# EXPANSIONS OF VORTEX SOLUTIONS FOR TWO VORTICES CLOSE TOGETHER

Eamon Kellegher
School of Mathematical Sciences
Dublin City University

Supervisor Dr J Burzlaff
A Dissertation submitted for the Degree of Master of Science

8 January 2001

# Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Master of Science is entirely my own work and has not been taken from the work of others save to the extent that such work has been cited and acknowledged within the text of my work

Signed ____________ ID No 96970812

Date 8 January 2001

# Acknowledgements


I would like to extend my gratitude to the following people

First and foremost to my supervisor Dr Jurgen Burzlaff, School of Mathematical Sciences, Dublin City University, for his help, guidance, support, understanding and friendship over the duration of this research

I would also like to thank all the staff in the School of Mathematical Sciences for such a pleasant place to work Thanks to my fellow researchers for their discussions

This research was supported by the Irish-American Partnership programme

To my parents and family, thanks for the continued support and encouragement

Special thanks to my wife, Claire, for her patience, love, and encourgement And to our children Meabh, Aileen and Paudi, who motivate me

# Dedications

To my parents, Maureen and Peter Kellegher, for their love and support They gave me an appreciation for knowledge

# Abstract

This thesis considers vortices in a superconductor and in a simple model and attempts to model two vortices close together In contrast to superimposed vortices which have been described explicitly, we look to describe the separation of two vortices close together in terms of an expansion in the parameters, which describe their relative location

A simple model is used to describe two static vortices close together An expansion in the parameters describing the relative position of the two vortices is derived in terms of trigonometric and exponential functions The series solutions are derived from solving the relevant partial differential equations and are studied up to third order

A more realistic model, the Ginzburg-Landau theory of a superconductor in a magnetic field is studied At the point between type-I and type-II superconductivity, this model has static vortex solutions, so-called Abrikosov vortices Starting with two vortices on top of each other, we derive an expansion, which describes these two vortices close together The expansion is studied up to third order

A similar pattern is found for the angular dependence in both models The first simple model is shown to have some peculiar features only two vortices can be superimposed and when pulled apart a singularity at third order develops In contrast, this does not happen in the Ginzburg-Landau model, which shows smooth solutions up to at least third order

# Contents

# 1 Introduction

Ever since 't Hooft[1] and Polyakov[2] found a monopole solution in the $SU(2)$ Yang-Mills-Higgs theory, solitons in field theories have been studied extensively. Our understanding of monopole solutions has been greatly enhanced by an extistence proof for static solutions by Taubes [3] and the construction of monopole solutions started by Ward [4]. The processs was not matched by quite the same progress in our understanding of the Abrikosov solutions of the Ginzburg-Landau theory, although one might have expected that the Abelian Higgs theory in $2+1$ dimensions is actually simpler than the $SU(2)$ Yang-Mills-Higgs theory in $3+1$ dimensions. Again an existence proof was given by Taubes [5]. However, only superimposed vortices can be described explicity and no explicit construction of separated vortices is known. We want to give the solutions for two vortices close together in terms of an expansion in the parameters which describe their relative location.

In Chapter 2 a simple model for one complex field is discussed. Its Lagrangian is given and the Euler-Lagrange equations, which minimize the action, are studied. These second-order equations are then replaced by first order Bogomol'nyi equations which saturate a topological lower bound of the action. The next Chapter discusses vortex solutions, first for two vortices on top of each other, then for two vortices close together. In Chapter 4 we discuss a realistic model of a superconductor in a magnetic field. This physical situation can be described by the Ginzburg-Landau model. At the point between type-I and type-II superconductivity, the model has static vortex solutions, so-called Abrikosov vortices. Starting with 2 vortices on top of each other, we give an expansion which describes vortices close together.

The main results in this thesis will be published in, J. Burzlaff and E. Kellegher, Journal of Mathematical Physics.

# 2 A model for one complex field

In this chapter a simple model for one complex field [6][7] is discussed Its Lagrangian is given and the Euler-Lagrange equations, which minimize the action, are studied These second-order equations are then replaced by first order Bogomol'nyi equations which saturate a topological lower bound of the action

## 2.1 Lagrangian and Euler-Lagrange equation

Our first model is a model for a pair of real fields, $\phi^a(\vec{x})$, a,b = 1,2, or equivalently, for a complex field $\phi = \phi_1 + i\phi_2$ The Lagrangian density of the model reads,

$$\mathcal{L} = \partial_{[i}\phi^a\partial_{j]}\phi^b\partial^{[i}\phi_a\partial^{j]}\phi_b + u^2(|\phi|^2), \tag{2 1}$$

where $a, b = 1, 2$ labels the components of the Higgs field and $i, j = 1, 2$ are the space indices The square brackets mean antisymmetrization,

$$\partial_{[i}\phi^a\partial_{j]}\phi^b = (\partial_i\phi^a)(\partial_j\phi^b) - (\partial_j\phi^a)(\partial_i\phi^b) \tag{2 2}$$

We are working in 2-dimensional Euclidean space, i e , the space indices can be raised and lowered without any change in the formulas The indices which label the components of the Higgs field can also be raised and lowered without any change

We now derive the Euler-Lagrange equation Since we will use similar arguments throughout the thesis we go through the derivation in great detail First, we obtain, using the product rule,

$$\frac{\partial\mathcal{L}}{\partial(\partial_i\phi^a)} = \partial^{[j}\phi_b\partial^{k]}\phi_c\frac{\partial}{\partial(\partial_i\phi^a)}\partial_{[j}\phi^b\partial_{k]}\phi^c + \partial_{[j}\phi^b\partial_{k]}\phi^c\frac{\partial}{\partial(\partial_i\phi^a)}\partial^{[j}\phi_b\partial^{k]}\phi_c \tag{2 3}$$

The two terms on the right-hand side differ in the position of the indices only However, in Euclidean space we can raise and lower indices without any change This reduces equation (2 3) to

$$\frac{\partial \mathcal{L}}{\partial(\partial_i \phi^a)} = 2\partial^{[j}\phi_b \partial^{k]}\phi_c \frac{\partial}{\partial(\partial_i \phi^a)} \partial_{[j}\phi^b \partial_{k]}\phi^c \tag{2 4}$$

If the right side of the above equation is expanded using equation (2 2), we obtain

$$\frac{\partial \mathcal{L}}{\partial(\partial_i \phi^a)} = 2\partial^{[j}\phi_b \partial^{k]}\phi_c \left(\frac{\partial}{\partial(\partial_i \phi^a)}(\partial_j \phi^b)(\partial_k \phi^c) - \frac{\partial}{\partial(\partial_i \phi^a)}(\partial_k \phi^b)(\partial_j \phi^c)\right) \tag{2 5}$$

The summation indices in the second term can be interchanged, and we can use the indentities

$$-\partial_{[j}\phi^a \partial_{i]}\phi^b = \partial_{[i}\phi^a \partial_{j]}\phi^b = \partial_{[j}\phi^b \partial_{i]}\phi^a \tag{2 6}$$

This yields

$$\frac{\partial \mathcal{L}}{\partial(\partial_i \phi^a)} = 4\ \partial^{[j}\phi_b \partial^{k]}\phi_c \frac{\partial}{\partial(\partial_i \phi^a)}((\partial_j \phi^b)(\partial_k \phi^c)) \tag{2 7}$$

We now perform the differentiation and obtain

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial(\partial_i \phi^a)} &= 4\ \partial^{[j}\phi_b \partial^{k]}\phi_c \left(\ (\partial_j \phi^b)\frac{\partial}{\partial(\partial_i \phi^a)}(\partial_k \phi^c) + (\partial_k \phi^c)\frac{\partial}{\partial(\partial_i \phi^a)}(\partial_j \phi^b)\ \right) \\ &= 4\ (\ \partial^{[j}\phi_b \partial^{k]}\phi_c (\delta^i_k \delta^c_a)(\partial_j \phi^b) + \partial^{[j}\phi_b \partial^{k]}\phi_c (\partial_k \phi^c)(\delta^i_j \delta^b_a)\ ) \end{aligned}$$

$$= 4\ (\ \partial^{[j}\phi_b\partial^{i]}\phi_a(\partial_j\phi^b) + \partial^{[i}\phi_a\partial^{k]}\phi_c(\partial_k\phi^c)\ )$$

$$= 8\ \partial^{[i}\phi_a\partial^{j]}\phi_b(\partial_j\phi^b) \tag{2.8}$$

Here $\delta_i^j$ is the Kronecker delta ( $\delta_i^j = 1$ if $i = j$ , $\delta_i^j = 0$ if $i \neq j$ ) and for the last step we have used (2.6) again. From (2.8),

$$\partial_i\frac{\partial\mathcal{L}}{\partial(\partial_i\phi^a)} = 8\partial_j\phi^b\partial_i(\partial^{[i}\phi_a\partial^{j]}\phi_b) \tag{2.9}$$

follows, since

$$\partial_i\partial_j\phi^b\partial^{[i}\phi_a\partial^{j]}\phi_b = \partial_j\partial_i\phi^b\partial^{[j}\phi_a\partial^{i]}\phi_b$$

$$= -\partial_j\partial_i\phi^b\partial^{[i}\phi_a\partial^{j]}\phi_b = -\partial_i\partial_j\phi^b\partial^{[i}\phi_a\partial^{j]}\phi_b \tag{2.10}$$

which therefore is zero. We also obtain

$$\frac{\partial\mathcal{L}}{\partial\phi^a} = 2u\frac{\partial u}{\partial\phi^a} \tag{2.11}$$

We now have all the terms in the Euler-Lagrange equation which is given by

$$\partial_i(\frac{\partial\mathcal{L}}{\partial(\partial_i\phi^a)}) - \frac{\partial\mathcal{L}}{\partial\phi^a} = 0 \tag{2.12}$$

The solutions of these equations are stationary points of the action

$$A = \int_{\mathbf{R}^2} \mathcal{L}(\phi^a, \partial_i \phi^b) d^2x \tag{2 13}$$

Substituting (2 9) and (2 11) into (2 12), yields

$$4(\partial_i \phi^b)\partial_j(\partial^{[i}\phi_b \partial^{j]}\phi_a) = u \frac{\partial u}{\partial \phi^a} \tag{2 14}$$

It is often convenient to express the Lagrangian and the Euler-Lagrange equation in terms of the complex field $\phi = \phi_1 + i\phi_2$ We have $\phi_1 = (\phi + \phi^*)/2$, $\phi_2 = i(\phi^* - \phi)/2$, and

$$\begin{aligned} \partial_{[i}\phi \partial_{j]}\phi^* &= \partial_i(\phi_1 + i\phi_2)\partial_j(\phi_1 - i\phi_2) - \partial_j(\phi_1 + i\phi_2)\partial_i(\phi_1 - i\phi_2) \\ &= 2i[(\partial_i \phi_2)(\partial_j \phi_1) - (\partial_i \phi_1)(\partial_j \phi_2)] \\ &= 2i\partial_{[i}\phi_2 \partial_{j]}\phi_1 \end{aligned} \tag{2 15}$$

Now,

$$\begin{aligned} \partial_{[i}\phi^a \partial_{j]}\phi^b \partial^{[i}\phi_a \partial^{j]}\phi_b &= 2\partial_{[i}\phi^1 \partial_{j]}\phi^2 \partial^{[i}\phi_1 \partial^{j]}\phi_2 \\ &= -\frac{1}{2}\partial_{[i}\phi \partial_{j]}\phi^* \partial^{[i}\phi \partial^{j]}\phi^*, \end{aligned} \tag{2 16}$$

and the Langrangian density of this model in terms of a complex field reads

$$\mathcal{L} = -\frac{1}{2}\partial_{[i}\phi\partial_{j]}\phi^*\partial^{[i}\phi\partial^{j]}\phi^* + u^2(|\phi|^2) \tag{2 17}$$

The Euler-Lagrange equation can be derived as before The same techniques of renaming, raising and lowering indices are applied to give the following

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial(\partial_i\phi)} &= -2\partial^{[j}\phi\partial^{k]}\phi^*\frac{\partial}{\partial(\partial_i\phi)}[(\partial_j\phi)(\partial_k\phi^*)] \\ &= -2(\partial_j\phi^*)\partial^{[i}\phi\partial^{j]}\phi^* \end{aligned} \tag{2 18}$$

and

$$\partial_i(\frac{\partial\mathcal{L}}{\partial(\partial_i\phi)}) = 2(\partial_j\phi^*)\partial_i(\partial^{[j}\phi\partial^{i]}\phi^*) \tag{2 19}$$

Again the other term we get applying the product rule vanishes because of the symmetry of $\partial_i\partial_j\phi$ and the antisymmetry of $\partial^{[i}\phi\partial^{j]}\phi^*$ This implies

$$\partial_i\phi^*\partial_j(\partial^{[i}\phi\partial^{j]}\phi^*) = u\frac{\partial u}{\partial\phi} \tag{2 20}$$

which is the Euler-Lagrange equation in terms of the complex field $\phi$

## 2.2 The Bogomol'nyi equation and a lower bound on the action

We will show now that any solution of the equation,

$$2\, det(\frac{\partial \phi^a}{\partial x^i}) = \pm u \tag{2 21}$$

solves the equations of motion (2 14) Note that equation (2 21) is a first order equation whereas equation (2 14) is of second order So we would expect that (2 21) is somewhat easier to solve than (2 14) For different types of models, this reduction of order was first introduced by Bogomol'nyi[8] That is why we call equation (2 21) the Bogomol'nyi equation here

To relate the two equations (2 14) and (2 21), we differentiate (2 21) and obtain

$$2\, \partial_j (\partial^{[i} \phi_1 \partial^{j]} \phi_2) = \pm \epsilon^{ij} \frac{\partial u}{\partial \phi^a} \partial_j \phi^a \tag{2 22}$$

Here $\epsilon_{ij}$ is the antisymmetric tensor given by $\epsilon_{12} = 1 = -\epsilon_{21}$, $\epsilon_{11} = \epsilon_{22} = 0$ Multiplication with $\partial_i \phi^1$ leads to

$$2(\partial_i \phi^b) \partial_j (\partial^{[i} \phi_b \partial^{j]} \phi_2) = \pm \epsilon^{ij} (\partial_i \phi^1)(\partial_j \phi^2) \frac{\partial u}{\partial \phi^2} \tag{2 23}$$

Using (2 21) on the right-hand side, we obtain (2 14) for $a = 2$ Analogously,

$$\begin{aligned} 2(\partial_i \phi^b) \partial_j (\partial^{[i} \phi_1 \partial^{j]} \phi_b) &= \pm \epsilon^{ij} (\partial_i \phi^2)(\partial_j \phi^1) \frac{\partial u}{\partial \phi^1} \\ &= -\frac{u}{2} \frac{\partial u}{\partial \phi_1} \end{aligned} \tag{2 24}$$

follows from (2 22) Equation (2 24) is equation (2 14) for $a = 1$

Any solution of (2 21) also attains the lower bound in the following inequality,

$$A = \int_{\mathbf{R}^2} \mathcal{L}\, d^2x \geq \frac{16\pi}{15}|Q|, \tag{2 25}$$

where

$$Q = \frac{15}{8\pi} \int_{\mathbf{R}^2} i\epsilon_{ij} u (\partial^i \phi)(\partial^j \phi^*) d^2x \tag{2 26}$$

Why Q is called the winding number and is explained below after a potential has been chosen

To show that a solution of (2 21) attains the lower bound of the action in (2 25), consider

$$M = (2\partial_{[i}\phi^1 \partial_{j]}\phi^2 \pm \epsilon_{ij} u)(2\partial^{[i}\phi_1 \partial^{j]}\phi_2 \pm \epsilon^{ij} u) \tag{2 27}$$

Note that M is a sum of nonnegative terms and therefore $M \geq 0$ Using the binomial formula we can express $M$ as

$$\begin{aligned} M &= 2\partial_{[i}\phi^a \partial_{j]}\phi^b \partial^{[i}\phi_a \partial^{j]}\phi_b \pm 4u\epsilon_{ij}\partial^{[i}\phi_1 \partial^{j]}\phi_2 + 2u^2 \\ &= 2\mathcal{L} \pm 4u\epsilon_{ij}\partial^{[i}\phi_1 \partial^{j]}\phi_2 \end{aligned} \tag{2 28}$$

A lower bound for the action is now obtained

$$\begin{aligned} A &= \int_{\mathbf{R}^2} \mathcal{L}\, d^2x \\ &= \frac{1}{2}\int_{\mathbf{R}^2} M\, d^2x \;\pm\; 2\int_{\mathbf{R}^2} u\epsilon_{ij}\partial^{[i}\phi_1\partial^{j]}\phi_2\, d^2x \\ &\geq \pm 2\int_{\mathbf{R}^2} u\epsilon_{ij}\partial^{[i}\phi_1\partial^{j]}\phi_2\, d^2x \\ &= \pm\frac{16\pi}{15}Q \end{aligned} \tag{2 29}$$

with Q given in (2 26) The last equality follows from the formula

$$i\epsilon_{ij}(\partial^i\phi)(\partial^j\phi^*) = 2\epsilon_{ij}(\partial^i\phi_1)(\partial^j\phi_2) \tag{2 30}$$

If Q $\geq$ 0 we take the upper sign in (2 27), if Q $\leq$ 0 we take the lower sign This implies

$$A \geq \frac{16\pi}{15}|Q| \tag{2 31}$$

For $Q \leq 0$, the lower bound is attained if and only if

$$2\partial_{[i}\phi^1\partial_{j]}\phi^2 - \epsilon_{ij}u \;=\; 0, \tag{2 32}$$

or

$$2(\partial_1\phi^1\partial_2\phi^2 - \partial_2\phi^1\partial_1\phi^2) \quad = \quad u \tag{2 33}$$

Equation (2 31) is the Bogomol'nyi equation (2 21) with the upper sign For $Q \geq 0$, the lower bound is attained in (2 31) if and only if

$$2\partial_{[i}\phi^1\partial_{j]}\phi^2 + \epsilon_{ij}u = 0 \tag{2 34}$$

which is equation (2 21) with lower sign

We now explain why Q in (2 26) is called a 'winding number' $\phi$ is a complex function on $\mathbf{R}^2$ If $|\phi|$ approaches a constant, $\eta$, at infinity, then a winding number can be defined in the following way if

$$\phi \to \phi_\infty(\theta) = \eta \exp^{i\sigma(\theta)} \quad as \;\; r \to \infty$$

then

$$\phi_\infty/\eta \quad S^1 \to U(1) \quad \theta \to e^{i\sigma} \tag{2 35}$$

i e $\phi_\infty/\eta$ maps the circle at infinity, $S^1$, to the unit circle in the complex plane, $U(1)$ To each continuous $\phi_\infty$ we can therefore associate a winding number

$$n = \frac{1}{2\pi}[\sigma(2\pi) - \sigma(0)] = \frac{i}{2\pi\eta^2}\int_0^{2\pi} \phi_\infty\partial_\theta\phi_\infty^* d\theta \tag{2 36}$$

To relate (2 36) to (2 26), we restrict our attention to a potential of the form

$$u = (1 - |\phi|^2)|\phi|, \tag{2 37}$$

Now the identity

$$\epsilon_{ij}\partial^i[\phi(\partial^j\phi^*)(c_1 - c_2|\phi|^2)|\phi|)] = \epsilon_{ij}(\partial^i\phi)(\partial^j\phi^*)(\frac{3}{2}c_1 - \frac{5}{2}c_2|\phi|^2)|\phi| \tag{2 38}$$

which holds for any constant $c_1$ and $c_2$, can be used For $c_1 = \frac{2}{3}, c_2 = \frac{2}{5}$, we have

$$\epsilon_{ij}\partial^i[\phi(\partial^j\phi^*)(\frac{2}{3} - \frac{2}{5}|\phi|^2)|\phi|] = \epsilon_{ij}(\partial^i\phi)(\partial^j\phi^*)u \tag{2 39}$$

and therefore

$$Q = \frac{15}{8\pi}i\int_{\mathbf{R}^2}\partial^i[\epsilon_{ij}\phi(\partial^j\phi^*)(\frac{2}{3} - \frac{2}{5}|\phi|^2)|\phi|\ ]dx \tag{2 40}$$

For our choice of potential (2 37), the condition of finite action implies $|\phi_\infty| = 0$ or $|\phi_\infty| = 1$, if $|\phi|$ converges as $r \to \infty$ In the following, we will concentrate on solutions $\phi$ with $|\phi_\infty| = 1$, i e , $\eta = 1$ in this case Using Green's Theorem, (2 40) can be written

$$Q = \frac{15}{8\pi} i \int_0^{2\pi} d\theta \, x^i \, \epsilon_{ij} \, \phi_\infty \, (\partial^j \phi^*_\infty) \, (\frac{2}{3} - \frac{2}{5}|\phi_\infty|^2) \, |\phi_\infty| \, d^2x \qquad (2\ 41)$$

since $\partial_\theta = x^i \epsilon_{ij} \partial^j$, we have

$$Q = \frac{1}{2\pi} i \int_0^{2\pi} d\theta \phi_\infty (\partial_\theta \phi^*_\infty) \qquad (2\ 42)$$

This shows that $Q = n$ for our choice of potential and explains why $Q$ in (2 26) is called a 'winding number' Equation (2 38) shows that the name 'winding number' is justified for a class of potentials, in general the integrand in (2 26) cannot be written as a divergence

We have seen that instead of solving the second order equation (2 14), we can solve the first order equation (2 31) To conclude this subsection we now show that *all* finite-action solutions actually solve the Bogomol'nyi equations, so that we do not miss out on any by concentrating on the first order equations Equation (2 20) above can be written

$$\partial_1 \phi^* \partial_2 (\partial^1 \phi \partial^2 \phi^* - \partial^2 \phi \partial^1 \phi^*) + \partial_2 \phi^* \partial_1 (\partial^2 \phi \partial^1 \phi^* - \partial^1 \phi \partial^2 \phi^*) = u \frac{\partial u}{\partial \phi} \qquad (2\ 43)$$

or by complex congugation as

$$\partial_1 \phi \partial_2 (\partial^1 \phi^* \partial^2 \phi - \partial^2 \phi^* \partial^1 \phi) + \partial_2 \phi \partial_1 (\partial^2 \phi^* \partial^1 \phi - \partial^1 \phi^* \partial^2 \phi) = u \frac{\partial u}{\partial \phi^*} \qquad (2\ 44)$$

We now multiply (2 43) by $\partial_1 \phi$, multiply (2 44) by $\partial_1 \phi^*$ and add the two resulting equations This leads to,

$$\partial_1 \phi \partial_2 \phi^* \partial_1 (\partial^2 \phi \partial^1 \phi^* - \partial^1 \phi \partial^2 \phi^*) + \partial_1 \phi^* \partial_2 \phi \partial_1 (\partial^2 \phi^* \partial^1 \phi - \partial^1 \phi^* \partial^2 \phi) = \frac{1}{2} \partial_1 u^2 \qquad (2\ 45)$$

By rewriting the left-hand side we see that

$$\partial_1(\partial^1\phi\partial^2\phi^* - \partial^1\phi^*\partial^2\phi)^2 = -\partial_1 u^2 \tag{2 46}$$

holds Multiplying (2 43) by $\partial_2\phi$ and (2 44) by $\partial_2\phi^*$ we obtain in a completely analogous manner,

$$\partial_2(\partial^1\phi\partial^2\phi^* - \partial^1\phi^*\partial^2\phi)^2 = -\partial_2 u^2 \tag{2 47}$$

Equations (2 46) and (2 47) tell us that

$$(\partial^1\phi\partial^2\phi^* - \partial^1\phi^*\partial^2\phi)^2 + u^2 = K \tag{2 48}$$

where $K$ is a constant, since the left-hand side is independent of $x^1$ according to (2 46) and also independent of $x^2$ according to equation (2 47) For a finite action solution we have (see equation (2 17))

$$u^2 \to 0 \ \ and \ \ (\partial^1\phi\partial^2\phi^* - \partial^1\phi^*\partial^2\phi)^2 \to 0 \ \ as \ \ r \to \infty, \tag{2 49}$$

so the constant K must be zero Therefore

$$\imath(\partial^1\phi\partial^2\phi^* - \partial^1\phi^*\partial^2\phi) = \pm u \tag{2 50}$$

holds Since $\phi = \phi_1 + \imath\phi_2$ we have

$$\imath(\partial^1\phi\partial^2\phi^* - \partial^1\phi^*\partial^2\phi) = 2det(\frac{\partial\phi^a}{\partial x^\imath}) \tag{2 51}$$

and we see that (2 50) is the Bogomol'nyi equation (2 21)

# 3 Vortex solutions

In this chapter we discuss vortex solutions, first for two vortices on top of each other, then for two vortices close together

## 3.1 Superimposed vortices

For the potential we have chosen,

$$u = (1 - |\phi|^2)|\phi|, \tag{3 1}$$

we seek a smooth finite-action solution of the form

$$\phi = f(r)e^{in\theta} \tag{3 2}$$

where $f(r) \to 1$ as $r \to \infty$ For this $\phi$, $\phi_\infty = e^{in\theta}$ holds, and according to formula (2 36), $\phi$ clearly has winding number $n$ If we go around the circle at infinity in anti-clockwise direction once, i e , $\theta$ goes from 0 to $2\pi$, then $\phi_\infty$ winds around $U(1)$ $n$ times, for positive $n$ in anti-clockwise direction and for negative $n$ in clockwise direction The ansatz (3 2) is the next best to a radially symmetric ansatz Since we want $\phi$ to have non zero winding number, we cannot assume that $\phi$ is a function of $r$ only So we build in the winding number $n$ in the most natural way by using $e^{in\theta}$ and are left with a real function of $r$ only In the next chapter we will discuss the Ginzburg-Landau theory where we will use the same ansatz (3 2) for a complex field There the ansatz (3 2) actually is radially symmetric in $\mathcal{R}^2 \setminus \{0\}$ according to the definition of radial symmetry in a gauge theory

We now seek to solve the Bogomol'nyi equation

$$\frac{\partial \phi^1}{\partial x^1}\frac{\partial \phi^2}{\partial x^2} - \frac{\partial \phi^1}{\partial x^2}\frac{\partial \phi^2}{\partial x^1} = \frac{1}{2}(1 - |\phi|^2)|\phi| \tag{3 3}$$

In polar coordinates, where $x^1 = r\cos\theta,\quad x^2 = r\sin\theta$, we have

$$\frac{\partial}{\partial x^1} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}, \qquad \frac{\partial}{\partial x^2} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} \tag{3 4}$$

Equation (3 3) now becomes,

$$(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta})(f(r)\cos(n\theta))(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta})(f(r)\sin(n\theta))$$

$$-(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta})(f(r)\cos(n\theta))(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta})(f(r)\sin(n\theta))$$

$$= \frac{1}{2}(1 - f^2)f \tag{3 5}$$

This can be simplified to,

$$\frac{nf(r)f'(r)}{r} = \frac{1}{2}(1 - f^2)f \tag{3 6}$$

The separation of variables in equation (3 6) leads to

$$2n\int\frac{1}{1-f^2}df = \int r dr \tag{3 7}$$

Performing the integration we obtain

$$\tanh^{-1} f = \frac{r^2}{4n} + c \tag{3 8}$$

where c is a constant of integration Now we want,

$$f \to 0 \ as \ r \to 0, \tag{3 9}$$

otherwise $\phi$ in (3 2) is not defined at the origin Hence, $c = 0$ and

$$f = \tanh(\frac{r^2}{4n}) \tag{3 10}$$

The solution $\phi$ in (3 2) with f(r) given by (3 10) is defined in the whole of $\mathbf{R}^2$ and is clearly a $C^\infty$ in $\mathbf{R}^2 \setminus \{0\}$ Since

$$f \approx 1 - 2\exp\frac{-r^2}{2n} \quad as \quad r \to \infty \tag{3 11}$$

$\phi$ has the correct asymptotic behaviour for a solution with winding number $n$

We still have to ensure that $\phi$ is $C^\infty$ at the origin At the origin we use the Taylor expansion of $f$,

$$f = \sum_{K=1}^{\infty} \frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!}(\frac{r^2}{4n})^{2k-1} = \frac{r^2}{4n} - \frac{1}{3}(\frac{r^2}{4n})^3 + \tag{3 12}$$

where $B_k$ is the $k^{th}$ Bernoulli number We also express

$$e^{in\theta} = (e^{i\theta})^n = (\frac{x^1}{r} + i\frac{x^2}{r})^n \tag{3 13}$$

as a polynomial in $\frac{x^i}{r}$ We see that for $n = 2$ and *only* for $n = 2$, $\phi$ is a polynomial in $x^i$ In this model, we have the (somewhat peculiar) situation that within the most natural ansatz (3 2), smooth finite action solutions exist *only* for $n = 2$ If we interpret $n$ as the number of vortices, we only have a solution of the form (3 2) for 2 vortices

The Langrangian (2 1), or the action density, for this solution is

$$\mathcal{L} = -\frac{16}{r^2} f^2 {f'}^2 + f^2(1 - f^2)^2 \qquad (3\ 14)$$

in terms of $f$ Because of equation (3 6) this expression can be simplified to

$$\mathcal{L} = 2f^2(r)(1 - f^2(r))^2 \qquad (3\ 15)$$

If we substitute $f$ from (3 10) $\mathcal{L}$ reads

$$\mathcal{L} = \frac{2 \sinh^2 \frac{r^2}{8}}{\cosh^6 \frac{r^2}{8}} \qquad (3\ 16)$$

This action density is radially symmetric about the origin We interpret our solution as two vortices superimposed at the origin

## 3.2 Zero modes

We have found the solution for two vortices sitting on top of each other, which we called $\hat{\phi}$ To extend our study to two vortices slightly apart we consider $\phi = \hat{\phi} + \gamma$, where $\gamma$ is very small Thus the solution we seek is of the form

$$\begin{aligned} \phi^1 &= f(r)\cos 2\theta + \gamma^1 \\ \phi^2 &= f(r)\sin 2\theta + \gamma^2 \end{aligned} \tag{3 17}$$

With this ansatz we try to solve the Bogomol'nyi equation, linearized in $\gamma$ That means we are looking for $\gamma$'s that do not increase the action to first order Such additions to the solution $\hat{\phi}$ are called zero modes

Using this ansatz and the expressions (3 4) in the Bogomol'nyi equation (3 3) we obtain

$$(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta})(f(r)\cos 2\theta + \gamma^1)(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta})(f(r)\sin 2\theta + \gamma^2)$$

$$-(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta})(f(r)\cos 2\theta + \gamma^1)(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta})(f(r)\sin 2\theta + \gamma^2)$$

$$= \frac{1}{2}(1 - |\phi|^2)|\phi| \tag{3 18}$$

Multiplying the terms out and differentiating, while leaving out higher order terms in $\gamma^1, \gamma^2$ simplifies the above equation to

$$
\begin{aligned}
\frac{2ff'}{r} &+ (f' \cos\theta\cos 2\theta + \frac{2}{r} f \sin\theta \sin 2\theta)\frac{\partial\gamma^2}{\partial x^2} \\
&+(f' \sin\theta\sin 2\theta + \frac{2}{r} f \cos\theta\cos 2\theta)\frac{\partial\gamma^1}{\partial x^1} \\
&-(f' \sin\theta\cos 2\theta - \frac{2}{r} f \cos\theta\sin 2\theta)\frac{\partial\gamma^2}{\partial x^1} \\
&-(f' \cos\theta\sin 2\theta - \frac{2}{r} f \sin\theta\cos 2\theta)\frac{\partial\gamma^1}{\partial x^2}
\end{aligned}
$$

$$
= \frac{1}{2}(1 - |\phi^2|)|\phi| \qquad (3\ 19)
$$

Since $\phi$ is now defined by equation (3 17) we have,

$$
|\phi|^2 = f^2 + 2f(\gamma^1 \cos 2\theta + \gamma^2 \sin 2\theta) \qquad (3\ 20)
$$

if the higher order terms $(\gamma^1)^2$ and $(\gamma^2)^2$ are ignored Now $|\phi|$ is calculated

$$
\begin{aligned}
|\phi| &= f(1 + \frac{2}{f}(\gamma^1 \cos 2\theta + \gamma^2 \sin 2\theta))^{\frac{1}{2}} \\
&= f + \gamma^1 \cos 2\theta + \gamma^2 \sin 2\theta \qquad (3\ 21)
\end{aligned}
$$

The right hand side of equation (3 19) is,

$$
\frac{1}{2}(1 - |\phi|^2)|\phi| = \frac{1}{2}f(1 - f^2) + \frac{1}{2}(1 - 3f^2)\ \ (\gamma^1 \cos 2\theta + \gamma^2 \sin 2\theta) \qquad (3\ 22)
$$

up to linear order in $\gamma$
Using equations (3 6) and (3 22), equation (3 19) becomes

$$\begin{aligned}
&(f' \cos\theta \cos 2\theta + \frac{2}{r} f \sin\theta \sin 2\theta)\frac{\partial\gamma^2}{\partial x^2} \\
&+(f' \sin\theta \sin 2\theta + \frac{2}{r} f \cos\theta \cos 2\theta)\frac{\partial\gamma^1}{\partial x^1} \\
&-(f' \sin\theta \cos 2\theta - \frac{2}{r} f \cos\theta \sin 2\theta)\frac{\partial\gamma^2}{\partial x^1} \\
&-(f' \cos\theta \sin 2\theta - \frac{2}{r} f \sin\theta \cos 2\theta)\frac{\partial\gamma^1}{\partial x^2} \\
&= \frac{1}{2}(1 - 3f^2)\ (\gamma^1 \cos 2\theta + \gamma^2 \sin 2\theta)
\end{aligned} \tag{3 23}$$

We will find a smooth solutions to this equation using the following two forms

$$\gamma^1 = h(r), \gamma^2 = 0 \tag{3 24}$$

or

$$\gamma^1 = 0, \gamma^2 = \tilde{h}(r) \tag{3 25}$$

Choosing the ansatz (3 24) equation (3 23) reduces to

$$\frac{2f(r)}{r}h'(r) = \frac{1}{2}(1 - 3f^2(r)\ )h(r) \tag{3 26}$$

Seperating the terms we form the integral equation

$$\int \frac{1}{h(r)} dh = \int \frac{r}{4f(r)}(1 - 3f^2(r)\ )dr \tag{3 27}$$

Integrating the above we obtain

$$\ln\ h(r) = \frac{1}{4}\int \frac{r}{\tanh\frac{r^2}{8}}dr - \frac{3}{4}\int r\tanh\frac{r^2}{8}\,dr$$

$$= \ln\sinh\frac{r^2}{8} - 3\ \ln\ \cosh\frac{r^2}{8} + C \tag{3 28}$$

C is a constant which we can set equal to zero if at this point we are interested in finding just one zero mode Hence, we have

$$h(r) = \frac{\sinh\frac{r^2}{8}}{\cosh^3\frac{r^2}{8}} \tag{3 29}$$

For the ansatz (3 25), equation (3 23) also reduces to (3 26) with $\tilde{h}$ instead of $h$ Therefore, we obtain the second zero modes

$$\gamma(r) = \imath\frac{\sinh\frac{r^2}{8}}{\cosh^3\frac{r^2}{8}} \tag{3 30}$$

$$\tag{3 31}$$

We have found a 2-parameter family of zero modes, which may be written

$$\gamma(r) = [\alpha+\beta+\imath(\alpha-\beta)]\frac{\sinh\frac{r^2}{8}}{\cosh^3\frac{r^2}{8}} \tag{3 32}$$

All these zero modes are $C^\infty$ functions which vanish exponentially at infinity They describe the different ways of seperating two vortices in the plane This can be seen by looking at the zeros of $|\phi|$ which have separated and moved away from the origin Note that by replacing $\vec{x}$ by $\vec{x} + \vec{a}$ we can introduce two more parameters which describe the displacement of the center of mass

## 3.3 The quadratic terms

We consider $\phi = \hat{\phi} + \gamma$, with $\gamma$ given in (3 32), as the first terms in the expansion about $(\alpha, \beta) = (0,0)$ To find the second order terms, we define

$$\phi = \hat{\phi} + \gamma + \delta \tag{3 33}$$

Now we equate the second order terms in the Bogomol'nyi equation (3 3), which, in polar coordinates, reads

$$\frac{1}{r}\left(\frac{\partial\phi^1}{\partial r}\frac{\partial\phi^2}{\partial\theta} - \frac{\partial\phi^1}{\partial\theta}\frac{\partial\phi^2}{\partial r}\right) = \frac{1}{2}(1 - |\phi|^2)|\phi| \tag{3 34}$$

This leads to the equation

$$\begin{aligned}
&\frac{2}{r}\left(f' \cos 2\theta\frac{\partial\delta^2}{\partial\theta} + 2f\sin 2\theta\frac{\partial\delta^2}{\partial r} - f' \sin 2\theta\frac{\partial\delta^1}{\partial\theta} + 2f\cos 2\theta\frac{\partial\delta^1}{\partial r}\right) \\
&= (\alpha^2+\beta^2) f h^2\left(\frac{1}{f^2} - 3\right) - \frac{1}{2} f h^2\left(3 + \frac{1}{f^2}\right) [\alpha^2(\cos 2\theta + \sin 2\theta)^2 \\
&\qquad +2\alpha\beta(\cos^2 2\theta - \sin^2 2\theta) + \beta^2(\cos 2\theta - \sin 2\theta)^2] \\
&\qquad +(1-3f^2)(\delta^1\cos 2\theta + \delta^2 \sin 2\theta)
\end{aligned} \tag{3 35}$$

with $f(r)$ given in (3 10) and $h(r)$ given in (3 29)

We now write out $\delta$ in the form

$$\delta = \alpha^2 F(r,\theta) + 2\alpha\beta G(r,\theta) + \beta^2 H(r,\theta) \tag{3 36}$$

and equate the $\alpha^2$, $\beta^2$ and $\alpha\beta$ terms in (3 35) Starting with the $\alpha^2$ terms, we obtain the following equation for $F(r,\theta)$

$$\begin{aligned}
&\frac{2}{r}\left(f' \cos 2\theta\frac{\partial F^2}{\partial\theta} + 2f\sin 2\theta\frac{\partial F^2}{\partial r} - f' \sin 2\theta\frac{\partial F^1}{\partial\theta} + 2f\cos 2\theta\frac{\partial F^1}{\partial r}\right) \\
&= h^2\left(\frac{1}{f} - 3f\right) - \frac{h^2}{2}\left(3f + \frac{1}{f}\right)(\cos 2\theta + \sin 2\theta)^2 \\
&\qquad +(1-3f^2)(F^1\cos 2\theta + F^2\sin 2\theta)
\end{aligned} \tag{3 37}$$

To solve equation (3 36) we seek a solution of the form

$$F = f_1(r)e^{i2\theta} - i f_2(r)e^{-i2\theta} \tag{3 38}$$

Substituting the ansatz (3 38) into equation (3 37) leads to the equation

$$\frac{4}{r}[f' f_1 + f f_1' + 2(f f_2' - f' f_2)\sin 2\theta\cos 2\theta] + (3f^2 - 1)(f_1 + 2f_2 \sin 2\theta \cos 2\theta)$$
$$= \frac{h^2}{f}(1 - 3f^2) - \frac{h^2}{2f}(1 + 3f^2)(1 + 2\sin 2\theta\cos 2\theta) \quad (3\ 39)$$

Equating the $\theta$ - independent terms and the $\sin 4\theta$ terms, we obtain two decoupled equations for $f_1$ and $f_2$ In terms of the variable $\xi = \frac{r^2}{8}$ they read

$$\frac{df_1}{d\xi} + \frac{1}{f}(3f^2 - 1 - \frac{df}{d\xi})f_1 = \frac{h^2}{2f^2}(1 - 9f^2) \quad (3\ 40)$$

$$\frac{df_2}{d\xi} + \frac{1}{f}(3f^2 - 1 + \frac{df}{d\xi})f_2 = -\frac{h^2}{2f}(1 + 3f^2) \quad (3\ 41)$$

Both equations (3 39) and (3 40) are of the form

$$\frac{dy}{d\xi} + p(\xi)y(\xi) = q(\xi) \quad (3\ 42)$$

In the case of equation 3 42), we have

$$p_1(\xi) = \frac{1}{f}(3f^2 - 1 + \frac{df}{d\xi}) = 2\tanh\xi, \quad (3\ 43)$$

$$q_1(\xi) = \frac{h^2}{2f^2}(1 - 9f^2) = \frac{9 - 8\cosh^2\xi}{2\cosh^6\xi} \quad (3\ 44)$$

For equation (3 39), we have

$$p_2(\xi) = \frac{1}{f}(3f^2 - 1 + \frac{df}{d\xi}) = \frac{2(\sinh^2\xi - 1)}{\sinh\xi\cosh\xi} \quad (3\ 45)$$

$$q_2(\xi) = \frac{-h^2}{2f^2}(1+3f^2) = -\frac{\cosh^2\xi + 3\sinh^2\xi}{2\cosh^6\xi} \tag{3 46}$$

Equation (3 40) has the solution

$$y(\xi) = e^{-\int_c^\xi p(s)ds} \int q(\xi) e^{\int_c^\xi p(s)ds} d\xi \tag{3 47}$$

where c is any constant for which the integral converges To determine $f_1$, we calculate

$$\int_0^\xi p_1(s)ds = \ln\cosh^2\xi \tag{3 48}$$

$$e^{-\int_0^\xi p_1(s)ds} = \frac{1}{\cosh^2\xi} \tag{3 49}$$

$$\int q_1(\xi) e^{\int_0^\xi p_1(s)ds} d\xi = \frac{3\sinh\xi}{2\cosh^3\xi} - \frac{\sinh\xi}{\cosh\xi} + C_1 \tag{3 50}$$

This gives a family of solutions to equation (3 39), namely,

$$f_1 = \frac{1}{\cosh^2\xi}\left(\frac{3\sinh\xi}{2\cosh^3\xi} - \frac{\sinh\xi}{\cosh\xi} + C_1\right) \tag{3 51}$$

The function $f_1$ in (3 51) is a $C^\infty$ function for $0 < \xi < \infty$ For $\xi \to 0$, however, $f_1 \to C_1$ holds This implies that $C_1 = 0$, otherwise $F$ in (3 38) is not defined at the origin $f_1$ reads

$$f_1 = \frac{3\sinh\xi}{2\cosh^5\xi} - \frac{\sinh\xi}{\cosh^3\xi} \tag{3 52}$$

The expansion of $f_1$ near the origin is of the form

$$f_1 = \sum_{k=1}^{\infty} a_k \xi^k = \sum_{k=1}^{\infty} a_k \left(\frac{r^2}{8}\right)^k \tag{3 53}$$

Hence, the first term in (3 36) is a $C^\infty$ function of $x^1$ and $x^2$ at the origin We also see that $f_1$ vanishes exponentially at infinity So its contribution to $\phi$ does not change the winding number (2 26) which is a multiple of the action

A similar calculation yields

$$\int_c^\xi p_2(s)ds = -\ln\frac{\sinh^2\xi}{\cosh^4\xi} + C_2 \qquad (3\ 54)$$

$$e^{-\int_c^\xi p_2(s)ds} = \frac{\sinh^2\xi}{\cosh^4\xi}e^{-C_2} \qquad (3\ 55)$$

$$\int q_2(\xi)e^{\int_c^\xi p_2(s)ds}d\xi = e^{C_2}(\frac{\cosh\xi}{2\sinh\xi} - \frac{3\sinh^3\xi}{2\cosh\xi} + C_3) \qquad (3\ 56)$$

This determines $f_2$ which is of the form

$$f_2 = \frac{\sinh\xi}{2\cosh^3\xi} - \frac{3\sinh^3\xi}{2\cosh^5\xi} + C_3\frac{\sinh^2\xi}{\cosh^4\xi} \qquad (3\ 57)$$

We again found a one parameter family of solutions In contrast to $f_1$ however, all the solutions $f_2$ are acceptable In fact, for all $C_3$, $f_2$ is of the form

$$f_2 = \sum_{k=1}^{\infty} b_k\xi^k = \sum_{k=1}^{\infty} b_k(\frac{r^2}{8})^k \qquad (3\ 58)$$

near the origin, and therefore the second term in (3 37) is in $C^\infty(\mathcal{R}^2)$ The winding number and the action are also not altered because $f_2$ decays exponentially at infinity If we are only interested in two paramter families describing different relative positions, we could set $C_3 = 0$ and work with

$$f_2 = \frac{\sinh\xi}{2\cosh^3\xi} - \frac{3\sinh^3\xi}{2\cosh^5\xi} \qquad (3\ 59)$$

To determine $H$ in (3 36), we equate the $\beta^2$ terms in (3 36) This yields

$$\frac{2}{r}(f^{'}\cos 2\theta\frac{\partial H^2}{\partial\theta} + 2f\sin 2\theta\frac{\partial H^2}{\partial r} - f^{'}\sin 2\theta\frac{\partial F^1}{\partial\theta}$$
$$+2f\cos 2\theta\frac{\partial F^1}{\partial r}) + (3f^2 - 1)(H^1\cos 2\theta + H^2\sin 2\theta)$$
$$= \frac{h^2}{f}(1 - 3f^2) - \frac{h^2}{2f}(1 + 3f^2)(\cos 2\theta - \sin 2\theta)^2 \qquad (3\ 60)$$

The only difference to equation (3 37) is the minus sign in the expression in the last bracket For this reason, we try an ansatz of the same form as (3 38), namely

$$H = h_1(r)e^{i2\theta} + ih_2(r)e^{-i2\theta} \qquad (3\ 61)$$

The ansatz leads to

$$\frac{dh_1}{d\xi} + \frac{1}{f}(3f^2 - 1 + \frac{df}{d\xi})h_1 = \frac{h^2}{2f^2}(1 - 9f^2) \qquad (3\ 62)$$

$$\frac{dh_2}{d\xi} + \frac{1}{f}(3f^2 - 1 - \frac{df}{d\xi})h_2 = \frac{h^2}{2f^2}(1 + 3f^2) \qquad (3\ 63)$$

These are the same equations as the ones for $f_1$ and $f_2$ except for the different sign on the right-hand side in the second equation So we can choose

$$h_1 = \frac{3\sinh\xi}{2\cosh^5\xi} - \frac{\sinh\xi}{\cosh^3\xi} \qquad (3\ 64)$$

and

$$h_2 = -\frac{\sinh\xi}{2\cosh^3\xi} + \frac{3\sinh^3\xi}{2\cosh^5\xi} + C_4\frac{\sinh^2\xi}{\cosh^4\xi} \qquad (3\ 65)$$

For $h_1$, we have no choice, to select $h_2$, we could again set an arbitary constant equal to zero

Finally we equate the $\alpha\beta$ terms in (3 35) This yields

$$\frac{2}{r}(f^{'}\cos 2\theta\frac{\partial G^2}{\partial\theta} + 2f\sin 2\theta\frac{\partial G^2}{\partial r} - f^{'}\sin 2\theta\frac{\partial G^1}{\partial\theta}$$
$$+2f\cos 2\theta\frac{\partial G^1}{\partial r} + (3f^2-1)(G^1\cos 2\theta + G^2\sin 2\theta)$$
$$= -\frac{h^2}{f}(1+3f^2)(\cos^2 2\theta - \sin^2 2\theta) \qquad (3\ 66)$$

This time our ansatz is

$$G = g(r)e^{-\imath 2\theta} \qquad (3\ 67)$$

For this ansatz, equation (3 66) is solved if and only if

$$\frac{dg}{d\xi} + \frac{1}{f}(3f^2 - 1 - \frac{df}{d\xi})g = -\frac{h^2}{2f^2}(1+3f^2) \qquad (3\ 68)$$

holds This is the same equation as that for $f_2$ (equation (3 41)) So we have

$$g = \frac{\sinh\xi}{2\cosh^3\xi} - \frac{3\sinh^3\xi}{2\cosh^5\xi} + C_5\frac{\sinh^2\xi}{\cosh^4\xi} \qquad (3\ 69)$$

If we put all our results together and set $C_3 = C_4 = C_5 = 0$, we obtain the second order terms (3 36)

$$\delta = (\alpha^2+\beta^2)f_1(r)e^{\imath 2\theta} + \imath(\alpha - \imath\beta)^2 f_2(r)e^{-\imath 2\theta} \qquad (3\ 70)$$

where $f_1$ and $f_2$ are given by (3 52) and (3 57) respectively

## 3.4 The cubic terms

We consider $\phi = \hat{\phi} + \gamma + \delta$, with $\gamma$ given in (3 32) and $\delta$ given in (3 70), as the terms up to second order in the expansion about $(\alpha, \beta) = (0, 0)$ To find the third order terms, we define

$$\phi = \hat{\phi} + \gamma + \delta + \epsilon, \tag{3 71}$$

and equate the third order terms in the Bogomol'nyi equation (3 3), which, in polar coordinates, reads

$$\frac{1}{r}\left(\frac{\partial\phi^1}{\partial r}\frac{\partial\phi^2}{\partial\theta} - \frac{\partial\phi^1}{\partial\theta}\frac{\partial\phi^2}{\partial r}\right) = \frac{1}{2}(1 - |\phi|^2)|\phi| \tag{3 72}$$

We write out $\epsilon$ in the form

$$\epsilon = \alpha^3 I(r,\theta) + \alpha^2\beta J(r,\theta) + \alpha\beta^2 K(r,\theta) + \beta^3 L(r,\theta) \tag{3 73}$$

but will concentrate on the case $\alpha \neq 0, \beta = 0$ first Since we will find that $I$ is not smooth, we will not carry our analysis any further

Substituting for $\epsilon$ from equation (3 73) and looking at the $\alpha^3$ terms only, we obtain

$$\begin{aligned}
&\tfrac{2}{r} \quad [f' \cos 2\theta\frac{\partial I^2}{\partial\theta} + 2f\sin 2\theta\frac{\partial I^2}{\partial r} - f'\sin 2\theta\frac{\partial I^1}{\partial\theta} + 2f\cos 2\theta\frac{\partial I^1}{\partial r} \\
&+ \quad h'(2f_1\cos 2\theta + 2f_2\sin 2\theta) + h'(2f_1\sin 2\theta + 2f_2\cos 2\theta] \quad = \\
&- \quad 3f^2(I^1\cos 2\theta + I^2\sin 2\theta) \\
&- \quad 3fh(\cos 2\theta + \sin 2\theta)(2f_1 - f_2 - 2f_2\cos 2\theta\sin 2\theta) \\
&- \quad 3(\cos 2\theta + \sin 2\theta)h^3 + I^1\cos 2\theta + I^2\sin 2\theta \\
&+ \quad \frac{h}{2}[f_1(\cos 2\theta + \sin 2\theta) - f_2(\cos 2\theta + \sin 2\theta)] \\
&+ \quad \frac{h^3}{2}(\cos 2\theta + \sin 2\theta)^3 - \frac{h^3}{f^2}(\cos 2\theta + \sin 2\theta) \\
&- \quad \frac{h}{f}(\cos 2\theta + \sin 2\theta)(f_1 - 2f_2\cos 2\theta\sin 2\theta) \\
&+ \quad \frac{h}{2f^2}(\cos 2\theta + \sin 2\theta)^3
\end{aligned} \tag{3 74}$$

with $f(r)$ given in (3 10), $h(r)$ given in (3 29), $f_1(r)$ given in (3 52) and $f_2(r)$ given in (3 57)

To solve equation (3 74) we seek a solution of the form $I = I' + \imath I^2$,

$$I^1 = g_1(\xi) + g_2(\xi)(\cos 4\theta - \sin 4\theta)$$
$$I^2 = g_1(\xi) + g_2(\xi)(-\cos 4\theta - \sin 4\theta) \quad (3\ 75)$$

in terms of the variable $\xi = \frac{r^2}{8}$ Using the ansatz (3 75) we obtain

$$\frac{1}{2}(\frac{df}{d\xi}\cos 2\theta\frac{\partial I^2}{\partial\theta} + 2f\sin 2\theta\frac{\partial I^2}{\partial\xi} - \frac{df}{d\xi}\sin 2\theta\frac{\partial I^1}{\partial\theta} + 2f\cos 2\theta\frac{\partial I^1}{\partial\xi})$$
$$-(1-3f^2)(I^1\cos 2\theta + I^2\sin 2\theta)$$

$$= (\cos 2\theta + \sin 2\theta)[f\frac{dg_1}{d\xi} - (1-3f^2)g_1]$$
$$+\frac{1}{2}[\frac{df}{d\xi}g_2(\cos 2\theta(4\sin 4\theta - 4\cos 4\theta) - \sin 2\theta(-4\sin 4\theta - 4\cos 4\theta))$$
$$+2f\frac{dg_2}{d\xi}(\sin 2\theta(-\cos 4\theta - \sin 4\theta) + \cos 2\theta(\cos 4\theta - \sin 4\theta))]$$
$$-(1-3f^2)g_2[\cos 2\theta(\cos 4\theta - \sin 4\theta) + \sin 2\theta(-\cos 4\theta - \sin 4\theta)]$$

$$= (\cos 2\theta + \sin 2\theta)\ [f\frac{dg_1}{d\xi} - (1-3f^2)g_1]$$
$$+(\cos 6\theta - \sin 6\theta)\ [-2\frac{df}{d\xi}g_2 + f\frac{dg_2}{d\xi} - (1-3f^2)g_2] \quad (3\ 76)$$

To rewrite the other terms in (3 73) we use the identities

$$(\cos 2\theta + \sin 2\theta)^3 = \frac{3}{2}(\cos 2\theta + \sin 2\theta) - \frac{1}{2}(\cos 6\theta - \sin 6\theta),$$
$$(\cos 2\theta + \sin 2\theta)\cos 2\theta\sin 2\theta = \frac{1}{4}(\cos 2\theta + \sin 2\theta) - \frac{1}{4}(\cos 6\theta - \sin 6\theta)(3\ 77)$$

Equating the $(\cos 2\theta + \sin 2\theta)$ terms and the $(\cos 6\theta - \sin 6\theta)$ terms, we obtain two decoupled equations for $g_1$ and $g_2$. They read

$$\frac{dg_1}{d\xi} + (-\frac{1}{f} + 3f)g_1 = -\frac{dh}{d\xi}\frac{f_1+f_2}{f} - 6hf_1 + \frac{9}{2}hf_2 - \frac{hf_2}{2f^2} - \frac{h^3}{4f^3} - \frac{9h^3}{4f} \quad (3.78)$$

$$\frac{dg_2}{d\xi} + (-\frac{1}{f} + 3f - \frac{2}{f}\frac{df}{d\xi})g_2 = -\frac{hf_2}{2f^2} - \frac{3hf_2}{2} - \frac{h^3}{4f^3} - \frac{h^3}{4f} \quad (3.79)$$

Both equations (3.78) and (3.79) are of the form

$$\frac{dy}{d\xi} + p(\xi)y(\xi) = q(\xi) \quad (3.80)$$

In the case of equation (3.78), we have

$$\begin{aligned} p_1(\xi) &= -\frac{1}{f} + 3f \\ &= -\frac{\cosh\xi}{\sinh\xi} + 3\frac{\sinh\xi}{\cosh\xi} \end{aligned} \quad (3.81)$$

$$\begin{aligned} q_1(\xi) &= -\frac{dh}{d\xi}(\frac{f_1+f_2}{f}) - 6hf_1 + \frac{9}{2}hf_2 - \frac{hf_2}{2f^2} - \frac{h^3}{4f^3} - \frac{9h^3}{4f} \\ &= -\frac{2}{\cosh^4\xi} + \frac{5}{\cosh^6\xi} - \frac{9}{2\cosh^8\xi} + C_3(\frac{6\sinh\xi}{\cosh^5\xi} - \frac{15\sinh\xi}{2\cosh^7\xi}) \end{aligned} \quad (3.82)$$

For equation (3.79), we have

$$\begin{aligned} p_2(\xi) &= -\frac{1}{f} + 3f - \frac{2}{f}\frac{df}{d\xi} \\ &= -\frac{\cosh\xi}{\sinh\xi} + \frac{3\sinh\xi}{\cosh\xi} - \frac{2}{\sinh\xi\cosh\xi} \end{aligned} \quad (3.83)$$

$$
\begin{aligned}
q_2(\xi) &= -\frac{hf_2}{2f^2} - \frac{3hf_2}{2} - \frac{h^3}{4f^3} - \frac{h^3}{4f} \\
&= \frac{2}{\cosh^4\xi} - \frac{5}{\cosh^6\xi} + \frac{5}{2\cosh^8\xi} - C_3\frac{\sinh\xi}{\cosh^5\xi}(2 - \frac{3}{2\cosh^2\xi}) \quad (3\,84)
\end{aligned}
$$

The general solution of equation (3 80) is

$$
y(\xi) = e^{-\int_c^\xi p(s)ds} \int q(\xi) e^{\int_c^\xi p(s)ds} d\xi \qquad (3\,85)
$$

where c is any constant for which the integral converges Note that (3 85) is independent of $c$ To determine $g_1$, we calculate

$$
\int_c^\xi p_1(s)ds = \ln(\frac{\cosh^3\xi}{\sinh\xi}) + c_1 \qquad (3\,86)
$$

and choose $c$ such that $c_1 = 0$ Then

$$
e^{-\int_c^\xi p_1(s)ds} = \frac{\sinh\xi}{\cosh^3\xi} \qquad (3\,87)
$$

and

$$
\begin{aligned}
\int q_1(\xi) e^{\int_0^\xi p_1(s)ds} d\xi = -\frac{3}{2}\ln(\frac{\sinh\xi}{\cosh\xi}) - \frac{1}{4\cosh^2\xi} - \frac{9}{8\cosh^4\xi} \\
+C_3(\frac{\sinh\xi}{\cosh\xi} - \frac{5\sinh\xi}{2\cosh^3\xi}) + C_6 \qquad (3\,88)
\end{aligned}
$$

This gives a family of solutions to equation (3 78), namely,

$$
\begin{aligned}
g_1 = -\frac{3\sinh\xi\ln\tanh\xi}{2\cosh^3\xi} - \frac{\sinh\xi}{4\cosh^5\xi} - \frac{9\sinh\xi}{8\cosh^7\xi} \\
+C_3(\frac{1}{\cosh^2\xi} - \frac{7}{2\cosh^4\xi} + \frac{5}{2\cosh^6\xi}) + C_6\frac{\sinh\xi}{\cosh^3\xi} \qquad (3\,89)
\end{aligned}
$$

A similar calculation yields

$$\int_c^\xi p_2(s)ds = \ln \frac{\sinh^5 \xi}{\cosh^3 \xi} + c_2 \tag{3 90}$$

where we can set $c_2 = 0$ Hence

$$e^{-\int_c^\xi p_2(s)ds} = \frac{\sinh^3 \xi}{\cosh^5 \xi} \tag{3 91}$$

and

$$\int q_2(\xi) e^{\int_c^\xi p_2(s)ds} d\xi = \frac{1}{4\sinh^2 \xi} - \frac{5}{4\cosh^2 \xi} + C_3 (\frac{\cosh \xi}{2 \sinh \xi} - \frac{3 \sinh \xi}{2 \cosh \xi}) + C_7 \tag{3 92}$$

Finally we obtain the solution

$$g_2 = \frac{\sinh \xi}{4\cosh^5 \xi} - \frac{5 \sinh^3 \xi}{4 \cosh^7 \xi} + c_3 (\frac{\sinh^2 \xi}{2\cosh^4 \xi} - \frac{3 \sinh^4 \xi}{2 \cosh^6 \xi}) + C_7 \frac{\sinh^3 \xi}{\cosh^5 \xi} \tag{3 93}$$

To see whether $g_1$ and $g_2$ have the required properties we study the cubic terms

$$\begin{aligned} I^1 &= g_1(\xi) + g_2(\xi)(\cos 4\theta - \sin 4\theta) \\ I^2 &= g_1(\xi) + g_2(\xi)(-\cos 4\theta - \sin 4\theta) \end{aligned} \tag{3 94}$$

The function $g_1$ given by (3 89) and $g_2$ given by (3 93) vanish exponentially at infinity Hence both have an acceptable asymptotic behaviour at infinity The leading terms at the origin, however, are

$$g_1 = -\frac{3}{8} r^2 \ln r + \tag{3 95}$$

$$g_2 = \frac{1}{32} r^2 + \qquad (3\ 96)$$

The asymptotic behaviour of $g_1$ and $g_2$ at the origin are both not acceptable if we require (3 94) to be a $C^\infty$ function Note that $g_1$ is not $C^\infty$ and that $g_2$ is not $O(r^4)$ as it should be to make the second term smooth in both equations (3 75)

# 4 Abrikosov vortices

In this chapter we discuss a realistic model of a superconductor in a magnetic field. This physical situation can be described by the Ginzburg-Landau model. At the point between type-I and type-II superconductivity, the model has static vortex solutions, so-called Abrikosov vortices. Starting with 2 vortices on top of each other, we give an expansion which describes vortices close together. This chapter gives a detailed account of the results presented in Ref. 9.

## 4.1 The Ginzburg-Landau model

The Ginzburg-Landau theory of a superconductor in a magnetic field in direction $z$ is given by the Lagrangian density

$$\mathcal{L} = \frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}(D_i\phi)(D^i\phi)^* + \frac{\lambda}{8}(\mid \phi \mid^2 - 1)^2, \tag{4.1}$$

where $\phi$ is the complex Higgs field, and

$$\begin{aligned} D_i\phi &= \partial_i\phi - iA_i\phi, \\ F_{ij} &= \partial_i A_j - \partial_j A_i \end{aligned}$$

in terms of the gauge potentials $A_i$, $i = 1, 2$.

To find the Euler-Langrange equations we calculate

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \phi} &= -\frac{i}{2}A_i(D^i\phi)^* + \frac{\lambda}{4}(\mid \phi \mid^2 - 1)\phi^*, \\ \partial_i \frac{\partial \mathcal{L}}{\partial \partial_i \phi} &= \partial_i \frac{1}{2}(D^i\phi)^* \end{aligned} \tag{4.2}$$

This implies

$$\partial_i \frac{\partial \mathcal{L}}{\partial \partial_i \phi} - \frac{\partial \mathcal{L}}{\partial \phi} = \frac{1}{2}(D_i D^i \phi)^* - \frac{\lambda}{4}\phi^*(\mid \phi \mid^2 - 1) \tag{4.3}$$

and therefore

$$D_i D^i \phi = \frac{\lambda}{2}\phi(|\phi|^2 - 1), \tag{4 4}$$

We also have,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial A_j} &= -\frac{i}{2}\phi(D^j\phi)^* + \frac{i}{2}\phi^* D^j \phi, \\ \partial_i \frac{\partial \mathcal{L}}{\partial \partial_i A_j} &= \partial_i F^{ij} \end{aligned} \tag{4 5}$$

and therefore

$$\partial_i F^{ij} = -\frac{i}{2}[\phi(D^j\phi)^* - \phi^* D^j \phi] \tag{4 6}$$

In the special case $\lambda = 1$, we can see that all solutions of the first-order Bogomol'nyi equations [8]

$$\begin{aligned} F_{12} &= \pm\frac{1}{2}(1 - \phi_1^2 - \phi_2^2), \\ D_1\phi &= \mp i D_2 \phi \end{aligned} \tag{4 7}$$

satisfy the equations of motion (4 4) and (4 6) In fact we have,

$$\begin{aligned} D_i D^i \phi &= \mp i(D^1 D_2 \phi - D^2 D_1 \phi) \\ &= \mp i[(\partial^1 - iA^1)(\partial_2\phi - iA_2\phi) - (\partial^2 - iA^2)(\partial_1\phi - iA_1\phi)] \\ &= \mp i[-i\partial^1(A_2\phi) + i\partial^2(A_1\phi) - iA^1\partial_2\phi + iA^2\partial_1\phi] \\ &= \mp\phi(\partial_1 A_2 - \partial_1 A_1) = \mp\phi F_{12} = \frac{\phi}{2}(|\phi|^2 - 1) \end{aligned} \tag{4 8}$$

which implies that equation (4 4) holds

We also have

$$\begin{aligned} \partial_i F^{i1} = \partial_2 F^{21} &= \pm\partial_2[\frac{1}{2}(\phi_1^2 + \phi_2^2 - 1)] \\ &= \pm(\phi_1 \partial_2 \phi_1 + \phi_2 \partial_2 \phi_2) \end{aligned} \tag{4 9}$$

and

$$\frac{\imath}{2}[\phi(D^1\phi)^* - \phi^* D^1\phi]$$

$$= \frac{\imath}{2}[(\phi_1 + \imath\phi_2)[\pm(\partial^2\phi_2 - A^2\phi_1) \pm \imath(\partial^2\phi_1 + A^2\phi_2)$$
$$-(\phi_1 - \imath\phi_2)[\pm(\partial^2\phi_2 - A^2\phi_1) \mp \imath(\partial^2\phi_1 + A^2\phi_2)]]$$

$$= \frac{\imath}{2}[\pm\imath\phi_2(\partial^2\phi_2 - A^2\phi_1) \pm \imath\phi_1(\partial^2\phi_1 + A^2\phi_2)$$
$$\pm\imath\phi_2(\partial^2\phi_2 - A^2\phi_1) \pm \imath\phi_1(\partial^2\phi_1 + A^2\phi_2)]$$

$$= \mp(\phi_2\partial_2\phi_2 + \phi_1\partial_2\phi_1) \qquad (4\ 10)$$

Furthermore,

$$\partial_\imath F^{\imath 2} = \partial_1 F^{12} = \mp(\phi_1\partial_1\phi 1 + \phi_2\partial_1\phi 2), \qquad (4\ 11)$$

and

$$\frac{\imath}{2}[\phi(D^2\phi)^* - \phi^* D^2\phi]$$

$$= \frac{\imath}{2}(\phi_1 + \imath\phi_2)[\mp(\partial^1\phi_2 - A^1\phi_1) \mp \imath(\partial^1\phi_1 + A^1\phi_2)$$
$$-(\phi_1 - \imath\phi_2)[\mp(\partial^1\phi_2 - A^1\phi_1)] \pm \imath(\partial^1\phi_1 + A^1\phi_2)]]$$

$$= \frac{\imath}{2}[\mp\imath\phi_2(\partial^1\phi_2 - A^1\phi_1) \mp \imath\phi_1(\partial^1\phi_1 + A^1\phi_2)$$
$$\mp\imath\phi_2(\partial^1\phi_2 - A^1\phi_1) \mp \imath\phi_1(\partial^1\phi_1 + A^1\phi_2)]$$

$$= \pm(\phi_1\partial^1\phi_1 + \phi_2\partial^2\phi_2) \qquad (4\ 12)$$

which means that (4 6) holds

It can also be shown [10] that all finite action solutions satisfy the first order Bogomol'nyi equations (4 7) Furthermore, it has been shown [10] that a $2n$-parameter family of solutions of (4 7) exists with winding number

$$n = \frac{1}{2\pi}\int_{\mathbf{R}^2} F_{12}\, d^2x \qquad (4\ 13)$$

This family describes $n$ vortices sitting at $n$ positions m space In the following we concentrate on the equations in (4 7) with the upper sign

## 4.2 Superimposed vortices and zero modes

Even for $n$ vortices sitting on top of each other, the solution is not known explicitly in terms of elementary functions It is known that this solution is of the form

$$\phi = f(r)e^{in\theta}, \qquad A_i = -\frac{na(r)}{r^2}\varepsilon_{ij}x^j \tag{4 14}$$

Using this ansatz we have,

$$D_1\phi = e^{in\theta}(\frac{x_1}{r}f' - in\frac{x_2}{r^2}f + i\frac{x_2}{r^2}naf) \tag{4 15}$$

$$= -iD_2\phi = -ie^{in\theta}(\frac{x_2}{r}f' + in\frac{x_1}{r^2}f - i\frac{x_1}{r^2}naf) \tag{4 16}$$

Here we have used

$$\partial_i = \frac{x_i}{r}\partial_r - \epsilon_{ij}\frac{x_j}{r^2}\partial_\theta \tag{4 17}$$

Hence,

$$rf' - n(1-a)f = 0 \tag{4 18}$$

Furthermore,

$$\begin{aligned} F_{12} &= n[\partial_1(x_1\frac{a(r)}{r^2}) + \partial_2(x_2\frac{a(r)}{r^2})] \\ &= 2n\frac{a(r)}{r^2} + n(\frac{(x^1)^2}{r} + \frac{(x^2)^2}{r})(\frac{a'}{r^2} - \frac{2a}{r^3}) \\ &= n\frac{a'}{r} = \frac{1}{2}(1- \mid \phi \mid^2) = \frac{1}{2}(1-f^2) \end{aligned} \tag{4 19}$$

So

$$\frac{2na'}{r} + f^2 - 1 = 0 \tag{4 20}$$

follows

It can be shown [11] that (4 18) and (4 21) have solutions with the correct asymtotic behaviour

$$f(0) = a(0) = 0, \quad \lim_{r\to\infty} f(r) = \lim_{r\to\infty} a(r) = 1 \tag{4 21}$$

In the following, we restrict our attention to $n = 2$ and use the solution (4 14) as the zero order term in an expansion in the separation parameters The first order terms are given by the two zero modes describing the separation of the vortices These were found by Weinberg [12] Using the above result and Weinberg's zero modes we can write up to linear terms,

$$\begin{aligned} \phi &= fe^{2\imath\theta} + 2(\alpha + \imath\beta)kf + \\ A_1 + \imath A_2 &= \imath\frac{2a}{r}e^{\imath\theta} - 2\imath(\alpha + \imath\beta)(k' + \frac{2k}{r})e^{-\imath\theta} + \end{aligned} \tag{4 22}$$

where

$$-\imath\frac{1}{r}(rk'' + k') + (f^2 + \frac{4}{r^2})k = 0 \tag{4 23}$$

In fact with this ansatz we find

$$\begin{aligned} \frac{1}{2}(1- \mid \phi \mid^2) = \frac{1}{2}[1 - f^2 - \alpha 4kf^2 \cos 2\theta \\ -\beta 4kf^2 \sin 2\theta \quad ] \end{aligned} \tag{4 24}$$

Also,

$$\begin{aligned} F_{12} &= \partial_1[\frac{2a}{r^2}x^1 - \alpha 2(\frac{k'}{r} + \frac{2k}{r^2})x^1 - \beta 2(\frac{k'}{r} + \frac{2k}{r^2})x^2 + \quad ] \\ &-\partial_2[-\frac{2a}{r^2}x^2 - \alpha 2(\frac{k'}{r} + \frac{2k}{r^2})x^2 + \beta 2(\frac{k'}{r} + \frac{2k}{r^2})x^1 + \quad ] \\ &= \frac{4a}{r^2} + (\frac{2a}{r^2})'r - \alpha 2(\frac{k'}{r} + \frac{2k}{r^2})'r\cos 2\theta - \beta 2(\frac{k'}{r} + \frac{2k}{r^2})'r \sin 2\theta + \end{aligned} \tag{4 25}$$

The $\alpha$ and $\beta$ independent terms cancel out as we have already seen above The linear terms cancel out because of equation (4 23) So the first Bogomol'nyi equation is satisfied up to linear terms by (4 22)

We also have

$$D_1\phi = \frac{x_1}{r}f' e^{2\imath\theta} - 2\imath\frac{x_2}{r^2}fe^{2\imath\theta} + \alpha 2\frac{x_1}{r}(kf)' + \beta 2\imath\frac{x_1}{r}(kf)' +$$

$$-\imath[-\frac{2a}{r^2}x^2 - \alpha 2(\frac{k'}{r} + \frac{2k}{r^2})x^2$$

$$+\beta 2(\frac{k'}{r} + \frac{2k}{r^2})x^1 + \quad ][fe^{2\imath\theta} + \alpha 2kf + \beta 2\imath kf \quad ]$$

$$= \frac{x_1}{r}f' e^{2\imath\theta} - 2\imath\frac{x_2}{r^2}fe^{2\imath\theta}$$

$$+\alpha 2\frac{x_1}{r}(kf' + fk') + \beta 2\imath\frac{x_1}{r}(kf' + fk') +$$

$$-\imath[-\frac{2af}{r^2}x^2e^{2\imath\theta} - \alpha 2f(\frac{k'}{r} + \frac{2k}{r^2})x^2e^{2\imath\theta} - \beta 2f(\frac{k'}{r} + \frac{2k}{r^2})x^1e^{2\imath\theta}$$

$$-\alpha\frac{4akf}{r^2}x^2 - \beta\imath\frac{4akf}{r^2}x^2 + \quad ] \qquad (4\ 26)$$

and

$$-\imath D_2\phi = -\imath[[\frac{x_2}{r}f' e^{2\imath\theta} + 2\imath\frac{x_1}{r^2}fe^{2\imath\theta} + \alpha 2\frac{x_2}{r}(kf)' + \beta 2\imath\frac{x_2}{r}(kf)' +$$

$$-\imath[\frac{2a}{r^2}x^1 - \alpha 2(\frac{k'}{r} + \frac{2k}{r^2})x^1$$

$$-\beta 2(\frac{k'}{r} + \frac{2k}{r^2})x^2 + \quad ][fe^{2\imath\theta} + \alpha 2kf + \beta 2\imath kf + \quad ]]$$

$$= -\imath[\frac{x_2}{r}f' e^{2\imath\theta} + 2\imath\frac{x_1}{r^2}fe^{2\imath\theta}$$

$$+\alpha 2\frac{x_2}{r}(kf' + fk') + \beta 2\imath\frac{x_2}{r}(kf' + fk') +$$

$$-\imath[\frac{2af}{r^2}x^1e^{2\imath\theta} - \alpha 2f(\frac{k'}{r} + \frac{2k}{r^2})x^1e^{2\imath\theta} - \beta 2f(\frac{k'}{r} + \frac{2k}{r^2})x^2e^{2\imath\theta}$$

$$+\alpha\frac{4akf}{r^2}x^1 + \beta\imath\frac{4akf}{r^2}x^1 + \quad ]] \ (4\ 27)$$

The $\alpha$ and $\beta$ independent terms cancel out, as we have seen above The linear terms cancel out because of equation (4 23)

## 4.3 The quadratic terms

Using the above results we can write up to quadratic terms,

$$\phi = fe^{2\imath\theta} + 2(\alpha + \imath\beta)kf + \alpha^2\psi + \alpha\beta\lambda + \beta^2\chi + \quad , \tag{4 28}$$

$$A_1 + \imath A_2 = \imath\frac{2a}{r}e^{\imath\theta} - 2\imath(\alpha + \imath\beta)(k' + \frac{2k}{r})e^{-\imath\theta}$$
$$+\alpha^2(B_1 + \imath B_2) + \alpha\beta(C_1 + \imath C_2) + \beta^2(E_1 + \imath E_2) + \tag{4 29}$$

Our task is to determine $\psi, \lambda, \chi, B_\imath, C_\imath, E_\imath$, which are functions of $r$ and $\theta$

Equating the $\alpha^2$-terms in the Bogomol'nyi equations (4 7), we obtain

$$(\partial_1 + \imath\partial_2)\psi + \frac{2a}{r}\psi e^{\imath\theta} - \imath f(B_1 + \imath B_2)e^{2\imath\theta} = 4kf(k' + \frac{2k}{r})e^{-\imath\theta}, \tag{4 30}$$

$$\partial_1 B_2 - \partial_2 B_1 + \frac{1}{2}(f\psi e^{-2\imath\theta} + f\psi^* e^{2\imath\theta}) = -2k^2 f^2 \tag{4 31}$$

A Fourier expansion with the minimal number of nonzero terms leads to the ansatz

$$\psi = g(r)f(r)e^{2\imath\theta} + \tilde{g}(r)e^{-2\imath\theta},$$
$$B_1 + \imath B_2 = \tilde{b}(r)e^{\imath\theta} + \imath b(r)f(r)e^{-3\imath\theta}, \tag{4 32}$$

and to equations for $g(r), \tilde{g}(r), b(r)$ and $\tilde{b}(r)$ The equations for $\tilde{g}(r)$ and $\tilde{b}(r)$ read

$$\tilde{g} = \frac{1+2a}{r}b - b', \qquad \tilde{b} = -\imath h' \tag{4 33}$$

The functions $g(r)$ and $b(r)$ must satisfy the equations

$$g'' + \frac{1}{r}g' - f^2 g = 2k^2 f^2, \tag{4 34}$$

$$b'' + \frac{1}{r}b' - (\frac{1+f^2}{2} + \frac{1+4a+4a^2}{r^2})b = -4kf(k' + \frac{2k}{r}) \tag{4 35}$$

Equating the $\alpha\beta$-terms in the Bogomol'nyi equation (4 7), we obtain

$$(\partial_1 + \imath\partial_2)\lambda + \frac{2a}{r}\lambda e^{\imath\theta} - \imath f(C_1 + \imath C_2)e^{2\imath\theta} = -\imath 8kf(k' + \frac{2k}{r})e^{-\imath\theta}, \tag{4 36}$$

$$\partial_1 C_2 - \partial_2 C_1 + \frac{1}{2}(f\lambda e^{-2\imath\theta} + f\lambda^* e^{2\imath\theta}) = 0 \qquad (4\ 37)$$

A Fourier expansion with the minimal number of nonzero terms leads to the ansatz

$$\lambda = 2\imath\tilde{g}(r)e^{-2\imath\theta},$$
$$C_1 + \imath C_2 = -2b(r)f(r)e^{-3\imath\theta}, \qquad (4\ 38)$$

and to equations for $\tilde{g}(r)$, and $\tilde{b}(r)$ The equation for $\tilde{g}(r)$ reads

$$\tilde{g} = \frac{1+2a}{r} - b', \qquad (4\ 39)$$

and $b(r)$ must satisfy the equation

$$b'' + \frac{1}{r}b' - (\frac{1+f^2}{2} + \frac{1+4a+4a^2}{r^2})b = -4kf(k' + \frac{2k}{r}) \qquad (4\ 40)$$

Equating the $\beta^2$-terms in the Bogomol'nyi equations (4 7), we obtain

$$(\partial_1 + \imath\partial_2)\chi + \frac{2a}{r}\chi e^{\imath\theta} - \imath f(E_1 + \imath E_2)e^{2\imath\theta} = -4kf(k' + \frac{2k}{r})e^{-\imath\theta}, \qquad (4\ 41)$$

$$\partial_1 E_2 - \partial_2 E_1 + \frac{1}{2}(f\chi e^{-2\imath\theta} + f\chi^* e^{2\imath\theta}) = -2k^2 f^2 \qquad (4\ 42)$$

A Fourier expansion with the minimal number of nonzero terms leads to the ansatz

$$\chi = g(r)f(r)e^{2\imath\theta} - \tilde{g}(r)e^{-2\imath\theta},$$
$$E_1 + \imath E_2 = \tilde{b}(r)e^{\imath\theta} - \imath b(r)f(r)e^{-3\imath\theta}, \qquad (4\ 43)$$

and to the equations for $g(r), \tilde{g}(r), b(r)$ and $\tilde{b}(r)$ given in (4 33) - (4 35)

Collecting all results, we can write the solution, up to quadratic terms, in the form

$$\phi = fe^{2\imath\theta} + 2(\alpha + \imath\beta)kf$$
$$+(\alpha^2 + \beta^2)gfe^{2\imath\theta} + (\alpha + \imath\beta)^2(\frac{1+2a}{r}b - b')e^{-2\imath\theta} +$$

$$A_1 + \imath A_2 = \imath \frac{2a}{r} e^{\imath\theta} - 2\imath(\alpha + \imath\beta)(k' + \frac{2k}{r})e^{-\imath\theta}$$
$$-\imath(\alpha^2 + \beta^2)g' e^{\imath\theta} + \imath(\alpha + \imath\beta)^2 bfe^{-3\imath\theta} + \quad (4\ 44)$$

It remains to be shown that the quadratic terms in (4 44) are $C^\infty$ functions on $\mathbf{R}^2$ which do not change the action (and the winding number) To this end we use the asymptotic expansions of $f, a$ and $k$ at zero [13],

$$f(r) = f_1 r^2 + \frac{1}{8} f_1 r^4 + \quad , \quad a(r) = \frac{1}{8} r^2 - \frac{1}{24} f_1^2 r^6 + \quad , \quad k(r) = r^{-2} + k_1 r^2 + \quad , \quad (4\ 45)$$

where $f_1 = 236$ and $k_1 = -025$ from the numerical analysis We find that the solutions of (4 34) and (4 35) have the following expansions at the origin,

$$g(r) = g_{-1} \log r + g_1 + \frac{1}{2} f_1^2 r^2 +$$
$$b(r) = b_{-1} r^{-1} + b_1 r + (\frac{1}{8} b_1 - 2 f_1 k_1) r^3 + \quad (4\ 46)$$

The higher order terms in $g(r)$ are even powers of $r$, whereas the higher order term in $b(r)$ are odd powers of $r$ Hence, the quadratic terms in (4 44) are $C^\infty$ near the origin if and only if $g_{-1} = b_{-1} = 0$ So far the constants $g_1$ and $b_1$ are arbitrary

For large $r$ the functions $f, a$, and $k$ have the following asymptotic behavior [13]

$$f(r) = 1 + \tilde{f}_1(r) e^{-r} + \quad ,$$
$$a(r) = 1 + \tilde{a}_1(r) e^{-r} + \quad , \quad (4\ 47)$$
$$k(r) = \tilde{k}_1(r) e^{-r} + \quad ,$$

with coefficient functions which are polynomially bounded This leads to the existence of exponentially decaying solutions which asymptotically are of the form

$$g(r) = \tilde{g}_1(r) e^{-r} + \quad , \quad b(r) = \tilde{b}_1(r) e^{-r} + \quad (4\ 48)$$

Here $\tilde{g}_1$ and $\tilde{b}_1$ are polynomially bounded

By numerical integration, the coefficients $g_1$ and $b_1$ which lead to an exponential fall-off at infinity, are found to be $g_1 = -144$ and $b_1 = -026$ The existence of such functions can be explained analytically as follows Equation (4 34) shows that for positive $g_1$, $g$ cannot have a maximum for any $r$ So the function diverges exponentially For very small $g_1$, the term on the right-hand side of (4 34) will force the function to cross the $r$-axis, and then, as before, diverge exponentially For very large negative $g_1$, the third term in (4 34) will force $g$ to go through a maximum for large $r$ After that, the function cannot have a minimum and must go to minus infinity Because of the continuous dependence on the initial data, we have an open set of data for which $g$ crosses the $r$-axis, and an open set of data for which $g$ goes through a maximum below the $r$-axis Therefore, we have at least one value of $g_1$ for which the function does neither This function must converge and does so to zero, exponentially

A similar argument explains the existence of an acceptable solution $b(r)$ to Eq (4 35) The right-hand side of that equation is positive So again $b$ cannot have a maximum above the $r$-axis Also, for very small negative $b_1$, the right-hand side will force $b$ to go through a minimum and then cross the $r$-axis For very large negative $b_1$, the third term in (4 35) prevents $b$ from going through a minimum In between these two possibilities we find the desired solution which goes through a minimum but does not cross the $r$-axis Such a solution must decay exponentially

## 4.4 The cubic terms

The cubic terms can be calculated in the same manner If we consider the third order terms

$$\phi = \quad + \alpha^3 \chi + \quad ,$$

$$A_1 + \imath A_2 = \quad + \alpha^3 B + \quad , \tag{4 49}$$

we obtain

$$\begin{aligned} &\imath(e^{\imath\theta}\partial_r B^* + e^{-\imath\theta}\partial_r B) - \frac{1}{r}(e^{\imath\theta}\partial_\theta B^* + e^{-\imath\theta}\partial_\theta B) \\ &+ f(e^{2\imath\theta}\chi^* + e^{-2\imath\theta}\chi) = -4kf(gf + \frac{1+2a}{r}b - b')\cos 2\theta \end{aligned} \tag{4 50}$$

and

$$\begin{aligned} &e^{\imath\theta}(\partial_r + \frac{\imath}{r}\partial_\theta)\chi - \imath f e^{2\imath\theta} B + \frac{2a}{r} e^{\imath\theta}\chi \\ &= 2(k' + \frac{2k}{r})(gfe^{\imath\theta} + (\frac{1+2a}{r}b - b')e^{-3\imath\theta}) \\ &\quad + 2kf(g' e^{\imath\theta} - bfe^{-3\imath\theta}) \end{aligned} \tag{4 51}$$

This time the Fourier expansion with the minimal number of terms is of the form

$$\begin{aligned} \chi &= h_1(r) + h_2(r)e^{-4\imath\theta} \\ B &= \imath C_1(r)e^{-\imath\theta} + \imath C_2(r)e^{-5\imath\theta} \end{aligned} \tag{4 52}$$

Equation (4 50) and (4 51) are now satisfied if the following equations hold

$$h_1' + fC_1 + \frac{2a}{r}h_1 = 2gf(k' + \frac{2k}{r}) + 2kfg' \tag{4 53}$$

$$h_2' + fC_2 + \frac{4+2a}{r}h_2 = 2(k^1 + \frac{2k}{r})(\frac{1+2a}{r}b - b') - 2kbf^2 \tag{4 54}$$

$$C_1^1 - \frac{1}{r}C_1 + fh_1 = -2kf(gf + \frac{1+2a}{r}b - b' \tag{4 55}$$

$$C_2' - \frac{5}{r}C_2 + fh_2 = 0 \tag{4 56}$$

Equations (4 56) and (4 53) can be solved for $h_2$ and $C_1$ respectively, and we are left with two second order equations for $h_1$ and $C_2$ respectively

The $\alpha^2\beta, \alpha\beta^2$ and $\beta^3$ terms can be calculated in the same way Putting all the results together, we find, at third order,

$$\phi = \quad + (\alpha + \imath\beta)(\alpha^2 + \beta^2)fh + (\alpha + \imath\beta)^3(-c' + \frac{3+2a}{r}c)e^{-4\imath\theta} + \quad ,$$

$$A_1 + \imath A_2 =$$

$$+\imath(\alpha+\imath\beta)(\alpha^2+\beta^2)[-h' - \frac{2}{r}h + 2g(k' + \frac{2k}{r}) + 2kg']e^{-\imath\theta} + \imath(\alpha+\imath\beta)^3 fce^{-5\imath\theta} + \tag{4 57}$$

The new radial functions, $h(r)$ and $c(r)$, satisfy the equations,

$$h'' + \frac{1}{r}h' - (f^2 + \frac{4}{r^2})h = 4k'g' + 2fk(2fk^2 + 3fg + \frac{1+2a}{r}b - b'), \tag{4 58}$$

$$c'' + \frac{1}{r}c' - (\frac{1+f^2}{2} + \frac{9+12a+4a^2}{r^2})c = 2kf^2b - 2(k' + \frac{2k}{r})(\frac{1+2a}{r}b - b') \tag{4 59}$$

Near the origin, Eq (4 58) has a series solution in powers of $r^2$ of the form

$$h(r) = f_1^2 + h_1r^2 + h_2r^4 + \tag{4 60}$$

The constant term is given in terms of the coefficient $f_1$ of the leading term in the expansion (4 45) of $f(r)$ The form of this term leads to the cancellation of the $r^{-1}$-terms in the radial function multiplying $e^{-\imath\theta}$ in (4 57), and thus ensures that this term is $C^\infty$ on $\mathbf{R}^2$ The series in odd powers of $r$ for $c(r)$ which solves Eq (4 59) near the origin, is

$$c(r) = c_1r^3 + c_2r^5 + \tag{4 61}$$

The form of the series solutions at the origin guarantees that the cubic terms in (4 23) are $C^\infty$ functions on $\mathbf{R}^2$ For large $r$, Eqs (4 58) and (4 59) have exponentially decaying solutions Hence the action and the winding number are unchanged by the inclusion of the third order terms

# 5 Conclusions

Our expansions show a simple $\theta$-dependence in terms of trigonometric functions. In both models, the expansion of $\phi$ exhibits the following pattern

$$
\begin{array}{cccccc}
 & & & & & e^{2i\theta} \\
 & & & & e^{0i\theta} & \\
 & & & e^{-2i\theta} & & e^{2i\theta} \\
 & & e^{-4i\theta} & & e^{0i\theta} & \\
 & e^{-6i\theta} & & e^{-2i\theta} & & e^{2i\theta} \\
e^{-8i\theta} & & e^{-4i\theta} & & e^{0i\theta} &
\end{array}
$$

Here the first line gives the $\theta$ dependence of the zero order term, the second line gives the first order term, and so on. We get a similar triangular pattern for the $\theta$ dependence of $A_1 + iA_2$ at any order. For the radial functions we find differences between the two models. In the model for one complex field, the radial functions can be given explicitly in terms of exponential functions. However, for the angular dependence (3.78), a singularity occurs at the origin. (We have found no solution to (3.77) which is not of the form (3.78), we have found no proof that there is none.)

For the Ginzburg-Landau theory on the other hand, the expansion is smooth, at least up to the order to which we carried out our calculations. In this model the radial functions are not given in terms of well-known functions. Having used the technique to calculate the terms up to third order, it is quite clear how to proceed to any order, and also how to proceed in the case of more than two vortices. We expect these expansions to converge for small separation parameters in the physical Ginzburg-Landau model. However, we do not have an estimate of the radius of convergence.

# 6 Bibliography

## References


[1] G t'Hooft, Nucl Phys B **79**, 276 (1974)

[2] A M Polyakov, JETT Lett **20**, 194 (1974)

[3] C Taubes, Commun Math Phys **80**, 343 (1981)

[4] R S Ward, Commun Math Phys **79**, 317 (1981)

[5] C Taubes, Commun Math Phys **72**, 277 (1980)

[6] D H Tchrakıan and H J W Muller-Kırsten, Phys Rev D **44**, 1204 (1991)

[7] B Pıette, D H Tchrakıan and W J Zakrzewskı, Z Phys C **54**, 497 (1992)

[8] E B Bogomol'nyı, Yad Fız **24**, 861 (1976) [Sov J Nucl Phys **24**, 449 (1976)]

[9] J Burzlaff, Proc XXXI Symposıum on Mathematıcal Physıcs, Torun 1999 , Rep Math Phys , to appear

[10] A Jaffe and Taubes, *Vortıces and Monopoles*, Bırkhauser, Boston 1980

[11] B Plohr, Doctoral Dıssertatıon, Prınceton Unıversıty 1980, J Math Phys **22**, 2184 (1981)

[12] E J Weınberg, Phys Rev D **19**, 3008 (1979)

[13] J Burzlaff and P McCarthy, J Math Phys **32**, 3376 (1991)