Worst-Case Optimal Investment with a Random Number of Crashes

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Abstract

We study a portfolio optimization problem in a market which is under the threat of crashes. At random times, the investor receives a warning that a crash in the risky asset might occur. We construct a strategy which renders the investor indifferent about an immediate crash of maximum size and no crash at all. We then verify that this strategy outperforms every other trading strategy using a direct comparison approach. We conclude with numerical examples and calculating the costs of hedging against crashes.

Keywords: optimal investment, market crashes, worst-case scenario, financial bubbles

1 Introduction

The aim of this paper is to study optimal investment decisions given that the financial market under consideration is subject to the threat of crashes in the risky asset. We take the point of view of an investor who aims at maximizing her expected utility of terminal wealth, while taking a worst-case perspective towards the impact of crashes. For this, we assume that at random times, the investor receives a warning about a potential crash together with an upper bound on the size of the crash. It is a key feature of our model that we do not make any further assumptions about the crash – i.e. we do not specify any distribution on its timing and size. Instead, we assume that the investor is extremely cautious and anticipates the worst-possible crash scenario corresponding to the trading strategy which she uses. That is, given a trading strategy π, we try to identify the worst-case crash scenario ϑ in the sense that expected utility of terminal wealth $X_T^{\pi,\vartheta}$ is minimized. Then we aim to find the trading strategy which yields the highest expected utility of terminal wealth if the corresponding worst-case scenario is realized.

We assume that the warnings occur at the jump times of an independent Poisson process and the maximum crash size is fixed and known to the investor. With this approach, we extend the model considered in Korn and Wilmott [10] (see also Korn and Menkens [8], Menkens [12], Korn and Steffensen [9], Seifried [16], Belak et al. [1], Desmettre et al. [4] for further extensions) to a setting in which the investor does not know the exact number of possible crashes a priori.

Our modelling approach can also be interpreted as a simple model for a financial market in the presence of bubbles. Whenever the investor receives a warning, she becomes aware that a bubble has formed in the market which may burst, and hence lead to a crash. Note, however, that this modelling approach is different from the related literature on financial bubbles (see e.g. Cox and Hobson [3], Jarrow et al. [6, 7], Heston et al. [5], Biagini et al. [2]), in the sense that we are more concerned with the effects of a burst of the bubble and not so much on its formation and price impacts.

This paper is organized as follows. In Section 2, we introduce the market model. In Section 3, we heuristically derive a candidate for the optimal strategies in the log utility case by an indifference

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argument and we verify the optimality in Section 4. We conclude in Section 5 with numerical examples, estimating the asymptotic behaviour, and computing the costs of hedging against crashes.

The main result is that the obtained optimal strategies are qualitatively different from the optimal strategies in the model of Korn and Wilmott [10]. While the optimal strategies in the Korn/Wilmott model will always converge to the classical Merton optimal strategy as the investment horizon goes to infinity, this does no longer hold for the optimal strategies obtained in the model of this paper. In other words, any given, fixed number of potential crashes has just a short term impact (which can be rather long), while a random number of crashes has a short term and a long term impact. The short term impact is similar to the short term impact in the Korn/Wilmott model if at most one crash can happen (however, it is different to the short term impact of the Korn/Wilmott model if more than one crash can happen). The long term impact is that the optimal strategies converge to a value which is strictly less than the classical optimal strategy of Merton (see Subsection 5.1).

2 Market model and problem formulation

We consider a financial market consisting of one risk-free asset (e.g. a *bond*), and one risky asset (e.g. a *stock*) with price evolutions as in the Black-Scholes model. That is, assume that the dynamics of the bond, denoted by $(B_t)_{t\geq 0}$, are given by $dB_t = 0$ with initial condition $B_0 = 1$.

To model the stock price, let W be a standard Brownian motion on a complete probability space $(\Omega, \mathbb{F}, \mathbb{P})$, and let $(T_k)_{k\in\mathbb{N}_0}$ denote the jump times of an independent Poisson process with parameter λ, where $T_0 = 0$. We denote the augmented filtration generated by W and the Poisson process by $(\mathcal{F}_t)_{t\geq 0}$. As described in the introduction, the sequence $(T_k)_{k\in\mathbb{N}_0}$ models the time points, at which we receive a warning about a potential crash in the market. Note that the sequence $(T_k)_{k\in\mathbb{N}_0}$ does not coincide with the crash times in general. These crash times are given by a sequence $(\tau_k)_{k\in\mathbb{N}_0}$ of $(T_k \wedge T, T] \cup \{+\infty\}$-valued stopping times with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, where $T > 0$ denotes the investor's investment horizon. We assume that whenever we have

$$T_k < \tau_k < T_{k+1},$$

a crash occurs at time τ_k. This condition means that there is at most one crash between every two warnings. In other words, before each crash, we get a warning. We interpret $\{\tau_k \geq T_{k+1}\}$ as the event that no crash occurs between the k-th and $(k+1)$-th warning. In other words, this setup is such that at any given time at most one crash warning is active. Moreover, note that $\tau_k = T_k$ is not allowed. This ensures that the investor has time to react to a crash warning. At each crash time τ_k, the stock price drops by a relative amount $0 \leq \kappa_k \leq \kappa^*$, where κ_k is a $\mathcal{F}_{\tau_k}$-measurable random variable and $\kappa^* \in (0,1)$ denotes the maximum (deterministic) crash height. We denote the set of all sequences $(\tau_k, \kappa_k)_{k\in\mathbb{N}_0}$ fulfilling the above requirements by $\mathcal{T}_1$. The subset of all crash scenarios such that no crash occurs until the first warning, that is $\tau_0 \geq T_1$, is denoted by $\mathcal{T}_0$.

Given a sequence $\vartheta = (\tau_k, \kappa_k)_{k\in\mathbb{N}_0} \in \mathcal{T}_1$, the corresponding asset price process $S = S^\vartheta$ is given by

$$dS_t = \alpha S_t dt + \sigma S_t dW_t, \qquad \text{for } t \text{ between each two crash times } (\tau_k, \tau_{k+1}),$$

where $\alpha, \sigma > 0$ denote the excess return and volatility of the asset price process. For the k-th crash time τ_k, we assume that

$$S_{\tau_k} = (1-\kappa_k) S_{\tau_k -} \text{ on } \{\tau_k < T_{k+1}\} \quad \text{and} \quad S_{\tau_k} = S_{\tau_k -}, \text{ on } \{\tau_k \geq T_{k+1}\}.$$

Now, we consider a portfolio optimization problem as follows. The investor acts according to the information given by the filtration $(\mathcal{F}_t)_{t\geq 0}$. Furthermore, the investor does not know the crash scenarios $(\tau_k, \kappa_k)_{k\in\mathbb{N}_0}$ a priori, but can observe each crash whenever it occurs. However, the investor knows κ^* and observes $(T_k)_{k\in\mathbb{N}_0}$ as well. For the investor, two different situations must be distinguished: whenever a crash has already happened and no new crash is announced – that is on an interval (τ_k, T_{k+1}) – the investor does not have to fear a crash and trades according to the strategy $(\pi_t^0)_{t\in[0,T]}$. At all other times, the investor must fear a crash and trades according to the strategy $(\pi_t^1)_{t\in[0,T]}$. Therefore, the trading strategies for the investor can be described by a pair $\pi = (\pi_t^0, \pi_t^1)_{t\in[0,T]}$, which is assumed to be predictable.

Given a crash scenario $\vartheta = (\tau_k, \kappa_k)_{k\in\mathbb{N}_0}$ and a trading strategy $\pi = (\pi_t^0, \pi_t^1)_{t\in[0,T]}$, the investor's wealth process $X = X^{\pi,\vartheta}$ is given by

$$\begin{aligned}
X_0 &= x, \\
dX_t &= \alpha\pi_t^1 X_t dt + \sigma\pi_t^1 X_t dW_t, && \text{on } [T_k, \tau_k) \text{ for each } k, \\
dX_t &= \alpha\pi_t^0 X_t dt + \sigma\pi_t^0 X_t dW_t, && \text{on } (\tau_k, T_{k+1}) \text{ for each } k, \\
X_{\tau_k} &= (1 - \pi_{\tau_k}^1 \kappa_k) X_{\tau_k -}, && \text{on } \{\tau_k < T_{k+1}\} \text{ for each } k.
\end{aligned}$$

If no confusion may occur, we drop one or both superscripts in the notation of the process $X^{\pi,\vartheta}$. Note that the explicit solution of the above SDE is given by

$$\begin{aligned}
X_t = x \prod_{k=0}^{\infty} \left(1 - \pi_{\tau_k}^1 \kappa_k \mathbb{1}_{\{\tau_k < T_{k+1}\wedge t\}}\right) \exp\left(\int_{T_k\wedge t}^{\tau_k\wedge t} \sigma\pi_s^1 \, dW_s + \int_{\tau_k\wedge t}^{T_{k+1}\wedge t} \sigma\pi_s^0 \, dW_s \right) \\
\cdot \exp\left(\int_{T_k\wedge t}^{\tau_k\wedge t} \left[\alpha\pi_s^1 - \frac{\sigma^2}{2}(\pi_s^1)^2\right] ds + \int_{\tau_k\wedge t}^{T_{k+1}\wedge t} \left[\alpha\pi_s^0 - \frac{\sigma^2}{2}(\pi_s^0)^2\right] ds \right). \quad (1)
\end{aligned}$$

Now, we denote the set of all trading strategies π that correspond to nonnegative wealth processes $X^{\pi,\vartheta}$ for all $\vartheta \in \mathcal{T}_1$ by $\mathcal{A}$. Note that the sign of the wealth process does not depend on the initial wealth $x > 0$, and that the strategy in the crash regime π_t^1 has to satisfy

$$\pi_t^1 \le \frac{1}{\kappa^*}, \qquad \text{for all } t.$$

If we consider the problem on a subinterval $[t, T]$, then we denote the corresponding strategies by $\mathcal{A}(t)$ and $\mathcal{T}_1(t)$, $\mathcal{T}_0(t)$, respectively.

Let us furthermore fix a utility function $U : [0, \infty) \to \mathbb{R}$. We consider the following worst-case optimization problem: the investor optimizes her expected utility under the worst possible crash scenario, that is we consider the problems

$$\sup_{\pi\in\mathcal{A}} \inf_{\vartheta\in\mathcal{T}_0} \mathbb{E}\left[U(X_T^{\pi,\vartheta})\right] \qquad \text{and} \qquad \sup_{\pi\in\mathcal{A}} \inf_{\vartheta\in\mathcal{T}_1} \mathbb{E}\left[U(X_T^{\pi,\vartheta})\right].$$

The first problem corresponds to the case that at time $t = 0$ we start in a situation without a crash warning. In the second problem, the first crash may occur immediately. We make the problem time-dependent by introducing the value functions

$$v_0(t,x) = \sup_{\pi\in\mathcal{A}(t)} \inf_{\vartheta\in\mathcal{T}_0(t)} \mathbb{E}_{(t,x)}\left[U(X_T^{\pi,\vartheta})\right], \qquad v_1(t,x) = \sup_{\pi\in\mathcal{A}(t)} \inf_{\vartheta\in\mathcal{T}_1(t)} \mathbb{E}_{(t,x)}\left[U(X_T^{\pi,\vartheta})\right].$$

Obviously, $v_1 \le v_0$ since the infimum is taken over a larger set.

In the sequel, we restrict our analysis to the case of log utility. Using the strict monotonicity of U and the explicit solution of the wealth process SDE in Equation (1), it follows immediately that the worst-possible crash size is always attained for $\kappa_k = \kappa^*$ for all $k \in \mathbb{N}_0$. Hence, it suffices to consider crash scenarios $\vartheta = (\tau_k, \kappa_k)_{k\in\mathbb{N}_0}$ satisfying $\kappa_k = \kappa^*$ for all $k \in \mathbb{N}_0$. Therefore, we also write $\vartheta = (\tau_k)_{k\in\mathbb{N}_0}$ (instead of $\vartheta = (\tau_k, \kappa^*)_{k\in\mathbb{N}_0}$) for short.

3 Heuristic derivation of the worst-case optimal strategies

In this section, we find a candidate solution $\pi^* = (\pi_t^{0,*}, \pi_t^{1,*})_{t\in[0,T]}$. By the usual pointwise maximization argument, it is immediately clear that in times with no crash warning, it is optimal for the investor to use the Merton strategy, that is

$$\pi_t^{0,*} := \frac{\alpha}{\sigma^2}, \qquad \text{for all } t \in [0, T].$$

If we start the process at time t with initial value x and no crash warning is given, by the memoryless property of the exponential distribution, the next warning arrives at an exponential time e_λ if this time is less than T. Therefore, by the Bellman principle, it is reasonable to assume that

$$v_0(t,x) = \mathbb{E}_{(t,x)}\Big[v_1(t+e_\lambda, X_{t+e_\lambda})\mathbb{1}_{\{t+e_\lambda<T\}} + \log X_T \mathbb{1}_{\{t+e_\lambda \geq T\}}\Big],$$

where X denotes the wealth process generated by $\pi^{0,*}$. We obtain

$$\begin{aligned} v_0(t,x) &= \int_0^{T-t} \mathbb{E}_{(t,x)}\big[v_1(t+s, X_{t+s})\big]\lambda e^{-\lambda s}\, ds + e^{-\lambda(T-t)}\mathbb{E}_{(t,x)}\big[\log X_T\big] \\ &= \int_0^{T-t} \mathbb{E}_{(t,x)}\big[v_1(t+s, X_{t+s})\big]\lambda e^{-\lambda s}\, ds + e^{-\lambda(T-t)}\left(\log x + \frac{\alpha^2}{2\sigma^2}(T-t)\right). \end{aligned}$$

To obtain an expression for v_1, we use the indifference approach as described in Korn and Wilmott [10] and formalized in Korn and Menkens [8]. For an initial value x and a time point t such that a crash could happen, we try to find a strategy $\pi^{1,*}$ such that the investor is indifferent between the scenarios *"A crash of height κ^* happens immediately"* and *"No crash happens until T"*. We expect this strategy to be optimal. For the first scenario, after the crash in t, the investor is faced with the problem without an active crash warning discussed above, so that

$$v_1(t,x) = v_0(t, (1-\pi_t^{1,*}\kappa^*)x).$$

On the other hand, in the second scenario (no crash at all), Itô's formula leads to

$$v_1(t,x) = \mathbb{E}_{(t,x)}\left[\log X_T^{\pi^{1,*}}\right] = \log x + \mathbb{E}\left[\int_t^T \left(\alpha\pi_s^{1,*} - \frac{\sigma^2}{2}(\pi_s^{1,*})^2\right) ds\right].$$

Putting the pieces together, we obtain

$$\begin{aligned} &\log x + \mathbb{E}\left[\int_t^T \left(\alpha\pi_s^{1,*} - \frac{\sigma^2}{2}(\pi_s^{1,*})^2\right) ds\right] \\ &\quad = v_1(t,x) = v_0(t,(1-\pi_t^{1,*}\kappa^*)x) \\ &\quad = \int_0^{T-t} \mathbb{E}_{(t,(1-\pi_t^{1,*}\kappa^*)x)}\left[v_1(t+s, X_{t+s})\right]\lambda e^{-\lambda s}\, ds \\ &\qquad + e^{-\lambda(T-t)}\left(\log x + \mathbb{E}\left[\log(1-\pi_t^{1,*}\kappa^*)\right] + \frac{\alpha^2}{2\sigma^2}(T-t)\right). \end{aligned}$$

Furthermore,

$$\begin{aligned} \mathbb{E}_{(t,(1-\pi_t^{1,*}\kappa^*)x)}\big[v_1(t+s, X_{t+s})\big] &= \mathbb{E}_{(t,(1-\pi_t^{1,*}\kappa^*)x)}\left[\log X_{t+s} + \int_{t+s}^T \left(\alpha\pi_r^{1,*} - \frac{\sigma^2}{2}(\pi_r^{1,*})^2\right) dr\right] \\ &= \mathbb{E}_{(t,(1-\pi_t^{1,*}\kappa^*)x)}\left[\log X_t + \int_t^{t+s} \left(\alpha\pi_r^{0,*} - \frac{\sigma^2}{2}(\pi_r^{0,*})^2\right) dr \right. \\ &\qquad \left. + \int_{t+s}^T \left(\alpha\pi_r^{1,*} - \frac{\sigma^2}{2}(\pi_r^{1,*})^2\right) dr\right] \\ &= \log x + \frac{\alpha^2}{2\sigma^2}s + \mathbb{E}_{(t,(1-\pi_t^{1,*}\kappa^*)x)}\left[\log(1-\pi_t^{1,*}\kappa^*)\right. \\ &\qquad \left. + \int_{t+s}^T \left(\alpha\pi_r^{1,*} - \frac{\sigma^2}{2}(\pi_r^{1,*})^2\right) dr\right], \end{aligned}$$

so that

$$\begin{aligned}\mathbb{E}\Big[\int_t^T \Big(\alpha\pi_s^{1,*} - \frac{\sigma^2}{2}(\pi_s^{1,*})^2\Big)\,ds\Big] &= \int_0^{T-t} \mathbb{E}\left[\log(1-\pi_t^{1,*}\kappa^*) + \int_{t+s}^T \Big(\alpha\pi_r^{1,*} - \frac{\sigma^2}{2}(\pi_r^{1,*})^2\Big)\,dr\right]\lambda e^{-\lambda s}\,ds \\ &\quad + \int_0^{T-t} \frac{\alpha^2}{2\sigma^2} s\lambda e^{-\lambda s}\,ds \\ &\quad + e^{-\lambda(T-t)}\left(\mathbb{E}\left[\log(1-\pi_t^{1,*}\kappa^*)\right] + \frac{\alpha^2}{2\sigma^2}(T-t)\right) \\ &= \int_0^{T-t} \mathbb{E}\left[\int_{t+s}^T \Big(\alpha\pi_r^{1,*} - \frac{\sigma^2}{2}(\pi_r^{1,*})^2\Big)\,dr\right]\lambda e^{-\lambda s}\,ds \\ &\qquad + \mathbb{E}\left[\log(1-\pi_t^{1,*}\kappa^*)\right] + \frac{\alpha^2}{2\sigma^2}\frac{1-e^{-\lambda(T-t)}}{\lambda}.\end{aligned}$$

Now, we make the ansatz that $\pi^{1,*}$ is deterministic and obtain the integral equation

$$\begin{aligned}\int_t^T \Big(\alpha\pi_s^{1,*} - \frac{\sigma^2}{2}(\pi_s^{1,*})^2\Big)\,ds = \log(1-\pi_t^{1,*}\kappa^*) + \int_0^{T-t}\left(\int_{t+s}^T \Big(\alpha\pi_r^{1,*} - \frac{\sigma^2}{2}(\pi_r^{1,*})^2\Big)\,dr\right)\lambda e^{-\lambda s}\,ds \\ + \frac{\alpha^2}{2\sigma^2}\frac{1-e^{-\lambda(T-t)}}{\lambda}.\end{aligned}$$

Integration by parts yields that

$$\begin{aligned}\int_0^{T-t}\left(\int_{t+s}^T \Big(\alpha\pi_r^{1,*} - \frac{\sigma^2}{2}(\pi_r^{1,*})^2\Big)\,dr\right)\lambda e^{-\lambda s}\,ds = \int_t^T \Big(\alpha\pi_s^{1,*} - \frac{\sigma^2}{2}(\pi_s^{1,*})^2\Big)\,ds \\ - \int_t^T e^{-\lambda(s-t)}\Big(\alpha\pi_s^{1,*} - \frac{\sigma^2}{2}(\pi_s^{1,*})^2\Big)\,ds,\end{aligned}$$

hence the integral equation read as

$$0 = e^{-\lambda t}\log(1-\pi_t^{1,*}\kappa^*) - \int_t^T e^{-\lambda s}\Big(\alpha\pi_s^{1,*} - \frac{\sigma^2}{2}(\pi_s^{1,*})^2\Big)\,ds + \frac{\alpha^2}{2\sigma^2}\frac{e^{-\lambda t}-e^{-\lambda T}}{\lambda}. \tag{2}$$

Note that for the degenerate case $\lambda \to 0$ (that is, the time until the second warning can occur is infinite, i.e. only one crash can happen) the equation simplifies to

$$\int_t^T \Big(\alpha\pi_s^{1,*} - \frac{\sigma^2}{2}(\pi_s^{1,*})^2\Big)\,ds = \log(1-\pi_t^{1,*}\kappa^*) + \frac{\alpha^2}{2\sigma^2}(T-t),$$

which indeed is the equation characterizing the optimal worst case strategy in the problem with only one crash, see Korn and Wilmott [10, Equation (A.5)].

Differentiation with respect to t in (2) and rearranging terms yields the following ordinary differential equation for $\pi^{1,*}$:

$$\begin{aligned}\frac{\partial}{\partial t}\pi_t^{1,*} &= \frac{1}{\kappa^*}(1-\kappa^*\pi_t^{1,*})\left(\alpha(\pi_t^{1,*}-\pi_t^{0,*}) - \frac{\sigma^2}{2}\big((\pi_t^{1,*})^2-(\pi_t^{0,*})^2\big) - \lambda\log(1-\kappa^*\pi_t^{1,*})\right) \\ &= -\frac{1}{\kappa^*}(1-\kappa^*\pi_t^{1,*})\left(\frac{\sigma^2}{2}\left(\pi_t^{1,*}-\pi_t^{0,*}\right)^2 + \lambda\log(1-\kappa^*\pi_t^{1,*})\right) \quad \text{with} \\ \pi_T^{1,*} &= 0,\end{aligned} \tag{3}$$

where as before $\pi_t^{0,*} = \alpha/\sigma^2$ for all $t \in [0,T]$. The existence and uniqueness of a solution of this differential equation is assured by arguments similar to the proof in Menkens [11, Theorem 2.5]. In particular, we have $\pi_t^{1,*} \leq 1/\kappa^*$ and $\pi^{1,*} \leq \pi^{0,*}$.

Again, note that for the degenerate case $\lambda = 0$ with only one jump, the previous equation reduces to

$$\begin{aligned}\frac{\partial}{\partial t}\pi_t^{1,*} &= \frac{1}{\kappa^*}(1-\kappa^*\pi_t^{1,*})\left(\alpha(\pi_t^{1,*}-\pi_t^{0,*}) - \frac{\sigma^2}{2}\left((\pi_t^{1,*})^2 - (\pi_t^{0,*})^2\right)\right) \\ &= -\frac{\sigma^2}{2\kappa^*}(1-\kappa^*\pi_t^{1,*})\left(\pi_t^{1,*}-\pi_t^{0,*}\right)^2,\end{aligned}$$

which is the equation obtained in Korn and Wilmott [10].

4 Direct verification

In this section, we verify directly that the indifference strategy $\pi^* = (\pi_t^{0,*}, \pi_t^{1,*})_{t\in[0,T]}$ constructed in the previous section is indeed optimal. For this, let $\hat{\vartheta} = (\hat{\tau}_k)_{k\in\mathbb{N}_0}$ denote the crash scenario such that no crash occurs at all, that is $\hat{\tau}_k \equiv \infty$ for all $k \in \mathbb{N}_0$. The next lemma shows that the strategy π^* leads to the same expected utility, no matter which crash scenario occurs.

Lemma 4.1. *For all crash scenarios $\vartheta = (\tau_k)_{k\in\mathbb{N}_0} \in \mathcal{T}_1$ it holds that*

$$\mathbb{E}\left[\log X_T^{\pi^*,\vartheta}\right] = \mathbb{E}\left[\log X_T^{\pi^*,\hat{\vartheta}}\right].$$

Proof. Writing $g(y) = \alpha y - 1/2\sigma^2 y^2$ for short, it holds that

$$\begin{aligned}&\mathbb{E}\left[\log X_T^{\pi^*,\vartheta}\right] \\ &= \mathbb{E}\left[\log X_0^{\pi^*,\vartheta}\right] + \mathbb{E}\left[\sum_{k\in\mathbb{N}_0}\int_{T_k\wedge T}^{\tau_k\wedge T} g(\pi_s^{1,*})\,ds\right] \\ &\quad + \mathbb{E}\left[\sum_{k\in\mathbb{N}_0}\left(\int_{\tau_k\wedge T}^{T_{k+1}\wedge T} g(\pi_s^{0,*})\,ds + \log(1-\pi_{\tau_k}^{1,*}\kappa^*)\mathbb{1}_{\{\tau_k<T_{k+1}\wedge T\}}\right)\right].\end{aligned} \tag{4}$$

By construction of the indifference strategy π^*, it holds that for each $k \in \mathbb{N}_0$ on the set $\{\tau_k < T_{k+1}\wedge T\}$

$$\mathbb{E}\left[\int_{\tau_k\wedge T}^{T_{k+1}\wedge T} g(\pi_s^{0,*})\,ds + \int_{T_{k+1}\wedge T}^{T} g(\pi_s^{1,*})\,ds + \log(1-\pi_{\tau_k}^{1,*}\kappa^*)\middle|\mathcal{F}_{\tau_k}\right] = \mathbb{E}\left[\int_{\tau_k\wedge T}^{T} g(\pi_s^{1,*})\,ds\middle|\mathcal{F}_{\tau_k}\right],$$

i.e. the investor is indifferent between a crash of size κ^* happening at time τ_k (left-hand side) and no crash happening (right-hand side). This can be rewritten as

$$\mathbb{E}\left[\int_{\tau_k\wedge T}^{T_{k+1}\wedge T} g(\pi_s^{0,*})\,ds + \log(1-\pi_{\tau_k}^{1,*}\kappa^*)\middle|\mathcal{F}_{\tau_k}\right] = \mathbb{E}\left[\int_{\tau_k\wedge T}^{T_{k+1}\wedge T} g(\pi_s^{1,*})\,ds\middle|\mathcal{F}_{\tau_k}\right]. \tag{5}$$

Hence, using Equation (4) together with Equation (5),

$$\begin{aligned}\mathbb{E}\left[\log X_T^{\pi^*,\vartheta}\right] &= \mathbb{E}\left[\log X_0^{\pi^*,\vartheta}\right] + \mathbb{E}\left[\sum_{k\in\mathbb{N}_0}\int_{T_k\wedge T}^{\tau_k\wedge T} g(\pi_s^{1,*})\,ds\right] + \mathbb{E}\left[\sum_{k\in\mathbb{N}_0}\left(\int_{\tau_k\wedge T}^{T_{k+1}\wedge T} g(\pi_s^{1,*})\,ds\right)\mathbb{1}_{\{\tau_k<T_{k+1}\}}\right] \\ &= \mathbb{E}\left[\log X_0^{\pi^*,\vartheta}\right] + \mathbb{E}\left[\int_0^T g(\pi_s^{1,*})\,ds\right] = \mathbb{E}\left[\log X_T^{\pi^*,\hat{\vartheta}}\right],\end{aligned}$$

which finishes the proof. □

Lemma 4.1 implies that the strategy π^* is indeed an indifference strategy, i.e. if the investor follows this strategy, she is indifferent about which crash scenario occurs (as long as the crash size is always equal to κ^*), since each scenario leads to the same expected utility. Note also, that since $\mathcal{T}_0 \subset \mathcal{T}_1$ and $\hat{\vartheta} \in \mathcal{T}_0$, the same result also applies for the case in which there is no crash warning at initial time $t = 0$. With this, it is easy to prove the optimality of π^*, since we now only need to find one crash scenario, in which the indifference strategy outperforms any other given strategy.

Proposition 4.2. *Let $\pi = (\pi^0, \pi^1) \in \mathcal{A}$ be arbitrary. Then*

$$\inf_{\vartheta \in \mathcal{T}_0} \mathbb{E}\left[\log X_T^{\pi,\vartheta}\right] \leq \inf_{\vartheta \in \mathcal{T}_0} \mathbb{E}\left[\log X_T^{\pi^*,\vartheta}\right]$$

and

$$\inf_{\vartheta \in \mathcal{T}_1} \mathbb{E}\left[\log X_T^{\pi,\vartheta}\right] \leq \inf_{\vartheta \in \mathcal{T}_1} \mathbb{E}\left[\log X_T^{\pi^*,\vartheta}\right].$$

Proof. By Lemma 4.1, the right-hand side is independent of ϑ. Therefore, to prove the first claim, it is enough to find at least one $\vartheta \in \mathcal{T}_0$ such that

$$\mathbb{E}\left[\log X_T^{\pi,\vartheta}\right] \leq \mathbb{E}\left[\log X_T^{\pi^*,\vartheta}\right].$$

For this, set $\tau_0 = T_1$ (since we start without a crash warning), define

$$\tau_k = \inf\{t > T_k : \pi_t^1 \geq \pi_t^{1,*}, t \leq T\} \wedge T_{k+1}, \qquad \text{for all } k \in \mathbb{N},$$

and let $\kappa_k = \kappa^*$ for all $k \in \mathbb{N}_0$ and consider $\vartheta = (\tau_k, \kappa_k)_{k \in \mathbb{N}_0} \in \mathcal{T}_0$. Then

$$\begin{aligned} \mathbb{E}\left[\log X_T^{\pi,\vartheta}\right] &= \log x + \mathbb{E}\left[\sum_{k \in \mathbb{N}_0} \int_{T_k \wedge T}^{\tau_k \wedge T} g(\pi_s^1)\, ds\right] \\ &\quad + \mathbb{E}\left[\sum_{k \in \mathbb{N}_0} \left(\int_{\tau_k \wedge T}^{T_{k+1} \wedge T} g(\pi_s^0)\, ds + \log(1 - \pi_{\tau_k}^1 \kappa^*) \mathbb{1}_{\{\tau_k < T_{k+1} \wedge T\}}\right)\right]. \end{aligned}$$

By noting that $g(\pi_s^1) \leq g(\pi_s^{1,*})$ on each time interval $[T_k \wedge T, \tau_k \wedge T]$ (since on this interval we have $\pi_s^1 \leq \pi_s^{1,*} \leq \pi_s^{0,*}$), $g(\pi_s^0) \leq g(\pi_s^{0,*})$ on $[\tau_k \wedge T, T_{k+1} \wedge T]$, and $\log(1 - \pi_{\tau_k}^1 \kappa^*) \leq \log(1 - \pi_{\tau_k}^{1,*} \kappa^*)$ on $\{\tau_k < T_{k+1} \wedge T\}$, we obtain

$$\mathbb{E}\left[\log X_T^{\pi,\vartheta}\right] \leq \mathbb{E}\left[\log X_T^{\pi^*,\vartheta}\right].$$

The proof for $\vartheta \in \mathcal{T}_1$ follows similarly. □

The optimal strategies are hence given by

$$\begin{aligned} \pi_t^{0,*} &= \frac{\alpha}{\sigma^2}, \\ \frac{\partial}{\partial t} \pi_t^{1,*} &= -\frac{1}{\kappa^*}(1 - \kappa^* \pi_t^{1,*}) \left(\frac{\sigma^2}{2}\left(\pi_t^{1,*} - \pi_t^{0,*}\right)^2 + \lambda \log(1 - \kappa^* \pi_t^{1,*})\right), \qquad \pi_T^{1,*} = 0. \end{aligned}$$

5 Numerical examples

We conclude this paper with numerical examples. If not stated otherwise, we assume the following parameters throughout this section:

$$\alpha = 0.096, \qquad \sigma = 0.4, \qquad \kappa^* = 0.3.$$

With these parameters, the optimal strategy in the absence of a crash warning is given by

$$\pi^{0,*} = 0.6.$$

We choose $\lambda = n/T$ such that we receive on average n warnings during the investment period with $n = 1$, 2, or 3 and $T = 25$ or 100. Clearly, an investor in our model will attain less expected utility than an investor in the classical Merton model, see Merton [14, 15]. Hence, we estimate this trade-off by determining which fraction η of the initial wealth x a crash hedging investor would require to obtain the same expected utility at terminal time T as a Merton investor – this is also known as efficiency. With this, it is straightforward to calculate the costs for using the crash hedging strategy instead of the classical optimal portfolio of Merton.

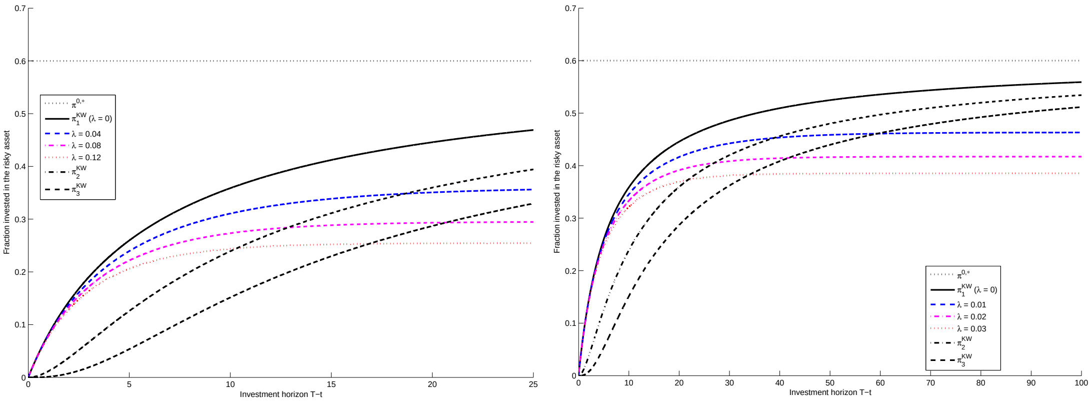

Figure 1: Optimal strategies for different λ vs. optimal strategies in Korn/Wilmott.

Figure 1 shows that the optimal strategy $\pi_t^{1,*}(\lambda)$ in the presence of a crash warning exhibits similar qualitative features in the short term (left figure) as the worst-case optimal strategy π_t^{KW} (the black solid line in Figure 1) given that at most one crash can occur in the related model by Korn and Wilmott [10], which is given as the solution of the ordinary differential equation

$$\frac{\partial}{\partial t}\pi_t^{KW} = -\frac{\sigma^2}{2\kappa^*}(1-\kappa^*\pi_t^{KW})\left(\pi_t^{KW} - \pi_t^{0,*}\right)^2, \qquad \pi_T^{KW} = 0,$$

which can formally obtained as the limiting case of our model as $\lambda \to 0$. As can be seen, the optimal strategy in our model is deterministic, increasing with increasing time to maturity, strictly positive for all $t < T$ and equal to zero if and only if $t = T$. It can also be seen that $\pi_t^{1,*}(\lambda)$ (the coloured lines in Figure 1) is more conservative than π_t^{KW} (the solid black line in Figure 1), which is due to the possibility of more than one crash in our model. Note that λ has been chosen such that 1 (dashed blue line), 2 (dash–dotted magenta line), and 3 (dotted red line) warnings are expected to happen within 25 years, respectively. These strategies should be compared with the optimal strategies of Korn/Wilmott where at most 1 (solid black line), 2 (dash–dotted black line), and 3 (dashed black line) crashes can happen. Clearly, the short term behaviour of the optimal strategies of Korn/Wilmott changes significantly the higher the maximum possible number n of potential crashes is (see the left figure of Figure 1) while the long term behaviour is the same for any n, that is, all strategies converge to the optimal Merton fraction $\pi^{0,*}$ in the classical crash free setting (see the right figure of Figure 1). Note that λ has been adjusted in the right figure to have 1, 2, and 3 expected crash warnings within an investment horizon of 100 years, respectively (instead of 25 years for the left figure). Clearly, the long term behaviour of $\pi_t^{1,*}(\lambda)$ is different and will be derived in the following subsection. Hence, more potential crashes impact the Korn/Wilmott strategies only in the short term (which can be rather long...) but not in the long term, while more potential crashes impact $\pi_t^{1,*}(\lambda)$ not so much in the short term, but more so in the long term. Randomizing the number of potential crash has therefore mainly a long term impact while the small term impact is minor if compared to the Korn/Wilmott strategy.

5.1 Asymptotic behaviour

It can be seen that $\pi_t^{1,*}(\lambda)$ is strictly smaller than the Korn/Wilmott strategy, and does not converge to $\pi^{0,*}$ for $\lambda > 0$ as the investment horizon tends to infinity (as opposed to the Korn/Wilmott strategy) (see Figure 1). This can be seen by taking a closer look at the differential equation for $\pi_t^{1,*}$ in Equation (3). At terminal time $T = 0$, we have

$$\left.\frac{\partial}{\partial t}\pi_t^{1,*}\right|_{t=T} = -\frac{\alpha^2}{2\kappa^*\sigma^2} < 0,$$

which implies that $\pi_t^{1,*}$ is increasing with increasing investment horizon until

$$0 = \frac{\sigma^2}{2}\left(\pi_t^{1,*} - \pi^{*,0}\right)^2 + \lambda \log\left(1 - \pi_t^{1,*}\kappa^*\right). \tag{6}$$

Since

$$\frac{\sigma^2}{2}\left(\pi_t^{1,*} - \pi^{*,0}\right)^2$$

is positive and decreasing in $\pi_t^{1,*}$ on $[0, \pi^{*,0})$ and equal to zero if $\pi_t^{1,*} = \pi^{*,0}$, and since

$$\lambda \log\left(1 - \pi_t^{1,*}\kappa^*\right)$$

is negative and increasing in $\pi_t^{1,*}$ on $(0, \pi^{*,0}]$, we see that $\pi_t^{1,*}$ is bounded away from $\pi^{0,*}$. This verifies what can be observed in Figure 1, that is, the long term behaviour of the optimal strategies is different from the long term behaviour of the optimal strategies in Korn/Wilmott.

5.2 Crash hedging costs

Clearly, for a fixed initial wealth x, an investor in our market model will obtain less expected utility at terminal time compared to an investor in the classical Merton model, see Merton [14, 15] if no crash occurs. In order to estimate this trade-off, we determine the efficiency $\eta(t)$, which is the fraction of the initial wealth x a worst-case investor requires at time t, to obtain the same expected utility as the Merton investor. To be more precise, we want to determine $\eta(t)$, such that

$$v_1(t, \eta(t)x) = v_M(t, x), \tag{7}$$

where v_M denotes the value function in the Merton model given by

$$v_M(t, x) = \log(x) + \frac{\alpha^2}{2\sigma^2}(T - t).$$

Plugging the optimal strategy $\pi^{1,*}$ and the no-crash scenario into $v_1(t, \eta(t)x)$ in Equation (7) and rearranging terms (using e.g. Menkens [12, p. 601]) yields

$$\eta(t) = \exp\left(\frac{\sigma^2}{2}\int_t^T \left[\pi_s^{1,*}(\lambda) - \pi^{0,*}\right]^2 ds\right).$$

Note that in the the crash model of Korn and Wilmott [10] and Korn and Menkens [8], the efficiency of the worst-case optimal strategy π_t^{KW} is given by the same formula if we replace $\pi_t^{1,*}(\lambda)$ by π_t^{KW}.

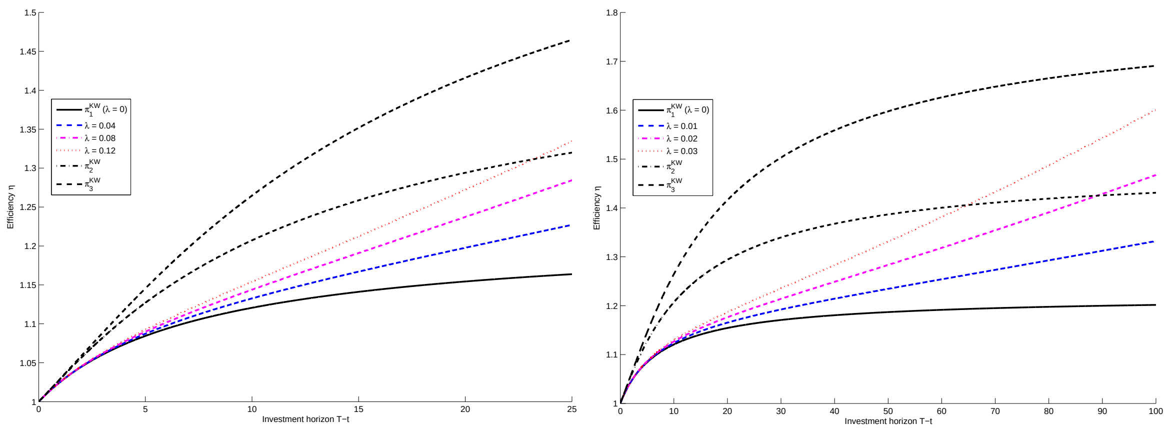

Figure 2: The efficiency $\eta(t)$ for terminal time $T = 25$ (left) and $T = 100$ (right).

On the left-hand side in Figure 2, we see the efficiency $\eta(t)$ with a maximum investment horizon of 25 years, whereas the right-hand side has a maximum investment horizon of 100 years. Clearly, the efficiency is bounded from below by 1 and is increasing in $T - t$. If the investment horizon is 25 years, the Korn/Wilmott worst-case investor with at most 1 crash requires about 16.38% of additional initial wealth if the Korn/Wilmott investor wants to get the same terminal expected utility as a Merton investor who ignores the possibility of a crash. Therefore, we will call this 16.38% the cost of worst-case scenario optimal investment (see Menkens [13]). The costs in the case of at most 2 and 3 crashes in the Korn/Wilmott setting are given by 31.99% and 46.47%, respectively. The corresponding costs for the investor in our market model are 22.71% (for $\lambda = 0.04$), 28.44% (for $\lambda = 0.08$), and 33.44% (for $\lambda = 0.12$) on an investment horizon of 25 years in order to obtain the same expected utility as a Merton investor.

On the right-hand side of Figure 2 is the corresponding figure for a maximum investment horizon of 100 years. The costs of crash hedging for an investment horizon of 100 are 20.16%, 43.11% and 69.08% for the Korn/Wilmott investor with at most 1, 2 and 3 crashes, respectively, and 33.22% (for $\lambda = 0.01$), 46.75% (for $\lambda = 0.02$), and 60.08% (for $\lambda = 0.03$) in our model.

Note that the costs of a Korn/Wilmott investor with n crashes have an upper bound, since

$$\eta_n^{KW}(t) = \prod_{i=1}^{n} \frac{1}{1 - \pi_i^{KW}(t)\kappa^*} \to \frac{1}{(1 - \pi^{0,*}\kappa^*)^n}, \qquad \text{for } T - t \to \infty.$$

This is because $\pi_i^{KW}(t) \to \pi^{0,*}$ as $T - t \to \infty$ (cf. Figure 1). The asymptotic behaviour (that is for $T - t \to \infty$) of the costs for the crash hedging strategy $\pi_t^{1,*}(\lambda)$ with $\lambda > 0$ is different: it is exponential in the investment horizon since $\pi_t^{1,*}(\lambda)$ is bounded away from $\pi^{0,*}$ uniformly in t (cf. Section 5.1). Note that this exponential growth takes a long time to be visible and is not to be mistaken by linear growth (see Figure 2).

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