact solution slightly different from a static configuration, otherwise the vortices remain in the initial location. Imposing suitable initial conditions to the solution and supposing that the initial velocity of the vortices is not zero, we hope to be able to understand the scattering of two vortices by means of the local proof we carried out, provided that they are sufficiently close together.

To conclude, we want to point out two limitations of our model to describe the physics of type-II superconductors. The first is that the model we considered assumes that the dimensionless conductivity σ and the Higgs self-coupling constant λ are both close to 1. The second is the fact that we took into account for our description only infinitely extended materials where no boundary effects appear.

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