On the Robustness of the Linear Quadratic Regulator via Perturbation Analysis of the Riccati Equation

Ci Chen

A dissertation submitted in partial fulfilment of

the requirements for the degree of

Doctor of Philosophy (Ph.D.)

to the

School of Electronic Engineering

Dublin City University

Supervisor: Dr. Anthony M. Holohan

July 2014

Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of the degree of Doctor of Philosophy is my own work, and that I have exercised reasonable care to ensure that the work is original, and does not to the best of my knowledge breach any law of copyright, and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work. I confirm that none of the published works contained within have previously been submitted for any other award.

Signed : ID No. : Date :

to my family

Acknowledgement

My greatest gratitude first and foremost goes to my thesis supervisor Dr. Anthony M. Holohan for his careful and beneficial guidance, timely advice and frequent encouragement throughout the entire process. His suggestions and comments, from the details to the organization, were always thoughtful, considerate and helpful. Without his help, this thesis could never have reached its present form. I would like to express my gratitude to the technical staffs at DCU, particularly Robert Clare and Conor Maguire for their assistance all the time. Thanks to Dr. Xiaojun Wang, who helped me a lot in the past five years. The internal examiner, Dr. Stephen Daniels, gave me great advice and a welcome diversity of perspectives. Dr. Paul Curran, graciously agreed to be my external examiner, and his detailed feedback in the final stages of my degree was invaluable.

My special thanks also go to my wife He Shen. She helped and encouraged me to undertake the writing of this thesis. Without her invaluable help and generous encouragement, the present thesis would not have been accomplished.

Besides, I gratefully acknowledge the School of Electronic Engineering for funding this project.

I should finally like to express my gratitude to my beloved parents and my grandmother who have always been helping me out of difficulties and supporting without a word of complaint.

List of publications

- C. Chen and A. M. Holohan, “A revisit of the stability robustness of linear quadratic regulators,” *25th IET Irish Signals and Systems Conference*, University of Limerick, 26-27 June 2014.

- C. Chen and A. M. Holohan, “Robustness results of LQR-based systems with structured uncertainty,” *25th IET Irish Signals and Systems Conference*, University of Limerick, 26-27 June 2014.

Abstract

The linear quadratic regulator (LQR) has been shown to have very attractive stability robustness properties. However, some authors have shown that LQR may suffer from poor robustness when special perturbations in its state-space formulation were introduced. This thesis continues the study of the stability robustness of LQ regulators. To acquire good stability robustness, weight selection is first investigated. For general cost weighting matrices, a new lower bound on the minimum singular value of the return difference is proposed. New guaranteed stability margins are also presented. This gives a formal mathematical basis for guidelines for the designer to improve stability robustness. As the weight on the plant's inputs approaches zero, the exact bound on the perturbations which ensures stability is compared with the guaranteed margins. It is shown that the stability robustness properties are preserved in a general sense. Then, a numerical analysis of the conditioning of the continuous-time algebraic Riccati equation (CARE) is presented. The condition numbers of the CARE are utilized to measure the sensitivity of LQR subject to parameter changes. It is shown that the condition numbers grow significantly when the weighting parameters on the system state and input matrices approach zero and infinity. This has application to the applied control situation because it can be used to detect hidden vulnerabilities in LQR systems.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Motivation of the thesis

In the context of the control problem, the basic idea is not only to stabilize a plant but also to involve the achievement of some desired performance specifications, such as bandwidth, disturbance rejection, noise reduction, reference tracking and so on. For those purposes, modern control design methods have been extensively used to acquire a great deal of fundamental and also empirically based knowledge of the systems. Linear optimal control is one approach which often gives the designer satisfactory results with respect to the stability and the performance of the controlled systems. One advantage of it is that the mathematical optimization methods are adopted so that a control law for a linear system can be readily derived based on a prescribed objective function. The resulting computational procedures may then often be applied to nonlinear systems. Moreover, if the plant states are all available, good robustness properties of the optimal regulators used can be clearly revealed in terms of the stability margins. A famous example is full state feedback design with linear quadratic regulator (LQR) design. As the state space representation is a natural way for system description by LQR the system performance can be managed and the plant inputs and the control input can be synthesized using an optimal control law by solving the Riccati equation. Meanwhile, the guaranteed stability robustness is automatically provided by LQR, unlike pole placement techniques, for instance. A comprehensive study was first reported by Kalman in 1960 [Kalman, 1960]. The stability and optimality conditions were analyzed in a mathematical way and his treatment was in the frequency domain in terms of the return difference. The last result nicely built a connection between classical control theory and modern control theory, which gave the designer some perspectives on the robust stability as in the classical point of view. The return difference plays the key role in the analysis of feedback properties. Moreover, another interesting analysis is the asymptotic behavior of the LQR as the weights approach the extreme cases. This further indicates that weight selection techniques can be used to improve the performance in the cheap control regulators on various systems. One approach involves minimizing the integral square error, instead of the system states, under the extreme condition of the input weights going to zero. The ideal characterizations provide us some valuable insights into whether the maximum accuracy can be achieved [Kwakernaak *et al.*, 1972]. The perfect regulation problem was also investigated in the context of the cheap regulator problem and the cheap servomechanism problem even for systems with non-minimum behaviours [Qiu *et al.*, 1993]. As is known, the right half plane (RHP) zeros of the open loop system always exert some limitations on the overall performance by the analysis of the sensitivity function and the complementary sensitivity function [Freudenberg *et al.*, 1985]. However, the limitations on the cheap regulators can be directly characterized by the complex plane plots of the RHP zeros. Anderson and Moore [Anderson *et al.*, 1970] have shown that LQR can have attractive stability margins, *i.e.* infinite gain margin, phase margins of $\pm 60^\circ$ and gain margin of 0.5 for single input single output (SISO) plants, based on the return difference equality [Kalman, 1964]. Such robustness results were then generalized to the multivariable case in a classic paper [Safonov *et al.*, 1977], which quantitatively characterized a great many tolerable perturbations, even nonlinear ones. Similar results were obtained in [Lehtomaki *et al.*, 1981], but their analysis was based on the multivariable Nyquist theorem, and so was quite different. In this way, the perturbations were not required to be stable, which was assumed in [Safonov *et al.*, 1977]. In general, both works showed that LQR has excellent stability robustness properties. To treat both the stability and the performance issues of LQR and linear quadratic Gaussian (LQG) in the multivariable feedback design, in [Doyle *et al.*, 1981] frequency domain methods were used to graphically interpret the limiting conditions of the stability and performance as well as the magnitude bound of the unstructured uncertainty in the complex plane of the loop transfer function. The singular values of the return difference matrix were used to quantitatively characterize the behaviour of LQR controllers. It is worth noting that the famous guaranteed stability margins were shown to be inadequate for tolerating some realistic perturbations. In other words, the stability robustness was also constrained by the asymptotic high frequency behaviour of LQR with respect to the crossover frequency of the loop. As a result, all the works above have shown that LQR possesses excellent stability robustness and ideal asymptotic responses, especially in non-minimum phase systems. However, a dissenting voice came from [Soroka *et al.*, 1984]. They argued that the optimal LQR may suffer from poor stability robustness when, even, small changes occur particularly in the input matrix. Their examples shows that the expectation of stability properties of LQR may be destroyed. Suggestions had also been made later by Grimble and Owens [Grimble *et al.*, 1986] in order to improve its robustness. It was noticed that the examples used by both of them were situated in the extreme circumstances when the input weight approached zero and regulation error was to be minimized, which was the "cheap" control problem with high feedback gains. In spite of this, it would be a disaster to have unstable modes in the slightly perturbed system. Thus, it leads us to think about a first question: whether this particular type of perturbation considered above is included in range of tolerable ones generalized in [Safonov *et al.*, 1977]? Recently, in [Zhang *et al.*, 1996] the well-known robustness properties of LQR were again questioned when the variations of the open-loop gain were considered. In this case, simultaneous parameter changes in the system matrix and the input matrix were treated. The deviations of the system states were to be minimized and a tuning parameter was included in the state weighting matrix. It was demonstrated that as the tuning parameter went to infinity, even a small perturbation may result in the instability of the closed-loop system. Hence, we further need to ask: when perturbations happen in the state space form of the plant, how would the perturbations affect the closed-loop transfer function? Will those perturbations be out of the wide range of those characterized by Safonov *et al.* [Safonov *et al.*, 1977]. Is there another way out to directly measure the sensitivity of LQR in the face of uncertainties in the plant? Instead of analyzing the perturbations in the feedback control framework, we turn to the basic and important optimization process of all LQR problems. We attempt to analyze the optimal and stable solutions of the continuous-time algebraic Riccati equation (CARE) which are used to synthesize the optimal LQR gain. One advantage in the perturbation analysis of the CARE lies in the fact that there is a good deal of work done on the theory and numerical solution of the CARE. Secondly, the primary purpose of perturbation analysis for the CARE is to find out how the Hermitian positive semi-definite (p.s.d) solutions of the CARE will behave when the coefficient matrices related to it suffer from perturbations. Therefore, it is natural to attempt to solve the sensitivity issue of LQR, due to perturbations occurring in the system matrices, in terms of numerical methods. In addition, it is, sometimes, not easy to examine the open loop structure which may be sensitive to the perturbations [Grimble *et al.*, 1986]. However, the analytic condition numbers of the CARE provides us with useful insights. Also, notable work was done by Sun [Sun, 1998]. He established a necessary condition for the perturbed CARE solution to be a stabilizing one using an approximation to first order. More interestingly, explicit expressions for the absolute condition numbers of the CARE were derived, as well as relative ones. Those values offer a quantitative measurement of the sensitivity of the CARE solution to the coefficient matrices subject to various errors. Furthermore, the sensitivity of the CARE solution with respect to each coefficient matrix can be clearly revealed separately by those numbers. So the next question to ask is: what behaviors will the CARE solutions of the two special exampless mentioned be like in terms of their corresponding condition numbers?

1.2 Research questions and hypotheses

In linear feedback control theory, a negative voice for using state-variable method came from Horowitz *et al.* [Horowitz *et al.*, 1975]. They questioned the feasibility of LQR in engineering designs. Khalil [Khalil, 1981; 1984] further showed that small modelling errors may destabilize the systems in output feedback control designs. When both plant and controller are subject to perturbations, Cobb [Cobb, 1987; 1988] argued that robust compensation can hardly be achieved. Meanwhile, Soroka *et al.* [Soroka *et al.*, 1984] argued that linear quadratic optimal regulators, especially for cheap control problems, may suffer from poor robustness when the open-loop plant is subject to small parameter variations. Improvements had been made by Grimble *et al.* [Grimble *et al.*, 1986] so as to mitigate the disastrous effects of these perturbations. Again, Zhang *et al.* [Zhang *et al.*, 1996] cautioned that the guaranteed stability margins by LQR with high weight on the state should be treated carefully when variations in the open-loop gain are considered.

Another key idea in the analysis of the robustness of LQR is to study the perturbation of the algebraic Riccati equation, which leads to the optimal control gain. In spite of the classical stability robustness of LQR against unstructured perturbations, it is requred to investigate the robustness properties of LQR when both the input and state weighting matrices are considered. It is believed that based on the return difference equality, the singular value method can be utilized for estimating the singular values of the return difference. In this way, guaranteed stability margins may be established. For structured perturbations, particularly in the application of cheap control, the distinction between unstructured and structured perturbations should be emphasized. The general robustness property, presented by Safonov *et al.* [Safonov *et al.*, 1977], is expected to cover the situation of parameter variations in cheap control. From a mathematical point of view, it is straightforward to investigate the perturbation of the algebraic Riccati equation. As the condition number provides an efficient tool for measuring the sensitivity of the solution of ARE, it is expected that it may be used to indicate the conditioning of the cheap control problem.

1.3 Contributions of the thesis

This thesis contributes to the following aspects in the linear quadratic optimal control area:

1. Proposal and evaluation of the guaranteed stability margins of LQR with re-

spect to general weighting matrices

2. Illustrating of the distinction between structured and unstructured perturbations in cheap control problems and evaluation of the condition numbers for detecting the system's vulnerability

1.4 Structure of the thesis

This thesis is organised as follows:

Chapter 2 presents a brief history of automatic control. Some history of the LQR theory is also presented.

Chapter 3 first gives a review of basic LQR theory. For analyzing the robustness properties of LQR, the return difference equality is developed. It nicely connects the weighting matrices and the system matrices. The eigenvalue and the singular value properties of the return difference are presented.

Chapter 4 first gives a review of stability margin analysis. The optimality and the stability of linear optimal control system are presented based on the Nyquist plot of the return difference. Also, the robustness problem of LQR is formulated. Following this, the general robustness results of LQR are presented. Eventually, the excellent stability margins of LQR are established using a special selection of the input weighting matrix.

Chapter 5 investigates the singular value properties of the return difference in relation to a more general class of weighting matrices. A new lower bound on the minimum singular value of the return difference is evaluated based on the return difference equality. This leads to the guaranteed stability margins of LQR for a wider class of weights. Simultaneously, the effect of tuning the state weighting matrix for improving the robustness is discussed. At last, some numerical examples are presented to demonstrate the effectiveness of our results.

Chapter 6 first gives a brief review of the cheap control problem. After that, the examples by Soroka *et al.* [Soroka *et al.*, 1984] and Zhang [Zhang *et al.*, 1984] are presented. The distinction between structured and unstructured perturbations is emphasized by comparing the perturbation bounds. Next, some history of the algebraic Riccati equation, as well as a concept of the condition number, is presented. The basic perturbation theory of the continuous-time algebraic Ricaati equation is also presented. The condition numbers of two examples are evaluated. By examining the condition numbers of some real control problem, condition numbers are proved to be useful for the detection of the vulnerability of LQR system.

Chapter 7 contains our conclusions. In this chapter, a summary, as well as the contributions of this thesis, are presented. Some suggestions for future work are then proposed.

Chapter 2

Background

In this chapter, some history of automatic control and the LQR theory is reviewed.

2.1 History of automatic control

In this section, a brief history of automatic control is presented.

The usage of automatic control dates back to ancient times from 300BC to 1200AD. In this period, the Greek and Arab [Mayr, 1970] civilizations created and developed float regulators for a water tank to track time accurately, which initialized the progess of feedback control. The French physicist Ren Antoine Ferchault de Raumur [Egerton, 2006], who established the famous Raumur scale, introduced the system for temperature control purposes in the 18th century. It is worth mentioning the Watt governor [Marsden, 2002], which is regarded as one of the greatest invention in the control area during that period. Many governors were built based on it at that time. In order to analyze the stability condition of the governor mechanism, mathematical control theory was developed. Several researchers, such as a British astronomer, Airy, [Airy, 1840] introduced the well-known differential equations to characterize the dynamic behaviour of the governor. Later, Maxwell [Maxwell, 1868] solved the governor problem by investigating the stability of the system via the roots of the characteristic equation. Earlier than this time, Hermite [1854] provided an effective procedure for determing whether a polynomial had certain properties for the stability of the a system. However, his work was not widely published. The Russian scientist Vyschnegradsky [Vyschnegradsky, 1876;1877] used a linear differential equation to investigate the behaviour of an automatic feedback system. Thus, the stability of the system could be determined by the roots of that equation. Both of three mentioned above identified how the stability of a system was determined by a polynomial having certain properties and came up with effective methods for determing whether a given polynomial actually possessed such properties. Later, Routh [Routh, 1877] proposed a solution on this problem and Hurwitz [Hurwitz, 1895] proposed a necessary and sufficient condition for all roots of an equation having negative real parts. A Russian mathematician Lyapunov [Lyapunov, 1893] developed a stability theory which has been extensively studied into modern times.

Proceeding to the middle 1900's, classical control began with the work by researchers at Bell Telephone Laboratories, who developed the frequency-domain methods for stability analysis as opposed to the time domain. In 1932, Nyquist [Nyquist, 1932] derived the famous Nyquist stability criterion, which is a graphic tool for analyzing the stability of control systems. In the 1940's, Bode [Bode, 1945] investigated the plots of the frequency response of the transfer function in the complex plane, known as Bode plots, and also introduced the classic notions of gain and phase margins. Another great contribution to classical control came from work done by the Radiation Laboratory at the Massachusetts Institute of Technology (M.I.T.). There, the frequency-domain approaches were proven to be successful when applied to practical design problems by Hall [Hall, 1946] in 1946. Also, the Nichols Chart was developed by Nichols in 1947.

With the developemnt of optimal control in the 1960's, going back to the analysis in the time-domain, the era of modern control began, which overcame some of the limitations of classical control. Among the outstanding researchers in this period were Pontryagin and Bellman. A notable work by Kalman and co-workers [Kalman, 1960; Kalman *et al.*, 1961] is regarded as one of the cornerstones of modern control theory. Their great contributions lay in the development of Lyapunov theory in the time-domain approach for both time-varing and nonlinear systems, the establishment of linear quadratic regulator theory and the proposal of the Kalman filter. A great attempt in combining modern and classic control was made in the 1970's. In this decade, two of the major contributions were from Rosenbrock [Rosenbrock, 1974] and Postlethwaite *et al.* [Postlethwaite *et al.*, 1977]. It is worth mentioning the work by Horowitz, who developed quantitative feedback theory (QFT). Papers published by Doyle *et al.* [Doyle *et al.*, 1981] and Safonov *et al.* [Safonov *et al.*, 1981] advanced robust modern control design. Based on the work of MacFarlane *et al.* [MacFarlane *et al.*, 1977], the singular value theory was incorporated into the robust design of multivariable systems.

2.2 Some history of the LQR theory

As mentioned in the last section, feedback loops have been used since ancient times [Bissell, 2009]. One of the earliest "modern" applications was Watt's governor for controlling the rotational velocity of steam engines. One natural approach to control theory is to appeal to mathematical optimisation. An early application of optimisation to feedback systems was from Hall [Hall, 1943]. Hall treated the linear servomechanism problem. He developed the theory of the servomechanism compensator based on the criterion of minimizing the integral squared error. Newton *et al.* [Newton *et al.*, 1957] further extended basic servomechanism theory to a broad category including nonlinear stochastic systems. The least squares control problem was mathematically formulated by Kalman [Kalman, 1960] in a rigorous way. His work formed the basic theory of the linear quadratic regulator (LQR). Later, Kalman studied the inverse problem of optimal control theory. The primary contribution in [Kalman, 1964] is the return difference equality (RDE), for single-input systems. The RDE is a rigorous frequency-domain characterization of optimality. The return difference inequality (RDI) in the multi-input case was studied by Anderson [Anderson, 1966]. To analyse the closed-loop sensitivity, a similar inequality relating the generalized return difference was derived by Kreindler [Kreindler, 1968]. In the light of ther RDE, MacFarlane [MacFarlane, 1970] used the eigenvalue properties of the return difference matrix to extend Kalman's optimality criterion [Kalman, 1964] to multivariable cases. Meanwhile, nonlinear optimal control problems were extensively studied in that time. The comprehensive study of linear optimal control began with the work by Anderson *et al.* [Anderson *et al.*, 1971]. They showed that LQR not only can have good stability margins, but also can have good tolerance to nonlinearities as shown in [Anderson, 1969]. Later, such robustness results of LQR including both gain and phase margins were generalized to the multivariable case by Safonov *et al.* [Safonov *et al.*, 1977], while authors in [Anderson, 1969; Wong, 1975; Wong *et al.*, 1977; Barnett *et al.*, 1966] only obtained the result relating to the gain margin. In [Safonov *et al.*, 1977], good tolerance to nonlinear perturbations was proved and a wide range of dynamic perturbations which can be tolerated by LQR was characterized. Unlike in [Safonov *et al.*, 1977], Lehtomaki *et al.* [Lehtomaki *et al.*, 1981] studied the stability robustness of LQR by examining the singular value properties of the return difference transfer function matrix. They also argued that to acquire good stability margins, the weight on the inputs should be chosen to be a scalar times the identity matrix, rather than any diagonal matrix or any matrix. Similar robustness results relating to control weighting was reported later by Anderson *et al.* [Anderson *et al.*, 1989] and Maciejowski [Maciejowski, 1989]. Stability margins for the discrete-time LQR were also comprehensively studied by Shaked [Shaked, 1986]. Recently, similar to the work by Safonov *et al.* [Safonov *et al.*, 1992], Arvanitis *et al.* [Arvanitis *et al.*, 2001] established new guaranteed stability margins for the discrete-time case.

Chapter 3

Linear quadratic regulator

Linear quadratic regulator has been proven to be successful in solving full state feedback problem since 1960. In this chapter, a brief historical study of LQR is first presented. Then the basics of LQR theory is presented. The property of the return difference of LQR is extensively studied at last.

3.1 Review of basic LQR theory

Consider a finite-dimensional, linear, time-invariant (FDLTI) system given in state space form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3.1}$$

$$y(t) = Cx(t) \tag{3.2}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$,$u(t) \in \mathbb{R}^m$. Here, $x(t)$, $u(t)$ and $y(t)$ denote the system state, the controlled input and the output,

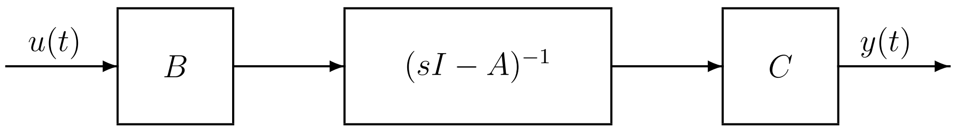

Figure 3.1: The open loop LQR system

respectively. LQR theory optimizes the performance index $J(x, u)$ given by

$$
\begin{aligned}
J(x,u) &= \int_0^\infty \left[x(t)^T Q x(t) + u(t)^T R u(t)\right] dt \\
& Q = Q^T \geq 0, \quad R = R^T > 0
\end{aligned}
\tag{3.3}
$$

The positive definite cost weighting matrix R is exerted on the control input so as to keep the control efforts be bounded over a specified period of time. At the same time, it is required to let $\int_0^\infty [x(t)^T Q x(t)]$ to be small, where the cost weighting matrix Q is symmetric positive semidefinite. It may be shown that [Kalman, 1960], the optimal control law is given by

$$u(t) = -Kx(t) \tag{3.4}$$

where $K = R^{-1}B^T P$ and $P = P^T > 0$ is the solution of the Riccati equation:

$$PA + A^T P + Q - PBR^{-1}B^T P = 0 \tag{3.5}$$

Assume the pair $[A, B]$ is stabilizable and $[A, Q^{\frac{1}{2}}]$ is observable. These conditions are enough to guarantee that there is a unique positive-definite solution P to the Riccati equation [Anderson *et al.*, 1989] and that the closed loop system is stable.

The plant without a feedback loop is shown in Fig. 3.1. The transfer function

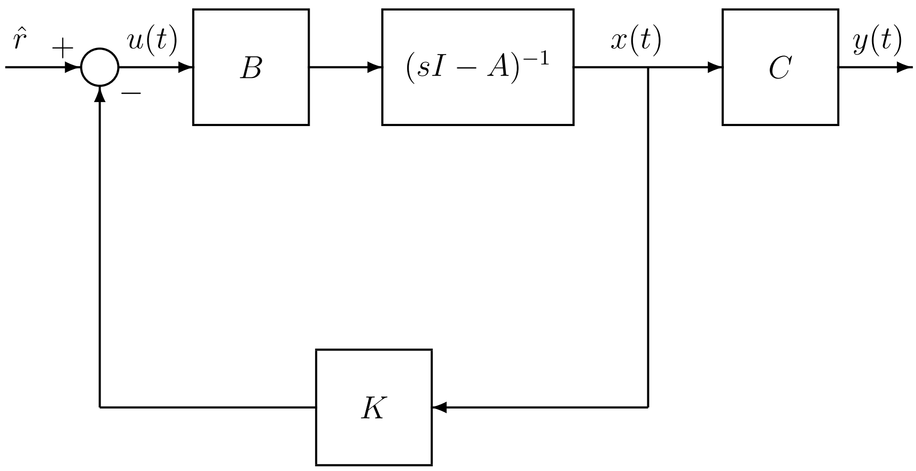

Figure 3.2: The closed loop LQR system

of the plant $V(s)$ is given by

$$V(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B \tag{3.6}$$

where $U(s)$ and $Y(s)$ are the Laplace transforms of the control input $u(t)$ and the output $y(t)$.

The plant with the optimal gain K is shown in Fig. 3.2. So the open loop transfer matrix $T_o(s)$ is given by

$$T_o(s) = K\Phi(s)B \tag{3.7}$$

where $\Phi(s) = (sI - A)^{-1}$. The closed loop transfer matrix $T_c(s)$ is given by

$$T_c(s) = K\Phi_K(s)B \tag{3.8}$$

where $\Phi_K(s) = (sI - A + BK)^{-1}$.

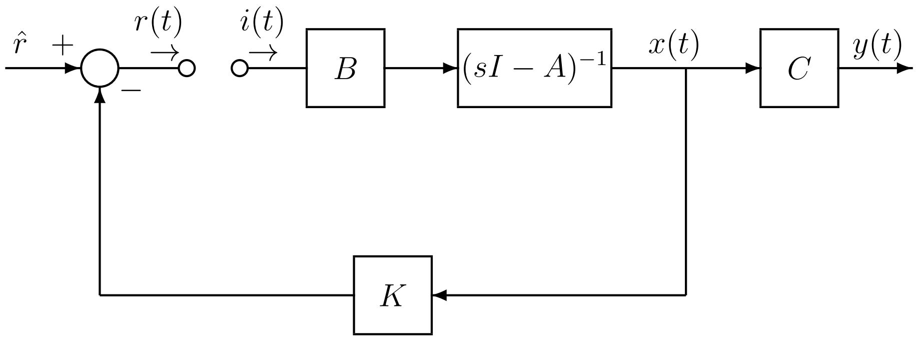

Figure 3.3: Breaking the closed loop LQR system in the input's side

3.2 The return difference

In this section, we shall derive the return difference transfer matrix at the plant's input.

Consider breaking a point at the input side of the plant shown in Fig. 3.3. Denote the injected signal as $I(s)$ and the response as $R(s)$ at that breakpoint. The return ratio $Z(s)$ is defined as the transfer function from the input $I(s)$ to the ouput signal $R(s)$, which is given by

$$Z(s) = \frac{R(s)}{I(s)} = -K(sI - A)^{-1}B \tag{3.9}$$

The return difference $F(s)$ is defined as the transfer function from the input $I(s)$ to the error signal $I(s) - R(s)$, which is given by

$$F(s) = \frac{I(s) - R(s)}{I(s)} = I + K(sI - A)^{-1}B \tag{3.10}$$

The closed loop and the open loop transfer function matrices are correlated by the return difference matrix as follows

$$T_c(s) = F^{-1}(s)T_o(s) = T_o(s)F^{-1}(s) \tag{3.11}$$

It has been shown that [Chen, 1968; Rosenbrock, 1969]

$$\det[F(s)] = \frac{\Psi_c(s)}{\Psi_o(s)} \tag{3.12}$$

where $\Psi_c(s) = \det(sI - A + BK)$ is the characteristic polynomials of the closed loop system and $\Psi_o(s) = \det(sI - A)$ is the characteristic polynomials of open loop system. $\det[F(s)]$ denotes the determinant of the return difference transfer function matrix.

3.3 The return difference equality

The return difference equality, derived from the algebraic Riccati equation, was first introduced by Kalman to represent the frequency domain characterization of optimality for single input LQR systems. It was later generalized to the multivriable case by MacFarlane [MacFarlane, 1970]. It has been widely used for the quantitative analysis of the stability and the performance of such systems. In this section, we shall present the derivation of the return difference equality.

Consider the algebraic Riccati equation by eqn. (3.5):

$$PA + A^TP + Q - PBR^{-1}B^TP = 0$$

where $P \geq 0$, $R > 0$ and $Q \geq 0$. Let the super scripts H and T denote the conjugate transpose of complex matrices and the transpose of real matrices, respectively. As

$(sI - A)^H = s^* I - A^T$, we have

$$P(sI - A) + (sI - A)^H P = 2\mathrm{Re}(s)P - PA - A^T P \tag{3.13}$$

By the fact that $PBR^{-1}B^T P = K^T RK$ and $PA + A^T P = PBR^{-1}B^T P - Q$ where $K = R^{-1}B^T P$, eqn. (3.13) equals to

$$P(sI - A) + (sI - A)^H P = 2\mathrm{Re}(s)P + Q - K^T RK \tag{3.14}$$

For all values of s for which $(sI - A)$ is invertible, multiply eqn. (3.14) on the left by $B^T(sI - A)^{-H}$ and on the right by $(sI - A)^{-1}B$

$$\begin{aligned}
\Rightarrow \quad & B^T(sI - A)^{-H}PB + B^T P(sI - A)^{-1}B \\
&= B^T(sI - A)^{-H}[2\mathrm{Re}(s)P + Q](sI - A)^{-1}B \\
&\quad -B^T(sI - A)^{-H}K^T RK(sI - A)^{-1}B
\end{aligned}$$

As $B^T P = RK$ and $PB = K^T R$,

$$\begin{aligned}
\Rightarrow \quad & B^T(sI - A)^{-H}K^T R + RK(sI - A)^{-1}B \\
&+B^T(sI - A)^{-H}K^T RK(sI - A)^{-1}B = \\
&B^T(sI - A)^{-H}[2\mathrm{Re}(s)P + Q](sI - A)^{-1}B
\end{aligned}$$

Recalling that $T_o(s) = K(sI - A)^{-1}B$ and $G(s) = (sI - A)^{-1}B$, we have

$$\begin{aligned}
\Rightarrow \quad & T_o^H(s)R + RT_o(s) + T_o^H(s)RT_o(s) \\
&= G^H(s)[2\mathrm{Re}(s)P + Q]G(s)
\end{aligned}$$

$$\Rightarrow \quad (I+T_o(s))^H R(I+T_o(s)) - R$$
$$= G^H(s)[2\mathrm{Re}(s)P+Q]G(s)$$

$$\begin{aligned} \Rightarrow \quad & (I+T_o(s))^H R\,(I+T_o(s)) \\ &= R + G^H(s)\,[2\mathrm{Re}(s)P+Q]\,G(s) \end{aligned} \tag{3.15}$$

When $s \in j\mathbb{R}$, eqn. (3.15) gives

$$F(s)^H R F(s) = R + G^H(s) Q G(s) \tag{3.16}$$

The above equation is called the multivariable return difference equality. It is noted that eqn. (3.15) is valid only for those values of s when $(sI-A)$ is invertible. When $s \in j\mathbb{R}$ and $(sI-A)$ is invertible, eqn. (3.16) is the return difference equality. If not specified, we assume that all s satisfying the condition that $(sI-A)$ is invertible throughout the dissertation.

For single input system, the input weight R is a scale parameter. It is convenient to express the input matrix B and the optimal gain K as vectors b and k, respectively. Assume $Q = qq'$, eqn. (3.16) becomes

$$r|1 + k'(sI-A)^{-1}b|^2 - r = |q'(sI-A)^{-1}b|^2 \tag{3.17}$$

$$\Rightarrow |1+t_o(s)|^2 = 1 + \frac{1}{r}|h(s)|^2 \tag{3.18}$$

where $t_o(s) = k'(sI-A)^{-1}b$ and $h(s) = q'(sI-A)^{-1}b$. This equation is often referred to as the scalar return difference equality.

3.3.1 The return difference inequality

The so-called return difference inequality, implied by the return difference equality, can be applied for analyzing the stability robustness and optimality of LQR systems. In the following, we shall present the return difference inequality in both single-input and multi-input cases.

For multivariable systems, the multivariable return difference equality eqn. (3.16) explicitly implies

$$F^H(s)RF(s) \geq R, \quad \forall s \in j\mathbb{R} \tag{3.19}$$

where $F(s) = I + K\,(sI - A)^{-1}\,B$. It is noted that the above matrix inequality is very useful for analyzing the robustness propeties of multivariable optimal systems. For single loop systems, the scalar return difference equality eqn. (3.18) gives

$$|f(s)| \geq 1, \quad \forall s \in j\mathbb{R} \tag{3.20}$$

where $f(s) = 1 + k'(sI - A)^{-1}b$. It is noted that eqn. (3.18) is known as Kalman's return difference inequality and it plays an important role in the inverse optimal control problem.

3.3.2 The eigenvalue property of the return difference

MacFarlane [MacFarlane, 1970] had established some neccesary conditions for optimality for linear optimal systems based on the eigenvalue property of the return difference matrix. We shall give a brief review of his main result here.

Recalling the multivariable return difference equality

$$F(s)^H RF(s) = R + G^H(s)QG(s), \quad \forall s \in j\mathbb{R} \tag{3.21}$$

Let $\gamma(s)$ be an eigenvector of $F(s)$ and $\lambda(s)$ be the corresponding eigenvalue, so that

$$
\begin{aligned}
F(s)\gamma(s) &= \lambda(s)\gamma(s) \\
\Rightarrow \gamma^H(s)F^H(s) &= \lambda(-s)\gamma^H(s)
\end{aligned}
\tag{3.22}
$$

Pre-multiply and post-multiply eqn. (3.21) by $\gamma^H(s)$ and $\gamma(s)$, respectively, we have

$$
\lambda(-s)\gamma^H(s)R\lambda(s)\gamma(s) = \gamma^H(s)R\gamma(s) + \gamma^H(s)G^H(s)QG(s)\gamma(s) \tag{3.23}
$$

As $Q \geq 0$, eqn. (3.23) implies that

$$
|\lambda(s)|^2\gamma^H(s)R\gamma(s) \geq \gamma^H(s)R\gamma(s) \tag{3.24}
$$

$$
\Rightarrow \left(|\lambda(s)|^2 - 1\right)\gamma^H(s)R\gamma(s) \geq 0 \tag{3.25}
$$

As $R > 0$ and consequently $\gamma^H(s)R\gamma(s) > 0$, to satisfy eqn. (3.25) it is necessary that

$$
|\lambda(s)| \geq 1, \quad \forall s \in j\mathbb{R} \tag{3.26}
$$

The above inequality implies that for any $Q \geq 0$ and $R > 0$, all the eigenvalues of the return difference transfer function matrix will be always greater than or equal to 1 for the whole frequency range.

3.3.3 The singular value property of the return difference

Compared to the result of the eigenvalue property of the return difference studied in the last section, the singular value property of the return difference can not be definitively treated. Instead, if the control weight matrix R is chosen as a scalar times the identity matrix, *i.e.* $R = \rho I$, the multivariable return difference inequality eqn. (3.19) becomes

$$F^H(s)F(s) \geq I \tag{3.27}$$

Let $\sigma(F)$ be any singular value of $F(s)$ and v and z are the corresponding left and right singular vectors, so we have

$$\begin{aligned} F \cdot v &= \sigma(F) \cdot z \\ v^* \cdot F^H &= \sigma(F) \cdot z^* \end{aligned} \tag{3.28}$$

Therefore, eqn. (3.27) implies that

$$\sigma[F(s)] \geq 1, \quad \forall s \in j\mathbb{R} \tag{3.29}$$

From eqn. (3.29), it is obvious that the minimum singular value $\underline{\sigma}[F(s)]$ is greater than or equal to 1. Next, we consider the situation for a general control matrix R. Using $R = R^{\frac{1}{2}} \cdot R^{\frac{1}{2}}$ and pre- and post-multiply eqn. (3.19) by $R^{-\frac{1}{2}}$, we have

$$R^{-\frac{1}{2}} F^H(s) R^{\frac{1}{2}} \cdot R^{\frac{1}{2}} F(s) R^{-\frac{1}{2}} \geq I \tag{3.30}$$

$$\Rightarrow \left[R^{\frac{1}{2}} F(s) R^{-\frac{1}{2}}\right]^H \cdot \left[R^{\frac{1}{2}} F(s)^{-\frac{1}{2}}\right] \geq I \tag{3.31}$$

$$\underline{\sigma}\left[R^{\frac{1}{2}} F(s) R^{-\frac{1}{2}}\right] \geq 1, \quad \forall s \in j\mathbb{R} \tag{3.32}$$

From eqn. (3.32), it is shown that the singular value property of the return difference is explicitly not revealed for the case of general R, meaning that we cannot conclude that $\underline{\sigma}[F(s)] \geq 1$.

3.4 Conclusion

In this chapter, it is shown that LQR theory has been extensively studied for a long time. With the introduction and the development of the return difference equality, the eigenvalue and singular value properties of the return difference are shown to be highly related to the selection of the cost weighting matrices.

Chapter 4

Stability robustness of LQR

In this chapter, stability margin analysis is first briefly reviewed. Based on the Nyquist stability criterion, the optimality, as well as the stability, characterization of LQR is graphically interpreted in terms of the frequency loci of the return difference. A general robustness result is revisited in a formal mathematical way. Finally, the classic stability margins, as well as the tolerance of the crossfeed perturbations, by LQR are presented when a special selection of the input weighting matrix is considered.

4.1 Brief review of stability margin analysis

In Section 2.1, it was shown that robust control theory has developed rapidly in the late 1970's and early 1980's. As a classical tool, stability margin analysis quantifies a control system's tolerance to model uncertainty. As compared with the classical stability margins of robustness design methods, such as the root locus and Nyquist-Bode theories, the multivariable stability margin (MSM) provides a more refined mathematical formulation for the analysis of robustness in feedback control systems, especially for multivariable systems. Stability margin theory has been widely studied since 1975 [Safonov, 2012]. However, the history of stability robustness dates back to 1945 [Bode, 1945].

The concepts of gain and phase margins were used to quantify the robustness of single-input single-ouput (SISO) feedback systems. Horowitz [Horowitz, 1963] later clarified the robust control problem in a mathematical way.

Until 1975, the focus on optimal control had been shifted to robust control due to the influential work by Wong and Athans [Wong *et al.*, 1975] (especially in North America).

In [Wong *et al.*, 1975], it was proven that multiloop linear quadratic regulators (LQR) can tolerate a 50% gain margin (decrease) and an infinite gain margin (increase) in each feedback loop against real uncertainties. Safonov *et al.* [Safonov *et al.*, 1976;1977] further showed a general robustness result which covered frequency-dependent complex uncertainties. They also showed that multivariable LQR can tolerate a phase margin of $\pm 60°$. Another major advance was the use of the singular value decomposition (SVD), a method which was introduced at the 1978 Allerton Conference.

Doyle [Doyle, 1979] used the singular value property of the return difference transfer function matrix to interpret the classical control notions for robustness analysis. A notion of principal gains, similar to singular values, was later presented by MacFarlane [MacFarlane, 1979]. Diagonal scaling was extensively investigated to reduce the conservativeness of singular value robustness by various authors. A great deal of research on the singular value method [Safonov *et al.*, 1981; Doyle *et al.*, 1981; Cruz *et al.*, 1981; Lehtomaki *et al.*, 1981] was published in 1981, particularly for LQG optimal control problems.

Meanwhile, to deal with structured uncertainty, the structured singular value μ was introduced by Doyle [Doyle, 1982] (Actually, it is the reciprocal of the MSM, which was introduced earlier by Safonov). Another successful application of the singular value decomposition was to investigate the guaranteed stability margins of discrete-time LQR systems proposed by Shaked [Shaked, 1986]. Similar stability margin results were reported by Hans *et al.* [Hans *et al.*, 1988] and Bourles *et al.* [Bourles *et al.*, 1994]. It is worth mentioning that Safonov *et al.* [Safonov *et al.*, 1992] gave a new insight into the singualr value property of LQR. Arvanitis *et al.* [Arvanitis *et al.*, 1997] studied the stability margins of discrete LQR with cross-product terms in the performance index. They established a relationship between weight selection and the guaranteed stability margins based on the return difference equality. Arvanitis *et al.* [Arvanitis *et al.*, 2001] further established some new robustness bounds using the singular value properties of the return difference transfer function matrix. Consequently, stability margin results were determined in relation to the selection of weights under less restrictive assumptions.

4.2 Nyquist plot for optimality

Before proceeding to the quantative analysis of the robustness of LQR, we shall give some graphical interpretations, for example Nyquist plots, of the eigenvalue and the singular value properties of the return difference based on the Nyquist stability criterion.

4.2.1 Kalman's criterion for optimality

Although in [Kalman, 1964] the return difference inequality was proposed and used for solving the inverse optimal control problem, it provides some advanced aspects

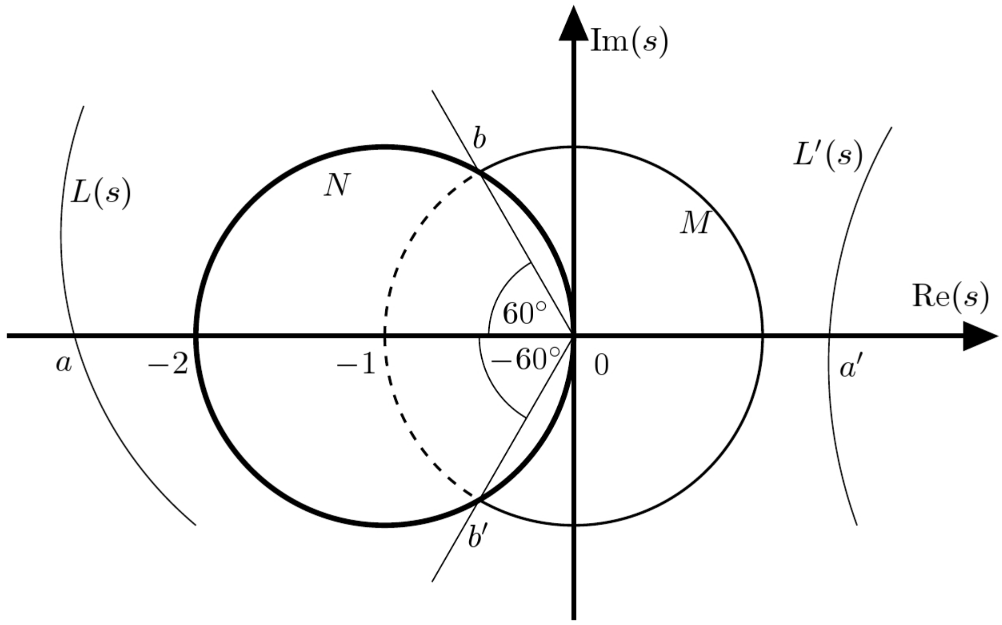

Figure 4.1: Illustration of the Nyquist stability of LQR

for analyzing the sensitivity and robustness of the LQR problem.

For example, for single input systems, based on the Nyquist criterion the scalar return difference inequality implies that the distance from -1 to the open-loop gain is always greater than or equal to 1 for all ω. This can be interpreted intuitively by examining the Nyquist plot of LQR. In Fig. 4.1, it is observed that the Nyquist diagram will never enter the unit circle N centered at the critical point $(-1, 0)$.

Assume that $L(s)$ and $L'(s)$ are the perturbation transfer functions configured in Fig. 4.1. When the Nyquist curves of $L(s)$ and $L'(s)$ hit the real axis at points $a \leq -2$ and $a' \geq 0$, $kL(s)$ or $kL'(s)$ will not penetrate the critical point $(-1, 0)$ if the perturbation gain k satisfies $k \geq \frac{1}{2}$.

The maximal phase tolerance corresponds to the phase angles of the intersection points b and b' between unit circles N and M. The latter circle M centered at the origin is introduced for the sake of interpreting the phase margins. It is clear that LQR controllers can tolerate a phase shift $|\phi| \leq 60°$ and retain stability.

As a result, the single input LQR controllers have a phase margin ϕ_i of $|\phi_i| \geq 60°$, a gain margin of 50% (decrease), and infinite gain (increase) margin.

4.2.2 Nyquist loci of the return difference

As was shown in the previous chapter that the singular value property of the return difference is not readily revealed for multivariable systems, MacFarlane [MacFarlane, 1970] instead used the eigenvalue property of the return difference to graphically interpret the optimality.

Recall that, regardless of any weighting matrices, the magnitude of the eigenvalue of the return difference is always greater than or equal to 1

$$|\lambda(s)| \geq 1, \quad \forall s = j\mathbb{R}$$

By the fact that

$$\det F(s) = \prod_{i=1}^{l} \lambda(s), \quad \forall s \in j\mathbb{R} \tag{4.1}$$

we have that

$$|\det F(s)| \geq 1, \quad \forall s = j\mathbb{R} \tag{4.2}$$

That is to say the complex plane contour of $|\det F(s)|$ will never penetrate the unit circle with the origin as its center.

4.3 Stability robustness of LQR

From the analysis of the previous chapter, LQR has good stability robustness, for example a gain margin of $[\frac{1}{2}, +\infty)$ and a phase margin of $[-60°, +60°]$, but only when the weight on the input is chosen as a scalar times the identity matrix. In this section, we shall present general robustness results of LQR against unstructured perturbations, for example multiplicative perturbations, based on the method used by Anderson *et al.* [Anderson *et al.*, 2007].

4.3.1 Introduction

To the author's knowledge, there are generally three main approaches to generalizing the robustness results for LQR. Firstly, a comprehensive treatment of both gain and phase margins was from Safonov *et al.* [Safonov *et al.*, 1977]. Although the method they used to present the main result appear to be old-fashioned, they presented the robustness result of LQR by tolerating both linear and nonlinear perturbations in a rigorous way. The second one came from Lehtomaki *et al.* [Lehtomaki *et al.*, 1981], who usded the multivariable Nyquist theorem for stability analysis of linear quadratic Guassian (LQG) control. The proof they reported was quite different from that used by Safonov, however it was further shown that the perturbation is not necessary to be assumed stable. The third one, which gives a simple and straightforward approach, was reported by Anderson *et al.* [Anderson *et al.*, 2007]. The key idea relied on the application of the multivariable return difference equality.

4.3.2 Problem formulation

In this section, we shall present the basis of the perturbation problem for LQR.

Assume the perturbed LQR system is illustrated in Fig. 4.2. The multiplicative perturbation transfer function is denoted as $L(s)$. The nominal LQR system is defined by eqns. (3.1) and (3.2). Consider a perturbed system which is given by

$$(\Gamma) \quad \begin{cases} \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u} \\ \tilde{u} = -K\tilde{x} = -R^{-1}B^T P\tilde{x} \end{cases} \tag{4.3}$$

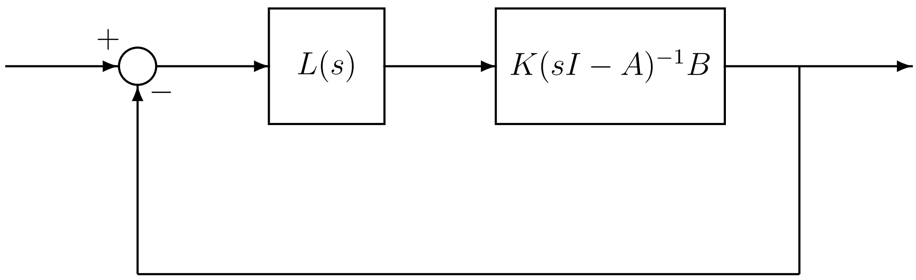

Figure 4.2: LQR system with multiplicative perturbation

where $\tilde{A}$ and $\tilde{B}$ are the perturbed system and input matrices, $P = P^T > 0$ is the solution of the nominal plant's algebraic Riccati equation by eqn. (3.5). The characteristic polynomial of the perturbed closed loop transfer function matrix is given by

$$\tilde{\Psi}_o(s) = \det(sI - \tilde{A}) \tag{4.4}$$

and similarily the perturbed closed loop transfer function matrix is obtained by

$$\tilde{\Psi}_c(s) = \det(sI - \tilde{A} + \tilde{B}K) \tag{4.5}$$

4.3.3 Classical results on the robustness of LQR

In this section, we shall present what is currently known about the general robustness of LQR. Excellent stability margins are presented for a special choice of R.

Theorem 1. *The perturbed LQR system shown in Fig. 4.2 will be asymptotically stable if*

i) The pair $[Q^{\frac{1}{2}}, \Gamma]$ *is detectable* *ii)* $L(s)$ *is a rational transfer function matrix*

iii) for all $s \in j\mathbb{R}$

$$L(j\omega)R^{-1} + R^{-1}L^{*}(j\omega) - R^{-1} \geq 0 \tag{4.6}$$

The above result quantifies a wide range of non-destabilizing perturbations in open-loop dynamics. See the prove by Anderson *et al.* [Anderson *et al.*, 2007].

4.3.4 The diagonality of R

In this section, we shall present a robustness result relating the diagonality of the input weight R.

When R is diagonal and L is also diagonal, so that

$$R = \begin{pmatrix} r_1 & 0 & \dots & 0 \\ 0 & r_2 & \dots & 0 \\ \vdots & \vdots & r_i & \vdots \\ 0 & 0 & \dots & r_m \end{pmatrix}, \quad L = \begin{pmatrix} l_1 & 0 & \dots & 0 \\ 0 & l_2 & \dots & 0 \\ \vdots & \vdots & l_i & \vdots \\ 0 & 0 & \dots & l_m \end{pmatrix} \tag{4.7}$$

the general robustness result of eqn. (4.6) implies that

$$l_i^H + l_i \geq 1 \tag{4.8}$$

This, in turn, implies that if l_i is real, the gain margin will be $[\frac{1}{2}, +\infty)$. Similarily, when $l_i = e^{i\theta}$, the phase margin will be $[-60^{\circ}, +60^{\circ}]$. Now, consider the cross coupling between any two input lines in L such as

$$L = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \tag{4.9}$$

and partitioning R into

$$R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \tag{4.10}$$

then to satisfy the inequality by eqn. (4.6) we have to let

$$\begin{pmatrix} R_1 & 0 \\ X^H R_1 & R_2 \end{pmatrix} + \begin{pmatrix} R_1 & R_1 X \\ 0 & R_2 \end{pmatrix} \geq \begin{pmatrix} R_1 & 0 \\ o & R_2 \end{pmatrix} \tag{4.11}$$

$$\Rightarrow \begin{pmatrix} R_1 & R_1 X \\ X^H R_1 & R_2 \end{pmatrix} \geq 0 \tag{4.12}$$

As $R_1 > 0$ and $R_2 > 0$, a neccesary and sufficient condition for satisfying eqn. (4.12) is to let

$$R_2 - X^H R_1 R_1^{-1} R_1 X > 0 \tag{4.13}$$

$$X^H R_1 X < R_2 \tag{4.14}$$

Thus, to satisfy eqn. (4.14) it is sufficient to let

$$\overline{\sigma}^2(X) < \frac{\lambda_{min}(R_2)}{\lambda_{max}(R_1)} \tag{4.15}$$

When any two diagonal entries of R are distinct from each other, especially $\lambda_{min}(R_2) < \lambda_{max}(R_1)$, a small bound on $\overline{\sigma}^2(X)$ will be expected. In other words, only when $R = \rho I$, LQR has good robustness, for example a gain margin of $[\frac{1}{2}, +\infty)$ and a phase margin of $[-60°, +60°]$. It is noted that eqn. (4.15) only gives a sufficient estimate of the bound of X and this may appear rather conservative. Nevertheless, choosing an input weighting matrix R which is far from a scalar times the identity would cause problems in obtaining good stability margins in each loop.

For non-diagonal R, Lehtomaki *et al.* [Lehtomaki *et al.*, 1981] and Anderson *et al.* [Anderson *et al.*, 2007] used a example to show that LQR may have arbitrarily small gain margins.

4.3.5 Classic stability margins

In the last section, it has been shown that the frequency-dependent perturbations tolerated by LQR are directly related to the control weight matrix R based on the multivariable return difference inequality. In this section, we shall present a robustness result for LQR in terms of the stability margins using the minimum singular value of the return difference transfer function matrix.

First, we shall introduce a robustness theorem by Lehtomaki *et al.* [Lehtomaki *et al.*, 1981] which established a relationship between the bound on the tolerable peturbations and the minimum singular value of the return difference matrix.

Theorem 2. *The characteristic polynomial of the perturbed closed loop system $\tilde{\Psi}_c(s)$ has no closed right half plane (CRHP) zeros if for all sufficiently large R_d:*

a) $\widetilde{\Psi}_o(s)$ *and* $\Psi_o(s)$ *have the same number of CRHP zeros.*

b) if $\widetilde{\Psi}_o(j\omega_0) = 0$*, then* $\Psi_o(j\omega_0) = 0$.

c) $\Psi_c(s)$ *has no CRHP zeros.*

d)

$$\overline{\sigma}[L^{-1}(s) - I] < \alpha(s) \triangleq \underline{\sigma}[I + T_o(s)], \quad s \in \Omega_{R_d} \tag{4.16}$$

where Ω_{R_d} is denoted as the Nyquist D-contour, $T_0(s)$ is the loop transfer function matrix and $L(s)$ is the perturbation transfer function matrix.

e) any one of the following is satisfied at each $s \in \Omega_{R_d}$:

i)

$$\alpha \leq 1 \tag{4.17}$$

ii)

$$L^H(s) + L(s) \geq 0 \tag{4.18}$$

iii) $$4(\alpha^2 - 1) \cdot \underline{\sigma}^2[L(s) - I] > \alpha^2 \cdot \overline{\sigma}^2[L(s) + L^H(s) - 2I] \tag{4.19}$$

From eqn. (4.16), two main results follow. Firstly, it may be shown that a sufficient condition for stability is to have the frequency-dependent loci of $\overline{\sigma}[L^{-1}(s) - I]$ stay below the $\underline{\sigma}$-plot of the return difference $F(s)$. Thus, the quantity $\alpha(s)$, as a function of $s \in \Omega_{R_d}$, is regarded as a reliable measure of the multivariable stability margin. $s \in \Omega_{R_d}$ is the Nyquist D-coutour. If we further denote α_0 as a global minimum singular value of the return difference, then a worst case perturbation bound is estimated by eqn. (4.17). Here, it is convenient to work with $\overline{\sigma}[L^{-1}(s) - I]$ instead of $\overline{\sigma}[L(s) - I]$ [Lehtomaki, 1981] for the stability margin analysis of LQR and LQG problems. A second result is that eqn. (4.18) gives the general robustness result of LQR presented in the previous sections.

Next, we shall introduce several results which give an explicit interpretation of the stability margins in relation to the minimum singular value of the return difference.

Corollary 1. *If all the conditions of Theorem 2 hold and*

$$\underline{\sigma}[I + T_o(s)] \geq \alpha_0, \quad \forall s \in \Omega_{R_d} \tag{4.20}$$

for some constant $\alpha_0 \leq 1$*, then the system has a gain margin* β

$$\beta = \frac{1}{1 \pm \alpha_0} \tag{4.21}$$

and a phase margin θ

$$\theta = \pm \cos^{-1} \left[1 - \frac{\alpha_0^2}{2} \right] \tag{4.22}$$

in each loop.

The above result implies that if a pure gain γ_i is inserted into the feedback loops of the system of Fig. 4.2, by eqn. (4.21) we have that

$$\frac{1}{1+\alpha_0} < \gamma_i < \frac{1}{1-\alpha_0} \tag{4.23}$$

is sufficient to ensure stability. Similarily, for the insertion of a phase $e^{j\phi_i}$ into the feedback loop, eqn. (4.22) provides a phase margin

$$|\phi_i| < \cos^{-1}\left[1 - \frac{\alpha_0^2}{2}\right] \tag{4.24}$$

For the system which is subject to the crossfeed perturbation, we have the following result

Corollary 2. *Suppose that all the conditions of Theorem 2 hold and*

$$\underline{\sigma}[I + T_o(s)] \geq \alpha_0, \quad \forall s \in \Omega_{R_d} \tag{4.25}$$

for some constant $\alpha_0 \leq 1$, then for the crossfeed perturbations

$$L(s) = \begin{pmatrix} I_k & X(s) \\ 0 & I_m \end{pmatrix} \textit{or} \begin{pmatrix} I_k & 0 \\ X(s) & I_m \end{pmatrix} \tag{4.26}$$

where I_k is the $k \times k$ identity matrix and

$$\overline{\sigma}[X(s)] < \alpha_0, \quad \forall s \in \Omega_{R_d} \tag{4.27}$$

the perturbed closed loop system will be stable.

In the previous sections, it was shown that when $R = \rho I$, then $\underline{\sigma}[I + T_o(s)] \geq 1$. We can readily obtain two robustness results for LQR with this special input weight.

Corollary 3. *If* $R = \rho I$ *and either* $Q > 0$ *or* $T_o(j\omega) \neq 0, \forall\omega$*, LQR has a gain margin of* $[0.5, +\infty)$ *and a phase margin of* $[-60°, +60°]$.

Corollary 4. *If* $R = \rho I$*, then LQR can tolerate crossfeed perturbations given by eqn.* (4.26)*, satisfying*

$$\overline{\sigma}[X(s)] < 1, \quad s \in \Omega_{R_d} \tag{4.28}$$

4.3.6 Conclusion

In this chapter, we gave a comprehensive review of the stability robustness of the LQR problem. It has been shown that the return difference inequality plays an important part in characterizing the robustness property of LQR. Combined with the Nyquist stability criterion, a general robustness result, as well as a graphic interpretation of the complex plane loci of the return difference, was presented. The stability margins and the tolerance of crossfeed perturbations by LQR are presented when $R = \rho I$.

Chapter 5

Guaranteed stability robustness

It has been shown that, based on the multivariable return difference inequality, the set of perturbations tolerated by LQR is highly related to the weighting matrix on the input. Specifically, when the input weighting matrix is selected as a scalar times the identity matrix, the LQ regulators have classic stability margins. However, for the situation in which non-diagonal cost weighting matrices in the performance index are considered, the robustness of LQR has not been studied yet. Inspired by the work on discrete-time systems [Shaked, 1986; Arvanitis *et al.*, 1997; Arvanitis *et al.*, 2001], new lower bounds on the minimum singular value of the return difference transfer function matrix is presented. It leads to the guaranteed stability margins, as well as the tolerance to crossfeed perturbations. We also investigate the effect of tuning the state cost weighting matrix so that the stability robustness can be improved.

5.1 Motivation

Previous works on the estimation of the minimum singular value of the return difference were mostly based on the multivariable return difference inequality. To obtain the classic stability margins, one has to select the input weighting matrix as $R = \rho I$. However, in most situations, the selection of the cost weighting matrices is not that restricted. Therefore, there is a requirement for guidelines for the designer to acquire good stability robustness by selecting appropriate cost weighting matrices. To deal with this problem, it is useful to consider the multivariable return difference equality, which nicely relates the return difference to the weights in the frequency domain. For exmaple, Shaked [Shaked, 1986] has obtained the guaranteed gain and phase margins for the discrete-time LQR system. More recently, Arvanitis *et al.* [Arvanitis *et al.*, 1997; 2001] have presented stability margin bounds for LQR in the discrete-time case for various special cases. It is believed that the singular valued decomposition technique can be successfully applied to the continuous-time case.

5.2 A new lower bound on the minimum singular value of the return difference transfer function matrix

In this section, a new frequency-dependent lower bound on the minimum singular value of the return difference is presented in terms of the system and cost weighting matrices. A new constant lower bound of the minimum singular value of the return difference is also presented with respect to the input cost weighting matrix.

Theorem 3. *For LQR controllers, the minimum singular value of the return difference transfer matrix $F(s)$ at the plant's input will satisfy:*

$$\underline{\sigma}[F(s)] \geq \sqrt{\frac{\underline{\sigma}^2(R^{\frac{1}{2}})}{\overline{\sigma}^2(R^{\frac{1}{2}})} + \frac{\underline{\sigma}^2[H(s)]}{\overline{\sigma}^2(R^{\frac{1}{2}})}} = \hat{\alpha}(s), \quad \forall s \in j\mathbb{R} \tag{5.1}$$

where $H(s) = Q^{\frac{1}{2}}(sI - A)^{-1}B$.

Proof. Consider that the return difference matrix F and its singular value decomposition (SVD)

$$\begin{aligned} F \cdot v &= \sigma(F) \cdot u \\ v^* \cdot F^H &= \sigma(F) \cdot u^* \end{aligned} \tag{5.2}$$

where v and u are the left and right singular vectors, respectively. Multiply eqn. (3.16) on the left by v^* and on the right by v, giving

$$v^* F^H R F v = v^*(R + G^H Q G)v \tag{5.3}$$

Combining eqns. (5.2) and (5.3) gives

$$\sigma^2(F) u^* R u = v^*(R + G^H Q G)v$$

$$\Rightarrow \quad \sigma^2(F) = \frac{v^*(R + G^H Q G)v}{u^* R u}$$

$$\Rightarrow \quad \sigma^2(F) = \frac{v^* R v + (v^* G^* Q^{\frac{1}{2}})(Q^{\frac{1}{2}} G v)}{u^* R u} \tag{5.4}$$

The facts that

$$\underline{\sigma}^2(R^{\frac{1}{2}}) \leq v^* R v \leq \overline{\sigma}^2(R^{\frac{1}{2}}), \quad \underline{\sigma}^2(R^{\frac{1}{2}}) \leq u^* R u \leq \overline{\sigma}^2(R^{\frac{1}{2}})$$
$$\underline{\sigma}^2(Q^{\frac{1}{2}}G) \leq (v^* G^* Q^{\frac{1}{2}})(Q^{\frac{1}{2}} G v) \leq \overline{\sigma}^2(Q^{\frac{1}{2}}G) \tag{5.5}$$

show that eqn. (5.4) implies that

$$\underline{\sigma}(F) \geq \sqrt{\frac{\underline{\sigma}^2(R^{\frac{1}{2}})}{\overline{\sigma}^2(R^{\frac{1}{2}})} + \frac{\underline{\sigma}^2(Q^{\frac{1}{2}}G)}{\overline{\sigma}^2(R^{\frac{1}{2}})}}$$

□

Theorem 3 gives a frequency-dependent lower bound on $\underline{\sigma}[F(s)]$ with respect to the cost weighting and system matrices.

Corollary 5. *For any $R > 0$, we have*

$$\underline{\sigma}[F(s)] \geq \frac{\underline{\sigma}(R^{\frac{1}{2}})}{\overline{\sigma}(R^{\frac{1}{2}})} = \hat{\alpha}_0, \quad \forall s = j\omega \in j\mathbb{R} \tag{5.6}$$

Proof. As $\underline{\sigma}^2[H(s)] \geq 0$, it follows trivially from eqn. (5.1). □

The above result estimates the lower bound of $\underline{\sigma}[F(s)]$ by the input cost weighting matrix R [Chen *et al.*, 2014]. It is shown that when $\underline{\sigma}(R^{\frac{1}{2}})$ is closed to $\overline{\sigma}(R^{\frac{1}{2}})$, the quantity $\hat{\alpha}_0$, as a single constant bound, goes close to 1. It is worth mentioning that if $\underline{\sigma}(R^{\frac{1}{2}})$ is far from $\overline{\sigma}(R^{\frac{1}{2}})$, we will have a very small $\hat{\alpha}_0$, which may appear rather conservative. To overcome this conservativeness, we shall also consider the effects contributed by $H(s)$, or simply by the state weighting matrix Q. However, when $R = \rho I$, it is again verified that all the singular values of the return difference are greater than or equal to 1.

Corollary 6. *If $R = \rho I$, then*

$$\underline{\sigma}[F(s)] \geq 1, \quad \forall s \in j\mathbb{R} \tag{5.7}$$

Proof. If R is diagonal, $\Rightarrow \underline{\sigma}(R^{\frac{1}{2}}) = \overline{\sigma}(R^{\frac{1}{2}})$. Thus, eqn. (5.7) follows trivially from eqn. (5.6). □

5.3 Guaranteed stability margins

As a new lower bound of the minimum singular value of the return difference has been established, we shall present the associated guaranteed stability margins in this section.

Corollary 7. *For any $R > 0$ and $Q \geq 0$, LQR has the guaranteed gain margin (GM) and phase margin (PM) given by*

$$GM = \frac{1}{1 \pm \hat{\alpha}_0}, \quad PM = \pm \cos^{-1}\left[1 - \frac{\hat{\alpha}_0^2}{2}\right] \tag{5.8}$$

where $\hat{\alpha}_0 \triangleq \underline{\sigma}(R^{\frac{1}{2}})/\overline{\sigma}(R^{\frac{1}{2}})$.

Proof. It follows trivially from Corollary 1 and Corollary 5. □

Compared to the classic stability margins stated in Corollary 3, this provides a more general result by considering the selection of an arbitrary input weighting matrix, rather than $R = \rho I$. It shows that when $\underline{\sigma}(R^{\frac{1}{2}})$ is close to $\overline{\sigma}(R^{\frac{1}{2}})$, we can expect relatively large stability margins. It should be pointed out that the guaranteed stability margins estimated by a simple lower bound on $\underline{\sigma}[F(s)]$ may be very conservative. Even so, this gives a formal mathematical basis for guidelines for the designer to improve stability robustness. As analyzed in the estimate of the lower bound of $\underline{\sigma}[F(s)]$, to obtain improved stability margins, we shall also consider how the $\underline{\sigma}$-plot of $H(s)$ will behave with respect to the state weighting matrix Q.

Corollary 8. *For any $R > 0$ and $Q \geq 0$, LQR will remain stable in the presence of the crossfeed perturbations satisfying*

$$\overline{\sigma}[X(s)] < \hat{\alpha}(s), \quad \forall s = j\omega \in j\mathbb{R} \tag{5.9}$$

where

$$\hat{\alpha}(s) \triangleq \sqrt{\frac{\underline{\sigma}^2(R^{\frac{1}{2}})}{\overline{\sigma}^2(R^{\frac{1}{2}})} + \frac{\underline{\sigma}^2[H(s)]}{\overline{\sigma}^2(R^{\frac{1}{2}})}} \tag{5.10}$$

Proof. It follows trivially from Theorem 3 and Corollary 2. □

From eqn. (5.9), it is shown that the upper bound on the crossfeed perturbations is estimated by the frequency-dependent bound of the minimum singular value of $\underline{\sigma}[F(s)]$. Compared to the existing result stated in Corollary 4, it gives an improved result by the inclusion of both the system matrices and the state weighting matrix Q.

5.4 The selection of Q

In the previous sections, new lower bounds on the minimum singular value of $\underline{\sigma}[F(s)]$, as well as the increasing guaranteed stability margins, were presented. It was shown that the minimum singular value behaviour of $H(s)$ also plays a key role in improving the robustness of LQR. In this section, we shall investigate how the state weighting matrix Q can be tuned so that the robustness of LQR is improved. It is shown that for low frequencies, by simply enlarging the minimum singular value of Q, the estimated lower bound increases, thereby improving the robustness.

We will first study the singular value behaviour of $H(j\omega)$ for high frequencies. As $\omega \to \infty$, then $(j\omega I - A)^{-1} \to 0$. Eventually, $H(j\omega)$ will vanish to 0, which makes no contribution to the bound. Assume A is invertible, for low frequencies $H(j\omega)$ will tend to $-Q^{\frac{1}{2}}A^{-1}B$. By the fact that for any three matrices $\overline{A}, \overline{B}, \overline{C}$ we have that $\underline{\sigma}(\overline{A} \cdot \overline{B} \cdot \overline{C}) \geq \underline{\sigma}(\overline{A}) \cdot \underline{\sigma}(\overline{B}) \cdot \underline{\sigma}(\overline{C})$, we have

$$\underline{\sigma}[Q^{\frac{1}{2}}A^{-1}B] \geq \underline{\sigma}(Q^{\frac{1}{2}}) \cdot \underline{\sigma}(A^{-1}) \cdot \underline{\sigma}(B) \tag{5.11}$$

$$\Rightarrow \underline{\sigma}[Q^{\frac{1}{2}}A^{-1}B] \geq \underline{\sigma}(Q^{\frac{1}{2}}) \cdot [\overline{\sigma}(A)]^{-1} \cdot \underline{\sigma}(B) \tag{5.12}$$

Based on Theorem 3, the above result gives a constant lower bound on the minimum singular value of $\underline{\sigma}[F(s)]$ for low frequencies. Then we have

Corollary 9. *For low frequencies, the minimum singular value of the return difference transfer matrix $F(s)$ at the plant's input will satisfy:*

$$\underline{\sigma}[F(s)] \geq \sqrt{\frac{\underline{\sigma}^2(R^{\frac{1}{2}})}{\overline{\sigma}^2(R^{\frac{1}{2}})} + \frac{\underline{\sigma}^2(Q^{\frac{1}{2}}) \cdot \underline{\sigma}^2(B)}{\overline{\sigma}^2(R^{\frac{1}{2}}) \cdot \overline{\sigma}^2(A)}} = \hat{\alpha}_{low} \tag{5.13}$$

The constant quantity $\hat{\alpha}_{low}$ in eqn. (5.13) gives a reliable estimate of the bound of $\underline{\sigma}[F(s)]$ for low frequencies. It gives a less conservative bound than $\hat{\alpha}_0$ presented in Corollary 5. Moreover, for a given open loop plant, if $\underline{\sigma}(Q^{\frac{1}{2}})$ is increased, it is observed that the bound $\hat{\alpha}_{low}$ will be increased, consequently improving the robustness of LQR.

5.5 Illustrative examples

In this section, several examples are presented for illustrating the effectiveness of our results. All the experiments are performed using MATLAB R2012a.

The open loop system matrices are randomly generated. All the weighting matrices in the performance index satisfy $R = R^T > 0$ and $Q = Q^T \geq 0$. In order to check whether the pair $[A, B]$ is stabilizable, the Hautus test is implemented. The test is also used to ensure the detectability of the pair $[A, Q^{\frac{1}{2}}]$. When all conditions above are satisfied, we use the MATLAB function *lqr* to compute the unique Riccati solution $P \geq 0$ satisfying eqn. (3.5). The frequency response of the return difference transfer matrix, $F(jw)$, is simulated in the frequency range between $[10^{-3}, 10^3]$. As a result, the $\underline{\sigma}$-plot of $F(jw)$ is captured. It should be mentioned that the σ sign shown in all Matlab figures represent the the minimum singular value.

Firstly, we investigate the characterisation of the minimum singular value of the return difference with a diagonal input weighting matrix, particularly a scalar times the identity. Secondly, we compare the frequency-dependent lower bound stated in Theorem 3 with the $\underline{\sigma}$-plot of the return difference $F(j\omega)$ with general cost weighting matrices selected. The global minimum of $\underline{\sigma}[F(j\omega)]$ is compared with the constant lower bound given by Corollary 5. Thirdly, we investigate the behaviour of $\underline{\sigma}[F(j\omega)]$ for low frequencies by comparing it with frequency-dependent bound generalized in Corollary 9. Finally, the behaviour of $\underline{\sigma}[F(j\omega)]$ is investigated with respect to the state weighting matrix Q.

1. The case of $R = \rho I$

   The open loop systems are randomly generated. We have simulated 1000 LQR systems with $R = \rho I$ and random $Q \geq 0$. All the results show that the global minimum of $\underline{\sigma}[F(j\omega)]$ is always greater than or equal to 1.

	$\underline{\sigma}(R^{\frac{1}{2}})$	$\overline{\sigma}(R^{\frac{1}{2}})$	$\underline{\sigma}(R^{\frac{1}{2}})/\overline{\sigma}(R^{\frac{1}{2}})$
Case 1	1.0496	3.0275	0.3467
Case 2	1.4816	2.5926	0.5715
Case 3	0.1812	2.4502	0.0740

Table 5.1: Illustration of the singular values of $R^{\frac{1}{2}}$ in three cases

2. The case of general R and Q

   (a) We shall first compare the new lower bound $\hat{\alpha}(s)$ with the minimum singular value of $F(s)$. All the system matrices and the cost weighting matrices are randomly generated. Some examples are shown in Fig. 5.1, Fig. 5.2 and Fig. 5.3. The $\underline{\sigma}$-plot of the return difference transfer matrix is plotted in the solid line, while the lower bound $\hat{\alpha}(s)$ is represented in the dashed line. Table 5.1 also lists the maximum and minimum singular values of $R^{\frac{1}{2}}$, as well as the ratio. Fig. 5.1, Fig. 5.2 and Fig. 5.3 show that the frequency-dependent bound $\hat{\alpha}(s)$ follows closely the $\underline{\sigma}$-plot of the return difference. Comparing them to Table 5.1, it is shown that the global minimum of $\underline{\sigma}[F(s)]$ is greater than the ratio of the singular values of $R^{\frac{1}{2}}$. However, for Case 3, it is shown that although the ratio is small, $\underline{\sigma}[F(s)]$ is above 0.6. This example shows that the bound can be conservative.

   (b) Secondly, we shall illustrate the effectiveness of the constant lower bound $\hat{\alpha}_0$. After simulating 1000 random LQR systems, it was found that the global minimum of $\underline{\sigma}[F(s)]$ is above $\hat{\alpha}_0$. Here, several examples are presented in Table 5.2.

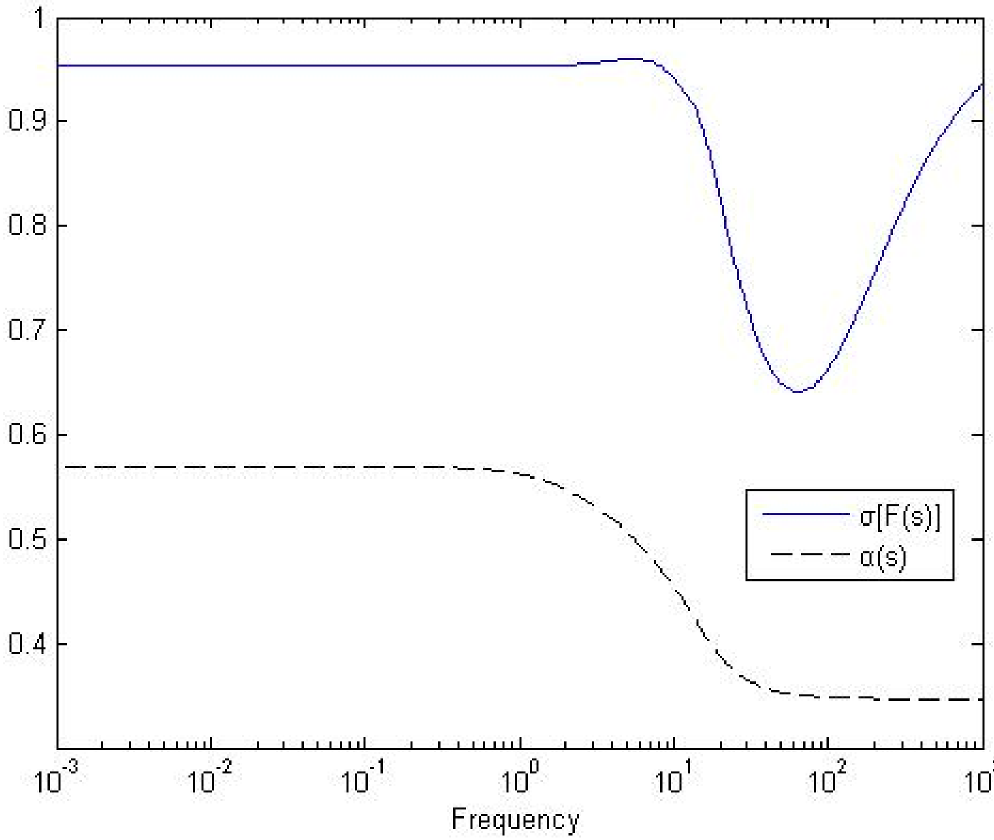

Figure 5.1: Illustration of the new lower bound on the minimum singular value of $F(s)$ of Case 1

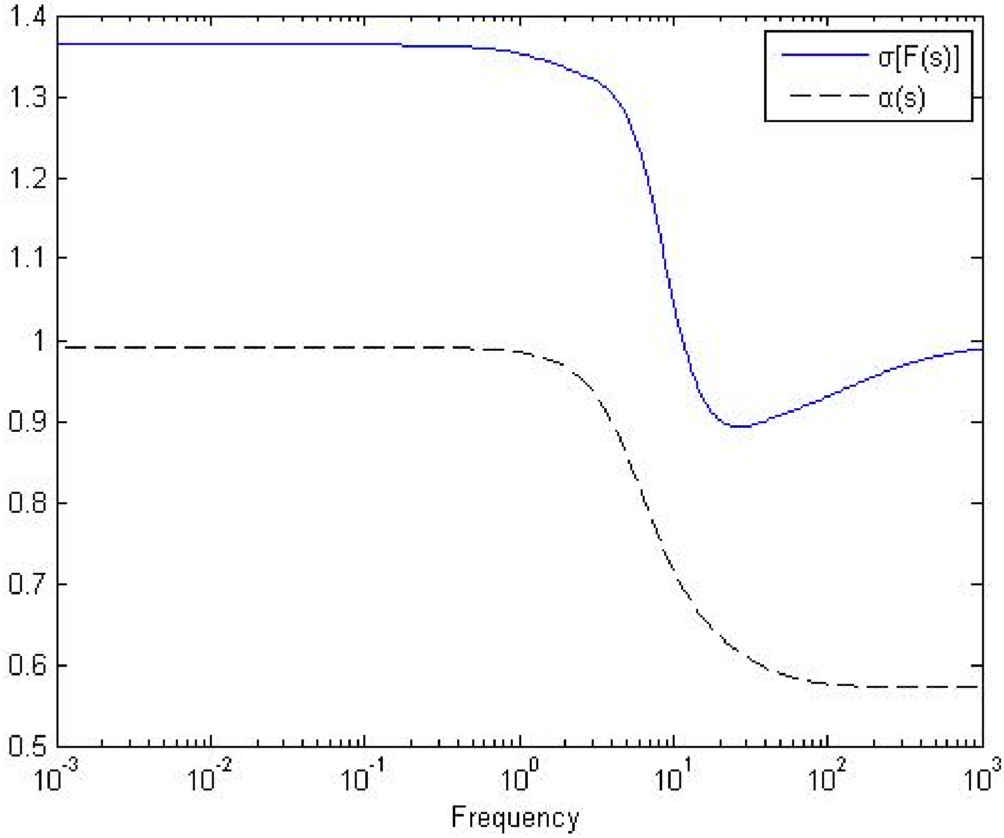

Figure 5.2: Illustration of the new lower bound on the minimum singular value of $F(s)$ of Case 2

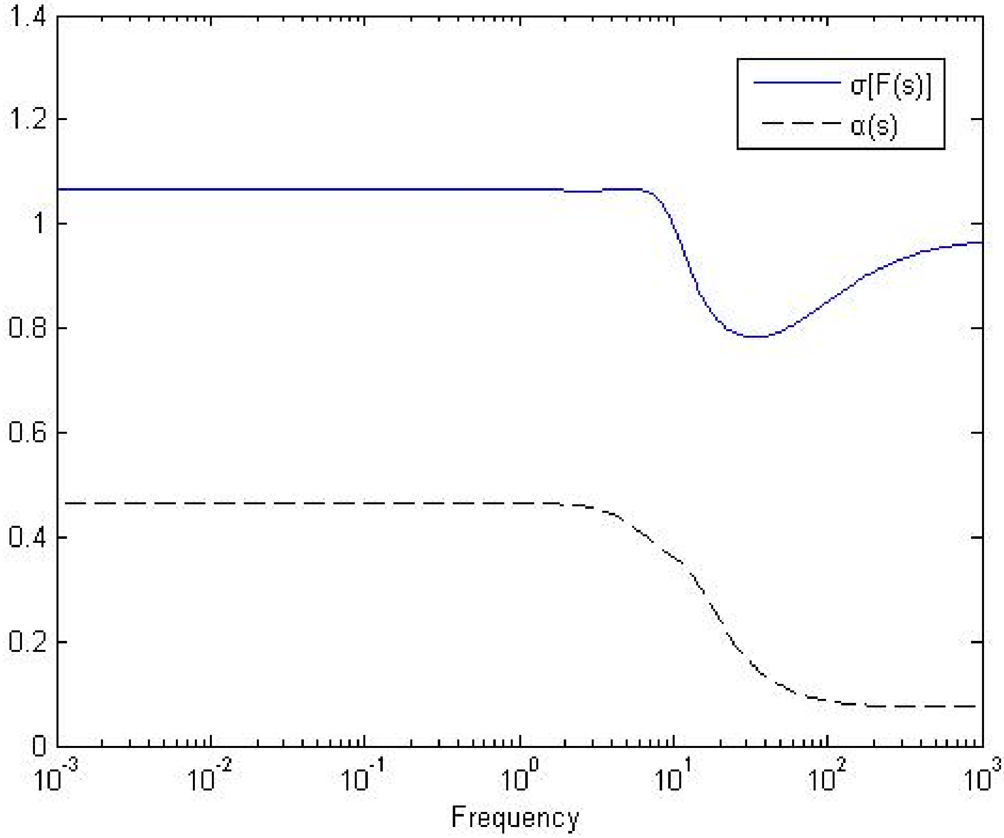

Figure 5.3: Illustration of the new lower bound on the minimum singular value of $F(s)$ of Case 3

(c) Thirdly, we shall demonstrate the validity of the bound $\hat{\alpha}_{low}$ for low frequencies. After testing 1000 random LQR systems, it was observed that the global minimum of $\underline{\sigma}[F(s)]$ in the low frequency range is above $\hat{\alpha}_{low}$.

$\hat{\alpha}_0$	0.283	0.315	0.618	0.153	0.512	0.364	0.353	0.312	0.159	0.527
$\underline{\sigma}[F(s)]$	0.645	0.840	0.958	0.681	0.888	0.692	0.849	0.873	0.688	0.894

Table 5.2: Illustration of the minimum singular value of $F(s)$ and $\hat{\alpha}_0$

Remarks: The simulation results show that all the bounds in terms of the system and the cost weighting matrices provide a reliable estimation of $\underline{\sigma}[F(s)]$. In particular, when $R = \rho I$ the $\underline{\sigma}$-plot of $F(s)$ is above one. This gives the classic stability margins. It should be noted that these bounds may be very conservative. The tuning of the $\underline{\sigma}$-plot of $H(s)$ needs to be further studied.

$\hat{\alpha}_{low}$	0.270	0.380	0.451	0.562	0.383	0.474	0.282	0.117	0.033	0.646
$\underline{\sigma}[F(s)]$	1.195	0.998	0.976	1.247	1.070	0.800	1.404	1.062	0.890	1.252

Table 5.3: Illustration of the minimum value of $F(s)$ and $\hat{\alpha}_{low}$ for low frequencies

3. The effect of Q

   Finally, we will examine the behaviour of the $\underline{\sigma}$-plot of $F(s)$ by tuning the weighting matrix Q. After simulating hundreds of randomly generated examples, it was found that by simply increasing $\underline{\sigma}(Q)$ the $\underline{\sigma}$-plot of $F(s)$ is improved. One of our simulation examples is illustrated in the following figures. Fig. 5.4 depicts the $\underline{\sigma}$-plot of $F(s)$ and the bound $\hat{\alpha}(s)$ when Q is unchanged. After decreasing the size of Q by a factor of 0.1, the result is shown in Fig. 5.5. It is shown that in the low frequency range, $\underline{\sigma}[F(s)]$ has dramatically decreased. However, when the size of Q is increased by a factor of 10, as shown in Fig. 5.6, for low frequencies the $\underline{\sigma}$-plot of $F(s)$ has been greatly improved. This suggests that in most cases, by rescaling the size of Q, $\underline{\sigma}[F(s)]$ can be improved, which results in good robustness of LQR.

5.5.1 Real control example

In this section, we shall investigate the stability robustness result of a real control problem when non-diagonal cost weighting matrices are selected.

The example we study here is a state variable model of two rolling carts [Dorf *et al.*, 2010] shown in Fig. 5.7. M_1 and M_2 defined on the figure are mass of carts. k_1 and k_2 are spring constants. b_1 and b_2 are damping coefficients. u_1 and u_2 are the external forces acting on the first cart and the second cart seperately. p and q are positions of carts. The carts are assumed to have negligible rolling friction.Any existing rolling friction are considered to be lumped into the damping coefficients. The free-body diagram of mass of two carts are shown in Fig. 5.8. $\dot{p}$ and $\dot{q}$ are velocities of M_1 and M_2, respectively.

Assume $\ddot{p}$ and $\ddot{q}$ are acceleration of M_1 and M_2, respectively. According to

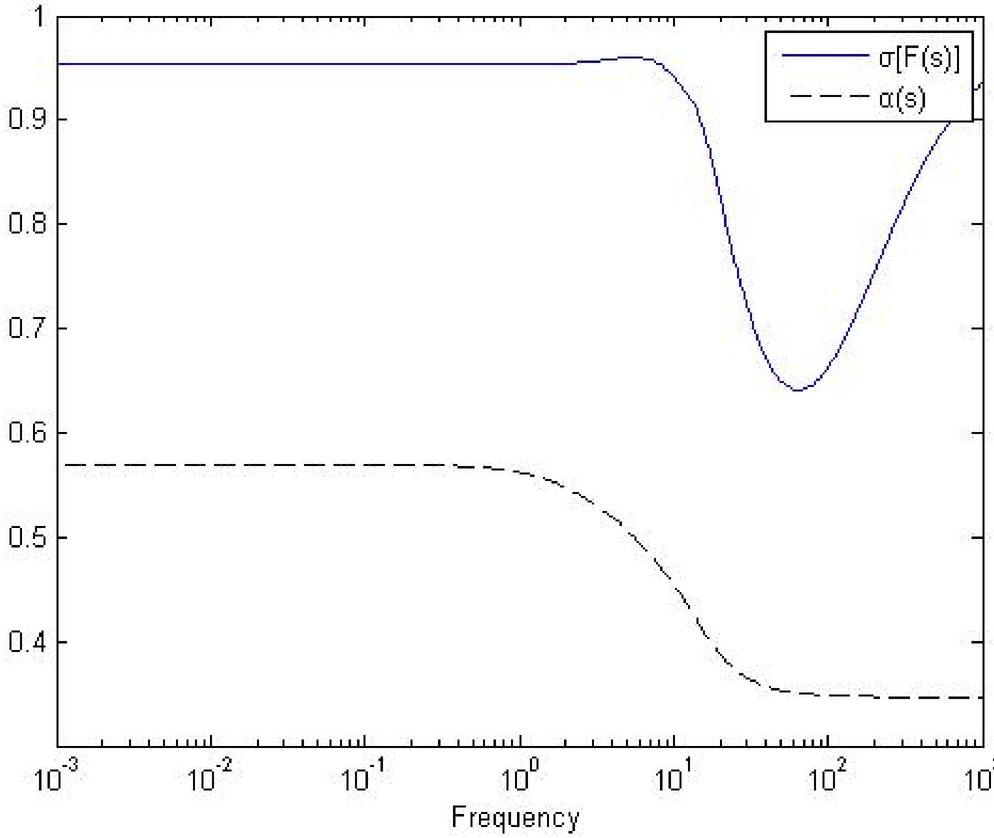

Figure 5.4: Illustration of the low frequency lower bound on the minimum singular values of $F(s)$ with Q unchanged

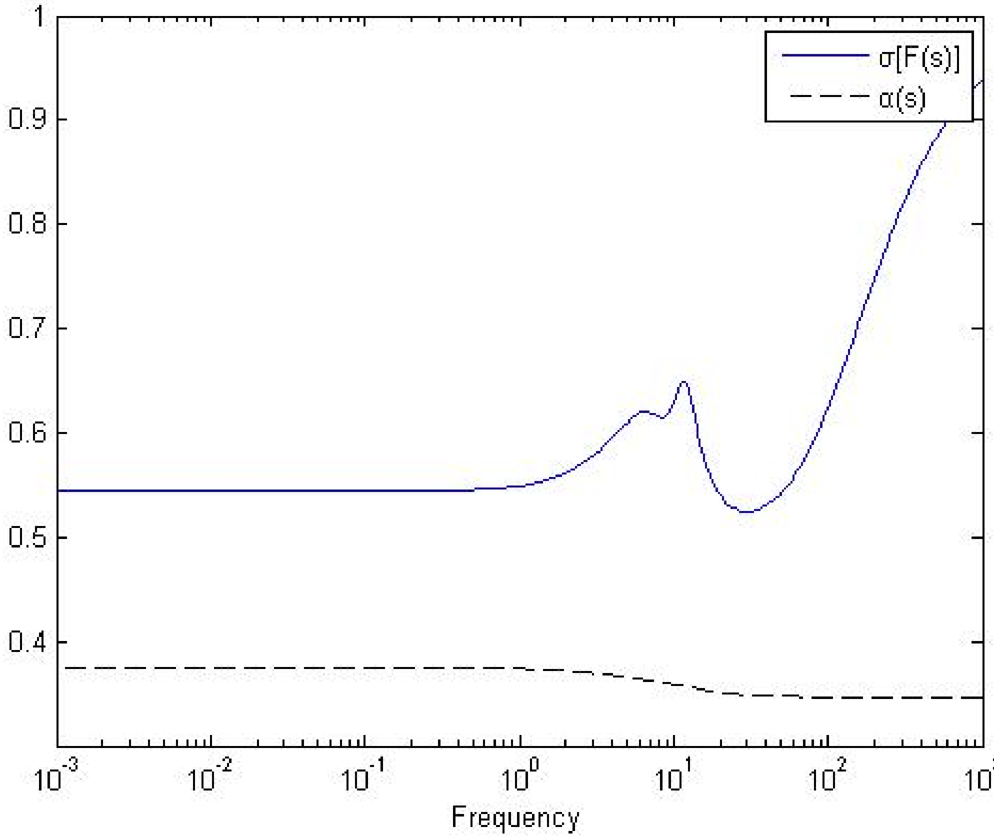

Figure 5.5: Illustration of the low frequency lower bound on the minimum singular values of $F(s)$ with decreased Q

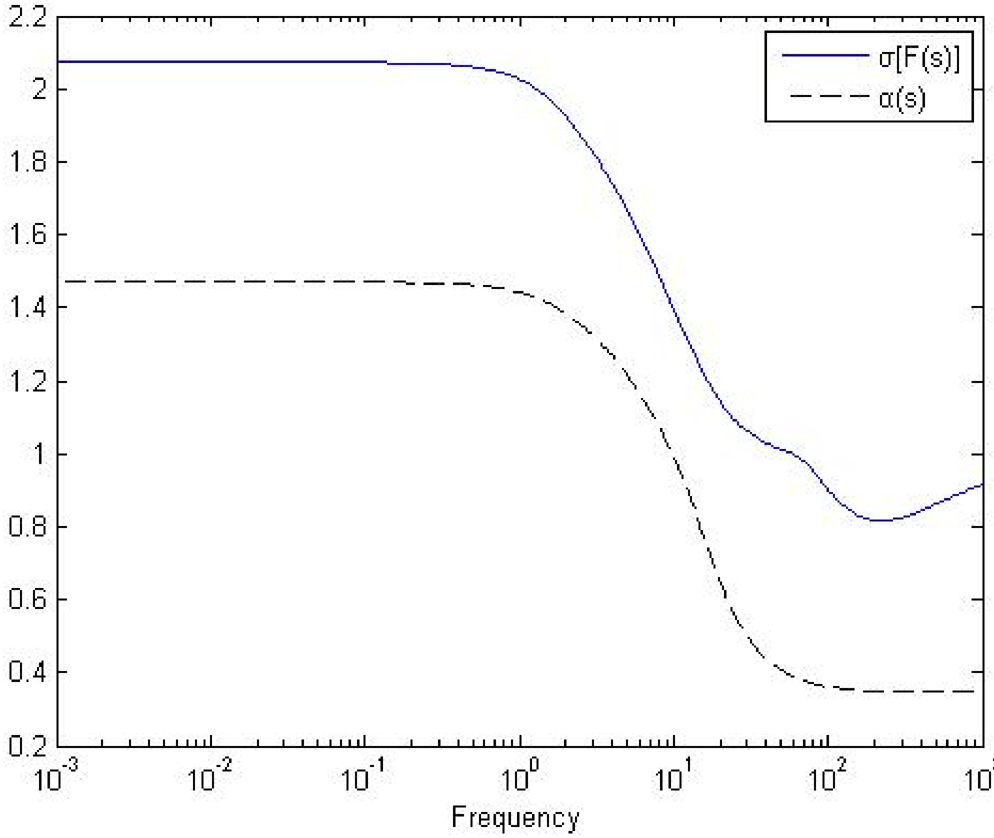

Figure 5.6: Illustration of the low frequency lower bound on the minimum singular values of $F(s)$ with increased Q

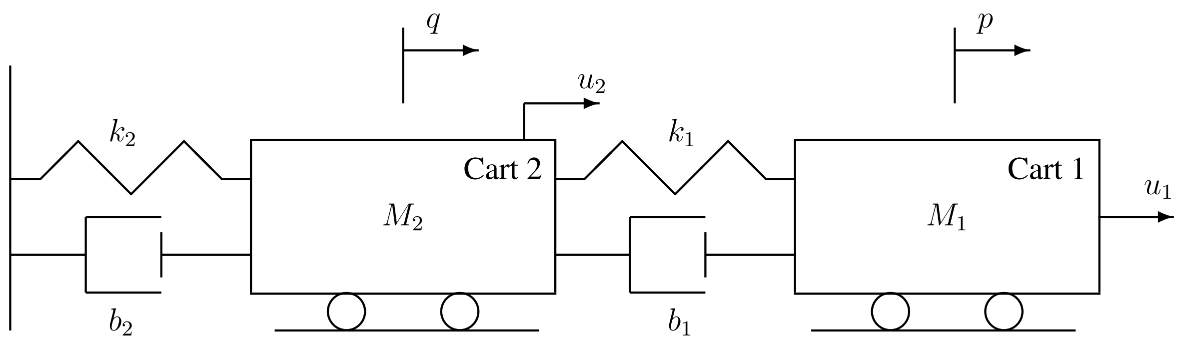

Figure 5.7: The model of two rolling carts

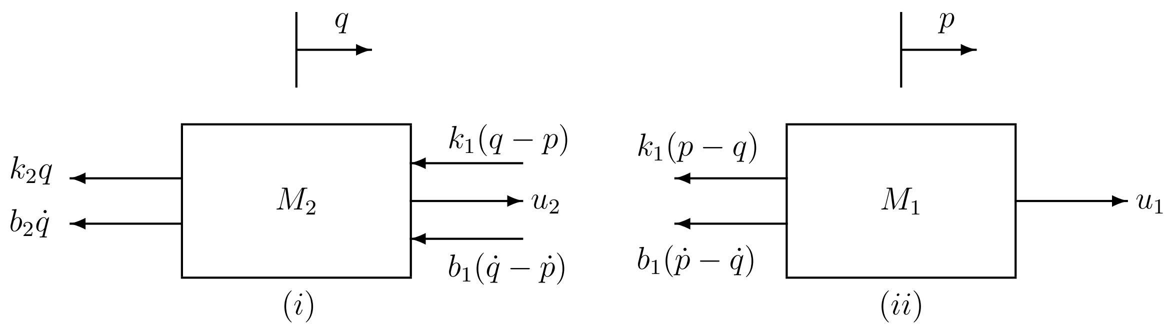

Figure 5.8: Free-body diagram of two rolling carts: (i) Cart 2; (ii) Cart 1

Newton's second law, for mass M_1 we have

$$M_1\ddot{p} + b_1\dot{p} + k_1 p = u_1 + k_1 q + b_1\dot{q} \tag{5.14}$$

For M_2 we have

$$M_2\ddot{q} + (k_1 + k_2)q + (b_1 + b_2)\dot{q} = u_2 + k_1 p + b_1\dot{p} \tag{5.15}$$

Define $x_1 = p$ and $x_2 = q$. Then define $x_3 = \dot{x}_1 = \dot{p}$ and $x_4 = \dot{x}_2 = \dot{q}$. So we have $\ddot{p} = \dot{x}_3$ and $\ddot{q} = \dot{x}_4$. By eqn. (5.14) and eqn. (5.15), the state-space model is obtained by

$$\dot{x}_1 = x_3 \tag{5.16}$$

$$\dot{x}_2 = x_4 \tag{5.17}$$

$$\dot{x}_3 = \frac{1}{M_1}(-k_1x_1 + k_1x_2 - b_1x_3 + b_1x_4 + u_1) \tag{5.18}$$

$$\dot{x}_4 = \frac{1}{M_2}[k_1x_1 - (k_1 + k_2)x_2 + b_1x_3 - (b_1 + b_2)x_4 + u_2] \tag{5.19}$$

Eqns. (5.16), (5.17), (5.18) and (5.19) can also be written as matrix form

$$\dot{x} = Ax + Bu \tag{5.20}$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} p \\ q \\ \dot{p} \\ \dot{q} \end{pmatrix}, u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{M_1} & \frac{k_1}{M_1} & -\frac{b_1}{M_1} & \frac{b_1}{M_1} \\ \frac{k_1}{M_2} & -\frac{k_1+k_2}{M_2} & \frac{b_1}{M_2} & -\frac{b_1+b_2}{M_2} \end{pmatrix} \tag{5.21}$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{M_1} & \frac{1}{M_2} \\ 0 & 0 \end{pmatrix} \tag{5.22}$$

If p is chosen as the output, then we have

$$y = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} = Cx \tag{5.23}$$

Suppose the following parameter values are specified for this two rolling carts model: $k_1 = 150N/m$, $k_2 = 700N/m$, $b_1 = 15N/m$, $b2 = 30N/m$, $M_1 = 5kg$ and $M_2 = 20kg$. The state space of the two rolling carts model is specified as

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -30 & 30 & -3 & 3 \\ 7.5 & -42.5 & 0.75 & -2.25 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0.2 & 0.05 \\ 0 & 0 \end{pmatrix} \tag{5.24}$$

It is clear that this system is controllable.

Next, we shall investigate the behavior of the minimum singular values of $F(s)$ when $\underline{\sigma}(R^{\frac{1}{2}})$ decreases in a continuous way. When the cost weighting matrix Q is the identity matrix, it is clear that the pair $[A, Q]$ is observable. As the cost weighting matrix R is a 2×2 matrix, we shall first let all the two singular values of R be equal to 1. The $\underline{\sigma}$-plot of the return difference transfer function matrix $F(s)$ is plotted as data 1 by the blue line in Fig. 5.9. It is seen that all the $\underline{\sigma}[F(s)]$ are above 1. Next, we decrease $\underline{\sigma}(R^{\frac{1}{2}})$ by a factor of 1.3. The simulation results of the minimum singular value of $F(s)$ are shown by the four lines with different colours. All of them are blow the blue line. The lowest points of those lines represent the global minimum of $\underline{\sigma}[F(s)]$. They are used to evaluate the stability margins. As we can see, all the minimum points of the following four curves (green line, red line, cyan line, magenta line) decrease in a continous way as $\underline{\sigma}(R^{\frac{1}{2}})$ decreases. Thus, according to (4.21) and (4.22) in Corollary 1, stability margins become progressively smaller. The exact gain margin and the guaranteed gain margin quantified in Corollary 7 are compared in Table 5.4. The exact phase margin and the guaranteed phase margin quantified in Corollary 7 are compared in Table 5.5.

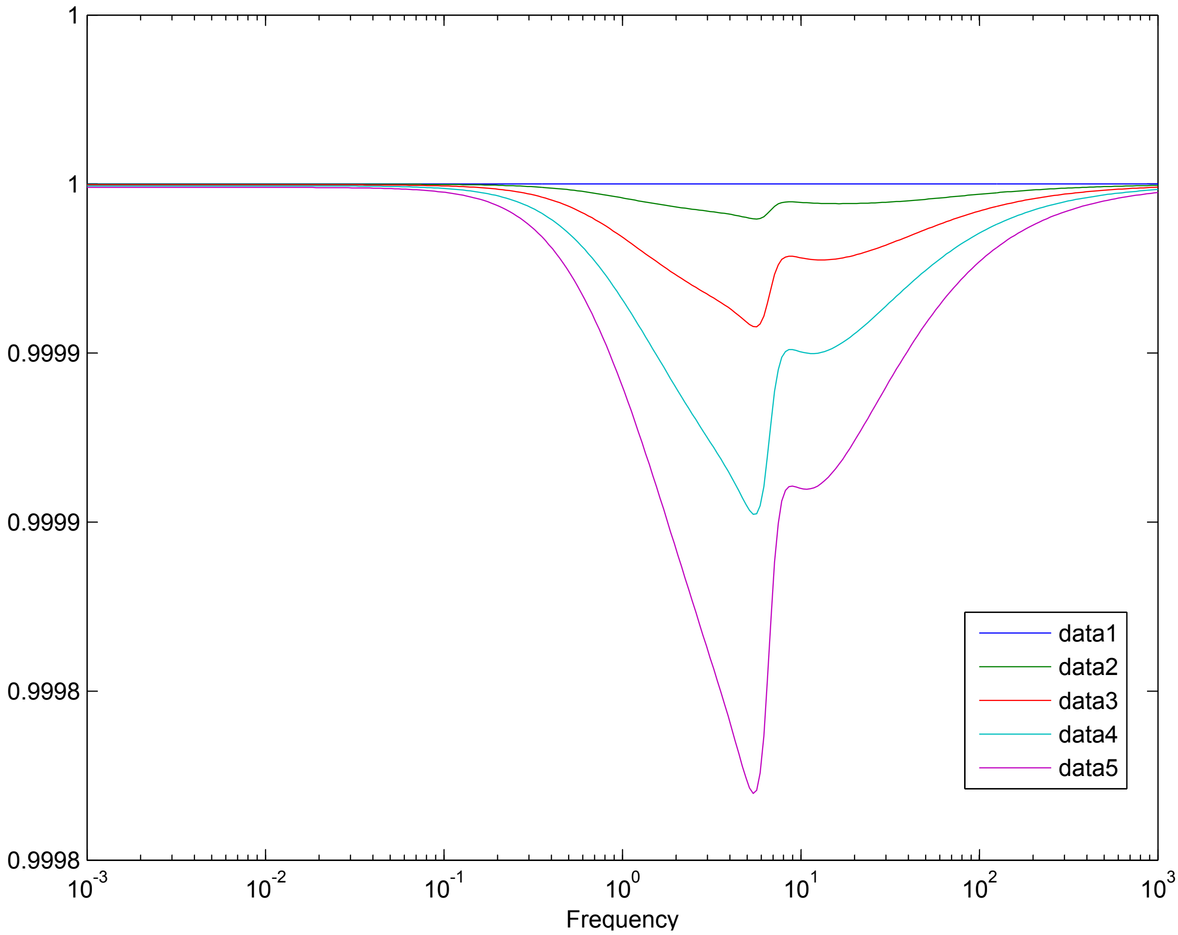

Figure 5.9: Illustration of the behavior of the minimum singular values of $F(s)$ when the minimum singular value of R decreases.

$\underline{\sigma}(R^{\frac{1}{2}})$	1.0	0.769231	0.591716	0.455166	0.350128
GM	0.5	0.500003	0.500011	0.500024	0.500045
Gm	0.5	0.532749	0.565217	0.597136	0.628253

Table 5.4: Comparison of the exact gain margin and the guaranteed gain margin when minimum singular value of $\sigma(R^{\frac{1}{2}})$ decrease

$\underline{\sigma}(R^{\frac{1}{2}})$	1.0	0.769231	0.591716	0.455166	0.350128
PM	$60°$	$59.99932°$	$59.9972°$	$59.99354°$	$59.98807°$
Pm	$60°$	$52.02013°$	$45.23973°$	$39.42857°$	$34.41799°$

Table 5.5: Comparison of the exact phase margin and the guaranteed phase margin when minimum singular value of $\sigma(R^{\frac{1}{2}})$ decrease

It is shown that when $\underline{\sigma}(R^{\frac{1}{2}}) = 1$, the classic stability margin (gain margin 0.5 and phase margin $60°$) are acquired. The exact stability margins are progressively getting smaller as $\underline{\sigma}(R^{\frac{1}{2}})$ decreases. The guaranteed stability margins are also decreasing. In other word, as the cost weighting matrix R deviate from the scalar times the identity matrix, the system will lose the stability margin smoothly. This will give control designers a guideline to choose the cost weighting matrices regarding the stability robustness of LQR.

5.6 Conclusion

In this chapter, new lower bounds on the minimum singular value of the return difference matrix were presented. The associated guaranteed stability margins, as well as the tolerance to crossfeed perturbations, were established. It was shown that to obtain relatively large margins, one may choose the input weighting matrix R so that $\underline{\sigma}(R^{\frac{1}{2}})$ is close to $\overline{\sigma}(R^{\frac{1}{2}})$. This gives a formal mathematical basis for guidelines for the designer to improve the robustness. A new constant lower bound on $\underline{\sigma}[F(j\omega)]$ for low frequencies was also presented with respect to the open loop system and the cost weighting matrices. It has been shown that when $\underline{\sigma}(R^{\frac{1}{2}})/\overline{\sigma}(R^{\frac{1}{2}}) = 1$, the classic stability margins are obtained. When the cost weighting matrix R is selected in which $\underline{\sigma}(R^{\frac{1}{2}})/\overline{\sigma}(R^{\frac{1}{2}})$ deviate from 1 progressively, the classic stability margins will be lost smoothly.

Chapter 6

Distinction between structured and unstructured perturbations

In the previous chapter, it was shown that LQ regulators can have classic stability robustness when a special class of the cost weighting matrices is selected. However, some authors [Soroka *et al.*, 1984; Zhang *et al.*, 1996] have argued that LQR may suffer from poor robustness when the system matrices are subject to small parameter changes. In particular, Soroka *et al.* [Soroka *et al.*, 1984] studied a cheap control problem in which the entries of the input matrix is pertubed by small variations. In this chapter, we revisit the robustness of cheap control against structured perturbations. The distinction between strucutred and unstructured perturbations is worth re-emphasizing [Fu *et al.*, 1989] for the stability analysis of LQR. We also investigate the perturbation issue of the algebraic Riccati equation in solving LQR problems. By means of condition numbers, the vulnerability of LQR system to structured perturbations can be detected.

6.1 Introduction

In classical control theory, linear quadratic regulators (LQR) are well-known to have excellent stability robustness. For instance, it has been shown that the multivariable LQR controllers have a tolerance of 50 percent gain reduction, infinite (increase) gain margin and $60°$ phase margin in each loop [Safonov *et al.*, 1977]. However, Soroka and Shaked [Soroka *et al.*, 1984], among others, argued that linear quadratic optimal regulators may suffer from poor robustness when the open-loop plant, especially the input matrix, is subject to small parameter variations. Improvements had been made by Grimble *et al.* [Grimble *et al.*, 1986] so as to mitigate the disastrous effects of these perturbations. Again, Zhang *et al.* [Zhang *et al.*, 1996] cautioned that the guaranteed stability margins should be carefully treated when variations in the open-loop gain are considered. Mainly, they were concerned with variations in the system's state-space matrices. Their analyses of robustness properties were based on the characteristic transfer functions. The return difference was used to compute the optimal gain and then the exact bounds of parameter changes were determined so that the closed-loop characteristic polynominals were Hurwitz.

There are two broad categories of robust stability results [Fu *et al.*, 1989]. One considers perturbations whose structure is known but with parameters whose exact values are unknown. This is the structured uncertainty approach. The other approach is the unstructured uncertainty approach, where one considers additive or multiplicative perturbation whose internal structure is unknown, but is stable and gain-bounded. It is important to keep the distinction between these two categories in mind.

Although the above-mentioned results are quite straightforward, it is not appropriate to directly compare the bounds on the allowable structured perturbations with the guaranteed stability margins. Stability margins presume the perturbation is in the open-loop. If not, stabiliity margins make no claim.

We attempt to analyze the robust stability bounds of the LQR problem in [Safonov *et al.*, 1977]. Instead of considering stability margins, the exact structured perturbation bounds are compared and contrasted with the unstructured bounds provided by the return difference inequality, bounds which are stronger than the stability margins described above. It will be shown that the robustness properties are preserved in a general sense. It will also be shown that the closed-loop may be very sensitive to variations in the state-space dynamical model of the plant, especially in case of low objective function weights on the plant's inputs. This gives a caution in designing high gain LQR controllers. Furthermore, the analysis provides guidance in choosing LQR weights.

In order to investigate the stability and vulnerability of LQR system against structured perturbations in general situations, we shall study the perturbation theory of the algebraic Riccati equation. Condition numbers are used to measure the sensitivity to the solution of the Riccati solution. It is also shown that they can be utilized to detect the vulnerability of the optimal control systems.

The rest of the chapter is organized as follows. In Section 6.2, a brief history of cheap control is presented. The example of Soroka *et al.* and the example of Zhang *et al.* are presented in Section 6.3. By comparing the perturbation bounds, the distinction between structured and unstructured perturbations is established in Section 6.4. In Section 6.5, the basic perturbation theory of the algebraic Riccati equation is reviewed and the concept of condition number is introduced. Some illustrating examples are presented in Section 6.6. In the last section, we give our conclusions.

6.2 Cheap control

In this section, we shall give a brief introduction of the so-called cheap control problem. It originates from the work by Kalman [Kalman, 1964]. The resulting asymptotic properties of the closed-loop poles [Graham *et al.*, 1953; Ashkenas *et al.*, 1962; Chang, 1961], were generalized to the multivariable case by Kwakernaak and Sivan [Kwakernaak *et al.*, 1972], which results in cheap control. Kwakernaak *et al.* [Kwakernaak *et al.*, 1969] and Friedland *et al.* [Friedland *et al.*, 1970] had earlier shown that the closed-loop system is insensitive to paramer variations in the sense of cheap control. Later Kwakernaak *et al.* [Kwakernaak *et al.*, 1972] presented necessary and sufficient condition for achieving perfect perfomance in cheap control problem. The cheap control problem was rigorously treated from a mathematical point of view by Jameson *et al.* [Jameson *et al.*,1975]. Francis [Francis *et al.*, 1978] generalized the result of [Kwakernaak *et al.*, 1972]. [Sannuti *et al.*, 1985; Saberi *et al.*, 1987] made comprehensive study of the cheap control problem.

Consider a FDLTI system in state space form

$$\dot{x}(t) = Ax(t) + Bu(t), t \geq 0 \tag{6.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$, $x(t) \in \mathbb{R}^n$,$u(t) \in \mathbb{R}^m$. We attempt to minimize a scalar cost functional

$$J(\epsilon) = \frac{1}{2}\int_0^\infty [x(t)^T Qx(t) + \epsilon^2 u(t)^T Ru(t)]dt, \epsilon > 0 \tag{6.2}$$

The state weighting matrix Q is symmetric positive semidefinite and the input weighting matrix R is symmetric positive definite. Here, ϵ is a small scalar parameter. As $\epsilon \to 0$, the cost of the control $u(t)$ is cheap relative to that of the state

$x(t)$. Therefore, the cheap control problem discussed above is also known as the low control weighting problem. It is equivalent to the hight state weighting problem with a scalar cost functional with respect to a positive parameter $\beta \to \infty$ instead

$$J(\beta) = \frac{1}{2}\int_0^{\infty} [\beta^2 x(t)^T Q x(t) + u(t)^T R u(t)] dt, \beta > 0 \tag{6.3}$$

6.3 Problem formulation

6.3.1 The case of Soroka *et al.*

In this section, we shall present the example considered by Soroka *et al.* [Soroka *et al.*, 1984].

Consider the following cheap control problem shown as follows:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, c = \begin{bmatrix} 1 & -1 \end{bmatrix} \tag{6.4}$$

It is noted that in [Soroka *et al.*, 1984] the vector c was shown as $c = \begin{bmatrix} 1 & 1 \end{bmatrix}$ rather than $c = \begin{bmatrix} 1 & -1 \end{bmatrix}$. Apparently it is a typographical error.

The performance index to be minimized is

$$J = \int_0^{\infty} \left[y^2(t) + ru^2(t)\right] dt, \; r > 0 \tag{6.5}$$

The well-known return difference equality is found to be

$$[1 + t_o(-s)]'[1 + t_o(s)] = 1 + \frac{1}{r} g(-s)' Q g(s) \tag{6.6}$$

where $g(s) = (sI - A)^{-1}b$, $Q = c'c$, $t_0(s) = k(sI - A)^{-1}b$ and k is the optimal gain. Note that Q is singular. From eqn. (6.6), the return difference is readily obtained by

$$f(s) = 1 + t_0(s) = \frac{s^2 + s\sqrt{5 + 2q} + q}{(s+1)(s+2)} \tag{6.7}$$

where $q \triangleq \sqrt{4 + \frac{1}{r}}$ and

$$t_0(s) = \frac{(\sqrt{5 + 2q} - 3)s + (q - 2)}{(s+1)(s+2)} \tag{6.8}$$

By eqn. (6.7) the optimal gain matrix is given by

$$k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}' = \begin{pmatrix} 1 + q - \sqrt{5 + 2q} \\ 2\sqrt{5 + 2q} - q - 4 \end{pmatrix}' \tag{6.9}$$

Consider now a small perturbation ϵ in b, namely $b_\epsilon = (1 + \epsilon, 1)'$. The return difference of the perturbed system is given by

$$f_\epsilon(s) = 1 + t_0^\epsilon(s) = \frac{s^2 + d_1 s + d_2}{(s+1)(s+2)} \tag{6.10}$$

where

$$t_0^\epsilon(s) = \frac{(d_1 - 3)s + (d_2 - 2)}{(s+1)(s+2)} \tag{6.11}$$

is the perturbed open-loop transfer function and the coefficients of the characteristic polynominal are

$$d_1 = \sqrt{5 + 2q} + \epsilon(1 + q - \sqrt{5 + 2q}) \tag{6.12}$$

$$d_2 = q + 2\epsilon(1 + q - \sqrt{5 + 2q}) \tag{6.13}$$

It is clear that necessary and sufficient conditions for stability are that (i) $d_1 > 0$

and (ii) $d_2 > 0$. By eqn. (6.12) and eqn. (6.13), it may be shown that a necessary and sufficient condition for stability for all ϵ satisfying

$$|\epsilon| < \epsilon_L = \min(\delta_1, \delta_2) \tag{6.14}$$

where

$$\delta_1 = \frac{\sqrt{5+2q}}{1+q-\sqrt{5+2q}} \tag{6.15}$$

$$\delta_2 = \frac{q/2}{1+q-\sqrt{5+2q}} \tag{6.16}$$

As $r \to 0$ then $q \to \infty$, $\delta_1 \to \sqrt{\frac{2}{q}}$ and $\epsilon_L = \delta_1$. Thus, for any $\epsilon < 0$ and $|\epsilon| > \sqrt{\frac{2}{q}}$ the closed-loop system will be unstable. That is to say, as the control weighting goes to zero, the bound on parameter variation for not destabilizing the closed loop system will be very small. As compared to the classic stability margins, for example a gain margin of $[0.5, +\infty)$, this result turns out to be somehow disappointing. In Soroka *et al.* [Soroka *et al.*, 1984], the individual entries in the system's b matrix are perturbed differently. Hence, from the very definition of gain margin, gain margin bounds make no claim. Gain margin bounds cover situations where the b matrix is perturbed by multiplication by a real gain. In other words, if both elements of b are perturbed by ϵ as $[1+\epsilon, 1+\epsilon]'$, the perturbation can be treated as a pure gain inserted into the nominal open loop. In this situation, it is clear that the bound ϵ_L for stability will satisfy $[0.5, +\infty)$. As a result, the remarkable result of unstructured perturbations in Corollary 3 is not applicable to the case of Soroka *et al.* who consider structured perturbations. Therefore, we shall investigate the perturbation bound based on the general robustness property of LQR characterized by Theorem 1. This bound is evaluated according to the H-infinity norm bound of the perturbation transfer function whose coefficients depend affinely on the perturbation parameters. By comparing this H-infinity norm bound with the exact perturbation bound given by eqn. (6.11), we are able to investigate whether the robustness properties of LQR are preserved in a general sense.

6.3.2 The case of Zhang *et al.*

In this section, we mainly study the “example” by Zhang *et al.*.

Unlike the example of Soroka *et al.* [Soroka *et al.*, 1984], Zhang *et al.* [Zhang *et al.*, 1996]’s example complicates things by the involvement of perturbations in both the system matrix and input matrix. The plant considered is

$$G(s) = k\,\frac{s-1}{s^2} \tag{6.17}$$

The performance index is

$$J = \int_0^\infty (x'q'qx + u^2)\,dt \tag{6.18}$$

where

$$q = [\sqrt{2r} - r, r] \tag{6.19}$$

and r is a positive tuning parameter. As there may have plenty of state-space realization of eqn. (6.17) a stabilizable and detectable one with respect to gain k is obtained by

$$A = \begin{bmatrix} 0 & 0 \\ -k & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ k \end{bmatrix}, \qquad c = [0, 1]. \tag{6.20}$$

The optimal input may be shown to be

$$u(t) = -h^T x(t) \tag{6.21}$$

Assume

$$L(s) = h^T (sI - A)^{-1} B \tag{6.22}$$

$$g(s) = q\,(sI - A)^{-1} B \tag{6.23}$$

The return difference equality is given by

$$[1 + L(s)]\,[1 + L(-s)] = 1 + g(s)\,g(-s) \tag{6.24}$$

The formula for the optimal gain h with respect to gain k and parameter r is found to be

$$h^T = [h_1, h_2] \tag{6.25}$$

where

$$h_1 = \sqrt{(\sqrt{2r} - r + rk)^2 + 2kr} + rk, \qquad h_2 = -r \tag{6.26}$$

The closed loop polynominal is then

$$t(s) = s^2 + (h_1 + h_2 k)s - h_2 k \tag{6.27}$$

If the nominal value of k is set to 1, the associated nominal h is given by

$$h = [h_1', h_2'], \qquad h_1' = 2\sqrt{r} + r, \qquad h_2' = -r \tag{6.28}$$

The nominal closed-loop polynominal is then

$$t'(s) = s^2 + 2\sqrt{r}s + r \tag{6.29}$$

which is always stable when $r > 0$.

Consider next perturbing this nominal k with small error e by

$$k' = 1 + e \tag{6.30}$$

Then the perturbed closed-loop system is

$$t''(s) = s^2 + (2\sqrt{r} - re)s + r(1 + e) \tag{6.31}$$

It is shown that if

$$2\sqrt{r} - re > 0 \tag{6.32}$$

and

$$r(1 + e) > 0 \tag{6.33}$$

then the system will be stable. The closed-loop system will lose its stability when very small errors show up

$$e = e^* = \frac{2}{\sqrt{r}} \to 0, \qquad \text{as} \qquad r \to \infty \tag{6.34}$$

By the stability theorem of Lehtomaki *et al.* [Lehtomaki *et al.*, 1981], it may be shown that either of the following two conditions must hold to ensure the stability

$$\text{Cond. 1} \qquad Q > 0 \qquad \text{and} \qquad r(s) \leq 0,\ s \in \Omega_R \tag{6.35}$$

$$\text{Cond. 2} \qquad \Phi_{OL}(jw) \neq 0,\ \forall w \qquad \text{and} \qquad r(s) < 1,\ s \in \Omega_R \tag{6.36}$$

As $Q = q'q$, condition 1 is violated. The denominator of $g(s)$ in eqn. (6.23) is s^2. Consequently, condition 2 is again violated. Hence, it cannot be argued that LQR may suffer from poor robustness because of this special example. However, if we replace the Q simply by a tuning parameter r just like in the case of Soroka *et al.*, then

$$Q = r \times I \tag{6.37}$$

All the conditions eqn. (6.35) and eqn. (6.36) will be satisfied.

6.4 Bounds comparison for the case of Soroka *et al.*

In this section, we investigate whether the robustness result in Theorem 1 covers the particular LQR problem of Soroka *et al*. The approach is to compare the bounds on ϵ for stability.

When $R = r$, the condition for the stability of the perturbed LQR system in Theorem 1 is

$$\mathrm{Re}[L(jw)] \geq \frac{1}{2}, \, \forall \omega \tag{6.38}$$

where $\mathrm{Re}[\cdot]$ denotes the real part of a complex number.

As is shown in Fig. 4.2, the perturbation $L(s)$ is readily given by

$$L(s) = \frac{t_0^\epsilon(s)}{t_0(s)} \tag{6.39}$$

where $t_0(s)$ and $t_0^\epsilon(s)$ are given in eqns.(6.8) and (6.11), respectively. By eqn. (6.38) and eqn. (6.39), the necessary and sufficient condition for stability for all ϵ satisfying

$$|\epsilon| \leq \epsilon_S = \min\left(\beta_1, \beta_2\right) \tag{6.40}$$

where

$$\beta_1 = \frac{\frac{1}{2}(\sqrt{5+2q}-3)}{1+q-\sqrt{5+2q}} \tag{6.41}$$

$$\beta_2 = \frac{\frac{1}{2}(\frac{q}{2}-1)}{1+q-\sqrt{5+2q}} \tag{6.42}$$

The derivation of the bounds β_1 and β_2 is detailed in appendix. As $r \to 0$ ($q \to \infty$), it is clear that $\epsilon_S = \beta_1$. Recall that in this circumstance the exact bound ϵ_L is found to be δ_1, as analyzed in Section 2. It is obvious that $\beta_1 < \delta_1$. In other words, the bound ϵ_S guaranteed by Theorem 1 is smaller than the exact bound ϵ_L. Therefore, the well-known robustness result of LQR shown in Theorem 1 is preserved in this particular case [Chen *et al.*, 2014]. However, the remarkable result of Corollary 1 is not applicable for this case. If both elements of b are perturbed by ϵ as $[1+\epsilon, 1+\epsilon]'$ can be treated as a pure gain inserted into the nominal open loop. In this situation, it is clear that the bound ϵ_L for stability is given by 0.5 based on Corollary 1. This then recovers the classic stability margins.

6.5 Perturbation theory of the CARE

In this section, we first present a brief review of the algebraic Riccati equation (ARE) and condition number. The robustness problem of LQR with respect to structured perturbations is investigated relying on numerical perturbation analysis of the continuous algebraic Riccati equation (CARE). We seek to use the condition numbers of the algebraic Riccati equations to detect the vulnerability of LQR systems.

6.5.1 Some history of algebraic Riccati equation

In this section, we will give a brief review of the algebraic Riccati equation. The history of the Riccati equation, in name of Riccati, begins in the early 18th century. Escherich [Escherich, 1898] used the eigenvector method to solve the Riccati equation. Kalman [Kalman, 1960] first introduced the Riccati differential equation for solving an optimal control problem. Later, Potter [Potter, 1966] and Kleinman [Kleinman, 1968] presented a numerical analysis of the solution of the ARE. A comprehensive treatment began with the book by Wonham [Wonham, 1970]. Willems [Willems, 1971] and Coppel [Coppel, 1974] studied the continuous-time algebraic Riccati equation (CARE). In the meantime, Martensson [Martensson, 1971] summarized the eigenvector approach for solving the ARE. A new algorithm to solve the ARE was proposed by Laub [Laub, 1979]. A generalized Schur approach was treated in [Pappas *et al.*,1980; Arnold *et al.*, 1984]. Some books [Bittanti *et al.*,1991; Mehrmann *et al.*, 1991; Lancaster *et al.*, 1995] extensively studied the application of the solution of the ARE to various areas, especially to the linear optimal control problem. Benchmark examples for solving AREs were set up by Benner *et al.* [Benner *et al.*, 1997] for comparison purposes.

6.5.2 Brief review of condition numbers

The conditioning of the ARE has been studied for a long time. A general concept of condition dates back to [Rice, 1966]. It was applicable to many areas by Geurts [Geurts, 1982]. Bucy [Bucy, 1975] introduced the notion of structural stability for Riccati equations. Laub [Laub, 1979] pointed out the relationship between the conditioning of the ARE and stability in optimal control problems. Based on the theory of [Rice, 1966], Byers [Byers, 1985] introduced the condition number for the ARE as a measure of the sensitivity of the solution of the ARE subject to perturbations in a rigorous way. The notion of stability radius for the perturbation analysis of ARE was generalized by Hinrichsen and Pritchard [Hinrichsen *et al.*, 1986]. Chen [Chen, 1988] sharpened the results of Byers. At the same time, Hewer and Kenny [Hewer *et al.*, 1988; Kenny *et al.*, 1990] related the condition number of the ARE to the damping property of the linear control problem. Similar results were reported by Xu [Xu, 1996], He [He, 1997] and Hewer *et al.* [Hewer *et al.*, 1998]. Sun [Sun 1998; 2002] made progress in computing the condition number in an explicit form. Zhou *et al.* [Zhou *et al.*, 2009] further presented tighter bounds for condition numbers of the ARE. The application of condition numbers of ARE to real control problems can be found in [Zietsman *et al.*, 2008].

6.5.3 Condition number

In this section, the concept of condition number is illustrated.

Consider an algebraic equation

$$Ax = B \tag{6.43}$$

If A and B are additively perturbed as $A + \triangle A$ and $B + \triangle B$, the solution $x + \triangle x$ satisfies

$$(A + \triangle A)(x + \triangle x) = B + \triangle B \tag{6.44}$$

$$\Rightarrow Ax + A\triangle x + (\triangle A)x + \triangle A\triangle x = B + \triangle B \tag{6.45}$$

It is, then, to investigate the upper bound on $\triangle x$. As is usual, the analysis is done to first order. Therefore, by eqns. (6.44) and (6.43) we are interested in

$$(\triangle A)x + \triangle A \triangle x = \triangle B \tag{6.46}$$

If A is invertible, eqn. (6.46) implies that

$$\triangle x = A^{-1}\triangle B - A^{-1}(\triangle A)x \tag{6.47}$$

Consequently, the upper bound of $\triangle x$ is established in terms of 2-norm

$$||\triangle x||_2 = \overline{\sigma}(A^{-1})||\triangle B - (\triangle A)x||_2 \leq \frac{1}{\underline{\sigma}(A)} \tag{6.48}$$

This shows that, as is well known, the sensitivity of this problem is determined by how close A is to being non-invertible.

6.5.4 The continuous-time algebraic Riccati equation (CARE)

Rather than consider the effects of various perturbations on the LQ full-state feedback system, numerical perturbation analysis of Riccati equations will be used as a tool to measure the sensitivity of the CARE solution. On that account, we will only concentrate on solving the CARE. Commonly, the CARE may be written as

$$Q + A^H X + XA - XGX = 0 \tag{6.49}$$

where

$$G = BR^{-1}B^H \tag{6.50}$$

$$A \in C^{n\times n}, \qquad B \in C^{n\times m}, \qquad Q \in H^{n\times n}, \qquad R \in H^{m\times m} \tag{6.51}$$

and

$$Q,\, G \geq 0 \tag{6.52}$$

It is noted here that the task is to seek a Hermitian positive semi-definite (p.s.d) solution X, which also makes $A - GX$ stable. It is known that [Laub, 1979] two conditions suffice for a unique p.s.d solution to the ARE to exist. The first one is to assume that (A, G) is a stabilizable pair. This requirement is to assure the optimality of the optimization problem. Since optimality does not imply stability [Kalman, 1964], the condition for the stability will need to be given by (C, A) is a detectable pair, where C is a full-rank factorization (FRF) of Q, i.e.

$$C^T C = Q \tag{6.53}$$

and $rank(C) = rank(Q)$. In this way there exists a uniqe Hermitian p.s.d. solution X of the CARE, and also a stable

$$A - GX \tag{6.54}$$

is ensured.

Next, we consider the effects of the perturbation on the solution of the CARE. To the end, assume $\overline{X}$ to be a solution to the perturbed CARE

$$\overline{Q} + \overline{A}^H \overline{X} + \overline{X}\,\overline{A} - \overline{X}\,\overline{B}\,\overline{R}^{-1} \overline{B}^H \overline{X} = 0 \tag{6.55}$$

where

$$\overline{A} = A + \triangle A, \qquad \overline{Q} = Q + \triangle Q, \qquad \overline{G} = G + \triangle G, \qquad \overline{X} = X + \triangle X \quad (6.56)$$

The perturbations considered here only occur in the system matrix A and input matrix B. Uncertainties in the matrix B lead to the errors in the coefficient matrix G. Now, the primary problem is to find upper bounds on

$$||\overline{X} - X|| \quad (6.57)$$

Also we are interested in whether the solution $\overline{X}$ is p.s.d.

Before proceeding, we shall define a set of linear operator as follows:

$$\mathfrak{L} : H^{n \times n} \to H^{n \times n} \quad (6.58)$$

Then for any $V \in H^{n \times n}$, $\Omega \in H^{n \times n}$, we have

$$\mathfrak{L}V = \Omega^H V + V\Omega \quad (6.59)$$

Next, we define a second family of linear operator as follows:

$$\mathfrak{P} : C^{n \times n} \to H^{n \times n} \quad (6.60)$$

Then for any $M \in C^{n \times n}$, $F \in H^{n \times n}$, we have

$$\mathfrak{P}M = \mathfrak{L}\,(FM + MF) \quad (6.61)$$

Thirdly, let $\mathfrak{J}$ be defined as

$$\mathfrak{J} : H^{n\times n} \rightarrow H^{n\times n} \tag{6.62}$$

Then for any $N \in C^{n\times n}$, we have

$$\mathfrak{J}N = \mathfrak{L}^{-1}\left(FNF\right) \tag{6.63}$$

The objective of following analysis is to seek a first-order bound on the solution X of the CARE [Sun, 1998]. By eqns. (6.49), (6.55) and (6.56), it may be shown that

$$\bigl(A - GX\bigr)^{H} \triangle X + \triangle X\bigl(A - GX\bigr) = -Z + d_1\triangle X + d_2\triangle X \tag{6.64}$$

where

$$Z = \triangle Q + \triangle A^{H} X + X\triangle A - X\triangle G \tag{6.65}$$

$$d_1\bigl(\triangle X\bigr) = -\Bigl[\bigl(\triangle A - \triangle GX\bigr)^{H} \triangle X + \triangle X\bigl(\triangle A - \triangle GX\bigr)\Bigr] \tag{6.66}$$

$$d_2\bigl(\triangle X\bigr) = \triangle X\bigl(G + \triangle G\bigr)\triangle X \tag{6.67}$$

The above derivation involves matrix algebra and letting second and higher order terms equal to zero. Recall that

$$A_{cl} = A - GX \tag{6.68}$$

By eqns. (6.58) and (6.59), eqn. (6.64) becomes

$$\mathfrak{L}\triangle X = -Z + d_1\bigl(\triangle X\bigr) + d_2\bigl(\triangle X\bigr) \tag{6.69}$$

As A_{cl} is invertible, the operator $\mathfrak{L}$ is invertible. By eqns. (6.61), (6.63) and (6.65), we have

$$\mathfrak{L}^{-1}Z = \mathfrak{L}^{-1}\triangle Q + \mathfrak{P}\triangle A - \mathfrak{J}\triangle G \tag{6.70}$$

Define l, q, j as

$$l = ||\mathfrak{L}^{-1}||^{-1}, \qquad p = ||\mathfrak{P}||, \qquad j = ||\mathfrak{J}|| \tag{6.71}$$

$$||\triangle A|| + ||\triangle G||_2||X|| \tag{6.72}$$

Let $s = ||G||_2$ and $\overline{s} = s + ||\triangle G||_2$. By eqns. (6.67) and (6.71), it may be shown that

$$||d_1 \triangle X|| \leq 2d||\triangle X|| \tag{6.73}$$

$$||d_2 \triangle X|| \leq \overline{s}||\triangle X||^2 \tag{6.74}$$

Define e as

$$e = \frac{1}{l}||\triangle Q|| + p||\triangle A|| + j||\triangle G|| \tag{6.75}$$

Therefore, by eqns (6.70), (6.73), (6.74) and (6.75), the upper bound on $\triangle X$ is found to be

$$||\triangle X|| \leq e + \frac{2d}{l}||\triangle X|| + \frac{\overline{s}}{l}||\triangle X||^2 \tag{6.76}$$

Let $y = ||\triangle X||$. Next, we need to solve

$$\overline{s}y^2 - \big(1 - 2d\big)y + le = 0 \tag{6.77}$$

To seek a positive solution of eqn. (6.77), it is to let

$$d < \frac{l}{2}, \qquad e \leq \frac{(l - 2d)^2}{4l\overline{s}} \tag{6.78}$$

Eqn. (6.78) can be equivalently expressed as

$$d + \sqrt{l\bar{s}e} < \frac{l}{2} \tag{6.79}$$

The solution to (6.77) is obtained by

$$w = \frac{2le}{l - 2d + \sqrt{\left(l - 2d\right)^2 - 4l\bar{s}e}} \tag{6.80}$$

If eqn. (6.79) is satisfied, the bound eqn. (6.80) will give a unique Hermitian p.s.d solution $\overline{X}$ with the constraints of nonnegative definite property of $Q, G, \overline{Q}, \overline{G}$. It is noted that for sufficiently small $||\big(\triangle Q, \triangle A, \triangle G\big)||$, the relative perturbation bound for the solution X will satisfy

$$\frac{||\overline{X} - X||}{||X||} \leq \frac{||Q||}{l||X||}\frac{||\triangle Q||}{||Q||} + \frac{p||A||}{||X||}\frac{||\triangle A||}{||A||} + \frac{j||G||}{||X||}\frac{||\triangle G||}{||G||} \tag{6.81}$$

As a consequence, condition numbers will be derived based on this first order bound on the CARE solution.

6.5.5 Condition number of CARE

Condition numbers are actually scalars which reflect the first order perturbation bound for the solution of the CARE as the perturbations approach zero. They are classified depending on how the perturbations go to zero. Based on the theory of condition numbers [Byers, 1985] define the condition number $\kappa(X)$ of the CARE solution X as

$$\kappa(X) = \lim_{\delta \to 0} \sup_{\rho\left(\triangle Q, \triangle A, \triangle G\right) \leq \delta} \frac{||\triangle X||_F}{h\delta} \tag{6.82}$$

where the perturbations are defined as

$$\rho\big(\triangle Q, \triangle A, \triangle G\big) = \left\| \left(\frac{\triangle Q}{a}, \frac{\triangle A}{b}, \frac{\triangle Q}{c}\right) \right\|_F \tag{6.83}$$

with positive numbers a, b, c. By inspecting eqn. (6.81), Sun [Sun, 2002] presented two explicit expressions for condition numbers with respect to coefficient matrices Q, A and G as $\kappa_Q(X)$, $\kappa_A(X)$, $\kappa_G(X)$. If we take it that

$$h = a = b = c = 1 \tag{6.84}$$

then

$$\kappa_Q\left(X\right) = \frac{1}{l}, \qquad \kappa_A\left(X\right) = p, \qquad \kappa_G\left(K\right) = j \tag{6.85}$$

Eqn. (6.85) are the expression of the absolute condition numbers.

Let $\kappa_Q^r(X)$, $\kappa_A^r(X)$ and $\kappa_G^r(X)$ denote the relative condition numbers with respect to each coefficient matrix. By selecting

$$a = \|Q\|_F, \qquad b = \|A\|_F, \qquad c = \|G\|_F, \qquad h = \|X\|_F \tag{6.86}$$

It may be shown that

$$\kappa_G^r\left(X\right) = \frac{j\|G\|}{\|X\|}, \qquad \kappa_A^r\left(X\right) = \frac{p\|A\|}{\|X\|}, \qquad \kappa_Q^r\left(X\right) = \frac{\|Q\|}{l\|X\|} \tag{6.87}$$

In addition, the overall relative condition number $\kappa^r(X)$ is given by

$$\kappa^r\left(X\right) = \frac{1}{\|X\|_F}\sqrt{\left(\frac{\|Q\|_F}{l}\right)^2 + \left(p\,\|A\|_F\right)^2 + \left(j\,\|G\|_F\right)^2} \tag{6.88}$$

Then

$$\frac{1}{\sqrt{3}}\kappa_{Byer}(X) \leq \kappa^{r}(X) \leq \kappa_{Byer}(X) \tag{6.89}$$

6.6 Numerical results

In this section, we implement the examples in [Soroka *et al.*, 1984] and [Zhang *et al.*, 1996] to illustrate the validity of our results. We also implement some real control problem to verify our results. All the calculations were performed using MATLAB R2012a.

6.6.1 The case of Soroka *et al.*

Consider the CARE with coefficient matrices

$$\begin{aligned} &A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad Q = cc' = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \\ &G = b * b'/r, \qquad \text{where} \qquad c = [1,\ -1] \end{aligned} \tag{6.90}$$

where r is a scalar positive tuning parameter. The pair (A, G) is stabilizable and the pair (A, Q) is detectable. Suppose that the perturbations in the coefficients are

$$\begin{aligned} &\triangle Q = 10^{-j} \begin{bmatrix} 5 & -2 \\ -2 & 4 \end{bmatrix}, \qquad \triangle A = 10^{-j} \begin{bmatrix} 0.3 & -0.2 \\ 0.1 & 0.1 \end{bmatrix}, \\ &\triangle G = 10^{-j} \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & -0.3 \end{bmatrix} \end{aligned} \tag{6.91}$$

where

$$\overline{Q} = \triangle Q + Q, \qquad A = \triangle A + A, \qquad G = \triangle G + G \tag{6.92}$$

and $j = 12$. Let $\overline{X}$ be the unique symmetric p.s.d solution of the perturbed CARE

$$\overline{Q} + \overline{A}^H \overline{X} + \overline{X}\,\overline{A} - \overline{X}\,\overline{B}\,\overline{R}^{-1} \overline{B}^H \overline{X} = 0 \tag{6.93}$$

r	$\frac{\|\bar{X}-X\|_F}{\|\bar{X}\|_F}$	$\frac{e}{\|X\|_F}$	$CQ(X)$	$CA(X)$	$CG(X)$	$Cr(X)$
1.00E-01	3.42E-12	6.38E-12	1.73E+00	1.53	3.66E+00	4.33E+00
1.00E-02	6.85E-12	1.55E-11	4.35E+00	1.39	1.69E+01	1.75E+01
1.00E-03	2.59E-11	7.12E-11	2.03E+01	1.60	6.71E+01	7.02E+01
1.00E-04	9.79E-11	2.77E-10	7.91E+01	1.83	2.45E+02	2.58E+02
1.00E-05	3.45E-10	9.76E-10	2.79E+02	1.99	8.45E+02	8.89E+02
1.00E-06	1.16E-09	3.27E-09	9.33E+02	2.09	2.81E+03	2.96E+03

Table 6.1: Comparison of the relative error of the CARE solution X with the estimated upper bound and the condition numbers with respect to each coefficient matrix in the case of Soroka *et al.*

We used MATLAB function `care.m` to compute the unique symmetric p.s.d solution X of the CARE eqn. (6.49) and the unique symmetric p.s.d solution $\overline{X}$ of the perturbed CARE eqn. (6.93). The exact relative error

$$\left\|\overline{X} - X\right\|_F / \left\|X\right\|_F \tag{6.94}$$

can be readily obtained. Next, we compute the condition numbers given in eqn. (6.87) and eqn. (6.88) as the parameter r decreases by a factor of 10. By eqn. (6.81) we are also able to compute the upper bound on $||\overline{X} - X||$. The numerical results are shown in Table 6.1. The parameter r is the cost weighting matrix on the control input. $\frac{e}{||X||_F}$ and $\frac{\|\bar{X}-X\|_F}{\|\bar{X}\|_F}$ are the estimated upper bound and exact relative error of the CARE solution X, respectively. $CQ(X)$, $CA(X)$ and $CG(X)$ are the relative condition numbers with respect to the coefficient matrices Q, A and G. The quantity $C_r(X)$ is the overall relative condition number. Comparing the approximate upper perturbation bounds in eqn. (6.81) with the exact relative error, it shows that the estimated upper bound is very close to the exact relative error. Therefore, this estimation is shown to be very effective and the condition numbers can serve as a good indication of the true perturbation bounds of the solution X. It is also noticed that the condition numbers with respect to the coefficient matrices Q and G increase dramatically as r decreases, while the condition numbers with respect to the coefficient matrix A remained almost the same. That is to say, as r approach zero small perturbations in the coefficient matrices Q and G may result in a large relative change in the CARE solutions. In addition, it is observed that the condition number of the coefficient matrix G is much larger than that of the coefficient matrix Q, which means that the perturbations in G is the dominating factor in affecting the CARE solution. By inspecting the overall condition numbers, it is concluded that as r decreases, the CARE becomes more sensitive to the perturbations in its coefficient matrices. In a word, the numerical results in terms of condition numbers correlate well with the perturbation analysis in terms of control methods analyzed in the previous chapters.

6.6.2 The case of Zhang *et al.*

Recall that

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{6.95}$$
$$Q = q'q, \qquad G = bb'$$

where

$$q = \left[\sqrt{2r} - r,\ r\right] \tag{6.96}$$

and r is a positive tuning parameter. The pair (A, G) is stabilizable and the pair (A, Q) is detectable. Assume we have the same perturbations given by eqn. (6.91) as the case of Soroka *et al.* [Soroka *et al.*, 1984]. We use the MATLAB function `care.m` to compute the unique symmetric p.s.d solution X of the CARE, eqn. (6.49), and the unique symmetric p.s.d solution X of the perturbed CARE, eqn. (6.93). The approximate perturbation upper bound, the exact relative bound and the condition numbers are compared in Table 6.2. Here, r is the tuning parameter with respect to the cost weighting matrix Q. It is seen that the approximate perturbation upper bounds follow the exact relative error quite well as the parameter r goes up. By comparing the condition numbers related to their corresponding coefficient matrices, it is found that the condition numbers with respect to the coefficient matrix G, as a dominant factor, were much larger than the other two and they grew significantly as r increases. Thus, it is concluded that the solution of the CARE are more sensitive to perturbations to the coefficient matrix G, which also resulted in a significant overall condition number. Eventually, as was analyzed in the previous chapters, small perturbations in b in this particular case may cause instability as r approaches infinity, the numerical results in terms of the condition numbers corresponds well with real control situations.

r	$\frac{\|\bar{X}-X\|_F}{\|\bar{X}\|_F}$	$\frac{e}{\|X\|_F}$	$CQ(X)$	$CA(X)$	$CG(X)$	$Cr(X)$
1	2.97E-12	9.69E-12	0.78	4.25	1.76E+01	1.81E+01
10	3.56E-13	2.95E-11	3.30	2.38	1.46E+02	1.47E+02
20	2.72E-13	5.61E-11	6.34	2.16	2.85E+02	2.85E+02
30	2.47E-13	8.27E-11	9.36	2.07	4.23E+02	4.23E+02
40	1.83E-13	1.09E-10	12.36	2.01	5.60E+02	5.60E+02
50	1.51E-13	1.36E-10	15.35	1.98	6.96E+02	6.97E+02

Table 6.2: Comparison of the relative error of the CARE solution X with the estimated upper bound and the condition numbers with respect to each coefficient matrix in the case of Soroka *et al.*

6.6.3 Real control example

We shall also analyse the vulnerability of real control problem to structured perturbations via condition numbers.

The example we use is the two rolling carts system, which has been studied in Section 5.5.1. Recall that the state space system

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -30 & 30 & -3 & 3 \\ 7.5 & -42.5 & 0.75 & -2.25 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0.2 & 0.05 \\ 0 & 0 \end{pmatrix}$$

The criterion is to minimize the cost function

$$J(\epsilon) = \int_0^\infty \left[x^T Q x + \epsilon u^T R u \right], \epsilon > 0 \tag{6.97}$$

where the cost weighting matrices Q, R are the identity matrices and ϵ is a positive tuning parameter. All the coefficient matrices in the CARE are perturbed by small variations. When the parameter ϵ decrease from 1, the simulation result is shown in Table 6.3. The parameter ϵ is the positive tuning variable which decreases by a factor of 10. The second and the third column represent the exact relative error and the estimated upper bound of the CARE solution X. By comparing these two columns, it is shown that the estimated upper bound is very close to the exact relative error. The last three columns represent the relative condition numbers with respect to the coefficient matrices Q, A and G. The relative condition numbers with respect to the coefficient matrix Q are kept unchanged. When $\epsilon = 1$, the relative condition number of the coefficient matrix A is fairly large. As ϵ decreases, the condition numbers of A decrease monotonically. Eventually, the relative condition number of A stay at round 80. The last column represent the relative condition number with respect to the coefficient matrix G.

ϵ	1.00E+00	1.00E-01	1.00E-02	1.00E-03	1.00E-04	1.00E-05
$C_r(X)$	1.33E+02	1.31E+02	1.20E+02	1.27E+02	8.31E+02	8.20E+03

Table 6.3: Illustration of the overall relative condition numbers in the two rolling carts model

ϵ	1.00E-06	1.00E-07	1.00E-08	1.00E-09	1.00E-10	1.00E-11
$C_r(X)$	8.18E+04	8.18E+05	8.18E+06	8.18E+07	8.18E+08	8.18E+09

Table 6.4: Illustration of the overall relative condition numbers in the two rolling carts model

It is shown that the condition numbers of G increase dramatically as ϵ decreases. It is concluded that the coefficient matrix G is more sensitive to the perturbations than the coefficient matrices Q and A. Specially, large relative condition numbers of A indicate A is also sensitive to the variations, but much less vulnerable than G. The overall relative condition number is presented in Table 6.3 and Table 6.4. It is shown that as ϵ decreases the overall relative condition number increase dramatically. That is to say, the system is very vulnerable to structured perturbaions in the state-space matrices. If we perturb the first entry of the nominal input matrix B by very small variation, it is shown that one of the poles of the perturbed closed loop system resides in the right half plane of the complex plane. Therefore, the closed loop system becomes unstable. This has further corroborated that condition numbers can be used to test the vulnerability of the system to strucutred perturbations.

ϵ	$\frac{\|\bar{X}-X\|_F}{\|\bar{X}\|_F}$	$\frac{e}{\|X\|_F}$	$CQ(X)$	$CA(X)$	$CG(X)$
1.00E+00	2.1E-13	1.14E-11	1.43	133.20	1.98E-01
1.00E-01	1.92E-13	1.11E-11	1.39	131.30	1.87E+00
1.00E-02	1.26E-13	9.34E-12	1.25	119.11	1.33E+01
1.00E-03	1.56E-13	9.18E-12	1.50	90.19	8.94E+01
1.00E-04	3.2E-13	1.02E-11	1.80	79.36	8.27E+02
1.00E-05	3.4E-13	1.04E-11	1.86	79.57	8.20E+03
1.00E-06	3.43E-13	1.04E-11	1.86	80.15	8.18E+04
1.00E-07	3.43E-13	1.04E-11	1.86	80.39	8.18E+05
1.00E-08	3.7E-13	1.04E-11	1.86	80.48	8.18E+06
1.00E-09	3.23E-13	1.04E-11	1.86	80.50	8.18E+07
1.00E-10	2.17E-13	1.04E-11	1.86	80.51	8.18E+08
1.00E-11	3.32E-13	1.04E-11	1.86	80.52	8.18E+09

Table 6.5: Comparison of the relative error of the CARE solution X with the estimated upper bound and the condition numbers with respect to each coefficient matrix in the two rolling carts model

6.6.4 Analysis

In general, the condition numbers of two examples in [Soroka *et al.*, 1984] and [Zhang *et al.*, 1996] were compared and the exact relative error, as well as the estimated upper perturbation bound, was investigated. It is found that the estimated upper perturbation bound follows quite well the exact relative error. Also as the weight increases or decreases, the condition numbers with respect to the coefficient matrix G becomes dominant, which contributes significantly to the increasing phenomenon of the overall condition number. The first observation indicates that this perturbation bound estimation reflects the real change of the CARE solution very well. As a result, the condition numbers based on this criterion will be treated as an effective way to measure the sensitivity of the CARE when the coefficient matrices are subject to various errors. Furthermore, by investigating the condition numbers of real control problem, it is concluded that numerical perturbation analysis of the CARE in terms of the condition numbers can be utilized to reflect the vulnerability of LQR system to strucutred perturbations.

6.7 Conclusion

In this chapter, the robustness property of LQR against structured perturbations was revisited in this chapter. In particular, we studied the counter-example of Soroka *et al.*. It was shown that the excellent robustness result regarding the stability margins is not applicable. The stability bounds on the perturbations determined based on the general robustness result are compared with the exact ones. It was shown that the robustness properties of LQR are preserved in a general sense. The distinction between structured and unstructured perturbations must be made. We then investigate the perturbation situation of the CARE's in the case of Soroka *et al.* [Soroka *et al.*, 1984] and the case of Zhang *et al.* [Zhang *et al.*, 1996] in terms of their condition numbers. The condition numbers for some real control problem are also investigated. It is shown numerically that the condition numbers provides an effective way to detect the vulnerability of LQR system.

Chapter 7

Summary and conclusion

In this chapter, a brief summary of this thesis is presented. The contributions in the area of mathematical optimal control theory are also presented. Finally, some suggestions are made for future research.

7.1 Summary of the thesis

The purpose of this thesis is to investigate the stability robustness of linear quadratic regulators against both unstructured and structured perturbations. In regard to unstructured perturbations, the singular value technique is applied to the return difference equality, which yields novel robustness results which involve both cost weighting matrices. When the choice of the cost weighting matrices are not quite obvious, this provides a guideline for LQR designers to acquire satisfying stability robustness. For structured perturbations in some cheap control problems, the distinction from unstructured perturbations shall be made with respect to the classic notion of stability margins. Condition numbers are utilized to detect the vulnerability of LQR systems to variations.

A summary of automatic control and the linear quadratic regulator

It was noted that automatic control has a long history dating back to ancient times. In the early and middle periods of the twentieth century, classical and modern control theory have enjoyed a rapid development. Due to the work by Kalman and other researhers [Kalman, 1960; Kalman *et al.*, 1961], optimal control plays an important part in modern control. As one of the major contributions in this area, the LQR theory has been extensively studied by many researchers. In the meantime, the stability margin analysis provides a mathematical formulation for analyzing the robustness in feedback control systems. Based on the return difference inequaliy, it has been shown that LQR can have excellent stability margins.

A summary of cheap control problem

With the advent of LQR, cheap control has gained much attention. It is shown that it has good stability robustness to parameter variations. Many authors have studied the conditions for achieving perfect performance in cheap control problem, which makes it more attractive.

A summary of the perturbation analysis of the algebraic Riccati equation

It was noted that the Riccati equation has been studied for centuries. For optimal control problems, it has proven to be a key in synthesizing the optimal gains. It has then been extensively studied and tested, and benchmark examples have been proposed. The concept of condition number has also been proposed for measuring the sensitivity of the solution of the algebraic Riccati equation subject to variations. The proposal of an explicit way to evaluate the condition number gives a quantatitive aspect to the conditioning of the ARE.

Chapters 2-4 Some history of automatic control, particularly of the LQR theory, is presented in Chapter 2. The return difference equality (RDE), which is comprehensively studied in Chapter 3, is the core idea for analyzing the robustness properties of LQR. In Chapter 4, it has been shown that both the eigenvalue and the singular value properties of the return difference can be used to study the optimality and stability robustness of optimal control systems. In there, it has also been shown that LQR can have a general robustness property based on the return difference inequality. By selecting a special input matrix, namely a scalar times the identity matrix, the classic stability margins are obtained. These results show that the linear quadratic controllers can have a very good tolerance of a wide variaty of perturbations.

Chapter 5 Although LQR can have excellent stability margins, such as infinite gain margin, phase margins of $\pm 60^{\circ}$ and downside margin of 0.5, Chapter 5 presents more general guaranteed stability margins with respect to both the plant's state and the input matrices. This can be achieved by directly analyzing the return difference equality using the singular value method. As a result, a new lower bound on the minimal singular value of the return difference is proposed and evaluated. Thus, this gives us new guaranteed stability margins for LQR. The effect of tuning the state weighting matrix is also investigated for improving the robustness. For example, it is shown that $\underline{\sigma}(F_i(j\omega))$ is improved when $\underline{\sigma}(Q)$ is increased. When $\frac{\underline{\sigma}(R^{\frac{1}{2}})}{\overline{\sigma}(R^{\frac{1}{2}})}$ deviate from 1 in a continous way, the classic stability margins will be lost progressively. This gives a formal mathematical basis for guidelines for the designer to improve stability robustness.

Chapter 6 Despite the fact that LQR can have excellent stability margins against unstructured perturbations, it is shown by Soroka *et al.* [Soroka *et al.*, 1984] and Zhang *et al.* [Zhang *et al.*, 1996] that LQR, particularly for some cheap control problems, may suffer from poor robustness when the input matrix is subject to small variations. However, it is not applicable to utilize the classic notion of stability margin to cover these situations of structured perturbations. The perturbation bound evaluated based on the general robustness property of LQR is compared with the exact stability perturbation bound. Consequently, it is shown that the stability robustness properties are preserved in a general sense. It is also shown that the closed-loop may be very sensitive to variations in the state-space dynamical model of the plant, especially in the case of low objective function weights on the plant's inputs. This gives a caution in designing high gain LQR controllers. Furthermore, the analysis provides guidance in choosing LQR weights.

The algebraic Riccati equation (ARE) plays an important part in linear optimal control theory. It is natural to analyse LQR problem with structured perturbations relying on the perturbation analysis of the continuous-time algebraic Riccati equation. The perturbation situations of the CARE's in the cases of Soroka *et al.* [Soroka *et al.*, 1984] and the case of Zhang *et al.* [Zhang *et al.*, 1996] are explored in terms of the condition numbers. The condition numbers are utilized to quantitatively measure the sensitivity of the CARE solutions. With their explicit formulae specified, numerical results showed that the condition numbers are good for the detection of the optimal system's vulnerability.

7.2 Contributions of the thesis

The contributions of this thesis in the area of mathematical optimal control theory are listed below.

Proposal of a new lower bound on $\underline{\sigma}(F(s))$ and new guaranteed stability margins

We have generalized this by considering the robustness properties of LQR when both weights are considered, especially for non-diagonal weights. By applying the SVD to the RDE, some relationships between the $\underline{\sigma}$-plot of $F_i(jw)$ and the LQR weights are obtained. A new lower bound on the minimum singular value of the return difference is established. The guaranteed stability margins are related to the singular values of R. The effect of tuning the state weighting matrix for improving the robustness is studied. It is shown that, by appropriately tuning the weights, the stability robustness of LQR can be improved. It is also shown that when the input weighting matrix R deviates from ρI, the classic stability margins may be lost progressively. This yields guidelines for LQR designs.

Distinction between structured and unstructured perturbations

The robustness properties of some special cheap control problems are revisited. For the example of Soroka *et al.* [Soroka *et al.*, 1984], the excellent robustness result regarding the stability margins is not applicable. The stability bounds on the perturbations determined based on the general robustness result are compared with the exact ones. This shows that the robustness properties are preserved in a general sense. Thus, one should be more cautious in the designs of high gain LQR controllers.

Two cheap control problems considered by Soroka *et al.* [Soroka *et al.*, 1984] and Zhang *et al.* [Zhang *et al.*, 1996] are revisited here. The condition numbers are evaluated when either increasing the state weighting parameter or decreasing the input weighting parameter. It is shown that the CARE's become more sensitive to perturbations in the coefficient matrices in both cases. It is found that the perturbation analysis correlates well with the classic stability robustness results. The conditioning of the Riccati equation for a textbook example is presented. The simulation result shows that condition number is useful for detecting hidden vulnerabilities in LQR problems.

Some new results in mathematical control theory:

1. Theorem 3
2. Corollary 5
3. Corollary 7
4. Corollary 8
5. Corollary 9

7.3 Suggestions for future work

Some suggestions for future research are listed below.

1. The new lower bound on the return difference was evaluated only in relation to the singular values of the state weighting and input weighting matrices. The bound, for example in Corollary 9, may appear to be very conservative. By inspecting the inequality shown in Theorem 3, it is worth considering the shape of $\underline{\sigma}[H(s)]$ instead of Q only. The effect of tuning Q requires more investigation.

2. We have pointed out that the distinction between unstructured and structured perturbations for some cheap control problem should be made. However, the case only covers one parameter variation in the input matrix. It is feasible to evaluate the perturbation bound based on the general robustness result of LQR. Howerever, when considering the case of Zhang *et al.* [Zhang *et al.*, 1996], which assumes two independent parameter changes in both the system and input matrix, it would be somehow difficult to directly calculate the $\mathcal{H}_\infty$ perturbation bounds. A vector block diagram technique can be utilized to separate the perturbation from the loop dynamics. Based on the Nyquist criterion, the stability robustness can be revealed. In addition, when the perturbations happen in the gain of the plant's transfer function, one could ask how do different types of the state space realization of the plant affect the robustness properties.

3. The relationship between the perturbation analysis of the continuous-time algebraic Riccati equation and the robustness properties of the control problem are not very strong. Recall that the optimal gain, as a link between them, is obtained by solving the Riccati equation. If we re-optimize the perturbed LQR system, the new optimal gain will ensure stability. This re-optimization procedure will create a new Hermitian positive semi-definite (p.s.d) CARE solution by solving the perturbed CARE. Our task is to relate the CARE solution change to the optimal gain change in the right way so as to resolve the robustness issues in real control situations.

4. Singular perturbation analysis can be applied to cheap control problems. The resulting high-gain closed-loop system is closely related to a standard singular perturbed system [Kokotovic *et al.*, 1999], one could instead investigate

the robustness of the latter system with uncertainty [Chen *et al.*, 1990; Shao, 2004]. The sensitivity of such systems to the uncertainty could be examined. This would give us another way to analyze the stability robustness of LQR.

Bibliography

B. D. O. Anderson, "Sensitivity improvement using optimal design," *Proc. IEE.*, vol. 113, no. 6, pp. 1084-1086, June 1966.

B. D. O. Anderson, "Stability results for optimal systems," *Electronics Letters*, vol. 5, no. 22, Oct. 1969.

B. D. O. Anderson and J. B. Moore, *"Linear Optimal Control,"* Englewood Cliffs, N. J.:Prentice-Hall, 1971.

B. D. O. Anderson and J. B. Moore, *"Optimal Control-Linear Quadratic Methods,"* Englewood Cliffs, N. J.:Prentice-Hall, 1989.

B. D. O. Anderson and J. B. Moore, *"Optimal Control: Linear Quadratic Methods,"* Dover Publications, Incorporated, 2007.

G. B. Airy, "On the regulator of the clock-work for effecting uniform movement of equatorials," *Mem. R. Astron. Soc. 11*, pp. 249-267, 1840.

K. G. Arvanitis and G. Kalogeropoulos, "Guaranteed stability margins for discrete-time LQ optimal regulators for the performance index with cross-product terms," *Circ. Syst. and Signal Proc.*, vol. 16, iss. 6, pp. 663-701, 1997.

K. G. Arvanitis, G. Kalogeropoulos and S. Giotopoulos, "Guaranteed stability margins and singular value properties of the discrete-time linear quadratic optimal regulator," *IMA J. Math. Control Info.*, vol 18, iss. 3, pp. 299-324, 2001.

I. L. Ashkenas and D. T. McRuer, "A Theory of Handling Qualities Derived from Pilot-vehicle System Considerations," *Aerospace Engineering*, 1962.

W. F. Arnold, III and A. J. Laub, "Generalized eigenproblem algorithms and software for algebraic Riccati equations," *Proceedings of the IEEE*, vol. 72, no. 12, pp. 1746-1754, Dec. 1984.

H. W. Bode, *"Network Analysis and Feedback Amplifier Design,"* Van Nostrand, Princeton, 1945.

C. C. Bissell, "A history of automatic control," In: Nof. S. Y. (ed.) Springer Handbook of Automation. Springer handbook series, vol. LXXVI, pp. 53-69. Springer, Heidelberg, 2009.

S. Barnett and C. Storey, "Insensitivity of optimal linear control systems to persistent changes in parameters," *Int. J. Contr.*, vol. 4, pp. 179-184, 1966.

H. Bourles and O. I. Kosmidou, "Gain and phase margins of the discrete-time guaranteed cost regulator," *Decision and Control, 1994., Proceedings of the 33rd IEEE Conference on*, vol. 4, pp. 3837-3839 vol. 4, 14-16 Dec. 1994.

S. Bittanti, P. Colaneri and G. D. Nicolao, *"The Periodic Riccati Equation,"* in The Riccati Equation, S. Bittanti, A.J. Laub, J.C. Willems eds., Springer-Verlag, 1991.

P. Benner, A.J. Laub and V. Mehrmann, "Benchmarks for the numerical solution of algebraic Riccati equations," *Control Systems, IEEE*, vol. 17, no. 5, pp. 18-28, Oct. 1997.

R. S. Bucy, "Structural stability for the Riccati equation," *J. SIAM on Contr.*, Vol. 13, Iss. 4, 1975.

R. Byers, "Numerical condition of the algebraic Riccati equation," *in: B.N. Datta (Ed.), Linear Algebra and Its Role in System Theory, AMS Contemporary Mathematics Series*, vol. 47, AMS, Providence, RI, pp. 3549, 1985.

J. D. Cobb, "An explicit characterization of destabilizing phenomena for an arbitrary control system," in *Proceedings of the 26th Conference on Decision and Control, Los Angeles, U.S.A*, pp. 445-448, Dec. 1987.

J. D. Cobb, "Linear compensator designs based exclusively on input-output information are never robust with respect to unmodeled dynamics," *IEEE Trans. Automat. Contr.*, vol. 33, no. 6, pp. 559-563, June 1988.

J. B. Cruz, J. S. Freudenberg and D. P. Looze, "A relationship between sensitivity and stability of multivariable feedback systems," *IEEE Trans. Automat. Contr.*, vol. 26, no. 1, pp. 66-74, Feb. 1981.

S. S. L. Chang, *"Synthesis of Optimum Control Systems*," McGraw-Hill, New York, 1961.

W. A. Coppel, "Matrix quadratic equations," *Bull. Austral. Math. Soc.*, vol. 10, pp. 377-401, 1974.

C. T. Chen, "Stability of linear multivariable feedback systems," *Proceedings of the IEEE*, vol. 56, no. 5, pp. 821-828, May 1968.

C. H. Chen, "Perturbation analysis for solutions of algebraic Riccati equations," *J. Comput. Math.*, 6:336-347, 1988.

B. S. Chen and C. L. Lin, "On the stability bounds of singularly perturbed systems," *IEEE Trans. Automat. Contr.*, vol. 35, no. 11, pp. 1265-1270, Nov. 1990.

C. Chen and A. M. Holohan, "A revisit of the stability robustness of linear quadratic regulators," *25th IET Irish Signals and Systems Conference*, University of Limerick, 26-27 June 2014.

C. Chen and A. M. Holohan, "Robustness results of LQR-based systems with structured uncertainty," *25th IET Irish Signals and Systems Conference*, University of Limerick, 26-27 June 2014.

J. C. Doyle, “Robustness of multiloop linear feedback systems,” *Decision and Control including the 17th Symposium on Adaptive Processes, 1978 IEEE Conference on*, vol. 17, pp. 12-18, Jan. 1978.

J. C. Doyle and G. Stein, “Multivariable feedback design: Concepts for a modern/classical synthesis,” *IEEE Trans. Automat. Contr.*, vol. 26, no. 1, pp. 4-16, Feb. 1981.

J. C. Doyle, “Analysis of feedback systems with structured uncertainties,” *Control Theory and Applications, IEE Proceedings D*, vol. 129, no. 6, pp. 242-250, Nov. 1982.

F. N. Egerton, “A History of the Ecological Sciences, Part 21: Raumur and His History of Insects,” Bulletin of the Ecological Society of America 87(3): 212224, 2006.

G. V. Escherich, “Die Zweite Variation der Einfachen Integrale”, Wiener Sitzungsberichte 8, pp. 1191-1250, 1898.

J. S. Freudenberg and D. Looze, “Right half plane poles and zeros and design tradeoffs in feedback systems,” *IEEE Trans. Automat. Contr.*, vol. 30, no. 6, pp. 555-565, June 1985.

M. Fu, A. W. Olbrot and M. P. Polis, “Introduction to the parametric approach to robust stability,” *Control Systems Magazine, IEEE*, vol. 9, no. 5, pp. 7-11, Aug. 1989.

B. Friedland and C. K. Ling, “Quasi-Optimum Design of Control Systems for Moving-Base Simulators”, National Aeronautics and Space Administration, Washington, D.C., number NASA CR-1613, Contractor Report, 1970.

B. A. Francis and K. Glover, “Bounded peaking in the optimal linear regulator with cheap control,” *IEEE Trans. Automat. Contr.*, vol. 23, no. 4, pp. 608-617, Aug. 1978.

M. J. Grimble and T. J. Owens, “On improving the robustness of LQ regulators,” *IEEE Trans. Automat. Contr.*, vol. 31, no. 1, pp. 54-55, Jan. 1986.

D. Graham and R. C. Lathrop, “The synthesis of optimum transient response: Criteria and standard forms,” *American Institute of Electrical Engineers, Part II: Applications and Industry, Transactions of the*, vol. 72, no. 5, pp. 273-288, Nov. 1953.

A. J. Geurts, “A contribution to the theory of condition,” *Numerische Mathematik*, vol. 39, pp. 85-96, 1982.

I. Horowitz, *“Synthesis of Feedback Systems,”* Academic Press, New York, 1963.

I. M. Horowitz and U. Shaked, “Superiority of transfer function over state-variable methods in linear time-invariant feedback system design,” *IEEE Trans. Automat. Contr.*, vol. 20, no. 1, pp. 84-97, Feb. 1975.

A. C. Hall, *“The Analysis and Synthesis of Linear Servomechanisms (restricted circulation),”* The Technology Press, Cambridge, 1943.

A. C. Hall, “Application of circuit theory to the design of servomechanisms,” *J. Franklin Inst. 242*, pp. 279307, 1946.

C. Hermite, “Sur le nombre des racines d’une equation algebrique comprises entre des limites donnees,” *J. Reine Angew, Math.,* Vol. 52, pp. 39-51, 1854

A. Hurwitz, “Uber die Bedingungen, unter welchen eine Gleichung nur Wurzeln mit negativen reelen Teilen besitz,” *Math. Ann.,* Vol. 46, pp. 273-284, 1895.

D. Hinrichsen, A.J. Pritchard, “Stability radius for structured perturbations and the algebraic Riccati equation,” *Systems Control Letters*, vol. 8, Iss. 2, pp. 105-113, Dec. 1986.

G. Hewer and C. Kenney, “The sensitivity of the stable Lyapunov equation,” *J. SIAM Contr. Optim.*, vol. 26, iss. 2, pp. 321-343, March 1988.

G. Hewer, C. Kenney and A. G. Laub, "Sensitivity of control computations," in *Proceedings of the American Control Conference Philadelphia, Pennsylvania*, June 1998.

C. He, "On the distance to uncontrollability and the distance to instability and their relation to some condition numbers in control," *Numer. Math.*, vol. 76, iss. 4, pp. 463- 477, June 1997.

A. Jameson and R. E. O' Malley, Jr., "Cheap control of the time-invariant regulator," *Applied Mathematics and Optimization*, vol. 1, iss. 4, pp. 337-354, 1975.

R. E. Kalman, "A new approach to linear filtering and prediction problems," *Trans. ASME J. Basic Eng. 82*, pp. 34-45, 1945.

R. E. Kalman, "Contributions to the theory of optimal control," *Bol. Soc. Mat. Mexicana 5*, pp. 102-119, 1960.

R. E. Kalman and R. S. Bucy, "New results in linear filtering and prediction theory," *Trans. ASME J. Basic Eng. 83*, pp. 95-108, 1961.

R. E. Kalman, "When is a linear control system optimal?" *Trans. ASME (J. Basic Eng.)*, vol. 86, pp. 51-60, 1964.

H. Kwakernaak, "Optimal low-sensitivity linear feedback systems," *Automatica*, vol. 5, iss. 3, pp. 279-285, May 1969.

H. Kwakernaak and R. Sivan, "The maximally achievable accuracy of linear optimal regulators and linear optimal filters," *IEEE Trans. Automat. Contr.*, vol. 17, no. 1, pp. 79- 86, Feb. 1972.

H. K. Khalil, "On the robustness of output feedback control methods to modeling errors," *IEEE Trans. Automat. Contr.*, vol. 26, no. 2, pp. 524-526, Apr. 1981.

H. K. Khalil, "A further note on the robustness of output feedback control methods to modeling errors," *IEEE Trans. Automat. Contr.*, vol. 29, no. 9, pp. 861-862, Sep. 1984.

E. Kreindler, “Closed-loop sensitivity reduction of linear optimal control systems,” *IEEE Trans. Automat. Contr.*, vol. 13, no. 3, pp. 254-262, June 1968.

D. L. Kleinman, “On an iterative technique for Riccati equation computations,” *IEEE Trans. Automat. Contr.*, vol. 13, no. 1, pp. 114-115, Feb. 1968.

C. Kenney and G. Hewer, “The sensitivity of the algebraic and differential Riccati equation,” *J. SIAM Contr. Optim.*, vol. 28, iss. 1, pp. 50-69, Jan. 1990.

P. Kokotovic, H. K. Khalil and J. O’Reilly, *“Singular Perturbation Methods in Control: Analysis and Design,”* SIAM, London: Academic Press, 1999.

A. M. Lyapunov, “Probleme general de la stabilite de mouvement,” *Ann. Fac. Sci. Toulouse*, Vol. 9, pp. 203-474, 1907. (French transl. of the Russian paper published in *Comm. Soc. Math. Kharkow.*)

N. A. Lehtomaki, N. R. Sandell Jr., and M. Athans, “Robustness results in linear-quadratic Gaussian based multivariable control designs,” *IEEE Trans. Automat. Contr.*, vol. 26, no. 1, pp. 75-93, Feb. 1981.

A. J. Laub, “A Schur method for solving algebraic Riccati equations,” *IEEE Trans. Automat. Contr.*, vol. 24, no. 6, pp. 913-921, Dec. 1979.

P. Lancaster and L. Rodman, *“Algebraic Riccati Equations,”* Oxford University Press, 1995.

O. Mayr, *“The Origins of Feedback Control,”* M.I.T, Cambridge, 1970.

B. Marsden, *“Watt’s Perfect Engine,”* Columbia Univ. Press, New York, 2002.

J. C. Maxwell, “On governors,” *Proc. Roy. Soc. London*, vol. 16, pp. 27-283, 1868.

A. G. J. MacFarlane, “Return-difference and return-ratio matrices and their use in analysis and design of multivariable feedback control systems,” *Proc. IEE.*, vol. 117, no. 10, pp. 2037-2049, Oct 1970.

A. G. J. MacFarlane and B. Kouvaritakis, “A design technique for linear multivariable feedback systems,” *Int. J. Contr.*, vol. 25, iss. 6, 1977.

A. G. J. MacFarlane, “The development of frequency-response methods in automatic control [Perspectives],” *IEEE Trans. Automat. Contr.*, vol. 24, no. 2, pp. 250-265, April 1979.

J. M. Maciejowski, *“Multivariable Feedback Design,”* Addison-Wesley, Wokingham, England, 1989.

V. L. Mehrmann, *“The Autonomous Linear Quadratic Control Problem: Theory and Numerical Solution,”* Lecture Notes in Control and Information Sciences, vol. 163, Springer, 1991.

K. Martensson, “On the matrix Riccati equation,” *Information Sci.*, vol. 3, iss. 1, pp. 17-49, 1971.

H. Nyquist, “Stabilized feedback amplifiers,” *Bell Syst. Tech. J. 13*, pp. 118, 1934.

G. C. Newton, Jr., L. A. Gould and J. F. Kaiser, *“Analytical Design of Linear Feedback Controls,”* New York: John Wiley and Sons, Inc., 1957.

H. H. Niemann and O. Jannerup, “Stability Margins for Discrete-Time Optimal Regulators,” *American Control Conference, 1988*, pp. 1506-1507, 15-17 June 1988.

J. E. Potter, “Matrix quadratic solutions,” *SIAM J. Appl. Math. 14*, pp.496-501, 1966.

T. Pappas, A. J. Laub and N. R. Sandell, “On the numerical solution of the discrete-time algebraic Riccati equation,” *IEEE Trans. Automat. Contr.*, vol. 25, no. 4, pp. 631-641, Aug. 1980.

I. Postlethwaite and A. G. J. MacFarlane, *“A Complex Variable Approach to the Analysis of Linear Multivariable Feedback Systems,”* Springer, Berlin, 1979.

L. Qiu and E. J. Davison, "Performance limitations of non-minimum phase systems in the servomechanism problem," *Automatica*, vol. 29, Iss. 2, pp. 337-349, March 1993.

E. J. Routh, *"A Treatise on the Stability of a Given State of Motion,"* Macmillan, London, 1877.

H. H. H. Rosenbrock, "Design of multivariable control systems using the inverse Nyquist array," *Proceedings of the Institution of Electrical Engineers*, vol. 116, no. 11, pp. 1929-1936, Nov. 1969.

H. H. H. Rosenbrock, *"Computer-Aided Control System Design,"* Academic Press, London, 1974.

J. R. Rice, "A theory of condition," *J. SIAM Numer. Anal.*, vol. 3, no. 2, pp. 287-310, 1966.

M. G. Safonov and M. Athans, "Gain and Phase Margin for Multiloop LQG Regulators," *Decision and Control including the 15th Symposium on Adaptive Processes, 1976 IEEE Conference on*, vol. 15, pp. 361-368, Dec. 1976.

M. G. Safonov and M. Athans, "Gain and Phase Margin for Multiloop LQG Regulators," *IEEE Trans. Automat. Contr.*, vol. 22, no. 2, pp. 173-179, Apr. 1977.

M. G. Safonov, A. J. Laub and G. L .Hartmann, "Feedback properties of multivariable systems-The role and use of the return difference matrix," *IEEE Trans. Automat. Contr.*, vol. 26, no. 1, pp. 47-65, Feb. 1981.

M. G. Safonov and W. Wang, "Singular value properties of LQ regulators," *IEEE Trans. Automat. Contr.*, vol. 37, no. 8, pp. 1210-1211, Aug. 1992.

M. G. Safonov, "Origins of robust control: Early history and future speculations," *Annual Reviews in Control*, vol. 36, iss. 2, pp. 173-181, Dec. 2012.

Z. H. Shao, "Robust stability of two-time-scale systems with nonlinear uncertainties," *IEEE Trans. Automat. Contr.*, vol. 49, no. 2, pp. 258-261, Feb. 2004.

E. Soroka and U. Shaked, “On the robustness of LQ regulators,” *IEEE Trans. Automat. Contr.*, vol. 29, no. 7, pp. 664-665, July 1984.

J. Sun, “Perturbation theory for algebraic Riccati equation,” *SIAM Journal on Matrix Analysis and Applications*, Iss. 19, pp. 39-65, 1998.

J. Sun, “Condition numbers of algebraic Riccati equations in the Frobenius norm,” *Linear Algebra and its Applications*, vol. 350, iss. 1-3, pp. 237-261, 15 July 2002.

U. Shaked, “Guaranteed stability margins for the discrete-time linear quadratic optimal regulator,” *IEEE Trans. Automat. Contr.*, vol. 31, no. 2, pp. 162-165, Feb. 1986.

P. Sannuti and H. Wason, “Multiple time-scale decomposition in cheap control problems–Singular control,” *IEEE Trans. Automat. Contr.*, vol. 30, no. 7, pp. 633-644, July 1985.

A. Saberi and P. Sannuti, “Cheap and singular controls for linear quadratic regulators,” *IEEE Trans. Automat. Contr.*, vol. 32, no. 3, pp. 208-219, Mar. 1987.

J. A. Vyschnegradsky, “Sur la theorie generale des regulateurs,” *Comptes Rendus*, vol. 83, pp. 318-321, 1876.

J. A. Vyschnegradsky, “Uber direkt wirkende regulatoren,” *Civilingenieur*, vol. 23, pp. 95-131, 1877.

P. K. Wong, “On the interaction structure of multi-input feedback control systems,” M. S. thesis. Massachusetts Inst. Technol., Cambridge, MA, Sept. 1975.

P. K. Wong and M. Athans, “Closed-loop structural stability for linear-quadratic optimal systems Control,” in *Proc. 1976 IEEE Conf: Decision and Control*, Dec. 1976; also *IEEE Trans. Automat. Contr.*, vol. 22, pp. 94-99, Feb. 1977.

W. M. Wonham, “Random differential equations in control theory,” *Prob. Methods in Appl. Math.,*, A. T. Bharucha-Reid (ed.), 2, New York, Academic Press, pp.

132-212, 1970.

J. C. Willems, “Least squares stationary optimal control and the algebraic Riccati equation,” *IEEE Trans. Automat. Contr.*, vol. 16, no. 6, pp. 621-634, Dec. 1971.

S. Xu, “Sensitivity analysis of the algebraic Riccati equations,” *Numer. Math.*, vol. 75, iss. 1, pp. 121-134, Nov. 1996.

C. Zhang and M. Fu, “A revisit to the gain and phase margins of linear quadratic regulators,” *IEEE Trans. Automat. Contr.*, vol. 41, no. 10, pp. 1527-1530, Oct. 1996.

L. Zhou, Y. Lin, Y. Wei and S. Qiao, “Perturbation analysis and condition numbers of symmetric algebraic Riccati equations,” *automatica*, vol. 45, pp. 1005-1011, 2009.

L. Zietsman, K. A. Evans, J. T. Brown and R. A. Idowu, “Riccati conditioning and sensitivity for a MinMax controlled cable-mass system,” *Decision and Control, 2008. CDC 2008. 47th IEEE Conference on*, pp. 4007-4011, 9-11 Dec. 2008.