Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of *Doctor of Philosophy in Applied Mathematical Sciences* is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

Fawzi Abdelwahid

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Dedicated to

my wife Mariam and my

children

Mabrouka, Ahmed, Maroua

and Adam.

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Abstract

The dynamics of magnetic monopoles is studied Since we set up an initial value problem compatible with the slow-motion approximation, our investigation requires a thorough understanding of the static solutions Therefore we review those aspects of SU(2) Yang-Mills-Higgs theory in (3+1)-dimensional Minkowski space-time necessary for our study, in particular, the invariance of the theory under SU(2) gauge transformation, the Bogomol'nyi-Prasad-Sommerfield limit, certain associated linear equations and their relation to the Riemann-Hilbert problem We review the ansatz for n-monopole solutions which leads to the existence of a GL(2,C) gauge transformation The construction of this transformation, which before was not given in explicit form, is our main contribution to setting-up of the initial value problem This gauge provides analytic real solutions of the monopole equations Our studies lead us to suitable series solutions which we use to construct a Cauchy problem guided by the idea of the slow-motion approximation Then the existence of a unique time-dependent series solution of this problem near the origin is shown by using the Cauchy-Kowalewskyi theorem Finally, we use Mathematica to find the leading terms of the solution which we then use to study the scattering of two monopoles Our most interesting finding is evidence of 90° scattering

Chapter 1

Introduction

In recent decades, these has been great interest in soliton solutions of nonlinear partial differential equations (see, e g , Ref 1) A soliton is a solution of a nonlinear equation or of a system of equations which retains its form over time even after interaction with other solutions The word soliton was coined by Zabusky and Kruskal after their dicovery that when two or more Korteweg-de Vries solitary waves collide they do not break up and disperse The solitary wave was first observed and described by the Scottish scientist and engineer, John Scott Russell Whilst observing the movement of a canal barge, he noticed a type of water wave on the surface of the canal which kept its shape for a very long time

More recently, starting with the work of Nielson and Oleson [2], ‘t Hooft [3] and Polyakov [4], soliton-like solutions have been found in the Yang-Mills-Higgs theories As fundamental theories for different gauge groups, we have the following examples of Yang-Mills-Higgs theories (I) The electromagnetic interactions· The gauge group G is the abelian unitary group U(1), and the Higgs field Φ and the Higgs potential $V(\Phi)$ are not present (II) The electromagnetic and weak intractions: The gauge group G is the non-abelian unitary group U(2), and the Higgs field Φ is a real four-component vector or a complex two-component vector (III) The strong intractions The gauge group G is the non-abelian group SU(3) and there is no Higgs field Φ

As an effective theory, we have the examples of the Ginzburg-Landau theory of superconductivity Here the group G is the unitary group U(1) and the Higgs field Φ is a two-component real field or one complex field Of great interst are also the Grand Unified Theories In these theories the gauge group G is a Lie group which contains the electro-weak group U(2) and the strong group SU(3) as subgroups For example, the Lie group SU(5) could be used and different Higgs fields have been tried to describe the experiments Both these theories have soliton-like solutions, the vortex solutions of the Ginzburg-Landau theory and the magnetic monopoles of the Grand Unified Theories [5] A simplified version of a Grand Unified Theory which still has monopole solutions, is the SU(2) theory where the gauge group G is the special unitary group SU(2) and the Higgs field Φ is a real 3-component vector

The magnetic monopole has a long history going back to Dirac [6] Dirac hypothesized the existence of separate magnetic poles and showed that the existence of a magnetic pole did not lead to any contradiction on principle with modern physical ideas He concluded " Under these circumstances one would be supriseed if Nature had no use of it" Monopoles reemerged as solutions of the nonlinear system of hyperbolic equations in Minkowski space in Yang-Mills-Higgs theory These monopoles are soliton-like objects because they have the same stability of solitons Monopoles, however, seem to generate radiation and do not seem to emerge unscathed from collisions

After Prasad and Sommerfield had found an explicit monopole solution [7], the construction of monopole solutions became very important (see, e g , Refs 8, 9 and 10). Ward [11-14] was the first to construct multi-monopole solutions. In this method the information contained in a self-dual gauge fields is coded in a certain analytic complex vector bundle in terms of a transition matrix This leads to a procedure for generating self-dual solutions of the Yang-Mills-Higgs equations Following the geometrical study by Atiyah and Ward [15], which leads to a series of Ansatze A_n for $n \geq 1$, Corrigan, Fairlie, Yates and Goddard [16] used the R_n gauge to write down the explicit construction of the Atiyah and Ward Ansatze This method is known as the splitting of the transition matrix or the solution of the Riemann-Hilbert problem

The axially symmetric time-independent solution describing two monopoles sitting on top of each other at the origin of R^3 was presented by Ward [12] His construction was not completed, however, the solutions were left in terms of complex-valued functions of x, y and z Ward showed the existence of a gauge transformation in which the solutions become real-valued functions, but such a gauge transformation was not given in explicit form Ward [13,14] also introduced a new sequence of Ansatze A_n which generates n monopoles solution These Ansatze lead to a (4n-1)-parameter family of solutions describing n-monopoles sitting somewhere in space For n=2, the solution is a function of one separation parameter p For p=0 the solution reduces to the axial symmetric solution For p different from zero, the solution describes two monopoles located at two distinct points in space, the distance between them being related to p This solution was found to be not symmetric about any axis in space, even about the line joining them Corrigan and Goddard [17] constructed a static monopole solution of 4n-1 degrees of freedom generalising the Ward two monopole solution

In recent years, there has been considerable interest in studying the scattering of solitons and soliton-like objects, like vortices or monopoles [18-21] These studies are based on the idea of the slow-motion approximation proposed by Manton [22] in the context of SU(2) monopoles [23] For vortices, Ruback [24] applied the idea that for $\lambda = 1$ and at low energies, the Bogomol'nyi solutions can be used to approximate time-dependent solutions In Ref. 25 we used a series solution approach to the study of the scattering of two vortices In that work we formulated a Cauchy problem and proved that a unique global finite-energy solution of the Ginzburg-Landau equations exists We studied the symmetry of the solution which leads to a left-right and up-down symmetry of the energy density Then we used the Cauchy-Kowalewskyi theorem to prove that a local solution of these equations exists near the origin We used this series solution to establish 90° scattering Using the same technique, π / n scattering can be established [26]

In this thesis, we will study scattering of monopoles in the SU(2) model The SU(2) model is chosen for the following reasons First, the SU(2) model has many features in common with Grand Unified Theories Second, the SU(2) model in the Prasad-Sommerfield limit, i e with $\lambda = 0$, and the Ginzburg-Landau theory with $\lambda = 1$ have many features in common This enables us to use some of the methods from Ref 25 Finally, to have a hope of success in applying our analytic techniques a fairly simple model has to be chosen

In the second chapter, we will introduce the Yang-Mills-Higgs theory in (3+1)-dimensional Minkowski space-time As an example, we study the SU(2) model and show that the Lagrangian of this model and the equations of motion are invariant under a SU(2) gauge transformation We discuss also the Bogomol'nyi-Prasad-Sommerfield limit in great detail and at the end of this chapter we will study the one-monopole solution In the third chapter, we introduce associated linear equations and prove that the compatibility conditions of these equations are the self-duality conditions We will then study the Riemann-Hilbert problem and the reality of the 2-monopole solutions These studies lead to the existence of a GL(2,C) gauge transformation which was not given before in explicit form At the end of this chapter, we will study this gauge transformation and write it down in explicit form

In the fourth chapter, we will introduce an ansatz given by Ward which leads to a (4n-1)-parameter family of solutions. These solutions describe n-monopoles sitting somewhere in space We will study this ansatz for n=2 to construct the two-monopole solution We will study also the reality and the regularity of these solutions and continue our discussion of the previous chapter At the end of this chapter we review the splitting of the transition matrix for n=2 In the fifth chapter, we will set up a Cauchy problem for the equations of motion Guided by the idea of the slow-motion approximation, we will find suitable initial data from the series expansions of two separated monopoles solutions In the sixth chapter, we will study the existence of a unique series solution of this Cauchy problem near the origin by applying the Cauchy-Kowalewskyi theorem Then we will use Mathematica to find the series solutions of this Cauchy problem near the origin At the end of this chapter we will use these series solutions to study the scattering of two monopoles This study shows 90° scattering

Chapter 2

Yang-Mills-Higgs Theory

In this chapter we introduce the Yang-Mills-Higgs theory in (3+1)-dimensional Minkowski space-time $\mathbf{M}^4$ We study the SU(2) model as an example and give the Euler-Lagrange equations (the equations of motion) We will also show that the Lagrangian of the SU(2) model and the equations of motion are invariant under SU(2) gauge transformations We discuss the Bogomol'nyi-Prasad-Sommerfield limit and introduce the Bogomol'nyi equations, which are known as the monopole equations Then we will prove that any solutions of these equations, with $A_0 = 0$, will solve the time-independent equations of motion At the end of this chapter the one-monopole solution is studied

2.1 The Model

In this section we discuss the Yang-Mills-Higgs model in (3+1)-dimensional Minkowski space-time $\mathbf{M}^4$ with Lagrangian

$$\mathcal{L}=-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a-\frac{1}{2}\left(D_\mu\Phi\right)^s\left(D^\mu\Phi\right)_s-V(\Phi). \tag{2 1}$$

Here $\mu,\nu=0,1,2,3$, where the indices μ and ν are raised and lowered with the metric g=diag(-1,+1,+1,+1) The gauge potentials A_μ take their values in the Lie algebra $\mathfrak{g}$ of a Lie group G. As a Lie algebra valued vector, we can write $A_\mu = A^a_\mu\tau^a$, a=1,2,3,...,ℓ, where ℓ is the dimension of the Lie algebra $\mathfrak{g}$, and where τ^a are the generators, which we chose to be hermitian with metric I_ℓ The Higgs field Φ takes its values in a vector space $\mathbf{L}$, i e Φ is a vector of m real functions Φ^s, s=1,2,3, ,m , where m is the dimension of the vector space $\mathbf{L}$ with metric I_m [27] The Higgs fields Φ^s are m real functions and the gauge potentials A^a_μ are 4ℓ real functions of the time variable t and the space variables x^i, i=1,2,3

The potential $V(\Phi)$ is taken to be a 4-th degree polynomial in Φ of the form

$$V(\Phi)=\frac{\lambda}{2}(|\Phi|^2-1)^2, \tag{2 2}$$

where the modulus of the Higgs field Φ is defined by

$$|\Phi|^2=\Phi^s\Phi_s \tag{2 3}$$

The gauge fields are defined as

$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu-C^a_{bc}A^b_\mu A^c_\nu, \tag{2 4}$$

in terms of the gauge potentials A^a_μ and the structure constants C^a_{bc} of the Lie algebra g, with $C^a_{bc} + C^a_{cb} = 0$ The generators τ^a of the Lie group G satisfy the commutation relation

$$[\tau^a, \tau^b] = iC^c_{ab}\tau^c, \qquad (2\ 5)$$

where the commutator (the poisson bracket), [x,y]=xy-yx, is skew-symmetry and satisfies the Jacobi identity

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

The gauge fields defined by (2 4) can also be express in the form

$$F_{\mu\nu} = F^a_{\mu\nu}\tau_a = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \qquad (2\ 6)$$

To define the covariant derivative of the Higgs field Φ we use an m-dimensional representation of the group and of its generators τ^a Then,

$$(D_\mu\vec{\Phi})^s = \partial_\mu\Phi^s + iA^a_\mu(\tau^a)_{sr}\Phi^r, \qquad (2\ 7)$$

is the covariant derivative For example, if we choose L = g, then the Higgs field Φ will take its values in the Lie algebra of the Lie group G, and as a Lie algebra valued vector $\Phi = \Phi^a\tau^a$, a=1,2,3, ,ℓ Then, the definition of the covariant derivative, (2 7), reads

$$(D_\mu\Phi)^a = \partial_\mu\Phi^a - C^c_{ab}A^b_\mu\Phi^c \qquad (2\ 8)$$

Next we will list some of the important examples of the Yang-Mills-Higgs theories, i e, we briefly describe important Yang-Mills-Higgs models for different gauge groups As fundamental theories, we have the following examples: (I) The electromagnatic interractions. The gauge group G is the abelian unitary group U(1), and the Higgs field Φ and the Higgs potential $V(\Phi)$ are not present. (II) The electromagnatic and weak intractions The gauge group G is the non-abelian unitary group U(2), and the Higgs field Φ is a real four-component vector or a complex two-component vector (III) The strong intractions: The gauge group G is the non-abelian group SU(3) and there is no Higgs field Φ

As an effective theory, we have the following example The Ginzburg-Landau theory of superconductivity: The group G is the unitary group U(1) and the Higgs field Φ is a two-component real field or one complex field Of great interest are also Grand Unified Theories The gauge group G is a Lie group which contains the electro weak group U(2) and the strong group SU(3) as subgroups For example, the Lie group SU(5) and different Higgs fields have been tried to describe the experiments A simple model is SU(2) theory The gauge group G is the special unitary group SU(2) and the Higgs field Φ is a real 3-component vector We chose to study this SU(2) model for the following reasons (1) The SU(2) model has many features of the Grand Unified Theories (2) The SU(2) model with $\lambda = 0$ and the Ginzburg-Landau theory with $\lambda = 1$ have many features in common

2.2 The SU(2) Model

In this thesis, as an example which has many features of the Grand Unified Theories, we chose the gauge group G to be the non-abelien group SU(2) which is of dimension $\ell=3$ We choose the Higgs field Φ to be a real 3-component vector Then the Lagrangian (2 1) will take the form

$$\mathcal{L}=-\frac{1}{4}F^{a}_{\mu\nu}F^{\mu\nu}_{a}-\frac{1}{2}\left(D_{\mu}\Phi\right)^{a}\left(D^{\mu}\Phi\right)_{a}-V(\Phi) \qquad (2\ 9)$$

If we express the generators of the gauge group in terms of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2 10}$$

then the commutation relation (2 5) will take the form

$$\left[\frac{1}{2}\sigma^a, \frac{1}{2}\sigma^b\right] = \frac{1}{2}\varepsilon^{abc}\sigma^c \tag{2.11}$$

The gauge fields (2 4) and the covariant derivatives (2 8) of the Higgs field Φ can be written in the form

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \varepsilon^{abc} A^b_\mu A^c_\nu, \tag{2 12}$$

$$(D_\mu \Phi)^a = \partial_\mu \Phi^a - \varepsilon^{abc} A^b_\mu \Phi^c, \tag{2 13}$$

respectively, with $\varepsilon_{123} = +1$, and the ε tensor totally antisymmetric Note that $\varepsilon^{abc}\varepsilon_{dbc} = 2\delta^a_d$, where δ is the Kronecker delta

Next we will prove that the Lagrangian of the SU(2) model (2 9) is invariant under the gauge transformation

$$(A^\mu, \Phi) \to (A'^\mu, \Phi') = (\omega A^\mu \omega^{-1} + \mathrm{i}(\partial^\mu \omega)\omega^{-1}, \omega \Phi \omega^{-1}), \tag{2 14}$$

where ω is an element of the Lie group SU(2) and $\Phi = \Phi^a \tau_a$ To prove the invariance of the Lagrangian (2 9), we use the fact that, $\omega\omega^{-1} = I$, which yields

$$\omega(\partial_\mu \omega^{-1}) = -(\partial_\mu \omega)\omega^{-1} \tag{2 15}$$

Making use of the identity (2 15), we can easily prove the following

$$(D_\mu\Phi)' = (\partial_\mu\Phi + i[A_\mu,\Phi])' = \omega(D_\mu\Phi)\omega^{-1}, \qquad (2\ 16\text{-a})$$

$$F'_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu])' = \omega F_{\mu\nu}\omega^{-1} \qquad (2\ 16\text{-b})$$

Hence (2 16-a,b), with the fact that $\mathrm{tr}(A\,B)=\mathrm{tr}(B\,A)$, yields

$$(D_\mu\Phi)'^a (D^\mu\Phi)'_a = 2\mathrm{tr}\Big((D_\mu\Phi)'(D^\mu\Phi)'\Big) = (D_\mu\Phi)^a (D^\mu\Phi)_a, \qquad (2\ 17\text{-a})$$

$$F'^a_{\mu\nu} F'^{\mu\nu}_a = 2\mathrm{tr}\Big(F'_{\mu\nu} F'^{\mu\nu}\Big) = F^a_{\mu\nu} F^{\mu\nu}_a, \qquad (2\ 17\text{-b})$$

$$|\Phi'|^2 = \Phi'^a \Phi'^a = 2\mathrm{tr}(\Phi'\Phi') = |\Phi|^2, \qquad (2\ 17\text{-c})$$

which establishes that the Lagrangian (2 9) is invariant under the SU(2) gauge transformation (2 14)

2.3 The Equations of Motion

In this section we will derive the equations of motion corresponding to the Lagrangian (2 9) To do that, we will use the usual variational technique, which in our case, leads to the variation equations

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \Phi^a_{,\mu}}\right) - \frac{\partial \mathcal{L}}{\partial \Phi^a} = 0, \qquad (2\ 18\text{-a})$$

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial A^a_{\nu,\mu}}\right) - \frac{\partial \mathcal{L}}{\partial A^a_\nu} = 0, \qquad (2\ 18\text{-b})$$

where

$$A^a_{\nu,\mu} = \frac{\partial A^a_\nu}{\partial x^\mu}, \qquad \Phi^a_{,\mu} = \frac{\partial \Phi^a}{\partial x^\mu}$$

First if we use the variation equation (2 18-a) and the Lagrangian (2 9), then we can derive the following

$$\frac{\partial \mathcal{L}}{\partial \Phi^a} = -(D^\mu\Phi)_d \frac{\partial}{\partial \Phi^a}(D_\mu\Phi)^d - \frac{\partial V}{\partial \Phi^a} = -(D^\mu\Phi)_d(-\varepsilon^{dbc}A^b_\mu\delta^a_c) - \frac{\partial V}{\partial \Phi^a}$$

$$= -\varepsilon^{abc}A^b_\mu(D^\mu\Phi)_c - \frac{\partial V}{\partial \Phi^a} \qquad (2\ 19)$$

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\Phi^a)}\right) = -\partial_\mu\left(\left(D^\rho\Phi\right)_d \frac{\partial}{\partial(\partial_\mu\Phi^a)}\left(\partial_\rho\Phi^d - \varepsilon^{def}A^e_\rho\Phi^f\right)\right),$$

which can be easily simplified to the form

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\Phi^a)}\right) = -\partial_\mu\left[\left(D^\rho\Phi\right)_d\delta^a_d\delta^\mu_\rho\right] = -\partial_\mu\left(D^\mu\Phi\right)_a \qquad (2\ 20)$$

Next the variation equation (2 18-b) and the Lagrangian (2 9) yield the following

$$\frac{\partial \mathcal{L}}{\partial A^a_\nu} = -\frac{1}{2}F^{\rho\sigma}_d \frac{\partial}{\partial A^a_\nu}F^d_{\rho\sigma} - (D^\rho\Phi)_d \frac{\partial}{\partial A^a_\nu}(D_\rho\Phi)^d,$$

and if we make use of $\varepsilon_{def}\varepsilon^{dbc} = \delta^b_e\delta^c_f - \delta^c_e\delta^b_f$, then we can easily write the above equation in the form

$$\frac{\partial \mathcal{L}}{\partial A_\nu^a} = \varepsilon^{abc} A_\mu^b F_c^{\nu\mu} - \varepsilon^{abc} (D^\nu \Phi)_b \Phi_c . \tag{2 21}$$

We can also derive

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu^a)} \right) = \partial_\mu \left(-\frac{1}{2} F_d^{\rho\sigma} \frac{\partial}{\partial(\partial_\mu A_\nu^a)} \left(\partial_\rho A_\sigma^d - \partial_\sigma A_\rho^d - \varepsilon^{dbc} A_\rho^b A_\sigma^c \right) \right),$$

which yields

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu^a)} \right) = \partial_\mu \left(-\frac{1}{2} F_d^{\rho\sigma} \left(\delta_\rho^\mu \delta_\sigma^\nu \delta_d^a - \delta_\sigma^\mu \delta_\rho^\nu \delta_d^a \right) \right) = \partial_\mu \left(F_a^{\nu\mu} \right) \tag{2 22}$$

Substituting (2 19) and (2 20) into the equation (2 18-a), yields

$$\partial_\mu \left(D^\mu \Phi \right)_a - \varepsilon^{abc} A_\mu^b (D^\mu \Phi)_c - \frac{\partial V}{\partial \Phi^a} = \left(D_\mu D^\mu \Phi \right)_a - \frac{\partial V}{\partial \Phi^a} = 0,$$

which can be written in the form

$$D_\mu D^\mu \Phi = \frac{\partial V(\Phi)}{\partial \Phi} \tag{2 23}$$

And substituting (2 21) and (2 22) into the equation (2 18-b), yields

$$\partial_\mu F_a^{\nu\mu} - \varepsilon^{abc} A_\mu^b F_c^{\nu\mu} + \varepsilon^{abc} (D^\nu \Phi)_b \Phi_c = \left(D_\mu F^{\nu\mu} \right)_a + \varepsilon^{abc} (D^\nu \Phi)_b \Phi_c = 0,$$

which can be written in the form

$$D_\mu F^{\nu\mu} = -i[\Phi, D^\nu\Phi]. \tag{2 24}$$

The equations (2 23) and (2 24) are the equations of motion of the Lagrangian (2.9), which are also known as the Euler-Lagrange equations Jaffe and Taubes [5] proved that finite energy ensures that solutions to these equations satisfy the boundary conditions

$$|\Phi| \to 1, \qquad |F_{\mu\nu}| \to 0, \qquad |D_\mu\Phi| \to 0, \tag{2 25}$$

uniformly as $|\vec{x}| \to \infty$ That the equations of motion are invariant under the gauge transformation (2.14) follows from the invariance of the Lagrangian (2 9) It can also be shown by using the same techniques we have used in the previous section We will consider any two solutions (A_μ, Φ) and (A'_μ, Φ') of these equations to describe the same physical situation if they are related to each other by the gauge transformation (2 14)

2.4 The Monopole Equations

In this section we will introduce the Bogomol'nyi-Prasad-Sommerfield (BPS) limit [28,7] and prove that any solution of the Bogomol'nyi equations, with $A_0 = 0$, will satisfy the equations of motion To do that, let us consider the potential energy

$$\mathcal{E}_{p} = \int d^3x \left\{ \frac{1}{2}\left(B^i_a B^i_a\right) + \frac{1}{2}\left(D^i\Phi\right)_a \left(D^i\Phi\right)_a + V(\Phi) \right\}, \tag{2 26}$$

where

$$B^i = \frac{1}{2}\varepsilon^{ijk} F_{jk}, \tag{2 27}$$

is the non-abelian magnetic field. Note that the spatial indices of ε^{ijk} are raised and lowered with the metric g=diag(+1,+1,+1) and that $\varepsilon_{123} = 1$.

Bogomol'nyi [28] pointed out that the potential energy $\mathcal{E}_{p}$ can be written in the form

$$\mathcal{E}_{p} = \int d^3x\left\{\frac{1}{2}\left[B^i_a \mp (D^i\Phi)_a\right]^2 + V(\Phi)\right\} \pm \int d^3x\left\{B^i_a\left(D^i\Phi\right)_a\right\} \tag{2.28}$$

The last integral is always an integer multiple of 4π [5]. This integer n is called the monopole number. Equation (2.28) implies that $\mathcal{E}_{p} \geq 4|n|\pi$. If we consider the equality

$$\mathcal{E}_{p} = 4|n|\pi, \tag{2.29}$$

then for $\lambda > 0$, the only solution of the equation (2.29) is $\mathcal{E}_{p} = n = 0$, and (2.29) can only be satisfied for $\lambda = 0$; if $|n| > 0$. $\lambda = 0$ is the Bogomol'nyi-Prasad-Sommerfield (BPS) limit [28,7]. In the Bogomol'nyi-Prasad-Sommerfield limit the potential energy (2.28) attains its lower bound if the fields satisfy the equations

$$B^i = \pm D^i\Phi, \tag{2.30}$$

with i=1,2,3, and for n>0 or n<0, respectively. Equations (2.30) are known as the Bogomol'nyi equations. In the following, we will concentrate on n>0. The n<0 case can be dealt with in a similar fashion.

In this limit any static solution of the equations (2.30), with A'_0=0, also satisfied the equations of motion. To prove this fact and to prepare for the monopole construction in the next chapter, we study the following two results in the BPS limit. (I) If we identify Φ with A_4 and assume that all the fields are x_4-

independent, then the Bogomol'nyi equations (2 30) for n>0 are equivalent to the self-duality conditions

$$F_{\mu\nu} = F^{*}_{\mu\nu} \tag{2 31}$$

Here $F^{*}_{\mu\nu}$ are the dual fields

$$F^{*}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

for $\mu,\nu,\rho,\sigma=1,2,3,4$, $\vec{x} \in \mathbf{R}^4$, with metric g=diag(+1,+1,+1,+1) We can write down the self-duality conditions in the form

$$F_{14} = F_{23}, \qquad F_{13} = -F_{24}, \qquad F_{12} = F_{34} \tag{2 32}$$

(II) Under the same assumptions, the equations of motion,

$$D^{\nu}F_{\mu\nu} = 0, \tag{2 33}$$

are equivalent to the time-independent equations of motion (2 23) and (2 24), where the equations (2 33) are the equations of motion of the Lagrangian of the Yang-Mills model in Euclidean space $\mathbf{R}^4$, which is of the form

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

To prove the first result, we write down the Bogomol'nyi equations (2 30) for i=1,2,3, and identify Φ with A_4 Hence, for example for i=1, the two sides of equations (2 30) can be written as follows

$$D^1\Phi = D_1\Phi = \partial_1\Phi + i[A_1,\Phi] = F_{14}, \tag{2.34-a}$$

$$B^1 = \frac{1}{2}\left(\varepsilon^{123}F_{23} + \varepsilon^{132}F_{32}\right) = F_{23} \tag{2.34-b}$$

Hence (2.34-a,b), yields the first condition of (2.32). Similarly, if we write down (2.30) for i=2,3, then we will get the other two conditions, which will establish the first result.

To prove the second result, we write down (2.33) for $\mu = 4$ and $\mu = 1,2,3$, as follows

$$D^{\nu}F_{4\nu} = 0, \qquad D^{\nu}F_{i\nu} = 0, \tag{2.35}$$

which can be rewritten in the form

$$D^{j}F_{4j} = 0, \tag{2.36-a}$$

$$D^4F_{i4} + D^jF_{ij} = 0, \tag{2.36-b}$$

respectively. If we identify A_4 with Φ, then we can write the following

$$F_{i4} = D_i\Phi, \tag{2.37}$$

which yields

$$D^4F_{i4} = i[\Phi, D_i\Phi] \tag{2.38}$$

Now (2.36-a) and (2.37) yield

$$D_jD^j\Phi = 0\,, \tag{2.39}$$

and if we make use of (3.37) and (3.38) then (3 36-b) can be written in the form

$$D_jF^{ij} = -i[\Phi, D^i\Phi]. \tag{2 40}$$

The equations (2 39) and (2 40) are just the time-independent equations of motion (2 23) and (2 24) in the BPS limit, which establishes the second result

The two results together with the Bianchi identity

$$D^{\nu}F^{*}_{\mu\nu} = 0, \tag{2 41}$$

establish the fact that the Bogomol'nyi equations, with A_0=0, are equivalent to the time-independent equations of motion (2 23) and (2 24) m the BPS limit This implies that any x_4-independent solutions of the self-duality conditions will satisfy the time-independent equations of motion In addition, any solutions of the Bogomol'nyi equations (2 30), with A_0=0, satisfying the boundary conditions (2 25) are known as the monopole solutions and hence the Bogomol'nyi equations (2 30) become the monopole equations

2.5 The Prasad-Sommerfield Monopole

For monopole number n=1, solutions of the equations of motion (2 23) and (2 24), and of the Bogomol'nyi equations (2 30), are not too difficult to find 't Hooft [3] and Polyakov [4] found a monopole solution of the equations of motion in terms of radial functions The existence of radial functions with the right properties was rigorously proven by Tyupkin et al [29] In the BPS limit, the solution can actually be given explicitly on terms of elementary functions This was done by Prasad and Sommerfield [7] without making use of the Bogomol'nyi equations (2.30) With the help of the Bogomol'nyi equations, the construction of the n=1 monopole solutions is much simpler [8], as we will show now.

The Higgs field Φ is a linear combination of the generators τ^a, where the coefficients are function of the space variable x^i The simplest combination of such a Lie algebra valued function is $x_a\tau^a$ Therefore, because of the first asymptotic condition in (2 25), we would try $x_a\tau^a/r$, where $r=\sqrt{x^i x_i}$. Furthermore, the simplest forms of possible gauge potentials A_i are, $\varepsilon_{aij}x^j\tau^a$, $x_i x^a\tau_a$, and τ_i Because of the third condition in (2.25), asymptotically we want

$$D_i\Phi=(D_i\Phi)^a\tau_a=(\partial_i\Phi^a-\varepsilon^{abc}A_i^b\Phi^c)\tau_a=0, \tag{2 42}$$

to hold We find that A_i of the form $A_i=-\varepsilon_{aij}x^j r^{-2}\tau^a$ satisfies equation (2 42)

The expression we have so far are not defined at the origin, so it is natural to try the following ansatz

$$\Phi=\Phi^a\tau^a=K(r)\frac{x^a}{r}\tau^a, \qquad A_i=A_i^a\tau^a=-\varepsilon_{aij}H(r)\frac{x^j}{r^2}\tau^a \tag{2 43}$$

This yields the following

$$D^i\Phi=\frac{K}{r}\tau^i+\left(\frac{K}{r}\right)'\frac{x^i x^a}{r}\tau^a-\frac{KH}{r}\tau^i+\frac{KH}{r^3}x^i x^a\tau^a, \tag{2 44-a}$$

$$B^i=\left(\frac{H}{r^2}\right)'r\tau^i-\left(\frac{H}{r^2}\right)'\frac{x^i x^a}{r}\tau^a+\frac{2H}{r^2}\tau^i-\frac{H^2}{r^4}x^i x^a\tau^a \tag{2 44-b}$$

Now the Bogomol'nyi equations (2 30) are used (for n>0), which leads to the following equations

$$r^2\frac{dK}{dr} = H(2-H), \qquad \frac{dH}{dr} = K(1-H) \tag{2 45}$$

After a change of variables,

$$K(r) = k(r) - \frac{1}{r}, \qquad H(r) = 1 - rh(r) \tag{2 46}$$

we have

$$k'(r) = -h^2(r), \qquad h'(r) = -h(r)k(r) \tag{2 47}$$

Equations (2 47) yield

$$\left(k^2(r) - h^2(r)\right)' = 0,$$

and because of the asymptotic condition, $k \to 1$, and $h \to 0$ as $r \to \infty$, $k^2 - h^2 = 1$ must hold Hence, finally we have $k' = 1 - k^2$, which yields

$$k(r) = \coth(r), \qquad h(r) = \frac{1}{\sinh(r)} \tag{2 48}$$

We have chosen those solutions of (2 30) which satisfy the asymptotic condition (2 25) at infinity, and have the correct behaviour at the origin to make them smooth solutions In fact,

$$\Phi = \left(\frac{1}{3} - \frac{1}{45}r^2 + \frac{2}{945}r^4 - \frac{1}{4725}r^6 + \quad\right) x^a \tau^a \tag{2 50-a}$$

$$A_i = -\varepsilon_{aij}\left(\frac{1}{6} - \frac{7}{360}r^2 + \frac{31}{15120}r^4 - \frac{127}{604800}r^6 + \dots\right) x^j \tau^a \qquad (2.50\text{-b})$$

are the first few terms in the Taylor series representation of the solutions This shows that $\Phi^a, A_i^a \in C^\infty(R^3)$

Equations (2 43), (2 46) and (2 48) also show that

$$|\Phi| = 0 \Leftrightarrow \vec{x} = 0$$

Hence the maximum of the Higgs potentials (2 2) is at the origin Fig. 1 shows that for the Prasad-Sommerfield monopole the maximum of the potential energy density

$$\mathcal{E} = \frac{K'^2}{2} + \frac{H'^2}{r^2} + \frac{K^2(1-H)^2}{r^2} + \frac{H^2(2-H)^2}{2r^4}$$

$$= \frac{1}{r^4} - \frac{4}{r}\coth(r)\operatorname{cosech}(r) + 2\coth^2(r)\operatorname{cosech}^2(r) + \operatorname{cosech}^4(r), \qquad (2\ 51)$$

is also at the origin In the following, we will sometimes refer to the zeros of $|\Phi|$ as the locations of the monopoles, although the location of an extended object is not strictly defined

In this section we have seen that the construction of the n=1 monopole solution is not too difficult The construction of monopoles with n>1 turned out to be much more involved We will turn to this construction in the next chapter

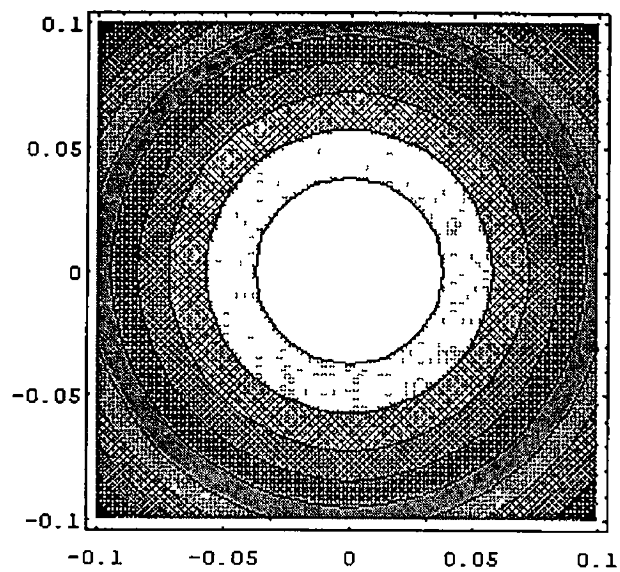

Fig 1

Contour plot of the energy density in the xy-plane

for the Prasad-Sommerfield monopole

Chapter 3

General Monopole Construction

In this chapter we introduce associated linear equations known as the Lax-pair equations and prove that the compatibility conditions of these equations are the self-duality conditions. To solve these equations we will study the Riemann-Hilbert problem and because the solution of this problem may not lead to real-valued solutions, we will study the reality of these solutions in great detail. This study will lead us to the existence of a GL(2,C) gauge transformation which before was not given in explicit form. We will study this gauge transformation and write it down in explicit form. At the end of this chapter, this gauge transformation is studied in detail.

3.1 The Associate Linear Equations

In this section we show that solutions of the self-duality equations (2.31) can be found by solving associated first order linear differential equations. These Lax-pair equations were given by Zakharov [30]. They also are the starting point of the Ward construction [11,12,13] To do this, we introduce first the associated linear equations

$$M_{\alpha'}k = i(D_{\alpha'}k) \tag{3 1}$$

Here $\alpha, \alpha' = 1,2$, $M_{\alpha'} = A_{\alpha'1} - \zeta A_{\alpha'2}$, and k is a 2x2 matrix The differential operator $D_{\alpha'}$ is defined by

$$D_{\alpha'} = \partial_{\alpha'1} - \zeta\partial_{\alpha'2}, \qquad \partial_{\alpha'\alpha} = \frac{\partial}{\partial x^{\alpha'\alpha}} \tag{3 2}$$

The coordinates $x^{\alpha'\alpha}$ and x^{μ} are related through the equation

$$\begin{pmatrix} x^{11} & x^{12} \\ x^{21} & x^{22} \end{pmatrix} = \begin{pmatrix} x^4 + ix^3 & x^2 + ix^1 \\ -x^2 + ix^1 & x^4 - ix^3 \end{pmatrix} \tag{3 3}$$

Until now x^{μ} were the standard coordinates on the Euclidian space $\mathbf{R}^4$ Now we treat x^{μ} and $x^{\alpha'\alpha}$ as the coordinates of the complexified Euclidian space At the end of the construction we, of course, have to go back to the real coordinates x^{μ} The fields $A_{\alpha'\alpha}$ and A_{μ} are related through the equation

$$A_{\alpha'\alpha}dx^{\alpha'\alpha} = A_{\mu}dx^{\mu} \tag{3 4}$$

Given the linear equations (3 1) it is easy to prove that their compatibility conditions are the self-duality conditions. To prove this fact we rewrite the linear equations (3.1) in the form

$$\left(M_{\alpha'} - iD_{\alpha'}\right)k = 0 \tag{3 5}$$

Then by operating on (3 5) with the operator $\varepsilon^{\alpha'\alpha}D_{\alpha}$, we can write down the following equation

$$\varepsilon^{\alpha\alpha'}D_{\alpha}\left(M_{\alpha'}k\right) - i\varepsilon^{\alpha\alpha'}D_{\alpha}D_{\alpha'}k = 0\,, \tag{3 6}$$

where $\varepsilon^{11} = \varepsilon^{22} = 0$, and $\varepsilon^{12} = -\varepsilon^{21} = 1$. The commutativity of the derivative D_{α} and the anti-symmetry of the symbol $\varepsilon^{\alpha'\alpha}$ lead to the following identity

$$\varepsilon^{\alpha'\alpha}D_{\alpha}D_{\alpha'}k = 0 \tag{3 7}$$

Making use of this identity and of (3 5), we can write down (3 6) in the form

$$\left(\varepsilon^{\alpha'\alpha}D_{\alpha}M_{\alpha'} + i\varepsilon^{\alpha'\alpha}M_{\alpha}M_{\alpha'}\right)k = 0\,, \tag{3 8}$$

where the left-hand side is a product of the matrix in brackets and the matrix k

If we choose the matrix k such that det k≠0, then the left matrix of equation (3 8) must be the zero matrix, which leads to the following equation

$$D_1M_2 - iM_2M_1 - D_2M_1 + iM_1M_2 = 0 \tag{3 9}$$

Also, by making use of

$$F_{\alpha'\alpha\,\beta'\beta} = \partial_{\alpha'\alpha}A_{\beta'\beta} - \partial_{\beta'\beta}A_{\alpha'\alpha} + i[A_{\alpha'\alpha}, A_{\beta'\beta}], \tag{3 10}$$

with $\beta, \beta' = 1,2$, we can easily write equation (3 9) as a polynomial of degree two in ζ as follows,

$$F_{11,21} - \zeta(F_{12,21} + F_{11,22}) + \zeta^2 F_{12,22} = 0 \tag{3 11}$$

Equation (3 11) implies the three conditions,

$$F_{11,21} = 0, \qquad F_{12,22} = 0, \qquad F_{12,21} + F_{11,22} = 0 \tag{3 12}$$

To prove that the conditions (3 12) are equivalent to the self-duality conditions (2 32), we use the relation (3 4) and express the fields $A_{\alpha\alpha'}$ in terms of A_μ as

$$\begin{aligned} A_{11} &= \frac{1}{2}(A_4 - iA_3), \qquad & A_{22} &= \frac{1}{2}(A_4 + iA_3), \\ A_{12} &= \frac{-1}{2}(A_1 + iA_2), \qquad & A_{21} &= \frac{-1}{2}(A_1 - iA_2) \end{aligned} \tag{3 13}$$

By using (3 3), we can express the derivatives $\partial_{\alpha\alpha'}$ in terms of ∂_μ as

$$\begin{aligned} \partial_{11} &= \frac{1}{2}(\partial_4 - i\partial_3), \qquad & \partial_{22} &= \frac{1}{2}(\partial_4 + i\partial_3), \\ \partial_{12} &= \frac{-1}{2}(\partial_1 + i\partial_2), \qquad & \partial_{21} &= \frac{-1}{2}(\partial_1 - i\partial_2) \end{aligned} \tag{3 14}$$

Substituting (3 13) and (3 14) into (3 12), we can write down these three conditions in the form of the self-duality conditions (2 32) This establishes that the compatibility conditions of the linear equations (3 1) are the self-duality conditions

3.2 The Riemann-Hilbert Problem

Any analytic nonsingular matrix function G of a complex variable λ defined in a neighborhood of the unit circle admits a decomposition $G(\lambda) = \phi_0 A \phi_\infty$, where ϕ_0 is analytic in the interior of the unit circle, ϕ_∞ in the exterior of the unit circle including ∞, and A is a diagonal matrix whose entries are integral powers of λ Any such decomposition is called a Birkhoff decomposition, which unfortunately is hard to find In the case where the diagonal matrix A is equal to the identity matrix, the Birkhoff decomposition is called the solution of the Riemann-Hilbert problem (R-H-P)

We have proved in the previous section that the problem of finding solutions of the self-duality conditions (2 32) can be reduce to finding matrices $A_{\alpha\alpha'}$ and k which satisfy the linear equations (3 1) The $A_{\alpha\alpha'}$ matrices can be obtained from a nonsingular matrix k if $(D_{\alpha'}k)k^{-1}$ is a linear function in ζ. In this section we will show that such a matrix k can be found by studying the solution of the R-H-P To do that, let us assume first that $\tilde{g}$ is an analytic matrix in the annular region $U_+ \cap U_-$ containing the unit circle $|\zeta| = 1$ Here U_+ contains the point $\zeta = 0$, and U_- contains the point $\zeta = \infty$ We assume also that $\det\tilde{g} = 1$, and that $\tilde{g}$ satisfies the equations

$$D_{\alpha'}\tilde{g} = 0, \qquad \alpha' = 1,2 \tag{3 15}$$

The R-H-P is to find two matrices $\tilde{k}_\pm$ with unit determinants, such that

$$\tilde{g} = \tilde{k}_-^{-1}\,\tilde{k}_+, \tag{3 16}$$

where $\tilde{k}_\pm$ are analytic matrices in U_+ and U_-, respectively Operating on both sides of (3 16) with the operator $D_{\alpha'}$ yields

$$(D_{\alpha'}\tilde{k}_+)\tilde{k}_+^{-1} = (D_{\alpha'}\tilde{k}_-)\tilde{k}_-^{-1}. \tag{3.17}$$

The left-hand side is clearly a Taylor series in ζ, and the Laurent expansion on the right-hand side can not contain any higher powers than ζ. Hence both sides of the equation (3 17) are a linear function in ζ, which establishes that solutions of the self-duality conditions can be found by solving the R-H-P

Furthermore, we would like to ensure that the solutions of the R-H-P, which solve the self-duality conditions, have the following properties:

(1) The gauge potentials $A_\mu(x)$ must take their values in the Lie algebra g of the Lie group SU(2), i e the fields $A_\mu(x)$ must satisfy,

$$\mathrm{tr}(A_\mu(x))=0, \tag{3 18-a}$$

$$A_\mu^*(x) = A_\mu(x) \tag{3 18-b}$$

Here A_μ^* is the hermitean conjugate of A_μ, defined as follows, $(A_\mu^*)_{ij} = (\overline{A}_\mu)_{ji}$, where $\overline{A}_\mu$ is the complex cojugate To ensure that the solution of the R-H-P leads to solutions satisfying the conditions (3 18-a,b) we will study in the next section the reality conditions in great detail and our study will continue in the next chapter (2) It is also required that the solutions of the self-duality conditions must be independent of the fourth component x^4 in order to be time-independent solutions of the equations of motion, i e , the solutions must satisfy the time independence condition

$$\partial_0 A_\mu = 0 \tag{3 19}$$

In the next chapter, we will see that this requirement can be incorporated in the construction of the transition matrix

(3) The solutions must have finite energy, or, according to (2.29), finite winding number n. This condition will also be discussed in the next chapter, where we construct the n=2 monopole.

(4) An essential requirement is that the fields $A_\mu(x)$ are $C^\infty(\mathbf{R}^3)$ functions. We will only need that the construction of the transitions matrix leads to C^∞ functions for monopoles close together [14] This will be shown below.

3.3 The Reality Condition

Given any transition matrix which allows for the decomposition (3 16), the solution of the R-H-P may not lead to real solutions. In this section we would like to impose conditions on the transition matrix such that, for real x^μ, the fields $A_\mu(x)$ take their values in the Lie algebra g of the Lie group SU(2) To do that, we prove the following If the transition matrix $\tilde{g}(\zeta, x)$ satisfies the conditions

$$[\tilde{g}(\zeta,x)]^* = \tilde{g}(-\bar{\zeta}^{-1},x), \tag{3 20-a}$$

$$\det(\tilde{g})=1, \tag{3 20-b}$$

then in some gauge, the fields $A_\mu(x)$ will take their values in the Lie algebra g of the Lie group SU(2), i e the $A_\mu(x)$ satisfy the properties (3 18-a) and (3 18-b)

We use the fact that both sides of (3 17) are linear functions in ζ and write,

$$A_{\alpha'1} - \zeta A_{\alpha'2} = i(D_{\alpha'}\tilde{k}_+)\tilde{k}_+^{-1} = i(D_{\alpha'}\tilde{k}_-)\tilde{k}_-^{-1}, \tag{3 21}$$

which is equivalent to the associate linear equations (3 1) First, if we choose the matrices $\tilde{k}_\pm$ such that $\det(\tilde{k}_\pm)=1$, then (3 21) implies that the trace of $A_\mu(x)$ is zero which establishes that the fields $A_\mu(x)$ have the property (3 18-a)

Next we will prove that in some gauge the fields $A_\mu(x)$ have the property (3.18-b). If we make use of the condition (3.20-a), then (3.16) yields

$$\widetilde{k}_-(\zeta,\vec{x})\left[\widetilde{k}_+(-\bar{\zeta}^{-1},x)\right]^* = \widetilde{k}_+(\zeta,\vec{x})\left[\widetilde{k}_-(-\bar{\zeta}^{-1},x)\right]^*.$$

The analyticity properties of $\widetilde{k}_+$ and $\widetilde{k}_-$ imply that both sides of the above equation are independent of ζ, which allows us to write the products in the form

$$\widetilde{k}_-(\zeta,x)\left[\widetilde{k}_+(-\bar{\zeta}^{-1},x)\right]^* = \widetilde{k}_+(\zeta,x)\left[\widetilde{k}_-(-\bar{\zeta}^{-1},x)\right]^* = M(x). \qquad (3\ 22)$$

Now ζ can be set equal to zero in (3 22), which implies that the matrix $M(x)$ is hermitian

The solution of the R-H-P does not determine $\widetilde{k}_+$ and $\widetilde{k}_-$ uniquely In fact, there exists the gauge freedom,

$$\widetilde{k}_+ \to \Omega\, \widetilde{k}_+, \qquad \widetilde{k}_- \to \Omega\, \widetilde{k}_- \qquad (3\ 23)$$

Here the matrix $\Omega(x)$ is an element of the general linear group GL(2,C) The gauge transformation (3 23) yields

$$A^\mu \to A'^\mu = \Omega A^\mu \Omega^{-1} + i(\partial^\mu \Omega)\Omega^{-1} \qquad (3\ 24)$$

The invariance of the Lagrangian (2 9) under this gauge transformation, which we proved in the previous chapter, imply that the gauge fields $A'^\mu(x)$ solve the equations of motion if the $A^\mu(x)$ do

Under the gauge transformation (3 23) the matrix $M(x)$ transforms as follows,

$$M(x) \to \Omega(x)\ M(x)\ \Omega^*(x) \tag{3 25}$$

Since we can choose $\Omega(x)$ such that after the transformation (3 23) we have

$$\Omega(x) M(x) \Omega^*(x) = \pm I, \tag{3 26}$$

we can achieve that $M(x)= \pm I$ holds in equation (3 22) Here I is the identity matrix We assume from now on that such a gauge transformation has been made

The hermiticity condition, $A_\mu^* = A_\mu$, can be written as

$$A_{11}^* = A_{22}, \qquad A_{21}^* = -A_{12}, \tag{3 27}$$

which is equivalent to

$$[A_{11} - \zeta A_{12}]^* = \overline{\zeta}[A_{21} - (-\overline{\zeta}^{-1})A_{22}] \tag{3 28}$$

If we make use of

$$\begin{aligned} A_{11} - \zeta A_{12} &= i[D_1^\zeta \widetilde{k}_+(\zeta)]\widetilde{k}_+^{-1}(\zeta), \\ A_{21} - (-\overline{\zeta}^{-1})A_{22} &= i[D_2^{-\overline{\zeta}^{-1}} \widetilde{k}_-(-\overline{\zeta}^{-1})]\widetilde{k}_-^{-1}(-\overline{\zeta}^{-1}), \end{aligned} \tag{3 29}$$

where

$$D_1^\zeta = \partial_{11} - \zeta\partial_{12}, \qquad D_2^{-\overline{\zeta}^{-1}} = \partial_{21} - (-\overline{\zeta}^{-1})\partial_{22}, \tag{3 30}$$

then the equation (3 28) can be written in the form

$$[(D_1^{\zeta}\widetilde{k}_+(\zeta))\widetilde{k}_+^{-1}(\zeta)]^* = -\bar{\zeta}[D_2^{-\bar{\zeta}^{-1}}\widetilde{k}_-(-\bar{\zeta}^{-1})]\widetilde{k}_-^{-1}(-\bar{\zeta}^{-1}). \tag{3.31}$$

We now want to show that (3.31), and that therefore the hermiticity condition, hold. Making use of (3.22) with M=±I, we get

$$\left[\widetilde{k}_+(\zeta)\right]^* = \pm\widetilde{k}_-^{-1}(-\bar{\zeta}^{-1}) \tag{3 32}$$

Together with the identities

$$\begin{aligned} &\overline{D_1^{\zeta}} = \bar{\zeta}\ D_2^{-\bar{\zeta}^{-1}}, \\ &(D_1^{\zeta}\widetilde{k}_+(\zeta))\widetilde{k}_+^{-1}(\zeta) = -\widetilde{k}_+(\zeta)(D_1^{\zeta}\widetilde{k}_+^{-1}(\zeta)), \end{aligned} \tag{3.33}$$

the right-hand side of the equation (3 31) can be obtained from the left-hand side This establishes that the fields $A_\mu(x)$ satisfy the property (3 18-b)

3.4 The Gauge Transformation

In this section we will find the gauge transformation (3 23) by writing down the gauge matrix Ω in explicit form To do that let us first write the matrix M in the form

$$M(x) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \tag{3 34-a}$$

Equation (3 22) implies that the matrix M is hermitian and

$$m_{11}m_{22} - m_{12}m_{21} = 1 \tag{3 34-b}$$

The eigenvalues of the matrix M are

$$\lambda_{1,2} = \frac{1}{2}\left\{(m_{11} + m_{22}) \pm \sqrt{(m_{11} + m_{22})^2 - 4}\right\}. \tag{3.35}$$

The fact that the matrix M is hermitian implies that the eigenvalues $\lambda_{1,2}$ are real functions of x. To ensure that $\lambda_{1,2} \in \mathbf{R}$ and $\lambda_1 \neq \lambda_2$, we have to restrict our attention to one of the cases

$$m_{11} + m_{22} > 2, \tag{3.36-a}$$

$$m_{11} + m_{22} < -2 \tag{3.36-b}$$

The eigenvalues given by (3 35) imply that the case (3 36-a) corresponds to the upper sign in (3 26), and the case (3 36-b) corresponds to the lower sign

We now solve our problem of finding an analytic matrix Ω satisfying the equation

$$\Omega^*(x)\,\Omega(x) = \pm\, M^{-1}(x) \tag{3 37}$$

If we choose to work with the upper sign, we can find the matrix Ω as follows First we assume that Ω is a hermitian matrix of the form

$$\Omega(x) = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}, \tag{3 38-a}$$

with

$$\Omega_{11}\Omega_{22} - \Omega_{12}\Omega_{21} = 1 \tag{3 38-b}$$

Then by substituting (3 34) and (3 38-a) into equation (3 37), we can write down the following equations,

$$\Omega^2_{11}+\Omega_{12}\Omega_{21}=m_{22}, \qquad \Omega^2_{22}+\Omega_{12}\Omega_{21}=m_{11}, \tag{3 39}$$

$$\Omega_{12}(\Omega_{11}+\Omega_{22})=-m_{12}, \qquad \Omega_{21}(\Omega_{11}+\Omega_{22})=-m_{21} \tag{3 40}$$

Solving equations (3 39) by making use of (3 38-b) yields

$$\Omega_{11}+\Omega_{22}=\Gamma_1, \qquad \Omega_{11}-\Omega_{22}=(m_{22}-m_{11})\Gamma_1^{-1}, \tag{3 41}$$

where

$$\Gamma_1=\pm\sqrt{2+m_{11}+m_{22}} \tag{3 42}$$

From the equations (3 40) we obtain for Ω_{12} and Ω_{21},

$$\Omega_{12}=-m_{12}\Gamma_1^{-1}, \qquad \Omega_{21}=-m_{21}\Gamma_1^{-1} \tag{3 43}$$

The equations (3 41) can be solved to give Ω_{11} and Ω_{22} in the form

$$\Omega_{11}=(1+m_{22})\Gamma_1^{-1}, \qquad \Omega_{22}=(1+m_{11})\Gamma_1^{-1} \tag{3 44}$$

Now the matrix $\Omega(x)$ in the gauge transformation (3 23) can be written

$$\Omega=\Gamma_1^{-1}\begin{pmatrix} 1+m_{22} & -m_{12} \\ -m_{21} & 1+m_{11} \end{pmatrix} \tag{3 45}$$

The gauge matrix (3.45) satisfies the equation (3.26) and hence the reality of the fields $A_\mu(x)$ is guaranteed. In the next chapter we will see that the matrix (3.45) leads to the reality of the solutions when we write down the series solutions of the monopole equations We will also see that the series expansion of the function Γ_1 of (3 42) does not vanish at the origin which will guarantee the regularity of the solutions This implies that Ω of (3 45) ensures both the reality and the regularity of the solutions This will become clear in the next section

In addition, if we choose to work with the upper sign of (3.37), it is also possible to assume that Ω is skew-hermitean, and M satisfies (3.36-a) In this case we found the matrix Ω has the form

$$\Omega=\Gamma_2^{-1}\begin{pmatrix}1-m_{22} & m_{12} \\ m_{21} & 1-m_{11}\end{pmatrix}, \tag{3 46}$$

where

$$\Gamma_2=\pm\sqrt{2-(m_{11}+m_{22})} \tag{3 47}$$

Furthermore, if we choose to work with the lower sign of (3 37), then we will have the following cases (1) If we assume that Ω is hermitean and M satisfies (3 36-b), then the matrix Ω can be found in the form (3 46) (2) If we assume that Ω is skew-hermitean and M satisfies (3 36-a), then the matrix Ω can be found in the form (3 45)

We will see m the next chapter that the series expansion of the function Γ_2 vanishes at the origin which will lead to the singularity of the series solutions To ensure regularity of the solutions, we will restrict our attention, in the next chapters, to a matrix Ω of the form (3 45)

3.5 Factorization of the Gauge Matrix

In this section we build up the gauge matrix (3.45), which we have found in the previous section, as a product of two matrices. This study will show that the gauge transformation (3.32) with Ω of the form (3 45) consists of two gauge transformations, the first is a GL(2,C)-gauge transformation to ensure the reality of the solutions and the second is a U(2)-gauge transformation to ensure the regularity of the solutions Therefore this study will clarify that the gauge transformation (3 32) with (3 45) is a two-in-one gauge transformation

Let us study again our problem of finding a matrix $\tilde{\Omega}$, such that

$$\tilde{\Omega} M \tilde{\Omega}^{*} = \pm I, \tag{3 48}$$

and use the fact that the matrix M can be diagonalized with a unitary matrix, say P This can be written as,

$$P^{*} M P = D, \tag{3 49}$$

where D is a diagonal matrix By using techniques of linear algebra we can find the matrix P as follows First we find the eigenvalues of the matrix M, as we did in the last section, and the corresponding normalized eigenvectors Then with the help of the identity

$$m_{11} - \lambda_1 = -(m_{22} - \lambda_2), \tag{3 50}$$

we can write the matrix P in the form

$$P = \frac{1}{N}\begin{pmatrix} (m_{22} - \lambda_1) & m_{12} \\ -m_{21} & -(m_{11} - \lambda_2) \end{pmatrix}, \tag{3 51}$$

where

$$N^2 = m_{22}\lambda_2 + m_{11}\lambda_1 - 2. \tag{3.52}$$

Using (3 51), the diagonal matrix D of (3 49) can be written in the form

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{3 53}$$

Equation (3 49) can be changed to

$$(P\Pi)^* M(P\Pi) = \pm I \tag{3.54}$$

If we restrict our attention to the case (3 36-a), the matrix Π can take the form

$$\Pi = \begin{pmatrix} \sqrt{\lambda_2} & 0 \\ 0 & \sqrt{\lambda_1} \end{pmatrix} \tag{3 55}$$

Now the equations (3 48) and (3 54) yield

$$\widetilde{\Omega} = \Pi P^* = \frac{1}{\sqrt{d_{11}}} \begin{pmatrix} (m_{22} - \lambda_1) & -m_{12} \\ \lambda_1 m_{21} & -\lambda_1(m_{11} - \lambda_2) \end{pmatrix}, \tag{3 56}$$

where

$$d_{11} = m_{22} - 2\lambda_1 + m_{11}\lambda_1^2 \tag{3 57}$$

Equation (3 57), as we will see in the next chapter, implies that the series expansion of the function d_{11} vanishes at the origin This will lead to the singularity of the solutions of the monopole equations. So the 'textbook' approach to the problem (3 48) leads to singular solutions, whereas the solution from section 3.4 is regular Of course, we can always turn one into the other by a unitary matrix Θ, which will ensure the regularity of the solutions. Without the result from the previous section such a unitary matrix is very hard to find

If we make use of the result of the previous section, it is possible to write this unitary matrix in explicit form To find this matrix, let us first write Θ as

$$\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix}, \tag{3 58}$$

$$\Theta_{11}\Theta_{22} - \Theta_{12}\Theta_{21} = 1, \quad \Theta_{11} = \overline{\Theta}_{22}, \qquad \Theta_{12} = -\overline{\Theta}_{21} \tag{3 59}$$

It follows from (3 26) and (3 48) that the unitary matrix Θ is

$$\Theta = \Omega\widetilde{\Omega}^{-1} \tag{3 60}$$

Finally, by substituting (3 45) and (3 56) into (3 60) we can write

$$\Theta = \frac{1}{\Gamma_1\sqrt{d_{11}}}\begin{pmatrix} (1+\lambda_2)(1-m_{11}\lambda_1) & m_{12}(1+\lambda_1) \\ -m_{21}(1+\lambda_1) & (1+\lambda_2)(1-m_{11}\lambda_1) \end{pmatrix} \tag{3 61}$$

Chapter 4

The n=2 Monopole Solutions

In this chapter we will introduce an ansatz $\widetilde{g}_n$ which leads to a (4n-1)-parameter family of solutions These solutions solve the time-independent equations of motion (2 23) and (2 24), and describe n-monopoles sitting somewhere in space We will study in great detail the ansatz $\widetilde{g}_n$ for n=2 and then construct 2-monopole solutions of the monopole equations which will be used to discuss the scattering of two monopoles We study also the reality of these solutions The regularity of these solutions will be investigated in great detail and at the end of this chapter the splitting of the transition matrix for n=2 is reviewed

4.1 The Ward Ansatz

In the previous chapter we showed that solutions of the equations of motion (2.23,24) can be found by studying the solution of the R-H-P for a suitable transition matrix $\tilde{g}_n$ In this section we will study an ansatz for $\tilde{g}_n$, given by Ward [12,13,14] and generalised in Refs 17,31, which leads to n-monopole solutions Ward works with the four complex coordinates $(\omega_1,\omega_2,\pi_1,\pi_2)$ on twistor space, which are related to the coordinates x_{pq} by

$$\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}=\begin{pmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix}\begin{pmatrix}\pi_1\\ \pi_2\end{pmatrix} \qquad (4\ 1)$$

Then the solutions of these equations form a complex projective line in $\mathbf{CP}^3$ space, so to each point in $\mathbf{R}^4$ there exists a complex line in $\mathbf{CP}^3$, and each point in the complex projective space $\mathbf{CP}^3$ lies on exactly one such line, unless $\pi_1 = \pi_2 = 0$, and in this case the twistor fibration sends it to infinity. If we remove the subspace $\pi_p = 0$ by assuming that π_1 and π_2 are not both equal to zero, and factor the π_p-space by the proportionality relation $\pi_p \sim \lambda\pi_p$, where λ is a non-zero complex constant, then we obtain the complex projective space $\mathbf{CP}^1$ This space can be covered with two patches U_+ and U_- If we set $\zeta = \pi_1 / \pi_2$, then we can assume that the patch U_+ contains the point $\zeta = 0$ ($\pi_1 = 0$), and the patch U_- contains the point $\zeta = \infty$ ($\pi_2 = 0$) We define also $\mathbf{U}_+ \cap \mathbf{U}_-$ to be the annular region containing the unit circle $|\zeta| = 1$

Next we write the equation (4 1) in the form,

$$\begin{aligned}\omega_1 &= x_{11}\pi_1 + x_{12}\pi_2,\\ \omega_2 &= x_{21}\pi_1 + x_{22}\pi_2,\end{aligned} \qquad (4\ 2)$$

and define new coordinates μ and ν,

$$\mu = i\omega_2 / \pi_2, \qquad \nu = i\omega_1 / \pi_1 \tag{4.3}$$

Making use of (4.2) yields,

$$\mu(\zeta, x) = ix_{21}\zeta + ix_{22}, \qquad \nu(\zeta, x) = ix_{12}\zeta^{-1} + ix_{11}, \tag{4.4}$$

where $\mu(\zeta, x)$ for fixed x is an analytic function of ζ in $\mathbf{U}_+$ and $\nu(\zeta, x)$ is an analytic function of ζ^{-1} in $\mathbf{U}_-$. Furthermore, for real x, $\mu(\zeta, x)$ and $\nu(\zeta, x)$ are related to each other through the reality condition,

$$\overline{\mu}(\zeta, x) = -\nu(-\overline{\zeta}^{-1}, x) \tag{4.5}$$

We also define

$$\omega(\zeta, x) = \mu(\zeta, x) - \nu(\zeta, x), \tag{4.6}$$

such that for fixed x, $\omega(\zeta, x)$ is an analytic function of ζ and ζ^{-1} in the annular region $\mathbf{U}_+ \cap \mathbf{U}_-$. It is clear that $\omega(\zeta, x)$ satisfies the reality condition,

$$\overline{\omega}(\zeta, x) = \omega(-\overline{\zeta}^{-1}, x) \tag{4.7}$$

Ward [12,13,14] and Corrigan and Goddard [17] suggested that a transition matrix of the form,

$$\widetilde{g}_n(\zeta, x) = \begin{pmatrix} \psi_n^{-1}(e^{\omega} + (-1)^n e^{-\omega}) & (-1)^n \zeta^n e^{-\omega} \\ \zeta^{-n} e^{-\omega} & \psi_n e^{-\omega} \end{pmatrix}, \tag{4.8}$$

is a good starting point to solve the R-H-P. Here the function $\psi_n(\zeta, x)$ is a polynomial of degree n in ζ and ζ^{-1} and satisfies the reality condition

$$\overline{\psi}_n(\zeta, x) = \psi_n(-\overline{\zeta}^{-1}, x) \tag{4 9}$$

The solution of the R-H-P for $\tilde{g}_n$ of the form (4.8) leads to n-monopole solutions of the time-independent equations of motion (2.23) and (2,24) These solutions belong to a (4n-1)-parameter family of solutions, where the (4n-1) parameters describe the position and the phase angle of each monopole [17].

The transition matrix $\tilde{g}_n(\zeta, x)$ is an analytic function for all $x \in \mathbf{M}$, where $\mathbf{M}$ is some region in $\mathbb{R}^4$ For fixed x, $\tilde{g}_n$ is an analytic function of ζ and ζ^{-1} only in some annular neighbourhood $\mathbb{N}$ [16,17], say

$$\mathbb{N} = \left\{\zeta \; \delta_+ < |\zeta| < \delta_-\right\} , \tag{4 10}$$

such that the neighbourhood $\mathbb{N}$ contains the unit circle $|\zeta| = 1$ Using conditions (4 7) and (4 9), it is easy to see that the transition matrix (4 8) satisfies the conditions (3 20-a,b), which leads to the reality of the fields A_μ The dependence of $\tilde{g}_n$ on x only through the combination (4 6) ensures the x^4-independence which guarantees that the solution of the R-H-P will lead to solutions satisfying the time-mdependence condition (3 19)

Because in this thesis we are interested in studying the scattering of two monopoles, we express $\tilde{g}_n$ for n=2, which yields

$$\tilde{g}_2(\zeta, x) = \begin{pmatrix} \psi_2^{-1}(e^{\omega} + e^{-\omega}) & \zeta^2 e^{-\omega} \\ \zeta^{-2} e^{-\omega} & \psi_2 e^{-\omega} \end{pmatrix} \tag{4 11}$$

The function $\psi_2(\zeta, x)$ is a polynomial of degree 2 in ζ and ζ^{-1}, which will be discussed below. In the remainder of this chapter we will show how to construct the 2-monopole solutions with the right properties from this transition matrix.

4.2 The Solution of the R-H-P

In this section we will construct the 2-monopole by solving the R-H-P for $\tilde{g}_2$ of the form (4 12) Because the solution of the R-H-P is very hard to find for the transition matrix $\tilde{g}_2$, let us introduce first an equivalence relation (~) as follows· A matrix g_2 is said to be equivalent to the matrix $\tilde{g}_2$ (written $g_2 \sim \tilde{g}_2$), if there exist two matrices Λ_+ and Λ_- both of them in the special linear group $SL(2,\mathbf{C})$, such that

$$\tilde{g}_2 = \Lambda_- g_2 \Lambda_+, \tag{4 12}$$

where Λ_+ is an analytic function in $\mathbf{M} \times \mathbf{U}_+$, and Λ_- is an analytic function in $\mathbf{M} \times \mathbf{U}_-$ If we choose

$$g_2 = \begin{pmatrix} \zeta^2 & \rho_2 \\ 0 & \zeta^{-2} \end{pmatrix}, \qquad \rho_2 = \frac{e^{2\mu} + e^{2\nu}}{\psi_2(\mu, \nu, \zeta)}, \tag{4 13}$$

it is easy to find the matrices Λ_+ and Λ_-, which are of the form

$$\Lambda_- = \begin{pmatrix} e^{-\nu} & 0 \\ 0 & e^{\nu} \end{pmatrix}, \qquad \Lambda_+ = \begin{pmatrix} 0 & -e^{\mu} \\ e^{-\mu} & \zeta^2 \psi_2 e^{-\mu} \end{pmatrix} \tag{4 14}$$

The function ρ_2 of (4 13), for fixed x, can be expanded as a Laurent series of the form

$$\rho_2 = \sum_{r=-\infty}^{\infty} \Delta_r \zeta^{-r}. \tag{4.15}$$

Here the coefficients Δ_r $(r = 0 \pm 1 \pm 2 \pm \ldots)$ are functions of x and the parameters These coefficients can also be found from the contour integral

$$\Delta_r = \frac{1}{2\pi i} \oint \zeta^{r-1} \rho_2 d\zeta, \tag{4 16}$$

where the contour integral is taken around the unit circle $|\zeta| = 1$ and is such that the coefficients Δ_r are analytic functions of x in the region $\mathbf{M}$. Furthermore, the fact that the transition matrix (4.13) satisfies $D_{\alpha'} g_2 = 0$ yields,

$$D_{\alpha'} \rho_2 = 0 \tag{4 17}$$

Equation (4 17) implies that the coefficients Δ_r must satisfy,

$$\partial_{\alpha' 1} \Delta_r - \partial_{\alpha' 2} \Delta_{r+1} = 0 \tag{4 18}$$

To solve the R-H-P, we first assume that the transition matrix (4 13) can be split as follows,

$$g_2 = k_-^{-1} k_+, \tag{4 19}$$

where the matrices k_+ and k_- are of the form,

$$k_+^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1, \tag{4 20}$$

$$k_-^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad \alpha\delta - \beta\gamma = 1 \tag{4 21}$$

Here a,b,c and d are analytic functions in $\mathbf{M} \times \mathbf{U}_+$, and α, β, γ and δ in $\mathbf{M} \times \mathbf{U}_-$.

By operating on the both sides of (4.19) with the operator (3.2) and making use of $D_{\alpha'} g_2 = 0$ we find

$$k_+(D_{\alpha'} k_+^{-1}) = k_-(D_{\alpha'} k_-^{-1}) = L_{\alpha'}(\zeta, x) \tag{4.22}$$

Substituting (4 20,21) into (4 22) yields

$$L_{\alpha'}(\zeta, x) = \begin{pmatrix} dD_{\alpha'}a - bD_{\alpha'}c & dD_{\alpha'}b - bD_{\alpha'}d \\ aD_{\alpha'}c - cD_{\alpha'}a & bD_{\alpha'}c - dD_{\alpha'}a \end{pmatrix}$$

$$= \begin{pmatrix} \delta D_{\alpha'}\alpha - \beta D_{\alpha'}\gamma & \delta D_{\alpha'}\beta - \beta D_{\alpha'}\delta \\ \alpha D_{\alpha'}\gamma - \gamma D_{\alpha'}\alpha & \beta D_{\alpha'}\gamma - \delta D_{\alpha'}\alpha \end{pmatrix} \tag{4 23}$$

The analyticity properties of the matrices k_+, k_- and the fact that the coefficients Δ_r satisfy the partial differential equation (4 18) imply that $L_{\alpha'}(\zeta, x)$ is a linear function in ζ only, which allows us to write

$$i\left[A_{\alpha'1} - \zeta A_{\alpha'2}\right] = L_{\alpha'}(\zeta, x) \tag{4 24}$$

If we choose the R_2 gauge, we can follow the splitting of Ref 16 for a matrix of the form (4 13), which will be explained in detail in the last section of this chapter The linear function $L_{\alpha'}(\zeta, x)$ can be given in the form

$$L_{\alpha'}(\zeta, x) = \frac{1}{2F}\begin{pmatrix} (\partial_{\alpha'1}F + \zeta\partial_{\alpha'2}F) & 2\zeta\partial_{\alpha'1}E \\ -2\partial_{\alpha'2}G & -(\partial_{\alpha'1}F + \zeta\partial_{\alpha'2}F) \end{pmatrix} \tag{4 25}$$

Here $A_{\alpha'\alpha}$ and $\partial_{\alpha'\alpha}$ are as defined by (3 13) and (3 14), respectively, and the functions E, F and G are given by

$$E = \frac{-\Delta_1}{\Delta}, \qquad F = \frac{\Delta_0}{\Delta}, \qquad G = \frac{-\Delta_{-1}}{\Delta} \tag{4 26-a}$$

Δ must be a non vanishing function of x, which is given by

$$\Delta = \Delta_0^2 - \Delta_1\Delta_{-1} \tag{4 26-b}$$

The equation (4 24) with $L_{\alpha'}(\zeta, x)$ of the form (4 25) can be easily expanded for $\alpha' = 1,2$ to give the fields $A_{\alpha'\alpha}$ in the form

$$A_{11} = \frac{-1}{2F}\begin{pmatrix} \partial_{11}F & 0 \\ -2\partial_{12}G & -\partial_{11}F \end{pmatrix}, \qquad A_{22} = \frac{-1}{2F}\begin{pmatrix} -\partial_{22}F & -2\partial_{21}E \\ 0 & \partial_{22}F \end{pmatrix},$$

$$A_{12} = \frac{-1}{2F}\begin{pmatrix} -\partial_{12}F & -2\partial_{11}E \\ 0 & \partial_{12}F \end{pmatrix}, \qquad A_{21} = \frac{-1}{2F}\begin{pmatrix} \partial_{21}F & 0 \\ -2\partial_{22}G & -\partial_{21}F \end{pmatrix} \tag{4 27}$$

Making use of (3 13,14), we can express the fields $A_{\alpha'\alpha}$ in terms of A_μ as follows,

$$A_1 = \frac{1}{2F}\begin{pmatrix} -\partial_2 F & -(\partial_4 - i\partial_3)E \\ -(\partial_4 + i\partial_3)G & \partial_2 F \end{pmatrix},$$

$$A_2 = \frac{1}{2F}\begin{pmatrix} \partial_1 F & i(\partial_4 - i\partial_3)E \\ -i(\partial_4 + i\partial_3)G & -\partial_1 F \end{pmatrix}, \tag{4 28-a}$$

$$A_3 = \frac{1}{2F}\begin{pmatrix} \partial_4 F & -i(\partial_1 - i\partial_2)E \\ i(\partial_1 - i\partial_2)G & -\partial_4 F \end{pmatrix},$$

$$A_4 = \frac{1}{2F}\begin{pmatrix} -\partial_3 F & (\partial_1 - i\partial_2)E \\ (\partial_1 + i\partial_2)G & \partial_3 F \end{pmatrix}. \tag{4 28-b}$$

The fields A_4 $(=\Phi)$ and A_j (j=1,2,3) solve the time-independent equations of motion (2 23,24), and for g_2 of the form (4 13) these solutions describe two monopoles sitting somewhere in space

4.3 The Reality of the Solutions

The fields Φ and A_j must satisfy (3 18-a,b) in order to be real-valued solutions but when we write them down as functions of x^1, x^2 and x^3, we discover that they are complex-valued functions In the previous chapter we proved that the gauge (3 23) exists and guarantees the reality of these fields In this section we will study the reality of these fields and continue our discussion from the previous chapter First we make use of (4 19) and write (4 12) in the form

$$\tilde{g}_2 = \Lambda_- k_-^{-1} k_+ \Lambda_+ \tag{4 29}$$

The fact that the transition matrix $\tilde{g}_2$ satisfies the reality condition (3 20-a) yields

$$(k_-)(\Lambda_-^{-1})(\Lambda_+)^*(k_+)^* = (k_+)(\Lambda_+)(\Lambda_-^{-1})^*(k_-)^*,$$

and the analyticity property of our matrices implies that both sides of the above equation are indeed independent of ζ This yields

$$(k_-)^\circ(\Lambda_-^{-1})^\circ(\Lambda_+^*)^\circ(k_+^*)^\circ = (k_+)^\circ(\Lambda_+)^\circ((\Lambda_-^{-1})^*)^\circ(k_-^*)^\circ = M, \qquad (4.30)$$

where the superscript zero labels the ζ-independent term of the corresponding Taylor expension in ζ^{-1} or ζ respectively, and the matrix M is a function of x.

To find the matrix M, we use the notation from section 4.2 and the results of the splitting in section 4.5, which yields,

$$\begin{aligned} M &= \begin{pmatrix} \delta_0 & -\beta_0 \\ -\gamma_0 & \alpha_0 \end{pmatrix} \begin{pmatrix} e^{\nu} & 0 \\ 0 & e^{-\nu} \end{pmatrix}^\circ \begin{pmatrix} 0 & e^{\nu} \\ -e^{-\nu} & \zeta^{-2}\psi_2 e^{\nu} \end{pmatrix}^\circ \begin{pmatrix} \bar{d}_0 & -\bar{c}_0 \\ -\bar{b}_0 & \bar{a}_0 \end{pmatrix}, \\ &= \begin{pmatrix} d_0 & -b_0 \\ -c_0 & a_0 \end{pmatrix} \begin{pmatrix} 0 & -e^{\mu} \\ e^{-\mu} & \zeta^2\psi_2 e^{-\mu} \end{pmatrix}^\circ \begin{pmatrix} e^{-\mu} & 0 \\ 0 & e^{\mu} \end{pmatrix}^\circ \begin{pmatrix} \bar{\delta}_0 & -\bar{\gamma}_0 \\ -\bar{\beta}_0 & \bar{\alpha}_0 \end{pmatrix}, \end{aligned} \qquad (4.31)$$

where

$$\begin{aligned} & d_0 = c_2 = \gamma_0 = 0, && c_0 = d_2 = \delta_0 = \sqrt{F}, \\ & c_1 = G/\sqrt{F}, && d_1 = E/\sqrt{F}, \\ & a_0 = -\sum_{i=0}^{2} c_i \Delta_{i-2}, && \beta_0 = \sum_{i=0}^{2} d_i \Delta_i \end{aligned} \qquad (4\ 32)$$

By substituting (4 32) into (4 31), we can easily find the matrix M, and by using this matrix we can write down the matrix Ω of (3 45) in explicit form The series expansion of the gauge matrix Ω will be discussed in great detail in the next chapter

To write the fields A_μ of (4 28-a,b) in the real gauge we make use of the gauge transformation (3 24) and

$$A'_\mu = \frac{1}{2} A'^a_\mu \sigma^a = \frac{1}{2}\begin{pmatrix} A'^3_\mu & A'^1_\mu - iA'^2_\mu \\ A'^1_\mu + iA'^2_\mu & -A'^3_\mu \end{pmatrix}. \tag{4.33}$$

Then the real fields Φ' and A'_j can be found and are given by

$$\Phi'^3 = \frac{-1}{F}\left\{\tilde{\Omega}\partial_3 F - \Omega_{12}\Omega_{22}\tilde{\partial}_{12} G + \Omega_{21}\Omega_{11}\tilde{\tilde{\partial}}_{12} E\right\},$$

$$\Phi'^1 - i\Phi'^2 = \frac{1}{F}\left\{2\Omega_{12}\Omega_{11}\partial_3 F - \Omega_{12}^2\tilde{\partial}_{12} G + \Omega_{11}^2\tilde{\tilde{\partial}}_{12} E\right\},$$

$$\Phi'^1 + i\Phi'^2 = \frac{1}{F}\left\{-2\Omega_{21}\Omega_{22}\partial_3 F + \Omega_{22}^2\tilde{\partial}_{12} G - \Omega_{21}^2\tilde{\tilde{\partial}}_{12} E\right\}, \tag{4 34-a}$$

$$A'^3_1 = \frac{-1}{F}\left\{\tilde{\Omega}\partial_2 F + \Omega_{12}\Omega_{22}\tilde{\partial}_{43} G - \Omega_{21}\Omega_{11}\tilde{\tilde{\partial}}_{43} E\right\} - 2i\left\{\Omega_{11}\partial_1\Omega_{22} - \Omega_{12}\partial_1\Omega_{21}\right\},$$

$$A'^1_1 - iA'^2_1 = \frac{1}{F}\left\{2\Omega_{12}\Omega_{11}\partial_2 F + \Omega_{12}^2\tilde{\partial}_{43} G - \Omega_{11}^2\tilde{\tilde{\partial}}_{43} E\right\} - 2i\left\{-\Omega_{11}\partial_1\Omega_{12} + \Omega_{12}\partial_1\Omega_{11}\right\},$$

$$A'^1_1 + iA'^2_1 = \frac{1}{F}\left\{-2\Omega_{21}\Omega_{22}\partial_2 F - \Omega_{22}^2\tilde{\partial}_{43} G + \Omega_{21}^2\tilde{\tilde{\partial}}_{43} E\right\} - 2i\left\{\Omega_{21}\partial_1\Omega_{22} - \Omega_{22}\partial_1\Omega_{21}\right\}, \tag{4.34-b}$$

$$A'^3_2 = \frac{1}{F}\left\{\tilde{\Omega}\partial_1 F - i\Omega_{12}\Omega_{22}\tilde{\partial}_{43} G - i\Omega_{21}\Omega_{11}\tilde{\tilde{\partial}}_{43} E\right\} - 2i\left\{\Omega_{11}\partial_2\Omega_{22} - \Omega_{12}\partial_2\Omega_{21}\right\},$$

$$A'^1_2 - iA'^2_2 = \frac{1}{F}\left\{-2\Omega_{12}\Omega_{11}\partial_1 F + i\Omega_{12}^2\tilde{\partial}_{43} G + i\Omega_{11}^2\tilde{\tilde{\partial}}_{43} E\right\} - 2i\left\{-\Omega_{11}\partial_2\Omega_{12} + \Omega_{12}\partial_2\Omega_{11}\right\},$$

$$A'^1_2 + iA'^2_2 = \frac{1}{F}\left\{2\Omega_{21}\Omega_{22}\partial_1 F - i\Omega_{22}^2\tilde{\partial}_{43} G - i\Omega_{21}^2\tilde{\tilde{\partial}}_{43} E\right\} - 2i\left\{\Omega_{21}\partial_2\Omega_{22} - \Omega_{22}\partial_2\Omega_{21}\right\}, \tag{4 34-c}$$

$$A'^3_3 = \frac{1}{F}\left\{\tilde{\Omega}\partial_4 F + i\Omega_{12}\Omega_{22}\tilde{\partial}_{12} G + i\Omega_{21}\Omega_{11}\tilde{\tilde{\partial}}_{12} E\right\} - 2i\left\{\Omega_{11}\partial_3\Omega_{22} - \Omega_{12}\partial_3\Omega_{21}\right\},$$

$$A'^1_3 - iA'^2_3 = \frac{1}{F}\left\{-2\Omega_{12}\Omega_{11}\partial_4 F - i\Omega_{12}^2\tilde{\partial}_{12} G - i\Omega_{11}^2\tilde{\tilde{\partial}}_{12} E\right\} - 2i\left\{-\Omega_{11}\partial_3\Omega_{12} + \Omega_{12}\partial_3\Omega_{11}\right\},$$

$$A'^1_3 + iA'^2_3 = \frac{1}{F}\left\{2\Omega_{21}\Omega_{22}\partial_4 F + i\Omega_{22}^2\tilde{\partial}_{12} G + i\Omega_{21}^2\tilde{\tilde{\partial}}_{12} E\right\} - 2i\left\{\Omega_{21}\partial_3\Omega_{22} - \Omega_{22}\partial_3\Omega_{21}\right\}, \tag{4 34-d}$$

where

$$\widetilde{\partial}_{12} = \partial_1 + i\partial_2, \qquad \widetilde{\partial}_{43} = \partial_4 + i\partial_3, \qquad \widetilde{\Omega} = \Omega_{11}\Omega_{22} + \Omega_{12}\Omega_{21}. \tag{4.35}$$

The fields A'_μ given by the formulas (4 34-a,b,c,d) are real-valued functions. Furthermore, the x^4-dependence will disappear at the last stage of the calculation as we will see later

4.4 The Regularity of the Solutions

From analytical considerations [5] we have the general result that all solutions are smooth in an appropriate gauge In this section we want to be more specific and study the regularity of the fields A'_μ given by the formulas (4.34-a,b,c,d). For a certain choice of ρ_2 we will show explicitly the existence of Taylor expansion of the transition matrix (4 13) as a function of the spatial coordinates and the parameters in some neighborhood of the origin We will show also that this expansion leads to power series solutions satisfying the monopole equations This study will show that the existence of such analytic solutions depends on the construction of the transition matrix (4 13) We do so because in the next chapter we are going to use the series solutions of the monopole equations to construct suitable initial conditions for the equations of motion (2 23) and (2 24) Series expansions were used before by Soper [32] to study the symmetries of 3 and 4 monopole solutions

It follows from the formulas (4 34-a,b,c,d) that the regularity of the functions E, F and G of (4 26-a) and the nonvanishing of the function F lead to the regularity of the fields A'_μ From the splitting of the transition matrix (4 13), which will be discussed in the next section, it is clear that the regularity of the functions E, F and G follows immediately from the regularity of the coefficients of the Laurent expansion (4 15) and the nonvanishing of the determinant (4.26-b)

To show the existence of Taylor series expansions of the Δ functions in some neighborhood of the origin for certain choices of the function ρ_2, let us first define the annular region $\mathbf{N}$ of (4.10) for n=2 by investigating the roots of the equation $\psi_2=0$ That the polynomial ψ_2 satisfies the reality condition (4.9) implies that the roots of the equation $\psi_2=0$ come in two pairs, $(\alpha,-\bar{\alpha}^{-1})$ and $(\beta,-\bar{\beta}^{-1})$ Therefore, we always have two roots α and β lying inside the unit circle $|\zeta| = 1$ This enables us to define δ_+ and δ_- of (4 10) as follows,

$$\delta_+ = \max\{|\alpha|,|\beta|\}, \qquad \delta_- = \min\{|\alpha^{-1}|,|\beta^{-1}|\}. \tag{4.36}$$

For certain data, we assume that an analytic series expansion of the function ρ_2 as a function of the spatial coordinates and the parameters exists in some neighbourhood, $\widetilde{\mathbf{M}}$ say, of the origin Then we can write this expansion as a Laurent series in ζ in some annular region $\widetilde{\mathbf{N}}$ contained in $\mathbf{N}$ It is clear that this Laurent series is analytic for every $(x,\zeta) \in \widetilde{\mathbf{M}} \times \widetilde{\mathbf{N}}$ with coefficients analytic in $\widetilde{\mathbf{M}}$ On the other hand, for a Laurent series of the function ρ_2 as a function of ζ in the annular region N, we can use the contour integral (4 16) But we found that proceeding in this order the expressions for the 2-monopole solutions are complex and not regular functions of x^1, x^2 and x^3 Furthermore, these expressions are very complicated, so any gauge transformation, as suggested in [12], to remove such singularities is very hard to find

To prove explicity the existence of analytic series solutions of the monopole equations in some neighbourhood of the origin, we consider the axially symmetric solutions [12,13] Ward chose to work with

$$\psi_2(\zeta, x) = \omega^2(\zeta, x) + \pi^2 / 4 \tag{4 37}$$

The function ρ_2 of (4.13) can be written in the form

$$\rho_2 = i e^{2ix^4}\left(\frac{1}{\omega - i(\pi/2)} - \frac{1}{\omega + i(\pi/2)}\right)\left(\frac{e^{2(-x_+\zeta+z)} + e^{2(-x_-\zeta^{-1}-z)}}{\pi}\right), \tag{4.38}$$

with $x_\pm = x^1 \pm ix^2$. The transition matrix (4.13) with ρ_2 of the form (4.38) leads to series solutions describing two monopoles sitting on top of each other at the origin.

With the help of Mathematica we expanded the function ρ_2 as a Taylor series of the spatial coordinates in the neighbourhood of the origin. Then we used this expansion to find the functions Δ_r for $r = 0,\pm1,\pm2$. Up to 3-rd order we find

$$\Delta_2 = \left((\frac{8}{\pi^2} - \frac{32}{\pi^4})x_+^2 + (\frac{16}{\pi^2} - \frac{128}{\pi^4})zx_+^2\right)e^{2ix^4},$$

$$\Delta_1 = \left(-\frac{8}{\pi^2}x_- - \frac{32}{\pi^4}x_+x_-^2 + (\frac{16}{\pi^2} - \frac{128}{\pi^4})zx_- - (\frac{16}{\pi^2} - \frac{128}{\pi^4})z^2x_-\right)e^{2ix^4},$$

$$\Delta_0 = \left(\frac{8}{\pi^2} + \frac{64}{\pi^4}x_+x_- + (\frac{16}{\pi^2} - \frac{128}{\pi^4})z^2\right)e^{2ix^4}, \tag{4.39}$$

$$\Delta_{-1} = \left(-\frac{8}{\pi^2}x_+ - \frac{32}{\pi^4}x_+^2x_- - (\frac{16}{\pi^2} - \frac{128}{\pi^4})zx_+ - (\frac{16}{\pi^2} - \frac{128}{\pi^4})z^2x_+\right)e^{2ix^4}$$

$$\Delta_{-2} = \left((\frac{8}{\pi^2} - \frac{32}{\pi^4})x_-^2 - (\frac{16}{\pi^2} - \frac{128}{\pi^4})zx_-^2\right)e^{2ix^4}$$

By making use of (4.26-b) and the series expansions (4.39) we find the series expansion of the determinant Δ in the form

$$\Delta = \left(\frac{64}{\pi^4} - (\frac{64}{\pi^4} - \frac{1024}{\pi^6}) x_+ x_- + (\frac{256}{\pi^4} - \frac{2048}{\pi^6}) z^2 + . \right) e^{4ix^4} \tag{4 40}$$

It is clear from (4 39) and (4.40) that the functions Δ_r for $r = 0, \pm 1, \pm 2$ have power series expansions, and additionally, can be shown to satisfy (4 18) The expansions (4.40) shows that the determinant (4 26-b) has power series expansion and this expansion does not vanish at least in some neighbourhood of the origin contained in $\widetilde{\mathbf{M}}$ This implies that the functions E, F and G have power series expansions in this neighbourhood From the series expansions (4 39) and (4 40), we can see that the function F does not vanish in some neighbourhood of the origin contained in $\widetilde{\mathbf{M}}$ either We found also Γ_1 of (3.42) to be a non vanishing function and the gauge matrix Ω of (3.45) to be an analytic function of x^1, x^2 and x^3 This guarantees the regularity of the fields A'_μ

To be sure that the Taylor series expansion of the function ρ_2 leads to analytic series solutions with the right properties, we went further and calculated the fields A'_μ The fields A'_μ are found to be real power series in x^1, x^2 and x^3 only, which, of course, satisfy the monopole equations (as can be easily checked using Mathematica) As an example, the series expansions of the Higgs fields Φ'^a up to 4-th order are given

$$\Phi'^1 = \left((2 - \frac{24}{\pi^2})(x^2 - y^2) + (2 + \frac{40}{3\pi^2} - \frac{320}{\pi^4})(x^4 - y^4) - (12 + \frac{48}{\pi^2} - \frac{1664}{\pi^4})(x^2 - y^2)z \right),$$

$$\Phi'^2 = \left((4 - \frac{48}{\pi^2})xy + (4 + \frac{80}{3\pi^2} - \frac{640}{\pi^4})(x^3 y + xy^3) - (24 + \frac{96}{\pi^2} - \frac{3328}{\pi^4})xyz^2 \right),$$

(4 41)

$$\Phi'^3 = \left((4 - \frac{32}{\pi^2})z + (8 - \frac{768}{\pi^4})(x^2 z + y^2 z) - (\frac{16}{3} - \frac{512}{\pi^4})z^3 \right)$$

The leading terms of the Higgs fields (4.41) have the same behaviour as the solutions discussed by Brown, Prasad and Rossi [33]. We conclude that the solutions of the monopole equations can be found in terms of analytic power series in the neighbourhood of the origin by starting with a function ρ_2 with the right properties and by choosing a suitable gauge

4.5 The Splitting of the Matrix $g_2(\zeta,x)$

In this section we will find the matrices k_+ and k_- of (4 20) and (4 21). Our study is a review of the splitting of Refs 13, 16, but we will use different techniques in some stages of the splitting. To find these matrices we substitute the matrices (4 20) and (4.21) into the equation (4.19). This substitution leads to the following equations

$$\alpha = a\zeta^2 + c\rho_2, \qquad \beta = b\zeta^2 + d\rho_2, \tag{4 42}$$

$$\gamma = c\zeta^{-2}, \qquad \delta = d\zeta^{-2} \tag{4 43}$$

We assumed that a, b, c and d are analytic functions in the region $M \times U_+$ and α, β, γ and δ in the region $M \times U_-$ This enables us to write the Taylor series of the functions a, b, c and d in the form

$$a = \sum_{k=0}^{\infty} a_k \zeta^k, \qquad b = \sum_{k=0}^{\infty} b_k \zeta^k, \qquad c = \sum_{k=0}^{\infty} c_k \zeta^k, \qquad d = \sum_{k=0}^{\infty} d_k \zeta^k, \tag{4 44}$$

and the functions α, β, γ and δ in the form

$$\alpha = \sum_{k=0}^{\infty} \alpha_k \zeta^{-k}, \qquad \beta = \sum_{i=0}^{\infty} \beta_k \zeta^{-k}, \qquad \gamma = \sum_{k=0}^{\infty} \gamma_k \zeta^{-k}, \qquad \delta = \sum_{k=0}^{\infty} \delta_k \zeta^{-k} \tag{4 45}$$

Here the coefficients of these Talylor series are analytic functions of x in the region **M**

Substituting (4.44) and (4.45) into (4.43) implies that the functions c and d are polynomials of degree two in ζ of the form

$$c = \sum_{L=0}^{2} c_L \zeta^L, \qquad d = \sum_{L=0}^{2} d_L \zeta^L, \tag{4.46}$$

and the functions γ and δ of (4 45) are polynomials of degree two in ζ^{-1} of the form

$$\gamma = \sum_{L=0}^{2} c_{2-L} \zeta^{-L}, \qquad \delta = \sum_{L=0}^{2} d_{2-L} \zeta^{-L} \tag{4 47}$$

So we have

$$\begin{aligned} &\gamma_0 = c_2, \qquad \gamma_1 = c_1, \qquad \gamma_2 = c_0, \\ &\delta_0 = d_2, \qquad \delta_1 = d_1, \qquad \delta_2 = d_0 \end{aligned} \tag{4 48}$$

If we substitute the functions (4 44) and (4 45) into the equations (4 42), we can express the coefficients a_k, b_k, α_k and β_k in terms of the coefficents c_L and d_L This substitution yields

$$a_k = -\sum_{L=0}^{2} c_L \Delta_{L-2-k}, \qquad b_k = -\sum_{L=0}^{2} d_L \Delta_{L-2-k}, \tag{4 49}$$

$$\alpha_k = \sum_{L=0}^{2} c_L \Delta_{L+k}, \qquad \beta_k = \sum_{L=0}^{2} d_L \Delta_{L+k} \tag{4 50}$$

The vanishing of the coefficients of ζ in the Laurent series of $a\zeta^2 + c\rho_2$ and $b\zeta^2 + d\rho_2$ yields

$$\sum_{i=0}^{2} c_i \Delta_{i-1} = 0, \qquad \sum_{i=0}^{2} d_i \Delta_{i-1} = 0. \tag{4 51}$$

The constraint $ad - bc = 1$ for the functions (4.44) implies

$$ad - bc = (a_0 d_0 - b_0 c_0) + \{(a_1 d_0 + a_0 d_1) - (b_1 c_0 + b_0 c_1)\}\zeta +$$

$$\sum_{k=2}^{\infty}\sum_{L=0}^{2}(a_{k-L} d_L - b_{k-L} c_L)\zeta^k = 1 \tag{4 52}$$

By making use of (4 49) and (4 51) we can prove that the coefficient of ζ in (4 52) is identically zero The coefficients of ζ^k for k>1 can be simply given by

$$\sum_{L=0}^{2}(a_{k-L} d_L - b_{k-L} c_L) = -\sum_{L=0}^{2}\sum_{i=0}^{2}(c_i d_L \Delta_{i-2-k+L}) + \sum_{L=0}^{2}\sum_{i=0}^{2}(d_i c_L \Delta_{i-2-k+L})$$

which also is clearly zero Now (4 52) is reduced to the following condition

$$a_0 d_0 - b_0 c_0 = 1 \tag{4 53}$$

We can use the same technique and show that the constraint $\alpha\delta - \beta\gamma = 1$ can be reduce to the condition

$$\alpha_0 d_2 - \beta_0 c_2 = 1 \tag{4 54}$$

If we substitute (4 44) and (4 45) into (4 23) and make use of (4 46) and (4 47), we can find the matrix elements of the matrix (4 23),

$$L_{\alpha'}(\zeta, x) = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} \hat{L}_{11} & \hat{L}_{12} \\ \hat{L}_{21} & \hat{L}_{22} \end{pmatrix}, \tag{4.55}$$

with $L_{22} = -L_{11}$ and $\hat{L}_{22} = -\hat{L}_{11}$. The matrix elements can be simplified as follows First we can write L_{11} in the form

$$\begin{aligned} L_{11} = {} & (d_0\partial_{\alpha'1}a_0 - b_0\partial_{\alpha'1}c_0) + \\ & \zeta(d_0\partial_{\alpha'1}a_1 - b_1\partial_{\alpha'1}c_0) + \zeta(d_1\partial_{\alpha'1}a_0 - b_0\partial_{\alpha'1}c_1) - \zeta(d_0\partial_{\alpha'2}a_0 - b_0\partial_{\alpha'2}c_0) + \\ & \zeta^2(d_0\partial_{\alpha'1}a_2 + d_1\partial_{\alpha'1}a_1 + d_2\partial_{\alpha'1}a_0 - d_0\partial_{\alpha'2}a_1 - d_1\partial_{\alpha'2}a_0) - \\ & \zeta^2(b_2\partial_{\alpha'1}c_0 + b_1\partial_{\alpha'1}c_1 + b_0\partial_{\alpha'1}c_2 - b_1\partial_{\alpha'2}c_0 - b_0\partial_{\alpha'2}c_1) + \\ & \sum_{k=3}^{\infty}\sum_{L=0}^{2}\left\{\left(d_L\partial_{\alpha'1}a_{k-L} - d_L\partial_{\alpha'2}a_{k-L-1}\right) - \left(b_{k-L}\partial_{\alpha'1}c_L - b_{k-L-1}\partial_{\alpha'2}c_L\right)\right\}\zeta^k \end{aligned} \tag{4 56}$$

Making use of (4 18), (4 49) and (4 51) we can prove that the coefficient of ζ^2 in (4 56) is identically zero By making use of (4 49), we can also write the coefficients of ζ^k for k>2 in the form

$$-\sum_{i=0}^{2}\sum_{L=0}^{2} c_i d_L\left(\partial_{\alpha'1}\Delta_{i-2-k+L} - d_L\partial_{\alpha'2}\Delta_{i-2-k+L+1}\right), \tag{4 57}$$

which is zero, because of (4 18)

Now the equation (4 56) is reduced to

$$\begin{aligned} L_{11} = {} & (d_0\partial_{\alpha'1}a_0 - b_0\partial_{\alpha'1}c_0) + \zeta(d_0\partial_{\alpha'1}a_1 - b_1\partial_{\alpha'1}c_0) + \zeta(d_1\partial_{\alpha'1}a_0 - b_0\partial_{\alpha'1}c_1) - \\ & \zeta(d_0\partial_{\alpha'2}a_0 - b_0\partial_{\alpha'2}c_0). \end{aligned} \tag{4 58}$$

Using the same techniques we can find L_{12} and L_{21} in the form

$$L_{12} = (d_0\partial_{\alpha'1}b_0 - b_0\partial_{\alpha'1}d_0) + \zeta(d_0\partial_{\alpha'1}b_1 - b_1\partial_{\alpha'1}d_0) + \zeta(d_1\partial_{\alpha'1}b_0 - b_0\partial_{\alpha'1}d_1) -$$

$$\zeta(d_0\partial_{\alpha'2}b_0 - b_0\partial_{\alpha'2}d_0), \tag{4 59}$$

$$L_{21} = (a_0\partial_{\alpha'1}c_0 - c_0\partial_{\alpha'1}a_0) + \zeta(a_1\partial_{\alpha'1}c_0 - c_0\partial_{\alpha'1}a_1) + \zeta(a_0\partial_{\alpha'1}c_1 - c_1\partial_{\alpha'1}a_0) -$$

$$\zeta(a_0\partial_{\alpha'2}c_0 - c_0\partial_{\alpha'2}a_0) \tag{4 60}$$

Again using the same techniques we can write $\hat{L}_{11}$, $\hat{L}_{12}$ and $\hat{L}_{21}$ in the form

$$\hat{L}_{11} = (d_2\partial_{\alpha'1}\alpha_0 - \beta_0\partial_{\alpha'1}c_2) - (d_1\partial_{\alpha'2}\alpha_0 - \beta_0\partial_{\alpha'2}c_1) - (d_2\partial_{\alpha'2}\alpha_1 - \beta_1\partial_{\alpha'2}c_2) -$$

$$\zeta(d_2\partial_{\alpha'2}\alpha_0 - \beta_0\partial_{\alpha'2}c_2) \tag{4 61}$$

$$\hat{L}_{12} = (\alpha_0\partial_{\alpha'1}c_2 - c_2\partial_{\alpha'1}\alpha_0) - (\alpha_0\partial_{\alpha'2}c_1 - c_1\partial_{\alpha'2}\alpha_0) - (\alpha_1\partial_{\alpha'2}c_2 - c_2\partial_{\alpha'2}\alpha_1) -$$

$$\zeta(\alpha_0\partial_{\alpha'2}c_2 - c_2\partial_{\alpha'2}\alpha_0) \tag{4 62}$$

$$\hat{L}_{21} = (\alpha_0\partial_{\alpha'1}c_2 - c_2\partial_{\alpha'1}\alpha_0) - (\alpha_0\partial_{\alpha'2}c_1 - c_1\partial_{\alpha'2}\alpha_0) - (\alpha_1\partial_{\alpha'2}c_2 - c_2\partial_{\alpha'2}\alpha_1) -$$

$$\zeta(\alpha_0\partial_{\alpha'2}c_2 - c_2\partial_{\alpha'2}\alpha_0) \tag{4 63}$$

For simplicity we follow Ref 16 and choose the R_2 gauge

$$d_0 = c_2 = 0, \qquad c_0 = d_2 = \sqrt{F} \tag{4 64}$$

In the R_2 gauge, (4 51) reduces to the two equations

$$c_0\Delta_{-1} + c_1\Delta_0 = 0, \qquad d_1\Delta_0 + d_2\Delta_1 = 0, \tag{4 65}$$

and the conditions (4 53) and (4 54) change into $b_0c_0 = -1$ and $\alpha_0 d_2 = 1$ Because $c_0 = d_2 = \sqrt{F}$ holds, (4 65) leads to

$$c_1 = \frac{G}{\sqrt{F}}, \qquad d_1 = \frac{E}{\sqrt{F}} \tag{4.66}$$

where E, F and G are as defined by (4 26-a)

Now (4.58), (4.59), (4 61) and (4 63) enable us to write the matrix (4.23) in the form

$$L_{\alpha'}(\zeta, x) = \begin{pmatrix} -(b_0\partial_{\alpha'1}c_0 + \zeta d_2\partial_{\alpha'2}\alpha_0) & \zeta(d_1\partial_{\alpha'1}b_0 - b_0\partial_{\alpha'1}d_1) \\ \\ -(\alpha_0\partial_{\alpha'2}c_1 - c_1\partial_{\alpha'2}\alpha_0) & (b_0\partial_{\alpha'1}c_0 + \zeta d_2\partial_{\alpha'2}\alpha_0) \end{pmatrix}. \tag{4 67}$$

Finally, if we make use of

$$b_0\partial_{\alpha'1}c_0 = \frac{-1}{2F}\partial_{\alpha'1}F, \qquad d_1\partial_{\alpha'1}b_0 - b_0\partial_{\alpha'1}d_1 = \frac{1}{F}\partial_{\alpha'1}E,$$

$$d_2\partial_{\alpha'2}\alpha_0 = \frac{-1}{2F}\partial_{\alpha'2}F, \qquad \alpha_0\partial_{\alpha'2}c_1 - c_1\partial_{\alpha'2}\alpha_0 = \frac{1}{F}\partial_{\alpha'2}G, \tag{4 68}$$

we can write the matrix (4 67) in the form (4 25)

Chapter 5

The Cauchy Problem

In thıs chapter we wıll set up an ınıtıal value problem for the equatıons of motıon (2 23) and (2 24), ın the BPS lımıt For the ınıtıal data we wıll use a transıtıon matrıx whıch leads to solutıons for two separated monopoles Guıded by the ıdea of the slow-motıon approxımatıon we wıll use these solutıons to construct suıtable ınıtıal condıtıons for the equatıons of motıon Thıs completes the constructıon of the Cauchy problem whıch we wıll solve ın the next chapter

5.1 Problem Formulation

The slow-motion approximation for monopoles was proposed by Manton [22] It was extensively used by Atiyah and Hitchin [23]. Recently, the validity of the approximation was rigorously shown under certain conditions [33] The slow-motion approximation starts with the (4n-1)-parameter family of minima $\Phi(x)$, $A_i(x)$ of the potential energy, i e with the solutions of the Bogomol'nyi equations The parameters are then allowed to become time dependent This means that within this approximation, the initial data are such that $\Phi(0,x)+t\partial_0\Phi(0,x)$ and $A_i(0,x)+t\partial_0 A_i(0,x)$ satisfy the Bogomol'nyi equations, up to linear terms in t. We will construct such data below but then use them in the full equations

To propagate the initial data in time, we use the equations of motion (2 23) and (2 24) Equation (2.23) can be written in the form

$$\partial_0^2\Phi^a = \partial_i^2\Phi^a + 2\varepsilon^{abc}A_0^b\partial_0\Phi^c - 2\varepsilon^{abc}A_i^b\partial_i\Phi^c + \varepsilon^{abc}\Phi^c\partial_0 A_0^b -$$

$$\varepsilon^{abc}\Phi^c\partial_i A_i^b - A_0^b A_0^a\Phi^b + A_i^b A_i^a\Phi^b + A_0^b A_0^b\Phi^a - A_i^b A_i^b\Phi^a \qquad (5\ 1)$$

Equation (2 24) for $\nu=1,2,3$ and $i,j=1,2,3$ reads

$$\partial_0^2 A_j^a = \partial_i^2 A_j^a + \partial_0\partial_j A_0^a - \partial_i\partial_j A_i^a - \varepsilon^{abc}A_j^b\partial_0 A_0^c + \varepsilon^{abc}A_j^b\partial_i A_i^c -$$

$$2\varepsilon^{abc}A_0^c\partial_0 A_j^b + 2\varepsilon^{abc}A_i^c\partial_i A_j^b - \varepsilon^{abc}A_0^b\partial_j A_0^c + \varepsilon^{abc}A_i^b\partial_j A_i^c +$$

$$A_0^b A_j^a A_0^b - A_i^b A_j^a A_i^b - A_0^b A_j^b A_0^a + A_i^b A_j^b A_i^a + \varepsilon^{abc}\Phi^b\partial_j\Phi^c -$$

$$\Phi^b A_j^a\Phi^b + \Phi^b A_j^b\Phi^a \qquad (5\ 2)$$

The remaining equation (2.24) for $\nu=0$ is the Gauss equation

$$\partial_0\partial_i A_i^a - \partial_i^2 A_0^a - \varepsilon_{abc}A_0^b\partial_i A_i^c - \varepsilon_{abc}A_i^b\partial_0 A_i^c + 2\varepsilon_{abc}A_i^b\partial_i A_0^c + A_0^a(A_i^b)^2 -$$

$$A_i^a(A_0^b A_i^b) - \varepsilon_{abc}\Phi_b\partial_0\Phi_c + A_0^a(\Phi_b\Phi^b) - \Phi_a(\Phi_b A_0^b) = 0 \tag{5.3}$$

This equation does not propagate A_0^a in time, but rather provides a constraint on A_0^a We will solve it to find a $A_0^a(0,x)$. Since we work with power series near the origin, we will actually only provide a power series solution of (5.3) for A_0^a which may not be the expansion at the origin of a solution with the right asymptotic behaviour at infinity to ensure finite energy. In this respect our initial data may be different from those of the slow-motion approximation To ensure finite energy, the asymptotic form of our data, given in terms of power series at the origin, has to be chosen appropriately We are, of course, free to choose any asymptotic form of our initial data if we do not mind loosing contact with the slow-motion approximation Differences in asymptotic behaviour, however, should make no difference to our local investigation near the origin

We still have no equation for the evolution of A_0^a We also still have to deal with the gauge freedom (2 14) The gauge freedom allows us to choose the Lorentz condition

$$\partial^\mu A_\mu = 0, \tag{5 4}$$

which we use to determine $\partial_0 A_0^a(0,x)$ The Lorentz condition (5 4) also provides the time evolution of A_0^a in the form

$$\partial_0^2 A_0^a = \partial_0\partial_i A_i^a \tag{5 5}$$

We can also use the Gauss *equation* (5.3) to write (5.5) in the form

$$\partial_0^2 A_0^a = \partial_i^2 A_0^a + \varepsilon_{abc} A_0^b \partial_i A_i^c + \varepsilon_{abc} A_i^b \partial_0 A_i^c - 2\varepsilon_{abc} A_i^b \partial_i A_0^c - A_0^a (A_i^b)^2 +$$

$$A_i^a (A_0^b A_i^b) + \varepsilon_{abc} \Phi_b \partial_0 \Phi_c - A_0^a (\Phi_b \Phi^b) + \Phi_a (\Phi_b A_0^b). \tag{5.6}$$

Here we have followed the strategy used for vortex scattering [25]. There it can be shown [34] that solutions to the second-order equations for Φ and A_μ exist and that the Lorentz condition and the Gauss equation are propagated. Since the global existence proof for monopoles [35] is not given in the Lorentz gauge, we will later on solve equations (5.1), (5.2) and (5 5) in terms of power series and check that the Lorentz condition and the Gauss equation hold for all time.

5.2 Expansion of the Transition Matrix

In this section we will introduce a transition matrix given in Ref 36 This transition matrix leads to solutions describing two separated monopoles The corresponding solutions will be used in the next section to construct suitable initial conditions for the equations (5 1), (5 2) and (5 5) In Ref 36 the transition matrix is written in the form

$$g(\zeta, x, \varepsilon) = \begin{pmatrix} f(\zeta, \mu, \nu) & \zeta^{-2} \\ -\zeta^2 & 0 \end{pmatrix}, \qquad f = f_+ + f_- = \frac{e^{(\pi/2)\hat{\mu}}}{\Psi} + \frac{e^{(\pi/2)\hat{\nu}}}{\Psi} \tag{5.7}$$

It leads to a solution describing two separated monopoles with the distance between them related to ε Here ε is a non-negative real number and the other parameters were removed by a rigid motion in $x^1x^2x^3$-space The functions $\hat{\mu}$ and $\hat{\nu}$ are analytic in $M \times U_+$ and in $M \times U_-$, respectively, and they are given by

$$\hat{\mu}-\hat{\nu}=2\omega\delta^{-1}, \qquad 2\omega=(\mu-\nu), \tag{5 8}$$

where μ and ν are given by (4.4). The function Ψ is a polynomial of degree two in ζ and ζ^{-1} of the form

$$\Psi=\omega^2+\delta^2, \qquad \delta^2=1-\tfrac{1}{4}\varepsilon(\zeta-\zeta^{-1})^2 \tag{5 9}$$

The transition matrix (5 7) is equivalent to the matrix (4 13) To see this, we set $\rho_2 = f(\mu,\nu,\varepsilon)$ Then by choosing λ_- and λ_+ in the form

$$\lambda_-=\begin{pmatrix} -\zeta^{-2}f_- & f_-^2-1 \\ 1 & -\zeta^2 f_- \end{pmatrix} \qquad \lambda_+=\begin{pmatrix} f_+^2-1 & \zeta^{-2}f_+ \\ -\zeta^2 f_+ & -1 \end{pmatrix}, \tag{5 10}$$

we can easily prove that $g=\lambda_- g_2 \lambda_+$ This implies that $g(\zeta,x,\varepsilon)$ is equivalent to the transition matrix (4 13), which enables us to write $g(\zeta,x,\varepsilon)$ in the form

$$g(\zeta,x,\varepsilon)=\begin{pmatrix} \zeta^2 & f(\zeta,\mu,\nu) \\ 0 & \zeta^{-2} \end{pmatrix} \tag{5 11}$$

We are now able to use the results of the previous chapter The fact that δ^2 is an analytic function of ε enables us to find the Taylor expansion of the function δ^{-1} in the form

$$\delta^{-1}(\varepsilon,\zeta)=C_0(\varepsilon)+C_+(\varepsilon,\zeta)+C_-(\varepsilon,\zeta) \tag{5 12}$$

For small ε, we find

$$C_0(\varepsilon)=1-\frac{1}{4}\varepsilon, \qquad C_+(\varepsilon,\zeta)=\frac{1}{8}\varepsilon\zeta^2, \qquad C_-(\varepsilon,\zeta)=\frac{1}{8}\varepsilon\zeta^{-2} \tag{5 13}$$

This means we can write $\hat{\mu}$ and $\hat{\nu}$ of (5.8) in the form

$$\hat{\mu}(\zeta,x) = C_0(\varepsilon)\mu + 2\omega C_+(\varepsilon,\zeta), \qquad \hat{\nu}(\zeta,x) = C_0(\varepsilon)\nu - 2\omega C_-(\varepsilon,\zeta). \tag{5.14}$$

In equation (5.13) $C_0(\varepsilon)$ is obviously a function of ε only, and the functions $C_+(\varepsilon,\zeta)$ and $C_-(\varepsilon,\zeta)$ are analytic functions in the regions $\mathbf{U}_+$ and $\mathbf{U}_-$, respectively. These functions are also related to each other through the reality condition

$$\overline{C}_+(\varepsilon,\zeta) = C_-(\varepsilon,-\overline{\zeta}^{-1}), \tag{5 15}$$

which yields

$$\overline{\hat{\mu}}(\zeta,x) = -\hat{\nu}(-\overline{\zeta}^{-1},x). \tag{5 16}$$

If we regard the transition matrix (5 11) as a function of the separation parameter ε, we can expand it as a power series of ε in the neighbourhood of the origin For small ε, this expansion can be written in the form

$$g(\zeta,x,\varepsilon) = \hat{g}(\zeta,x) + \varepsilon\ \hat{\hat{g}}(\zeta,x), \tag{5 17}$$

where $\hat{g}(\zeta,x)$ and $\hat{\hat{g}}(\zeta,x)$ are functions of x and ζ only For ε =0, the transition matrix $\hat{g}(\zeta,x)$ leads to solutions equivalent to the Ward axisymmetric solutions studied in the previous chapter The Taylor expansion (5 17) leads to the solution

$$A^a_\mu(x) = \hat{A}^a_\mu(x) + \varepsilon\ \hat{\hat{A}}^a_\mu(x), \tag{5 18}$$

up to first order in ε

Multiplying the transition matrix (5 11) on the left and on the right by

$$\Pi_- = \begin{pmatrix} e^{-(\pi/4)\hat{v}} & 0 \\ 0 & e^{(\pi/4)\hat{v}} \end{pmatrix} \qquad \Pi_+ = \begin{pmatrix} 0 & -e^{(\pi/4)\hat{\mu}} \\ e^{-(\pi/4)\hat{\mu}} & \zeta^2 \Psi e^{-(\pi/4)\hat{\mu}} \end{pmatrix},$$

respectively, yields

$$\tilde{g} = \begin{pmatrix} \dfrac{e^{(\pi\backslash 2)\omega\delta^{-1}} + e^{-(\pi\backslash 2)\omega\delta^{-1}}}{\Psi} & \zeta^2 e^{-(\pi\backslash 2)\omega\delta^{-1}} \\ \zeta^{-2} e^{-(\pi\backslash 2)\omega\delta^{-1}} & \Psi e^{-(\pi\backslash 2)\omega\delta^{-1}} \end{pmatrix} \tag{5 19}$$

which satisfies the conditions (3 20-a,b). This will guarantee the reality of the solution (5 18) of the monopole equations To guarantee the regularity of the solutions we have to be sure that Taylor series expansions of the functions $\Delta_r(\varepsilon, x)$ $(r = 0, \pm 1, \pm 2)$ exist and lead to non vanishing $\Delta(\varepsilon, x)$ and F For $\varepsilon = 0$, the series solutions of the monopole equations reduce to the axisymmetric series solutions discussed in section 4 4 In this limit we showed explicitly that the series expansion of the transition matrix leads to series solutions in some neighbourhood of the origin For small values of ε, the transition matrix (5 11) is an analytic function of ε This fact will guarantee the existence of the series solutions for small ε in some neighbourhood of the origin and the non-vanishing of $\Delta(\varepsilon, x)$ and F This is sufficient for our discussion The general result, that all solutions are smooth in an appropriate gauge, can be found in Ref 5

5.3 Expansion of the Matrices M and Ω

In this section we will write down the Taylor series expansions of the matrices M and Ω, corresponding to (5 11), in the neighbourhood of the origin as functions of x To do that, we will make use of (4 31) and the splitting of section 4 5 This enables us to write the matrix M, with m_{11}, m_{22}, m_{12} and m_{21} in the form

$$m_{11} = \frac{\sqrt{F}}{\sqrt{\overline{F}}} e^{-(\pi/2)C_0(\varepsilon)z} - \left(\sqrt{F}\Delta_2 + \frac{E}{\sqrt{F}}\Delta_1\right)\frac{(x_+^2 - \varepsilon)}{4\sqrt{F}}$$

$$= \frac{\sqrt{\overline{F}}}{\sqrt{F}} e^{-(\pi/2)C_0(\varepsilon)z} - \left(\sqrt{\overline{F}}\overline{\Delta}_2 + \frac{\overline{E}}{\sqrt{\overline{F}}}\overline{\Delta}_1\right)\frac{(x_-^2 - \varepsilon)}{4\sqrt{\overline{F}}},$$

$$m_{22} = \frac{\sqrt{\overline{F}}}{\sqrt{F}} e^{(\pi/2)C_0(\varepsilon)z} - \left(\sqrt{\overline{F}}\overline{\Delta}_{-2} + \frac{\overline{G}}{\sqrt{\overline{F}}}\overline{\Delta}_{-1}\right)\frac{(x_+^2 - \varepsilon)}{4\sqrt{\overline{F}}} \tag{5 20}$$

$$= \frac{\sqrt{F}}{\sqrt{\overline{F}}} e^{(\pi/2)C_0(\varepsilon)z} - \left(\sqrt{F}\Delta_{-2} + \frac{G}{\sqrt{F}}\Delta_{-1}\right)\frac{(x_-^2 - \varepsilon)}{4\sqrt{F}},$$

$$m_{12} = \overline{a}_0\sqrt{F}e^{-(\pi/2)C_0(\varepsilon)z} - \beta_0\sqrt{\overline{F}}e^{(\pi/2)C_0(\varepsilon)z} - \frac{\overline{a}_0\beta_0}{4}(x_+^2 - \varepsilon) = \frac{1}{4\sqrt{F\overline{F}}}(x_-^2 - \varepsilon),$$

$$m_{21} = \frac{1}{4\sqrt{F\overline{F}}}(x_+^2 - \varepsilon) = a_0\sqrt{\overline{F}}e^{-(\pi/2)C_0(\varepsilon)z} - \overline{\beta}_0\sqrt{F}e^{(\pi/2)C_0(\varepsilon)z} - \frac{a_0\overline{\beta}_0}{4}(x_-^2 - \varepsilon)$$

Here $x_\pm = x^1 \pm ix^2$ and $z = x^3$ It is clear from (5 20) that m_{11}, m_{22} are real and $m_{12} = \overline{m}_{21}$

With the help of Mathematica we found the Taylor series expansions of the matrix M of (5 20) and the gauge matrix Ω of (3 45), which we used to guarantee the reality of the series solutions of the Bogomol'nyi equations Using the results of chapter 4, we can easily show that the series expansion of the matrix M does indeed exist in some neighbourhood of the origin and that the inequality $m_{11} + m_{22} + 2 > 0$ holds, which guarantees the non-vanishing of the gauge matrix Ω Here we will write down these expansions up to 4-th order In order to find the time-dependent solutions up to 4-th order, we had to find the series expansions of the matrices M and Ω up to the 6-th order, but we will not write down the higher order terms.

If we make use of the Taylor series expansion (5.17) and the splitting discussed in section 4.5, then the series expansion of the matrix M can be written up to any order. Up to the 4-th order, we find in terms of $x_\pm = x^1 \pm ix^2$ and $z = x^3$,

$$m_{11} = 1 + \frac{1}{8}x_+^2x_-^2 - \frac{\pi}{2}z + \frac{\pi^2}{8}z^2 - \frac{\pi^3}{48}z^3 + \frac{\pi^4}{384}z^4 + \ldots - \frac{\varepsilon}{8}(x_+^2 + x_-^2 + x_+^2x_-^2) -$$
$$\varepsilon(\frac{1}{8} - \frac{\pi^2}{64})x_+^3x_- - \varepsilon(\frac{1}{8} - \frac{\pi^2}{64})x_+x_-^3 + \frac{\varepsilon\pi}{8}z - \varepsilon(\frac{3\pi}{64} - \frac{\pi^3}{128})x_+^2z -$$
$$\varepsilon(\frac{3\pi}{64} - \frac{\pi^3}{128})x_-^2z - \frac{\varepsilon\pi^2}{16}z^2 + \varepsilon(\frac{1}{4} + \frac{\pi^2}{128} - \frac{\pi^4}{256})x_+^2z^2 +$$
$$\varepsilon(\frac{1}{4} + \frac{\pi^2}{128} - \frac{\pi^4}{256})x_-^2z^2 + \frac{\varepsilon\pi^3}{64}z^3 - \frac{\varepsilon\pi^4}{384}z^4 + \quad , \qquad (5.21\text{-a})$$

$$m_{22} = 1 + \frac{1}{8}x_+^2x_-^2 + \frac{\pi}{2}z + \frac{\pi^2}{8}z^2 + \frac{\pi^3}{48}z^3 + \frac{\pi^4}{384}z^4 + \quad - \frac{\varepsilon}{8}(x_+^2 + x_-^2 + x_+^2x_-^2) -$$
$$\varepsilon(\frac{1}{8} - \frac{\pi^2}{64})x_+^3x_- - \varepsilon(\frac{1}{8} - \frac{\pi^2}{64})x_+x_-^3 - \frac{\varepsilon\pi}{8}z + \varepsilon(\frac{3\pi}{64} - \frac{\pi^3}{128})x_+^2z +$$
$$\varepsilon(\frac{3\pi}{64} - \frac{\pi^3}{128})x_-^2z - \frac{\varepsilon\pi^2}{16}z^2 + \varepsilon(\frac{1}{4} + \frac{\pi^2}{128} - \frac{\pi^4}{256})x_+^2z^2 +$$
$$\varepsilon(\frac{1}{4} + \frac{\pi^2}{128} - \frac{\pi^4}{256})x_-^2z^2 - \frac{\varepsilon\pi^3}{64}z^3 - \frac{\varepsilon\pi^4}{384}z^4 + \quad , \qquad (5\ 21\text{-b})$$

$$m_{12} = \overline{m}_{12} = \frac{1}{2}x_-^2 + (\frac{1}{4} - \frac{\pi^2}{32})x_+x_-^3 - (\frac{1}{2} - \frac{\pi^2}{16})x_-^2z^2 + \quad - \frac{\varepsilon}{2}(1 + \frac{1}{2}x_-^2) -$$
$$\varepsilon(\frac{1}{16} - \frac{\pi^2}{256})x_-^4 - \varepsilon(\frac{1}{4} - \frac{5\pi^2}{256})x_+^2x_-^2 - \varepsilon(\frac{1}{4} - \frac{\pi^2}{32})x_+x_- - \varepsilon(\frac{1}{4} - \frac{\pi^2}{32})x_+x_-^3 +$$
$$\varepsilon(\frac{1}{2} - \frac{\pi^2}{16})z^2 + \varepsilon(\frac{1}{2} - \frac{\pi^2}{16})x_-^2z^2 + \varepsilon(1 - \frac{\pi^2}{16} - \frac{\pi^4}{256})x_+x_-z^2 -$$
$$\varepsilon(\frac{1}{2} - \frac{\pi^2}{16} + \frac{\pi^4}{768})z^4 + \qquad (5\ 21\text{-c})$$

Next by substituting (5 21-a,b,c) into (3.45) we can write down the series expansion of the gauge matrix Ω in the form

$$\Omega_{11}=1+\frac{1}{32}x_+^2x_-^2+\frac{\pi}{4}z+\frac{\pi^2}{32}z^2+\frac{\pi^3}{384}z^3+\frac{\pi^4}{6144}z^4+\ -\frac{\varepsilon}{32}(x_+^2+x_-^2+x_+^2x_-^2)-$$

$$\varepsilon(\frac{1}{32}-\frac{\pi^2}{256})x_+^3x_- -\varepsilon(\frac{1}{32}-\frac{\pi^2}{256})x_+x_-^3-\frac{\varepsilon\pi}{16}z+\varepsilon(\frac{\pi}{32}-\frac{\pi^3}{256})x_+^2z+$$

$$\varepsilon(\frac{\pi}{32}-\frac{\pi^3}{256})x_-^2z-\frac{\varepsilon\pi^2}{64}z^2+\varepsilon(\frac{1}{16}+\frac{3\pi^2}{1024}-\frac{\pi^4}{1024})x_+^2z^2+$$

$$\varepsilon(\frac{1}{16}+\frac{3\pi^2}{1024}-\frac{\pi^4}{1024})x_-^2z^2-\frac{\varepsilon\pi^3}{512}z^3-\frac{\varepsilon\pi^4}{6144}z^4+. \ , \tag{5 22-a}$$

$$\Omega_{22}=1+\frac{1}{32}x_+^2x_-^2-\frac{\pi}{4}z+\frac{\pi^2}{32}z^2-\frac{\pi^3}{384}z^3+\frac{\pi^4}{6144}z^4+\ -\frac{\varepsilon}{32}(x_+^2+x_-^2+x_+^2x_-^2)-$$

$$\varepsilon(\frac{1}{32}-\frac{\pi^2}{256})x_+^3x_- -\varepsilon(\frac{1}{32}-\frac{\pi^2}{256})x_+x_-^3+\frac{\varepsilon\pi}{16}z-\varepsilon(\frac{\pi}{32}-\frac{\pi^3}{256})x_+^2z-$$

$$\varepsilon(\frac{\pi}{32}-\frac{\pi^3}{256})x_-^2z-\frac{\varepsilon\pi^2}{64}z^2+\varepsilon(\frac{1}{16}+\frac{3\pi^2}{1024}-\frac{\pi^4}{1024})x_+^2z^2+$$

$$\varepsilon(\frac{1}{16}+\frac{3\pi^2}{1024}-\frac{\pi^4}{1024})x_-^2z^2+\frac{\varepsilon\pi^3}{512}z^3-\frac{\varepsilon\pi^4}{6144}z^4+ \ , \tag{5 22-b}$$

$$\Omega_{12}=\overline{\Omega}_{21}=-\frac{1}{4}x_-^2-(\frac{1}{8}-\frac{\pi^2}{64})x_+x_-^3+(\frac{1}{4}+\frac{\pi^2}{128})x_-^2z^2+\ -\frac{\varepsilon}{4}(1+\frac{1}{2}x_-^2)+$$

$$\varepsilon(\frac{3}{128}-\frac{\pi^2}{512})x_-^4+\varepsilon(\frac{7}{64}-\frac{5\pi^2}{512})x_+^2x_-^2+\varepsilon(\frac{1}{8}-\frac{\pi^2}{64})x_+x_- +\varepsilon(\frac{1}{8}-\frac{\pi^2}{64})x_+x_-^3-$$

$$\varepsilon(\frac{1}{4}-\frac{3\pi^2}{128})z^2-\varepsilon(\frac{1}{4}-\frac{3\pi^2}{128})x_-^2z^2-\varepsilon(\frac{1}{2}-\frac{7\pi^2}{256}-\frac{5\pi^4}{2048})x_+x_-z^2+$$

$$\varepsilon(\frac{1}{4}-\frac{3\pi^2}{128}-\frac{\pi^4}{8192})z^4+ \tag{5 22-c}$$

5.4 Expansion of the Initial Data

In this section we will complete the construction of the Cauchy problem by choosing suitable initial conditions for the equations (5.1), (5.2) and (5.5). We will choose conditions which are compatible with the slow-motion approximation. Using the solution (5.18), we can write down the initial conditions

$$\Phi^a(0,x) = \hat{A}^a_4(x), \qquad A^a_i(0,x) = \hat{A}^a_i(x), \tag{5.23}$$

$$\partial_0\Phi^a(0,x) = \hat{\hat{A}}^a_4(x), \qquad \partial_0 A^a_i(0,x) = \hat{\hat{A}}^a_i(x), \tag{5.24}$$

where we have used $A^a_4 = \Phi^a$, and $t = x^0 (= -x_0)$. Up to linear terms in t, the corresponding functions $A^a_i(t,x)$, $\Phi^a(t,x)$ are minima of the potential energy and satisfy the Bogomol'nyi equations (2.30).

To write the initial conditions (5.23) and (5.24) as Taylor series, we expand the transitions matrices $\hat{g}(\zeta,x)$ and $\hat{\hat{g}}(\zeta,x)$ as functions of x in the neighbourhood of the origin. These expansions lead to the solutions (5.18). The reality and the regularity of the solutions (5.18), which we have proved, guarantee the reality and the regularity of the initial conditions (5.23) and (5.24). The Taylor expansions of the matrices $\hat{g}(\zeta,x)$ and $\hat{\hat{g}}(\zeta,x)$ in the neighbourhood of the origin were found using Mathematica. From these expansions we found the functions Δ_r (r= 0,±1,±2) which are analytic and satisfy the equation (4.18). We found also that the determinant Δ and F are non-vanishing functions of x in some neighbourhood of the origin. This guarantees the regularity of the solutions (5.18) and hence the regularity of the initial data (5.23) and (5.24).

In the previous section we wrote down the leading terms of the Taylor expansion of the gauge matrix Ω. Next by making use of this expansion and (4.34-a,b,c,d) we found the time-independent series solutions of the equations of motion (2.23) and (2 24) to be analytic real valued functions of x,y and z only, where we have replaced the coordinates x^1, x^2 and x^3 with x,y and z, respectively The initial conditions of the equations (5.1), (5.2) and (5.5) as Taylor series can be written down by making use of (5.23) and (5.24). First by making use of (5 23) we can write the initial conditions for the Higgs field Φ^a at t=0 in the form

$$\Phi^1(0,x) = (\frac{-3\pi}{8}+\frac{\pi^3}{32})x^2 + (\frac{-5\pi}{16}+\frac{5\pi^3}{384}+\frac{\pi^5}{512})x^4 + (\frac{5\pi}{16}-\frac{5\pi^3}{384}-\frac{\pi^5}{512})y^4 +$$

$$(\frac{3\pi}{8}-\frac{\pi^3}{32})y^2 + (\frac{13\pi}{8}-\frac{3\pi^3}{64}-\frac{3\pi^5}{256})x^2z^2 + (\frac{-13\pi}{8}+\frac{3\pi^3}{64}+\frac{3\pi^5}{256})y^2z^2,$$

(5 25-a)

$$\Phi^2(0,x) = (\frac{-3\pi}{4}+\frac{\pi^3}{16})xy + (\frac{-5\pi}{8}+\frac{5\pi^3}{192}+\frac{\pi^5}{256})x^3y + (\frac{-5\pi}{8}+\frac{5\pi^3}{192}+\frac{\pi^5}{256})xy^3 +$$

$$(\frac{13\pi}{4}-\frac{3\pi^3}{32}-\frac{3\pi^5}{128})xyz^2,$$

$$\Phi^3(0,x) = (-2+\frac{\pi^2}{4})z + (-3+\frac{\pi^4}{32})x^2z + (-3+\frac{\pi^4}{32})y^2z + (2-\frac{\pi^4}{48})z^3$$

For the gauge potentials A_i^a we obtain

$$A_i^1(0,x) = (\frac{5\pi^2}{4}-\frac{\pi^3}{8})yz + (2\pi-\frac{\pi^3}{8}-\frac{\pi^5}{128})x^2yz + (2\pi-\frac{\pi^3}{8}-\frac{\pi^5}{128})y^3z +$$

$$(\frac{-5\pi}{4}-\frac{25\pi^3}{192}+\frac{5\pi^5}{192})yz^3,$$

(5 25-b)

$$A_i^2(0,x) = (\frac{-5\pi^2}{4}+\frac{\pi^3}{8})xz + (-2\pi+\frac{\pi^3}{8}+\frac{\pi^5}{128})x^3z + (-2\pi+\frac{\pi^3}{8}+\frac{\pi^5}{128})xy^2z +$$

$$(\frac{5\pi}{4}+\frac{25\pi^3}{192}-\frac{5\pi^5}{192})xz^3,$$

$$A_i^3(0,x) = (1-\frac{\pi^2}{8})y + (\frac{3}{4}-\frac{\pi^4}{128})x^2y + (\frac{3}{4}-\frac{\pi^4}{128})y^3 + (-3+\frac{\pi^4}{32})yz^2,$$

$$A_2^1(0,x) = (\frac{5\pi^2}{4} - \frac{\pi^3}{8})xz + (2\pi - \frac{\pi^3}{8} - \frac{\pi^5}{128})x^3z + (2\pi - \frac{\pi^3}{8} - \frac{\pi^5}{128})xy^2z +$$

$$(\frac{-5\pi}{4} - \frac{25\pi^3}{192} + \frac{5\pi^5}{192})xz^3,$$

$$\text{(5.25-c)}$$

$$A_2^2(0,x) = (\frac{5\pi^2}{4} - \frac{\pi^3}{8})yz + (2\pi - \frac{\pi^3}{8} - \frac{\pi^5}{128})x^2yz + (2\pi - \frac{\pi^3}{8} - \frac{\pi^5}{128})y^3z +$$

$$(\frac{-5\pi}{4} - \frac{25\pi^3}{192} + \frac{5\pi^5}{192})yz^3,$$

$$A_2^3(0,x) = (-1 + \frac{\pi^2}{8})x + (\frac{-3}{4} + \frac{\pi^4}{128})x^3 + (\frac{-3}{4} + \frac{\pi^4}{128})xy^2 + (3 - \frac{\pi^4}{32})xz^2.$$

$$A_3^1(0,x) = (\frac{\pi}{2} - \frac{\pi^3}{16})xy + (\frac{\pi}{2} - \frac{\pi^3}{96} - \frac{\pi^5}{256})x^3y + (\frac{\pi}{2} - \frac{\pi^3}{96} - \frac{\pi^5}{256})xy^3 +$$

$$(-3\pi + \frac{5\pi}{64} + \frac{3\pi^5}{128})xyz^2,$$

$$\text{(5 25-d)}$$

$$A_3^2(0,x) = (\frac{-\pi}{4} + \frac{\pi^3}{32})x^2 + (\frac{-\pi}{4} + \frac{\pi^3}{192} + \frac{\pi^5}{512})x^4 + (\frac{\pi}{4} - \frac{\pi^3}{32})y^2 +$$

$$(\frac{\pi}{4} - \frac{\pi^3}{192} - \frac{\pi^5}{512})y^4 + (\frac{3\pi}{2} - \frac{5\pi^3}{128} - \frac{3\pi^5}{256})x^2z^2 + (\frac{-3\pi}{2} + \frac{5\pi^3}{128} + \frac{3\pi^5}{256})y^2z^2,$$

$$A_3^3(0,x) = 0$$

By making use of (5 24), the initial conditions for the time derivative of the Higgs field Φ^a is found to be

$$\Phi_t^1(0,x) = \frac{\pi}{16} + (\frac{15\pi}{32} - \frac{11\pi^3}{256})x^2 + (\frac{175\pi}{256} - \frac{5\pi^3}{192} - \frac{55\pi^5}{12288})x^4 - (\frac{3\pi}{32} - \frac{\pi^3}{256})y^2 -$$

$$(\frac{25\pi}{256} - \frac{5\pi^3}{768} - \frac{5\pi^5}{12288})y^4 + (\frac{19\pi}{128} - \frac{5\pi^3}{256} + \frac{\pi^5}{2048})x^2y^2 +$$

$$(\frac{-121\pi}{32} + \frac{37\pi^3}{256} + \frac{49\pi^5}{2048})x^2z^2 + (\frac{9\pi}{32} + \frac{7\pi^3}{256} - \frac{11\pi^5}{2048})y^2z^2 +$$

$$(\frac{-\pi}{4}+\frac{3\pi^3}{128})z^2+(\frac{\pi}{2}-\frac{5\pi^3}{192}-\frac{5\pi^5}{2048})z^4,$$

$$\Phi_t^2(0,x)=(\frac{9\pi}{16}-\frac{3\pi^3}{64})xy+(\pi-\frac{25\pi^3}{768}-\frac{\pi^5}{1536})x^3y+(\frac{9\pi}{16}-\frac{25\pi^3}{768}-\frac{\pi^5}{384})xy^3+$$

$$(\frac{-65\pi}{16}+\frac{15\pi^3}{128}+\frac{15\pi^5}{512})xyz^2,$$

(5.26-a)

$$\Phi_t^3(0,x)=(1-\frac{\pi^2}{8})z+(\frac{9}{2}+\frac{5\pi^2}{128}-\frac{13\pi^4}{256})x^2z+(\frac{3}{2}-\frac{5\pi^2}{128}-\frac{3\pi^4}{256})y^2z+$$

$$(-2+\frac{\pi^4}{48})z^3.$$

The analogous formulas for the gauge potentials A_i^a read

$$A^1_{1,t}(0,x)=(\frac{-5\pi^2}{8}+\frac{\pi^3}{16})yz+(\frac{-203\pi}{64}+\frac{5\pi^3}{32}+\frac{17\pi^5}{1024})x^2yz+$$

$$(\frac{-59\pi}{64}+\frac{9\pi^3}{128}+\frac{7\pi^5}{3072})y^3z+(\frac{5\pi}{8}+\frac{47\pi^3}{256}-\frac{77\pi^5}{3072})yz^3,$$

(5.26-b)

$$A^2_{1,t}(0,x)=(\frac{5\pi^2}{4}-\frac{\pi^3}{8})xz+(4\pi-\frac{15\pi^3}{64}-\frac{53\pi^5}{3072})x^3z+$$

$$(\frac{7\pi}{4}-\frac{19\pi^3}{128}-\frac{3\pi^5}{1024})xy^2z+(\frac{-5\pi}{2}-\frac{109\pi^3}{768}+\frac{41\pi^5}{1024})xz^3,$$

$$A^3_{1,t}(0,x)=(\frac{-1}{4}+\frac{\pi^2}{32})y+(\frac{-3}{4}+\frac{\pi^4}{128})x^2y+(\frac{-3}{4}+\frac{\pi^2}{64}+\frac{\pi^4}{768})y^3+$$

$$(\frac{3}{2}+\frac{5\pi^2}{64}-\frac{3\pi^4}{128})yz^2,$$

$$A^1_{2,t}(0,x)=(\frac{-5\pi^2}{4}+\frac{\pi^3}{8})xz+(\frac{-261\pi}{64}+\frac{31\pi^3}{8}+\frac{53\pi^5}{3072})x^3z+$$

$$(\frac{-117\pi}{64}+\frac{5\pi^3}{32}+\frac{3\pi^5}{1024})xy^2z+(\frac{5\pi}{2}+\frac{109\pi^3}{768}-\frac{41\pi^5}{1024})xz^3,$$

$$A^2_{2,t}(0,x) = (\frac{-5\pi}{8}+\frac{\pi^3}{16})yz+(\frac{-13\pi}{4}+\frac{21\pi^3}{128}+\frac{17\pi^5}{1024})x^2yz+$$

$$(-\pi+\frac{5\pi^3}{64}+\frac{7\pi^5}{3072})y^3z+(\frac{5\pi}{8}+\frac{47\pi^3}{256}-\frac{77\pi^5}{3072})yz^3,$$

(5.26-c)

$$A^3_{2,t}(0,x) = (\frac{3}{4}-\frac{3\pi^2}{32})x+(\frac{5}{4}+\frac{\pi^2}{64}-\frac{11\pi^4}{768})x^3+(\frac{3}{4}-\frac{\pi^4}{128})xy^2-$$

$$(\frac{9}{2}-\frac{5\pi^2}{64}-\frac{5\pi^4}{128})xz^2$$

$$A^1_{3,t}(x,0) = (\frac{-3\pi}{8}+\frac{3\pi^3}{64})xy+(\frac{-57\pi}{64}+\frac{13\pi^3}{768}+\frac{11\pi^5}{1536})x^3y+$$

$$(\frac{-23\pi}{64}+\frac{7\pi^3}{768}+\frac{\pi^5}{384})xy^3+(\frac{15\pi}{4}-\frac{25\pi^3}{265}-\frac{15\pi^5}{512})xyz^2,$$

(5 26-d)

$$A^2_{3,t}(0,x) = \frac{-\pi}{4}+(\frac{5\pi}{16}-\frac{9\pi^3}{256})x^2+(\frac{141\pi}{256}-\frac{\pi^3}{96}-\frac{55\pi^5}{12288})x^4+$$

$$(\frac{-\pi}{16}+\frac{3\pi^3}{256})y^2+(\frac{-19\pi}{256}+\frac{\pi^3}{384}+\frac{5\pi^5}{12288})y^4+(\frac{-7\pi}{128}+\frac{\pi^5}{2048})x^2y^2+$$

$$(\frac{-27\pi}{8}+\frac{63\pi^3}{512}+\frac{23\pi^5}{1024})x^2z^2+(\frac{3\pi}{8}+\frac{13\pi^3}{512}-\frac{7\pi^5}{1024})y^2z^2+$$

$$(\frac{-\pi}{8}+\frac{\pi^3}{64})z^2+(\frac{3\pi^2}{8}-\frac{7\pi^3}{384}-\frac{13\pi^5}{6144})z^4,$$

$$A^3_{3,t}(x,0) = (\frac{5\pi^2}{64}-\frac{\pi^4}{128})xyz$$

To write down the initial condition for the field A^a_0, we assume first that at t=0, A^a_0 has the form

$$A^a_0(0,x) = \sum_{i,j,k=0}^{\infty} e_{ijk}x^iy^jz^k \qquad (5\ 27)$$

To find the coefficients of the power series (5.27), we make use of the fact that the initial conditions satisfy the Gauss equation (5.3). Substituting the power series (5.27) and the initial conditions found above into the Gauss equation, we can find the coefficients of (5.27) up to any order. Up to 4 th order we found

$$
\begin{aligned}
A_0^1(0,x) &= 0, \\
A_0^2(0,x) &= \left(\frac{\pi}{8}-\frac{3\pi^3}{256}\right)x^2 z+\left(\frac{\pi}{8}-\frac{3\pi^3}{256}\right)y^2 z, \\
A_0^3(0,x) &= \left(\frac{\pi^2}{192}-\frac{\pi^4}{1536}\right)x^3 y+\left(\frac{\pi^2}{192}-\frac{\pi^4}{1536}\right)y^3 x
\end{aligned}
\tag{5.28-a}
$$

The coefficients of the power series (5.27) are not determined uniquely by (5.3) We have chosen a simple solution for which the Gauss equation is satisfied

Next, by using the Lorentz condition (5.4), we can write down the initial condition for $A^a_{0,t}$ which is

$$
\begin{aligned}
A^1_{0,t}(0,x) &= \left(2\pi-\frac{11\pi^3}{32}+\frac{\pi^5}{64}\right)xyz, \\
A^2_{0,t}(0,x) &= -\left(\pi-\frac{11\pi^3}{64}+\frac{\pi^5}{128}\right)x^2 z+\left(\pi-\frac{11\pi^3}{64}+\frac{\pi^5}{128}\right)y^2 z, \\
A^3_{0,t}(0,x) &= 0
\end{aligned}
\tag{5.28-b}
$$

This given us a complete set of initial data

Chapter 6

Time-Dependent Series Solutions

In this chapter we will study the Cauchy-Kowalewskyi theorem This study leads to the existence of a unique series solution of the Cauchy problem, formulated in the previous chapter, in the neighbourhood of the origin We will use Mathematica to write down the leading terms of this time-dependent series solution We will study also the symmetry of this solution under reflexion on the coordinates planes At the end of this chapter the position of the two monopoles is studied This study shows 90° scattering

6.1 The Cauchy-Kowalewskyi Theorem

In this section we will show the existence of a unique series solution of our initial value problem (5.1,2,5), (5.25,a,b,c,d), (5.26,a,b,c,d) and (5.28,a,b). We do this by using the Cauchy-Kowalewskyi theorem which assures the existence of a unique analytic solution of a certain class of Cauchy problems. We first introduce a Cauchy problem which will help us to prove the existence of a unique analytic solution of our initial value problem. We choose the equations

$$\frac{\partial u_i}{\partial t}=\sum_{k=1}^{n}\sum_{j=1}^{m}K_{ijk}(u_1,\ldots,u_m)\frac{\partial u_j}{\partial x_k},\qquad i=1,\ldots,m \tag{6.1}$$

The equations (6 1) describe a system of first order partial differential equations Here we assume that the $K_{ijk}(u_1,\ldots,u_m)$ are analytic functions of their arguments in some neighbourhood of the origin We also assume that the initial data of the unknown functions u_i at t=0,

$$u_i(0,x_1,\ldots,x_n)=\widetilde{K}_i(x_1,\ldots,x_n), \tag{6 2}$$

are such that the $\widetilde{K}_i(x^1,\ldots,x^n)$ are analytic functions of $(x^1,\ldots,x^n)$ in some neighbourhood of the origin with $\widetilde{K}_i(0,\ldots,0)=0$ The last condition can be always achieved for any initial data, and hence for our system, by substituting $u_i-\widetilde{K}_i(0,\ldots,0)$ for u_i

The system (6 1) with the initial conditions (6 2) constitutes a Cauchy problem A unique analytic solution, in some neighbourhood of the origin, say O, of this problem exists according to the Cauchy-Kowalewskyi theorem To illustrate this, let us assume that the solution of this problem has the series expansions

$$u_i(t,x_1,\dots,x_n) = \sum_{j_0,j_1,\dots,j_n=0}^{\infty} C^i_{j_0 j_1 \dots j_n} t^{j_0} x_1^{j_1} \dots x_n^{j_n}. \tag{6.3}$$

Uniqueness of an analytic solution of this problem in some neighbourhood of the origin means that any two analytic solutions must coincide in this neighbourhood. Now the first terms in the Taylor series expansions about the origin are determined by the initial conditions (6.2) and the other coefficients will be determined uniquely from the conditions (6.1). This establishes the uniqueness of the solution.

To prove the existence of a solution of the Cauchy problem (6.1) and (6.2) it is enough to prove that the power series (6.3) is convergent in O and satisfies the equations (6.1) and the initial data (6.2). The last two requirememts follow immediately from the fact that the coefficients of the power series (6.3) are determined uniquely from the initial data (6.2) and recursion relations following from (6.1). The convergence of the power series (6.3) can be proved by using the method of majorants. Here we only give the idea of the proof, which can be found in Ref. 37 and for a system of the form (6.1) can by found in detail in Ref. 38. The idea of this method is based on the fact that for each function analytic in some neighbourhood of the origin there exists a function which majorizes it; i.e. for any function, say Ψ, analytic in some neighbourhood of the origin there exists a function analytic in the same neighbourhood such that the coefficients of its expansion in powers of its arguments are non-negative and not smaller than the absolute values of the corresponding coefficients of the power series expansion of the function Ψ. Using this fact and some elementary techniques one can easily establish the convergence of the power series (6.3).

Finally, our Cauchy problem (5.1,2,5), (5.25,a,b,c,d), (5.26,a,b,c,d), (5.28,a,b) can be easily reduced to a Cauchy problem of the form (6.1), (6.2). Hence, by making use of the fact that our initial data (5.25,a,b,c,d) and (5.26,a,b,c,d) and (5.28,a,b) represent power seies expansions of analytic functions of the form $\tilde{K}_1$, we can easily establish the existence of a unique series solution of our Cauchy problem.

6.2 Local Series Solutions

In this section we will find the series solution of the Cauchy problem (5.1,2,5), (5.25,a,b,c,d), (5.26,a,b,c,d) and (5 28,a,b) in some neighbourhood of the origin With the help of Mathematica we will write down the leading terms of this series solution. First, let us assume that the series solutions of our Cauchy problem have the form

$$\Phi^a(t,x,y,z) = \sum_{i,j,p,q=0}^{\infty} u^a[i,j,p,q]x^i y^j z^p t^q, \qquad (6\,4\text{-a})$$

$$A_r^a(t,x,y,z) = \sum_{i,j,p,q=0}^{\infty} v_r^a[i,j,p,q]x^i y^j z^p t^q, \qquad (6\,4\text{-b})$$

with r=0,1,2,3

The unknown coefficients of the power series (6 4-a,b) can be found as follows First substituting (6 4-a,b) into (5 1), yields

$$u^a[i,j,p,q+2] = \left((q+1)(q+2)\right)^{-1}\{(i+1)(i+2)*u^a[i+2,j,p,q]+$$

$$(j+1)(j+2)*u^a[i,j+2,p,q]+(p+1)(p+2)*u^a[i,j,p+2,q]+$$

$$\sum_{i_1,j_1,p_1,q_1=0}^{i,j,p,q}\Big[2\varepsilon^{abc}(q+1)*v_0^b[i-i_1,j-j_1,p-p_1,q-q_1]*u^c[i_1,j_1,p_1,q_1+1]-$$

$$2\varepsilon^{abc}(i+1)*v_1^b[i-i_1,j-j_1,p-p_1,q-q_1]*u^c[i_1+1,j_1,p_1,q_1]-$$

$$2\varepsilon^{abc}(j+1)*v_2^b[i-i_1,j-j_1,p-p_1,q-q_1]*u^c[i_1,j_1+1,p_1,q_1]-$$

$$2\varepsilon^{abc}(p+1)*v_3^b[i-i_1,j-j_1,p-p_1,q-q_1]*u^c[i_1,j_1,p_1,q_1+1]+$$

$$\varepsilon^{abc}(q+1)*u^c[i-i_1,j-j_1,p-p_1,q-q_1]*v_0^b[i_1,j_1,p_1,q_1+1]+$$

$$\varepsilon^{abc}(i+1)*u^c[i-i_1,j-j_1,p-p_1,q-q_1]*v_1^b[i_1+1,j_1,p_1,q_1]+$$

$$\varepsilon^{abc}(j+1)*u^c[i-i_1,j-j_1,p-p_1,q-q_1]*v_2^b[i_1,j_1+1,p_1,q_1]+$$

$$\varepsilon^{abc}(p+1)*u^c[i-i_1,j-j_1,p-p_1,q-q_1]*v_3^b[i_1,j_1,p_1+1,q_1]-$$

$$\sum_{i_2,j_2,p_2,q_2=0}^{i-i_1,j-j_1,p-p_1,q-q_1}\Big(v_0^b[i_1,j_1,p_1,q_1]*v_0^a[i_2,j_2,p_2,q_2]-v_r^b[i_1,j_1,p_1,q_1]*$$

$$v_r^a[i_2,j_2,p_2,q_2]\Big)*u^b[i-i_1-i_2,j-j_1-j_2,p-p_1-p_2,q-q_1-q_2]+$$

$$\sum_{i_2,j_2,p_2,q_2=0}^{i-i_1,j-j_1,p-p_1,q-q_1}\Big(v_0^b[i_1,j_1,p_1,q_1]*v_0^b[i_2,j_2,p_2,q_2]-v_r^b[i_1,j_1,p_1,q_1]*$$

$$v_r^b[i_2,j_2,p_2,q_2]\Big)*u^a[i-i_1-i_2,j-j_1-j_2,p-p_1-p_2,q-q_1-q_2]\Big]\Big\} \qquad (6.5)$$

Substitution into (5 2) leads to the next nine formulas For j=1 we find

$$v_1^a[i,j,p,q+2]=\big((q+1)(q+2)\big)^{-1}\Big\{(i+1)(q+1)v_0^a(i+1,j,p,q+1)-$$

$$(i+1)(j+1)v_2^a(i+1,j+1,p,q)-(i+1)(p+1)v_3^a(i+1,j,p+1,q)+$$

$$(j+1)(j+2)v_1^a(i,j+2,p,q)+(p+1)(p+2)v_1^a(i,j,p+2,q)-$$

$$\sum_{i_1,j_1,p_1,q_1=0}^{i,j,p,q}\Big[\varepsilon^{abc}(q+1)*v_1^b[i-i_1,j-j_1,p-p_1,q-q_1]*v_0^c[i_1,j_1,p_1,q_1+1]-$$

$$\varepsilon^{abc}(i+1)*v_1^b[i-i_1,j-j_1,p-p_1,q-q_1]*v_1^c[i_1+1,j_1,p_1,q_1]-$$

$$\varepsilon^{abc}(i+1)*v_1^b[i-i_1,j-j_1,p-p_1,q-q_1]*v_2^c[i_1,j_1+1,p_1,q_1]-$$

$$\varepsilon^{abc}(i+1)*v_1^b[i-i_1,j-j_1,p-p_1,q-q_1]*v_3^c[i_1,j_1,p_1+1,q_1]+$$

$$2\varepsilon^{abc}(q+1)*v_0^c[i-i_1,j-j_1,p-p_1,q-q_1]*v_1^b[i_1,j_1,p_1,q_1+1]-$$

$$2\varepsilon^{abc}(i+1)*v_1^c[i-i_1,j-j_1,p-p_1,q-q_1]*v_1^b[i_1+1,j_1,p_1,q_1]-$$

$$2\varepsilon^{abc}(j+1) * v_2^c[i-i_1, j-j_1, p-p_1, q-q_1] * v_1^b[i_1, j_1+1, p_1, q_1] -$$

$$2\varepsilon^{abc}(p+1) * v_3^c[i-i_1, j-j_1, p-p_1, q-q_1] * v_1^b[i_1, j_1, p_1+1, q_1] +$$

$$\varepsilon^{abc}(i+1) * v_0^b[i-i_1, j-j_1, p-p_1, q-q_1] * v_0^c[i_1+1, j_1, p_1, q_1] -$$

$$\varepsilon^{abc}(i+1) * v_1^b[i-i_1, j-j_1, p-p_1, q-q_1] * v_1^c[i_1+1, j_1, p_1, q_1] -$$

$$\varepsilon^{abc}(i+1) * v_2^b[i-i_1, j-j_1, p-p_1, q-q_1] * v_2^c[i_1+1, j_1, p_1, q_1] -$$

$$\varepsilon^{abc}(i+1) * v_3^b[i-i_1, j-j_1, p-p_1, q-q_1] * v_3^c[i_1+1, j_1, p_1, q_1] -$$

$$\varepsilon^{abc}(i+1) * u^b[i-i_1, j-j_1, p-p_1, q-q_1] * u^c[i_1+1, j_1, p_1, q_1] -$$

$$\sum_{i_2, j_2, p_2, q_2 = 0}^{i-i_1, j-j_1, p-p_1, q-q_1} \Big(v_0^b[i_1, j_1, p_1, q_1] * v_0^b[i_2, j_2, p_2, q_2] - v_r^b[i_1, j_1, p_1, q_1] *$$

$$v_r^b[i_2, j_2, p_2, q_2] \Big) * v_1^a[i-i_1-i_2, j-j_1-j_2, p-p_1-p_2, q-q_1-q_2] -$$

$$\sum_{i_2, j_2, p_2, q_2 = 0}^{i-i_1, j-j_1, p-p_1, q-q_1} \Big(v_0^b[i_1, j_1, p_1, q_1] * v_0^a[i_2, j_2, p_2, q_2] - v_r^b[i_1, j_1, p_1, q_1] *$$

$$v_r^a[i_2, j_2, p_2, q_2] \Big) * v_1^b[i-i_1-i_2, j-j_1-j_2, p-p_1-p_2, q-q_1-q_2] -$$

$$\sum_{i_2, j_2, p_2, q_2 = 0}^{i-i_1, j-j_1, p-p_1, q-q_1} \Big(u^b[i_1, j_1, p_1, q_1] * v_1^a[i_2, j_2, p_2, q_2] - u^a[i_1, j_1, p_1, q_1] *$$

$$v_1^a[i_2, j_2, p_2, q_2] \Big) * u^b[i-i_1-i_2, j-j_1-j_2, p-p_1-p_2, q-q_1-q_2] \Big] \Big\} \qquad (6.6)$$

For j=2,3 we can find similar formulas for $v_2^a[i,j,p,q+2]$ and $v_3^a[i,j,p,q+2]$ To find the other three formulas, we substitute the power series (6 4-a,b) into the equation (5 5) This gives

$$v_0^a[i,j,p,q+2] = \big((q+1)(q+2)\big)^{-1} \Big\{ (i+1)(q+1) * v_1^a(i+1,j,p,q+1) +$$

$$(j+1)(q+1) * v_2^a(i,j+1,p,q+1) + (p+1)(q+1) * v_3^a(i,j,p+1,q+1) \Big\} \qquad (6.7)$$

Making use of the initial data (5.25,a,b,c,d), (5.26,a,b,c,d) and (5.28,a,b) we can find the coefficients $u^a[i,j,p,q]$ and $v_r^a[i,j,p,q]$ for i,j,p=1,2,3,..., q=0,1, then by backsubstitution the unknown coefficients for i,j,p=1,2,3,...; q=2,3, .; can be found easily. This will enable us to write the leading terms of the series solution of our Cauchy problem. To do that, we use Mathematica. Up to fourth order we found

$$\Phi^1(t,x,y,z)=\frac{\pi}{16}t+(\frac{\pi}{96}-\frac{\pi^3}{768})t^4-(\frac{3\pi}{8}-\frac{\pi^3}{32})x^2+(\frac{3\pi}{8}-\frac{\pi^3}{32})y^2- \tag{6.8-a}$$
$$(\frac{5\pi}{16}-\frac{5\pi^3}{384}-\frac{\pi^5}{512})x^4+(\frac{5\pi}{16}-\frac{5\pi^3}{384}-\frac{\pi^5}{512})y^4+(\frac{13\pi}{8}-\frac{3\pi^3}{64}-\frac{3\pi^5}{256})x^2z^2-$$
$$(\frac{13\pi}{8}-\frac{3\pi^3}{64}-\frac{3\pi^5}{256})y^2z^2+(\frac{15\pi}{32}-\frac{11\pi^3}{256})tx^2-(\frac{3\pi}{32}-\frac{\pi^3}{256})ty^2-(\frac{\pi}{4}-\frac{3\pi^3}{128})tz^2,$$

$$\Phi^2(t,x,y,z)=-(\frac{3\pi}{4}-\frac{\pi^3}{16})xy-(\frac{5\pi}{8}-\frac{5\pi^3}{192}-\frac{\pi^5}{256})x^3y- \tag{6 8-b}$$
$$(\frac{5\pi}{8}-\frac{5\pi^3}{192}-\frac{\pi^5}{256})xy^3+(\frac{13\pi}{4}-\frac{3\pi^3}{32}-\frac{3\pi^5}{128})xyz^2+(\frac{9\pi}{16}-\frac{3\pi^3}{64})txy,$$

$$\Phi^3(t,x,y,z)=-(2-\frac{\pi^2}{4})z-(3-\frac{\pi^4}{32})x^2z-(3-\frac{\pi^4}{32})y^2z+(2-\frac{\pi^4}{48})z^3+ \tag{6 8-c}$$
$$(1-\frac{\pi^2}{8})tz+(\frac{9}{2}+\frac{5\pi^2}{128}-\frac{13\pi^4}{256})tx^2z+(\frac{3}{2}-\frac{5\pi^2}{128}-\frac{3\pi^4}{256})ty^2z-(2-\frac{\pi^4}{48})tz^3,$$

$$A_1^1(t,x,y,z)=(\frac{5\pi^2}{4}-\frac{\pi^3}{8})yz+(2\pi-\frac{\pi^3}{8}-\frac{\pi^5}{128})x^2yz+(2\pi-\frac{\pi^3}{8}-\frac{\pi^5}{128})y^3z- \tag{6 9-a}$$
$$(\frac{5\pi}{4}+\frac{25\pi^3}{192}-\frac{5\pi^5}{192})yz^3+(\frac{5\pi^2}{8}-\frac{\pi^3}{16})tyz+(\pi-\frac{11\pi^3}{64}+\frac{\pi^5}{128})t^2yz,$$

$$A_1^2(t,x,y,z)=-(\frac{5\pi^2}{4}-\frac{\pi^3}{8})xz-(2\pi-\frac{\pi^3}{8}-\frac{\pi^5}{128})xy^2z-(2\pi-\frac{\pi^3}{8}-\frac{\pi^5}{128})x^3z+ \tag{6 9-b}$$
$$(\frac{5\pi}{4}+\frac{25\pi^3}{192}-\frac{5\pi^5}{192})xz^3+(\frac{5\pi^2}{4}-\frac{\pi^3}{8})txz-(\pi-\frac{11\pi^3}{64}+\frac{\pi^5}{128})t^2xz,$$

$$A_1^3(t,x,y,z)=(1-\frac{\pi^2}{8})y+(\frac{3}{4}-\frac{\pi^4}{128})x^2y+(\frac{3}{4}-\frac{\pi^4}{128})y^3-(3-\frac{\pi^4}{32})yz^2-$$

$$(\frac{1}{4}-\frac{\pi^2}{32})ty-(\frac{3}{4}-\frac{\pi^4}{128})tx^2y-(\frac{3}{4}-\frac{\pi^2}{64}-\frac{\pi^4}{768})ty^3+$$

$$(\frac{3}{2}+\frac{5\pi^2}{64}-\frac{3\pi^4}{128})tyz^2+(\frac{5\pi^2}{384}-\frac{3\pi^4}{768})t^3y, \tag{6 9-c}$$

$$A_2^1(t,x,y,z)=(\frac{5\pi^2}{4}-\frac{\pi^3}{8})xz+(2\pi-\frac{\pi^3}{8}-\frac{\pi^5}{128})x^3z+(2\pi-\frac{\pi^3}{8}-\frac{\pi^5}{128})xy^2z-$$

$$(\frac{5\pi}{4}+\frac{25\pi^3}{192}-\frac{5\pi^5}{192})xz^3-(\frac{5\pi^2}{4}-\frac{\pi^3}{8})txz+(\pi-\frac{11\pi^3}{64}+\frac{\pi^5}{128})t^2xz, \tag{6.10-a}$$

$$A_2^2(t,x,y,z)=(\frac{5\pi^2}{4}-\frac{\pi^3}{8})yz+(2\pi-\frac{\pi^3}{8}-\frac{\pi^5}{128})x^2yz+(2\pi-\frac{\pi^3}{8}-\frac{\pi^5}{128})y^3z-$$

$$(\frac{5\pi}{4}+\frac{25\pi^3}{192}-\frac{5\pi^5}{192})yz^3-(\frac{5\pi}{8}-\frac{\pi^3}{16})tyz+(\pi-\frac{11\pi^3}{64}+\frac{\pi^5}{128})t^2yz, \tag{6 10-b}$$

$$A_2^3(t,x,y,z)=-(1-\frac{\pi^2}{8})x-(\frac{3}{4}-\frac{\pi^4}{128})x^3-(\frac{3}{4}-\frac{\pi^4}{128})xy^2+(3-\frac{\pi^4}{32})xz^2+$$

$$(\frac{3}{4}-\frac{3\pi^2}{32})tx+(\frac{5}{4}+\frac{\pi^2}{64}-\frac{11\pi^4}{768})tx^3+(\frac{3}{4}-\frac{\pi^4}{128})txy^2-$$

$$(\frac{9}{2}-\frac{5\pi^2}{64}-\frac{5\pi^4}{128})txz^2+(\frac{5\pi^2}{384}-\frac{\pi^4}{768})t^3x, \tag{6 10-c}$$

$$A_3^1(t,x,y,z)=(\frac{\pi}{2}-\frac{\pi^3}{16})xy+(\frac{\pi}{2}-\frac{\pi^3}{96}-\frac{\pi^5}{256})x^3y+(\frac{\pi}{2}-\frac{\pi^3}{96}-\frac{\pi^5}{256})xy^3-$$

$$(3\pi-\frac{5\pi}{64}-\frac{3\pi^5}{128})xyz^2-(\frac{3\pi}{8}-\frac{3\pi^3}{64})txy+(\pi-\frac{11\pi^3}{64}+\frac{\pi^5}{128})t^2xz, \tag{6 11-a}$$

$$A_3^2(t,x,y,z)=-(\frac{\pi}{4}-\frac{\pi^3}{32})x^2-(\frac{\pi}{4}-\frac{\pi^3}{192}-\frac{\pi^5}{512})x^4+(\frac{\pi}{4}-\frac{\pi^3}{32})y^2+$$

$$(\frac{\pi}{4}-\frac{\pi^3}{192}-\frac{\pi^5}{512})y^4+(\frac{3\pi}{2}-\frac{5\pi^3}{128}-\frac{3\pi^5}{256})x^2z^2-(\frac{3\pi}{2}-\frac{5\pi^3}{128}-\frac{3\pi^5}{256})y^2z^2- \tag{6 11-b}$$

$$\frac{\pi}{4}t+(\frac{5\pi}{16}-\frac{9\pi^3}{256})tx^2-(\frac{\pi}{16}-\frac{3\pi^3}{256})ty^2-(\frac{\pi}{8}-\frac{\pi^3}{64})tz^2-(\frac{\pi}{2}-\frac{11\pi^3}{128}-\frac{\pi^5}{256})t^2x^2+$$

$$(\frac{\pi}{2}-\frac{11\pi^3}{128}-\frac{\pi^5}{256})t^2y^2+(\frac{\pi}{16}-\frac{\pi^3}{192})t^3-(\frac{\pi}{192}-\frac{\pi^3}{1536})t^4,$$

$$A_3^3(t,x,y,z)=(\frac{5\pi^2}{64}-\frac{\pi^4}{128})txyz \quad (6.11\text{-c})$$

$$A_0^1(t,x,y,z)=(2\pi-\frac{11\pi^3}{32}+\frac{\pi^5}{64})txyz, \quad (6.12\text{-a})$$

$$A_0^2(t,x,y,z)=(\frac{\pi}{8}-\frac{3\pi^3}{256})x^2z+(\frac{\pi}{8}-\frac{3\pi^3}{256})y^2z-$$

$$(\pi-\frac{11\pi^3}{64}+\frac{\pi^5}{128})tx^2z+(\pi-\frac{11\pi^3}{64}+\frac{\pi^5}{128})tx^2z+(\frac{3\pi}{16}-\frac{\pi^3}{64})t^2z, \quad (6.12\text{-b})$$

$$A_0^3(t,x,y,z)=(\frac{\pi^2}{192}-\frac{\pi^4}{1536})x^3y+(\frac{\pi^2}{192}-\frac{\pi^4}{1536})y^3x+(\frac{5\pi^2}{128}-\frac{\pi^4}{256})t^2xy \quad (6\ 12\text{-b})$$

6.3 The Symmetry of the Solutions

In this section we study the symmetry of the time-dependent series solutions found in the previous section under reflexion on the coordinate planes First under the transformation

$$(t,x,y,z)\rightarrow(t,-x,y,z), \quad (6\ 13)$$

we found the time-dependent series solutions (6 8-a,b,c), (6 9-a,b,c), (6 10-a,b,c), (6 11-a,b,c) and (6 12-a,b,c) change as follows

$$\begin{pmatrix}\Phi^1\\ \Phi^2\\ \Phi^3\end{pmatrix}\rightarrow\begin{pmatrix}+\Phi^1\\ -\Phi^2\\ +\Phi^3\end{pmatrix},\quad \begin{pmatrix}A_1^1\\ A_1^2\\ A_1^3\end{pmatrix}\rightarrow\begin{pmatrix}+A_1^1\\ -A_1^2\\ +A_1^3\end{pmatrix},\quad \begin{pmatrix}A_2^1\\ A_2^2\\ A_2^3\end{pmatrix}\rightarrow\begin{pmatrix}-A_2^1\\ +A_2^2\\ -A_2^3\end{pmatrix},$$

$$\begin{pmatrix} A_3^1 \\ A_3^2 \\ A_3^3 \end{pmatrix} \to \begin{pmatrix} -A_3^1 \\ +A_3^2 \\ -A_3^3 \end{pmatrix}, \qquad \begin{pmatrix} A_0^1 \\ A_0^2 \\ A_0^3 \end{pmatrix} \to \begin{pmatrix} -A_0^1 \\ +A_0^2 \\ -A_0^3 \end{pmatrix}. \tag{6 14}$$

We now show that this symmetry of the solution is a consequence of the symmetry of the equations of motion and of the initial data Define

$$\widetilde{\Phi}^a(t,x,y,z) = (-1)^{a-1}\Phi^a(t,-x,y,z), \quad \widetilde{A}_1^a(t,x,y,z) = (-1)^{a-1}A_1^a(t,-x,y,z),$$

$$\widetilde{A}_2^a(t,x,y,z) = (-1)^a A_2^a(t,-x,y,z), \quad \widetilde{A}_3^a(t,x,y,z) = (-1)^a A_3^a(t,-x,y,z),$$

$$\widetilde{A}_0^a(t,x,y,z) = (-1)^a A_0^a(t,-x,y,z), \tag{6 15}$$

where Φ^a, A_μ^a is a solution. Now consider (5.1) for a=1,

$$\begin{aligned}\partial_0^2\Phi_1 = {} & \partial_1^2\Phi_1 + \partial_2^2\Phi_1 + \partial_3^2\Phi_1 + 2(A_{02}\partial_0\Phi_3 - A_{03}\partial_0\Phi_2) - 2(A_{12}\partial_1\Phi_3 - A_{13}\partial_1\Phi_2) - \\ & 2(A_{22}\partial_2\Phi_3 - A_{23}\partial_2\Phi_2) - 2(A_{32}\partial_3\Phi_3 - A_{33}\partial_3\Phi_2) + (\Phi_3\partial_0A_{02} - \Phi_2\partial_0A_{03}) - \\ & (\Phi_3\partial_1A_{12} - \Phi_2\partial_1A_{13}) - (\Phi_3\partial_2A_{22} - \Phi_2\partial_2A_{23}) - (\Phi_3\partial_3A_{32} - \Phi_2\partial_3A_{33}) - \\ & \Phi_2(A_{01}A_{02} - A_{11}A_{12} - A_{21}A_{22} - A_{31}A_{32}) - \\ & \Phi_3(A_{01}A_{03} - A_{11}A_{13} - A_{21}A_{23} - A_{31}A_{33}) + \\ & \Phi_1(A_{02}^2 + A_{03}^2 - A_{12}^2 - A_{13}^2 - A_{22}^2 - A_{23}^2 - A_{32}^2 - A_{33}^2)\end{aligned} \tag{6 16}$$

By substituting (6 15) into (6 16), we can easily see that the solution $\widetilde{\Phi}^a$, $\widetilde{A}_\mu^a$ also satisfies the equation (6 16) and the initial data (5 25) and (5 26) and (5 28) Using the same techniques we can easily complete the proof by showing $\widetilde{\Phi}^a$, $\widetilde{A}_\mu^a$ is a solution of the other equations too Uniqueness now implies that $\widetilde{\Phi}^a = \Phi^a$ and $\widetilde{A}_\mu^a = A_\mu^a$ This general result is, of course, reflected in the symmetry (6 14) of the series solutions.

Next, under the transformation

$$(t,x,y,z) \to (t,x,-y,z), \tag{6.17}$$

the time-dependent series solutions change as follows:

$$\begin{pmatrix}\Phi^1\\ \Phi^2\\ \Phi^3\end{pmatrix} \to \begin{pmatrix}+\Phi^1\\ -\Phi^2\\ +\Phi^3\end{pmatrix}, \quad \begin{pmatrix}A_1^1\\ A_1^2\\ A_1^3\end{pmatrix} \to \begin{pmatrix}-A_1^1\\ +A_1^2\\ -A_1^3\end{pmatrix}, \quad \begin{pmatrix}A_2^1\\ A_2^2\\ A_2^3\end{pmatrix} \to \begin{pmatrix}+A_2^1\\ -A_2^2\\ +A_2^3\end{pmatrix},$$

$$\begin{pmatrix}A_3^1\\ A_3^2\\ A_3^3\end{pmatrix} \to \begin{pmatrix}-A_3^1\\ +A_3^2\\ -A_3^3\end{pmatrix}, \quad \begin{pmatrix}A_0^1\\ A_0^2\\ A_0^3\end{pmatrix} \to \begin{pmatrix}-A_0^1\\ +A_0^2\\ -A_0^3\end{pmatrix}. \tag{6.18}$$

Under the transformation

$$(t,x,y,z) \to (t,x,y,-z), \tag{6.19}$$

we have the following behaviour of the time-dependent series solutions:

$$\begin{pmatrix}\Phi^1\\ \Phi^2\\ \Phi^3\end{pmatrix} \to \begin{pmatrix}+\Phi^1\\ +\Phi^2\\ -\Phi^3\end{pmatrix}, \quad \begin{pmatrix}A_1^1\\ A_1^2\\ A_1^3\end{pmatrix} \to \begin{pmatrix}-A_1^1\\ -A_1^2\\ +A_1^3\end{pmatrix}, \quad \begin{pmatrix}A_2^1\\ A_2^2\\ A_2^3\end{pmatrix} \to \begin{pmatrix}-A_2^1\\ -A_2^2\\ +A_2^3\end{pmatrix},$$

$$\begin{pmatrix}A_3^1\\ A_3^2\\ A_3^3\end{pmatrix} \to \begin{pmatrix}+A_3^1\\ +A_3^2\\ -A_3^3\end{pmatrix}, \quad \begin{pmatrix}A_0^1\\ A_0^2\\ A_0^3\end{pmatrix} \to \begin{pmatrix}-A_0^1\\ -A_0^2\\ +A_0^3\end{pmatrix}. \tag{6.20}$$

Using the same techniques we have used for the transformation (6.13), we can show that (6.18) and (6.20) follow from the symmetry of the equations of motion and of the initial data. The important consequence is that

$$\mathcal{E} = \frac{1}{2}\left|D_0\Phi\right|^2 + \frac{1}{2}\left|F_{0i}\right|^2 + \frac{1}{2}\left|D_i\Phi\right|^2 + \frac{1}{4}\left|F_{ij}\right|^2, \tag{6.21}$$

remains invariant under the transformations (6.13), (6.17) and (6.19). These transformations represent reflexions on the coordinates planes. This means that the two monopoles are forced to lie on any one of the coordinates axis. This rules out all cases other than $0°$, $90°$ or $180°$scattering.

6.4 The Monopole Locations

In this section we will investigate the positions of the the two monopoles for different times. We will do that by studying the zeros of the modulus of the Higgs field Φ. First using (6.8-a,b,c) we can write $|\Phi|^2$ up to fourth order in the form

$$
\begin{aligned}
|\Phi|^2 = {} & \frac{\pi^2}{256}t^2 - \left(\frac{3\pi^2}{64} - \frac{\pi^4}{256}\right)tx^2 + \left(\frac{3\pi^2}{64} - \frac{\pi^4}{256}\right)ty^2 + \left(\frac{9\pi^2}{64} - \frac{3\pi^4}{128} + \frac{\pi^6}{1024}\right)x^4 + \\
& \left(\frac{9\pi^2}{64} - \frac{3\pi^4}{128} + \frac{\pi^6}{1024}\right)y^4 + \left(\frac{15\pi^2}{256} - \frac{11\pi^4}{2048}\right)t^2x^2 - \left(\frac{3\pi^2}{256} - \frac{\pi^4}{2048}\right)t^2y^2 + \\
& \left(\frac{9\pi^2}{32} - \frac{3\pi^4}{64} + \frac{\pi^5}{512}\right)x^2y^2 + \left(4 - \pi^2 + \frac{\pi^4}{16}\right)z^2 - \left(4 - \pi^2 + \frac{\pi^4}{16}\right)tz^2 + \\
& \left(1 - \frac{9\pi^2}{32} + \frac{19\pi^4}{1024}\right)t^2z^2 + \left(12 - \frac{3\pi^2}{2} - \frac{\pi^4}{8} + \frac{\pi^6}{64}\right)x^2z^2 + \\
& \left(12 - \frac{3\pi^2}{2} - \frac{\pi^4}{8} + \frac{\pi^6}{64}\right)y^2z^2 - \left(8 - \pi^2 + \frac{\pi^4}{12} - \frac{\pi^6}{96}\right)z^4, \qquad (6.22)
\end{aligned}
$$

Next we used Mathematica to plot the level surfaces of $|\Phi|^2$ in the xy-plane for different times. First for t=-0.16, we found that the two monopoles lie on the y-axis as shown in Fig. 2. As t increases the two monopoles start moving towards the origin (Fig. 3). At t=0 they will sit on top of each other at the origin as shown in Fig. 4. For t>0, the two monopoles start separating on the x-axis (Figs. 5,6). This study shows that the two monopoles scatter at 90°. Different from the scattering of soliton-like objects such as vortices [25], the scattering is non-trivial in the sense that the distance between the two monopoles is not an even function of time The distance between the monopoles a unit of time before the collision is different from the distance a unit time after. This shows that their velocities have changed. This has also been seen in the slow-motion approximation [23].

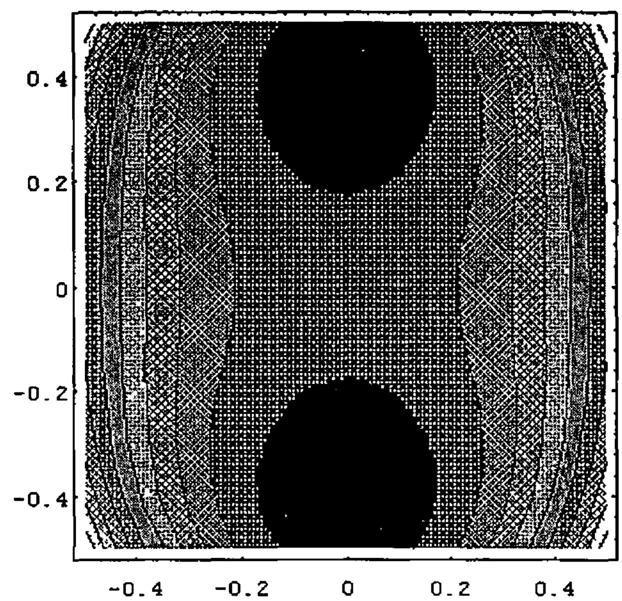

Fig 2

Contour plot of $|\Phi|^2$ in the xy-plane for t=-0 16

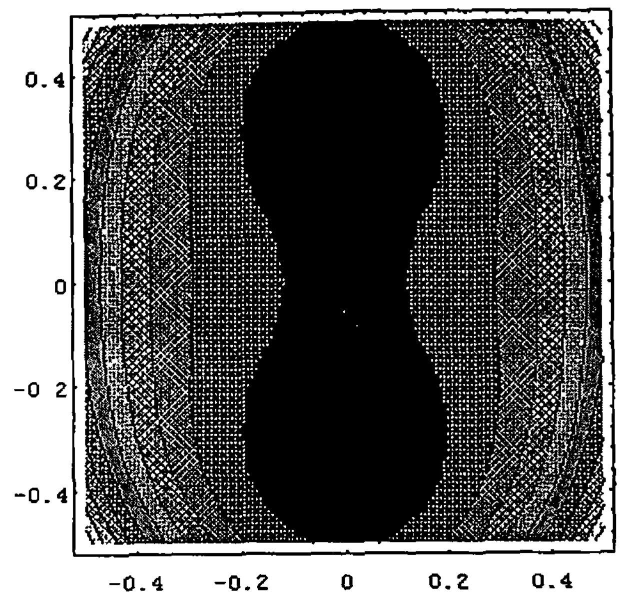

Fig 3

Contour plot of $|\Phi|^2$ in the

xy-plane for t=-0 12

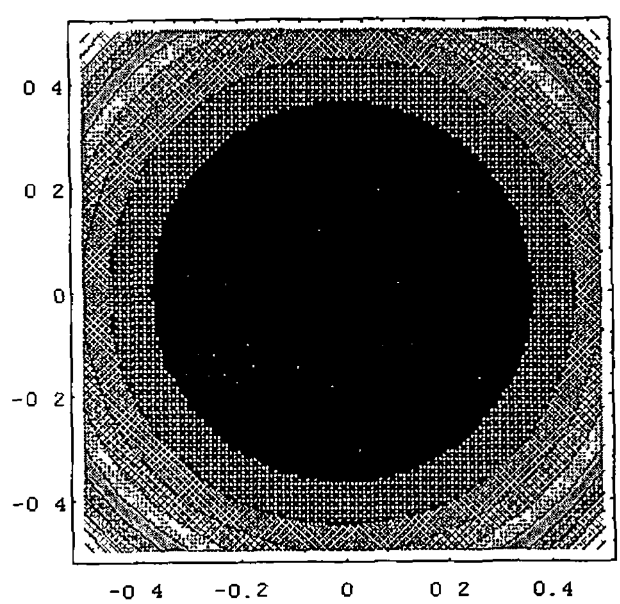

Fig 4

Contour plot of $|\Phi|^2$ in the

xy-plane for t=0

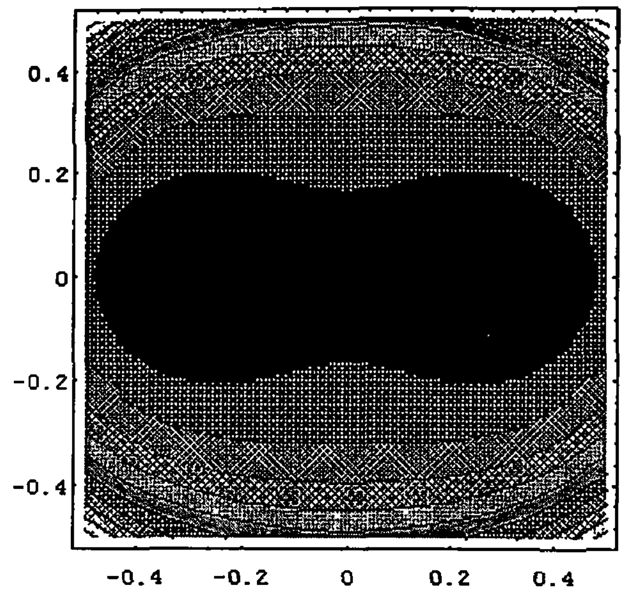

Fig 5

Contour plot of $|\Phi|^2$ in the

xy-plane for t=0 12

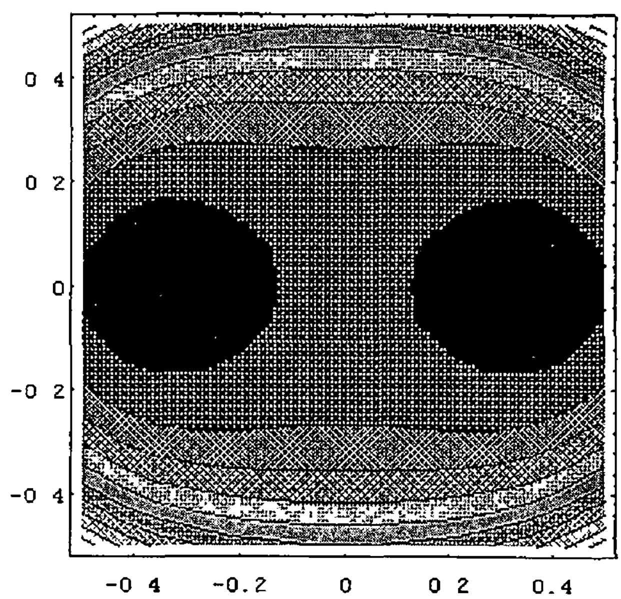

Fig 6

Contour plot of $|\Phi|^2$ in the

xy-plane for t=0 16

Chapter 7

Conclusions

Guided by the idea of the slow-motion approximation and encouraged by our results of Ref 25, our aim in this thesis was first to find suitable initial data for the equations of motion of the SU(2) Yang-Mills-Higgs model In order to study and understand the scattering of two monopoles we then used the time-dependent solutions of the constructed initial value problem The results of this thesis add to a long list of results on scattering of solitons and soliton-like objects The techniques used, which allow an investigation at the centre of the scattering process, complement other techniques, in particular those of the slow-motion approximation

In the previous chapters we studied in detail the SU(2) model as an example of the Yang-Mills-Higgs theory in (3+1)-dimensional Minkowski space-time We discussed the Bogomol'nyi-Prasad-Sommerfield limit and introduced associated linear equations. We proved that the compatibility conditions of these equations are the self-duality conditions. We also studied the Riemann-Hilbert problem and the reality of the 2-monopole solutions This study led to the existence of a GL(2,C) gauge transformation which was not given before in explicit form.

The first new contribution to the theory was to find this gauge transformation and to write it down in explicit form Applying the standard techniques of linear algebra we found a gauge matrix which was a vanishing function at the origin This led to the singularity of the time-independent solutions Theoretically it is known that a unitary gauge transformation can be used to remove this singularity but this transformation is very hard to find Using different techniques in section 3 4 we found a gauge matrix which guaranteed the regularity and the reality of the time-independent solutions Using this gauge matrix we found the unitary gauge transformation. This result was essential to set up a Cauchy problem with analytic data for the equations of motion

We then proved the existence of a unique series solution of this Cauchy problem near the origin We used Mathematica to find the series solutions of the Cauchy problem near the origin Using this series solution we were able to show that the two monopoles lie on any one of the coordinates axis which ruled out all cases other than 0°, 90° or 180° scattering Next we used Mathematica to plot the level surfaces of $|\Phi|^2$ in the xy-plane for different times This study showed that the two monopoles scatter at 90° It showed also that the scattering is non-trivial in the sense that the distance between the two monopoles is not an even function of time This proved that their velocities have changes, a result also seen in the slow-motion approximation Our results show the power of the series solution approach Its obvious limitation is, of course, that it only allows for local investigations

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