Robust and Optimal Adaptive Meshes for Non-Linear Differential Equations with Finite-Time Singularities: Motivated by Finance

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To Mam and Dad

Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Doctor of Philosophy is entirely my own work, and that I have exercised reasonable care to ensure that the work is original, and does not to the best of my knowledge breach any law of copyright, and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

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PhDs are difficult but impossible without support from many people throughout the process. The fact this thesis is now finished is thanks to the many people who have been there for me from the very beginning. It's now my turn to make sure they get the recognition they rightly deserve.

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Abstract

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This thesis studies the related problems of modelling highly non-linear Ordinary and Stochastic equations whose solutions remain positive but either converge to an equilibrium point or blow-up. Neither a metric nor rigorous results in continuous time to characterise these solutions asymptotic behaviour exist. Direct discretisations of the equation using fixed-step numerical schemes fail to reproduce important qualitative properties such as positivity. The thesis develops a suitable metric, a generalised Liapunov exponent, to describe the asymptotic convergence and reliable adaptive numerical schemes that are optimal for both ODEs and SDEs. The schemes are optimal in the sense of minimising computational effort by taking the largest step-size possible whilst preserving the qualitative properties and correct asymptotic behaviour of the continuous-time solution.

The schemes recover the qualitative properties and asymptotic rates of convergence under assumptions of monotonicity and regular variation. The critical rate of decay for the step-size is identified. The work shows the resulting error in the convergence rate is insensitive to the assumption of regular variation.

Transforming the co-ordinate system is essential to preserving positivity in the case of SDEs. We determine the class of suitable transforms to use and identify that a logarithmic pre-transformation is optimal for ODEs. The class of suitable transformation shows that the problems of hitting an equilibrium and explosion in solutions for ODEs are not equivalent problems in terms of numeric schemes. We develop a quasi-adaptive scheme that can revert to a fixed-step when less computational effort is needed for SDEs. This quasi-adaptive scheme is universal: the scheme works on the highly non-linear problems covered by the thesis and on more standard problems with non-positive solutions, exponential or sub-exponential convergence.

The Implicit, Explicit and Transformed schemes can be ranked as measured by the error in convergence rates. No scheme is superior in all circumstances but a ranking can always be achieved.

Contents

Chapter 1

Introduction

1.1 Synopsis of Research

This research focuses on highly non-linear Ordinary Differential Equations ,“ODEs”, and Stochastic Differential Equations, “SDEs”, whose solutions either reach an equilibrium at zero in finite-time or do not hit zero but approach it quicker than any negative exponential function. We determine rigorous results which characterise the solutions' asymptotic behaviour near the equilibrium and prove that adaptive numerical methods reproduce its qualitative behaviour. Most of the research is devoted to determining which mesh size to use in numerical schemes and in making this optimal. The mesh size is optimal in the sense that it is the largest one possible while still recovering the qualitative and asymptotic behaviour of the underlying ODE. This optimal mesh minimises the computational effort, number of iterations of the Euler scheme and run-time required to reproduce this behaviour in computer simulations. For both continuous and discrete problems, the precise asymptotic behaviour (either decay rate to zero or asymptotics at the finite hitting time) of solutions is captured.

1.2 Continuous Non-Linear ODEs

We examine the asymptotic and qualitative behaviour of Euler discretisations of the scalar non-linear ODE:

$$x'(t) = -f(x(t)), \quad t > 0, \quad x(0) = \xi > 0, \tag{1.1}$$

which has a unique globally stable equilibrium at zero. Solutions are very strongly attracted to zero as they approach the equilibrium, as measured by $f'(x) \to \infty$ as $x \to 0^+$. It is possible to characterise whether:

- x hits zero in finite-time at time T_ξ; or
- x remains positive for all time and approaches zero quicker than any negative exponential function as $t \to \infty$.

The nature of this finite-time "hitting" of the equilibrium is better termed "kissing" because the solution smoothly approaches the equilibrium which we term a "soft landing". The soft landing is captured by the asymptotic relation:

$$\lim_{t \to T_\xi^-} x'(t) = 0.$$

In the case that the solution converges quicker than any negative exponential function we say that the convergence is "super-exponential" and that the zero solution is "super-exponentially stable". We use the term "stable" informally throughout, understanding that it is shorthand for asymptotically stable. With f obeying $f'(0^+) = \infty$ the solution of (1.1) has an infinite negative Liapunov exponent, expressed as:

$$\lim_{t \to \infty} \frac{\log x(t)}{t} = -\infty,$$

in the case of super-exponential stability: indeed it is this convergence to zero more quickly than any exponential function (which always have a finite negative Liapunov exponent) that gives rise to the terminology super-exponential convergence or stability. The infinite negative Liapunov exponent characterises super-exponential convergence but does not give particularly refined asymptotic information about the speed of convergence. In order to capture exact asymptotic behaviour comprehensively, we develop analogues of Liapunov exponents for non-linear ODEs.

We investigate Euler schemes in which the mesh size varies according to the state of the system and tends to zero as the discrete solution gets closer to the equilibrium. We examine whether the Euler schemes preserve the asymptotic and qualitative behaviour of interest, namely:

- The positivity, monotonicity and asymptotic convergence of solutions in all cases;
- Reproducing the finite hitting time and gentle approach to the equilibrium when T_ξ is finite; and
- Super-exponential convergence to zero when T_ξ is infinite.

We will assume throughout that $f : [0, \infty) \to \mathbb{R}$ is a continuous function with the following properties:

$$f(x) > 0 \text{ for all } x > 0; \tag{1.2}$$

$$f(0) = 0; \tag{1.3}$$

$$f \text{ is locally Lipschitz continuous on } [\epsilon, \infty) \text{ for every } \epsilon > 0; \tag{1.4}$$

$$f \in C([0, \infty); \mathbb{R}); \text{ and} \tag{1.5}$$

$$\lim_{x \to 0^+} \frac{f(x)}{x} = \infty. \tag{1.6}$$

As we mentioned, super-exponential convergence is characterised by having an infinite derivative of f at the equilibrium, see (1.6). When the derivative of f at the equilibrium is finite and non-zero then the decay is exponential and when the derivative is zero then the decay is sub-exponential. In these cases, conventional fixed step numerical methods will reproduce satisfactory performance. So for deterministic problems at least, we will generally impose (1.6) and not study numerically ODEs for which $f'(0^+)$ is finite. Under condition (1.4) on f the Initial Value Problem (1.1) has a unique continuous solution on a maximal interval of existence $[0, T_\xi)$. On this interval of existence, x is positive and decreasing. In the case that

$$\int_{0^+}^{1} \frac{1}{f(u)}\, du < \infty, \tag{1.7}$$

it follows that $T_\xi < \infty$. A formula for T_ξ is given by

$$T_\xi = \int_0^\xi \frac{1}{f(u)}\, du, \tag{1.8}$$

and $\lim_{t \to T_\xi^-} x(t) = 0$. In the case that

$$\int_{0^+}^{1} \frac{1}{f(u)}\, du = \infty, \tag{1.9}$$

it follows that $T_\xi = \infty$ and that $\lim_{t\to\infty} x(t) = 0$. These results are easily established by integration.

We introduce some auxiliary functions to determine the asymptotic behaviour of x. In the case that f obeys (1.7), the function $\bar{F}$ given by

$$\bar{F}(x) = \int_0^x \frac{1}{f(u)}\, du, \quad x > 0, \tag{1.10}$$

is well-defined. In the case that f obeys (1.9), the function F given by

$$F(x) = \int_x^1 \frac{1}{f(u)}\, du, \quad x > 0, \tag{1.11}$$

is well-defined. By construction and (1.2), both (1.10) and (1.11) are positive. Moreover, (1.10) is increasing while (1.11) is decreasing, so both functions have an inverse, denoted $\bar{F}^{-1}(t)$ and $F^{-1}(t)$ respectively. Integration of (1.1) over the appropriate time interval leads to the following well-known results which we present as a theorem. Our motivation for presenting them is to help compare the qualitative properties of the solution of (1.1) against the qualitative success of the numerical schemes we consider.

Theorem 1. *Suppose that f obeys* (1.2) *and that $\xi > 0$. Then there exists a unique* *continuous solution of* (1.1) x *on the interval* $I_\xi = [0, T_\xi)$. *Moreover* x *is positive and decreasing on* I_ξ *with*

$$\lim_{t \to T_\xi^-} x(t) = 0. \tag{1.12}$$

(i) If f *obeys* (1.9), *then* $T_\xi = \infty$ *and* x *obeys*

$$\lim_{t \to \infty} \frac{x'(t)}{x(t)} = -\infty, \tag{1.13}$$

$$F(x(t)) = t + F(\xi), \quad t \geq 0 \quad \textit{and moreover} \quad \lim_{t \to \infty} \frac{F(x)}{t} = 1. \tag{1.14}$$

(ii) If f *obeys* (1.7), *then* T_ξ *is finite and given by* (1.8). *Moreover*

$$\lim_{t \to T_\xi^-} x'(t) = 0, \tag{1.15}$$

$$\bar{F}(x(t)) = T_\xi - t, \quad 0 \leq t < T_\xi \quad \textit{and moreover} \quad \lim_{t \to T_\xi^-} \frac{\bar{F}(x(t))}{T_\xi - t} = 1. \tag{1.16}$$

1.3 Outline of Continuous Non-Linear SDEs

The second part of the thesis extends the same analysis of the ODE's asymptotic and qualitative behaviour to a scalar time-homogeneous diffusion process $X(t)$ whose sample paths obey the SDE:

$$dX(t) = f(X(t))\,dt + g(X(t))\,dB(t), \tag{1.17}$$

and has positive solution on a maximal interval of existence and a unique equilibrium at zero. We assume also that

$$X(0) = \zeta \text{ and } \zeta > 0 \text{ is deterministic.} \tag{1.18}$$

Requesting a unique equilibrium forces the hypothesis and continuity requirements on f and g

$$f, g \in C([0, \infty); \mathbb{R}), \quad \text{with } f(0) = g(0) = 0. \tag{1.19}$$

We assume the so-called "non-degeneracy condition"

$$g^2(x) > 0, \quad \text{for all } x > 0, \tag{1.20}$$

holds for convenience, so that there are no equilibria other than the zero equilibrium on $[0,\infty)$. The non-degeneracy condition corresponds to the noise never "switching off". The continuity restriction on f and g, and the positivity of g ensures that the local integrability restriction

$$\int_{x-\epsilon}^{x+\epsilon} \frac{1+|f(y)|}{g(y)^2}\,dy < \infty, \quad \text{for all } x>0 \text{ and some } \epsilon>0,$$

holds. These assumptions yield $X(0)=0$ implies $X(t)=0$ for all $t\geq 0$. Assuming local Lipschitz continuity away from zero and infinity according to

$$f \text{ and } g \text{ are locally Lipschitz continuous on } [1/k,k], \text{ for every } k\in\mathbb{N}, \tag{1.21}$$

ensures that there is a unique continuous adapted process which obeys (1.17) on $[0,T)$ where

$$T=\inf\{t>0\,:\,X(t)\notin(0,\infty)\}. \tag{1.22}$$

T can be thought of as the "explosion time" or "time of hitting zero". We define $T=\infty$ if $\inf\{t>0\,:\,X(t)\notin(0,\infty)\}=\emptyset$. More precisely, we have for the sequence of stopping times $\tau_k=\inf\{t>0: X(t)=k \text{ or } X(t)=1/k\}$ that X obeys

$$X(\min(t,\tau_k))=\zeta+\int_0^{\min(t,\tau_k)} f(X(s))\,ds+\int_0^{\min(t,\tau_k)} g(X(s))\,dB(s), \quad 0\leq t<\infty, \quad \text{a.s.}, \tag{1.23}$$

where $\tau_k\to T$ as $k\to\infty$. We wish to consider situations in which X is a.s. attracted to zero, in the sense that

$$X(t)>0\text{for all } t\in[0,T) \quad \textit{and} \quad \lim_{t\to T^-} X(t)=0, \quad \text{a.s.}.$$

When the drift of (1.17) dominates, in a certain sense, the dynamics of (1.17) are governed by an auxiliary ODE which is in fact the noiseless unperturbed ODE

$$x'(t)=f(x(t)), \tag{1.24}$$

while in the case when the diffusion dominates, the dynamics are governed by another auxiliary ODE

$$x'(t)=-c\frac{g^2(x(t))}{2x(t)},$$

for some positive constant c. If the limit

$$\lim_{x\to 0^+}\frac{xf(x)}{g^2(x)}=:L, \tag{1.25}$$

exists, then L describes these two regimes of asymptotic behaviour. This general claim holds irrespective of whether solutions remain positive or tend to zero in finite-time. When $L = -\infty$, we recover the same asymptotics as the ODE (1.24) because the contribution of the noise term to the asymptotic behaviour is small relative to the drift. In particular, under certain monotonicity conditions on f at zero, we will have that

$$\int_{0+}^{1} \frac{1}{|f(u)|}\,du < \infty, \tag{1.26}$$

implies $T < \infty$ a.s.. In the case that

$$\int_{0+}^{1} \frac{1}{|f(u)|}\,du = \infty, \tag{1.27}$$

we have $T = \infty$ a.s.. We introduce some auxiliary functions to determine the asymptotic behaviour of X. In the case that f obeys (1.26), the function $\bar{F}$ given by

$$\bar{F}(x) = \int_{0}^{x} \frac{1}{|f(u)|}\,du, \quad x > 0, \tag{1.28}$$

is well-defined. In the case that f obeys (1.27), the function F given by

$$F(x) = \int_{x}^{1} \frac{1}{|f(u)|}\,du, \quad x > 0, \tag{1.29}$$

is well-defined. When $-\infty < L < 1/2$, the noise contribution dominates and the asymptotic behaviour is the same as that of solutions of the deterministic equation

$$x'(t) = -\left(\tfrac{1}{2} - L\right)\frac{g^2(x(t))}{x(t)}, \quad t > 0. \tag{1.30}$$

In particular it follows that

$$\int_{0+}^{1} \frac{u}{g^2(u)}\,du < \infty, \tag{1.31}$$

implies $T < \infty$ a.s.. In the case that

$$\int_{0+}^{1} \frac{u}{g^2(u)}\,du = \infty, \tag{1.32}$$

we have $T = \infty$ a.s.. We use analogous auxiliary functions to determine the asymptotic behaviour of X. In the case that g obeys (1.31), the function $\bar{G}$ given by

$$\bar{G}(x) = \int_{0}^{x} \frac{u}{g^2(u)}\,du, \quad x > 0, \tag{1.33}$$

is well-defined. In the case that g obeys (1.32), the function G given by

$$G(x) = \int_{x}^{1} \frac{u}{g^2(u)}\,du, \quad x > 0, \tag{1.34}$$

is well-defined.

1.4 Regularly Varying Functions

Regularly varying functions are an important class to examine because f for many ODEs, both linear and non-linear, is specified in terms of them. They are a natural enlargement of the class of power functions which arises in many applications. A measurable function $f : (0, \infty) \to (0, \infty)$ with $f(x) > 0$ for $x > 0$ is said to be regularly varying at 0 with index $\beta \in \mathbb{R}$ if

$$\lim_{x \to 0^+} \frac{f(\lambda x)}{f(x)} = \lambda^\beta, \quad \text{for all } \lambda > 0. \tag{1.35}$$

We use the notation $f \in RV_0(\beta)$. If $\beta = 0$, f is said to be slowly varying at zero and we denote this by $f \in \text{RV}_0(0)$ or $f \in \text{SV}_0(0)$. Regular variation at infinity arises when the limit in (1.35) is taken as $x \to \infty$, and we write $f \in \text{RV}_\infty(\beta)$ in this instance.

To motivate the use of regularly varying functions in our application, we see when $\beta > 0$, if f obeying (1.35) is also continuous, that $f(0) = 0$. Moreover it is the case that for $\beta > 0$, f is asymptotic to an increasing function, and for $\beta < 1$, we have $f(x)/x \to \infty$ as $x \to 0^+$, but for $\beta > 1$ we have $f(x)/x \to 0$ as $x \to 0^+$. From these simple observations, it can be seen that it is quite natural to take f in (1.1) to be a function in $\text{RV}_0(\beta)$ for $\beta \in (0, 1)$, although the cases $\beta = 0$ and $\beta = 1$ are also possible.

The convergence in (1.35) is uniform in λ; this result is called the uniform convergence theorem for regularly varying functions. Regular variation is an important quantitative property because we can quantify the change in value of the function when the argument is scaled by a factor of λ. Important results about regular variation that we use in this work can be found in the monograph of Bingham, Goldie and Teugels [12]. The literature is written in terms of functions regularly varying at infinity. However, regular variation can also be defined at any point $x_0 \in \mathbb{R}$ by requiring that $f(x_0 - x^{-1})$ is regularly varying at infinity.

The sign and size of β give information about the qualitative behaviour of f. The larger the value of β then the quicker the rate-of-increase as $x \to \infty$, but the slower the rate-of-increase as we increase away from 0. The functions

$$x^\beta, \quad x^\beta \log(1/x), \quad (x \log(1/x))^\beta,$$

are regularly varying at zero with index β. Typical examples of slowly varying functions are positive constants, functions converging to a positive constant, logarithms and iterated logarithms.

We restrict our analysis to functions with an index between 0 and 1. The integral defined by

$$\int_x^1 \frac{1}{f(u)}\, du,$$

is guaranteed to converge as $x \to 0^+$ when $\beta < 1$. When $\beta = 1$ the integral will converge for some f's and diverge for others. Super-exponential convergence is impossible in the solution x of (1.1) when $\beta > 1$ because x converges sub-exponentially.

1.5 Summary of Main Results

Most non-linear ODEs cannot be solved analytically and do not have explicit solutions. Even when it is possible to find a closed-form solution, we may still be faced with equations of enormous complexity and size making the closed-form solution useless for most practical purposes. Solutions must be approximated numerically as a result. Many numerical methods exist for solving ODEs but differ in terms of accuracy, performance and applicability. The one-step Explicit and Implicit Euler methods with constant step-size are among the simplest methods.

Let $h > 0$. The Explicit Euler scheme for equation (1.1) with constant step-size h is given by

$$x_{n+1} = x_n - hf(x_n), \quad n \geq 0, \quad x_0 = \xi > 0, \tag{1.36}$$

and the Implicit Euler scheme with constant step-size h is given by

$$x_{n+1} = x_n - hf(x_{n+1}), \quad n \geq 0, \quad x_0 = \xi > 0. \tag{1.37}$$

The increment h is called the "mesh" or "step-size". The step-size is a parameter of the method and determines the accuracy of the approximation. The smaller the step-size then the more accurate the approximation. A constant step-size produces unsatisfactory results when used with both the Explicit and Implicit Euler schemes to discretise the non-linear ODE (1.1) when f obeys (1.2)-(1.6). An Explicit scheme becomes negative after a finite number of time steps so the positivity of a solution is not preserved, except for a small set of initial conditions. This is made precise below.

Proposition 1. *Suppose that f obeys* (1.2), (1.3), (1.5) *and* (1.6). *Suppose f in C^2 with $f''(x) < 0$ for all $x > 0$. Let (x_n) be the solution of* (1.36). *Then for each $h > 0$ there is a set $\Lambda(h)$ such that $\Lambda(h) = (0,\infty) \setminus C_h$ where C_h is an at most countable subset of $(0,\infty)$, such that if $\xi \in \Lambda(h)$, then there exists an $N = N(h) \in \mathbb{N}$ such that $x_{N(h)} < 0$.*

Without the C^2 assumption on f, we can say that for all initial conditions, the solution will become non-positive after a finite number of steps. This is easily seen. Suppose $x_n > 0$ for some initial value $x_0 = \xi > 0$. Then (x_n) is decreasing and bounded below, so has a limit $L \in [0, \infty)$. If $L > 0$, by continuity of f we get the contradiction $L = L - hf(L)$. Hence $L = 0$. But now, if we write

$$\frac{x_{n+1}}{x_n} = 1 - h\frac{f(x_n)}{x_n},$$

and take limits as $n \to \infty$, we get $\lim_{n\to\infty} x_{n+1}/x_n = -\infty$, which violates the hypothesis that (x_n) remains positive.

Implicit methods do not suffer from this problem. In fact, solutions of (1.37) remain positive and tend monotonically to zero. However, an Implicit scheme predicts $[0, \infty)$ as the interval of positivity of the solution and that therefore $T_\xi = \infty$, regardless of whether the underlying ODE has a finite interval of existence or not. This means it would, incorrectly, not recover a finite-time hitting of zero if the solution of the underlying ODE has that property.

Proposition 2. *Suppose that f obeys* (1.2), (1.3), (1.5). *Then there exists at least one sequence (x_n) which obeys* (1.37). *Moreover, all non-negative solutions of* (1.37) *are decreasing, positive and obey* $\lim_{n\to\infty} x_n = 0$.

These unsatisfactory results motivate using a mesh which changes size in line with the state of the system, "an adaptive mesh". The qualitative properties of $h(x)$ are that is should be positive and continuous so it correctly models the time index. We will mostly suppose that $h(x)$ is asymptotic to $x/f(x)$, but ask what happens if $h(x)$ tends to zero more quickly or slowly than $x/f(x)$ as $x \to 0^+$ also. These are made precise below:

$$h(x) > 0 \text{ for all } x > 0; \tag{1.38}$$

$$h \in C([0, \infty); \mathbb{R}); \text{ and} \tag{1.39}$$

$$\lim_{x\to 0^+} \frac{h(x)f(x)}{x} = \Delta \in [0, \infty]. \tag{1.40}$$

Ideally, $h(x)$ should tend to zero as the solution approaches an equilibrium point to improve the accuracy of the approximation. We approximate $x(t_n)$ by x_n, where $x(t_n)$ is the solution x of (1.1) at time t_n. The associated Explicit Euler scheme is

$$x_{n+1} = x_n - h(x_n)f(x_n), \quad n \geq 0, \quad x_0 = \xi > 0, \tag{1.41}$$

where

$$t_{n+1} = \sum_{j=0}^{n} h(x_j), \quad n \geq 0, \quad t_0 = 0. \tag{1.42}$$

The associated Implicit Euler scheme is

$$x_{n+1} = x_n - h(x_{n+1})f(x_{n+1}), \quad n \geq 0, \quad x_0 = \xi > 0, \tag{1.43}$$

where

$$t_{n+1} = \sum_{j=0}^{n} h(x_{j+1}), \quad n \geq 0, \quad t_0 = 0. \tag{1.44}$$

Recalling Theorem 1, if we write

$$F(x) = \int_x^1 \frac{1}{f(u)}\, du,$$

then the solution of (1.1) obeys

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 1, \tag{1.45}$$

when $F(x) \to \infty$ as $x \to 0^+$ and

$$\lim_{t\to T_\xi^-} \frac{\bar{F}(x(t))}{T_\xi - t} = 1, \tag{1.46}$$

where T_ξ is the finite number $\int_0^\xi 1/f(u)\, du$ when $F(x)$ tends to a finite limit as $x \to 0^+$. We prove in Chapter 4 that when f is regularly varying and

$$h(x) \sim \frac{\Delta x}{f(x)}, \qquad \text{as } x \to 0^+,$$

then the numerical method obeys the same asymptotic behaviour as the continuous solution, namely

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = c_1(\Delta) \to 1, \qquad \text{as } \Delta \to 0^+, \tag{1.47}$$

where $t_n = \sum_{j=0}^{n-1} h(x_j)$ for $n \geq 1$ and $F(x) \to \infty$ as $x \to 0^+$. If $\bar{F}(x)$ tends to a finite limit as $x \to 0^+$

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = c_2(\Delta) \to 1, \qquad \text{as } \Delta \to 0^+, \tag{1.48}$$

where $\hat{T}_h := \sum_{n=0}^{\infty} h(x_n) < \infty$. The rate is optimal when

$$h(x) = o(x/f(x)), \qquad \text{as } x \to 0^+,$$

for in this case the $c_i(\Delta)$'s would be exactly 1, with $\Delta = 0$. On the other hand, for the Explicit Scheme when

$$\lim_{x\to 0^+} \frac{h(x)f(x)}{x} = \infty,$$

the discretised solution will oscillate in sign, violating the positivity and monotonicity of the solution. The exact features of the asymptotic behaviour are recovered if the step-size tends to zero more rapidly than $x/f(x)$. However, the numerical method gives a solution which violates important properties of the continuous equation if the step-size tends to zero more slowly than $x/f(x)$. These identify the relative rates of growth of $h(x)$ and $x/f(x)$ as being critical.

These results are developments of work in Chapter 3, which merely assume certain types of monotonicity on f. The corresponding asymptotic results are typically weaker, with limits inferior and superior being found in place of limits. However, these limits are within $O(\Delta)$ of true limits as $\Delta \to 0^+$, as are estimates on the first hitting time of zero, T_ξ, when it is finite. In this sense, the advantage of working with regular variation is that it demonstrates that working with one-step methods, one cannot take asymptotically larger step-sizes without destroying some qualitative or quantitative aspect of the solution.

The situation is more delicate in the case when the index of regular variation is zero or unity, and it may be possible to take asymptotically larger step-sizes in these cases. We investigate this in more depth for slowly varying f in the case of the Implicit method in Chapter 4, as well as the possibility that multi-step methods may give enhanced performance. This investigation is taken up in Chapter 7 for finite-time stability problems and for explosion problems in Chapter 8. There is evidence that the midpoint method may out-perform the one-step methods, in that the exponent in (1.48) is accurate to $O(\Delta^2)$ for explosion problems.

The Explicit method has the advantage of computational speed, but there are restrictions on the parameter. The Implicit method has no restrictions on the parameter but a non-linear equation must be solved at each time step. A method which has the advantages of both methods is an explicit method in which the state space of the original ODE is transformed, so that after simulation (with a step-size of the same order) and recovery of the value in the old co-ordinate system, the solution is still guaranteed to be positive. In Chapter 5 we show that a smart choice of transformation from the solution $x = x(t)$ of the original ODE to the new ODE $z = z(t)$ is $z(t) = T(x(t))$ where $T(x) = -\log x$. This choice is prudent both on practical and theoretical grounds: it is practical because it is easy to apply and to invert, and also because this choice tends to minimise errors in the exponent in (1.47) and (1.48). In fact, we preserve monotonicity, positivity, the presence of absence of a finite stability time for all values of $\Delta > 0$, as well as estimating exponents to within $O(\Delta)$. This unconditional positivity and stability is a signal advantage of the Transformed method, as such qualitatively satisfactory behaviour without condition on the convergence parameter is most usually associated with implicit methods. Furthermore, in the case that (1.47) holds, we can show for any $\Delta > 0$ we have a unit exponent. This means that it may be possible to take asymptotically larger step-sizes while still maintaining finite exponents and, under some extra restrictions on f, an optimal condition on the larger step-sizes is identified.

Chapter 6 compares errors in the exponents for the three methods employed, when f is regularly varying with index β. Very roughly, it is shown that the Explicit Transformed and Implicit method are superior to the direct Explicit method, and that for small Δ and $\beta > 1/2$, the Transformed scheme is superior, while for $\beta < 1/2$, the

Implicit scheme is superior. Nevertheless, for all schemes, the errors in the exponents are all $O(\Delta)$ as $\Delta \to 0^+$ so the performance differences between the schemes are small. Therefore, on grounds of computational convenience, unconditional recovery of important properties (e.g. positivity) we are lead to prefer the Transformed scheme when we turn our attention to simulating solutions of SDEs.

Chapter 8 explores explosions and super-exponential growth in the ODE $x'(t) = f(x(t))$. We are able to show, under Explicit and Pre-Transformed schemes, that there is a sufficiently small step-size which recovers explosions and exponents to within $O(\Delta)$ as $\Delta \to 0^+$. It is

$$h(x) = \frac{\Delta}{f(x + \Delta f'(x)/f(x))}.$$

Once again, optimality is established by considering non-linearities f in special function classes. The class of regularly varying functions is an obvious choice. but it is perfectly reasonable for the growth of f to be faster in the explosive case (e.g. we could have $f(x) = e^x$ for example). A class of very rapidly growing functions which are important in this case are the class Γ for which

$$\lim_{x\to\infty} \frac{f'(x)\int_0^x \int_0^y f(z)\,dz\,dy}{f^2(x)} = 1.$$

Such functions are abundant and $f(x) = e^x$ is but one example. Strengthening slightly our assumptions on f so that

$$\lim_{x\to\infty} \frac{f''(x)f(x)}{f'(x)^2} = 1,$$

we find that the optimal step-size for logarithmic pre-transformation is

$$h(x) \sim \frac{\Delta}{f'(x)}, \quad \text{as } x \to \infty,$$

and this is also optimal if f' is regularly varying.

We have not, in this introduction, been greatly exercised as to whether the limits recorded in (1.45) and (1.46) are a good way to express the asymptotic behaviour of solutions of (1.1), rather than a merely convenient one which enables us to compare discrete and continuous asymptotics via (1.47) or (1.48). In Chapter 2, we carefully examine this question. In rough terms, the measure (1.45) is hard to improve upon in the super-exponential case, and the evidence of Chapter 2 suggests it is likely to be a robust measure for SDEs also. A reasonable competing measure, which is also inspired by the Liapunov exponent, is

$$\lim_{t\to\infty} \frac{-\log x(t)}{(-\log \circ F^{-1})(t)} = 1,$$

but it can be shown that (1.47) holds more generally.

As for the measure (1.48) the situation is more nuanced; in some situations, if a function x (not necessarily the solution of an ODE) obeys (1.48) it will also obey the Liapunov-like estimate

$$\lim_{t\to T^-} \frac{-\log x(t)}{(-\log\circ \bar{F}^{-1})(T-t)} = 1, \tag{1.49}$$

while in other circumstances this last limit implies (1.48). These circumstances are explored carefully in Chapter 2. However, the measure (1.48) implies (1.49) in the case that $f \in \text{RV}_0(\beta)$ for $\beta \in [0,1)$, suggesting it is preferable for the bulk of finite-time stability problems. Despite this, in the case of SDEs, we are often happy to determine asymptotic behaviour in the form (1.49) for finite-time stability problems; in certain cases, we show how this can be strengthened to get a result of the form (1.48).

In very rough terms, we show in Chapter 9 that for many functions f and g, that the long run behaviour of solutions (and in particular their ultimate positivity and convergence to zero) can be captured by a few simple parameters and the finiteness (or not) of certain integrals. This specialises a general result on the classification of solutions of (1.17), often called Feller's test (see e.g. [34]), but our reformulation and specialisation identifies, under modest monotonicity restrictions, that when the drift of (1.17) dominates, in a certain sense, the dynamics of (1.17) are governed by those of the ODE (1.24) viz.,

$$x'(t) = f(x(t)),$$

and in the case the diffusion of (1.17) dominates, the dynamics are governed by the ODE

$$x'(t) = -c\frac{g^2(x(t))}{2x(t)}.$$

for some positive constant c. More specifically, if we wish to consider situations in which X is a.s. attracted to zero, in the sense that

$$X(t) > 0 \text{ for all } t \in [0,T) \quad \textit{and} \quad \lim_{t\to T} X(t) = 0, \quad \text{a.s.,}$$

it suffices to assume that

$$\sup_{x>0} \frac{xf(x)}{g^2(x)} < \frac{1}{2}. \tag{1.50}$$

Furthermore, we can recover in the super-exponential case that when f obeys (1.27) and $L = -\infty$ then

$$\lim_{t\to\infty} \frac{F(X(t))}{t} = 1, \quad \text{a.s.,}$$

where F is defined by (1.29). On the other hand when g obeys (1.32) and $L \in (-\infty, 1/2)$ then

$$\lim_{t\to\infty} \frac{G(X(t))}{t} = \frac{1}{2} - L, \quad \text{a.s.,}$$

where G is defined by (1.34). In the case that the integrals are finite, we still obtain the asymptotic behaviour as $t \to T^- < \infty$: in this case, we obtain results like

$$\lim_{t \to T^-} \frac{\bar{G}(X(t))}{T-t} = \frac{1}{2} - L, \quad \text{a.s.,} \tag{1.51}$$

or

$$\lim_{t \to T^-} \frac{-\log X(t)}{(-\log \circ \bar{G}^{-1})((1/2 - L)(T-t))} = 1, \quad \text{a.s.,} \tag{1.52}$$

according to the deterministic asymptotic properties of g, where $\bar{G}$ is defined by (1.33).

In Chapter 10, we show that the insight into the optimal step-size for the auxiliary ODEs (1.24) (for $L = -\infty$) or (1.30) (for $L \in (-\infty, 1/2)$ is optimal for the SDE (1.17). We assume that (1.50) still holds. Following the logarithmic pre-transformation employed in Chapter 5, our method is to consider the SDE obeyed by $Z(t) = -\log X(t)$. By Itô's Lemma, we can find in closed form $\tilde{f}$ and $\tilde{g}$ such that

$$dZ(t) = \tilde{f}(Z(t))\, dt + \tilde{g}(Z(t))\, dB(t).$$

We now seek to discretise Z. In the case when $L = -\infty$, the auxiliary ODE (1.24) suggests a step-size

$$h_{\text{det}}(x) \sim \frac{\Delta x}{|f(x)|}, \quad \text{as } x \to 0^+,$$

and when $L \in (-\infty, 1/2)$ the auxiliary ODE (1.30) suggests a step-size

$$h_{\text{det}}(x) \sim \frac{\Delta x^2}{g^2(x)}, \quad \text{as } x \to 0^+.$$

Furthermore, there is no need to take very short step-sizes when $|f|$ and g are linearly bounded. In that case, we may take a constant step-size, Δ, so overall we consider a step-size of the form

$$h(x) = \Delta \min\left(1, \frac{x}{|f(x)|}, \frac{x^2}{g^2(x)}\right), \tag{1.53}$$

when the simulated value of $X(t) = x$. Hence, if the simulated value at time $t = t_n$ of $Z(t)$ is Z_n, we have $X_n = e^{-Z_n}$ and take a step-size of $h(e^{-Z_n}) = h(X_n)$. Therefore $t_{n+1} = t_n + h(X_n)$ and

$$Z_{n+1} = Z_n + h(X_n)\tilde{f}(Z_n) + \sqrt{h(X_n)}\tilde{g}(X_n)\xi_{n+1},$$

where $(\xi_n)_{n\geq 1}$ is a sequence of independent and identically distributed Standard Normal random variables.

In Chapter 10, we show that $X_n > 0$ for all $n \geq 0$ a.s. and that $X_n \to 0$ as $n \to \infty$ a.s. under (1.50). Moreover, we can show under the monotonicity conditions that hold for f and g in Chapter 9, that

$$\lim_{n\to\infty} t_n =: \hat{T}_h < \infty \quad \text{a.s.} \quad \Leftrightarrow \quad T < \infty \quad \text{a.s.},$$

and that this holds *regardless of the value of* $\Delta > 0$. In other words, we recover unconditionally on the discretisation parameter Δ the asymptotic stability, positivity, and presence or absence of finite-time stability.

We can also recover the rates of convergence, whether these are super-exponential or the discrete solution reaches zero in finite-time. For instance, in the case when $L \in (-\infty, 1/2)$, we can prove in the case that $\hat{T}_h = \infty$ that

$$\lim_{n\to\infty} \frac{G(X_n)}{t_n} = \frac{1}{2} - L,$$

and in the case when $\hat{T}_h < \infty$ that either

$$\lim_{n\to\infty} \frac{\bar{G}(X_n)}{\hat{T}_h - t_n} = \frac{1}{2} - L,$$

in the case that the solution of (1.17) obeys (1.51), and

$$\lim_{n\to\infty} \frac{-\log X_n}{(-\log\circ\bar{G})((1/2 - L)(\hat{T}_h - t_n))} = 1,$$

in the case that the solution of (1.17) obeys (1.52).

These results suggest that the scheme works very well, but the fact that unit limits are preserved in (1.51) and (1.52) leave open the question that the scheme may be working harder than necessary in order to recover the desired asymptotic behaviour. Furthermore, we would be interested in understanding whether it is possible, in the presence of noise, for the solution of (1.17) to obey the deterministic-like asymptotic behaviour

$$\lim_{t\to T^-} \frac{\bar{F}(X(t))}{T - t} = 1, \quad \text{a.s.}, \tag{1.54}$$

under appropriate conditions.

In Chapter 11, we show that when the drift $f(x)$ is always negative, then (1.54) holds under the "small noise" condition:

$$\text{there exists } \theta > 0 \text{ such that } \limsup_{x\to 0^+} \frac{g^2(x)}{x^{1+\theta}|f(x)|} < \infty. \tag{1.55}$$

Furthermore, this result is preserved to $O(\Delta)$ as $\Delta \to 0^+$ under monotonicity conditions on f, in the sense that

$$\liminf_{n\to\infty} \frac{\bar{F}(X_n)}{\hat{T}_h - t_n} = 1 + O(\Delta) \quad \text{and} \quad \limsup_{n\to\infty} \frac{\bar{F}(X_n)}{\hat{T}_h - t_n} = 1 + O(\Delta), \quad \text{as } \Delta \to 0^+,$$

and in the case that $|f| \in \mathrm{RV}_0(\beta)$ we even get

$$\lim_{n\to\infty} \frac{\bar{F}(X_n)}{\hat{T}_h - t_n} = c_4(\Delta),$$

where $c_4(\Delta)$ is deterministic and non-unit for $\beta \neq 1$ and $c_4(\Delta) = 1$ for $\beta = 1$. This shows that if the step-size is set asymptotically smaller, then we recover the asymptotic behaviour exactly but with the cost of increased computational effort. On the other hand, we have for $\beta \neq 1$ that $c_4(\Delta) \to 0$ as $\Delta \to \infty$, showing that the method loses good quantitative features if the step-size is taken larger than the rate $x/|f(x)|$ as $x \to 0^+$. These results hold both for power and logarithmic pre-transformations.

In Chapter 12, we show that our method with step-size h chosen according to (1.53) also works if the equilibrium of (1.17) gives rise to subexponential solutions. This necessitates new results on the asymptotic behaviour of the SDE (1.17) in the case that $f(x)/x \to 0$ and $g^2(x)/x^2 \to 0$ as $x \to 0^+$. Once again, if we assume for the sake of simplicity that (1.50) holds and L in (1.25) exists, then we have that $T = \infty$ a.s., $X(t) > 0$ for all $t \geq 0$ a.s. and $X(t) \to 0$ as $t \to \infty$ a.s.. Moreover, if $L = -\infty$ and the appropriate monotonicity conditions hold for f, then

$$\lim_{t\to\infty} \frac{-\log X(t)}{(-\log \circ F^{-1})(t)} = 1, \quad \text{a.s.},$$

while $L \in (-\infty, 1/2)$ implies

$$\lim_{t\to\infty} \frac{-\log X(t)}{(-\log \circ G^{-1})((1/2 - L)t)} = 1, \quad \text{a.s.}.$$

These results are recovered by the logarithmically pre-transformed scheme: for all $\Delta > 0$ we have that $t_n \to \infty$, $X_n > 0$ and $\lim_{n\to\infty} X_n = 0$ and we have

$$L = -\infty \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{-\log X_n}{(-\log \circ F^{-1})(t_n)} = 1,$$
$$L \in (-\infty, 1/2) \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{-\log X_n}{(-\log \circ G^{-1})((1/2 - L)t_n)} = 1.$$

In Chapter 13, we consider the case when solutions of (1.17) are subexponentially stable and the small noise condition (1.55) holds along with $f(x) < 0$ for all $x > 0$. In this case, very roughly speaking, the solution of the SDE obeys

$$\lim_{t\to\infty} \frac{F(X(t))}{t} = 1, \quad \text{a.s.},$$

in the case that solutions tend to zero faster than any negative power of t, which is generated by a new monotonicity condition on f of the form that $x \mapsto |f(x)|/x^{1+\theta}$ is decreasing close to zero for all $\theta > 0$ sufficiently small. If some extra smoothness on f

is granted, we can prove the stronger statement

$$\lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} = 1, \quad \text{a.s..}$$

This can also be established when $x \mapsto |f(x)|/x^{1+\theta}$ is increasing close to zero for some $\theta > 0$. In broad terms, we show that these refined asymptotic estimates hold for the discretised equations, under both power and logarithmic pre-transformations, although we sometimes need additional regularity on f to achieve this.

Chapter 14, which concludes the thesis, enumerates some extensions of the results presented, chiefly considering developments of results from Chapters 9 to 13. In particular, we conjecture that results for growth, explosion, recurrence and non-positive solutions can be proven.

1.6 Comparison with Works in the Literature

1.6.1 Overall Goals and Philosophy

The analysis in this work seeks to develop the existing literature in three directions:

(1) We seek new results in the analysis of the asymptotic behaviour of discretisations of ODEs. This is a classical subject and much of what we say that is new concerns the asymptotic behaviour or the computational optimality of our efforts, as results which talk about preserving explosions, finite-time stability or approximating the explosion time (or finite stability time) abound. Another concern we have is to demonstrate that rapidly growing solutions of ODEs are not incorrectly classified as being explosive.

(2) We seek new asymptotic results on the solutions of SDEs, covering the cases when solutions converge to the equilibrium subexponentially, super-exponentially, or in finite time. Our new results in this direction chiefly concern the determination of exact asymptotic rates of decay and of establishing critical levels of the noise at which there are "bifurcations" in the decay rate. For each type of convergence, we seek refined results when the noise is "small". We do not believe these results to be completely canonical, but to the best of our knowledge, they seem among the best available in the literature to date. Furthermore, they also prove to be of great value in establishing that the numerical methods we employ are in many cases computationally optimal, in the sense that diminishing the computational effort leads to unacceptable qualitative errors in the long-time discrete dynamics.

(3) We wish to prove new results on the long time dynamics of SDEs which are good models for solutions of SDEs. Since this is the part of our mathematical analysis

which is the least classical and which has undergone rapid development in the last few years, it is understandable that we have more sub-goals to target here:

(i) We wish to use explicit methods in order to reduce the computational effort for these highly non-linear equations. Our work on implicit deterministic numerical methods for equations with super-exponentially stable or finite-time attracting equilibria suggest that there is only a marginal increase in the step-size needed for explicit schemes to recover all important asymptotic behaviour. Since non-linear solving is unnecessary explicit methods would seem to be very attractive and the highly competitive performance of the Explicit Pre-Transformed scheme compared to the Implicit scheme in the deterministic case promotes it as the scheme of choice for SDEs, especially as it automatically preserves positive solutions.

(ii) We wish the schemes to possess, unconditionally, good qualitative properties for all values of the convergence parameter. Such properties include asymptotic stability, the presence or absence of a finite-time explosion or finite-time stability, as well as positivity. In fact, we suggest in the final chapter that other properties such as recurrence or non-positivity of solutions may also be unconditionally recovered by our new method.

(iii) We do not want the scheme to generate spurious yet interesting "false positives" such as producing a numerical explosion when no explosion is present in the solution of the original SDE.

(iv) We want the quantitative characteristics of our schemes to improve as the convergence parameter tends to zero and, if possible, show that the performance becomes quantitatively bad as the parameter tends to infinity. Furthermore, with a view to the computational efficiency of the methods, it would be desirable if the performance slightly departs from the behaviour of the SDE when the parameter is non-zero, as this shows that the amount of computational effort is well-judged and excessive effort is not used to achieve a given performance.

All these considerations place restrictions on the classes of problems we wish to study. We seek to work with scalar autonomous equations so that we can benchmark the performance of the numerical schemes against known theoretical long time behaviour. However, even for scalar SDEs, many aspects of the asymptotic behaviour (especially in the direction of precise rates of growth and decay) are not known, and this has necessitated the proof of new results.

In order to get good results for SDEs, we have typically imposed two different types of regularity on the drift and diffusion coefficient. One type of constraint involves imposing monotonicity (so that certain functions are asymptotically increasing or decreasing close to zero). In the context of the problems studied this seems quite reasonable and excludes certain rather pathological behaviour which might present difficulty in the simulations. The other involves asking that certain functions are regularly varying. This can be justified in several ways:

(i) it allows us to generalise from the case that certain functions have Taylor series approximations close to the equilibrium or have power law asymptotic behaviour;

(ii) the theory of regular variation extends to so-called "slow" or "rapid variation", allowing us to deal with non-linearities, such as logarithms or exponential functions or iterates thereof, which do not have behaviour which is of power-law type; and

(iii) powerful convergence and representation theorems for regularly varying (and certain subclasses of rapidly varying) functions enable us to produce extremely sharp results concerning our numerical approximations.

This demonstrates that our methods possess a certain computational efficiency, while at the same time we are not sacrificing greatly the generality of our results, as the class of regularly varying functions covers most important non-linearities used in real-world applications.

Intuitively, these monotonicity and regular variation assumptions work well with our numerical results because they tell us that behaviour of a function at a point will be representative of its behaviour over a certain interval. Such a property is highly likely to be helpful in numerical analysis of differential equations, which seeks at its simplest, to makes inferences concerning the behaviour of a function over an interval based on an approximation at a point.

1.6.2 Review of Relevant Literature

Works which study the qualitative behaviour of numerical simulations of solutions of differential equations and which seek conditions under which this behaviour is preserved by discretisation, have become more commonplace in recent years, but the monographs of Stuart and Humphries [57] and Mickens [44] are among the first comprehensive treatments. It has long been known that differential equations with finite-time singularities present special problems for numerical analysis. A classic text which examines stiff systems of this type is Hairer and Wanner [26]. Among the first papers that consider state-dependent time-steps in ODEs, in order to capture explosive behaviour, is Hocking et al. [28] but one of the first comprehensive treatments is that of Stuart and Floater [56] for both ordinary and partial differential equations. [56] identifies time-steps of the order $x/f(x)$ as being sufficient to recover the presence of an explosion in the ODE $x'(t) = f(x(t))$ when f is of polynomial order and of order $1/f(x)$ if f is of exponential type, although typically explosion asymptotics are not established. These types of time-step are shown to be optimal in this work in the polynomial case, but we also show that smaller step-sizes can be taken in the rapid growth case. In the PDE literature, many papers consider blow-up asymptotics but typically the step-size taken is proportional to $1/f(x)$: see for instance [1, 15, 16]. Care has also been taken to avoid spurious explosions in the simulation, as in Bonder and Rossi [13]. The literature on the recovery of finite-time stability is less extensive, but many results can be recovered from the blow-up case. There is nevertheless a significant literature on deterministic finite-time stability: representative papers include Bhat et al [11], Hong [29] and Moulay and Perruquetti [46], with often an emphasis on controlling the solution to reach the equilibrium in finite time.

Our contribution here would therefore appear to be five-fold:

(1) we establish that the step-size recovers the *asymptotic behaviour* faithfully and non-spuriously;

(2) we have identified the *optimal size* of mesh for explicit methods, in the sense that asymptotically smaller meshes (with $\Delta = 0$) correctly identify asymptotics but that larger mesh sizes (with $\Delta = \infty$) misspecify them;

(3) that *transforming* the state space can enable explicit methods to be used without restriction on the step-size;

(4) for certain non-linearities, larger step-sizes can be used without loss of asymptotic performance; and

(5) midpoint methods allow the asymptotic behaviour to be captured to higher order.

Superexponential stability in autonomous SDEs was comprehensively studied in Appleby et al [5] with rates of convergence determined contingent on the dominant non-linearities being regularly varying with unit index. The literature on stochastic finite-time stability is in its infancy. In the stochastic automatic control literature, a body of results starts to emerge (see e.g. [18, 60, 61]). However, the asymptotic behaviour of solutions close to the stability time, the connection with ODEs and the interplay between the noise and drift that we detail here does not seem to be generally known. Feller's test, which gives conditions on the scale and v function under which finite-time explosion or stability result are given in e.g. Karatzas and Shreve [34]: our connection between these classic results and Osgood-like conditions for stability or blow up in related ODEs appears however to be new. Stochastic explosions have been extensively studied, especially when the diffusion term g is $\text{o}(f)$, particularly in the context of fracture dynamics (see e.g. Sobczyk [52]). The numerical analysis of these explosions (using step-sizes of the order $1/f(x)$ when $g = o(f)$), as well as continuity of the explosion time in the initial data have been studied by Davila, Groissman, Rossi et al in a series of papers which include [14, 19, 25].

Precise asymptotic results on the non-exponential growth or decay in solutions of autonomous SDEs were pioneered by Gikhman and Skorohod [24], with follow-up work by Zhang and Tsoi [64, 65]. The paper of Appleby, Rodkina and Schurz [9] relaxed significantly the requirements on the size of the diffusion term so that exact rates of decay could be recovered, but with additional regular variation hypotheses being needed on the non-linearities. The contribution of this work has been to replace these quantitative, regular variation hypotheses with monotonicity hypotheses, and to prove new results in the case of super-exponential convergence and subexponential stability. In addition, the results for the asymptotic behaviour in the neighbourhood of the finite-time stability time under condition (1.25) appear also to be new and again do not need regular variation to proceed. Furthermore, the precise asymptotic results recorded for subexponential stability in Chapter 13 and for finite-time stability in Chapter 11 allow for larger noise contributions than in the existing literature. Nevertheless, we are still short of determining necessary and sufficient conditions for the preservation of deterministic rates of convergence in the presence of “small noise”, as has been achieved for SDEs with state-independent noise in Appleby and Patterson [6] in the case that $|f| \in \text{RV}_0(\beta)$ for $\beta \in (0, 1)$. There are many works in the deterministic literature which exploit regular variation in order to establish sharp asymptotic results, and a nice monograph summarising some of this work is by Maric [43].

Concerning the numerical methods for SDEs, our works are more in the spirit of dynamic consistency of Appleby, Berkoliako and Rodkina [2], which determines rates of (subexponential) convergence to zero of difference schemes modelling the solution of (1.17). However, that work does not prevent solutions of the difference equation from changing sign, and does not recover the asymptotic behaviour unconditionally in the convergence parameter Δ. Moreover, it is unclear whether the scheme in that paper would recognise if the solution of the SDE tended to infinity, and the rate at which that would arise, as can be done in the work here. Finally, we are able to study general decay rates, rather than the power decay rates which are studied most extensively in [2].

Positivity preservation for explicit schemes with fixed step-sizes (see e.g. [3, 4]) can generally only be achieved with positive probability, and implicit schemes have been developed which overcome this problem, such as that in Szpruch, Mao, Higham and Pan [54] especially in the context of simulating solutions of financial problems, such as the Cox-Ingersoll-Ross, constant elasticity of variance or Ait-Sahalia models. In these cases, the authors are interested both in preserving positivity with probability one and with the simulation being a strong approximations on a compact interval. Our decision to make logarithmic transformations makes the recovery of strong approximations difficult, but we seem to gain by being able to recover other qualitative features of the dynamics on infinite time domains with neither a restriction on our step-size parameter nor on the strength of the non-linearity.

The paper of Szpruch and Neuenkirch [53] is especially germane, as they apply the philosophy presented here by pre-transforming the SDE using a Lamperti transformation, simulating the solution with a constant step-size in the new co-ordinate system, and then recovering the solution by undoing the co-ordinate transformation. However, this method works only when the function

$$j(x) = \int_1^x \frac{1}{g(u)}\, du,$$

and its inverse are known in closed form. By contrast, our method does not rely on a particular transformation which is dependent on the structure of the SDE, nor on the existence of certain integrals or their inverses in closed form. A similar approach to keeping the solution in a given domain is presented in [20]. Pre-transformation with a view to preserving asymptotic behaviour, as well as strong convergence, is nicely treated in Szpruch and Zhang [55], though the methods there would not be able to deal with faster than exponential convergence nor with finite-time stability.

Regarding explicit schemes, another approach which deals attractively with the problem of loss of positivity is advanced by Mao and Liu [39] by stopping the simulation as soon as a negative value of the solution is obtained and they are able to show strong convergence of the solutions up to this crossing time.

The use of variable step-size methods for highly non-linear SDEs has been appreciated in recent years. It seems work of Higham, Mao and Stuart [27], and then of Hutzenthaler, Jentzen, Kloeden and Neuenkirch in a series of papers [32, 33] identified this problem for fixed-step methods in explicit problems, showing that strong convergence could not be obtained. One approach to obviate this is to employ fixed-step implicit or semi-implicit methods (see works of Mao and Szpruch[41], Milstein et al.[45] and Schurz [50, 51], but in the higher dimensional cases this is computationally expensive. A method of controlling the drift and diffusion coefficients by suitable mollifiers, while still using an explicit scheme, and recovering strong convergence is the so-called “Tamed Euler method” first proposed and studied in the papers and monograph of Hutzenthaler, Jentzen and Kloeden [30, 31], and further developed by Sabanis [48, 49]. Nevertheless, it appears that some long-time dynamical features of the tamed scheme may not be acceptable (see e.g. [58]) and, for this reason, adaptive time-stepping can sometimes be an attractive option.

The first generation of works with adaptive time-stepping in SDEs include Gaines and Lyons [23], Burrage et al [17] and Lamba et al. [38]. However, the works that are closer in spirit to our own are those of Fang and Giles [21, 22], Kelly and Lord [35], Kelly et al. [37] and Liu and Mao [40], as well as Davila et al. [19]. In each of these works, the goal is to recover the long-time behaviour of the solution of an SDE and perhaps strong convergence on compacts, such as in [21, 35, 40]. The works [40] and [37] pay more care to the discretisation of the Itô integral than we do here. This enables strong convergence to be obtained in certain cases.

The work [37] in particular shows that the condition $L > 1/2$ in (1.25) does not give stability, while (1.50) gives convergence of the solutions with probability one, provided the convergence parameter is sufficiently small. Also, the simulated solutions remain positive for a certain number of time steps with probability arbitrarily close to unity provided the convergence parameter is small enough. All this is achieved with a step-size which would be asymptotic to ours in (1.53). However, in the case of our work stability and positivity are achieved independently of the convergence parameter and the results in [37]. Furthermore, in [37] questions as to the convergence or divergence of the sequence (t_n), which are of great concern to us, are of less worry in [37] and are not studied. This is potentially of importance, as in [37] positivity with high probability is ensured for a fixed number of steps, but if the t_n tends to a finite limit, positivity may not be ensured as the finite stability time is approached.

We should mention of course that there are numerous other works concerning the preservation of asymptotic features in discretisation of SDEs. Apart from simple Euler methods and preservation of stability or equilibria, researchers have examined more complicated numerical methods (e.g. Milstein methods [36] or θ-methods [10]) as well as more complicated features such as stationary distributions (see e.g. the series of papers of Mao, Yuan and Yin [42, 63, 62]).

Chapter 2

Ranking of Asymptotic Convergence Measures

2.1 Introduction

In this chapter we discuss the connections between various ways in which we can obtain limiting behaviour for the solutions of the ODE (1.1) or the SDE (1.17), both in the case of super-exponential stability and finite-time stability.

There are several plausible measures for the asymptotic behaviour and in this introduction we focus on the super-exponentially stable cases since the necessary considerations are similar for the case of finite-time stability. In particular we identify three plausible measures and to see how these measures arise, we momentarily consider an ODE which will have exponential-type decay. Let $x'(t) = f(x(t))$ where $f(x) \sim -ax$ as $x \to 0^+$ and $a > 0$. Define $F(x) := \int_x^1 1/f(u)\,du$. Then we notice that

$$\lim_{t\to\infty} \frac{\log x(t)}{t} = -a = -a \lim_{t\to\infty} \frac{F(x(t))}{t},$$

so that

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = \lim_{t\to\infty} \frac{-\log x(t)}{(-\log \circ F^{-1})(t)} = 1.$$

Therefore in the case of asymptotically linear f, the measures $\lim_{t\to\infty} F(x(t))/t$ and $\lim_{t\to\infty} -\log(x(t))/(-\log \circ F^{-1})(t)$ capture the negative Liapunov exponent of x. If $x_0 = \xi$, we also have that $x(t) = F^{-1}(F(\xi) + t)$ and it is natural to ask whether $x(t) \sim cF^{-1}(t)$ as $t \to \infty$ for some $c > 0$. This certainly holds when $f(x) = -ax$ and can also hold when $f(x) + ax$ tends to zero sufficiently rapidly as x tends to zero. This suggests for general non-linear ODEs that we consider also the measure

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)}.$$

In this chapter, we examine the reliability of these measures for fixed f and different initial conditions on one hand and for different classes of functions f on the other.

2.2 Super-Exponential Stability

The function F, defined by (1.11), plays a key role in determining the rate of convergence because solutions of (1.1) are given by

$$F(x(t,\xi)) = F(\xi) + t, \quad t > 0,$$

or $x(t,\xi) = F^{-1}(F(\xi) + t)$. Moreover, solutions always obey

$$\lim_{t\to\infty} \frac{F(x(t,\xi))}{t} = 1, \tag{2.1}$$

which gives an implicit and ξ-independent estimate of the rate of convergence. This is a natural analogue to the Liapunov exponent because it considers the convergence of a function of the solution rather than the solution itself, relative to linear growth in time.

Comparing the convergence of the solution itself for different initial conditions gives poor asymptotics because of super-exponential convergence. If $x(t)$ is the solution of (1.1) with $\xi = 1$ then the solution is $x(t) = F^{-1}(t)$ where $\lim_{t\to\infty} F^{-1}(t) = 0$. By (1.13)

$$\lim_{t\to\infty} \frac{x'(t)}{x(t)} = \lim_{t\to\infty} \frac{-f(x(t))}{x(t)} = -\infty.$$

Therefore for $c > 0$ then

$$\lim_{t\to\infty} \frac{F^{-1}(t+c)}{F^{-1}(t)} = \lim_{t\to\infty} \frac{x(t+c)}{x(t)} = 0.$$

Letting $\xi_1 < \xi_2$ then

$$\lim_{t\to\infty} \frac{x(t,\xi_1)}{x(t,\xi_2)} = \lim_{t\to\infty} \frac{F^{-1}(t+F(\xi_1))}{F^{-1}(t+F(\xi_2))} = \lim_{T\to\infty} \frac{F^{-1}(T+F(\xi_1)-F(\xi_2))}{F^{-1}(T)} = 0.$$

There is another natural analogue of the Liapunov exponent, where we retain the logarithm dependence on x in the numerator. Using $\log x(t)$ in the numerator of the metric calculates rates of convergence that depend on the initial condition when f is $O(x\log(1/x))$ as $x \to 0^+$. However, when f is $o(x\log(1/x))$ as $x \to 0^+$ the calculated rates are independent. As a result, $\log x(t)$ is only suitable for a limited class of rate functions f. This is made precise in the lemma below.

Lemma 1. *Suppose F is the function defined by* (1.11). *Let*

$$\lim_{x\to 0^+} \frac{f(x)}{x\log(1/x)} =: c.$$

(i) If $c = 0$, then

$$\lim_{t\to\infty} \frac{-\log x(t,\xi_1)}{(-\log \circ F^{-1})(t)} = 1, \tag{2.2}$$

thus

$$\lim_{t\to\infty} \frac{-\log x(t,\xi_1)}{-\log x(t,\xi_2)} = 1.$$

(ii) If $c \neq 0$, then

$$\lim_{t\to\infty} \frac{-\log x(t,\xi_1)}{(-\log \circ F^{-1})(t)} = e^{cF(\xi_1)},$$

thus

$$\lim_{t\to\infty} \frac{-\log x(t,\xi_1)}{-\log x(t,\xi_2)} = e^{c(F(\xi_1)-F(\xi_2))}.$$

Proof. If $x(t)$ is the solution of (1.1) with $x(0) = 1$ then $x(t) = F^{-1}(t)$ with $F^{-1}(t) \to 0$ as $t \to \infty$. Define $a(t) := (-\log \circ F^{-1})(t)$. Then $-\log x(t,\xi_1) = (-\log \circ F^{-1})(F(\xi_1) + t) = a(t + F(\xi_1))$. Note that

$$\frac{a'(t)}{a(t)} = \frac{f(F^{-1}(t))}{F^{-1}(t)\log(1/F^{-1}(t))}.$$

Thus

$$\lim_{t\to\infty} \frac{f(F^{-1}(t))}{F^{-1}(t)\log(1/F^{-1}(t))} = \lim_{x\to 0} \frac{f(x)}{x\log(1/x)} = c.$$

Thus

$$\lim_{t\to\infty} \frac{-\log x(t,\xi_1)}{(-\log \circ F^{-1})(t)} = \lim_{t\to\infty} \frac{a(t+F(\xi_1))}{a(t)} = e^{cF(\xi_1)}.$$

If $c = 0$ then

$$\lim_{t\to\infty} \frac{-\log x(t,\xi_1)}{(-\log \circ F^{-1})(t)} = 1,$$

thus

$$\lim_{t\to\infty} \frac{-\log x(t,\xi_1)}{-\log x(t,\xi_2)} = \lim_{t\to\infty} \left(\frac{-\log x(t,\xi_1)}{(-\log \circ F^{-1})(t)} \cdot \frac{(-\log \circ F^{-1})(t)}{-\log x(t,\xi_2)} \right) = 1.$$

If $c \neq 0$ then

$$\lim_{t\to\infty} \frac{-\log x(t,\xi_1)}{(-\log \circ F^{-1})(t)} = e^{cF(\xi_1)} \neq 1.$$

thus

$$\lim_{t\to\infty} \frac{-\log x(t,\xi_1)}{-\log x(t,\xi_2)} = e^{c(F(\xi_1)-F(\xi_2))} \neq 1,$$

as claimed. □

Remark 1. The analysis above shows that

$$\lim_{t\to\infty} \frac{\log x(t,\xi_1)}{\log x(t,\xi_2)} = 1,$$

in the case of sub-exponential convergence since $f(x)/x \to 0$ as $x \to 0^+$. □

In the case when

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 1 \quad \text{and} \quad \lim_{t\to\infty} \frac{-\log x(t)}{(-\log\circ F^{-1})(t)} = 1,$$

it is interesting to ask whether we might prefer to use the second metric. However, the following result suggests that the first metric is more fundamental.

Lemma 2. *Suppose x is a function with*

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 1,$$

and $a(t) = (-\log\circ F^{-1})(t)$ is regularly varying at infinity. Then

$$\lim_{t\to\infty} \frac{-\log x(t)}{(-\log\circ F^{-1})(t)} = 1. \tag{2.3}$$

Proof. An $\epsilon - T(\epsilon)$ argument gives for any $t \geq T(\epsilon)$, $F^{-1}((1-\epsilon)t) > x(t) > F^{-1}((1+\epsilon)t)$. Therefore

$$\frac{a((1-\epsilon)t)}{a(t)} < \frac{-\log x(t)}{(-\log\circ F^{-1})(t)} < \frac{a((1+\epsilon)t)}{a(t)}, \quad t \geq T(\epsilon).$$

Letting $t \to \infty$ and $\epsilon \to 0^+$ yields (2.3). □

With $a(t) = (-\log\circ F^{-1})(t)$ and $x = F^{-1}(t)$, we get

$$\frac{ta'(t)}{a(t)} = \frac{tf(F^{-1}(t))}{F^{-1}(t)\log(1/F^{-1}(t))} = \frac{F(x)f(x)}{x\log(1/x)}.$$

Therefore a sufficient condition for a to be regularly varying is

$$\lim_{t\to\infty} \frac{ta'(t)}{a(t)} = \lim_{x\to 0^+} \left(\frac{f(x)}{x\log(1/x)} \int_x^1 \frac{1}{f(u)}\,du \right) =: c < \infty. \tag{2.4}$$

We see that we need $f(x) = o(x\log(1/x))$ as $x \to 0^+$ since $F(x) \to \infty$ as $x \to 0^+$. We now investigate when (2.4) holds and identify a critical non-linearity at which this hypothesis ceases to hold.

Proposition 3. *Suppose $M \in C^1((0,\infty);(0,\infty))$ and*

$$\lim_{x\to 0^+} M(x)\log\log\left(\tfrac{1}{x}\right) = \infty, \tag{2.5}$$

$$\lim_{x\to 0^+} \left(M(x) - M'(x)x\log\left(\tfrac{1}{x}\right)\log\log\left(\tfrac{1}{x}\right)\right) =: M^* \in [0,\infty]. \tag{2.6}$$

Then

$$M(x) := \frac{\log\left(f(x)/(x\log(1/x))\right)}{\log\log(1/x)},$$

obeys

$$\lim_{x\to 0^+} \frac{f(x)}{x\log(1/x)} \int_x^\delta \frac{1}{f(u)}\,du = \frac{1}{M^*}, \tag{2.7}$$

and hence $a \in RV_\infty(1/M^*)$ *if* $M^* \in (0,\infty]$.

Proof. Writing $y := \log\log(1/x)$ we get

$$\begin{aligned} \frac{f(x)}{x\log(1/x)} \int_x^\delta \frac{1}{f(u)}\,du &= e^{-M(x)\log\log(1/x)} \int_x^\delta \frac{e^{M(u)\log\log(1/u)}}{u\log(1/u)}\,du \\ &= e^{-M(\exp(-e^y))y} \int_{\log\log(1/\delta)}^y e^{M(\exp(-e^v))v}\,dv. \end{aligned}$$

Thus

$$\lim_{x\to 0^+} \frac{f(x)}{x\log(1/x)} \int_x^\delta \frac{1}{f(u)}\,du = \lim_{y\to\infty} \left(e^{-\tilde{M}(y)y} \int_{\delta'}^y e^{\tilde{M}(v)v}\,dv \right),$$

where $\tilde{M}(x) := M(\exp(-e^x))$. Since $M(x)\log\log(1/x) \to \infty$ as $x \to 0^+$ then $\tilde{M}(y)y = M(\exp(-e^y))y = M(x)\log\log(1/x) \to \infty$ as $y \to \infty$ where $y = \log\log(1/x)$. By L'Hôpital's Rule

$$\lim_{y\to\infty} \frac{\int_{\delta'}^y e^{\tilde{M}(v)v}\,dv}{e^{\tilde{M}(y)y}} = \lim_{y\to\infty} \frac{e^{\tilde{M}(y)y}}{e^{\tilde{M}(y)y}(\tilde{M}(y)y)'} = \lim_{y\to\infty} \frac{1}{\tilde{M}(y) + y\tilde{M}'(y)},$$

where

$$\begin{aligned} \tilde{M}(y) + y\tilde{M}'(y) &= M\left(\exp(-e^y)\right) + y \cdot M'\left(\exp(-e^y)\right) \cdot \exp(-e^y) \cdot -e^y \\ &= M(x) + \log\log\left(\tfrac{1}{x}\right) \cdot M'(x) \cdot x \cdot -\log\left(\tfrac{1}{x}\right) \\ &= M(x) - M'(x)x\log\left(\tfrac{1}{x}\right)\log\log\left(\tfrac{1}{x}\right). \end{aligned}$$

By (2.6), $\tilde{M}(y) + y\tilde{M}'(y) \to M^*$ as $y \to \infty$ and so (2.7) holds. □

The condition (2.6) is cumbersome and opaque. In the presence of monotonicity matters simplify as the following corollary demonstrates.

Corollary 1. *Let M obey (2.5). Then*

(i) If M is in $C^1(0,\infty)$, then $M(x) \to M^* \in (0,\infty)$ as $x \to 0^+$ then $a \in RV_\infty(1/M^*)$.

(ii) If M is decreasing and $M(x) \to \infty$ as $x \to 0^+$, then $a \in RV_\infty(0)$.

(iii) If M is increasing and $M(x) \to 0$ as $x \to 0^+$, then a is not regularly varying at infintiy.

Proof. Note that

$$\lim_{x\to 0^+} x\log\left(\tfrac{1}{x}\right)\log\log\left(\tfrac{1}{x}\right) = \lim_{y\to\infty} \exp\left(-e^y\right) \cdot e^y \cdot y = \lim_{y\to\infty} \frac{ye^y}{\exp\left(e^y\right)} = 0.$$

Thus if $M \in C^1(0,\infty)$ then

$$\lim_{x\to 0^+} \frac{f(x)}{x\log(1/x)} \int_x^\delta \frac{1}{f(u)}\,du = \lim_{y\to\infty} \frac{1}{\tilde{M}(y) + y\tilde{M}'(y)} = \lim_{x\to 0^+} \frac{1}{M(x)} = \frac{1}{M^*},$$

as claimed. The proofs of the other parts are similar to Lemma 5 and hence are omitted. □

We have three competing metrics, mainly

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(x(t))} = 1, \quad \lim_{t\to\infty} \frac{F(x(t))}{t} = 1, \quad \lim_{t\to\infty} \frac{-\log x(t)}{(-\log\circ F^{-1})(t)} = 1.$$

We have shown that measure one seldom prevails when x is the solution of an ODE, that measure two always prevails and measure three prevails for relatively weak f's that generate super-exponential convergence but not prevail for very strong f's.

We will later show for super-exponentially stable solutions that measure two always applies for SDEs but that measure three applies for relatively weak non-linearities. However, it seems that measure three might not apply for SDEs with stronger non-linearities.

Based on our experience for ODEs, which places the second metric as being extremely reliable but the first and third metric being at least less univeral, we will prefer for SDEs to use metric two. Metric two has further advantages for numerical methods. Since we will wish any good numerical method to recover a discrete analogue of metric two. Supposing that x_n is the simulated value of the solution at the n^{th} mesh point, which we suppose to be at t_n, we will want to show that $F(x_n)/t_n$ does not deport appreciably from unity for large n. Of course

$$F(x_n) = F(x_0) + \sum_{j=0}^{n-1} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du,$$

a good approximation of $F(x_n)$ will result from good control over the summand $\int_{x_{j+1}}^{x_j} 1/f(u)\,du$. We will show under mild monotonicity restrictions on f that these summands can be well controlled for appropriate numerical methods. Thus it becomes a practical proposition to demonstrate that the discrete version of metric two behaves appropriately for suitable numerical schemes. Similar control with natural control on f seems harder to achieve for discrete analogues of the other metrics.

2.3 Finite-Time Stability

In the case of finite-time stability, solutions of (1.1) are given by

$$\bar{F}(x(t,\xi)) = T_\xi - t, \quad t \in [0, T_\xi),$$

or $x(t,\xi) = \bar{F}^{-1}(T_\xi - t)$ where $T_\xi = \int_0^\xi 1/f(u)\,du$ is an explicit function of ξ. Thus $-\log x(t,\xi) = (-\log\circ \bar{F}^{-1})(T_\xi - t) = a(T_\xi - t)$ where $a(x) := (-\log\circ\bar{F}^{-1})(x)$. Since $\bar{F}(x)\to 0$ as $x\to 0^+$, $\bar{F}^{-1}(x)\to 0$ as $x\to 0^+$. Therefore $a(x)\to\infty$ as $x\to 0^+$. As a result

$$\bar{F}(x(T_\xi - t,\xi)) = t, \quad 0 < t \le T_\xi,$$

$$-\log x(T_\xi - t,\xi) = (-\log\circ\bar{F}^{-1})(t), \quad 0 < t \le T_\xi,$$

and

$$x(T_{\xi_1} - t,\xi_1) = x(T_{\xi_2} - t,\xi_2), \quad 0 < t < \max(T_{\xi_1}, T_{\xi_2}).$$

Therefore, there is no distinction between the measures

$$\lim_{t\to 0^+} \frac{\bar{F}(x(T-t))}{t} = 1, \tag{2.8}$$

$$\lim_{t\to 0^+} \frac{-\log x(T-t)}{(-\log\circ\bar{F}^{-1})(t)} = 1, \tag{2.9}$$

and

$$\lim_{t\to 0^+} \frac{x_1(T_1-t)}{x_2(T_2-t)} = 1, \tag{2.10}$$

where x, x_1 and x_2 are solutions of (1.1) with finite-time stability times T, T_1 and T_2 respectively. All are invariant with respect to initial conditions and path. This is in clear contrast from the super-exponential case where a ranking exists.

Suppose now we have asymptotic information regarding a positive function $x(t)$ (which may not be the solution to an ODE) for which $\lim_{t\to T^-} x(t) = 0$. The question is whether a ranking now exists with respect to the measures (2.8) and (2.9). We now show that in some cases (2.9) implies (2.8) and in others (2.8) implies (2.9). This is of particular interest later when x is the solution of an SDE, and also when a discretisation of the solution is considered. The main conclusion to be drawn from the analysis in the SDE case is that we can always arrive at a measure which is invariant with respect to initial conditions and sample paths and moreover the same measure will preserve this invariance or robustness under suitable discretisation. These observations are made precise by the following lemma whose result can be inferred from Lemmas 37 and 38 in Chapter 9; accordingly we postpone the proofs until that point.

Lemma 3. *Let x be a positive function on $[0,T)$ with $x(T^-) = 0$.*

(i) If

$$\lim_{t\to 0^+} \frac{-\log x(T-t)}{(-\log\circ\bar{F}^{-1})(t)} = 1, \tag{2.11}$$

and

$$\lim_{\lambda\to 1^+}\lim_{x\to 0^+} \frac{(-\log\circ\bar{F}^{-1})(\lambda x)}{(-\log\circ\bar{F}^{-1})(x)} = 1,$$

then

$$\lim_{t\to 0^+} \frac{\bar{F}(x(T-t))}{T-t} = 1. \tag{2.12}$$

(ii) If (2.12) *holds and*

$$\lim_{\lambda\to 1^+} \lim_{x\to 0^+} \frac{(-\log\circ \bar{F}^{-1})(\lambda x)}{(-\log\circ \bar{F}^{-1})(x)} = \infty,$$

then (2.11) *holds.*

The analogues of the limits (2.1) and (2.2) in the super-exponential case are

$$\lim_{t\to T_\xi^-} \frac{\bar{F}(x(t,\xi))}{T_\xi - t} = 1, \tag{2.13}$$

and

$$\lim_{t\to T_\xi^-} \frac{-\log x(t,\xi)}{(-\log\circ \bar{F}^{-1})(T_\xi - t)} = 1. \tag{2.14}$$

Equation (2.13) always prevails. However, in contrast to (2.1), the denominator in (2.13) is ξ-dependent. Nevertheless, (2.13) can still be viewed as a robust metric for the asymptotic behaviour of (1.1) close to the finite stability time T_ξ. Note that $T_\xi - t$ is the time remaining before the solution reaches the equilibrium and if this is used as our measure of time, the asymptotic behaviour of $\bar{F}(x(t,\xi))$ is ξ-independent, as measured by (2.13). Furthermore, this measure need not be viewed as a technical contrivance but is, we claim, a meaningful and natural quantity to study in applications. This is because in finite-time stability (or that matter explosion) problems, it is natural to consider asymptotic behaviour as the time to the singularity, $T_\xi - t$, approaches zero. Indeed, the measure of time in this denominator in (2.13) is linear, just as in (2.1), albeit that in the former case we measure time remaining and in the latter time elapsed.

In the case of SDEs, we would prefer if our measures were invariant with respect to the path, and since finite stability times are likely to be path-dependent, measures such as (2.13) and (2.14), which depend on the time remaining to the finite stability times but are otherwise independent of the initial data or path, are of value.

The question addressed previously was whether (2.1) implies (2.2). We now ask whether (2.13) implies (2.14). This is addressed in the following lemma.

Lemma 4. *Suppose*

$$\lim_{t\to T_\xi^-} \frac{\bar{F}(x(t))}{T_\xi - t} = 1,$$

and let $a(x) := (-\log\circ \bar{F}^{-1})(x)$ *be regularly varying at 0. Then*

$$\lim_{t\to T_\xi^-} \frac{-\log x(t)}{(-\log\circ \bar{F}^{-1})(T_\xi - t)} = 1. \tag{2.15}$$

Proof. For all $\epsilon \in (0,1)$ and any t sufficiently close to T_ξ we have $(1-\epsilon)\cdot(T_\xi - t) < \bar{F}(x(t)) < (1+\epsilon)\cdot(T_\xi - t)$. An $\epsilon - T_\xi(\epsilon)$ argument gives for any $t \geq T_\xi - \delta_\xi$

$$a((1-\epsilon)(T_\xi - t)) > -\log x(t) > (-\log \circ \bar{F}^{-1})((1+\epsilon)(T_\xi - t)) = a((1+\epsilon)(T_\xi - t)).$$

Therefore

$$\frac{a((1-\epsilon)(T_\xi - t))}{a((T_\xi - t))} > \frac{-\log x(t)}{(-\log \circ \bar{F}^{-1})(T_\xi - t)} > \frac{a((1+\epsilon)(T_\xi - t))}{a((T_\xi - t))}.$$

Now let $t \to T_\xi^-$ and then $\epsilon \to 0^+$ to get (2.15). $\square$

In the case that

$$\lim_{x\to 0^+} \frac{xa'(x)}{a(x)} = \lim_{x\to 0^+} \frac{-xf(\bar{F}^{-1}(x))}{\bar{F}^{-1}(x)\log(1/\bar{F}^{-1}(x))} = \lim_{u\to 0^+} \frac{-\bar{F}(u)f(u)}{u\log(1/u)}$$
$$= \lim_{u\to 0^+} \frac{-f(u)}{u\log(1/u)} \int_0^u \frac{1}{f(v)}\,dv =: -c,$$

a will be regularly varying as required by Lemma 4. Therefore

$$\lim_{x\to 0^+} \frac{f(x)}{x\log(1/x)} \int_0^x \frac{1}{f(u)}\,du =: c, \tag{2.16}$$

gives the right behaviour and we now investigate when (2.16) holds.

Lemma 5. *Define* $a(x) := (-\log \circ \bar{F}^{-1})(x)$. *Suppose*

$$M(x) := \frac{\log\left(f(x)/(x\log(1/x))\right)}{\log\log(1/x)}.$$

(i) If $M(x) \to \infty$ *as* $x \to 0^+$ *and* M *is decreasing, then* $a \in RV_0(0)$.

(ii) If $M(x) \to M^*$ *as* $x \to 0^+$ *and* $xM'(x)\log\log(1/x)\log(1/x) \to 0$ *as* $x \to 0^+$, *then* $a \in RV_0(1/M^*)$.

(iii) If $M(x) \to 0$ *as* $x \to 0^+$, *then* $xa'(x)/a(x) \to -\infty$ *as* $x \to 0^+$ *and* a *is not regularly varying.*

Proof. Putting $y := \log\log(1/x)$ we get

$$\frac{f(x)}{x\log(1/x)} \int_0^x \frac{1}{f(u)}\,du = e^{M(x)\log\log(1/x)} \int_0^x \frac{e^{-M(u)\log\log(1/u)}}{u\log(1/u)}\,du$$
$$= e^{M(\exp(-e^y))y} \int_y^\infty e^{-M(\exp(-e^v))v}\,dv.$$

Then with $\tilde{M}(y) := M(\exp(-e^y))$,

$$\limsup_{x\to 0^+} \frac{f(x)\bar{F}(x)}{x\log(1/x)} = \limsup_{y\to\infty}\left(e^{\tilde{M}(y)y}\int_y^\infty e^{-\tilde{M}(v)v}\,dv\right).$$

If M is decreasing then $y \mapsto \tilde{M}(y)$ is increasing. For $v > y$, then $\tilde{M}(v) > \tilde{M}(y)$, so $v\tilde{M}(v) > v\tilde{M}(y)$ and hence $\exp\left(-v\tilde{M}(v)\right) < \exp\left(-v\tilde{M}(y)\right)$. Thus

$$\int_y^\infty e^{-v\tilde{M}(v)}\,dv \le \int_y^\infty e^{-v\tilde{M}(y)}\,dv = \frac{e^{-y\tilde{M}(y)}}{\tilde{M}(y)}.$$

Therefore

$$0 \le \limsup_{x\to 0^+} \frac{f(x)\bar{F}(x)}{x\log(1/x)} \le \limsup_{y\to\infty}\frac{1}{\tilde{M}(y)} = 0,$$

as needed in part (i). For part (ii), by L'Hôpital's Rule

$$\lim_{y\to\infty}\frac{\int_y^\infty e^{-\tilde{M}(v)v}\,dv}{e^{-\tilde{M}(y)y}} = \lim_{y\to\infty}\frac{-e^{-\tilde{M}(y)y}}{e^{-\tilde{M}(y)y}\left(-\tilde{M}'(y)y-\tilde{M}(y)\right)} = \lim_{y\to\infty}\frac{1}{y\tilde{M}'(y)+\tilde{M}(y)} = \frac{1}{M^*},$$

where

$$\begin{aligned}
\tilde{M}(y)+y\tilde{M}'(y) &= M\left(\exp(-e^y)\right) + y\cdot M'\left(\exp(-e^y)\right)\cdot\exp(-e^y)\cdot -e^y\\
&= M(x) + \log\log\left(\tfrac{1}{x}\right)\cdot M'(x)\cdot x\cdot -\log\left(\tfrac{1}{x}\right)\\
&= M(x) - M'(x)x\log\left(\tfrac{1}{x}\right)\log\log\left(\tfrac{1}{x}\right).
\end{aligned}$$

If $M(x)\to M^*$ as $x\to 0^+$ and $M'(x)x\log\left(\frac{1}{x}\right)\log\log\left(\frac{1}{x}\right)\to 0$ as $x\to 0^+$ then

$$\lim_{y\to\infty}\frac{\int_y^\infty e^{-\tilde{M}(v)v}\,dv}{e^{-\tilde{M}(y)y}} = \lim_{y\to\infty}\frac{1}{y\tilde{M}(y)+\tilde{M}(y)} = \frac{1}{M^*},$$

hence

$$\lim_{x\to 0^+}\frac{xa'(x)}{a(x)} = \frac{f(x)}{x\log(1/x)}\int_0^x\frac{1}{f(u)}\,du = \frac{1}{M^*},$$

and thus $a \in RV_0(1/M^*)$. For part (iii), if M is increasing, $\tilde{M}$ is decreasing, then for $v > y$, $v\tilde{M}(v) < v\tilde{M}(y)$ so $\exp(-v\tilde{M}(v)) > \exp(-v\tilde{M}(y))$. Thus

$$e^{\tilde{M}(y)y}\int_y^\infty e^{-\tilde{M}(v)v}\,dv \ge e^{\tilde{M}(y)y}\int_y^\infty e^{-\tilde{M}(y)v}\,dv = \frac{e^{\tilde{M}(y)y}\cdot e^{-\tilde{M}(y)y}}{\tilde{M}(y)} = \frac{1}{\tilde{M}(y)}.$$

Thus

$$\liminf_{x\to 0^+}\frac{f(x)\bar{F}(x)}{x\log(1/x)} \ge \liminf_{y\to\infty}\frac{1}{\tilde{M}(y)} = \infty.$$

Hence

$$\lim_{x \to 0^+} \frac{f(x)\bar{F}(x)}{x \log(1/x)} = \infty.$$

Then $xa'(x)/a(x) \to -\infty$ as $x \to 0^+$. Now $a(x) = (-\log \circ \bar{F}^{-1})(x) \to \infty$. Thus for any $N > 1$ there is an $x^*(N) > 0$ such that $\forall\, x < x^*(N)$, $xa'(x)/a(x) < -N$ and so for fixed $\lambda > 1$ and $\lambda x < x^*(N)$

$$\log\left(\frac{a(\lambda x)}{a(x)}\right) = \int_x^{\lambda x} \frac{ua'(u)}{a(u)} \cdot \frac{1}{u}\, du < -N \int_x^{\lambda x} \frac{1}{u}\, du < -N \log \lambda.$$

Hence

$$\limsup_{x \to 0^+} \log\left(\frac{a(\lambda x)}{a(x)}\right) \leq -N \log \lambda.$$

Thus

$$\limsup_{x \to 0^+} \log\left(\frac{a(\lambda x)}{a(x)}\right) = -\infty,$$

and hence

$$\lim_{x \to 0^+} \frac{a(\lambda x)}{a(x)} = 0, \quad \forall\, \lambda > 1,$$

so a is not regularly varying, as we claimed. □

Chapter 3

Asymptotic Behaviour with Monotonicity Assumptions on the Non-Linearity

3.1 Introduction

As stated in the introduction, the central theme of this thesis is developing numerical methods which preserve important qualitative properties, such as positivity, monotonicity and convergence to an equilibrium, and quantitative measures of this phenomena, such as estimates for exit times i.e. explosion and finite stability time and precise estimates of asymptotic behaviour of the numerical simulations in the temporal vicinity of these exit times.

In this chapter, we start this analysis by considering the simplest class of ODEs which will generate these diverse phenomena. In fact we specialise to consider scalar autonomous ODEs with positive initial values which possess a unique and globally attracting equilibrium at zero. More particularly we consider the differential equation (1.1) viz.,

$$x'(t) = -f(x(t)), \quad t > 0, \quad x(0) = \xi > 0,$$

for which $f(x) > 0$ for all $x > 0$, $f(0) = 0$ and f is continuous.

In the case when f is well-behaved, in the sense it obeys a global Lipschitz condition, standard fixed-step numerical methods will recover the important asymptotic behaviour of the solution both on compact and infinite intervals. However, in the absence of such global Lipschitz conditions and especially in the case when f has infinite one-sided derivative at zero, both Implicit and Explicit fixed-step methods will fail to recover important features, such as global positivity in the case of Explicit methods and finite-time stability in the case of Implicit methods.

In this chapter, we will consider one-step Implicit and Explicit methods for simulating the solution to (1.1) in which the time-step will depend solely on the state of the system. We impose monotonicity conditions on f which are irrestrictive and likely to be true in the case where solutions approach the equilibrium more rapidly than an exponential function or reach the equilibrium in a finite-time interval. Under these monotonicity assumptions, we show in broad terms that taking step-sizes which preserve positivity and which behave asymptotically according to

$$\lim_{x\to 0^+} \frac{h(x)f(x)}{x} = \Delta,$$

will recover all important qualitative and quantitative properties for sufficiently small Δ in the case of Explicit methods and without restriction on Δ for Implicit methods. Furthermore, important quantitative measures, such as estimates for the time at which solutions of (1.1) hits zero as well as generalisations of the Liapunov exponent which are tailored to these non-linear problems, are estimated to within $O(\Delta)$ as $\Delta \to 0^+$.

The results hold rather generally but these are certain exceptions which suggest that it may be possible in some cases to choose a larger step-size without appreciable loss of performance. Furthermore, the $O(\Delta)$ error estimates that we develop tend often to come in the form of inequalities leaving open the possibility that the theoretical analysis we present may be too conservative. These features of our general results prompt in later chapters further analysis on equations which deal with a rich but more limited class of non-linearities. In this class of so called "regularly varying functions" we will later show that the general analysis presented in this chapter is in fact sharp and that the choice of step-size of $h(x) \sim \Delta x/f(x)$ as $x \to 0^+$ for $\Delta > 0$ and small is in many cases optimal. In fact we will see in the second half of the thesis when autonomous SDEs are considered that time-steps of this order of magnitude are also optimal for preserving the important quantitative and qualitative features we have discussed above. We feel these facts justify the careful analysis we present for these simple equations as that analysis helps build intuition for work on more complicated problems.

3.2 Preliminary Analysis for Explicit Schemes

Our goal is to simulate the solution of the Initial Value Problem (1.1) viz.,

$$x'(t) = -f(x(t)), \quad t > 0, \quad x(0) = \xi > 0.$$

We want a continuous solution of (1.1) to exist, for the solution x to be monotone decreasing on its maximal interval of existence $[0, T_\xi)$ and that

$$\lim_{t\to T_\xi^-} x(t) = 0,$$

the question as to whether T_ξ is finite or infinite being temporarily put to one side. In order to satisfy these qualitative requirements it is natural to impose the following hypothesis on f:

$$f \in C([0,\infty);[0,\infty)),\ f(x) > 0 \text{ for all } x > 0,\ f(0) = 0. \tag{3.1}$$

We now consider a discrete approximation to x. We compute this approximation at an increasing sequence of times $(t_n)_{n\geq 0}$ with $t_0 = 0$ and let x_n be the approximation to $x(t_n)$ for $n \geq 0$. At every state y, we decide *a priori* how big a time-step we shall take which depends solely on the state y. Therefore, if the time-step is to be $h(y)$ at y, we should define (t_n) by

$$t_{n+1} := t_n + h(x_n), \quad n \geq 0, \quad t_0 = 0.$$

This suggests we make the assumption

$$h \in C([0,\infty);[0,\infty)),\ h(x) > 0 \text{ for all } x > 0. \tag{3.2}$$

The assumption of continuity and existence of the one-sided limit

$$\lim_{x\to 0^+} h(x) = h(0^+) < \infty,$$

are technical but essential in generating qualitatively satisfactory solutions. The one-step Explicit Euler scheme based on these precepts is

$$x_{n+1}(\xi) = x_n(\xi) - h(x_n(\xi))f(x_n(\xi)), \quad n = 0,\ldots,N_\xi - 1, \quad x_0(\xi) = \xi > 0, \tag{3.3}$$

where $(x_n(\xi))$ is the sequence for a given initial value ξ and

$$Q_\xi = \{n \geq 0 : x_n(\xi) \leq 0\} \quad \text{and} \quad N_\xi := \inf Q_\xi \tag{3.4}$$

are the set of all n's for which the solution is non-positive and first time the solution becomes non-positive. Define

$$x_\xi := \left\{x_n : n \in Q_\xi^c\right\}. \tag{3.5}$$

Note if $Q_\xi = \emptyset$ then $x_n(\xi) > 0\, \forall\, n \geq 0$ and we set $N_\xi = \infty$. Define also

$$P := \{(y_n)_{n\geq 0} : y_n > 0\, \forall\, n \geq 0\}. \tag{3.6}$$

where P is the set of all positive sequences. For $\xi > 0$, $x_\xi \in P$ is equivalent to $x_n(\xi) > 0 \,\forall\, n \geq 0$. We will show that the condition

$$\frac{h(x)f(x)}{x} < 1, \quad \forall\, x > 0, \tag{3.7}$$

is necessary and sufficient to ensure the positivity of the computed solution.

Theorem 2. *Suppose that f obeys (3.1) and h obeys (3.2). Let x_ξ be the real sequence defined by (3.5) where N_ξ and Q_ξ are given by (3.3) and (3.4). Then the following are equivalent*

(a) h and f obey (3.7);

(b) $\forall\, \xi > 0$, $x_\xi \in P$.

Proof. Suppose (a) holds. Let $\xi > 0$ be arbitrary. Then $x_0(\xi) = \xi > 0$. Clearly

$$x_1(\xi) = \xi - h(\xi)f(\xi) > 0.$$

Suppose now we make the hypothesis

$$x_j(\xi) > 0, \qquad j = 0, 1, \ldots, n. \tag{H_n}$$

Then (H_0) and (H_1) hold. Suppose (H_n) holds. Then

$$x_{n+1}(\xi) = x_n(\xi) - h(x_n(\xi))f(x_n(\xi)) = x_n(\xi)\left(1 - \frac{h(x_n(\xi))f(x_n(\xi))}{x_n(\xi)}\right) > 0.$$

Therefore (H_{n+1}) holds and thus (H_n) holds for all $n \geq 0$ i.e. $x_n(\xi) > 0 \,\forall\, n \geq 0$. This implies that $Q_\xi = \emptyset$, $N_\xi = \infty$ and therefore $x_\xi \in P$. Hence (a) implies (b). To show (b) implies (a), by hypothesis $x_\xi \in P$. This means that $x_n(\xi) > 0 \,\forall\, n \geq 0$. In particular $x_1(\xi) > 0$. But as $x_0(\xi) = \xi$ we have

$$0 < x_1(\xi) = x_0(\xi) - h(x_0(\xi))f(x_0(\xi)) = \xi\left(1 - \frac{h(\xi)f(\xi)}{\xi}\right),$$

so

$$\frac{h(\xi)f(\xi)}{\xi} < 1. \tag{3.8}$$

But since $\xi > 0$ was chosen arbitrarily, (3.8) holds for all $\xi > 0$ which is precisely (3.7). Hence (b) implies (a) and (a) and (b) are equivalent. □

The condition (3.7) is not only necessary and sufficient to ensure positivity; it also guarantees, in conjunction with (3.1) and (3.2), the monotonicity and convergence of $(x_n)_{n\geq 0}$ to zero as $n \to \infty$.

Theorem 3. *Suppose that f obeys* (3.1) *and h obeys* (3.2) *and that* (3.7) *holds. Let x_ξ be the real sequence defined by* (3.5) *where N_ξ and Q_ξ are given by* (3.3) *and* (3.4). *Then for any $\xi > 0$:*

(i) $x_n(\xi) > 0 \,\forall\, n \geq 0$.

(ii) $x_{n+1}(\xi) < x_n(\xi)$, $n \geq 0$.

(iii) $\lim_{n\to\infty} x_n(\xi) = 0$.

Proof. Part (i) is the forward implication in the previous theorem. Since $x_n(\xi) > 0$ for each $n \geq 0$ and $\xi > 0$ we have that $f(x_n(\xi)) > 0$ and $h(x_n(\xi)) > 0$. Hence

$$x_{n+1}(\xi) = x_n(\xi) - h(x_n(\xi))f(x_n(\xi)) < x_n(\xi),$$

which is part (ii). Since $(x_n(\xi))_{n\geq 0}$ is a positive sequence which is bounded below, it must have a limit as $n \to \infty$. Let $L_\xi = \lim_{n\to\infty} x_n(\xi)$. Then $L_\xi \in [0, \infty)$. Also as f and h are continuous on $[0, \infty)$, we have

$$L_\xi = \lim_{n\to\infty} x_{n+1}(\xi) = \lim_{n\to\infty} \{x_n(\xi) - h(x_n(\xi))f(x_n(\xi))\} = L_\xi - h(L_\xi)f(L_\xi),$$

and this is valid even when $L_\xi = 0$ (in which case part (iii) is true). Clearly $L_\xi \in [0, \infty)$ must be such that $h(L_\xi)f(L_\xi) = 0$. Suppose $L_\xi > 0$. Then $h(L_\xi)f(L_\xi) > 0$ a contradiction so it must follow that $L_\xi = 0$, as required. □

Since we are going to impose (3.7) in what follows for the solutions of the Explicit Euler method in order to make the presentation more digestible, we will suppress the careful notation and constructions of this section and talk freely about the solution $(x_n)_{n\geq 0}$ of the difference equation

$$x_{n+1} = x_n - h(x_n)f(x_n), \quad n \geq 0, \quad x_0 = \xi > 0,$$

where

$$t_{n+1} = \sum_{j=0}^{n} h(x_j), \quad n \geq 0, \quad t_0 = 0.$$

Under (3.7), these sequences are well-defined without further qualification.

3.3 Monotonicity Estimates and Standing Assumptions

We will make the following standing assumptions throughout this chapter when seeking to simulate the solution of (1.1)

$$f \in C([0,\infty) : [0,\infty)),\ f(x) > 0 \text{ for all } x > 0,\ f(0) = 0;$$
$$h \in C([0,\infty) : [0,\infty)),\ h(x) > 0 \text{ for all } x > 0;$$
$$f \text{ is asymptotic to an increasing function; and} \tag{3.9}$$
$$x \mapsto x/f(x) \text{ is asymptotic to an increasing function.} \tag{3.10}$$

Other assumptions such as (3.7) may be needed for different numerical schemes.

We make the following observations which will be of use in several of our proofs. Suppose (x_n) is a decreasing positive sequence such that $x_n \to 0$ as $n \to \infty$. Suppose $\int_{0+}^1 1/f(u)\,du = \infty$. If F is defined by (1.11) then $F(x) \to \infty$ as $x \to 0^+$, so $F(x_n) \to \infty$ as $n \to \infty$. Then for $n \geq 1$

$$F(x_n) = \int_{x_n}^1 \frac{1}{f(u)}\,du \quad = \quad \int_{x_0}^1 \frac{1}{f(u)}\,du + \int_{x_n}^{x_0} \frac{1}{f(u)}\,du = F(x_0) + \sum_{j=0}^{n-1} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du.$$

If $F(x_n) \to \infty$ as $n \to \infty$ then

$$\sum_{j=0}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du = \infty, \tag{3.11}$$

since $F(x_0)$ is finite. Suppose $\int_{0+}^1 1/f(u)\,du < \infty$. Then $F(x) \to L \in [0,\infty)$ as $x \to 0^+$, so $F(x_n) \to L$ as $n \to \infty$. Hence

$$\sum_{j=0}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du < \infty. \tag{3.12}$$

If T_ξ is defined by (1.8) then for $n \geq 0$

$$T_\xi = \int_0^\xi \frac{1}{f(u)}\,du = \sum_{j=0}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du.$$

Equations (3.11) and (3.12) show that T_ξ is finite or infinite according to whether $F(x)$ is finite or infinite. If $\bar{F}$ is defined by (1.10) then $\bar{F}(x) \to 0$ as $x \to 0^+$ so $\bar{F}(x_n) \to 0$ as $n \to \infty$. Then for $n \geq 0$

$$\bar{F}(x_n) = \int_0^{x_n} \frac{1}{f(u)}\,du = \sum_{j=n}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du.$$

The closed-form expressions for $F(x_n)$, $\bar{F}(x_n)$ and T_ξ identify the summand in the last identity as the key sequence in our analysis. We derive an asymptotic inequality for this sequence by imposing monotonicity assumptions on f and $x/f(x)$. By (3.9) and (3.10) for every $\epsilon \in (0,1)$, there is $x_1(\epsilon) > 0$ such that

$$1-\epsilon < \frac{f(x)}{\phi(x)} < 1+\epsilon, \quad \forall\, x < x_1(\epsilon), \tag{3.13}$$

$$1-\epsilon < \frac{x/f(x)}{\psi(x)} < 1+\epsilon, \quad \forall\, x < x_1(\epsilon), \tag{3.14}$$

where ϕ and ψ are increasing functions. Next as $x_n \to 0$ as $n \to \infty$, then $x_n < x_1(\epsilon)$ for all $n \geq N_1(\epsilon)$. Let $n \geq N_1(\epsilon)$ and $x_{n+1} < u < x_n$. Then

$$(1-\epsilon)\cdot\psi(u) < \frac{u}{f(u)} < (1+\epsilon)\cdot\psi(u),$$

and since ψ is increasing

$$(1-\epsilon)\cdot\frac{\psi(x_{n+1})}{u} \leq \frac{1}{f(u)} \leq (1+\epsilon)\cdot\frac{\psi(x_n)}{u}.$$

Therefore for all $n \geq N_1(\epsilon)$

$$(1-\epsilon)\cdot\psi(x_{n+1})\log\left(\frac{x_n}{x_{n+1}}\right) \leq \int_{x_{n+1}}^{x_n}\frac{1}{f(u)}\,du \leq (1+\epsilon)\cdot\psi(x_n)\log\left(\frac{x_n}{x_{n+1}}\right). \tag{3.15}$$

The form of (3.15) changes according to whether we discretise (1.1) using an Explicit or Implicit Euler scheme. We determine the applicable form in the relevant section.

3.3.1 Preserving Soft Landings and Super-Exponential Stability

The Explicit Euler scheme, defined by equation (1.41), reproduces the finite hitting time of the equilibrium at zero when T_ξ is finite. When T_ξ is infinite, the Euler scheme reproduces the super-exponential convergence to zero. Since $(h(x_n))$ is a positive sequence the limit

$$\lim_{n\to\infty} t_n =: \hat{T}_h = \sum_{j=0}^{\infty} h(x_j), \tag{3.16}$$

exists but can be finite or infinite.

We first determine the form of (3.15) when an Explicit Euler scheme is used to discretise (1.1). The resulting equations will be used in out later proofs of preserving soft landings, super-exponential stability and determining asymptotic convergence rates. The following quantitative estimate on the step-size as we approach the equilibrium is central to the analysis in this thesis:

$$\lim_{x\to 0^+} \frac{h(x)f(x)}{x} = \Delta. \tag{3.17}$$

Lemma 6. *Suppose* (x_n) *is a positive decreasing sequence and the solution of* (1.41). *If* f *obeys* (3.1), (3.7) *and* (3.10) *while* h *obeys* (3.2) *and* (3.17) *with* $\Delta \in [0,1)$ *then* (3.15) *holds viz.,*

$$(1-\epsilon)\cdot\psi(x_{n+1})\log\left(\frac{x_n}{x_{n+1}}\right) \le \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\,du \le (1+\epsilon)\cdot\psi(x_n)\log\left(\frac{x_n}{x_{n+1}}\right).$$

Furthermore,

(i) if $\Delta = 0$ *and* f *obeys* (3.9), *then for all* $\epsilon \in (0,1)$ *and all* n *sufficiently large*

$$\frac{(1-\epsilon)^4}{(1+\epsilon)^2}\cdot h(x_n) \le \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\,du \le \frac{(1+\epsilon)^2}{1-\epsilon}\cdot h(x_n). \tag{3.18}$$

(ii) if $\Delta \in (0,1)$, *then for all* $\epsilon \in (0,1)$ *and all* n *sufficiently large*

$$\frac{(1-\epsilon)^2}{(1+\epsilon)^2}\cdot\frac{1}{\Delta}\log\left(\frac{1}{1-\Delta}\right)h(x_{n+1}) \le \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\,du \le \frac{(1+\epsilon)^2}{(1-\epsilon)^2}\cdot\frac{1}{\Delta}\log\left(\frac{1}{1-\Delta}\right)h(x_n). \tag{3.19}$$

Proof. We prove part (i) first. Let $\Delta = 0$. By virtue of (3.9), (3.14) and the constructions thereafter are valid and we have

$$\frac{x_{n+1}}{(1+\epsilon)f(x_{n+1})} < \psi(x_{n+1}) \quad \text{and} \quad \psi(x_n) < \frac{x_n}{(1-\epsilon)f(x_n)}.$$

Substituting these expressions into (3.15) yields for $n \ge N_1(\epsilon)$

$$\frac{1-\epsilon}{1+\epsilon}\cdot\frac{x_{n+1}}{f(x_{n+1})}\cdot\log\left(\frac{x_n}{x_{n+1}}\right) \le \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\,du \le \frac{1+\epsilon}{1-\epsilon}\cdot\frac{x_n}{f(x_n)}\cdot\log\left(\frac{x_n}{x_{n+1}}\right). \tag{3.20}$$

Define

$$a_n := \log\left(\frac{x_n}{x_{n+1}}\right) = -\log\left(1 - \frac{h(x_n)f(x_n)}{x_n}\right) \sim \frac{h(x_n)f(x_n)}{x_n},$$

as $n \to \infty$, since (3.17) holds with $\Delta = 0$ and $-\log(1-x) \sim x$ as $x \to 0^+$. Thus there is an $N_2(\epsilon) \in \mathbb{N}$ such that for $n \ge N_2(\epsilon)$

$$(1-\epsilon)\cdot\frac{h(x_n)f(x_n)}{x_n} < a_n < (1+\epsilon)\cdot\frac{h(x_n)f(x_n)}{x_n}.$$

Let $N_3(\epsilon) := \max(N_1(\epsilon), N_2(\epsilon))$. Then for $n \geq N_3(\epsilon)$

$$\frac{(1-\epsilon)^2}{1+\epsilon} \cdot \frac{x_{n+1}}{f(x_{n+1})} \cdot \frac{h(x_n)f(x_n)}{x_n} \leq \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\, du \leq \frac{(1+\epsilon)^2}{1-\epsilon} \cdot \frac{x_n}{f(x_n)} \cdot \frac{h(x_n)f(x_n)}{x_n}.$$

Thus for $n \geq N_3(\epsilon)$

$$\frac{(1-\epsilon)^2}{1+\epsilon} \cdot \frac{x_{n+1}}{x_n} \cdot \frac{f(x_n)}{f(x_{n+1})} \cdot h(x_n) \leq \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\, du \leq \frac{(1+\epsilon)^2}{1-\epsilon} \cdot h(x_n). \tag{3.21}$$

Equation (3.13) implies

$$1-\epsilon < \frac{f(x_n)}{\phi(x_n)} < 1+\epsilon \quad \text{and} \quad 1-\epsilon < \frac{f(x_{n+1})}{\phi(x_{n+1})} < 1+\epsilon.$$

Thus $(1-\epsilon) \cdot \phi(x_n) < f(x_n)$ and $f(x_{n+1}) < (1+\epsilon) \cdot \phi(x_{n+1})$. So

$$\frac{f(x_n)}{f(x_{n+1})} > \frac{1-\epsilon}{1+\epsilon} \cdot \frac{\phi(x_n)}{\phi(x_{n+1})} > \frac{1-\epsilon}{1+\epsilon}, \tag{3.22}$$

since ϕ is increasing and (x_n) is decreasing. Substituting this into (3.21) yields

$$\frac{(1-\epsilon)^3}{(1+\epsilon)^2} \cdot \frac{x_{n+1}}{x_n} \cdot h(x_n) \leq \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\, du \leq \frac{(1+\epsilon)^2}{1-\epsilon} \cdot h(x_n),$$

for $n \geq N_3(\epsilon)$. Since

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \left(1 - \frac{h(x_n)f(x_n)}{x_n}\right) = 1,$$

there is $n \geq N_4(\epsilon)$ such that

$$1-\epsilon < \frac{x_{n+1}}{x_n} < 1+\epsilon.$$

Let $N_5(\epsilon) := \max(N_3(\epsilon), N_4(\epsilon))$. Then for $n \geq N_5(\epsilon)$

$$\frac{(1-\epsilon)^4}{(1+\epsilon)^2} \cdot h(x_n) \leq \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\, du < \frac{(1+\epsilon)^2}{1-\epsilon} \cdot h(x_n),$$

which is equation (3.18) and proves part (i). To prove part (ii), we have $\Delta \in (0,1)$: the estimates $x_n < x_1(\epsilon)$ for all $n > N_1(\epsilon)$ and (3.20) still pertain:

$$\frac{1-\epsilon}{1+\epsilon} \cdot \frac{x_{n+1}}{f(x_{n+1})} \cdot \log\left(\frac{x_n}{x_{n+1}}\right) \leq \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\, du \leq \frac{1+\epsilon}{1-\epsilon} \cdot \frac{x_n}{f(x_n)} \cdot \log\left(\frac{x_n}{x_{n+1}}\right).$$

Define

$$a_n := \log\left(\frac{x_n}{x_{n+1}}\right) = -\log\left(1 - \frac{h(x_n)f(x_n)}{x_n}\right).$$

Since (3.17) holds and $\Delta \in (0,1)$ then

$$\lim_{n\to\infty} a_n = -\lim_{n\to\infty} \log\left(1 - \frac{h(x_n)f(x_n)}{x_n}\right) = -\log(1-\Delta) = \log\left(\frac{1}{1-\Delta}\right).$$

Thus there is $n \geq N_2(\epsilon) \in \mathbb{N}$ such that

$$(1-\epsilon)\cdot\log\left(\frac{1}{1-\Delta}\right) < a_n < (1+\epsilon)\cdot\log\left(\frac{1}{1-\Delta}\right).$$

Let $N_3(\epsilon) := \max(N_1(\epsilon), N_2(\epsilon))$. Then for $n \geq N_3(\epsilon)$

$$\frac{(1-\epsilon)^2}{1+\epsilon}\cdot\frac{1}{\Delta}\log\left(\frac{1}{1-\Delta}\right)\cdot\frac{\Delta x_{n+1}}{f(x_{n+1})} \leq \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\,du \leq \frac{(1+\epsilon)^2}{1-\epsilon}\cdot\frac{1}{\Delta}\log\left(\frac{1}{1-\Delta}\right)\cdot\frac{\Delta x_n}{f(x_n)}.$$

Since $h(x_n) \sim \Delta x_n / f(x_n)$, so there is $N_4(\epsilon) \in \mathbb{N}$ such that for $n \geq N_4(\epsilon)$ we have

$$(1-\epsilon)\cdot\frac{\Delta x_n}{f(x_n)} < h(x_n) < (1+\epsilon)\cdot\frac{\Delta x_n}{f(x_n)}.$$

Let $N_5(\epsilon) := \max(N_4(\epsilon), N_3(\epsilon))$ and $n \geq N_5(\epsilon)$. Then

$$\frac{(1-\epsilon)^2}{(1+\epsilon)^2}\cdot\frac{1}{\Delta}\log\left(\frac{1}{1-\Delta}\right)\cdot h(x_{n+1}) \leq \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\,du \leq \frac{(1+\epsilon)^2}{(1-\epsilon)^2}\cdot\frac{1}{\Delta}\log\left(\frac{1}{1-\Delta}\right)\cdot h(x_n),$$

which is equation (3.19) and proves part (ii). □

In our next result, we show that $\hat{T}_h$ is finite or infinite according to whether T_ξ defined by (1.8) is finite or infinite.

Theorem 4. *Suppose f obeys* (3.1), (3.7), (3.9) *and* (3.10) *while h obeys* (3.2) *and* (3.17) *with $\Delta \in [0,1)$. Let (t_n) and $\hat{T}_h$ be defined* (1.42) *and* (3.16).

(i) If f obeys (1.7), *then $\hat{T}_h < \infty$.*

(ii) If f obeys (1.9), *then $\hat{T}_h = \infty$.*

Proof. By (1.7), $\int_{0^+}^1 1/f(u)\,du < \infty$ then $T_\xi < \infty$ from (3.12) since

$$\sum_{j=0}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du = \int_0^\xi \frac{1}{f(u)}\,du = T_\xi < \infty.$$

When $\Delta = 0$, the Comparison Test applied to (3.18) shows the summability of the summand, $(\int_{x_{n+1}}^{x_n} 1/f(u)\,du)$, implies that of $(h(x_n))$. When $\Delta \in (0,1)$, the Comparison Test applied to (3.19) shows the summability of $(\int_{x_{n+1}}^{x_n} 1/f(u)\,du)$ implies that of $(\log\left(\frac{1}{1-\Delta}\right) h(x_n)/\Delta)$ and hence $(h(x_n))$ since $\log\left(\frac{1}{1-\Delta}\right)/\Delta$ is finite when $\Delta \in (0,1)$. Hence in both cases $(h(x_n))$ is summable and we have that $t_n = \sum_{j=0}^{n-1} h(x_j)$ for $n \geq 1$

obeys $t_n \to \hat{T}_h := \sum_{j=0}^{\infty} h(x_j) < \infty$ as $n \to \infty$.

By (1.9), $\int_{0+}^{1} 1/f(u)\,du = \infty$ then $T_\xi = \infty$ from (3.11) since

$$\sum_{j=0}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du = \int_0^\xi \frac{1}{f(u)}\,du = T_\xi = \infty.$$

The Comparison Test applied to (3.18) and (3.19) shows that $(h(x_n))$ is not summable and obeys $t_n = \sum_{j=0}^{n-1} h(x_j) \to \infty$ as $n \to \infty$. Therefore $t_n \to \infty$ as $n \to \infty$. □

The next result shows that, once h obeys (3.17), the Euler scheme (1.41) recovers the exact rate of asymptotic convergence when $\Delta = 0$ but not when $\Delta \in (0,1)$ despite preserving finite-time and super-exponential stability. We tackle the case of super-exponential convergence first.

Theorem 5. *Suppose f obeys* (1.9), (3.1), (3.7) *and* (3.10) *while h obeys* (3.2) *and* (3.17) *with $\Delta \in [0,1)$. Let F and (t_n) be defined by* (1.11) *and* (1.42).

(i) If $\Delta = 0$ and f obeys (3.9), *then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \infty$ as $n \to \infty$ and*

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = 1.$$

(ii) If $\Delta \in (0,1)$, then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = \frac{1}{\Delta}\log\left(\frac{1}{1-\Delta}\right) =: \lambda_E(\Delta).$$

Proof. The positivity, monotonicity and convergence of (x_n) have been addressed in Theorem 3. Since f obeys (1.9) then $\int_{0+}^{1} 1/f(u)\,du = \infty$ and $t_n \to \infty$ as $n \to \infty$ by Theorem 4. We prove part (i) first. Letting $n \geq N_5(\epsilon) + 1$ in (3.18) yields

$$\frac{(1-\epsilon)^4}{(1+\epsilon)^2} \sum_{j=N_5(\epsilon)}^{n-1} h(x_j) \leq \sum_{j=N_5(\epsilon)}^{n-1} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du \leq \frac{(1+\epsilon)^2}{1-\epsilon} \sum_{j=N_5(\epsilon)}^{n-1} h(x_j).$$

Thus for $n \geq N_5(\epsilon) + 1$

$$\frac{(1-\epsilon)^4}{(1+\epsilon)^2} \cdot (t_n - t_{N_5(\epsilon)}) \leq F(x_n) - F(x_{N_5(\epsilon)}) \leq \frac{(1+\epsilon)^2}{1-\epsilon} \cdot (t_n - t_{N_5(\epsilon)}).$$

Therefore, as $t_n \to \infty$ as $n \to \infty$, by dividing by t_n and letting $n \to \infty$ we get

$$\frac{(1-\epsilon)^4}{(1+\epsilon)^2} \leq \liminf_{n\to\infty} \frac{F(x_n)}{t_n} \leq \limsup_{n\to\infty} \frac{F(x_n)}{t_n} \leq \frac{(1+\epsilon)^2}{1-\epsilon}.$$

Letting $\epsilon \to 0^+$ yields

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = 1,$$

the desired limit in part (i), as claimed. We now prove part (ii). Letting $n \geq N_5(\epsilon)+1$ in (3.19) yields

$$\frac{(1-\epsilon)^2}{(1+\epsilon)^2}\cdot \lambda_E(\Delta) \sum_{j=N_5(\epsilon)}^{n-1} h(x_{j+1}) \leq \sum_{j=N_5(\epsilon)}^{n-1} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du \leq \frac{(1+\epsilon)^2}{(1-\epsilon)^2}\cdot \lambda_E(\Delta) \sum_{j=N_5(\epsilon)}^{n-1} h(x_j).$$

Therefore

$$\frac{(1-\epsilon)^2}{(1+\epsilon)^2}\cdot \lambda_E(\Delta) \sum_{j=N_5(\epsilon)+1}^{n} h(x_j) \leq F(x_n) - F(x_{N_5(\epsilon)}) \leq \frac{(1+\epsilon)^2}{(1-\epsilon)^2}\cdot \lambda_E(\Delta)\cdot (t_n - t_{N_5(\epsilon)}),$$

or

$$\frac{(1-\epsilon)^2}{(1+\epsilon)^2}\cdot \lambda_E(\Delta)\cdot (t_{n+1} - t_{N_5(\epsilon)+1}) \leq F(x_n) - F(x_{N_5(\epsilon)}) \leq \frac{(1+\epsilon)^2}{(1-\epsilon)^2}\cdot \lambda_E(\Delta)\cdot (t_n - t_{N_5(\epsilon)}). \tag{3.23}$$

Since $h(x_n) \sim \Delta x_n/f(x_n)$ then $h(x_n) \to 0$ as $n \to \infty$ by (1.6). By (1.42), $t_{n+1} = t_n + h(x_n)$ so $t_{n+1}/t_n \to 1$ as $n \to \infty$. Therefore as $t_n \to \infty$ as $n \to \infty$ dividing (3.23) by t_n and letting $n \to \infty$ yields

$$\frac{(1-\epsilon)^2}{(1+\epsilon)^2}\cdot \lambda_E(\Delta) \leq \liminf_{n\to\infty} \frac{F(x_n)}{t_n} \leq \limsup_{n\to\infty} \frac{F(x_n)}{t_n} \leq \frac{(1+\epsilon)^2}{(1-\epsilon)^2}\cdot \lambda_E(\Delta).$$

Letting $\epsilon \to 0^+$ yields

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = \lambda_E(\Delta),$$

the desired limit in part (ii), as claimed. □

Remark 2. Theorem 5 part (ii) shows that $\Delta \geq 1$ gives spurious asymptotic behaviour since $\lambda_E(\Delta)$ is undefined. □

The next results shows the Euler scheme correctly predicts the precise asymptotic behaviour by imposing a monotonicity condition on $h(x)f(x)/x$ instead of f.

Corollary 2. *Suppose f obeys* (1.9), (3.1), (3.7) *and* (3.10) *while h obeys* (3.2) *and* (3.17) *with $\Delta = 0$. Let F and (t_n) be defined by* (1.11) *and* (1.42). *If*

$$x \mapsto \frac{h(x)f(x)}{x} \quad \textit{is asymptotically increasing,} \tag{3.24}$$

holds instead of (3.9) *then*

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = 1.$$

Proof. The positivity, monotonicity and convergence of (x_n) still prevail. Since f obeys (1.9) then $\int_{0+}^1 1/f(u)\,du = \infty$ and $t_n \to \infty$ as $n \to \infty$ by Theorem 4. By the second inequality of (3.21), for $n \geq N_3(\epsilon)+1$

$$F(x_n) - F(x_{N_3(\epsilon)}) = \sum_{j=N_3(\epsilon)}^{n-1} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du \leq \frac{(1+\epsilon)^2}{1-\epsilon} \sum_{j=N_3(\epsilon)}^{n-1} h(x_j).$$

Thus

$$F(x_n) \leq F(x_{N_3(\epsilon)}) + \frac{(1+\epsilon)^2}{1-\epsilon} \cdot (t_n - t_{N_3(\epsilon)}).$$

Therefore, dividing by t_n, letting $n \to \infty$ and then $\epsilon \to 0^+$ yields

$$\limsup_{n\to\infty} \frac{F(x_n)}{t_n} \leq 1. \tag{3.25}$$

Since $\Delta = 0$, $x_{n+1}/x_n \to 1$ as $n \to \infty$ and so there is $N_4(\epsilon) \in \mathbb{N}$ such that for all $n \geq N_4(\epsilon)$ we have

$$1-\epsilon < \frac{x_{n+1}}{x_n} < 1+\epsilon.$$

The first inequality of (3.21) implies for $n \geq N_5(\epsilon) := \max(N_3(\epsilon), N_4(\epsilon))$

$$\frac{(1-\epsilon)^3}{1+\epsilon} \cdot \frac{f(x_n)}{f(x_{n+1})} \cdot h(x_n) \leq \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\,du. \tag{3.26}$$

Calling θ the increasing function asymptotic to $h(x)f(x)/x$ we have for every $\epsilon \in (0,1)$ there is $x_2(\epsilon) > 0$ such that for $x < x_2(\epsilon)$

$$(1-\epsilon)\cdot\theta(x) < \frac{h(x)f(x)}{x} < (1+\epsilon)\cdot\theta(x).$$

Let $N_6(\epsilon)$ be so large that $n \geq N_6(\epsilon)$ implies $x_n < x_2(\epsilon)$. Then, as (x_n) is decreasing, for $n \geq N_6(\epsilon)$

$$\frac{h(x_n)f(x_n)}{x_n} > (1-\epsilon)\cdot\theta(x_n) \quad > \quad (1-\epsilon)\cdot\theta(x_{n+1}) > \frac{1-\epsilon}{1+\epsilon}\cdot\frac{h(x_{n+1})f(x_{n+1})}{x_{n+1}}.$$

Therefore for $n \geq N_6(\epsilon)$

$$\frac{h(x_n)f(x_n)}{f(x_{n+1})} > \frac{1-\epsilon}{1+\epsilon}\cdot\frac{x_n}{x_{n+1}}\cdot h(x_{n+1}).$$

Let $N_7(\epsilon) := \max(N_5(\epsilon), N_6(\epsilon))$. Then for $n \geq N_7(\epsilon)$ then (3.26) implies

$$\frac{(1-\epsilon)^5}{(1+\epsilon)^2}\cdot h(x_{n+1}) < \frac{(1-\epsilon)^4}{(1+\epsilon)^2}\cdot\frac{x_n}{x_{n+1}}\cdot h(x_{n+1}) \leq \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\,du.$$

Hence for $n \geq N_7(\epsilon) + 1$

$$F(x_n) - F(x_{N_7(\epsilon)}) = \sum_{j=N_7(\epsilon)}^{n-1} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\, du \;\geq\; \frac{(1-\epsilon)^5}{(1+\epsilon)^2} \sum_{j=N_7(\epsilon)}^{n-1} h(x_{j+1})$$
$$= \frac{(1-\epsilon)^5}{(1+\epsilon)^2} \sum_{j=N_7(\epsilon)+1}^{n} h(x_j).$$

Thus for $n \geq N_7(\epsilon) + 1$

$$F(x_n) \geq F(x_{N_7(\epsilon)}) + \frac{(1-\epsilon)^5}{(1+\epsilon)^2} \cdot (t_{n+1} - t_{N_7(\epsilon)+1}). \tag{3.27}$$

Since $h(x) = o(x/f(x))$ and $x/f(x)$ is asymptotic to an increasing function then $h(x) \to 0$ as $x \to 0^+$. Thus $t_{n+1}/t_n \to 1$ as $n \to \infty$. Hence

$$\liminf_{n\to\infty} \frac{F(x_n)}{t_n} \geq 1.$$

Combining with (3.25) yields

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = 1,$$

as claimed. □

Remark 3. By assuming (1.6) in Theorem 5 part (ii) and Corollary 2 the solution of (1.1) convergences super-exponentially. However, we are able to recover the correct asymptotic behaviour when there is exponential convergence if (1.6) is replaced by $f(x)/x$ tending to a positive finite limit. The proofs are the same up to (3.23) and (3.27). However, in the case of exponential convergence $h(x_n) \sim \Delta x_n/f(x_n) \sim \Delta\psi(x_n)$ tends to a finite limit as $n \to \infty$ since (x_n) is decreasing and ψ is increasing. Therefore, as $t_n \to \infty$ as $n \to \infty$, then $t_{n+1}/t_n \to 1$ as $n \to \infty$. Hence, dividing (3.23) by t_n and letting $n \to \infty$ yields the desired results. □

We now tackle the case when there is finite-time stability or a soft landing.

Theorem 6. *Suppose f obeys* (1.7), (3.1), (3.7) *and* (3.10) *while h obeys* (3.2) *and* (3.17) *with $\Delta \in [0, 1)$. Let $\bar{F}$, (t_n) and $\hat{T}_h$ be defined by* (1.10), (1.42) *and* (3.16).

(i) If $\Delta = 0$ and f obeys (3.9), *then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and*

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = 1.$$

(ii) If $\Delta \in (0, 1)$, then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$,

$t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and

$$\lambda_E(\Delta) \le \liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_{n+1}} \quad \text{and} \quad \limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \le \frac{1}{\Delta}\log\left(\frac{1}{1-\Delta}\right) =: \lambda_E(\Delta).$$

If in addition, f obeys (3.9) and $0 < \Delta < 1 - 1/e$ then

$$\lambda_E(\Delta)(1 - \Delta\lambda_E(\Delta)) \le \liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \le \limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \le \lambda_E(\Delta).$$

Proof. The positivity, monotonicity and convergence of (x_n) have been addressed in Theorem 3. Since f obeys (1.7) then $\int_{0+}^1 1/f(u)\,du < \infty$ and $t_n \to \hat{T}_h := \sum_{j=0}^\infty h(x_j) < \infty$ by Theorem 4. Hence $\hat{T}_h - t_n = \sum_{j=n}^\infty h(x_j) \to 0$ as $n \to \infty$. We prove part (i) first. Letting $n \ge N_5(\epsilon)$ in (3.18) yields

$$\frac{(1-\epsilon)^4}{(1+\epsilon)^2}\sum_{j=n}^\infty h(x_j) \le \sum_{j=n}^\infty \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du \le \frac{(1+\epsilon)^2}{1-\epsilon}\sum_{j=n}^\infty h(x_j).$$

Thus for $n \ge N_5(\epsilon)$

$$\frac{(1-\epsilon)^4}{(1+\epsilon)^2} \cdot (\hat{T}_h - t_n) \le \bar{F}(x_n) \le \frac{(1+\epsilon)^2}{1-\epsilon} \cdot (\hat{T}_h - t_n).$$

Therefore, as $\hat{T}_h - t_n \to 0$, dividing by $\hat{T}_h - t_n$ and letting $n \to \infty$ yields

$$\frac{(1-\epsilon)^4}{(1+\epsilon)^2} \le \liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \le \limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \le \frac{(1+\epsilon)^2}{1-\epsilon}.$$

Letting $\epsilon \to 0^+$ yields

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = 1,$$

as claimed in part (i). We now prove part (ii). Letting $n \ge N_5(\epsilon)$ in (3.19) yields

$$\frac{(1-\epsilon)^2}{(1+\epsilon)^2} \cdot \lambda_E(\Delta)\sum_{j=n}^\infty h(x_{j+1}) \le \sum_{j=n}^\infty \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du \le \frac{(1+\epsilon)^2}{(1-\epsilon)^2} \cdot \lambda_E(\Delta)\sum_{j=n}^\infty h(x_j).$$

Thus for $n \ge N_5(\epsilon)$

$$\frac{(1-\epsilon)^2}{(1+\epsilon)^2} \cdot \lambda_E(\Delta) \cdot (\hat{T}_h - t_{n+1}) \le \bar{F}(x_n) \le \frac{(1+\epsilon)^2}{(1-\epsilon)^2} \cdot \lambda_E(\Delta) \cdot (\hat{T}_h - t_n).$$

Therefore

$$\lambda_E(\Delta) \le \liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_{n+1}} \quad \text{and} \quad \limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \le \lambda_E(\Delta),$$

which proves the first estimate in part (ii) as claimed. For the second estimate in (ii), define $\delta(x) := h(x)f(x)/x$ where $\delta(x) \to \Delta$ as $x \to 0^+$. Thus

$$\begin{aligned}
\liminf_{n\to\infty} \frac{\bar{F}(x_{n+1})}{\hat{T}_h - t_{n+1}} &= \liminf_{n\to\infty} \left(\frac{\bar{F}(x_{n+1})}{\bar{F}(x_n)} \cdot \frac{\bar{F}(x_n)}{\hat{T}_h - t_{n+1}} \right) \\
&\geq \lambda_E(\Delta) \cdot \liminf_{n\to\infty} \frac{\bar{F}(x_{n+1})}{\bar{F}(x_n)} \\
&= \lambda_E(\Delta) \cdot \liminf_{n\to\infty} \frac{\bar{F}(x_n(1-\delta(x_n)))}{\bar{F}(x_n)} \\
&\geq \lambda_E(\Delta) \cdot \liminf_{x\to 0^+} \frac{\bar{F}(x(1-\delta(x)))}{\bar{F}(x)}. \qquad (3.28)
\end{aligned}$$

Suppose $x < x_1(\epsilon)$. Then for $u < x < x_1(\epsilon)$, by (3.14), $u/f(u) < (1+\epsilon) \cdot \psi(u) < (1+\epsilon) \cdot \psi(x)$. Hence

$$\begin{aligned}
0 < \bar{F}(x) - \bar{F}(x(1-\delta(x))) = \int_{x(1-\delta(x))}^{x} \frac{1}{u} \cdot \frac{u}{f(u)}\, du &\leq (1+\epsilon) \cdot \psi(x) \int_{x(1-\delta(x))}^{x} \frac{1}{u}\, du \\
&= (1+\epsilon) \cdot \psi(x) \cdot \log\left(\frac{1}{1-\delta(x)}\right).
\end{aligned}$$

Therefore for $x < x_1(\epsilon)$

$$0 < 1 - \frac{\bar{F}(x(1-\delta(x)))}{\bar{F}(x)} \leq (1+\epsilon) \cdot \log\left(\frac{1}{1-\delta(x)}\right) \cdot \frac{\psi(x)}{\bar{F}(x)}.$$

For $u < x < x_1(\epsilon)$, by (3.13), $f(u) < (1+\epsilon) \cdot \phi(u) < (1+\epsilon) \cdot \phi(x)$. Thus

$$\frac{\bar{F}(x)}{\psi(x)} = \frac{1}{\psi(x)} \int_0^x \frac{1}{f(u)}\, du > \frac{1}{\psi(x)} \int_0^x \frac{1}{(1+\epsilon)\phi(x)}\, du = \frac{1}{1+\epsilon} \cdot \frac{x}{\psi(x)\phi(x)}.$$

Thus

$$\liminf_{x\to 0^+} \frac{\bar{F}(x)}{\psi(x)} \geq \frac{1}{1+\epsilon},$$

and letting $\epsilon \to 0^+$ yields

$$\liminf_{x\to 0^+} \frac{\bar{F}(x)}{\psi(x)} \geq 1.$$

Hence there is $x_2(\epsilon) > 0$ such that

$$\frac{\bar{F}(x)}{\psi(x)} > \frac{1}{1+\epsilon}, \quad \forall\, x < x_2(\epsilon).$$

Let $x_3(\epsilon) := \min(x_1(\epsilon), x_2(\epsilon))$. Then for $x < x_3(\epsilon)$

$$0 < 1 - \frac{\bar{F}(x(1-\delta(x)))}{\bar{F}(x)} \leq (1+\epsilon)^2 \cdot \log\left(\frac{1}{1-\delta(x)}\right).$$

Rearranging yields

$$\frac{\bar{F}(x(1-\delta(x)))}{\bar{F}(x)} \geq 1-(1+\epsilon)^2 \cdot \log\left(\frac{1}{1-\delta(x)}\right).$$

Hence

$$\liminf_{x\to 0^+} \frac{\bar{F}(x(1-\delta(x)))}{\bar{F}(x)} \geq 1-(1+\epsilon)^2 \cdot \log\left(\frac{1}{1-\Delta}\right).$$

Letting $\epsilon \to 0^+$ yields

$$\liminf_{x\to 0^+} \frac{\bar{F}(x(1-\delta(x)))}{\bar{F}(x)} \geq 1-\log\left(\frac{1}{1-\Delta}\right). \tag{3.29}$$

This bound is useful if $1-\log(1/(1-\Delta)) > 0$ or $1/e < 1-\Delta$. This implies $\Delta < 1-1/e$. Combining (3.28) and (3.29) for $\Delta \in (0, 1-1/e)$ yields

$$\liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \;\geq\; \lambda_E(\Delta)\left(1-\Delta\cdot\frac{1}{\Delta}\log\left(\frac{1}{1-\Delta}\right)\right) = \lambda_E(\Delta)\left(1-\Delta\lambda_E(\Delta)\right),$$

as claimed. □

Remark 4. In Theorems 5 and 6 the Explicit scheme does not recover the exact asymptotic convergence rate when $\Delta \in (0,1)$. However, the error between the rate predicted by the scheme and the true rate of unity can be approximated to within $O(\Delta)$ as $\Delta \to 0^+$. To see this define

$$\lambda_1^*(\Delta) := \liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \quad \text{and} \quad \lambda_2^*(\Delta) := \limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n}.$$

Then

$$\lambda_E(\Delta)(1-\Delta\lambda_E(\Delta)) \leq \lambda_1^*(\Delta) \leq \lambda_2^*(\Delta) \leq \lambda_E(\Delta).$$

From this inequality we can infer that the error in these upper and lower exponents from the true exponent of unity is given by

$$|\lambda_i^*(\Delta) - 1| \leq \max\left(\lambda_E(\Delta)-1, 1-\lambda_E(\Delta)(1-\Delta\lambda_E(\Delta))\right), \quad i=1,2.$$

From the Taylor Series of $\log(1+x)$ about zero, the error in the exponent can be bounded by

$$|\lambda_E(\Delta)-1| = \lambda_E(\Delta)-1 \;=\; 1+\frac{\Delta}{2}+\frac{\Delta^2}{3}+O(\Delta^3)-1 = O(\Delta).$$

The error in the upper bound can be bounded by

$$|\lambda_E(\Delta)\left(1-\Delta\lambda_E(\Delta)\right)-1| = 1-\lambda_E(\Delta)\left(1-\Delta\lambda_E(\Delta)\right) = 1-1+\frac{\Delta}{2}+\frac{2\Delta^2}{3}+O(\Delta^3) = O(\Delta).$$

□

Remark 5. In the case when $\Delta = 0$, we can replace the hypothesis (3.9) on f by (3.24) and can conclude that

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = 1,$$

by emulating Corollary 2. □

3.4 Implicit Euler Scheme with Adaptive Step Size

We approximate $x(t_n)$ by x_n, where $x(t_n)$ is the solution x of (1.1) at time t_n. The sequences (x_n), (t_n) and $(h(x_n))$ are defined by (1.43), (1.44) and (3.17):

$$x_{n+1} = x_n - h(x_{n+1})f(x_{n+1}), \quad n \geq 0, \quad x_0 = \xi > 0,$$

where

$$t_{n+1} = \sum_{j=0}^{n} h(x_{j+1}), \quad n \geq 0, \quad t_0 = 0,$$

and

$$\lim_{x \to 0^+} \frac{h(x)f(x)}{x} = \Delta \in [0, \infty].$$

3.4.1 Preserving Positivity, Monotonicity and Convergence

The following results guarantee the existence, positivity and convergence of the solutions of (1.43).

Lemma 7. *Suppose f obeys* (3.1) *and h obeys* (3.2)*. If $x > 0$, the equation*

$$y + h(y)f(y) = x, \tag{3.30}$$

has at least one solution in $(0, x)$ and no solutions in $[x, \infty)$. If $y(x)$ is a solution of (3.30), then $y(x) \to 0$ as $x \to 0^+$. With $x = 0$, (3.30) has a unique solution $y = 0$.

Proof. Define for each $x > 0$ and all $y \geq 0$

$$K(y) := y + h(y)f(y) - x.$$

Then $K(0) = -x < 0$ and $K(x) = h(x)f(x) > 0$. Since $K : [0, \infty) \to \mathbb{R}$ is continuous, $K(y) = 0$ has at least one solution in $(0, x)$. For $y > x$

$$K(y) = y + h(y)f(y) - x > y - x > 0,$$

and $K(x) > 0$. Thus $K(y) > 0$ for all $y \geq x$, and (3.30) has no solutions in $[x, \infty)$. Since any solution $y(x)$ of (3.30) obeys $0 < y(x) < x$, it follows that $y(x) \to 0$ as $x \to 0^+$ by The Squeeze Theorem. If $x = 0$, $K(y) > 0$ for all $y > 0$, and $K(0) = 0$. □

Proposition 4. *Suppose f obeys* (3.1) *and h obeys* (3.2)*. There exists at least one positive sequence (x_n) which obeys* (1.43) *and any such sequence is decreasing and obeys $x_n \to 0$ as $n \to \infty$.*

Proof. The existence and positivity of the sequence is implied by the root of (3.30) in Lemma 7. Since the solution $y(x) \in (0, x)$ then $x_n > 0$ for all $n \geq 0$ implies

$0 < x_{n+1} < x_n$. Since (x_n) is decreasing, we have $x_n \to L \in [0, \infty)$. Therefore if $L > 0$ then

$$L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \{x_n - h(x_{n+1})f(x_{n+1})\} = L - h(L)f(L),$$

by (3.1) and (3.2) and $h(L)f(L) = 0$ which is impossible by (3.1) and (3.2). Hence $x_n \to 0$ as $n \to \infty$. □

Remark 6. The ODE (1.1) has a unique solution for a given initial value. It would be desirable that the sequence used to model the continuous solution is also unique to reflect this important property. However, Proposition 4 does not imply the sequence is unique. A sufficient condition to ensure this is that $h(x)f(x)$ is increasing. □

A necessary condition to maintain positivity is that $h(x)f(x) \to 0$ as $x \to 0^+$. Otherwise, the Implicit Euler scheme becomes negative after a finite number of steps, as the Explicit Euler scheme does. We make this standing assumption throughout our analysis. The following result shows the necessity of a condition on h and f of this type.

Theorem 7. *Suppose f obeys* (3.1) *and h obeys* (3.2)*. Suppose also that*

$$\lim_{x\to 0^+} h(x)f(x) = \Lambda \in [0, \infty]. \tag{3.31}$$

(i) If $\Lambda = 0$, there is a monotone positive solution of (1.43) such that $x_n \to 0$ as $n \to \infty$.

(ii) If $\Lambda \in (0, \infty]$, there is $N > 0$ such that $x_N \leq 0$.

(iii) Analogously, if (3.31) *holds and $x_n > 0\ \forall n \geq 0$, then $\Lambda = 0$.*

Proof. We prove part (i) first where $\Lambda = 0$. Define for each $x > 0$

$$K(y) := y + h(y)f(y) - x.$$

Then K is continuous on $[0, x]$. Also by (3.31)

$$\lim_{y\to 0^+} K(y) = -x < 0 \quad \text{and} \quad K(x) = h(x)f(x) > 0.$$

Thus, by the Intermediate Value Theorem, there is a solution in $(0, x)$. However, if $y > x$,

$$K(y) = (y - x) + h(y)f(y) > 0,$$

so $K(y) = 0, x > 0$ implies $y \in (0, x)$. Hence $x_n > 0$ implies $0 < x_{n+1} < x_n$. Since x_n is decreasing, we have $x_n \to L \in [0, \infty)$ as $n \to \infty$. Therefore if $L > 0$ by (3.1) and (3.2) $L = L - h(L)g(L)$ so $h(L)f(L) = 0$ which is impossible by (3.1) and (3.2). Hence

$x_n \to 0$ as $n \to \infty$ as claimed. We now prove part (ii). We suppose $x_n > 0 \; \forall n \geq 0$. Then

$$0 < x_{n+1} = x_n - h(x_{n+1})f(x_{n+1}) < x_n.$$

Hence $x_n \to L \in [0, \infty)$ as $n \to \infty$. Once again, this implies $L = 0$. Then by (3.31),

$$0 = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \{x_n - h(x_{n+1})f(x_{n+1})\} = -\Lambda < 0,$$

a contradiction. Hence there must exist $N > 0$ so that $x_N \leq 0$ as claimed.

For part (iii), if $x_n > 0$, we may argue as in part (ii), to get $\Lambda = 0$ as claimed. □

Hence, we take as a standing assumption

$$\lim_{x\to 0^+} h(x)f(x) = 0. \tag{3.32}$$

3.4.2 Preserving Soft Landings and Super-Exponential Stability

The Implicit Euler scheme, defined by equation (1.43), correctly recovers the presence, or absence, of a finite hitting time according to whether the solution of the underlying ODE has that property or not. Since $(h(x_{n+1}))$ is a positive sequence, the limit defined by

$$\lim_{n\to\infty} t_n =: \hat{T}_h = \sum_{j=0}^{\infty} h(x_{j+1}), \tag{3.33}$$

exists but can be finite or infinite.

We first determine the form of (3.15) when an Implicit Euler scheme is used to discretise (1.1). This equation will be used in our later proofs of preserving soft landings, super-exponential stability and asymptotic convergence rates.

Lemma 8. *Suppose (x_n) is a positive decreasing sequence and the solution of* (1.43)*. If f obeys* (3.1), (3.10) *and* (3.32) *while h obeys* (3.2) *and* (3.17) *with $\Delta \in [0, \infty)$ then* (3.15) *holds viz.,*

$$(1-\epsilon)\cdot\psi(x_{n+1})\log\left(\frac{x_n}{x_{n+1}}\right) \leq \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\,du \leq (1+\epsilon)\cdot\psi(x_n)\log\left(\frac{x_n}{x_{n+1}}\right).$$

Furthermore,

(i) if $\Delta = 0$ and f obeys (3.9)*, then for all $\epsilon \in (0,1)$ and all n sufficiently large*

$$\frac{(1-\epsilon)^2}{1+\epsilon}\cdot h(x_{n+1}) \leq \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\,du \leq \frac{(1+\epsilon)^4}{(1-\epsilon)^2}\cdot h(x_{n+1}). \tag{3.34}$$

(ii) if $\Delta \in (0,\infty)$, then for all $\epsilon \in (0,1)$ and all n sufficiently large

$$\frac{(1-\epsilon)^2}{(1+\epsilon)^2} \cdot \frac{\log(1+\Delta)}{\Delta} h(x_{n+1}) \le \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\,du \le \frac{(1+\epsilon)^2}{(1-\epsilon)^2} \cdot \frac{\log(1+\Delta)}{\Delta} h(x_n). \tag{3.35}$$

Proof. For part (i), since $x_n \to 0$ as $n \to \infty$ and (x_n) is decreasing for every $\epsilon \in (0,1)$ there is $x_1(\epsilon) > 0$ and $n \ge N_1(\epsilon) \in \mathbb{N}$ such that $x_n < x_1(\epsilon)$ for all $n \ge N_1(\epsilon)$, so the estimate (3.20) still pertains:

$$\frac{1-\epsilon}{1+\epsilon} \cdot \frac{x_{n+1}}{f(x_{n+1})} \cdot \log\left(\frac{x_n}{x_{n+1}}\right) \le \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\,du \le \frac{1+\epsilon}{1-\epsilon} \cdot \frac{x_n}{f(x_n)} \cdot \log\left(\frac{x_n}{x_{n+1}}\right).$$

Define

$$a_n := \log\left(\frac{x_n}{x_{n+1}}\right) = \log\left(\frac{x_{n+1} + h(x_{n+1})f(x_{n+1})}{x_{n+1}}\right) = \log\left(1 + \frac{h(x_{n+1})f(x_{n+1})}{x_{n+1}}\right).$$

Since (3.17) holds and $\Delta = 0$ then

$$a_n = \log\left(1 + \frac{h(x_{n+1})f(x_{n+1})}{x_{n+1}}\right) \sim \frac{h(x_{n+1})f(x_{n+1})}{x_{n+1}}, \quad \text{as } n \to \infty, \tag{3.36}$$

since $\log(1+x) \sim x$ as $x \to 0^+$. Thus for every $\epsilon \in (0,1)$ there is an $n \ge N_2(\epsilon)$ such that

$$(1-\epsilon) \cdot \frac{h(x_{n+1})f(x_{n+1})}{x_{n+1}} < a_n < (1+\epsilon) \cdot \frac{h(x_{n+1})f(x_{n+1})}{x_{n+1}}.$$

Let $N_3(\epsilon) := \max(N_1(\epsilon), N_2(\epsilon))$. Then for $n \ge N_3(\epsilon)$

$$\frac{(1-\epsilon)^2}{1+\epsilon} \cdot \frac{x_{n+1}}{f(x_{n+1})} \cdot \frac{h(x_{n+1})f(x_{n+1})}{x_{n+1}} \le \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\,du \le \frac{(1+\epsilon)^2}{1-\epsilon} \cdot \frac{x_n}{f(x_n)} \cdot \frac{h(x_{n+1})f(x_{n+1})}{x_{n+1}}.$$

Thus for $n \ge N_3(\epsilon)$

$$\frac{(1-\epsilon)^2}{1+\epsilon} \cdot h(x_{n+1}) \le \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\,du \le \frac{(1+\epsilon)^2}{1-\epsilon} \cdot \frac{x_n}{x_{n+1}} \cdot \frac{f(x_{n+1})}{f(x_n)} \cdot h(x_{n+1}). \tag{3.37}$$

Arguing as in (3.22), (3.13) implies

$$\frac{f(x_n)}{f(x_{n+1})} > \frac{1-\epsilon}{1+\epsilon} \quad \text{or} \quad \frac{f(x_{n+1})}{f(x_n)} < \frac{1+\epsilon}{1-\epsilon},$$

for all $n \ge N_3'(\epsilon)$. Substituting this into (3.37) yields $n \ge \max(N_3'(\epsilon), N_3(\epsilon))$

$$\frac{(1-\epsilon)^2}{1+\epsilon} \cdot h(x_{n+1}) \le \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\,du \le \frac{(1+\epsilon)^3}{(1-\epsilon)^2} \cdot \frac{x_n}{x_{n+1}} \cdot h(x_{n+1}).$$

Since

$$\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \left(1 + \frac{h(x_{n+1})f(x_{n+1})}{x_{n+1}}\right) = 1,$$

for every $\epsilon \in (0,1)$ there is $N_4(\epsilon)$ such that for $n \geq N_4(\epsilon)$

$$1 - \epsilon < \frac{x_n}{x_{n+1}} < 1 + \epsilon.$$

Let $N_5(\epsilon) := \max(N_3'(\epsilon), N_3(\epsilon), N_4(\epsilon))$. Then for $n \geq N_5(\epsilon)$

$$\frac{(1-\epsilon)^2}{1+\epsilon} \cdot h(x_{n+1}) \leq \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\, du \leq \frac{(1+\epsilon)^4}{(1-\epsilon)^2} \cdot h(x_{n+1}),$$

which is the inequality (3.34). We now prove part (ii). Let $\Delta \in (0,\infty)$. The estimate $x_n < x_1(\epsilon)$ for all $n > N_1(\epsilon)$ and (3.20) still pertains:

$$\frac{1-\epsilon}{1+\epsilon} \cdot \frac{x_{n+1}}{f(x_{n+1})} \cdot \log\left(\frac{x_n}{x_{n+1}}\right) \leq \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\, du \leq \frac{1+\epsilon}{1-\epsilon} \cdot \frac{x_n}{f(x_n)} \cdot \log\left(\frac{x_n}{x_{n+1}}\right).$$

Define

$$a_n := \log\left(\frac{x_n}{x_{n+1}}\right) = \log\left(\frac{x_{n+1} + h(x_{n+1})f(x_{n+1})}{x_{n+1}}\right) = \log\left(1 + \frac{h(x_{n+1})f(x_{n+1})}{x_{n+1}}\right).$$

Since (3.17) holds and $\Delta \in (0,\infty)$ then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \log\left(1 + \frac{h(x_{n+1})f(x_{n+1})}{x_{n+1}}\right) = \log(1+\Delta).$$

Thus there is an $N_2(\epsilon) \in \mathbb{N}$ such that for all $n \geq N_2(\epsilon)$

$$(1-\epsilon) \cdot \log(1+\Delta) < a_n < (1+\epsilon) \cdot \log(1+\Delta).$$

Let $n \geq N_3(\epsilon) := \max(N_1(\epsilon), N_2(\epsilon))$. Then for $n \geq N_3(\epsilon)$

$$\frac{(1-\epsilon)^2}{1+\epsilon} \cdot \frac{\log(1+\Delta)}{\Delta} \cdot \frac{\Delta x_{n+1}}{f(x_{n+1})} \leq \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\, du \leq \frac{(1+\epsilon)^2}{1-\epsilon} \cdot \frac{\log(1+\Delta)}{\Delta} \cdot \frac{\Delta x_n}{f(x_n)}.$$

Since $h(x_n) \sim \Delta x_n / f(x_n)$ as $n \to \infty$, for every $\epsilon \in (0,1)$ there is an $N_4(\epsilon) \in \mathbb{N}$ such that $n \geq N_4(\epsilon)$ implies

$$(1-\epsilon) \cdot \frac{\Delta x_n}{f(x_n)} < h(x_n) < (1+\epsilon) \cdot \frac{\Delta x_n}{f(x_n)}.$$

Let $N_5(\epsilon) := \max(N_4(\epsilon), N_3(\epsilon))$ and $n \geq N_5(\epsilon)$. Then

$$\frac{(1-\epsilon)^2}{(1+\epsilon)^2} \cdot \frac{\log(1+\Delta)}{\Delta} \cdot h(x_{n+1}) \leq \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\, du \leq \frac{(1+\epsilon)^2}{(1-\epsilon)^2} \cdot \frac{\log(1+\Delta)}{\Delta} \cdot h(x_n),$$

which is the inequality (3.35). □

In our next result, we show that $\hat{T}_h$ is finite or infinite according to whether T_ξ defined by (1.8) is finite or infinite.

Theorem 8. *Suppose f obeys* (3.1), (3.9), (3.10) *and* (3.32) *while h obeys* (3.2) *and* (3.17) *with $\Delta \in [0, \infty)$. Let (t_n) and $\hat{T}_h$ be defined* (1.44) *and* (3.33).

(i) If f obeys (1.7), *then $\hat{T}_h < \infty$.*

(ii) If f obeys (1.9), *then $\hat{T}_h = \infty$.*

Proof. By (1.7), $\int_{0+}^1 1/f(u)\,du < \infty$ then $T_\xi < \infty$ from (3.12) since

$$\sum_{j=0}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du = \int_0^\xi \frac{1}{f(u)}\,du = T_\xi < \infty.$$

When $\Delta = 0$, the Comparison Test applied to equation (3.34) shows the summability of $(\int_{x_{n+1}}^{x_n} 1/f(u)\,du)$ implies that of $(h(x_n))$. When $\Delta \in (0, \infty)$, the Comparison Test applied to (3.35) shows the summability of $(\int_{x_{n+1}}^{x_n} 1/f(u)\,du)$ implies that of $(\log(1+\Delta)\,h(x_n)/\Delta)$ and hence the summability of $(h(x_n))$ when $\Delta \in (0, \infty)$. Hence in both cases $(h(x_n))$ is summable and we have that $t_n = \sum_{j=0}^{n-1} h(x_j)$ for $n \geq 1$ obeys $t_n \to \hat{T}_h := \sum_{j=0}^{\infty} h(x_j) < \infty$ as $n \to \infty$.

By (1.9), $\int_{0+}^1 1/f(u)\,du = \infty$ then $T_\xi = \infty$ from (3.11) since

$$\sum_{j=0}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du = \int_0^\xi \frac{1}{f(u)}\,du = T_\xi = \infty.$$

The Comparison Test applied to (3.34) and (3.35) shows that $(h(x_n))$ is not summable and obeys $t_n = \sum_{j=0}^{n-1} h(x_j) \to \infty$ as $n \to \infty$. Therefore $t_n \to \infty$ as $n \to \infty$. □

The next result shows that, once h obeys (3.17), the Implicit Euler scheme (1.43) recovers the exact rate of asymptotic convergence when $\Delta = 0$ but not when $\Delta \in (0, \infty)$ despite preserving finite-time and super-exponential stability. We tackle the case of super-exponential convergence first.

Theorem 9. *Suppose f obeys* (1.9), (3.1), (3.10) *and* (3.32) *while h obeys* (3.2) *and* (3.17) *with $\Delta \in [0, \infty)$. Let F and (t_n) be defined by* (1.11) *and* (1.44).

(i) If $\Delta = 0$ and f obeys (3.9), *then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \infty$ as $n \to \infty$ and*

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = 1.$$

(ii) If $\Delta \in (0,\infty)$, then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = \frac{\log(1+\Delta)}{\Delta} =: \lambda_I(\Delta).$$

Proof. The positivity, monotonicity and convergence of (x_n) have been addressed in Lemma 7 and Proposition 4. Since f obeys (1.9) then $\int_{0^+}^1 1/f(u)\,du = \infty$ and $t_n \to \infty$ as $n \to \infty$ by Theorem 8. We prove part (i) first. Letting $n \geq N_5(\epsilon)+1$ in (3.34) yields

$$\frac{(1-\epsilon)^2}{1+\epsilon} \sum_{j=N_5(\epsilon)}^{n-1} h(x_{n+1}) \leq \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\,du \leq \frac{(1+\epsilon)^4}{(1-\epsilon)^2} \sum_{j=N_5(\epsilon)}^{n-1} h(x_{n+1}).$$

Thus for $n \geq N_5(\epsilon)+1$

$$\frac{(1-\epsilon)^2}{1+\epsilon} \cdot (t_n - t_{N_5(\epsilon)}) \leq F(x_n) - F(x_{N_5(\epsilon)}) \leq \frac{(1+\epsilon)^4}{(1-\epsilon)^2} \cdot (t_n - t_{N_5(\epsilon)}).$$

Therefore, as $t_n \to \infty$ as $n \to \infty$ dividing by t_n and letting $n \to \infty$ yields

$$\frac{(1-\epsilon)^2}{1+\epsilon} \leq \liminf_{n\to\infty} \frac{F(x_n)}{t_n} \leq \limsup_{n\to\infty} \frac{F(x_n)}{t_n} \leq \frac{(1+\epsilon)^4}{(1-\epsilon)^2}.$$

Letting $\epsilon \to 0^+$ yields

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = 1,$$

the desired limit in part (i) as claimed. We now prove part (ii). Letting $n \geq N_5(\epsilon)+1$ in (3.35) yields

$$\frac{(1-\epsilon)^2}{(1+\epsilon)^2} \cdot \lambda_I(\Delta) \sum_{j=N_5(\epsilon)}^{n-1} h(x_{j+1}) \leq \sum_{j=N_5(\epsilon)}^{n-1} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du \leq \frac{(1+\epsilon)^2}{(1-\epsilon)^2} \cdot \lambda_I(\Delta) \sum_{j=N_5(\epsilon)}^{n-1} h(x_j).$$

Therefore

$$\frac{(1-\epsilon)^2}{(1+\epsilon)^2} \cdot \lambda_I(\Delta) \sum_{j=N_5(\epsilon)+1}^{n} h(x_j) \leq \sum_{j=N_5(\epsilon)}^{n-1} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du \leq \frac{(1+\epsilon)^2}{(1-\epsilon)^2} \cdot \lambda_I(\Delta) \sum_{j=N_5(\epsilon)}^{n-1} h(x_j),$$

or

$$\frac{(1-\epsilon)^2}{(1+\epsilon)^2} \cdot \lambda_I(\Delta) \cdot (t_n - t_{N_5(\epsilon)}) \leq F(x_n) - F(x_{N_5(\epsilon)}) \leq \frac{(1+\epsilon)^2}{(1-\epsilon)^2} \cdot \lambda_I(\Delta) \cdot (t_{n-1} - t_{N_5(\epsilon)}). \tag{3.38}$$

Since $h(x_n) \sim \Delta x_n / f(x_n)$ as $n \to \infty$ then $h(x_n) \to 0$ as $n \to \infty$ by (1.6). By (1.44), $t_n = t_{n-1} + h(x_n)$ so $t_{n-1}/t_n \to 1$. Therefore, as $t_n \to \infty$ as $n \to \infty$ dividing (3.38) by

t_n and letting $n \to \infty$ yields

$$\frac{(1-\epsilon)^2}{(1+\epsilon)^2} \cdot \lambda_I(\Delta) \le \liminf_{n\to\infty} \frac{F(x_n)}{t_n} \le \limsup_{n\to\infty} \frac{F(x_n)}{t_n} \le \frac{(1+\epsilon)^2}{(1-\epsilon)^2} \cdot \lambda_I(\Delta).$$

Letting $\epsilon \to 0^+$ yields

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = \lambda_I(\Delta),$$

the desired limit in part (ii), as claimed. □

The next result shows the Implicit Euler scheme correctly predicts the precise asymptotic behaviour by imposing a monotonicity condition on $h(x)f(x)/x$ instead of f.

Corollary 3. *Suppose f obeys* (1.9), (3.1), (3.10) *and* (3.32) *while h obeys* (3.2) *and* (3.17) *with $\Delta = 0$. Let F and (t_n) be defined by* (1.11) *and* (1.44). *If*

$$x \mapsto \frac{h(x)f(x)}{x} \quad \textit{is asymptotically increasing,} \tag{3.39}$$

holds instead of (3.9) then

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = 1.$$

Proof. The positivity, monotonicity and convergence of (x_n) still prevail. Since f obeys (1.9) then $\int_{0^+}^1 1/f(u)\,du = \infty$ and $t_n \to \infty$ as $n \to \infty$ by Theorem 8. By the first inequality of (3.37), for $n \ge N_3(\epsilon) + 1$

$$F(x_n) - F(x_{N_3(\epsilon)}) = \sum_{j=N_3(\epsilon)}^{n-1} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du \ge \frac{(1+\epsilon)^2}{1-\epsilon} \sum_{j=N_3(\epsilon)}^{n-1} h(x_j).$$

Thus

$$F(x_n) \ge F(x_{N_3(\epsilon)}) + \frac{(1+\epsilon)^2}{1-\epsilon} \cdot (t_n - t_{N_3(\epsilon)}).$$

Therefore dividing by t_n, letting $n \to \infty$ and then $\epsilon \to 0^+$ yields

$$\liminf_{n\to\infty} \frac{F(x_n)}{t_n} \ge 1 \tag{3.40}$$

Since $\Delta = 0$, $x_{n+1}/x_n \to 1$ as $n \to \infty$ and so there is $N_4(\epsilon) \in \mathbb{N}$ such that for all $n \ge N_4(\epsilon)$ we have

$$1 - \epsilon < \frac{x_n}{x_{n+1}} < 1 + \epsilon.$$

The second inequality of (3.37) reads for $n \ge N_5(\epsilon) := \max(N_3(\epsilon), N_4(\epsilon))$

$$\int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\,du \le \frac{(1+\epsilon)^2}{1-\epsilon} \cdot \frac{x_n}{x_{n+1}} \cdot \frac{f(x_{n+1})}{f(x_n)} \cdot h(x_{n+1}). \tag{3.41}$$

Calling θ the increasing function asymptotic to $h(x)f(x)/x$ we have for every $\epsilon \in (0,1)$ there is $x_2(\epsilon) > 0$ such that for $x < x_2(\epsilon)$

$$(1-\epsilon)\cdot\theta(x) < \frac{h(x)f(x)}{x} < (1+\epsilon)\cdot\theta(x).$$

Let $N_6(\epsilon)$ be so large that $n \ge N_6(\epsilon)$ implies $x_n < x_2(\epsilon)$. Then, as (x_n) is decreasing, for $n \ge N_6(\epsilon)$

$$\frac{h(x_{n+1})f(x_{n+1})}{x_{n+1}} < (1+\epsilon)\cdot\theta(x_{n+1}) \quad < \quad (1+\epsilon)\cdot\theta(x_n) < \frac{1+\epsilon}{1-\epsilon}\cdot\frac{h(x_n)f(x_n)}{x_n}.$$

Therefore for $n \ge N_6(\epsilon)$

$$\frac{h(x_{n+1})f(x_{n+1})}{f(x_n)} < \frac{1+\epsilon}{1-\epsilon}\cdot\frac{x_{n+1}}{x_n}\cdot h(x_n).$$

Let $N_7(\epsilon) := \max(N_5(\epsilon), N_6(\epsilon))$. Then for $n \ge N_7(\epsilon)$ then (3.41) implies

$$\int_{x_{n+1}}^{x_n}\frac{1}{f(u)}\,du \le \frac{(1+\epsilon)^3}{(1-\epsilon)^2}\cdot\frac{x_{n+1}}{x_n}\cdot h(x_n) \le \frac{(1+\epsilon)^3}{(1-\epsilon)^3}\cdot h(x_n).$$

Hence for $n \ge N_7(\epsilon)+1$

$$F(x_n) - F(x_{N_7(\epsilon)}) = \sum_{j=N_7(\epsilon)}^{n-1}\int_{x_{j+1}}^{x_j}\frac{1}{f(u)}\,du \le \frac{(1+\epsilon)^3}{(1-\epsilon)^3}\sum_{j=N_7(\epsilon)}^{n-1} h(x_j).$$

Thus for $n \ge N_7(\epsilon)+1$

$$F(x_n) \le F(x_{N_7(\epsilon)}) + \frac{(1+\epsilon)^3}{(1-\epsilon)^3}\cdot(t_{n-1} - t_{N_7(\epsilon)-1}). \tag{3.42}$$

Since $h(x) = o(x/f(x))$ as $x \to 0^+$ and $x/f(x)$ is asymptotic to an increasing function then $h(x) \to 0$ as $x \to 0^+$. Thus $t_{n-1}/t_n \to 1$ as $n \to \infty$. Hence

$$\limsup_{n\to\infty}\frac{F(x_n)}{t_n} \le 1.$$

Combining with (3.40) yields

$$\lim_{n\to\infty}\frac{F(x_n)}{t_n} = 1,$$

as claimed. □

Remark 7. By assuming (1.6), in Theorem 9 part (ii) and Corollary 3 the solution of (1.1) convergences super-exponentially. However, we are able to recover the correct asymptotic behaviour when there is exponential convergence if (1.6) is replaced by $f(x)/x$ tending to a positive finite limit. The proofs are the same up to (3.38) and

(3.42). However, in the case of exponential convergence $h(x_n) \sim \Delta x_n/f(x_n) \sim \Delta\psi(x_n)$ tends to a finite limit as $n \to \infty$ since (x_n) is decreasing and ψ is increasing. Therefore, as $t_n \to \infty$ as $n \to \infty$, then $t_{n-1}/t_n \to 1$ as $n \to \infty$. Hence, dividing (3.38) by t_n and letting $n \to \infty$ yields the desired results. □

We now tackle the case of finite-time stability.

Theorem 10. *Suppose f obeys* (1.7), (3.1), (3.10) *and* (3.32) *while h obeys* (3.2) *and* (3.17) *with $\Delta \in [0,\infty)$. Let $\bar{F}$, (t_n) and $\hat{T}_h$ be defined by* (1.10), (1.44) *and* (3.33).

(i) If $\Delta = 0$ and f obeys (3.9), *then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and*

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = 1.$$

(ii) If $\Delta = (0,\infty)$, then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and

$$\lambda_I(\Delta) \leq \liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \quad \textit{and} \quad \limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_{n-1}} \leq \frac{\log(1+\Delta)}{\Delta} =: \lambda_I(\Delta).$$

If in addition, f obeys (3.9) and $\Delta < e - 1$, then

$$\lambda_I(\Delta) \leq \liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq \limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq \frac{\lambda_I(\Delta)}{1 - \Delta\lambda_I(\Delta)}.$$

Proof. The positivity, monotonicity and convergence of (x_n) have been addressed in Lemma 7 and Proposition 4. Since f obeys (1.7) then $\int_{0+}^1 1/f(u)\,du < \infty$ and $t_n \to \hat{T}_h := \sum_{j=0}^{\infty} h(x_{j+1}) < \infty$ by Theorem 8. Hence $\hat{T}_h - t_n = \sum_{j=n}^{\infty} h(x_{j+1}) \to 0$ as $n \to \infty$. We prove part (i) first. Letting $n \geq N_5(\epsilon)$ in (3.34) yields

$$\frac{(1-\epsilon)^2}{1+\epsilon} \sum_{j=n}^{\infty} h(x_{j+1}) \leq \sum_{j=n}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du \leq \frac{(1+\epsilon)^4}{(1-\epsilon)^2} \sum_{j=n}^{\infty} h(x_{j+1}).$$

Thus for $n \geq N_5(\epsilon)$

$$\frac{(1-\epsilon)^2}{1+\epsilon} \cdot (\hat{T}_h - t_n) \leq \bar{F}(x_n) \leq \frac{(1+\epsilon)^4}{(1-\epsilon)^2} \cdot (\hat{T}_h - t_n).$$

Therefore, as $\hat{T}_h - t_n \to 0$ as $n \to \infty$ dividing by $\hat{T}_h - t_n$ and letting $n \to \infty$ implies

$$\frac{(1-\epsilon)^2}{1+\epsilon} \leq \liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq \limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq \frac{(1+\epsilon)^4}{(1-\epsilon)^2}.$$

Letting $\epsilon \to 0^+$ yields

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = 1,$$

as claimed in part (i). We now prove part (ii). Letting $n \geq N_5(\epsilon)$ in (3.35) yields

$$\frac{(1-\epsilon)^2}{(1+\epsilon)^2} \cdot \lambda_I(\Delta) \sum_{j=n}^{\infty} h(x_{j+1}) \leq \sum_{j=n}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\, du \leq \frac{(1+\epsilon)^2}{(1-\epsilon)^2} \cdot \lambda_I(\Delta) \sum_{j=n}^{\infty} h(x_j).$$

Thus for $n \geq N_5(\epsilon)$

$$\frac{(1-\epsilon)^2}{(1+\epsilon)^2} \cdot \lambda_I(\Delta) \cdot (\hat{T}_h - t_n) \leq \bar{F}(x_n) \leq \frac{(1+\epsilon)^2}{(1-\epsilon)^2} \cdot \lambda_I(\Delta) \cdot (\hat{T}_h - t_{n-1}).$$

Therefore

$$\lambda_I(\Delta) \leq \liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \quad \text{and} \quad \limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_{n-1}} \leq \lambda_I(\Delta),$$

which proves the first estimate in part (ii), as claimed. For the second estimate in (ii), define $\delta(x) := h(x)f(x)/x$ where $\delta(x) \to \Delta$ as $x \to 0^+$. Next

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\bar{F}(x_{n-1})}{\hat{T}_h - t_{n-1}} &= \limsup_{n\to\infty} \left(\frac{\bar{F}(x_{n-1})}{\bar{F}(x_n)} \cdot \frac{\bar{F}(x_n)}{\hat{T}_h - t_{n-1}} \right) \\
&\leq \lambda_I(\Delta) \cdot \limsup_{n\to\infty} \frac{\bar{F}(x_{n-1})}{\bar{F}(x_n)} \\
&= \lambda_I(\Delta) \cdot \limsup_{n\to\infty} \frac{\bar{F}(x_n(1+\delta(x_n)))}{\bar{F}(x_n)} \\
&\leq \lambda_I(\Delta) \cdot \limsup_{x\to 0^+} \frac{\bar{F}(x(1+\delta(x)))}{\bar{F}(x)}. \qquad (3.43)
\end{aligned}$$

Suppose $x(1+\delta(x)) < x_1(\epsilon)$. Then for $u < x(1+\delta(x)) < x_1(\epsilon)$, by (3.14), $u/f(u) < (1+\epsilon) \cdot \psi(u) < (1+\epsilon) \cdot \psi(x(1+\delta(x)))$. Hence

$$\begin{aligned}
0 < \bar{F}(x(1+\delta(x))) - \bar{F}(x) &\leq \int_x^{x(1+\delta(x))} \frac{1}{u} \cdot \frac{u}{f(u)}\, du \\
&\leq (1+\epsilon) \cdot \psi(x(1+\delta(x))) \int_x^{x(1+\delta(x))} \frac{1}{u}\, du \\
&= (1+\epsilon) \cdot \psi(x(1+\delta(x))) \cdot \log(1+\delta(x)).
\end{aligned}$$

For $x(1+\delta(x)) < x_1(\epsilon)$, by (3.13), $f(u) < (1+\epsilon)\cdot\phi(u) < (1+\epsilon)\cdot\phi(x(1+\delta(x)))$. Thus

$$\begin{aligned}\bar{F}(x(1+\delta(x))) = \int_0^{x(1+\delta(x))} \frac{1}{f(u)}\,du \geq \int_0^{x(1+\delta(x))} \frac{1}{(1+\epsilon)\phi(u)}\,du \quad &\geq \quad \frac{1}{1+\epsilon}\cdot\frac{x(1+\delta(x))}{\phi(x(1+\delta(x)))} \\ &> \quad \frac{1-\epsilon}{1+\epsilon}\cdot\frac{x(1+\delta(x))}{f(x(1+\delta(x)))} \\ &> \quad \frac{(1-\epsilon)^2}{1+\epsilon}\cdot\psi(x(1+\delta(x))).\end{aligned}$$

Hence for $x(1+\delta(x)) < x_1(\epsilon)$

$$0 < \bar{F}(x(1+\delta(x))) - \bar{F}(x) < \frac{(1+\epsilon)^2}{(1-\epsilon)^2}\cdot\log(1+\delta(x))\cdot\bar{F}(x(1+\delta(x))).$$

Rearranging yields

$$0 < \bar{F}(x(1+\delta(x)))\left\{1 - \frac{(1+\epsilon)^2}{(1-\epsilon)^2}\cdot\log(1+\delta(x))\right\} < \bar{F}(x),$$

provided

$$1 - \frac{(1+\epsilon)^2}{(1-\epsilon)^2}\cdot\log(1+\delta(x)) > 0.$$

Thus

$$\frac{\bar{F}(x(1+\delta(x)))}{\bar{F}(x)} < \frac{1}{1-\frac{(1+\epsilon)^2}{(1-\epsilon)^2}\cdot\log(1+\delta(x))}. \tag{3.44}$$

Suppose now that $\epsilon > 0$ is so small that

$$1 - \frac{(1+\epsilon)^2}{(1-\epsilon)^2}\cdot\log(1+\Delta) > 0.$$

This is true provided $1-\log(1+\Delta) > 0$, or $\Delta < e-1$. Then, as $\delta(x)\to\Delta$ as $x\to 0^+$, it follows for all ϵ sufficiently small that there is $x_2(\epsilon) > 0$ such that for all $\epsilon\in(0,1)$ such that for all $x < x_2(\epsilon)$ we have

$$1 - \frac{(1+\epsilon)^2}{(1-\epsilon)^2}\cdot\log(1+\delta(x)) > 0.$$

Therefore

$$\limsup_{x\to 0^+}\frac{\bar{F}(x(1+\delta(x)))}{\bar{F}(x)} \quad\leq\quad \limsup_{x\to 0^+}\frac{1}{1-\frac{(1+\epsilon)^2}{(1-\epsilon)^2}\cdot\log(1+\delta(x))} = \frac{1}{1-\frac{(1+\epsilon)^2}{(1-\epsilon)^2}\cdot\log(1+\Delta)}.$$

Letting $\epsilon\to 0^+$ yields

$$\limsup_{x\to 0^+}\frac{\bar{F}(x(1+\delta(x)))}{\bar{F}(x)} \leq \frac{1}{1-\log(1+\Delta)}.$$

Hence for $\Delta < e - 1$,

$$\limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq \frac{\lambda_I(\Delta)}{1 - \log(1+\Delta)} = \frac{\lambda_I(\Delta)}{1 - \Delta\log(1+\Delta)/\Delta} = \frac{\lambda_I(\Delta)}{1 - \Delta\lambda_I(\Delta)},$$

as required. □

Remark 8. In Theorems 9 and 10 the Implicit scheme does not recover the exact asymptotic convergence rate when $\Delta \in (0, \infty)$. However, the error between the rate predicted by the scheme and the true rate of unity can be approximated to within $O(\Delta)$. To see this define

$$\lambda_1^*(\Delta) := \liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \quad \text{and} \quad \lambda_2^*(\Delta) := \limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n}.$$

Then

$$\lambda_I(\Delta) \leq \lambda_1^*(\Delta) \leq \lambda_2^*(\Delta) \leq \frac{\lambda_I(\Delta)}{1 - \Delta\lambda_I(\Delta)}.$$

From this inequality we can infer that the error in these upper and lower exponents from the true exponent of unity is given by

$$|\lambda_i^*(\Delta) - 1| \leq \max\left(1 - \lambda_I(\Delta), \frac{\lambda_I(\Delta)}{1 - \Delta\lambda_I(\Delta)} - 1\right), \quad i = 1, 2.$$

From the Taylor Series of $\log(1 + x)$ about zero, the error in the exponent can be bounded by

$$|\lambda_I(\Delta) - 1| = 1 - \lambda_I(\Delta) = 1 - 1 + \frac{\Delta}{2} - \frac{\Delta^2}{3} + O(\Delta^3) = O(\Delta).$$

The error in the upper bound can be bounded by

$$\left|\frac{\lambda_I(\Delta)}{1 - \Delta\lambda_I(\Delta)} - 1\right| = \frac{\lambda_I(\Delta)}{1 - \Delta\lambda_I(\Delta)} - 1 = 1 + \frac{\Delta}{2} + \frac{\Delta^2}{3} + O(\Delta^3) - 1 = O(\Delta).$$

□

Remark 9. We have already shown that

$$\bar{E}_E(\Delta) := \max\left(\frac{\Delta}{2} + \frac{\Delta^2}{3} + O(\Delta^3), \frac{\Delta}{2} + \frac{2\Delta^2}{3} + O(\Delta^3)\right) = \frac{\Delta}{2} + \frac{2\Delta^2}{3} + O(\Delta^3),$$
$$\bar{E}_I(\Delta) := \max\left(\frac{\Delta}{2} - \frac{\Delta^2}{3} + O(\Delta^3), \frac{\Delta}{2} + \frac{\Delta^2}{3} + O(\Delta^3)\right) = \frac{\Delta}{2} + \frac{\Delta^2}{3} + O(\Delta^3).$$

Thus $\bar{E}_E(\Delta) > \bar{E}_I(\Delta)$ and therefore there is evidence that the Implicit scheme outperforms the Explicit to $O(\Delta^2)$ to $\Delta \to 0^+$. In Chapter 3 when f is a regularly varying function we are able to estimate these errors exactly and we can show that the Implicit scheme always outperforms the Explicit scheme within the class of regularly varying functions, as we will see in Chapter 5. □

Remark 10. In the case when $\Delta = 0$, we can replace the hypothesis (3.9) on f by (3.39) and can conclude that

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = 1,$$

by emulating Corollary 3. □

3.5 Approximation of Finite Stability Time

Under the condition (1.4) on f the Initial Value Problem (1.1) viz.,

$$x'(t) = -f(x(t)), \quad t > 0, \quad x(0) = \xi > 0,$$

has a unique and continuous solution on a maximal interval of existence $I_\xi = [0, T_\xi)$, cf. Theorem 1. Furthermore, the solution is positive and decreasing on this interval. In the case of finite-time stability, T_ξ is finite. The formula for T_ξ is given by (1.8)

$$T_\xi = \int_0^\xi \frac{1}{f(u)}\, du,$$

when f obeys (1.7). We cannot compute $\hat{T}(\Delta)$, defined by (3.16), exactly as it involves computing an infinite sum, namely

$$\lim_{n\to\infty} t_n =: \hat{T}(\Delta) = \sum_{j=0}^{\infty} h(x_j) < \infty.$$

However, the finite sum

$$\hat{T}_n(\Delta) = \sum_{j=0}^{n-1} h(x_j), \quad n \geq 1,$$

can be computed. We obtain estimates for the error in approximating the finite stability time, $T_\xi - \hat{T}(\Delta)$, the truncated error, $T_\xi - \hat{T}_n(\Delta)$, and show that we can approximate T_ξ to within $O(\Delta)$ as $\Delta \to 0^+$. Suppose the following conditions on f and h hold:

$$f \text{ is increasing,} \tag{3.45}$$

$$\text{there is } \Delta > 0 \text{ such that for all } x > 0,\ h(x) = \frac{\Delta x}{f(x)}. \tag{3.46}$$

We approximate $x(t_n)$ by x_n, where $x(t_n)$ is the solution x of (1.1) at time t_n. The sequences (x_n), (t_n) and $(h(x_n))$ are defined by (1.41), (1.42) and (3.46):

$$x_{n+1} = x_n - h(x_n)f(x_n), \quad n \geq 0, \quad x_0 = \xi > 0,$$

where

$$t_{n+1} = \sum_{j=0}^{n} h(x_j), \quad n \geq 0, \quad t_0 = 0.$$

Theorem 11. *Suppose* (x_n) *is the solution of* (1.41). *Suppose also* f *obeys* (1.7), (3.1), (3.7) *and* (3.45) *while* h *obeys* (3.2) *and* (3.46) *with* $\Delta \in [0, 1)$. *Let* $\hat{T}_h$ *be defined by* (3.16). *Then*

$$0 < T_\xi - \hat{T}(\Delta) < \Delta \left(T_\xi - \frac{\xi}{f(\xi)} \right), \tag{3.47}$$

$$\int_0^{\xi(1-\Delta)^n} \frac{1}{f(u)}\, du < T_\xi - \hat{T}_n(\Delta) < \Delta \left(T_\xi - \frac{\xi}{f(\xi)} \right) + (1-\Delta) \int_0^{\xi(1-\Delta)^{n-1}} \frac{1}{f(u)}\, du. \tag{3.48}$$

Proof. Since (x_n) is positive and decreasing and f obeys (3.45) then for $x_{j+1} < u < x_j$ we have

$$\frac{1}{f(x_j)} < \frac{1}{f(u)} < \frac{1}{f(x_{j+1})}.$$

Integrating over $[x_{j+1}, x_j]$ and (1.41) yields

$$h(x_j) = \frac{x_j - x_{j+1}}{f(x_j)} < \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\, du < \frac{x_j - x_{j+1}}{f(x_{j+1})} = \frac{1}{1-\Delta} \cdot h(x_{j+1}),$$

since $(x_j - x_{j+1}) = (x_{j+1} - x_{j+2})/(1-\Delta)$ because $x_n = \xi(1-\Delta)^n$ by (3.46). Hence for $j \geq 0$

$$\hat{T}(\Delta) = \sum_{j=0}^{\infty} h(x_j) < \sum_{j=0}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\, du < \frac{1}{1-\Delta} \sum_{j=0}^{\infty} h(x_{j+1}). \tag{3.49}$$

Since $\sum_{j=0}^{\infty} h(x_{j+1}) = \sum_{j=1}^{\infty} h(x_j) = \sum_{j=0}^{\infty} h(x_j) - h(x_0)$, (3.49) yields

$$\hat{T}(\Delta) < T_\xi < \frac{1}{1-\Delta} \left(\hat{T}(\Delta) - \frac{\Delta\xi}{f(\xi)} \right).$$

The first inequality establishes $0 < T_\xi - \hat{T}(\Delta)$, while the second establishes $(1-\Delta)T_\xi < \hat{T}(\Delta) - \Delta\xi/f(\xi)$. Rearranging the second inequality yields

$$T_\xi - \hat{T}(\Delta) < \Delta \left(T_\xi - \frac{\xi}{f(\xi)} \right).$$

Combining both inequalities yields

$$0 < T_\xi - \hat{T}(\Delta) < \Delta \left(T_\xi - \frac{\xi}{f(\xi)} \right),$$

which establishes (3.47). To prove (3.48), letting $n \geq 1$ in (3.49) yields

$$\hat{T}_n(\Delta) = \sum_{j=0}^{n-1} h(x_j) < \sum_{j=0}^{n-1} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\, du < \frac{1}{1-\Delta} \sum_{j=0}^{n-1} h(x_{j+1}).$$

Therefore

$$\hat{T}_n(\Delta) < \int_{\xi(1-\Delta)^n}^{\xi} \frac{1}{f(u)}\, du < \frac{1}{1-\Delta} \sum_{j=1}^{n} h(x_j).$$

Since $\sum_{j=1}^n h(x_j) = \sum_{j=0}^n h(x_j) - h(x_0)$ we have for $n \geq 1$ that

$$\hat{T}_n(\Delta) < T_\xi - \int_0^{\xi(1-\Delta)^n} \frac{1}{f(u)}\, du < \frac{1}{1-\Delta}\left(\hat{T}_{n+1}(\Delta) - \frac{\Delta\xi}{f(\xi)}\right).$$

Rearranging the first inequality yields

$$\int_0^{\xi(1-\Delta)^n} \frac{1}{f(u)}\, du < T_\xi - \hat{T}_n(\Delta).$$

Rearranging the second inequality yields

$$(1-\Delta)T_\xi - (1-\Delta)\int_0^{\xi(1-\Delta)^n} \frac{1}{f(u)}\, du < \hat{T}_{n+1}(\Delta) - \frac{\Delta\xi}{f(\xi)},$$

and so

$$T_\xi - \hat{T}_{n+1}(\Delta) < \Delta\left(T_\xi - \frac{\xi}{f(\xi)}\right) + (1-\Delta)\int_0^{\xi(1-\Delta)^n} \frac{1}{f(u)}\, du.$$

Combining both ineqaulities yields

$$\int_0^{\xi(1-\Delta)^n} \frac{1}{f(u)}\, du < T_\xi - \hat{T}_n(\Delta) < \Delta\left(T_\xi - \frac{\xi}{f(\xi)}\right) + (1-\Delta)\int_0^{\xi(1-\Delta)^{n-1}} \frac{1}{f(u)}\, du,$$

which is (3.48). □

Remark 11. Equation (3.47) shows the Explicit scheme under-estimates the finite hitting time of the equilibrium at zero because this scheme under-estimates the solution. □

Remark 12. The extra terms in (3.48) compared to (3.47) represent upper and lower bounds on the difference between $\hat{T}(\Delta) - \hat{T}_n(\Delta)$ or alternatively $\sum_{j=n}^{\infty} h(x_j)$. By the definition of t_n and $\hat{T}_h$, we have

$$\hat{T}_h - t_n = \sum_{j=0}^{\infty} h(x_j) - \sum_{j=0}^{n-1} h(x_j) = \sum_{j=n}^{\infty} h(x_j).$$

Moreover as $\hat{T}_h < \infty$, we have that $\sum_{j=n}^{\infty} h(x_j) \to 0$ as $n \to \infty$. The extra terms in (3.48) tend to zero as $n \to \infty$ since $x_n \to 0$ as $n \to \infty$. □

The estimate (3.47) suggests that $T_\xi - \hat{T}(\Delta)$ is $O(\Delta)$ as $\Delta \to 0^+$. We show that this estimate is sharp in the case of a power non-linearity.

Example 12. Let $f(x) = x^\beta$, $\beta \in (0,1)$. Then

$$T_\xi = \int_0^\xi \frac{1}{f(u)}\,du = \int_0^\xi u^{-\beta}\,du = \frac{\xi^{1-\beta}}{1-\beta}.$$

Since $h(x) = \Delta x/f(x)$ and $x_j = \xi(1-\Delta)^j$, we have

$$\hat{T}(\Delta) = \sum_{j=0}^{\infty} h(x_j) = \sum_{j=0}^{\infty} \Delta x_j^{1-\beta} = \frac{\xi^{1-\beta}}{1-\beta} \cdot \frac{(1-\beta)\Delta}{1-(1-\Delta)^{1-\beta}}.$$

Thus

$$T_\xi - \hat{T}(\Delta) = \frac{\Delta\xi^{1-\beta}}{1-\beta} \cdot \left(1 - \frac{(1-\beta)\Delta}{1-(1-\Delta)^{1-\beta}}\right) \cdot \frac{1}{\Delta}.$$

A Taylor expansion of $g(\Delta) := 1-(1-\Delta)^{1-\beta}$ at $\Delta = 0$ gives

$$1-(1-\Delta)^{1-\beta} - (1-\beta)\Delta = \frac{\beta(1-\beta)}{2}\Delta^2 + O(\Delta^3),$$

and

$$\Delta(1-(1-\Delta)^{1-\beta}) = (1-\beta)\Delta^2 + O(\Delta^3).$$

Thus

$$\lim_{\Delta\to 0^+} \frac{1-(1-\Delta)^{1-\beta}-(1-\beta)\Delta}{\Delta(1-(1-\Delta)^{1-\beta})} = \lim_{\Delta\to 0^+} \frac{(\beta(1-\beta)/2)\Delta^2 + O(\Delta^3)}{(1-\beta)\Delta^2 + O(\Delta^3)} = \frac{\beta}{2},$$

which gives

$$\lim_{\Delta\to 0^+} \frac{T_\xi - \hat{T}(\Delta)}{\Delta} = \frac{\xi^{1-\beta}}{1-\beta} \cdot \frac{\beta}{2}.$$

Similarly we can compute

$$\begin{aligned} T_\xi - \hat{T}_n(\Delta) &= \frac{\xi^{1-\beta}}{1-\beta} - \sum_{j=0}^{n-1} \Delta\left(\xi(1-\Delta)^j\right)^{1-\beta} \\ &= T_\xi - \hat{T}(\Delta) + \xi^{1-\beta}\Delta \cdot \frac{\left((1-\Delta)^{1-\beta}\right)^n}{1-(1-\Delta)^{1-\beta}} \\ &= T_\xi - \hat{T}(\Delta) + \xi^{1-\beta} \cdot \frac{\Delta}{1-(1-\Delta)^{1-\beta}} \cdot (1-\Delta)^{n(1-\beta)}, \end{aligned}$$

where $T_\xi - \hat{T}(\Delta)$ is $O(\Delta)$ as $\Delta \to 0^+$. Notice that the computable error, $\hat{T}_n(\Delta)$, converges geometrically in the number of time-steps considered to the error that could be theoretically achieved, $\hat{T}(\Delta)$, if all the terms in the sum were considered. This completes the example. □

Consider the Implicit Euler scheme. We approximate $x(t_n)$ by x_n, where $x(t_n)$ is the solution x of (1.1) at time t_n. The sequences (x_n), (t_n) and $(h(x_n))$ are defined by (1.43), (1.44) and (3.46):

$$x_{n+1} = x_n - h(x_{n+1})f(x_{n+1}), \quad n \geq 0, \quad x_0 = \xi > 0,$$

where

$$t_{n+1} = \sum_{j=0}^{n} h(x_{j+1}), \quad n \geq 0, \quad t_0 = 0.$$

Thus $\hat{T}(\Delta) := \sum_{j=0}^{\infty} h(x_{j+1})$ and $\hat{T}_n(\Delta) := \sum_{j=0}^{n-1} h(x_{j+1})$ are the analogues of the approximations of the finite stability time and its truncation in the Implicit case.

Theorem 13. *Suppose* (x_n) *is the solution of* (1.43)*. Suppose also* f *obeys* (1.7)*,* (3.1)*,* (3.32) *and* (3.45) *while* h *obeys* (3.2) *and* (3.46) *with* $\Delta \in [0, \infty)$*. Let* $\hat{T}_h$ *be defined by* (3.33)*. Then*

$$0 < \hat{T}(\Delta) - T_\xi < \Delta \left(T_\xi - \frac{\xi}{f(\xi)} \right), \tag{3.50}$$

$$-\int_0^{\xi(1+\Delta)^{-n}} \frac{1}{f(u)}\,du < \hat{T}_n(\Delta) - T_\xi < \Delta \left(T_\xi - \frac{\xi}{f(\xi)} \right) - (1+\Delta) \int_0^{\xi(1+\Delta)^{-(n+1)}} \frac{1}{f(u)}\,du. \tag{3.51}$$

Proof. Since (x_n) is positive and decreasing and f obeys (3.45) then for $x_{j+1} < u < x_j$ we have

$$\frac{1}{f(x_j)} < \frac{1}{f(u)} < \frac{1}{f(x_{j+1})}.$$

Integrating over $[x_{j+1}, x_j]$ and (1.43) yields

$$\frac{1}{1+\Delta} \cdot h(x_j) = \frac{x_{j-1} - x_j}{f(x_j)} < \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du < \frac{x_j - x_{j+1}}{f(x_{j+1})} = h(x_{j+1}),$$

since $(x_j - x_{j+1}) = (x_{j-1} - x_j)/(1+\Delta)$ because $x_n = \xi(1+\Delta)^{-n}$ by (3.46). Hence for $j \geq 0$

$$\frac{1}{1+\Delta} \sum_{j=0}^{\infty} h(x_j) < \sum_{j=0}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du < \sum_{j=0}^{\infty} h(x_{j+1}) = \hat{T}(\Delta). \tag{3.52}$$

Since $\sum_{j=0}^{\infty} h(x_j) = h(x_0) + \sum_{j=0}^{\infty} h(x_{j+1})$, (3.52) yields

$$\frac{1}{1+\Delta} \left(\frac{\Delta\xi}{f(\xi)} + \hat{T}(\Delta) \right) < T_\xi < \hat{T}(\Delta).$$

The second inequality establishes $0 < \hat{T}(\Delta) - T_\xi$, while the first establishes $\Delta\xi/f(\xi) +$

$\hat{T}(\Delta) < (1+\Delta)T_\xi$. Rearranging the first inequality yields

$$\hat{T}(\Delta) - T_\xi < \Delta\left(T_\xi - \frac{\xi}{f(\xi)}\right).$$

Combining both inequalities yields

$$0 < \hat{T}(\Delta) - T_\xi < \Delta\left(T_\xi - \frac{\xi}{f(\xi)}\right),$$

which establishes (3.50). To prove (3.51), letting $n \geq 1$ in (3.52) yields

$$\frac{1}{1+\Delta}\sum_{j=0}^{n-1} h(x_j) < \sum_{j=0}^{n-1}\int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du < \sum_{j=0}^{n-1} h(x_{j+1}) = \hat{T}_n(\Delta).$$

Therefore

$$\frac{1}{1+\Delta}\sum_{j=0}^{n-1} h(x_j) < \int_{\xi(1+\Delta)^{-n}}^{\xi} \frac{1}{f(u)}\,du < \hat{T}_n(\Delta).$$

Since $\sum_{j=0}^{n-1} h(x_j) = h(x_0) + \sum_{j=0}^{n-2} h(x_{j+1})$ we have for $n \geq 1$

$$\frac{1}{1+\Delta}\left(\frac{\Delta\xi}{f(\xi)} + \hat{T}_{n-1}(\Delta)\right) < T_\xi - \int_0^{\xi(1+\Delta)^{-n}} \frac{1}{f(u)}\,du < \hat{T}_n(\Delta).$$

Rearranging the second inequality yields

$$-\int_0^{\xi(1+\Delta)^{-n}} \frac{1}{f(u)}\,du < \hat{T}_n(\Delta) - T_\xi.$$

Rearranging the first inequality yields and hence

$$\frac{\Delta\xi}{f(\xi)} + \hat{T}_{n-1}(\Delta) < (1+\Delta)\left(T_\xi - \int_0^{\xi(1+\Delta)^{-n}} \frac{1}{f(u)}\,du\right),$$

and thus

$$\hat{T}_{n-1}(\Delta) - T_\xi < \Delta\left(T_\xi - \frac{\xi}{f(\xi)}\right) - (1+\Delta)\int_0^{\xi(1+\Delta)^{-n}} \frac{1}{f(u)}\,du.$$

Combining the estimates gives

$$-\int_0^{\xi(1+\Delta)^{-n}} \frac{1}{f(u)}\,du < \hat{T}_n(\Delta) - T_\xi < \Delta\left(T_\xi - \frac{\xi}{f(\xi)}\right) - (1+\Delta)\int_0^{\xi(1+\Delta)^{-(n+1)}} \frac{1}{f(u)}\,du,$$

which is (3.51). □

Remark 13. Equation (3.50) shows the Implicit scheme over-estimates the finite hitting time of the equilibrium at zero because this scheme over-estimates the solution. □

Remark 14. The extra terms in (3.51) compared to (3.50) represent upper and lower bounds on the difference between $\hat{T}(\Delta) - \hat{T}_n(\Delta)$ or alternatively $\sum_{j=n}^{\infty} h(x_{j+1})$. By the definition of t_n and $\hat{T}_h$, we have

$$\hat{T}_h - t_n = \sum_{j=0}^{\infty} h(x_{j+1}) - \sum_{j=0}^{n-1} h(x_{j+1}) = \sum_{j=n}^{\infty} h(x_{j+1}).$$

Moreover as $\hat{T}_h < \infty$, we have that $\sum_{j=n}^{\infty} h(x_{j+1}) \to 0$ as $n \to \infty$. The extra terms in (3.51) tend to zero as $n \to \infty$ since $x_n \to 0$ as $n \to \infty$. □

3.6 Implicit Euler Scheme with Step-Size $O(1/f'(x))$

We have chosen up to now to take step-sizes $O(x/f(x))$ as $x \to 0^+$. We show in what follows, for the Implicit Scheme that taking step-sizes $O(1/f'(x))$ as $x \to 0^+$ can also be effective. This is of interest for some non-linearity for which $1/f'(x)$ tends to zero more slowly than $x/f(x)$ as $x \to 0^+$, meaning that larger step-sizes can be taken without an appreciable loss of performance. Suppose

$$f' \text{ is continuous, } f' \text{ is decreasing on } (0, x^*),\ f(x) > 0 \,\forall\, x > 0 \text{ and } f \text{ is increasing} \tag{3.53}$$

$$\Delta \text{ is continuous, } \Delta(x) > 0,\ \Delta(x) \to \Delta \in [0, \infty) \text{ as } x \to 0^+ \tag{3.54}$$

$$h(x) := \frac{\Delta(x)}{f'(x)} \quad \text{for all } x > 0. \tag{3.55}$$

We approximate $x(t_n)$ by x_n, where $x(t_n)$ is the solution x of (1.1) at time t_n. The sequences (x_n), (t_n) and $(h(x_n))$ are defined by (1.43), (1.44) and (3.55)

$$x_{n+1} = x_n - h(x_{n+1}) f(x_{n+1}), \quad n \geq 0, \quad x_0 = \xi > 0,$$

where $t_{n+1} = \sum_{j=0}^{n} h(x_{j+1}), \quad n \geq 0, \quad t_0 = 0.$

Theorem 14. *Suppose f, Δ and h obey* (3.53), (3.54) *and* (3.55). *Let $\bar{F}$, F, (t_n) and $\hat{T}_h$ be defined by* (1.10), (1.11), (1.44) *and* (3.33).

(i) If f obeys (1.7), *then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and*

$$\frac{1}{1+\Delta} \leq \liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq \limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq 1.$$

(ii) If f obeys (1.9), *then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$,*

$t_n \to \infty$ as $n \to \infty$ and

$$\frac{1}{1+\Delta} \leq \liminf_{n\to\infty} \frac{F(x_n)}{t_n} \leq \limsup_{n\to\infty} \frac{F(x_n)}{t_n} \leq 1.$$

Proof. Suppose for the moment that $h(x)f(x) \to 0$ as $x \to 0^+$. Since f is increasing for $u \in [x, x+h(x)f(x)]$, then we have that $f(x) < f(u) < f(x+h(x)f(x))$. Therefore for $x > 0$

$$\frac{f(x)}{f(x+h(x)f(x))} < \frac{1}{h(x)} \int_x^{x+h(x)f(x)} \frac{1}{f(u)}\,du < 1.$$

By the Mean Value Theorem, there is $\theta_x \in (0,1)$ such that

$$f(x+h(x)f(x)) = f(x) + f'(x+\theta_x h(x)f(x)) \cdot h(x)f(x).$$

Hence

$$\frac{f(x+h(x)f(x))}{f(x)} = 1 + f'(x+\theta_x h(x)f(x)) \cdot h(x).$$

Since $x \leq x + \theta_x h(x)f(x) < x + h(x)f(x)$ then by (3.53) for all x sufficiently small $0 < f'(x+\theta_x h(x)f(x)) < f'(x)$. Thus

$$\frac{f(x+h(x)f(x))}{f(x)} \leq 1 + h(x)f'(x) = 1 + \Delta(x).$$

Hence for all x sufficiently small

$$\frac{1}{1+\Delta(x)} < \frac{1}{h(x)} \int_x^{x+h(x)f(x)} \frac{1}{f(u)}\,du < 1. \tag{3.56}$$

Before proceeding further we now verify that $h(x)f(x) \to 0$ as $x \to 0^+$

$$\lim_{x\to 0^+} h(x)f(x) = \lim_{x\to 0^+} \frac{\Delta(x)f(x)}{f'(x)} = \lim_{x\to 0^+} \Delta(x) \cdot \lim_{x\to 0^+} \frac{f(x)}{f'(x)} = \Delta \cdot \frac{0}{f'(0^+)} = 0,$$

since $\lim_{x\to 0^+} f'(x) = f'(0^+) \in (0,\infty]$. Next as (x_n) decreases to zero as $n \to \infty$ for $n \geq N_1(\epsilon)$, x_n will be sufficiently small such that estimate (3.56) holds for $x = x_{n+1}$. Thus for $n \geq N_1(\epsilon)$

$$\frac{1}{1+\Delta(x_{n+1})} < \frac{1}{h(x_{n+1})} \int_{x_{n+1}}^{x_{n+1}+h(x_{n+1})f(x_{n+1})} \frac{1}{f(u)}\,du < 1.$$

Now $x_{n+1} + h(x_{n+1})f(x_{n+1}) = x_n$, so for $n \geq N_1(\epsilon)$

$$\frac{1}{1+\Delta(x_{n+1})} < \frac{1}{h(x_{n+1})} \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\,du < 1. \tag{3.57}$$

Since $x_n \to 0$ as $n \to \infty$ and $\Delta(x) \to \Delta \in [0,\infty)$ as $x \to 0^+$, we get that $(\int_{x_{n+1}}^{x_n} 1/f(u)\,du)$

is summable if and only if $(h(x_{n+1}))$ is summable. If f obeys (1.7) then $\int_{0^+}^1 1/f(u)\,du < \infty$ and $\sum_{n=0}^\infty \int_{x_{n+1}}^{x_n} 1/f(u)\,du = \int_0^\xi 1/f(u)\,du < \infty$ and so $(h(x_{n+1}))$ is summable. Hence $\hat{T}_h := \sum_{j=0}^\infty h(x_{j+1}) < \infty$. Thus $\hat{T}_h - t_n = \sum_{j=0}^\infty h(x_{j+1}) - \sum_{j=0}^{n-1} h(x_{j+1}) = \sum_{j=n}^\infty h(x_{j+1})$. Thus with $\bar{F}(x) = \int_0^x 1/f(u)\,du$, we have by (3.57) for $n \geq N_1(\epsilon)$

$$\sum_{j=n}^\infty \frac{h(x_{j+1})}{1+\Delta(x_{j+1})} < \bar{F}(x_n) = \sum_{j=n}^\infty \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du < \sum_{j=n}^\infty h(x_{j+1}) = \hat{T}_h - t_n.$$

Hence

$$\limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq 1.$$

By Toeplitz's Lemma and $\Delta(x_j) \to \Delta$ as $j \to \infty$,

$$\liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \geq \liminf_{n\to\infty} \frac{\sum_{j=n}^\infty h(x_{j+1})/(1+\Delta(x_{j+1}))}{\sum_{j=n}^\infty h(x_{j+1})} = \frac{1}{1+\Delta}.$$

In the case $\int_{0^+}^1 1/f(u)\,du = \infty$, we have that $\sum_{n=0}^\infty \int_{x_{n+1}}^{x_n} 1/f(u)\,du = \int_0^\xi 1/f(u)\,du = \infty$. Thus $(h(x_{n+1}))$ is not summable. Hence $t_n \to \infty$ as $n \to \infty$, where $t_n = \sum_{j=0}^{n-1} h(x_{j+1})$. Let $n \geq N_1(\epsilon)$, by (3.57)

$$F(x_n) = F(x_{N_1(\epsilon)}) + \sum_{j=N_1(\epsilon)}^{n-1} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du \leq F(x_{N_1(\epsilon)}) + \sum_{j=N_1(\epsilon)}^{n-1} h(x_{j+1}).$$

Hence

$$\limsup_{n\to\infty} \frac{F(x_n)}{t_n} \leq \limsup_{n\to\infty} \left(\frac{F(x_{N_1(\epsilon)})}{t_n} + \frac{\sum_{j=N_1(\epsilon)}^{n-1} h(x_{j+1})}{\sum_{j=0}^{n-1} h(x_{j+1})} \right) = 1.$$

On the other hand for $n \geq N_1(\epsilon)$, by (3.57)

$$F(x_n) \geq F(x_{N_1(\epsilon)}) + \sum_{j=N_1(\epsilon)}^{n-1} \frac{h(x_{j+1})}{1+\Delta(x_{j+1})}.$$

Thus by Toeplitz's Lemma and that $\Delta(x_{j+1}) \to \Delta$ as $j \to \infty$, we get

$$\liminf_{n\to\infty} \frac{F(x_n)}{t_n} \geq \liminf_{n\to\infty} \left(\frac{F(x_{N_1(\epsilon)})}{t_n} + \frac{\sum_{j=N_1(\epsilon)}^{n-1} h(x_{j+1})/(1+\Delta(x_{j+1}))}{\sum_{j=0}^{n-1} h(x_{j+1})} \right) = \frac{1}{1+\Delta}.$$

□

Remark 15. We know that if $f \in RV_0(\beta)$, $\beta \in [0,1]$ and f' is monotone, then $xf'(x)/f(x) \to \beta$ as $x \to 0^+$. Also if $h(x) \sim \Delta x/f(x)$ as $x \to 0^+$ for $\Delta \in (0,\infty)$ we will show in the next chapter that

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \frac{1}{\Delta} \int_1^{1+\Delta} \lambda^{-\beta}\,d\lambda.$$

For $\beta = 0$ and $\Delta \in (0, \infty)$ the limit on the right-hand side is unity, while for $\beta > 0$ the quantity on the right-hand side is with $O(\Delta)$ of unity as $x \to 0^+$. Theorem 14 specifies $h(x) \sim \Delta/f'(x)$ as $x \to 0^+$, so

$$\lim_{x \to 0^+} \frac{f(x)h(x)}{x} = \frac{\Delta}{\beta} = \begin{cases} \infty, & \text{if } \beta = 0, \\ (0, \infty), & \text{if } \beta > 0. \end{cases}$$

If $\beta > 0$, we have the choice of h in Theorem 14 being of the same order as chosen up to now. If $\beta = 0$ then $h(x) = o(1/f'(x))$ as $x \to 0^+$ and we get the perfect rate as predicted. Our new method is taking asymptotically larger step-sizes and yet still predicting exponents to within $O(\Delta)$ as $x \to 0^+$. Furthermore, Theorem 14 anticipates our later results because if we take $h(x) \sim \Delta x/f(x)$ as $x \to 0^+$ then $h(x) = o(1/f'(x))$ as $x \to 0^+$ and so Theorem 14 part (i) implies

$$\lim_{n \to \infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = 1.$$

□

Chapter 4

Asymptotic Behaviour of Discretisation with Regularly Varying Non-Linearity

4.1 Introducton

In this chapter, we seek to refine the results from Chapter 3 by assuming that f is regularly varying at zero. In Chapter 3 we are typically unable, for positive values of the convergence parameter Δ, to obtain in the case of finite-time stability a limit of the form

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = c(\Delta).$$

In this chapter we show for $f \in RV_0(\beta)$ that such limits obtain and generally $c(\Delta) \neq 1$ for $\Delta > 0$. We also show that $c(\Delta) \to 1$ as $\Delta \to 0^+$ for both Explicit and Implicit schemes, that $c(\Delta) \to 0$ as $\Delta \to \infty$ for the Implicit scheme only and that $\Delta \mapsto c(\Delta)$ is increasing and $c(\Delta) > 1$ for $\Delta \in (0,1)$.

These results show that taking a step-size of $O(x/f(x))$ as $x \to 0^+$ is indeed optimal for ODEs with regularly varying non-linearities because taking asymptotically larger step-sizes will lead to spurious asymptotic convergence rates for (x_n). Likewise taking step-sizes which obey $h(x) = o(x/f(x))$ as $x \to 0^+$, will recover asymptotic behaviour exactly, but will do so at a greatly increased computational cost owing to the asymptotically smaller step-size required.

There are two situations which require further analysis, namely the case when $\beta = 0$ for both schemes. In this case

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = 1, \tag{4.1}$$

for the Implicit scheme we show that it is possible to take an asymptotically larger step-size and still recover a finite limit in (4.1). We also give conditions which help us determine for a given f what classes of asymptotic step-sizes larger than $O(x/f(x))$ as $x \to 0^+$ will deliver a finite or unit limit in (4.1).

4.2 Comparing Behaviour of Numerical Scheme and ODE

4.2.1 Explicit Euler Scheme with Adaptive Step Size

We approximate $x(t_n)$ by x_n, where $x(t_n)$ is the solution x of (1.1) at time t_n. As before, (x_n), (t_n) and $(h(x_n))$ are defined by (1.41), (1.42) and (3.17):

$$x_{n+1} = x_n - h(x_n)f(x_n), \quad n \geq 0, \quad x_0 = \xi > 0,$$

where

$$t_{n+1} = \sum_{j=0}^{n} h(x_j), \quad n \geq 0, \quad t_0 = 0,$$

and

$$\lim_{x \to 0^+} \frac{h(x)f(x)}{x} = \Delta \in [0, 1].$$

We make the following observations which will be of use in several of our proofs. These observations are of a similar character to those in the previous chapter. Suppose $\int_{0^+}^{1} 1/f(u)\, du = \infty$. If F is defined by (1.11) then $F(x) \to \infty$ as $x \to 0^+$, so $F(x_n) \to \infty$ as $n \to \infty$. Then for $n \geq 1$

$$F(x_n) = \int_{x_n}^{1} \frac{1}{f(u)}\, du = F(x_0) + \sum_{j=0}^{n-1} \int_{x_j - h(x_j)f(x_j)}^{x_j} \frac{1}{f(u)}\, du.$$

If $F(x_n) \to \infty$ as $n \to \infty$ then

$$\sum_{j=0}^{n-1} \int_{x_j - h(x_j)f(x_j)}^{x_j} \frac{1}{f(u)}\, du = \infty, \tag{4.2}$$

since $F(x_0)$ is finite. Suppose $\int_{0^+}^{1} 1/f(u)\, du < \infty$ then $F(x) \to L \in (0, \infty)$ as $x \to 0^+$ so $F(x_n) \to L$ as $n \to \infty$. Hence

$$\sum_{j=0}^{n-1} \int_{x_j - h(x_j)f(x_j)}^{x_j} \frac{1}{f(u)}\, du < \infty. \tag{4.3}$$

If T_ξ is defined by (1.8) then for $n \geq 0$

$$T_\xi = \int_{0}^{\xi} \frac{1}{f(u)}\, du = \sum_{j=0}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\, du = \sum_{j=0}^{n-1} \int_{x_j - h(x_j)f(x_j)}^{x_j} \frac{1}{f(u)}\, du.$$

Equations (4.2) and (4.3) show that T_ξ is finite or infinite according to whether $F(x)$ is finite or infinite. If $\bar{F}$ is defined by (1.10) then $\bar{F}(x) \to 0$ as $x \to 0^+$ so $\bar{F}(x_n) \to 0$ as $n \to \infty$. Then for $n \geq 0$

$$\bar{F}(x_n) = \int_0^{x_n} \frac{1}{f(u)}\,du = \sum_{j=n}^{\infty} \int_{x_j - h(x_j)f(x_j)}^{x_j} \frac{1}{f(u)}.$$

The closed-form expressions for $F(x_n)$, $\bar{F}(x_n)$ and T_ξ identify the summand in the last identity as the key sequence in our analysis.

4.3 Preserving Soft Landings and Super-Exponential Stability

In this section, we assume that $f \in \text{RV}_0(\beta)$ for some $\beta \in [0,1]$. The Explicit scheme defined by equation (1.41) preserves the properties of the soft landing (1.15) under the condition (1.7) while the property of super-exponentially stable solutions (1.13) is preserved under the condition (1.9). Since $(h(x_n))$ is a positive sequence the limit

$$\lim_{n\to\infty} t_n =: \hat{T}_h = \sum_{j=0}^{\infty} h(x_j)$$

exists, but can be finite or infinite. In our next result, we show that $\hat{T}_h$ is finite or infinite according to whether T_ξ defined by (1.8) is finite or infinite.

Theorem 15. *Suppose f obeys* (3.1), (3.7) *and that $f \in RV_0(\beta)$ for some $\beta \in (0,1]$ while h obeys* (3.2) *and* (3.17) *with $\Delta \in [0,1]$. Let (t_n) and $\hat{T}_h$ be defined* (1.42) *and* (3.16).

(i) If f obeys (1.7), *then $\hat{T}_h < \infty$.*

(ii) If f obeys (1.9), *then $\hat{T}_h = \infty$.*

Proof. Since $f \in RV_0(\beta)$, for some $\beta \in (0,1]$, it follows that there is $\phi(x) \sim f(x)$ as $x \to 0^+$ where $\phi \in RV_0(\beta)$ and ϕ is increasing as $x \to 0^+$. Since (x_n) is positive and decreasing then for $x_{j+1} < u < x_j$ and with ϕ increasing

$$\frac{1}{\phi(x_j)} < \frac{1}{\phi(u)} < \frac{1}{\phi(x_{j+1})}.$$

Integrating over $[x_{j+1}, x_j]$ and (1.41) implies

$$\tilde{h}(x_j) := \frac{x_{j+1} - x_j}{\phi(x_j)} < \int_{x_{j+1}}^{x_j} \frac{1}{\phi(u)}\,du < \frac{x_{j+1} - x_j}{\phi(x_{j+1})} = \frac{\phi(x_j)}{\phi(x_{j+1})} \cdot \tilde{h}(x_j). \qquad (4.4)$$

By (1.7), $\int_{0^+}^1 1/f(u)\,du < \infty$ then $T_\xi < \infty$ from (4.3). Since $\phi(x) \sim f(x)$ as $x \to 0^+$ then

$$\sum_{j=0}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{\phi(u)}\,du = \int_0^{\xi} \frac{1}{\phi(u)}\,du < \infty.$$

The Comparison Test applied to (4.4) shows the summability of $(\int_{x_{n+1}}^{x_n} 1/\phi(u)\,du)$ implies that of $(\tilde{h}(x_j))$. Hence $(\tilde{h}(x_n))$ is summable and so is $(h(x_n))$ since $f(x) \sim \phi(x)$ as $x \to 0^+$. Thus $t_n = \sum_{j=0}^{n-1} h(x_j)$ for $n \geq 1$ obeys $t_n \to \hat{T}_h := \sum_{j=0}^{\infty} h(x_j) < \infty$ as $n \to \infty$.

By (1.9) $\int_{0^+}^1 1/f(u)\,du = \infty$ then $T_\xi = \infty$ from (4.2). Since $f(x) \sim \phi(x)$ as $x \to 0^+$ then

$$\sum_{j=0}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{\phi(u)}\,du = \int_0^{\xi} \frac{1}{\phi(u)}\,du = \infty.$$

The Comparison Test applied to (4.4) shows that $(\phi(x_j)/\phi(x_{j+1}) \cdot \tilde{h}(x_j))$ is not a summable sequence. Define

$$\lambda_j := \frac{x_{j+1}}{x_j} = 1 - \frac{h(x_j)f(x_j)}{x_j}.$$

Since (3.17) holds then

$$\lim_{j\to\infty} \lambda_j = \lim_{j\to\infty} \left(1 - \frac{h(x_j)f(x_j)}{x_j}\right) = 1 - \Delta.$$

Since $\phi \in RV_0(\beta)$ then

$$\lim_{j\to\infty} \frac{\phi(x_j)}{\phi(x_{j+1})} = \lim_{j\to\infty} \frac{\phi(x_j)}{\phi(\lambda_j x_j)} = (1-\Delta)^{-\beta}.$$

Thus $(\tilde{h}(x_{j+1}))$ is not summable since

$$\lim_{j\to\infty} \frac{\tilde{h}(x_j)}{\phi(x_j)/\phi(x_{j+1}) \cdot \tilde{h}(x_j)} = (1-\Delta)^{-\beta},$$

and thus $(h(x_n))$ obeys $t_n = \sum_{j=0}^{n-1} h(x_j) \to \infty$ as $n \to \infty$. Therefore $t_n \to \infty$ as $n \to \infty$. □

Remark 16. In the case when $\beta = 0$, we must additionally assume that f is asymptotically monotone. The function ϕ to which f is asymptotically monotone is therefore in $RV_0(0)$ and therefore the proof above for $\beta > 0$ holds in all regards. □

4.4 Preserving Asymptotic Convergence Rates

Our next result shows that when f is regularly varying not only is the absence of a finite hitting time correctly predicted but the precise asymptotic behaviour is also recovered once h obeys (3.17) with $\Delta \in [0,1]$.

Lemma 9. *Suppose f obeys* (3.1), (3.7) *and that $f \in RV_0(\beta)$ where $\beta \in [0,1]$ while h obeys* (3.2) *and* (3.17) *with $\Delta \in [0,1]$.*

(i) If $\Delta = 0$, then

$$\lim_{x\to 0^+} \frac{1}{h(x)} \int_{x-h(x)f(x)}^{x} \frac{1}{f(u)}\,du = 1.$$

(ii) If $\Delta \in (0,1)$, then

$$\lim_{x\to 0^+} \frac{1}{h(x)} \int_{x-h(x)f(x)}^{x} \frac{1}{f(u)}\,du = \frac{1}{\Delta}\int_{1-\Delta}^{1} \lambda^{-\beta}\,d\lambda.$$

(iii) If $\Delta = 1$, then

(a) $\beta \in [0,1)$ implies

$$\lim_{x\to 0^+} \frac{1}{h(x)} \int_{x-h(x)f(x)}^{x} \frac{1}{f(u)}\,du = \frac{1}{1-\beta}.$$

(b) $\beta = 1$ implies

$$\lim_{x\to 0^+} \frac{1}{h(x)} \int_{x-h(x)f(x)}^{x} \frac{1}{f(u)}\,du = \infty.$$

Proof. It is useful in the proof to express the integral in terms of regularly varying functions as follows:

$$\begin{aligned}
\frac{1}{h(x)} \int_{x-h(x)f(x)}^{x} \frac{1}{f(u)}\,du &= \frac{1}{h(x)f(x)} \int_{x-h(x)f(x)}^{x} \frac{f(x)}{f(u)}\,du \\
&= \frac{x}{h(x)f(x)} \int_{1-h(x)f(x)/x}^{1} \frac{f(x)}{f(\lambda x)}\,d\lambda \\
&= \frac{1}{\Delta(x)} \int_{1-\Delta(x)}^{1} \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)}\,d\lambda,
\end{aligned}$$

where $\Delta(x) := h(x)f(x)/x$ and $\tilde{f}(x) := 1/f(x)$. It follows that $\tilde{f} \in RV_0(-\beta)$. We start with the proof of part (iii). Put $y = 1/x$ and let $x \to 0$ or equivalently $y \to \infty$: then

$$\begin{aligned}\frac{1}{h(x)}\int_{x-h(x)f(x)}^{x}\frac{1}{f(u)}\,du = \frac{1}{\Delta(x)}\int_{1-\Delta(x)}^{1}\frac{\tilde{f}(\lambda x)}{\tilde{f}(x)}\,d\lambda \;&\sim\; \int_{1-\Delta(x)}^{1}\frac{\tilde{f}(\lambda x)}{\tilde{f}(x)}\,d\lambda \\ &= \int_{1-\Delta(1/y)}^{1}\frac{\tilde{f}(\lambda\cdot 1/y)}{\tilde{f}(1/y)}\,d\lambda \\ &= \int_{1-\Delta^*(y)}^{1}\frac{f^*(\lambda^{-1}y)}{f^*(y)}\,d\lambda,\end{aligned}$$

as $y \to \infty$. In making these estimates we have taken $f^*(x) := \tilde{f}(1/x)$ so $f^* \in RV_\infty(\beta)$, $\Delta^*(y) := \Delta(1/y)$ so $\Delta^*(y) \to 1$ as $y \to \infty$. It remains to obtain the limit of the last displayed quantity as $y \to \infty$ according to

$$\begin{aligned}\int_{1-\Delta^*(y)}^{1}\frac{f^*(\lambda^{-1}y)}{f^*(y)}\,d\lambda &= \int_{(1-\Delta^*(y))^{-1}}^{1}\frac{f^*(\mu y)}{f^*(y)}\cdot\frac{-1}{\mu^2}\,d\mu \\ &= \int_{1}^{(1-\Delta^*(y))^{-1}}\frac{f^*(\mu y)}{f^*(y)}\cdot\frac{1}{\mu^2}\,d\mu \\ &\le \int_{1}^{\infty}\frac{f^*(\mu y)}{f^*(y)}\cdot\frac{1}{\mu^2}\,d\mu.\end{aligned}$$

By the Representation Theorem for Regularly Varying functions (see Theorem 1.3.1 in [12]) we have that

$$f^*(y) = y^\beta c(y)\exp\left(\int_a^y \frac{\epsilon(u)}{u}\,du\right),$$

where $c(y) \to c > 0$ and $\epsilon(y) \to 0$ as $y \to \infty$. Let $x \ge 1$. Now there exists $y_0 > 0$ such that

$$\frac{c(xy)}{c(y)} < 2, \quad \text{for all } y > y_0,$$

and a $y(\epsilon)$ such that for all $y > y(\epsilon)$

$$\exp\left(\int_y^{xy}\frac{\epsilon(u)}{u}\,du\right) \le x^\epsilon.$$

Thus for every $\epsilon > 0$, there is $y(\epsilon) > 0$ such that for all $y > \max(y(\epsilon), y_0)$ and all $x \ge 1$, we have

$$\begin{aligned}\frac{f^*(xy)}{f^*(y)} = \frac{(xy)^\beta c(xy)\exp\left(\int_a^{xy}\epsilon(u)/u\,du\right)}{y^\beta c(y)\exp\left(\int_a^{y}\epsilon(u)/u\,du\right)} &= x^\beta\cdot\frac{c(xy)}{c(x)}\cdot\exp\left(\int_y^{xy}\frac{\epsilon(u)}{u}\,du\right) \\ &\le 2x^{\beta+\epsilon}.\end{aligned}$$

Let $X > 1$. Then for $y \geq \max(y(\epsilon), y_0)$ we have

$$\begin{aligned}\int_1^\infty \frac{f^*(\mu y)}{f^*(y)} \cdot \frac{1}{\mu^2}\, d\mu &= \int_1^X \frac{f^*(\mu y)}{f^*(y)} \cdot \frac{1}{\mu^2}\, d\mu + \int_X^\infty \frac{f^*(\mu y)}{f^*(y)} \cdot \frac{1}{\mu^2}\, d\mu \\ &\leq \int_1^X \frac{f^*(\mu y)}{f^*(y)} \cdot \frac{1}{\mu^2}\, d\mu + \int_X^\infty 2\mu^{\beta+\epsilon-2}\, d\mu.\end{aligned}$$

Thus by the uniform convergence theorem for regularly varying functions (see Theorem 1.5.2 in [12])

$$\begin{aligned}\limsup_{y\to\infty} \int_1^\infty \frac{f^*(\mu y)}{f^*(y)} \cdot \frac{1}{\mu^2}\, d\mu &\leq \lim_{y\to\infty} \left(\int_1^X \frac{f^*(\mu y)}{f^*(y)} \cdot \frac{1}{\mu^2}\, d\mu + \int_X^\infty 2\mu^{\beta+\epsilon-2}\, d\mu \right) \\ &\leq \int_1^X \mu^{\beta-2}\, d\mu + \int_X^\infty 2\mu^{\beta+\epsilon-2}\, d\mu.\end{aligned}$$

Letting $X \to \infty$ yields

$$\limsup_{x\to 0^+} \int_{1-\Delta(x)}^1 \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)}\, d\lambda \leq \frac{1}{1-\beta}. \tag{4.5}$$

We now seek to find a corresponding lower bound. Let $y = 1/x$ and let $X > 1$. Since $(1 - \Delta^*(y))^{-1} \to \infty$ as $y \to \infty$ it follows that there is a $y(X) > 0$ such that $(1 - \Delta^*(y))^{-1} > X$ for all $y > y(X)$. Thus for $y \geq y(X)$

$$\begin{aligned}\int_{1-\Delta(x)}^1 \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)}\, d\lambda &= \int_1^{(1-\Delta^*(y))^{-1}} \frac{f^*(\mu y)}{f^*(y)} \cdot \frac{1}{\mu^2}\, d\mu \\ &= \int_1^X \frac{f^*(\mu y)}{f^*(y)} \cdot \frac{1}{\mu^2}\, d\mu + \int_X^{(1-\Delta^*(y))^{-1}} \frac{f^*(\mu y)}{f^*(y)} \cdot \frac{1}{\mu^2}\, d\mu \\ &\geq \int_1^X \frac{f^*(\mu y)}{f^*(y)} \cdot \frac{1}{\mu^2}\, d\mu.\end{aligned}$$

Thus by the uniform convergence theorem for regularly varying functions

$$\liminf_{x\to 0^+} \int_{1-\Delta(x)}^1 \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)}\, d\lambda \geq \liminf_{y\to\infty} \int_1^X \frac{f^*(\mu y)}{f^*(y)} \cdot \frac{1}{\mu^2}\, d\mu = \int_1^X \mu^{\beta-2}\, d\mu.$$

Letting $X \to \infty$ yields that

$$\liminf_{x\to 0^+} \int_{1-\Delta(x)}^1 \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)}\, d\lambda \geq \frac{1}{1-\beta}.$$

Combining this with (4.5) yields

$$\lim_{x\to 0^+} \int_{1-\Delta(x)}^1 \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)}\, d\lambda = \frac{1}{1-\beta},$$

and hence the proof of part (iii)(a) is complete. In the case when $\beta = 1$ it suffices to consider the lower bound and arguing as before we obtain

$$\liminf_{x\to 0^+} \int_{1-\Delta(x)}^{1} \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)}\, d\lambda \geq \int_{1}^{X} \mu^{\beta-2}\, d\mu.$$

Therefore

$$\liminf_{x\to 0^+} \int_{1-\Delta(x)}^{1} \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)}\, d\lambda \geq \log X.$$

Letting $X \to \infty$ yields that

$$\lim_{x\to 0^+} \int_{1-\Delta(x)}^{1} \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)}\, d\lambda = \infty,$$

and hence the proof of part (iii)(b) is complete. We now prove part (ii). Since $\Delta(x) \to \Delta$ as $x \to 0^+$ for all x sufficiently small

$$\begin{aligned}\int_{1-\Delta(x)}^{1} \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)}\, d\lambda \leq \int_{1-\Delta-\epsilon}^{1} \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)}\, d\lambda &= \int_{1-\Delta-\epsilon}^{1} \left(\frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - \lambda^{-\beta}\right) + \lambda^{-\beta}\, d\lambda \\ &= \int_{1-\Delta-\epsilon}^{1} \left(\frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - \lambda^{-\beta}\right) d\lambda + \int_{1-\Delta-\epsilon}^{1} \lambda^{-\beta}\, d\lambda.\end{aligned}$$

Hence by the Uniform Convergence Theorem for Regularly Varying functions we get

$$\limsup_{x\to 0^+} \int_{1-\Delta(x)}^{1} \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)}\, d\lambda \leq \int_{1-\Delta-\epsilon}^{1} \lambda^{-\beta}\, d\lambda.$$

Letting $\epsilon \to 0$ yields

$$\limsup_{x\to 0^+} \int_{1-\Delta(x)}^{1} \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)}\, d\lambda \leq \int_{1-\Delta}^{1} \lambda^{-\beta}\, d\lambda.$$

Similarly for all x sufficiently small the corresponding lower bound is

$$\begin{aligned}\int_{1-\Delta(x)}^{1} \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)}\, d\lambda \geq \int_{1-\Delta+\epsilon}^{1} \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)}\, d\lambda &= \int_{1-\Delta+\epsilon}^{1} \left(\frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - \lambda^{-\beta}\right) + \lambda^{-\beta}\, d\lambda \\ &= \int_{1-\Delta+\epsilon}^{1} \left(\frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - \lambda^{-\beta}\right) d\lambda + \int_{1-\Delta+\epsilon}^{1} \lambda^{-\beta}\, d\lambda.\end{aligned}$$

Hence

$$\liminf_{x\to 0^+} \int_{1-\Delta(x)}^{1} \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)}\, d\lambda \geq \int_{1-\Delta+\epsilon}^{1} \lambda^{-\beta}\, d\lambda.$$

Letting $\epsilon \to 0$ yields

$$\liminf_{x\to 0^+} \int_{1-\Delta(x)}^{1} \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)}\, d\lambda \geq \int_{1-\Delta}^{1} \lambda^{-\beta}\, d\lambda.$$

Combining the upper and lower bounds yields

$$\lim_{x\to 0^+} \frac{1}{h(x)} \int_{x-h(x)f(x)}^{x} \frac{1}{f(u)}\, du = \frac{1}{\Delta} \int_{1-\Delta}^{1} \lambda^{-\beta}\, d\lambda,$$

as claimed. To prove part (i), we start by writing

$$\begin{aligned}
\frac{1}{h(x)} \int_{x-h(x)f(x)}^{x} \frac{1}{f(u)}\, du &= \frac{1}{\Delta(x)} \int_{1-\Delta(x)}^{1} \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)}\, d\lambda \\
&= \frac{1}{\Delta(x)} \int_{1-\Delta(x)}^{1} \left\{ \left(\frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - \lambda^{-\beta} \right) + \lambda^{-\beta} \right\} d\lambda \\
&= \frac{1}{\Delta(x)} \int_{1-\Delta(x)}^{1} \left(\frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - \lambda^{-\beta} \right) d\lambda + \frac{1}{\Delta(x)} \int_{1-\Delta(x)}^{1} \lambda^{-\beta}\, d\lambda.
\end{aligned}$$

For all $x < x(\epsilon)$, we have $\Delta(x) < \epsilon$ so

$$\begin{aligned}
\left| \frac{1}{\Delta(x)} \int_{1-\Delta(x)}^{1} \left(\frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - \lambda^{-\beta} \right) d\lambda \right| &\leq \frac{1}{\Delta(x)} \int_{1-\Delta(x)}^{1} \left| \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - \lambda^{-\beta} \right| d\lambda \\
&\leq \frac{1}{\Delta(x)} \cdot \Delta(x) \sup_{1-\Delta(x)\leq\lambda\leq 1} \left| \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - \lambda^{-\beta} \right| \\
&\leq \sup_{1-\epsilon\leq\lambda\leq 1} \left| \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - \lambda^{-\beta} \right|.
\end{aligned}$$

Hence by the Uniform Convergence Theorem for Regularly Varying functions

$$\limsup_{x\to 0^+} \left| \frac{1}{\Delta(x)} \int_{1-\Delta(x)}^{1} \left(\frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - \lambda^{-\beta} \right) d\lambda \right| = 0.$$

The second term on the right hand side has zero limit since L'Hôpital's Rule shows that

$$\lim_{y\to 0^+} \frac{1}{y} \int_{1-y}^{1} \lambda^{-\beta}\, d\lambda = 0,$$

and because $\Delta(x) \to 0$ as $x \to 0^+$ we have that

$$\lim_{x\to 0^+} \frac{1}{\Delta(x)} \int_{1-\Delta(x)}^{1} \lambda^{-\beta}\, d\lambda = 0.$$

Hence

$$\frac{1}{h(x)}\int_{x-h(x)f(x)}^{x}\frac{1}{f(u)}\,du = 1,$$

as claimed. □

If $f \in \mathrm{RV}_0(\beta)$ it can only happen that x does not hit zero in finite-time when $\beta = 1$. We now consider the asymptotic behaviour in that case.

Theorem 16. *Suppose f obeys (1.9), (3.1), (3.7) and that $f \in RV_0(1)$ while h obeys (3.2) and (3.17) with $\Delta \in [0,1]$. Let F and (t_n) be defined by (1.11) and (1.42).*

(i) If $\Delta = 0$, then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty}\frac{F(x_n)}{t_n} = 1.$$

(ii) If $0 < \Delta < 1$, then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty}\frac{F(x_n)}{t_n} = \frac{1}{\Delta}\int_{1-\Delta}^{1}\lambda^{-1}\,d\lambda.$$

(iii) If $\Delta = 1$, then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$ and either: $t_n \to \hat{T}_h < \infty$ as $n \to \infty$; or $t_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty}\frac{F(x_n)}{t_n} = \infty.$$

Proof. The positivity, monotonicity and convergence of (x_n) have been addressed in Theorem 3. Since f obeys (1.9) then $\int_{0^+}^{1} 1/f(u)\,du = \infty$ and $t_n \to \infty$ as $n \to \infty$ by Theorem 15. If $t_n = \sum_{j=0}^{n-1} h(x_j) \to \infty$ as $n \to \infty$, which is true if $\Delta \in (0,1)$, dividing by t_n and letting $n \to \infty$ yields

$$\begin{aligned}
\lim_{n\to\infty}\frac{F(x_n)}{t_n} &= \lim_{n\to\infty}\frac{F(x_0)+\sum_{j=0}^{n-1}\int_{x_j-h(x_j)f(x_j)}^{x_j} 1/f(u)\,du}{\sum_{j=0}^{n-1} h(x_j)}\\
&= \lim_{n\to\infty}\frac{\sum_{j=0}^{n-1}\int_{x_j-h(x_j)f(x_j)}^{x_j} 1/f(u)\,du}{\sum_{j=0}^{n-1} h(x_j)}\\
&= \lim_{j\to\infty}\frac{\int_{x_j-h(x_j)f(x_j)}^{x_j} 1/f(u)\,du}{h(x_j)}\\
&= \lim_{x\to 0^+}\frac{1}{h(x)}\int_{x-h(x)f(x)}^{x}\frac{1}{f(u)}\,du,
\end{aligned}$$

by Toeplitz's Lemma. Therefore, the proof of parts (i) and (ii) comes from combining the above limit and the relevant part of Lemma 9. For part (iii) if $t_n \to \infty$ as $n \to \infty$, we may argue as above. Otherwise, t_n tends to a finite limit covering the other possibility in the statement of part (iii). □

The next result shows that when the ODE (1.1) hits zero in finite-time, the numerical method will detect the correct asymptotic behaviour, provided h obeys (3.2) and (3.17).

Theorem 17. *Suppose f obeys (1.7), (3.1), (3.7) and that $f \in RV_0(\beta)$ for some $\beta \in [0,1]$ while h obeys (3.2) and (3.17) with $\Delta \in [0,1]$. Let $\bar{F}$ and $\hat{T}_h$ be defined by (1.10) and (3.16).*

(i) If $\Delta = 0$, then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = 1.$$

(ii) If $0 < \Delta < 1$, then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \frac{1}{\Delta}\int_{1-\Delta}^{1} \lambda^{-\beta}\, d\lambda.$$

(iii) If $\Delta = 1$, then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and

(a) if $\beta \in [0,1)$ then

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \frac{1}{\Delta}\int_{1-\Delta}^{1} \lambda^{-\beta}\, d\lambda.$$

(b) if $\beta = 1$ then

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \infty.$$

Proof. The positivity, monotonicity and convergence of (x_n) have been addressed in Theorem 3. Since f obeys (1.7) then $\int_{0^+}^{1} 1/f(u)\, du < \infty$ and $t_n \to \hat{T}_h := \sum_{j=0}^{\infty} h(x_j) < \infty$ by Theorem 15. Hence $\hat{T}_h - t_n = \sum_{j=n}^{\infty} h(x_j) \to 0$ as $n \to \infty$. As $\bar{F}(x_n) \to 0$ as $n \to \infty$ dividing by $\hat{T}_h - t_n = \sum_{j=n}^{\infty} h(x_j)$ and letting $n \to \infty$ yields

$$\begin{aligned}
\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} &= \lim_{n\to\infty} \frac{\sum_{j=n}^{\infty} \int_{x_j - h(x_j)f(x_j)}^{x_j} 1/f(u)\, du}{\sum_{j=n}^{\infty} h(x_j)} \\
&= \lim_{j\to\infty} \frac{\int_{x_j - h(x_j)f(x_j)}^{x_j} 1/f(u)\, du}{h(x_j)} \\
&= \lim_{x\to 0^+} \frac{1}{h(x)} \int_{x-h(x)f(x)}^{x} \frac{1}{f(u)}\, du,
\end{aligned}$$

by Toeplitz's Lemma. The proof for each part comes from combining the above limit and the relevant part of Lemma 9. □

4.4.1 Implicit Euler Scheme with Adaptive Step Size

We approximate $x(t_n)$ by x_n, where $x(t_n)$ is the solution x of (1.1) at time t_n. The sequences (x_n), (t_n) and $(h(x_n))$ are defined by (1.43), (1.44) and (3.17):

$$x_{n+1} = x_n - h(x_{n+1})f(x_{n+1}), \quad n \geq 0, \quad x_0 = \xi > 0,$$

where

$$t_{n+1} = \sum_{j=0}^{n} h(x_{j+1}), \quad n \geq 0, \quad t_0 = 0,$$

and

$$\lim_{x \to 0^+} \frac{h(x)f(x)}{x} = \Delta \in [0, \infty].$$

We make the following observations which will be of use in several of our proofs. Suppose $\int_{0+}^{1} 1/f(u)\, du = \infty$. If F is defined by (1.11) then $F(x) \to \infty$ as $x \to 0^+$, so $F(x_n) \to \infty$ as $n \to \infty$. Then for $n \geq 1$

$$F(x_n) = \int_{x_n}^{1} \frac{1}{f(u)}\, du = F(x_0) + \sum_{j=1}^{n} \int_{x_j}^{x_j + h(x_j)f(x_j)} \frac{1}{f(u)}\, du$$

If $F(x_n) \to \infty$ as $n \to \infty$ then

$$\sum_{j=1}^{n} \int_{x_j}^{x_j + h(x_j)f(x_j)} \frac{1}{f(u)}\, du = \infty. \tag{4.6}$$

since $F(x_0)$ is finite. Suppose $\int_{0+}^{1} 1/f(u)\, du < \infty$. Then $F(x) \to L \in [0, \infty)$ as $x \to 0^+$, so $F(x_n) \to L$ as $n \to \infty$. Hence

$$\sum_{j=1}^{n} \int_{x_j}^{x_j + h(x_j)f(x_j)} \frac{1}{f(u)}\, du < \infty. \tag{4.7}$$

If T_ξ is defined by (1.8) then for $n \geq 0$

$$T_\xi = \int_{0}^{\xi} \frac{1}{f(u)}\, du = \sum_{j=0}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\, du = \sum_{j=1}^{n} \int_{x_j}^{x_j + h(x_j)f(x_j)} \frac{1}{f(u)}\, du.$$

Equations (4.6) and (4.7) show that T_ξ is finite or infinite according to whether $F(x)$ is finite or infinite. If $\bar{F}$ is defined by (1.10) then $\bar{F}(x) \to 0$ as $x \to 0^+$ so $\bar{F}(x_n) \to 0$ as $n \to \infty$. Then for $n \geq 0$

$$\bar{F}(x_n) = \int_{0}^{x_n} \frac{1}{f(u)}\, du = \sum_{j=n+1}^{\infty} \int_{x_j}^{x_j + h(x_j)f(x_j)} \frac{1}{f(u)}\, du.$$

The closed-form expressions for $F(x_n)$, $\bar{F}(x_n)$ and T_ξ identify the summand in the last identity as the key sequence in our analysis.

4.5 Preserving Finite-Time Stability

The main advantage of the Implicit scheme is that if h obeys (3.17), there is no restriction on the size of finite Δ. We make this precise below. In our next result, we show that $\hat{T}_h$ is finite or infinite according as to whether T_ξ defined by (1.8) is finite or infinite.

Theorem 18. *Suppose f obeys (3.1), (3.32) and that $f \in RV_0(\beta)$ for some $\beta \in [0,1]$ while h obeys (3.2) and (3.17) with $\Delta \in [0,\infty]$. Let (t_n) and $\hat{T}_h$ be defined (1.44) and (3.33).*

(i) If f obeys (1.7), then $\hat{T}_h < \infty$.

(ii) If f obeys (1.9), then $\hat{T}_h = \infty$.

Proof. Since $f \in RV_0(\beta)$, for some $\beta \in (0,1]$, it follows that there is $\phi(x) \sim f(x)$ as $x \to 0^+$ where $\phi \in RV_0(\beta)$ and ϕ is increasing. We tackle the case of $\beta = 0$ later. Since (x_n) is positive and decreasing then for $x_{j+1} < u < x_j$ and with ϕ increasing

$$\frac{1}{\phi(x_j)} < \frac{1}{\phi(u)} < \frac{1}{\phi(x_{j+1})}.$$

Hence

$$\frac{\phi(x_{j+1})}{\phi(x_j)} \cdot \tilde{h}(x_{j+1}) = \frac{x_j - x_{j+1}}{\phi(x_j)} < \int_{x_{j+1}}^{x_j} \frac{1}{\phi(u)}\,du < \frac{x_j - x_{j+1}}{\phi(x_{j+1})} =: \tilde{h}(x_{j+1}). \tag{4.8}$$

By (1.7), $\int_{0^+}^1 1/f(u)\,du < \infty$ then $T_\xi < \infty$ from (4.7). Since $\phi(x) \sim f(x)$ as $x \to 0^+$ then

$$\sum_{j=0}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{\phi(u)}\,du = \int_0^\xi \frac{1}{\phi(u)}\,du < \infty.$$

The Comparison Test applied to (4.8) and the summability of $(\int_{x_{n+1}}^{x_n} 1/\phi(u)\,du)$ implies that of $(\phi(x_{j+1})/\phi(x_j) \cdot \tilde{h}(x_{n+1}))$.
Define

$$\lambda_{j+1} := \frac{x_{j+1}}{x_j} = \left(\frac{x_j}{x_{j+1}}\right)^{-1} = \left(\frac{x_{j+1} + h(x_{j+1})f(x_{j+1})}{x_{j+1}}\right)^{-1} = \left(1 + \frac{h(x_{j+1})f(x_{j+1})}{x_{j+1}}\right)^{-1}.$$

Since (3.17) holds then

$$\lim_{j\to\infty} \lambda_{j+1} = \lim_{j\to\infty} \left(1 + \frac{h(x_{j+1})f(x_{j+1})}{x_{j+1}}\right)^{-1} = (1+\Delta)^{-1}.$$

Since $\phi \in RV_0(\beta)$ then

$$\lim_{j\to\infty}\frac{\phi(x_{j+1})}{\phi(x_j)} = \lim_{j\to\infty}\frac{\phi(\lambda_{j+1}x_j)}{\phi(x_j)} = \left((1+\Delta)^{-1}\right)^{\beta} = (1+\Delta)^{-\beta}.$$

Thus $(\tilde{h}(x_{j+1}))$ is summable since

$$\lim_{j\to\infty}\frac{\tilde{h}(x_{j+1})}{\phi(x_{j+1})/\phi(x_j)\cdot\tilde{h}(x_{j+1})} = (1+\Delta)^{-\beta}.$$

Hence $(\tilde{h}(x_{n+1}))$ is summable and so is $(h(x_{n+1}))$ since $f(x) \sim \phi(x)$ as $x \to 0^+$. Thus $t_n = \sum_{j=0}^{n-1} h(x_{j+1})$ for $n \geq 1$ obeys $t_n \to \hat{T}_h := \sum_{j=0}^{\infty} h(x_{j+1}) < \infty$ as $n \to \infty$. In the case when $\beta = 0$ we have that $x \mapsto f(x)/x \in RV_0(-1)$ and therefore is asymptotically decreasing. This can be used as in the monotonicity section to demonstrate the summability of $(h(x_j))$.

By (1.9), $\int_{0^+}^1 1/f(u)\,du = \infty$ then $T_\xi = \infty$ from (4.6). Since $f(x) \sim \phi(x)$ as $x \to 0^+$ then

$$\sum_{j=0}^{\infty}\int_{x_{j+1}}^{x_j}\frac{1}{\phi(u)}\,du = \int_0^{\xi}\frac{1}{\phi(u)}\,du = \infty.$$

The Comparison Test applied to (4.8) shows that $(\tilde{h}(x_{n+1}))$ is not summable and thus $(h(x_{n+1}))$ obeys $t_n = \sum_{j=0}^{n-1} h(x_{j+1}) \to \infty$ as $n \to \infty$. Therefore $t_n \to \infty$ as $n \to \infty$. □

A consequence of this result is that the scheme defined by (1.43) preserves finite-time stability under the condition (1.7) while positivity is preserved under the condition (1.9).

4.6 Preserving Asymptotic Convergence Rates

We show that the Euler scheme (1.43) reproduces the exact asymptotic behaviour of the solution to (1.1) when there is a soft landing and when there is super-exponential convergence. The scheme does not recover the exact rate of convergence because the Implicit scheme over-estimates the solution despite preserving finite-time stability and super-exponential convergence. We also see that if $\Delta = \infty$ in (1.40), then exact convergence rates may not be recovered. We tackle the case of super-exponential convergence first.

Lemma 10. *Suppose f obeys* (3.1), (3.32) *and that $f \in RV_0(\beta)$ where $\beta \in (0,1]$ while h obeys* (3.2) *and* (3.17) *with $\Delta \in [0,\infty]$.*

(i) If $\Delta = 0$, then

$$\lim_{x\to 0^+}\frac{1}{h(x)}\int_x^{x+h(x)f(x)}\frac{1}{f(u)}\,du = 1.$$

(ii) If $\Delta \in (0,\infty)$, *then*

$$\lim_{x\to 0^+} \frac{1}{h(x)} \int_x^{x+h(x)f(x)} \frac{1}{f(u)}\,du = \frac{1}{\Delta}\int_1^{1+\Delta} \lambda^{-\beta}\,d\lambda.$$

(iii) If $\Delta = \infty$, *then*

$$\lim_{x\to 0^+} \frac{1}{h(x)} \int_x^{x+h(x)f(x)} \frac{1}{f(u)}\,du = 0.$$

Proof. It is useful in the following proof to express the integral in terms of regularly varying functions as follows

$$\begin{aligned}
\frac{1}{h(x)} \int_x^{x+h(x)f(x)} \frac{1}{f(u)}\,du &= \frac{1}{h(x)f(x)} \int_x^{x+h(x)f(x)} \frac{f(x)}{f(u)}\,du \\
&= \frac{x}{h(x)f(x)} \int_1^{1+h(x)f(x)/x} \frac{f(x)}{f(\lambda x)}\,d\lambda \\
&= \frac{1}{\Delta(x)} \int_1^{1+\Delta(x)} \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)}\,d\lambda,
\end{aligned}$$

where $\Delta(x) := h(x)f(x)/x$ and $\tilde{f}(x) := 1/f(x)$. It follows that $\tilde{f} \in RV_0(-\beta)$. We prove part (iii). Since $\tilde{f} \in RV_0(-\beta)$ there is a decreasing $\tilde{\phi} \in RV_0(-\beta)$ such that $\tilde{\phi}(x) \sim \tilde{f}(x)$ as $x \to 0^+$. Now we write

$$\frac{1}{h(x)} \int_x^{x+h(x)f(x)} \frac{1}{f(u)}\,du = \frac{\tilde{\phi}(x)}{\tilde{f}(x)} \cdot \frac{1}{\Delta(x)} \int_1^{1+\Delta(x)} \frac{\tilde{f}(\lambda x)}{\tilde{\phi}(\lambda x)} \cdot \frac{\tilde{\phi}(\lambda x)}{\tilde{\phi}(x)}\,d\lambda.$$

If $\lambda \le 1 + \Delta(x)$, then $\lambda x \le x + h(x)f(x) \to 0$ as $x \to 0^+$. Now, there exists $\delta_1 > 0$ such that

$$\frac{1}{2} < \frac{\tilde{f}(x)}{\tilde{\phi}(x)} < 2, \quad \text{for all } x < \delta_1,$$

and a δ_2 such that

$$x + h(x)f(x) < \delta_1, \quad \text{for all } x < \delta_2.$$

Let $\delta_3 := \min(\delta_1, \delta_2)$. For $x < \delta_3$ then $\lambda x \le x + h(x)f(x) < \delta_1$ and thus

$$\frac{1}{2} < \frac{\tilde{f}(\lambda x)}{\tilde{\phi}(\lambda x)} < 2, \quad \text{for all } x < \delta_3.$$

Thus for $x < \delta_3$

$$\frac{1}{h(x)} \int_x^{x+h(x)f(x)} \frac{1}{f(u)}\,du \le \frac{4}{\Delta(x)} \int_1^{1+\Delta(x)} \frac{\tilde{\phi}(\lambda x)}{\tilde{\phi}(x)}\,d\lambda.$$

Since $\Delta(x) \to \infty$ as $x \to 0$, for every $\epsilon \in (0,1)$, there is $x_1(\epsilon) > 0$ such that $\Delta(x) > 1/\epsilon^2$, for all $x < x_1(\epsilon)$. Hence $\epsilon\Delta(x) > \epsilon^2\Delta(x) > 1$ for $x < x_1(\epsilon)$. For $1 \le \lambda$, then

$\tilde{\phi}(x) > \tilde{\phi}(\lambda x)$ and

$$\begin{aligned}
&\frac{1}{\Delta(x)} \int_1^{1+\Delta(x)} \frac{\tilde{\phi}(\lambda x)}{\tilde{\phi}(x)}\, d\lambda \\
&\quad = \frac{1}{\Delta(x)} \left(\int_1^{\epsilon\Delta(x)} \frac{\tilde{\phi}(\lambda x)}{\tilde{\phi}(x)}\, d\lambda + \int_{\epsilon\Delta(x)}^{\Delta(x)} \frac{\tilde{\phi}(\lambda x)}{\tilde{\phi}(x)}\, d\lambda + \int_{\Delta(x)}^{1+\Delta(x)} \frac{\tilde{\phi}(\lambda x)}{\tilde{\phi}(x)}\, d\lambda \right) \\
&\quad \le \frac{1}{\Delta(x)} \left(\int_1^{\epsilon\Delta(x)} 1\, d\lambda + \int_{\Delta(x)}^{1+\Delta(x)} 1\, d\lambda + \int_{\epsilon\Delta(x)}^{\Delta(x)} \frac{\tilde{\phi}(\lambda x)}{\tilde{\phi}(x)}\, d\lambda \right) \\
&\quad = \frac{(\epsilon\Delta(x) - 1)}{\Delta(x)} + \frac{(1 + \Delta(x) - \Delta(x))}{\Delta(x)} + \frac{1}{\Delta(x)} \int_{\epsilon\Delta(x)}^{\Delta(x)} \frac{\tilde{\phi}(\lambda x)}{\tilde{\phi}(x)}\, d\lambda \\
&\quad = \epsilon + \frac{1}{\Delta(x)} \int_{\epsilon\Delta(x)}^{\Delta(x)} \frac{\tilde{\phi}(\lambda x)}{\tilde{\phi}(x)}\, d\lambda.
\end{aligned}$$

For $\epsilon\Delta(x) \le \lambda \le \Delta(x)$, then $\epsilon x\Delta(x) \le \lambda x \le x\Delta(x)$, so as $\tilde{\phi}$ is decreasing then $\tilde{\phi}(\epsilon x\Delta(x)) \ge \tilde{\phi}(\lambda x) \ge \tilde{\phi}(x\Delta(x))$. Thus

$$\frac{1}{\Delta(x)} \int_{\epsilon\Delta(x)}^{\Delta(x)} \frac{\tilde{\phi}(\lambda x)}{\tilde{\phi}(x)}\, d\lambda \le \frac{(1-\epsilon)\Delta(x)}{\Delta(x)} \cdot \frac{\tilde{\phi}(\epsilon x\Delta(x))}{\tilde{\phi}(x)} \le \frac{\tilde{\phi}(\epsilon x\Delta(x))}{\tilde{\phi}(x)}.$$

For $x < x_1(\epsilon)$, then $\epsilon\Delta(x) > 1/\epsilon$, thus $\epsilon x\Delta(x) > x/\epsilon$ so as $\tilde{\phi}$ is decreasing then $\tilde{\phi}(\epsilon x\Delta(x)) < \tilde{\phi}(x/\epsilon)$. Thus for $x < x_1(\epsilon)$

$$\frac{1}{\Delta(x)} \int_{\epsilon\Delta(x)}^{\Delta(x)} \frac{\tilde{\phi}(\lambda x)}{\tilde{\phi}(x)}\, d\lambda \le \frac{\tilde{\phi}(x/\epsilon)}{\tilde{\phi}(x)}.$$

Hence since $\tilde{\phi} \in RV_0(-\beta)$

$$\limsup_{x\to 0^+} \frac{1}{\Delta(x)} \int_{\epsilon\Delta(x)}^{\Delta(x)} \frac{\tilde{\phi}(\lambda x)}{\tilde{\phi}(x)}\, d\lambda \le \lim_{x\to 0^+} \frac{\tilde{\phi}(\epsilon^{-1}x)}{\tilde{\phi}(x)} = (\epsilon^{-1})^{-\beta} = \epsilon^{\beta}.$$

Therefore

$$\limsup_{x\to 0^+} \frac{1}{h(x)} \int_x^{x+h(x)f(x)} \frac{1}{f(u)}\, du \le 4\left(\epsilon + \epsilon^{\beta}\right).$$

Letting $\epsilon \to 0^+$ yields the desired result. We now prove part (ii). Using the opening considerations and partitioning the integrals we obtain

$$\begin{aligned}
\frac{1}{h(x)} \int_x^{x+h(x)f(x)} \frac{1}{f(u)}\, du - \frac{1}{\Delta} \int_1^{1+\Delta} \lambda^{-\beta}\, d\lambda &= \frac{1}{\Delta(x)} \int_1^{1+\Delta(x)} \left(\frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - \lambda^{-\beta} \right) d\lambda \\
&+ \frac{1}{\Delta(x)} \int_1^{1+\Delta(x)} \lambda^{-\beta}\, d\lambda - \frac{1}{\Delta} \int_1^{1+\Delta} \lambda^{-\beta}\, d\lambda.
\end{aligned}$$

Taken together the second and third terms on the right-hand side have zero limit since

$\Delta(x) \to \Delta$ as $x \to 0^+$. For $x < \delta$, then $\Delta/2 < \Delta(x) < 3\Delta/2$. Thus for $x < \delta$

$$\begin{aligned}
\left| \frac{1}{\Delta(x)} \int_1^{1+\Delta(x)} \left(\frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - \lambda^{-\beta} \right) d\lambda \right| &\leq \frac{1}{\Delta(x)} \int_1^{1+\Delta(x)} \left| \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - \lambda^{-\beta} \right| d\lambda \\
&\leq \frac{1}{\Delta/2} \cdot \frac{3\Delta}{2} \sup_{1 \leq \lambda \leq 1+\frac{3\Delta}{2}} \left| \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - \lambda^{-\beta} \right| \\
&= 3 \sup_{1 \leq \lambda \leq 1+\frac{3\Delta}{2}} \left| \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - \lambda^{-\beta} \right|.
\end{aligned}$$

Hence by the Uniform Convergence Theorem we have

$$\lim_{x \to 0^+} \frac{1}{\Delta(x)} \int_1^{1+\Delta(x)} \left(\frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - \lambda^{-\beta} \right) d\lambda = 0,$$

and hence

$$\lim_{x \to 0^+} \frac{1}{h(x)} \int_x^{x+h(x)f(x)} \frac{1}{f(u)}\, du = \frac{1}{\Delta} \int_1^{1+\Delta} \lambda^{-\beta}\, d\lambda,$$

as claimed. We now prove part (i). We start with the identity

$$\begin{aligned}
\frac{1}{h(x)} \int_x^{x+h(x)f(x)} \frac{1}{f(u)}\, du &= \frac{1}{\Delta(x)} \int_1^{1+\Delta(x)} \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)}\, d\lambda \\
&= \frac{1}{\Delta(x)} \int_1^{1+\Delta(x)} \left\{ 1 + \left(\frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - \lambda^{-\beta} \right) + \left(\lambda^{-\beta} - 1 \right) \right\} d\lambda,
\end{aligned}$$

which gives

$$\begin{aligned}
\frac{1}{h(x)} \int_x^{x+h(x)f(x)} \frac{1}{f(u)}\, du - 1 = \frac{1}{\Delta(x)} \int_1^{1+\Delta(x)} \left(\frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - \lambda^{-\beta} \right) d\lambda \\
+ \frac{1}{\Delta(x)} \int_1^{1+\Delta(x)} \left(\lambda^{-\beta} - 1 \right) d\lambda.
\end{aligned}$$

For all $x < x_1$, $\Delta(x) < 1$. Hence for all $x < x_1$

$$\begin{aligned}
\left| \frac{1}{\Delta(x)} \int_1^{1+\Delta(x)} \left(\frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - \lambda^{-\beta} \right) d\lambda \right| &\leq \frac{1}{\Delta(x)} \int_1^{1+\Delta(x)} \left| \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - \lambda^{-\beta} \right| d\lambda \\
&\leq \frac{1}{\Delta(x)} \cdot \Delta(x) \sup_{1 \leq \lambda \leq 1+\Delta(x)} \left| \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - \lambda^{-\beta} \right| \\
&\leq \sup_{1 \leq \lambda \leq 2} \left| \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - \lambda^{-\beta} \right|.
\end{aligned}$$

Hence

$$\lim_{x\to 0^+} \frac{1}{\Delta(x)} \int_1^{1+\Delta(x)} \left(\frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - \lambda^{-\beta} \right) d\lambda = 0,$$

by The Uniform Convergence Theorem. The second term on the right-hand side has zero limit since L'Hôpital's Rule shows that

$$\lim_{y\to 0^+} \frac{1}{y} \int_1^{1+y} \left(\lambda^{-\beta} - 1 \right) d\lambda = 0.$$

Since $\Delta(x) \to 0$ as $x \to 0^+$ then

$$\lim_{x\to 0^+} \frac{1}{\Delta(x)} \int_1^{1+\Delta(x)} \left(\lambda^{-\beta} - 1 \right) d\lambda = 0.$$

Hence

$$\frac{1}{h(x)} \int_x^{x+h(x)f(x)} \frac{1}{f(u)}\, du = 1,$$

as claimed. □

We now tackle the case of super-exponential stability. We only consider $\beta = 1$ since the integral defined by (1.11) is guaranteed to converge when $\beta < 1$.

Theorem 19. *Suppose f obeys* (1.9), (3.1), (3.32) *and that $f \in RV_0(1)$ while h obeys* (3.2) *and* (3.17) *with $\Delta \in [0, \infty]$. Let F and (t_n) be defined by* (1.11) *and* (1.44).

(i) If $\Delta = 0$, then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = 1.$$

(ii) If $\Delta \in (0, \infty)$, then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = \frac{\log(1+\Delta)}{\Delta}.$$

(iii) If $\Delta = \infty$, then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = 0.$$

Proof. The positivity, monotonicity and convergence of (x_n) have been addressed in Lemma 7 and Proposition 4. By Theorem 18 we are guaranteed that $t_n \to \infty$ as $n \to \infty$ in all cases. Therefore, as $t_n = \sum_{j=0}^{n-1} h(x_{j+1}) = \sum_{j=1}^{n} h(x_j) \to \infty$ as $n \to \infty$, dividing by t_n and letting $n \to \infty$ yields

$$\begin{aligned}\lim_{n\to\infty} \frac{F(x_n)}{t_n} &= \lim_{n\to\infty} \frac{F(x_0) + \sum_{j=1}^{n} \int_{x_j}^{x_j+h(x_j)f(x_j)} 1/f(u)\,du}{\sum_{j=1}^{n} h(x_j)} \\ &= \lim_{j\to\infty} \frac{\int_{x_j}^{x_j+h(x_j)f(x_j)} 1/f(u)\,du}{h(x_j)} \\ &= \lim_{x\to 0^+} \frac{1}{h(x)} \int_{x}^{x+h(x)f(x)} \frac{1}{f(u)}\,du,\end{aligned}$$

by Toeplitz's Lemma. The proof for each part comes from combining the above limit and the relevant part of Lemma 10. □

We now tackle the case of finite-time stability.

Theorem 20. *Suppose f obeys (1.7), (3.1), (3.32) and that $f \in RV_0(\beta)$ where $\beta \in (0,1]$ while h obeys (3.2) and (3.17) with $\Delta \in [0,\infty]$. Let $\bar{F}$, (t_n) and $\hat{T}_h$ be defined by (1.10), (1.44) and (3.33).*

(i) If $\Delta = 0$, then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = 1.$$

(ii) If $\Delta \in (0,\infty)$, then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \frac{1}{\Delta} \int_1^{1+\Delta} \lambda^{-\beta}\,d\lambda.$$

(iii) If $\Delta = \infty$, then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = 0.$$

Proof. The positivity, monotonicity and convergence of (x_n) have been addressed in Lemma 7 and Proposition 4. By Theorem 18 we are guaranteed that $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ in all cases. As $\hat{T}_h - t_n = \sum_{j=n+1}^{\infty} h(x_j) \to 0$ as $x \to 0^+$ dividing by $\hat{T}_h - t_n$

and letting $n \to \infty$ yields

$$\begin{aligned}\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} &= \lim_{n\to\infty} \frac{\sum_{j=n+1}^{\infty} \int_{x_j}^{x_j+h(x_j)f(x_j)} 1/f(u)\,du}{\sum_{j=n+1}^{\infty} h(x_j)} \\ &= \lim_{j\to\infty} \frac{\int_{x_j}^{x_j+h(x_j)f(x_j)} 1/f(u)\,du}{h(x_j)} \\ &= \lim_{x\to 0^+} \frac{1}{h(x)} \int_x^{x+h(x)f(x)} \frac{1}{f(u)}\,du,\end{aligned}$$

by Toeplitz's Lemma. The proof for each part comes from combining the above limit and the relevant part of Lemma 10. □

4.7 Treatment of $RV_0(0)$

By virtue of condition (3.17) on h, if $f \in \mathrm{RV}_0(\beta)$, it would appear that $h \in \mathrm{RV}_0(1-\beta)$ and decaying sufficiently rapidly to zero are necessary conditions to recover all of the qualitative properties we want. However, we can allow $h(x) \to 0$ as $x \to 0^+$ as slowly as an $\mathrm{RV}_0(0)$ function if we wish to preserve the property of hitting zero in finite-time, provided an extra integral condition on h is satisfied. Naturally, we must check independently whether other asymptotic properties still hold.

When $h \in \mathrm{RV}_0(0)$, this represents as close as you can go to a constant step-size while still preserving positivity of the solution, bearing in mind that constant functions are in $\mathrm{RV}_0(0)$. Also, the step-size is larger when $h \in \mathrm{RV}_0(0)$ than when $h \in \mathrm{RV}_0(\alpha)$ for $\alpha > 0$.

The proof of the following two theorems is helped by the following lemma.

Lemma 11. *Suppose h obeys* (3.2) *and is increasing.*

(i) If $\int_1^\infty h(\exp(-e^x))\,dx < \infty$, then $\sum_{j=1}^\infty h(\exp(-e^{\lambda n})) < \infty$ for all $\lambda > 0$.

(ii) If $\int_1^\infty h(\exp(-e^x))\,dx = \infty$, then $\sum_{j=1}^\infty h(\exp(-e^{\lambda n})) = \infty$ for all $\lambda > 0$.

Proof. Let $\lambda n \le x \le \lambda(n+1)$: since h is increasing

$$h(\exp(-e^{\lambda(n+1)})) \le h(\exp(-e^x)) \le h(\exp(-e^{\lambda n})).$$

Hence

$$\lambda h(\exp(-e^{\lambda(n+1)})) \le \int_{\lambda n}^{\lambda(n+1)} h(\exp(-e^x))\,dx \le \lambda h(\exp(-e^{\lambda n})).$$

Suppose $\int_1^\infty h(\exp(-e^x))\,dx < \infty$: then

$$\lambda \sum_{n=1}^{\infty} h(\exp(-e^{\lambda(n+1)})) \le \int_{\lambda}^{\infty} h(\exp(-e^x))\,dx < \infty,$$

thus $\sum_{n=2}^{\infty} h(\exp(-e^{\lambda n})) < \infty$. Suppose $\int_1^\infty h(\exp(-e^x))\,dx = \infty$: then

$$\infty = \int_\lambda^\infty h(\exp(-e^x))\,dx \leq \lambda \sum_{n=1}^{\infty} h(\exp(-e^{\lambda n})),$$

thus $\sum_{n=1}^{\infty} h(\exp(-e^{\lambda n})) = \infty$. □

Theorem 21. *Suppose $f \in RV_0(\beta)$ where $\beta \in (0,1)$ and $h \in RV_0(0)$. Let (t_n) and $\hat{T}_h$ be defined by (1.44) and (3.33).*

(i) If $\int_1^\infty h(\exp(-e^x))\,dx < \infty$, then $t_n \to \hat{T}_h < \infty$ as $n \to \infty$.

(ii) If $\int_1^\infty h(\exp(-e^x))\,dx = \infty$, then $t_n \to \infty$ as $n \to \infty$.

Proof. Define

$$K(x_{n+1}) := x_{n+1} + h(x_{n+1})f(x_{n+1}) = x_n.$$

Since regularly varying functions are closed under multiplication and addition then K is regularly varying with index $\min(1, 0+\beta) = \beta$ when $\beta \in (0,1)$. Hence $K \in RV_0(\beta)$, so $1/K \in RV_0(-\beta)$ and so

$$\lim_{n\to\infty} \frac{\log(1/x_n)}{\log(1/x_{n+1})} = \lim_{n\to\infty} \frac{\log(1/K(x_{n+1}))}{\log(1/x_{n+1})} = \lim_{n\to\infty} \frac{-\log(1/K(x_{n+1}))}{\log x_{n+1}} = \beta.$$

Thus

$$\lim_{n\to\infty} \frac{\log(1/x_{n+1})}{\log(1/x_n)} = \frac{1}{\beta} > 1.$$

Define $y_n := \log(1/x_n)$. Thus

$$\lim_{n\to\infty} \frac{\log y_n}{n} = \lim_{n\to\infty} \left\{ \frac{\log y_0}{n} + \frac{1}{n} \sum_{j=1}^{n} \log\left(\frac{y_j}{y_{j-1}}\right) \right\} = \log(1/\beta) > 0.$$

Thus

$$\lim_{n\to\infty} \frac{\log\log(1/x_n)}{n} = \log(1/\beta). \tag{4.9}$$

Hence for every $\epsilon \in (0,1)$ there is $N(\epsilon) \in \mathbb{N}$ such that

$$(1-\epsilon)\cdot\log(1/\beta) < \frac{\log\log(1/x_n)}{n} < (1+\epsilon)\cdot\log(1/\beta).$$

Hence for $n \geq N(\epsilon)$

$$\exp(-e^{\lambda_+(\epsilon)n}) < x_n < \exp(-e^{\lambda_-(\epsilon)n}),$$

and

$$h(\exp(-e^{\lambda_+(\epsilon)n})) < h(x_n) < h(\exp(-e^{\lambda_-(\epsilon)n})),$$

where $\lambda_\pm(\epsilon) := (1\pm\epsilon)\log(1/\beta)$. Suppose $\int_1^\infty h(\exp(-e^x))\,dx < \infty$. Then for $\lambda > 0$

and for $n \geq N(\epsilon)$

$$\sum_{j=N(\epsilon)}^{\infty} h(x_j) \leq \sum_{j=N(\epsilon)}^{\infty} h(\exp(-e^{\lambda_-(\epsilon)j})) < \int_{\lambda}^{\infty} h(\exp(-e^x))\,dx < \infty,$$

so $t_n \to \hat{T}_h < \infty$. Suppose $\int_1^\infty h(\exp(-e^x))\,dx = \infty$. Then $\sum_{j=1}^\infty h(\exp\left(-e^{\lambda j}\right)) = \infty$ and for $n \geq N(\epsilon)$

$$\sum_{j=N(\epsilon)}^{\infty} h(x_j) \geq \sum_{j=N(\epsilon)}^{\infty} h(\exp(-e^{\lambda_+(\epsilon)j})) > \infty,$$

so $t_n \to \infty$ as $n \to \infty$. □

Notice that if $\beta \in (0,1)$, then $T_\xi < \infty$, so the condition on h in (ii) yields spurious behaviour. Theorem 21 does not cover the case when $\beta = 0$. The next result shows that if $f \in \text{RV}_0(0)$ and h tends to zero more rapidly than an $\text{RV}_0(0)$ function, then we correctly predict finite-time stability.

Theorem 22. *Suppose $h \in RV_0(\beta)$ where $\beta \in (0,1)$ and $f \in RV_0(0)$. Let (t_n) and $\hat{T}_h$ be defined by* (1.44) *and* (3.33)*. Then $t_n \to \hat{T}_h < \infty$ as $n \to \infty$.*

Proof. Since $h \in \text{RV}_0(\beta)$, $\int_1^\infty h(\exp(-e^x))\,dx < \infty$. Define

$$K(x_{n+1}) := x_{n+1} + h(x_{n+1})f(x_{n+1}) = x_n.$$

Since regularly varying functions are closed under multiplication and addition then K is regularly varying with index $\min(1, 0+\beta) = \beta$ when $\beta \in (0,1)$. Hence $K \in RV_0(\beta)$. Therefore

$$\lim_{n\to\infty} \frac{\log(1/x_n)}{\log(1/x_{n+1})} = \lim_{n\to\infty} \frac{\log x_n}{\log x_{n+1}} = \lim_{n\to\infty} \frac{\log K(x_{n+1})}{\log x_{n+1}} = \beta.$$

Hence

$$\lim_{n\to\infty} \frac{\log(1/x_{n+1})}{\log(1/x_n)} = \frac{1}{\beta} > 1.$$

Now $\int_1^\infty h(\exp(-e^x))\,dx < \infty$ and arguing by Theorem 21 from (4.9) to the end and using the finiteness of $\int_1^\infty h(\exp(-e^x))\,dx$ we may use Theorem 21 part (i) to conclude $t_n \to \hat{T}_h < \infty$ as $n \to \infty$. □

The condition on μ implies that

$$\mu = \lim_{n\to\infty} \frac{h(x_n)}{h(x_{n+1})}.$$

Since (x_n) is decreasing and h increasing, it is clear that $\mu \geq 1$. If $\mu > 1$, then the step-size decays to zero at least geometrically fast, so there is sufficient computational effort to recover the presence of a finite hitting time of zero. On the other hand, if $\mu = 1$, it may be that insufficient effort has been expended and the solution may remain spuriously positive. This is decided by the finiteness of the integral J in part (iii).

Theorem 23. *Suppose f obeys (3.32) and $f \in RV_0(\beta)$, $\beta \in [0,1]$ while $x \mapsto h(x)f(x)$ is increasing and $x \mapsto h(x)$ is increasing. Suppose also there exists $\mu \in [0,\infty]$ such that*

$$\mu := \lim_{x\to 0^+} \frac{h(x+h(x)f(x))}{h(x)}.$$

Then:

(i) $\mu \geq 1$.

(ii) If $\mu \in (1,\infty]$, then $t_n \to \hat{T}_h < \infty$ as $n \to \infty$.

(iii) If $\mu = 1$, let h, f be in $C^1(0,\delta)$ for some $\delta > 0$, define

$$J := \int_0^\delta \frac{h(K(z))}{h(K(z)) - h(z)} \cdot (h \circ K)'(z)\, dz,$$

where $K(z) := z + h(z)f(z)$.

(a) If $J = \infty$, then $t_n \to \infty$ as $n \to \infty$.

(b) If $J < \infty$, then $t_n \to \hat{T}_h < \infty$ as $n \to \infty$.

Proof. Note that K is increasing since hf is increasing and $x_{n+1} = x_n - h(x_{n+1})f(x_{n+1})$ so $K(x_{n+1}) = x_n$. Hence $x_{n+1} = K^{-1}(x_n)$. Therefore $s_n := h(x_n)$ obeys

$$s_{n+1} = h(x_{n+1}) = \left(h \circ K^{-1}\right)(x_n) = \left(h \circ K^{-1} \circ h^{-1}\right)(s_n),$$

because h is increasing. Note that (s_n) is decreasing, as (x_n) is decreasing and h is increasing. Then

$$s_{n+1} = s_n - \left(s_n - \left(h \circ K^{-1} \circ h^{-1}\right)(s_n)\right) = s_n - k(s_n),$$

where $k(x) := x - (h \circ K^{-1} \circ h^{-1})(x)$ with $k(0) = 0$. Note that $K(x) > x$, so $x > K^{-1}(x)$ for all $x > 0$. Hence $h(x) > h(K^{-1}(x))$, or $x = h(h^{-1}(x)) > h(K^{-1}(h^{-1}(x)))$ so $x > (h \circ K^{-1} \circ h^{-1})(x)$, so $k(x) > 0$. Furthermore

$$\begin{aligned}\lim_{x\to 0^+} \frac{k(x)}{x} = \lim_{x\to 0^+} \left\{1 - \frac{h(K^{-1}(h^{-1}(x)))}{x}\right\} &= 1 - \lim_{z\to 0^+} \frac{h(z)}{h(K(z))} \\ &= 1 - \lim_{z\to 0^+} \frac{h(z)}{h(z+h(z)f(z))} \\ &= 1 - \frac{1}{\mu}.\end{aligned}$$

Part (i) is true, because h is increasing, so $h(x + h(x)f(x)) > h(x)$. For part (ii), if $\mu \in (1, \infty]$, then

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = 1 - \lim_{n\to\infty} \frac{k(s_n)}{s_n} = 1 - \left(1 - \frac{1}{\mu}\right) = \frac{1}{\mu},$$

where $1/\mu \in [0, 1)$. Thus (s_n) is dominated by a geometric sequence with common ratio < 1. Hence (s_n) is summable. Therefore $\lim_{n\to\infty} t_n = \hat{T}_h = \sum_{j=0}^{\infty} h(x_{j+1}) = \sum_{j=0}^{\infty} s_{j+1} < \infty$. In part (iii), we have $s_{n+1}/s_n \to 1$ as $n \to \infty$. Since h and f are in $C^1(0, \delta)$, then $K \in C^1(0, \delta)$. Therefore as $k(x)/x \to 0$ as $x \to 0^+$, $k(0) = 0$, and $k(x) = x - (h \circ K^{-1} \circ h^{-1})(x)$, we have that $k \in C^1(0, \delta)$ with $k'(x) \to 0$ as $x \to 0^+$. We now determine the asymptotic behaviour of (s_n). Define

$$L(x) := \int_x^1 \frac{1}{k(u)}\, du.$$

Since $k(x)/x \to 0$, we have $L(x) \to \infty$, as $x \to 0^+$. By the Mean Value Theorem there is $\theta_n \in (0, 1)$ such that

$$L(s_{n+1}) = L(s_n - k(s_n)) = L(s_n) + L'(s_n - \theta_n k(s_n)) \cdot -k(s_n),$$

or

$$L(s_{n+1}) - L(s_n) = \frac{k(s_n)}{k(s_n - \theta_n k(s_n))}.$$

Again by the Mean Value Theorem there is $\bar{\theta}_n \in (0, \theta_n)$ such that

$$k(s_n - \theta_n k(s_n)) = k(s_n) + k'(s_n - \bar{\theta}_n k(s_n)) \cdot -\theta_n k(s_n).$$

We have

$$\lim_{n\to\infty} \frac{k(s_n - \theta_n k(s_n))}{k(s_n)} = 1 - \lim_{n\to\infty} \theta_n k'(s_n - \bar{\theta}_n k(s_n)) = 1,$$

since $k'(x) \to 0$ as $x \to 0^+$. Hence

$$\lim_{n\to\infty} (L(s_{n+1}) - L(s_n)) = 1.$$

Thus

$$\lim_{n\to\infty} \frac{L(s_n)}{n} = 1.$$

Hence, for every $\epsilon \in (0, 1)$, there is $N(\epsilon) \in \mathbb{N}$ such that for $n \geq N(\epsilon)$

$$(1 - \epsilon) \cdot n < L(s_n) < (1 + \epsilon) \cdot n.$$

As L is decreasing since $L'(x) = -1/k(x) < 0$ then L^{-1} is decreasing and hence for

$n \geq N(\epsilon)$

$$L^{-1}((1-\epsilon)n) > s_n > L^{-1}((1+\epsilon)n).$$

Now, suppose that $\int_1^\infty L^{-1}(x)\,dx < \infty$. Then, as L^{-1} is decreasing by the Improper Integral Test, we have $\sum_{n=1}^\infty L^{-1}((1-\epsilon)n) < \infty$. Hence

$$\sum_{n=N(\epsilon)}^{\infty} s_n < \sum_{n=N(\epsilon)}^{\infty} L^{-1}((1-\epsilon)n) < \infty,$$

and therefore $t_n \to \hat{T}_h < \infty$ as $n \to \infty$. If $\int_1^\infty L^{-1}(x)\,dx = \infty$, because L^{-1} is decreasing, by the Improper Integral Test, we have $\sum_{n=1}^\infty L^{-1}((1+\epsilon)n) = \infty$. Then

$$\sum_{n=N(\epsilon)}^{\infty} s_n > \sum_{n=N(\epsilon)}^{\infty} L^{-1}((1+\epsilon)n) = \infty,$$

and so $t_n \to \infty$ as $n \to \infty$. Define

$$I := \int_C^\infty L^{-1}(x)\,dx$$

We note that $\lim_{x\to 0^+} L(x) = \infty$, so $L^{-1}(\infty) = 0$.

$$\begin{aligned} I = \int_{L^{-1}(C)}^{L^{-1}(\infty)} uL'(u)\,du &= \int_0^{L^{-1}(C)} \frac{u}{k(u)}\,du \\ &= \int_0^{L^{-1}(C)} \frac{u}{u - (h\circ K^{-1}\circ h^{-1})(u)}\,du \\ &= \int_{h^{-1}(0)}^{h^{-1}(L^{-1}(C))} \frac{h(v)h'(v)}{h(v) - (h\circ K^{-1})(v)}\,dv \\ &= \int_0^{K^{-1}(h^{-1}(L^{-1}(C)))} \frac{h(K(z))}{h(K(z)) - h(z)} \cdot (h\circ K)'(z)\,dz =: J \end{aligned}$$

Therefore, the finiteness of J and I are equivalent, and part (iii) is proven. □

Remark 17. In part (ii) of the theorem, we have a result that the infiniteness of a certain integral is equivalent to a (simulated) solution of an ODE remaining positive for all time, while if that integral is finite the (simulated) solution tends to zero in finite-time. Therefore the condition on J if highly reminiscent of an Osgood criterion.

It is interesting to ask whether a constant step-size is covered by the framework of Theorem 23; clearly $h(x) = \Delta\,\forall\, x$ does not pass the monotonicity restriction, so the theorem is not directly applicable, but if we perturb h slightly so that an asymptotically constant (but increasing) step-size is assumed, we can show that $J = \infty$, as would be anticipated. Supposing that $f \in C^1$, we have

$$J = \int_0^\delta \frac{1+f(x)}{f(x)} \left(1 + \Delta f'(x) + x f'(x) + f(x)\right) dx,$$

and if for instance f is increasing, we have

$$J \geq \int_0^\delta \Delta \frac{f'(x)}{f(x)} \, dx = \infty,$$

so $J = \infty$. $\square$

Finally, if $\Delta = \infty$ and $f \in RV_0(0)$, we may still be able to reproduce all acceptable aspects of the asymptotic behaviour. Theorem 24, which is supported by Lemma 12, makes this precise. If λ in (4.10) is finite, we get acceptable behaviour but if $\lambda = \infty$, we do not. This once again places sharp restrictions on the relative size of h and f.

Lemma 12. *Suppose f obeys (3.1), $f \in RV_0(0)$ and is increasing. Suppose also $h(x)f(x)/x \to \infty$ as $x \to 0^+$, $h(x)f(x) \to 0$ as $x \to 0^+$ and*

$$\lim_{x \to 0^+} \frac{f(h(x)f(x))}{f(x)} =: \lambda. \tag{4.10}$$

Then

$$\lim_{x \to 0^+} \frac{1}{h(x)} \int_x^{x+h(x)f(x)} \frac{1}{f(u)} \, du = \frac{1}{\lambda}.$$

Remark 18. $\lambda \in [1, \infty]$ because $h(x)f(x) > x$ for all x sufficiently small and f is increasing. $\square$

Proof. Define $\Delta(x) := h(x)f(x)/x$, $\Delta_*(x) := \Delta(1/x)$, $\tilde{f}(x) := 1/f(x)$ and $f_*(x) := \tilde{f}(1/x) = 1/f(1/x)$. Since $\Delta(x) = h(x)f(x)/x \to \infty$ as $x \to 0$, so $\Delta_*(x) = \Delta(1/x) \to \infty$ as $x \to \infty$ and $x\Delta(x) = h(x)f(x) \to 0$ as $x \to 0^+$. Also $f_*(x) = 1/f(1/x) \to \infty$ as $x \to \infty$ and f_* is increasing. We have already used the identity

$$\frac{1}{h(x)} \int_x^{x+h(x)f(x)} \frac{1}{f(u)} \, du = \frac{1}{\Delta(x)} \int_1^{1+\Delta(x)} \frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} \, d\lambda =: I(x).$$

We want $I(x) \to \lambda$ as $x \to 0^+$ or $I_*(x) = I(1/x) \to \lambda$ as $x \to \infty$. Thus we may write

$$\begin{aligned} I_*(x) = \frac{1}{\Delta(1/x)} \int_1^{1+\Delta(1/x)} \frac{\tilde{f}(\lambda/x)}{\tilde{f}(1/x)} \, d\lambda &= \frac{1}{\Delta_*(x)} \int_1^{1+\Delta_*(x)} \frac{\tilde{f}(1/(\lambda^{-1}x))}{\tilde{f}(1/x)} \, d\lambda \\ &= \frac{1}{\Delta_*(x)} \int_1^{1+\Delta_*(x)} \frac{f_*(\lambda^{-1}x)}{f_*(x)} \, d\lambda. \end{aligned}$$

Since $\Delta_*(x) \to \infty$ as $x \to \infty$, for every $\epsilon \in (0,1)$, there is $x_1(\epsilon) > 0$ such that $\Delta_*(x) > 1/\epsilon^2$, for all $x > x_1(\epsilon)$. Hence $\epsilon\Delta_*(x) > \epsilon^2\Delta_*(x) > 1$ for $x > x_1(\epsilon)$. For $1 \leq \lambda$,

$\lambda^{-1}x < x$, so as f_* is increasing, $f_*(\lambda^{-1}x) < f_*(x)$ and so for $x > x_1(\epsilon)$ we get

$$\begin{aligned}
I_*(x) &= \frac{1}{\Delta_*(x)}\left(\int_1^{\epsilon\Delta_*(x)} \frac{f_*(\lambda^{-1}x)}{f_*(x)}\,d\lambda + \int_{\epsilon\Delta_*(x)}^{\Delta_*(x)} \frac{f_*(\lambda^{-1}x)}{f_*(x)}\,d\lambda + \int_{\Delta_*(x)}^{1+\Delta_*(x)} \frac{f_*(\lambda^{-1}x)}{f_*(x)}\,d\lambda\right) \\
&\le \frac{1}{\Delta_*(x)}\left(\int_1^{\epsilon\Delta_*(x)} 1\,d\lambda + \int_{\Delta_*(x)}^{1+\Delta_*(x)} 1\,d\lambda + \int_{\epsilon\Delta_*(x)}^{\Delta_*(x)} \frac{f_*(\lambda^{-1}x)}{f_*(x)}\,d\lambda\right) \\
&= \frac{\epsilon\Delta_*(x)-1}{\Delta_*(x)} + \frac{1+\Delta_*(x)-\Delta_*(x)}{\Delta_*(x)} + \frac{1}{\Delta_*(x)}\int_{\epsilon\Delta_*(x)}^{\Delta_*(x)} \frac{f_*(\lambda^{-1}x)}{f_*(x)}\,d\lambda \\
&= \epsilon + \frac{1}{\Delta_*(x)}\int_{\epsilon\Delta_*(x)}^{\Delta_*(x)} \frac{f_*(\lambda^{-1}x)}{f_*(x)}\,d\lambda.
\end{aligned}$$

For $x > x_1(\epsilon)$ the last integral above can be expressed as follows:

$$\frac{1}{\Delta_*(x)}\int_{\epsilon\Delta_*(x)}^{\Delta_*(x)} \frac{f_*(\lambda^{-1}x)}{f_*(x)}\,d\lambda = \frac{f_*\left(x/\Delta_*(x)\right)}{f_*(x)} \cdot \int_\epsilon^1 \frac{f_*\left(x/(\mu\Delta_*(x))\right)}{f_*\left(x/\Delta_*(x)\right)}\,d\mu.$$

Since $x/\Delta_*(x) = x/\Delta(1/x)$, then

$$\lim_{x\to\infty} \frac{x}{\Delta_*(x)} = \lim_{y\to 0^+} \frac{1}{y\Delta(y)} = \infty.$$

Hence

$$\lim_{x\to\infty}\int_\epsilon^1 \frac{f_*\left(x/(\mu\Delta_*(x))\right)}{f_*\left(x/\Delta_*(x)\right)}\,d\mu = \lim_{z\to\infty}\int_\epsilon^1 \frac{f_*\left(z/\mu\right)}{f_*(z)}\,dz = (1-\epsilon)\cdot 1 = 1-\epsilon,$$

by the Uniform Convergence Theorem applied to $f_* \in RV_\infty(0)$. Furthermore

$$\begin{aligned}
\lim_{x\to\infty} \frac{f_*\left(x/\Delta_*(x)\right)}{f_*(x)} \lim_{y\to 0^+} \frac{f_*\left(1/y\cdot 1/\Delta_*(1/y)\right)}{f_*\left(1/y\right)} &= \lim_{y\to 0^+} \frac{\tilde{f}\left(y\Delta_*(1/y)\right)}{\tilde{f}(y)} \\
&= \lim_{y\to 0^+} \frac{\tilde{f}\left(y\Delta(y)\right)}{\tilde{f}(y)} \\
&= \lim_{y\to 0^+} \frac{1/f(h(y)f(y))}{1/f(y)} = \frac{1}{\lambda}.
\end{aligned}$$

Combining these results gives

$$\begin{aligned}
\lim_{x\to\infty} \frac{1}{\Delta_*(x)}\int_{\epsilon\Delta_*(x)}^{\Delta_*(x)} \frac{f_*(\lambda^{-1}x)}{f_*(x)}\,d\lambda &= \lim_{x\to\infty}\left(\frac{f_*\left(x/\Delta_*(x)\right)}{f_*(x)} \cdot \int_\epsilon^1 \frac{f_*\left(x/\mu\Delta_*(x)\right)}{f_*\left(x/\Delta_*(x)\right)}\,d\mu\right) \\
&= \lim_{x\to\infty} \frac{f_*\left(x/\Delta_*(x)\right)}{f_*(x)} \cdot \lim_{x\to\infty}\int_\epsilon^1 \frac{f_*\left(x/\mu\Delta_*(x)\right)}{f_*\left(x/\Delta_*(x)\right)}\,d\mu \\
&= \frac{1-\epsilon}{\lambda}.
\end{aligned}$$

Thus

$$\frac{1-\epsilon}{\lambda} \le \liminf_{x\to\infty} I_*(x) \le \limsup_{x\to\infty} I_*(x) \le \epsilon + \frac{1-\epsilon}{\lambda}.$$

Letting $\epsilon \to 0$ yields

$$\lim_{x\to\infty} I_*(x) = \lim_{x\to 0^+} I(x) = \frac{1}{\lambda},$$

as claimed. □

Theorem 24. *Suppose f obeys* (1.7), (3.1), (3.32), (4.10), $f \in RV_0(0)$ *and is increasing. Suppose also h obeys* (3.17) *with $\Delta \in [0,\infty]$. Let t_n and $\hat{T}_h$ be defined by* (1.44) *and* (3.33).

(i) If $\lambda \in [1,\infty)$, then $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \frac{1}{\lambda}.$$

(ii) If $\lambda = \infty$, then either: (a) $t_n \to \infty$ as $n \to \infty$; or (b) $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = 0.$$

Proof. Since f obeys (1.7) then

$$\bar{F}(x_n) = \sum_{j=n+1}^{\infty} \int_{x_j}^{x_j+h(x_j)f(x_j)} \frac{1}{f(u)}\,du =: \sum_{j=n+1}^{\infty} a_j < \infty.$$

Then by Lemma 12

$$\lim_{j\to\infty} \frac{a_j}{h(x_j)} = \lim_{j\to\infty} \frac{1}{h(x_j)} \int_{x_j}^{x_j+h(x_j)f(x_j)} \frac{1}{f(u)}\,du = \frac{1}{\lambda}.$$

Since (a_j) is summable, $(h(x_j))$ is summable. Hence $t_n \to \hat{T}_h := \sum_{j=0}^{\infty} h(x_j) < \infty$ as $n \to \infty$. Thus $\hat{T}_h - t_n = \sum_{j=0}^{\infty} h(x_{j+1}) - \sum_{j=0}^{n-1} h(x_{j+1}) = \sum_{j=n}^{\infty} h(x_{j+1}) = \sum_{j=n+1}^{\infty} h(x_j)$. Thus by Toeplitz's Lemma

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \lim_{n\to\infty} \frac{\sum_{j=n+1}^{\infty} a_j}{\sum_{j=n+1}^{\infty} h(x_j)} = \lim_{j\to\infty} \frac{a_j}{h(x_j)} = \frac{1}{\lambda},$$

as claimed. For part (ii), suppose $t_n \to \hat{T}_h$ as $n \to \infty$. Then once again,

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \lim_{n\to\infty} \frac{\sum_{j=n+1}^{\infty} a_j}{\sum_{j=n+1}^{\infty} h(x_j)} = \lim_{j\to\infty} \frac{a_j}{h(x_j)} = \frac{1}{\lambda} = 0,$$

when $\lambda = \infty$. Otherwise, $t_n \to \infty$ as $n \to \infty$, as claimed. □

Clearly in the case when $\lambda = \infty$ we do not recover the appropriate asymptotic behaviour: either the finite-time stability is recovered but the rate is incorrect as in part (b); or the finite-time stability is not recovered as in part (a). On the other hand we see in part (i) that for finite λ the finite-time stability is recovered and the exponent is $1/\lambda$ rather than unity.

Scrutinising Lemma 12 one might question whether it is possible to find a function h which satisfies all the hypotheses for a given f. Lemma 15 shows that such an h always exists and we show this by constructing a suitable h in the next result. It relies on two known results from slow variation theory. There are mentioned next. The following result is Theorem 2.3.1. in [12].

Lemma 13. *Let $l \in RV_\infty(0)$. If there exists $\lambda_0 > 1$ so that*

$$\lim_{x\to\infty}\left(\frac{l(\lambda_0 x)}{l(x)} - 1\right)\log\eta(x) = 0, \tag{4.11}$$

and $x \mapsto x^\gamma \eta(x)$ is increasing, then

$$\lim_{x\to\infty}\frac{l(x\eta(x)^\nu)}{l(x)} = 1,$$

uniformly in $\nu \in [0, c]$, where $0 < c < 1/\gamma$.

The following result is Proposition 2.3.2. in [12].

Lemma 14. *Suppose $\eta > 1$, η is increasing and $l \in C^1$ with*

$$\frac{l'(x)x}{l(x)} = o\left(\frac{1}{\log\eta(x)}\right).$$

Then (4.11) holds.

Lemma 15. *Suppose f obeys (3.1), $f \in RV_0(0)$ and is increasing. Suppose also $x \mapsto xf'(x)/f(x)$ is increasing. Then we can find h so that:*

(i)

$$\lim_{x\to 0^+} h(x)f(x) = 0.$$

(ii)

$$\lim_{x\to 0^+}\frac{h(x)f(x)}{x} = \infty.$$

(iii)

$$\lim_{x\to 0^+}\frac{f(h(x)f(x))}{f(x)} = 1. \tag{4.12}$$

Proof. Define $\tilde{f}(x) := 1/f(x)$, $f_*(x) := \tilde{f}(1/x) = 1/f(1/x) = (f(1/x))^{-1}$. Then

$$\frac{xf_*'(x)}{f_*(x)} = \frac{1/xf'(1/x)}{f(1/x)} = \rho\left(1/x\right),$$

where we defined $\rho(x) := xf'(x)/f(x)$. Then $x \mapsto \rho(x)$ is increasing and so $x \mapsto xf_*'(x)/f_*(x) = \rho(1/x)$ is decreasing with

$$\lim_{x\to\infty} \frac{xf_*'(x)}{f_*(x)} = \lim_{x\to\infty} \rho\left(\tfrac{1}{x}\right) = \lim_{y\to 0^+} \rho(y) =: \rho^*.$$

This implies that $f_* \in RV_\infty(\rho^*)$ and since $f \in RV_0(0)$ so we must have $\rho^* = 0$. Define $\mu(x) := f_*(x)/f_*'(x)$. Now $x \mapsto f_*(x)/(xf_*'(x)) =: g(x)$ is increasing, so $x \mapsto \mu(x)$ is increasing. Moreover,

$$\lim_{x\to\infty} \frac{\mu(x)}{x} = \frac{f_*(x)}{xf_*'(x)} = \infty.$$

Hence, $\mu(x) \to \infty$ as $x \to \infty$. Thus μ^{-1} exists and $\mu^{-1}(x) \to \infty$ as $x \to \infty$. Define $n_*(x) := x/\mu^{-1}(x)$. Since $\mu^{-1}(x) \to \infty$ as $x \to \infty$

$$\lim_{x\to\infty} \frac{\mu(\mu^{-1}(x))}{\mu^{-1}(x)} = \infty.$$

Thus $x/\mu^{-1}(x) \to \infty$ as $x \to \infty$. Therefore $n_*(x) \to \infty$ as $x \to \infty$. Also, $n_*(x)/x = 1/\mu^{-1}(x) \to 0$ as $x \to \infty$. Next define $m(x) := \mu^{-1}(x)$, $n(x) := n_*(1/x)$ and $h(x) := xn(x)/f(x)$ so $h(x)f(x) = xn(x)$. Thus

$$\lim_{x\to 0^+} h(x)f(x) = \lim_{x\to 0^+} xn(x) = \lim_{x\to 0^+} xn_*\left(\tfrac{1}{x}\right) = \lim_{y\to\infty} \frac{n_*(y)}{y} = 0,$$

and

$$\lim_{x\to 0^+} \frac{h(x)f(x)}{x} = \lim_{x\to 0^+} n(x) = \lim_{x\to 0^+} n_*\left(\tfrac{1}{x}\right) = \lim_{y\to\infty} n_*(y) = \infty.$$

Next, g is increasing and $g(x) \to \infty$ as $x \to \infty$. Thus

$$\frac{xf_*'(x)}{f_*(x)} = \frac{1}{g(x)} = o\left(\frac{1}{\log g(x)}\right), \quad \text{as } x \to \infty.$$

By Lemma 14, we have

$$\lim_{x\to\infty} \left(\frac{f_*(\lambda x)}{f_*(x)} - 1\right) \log g(x) = 0.$$

Let $\delta = 1$, $\gamma = 1/2$, $\Delta = 3/2$. Then $x \mapsto x^{1/2} g(x) = x^\gamma g(x)$ is increasing, because g is increasing. Thus $0 < \Delta = 3/2 < 2 = 1/\gamma$ and $\delta = 1 \in [0, 3/2] = [0, \Delta]$. Therefore by Lemma 13

$$\lim_{x\to\infty} \frac{f_*(xg(x))}{f_*(x)} = 1.$$

Now $xg(x) = f_*(x)/f_*'(x) = \mu(x)$. Thus

$$\lim_{x\to\infty} \frac{f_*(\mu(x))}{f_*(x)} = 1.$$

Since $m(x) = \mu^{-1}(x)$ and $\mu(x) \to \infty$ and $\mu^{-1}(x) \to \infty$ as $x \to \infty$, we have

$$\lim_{x\to\infty} \frac{f_*(x)}{f_*(m(x))} = \lim_{x\to\infty} \frac{f_*(x)}{f_*(\mu^{-1}(x))} = \lim_{x\to\infty} \frac{f_*(\mu(\mu^{-1}(x)))}{f_*(\mu^{-1}(x))} = 1.$$

Now $n_*(x) = x/\mu^{-1}(x) = x/m(x)$, so $m(x) = x/n_*(x)$. Thus

$$\lim_{x\to\infty} \frac{f_*(x)}{f_*(x/n_*(x))} = 1.$$

Hence to get the limit in (4.12) we write

$$\begin{aligned} 1 = \lim_{x\to 0^+} \frac{f_*\left(1/x\right)}{f_*\left((1/x)/n_*(1/x)\right)} = \lim_{x\to 0^+} \frac{1/f(x)}{f_*\left(1/x \cdot 1/n(x)\right)} &= \lim_{x\to 0^+} \frac{1/f(x)}{f_*\left(1/(xn(x))\right)} \\ &= \lim_{x\to 0^+} \frac{1/f(x)}{1/f(xn(x))}. \end{aligned}$$

Thus

$$\lim_{x\to 0^+} \frac{f(h(x)f(x))}{f(x)} = \lim_{x\to 0^+} \frac{f(xn(x))}{f(x)} = 1,$$

as claimed. □

Remark 19. Lemma 15 demonstrates that we can have $h(x)f(x)/x \to \infty$ as $x \to 0^+$ for a wide class of $f \in RV_0(0)$ and still have $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = 1,$$

because for the f in Lemma 15, we have $\lambda = 1$. □

Remark 20. Limits other than $\lambda = 1$ are possible. For instance, let $f(x) = 1/\log(1/x)$ and for $\lambda > 1$, let $h_\lambda(x) = x^{1/\lambda}\log(1/x)$. Then

(i)
$$\lim_{x\to 0^+} h_\lambda(x) = \lim_{x\to 0^+} x^{1/\lambda}\log(1/x) = \lim_{y\to\infty} \frac{\log y}{y^{1/\lambda}} = 0.$$

(ii)
$$\lim_{x\to 0^+} h_\lambda(x)f(x) = \lim_{x\to 0^+} x^{1/\lambda} = 0.$$

(iii)
$$\lim_{x\to 0^+} \frac{h_\lambda(x)f(x)}{x} = \lim_{x\to 0^+} x^{-(1-1/\lambda)} = \infty.$$

(iv)

$$\frac{f(h_\lambda(x)f(x))}{f(x)} = \frac{f(x^{1/\lambda})}{f(x)} = \frac{1/\log\left(1/x^{1/\lambda}\right)}{1/\log(1/x)} = \frac{\log(1/x)}{\log\left(1/x^{1/\lambda}\right)} = \frac{\log(1/x)}{1/\lambda\cdot\log(1/x)} = \lambda.$$

Thus

$$\lim_{x\to 0^+}\frac{f(h_\lambda(x)f(x))}{f(x)} = \lambda \in (1,\infty).$$

We now give an example when $\lambda = 1$. Defining $h_1(x) := x\left(\log\left(1/x\right)\right)^2$ and $f(x)$ as before gives:

(i)

$$\lim_{x\to 0^+} h_1(x) = \lim_{x\to 0^+} x\left(\log\left(1/x\right)\right)^2 = \lim_{y\to\infty}\frac{(\log y)^2}{y} = 0.$$

(ii)

$$\lim_{x\to 0^+} h_1(x)f(x) = \lim_{x\to 0^+} x\log(1/x) = \lim_{y\to\infty}\frac{\log y}{y} = 0.$$

(iii)

$$\lim_{x\to 0^+}\frac{h_1(x)f(x)}{x} = \lim_{x\to 0^+}\log(1/x) = \infty.$$

(iv)

$$\begin{aligned}\frac{f(h_1(x)f(x))}{f(x)} &= \frac{f(x\log(1/x))}{f(x)} = \frac{1/\log\left(1/(x\log(1/x))\right)}{1/\log(1/x)}\\ &= \frac{\log(1/x)}{\log(1/x)-\log\log(1/x)}.\end{aligned}$$

Thus

$$\lim_{x\to 0^+}\frac{f(h_1(x)f(x))}{f(x)} = \lim_{x\to 0^+}\frac{\log(1/x)}{\log(1/x)-\log\log(1/x)} = \lim_{y\to\infty}\frac{1}{1-\log y/y} = 1.$$

Finally we give an example when $\lambda = \infty$. Define

$$h(x) := \frac{\log\log(1/x)}{\log(1/x)}.$$

Then

(i)

$$\lim_{x\to 0^+} h(x) = \lim_{x\to 0^+}\frac{\log\log(1/x)}{\log(1/x)} = \lim_{y\to\infty}\frac{\log y}{y} = 0.$$

(ii)

$$\lim_{x\to 0^+} h(x)f(x) = \frac{\log\log(1/x)}{\left(\log(1/x)\right)^2} = \lim_{y\to\infty}\frac{\log y}{y^2} = 0.$$

(iii)

$$\lim_{x\to 0^+}\frac{h(x)f(x)}{x}=\lim_{x\to 0^+}\frac{\log\log(1/x)}{x\left(\log(1/x)\right)^2}=\lim_{y\to\infty}\frac{e^y\log y}{y^2}=\infty.$$

(iv)

$$\begin{aligned}\frac{f(x)}{f(h(x)f(x))}=\frac{1/\log(1/x)}{1/\log\left(1/(h(x)f(x))\right)} &= \frac{\log\left(1/h(x)\right)}{\log\left(1/x\right)}+\frac{\log\left(1/f(x)\right)}{\log\left(1/x\right)}\\ &= \frac{\log\left(1/h(x)\right)}{\log\left(1/x\right)}+\frac{\log\log\left(1/x\right)}{\log\left(1/x\right)}.\end{aligned}$$

Clearly

$$\lim_{x\to 0^+}\frac{\log\log\left(1/x\right)}{\log\left(1/x\right)}=0.$$

Also

$$\begin{aligned}\lim_{x\to 0^+}\frac{\log\left(1/h(x)\right)}{\log\left(1/x\right)} &= \lim_{x\to 0^+}\frac{\log\left(\log(1/x)/\log\log(1/x)\right)}{\log(1/x)}\\ &= \lim_{y\to\infty}\frac{\log\left(y/\log y\right)}{y}\\ &= \lim_{y\to\infty}\frac{\log y-\log\log y}{y}=0.\end{aligned}$$

Thus

$$\lim_{x\to 0^+}\frac{f(x)}{f(h(x)f(x))}=\lim_{x\to 0^+}\frac{\log\left(1/h(x)\right)}{\log\left(1/x\right)}+\frac{\log\log\left(1/x\right)}{\log\left(1/x\right)}=0.$$

Hence

$$\lim_{x\to 0^+}\frac{f(h(x)f(x))}{f(x)}=\infty,$$

as needed. □

Lemma 15 does not give a recipe for explicitly constructing a h in terms of f so that the condition of Theorem 24 are fulfilled. Furthermore, Lemma 15 generates a h for which the limit λ is one and such a h may constitute an overly-conservative step-size. In the following theorem we attempt to determine directly a class of functions Δ (and therefore h) in terms of f which fulfill the conditions of Lemma 15 with non-unit λ. Such functions Δ will constitute optimally chosen step-sizes for preserving the asymptotic behaviour of the finite-time stability.

Theorem 25. *Suppose f obeys (3.1), $f'\in RV_0(-1)$ and f is increasing. Let Δ obey*

$$xf'(x)/f(x)\cdot\log\Delta(x)\to c>0 \text{ as } x\to 0^+,\ x\mapsto\Delta(x)$$
$$\text{is decreasing as } x\to 0^+ \text{ and } \Delta(x)\to\infty \text{ as } x\to 0^+, \quad (4.13)$$

and

$$x\Delta(x) \to 0 \text{ as } x \to 0^+. \tag{4.14}$$

Then

(i) f *obeys*

$$\liminf_{x\to 0^+} \frac{f(x\Delta(x))}{f(x)} \geq e^c > 1.$$

(ii) If in addition

$$x \mapsto \log \Delta(e^{-x}) \text{ is self-neglecting,} \tag{4.15}$$

then

$$\lim_{x\to 0^+} \frac{f(x\Delta(x))}{f(x)} = e^c. \tag{4.16}$$

(iii) If for some $\gamma \in (0,1)$, $x \mapsto x^\gamma \Delta(x)$ *is increasing, then*

$$\lim_{x\to 0^+} \frac{f(x\Delta(x))}{f(x)} = e^c.$$

Remark 21. Recall that a function $\bar{g}$ said to be self-neglecting if $\bar{g}(x) = o(x)$ as $x \to \infty$ and $\bar{g}(x + t\bar{g}(x)) \sim \bar{g}(x)$, $\forall\, t \in \mathbb{R}$ as $x \to \infty$. □

Proof of Theorem 25. Define

$$f_1(x) := \frac{xf'(x)}{f(x)}, \tag{4.17}$$

so $f_1(x) \to 0$ as $x \to 0^+$. Define $\tilde{f}(x) := 1/f(1/x)$ and $\tilde{\Delta}(x) := \Delta(1/x)$. We have that $\tilde{f}(x) \to \infty$ as $x \to \infty$, $\tilde{f} \in RV_\infty(0)$ and with $x = 1/y$, $f_1(1/y)\log\tilde{\Delta}(y) \to c$ as $y \to \infty$. Also

$$\frac{x\tilde{f}'(x)}{\tilde{f}(x)} = \frac{1/x \cdot f'(1/x)}{f(1/x)} = f_1\left(\tfrac{1}{x}\right).$$

Thus

$$\lim_{y\to\infty} \frac{y\tilde{f}'(y)}{f(y)} \log\tilde{\Delta}(y) = c > 0. \tag{4.18}$$

By (4.13) since $f_1 \in RV_0(0)$, $x \mapsto \log\Delta(x) \in RV_0(0)$ and therefore $y \mapsto \log\tilde{\Delta}(y) \in RV_\infty(0)$. Let $\lambda > 1$ and estimate as $x \to \infty$ as follows

$$\begin{aligned}\log\left(\frac{\tilde{f}(\lambda x)}{\tilde{f}(x)}\right) = \int_x^{\lambda x} \frac{\tilde{f}'(t)}{\tilde{f}(t)}\,dt = \int_x^{\lambda x} \frac{t\tilde{f}'(t)}{\tilde{f}(t)} \cdot \frac{1}{t}\,dt \quad &\sim \quad \int_x^{\lambda x} \frac{c}{\log\tilde{\Delta}(t)} \cdot \frac{1}{t}\,dt \\ &\sim \quad \frac{c}{\log\tilde{\Delta}(x)} \int_x^{\lambda x} \frac{1}{t}\,dt \\ &= \quad \frac{c\log\lambda}{\log\tilde{\Delta}(x)} \quad \text{as } x \to \infty.\end{aligned}$$

Therefore as $\tilde{f}(\lambda x)/\tilde{f}(x) \to 1$ as $x \to \infty$

$$\frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - 1 \sim \log\left(1 + \left(\frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - 1\right)\right) = \log\left(\frac{\tilde{f}(\lambda x)}{\tilde{f}(x)}\right) \sim \frac{c\log\lambda}{\log\tilde{\Delta}(x)}, \quad \text{as } x \to \infty.$$

Hence

$$\left(\frac{\tilde{f}(\lambda x)}{\tilde{f}(x)} - 1\right)\log\tilde{\Delta}(x) \to c\log\lambda, \quad \text{as } x \to \infty,\ \lambda > 1.$$

Put $y = \lambda x$. With $\mu := 1/\lambda < 1$ and since $\log\tilde{\Delta} \in RV_\infty(0)$

$$\log\left(\frac{\tilde{f}(y)}{\tilde{f}(\mu y)}\right) \sim \frac{c\log(1/\mu)}{\log\tilde{\Delta}(y/\lambda)} \sim \frac{c\log(1/\mu)}{\log\tilde{\Delta}(y)}, \quad \text{as } y \to \infty.$$

Define $\bar{h}(x) := \log\tilde{f}(e^x)$, $\bar{g}(x) := \log\tilde{\Delta}(e^x)$ and $u := \log\lambda > 0$. Then as $x \to \infty$

$$\begin{aligned}\bar{h}(x) - \bar{h}(x-u) = \log\tilde{f}(e^x) - \log\tilde{f}(e^x e^{-u}) &= \log\left(\frac{\tilde{f}(e^x)}{\tilde{f}(e^x e^{-u})}\right) \\ &\sim \frac{c\log(1/e^{-u})}{\log\tilde{\Delta}(e^x)} = \frac{cu}{\bar{g}(x)} \qquad (4.19)\end{aligned}$$

Since $x\Delta(x) \to 0$ as $x \to 0^+$, $1/y \cdot \Delta(1/y) \to 0$ as $y \to \infty$, or $\tilde{\Delta}(y)/y \to 0$ as $y \to \infty$. Hence for all $\epsilon > 0$ there exists $y_1(\epsilon) > 0$ so that $\tilde{\Delta}(y) < \epsilon y$ for all $y \geq y_1(\epsilon)$. Hence $\bar{g}(x) = \log\tilde{\Delta}(e^x) < \log(\epsilon e^x) = \log\epsilon + x$ for $x \geq \log y_1(\epsilon) =: x_1(\epsilon)$. Thus $x - \bar{g}(x) > \log(1/\epsilon)\ \forall\, x \geq x_1(\epsilon)$. This implies $x - \bar{g}(x) \to \infty$ as $x \to \infty$. Next write

$$\frac{f(y\Delta(y))}{f(y)} = \frac{1/\tilde{f}(1/(y\Delta(y)))}{1/\tilde{f}(1/y)} = \frac{\tilde{f}(1/y)}{\tilde{f}(1/y \cdot 1/\Delta(y))}.$$

Thus

$$\lim_{y\to 0^+}\frac{f(y\Delta(y))}{f(y)} = \lim_{x\to\infty}\frac{\tilde{f}(x)}{\tilde{f}(x/\Delta(1/x))} = \lim_{x\to\infty}\frac{\tilde{f}(x)}{\tilde{f}(x/\tilde{\Delta}(x))}.$$

Write

$$\begin{aligned}\log\left(\frac{\tilde{f}(e^x)}{\tilde{f}(e^x/\tilde{\Delta}(e^x))}\right) = \log\tilde{f}(e^x) - \log\tilde{f}\left(\frac{e^x}{\tilde{\Delta}(e^x)}\right) &= \bar{h}(x) - \log\tilde{f}\left(e^x e^{-\bar{g}(x)}\right) \\ &= \bar{h}(x) - \bar{h}(x - \bar{g}(x)). \quad (4.20)\end{aligned}$$

Therefore, in order for (4.16) to be true we wish to show that

$$\lim_{x\to\infty}\left(\bar{h}(x) - \bar{h}(x - \bar{g}(x))\right) = c. \qquad (4.21)$$

Now set $\eta_x := \lfloor \bar{g}(x)/u_0 \rfloor$. Then we have

$$\bar{h}(x)-\bar{h}(x-\bar{g}(x)) = \sum_{k=1}^{\eta_x} \left(\bar{h}(x-(k-1)u_0) - \bar{h}(x-ku_0)\right) + \left(\bar{h}(x-\eta_x u_0) - \bar{h}(x-\bar{g}(x))\right).$$

Note that $\eta_x \le \bar{g}(x)/u_0$, $\eta_x + 1 > \bar{g}(x)/u_0$ so we have from (4.19) that for all $y > y(\epsilon)$

$$(1-\epsilon)\cdot cu_0 < \left(\bar{h}(y) - \bar{h}(y-u_0)\right)\bar{g}(y) < (1+\epsilon)\cdot cu_0.$$

Take $X(\epsilon) < x$ so large that $x - \eta_x u_0 \ge x - \bar{g}(x) > y(\epsilon)$. Then $x - ku_0 > y(\epsilon)$ $\forall\, k \in \{1, \ldots, \eta_x\}$ and so

$$(1-\epsilon)\cdot cu_0 < \left(\bar{h}(x-(k-1)u_0) - \bar{h}(x-ku_0)\right)\bar{g}(x-(k-1)u_0) < (1+\epsilon)\cdot cu_0.$$

Also $\forall\, k \in \{1, \ldots, \eta_x\}$, $\bar{g}(x-(\eta_x-1)u_0) \le \bar{g}(x-(k-1)u_0) < \bar{g}(x)$. Thus

$$\bar{h}(x) - \bar{h}(x-\bar{g}(x)) \ge \eta_x \frac{(1-\epsilon)cu_0}{\bar{g}(x)} > \left(\frac{\bar{g}(x)}{u_0} - 1\right)\frac{(1-\epsilon)cu_0}{\bar{g}(x)} = (1-\epsilon)c - \frac{(1-\epsilon)cu_0}{\bar{g}(x)}$$

Therefore $\liminf_{x\to\infty}\left\{\bar{h}(x) - \bar{h}(x-\bar{g}(x))\right\} \ge c$ which by (4.20) gives part (i). To get the upper estimate in (ii), we start by noting that

$$\bar{h}(x-\eta_x\mu_0) - \bar{h}(x-\bar{g}(x)) \le h(x-\eta_x u_0) - h(x-(\eta_x+1)u_0) \le \frac{(1+\epsilon)cu_0}{\bar{g}(x-\eta_x u_0)},$$

because $\bar{g}(x) < u_0(\eta_x+1)$ implies that $x-\bar{g}(x) > x-u_0(\eta_x+1)$ and hence $\bar{h}(x-\bar{g}(x)) > \bar{h}(x-u_0(\eta_x+1))$. Therefore

$$\begin{aligned}
\bar{h}(x) - \bar{h}(x-\bar{g}(x)) &\le \sum_{k=1}^{\eta_x} \frac{(1+\epsilon)cu_0}{\bar{g}(x-(k-1)u_0)} + \frac{(1+\epsilon)cu_0}{\bar{g}(x-\eta_x u_0)} \\
&\le \frac{(1+\epsilon)cu_0}{\bar{g}(x-(\eta_x-1)u_0)}\cdot\eta_x + \frac{(1+\epsilon)cu_0}{\bar{g}(x-\eta_x u_0)} \\
&\le \frac{(1+\epsilon)cu_0}{\bar{g}(x-(\eta_x-1)u_0)}\cdot\frac{\bar{g}(x)}{u_0} + \frac{(1+\epsilon)cu_0}{\bar{g}(x-\bar{g}(x))} \\
&= (1+\epsilon)\cdot\frac{c\bar{g}(x)}{\bar{g}(x-(\eta_x-1)u_0)} + o(1), \quad \text{as } x\to\infty.
\end{aligned}$$

Now $\eta_x u_0 \le \bar{g}(x)$ and $\eta_x u_0 > \bar{g}(x) - u_0$. Thus $x - \bar{g}(x) - 2u_0 > x - \eta_x u_0 + u_0 > x - \bar{g}(x) + u_0$ and so

$$\bar{g}(x-\bar{g}(x)+2u_0) > \bar{g}(x-(\eta_x-1)u_0) > \bar{g}(x-\bar{g}(x)+u_0).$$

Hence

$$\frac{\bar{g}(x)}{\bar{g}(x-\bar{g}(x)+2u_0)} < \frac{\bar{g}(x)}{\bar{g}(x-(\eta_x-1)u_0)} < \frac{\bar{g}(x)}{\bar{g}(x-\bar{g}(x)+u_0)}.$$

Therefore as $\bar{g}$ is self-neglecting

$$\limsup_{x\to\infty}\left(\bar{h}(x)-\bar{h}(x-\bar{g}(x))\right)\leq(1+\epsilon)\cdot c\limsup_{x\to\infty}\frac{\bar{g}(x)}{\bar{g}(x-\bar{g}(x)+u_0)}\leq(1+\epsilon)\cdot c.$$

Thus

$$\limsup_{x\to\infty}\left(\bar{h}(x)-\bar{h}(x-\bar{g}(x))\right)\leq c.$$

We have already shown that

$$\liminf_{x\to\infty}\left(\bar{h}(x)-\bar{h}(x-\bar{g}(x))\right)\geq c,$$

so hence (ii) is true. We now prove part (iii). Bounding as above yields

$$\bar{h}(x)-\bar{h}(x-\bar{g}(x))\leq(1+\epsilon)\cdot cu_0\sum_{k=1}^{\eta_x}\frac{1}{\bar{g}(x-(k-1)u_0)}+o(1),\quad\text{as } x\to\infty.$$

Since $y\mapsto y^{\gamma}\Delta(y)$ is increasing for all $0<\gamma<1$ on $(0,1/X_1(\gamma)]$ then the function $x\mapsto(1/x)^{\gamma}\Delta(1/x)$ is decreasing for all $0<\gamma<1$ on $[X_1(\gamma),\infty)$. Hence $x\mapsto\tilde{\Delta}(x)/x^{\gamma}$ is decreasing for all $0<\gamma<1$ on $[X_1(\gamma),\infty)$ and so $x\mapsto\log\left(\tilde{\Delta}(e^x)/(e^x)^{\gamma}\right)=\log\tilde{\Delta}(e^x)-\log(e^{\gamma x})$ is decreasing for all γ. Therefore $\bar{g}_{\gamma}(x):=\bar{g}(x)-\gamma x$ is decreasing for all $\gamma\in(0,1)$. Hence for all k, $\bar{g}_{\gamma}(x)\leq\bar{g}_{\gamma}(x-(k-1)u_0)$ which implies for all k, $\bar{g}(x)-\gamma x\leq\bar{g}(x-(k-1)u_0)-\gamma(x-(k-1)u_0)$ and $\bar{g}(x)\leq\bar{g}(x-(k-1)u_0)+\gamma(k-1)u_0$. Thus

$$\begin{aligned}\bar{g}(x-(k-1)u_0)\geq\bar{g}(x)-\gamma(k-1)u_0\geq\bar{g}(x)-\gamma(\eta_x-1)u_0 &= \bar{g}(x)-\gamma\eta_x u_0+\gamma u_0\\ &\geq \bar{g}(x)-\gamma\bar{g}(x)+\gamma u_0.\end{aligned}$$

Thus

$$\bar{h}(x)-\bar{h}(x-\bar{g}(x))\leq\frac{(1+\epsilon)cu_0\eta_x}{\bar{g}(x)(1-\gamma)+\gamma u_0}+o(1)\leq\frac{(1+\epsilon)c\bar{g}(x)}{\bar{g}(x)(1-\gamma)+\gamma u_0}+o(1),\quad\text{as } x\to\infty.$$

Thus

$$\limsup_{x\to\infty}\left(\bar{h}(x)-\bar{h}(x-\bar{g}(x))\right)\leq\frac{c}{1-\gamma}.$$

Hence

$$\limsup_{x\to\infty}\frac{f(x\Delta(x))}{f(x)}\leq e^{c/(1-\gamma)}.$$

Since γ is arbitrary, we let $\gamma\to0^+$ and conclude that

$$\limsup_{x\to\infty}\frac{f(x\Delta(x))}{f(x)}\leq e^{c},$$

which combined with the lim inf gives the desired limit. □

The condition $x\Delta(x) \to 0$ as $x \to 0^+$, which is essential for the Implicit method to make sense, places a restriction on the rate of growth of f when the optimal choice of Δ is made in (4.13).

Proposition 5. *If $x\Delta(x) \to 0$ as $x \to 0^+$ and (4.13) holds with $c > 0$ then*

$$\liminf_{x\to 0^+} \frac{\log(1/f(x))}{\log\log(1/x)} \geq c.$$

Proof. Since $xf'(x)\log\Delta(x)/f(x) \to c$ as $x \to 0^+$ and $x\Delta(x) \to 0$ as $x \to 0^+$, we have

$$\begin{aligned} \frac{xf'(x)}{f(x)}\log\left(\frac{1}{x}\right) &= \frac{xf'(x)}{f(x)}\left(\log\Delta(x) - \log(x\Delta(x))\right) \\ &= \frac{xf'(x)}{f(x)}\log\left(\frac{1}{x\Delta(x)}\right) + \frac{xf'(x)}{f(x)}\log\Delta(x). \end{aligned}$$

Therefore

$$\liminf_{x\to 0^+} \frac{xf'(x)}{f(x)}\log\left(\frac{1}{x}\right) \geq c.$$

Thus for every $\epsilon \in (0, c)$ there is $x_1(\epsilon) > 0$ such that $\forall\, x < x_1(\epsilon) < 1$

$$\frac{xf'(x)}{f(x)}\log\left(\frac{1}{x}\right) > c - \epsilon > 0,$$

and so $\forall\, x < x_1(\epsilon) < 1$

$$\frac{f'(x)}{f(x)} > \frac{c-\epsilon}{x\log(1/x)}.$$

Therefore for all $x < x_1(\epsilon)$

$$\begin{aligned} \log\left(\frac{f(x_1(\epsilon))}{f(x)}\right) = \int_x^{x_1(\epsilon)} \frac{f'(u)}{f(u)}\,du &\geq (c-\epsilon)\int_x^{x_1(\epsilon)} \frac{1}{u\log(1/u)}\,du \\ &= (c-\epsilon)\int_{\log(1/x)}^{\log(1/x_1(\epsilon))} e^v \cdot \frac{1}{v} \cdot -e^{-v}\,dv \\ &= (c-\epsilon)\left(\log\log\left(\frac{1}{x}\right) - \log\log\left(\frac{1}{x_1(\epsilon)}\right)\right). \end{aligned}$$

Letting $x \to 0^+$ and then $\epsilon \to 0^+$ gives the result. □

Therefore f cannot grow too rapidly at zero. If it does then $x\Delta(x) \to 0$ as $x \to 0^+$ and (4.13) are incompatible. In that case, we must have $c = 0$ in (4.13) and request that $x \mapsto x^\gamma\Delta(x)$ be increasing for some $\gamma \in (0, 1)$ in order to guarantee that $f(x\Delta(x))/f(x)$ has a unit limit. The other way forward is to try to find directly a Δ which yields $f(x\Delta(x))/f(x) \to \lambda \in (1, \infty)$ and $x\Delta(x) \to 0$ as $x \to 0^+$. However, this involves considering the asymptotic behaviour of the inverse of f which may not be possible in some cases.

The stipulation that $x \mapsto x^\gamma \Delta(x)$ be increasing together with "rapid" growth in f covers the condition in (4.13) with $c = 0$. Therefore in practice we take $\Delta(x) = x^{-\gamma}$ for small γ in the case $f(x)$ grows more rapidly than $1/\log(1/x)$.

Proposition 6. *Suppose there is $\gamma \in (0, 1)$ such that $x \mapsto x^\gamma \Delta(x)$ is increasing and that*

$$\lim_{x \to 0^+} \frac{x f'(x)}{f(x)} \log\left(\frac{1}{x}\right) = 0.$$

Then

(i)

$$\lim_{x \to 0^+} \frac{1/f(x)}{\log\log(1/x)} = 0.$$

(ii)

$$\lim_{x \to 0^+} \frac{x f'(x)}{f(x)} \log \Delta(x) = 0.$$

Proof. First, for every $\epsilon > 0$ there is $x_2(\epsilon) > 0$ such that $\forall\, x < x_2(\epsilon)$

$$\frac{x f'(x)}{f(x)} \log\left(\frac{1}{x}\right) < \epsilon.$$

This implies for all $x < x_2(\epsilon)$

$$\begin{aligned} \log\left(\frac{f(x_2(\epsilon))}{f(x)}\right) = \int_x^{x_2(\epsilon)} \frac{f'(u)}{f'(u)}\, du \quad &\leq \quad \epsilon \int_x^{x_2(\epsilon)} \frac{1}{u \log(1/u)}\, du \\ &= \quad \epsilon\left(\log\log\left(\frac{1}{x}\right) - \log\log\left(\frac{1}{x_2(\epsilon)}\right)\right), \end{aligned}$$

giving part (i). For part (ii), note that $x < x_1(\epsilon)$ implies $x^\gamma \Delta(x) \leq K$, for some $K > 0$. Thus $\gamma \log x + \log \Delta(x) \leq \log K$. Since $f_1(x) = x f'(x)/f(x) > 0$ then $\gamma f_1(x) \log x + f_1(x) \log \Delta(x) \leq f_1(x) \log K$. Thus for $x < x_1(\epsilon)$,

$$0 < f_1(x) \log \Delta(x) < f_1(x) \log K + \gamma f_1(x) \log\left(\tfrac{1}{x}\right) \to 0, \quad \text{as } x \to 0^+,$$

as claimed. ∎

Chapter 5

Transformed Co-ordinate System

5.1 Introduction

This section shows that it is possible to exactly mimic the rate of decay of solutions of the continuous equation by making a pre-transformation of the original ODE and discretising the resulting ODE. This result applies even when the rate of decay to the equilibrium is arbitrarily fast. This Transformed Explicit method outperforms both Explicit and Implicit methods applied directly to the original equation for the same computational effort in the case of super-exponential convergence. However, this transformation does not give any extra benefits to the efficiency of the algorithm when the equation hits zero in finite-time. The performance of the algorithm is the same as that when the original ODE is discretised directly. This is contingent on the step-size in the new co-ordinate system decaying in length to zero at the same rate as that shown to be optimal in previous sections.

Our aim is to make a transformation of the co-ordinate system with a view to the numerical simulation being more straightforward and efficient in the new co-ordinate system whilst being sufficiently tractable to allow the values of the approximations in the original co-ordinate system to be simultaneously and simply computed. A key feature of the method is that it retains the unconditional stability of the Implicit method, in the sense that the solution tends to zero for all positive values of the control parameter Δ. It also has the advantage, which it shares with the Explicit method, that it is unnecessary to perform non-linear solving at each step since it also an Explicit method.

We consider once again the ODE (1.1) viz.,

$$x'(t) = -f(x(t)), \quad t > 0, \quad x(0) = \xi,$$

where $f(x) > 0$ for all $x > 0$, $f(0) = 0$. Define $z(t) := T(x(t))$. We suppose that

$T:(0,\infty)\to\mathbb{R}$ is a continuous function with the following properties:

$$T:(0,\infty)\to\mathbb{R}\,\text{is in } C^1; \tag{5.1}$$
$$T\,\text{is decreasing with } \lim_{x\to 0^+} T(x)=\infty; \tag{5.2}$$
$$x\mapsto -T'(x)f(x)\,\text{is decreasing; and} \tag{5.3}$$
$$-T'\in RV_0(-1). \tag{5.4}$$

We will justify the selection of these properties as we proceed. Since T obeys (5.1) then $z'(t)=T'(x(t))x'(t)=-T'(x(t))f(x(t))$. Therefore since $x(t)=T^{-1}(z(t))$ then

$$z'(t)=-T'\left(T^{-1}\left(z(t)\right)\right)f\left(T^{-1}\left(z(t)\right)\right)=\eta(z(t))$$

where $\eta(z):=-T'\left(T^{-1}\left(z\right)\right)f\left(T^{-1}\left(z\right)\right)=(-T'f)(T^{-1}(z))=((-T'f)\circ T^{-1})(z)$. The associated Explicit Euler scheme is

$$z_{n+1}=z_n+\tilde{h}(z_n)\eta(z_n),\quad n\geq 0,\quad z_0=T(\xi), \tag{5.5}$$

where

$$t_{n+1}=\sum_{j=0}^{n}\tilde{h}(z_j),\quad n\geq 0,\quad t_0=0, \tag{5.6}$$

and

$$\tilde{h}(z):=h(T^{-1}(z))=\frac{\Delta T^{-1}(z)}{f(T^{-1}(z))}, \tag{5.7}$$

where for simplicity we take $h(x)=\Delta x/f(x)$, with $\Delta>0$. Applying the definitions of η, z_n and $\tilde{h}(z_n)$, the sequences (z_n) and (x_n) may therefore be given by

$$z_{n+1}=z_n+\Delta T^{-1}(z_n)(-T')(T^{-1}(z_n)),\quad n\geq 0,\quad z_0=T(\xi), \tag{5.8}$$
$$x_{n+1}=T^{-1}(z_{n+1}),\quad n\geq 0,\quad x_0=\xi. \tag{5.9}$$

Notice that

$$t_{n+1}=\sum_{j=0}^{n}h(x_j).$$

so that (t_n) still obeys (3.16). Therefore, from (5.5) it can be seen that z_n approximates $z(t_n)$ and hence from (5.9) that x_n approximates $x(t_n)$.

5.2 Asymptotic Analysis of Pre-Transformed Scheme with Standard Step-Size

Proposition 7. *Suppose T obeys* (5.1), (5.2) *and* (5.3). *There exists a unique positive sequence (z_n) which obeys* (5.5) *and this sequence is increasing and obeys $z_n\to\infty$ as*

$n \to \infty$. *Moreover, the sequence* (x_n) *defined by* (5.9) *is positive, decreasing and obeys* $x_n \to 0$ *as* $n \to \infty$.

Proof. The sequence exists and is unique by construction. By (5.2), T is decreasing and hence so is T^{-1}. By (5.3), $z \mapsto ((-T'f) \circ T^{-1})(z) =: \eta(z)$ is increasing. Since $z \mapsto \eta(z)$ is increasing and $\tilde{h}(z) > 0$ for all $z \in \mathbb{R}$ then (z_n) is an increasing sequence by (5.8). Since $T^{-1}(z) > 0$ for all $z \in \mathbb{R}$, by (5.1), and T is decreasing then $-T'(T^{-1}(z))) > 0$ for all $z \in \mathbb{R}$. Thus $-\Delta T'(T^{-1}(z))T^{-1}(z) > 0$ for all $z \in \mathbb{R}$. Therefore, it must be that $z_n \to \infty$ as $n \to \infty$ because if it tends to a finite limit, L say, then from (5.8)

$$L = \lim_{n\to\infty} z_{n+1} = \lim_{n\to\infty} \left\{ z_n + \tilde{h}(z_n)\eta(z_n) \right\} = L + \Delta T^{-1}(z)(-T')(T^{-1}(z)),$$

thus $-\Delta T'(T^{-1}(z))T^{-1}(z) = 0$, a contradiction. Since $T^{-1} : \mathbb{R} \to (0, \infty)$ by (5.1) it follows that $x_n > 0$ for all $n \geq 0$. The sequence (x_n) is decreasing since $x \mapsto T^{-1}(x)$ is decreasing and $x_n \to 0$ as $n \to \infty$ because $\lim_{n\to\infty} x_n = \lim_{n\to\infty} T^{-1}(z_n) = 0$ since $T^{-1}(z) \to 0$ as $z \to \infty$, by (5.2). Therefore we have that

$$x_n > 0 \text{ for all } n \geq 0, \quad (x_n) \text{ is decreasing} \quad \text{and} \quad x_n \to 0 \text{ as } n \to \infty, \tag{5.10}$$

as claimed. □

It is clear that the conditions (5.1) and (5.2) are essentially necessary in order to affect a transformation that gives rise to a sequence (x_n) satisfying (5.10). Clearly the role of (5.3) is to imbue η, the rate function in the transformed system, with additional monotonicity. The condition (5.4) however has not yet been employed and practical questions such as the existence of a T^{-1} expressible in closed-form have not yet been addressed. One advantageous and notable feature of the sequence (x_n) in (5.10) is that it retains its positivity, monotonicity and tends to zero *for all values of* $\Delta > 0$. We will now explore how our new scheme preserves finite-time stability, global positivity and asymptotic behaviour of the ODE (1.1).

Lemma 16. *Let* (x_n) *be a positive decreasing sequence such that* $x_{n+1} \sim e^{-\Delta}x_n$ *as* $n \to \infty$ *and suppose* $\phi \in RV_0(0)$. *Then*

$$\lim_{n\to\infty} \frac{\phi(x_{n+1})}{\phi(x_n)} = 1.$$

Proof. Define $\lambda_n := x_{n+1}/x_n \to e^{-\Delta}$ as $n \to \infty$. For every $\epsilon \in (0, 1)$, there is $N_1(\epsilon) \in \mathbb{N}$ such that

$$(1 - \epsilon) \cdot e^{-\Delta} < \lambda_n < (1 + \epsilon) \cdot e^{-\Delta}.$$

Hence for $n \geq N_1(\epsilon)$

$$\left| \frac{\phi(x_{n+1})}{\phi(x_n)} - 1 \right| = \left| \frac{\phi(\lambda_n x_n)}{\phi(x_n)} - 1 \right| \leq \sup_{\lambda\in[(1-\epsilon)e^{-\Delta},(1+\epsilon)e^{-\Delta}]} \left| \frac{\phi(\lambda x_n)}{\phi(x_n)} - 1 \right|.$$

By the Uniform Convergence Theorem for Slowly Varying Functions

$$0 \le \limsup_{n\to\infty} \left| \frac{\phi(x_{n+1})}{\phi(x_n)} - 1 \right| \le \limsup_{n\to\infty} \sup_{\lambda_n \in [(1-\epsilon)e^{-\Delta},(1+\epsilon)e^{-\Delta}]} \left| \frac{\phi(\lambda_n x_n)}{\phi(x_n)} - 1 \right| \le$$
$$\limsup_{x\to 0^+} \sup_{\lambda \in [(1-\epsilon)e^{-\Delta},(1+\epsilon)e^{-\Delta}]} \left| \frac{\phi(\lambda x)}{\phi(x)} - 1 \right| = 0,$$

as required. □

Lemma 17. *Let $z_n = T(x_n)$ for $n \ge 0$ be a positive increasing sequence such that $z_n \to \infty$ as $n \to \infty$ and the solution of* (5.8)*. Suppose T obeys* (5.4)*. Then*

$$\lim_{n\to\infty} \frac{z_n - z_{n-1}}{z_{n+1} - z_n} = 1. \tag{5.11}$$

Proof. From (5.8)

$$z_{n+1} = z_n - \Delta T'(T^{-1}(z_n))T^{-1}(z_n) = z_n + k(z_n),$$

where $k(z) := -\Delta T'(T^{-1}(z))T^{-1}(z) > 0$. Therefore (5.11) is equivalent to

$$\lim_{n\to\infty} \frac{k(z_{n-1})}{k(z_n)} = 1.$$

Note $k(T(z)) = -\Delta T'(z)z$. By (5.4), $-T' \in RV_0(-1)$. Then there is τ such that

$$\lim_{x\to 0^+} \frac{\tau(x)}{-T'(x)} = 1 \quad \text{and} \quad \lim_{x\to 0^+} \frac{x\tau'(x)}{\tau(x)} = -1.$$

Define $\kappa(T(x)) := \Delta\tau(x)x$. Then $\kappa'(T(x))T'(x) = \Delta\left(x\tau'(x) + \tau(x)\right)$ so

$$\lim_{x\to 0^+} \kappa'(T(x)) = \lim_{x\to 0^+} \Delta \left(\frac{x\tau'(x)}{\tau(x)} \cdot \frac{\tau(x)}{T'(x)} + \frac{\tau(x)}{T'(x)} \right) = 0.$$

Since $\kappa(T(x)) = \Delta\tau(x)x$ and $k(T(x)) = -\Delta T'(x)x$ then $k(z) \sim \kappa(z)$ as $z \to \infty$ and $\kappa'(z) \to 0$ as $z \to \infty$. Set

$$\begin{aligned} K(z) := \int_{z^*}^{z} \frac{1}{k(u)}\, du = \int_{z^*}^{z} \frac{1}{-\Delta T'(T^{-1}(u))T^{-1}(u)}\, du &= \int_{T^{-1}(z^*)}^{T^{-1}(z)} \frac{T'(v)}{-\Delta T'(v)v}\, dv \\ &= \frac{-1}{\Delta} \int_{T^{-1}(z^*)}^{T^{-1}(z)} \frac{1}{v}\, dv \\ &= \frac{-1}{\Delta} \log\left(\frac{T^{-1}(z)}{T^{-1}(z^*)} \right) \end{aligned} \tag{5.12}$$

By the Mean Value Theorem there is $\theta_n \in (0,1)$ such that

$$K(z_{n+1}) = K(z_n + k(z_n)) = K(z_n) + K'(z_n + \theta_n k(z_n)) \cdot k(z_n).$$

Hence

$$K(z_{n+1}) - K(z_n) = \frac{k(z_n)}{k(z_n + \theta_n k(z_n))}. \tag{5.13}$$

Next write

$$\frac{k(z_n)}{k(z_n + \theta_n k(z_n))} = \frac{k(z_n)}{\kappa(z_n)} \cdot \frac{\kappa(z_n)}{\kappa(z_n + \theta_n k(z_n))} \cdot \frac{\kappa(z_n + \theta_n k(z_n))}{k(z_n + \theta_n k(z_n))}. \tag{5.14}$$

Similarly, there is $\bar{\theta}_n \in (0, \theta_n)$ such that

$$\kappa(z_n + \theta_n k(z_n)) = \kappa(z_n) + \kappa'(z_n + \bar{\theta}_n \kappa(z_n)) \cdot \theta_n k(z_n).$$

Since $\kappa(z_n) \sim k(z_n)$ as $n \to \infty$ and $\kappa'(z) \to 0$ as $z \to \infty$, from this identity we get

$$\frac{\kappa(z_n + \theta_n k(z_n))}{\kappa(z_n)} = 1 + \frac{\kappa'(z_n + \bar{\theta}_n \kappa(z_n)) \cdot \theta_n k(z_n)}{\kappa(z_n)},$$

so $\kappa(z_n + \theta_n k(z_n)) \sim \kappa(z_n)$ as $n \to \infty$. Then from (5.12), (5.13) and (5.14)

$$1 = \lim_{n\to\infty} (K(z_{n+1}) - K(z_n)) = \frac{-1}{\Delta} \lim_{n\to\infty} \log\left(\frac{T^{-1}(z_{n+1})}{T^{-1}(z_n)}\right).$$

Therefore as $x_n = T^{-1}(z_n)$, we have

$$\frac{-1}{\Delta} \lim_{n\to\infty} \log\left(\frac{T^{-1}(z_{n+1})}{T^{-1}(z_n)}\right) = \frac{-1}{\Delta} \lim_{n\to\infty} \log\left(\frac{x_{n+1}}{x_n}\right) = 1,$$

and thus $x_{n+1}/x_n \to e^{-\Delta}$ as $n \to \infty$. Now

$$z_{n+1} - z_n = k(T(x_n)) = -\Delta T'(x_n) x_n = \phi(x_n)$$

where $\phi(x) := -\Delta T'(x) x \in RV_0(0)$. Now $x_{n+1}/x_n \to e^{-\Delta}$ as $n \to \infty$ and as $\phi \in RV_0(0)$ then

$$\lim_{n\to\infty} \frac{z_{n+2} - z_{n+1}}{z_{n+1} - z_n} = \lim_{n\to\infty} \frac{\phi(x_{n+1})}{\phi(x_n)} = 1,$$

by Lemma 16. Therefore, (5.11) holds with the obvious change in indices. □

The Transformed Explicit scheme defined by equation (5.8) preserves the properties of the soft landing (1.15) under the condition (1.7) while the property of super-exponentially stable solutions (1.13) is preserved under the condition (1.9). Since $(\tilde{h}(x_n))$ is a positive sequence the limit

$$\lim_{n\to\infty} t_n =: \hat{T}_h = \sum_{j=0}^{\infty} h(x_j),$$

exists and equals (3.16). In our next result, we show that $\hat{T}_h$ is finite or infinite according to whether T_ξ defined by (1.8) is finite or infinite.

Theorem 26. *Suppose f obeys* (3.1)*, $\tilde{h}$ obeys* (5.7) *and T obeys* (5.1)*,* (5.2) *and* (5.3)*. Let (t_n) and $\hat{T}_h$ be defined* (5.6) *and* (3.16)*.*

(i) If f obeys (1.7)*, then $\hat{T}_h < \infty$.*

(ii) If f obeys (1.9)*, then $\hat{T}_h = \infty$.*

Proof. Note that

$$\begin{aligned}\int_{z_j}^{z_{j+1}} \frac{1}{\eta(z)}\,dz = \int_{z_j}^{z_{j+1}} \frac{1}{-T'(T^{-1}(z))f(T^{-1}(z))}\,dz &= \int_{T^{-1}(z_j)}^{T^{-1}(z_{j+1})} \frac{T'(u)}{-T'(u)f(u)}\,du \\ &= \int_{x_j}^{x_{j+1}} \frac{1}{-f(u)}\,du \\ &= \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du. \qquad (5.15)\end{aligned}$$

For $z_j \le z \le z_{j+1}$ with η increasing we have $\eta(z_j) \le \eta(z) \le \eta(z_{j+1})$ and

$$\frac{1}{\eta(z_{j+1})} \le \frac{1}{\eta(z)} \le \frac{1}{\eta(z_j)}.$$

Hence by (5.5) and (5.15), integrating over $[z_j, z_{j+1}]$ gives

$$\frac{z_{j+1}-z_j}{\eta(z_{j+1})} \le \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du \le \frac{z_{j+1}-z_j}{\eta(z_j)}.$$

Now

$$\frac{z_{j+1}-z_j}{z_{j+2}-z_{j+1}} \cdot \tilde{h}(z_{j+1}) = \frac{z_{j+1}-z_j}{z_{j+2}-z_{j+1}} \cdot \frac{z_{j+2}-z_{j+1}}{\eta(z_{j+1})} \le \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du \le \frac{z_{j+1}-z_j}{\eta(z_j)} = \tilde{h}(z_j).$$

Therefore

$$\frac{z_{j+1}-z_j}{z_{j+2}-z_{j+1}} \cdot \tilde{h}(z_{j+1}) \le \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du \le \tilde{h}(z_j). \qquad (5.16)$$

By (1.7), $\int_{0^+}^{1} 1/f(u)\,du < \infty$ then $T_\xi < \infty$ from (3.12) since

$$\sum_{j=0}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du = \int_0^{\xi} \frac{1}{f(u)}\,du < \infty.$$

The Comparison Test applied to (5.16) shows the summability of $(\int_{x_{n+1}}^{x_n} 1/f(u)\,du)$ implies that of $((z_{j+1}-z_j)/(z_{j+2}-z_{j+1}) \cdot \tilde{h}(z_{j+1}))$. Thus $(\tilde{h}(z_{n+1}))$ is summable since

$$\lim_{j\to\infty} \frac{(z_{j+1}-z_j)/(z_{j+2}-z_{j+1}) \cdot \tilde{h}(z_{j+1})}{\tilde{h}(z_{j+1})} = 1,$$

by (5.11). Thus $t_n = \sum_{j=0}^{n-1} \tilde{h}(z_j)$ for $n \geq 1$ obeys $t_n \to \hat{T}_h := \sum_{j=0}^{\infty} \tilde{h}(z_j) < \infty$ as $n \to \infty$.

By (1.9), $\int_{0+}^{1} 1/f(u)\, du = \infty$ then $T_\xi = \infty$ from (3.11) since

$$\sum_{j=0}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\, du = \int_0^\xi \frac{1}{f(u)}\, du = \infty.$$

The Comparison Test applied to (5.16) shows that $(\tilde{h}(z_n))$ is not summable and thus $(\tilde{h}(z_n))$ obeys $t_n = \sum_{j=0}^{n-1} \tilde{h}(z_j) \to \infty$ as $n \to \infty$. Therefore $t_n \to \infty$ as $n \to \infty$. □

Theorem 27. *Suppose T obeys* (5.1), (5.2), (5.3) *and* (5.4), *f obeys* (3.1) *and $\tilde{h}$ obeys* (5.7). *Let $\bar{F}$, F, (t_n) and $\hat{T}_h$ be defined by* (1.10), (1.11), (5.6) *and* (3.16).

(i) If f obeys (1.9), *then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \infty$ as $n \to \infty$ and*

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = 1.$$

(ii) If f obeys (1.7), *then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and*

$$1 \leq \liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_{n+1}} \quad \text{and} \quad \limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq 1. \tag{5.17}$$

If in addition, f is increasing and $\Delta > 0$ then

$$\frac{1 - e^{\Delta}}{\Delta} \leq \liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq \limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq 1. \tag{5.18}$$

Proof. The positivity, monotonicity and convergence of (x_n) have been addressed in Proposition 7. If f obeys (1.9) then $\int_{0+}^{1} 1/f(u)\, du = \infty$ and $t_n \to \infty$ as $n \to \infty$ by Theorem 26. We now prove part (i). By the second inequality of (5.16), for $n \geq 1$

$$F(x_n) - F(x_0) = \sum_{j=0}^{n-1} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\, du \leq \sum_{j=0}^{n-1} \tilde{h}(z_j) = t_n.$$

Therefore dividing by t_n and letting $n \to \infty$ yields

$$\limsup_{n\to\infty} \frac{F(x_n)}{t_n} \leq 1. \tag{5.19}$$

By the first inequality of (5.16), for $n \geq 1$

$$F(x_n) - F(x_0) = \sum_{j=0}^{n-1} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\, du \geq \sum_{j=0}^{n-1} \frac{z_{j+1} - z_j}{z_{j+2} - z_{j+1}} \cdot \tilde{h}(z_{j+1}) = \sum_{j=1}^{n} \frac{z_j - z_{j-1}}{z_{j+1} - z_j} \cdot \tilde{h}(z_j). \tag{5.20}$$

Therefore, dividing by t_n and letting $n \to \infty$ yields

$$\liminf_{n\to\infty} \frac{F(x_n)}{t_n} \geq \liminf_{n\to\infty} \frac{\sum_{j=1}^{n} \frac{z_j - z_{j-1}}{z_{j+1} - z_j} \cdot \tilde{h}(z_j)}{t_n}.$$

As $\sum_{j=1}^{n} \tilde{h}(z_j) = t_{n+1} - t_1 \to \infty$ as $n \to \infty$ and $\tilde{h}(z_n) \to 0$ as $n \to \infty$ then $(t_{n+1} - t_1)/t_n \to 1$ and by Toeplitz's Lemma and (5.11)

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sum_{j=1}^{n} \frac{z_j - z_{j-1}}{z_{j+1} - z_j} \cdot \tilde{h}(z_j)}{t_n} &= \lim_{n\to\infty} \left(\frac{\sum_{j=1}^{n} \frac{z_j - z_{j-1}}{z_{j+1} - z_j} \cdot \tilde{h}(z_j)}{\sum_{j=1}^{n} \tilde{h}(z_j)} \cdot \frac{t_{n+1} - t_1}{t_n} \right) \\
&= \lim_{j\to\infty} \frac{\frac{z_j - z_{j-1}}{z_{j+1} - z_j} \cdot \tilde{h}(z_j)}{\tilde{h}(z_j)} \\
&= \lim_{j\to\infty} \frac{z_j - z_{j-1}}{z_{j+1} - z_j} = 1.
\end{aligned}$$

Hence

$$\liminf_{n\to\infty} \frac{F(x_n)}{t_n} \geq 1. \tag{5.21}$$

Combining (5.19) and (5.21) yields

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = 1,$$

and hence part (i) is true. Now we prove part (ii). If f obeys (1.7) then $\int_{0^+}^{1} 1/f(u)\, du < \infty$ and $t_n \to \hat{T}_h := \sum_{j=0}^{\infty} \tilde{h}(z_j) < \infty$ as $n \to \infty$ by Theorem 26. Hence $\hat{T}_h - t_n = \sum_{j=n}^{\infty} \tilde{h}(z_j) \to 0$ as $n \to \infty$. By the second inequality of (5.16)

$$\bar{F}(x_n) = \sum_{j=n}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\, du \leq \sum_{j=n}^{\infty} \tilde{h}(z_j) = \hat{T}_h - t_n.$$

Therefore, dividing by $\hat{T}_h - t_n$ and letting $n \to \infty$ yields

$$\limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq 1.$$

By the first inequality of (5.16)

$$\bar{F}(x_n) = \sum_{j=n}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\, du \geq \sum_{j=n}^{\infty} \frac{z_{j+1} - z_j}{z_{j+2} - z_{j+1}} \cdot \tilde{h}(z_{j+1}).$$

By (5.11), $(z_{j+1} - z_j)/(z_{j+2} - z_{j+1}) \to 1$ as $j \to \infty$ therefore

$$\bar{F}(x_n) \geq \sum_{j=n}^{\infty} \frac{z_{j+1} - z_j}{z_{j+2} - z_{j+1}} \cdot \tilde{h}(z_{j+1}) \sim \sum_{j=n}^{\infty} \tilde{h}(z_{j+1}) = \sum_{j=n+1}^{\infty} \tilde{h}(z_j) = \hat{T}_h - t_{n+1}.$$

Therefore dividing by $\hat{T}_h - t_{n+1}$ and letting $n \to \infty$ yields

$$\liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_{n+1}} \geq 1.$$

Therefore

$$1 \leq \liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_{n+1}} \quad \text{and} \quad \limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq 1,$$

which is (5.17). To prove (5.18), since f is increasing for $x_{j+1} \leq x \leq x_j$ then $f(x_{j+1}) \leq f(x) \leq f(x_j)$, so

$$\frac{x_j - x_{j+1}}{f(x_j)} \leq \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\, du \leq \frac{x_j - x_{j+1}}{f(x_{j+1})}. \tag{5.22}$$

Recalling that $\tilde{h}(z) = \Delta T^{-1}(z)/f(T^{-1}(z))$, then

$$\frac{x_j - x_{j+1}}{f(x_j)} = \frac{1}{\Delta}\left(1 - \frac{x_{j+1}}{x_j}\right)\frac{\Delta x_j}{f(x_j)} = \frac{1}{\Delta}\left(1 - \frac{x_{j+1}}{x_j}\right)\tilde{h}(z_j), \tag{5.23}$$

because $x_j = T^{-1}(z_j)$. Thus

$$\int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\, du \geq \frac{x_j - x_{j+1}}{f(x_j)} = \frac{1}{\Delta}\left(1 - \frac{x_{j+1}}{x_j}\right)\tilde{h}(z_j) =: a_j \tilde{h}(z_j).$$

As $x_{j+1}/x_j \to e^{-\Delta}$ as $j \to \infty$, then $a_j \to \left(1 - e^{-\Delta}\right)/\Delta$ as $j \to \infty$. Thus as $\int_{0^+}^{1} 1/f(u)\, du < \infty$, the summability of $(\int_{x_{j+1}}^{x_j} 1/f(u)\, du)$ implies the summability of $(a_j \tilde{h}(z_j))$ and hence of $(\tilde{h}(z_j))$. Therefore as $n \to \infty$

$$\bar{F}(x_n) = \sum_{j=n}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\, du \geq \sum_{j=n}^{\infty} a_j \tilde{h}(z_j) \sim \frac{\left(1 - e^{-\Delta}\right)}{\Delta} \sum_{j=n}^{\infty} \tilde{h}(z_j) = \frac{\left(1 - e^{-\Delta}\right)}{\Delta}\left(\hat{T}_h - t_n\right).$$

Hence

$$\liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \geq \frac{1 - e^{-\Delta}}{\Delta},$$

as needed. The other estimates are the same. □

The result in Theorem 27 part(i) is very striking because it recovers exactly the rate of decay for solutions to the ODE (1.1) namely

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 1.$$

This occurs for every value of Δ. Therefore the unconditional stability, global positivity and monotonicity we observed in Proposition 7 and the unconditional recovery of finite-time stability we saw in Theorem 26 are recovered. Such qualitative properties have been recovered unconditionally for the Implicit scheme but we have not and do not expect to recover them for an Explicit scheme. However, the Transformed Explicit scheme recovers these qualitative properties unconditionally, as well as the quantitative decay rate property. Also, the asymptotic behaviour in the vicinity of the finite stability time is recovered unconditionally on Δ and the exponent is accurate to $O(\Delta)$ as $\Delta \to 0^+$.

It is now reasonable to ask if the hypothesis $-T' \in RV_0(-1)$ is responsible for the successful unconditional asymptotic behaviour recorded in Proposition 7 and Theorem 27. We will show by requesting that $-T' \in RV$ with an index other than -1, that the choice of exponent -1 is optimal in a sense. To do this we specialise to the case when f is regularly varying.

Theorem 28. *Suppose T obeys* (5.1), (5.2) *and* (5.3) *while $f \in RV_0(\beta)$ where $\beta \in [0,1]$. Let $\bar{F}$, F, (t_n) and $\hat{T}_h$ be defined by* (1.10), (1.11), (5.6) *and* (3.16).

(a) Let $-T' \in RV_0(-\mu-1)$, $\mu > 0$.

(i) If f obeys (1.9), *then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \infty$ as $n \to \infty$ and*

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = \frac{1}{\Delta}\int_{\left(\frac{1}{1+\mu\Delta}\right)^{1/\mu}}^{1} \lambda^{-1}\, d\lambda = \frac{\log(1+\mu\Delta)}{\mu\Delta} \neq 1.$$

(ii) If f obeys (1.7), *then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and*

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \frac{1}{\Delta}\int_{\left(\frac{1}{1+\mu\Delta}\right)^{1/\mu}}^{1} \lambda^{-\beta}\, d\lambda \neq 1.$$

(b) Let $-T' \in RV_0(-1)$.

(i) If f obeys (1.9), *then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \infty$ as $n \to \infty$ and*

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = 1.$$

(ii) If f obeys (1.7), *then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and*

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \frac{1}{\Delta}\int_{e^{-\Delta}}^{1} \lambda^{-\beta}\, d\lambda = \begin{cases} 1, & \text{if } \beta = 1, \\ \frac{\left(1-e^{-\Delta(1-\beta)}\right)}{\Delta(1-\beta)}, & \text{if } 0 \leq \beta < 1. \end{cases}$$

Proof. First write

$$\frac{1}{\tilde{h}(z_j)}\int_{x_{j+1}}^{x_j}\frac{1}{f(u)}\,du = \frac{1}{\Delta x_j/f(x_j)}\int_{x_{j+1}}^{x_j}\frac{1}{f(u)}\,du = \frac{f(x_j)}{\Delta x_j}\int_{x_{j+1}/x_j}^{1}\frac{x_j}{f(\lambda x_j)}\,d\lambda = \frac{1}{\Delta}\int_{x_{j+1}/x_j}^{1}\frac{\tilde{f}(\lambda x_j)}{\tilde{f}(x_j)}\,d\lambda,$$

since $\tilde{h}(z_j) = \Delta T^{-1}(z_j)/f(T^{-1}(z_j)) = \Delta x_j/f(x_j)$ and $x_j = T^{-1}(z_j)$ and where $\tilde{f} := 1/f \in RV_0(-\beta)$. Alternatively

$$\frac{1}{\tilde{h}(z_j)}\int_{x_{j+1}}^{x_j}\frac{1}{f(u)}\,du = \frac{1}{\Delta}\int_{x_{j+1}/x_j}^{1}\left(\frac{\tilde{f}(\lambda x_j)}{\tilde{f}(x_j)} - \lambda^{-\beta}\right)d\lambda + \frac{1}{\Delta}\int_{x_{j+1}/x_j}^{1}\lambda^{-\beta}\,d\lambda.$$

The first term has zero limit by The Uniform Convergence Theorem for Regularly Varying functions. Thus

$$\lim_{j\to\infty}\frac{1}{\tilde{h}(z_j)}\int_{x_{j+1}}^{x_j}\frac{1}{f(u)}\,du = \frac{1}{\Delta}\int_{\lambda(\Delta)}^{1}\lambda^{-\beta}\,d\lambda,$$

contingent upon

$$\lim_{j\to\infty}\frac{x_{j+1}}{x_j} =: \lambda(\Delta) \in (0,1). \tag{5.24}$$

We know from Lemma 17 that if $-T' \in RV_0(-1)$, that $\lambda(\Delta) = e^{-\Delta}$. When f obeys (1.9) then $\int_{0^+}^1 1/f(u)\,du = \infty$ and $(\int_{x_{j+1}}^{x_j} 1/f(u)\,du)$ is a divergent series and so is $(\tilde{h}(z_j))$. Hence by Toeplitz's Lemma

$$\lim_{n\to\infty}\frac{F(x_n)}{t_n} = \lim_{n\to\infty}\frac{F(x_0)+\sum_{j=0}^{n-1}\int_{x_{j+1}}^{x_j}1/f(u)\,du}{\sum_{j=0}^{n-1}\tilde{h}(z_j)} = \lim_{j\to\infty}\frac{1}{\tilde{h}(z_j)}\int_{x_{j+1}}^{x_j}\frac{1}{f(u)}\,du = \frac{1}{\Delta}\int_{\lambda(\Delta)}^{1}\lambda^{-\beta}\,d\lambda,$$

and in the case that f obeys (1.7) then $\int_{0^+}^1 1/f(u)\,du < \infty$ and $(\int_{x_{j+1}}^{x_j} 1/f(u)\,du)$ is a convergent series and so is $(\tilde{h}(z_j))$. Hence by Toeplitz's Lemma

$$\lim_{n\to\infty}\frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \lim_{n\to\infty}\frac{\sum_{j=n}^{\infty}\int_{x_{j+1}}^{x_j}1/f(u)\,du}{\sum_{j=n}^{\infty}\tilde{h}(z_j)} = \lim_{j\to\infty}\frac{1}{\tilde{h}(z_j)}\int_{x_{j+1}}^{x_j}\frac{1}{f(u)}\,du = \frac{1}{\Delta}\int_{\lambda(\Delta)}^{1}\lambda^{-\beta}\,d\lambda,$$

which completes the proof of part(b)(i) and (ii).

If $-T' \in RV_0(-\mu - 1)$, then there is τ such that

$$\lim_{x\to 0^+} \frac{\tau(x)}{-T'(x)} = 1 \quad \text{and} \quad \lim_{x\to 0^+} \frac{x\tau'(x)}{\tau(x)} = -\mu - 1.$$

Define $k(z) := -\Delta T'(T^{-1}(z))T^{-1}(z)$. Then $k(T(x)) = -\Delta T'(x)x$. Also define $\kappa(T(x)) := \Delta\tau(x)x$. Then $k(T(x)) \sim \kappa(T(x))$ as $x \to 0^+$, or $k(z) \sim \kappa(z)$ as $z \to \infty$ since $T(z) \to \infty$ as $z \to 0^+$. Next

$$\frac{z_{n+1}}{z_n} = 1 + \frac{k(z_n)}{z_n} = 1 + \frac{k(z_n)}{\kappa(z_n)} \cdot \frac{\kappa(z_n)}{z_n}.$$

Now

$$\lim_{z\to\infty} \frac{\kappa(z)}{z} = \lim_{z\to\infty} \frac{\Delta\tau(T^{-1}(z))T^{-1}(z)}{T(T^{-1}(z))} = \Delta \lim_{x\to 0^+} \frac{\tau(x)x}{T(x)}.$$

Since $T \in RV_0(-\mu)$ and $x \mapsto x\tau(x) \in RV_0(-\mu)$ the last limit, if it exists, is indeterminate. However, by L'Hôpital's rule and definition of τ

$$\lim_{x\to 0^+} \frac{\tau(x)x}{T(x)} = \lim_{x\to 0^+} \frac{\tau'(x)x + \tau(x)}{T'(x)} = \lim_{x\to 0^+} \frac{\tau'(x)x/\tau(x) + 1}{T'(x)/\tau(x)} = \frac{-\mu - 1 + 1}{-1} = \mu.$$

Hence $k(z)/z \to \Delta\mu$ as $z \to \infty$ and hence

$$\lim_{j\to\infty} \frac{T(x_{j+1})}{T(x_j)} = \lim_{j\to\infty} \frac{z_{j+1}}{z_j} = 1 + \lim_{j\to\infty} \frac{k(z_j)}{z_j} = 1 + \mu\Delta.$$

Since $T^{-1} \in RV_\infty(-1/\mu)$ then as $j \to \infty$

$$\begin{aligned} x_{j+1} = T^{-1}(T(x_{j+1})) \sim T^{-1}\left((1 + \Delta\mu)\, T(x_j)\right) \quad &\sim \quad (1 + \Delta\mu)^{-1/\mu}\, T^{-1}(T(x_j)) \\ &= \quad x_j \left(\frac{1}{1 + \Delta\mu}\right)^{1/\mu}. \end{aligned}$$

Hence

$$\lim_{j\to\infty} \frac{x_{j+1}}{x_j} = \left(\frac{1}{1 + \Delta\mu}\right)^{1/\mu} =: \lambda(\Delta) \in (0, 1),$$

proving (5.24) with $\lambda(\Delta) = \left(\frac{1}{1+\Delta\mu}\right)^{1/\mu}$. This gives part (a) of the theorem. If $-T' \in RV_0(-1)$, then $\lambda(\Delta) = e^{-\Delta}$ from Lemma 17 so we have part (b). □

Remark 22. If $\beta = 1$, then the limit in Theorem 28 part (b)(ii) agrees with lim sup bound in Theorem 27 part (ii). If $\beta = 0$, then the limit in Theorem 28 part (b)(ii) agrees with lim inf bound in Theorem 27 part (ii). This demonstrates that our analysis for general f is quite sharp. □

Remark 23. The exponent in Theorem 28 part (b) is always closer to the true exponent of unity than in part (a). This demonstrates that when $-T' \in RV_0(-1)$ that the method outperforms the transformation when $-T' \in RV_0(-\mu-1)$ *for any other choice of* μ. $\square$

Remark 24. $T \in RV_0(0)$ gives better asymptotic performance than when $T \in RV_0(-\mu)$ but worse performance than the true exponent of 1.

Proof. Since $e^x > 1+x$ for all $x \geq 0$ then $e^{\mu\Delta} > 1+\mu\Delta$ and $e^{-\Delta} < \left(\frac{1}{1+\mu\Delta}\right)^{1/\mu}$. Hence

$$\lambda_\beta(\mu) := \frac{1}{\Delta}\int_{\left(\frac{1}{1+\mu\Delta}\right)^{1/\mu}}^{1} \lambda^{-\beta}\,d\lambda < \frac{1}{\Delta}\int_{e^{-\Delta}}^{1} \lambda^{-\beta}\,d\lambda =: \lambda_\beta(0).$$

For $0 \leq \beta < 1$ then

$$\lambda_\beta(0) = \frac{1}{\Delta}\int_{e^{-\Delta}}^{1} \lambda^{-\beta}\,d\lambda = \frac{1-e^{-\Delta(1-\beta)}}{\Delta(1-\beta)} < 1.$$

Thus

$$\lim_{t\to T_\xi^-} \frac{\bar{F}(x(t))}{T_\xi - t} = 1 > \lambda_\beta(0) > \lambda_\beta(\mu).$$

Thus $T \in RV_0(0)$ gives better asymptotic performance than $T \in RV_0(\mu)$. Also $\lim_{\mu\to 0^+}\lambda_\beta(\mu) = \lambda_\beta(0)$ and $\mu \mapsto \lambda_\beta(\mu)$ is decreasing. $\square$

The theoretical asymptotic performance of transformed methods suggests, at least in the class of regularly varying transforms, that the best transformations should be those for which $-T' \in RV_0(-1)$. This forces $T \in RV_0(0)$ and $\lim_{x\to 0^+} T(x) = \infty$. However, to date a vital practical consideration has been omitted; in order to recover the values of $x(t)$ at $t = t_n$ it is necessary to calculate $x_n = T^{-1}(z_n)$. In other words, in order that the algorithm be practicable it is necessary that $z \mapsto T^{-1}(z)$ should be computable in closed-form for all z and that $(-T'f)$ is a decreasing function. A simple choice of T which places an irrestrictive condition on f is $T(z) := -\log z$ for which $T^{-1}(z) = e^{-z}$ and $x \mapsto f(x)/x$ is decreasing. Henceforth, we shall use this choice of T and refer to our transformation scheme as being "logarithmically pre-transformed". For this transformation we now show that the numerical method recovers the value of the finite hitting time to within $O(\Delta)$ as $\Delta \to 0^+$.

To emphasise the dependence on the initial data, we are writing $\hat{T}_\xi$ in place of $\hat{T}_h$ in the theorem and its proof.

Theorem 29. *Suppose* f *obeys* (1.7) *while* $h(x) = \Delta x/f(x)$, $\Delta > 0$. *Let* T_ξ *and* $\hat{T}_\xi$ *be defined by* (1.8) *and* (3.16) *while* $T(x) := -\log x$.

(i) If f *is increasing, then*

$$\frac{\Delta\xi}{f(\xi)} + \left(\frac{\Delta}{e^{\Delta}-1}\right)T_\xi < \hat{T}_\xi < \left(\frac{\Delta}{1-e^{-\Delta}}\right)T_\xi.$$

(ii) If $x \mapsto x/f(x)$ is increasing, then

$$0 < \hat{T}_\xi - T_\xi < \frac{\Delta\xi}{f(\xi)}.$$

In both cases $\lim_{\Delta\to 0^+} \hat{T}_\xi = T_\xi$ and $|\hat{T}_\xi - T_\xi| = O(\Delta)$ as $\Delta \to 0^+$.

Proof. Since $T(x) := -\log x$ then $T^{-1}(x) = e^{-x}$, and $-T'(x) = 1/x$. Thus

$$z_{j+1} = z_j + \Delta T^{-1}(z_j)(-T')(T^{-1}(z_j)) = z_j + \Delta.$$

Therefore $z_n = z_0 + n\Delta$ and hence $x_n = \xi e^{-n\Delta}$. Thus $x_n/x_{n+1} = e^{\Delta}$ and $t_n = \sum_{j=0}^{n-1} h(x_j)$ obeys $\hat{T}_\xi := \lim_{n\to\infty} t_n < \infty$ since $\int_{0^+}^{1} 1/f(u)\,du < \infty$ because f obeys (1.7). Thus

$$\hat{T}_\xi = \sum_{j=0}^{\infty} h(x_j) = \frac{\Delta}{1-e^{-\Delta}} \sum_{j=0}^{\infty} \frac{x_j - x_{j+1}}{f(x_j)}.$$

If f is increasing for $x_{j+1} < u < x_j$ then we have

$$\frac{1}{f(x_j)} < \frac{1}{f(u)} < \frac{1}{f(x_{j+1})}. \tag{5.25}$$

while if $x \mapsto x/f(x)$ is increasing $x_{j+1} < u < x_j$ then

$$\frac{x_{j+1}}{f(x_{j+1})} \cdot \frac{1}{u} < \frac{1}{f(u)} < \frac{x_j}{f(x_j)} \cdot \frac{1}{u}. \tag{5.26}$$

Integrating (5.25) over $[x_{j+1}, x_j]$ yields

$$\frac{x_j - x_{j+1}}{f(x_j)} < \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du < \frac{x_j - x_{j+1}}{f(x_{j+1})}, \tag{5.27}$$

while integrating (5.26) over $[x_{j+1}, x_j]$ yields

$$\frac{\Delta x_{j+1}}{f(x_{j+1})} = \frac{x_{j+1}}{f(x_{j+1})} \log\left(\frac{x_j}{x_{j+1}}\right) < \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du < \frac{x_j}{f(x_j)} \log\left(\frac{x_j}{x_{j+1}}\right) = \frac{\Delta x_j}{f(x_j)}. \tag{5.28}$$

If $x \mapsto x/f(x)$ is increasing by (5.28)

$$T_\xi = \int_0^\xi \frac{1}{f(u)}\,du = \sum_{j=0}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du < \sum_{j=0}^{\infty} \frac{\Delta x_j}{f(x_j)} = \hat{T}_\xi,$$

and

$$\hat{T}_\xi = \frac{\Delta\xi}{f(\xi)} + \sum_{j=1}^{\infty} \frac{\Delta x_j}{f(x_j)} = \frac{\Delta\xi}{f(\xi)} + \sum_{j=0}^{\infty} \frac{\Delta x_{j+1}}{f(x_{j+1})} < \frac{\Delta\xi}{f(\xi)} + \sum_{j=0}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du.$$

The first inequality establishes $0 < \hat{T}_\xi - T_\xi$ while the second establishes $\hat{T}_\xi \leq \Delta\xi/f(\xi) + T_\xi$. Combining both inequalities yields

$$0 < \hat{T}_\xi - T_\xi < \frac{\Delta\xi}{f(\xi)},$$

which completes the proof of part (ii). Clearly in this case $\hat{T}_\xi - T_\xi$ is $O(\Delta)$ as $\Delta \to 0^+$.

If f is increasing by (5.27)

$$\hat{T}_\xi = \frac{\Delta}{1-e^{-\Delta}} \sum_{j=0}^{\infty} \frac{x_j - x_{j+1}}{f(x_j)} < \frac{\Delta}{1-e^{-\Delta}} \sum_{j=0}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du = \left(\frac{\Delta}{1-e^{-\Delta}}\right) T_\xi,$$

and also by (5.27) we have

$$\begin{aligned}
\hat{T}_\xi = \frac{\Delta}{1-e^{-\Delta}} \left\{ \frac{x_0 - x_1}{f(x_0)} + \sum_{j=1}^{\infty} \frac{x_j - x_{j+1}}{f(x_j)} \right\} \quad &= \quad \frac{\Delta\xi}{f(\xi)} + \frac{\Delta}{1-e^{-\Delta}} \cdot \frac{1-e^{-\Delta}}{e^{\Delta}-1} \sum_{j=0}^{\infty} \frac{x_j - x_{j+1}}{f(x_{j+1})} \\
&> \quad \frac{\Delta\xi}{f(\xi)} + \frac{\Delta}{e^{\Delta}-1} \sum_{j=0}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du \\
&= \quad \frac{\Delta\xi}{f(\xi)} + \left(\frac{\Delta}{e^{\Delta}-1}\right) T_\xi.
\end{aligned}$$

Hence

$$\frac{\Delta\xi}{f(\xi)} + \left(\frac{\Delta}{e^{\Delta}-1}\right) T_\xi < \hat{T}_\xi < \left(\frac{\Delta}{1-e^{-\Delta}}\right) T_\xi,$$

which is part (ii). Finally to see that $|\hat{T}_\xi - T_\xi|$ is $O(\Delta)$ as $\Delta \to 0^+$ consider

$$\frac{\Delta\xi}{f(\xi)} + \left(\frac{\Delta}{e^{\Delta}-1} - 1\right) T_\xi < \hat{T}_\xi - T_\xi < \left(\frac{\Delta}{1-e^{-\Delta}} - 1\right) T_\xi.$$

Therefore

$$\frac{\Delta\xi}{f(\xi)} + \left(\frac{\Delta - (e^{\Delta}-1)}{e^{\Delta}-1}\right) T_\xi < \hat{T}_\xi - T_\xi < \left(\frac{\Delta - (1-e^{-\Delta})}{1-e^{-\Delta}}\right) T_\xi.$$

Taylor expansions of e^{Δ} and $e^{-\Delta}$ as $\Delta \to 0^+$ imply that

$$e^{\Delta} - 1 = \Delta + \frac{\Delta^2}{2} + O(\Delta^3) \quad \text{and} \quad 1 - e^{-\Delta} = \Delta - \frac{\Delta^2}{2} + O(\Delta^3).$$

Hence as $\Delta \to 0^+$

$$\frac{\Delta - (e^{\Delta}-1)}{e^{\Delta}-1} = \frac{-\Delta}{2} + o(\Delta) \quad \text{and} \quad \frac{\Delta - (1-e^{-\Delta})}{1-e^{-\Delta}} = \frac{\Delta}{2} + o(\Delta).$$

Thus

$$\frac{\Delta\xi}{f(\xi)} + \left(\frac{-\Delta}{2} + o(\Delta)\right) T_\xi < \hat{T}_\xi - T_\xi < \left(\frac{\Delta}{2} + o(\Delta)\right) T_\xi,$$

as needed. □

One may object to the supposition (5.3) namely that $x \mapsto (-T^{-1}f)(x)$ is decreasing because this creates an interaction between the transformation and the non-linearity f and we might prefer to make transformations which are to an extent f-independent. In the next theorem we merely assume that f is increasing and show that it is possible to obtain good quality results which are qualitatively independent of $\Delta > 0$ but which improve quantitatively as $\Delta \to 0^+$.

Theorem 30. *Suppose T obeys (5.4) while f obeys (3.1) and is increasing but with (5.3) suppressed. Let $\bar{F}$, F, (t_n) and $\hat{T}_h$ be defined by (1.10), (1.11), (5.6) and (3.16).*

(i) If f obeys (1.9), then

$$\frac{1-e^{-\Delta}}{\Delta} \le \liminf_{n\to\infty} \frac{F(x_n)}{t_n} \le \limsup_{n\to\infty} \frac{F(x_n)}{t_n} \le \frac{e^{\Delta}-1}{\Delta}.$$

(ii) If f obeys (1.7), then

$$\frac{1-e^{-\Delta}}{\Delta} \le \liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \le \limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \le \frac{e^{\Delta}-1}{\Delta}.$$

Proof. If f is increasing only

$$\int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du \ge \frac{x_j - x_{j+1}}{f(x_j)} = \frac{x_j - x_{j+1}}{\Delta x_j} \cdot \frac{\Delta x_j}{f(x_j)} = \frac{1}{\Delta}\left(1 - \frac{x_{j+1}}{x_j}\right) h(x_j).$$

Similarly

$$\int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du \le \frac{x_j - x_{j+1}}{f(x_{j+1})} = \frac{1}{\Delta}\left(\frac{x_j}{x_{j+1}} - 1\right) h(x_{j+1}).$$

Thus

$$\frac{1}{\Delta}\left(1 - \frac{x_{j+1}}{x_j}\right) h(x_j) \le \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du \le \frac{1}{\Delta}\left(\frac{x_j}{x_{j+1}} - 1\right) h(x_{j+1}).$$

Since $-T' \in RV_0(-1)$ then $x_{j+1}/x_j \to e^{-\Delta}$ as $j \to \infty$ so $x_j/x_{j+1} \to e^{\Delta}$ as $j \to \infty$. We are now in a position to prove part (ii). If f obeys (1.7) then $(\int_{x_{j+1}}^{x_j} 1/f(u)\,du)$ is summable so $((1 - x_{j+1}/x_j)\,h(x_j)/\Delta)$ is summable and as $x_{j+1}/x_j \to e^{-\Delta}$, then $(h(x_j))$ is summable. Thus in the case that $\int_{0^+}^1 1/f(u)\,du < \infty$, we have as $n \to \infty$

$$\bar{F}(x_n) = \sum_{j=n}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du \ge \frac{1}{\Delta}\sum_{j=n}^{\infty}\left(1 - \frac{x_{j+1}}{x_j}\right) h(x_j) \sim \frac{\left(1-e^{-\Delta}\right)}{\Delta}\left(\hat{T}_h - t_n\right).$$

Thus

$$\liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \ge \frac{1-e^{-\Delta}}{\Delta}.$$

Similarly as $n \to \infty$

$$\bar{F}(x_{n+1}) < \bar{F}(x_n) \le \frac{1}{\Delta}\sum_{j=n}^{\infty}\left(\frac{x_j}{x_{j+1}} - 1\right) h(x_{j+1}) \sim \frac{\left(e^{\Delta}-1\right)}{\Delta}\left(\hat{T}_h - t_{n+1}\right).$$

Thus

$$\limsup_{n\to\infty} \frac{\bar{F}(x_{n+1})}{\hat{T}_h - t_{n+1}} \le \frac{e^{\Delta}-1}{\Delta},$$

and therefore

$$\limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \le \frac{e^{\Delta}-1}{\Delta}.$$

Combining the results yields

$$\frac{1-e^{-\Delta}}{\Delta} \le \liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \le \limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \le \frac{e^{\Delta}-1}{\Delta},$$

proving part (ii). To prove part (i), if f obeys (1.9) then $\int_{0+}^{1} 1/f(u)\,du = \infty$ and $(\int_{x_{j+1}}^{x_j} 1/f(u)\,du)$ is not summable so $((x_j/x_{j+1} - 1)\,h(x_{j+1})/\Delta)$ is not summable and hence $t_n \to \infty$ as $n \to \infty$. Thus as $n \to \infty$

$$\begin{aligned} F(x_n) - F(x_0) &= \sum_{j=0}^{n-1}\int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du \le \frac{1}{\Delta}\sum_{j=0}^{n-1}\left(\frac{x_j}{x_{j+1}} - 1\right) h(x_{j+1}) \\ &\sim \frac{\left(e^{\Delta}-1\right)}{\Delta}\sum_{j=1}^{n} h(x_j) \sim \frac{\left(e^{\Delta}-1\right)}{\Delta}\cdot t_{n+1} \sim \frac{\left(e^{\Delta}-1\right)}{\Delta}\cdot t_n. \end{aligned}$$

Thus

$$\limsup_{n\to\infty} \frac{F(x_n)}{t_n} \le \frac{e^{\Delta}-1}{\Delta}.$$

A similar argument for the lower estimate yields as $n \to \infty$

$$F(x_n) - F(x_0) \ge \frac{1}{\Delta}\sum_{j=0}^{n-1}\left(1 - \frac{x_{j+1}}{x_j}\right) h(x_j) \sim \frac{\left(1-e^{-\Delta}\right)}{\Delta}\cdot t_n.$$

Thus

$$\liminf_{n\to\infty} \frac{F(x_n)}{t_n} \ge \frac{1-e^{-\Delta}}{\Delta},$$

which combined with the upper estimate gives.

$$\frac{1-e^{-\Delta}}{\Delta} \le \liminf_{n\to\infty} \frac{F(x_n)}{t_n} \le \limsup_{n\to\infty} \frac{F(x_n)}{t_n} \le \frac{e^{\Delta}-1}{\Delta},$$

which is part (i). □

5.3 Logarithmic Transformation with Larger Step-Size

In the previous section we showed in the case when $x \mapsto f(x)/x$ is decreasing and $\int_{0+}^{1} 1/f(u) = \infty$ that

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = 1,$$

under logarithmic pre-transformation with step-size $h(x) = \Delta x/f(x)$. Since this recovers exactly the asymptotic rate of decay of the solution of the ODE, one naturally asks whether it might be possible to take a larger vanishing step-size and still recover a satisfactory recovery of the rate of decay such as

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = \lambda \in (0,\infty),$$

or even $\lambda = 1$. We reconsider the ODE (1.1). Define $z(t) := -\log x(t)$ with $\eta(z) := f(e^{-z})/e^{-z}$, the transformed ODE is

$$z'(t) = \eta(z(t)), \quad t > 0, \quad z(0) = -\log \xi > 0.$$

The associated Explicit Euler scheme is

$$z_{n+1} = z_n + \tilde{h}(z_n)\eta(z_n), \quad n \geq 0, \quad z_0 = -\log \xi,$$
$$x_{n+1} = e^{-z_{n+1}}, \quad n \geq 0, \quad x_0 = \xi > 0,$$

where

$$t_{n+1} = \sum_{j=0}^{n} \tilde{h}(z_j), \quad n \geq 0, \quad t_0 = 0, \tag{5.29}$$

and $\tilde{h}(z) := h(e^{-z})$ where $z \in \mathbb{R}$ and h obeys (3.2). Thus $t_{n+1} = \sum_{j=0}^{n} \tilde{h}(z_j) = \sum_{j=0}^{n} h(e^{-z_j}) = \sum_{j=0}^{n} h(x_j)$. Define

$$\Delta(x) := \frac{f(x)h(x)}{x}, \quad x > 0, \tag{5.30}$$

and $\tilde{\Delta}(z) := \Delta(e^{-z})$. We suppose

$$\lim_{x\to 0^+} \frac{f(x)h(x)}{x} = \infty; \tag{5.31}$$

$$x \mapsto x/f(x) \text{ is increasing.} \tag{5.32}$$

Thus by (5.30)

$$t_{n+1} = \sum_{j=0}^{n} h(x_j) = \sum_{j=0}^{n} \frac{\Delta(x_j)x_j}{f(x_j)}, \quad n \geq 0. \tag{5.33}$$

Then $z_{n+1} = z_n + \tilde{h}(z_n)\eta(z_n)$ implies

$$-\log x_{n+1} = z_{n+1} = z_n + h(e^{-z_n}) \cdot \frac{f(e^{-z_n})}{e^{-z_n}} \quad = \quad -\log x_n + \Delta(x_n).$$

Therefore

$$x_{n+1} = x_n e^{-\Delta(x_n)}, \quad n \geq 0, \quad x_0 = \xi > 0. \tag{5.34}$$

Proposition 8. *Suppose f obeys* (3.1) *and h obeys* (3.2)*. There exists a unique positive sequence (x_n) which obeys* (5.34) *and any such sequence is decreasing and obeys $x_n \to 0$ as $n \to \infty$.*

Proof. The sequence exists by construction. Since $\Delta(x) > 0$ for all $x > 0$, we have that (x_n) is a positive decreasing sequence. Since (x_n) is decreasing, we have as $n \to \infty$ that $x_n \to L \in [0, \infty)$. If $L > 0$ since Δ is continuous, (5.34) yields

$$L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n e^{-\Delta(x_n)} = Le^{-\Delta(L)},$$

thus $\Delta(L) = 0$ which is impossible by (3.1) and (3.2). Hence $x_n \to 0$ as $n \to \infty$. □

Theorem 31. *Suppose f obeys* (1.9) *and* (5.31) *holds. Let (x_n), (t_n), F and Δ be defined by* (5.34), (5.33), (1.11) *and* (5.30)*. If*

$$\lim_{x\to 0^+} \frac{1}{h(x)} \int_{xe^{-\Delta(x)}}^{x} \frac{1}{f(u)}\, du = 1, \tag{5.35}$$

then (x_n) for all $n \geq 0$, $x_n \to 0$ as $n \to \infty$, $t_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = 1.$$

Proof. Since f obeys (1.9) then $\int_{0^+}^{1} 1/f(u)\, du = \infty$. If F is defined by (1.11) then $F(x) \to \infty$ as $x \to 0^+$, so $F(x_n) \to \infty$ as $n \to \infty$. Then, as $x_n \to 0^+$ as $n \to \infty$, we have from (5.34) and (5.35)

$$\lim_{n\to\infty} \frac{\int_{x_{n+1}}^{x_n} 1/f(u)\, du}{\Delta(x_n)x_n/f(x_n)} = \lim_{n\to\infty} \frac{\int_{x_n e^{-\Delta(x_n)}}^{x_n} 1/f(u)\, du}{\Delta(x_n)x_n/f(x_n)} = \lim_{x\to 0^+} \frac{1}{h(x)} \int_{xe^{-\Delta(x)}}^{x} \frac{1}{f(u)}\, du = 1. \tag{5.36}$$

Therefore, it follows that $\sum_{j=0}^{n-1} \Delta(x_j)x_j/f(x_j) \to \infty$ as $n \to \infty$. Moreover, this implies $t_n \to \infty$ as $n \to \infty$, and by Toeplitz's Lemma, (5.33) and (5.36)

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = \lim_{n\to\infty} \left(\frac{F(x_0)}{t_n} + \frac{\sum_{j=0}^{n-1} \int_{x_{j+1}}^{x_j} 1/f(u)\, du}{\sum_{j=0}^{n-1} \Delta(x_j)x_j/f(x_j)} \right) = 1.$$

as claimed. □

Our goal now is to determine explicit asymptotic conditions on Δ in terms of f for which the limit (5.35) holds. Clearly, as presently stated (5.35) gives only implicit information as to how Δ might be so chosen. In the following lemma we start to rectify this problem.

Lemma 18. *Suppose f obeys* (3.1) *and $x \mapsto x/f(x) =: \delta(x)$ is asymptotically increasing. If*

$$\lim_{x\to 0^+} \frac{\delta(xe^{-\Delta(x)})}{\delta(x)} = 1, \tag{5.37}$$

then (5.35) *holds.*

Proof. Note that

$$\int_{xe^{-\Delta(x)}}^{x} \frac{1}{f(u)}\,du = F(xe^{-\Delta(x)}) - F(x).$$

Define $\tilde{F}(z) := F(e^{-z})$ and $\tilde{\delta}(z) := \delta(e^{-z})$. Notice that $\tilde{\delta}$ is asymptotically decreasing, so that $\tilde{\delta}(z) \sim \tilde{\delta}_1(z)$ and $\tilde{\delta}_1$ is decreasing as $z \to \infty$. Since f is continuous, then $F'(x) = -1/f(x)$ and $\tilde{F}'(z) = \tilde{\delta}(z)$. Let $x = e^{-z}$. Then by the Mean Value Theorem for every $z \in \mathbb{R}$ there is $c_z \in (0,1)$ so that

$$\begin{aligned} F(xe^{-\Delta(x)}) - F(x) = \tilde{F}(z+\tilde{\Delta}(z)) - \tilde{F}(z) &= \tilde{F}'(z + c_z\tilde{\Delta}(z))\cdot\tilde{\Delta}(z) \\ &= \tilde{\delta}(z + c_z\tilde{\Delta}(z))\cdot\tilde{\Delta}(z). \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{h(x)}\int_{xe^{-\Delta(x)}}^{x} \frac{1}{f(u)}\,du = \frac{F(xe^{-\Delta(x)}) - F(x)}{\Delta(x)x/f(x)} &= \frac{\tilde{\delta}(z + c_z\tilde{\Delta}(z))\cdot\tilde{\Delta}(z)}{\Delta(e^{-z})\cdot\delta(e^{-z})} \\ &= \frac{\tilde{\delta}(z + c_z\tilde{\Delta}(z))}{\tilde{\delta}(z)}. \end{aligned} \tag{5.38}$$

Since $\tilde{\Delta}(z) > 0$ and $c_z > 0$ then $z + c_z\tilde{\Delta}(z) \to \infty$ as $z \to \infty$. We write

$$\frac{\tilde{\delta}(z + c_z\tilde{\Delta}(z))}{\tilde{\delta}(z)} = \frac{\tilde{\delta}(z + c_z\tilde{\Delta}(z))}{\tilde{\delta}_1(z + c_z\tilde{\Delta}(z))}\cdot\frac{\tilde{\delta}_1(z + c_z\tilde{\Delta}(z))}{\tilde{\delta}_1(z)}\cdot\frac{\tilde{\delta}_1(z)}{\tilde{\delta}(z)}.$$

Since $\tilde{\delta}_1$ is decreasing we have

$$\limsup_{z\to\infty} \frac{\tilde{\delta}(z + c_z\tilde{\Delta}(z))}{\tilde{\delta}(z)} \le \lim_{z\to\infty}\left(\frac{\tilde{\delta}(z + c_z\tilde{\Delta}(z))}{\tilde{\delta}_1(z + c_z\tilde{\Delta}(z))}\cdot\frac{\tilde{\delta}_1(z)}{\tilde{\delta}(z)}\right) = 1.$$

Thus from (5.38)

$$\limsup_{x\to 0^+} \frac{1}{h(x)}\int_{xe^{-\Delta(x)}}^{x} \frac{1}{f(u)}\,du \le 1. \tag{5.39}$$

On the other hand, as $\tilde{\delta}_1$ is decreasing, and $c_z \in (0,1)$ then $\tilde{\delta}_1(z + c_z\tilde{\Delta}(z)) \ge \tilde{\delta}_1(z +$

$\tilde{\Delta}(z))$. Thus

$$\liminf_{z\to\infty} \frac{\tilde{\delta}(z+c_z\tilde{\Delta}(z))}{\tilde{\delta}(z)} \geq \lim_{z\to\infty} \left(\frac{\tilde{\delta}(z+c_z\tilde{\Delta}(z))}{\tilde{\delta}_1(z+c_z\tilde{\Delta}(z))} \cdot \frac{\tilde{\delta}_1(z)}{\tilde{\delta}_1(z)} \cdot \frac{\tilde{\delta}_1(z+\tilde{\Delta}(z))}{\tilde{\delta}_1(z)} \right) = 1, \qquad (5.40)$$

provided $\tilde{\delta}_1(z+\tilde{\Delta}(z))/\tilde{\delta}_1(z) \to 1$ as $z \to \infty$. This limit is implied by $\tilde{\delta}(z+\tilde{\Delta}(z))/\tilde{\delta}(z) \to 1$ as $z \to \infty$. We show momentarily that this limit follows from (5.37). From (5.38), the lim inf in (5.40) implies

$$\liminf_{x\to\infty} \frac{1}{h(x)} \int_{xe^{-\Delta(x)}}^{x} \frac{1}{f(u)}\, du \geq 1,$$

which combined with (5.38), gives (5.35). Lastly, with $x = e^{-z}$ we see that

$$1 = \lim_{z\to\infty} \frac{\tilde{\delta}(z+\tilde{\Delta}(z))}{\tilde{\delta}(z)} = \lim_{z\to\infty} \frac{\delta(e^{-z}e^{-\Delta(e^{-z})})}{\delta(e^{-z})} = \lim_{x\to 0^+} \frac{\delta(xe^{-\Delta(x)})}{\delta(x)},$$

by (5.37). This completes the proof. □

Theorem 32. *Suppose f obeys* (1.9) *and $x \mapsto x/f(x) =: \delta(x)$ is asymptotically increasing. Let (x_n), (t_n), F and Δ be defined by* (5.34), (5.33), (1.11) *and* (5.30). *If Δ obeys*

$$\lim_{x\to 0^+} \frac{\delta(xe^{-\Delta(x)})}{\delta(x)} = 1,$$

then, $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = 1.$$

Proof. The proof comes from combining Theorem 31 and Lemma 18. □

Remark 25. We are allowed to have $\Delta(x) \to \infty$ as $x \to 0^+$, once (5.30) and (5.32) hold. □

The condition (5.37) gives an implicit description of how rapidly Δ can be allowed to grow without changing asymptotic behaviour. Under stronger conditions on f, we can give an estimate on the allowable growth of Δ which is explicit. We will show presently that the estimate on Δ supplied by this result is sharp.

Proposition 9. *Suppose $f \in RV_0(1)$ and $f \in C^1(0,\delta)$. Define $f_1(x) := f(x)/x \to \infty$, and suppose that $f_1' < 0$ and*

$$\Delta \text{ is decreasing on } (0,\delta'),\ \Delta(x) \to \infty \text{ as } x \to 0^+; \qquad (5.41)$$

$$\lim_{x\to 0^+} \frac{\Delta(x)}{-f_1(x)/(xf_1'(x))} = 0. \qquad (5.42)$$

Then (5.37) *holds and hence* (5.35) *holds.*

The next result removes the smoothness restriction on f in Proposition 9 and demonstrates that Δ can be chosen tending to infinity in (5.41) and (5.42).

Proposition 10. *Suppose $f \in RV_0(1), x \mapsto f(x)/x$ is asymptotically decreasing. Then there exists $f_1 \in C^1(0,\delta)$ such that $f_1(x) \sim f(x)/x$ and $f_1'(x) < 0$ for $x \in (0,\delta)$ and*

$$\lim_{x \to 0^+} \frac{f_1(x)}{-x f_1'(x)} = \infty.$$

Moreover, if Δ obeys (5.41) *and* (5.42)*, then* (5.37) *holds and hence* (5.35) *holds.*

Proof. Suppose $f \in \mathrm{RV}_0(1)$, $x \mapsto f(x)/x$ is asymptotically decreasing. Suppose $\psi(x) \sim f(x)/x$ as $x \to 0^+$, is such that ψ is decreasing. Now we mimic the proof of Theorem 1.3.3. in [12]. Define $l_0(x) := \psi(1/x)$. Then l_0 is increasing. Set

$$q(x) = \begin{cases} \exp(-x^{-1} - (1-x)^{-1}), & 0 < x < 1, \\ 0, & \text{otherwise} . \end{cases}$$

Set

$$p(x) = \frac{q(x)}{\int_{0^+}^1 q(t)\,dt}, \quad x \in [0,1].$$

Let $h_0(x) := \log l_0(e^x)$ and $e(x) = [h(n+1) - h(n)]\, p(x-n)$ for $n \le x \le n+1$ for $n \in \mathbb{N}$ large enough for $[n,\infty)$ to lie in the domain of definition of l_0, $n \ge B$ say. Then e is C^∞ in each interval and also at the end points. Moreover $e^{(k)}(x) \to 0$ as $x \to \infty$ for all $k = 1, 2 \dots$. Define

$$h_1(x) := h(B) + \int_B^x e(t)\,dt.$$

Then $h_1^{(k)}(x) \to 0$ as $x \to \infty$ for all $k = 1, 2 \dots$. Also $h(x) - h_1(x) \to 0$ as $x \to \infty$. Then $l_0(x) \sim l_1(x) := \exp(h_1(\log x))$, and $h_1'(x) \to 0$ as $x \to \infty$. Now as l_0 is increasing, h is also. Thus $e(x) > 0$ for all x. Therefore $h_1'(x) > 0$ for all $x \ge B$. Hence

$$e_1'(x) = \frac{\exp'(h_1(\log x)) \cdot h_1'(\log x)}{x} > 0,$$

for all x sufficiently large. Since $h_1(x) = \log(l_1(e^x))$ and $h_1'(x) \to 0$ as $x \to \infty$. Thus

$$\lim_{x \to \infty} \frac{l_1'(x) x}{l_1(x)} = 0.$$

Moreover $l_1'(x) > 0$ for x sufficiently large. Now, define $f_1(x) := l_1(1/x)$. Then because $h_1'(x) > 0$, $f_1'(x) = -l_1'(1/x)/x^2 < 0$ and

$$\lim_{x \to 0^+} \frac{x f_1'(x)}{f_1(x)} = \lim_{x \to 0^+} \frac{x \cdot -l_1'(1/x)/x^2}{l_1(1/x)} = 0.$$

Also $f_1 \in RV_0(0)$ and

$$\frac{f_1(x)}{f(x)/x} \sim \frac{l_1(1/x)}{\psi(x)} = \frac{l_1(1/x)}{l_0(1/x)} \to 1, \quad \text{as } x \to 0^+.$$

Therefore if $f \in RV_0(1)$ and $x \mapsto f(x)/x$ is asymptotically decreasing, then there exists a number $\delta' > 0$, $f_1 \in C^1(0, \delta')$ such that

(i) $f_1 \in \mathrm{RV}_0(0)$, $f_1(x) \sim f(x)/x$ as $x \to 0^+$.

(ii) $f_1'(x) < 0$ for all $x \in (0, \delta')$, $-xf_1'(x)/f_1(x) \to 0$ as $x \to 0^+$.

Note that because $-xf_1'(x)/f_1(x) > 0$, $\forall\, x < 1/B$ and

$$\lim_{x\to 0^+} \frac{-f_1'(x)}{xf_1(x)} = 0,$$

then

$$\lim_{x\to 0^+} \frac{f_1(x)}{-xf_1'(x)} = \infty.$$

We next suppose that Δ is decreasing and that $\Delta(x) \to \infty$ as $x \to 0^+$ and $\Delta(x) = o\left(f_1(x)/(-xf_1'(x))\right)$ as $x \to 0^+$. Clearly, this second hypothesis does not prevent $\Delta(x) \to \infty$ as $x \to 0^+$. Also, as $x \to 0^+$

$$h(x) = \frac{\Delta(x)x}{f(x)} = o\left(\frac{f_1(x)}{-xf_1'(x)} \cdot \frac{x}{f(x)}\right) = o\left(\frac{1}{-xf_1'(x)}\right).$$

Next, write $l(x) = f_1(1/x)$. Since $f_1 \in C^1$, we have $l \in C^1$. Moreover $l(x) \to \infty$ as $x \to \infty$ and

$$\frac{l'(x)x}{l(x)} = \frac{-f_1'(1/x)/x}{f_1(1/x)}.$$

Define $\tilde{\eta}(x) := \Delta(1/x)$, $\eta(x) := \exp\left(\tilde{\eta}(x)\right)$. Since Δ is decreasing on $(0,\delta')$, $\tilde{\eta}$ is increasing on $(1/\delta', \infty)$ and hence so is η. Also $\eta(x) > 1$, and $\eta(x) \to \infty$ as $x \to \infty$ because $\Delta(x) \to \infty$ as $x \to 0^+$. Thus by (5.42)

$$\lim_{y\to\infty} \frac{\Delta(1/y)}{f_1(1/y)/(1/y \cdot -f_1'(1/y))} = 0,$$

so

$$\lim_{x\to\infty} \frac{l'(x)x/l(x)}{1/\log\eta(x)} = \lim_{x\to\infty} \frac{(-f_1'(1/x) \cdot 1/x)/f_1(1/x)}{1/\Delta(1/x)} = 0.$$

Thus

$$\frac{l'(x)x}{l(x)} = o\left(\frac{1}{\log\eta(x)}\right), \quad \text{as } x \to \infty.$$

Hence all properties of Lemma 14 hold, so statement (4.11) in Lemma 13 holds with $l(x) = f_1(1/x)$. Since η is increasing from Lemma 13, $l(x\eta(x)^c)/l(x) \to 1$ as $x \to \infty$ for any $c > 0$ (as we may take γ arbitrarily small in Lemma 13). Take $c = 1$, so that

$l(x\eta(x))/l(x) \to 1$ as $x \to \infty$. Also we have

$$l(x) = f_1(1/x) \sim \frac{f(1/x)}{1/x} = \frac{1}{\delta(1/x)}, \quad \text{as } x \to \infty.$$

Hence with $x = 1/y$, as $x \to 0^+$ (or $y \to \infty$), we have

$$\frac{\delta(x)}{\delta(xe^{-\Delta(x)})} = \frac{\delta(1/y)}{\delta\left(e^{-\Delta(1/y)}/y\right)} \sim \frac{1/l(y)}{1/l(ye^{\Delta(1/y)})} = \frac{l(ye^{\Delta(1/y)})}{l(y)} = \frac{l(y\eta(y))}{l(y)} \to 1,$$

since $\eta(y) = e^{\Delta(1/y)}$, as required. □

By Proposition 10 if

$$\Delta(x) = o\left(\frac{\delta(x)}{x\delta'(x)}\right), \quad \text{as } x \to 0^+,$$

where $\delta(x) := x/f(x)$ we have, under some extra conditions, that

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = 1.$$

It is natural to ask whether this size restriction on $\Delta(x)$ is sharp. In the next theorem we show under strengthened conditions on $\Delta(x)$ if there exists $K \in (0, \infty)$ such that $\Delta(x) \sim K\delta(x)/(x\delta'(x))$ then

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = \int_{0^+}^{1} e^{-Kv}\, dv.$$

This result can be easily read-off from Theorem 33 and the proof of Theorem 31. Theorem 33 part (iv) requires the following lemma which we present now.

Lemma 19. *If*

$$x \mapsto \Delta(x) - \gamma \log(1/x) \textit{ is increasing for some } \gamma > 0, \tag{5.43}$$

and $\bar{g}(x) := \bar{\Delta}(e^x)$, $\bar{\Delta}(x) := \Delta(1/x)$, *then*

$$g_\gamma(x) := \bar{g}(x) - \gamma x \textit{ is decreasing for some } \gamma > 0.$$

Proof. By construction

$$g_\gamma(x) = \Delta(e^{-x}) - \gamma x = \Delta(e^{-x}) + \gamma \log(e^{-x}) = \Delta(y) + \gamma \log(y)\Big|_{y=e^{-x}} = G_\gamma(e^{-x}),$$

where $G_\gamma(x) := \Delta(x) + \gamma \log x = \Delta(x) - \gamma \log(1/x)$. Now G_γ is increasing if and only if g_γ is decreasing, as claimed. □

Theorem 33. *Suppose that* $\delta' \in RV_0(-1)$ *where*

$$x \mapsto x/f(x) = \delta(x) \text{ is increasing;}$$
$$x \mapsto \Delta(x) \text{ is decreasing, } \Delta(x) \to \infty, \text{ as } x \to 0^+\text{; and}$$
$$\lim_{x\to 0^+} \frac{x\delta'(x)}{\delta(x)}\Delta(x) = K \in [0,\infty] \tag{5.44}$$

(i) If $K = 0$, *then*

$$\lim_{x\to 0^+} \frac{F(xe^{-\Delta(x)}) - F(x)}{\Delta(x)x/f(x)} = 1.$$

(ii) If $K > 0$, *then*

$$\int_{0^+}^1 e^{-Kv}\,dv \le \liminf_{x\to 0^+} \frac{F(xe^{-\Delta(x)}) - F(x)}{\Delta(x)x/f(x)} \le \limsup_{x\to 0^+} \frac{F(xe^{-\Delta(x)}) - F(x)}{\Delta(x)x/f(x)} \le 1. \tag{5.45}$$

(iii) If $K > 0$, *and*

(a) $\bar{g}$ *is self-neglecting; or*

(b) $x \mapsto \Delta(x) - \gamma\log(1/x)$ *is increasing for all* $\gamma > 0$

then

$$\lim_{x\to 0^+} \frac{F(xe^{-\Delta(x)}) - F(x)}{\Delta(x)x/f(x)} = \int_{0^+}^1 e^{-Kv}\,dv. \tag{5.46}$$

(iv) If $K > 0$ *and* $x \mapsto \Delta(x) - \gamma\log(1/x)$ *is increasing for some* $\gamma > 0$ *then*

$$\int_{0^+}^1 e^{-Kv}\,dv \le \liminf_{x\to 0^+} \frac{F(xe^{-\Delta(x)}) - F(x)}{\Delta(x)x/f(x)} \le \limsup_{x\to 0^+} \frac{F(xe^{-\Delta(x)}) - F(x)}{\Delta(x)x/f(x)} \le \int_{0^+}^1 e^{-Kv/(1+v\gamma)}\,dv \le 1.$$

Proof. Define $\tilde{F}(z) := F(e^{-z})$, $\tilde{\delta}(z) := \delta(e^{-z})$ and $\tilde{\Delta}(z) := \Delta(e^{-z})$. Let $x = e^{-z}$; then $x \to 0$ if $z \to \infty$. Write

$$\begin{aligned} F\left(xe^{-\Delta(x)}\right) - F(x) = F\left(e^{-z}e^{-\Delta(e^z)}\right) - F\left(e^{-z}\right) &= F\left(e^{-(z\tilde{\Delta}(z))}\right) - F\left(e^{-z}\right) \\ &= \tilde{F}\left(z + \tilde{\Delta}(z)\right) - \tilde{F}(z). \end{aligned}$$

Thus

$$\frac{F\left(xe^{-\Delta(x)}\right) - F(x)}{\Delta(x)x/f(x)} = \frac{\tilde{F}\left(z+\tilde{\Delta}(z)\right) - \tilde{F}(z)}{\Delta(e^{-z})e^{-z}/f(e^{-z})} = \frac{\tilde{F}\left(z+\tilde{\Delta}(z)\right) - \tilde{F}(z)}{\tilde{\Delta}(z)\tilde{\delta}(z)}.$$

Hence

$$\tilde{F}(z) = F(e^{-z}) = \int_{e^{-z}}^{1} \frac{1}{f(u)}\,du = \int_{z}^{0} \frac{-e^{-w}}{f(e^{-w})}\,dw = \int_{0}^{z} \tilde{\delta}(w)\,dw.$$

Our goal is to show that

$$\tilde{\lambda}(z) := \frac{\tilde{F}\left(z+\tilde{\Delta}(z)\right) - \tilde{F}\left(z\right)}{\tilde{\Delta}(z)\tilde{\delta}(z)} = \frac{\int_{z}^{z+\tilde{\Delta}(z)} \tilde{\delta}(w)\,dw}{\tilde{\Delta}(z)\tilde{\delta}(z)} = \int_{0^+}^{1} \frac{\tilde{\delta}(z+v\tilde{\Delta}(z))}{\tilde{\delta}(z)}\,dv, \qquad (5.47)$$

tends to a limit. We attempt to prove that

$$\lim_{z\to\infty} \frac{\tilde{\delta}(z+v\tilde{\Delta}(z))}{\tilde{\delta}(z)} = e^{-Kv}, \quad \text{uniformly for } v \in [0,1]. \qquad (5.48)$$

This implies (5.46). We find it useful to record the identity

$$\frac{\tilde{\delta}(z+v\tilde{\Delta}(z))}{\tilde{\delta}(z)} = \frac{\delta(e^{-(z+v\tilde{\Delta}(z))})}{\delta(e^{-z})} = \frac{\delta(e^{-z}e^{-v\Delta(e^{-z})})}{\delta(e^{-z})} = \frac{\delta(xe^{-v\Delta(x)})}{\delta(x)}.$$

Next define $\delta_1(x) := x\delta'(x)/\delta(x)$, $\bar{\delta}(x) := 1/\delta(1/x)$ and $\bar{\Delta}(x) := \Delta(1/x)$. Then (5.44) implies $\Delta \in RV_0(0)$ if $K > 0$; $\bar{\Delta}(x)$ and $\bar{\delta}(x)$ both increase to infinity as $x \to 0^+$. Hence $x\bar{\delta}'(x)/\bar{\delta}(x) = \delta_1(1/x)$. Thus by (5.44)

$$\lim_{z\to\infty} \frac{z\bar{\delta}'(z)}{\bar{\delta}(z)}\bar{\Delta}(z) = \lim_{z\to\infty} \delta_1\left(\tfrac{1}{z}\right)\Delta\left(\tfrac{1}{z}\right) = \lim_{x\to 0^+} \delta_1(x)\,\Delta(x) = K.$$

Then for $\lambda > 1$ as $z \to \infty$ we get

$$\begin{aligned}\log\left(\frac{\bar{\delta}(\lambda z)}{\bar{\delta}(x)}\right) = \int_{z}^{\lambda z} \frac{\bar{\delta}'(u)}{\bar{\delta}(u)}\,du \quad &= \quad \int_{z}^{\lambda z} \frac{u\bar{\delta}'(u)}{\bar{\delta}(u)} \cdot \frac{1}{u}\,du \\ &\sim \quad \int_{z}^{\lambda z} \frac{1}{u} \cdot \frac{K}{\bar{\Delta}(u)}\,du \sim \frac{K}{\bar{\Delta}(z)} \int_{z}^{\lambda z} \frac{1}{u}\,du,\end{aligned}$$

since $\bar{\Delta} \in RV_\infty(0)$. Therefore $\log\left(\bar{\delta}(\lambda z)/\bar{\delta}(z)\right) \sim K\log\lambda/\bar{\Delta}(z)$ as $z \to \infty$. Define $\bar{h}(z) := \log\bar{\delta}(e^z)$, $\bar{g}(z) := \bar{\Delta}(e^z)$ and $u_0 := \log\lambda$. Then as $z \to \infty$

$$\begin{aligned}\bar{h}(z+u_0) - \bar{h}(z) = \log\bar{\delta}(e^{z+u_0}) - \log\bar{\delta}(e^z) = \log\left(\frac{\bar{\delta}(e^{z+u_0})}{\bar{\delta}(e^z)}\right) \quad &= \quad \frac{K\log\lambda}{\bar{g}(z)} \\ &= \quad \frac{Ku_0}{\bar{g}(z)}. \qquad (5.49)\end{aligned}$$

Next we have the identity

$$\begin{aligned}\log\left(\frac{\bar{\delta}(e^z)}{\bar{\delta}(e^z e^{v\bar{\Delta}(e^z)})}\right) = \log\bar{\delta}(e^z) - \log\bar{\delta}(e^z e^{v\bar{\Delta}(e^z)}) \quad &= \quad \log\bar{\delta}(e^z) - \log\bar{\delta}(e^z e^{v\bar{g}(z)}) \\ &= \quad \bar{h}(z) - \bar{h}(x+v\bar{g}(z)).\end{aligned}$$

Also
$$\frac{\tilde{\delta}(z+v\tilde{\Delta}(z))}{\tilde{\delta}(z)} = \frac{\delta(xe^{-v\Delta(x)})}{\delta(x)} = \frac{\bar{\delta}(1/x)}{\bar{\delta}(1/x\cdot e^{v\bar{\Delta}(1/x)})} = \frac{\bar{\delta}(e^z)}{\bar{\delta}(e^z e^{v\bar{\Delta}(z)})}.$$

Hence
$$\log\left(\frac{\tilde{\delta}(z+v\tilde{\Delta}(z))}{\tilde{\delta}(z)}\right) = \bar{h}(z) - \bar{h}(z+v\bar{g}(z)). \tag{5.50}$$

Let $\eta_z := \lfloor \bar{g}(z)v/u_0 \rfloor$, $\eta_z \le \bar{g}(z)v/u_0$, $\eta_z + 1 > \bar{g}(z)v/u_0$. Thus

$$\bar{h}(z+v\bar{g}(x)) - \bar{h}(z) \sum_{k=1}^{\eta_z} \left(\bar{h}(z+ku_0) - \bar{h}(z+(k-1)u_0)\right) + \left(\bar{h}(z+v\bar{g}(z)) - \bar{h}(z+\eta_z u_0)\right).$$

Thus by (5.49) for every $\epsilon \in (0,1)$, there is $z(\epsilon) > 0$ such that $\forall\, z > z(\epsilon)$

$$(1-\epsilon)\cdot\frac{Ku_0}{\bar{g}(z)} < \bar{h}(z+u_0) - \bar{h}(z) < (1+\epsilon)\cdot\frac{Ku_0}{\bar{g}(z)},$$

Thus for $z > z(\epsilon)$, as $\bar{h}$ is increasing

$$\bar{h}(z+v\bar{g}(x)) - \bar{h}(z) = (1+\epsilon)\cdot Ku_0 \sum_{k=1}^{\eta_z} \frac{1}{\bar{g}(z+(k-1)u_0)} + (1+\epsilon)\cdot\frac{Ku_0}{\bar{g}(z+\eta_z u_0)}. \tag{5.51}$$

and similarly

$$\bar{h}(z+v\bar{g}(x)) - \bar{h}(z) \ge (1-\epsilon)\cdot Ku_0 \sum_{k=1}^{\eta_z} \frac{1}{\bar{g}(z+(k-1)u_0)}. \tag{5.52}$$

Now $\bar{g}(z) < \bar{g}(z+(k-1)u_0) < \bar{g}(z+(\eta_z-1)u_0)$, so

$$\frac{1}{\bar{g}(z)} > \frac{1}{\bar{g}(z+(k-1)u_0)} > \frac{1}{\bar{g}(z+(\eta_z-1)u_0)}.$$

Therefore from (5.51)

$$\begin{aligned}\bar{h}(z+v\bar{g}(z)) - \bar{h}(z) < (1+\epsilon)\cdot\frac{Ku_0(\eta_z+1)}{\bar{g}(z)} &< (1+\epsilon)\cdot\frac{Ku_0(\eta_z+1)}{\bar{g}(z)}\left(\frac{\bar{g}(z)v}{u_0}+1\right)\\ &< (1+\epsilon)\cdot Kv + (1+\epsilon)\cdot\frac{Ku_0}{\bar{g}(z)}.\end{aligned}$$

Thus
$$\bar{h}(z+v\bar{g}(z)) - \bar{h}(z) < K(1+\epsilon)v + \epsilon, \quad \forall\, z > z(\epsilon). \tag{5.53}$$

From (5.53), (5.50) and (5.47) we can readily establish the lower limit in part (ii) which together with the trivial unit upper bound (5.39) completes the proof of part (ii). To prove part (iii) we prepare estimates that enable us to exploit the self-neglecting character of $\bar{g}$. For $z > Z(\epsilon)$, $\eta_z \le \bar{g}(z)v/u_0$, $\eta_z + 1 > \bar{g}(z)v/u_0$ and so from (5.52)

$$\begin{aligned}
\bar{h}(z+v\bar{g}(z)) - \bar{h}(z) &\ge (1-\epsilon)\cdot Ku_0 \sum_{k=1}^{\eta_z} \frac{1}{\bar{g}(z+(k-1)u_0)} \\
&\ge (1-\epsilon)\cdot \frac{Ku_0\eta_z}{\bar{g}(z+(\eta_z-1)u_0)} \\
&\ge (1-\epsilon)\cdot Ku_0 \left(\frac{\bar{g}(z)v}{u_0} - 1\right) \frac{1}{\bar{g}(z+(\eta_z-1)u_0)} \\
&\ge (1-\epsilon)\cdot Kv \frac{\bar{g}(z)}{\bar{g}(z+(\eta_z-1)u_0)} - (1-\epsilon)\cdot \frac{Ku_0}{\bar{g}(z+(\eta_z-1)u_0)}.
\end{aligned}$$

Now

$$u_0\left(\frac{\bar{g}(z)v}{u_0} - 2\right) < u_0\,(\eta_z - 1) \le u_0\left(\frac{\bar{g}(z)v}{u_0} - 1\right).$$

Now for $0 \le v \le 1$

$$\bar{g}\,(z-2u_0) \le \bar{g}\,(z+\bar{g}(z)v-2u_0) < \bar{g}\,(z+u_0(\eta_z-1)) < \bar{g}\,(z+\bar{g}(z)v-u_0) < \bar{g}\,(z+\bar{g}(z)-u_0)\,.$$

Hence

$$\frac{1}{\bar{g}\,(z-2u_0)} > \frac{1}{\bar{g}\,(z+u_0(\eta_z-1))} > \frac{1}{\bar{g}\,(z+\bar{g}(z)-u_0)}.$$

Thus

$$\bar{h}(x+v\bar{g}(x)) - \bar{h}(x) > (1-\epsilon)\cdot Kv \frac{\bar{g}(x)}{\bar{g}\,(x+\bar{g}(x)-u_0)} - (1-\epsilon)\cdot \frac{Ku_0}{\bar{g}(x-2u_0)}. \tag{5.54}$$

Since

$$\log\left(\frac{\tilde{\delta}(z+v\tilde{\Delta}(z))}{\tilde{\delta}(z)}\right) = \bar{h}(z) - \bar{h}(z+v\bar{g}(z)),$$

we have from (5.53) that

$$\log\left(\frac{\tilde{\delta}(z+v\tilde{\Delta}(z))}{\tilde{\delta}(z)}\right) > -(1+\epsilon)\cdot Kv + \epsilon, \quad \forall\, z > z_1(\epsilon) \text{ uniformly in } v \in [0,1]. \tag{5.55}$$

From (5.54) if g is self-neglecting

$$\log\left(\frac{\tilde{\delta}(z+v\tilde{\Delta}(z))}{\tilde{\delta}(z)}\right) < -((1-\epsilon)^2\cdot Kv - \epsilon), \quad \forall\, z > z_1(\epsilon) \text{ uniformly in } v \in [0,1].$$

Thus combining this and (5.55) we get

$$\lim_{z\to\infty} \frac{\tilde{\delta}(z+v\tilde{\Delta}(z))}{\tilde{\delta}(z)} = e^{-Kv}, \quad \text{uniformly in } v \in [0,1],$$

which is (5.48) and hence we have part (iii)(a). We now prove part (iv). (5.43) implies g_γ is decreasing from Lemma 19. Hence $g_\gamma(z+(k-1)u_0) < g_\gamma(z)$, $\forall\, k \geq 1$. Then for $k \in \{1, \ldots, \eta_z\}$

$$\begin{aligned}\bar{g}(z+(k-1)u_0) < \bar{g}(z)+(k-1)u_0 \leq \bar{g}(z)+\gamma(\eta_z-1)u_0 &< \bar{g}(z)+\gamma\eta_z u_0 \\ &\leq \bar{g}(z)+\gamma\frac{\bar{g}(z)}{u_0}vu_0.\end{aligned}$$

Thus $\forall\, k \in \{1, \ldots, \eta_z\}$, $\bar{g}(z+(k-1)u_0) \leq \bar{g}(z)(1+\gamma v)$, $\forall\, v \in [0,1]$. Thus

$$\begin{aligned}\bar{h}(z+v\bar{g}(z))-\bar{h}(z) &> (1-\epsilon)\cdot Ku_0\sum_{k=1}^{\eta_z}\frac{1}{\bar{g}(z+(k-1)u_0)} \\ &> (1-\epsilon)\cdot\frac{Ku_0\eta_z}{\bar{g}(z)(1+v\gamma)} \\ &> (1-\epsilon)\cdot Ku_0\left(\frac{\bar{g}(z)v}{u_0}-1\right)\frac{1}{\bar{g}(z)(1+v\gamma)} \\ &> (1-\epsilon)\cdot\frac{Kv}{1+v\gamma}-(1-\epsilon)\cdot\frac{Ku_0}{\bar{g}(z)}\cdot\frac{1}{1+v\gamma}.\end{aligned}$$

Hence

$$\bar{h}(z+v\bar{g}(z))-\bar{h}(z) > (1-\epsilon)\cdot\frac{Kv}{1+v\gamma}-\epsilon, \quad \forall\, v \in [0,1].$$

Hence $\forall\, z \geq Z_3(\epsilon)$

$$\frac{\tilde{\delta}(z+v\tilde{\Delta}(z))}{\tilde{\delta}(z)} \leq \exp\left(\frac{-(1-\epsilon)Kv}{1+v\gamma}\right)\exp(\epsilon), \quad \forall\, v \in [0,1].$$

We have for all x sufficiently small that

$$\frac{F(xe^{-\Delta(x)})-F(x)}{\Delta(x)x/f(x)} = \int_{0^+}^{1}\frac{\tilde{\delta}(z+v\tilde{\Delta}(z))}{\tilde{\delta}(z)}\,dv \leq e^{\epsilon}\int_{0^+}^{1}\exp\left(\frac{-(1-\epsilon)Kv}{1+v\gamma}\right)\,dv.$$

Hence

$$\limsup_{x\to 0^+}\frac{F(xe^{-\Delta(x)})-F(x)}{\Delta(x)x/f(x)} \leq e^{\epsilon}\int_{0^+}^{1}\exp\left(\frac{-(1-\epsilon)Kv}{1+v\gamma}\right)\,dv.$$

Letting $\epsilon \to 0^+$ yields

$$\limsup_{x\to 0^+}\frac{F(xe^{-\Delta(x)})-F(x)}{\Delta(x)x/f(x)} \leq \int_{0^+}^{1}\exp\left(\frac{-Kv}{1+v\gamma}\right)\,dv.$$

This proves part (iv). To prove part (iii)(b) observe that γ is arbitrary in the last limit and so letting $\gamma \to 0^+$ in that limit

$$\limsup_{x\to 0^+}\frac{F(xe^{-\Delta(x)})-F(x)}{\Delta(x)x/f(x)} \leq \int_{0^+}^{1}e^{-Kv}\,dv.$$

Combining this with (5.55) implies

$$\liminf_{x\to 0^+} \frac{F(xe^{-\Delta(x)}) - F(x)}{\Delta(x)x/f(x)} \geq \int_{0^+}^{1} e^{-Kv}\, dv,$$

and we obtain (5.45). □

5.4 Examples

We consider a number of examples where we can verify by direct computation the sharpness of Propositions 9, 10 and Theorem 33.

Proposition 11. *Suppose*

$$\lim_{x\to 0^+} \Delta(x) = \infty \quad \textit{and} \quad \lim_{x\to 0^+} \frac{\Delta(x)x}{f(x)} = 0,$$

and

$$f(x) = x\log\left(\tfrac{1}{x}\right)\left(\log\log\left(\tfrac{1}{x}\right)\right)^{\alpha}, \quad x < e^{-e}, \quad \alpha \in (0,1].$$

Define

$$\lim_{x\to 0^+} \frac{\Delta(x)}{\log(1/x)} =: c.$$

(i) If $c = 0$, then

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = \lim_{x\to 0^+} \frac{\int_{xe^{-\Delta(x)}}^{x} 1/f(u)\, du}{\Delta(x)x/f(x)} = 1.$$

(ii) If $c \in (0,\infty)$, then

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = \lim_{x\to 0^+} \frac{\int_{xe^{-\Delta(x)}}^{x} 1/f(u)\, du}{\Delta(x)x/f(x)} = \frac{\log(1+c)}{c}.$$

(iii) If $c = \infty$, then

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = \lim_{x\to 0^+} \frac{\int_{xe^{-\Delta(x)}}^{x} 1/f(u)\, du}{\Delta(x)x/f(x)} = 0.$$

Proof. Define

$$\begin{aligned} F(x) := \int_x^{e^{-e}} \frac{1}{f(u)}\, du &= \int_x^{e^{-e}} \frac{1}{u\log(1/u)\left(\log\log(1/u)\right)^{\alpha}}\, du \\ &= \int_e^{\log(1/x)} \frac{1}{v\left(\log v\right)^{\alpha}}\, dv = \int_1^{\log\log(1/x)} w^{-\alpha}\, dw. \end{aligned}$$

Thus

$$h(x) = \frac{\Delta(x)x}{f(x)} = \frac{\Delta(x)x}{x\log(1/x)(\log\log(1/x))^{\alpha}} = \frac{\Delta(x)}{\log(1/x)(\log\log(1/x))^{\alpha}}.$$

We consider the case when $\alpha \in (0,1)$ first. Thus

$$F(x) = \frac{1}{1-\alpha}\left[\left(\log\log\left(\frac{1}{x}\right)\right)^{1-\alpha} - 1\right].$$

For $\alpha \in (0,1)$

$$F(xe^{-\Delta(x)}) - F(x) = \frac{1}{1-\alpha}\left[\left(\log\left(z+\tilde{\Delta}(z)\right)\right)^{1-\alpha} - (\log z)^{1-\alpha}\right],$$

and

$$h(x) = \frac{\Delta(x)}{\log(1/x)(\log\log(1/x))^{\alpha}} = \frac{\tilde{\Delta}(z)}{z(\log z)^{\alpha}}, \tag{5.56}$$

where $z := \log(1/x)$ and $\tilde{\Delta}(z) := \Delta(e^{-z}) = \Delta(x)$. Note that $\tilde{\Delta}(z)/z \to c$ as $z \to \infty$ is equivalent to $\Delta(x)/\log(1/x) \to c$ as $x \to 0^+$. By the Mean Value Theorem, there is $c_z \in (0,1)$ such that

$$\begin{aligned} F(xe^{-\Delta(x)}) - F(x) &= \left(\log\left(z + c_z\tilde{\Delta}(z)\right)\right)^{-\alpha}\log\left(1+\frac{\tilde{\Delta}(z)}{z}\right) \\ &= \left(\log z + \log\left(1+c_z\frac{\tilde{\Delta}(z)}{z}\right)\right)^{-\alpha}\log\left(1+\frac{\tilde{\Delta}(z)}{z}\right). \end{aligned} \tag{5.57}$$

(i) If $c = 0$ then $\tilde{\Delta}(z)/z \to 0$ as $z \to \infty$ so from (5.57) we have

$$F(xe^{-\Delta(x)}) - F(x) \sim \frac{\tilde{\Delta}(z)}{z(\log z)^{\alpha}}, \quad \text{as } z \to \infty.$$

Hence as $\Delta(x)/\log(1/x) \to 0$ as $x \to 0^+$ we get from (5.56)

$$\lim_{x\to 0^+}\frac{1}{h(x)}\int_{xe^{-\Delta(x)}}^{x}\frac{1}{f(u)}\,du = \lim_{x\to 0^+}\frac{F(xe^{-\Delta(x)}) - F(x)}{\Delta(x)x/f(x)} = 1.$$

(ii) If $c \in (0,\infty)$ then $\tilde{\Delta}(z)/z \to c$ as $z \to \infty$ so from (5.57) we have $F(xe^{-\Delta(x)}) - F(x) \sim (\log z)^{-\alpha}\log(1+c)$ as $z \to \infty$. Also from (5.56), $h(x) = \tilde{\Delta}(z)/(z(\log z)^{\alpha}) \sim c(\log z)^{-\alpha}$ as $z \to \infty$. Hence

$$\lim_{x\to 0^+}\frac{1}{h(x)}\int_{xe^{-\Delta(x)}}^{x}\frac{1}{f(u)}\,du = \lim_{z\to\infty}\frac{(\log z)^{-\alpha}\log(1+c)}{c(\log z)^{-\alpha}} = \frac{\log(1+c)}{c}.$$

(iii) If $c = \infty$ then $\tilde{\Delta}(z)/z \to c$ as $z \to \infty$ so we have

$$\begin{aligned} F(xe^{-\Delta(x)}) - F(x) &= \frac{1}{1-\alpha}\left[\left(\log\left(z + \tilde{\Delta}(z)\right)\right)^{1-\alpha} - (\log z)^{1-\alpha}\right] \\ &= \frac{1}{1-\alpha}\left[\left(\log z + \log\left(1 + \frac{\tilde{\Delta}(z)}{z}\right)\right)^{1-\alpha} - (\log z)^{1-\alpha}\right] \\ &= \frac{1}{1-\alpha}\left[(w + \log(1 + \tilde{a}(w)))^{1-\alpha} - w^{1-\alpha}\right], \end{aligned}$$

where $w := \log z$ and $\tilde{a}(w) := a(e^w) = a(z)$. By the Mean Value Theorem there is $\theta_w \in (0,1)$ such that

$$\frac{(w + \log(1 + \tilde{a}(w)))^{1-\alpha} - w^{1-\alpha}}{1-\alpha} = (w + \theta_w \log(1 + \tilde{a}(w)))^{-\alpha} \cdot \log(1 + \tilde{a}(w)).$$

We have $\tilde{a}(w) = a(e^w) \to \infty$ as $w \to \infty$. Since

$$\Delta(x) = o\left(\log\left(\tfrac{1}{x}\right)\left(\log\log\left(\tfrac{1}{x}\right)\right)^{\alpha}\right), \quad \text{as } x \to 0^+,$$

we have as $w \to \infty$

$$\tilde{a}(w) = \frac{\Delta(\exp(-e^w))}{e^w} = o\left(e^{-w}\log(\exp(e^w))(\log\log(\exp(e^w)))^{\alpha}\right) = o(w^{\alpha}).$$

Thus for every $\epsilon > 0$ there is $w^*(\epsilon)$ such that $\tilde{a}(w) < \epsilon w^{\alpha}$ for all $w > w^*(\epsilon)$. Thus

$$\log \tilde{a}(w) < \log \epsilon + \alpha \log w.$$

Hence $\log \tilde{a}(w) = O(\log w)$ as $w \to \infty$ and so $\log(1+\tilde{a}(w)) = O(\log w)$ as $w \to \infty$ since $\tilde{a}(w) \to \infty$ as $w \to \infty$. Hence

$$w + \theta_w \log(1 + \tilde{a}(w)) \sim w, \quad \text{as } w \to \infty,$$

and therefore as $w \to \infty$

$$\frac{1}{1-\alpha}\left[(w + \log(1 + \tilde{a}(w)))^{1-\alpha} - w^{1-\alpha}\right] \sim w^{-\alpha}\log(1 + \tilde{a}(w)) \sim w^{-\alpha}\log \tilde{a}(w).$$

Hence as $w \to \infty$ with $w = \log z$ and $z = \log(1/x)$

$$F(xe^{-\Delta(x)}) - F(x) \sim w^{-\alpha}\log \tilde{a}(w) = o(w^{-\alpha}\tilde{a}(w)) = o(h(x)), \quad \text{as } x \to 0^+,$$

since $h(x) = w^{-\alpha}\tilde{a}(w)$. Hence

$$\lim_{x \to 0^+} \frac{1}{h(x)} \int_{xe^{-\Delta(x)}}^{x} \frac{1}{f(u)}\, du = 0.$$

We now consider the case when $\alpha = 1$. Thus

$$F(x) = \int_1^{\log\log(1/x)} v^{-1}\, dv = \log\log\log\left(\tfrac{1}{x}\right).$$

Thus $z = \log(1/x)$ we get

$$F(xe^{-\Delta(x)}) - F(x) \;=\; \log\log\left(z + \tilde{\Delta}(z)\right) - \log\log z,$$

and

$$h(x) = \frac{\tilde{\Delta}(z)}{z\log z}.$$

(i) If $\tilde{\Delta}(z)/z \to 0$ as $z \to \infty$ then by the Mean Value Theorem there is $c_z \in (0,1)$ such that as $z \to \infty$

$$F(xe^{-\Delta(x)}) - F(x) = \frac{\tilde{\Delta}(z)}{(z + c_z\tilde{\Delta}(z))\log(z + c_z\tilde{\Delta}(z))} \sim \frac{\tilde{\Delta}(z)}{z} = h(z).$$

Thus

$$\lim_{x\to 0^+} \frac{1}{h(x)} \int_{xe^{-\Delta(x)}}^{x} \frac{1}{f(u)}\, du = 1.$$

(ii) If $\tilde{\Delta}(z)/z \to c \in (0,\infty)$ as $z \to \infty$, then

$$\begin{aligned} F(xe^{-\Delta(x)}) - F(x) &= \log\log\left(z + \tilde{\Delta}(z)\right) - \log\log z \\ &= \log\left(\log z + \log\left(1 + \frac{\tilde{\Delta}(z)}{z}\right)\right) - \log\log z. \end{aligned}$$

Define

$$\lambda(z) := \log\left(1 + \frac{\tilde{\Delta}(z)}{z}\right) \to \log(1+c), \quad \text{as } z \to \infty.$$

Set $\tilde{\lambda}(z) := \lambda(e^z)$ and $w := \log z$. Then $\tilde{\lambda}(w) \to \log(1+c)$ as $w \to \infty$. We have, with $h_1(x) = \log x$ and for some $c_w \in (0,1)$ and any w sufficiently large

$$\log\log(z + \tilde{\Delta}(z)) - \log\log z = \log(w + \tilde{\lambda}(w)) - \log w = \frac{\tilde{\lambda}(w)}{w + c_w\tilde{\lambda}(w)}.$$

Hence

$$F(xe^{-\Delta(x)}) - F(x) = \frac{\tilde{\lambda}(w)}{w + c_w\tilde{\lambda}(w)} \sim \frac{\log(1+c)}{w} = \frac{\log(1+c)}{\log z}, \quad \text{as } z \to \infty.$$

Since

$$h(x) = \frac{\tilde{\Delta}(z)}{z\log z} \sim \frac{c}{\log z}, \quad \text{as } z \to \infty,$$

we have as $x \to 0^+$

$$\frac{F(xe^{-\Delta(x)}) - F(x)}{\Delta(x)x/f(x)} \sim \frac{\log(1+c)/\log z}{c/\log z} = \frac{\log(1+c)}{c}.$$

Hence

$$\lim_{x\to 0^+} \frac{1}{h(x)} \int_{xe^{-\Delta(x)}}^{x} \frac{1}{f(u)}\,du = \frac{\log(1+c)}{c}.$$

(iii) If $\tilde{\Delta}(z)/z \to \infty$ then $a(z) := \tilde{\Delta}(z)/z \to \infty$ as $z \to \infty$ and

$$\begin{aligned} F(xe^{-\Delta(x)}) - F(x) &= \log\log\left(z + \tilde{\Delta}(z)\right) - \log\log z \\ &= \log\left(\log z + \log\left(1 + a(z)\right)\right) - \log\log z \\ &= \log\left(w + \log\left(1 + \tilde{a}(w)\right)\right) - \log w, \end{aligned}$$

where $w := \log z$ and $\tilde{a}(w) = a(e^w) = a(z)$. By the Mean Value Theorem there is $\theta_w \in (0,1)$ such that

$$\log\left(w + \log\left(1 + \tilde{a}(w)\right)\right) - \log w = \frac{\log\left(1 + \tilde{a}(w)\right)}{w + \theta_w \log\left(1 + \tilde{a}(w)\right)}.$$

Since $\tilde{a}(w) \to \infty$ as $w \to \infty$ then

$$\log\left(1 + \tilde{a}(w)\right) \sim \log \tilde{a}(w) = \log a(e^w) = \log\left(\frac{\tilde{\Delta}(e^w)}{e^w}\right) =: \log a^*(w).$$

Next $\tilde{\Delta}(z) = o(z \log z)$ as $z \to \infty$. Thus $\tilde{\Delta}(e^w) = o(e^w \log e^w) = o(e^w w)$ as $w \to \infty$. Then $a^*(w) = \tilde{\Delta}(e^w)/e^w = o(w)$ as $w \to \infty$. Thus for every $\epsilon > 0$ there is $w^*(\epsilon) > 0$ such that $a^*(w) < \epsilon w^\alpha$ for all $w > w^*(\epsilon)$. Thus

$$\log \tilde{a}(w) = \log a^*(w) < \log \epsilon + \alpha \log w.$$

Hence $\log \tilde{a}(w) = O(\log w)$ as $w \to \infty$. Therefore

$$F(xe^{-\Delta(x)}) - F(x) \sim \frac{\log \tilde{a}(w)}{w} = o\left(\frac{\tilde{a}(w)}{w}\right) = o(h(x)), \quad \text{as } x \to 0^+,$$

since

$$h(x) = \frac{\tilde{\Delta}(z)}{z \log z} = \frac{a(z)}{\log z} = \frac{a(e^w)}{w} = \frac{\tilde{a}(w)}{w}.$$

Hence

$$\lim_{x\to 0^+} \frac{1}{h(x)} \int_{xe^{-\Delta(x)}}^{x} \frac{1}{f(u)}\,du = 0.$$

□

Remark 26. For f in the example above $h(x) \sim c\left(\log\log\left(\frac{1}{x}\right)\right)^{-\alpha}$ as $x \to 0^+$. Thus we can establish the critical rate of convergence of h. The finite values of c distinguish between

$$h(x) = O\left(\left(\log\log\left(\tfrac{1}{x}\right)\right)^{-\alpha}\right) \quad \text{and} \quad h(x) = o\left(\frac{1}{(\log\log(1/x))^{\alpha}}\right), \quad \text{as } x \to 0^+.$$

□

Remark 27. The criterion (5.37) is

$$\lim_{x\to 0^+} \frac{\delta(xe^{-\Delta(x)})}{\delta(x)} = 1.$$

Since

$$\begin{aligned} \frac{\delta(xe^{-\Delta(x)})}{\delta(x)} &= \frac{xe^{-\Delta(x)}/f(xe^{-\Delta(x)})}{x/f(x)} = \frac{e^{-\Delta(x)}f(x)}{f(xe^{-\Delta(x)})} \\ &= \frac{x\log(1/x)\,(\log\log(1/x))^{\alpha}\,e^{-\Delta(x)}}{xe^{-\Delta(x)}\log(1/xe^{-\Delta(x)})\,(\log\log(1/xe^{-\Delta(x)}))^{\alpha}} \\ &= \frac{\log(1/x)}{\log(1/x)+\Delta(x)} \cdot \left\{\frac{\log\log(1/x)}{\log(\log(1/x)+\Delta(x))}\right\}^{\alpha}. \end{aligned}$$

Thus

$$\lim_{x\to 0^+} \frac{\delta(xe^{-\Delta(x)})}{\delta(x)} = \lim_{x\to 0^+} \frac{\log(1/x)}{\log(1/x)+\Delta(x)} \cdot \left\{\frac{\log\log(1/x)}{\log(\log(1/x)+\Delta(x))}\right\}^{\alpha} = 1,$$

if $\Delta(x)/\log(1/x) \to 0$ as $x \to 0^+$. Therefore, the condition (5.37) which guarantees the exact rate in Theorem 32 is a sharp estimate on the maximal allowable rate of growth in Δ for this example, since $\Delta(x)/\log(1/x) \to c$ as $x \to 0^+$ implies

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = \lim_{x\to 0^+} \frac{\int_{xe^{-\Delta(x)}}^{x} 1/f(u)\,du}{\Delta(x)x/f(x)} = \frac{\log(1+c)}{c},$$

and scrutinising the proof of Theorem 32 we see that $\Delta(x) \sim c\log(1/x)$ as $x \to 0^+$ where $c \in (0,\infty)$ implies $x_n \to 0$, $t_n \to \infty$ and

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = \lim_{x\to 0^+} \frac{\int_{xe^{-\Delta(x)}}^{x} 1/f(u)\,du}{\Delta(x)x/f(x)} = \frac{\log(1+c)}{c}.$$

Looking at Theorem 33, rather than doing the calculations, directly we can only obtain non-unit bounds on $\liminf_{n\to\infty} F(x_n)/t_n$ and $\limsup_{n\to\infty} F(x_n)/t_n$.

Proposition 12. *Suppose*

$$\lim_{x\to 0^+} \Delta(x) = \infty \quad \text{and} \quad \lim_{x\to 0^+} \frac{\Delta(x)x}{f(x)} = 0,$$

and

$$f(x) = x\left(\log\left(\tfrac{1}{x}\right)\right)^{\alpha}, \quad x < 1/e, \quad \alpha \in (0,1].$$

Define

$$\lim_{x\to 0^+} \frac{\Delta(x)}{\log(1/x)} =: c.$$

(i) *If $c = 0$, then*

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = 1.$$

(ii) (a) *If $\alpha \in (0,1)$ and $c \in (0,\infty)$, then*

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = \frac{(1+c)^{1-\alpha} - 1}{(1-\alpha)c}.$$

(b) *If $\alpha = 1$ and $c \in (0,\infty)$, then*

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = \frac{\log(1+c)}{c}.$$

(iii) *If $c = \infty$, then*

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = 0.$$

Proof. Define

$$F(x) := \int_x^{1/e} \frac{1}{f(u)}\,du = \int_1^{\log(1/x)} v^{-\alpha}\,dv.$$

Thus

$$h(x) = \frac{\Delta(x)x}{f(x)} = \frac{\Delta(x)x}{x\left(\log(1/x)\right)^{\alpha}} = \frac{\Delta(x)}{\left(\log(1/x)\right)^{\alpha}}.$$

We consider the case when $\alpha \in (0,1)$ first. Thus

$$F(x) = \frac{1}{1-\alpha}\left[\left(\log\left(\frac{1}{x}\right)\right)^{1-\alpha} - 1\right].$$

For $\alpha \in (0,1)$

$$F(xe^{-\Delta(x)}) - F(x) \;=\; \frac{1}{1-\alpha}\left[\left(z + \tilde{\Delta}(z)\right)^{1-\alpha} - z^{1-\alpha}\right],$$

and

$$h(x) = \frac{\Delta(x)}{\left(\log(1/x)\right)^{\alpha}} = \tilde{\Delta}(z)z^{-\alpha},$$

where $z := \log(1/x)$ and $\tilde{\Delta}(z) := \Delta(e^{-z}) = \Delta(x)$. Note that $\tilde{\Delta}(z)/z \to c$ as $z \to \infty$ is equivalent to $\Delta(x)/\log(1/x) \to c$ as $x \to 0^+$.

(i) If $\tilde{\Delta}(z)/z \to 0$ as $z \to \infty$ then by the Mean Value Theorem there is $c_z \in (0,1)$ such that

$$F(xe^{-\Delta(x)}) - F(x) \quad = \quad \left[z + c_z\tilde{\Delta}(z)\right]^{-\alpha} \tilde{\Delta}(z) \sim z^{-\alpha}\tilde{\Delta}(z), \quad \text{as } z \to \infty.$$

Hence as $h(x) = \tilde{\Delta}(z)z^{-\alpha}$ then

$$\lim_{x\to 0^+} \frac{1}{h(x)} \int_{xe^{-\Delta(x)}}^{x} \frac{1}{f(u)}\,du = 1.$$

(ii) If $\tilde{\Delta}(z)/z \to c \in (0,\infty)$ as $z \to \infty$ then

$$\begin{aligned} F(xe^{-\Delta(x)}) - F(x) &= \frac{1}{1-\alpha}\left(\left(z + \tilde{\Delta}(z)\right)^{1-\alpha} - z^{1-\alpha}\right) \\ &= \frac{z^{1-\alpha}}{1-\alpha}\left(\left(1 + \frac{\tilde{\Delta}(z)}{z}\right)^{1-\alpha} - 1\right) \\ &\sim \frac{z^{1-\alpha}\left((1+c)^{1-\alpha} - 1\right)}{1-\alpha}, \quad \text{as } z \to \infty, \end{aligned}$$

and

$$h(x) = \frac{\tilde{\Delta}(z)}{z} \cdot z^{1-\alpha} \sim cz^{1-\alpha}, \quad \text{as } z \to \infty.$$

Hence

$$\lim_{x\to 0^+} \frac{1}{h(x)} \int_{xe^{-\Delta(x)}}^{x} \frac{1}{f(u)}\,du = \frac{((1+c)^{1-\alpha} - 1)}{(1-\alpha)c}.$$

(iii) If $\tilde{\Delta}(z)/z \to \infty$ as $z \to \infty$ then

$$\begin{aligned} F(xe^{-\Delta(x)}) - F(x) &= \frac{1}{1-\alpha}\left(\left(z + \tilde{\Delta}(z)\right)^{1-\alpha} - z^{1-\alpha}\right) \\ &= \frac{\tilde{\Delta}(z)^{1-\alpha}}{1-\alpha}\left(\left(1 + \frac{z}{\tilde{\Delta}(z)}\right)^{1-\alpha} - \left(\frac{z}{\tilde{\Delta}(z)}\right)^{1-\alpha}\right) \\ &\sim \frac{\tilde{\Delta}(z)^{1-\alpha}}{1-\alpha}, \quad \text{as } z \to \infty, \end{aligned}$$

and

$$h(x) = \frac{\Delta(x)x}{f(x)} = \tilde{\Delta}(z)z^{-\alpha}, \quad \text{as } z \to \infty.$$

Hence

$$\lim_{x\to 0^+} \frac{1}{h(x)} \int_{xe^{-\Delta(x)}}^{x} \frac{1}{f(u)}\,du = \lim_{z\to\infty} \frac{\tilde{\Delta}(z)^{1-\alpha}}{(1-\alpha)\tilde{\Delta}(x)z^{-\alpha}} = \frac{1}{1-\alpha} \lim_{z\to\infty} \left(\frac{z}{\tilde{\Delta}(z)}\right)^{\alpha} = 0.$$

We now consider the case when $\alpha = 1$. Thus

$$F(x) = \int_1^{\log(1/x)} v^{-1}\, dv = \log\log\left(\tfrac{1}{x}\right).$$

Thus

$$F(xe^{-\Delta(x)}) - F(x) \;=\; \log\left(1 + \frac{\tilde{\Delta}(z)}{z}\right),$$

and $h(x) = \tilde{\Delta}(z)/z$.

(i) If $\tilde{\Delta}(z)/z \to 0$ as $z \to \infty$ then

$$F(xe^{-\Delta(x)}) - F(x) = \log\left(1 + \frac{\tilde{\Delta}(z)}{z}\right) \sim \frac{\tilde{\Delta}(z)}{z}, \quad \text{as } x \to 0^+.$$

or $F(xe^{-\Delta(x)}) - F(x) \sim h(x)$ as $x \to 0^+$. Hence

$$\lim_{x\to 0^+} \frac{1}{h(x)} \int_{xe^{-\Delta(x)}}^{x} \frac{1}{f(u)}\, du = 1.$$

(ii) If $\tilde{\Delta}(z)/z \to c \in (0, \infty)$ as $z \to \infty$ then

$$F(xe^{-\Delta(x)}) - F(x) = \log\left(1 + \frac{\tilde{\Delta}(z)}{z}\right) \to \log(1 + c), \quad \text{as } x \to 0^+.$$

Hence then $h(x) = \Delta(x)x/f(x) \to c$ as $x \to 0^+$ and

$$\lim_{x\to 0^+} \frac{1}{h(x)} \int_{xe^{-\Delta(x)}}^{x} \frac{1}{f(u)}\, du = \frac{\log(1+c)}{c}.$$

(iii) If $\tilde{\Delta}(z)/z \to \infty$ as $z \to \infty$ then

$$\frac{F(xe^{-\Delta(x)}) - F(x)}{h(x)} = \lim_{z\to\infty} \frac{\log\left(1 + \tilde{\Delta}(z)/z\right)}{\tilde{\Delta}(z)/z} = \lim_{y\to\infty} \frac{\log(1+y)}{y} = 0.$$

Hence

$$\lim_{x\to 0^+} \frac{1}{h(x)} \int_{xe^{-\Delta(x)}}^{x} \frac{1}{f(u)}\, du = 0.$$

□

Chapter 6

Comparison of Errors in Estimated Rates of Convergence

6.1 Introduction

This section compares the error in the rates of asymptotic convergence estimated by the Explicit, Implicit and Transformed Explicit Euler schemes. We compare the schemes for $\Delta \in (0,1)$ and $\beta \in (0,1)$ only because for $\Delta \geq 1$ the Explicit scheme violates positivity and gives poor asymptotics. When $\beta < 1$, the integral

$$\int_x^1 \frac{1}{f(u)}\,du,$$

is guaranteed to converge as $x \to 0^+$ but will either diverge or converge when $\beta = 1$. As a result, $\beta < 1$ relates to the case of finite-time stability while $\beta = 1$ can relate to either finite-time stability or super-exponential stability.

We start with a brief summary of the estimated rates of converge and the resulting error for each scheme. For the Explicit scheme with explicit step-size, the estimated rate of convergence from Theorem 17 part (ii) is

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \frac{1}{\Delta}\int_{1-\Delta}^{1} \lambda^{-\beta}\,d\lambda =: \lambda_E(\Delta) > 1.$$

The error between the rate predicted by the scheme and the true rate of unity is

$$Error_E(\Delta) := |\lambda_E(\Delta) - 1| =: \lambda_E(\Delta) - 1 \quad = \quad \frac{1}{\Delta}\int_{1-\Delta}^{1} \lambda^{-\beta}\,d\lambda - 1 = \frac{1-(1-\Delta)^{1-\beta}}{\Delta(1-\beta)} - 1.$$

For the Implicit scheme, the estimated rate of convergence from Theorem 20 part (ii) is

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \frac{1}{\Delta}\int_{1}^{1+\Delta} \lambda^{-\beta}\,d\lambda =: \lambda_I(\Delta) < 1.$$

The error between the rate predicted by the scheme and the true rate of unity is

$$Error_I(\Delta) := |\lambda_I(\Delta) - 1| =: 1 - \lambda_I(\Delta) \quad = \quad 1 - \frac{1}{\Delta}\int_1^{1+\Delta} \lambda^{-\beta}\, d\lambda = 1 - \frac{(1+\Delta)^{1-\beta} - 1}{\Delta(1-\beta)}.$$

For the Transformed Explicit scheme, the estimated rate of convergence from Theorem 28 part (b)(ii) is

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \frac{1}{\Delta}\int_{e^{-\Delta}}^{1} \lambda^{-\beta}\, d\lambda =: \lambda_T(\Delta) < 1.$$

The error between the rate predicted by the scheme and the true rate of unity is

$$Error_T(\Delta) := |\lambda_T(\Delta) - 1| =: 1 - \lambda_T(\Delta) \quad = \quad 1 - \frac{1}{\Delta}\int_{e^{-\Delta}}^{1} \lambda^{-\beta}\, d\lambda = 1 - \frac{1 - e^{-\Delta(1-\beta)}}{\Delta(1-\beta)}.$$

6.2 Boundary Cases

If $\beta = 0$, then estimates rates from each of the three schemes is as follows:

$$\lambda_E(\Delta) = \lambda_I(\Delta) = 1 \quad \text{and} \quad \lambda_T(\Delta) = \frac{1}{\Delta}\int_{e^{-\Delta}}^{1} 1\, d\lambda = \frac{1 - e^{-\Delta}}{\Delta} < 1.$$

Thus the Explicit and Implicit schemes give equal performance by predicting the true rate of unity and both outperform the Transformed Explicit scheme, summarised as

$$0 = Error_E(\Delta) = Error_I(\Delta) < Error_T(\Delta). \tag{6.1}$$

When $\beta = 1$ then we have either super-exponential convergence or finite-time stability. For the Transformed Explicit scheme

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \lim_{n\to\infty} \frac{F(x_n)}{t_n} = \lambda_T(\Delta) = 1,$$

with associated error $Error_T(\Delta) = 1 - 1 = 0$. For the Explicit scheme

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \lim_{n\to\infty} \frac{F(x_n)}{t_n} = \lambda_E(\Delta) = \frac{1}{\Delta}\log\left(\frac{1}{1-\Delta}\right) > 1,$$

with associated error

$$Error_E(\Delta) = \frac{1}{\Delta}\log\left(\frac{1}{1-\Delta}\right) - 1.$$

For the Implicit scheme

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \lim_{n\to\infty} \frac{F(x_n)}{t_n} = \lambda_I(\Delta) = \frac{\log(1+\Delta)}{\Delta} < 1,$$

with associated error

$$Error_I(\Delta) = 1 - \frac{\log(1+\Delta)}{\Delta}.$$

Theorem 34. *Let $\beta = 1$. Then*

$$Error_E(\Delta) > Error_I(\Delta), \quad \forall\, \Delta \in (0,1).$$

Proof. Define

$$k(\Delta) \quad := \quad Error_E(\Delta) - Error_I(\Delta) = \frac{\tilde{k}(\Delta)}{\Delta},$$

where $\tilde{k}(\Delta) := -\log(1-\Delta) - \log(1+\Delta) - 2\Delta$ with $\tilde{k}(0) = 0$ and $\tilde{k}'(\Delta) = 2\Delta^2/(1-\Delta^2) > 0$. Hence $\tilde{k}(\Delta) > 0$ for all $\Delta \in (0,1)$. Thus $Error_E(\Delta) > Error_I(\Delta)$. □

Overall, when $\beta = 1$, the Transformed Explicit scheme outperforms both the Explicit and Implicit schemes and estimates the perfect rate of unity, summarised as

$$Error_E(\Delta) > Error_I(\Delta) > Error_T(\Delta) = 0. \tag{6.2}$$

When $\beta \in (0,1)$, the result is not as clear as the following sections demonstrate.

6.3 Explicit and Transformed Euler Schemes

The following theorem ranks the performance of the Explicit and Transformed schemes.

Theorem 35. *Let $\beta = \beta_1^*$ be the unique number in $(0,1)$ such that $2\beta = e^{-(1-\beta)}$. Then $\beta_1^* \in (0,1/2)$ and the following hold:*

(i) If $\beta < \beta_1^*$, then

$$Error_E(\Delta) < Error_T(\Delta), \quad \forall\, \Delta \in (0,1).$$

(ii) If $\beta \in (\beta_1^*, 1/2)$, then there exists $\Delta_3 = \Delta_3(\beta) \in (0,1)$ such that

$$Error_E(\Delta) < Error_T(\Delta), \quad \forall\, \Delta < \Delta_3,$$
$$Error_E(\Delta) > Error_T(\Delta), \quad \forall\, \Delta > \Delta_3,$$

where $\Delta = \Delta_3$ obeys $(1-\Delta)^{1-\beta} + e^{-\Delta(1-\beta)} + 2(1-\beta)\Delta = 2$.

(iii) If $\beta > 1/2$, then

$$Error_E(\Delta) > Error_T(\Delta), \quad \forall\, \Delta \in (0,1).$$

Proof. Define

$$k(\Delta) \;:=\; Error_E(\Delta) - Error_T(\Delta) = \frac{\tilde{k}(\Delta)}{(1-\beta)\Delta},$$

where $\tilde{k}(\Delta) := 2 - (1-\Delta)^{1-\beta} - e^{-\Delta(1-\beta)} - 2(1-\beta)\Delta$ with $\tilde{k}(0) = 0$ and $\tilde{k}(1) = 2\beta - e^{-1(1-\beta)}$. Thus $k(\Delta) > 0$ if $\tilde{k}(\Delta) > 0$ since $(1-\beta)\Delta > 0$. Define $k_2(\beta) := 2\beta - e^{-1(1-\beta)}$. Since $k_2(0) = -e^{-1} < 0$, $k_2(1/2) = 1 - e^{-1/2} > 0$ and $k_2'(\beta) = 2 - e^{-1(1-\beta)} > 1$ for all $\beta \in (0,1)$, there is a unique $\beta = \beta_1^* \in (0, 1/2)$ such that $k_2(\beta_1^*) = 0$. Moreover, $k_2(\beta) = \tilde{k}(1) < 0$ for all $\beta < \beta_1^*$ and $k_2(\beta) = \tilde{k}(1) > 0$ for all $\beta > \beta_1^*$. Furthermore, $\beta_1^* \simeq 0.231961$. Note that

$$\tilde{k}'(\Delta) = (1-\beta)\left((1-\Delta)^{-\beta} + e^{-\Delta(1-\beta)} - 2\right),$$

with $\tilde{k}'(0) = 0$, $\lim_{\Delta\to 1^-} \tilde{k}'(\Delta) = \infty$ and

$$\tilde{k}''(\Delta) = (1-\beta)\left(\beta(1-\Delta)^{-(\beta+1)} - (1-\beta)e^{-\Delta(1-\beta)}\right),$$

with $\tilde{k}''(0) = (1-\beta)(2\beta-1)$. Moreover, $\tilde{k}''(0) > 0$ for $\beta > 1/2$ and $\tilde{k}''(0) < 0$ for $\beta < 1/2$. Thus $\tilde{k}''(\Delta) > 0$ is equivalent to

$$(1-\Delta)^{-(\beta+1)} > \frac{1-\beta}{\beta} e^{-\Delta(1-\beta)}, \tag{6.3}$$

which is equivalent to $k_3(\Delta) < 0$ for all $\Delta \in (0,1)$ where

$$k_3(\Delta) := \log\left(\frac{1-\beta}{\beta}\right) - (1-\beta)\Delta + (1+\beta)\log(1-\Delta),$$

with $k_3(0) := \log\left(\frac{1-\beta}{\beta}\right)$ and $\lim_{\Delta\to 1^-} k_3(\Delta) = -\infty$. Note that $k_3(0) > 0$ if $\beta < 1/2$, $k_3(0) < 0$ if $\beta > 1/2$ and

$$k_3'(\Delta) = -\left((1-\beta) + \frac{1+\beta}{1-\Delta}\right) < 0,$$

for all $\Delta \in (0,1)$ and $\beta \in (0,1)$. If $k_3(\Delta) < 0$ for all $\Delta \in (0,1)$ then $k''(\Delta) > 0$ for all $\Delta \in (0,1)$. Then as $\tilde{k}'(0) = 0$, $\tilde{k}'(\Delta) > 0$ for all $\Delta \in (0,1)$ and thus $\tilde{k}(\Delta) > 0$ since $\tilde{k}(0) = 0$.

When $\beta < 1/2$ then $k_3(0) > 0$, $k_3'(\Delta) < 0$ and $\lim_{\Delta\to 1^-} k_3(\Delta) = -\infty$. Thus there is $\Delta_1 \in (0,1)$ such that $k_3(0) > 0$ and $k''(\Delta) < 0$ for all $\Delta \in (0, \Delta_1)$ while $k_3(0) < 0$ and $k''(\Delta) > 0$ for all $\Delta \in (\Delta_1, 1)$. When $\beta < 1/2$, $k''(\Delta) < 0$. Since $\tilde{k}'(0) = 0$ and $\tilde{k}'(1) = \infty$ then $\tilde{k}'(\Delta)$ is decreasing over $(0, \Delta_1)$ and increasing over $(\Delta_1, 1)$. Thus there is a unique $\Delta_2 \in (\Delta_1, 1)$ such that $\tilde{k}'(\Delta_2) = 0$. Furthermore $\tilde{k}'(\Delta) < 0$ for all

$\Delta \in (0, \Delta_2)$ and $\tilde{k}'(\Delta) > 0$ for all $\Delta \in (\Delta_2, 1)$. Similarly, $\tilde{k}(\Delta)$ is decreasing over $(0, \Delta_2)$ and increasing over $(\Delta_2, 1)$. For $\beta \in (\beta_1^*, 1/2)$, $\tilde{k}(1) = k_2(\beta) > 0$. Since $\tilde{k}(0) = 0$, $\tilde{k}'(\Delta) < 0$ over $(0, \Delta_2)$ and $\tilde{k}'(\Delta) > 0$ over $(\Delta_2, 1)$ there is a unique $\Delta_3 \in (\Delta_2, 1)$ such that $\tilde{k}(\Delta) = 0$ and $\tilde{k}(\Delta) < 0$ for all $\Delta \in (0, \Delta_3)$ and $\tilde{k}(\Delta) > 0$ for all $\Delta \in (\Delta_3, 1)$. Thus when $\beta \in (\beta_1^*, 1/2)$

$$Error_E(\Delta) < Error_T(\Delta), \quad \forall\, \Delta < \Delta_3, \tag{6.4}$$

$$Error_E(\Delta) > Error_T(\Delta), \quad \forall\, \Delta > \Delta_3. \tag{6.5}$$

When $\beta \in (0, \beta_1^*)$ then the situation is similar but $\tilde{k}(1) = k_2(\beta) < 0$. Thus $\tilde{k}(\Delta) = 0$ with $\tilde{k}(\Delta)$ decreasing over $(0, \Delta_2)$ and increasing over $(\Delta_2, 1)$. Thus $\tilde{k}(\Delta) < 0$ for all $\Delta \in (0, 1)$. Therefore when $\beta \in (0, \beta_1^*)$

$$Error_E(\Delta) < Error_T(\Delta), \quad \forall\, \Delta \in (0, 1). \tag{6.6}$$

When $\beta > 1/2$, then $k_3(0) < 0$, $k_3'(0) < 0$ and $k_3(1) = -\infty$. Thus $\tilde{k}(\Delta) > 0$ and therefore

$$Error_E(\Delta) > Error_T(\Delta), \quad \forall\, \beta \in (1/2, 1), \quad \forall\, \Delta \in (0, 1), \tag{6.7}$$

as claimed. □

Equations (6.4) and (6.5) identify that the Explicit Euler scheme outperforms the Transformed Explicit scheme when $\Delta < \Delta_3$, and vice versa for $\Delta > \Delta_3$, where Δ_3 is the unique root of

$$(1-\Delta)^{1-\beta} + e^{-\Delta(1-\beta)} + 2(1-\beta)\Delta = 2.$$

We now show that $\Delta_3 = \Delta_3(\beta)$ is a decreasing function of β where $\beta \in (\beta_1^*, 1/2)$. Letting $\beta = 1/2$, then we may define $\Delta_3(1/2) := 0$ because for $\beta = 1/2$

$$(1-\Delta_3(\beta))^{1-\beta} + e^{-\Delta_3(\beta)(1-\beta)} + 2(1-\beta)\Delta_3(\beta) = (1-0)^{1-1/2} + e^{-0(1-1/2)} + 2(1-1/2)\cdot 0 = 2.$$

Similarly, we can define $\Delta_3(\beta_1^*) := 1$ because

$$(1-\Delta_3(\beta_1^*))^{1-\beta_1^*} + e^{-\Delta_3(\beta_1^*)(1-\beta_1^*)} + 2(1-\beta_1^*)\Delta_3(\beta_1^*) = e^{-(1-\beta_1^*)} + 2 - 2\beta_1^* = 2,$$

by definition of β_1^*. The following lemma shows that $\beta \mapsto \Delta_3(\beta)$ is a decreasing function.

Lemma 20. *Let $\beta = \beta_1^*$ be the unique number in $(0, 1)$ such that $2\beta = e^{-(1-\beta)}$ and Δ_3 obeys*

$$2 = (1-\Delta)^{1-\beta} + e^{-\Delta(1-\beta)} + 2(1-\beta)\Delta,$$

then $\Delta \mapsto \Delta_3(\beta)$ *is decreasing.*

Proof. Define

$$c(\Delta, \beta) := (1-\Delta)^{1-\beta} + e^{-\Delta(1-\beta)} + 2(1-\beta)\Delta - 2.$$

Then $(\Delta, \beta) \mapsto c(\Delta, \beta)$ is in C^2 in both Δ and β for $\beta \in (\beta_1^*, 1/2)$ and $\Delta \in (0,1)$. By the Implicit Function Theorem $\beta \mapsto \Delta_3(\beta)$ is C^2. Since $c(\Delta_3(\beta), \beta) = 0$ and $c(\Delta, \beta) \neq 0$ for $\Delta \neq \Delta_3(\beta)$ hence

$$\frac{d}{d\beta} c(\Delta_3(\beta), \beta) = \frac{\partial}{\partial \Delta} c(\Delta_3(\beta), \beta) \cdot \Delta_3'(\beta) + \frac{\partial}{\partial \beta} c(\Delta_3(\beta), \beta) = 0.$$

If $\frac{\partial}{\partial \Delta} c(\Delta_3(\beta), \beta) < 0$ and $\frac{\partial}{\partial \beta} c(\Delta_3(\beta), \beta) < 0$ then $\Delta_3'(\beta) = \frac{-\partial c/\partial \beta}{\partial c/\partial \Delta} < 0$ as required. Note that

$$\begin{aligned} \frac{\partial}{\partial \Delta} c(\Delta, \beta) &= \frac{d}{d\Delta}\Big((1-\Delta)^{1-\beta} + e^{-\Delta(1-\beta)} + 2(1-\beta)\Delta\Big) \\ &= (1-\beta)\left(2 - e^{-\Delta(1-\beta)} - (1-\Delta)^{-\beta}\right) =: -\tilde{k}'(\Delta). \end{aligned}$$

From Theorem 35 part (ii), when $\beta \in (\beta_1^*, 1/2)$, then $\tilde{k}'(\Delta) < 0$ for all $\Delta \in (\Delta_1, \Delta_2)$ and $\tilde{k}'(\Delta) > 0$ for all $\Delta \in (\Delta_2, 1)$. Since $\Delta_3 > \Delta_2$, then $\tilde{k}'(\Delta_3) > 0$, so $\frac{\partial}{\partial \Delta} c(\Delta_3, \beta) < 0$. Now

$$c(\Delta, \beta) = \;=\; (1-\Delta)e^{-\beta \log(1-\Delta)} + e^{-\Delta} e^{\beta\Delta} + 2\Delta - 2\beta\Delta - 2.$$

Thus

$$\begin{aligned} \frac{\partial}{\partial \beta} c(\Delta, \beta) &= -(1-\Delta)\log(1-\Delta) e^{-\beta \log(1-\Delta)} + e^{-\Delta}\Delta e^{\beta\Delta} - 2\Delta \\ &= -(1-\Delta)^{1-\beta}\log(1-\Delta) + \Delta\left(e^{-\Delta(1-\beta)} - 2\right). \end{aligned}$$

Since $c(\Delta_3(\beta), \beta) = 0$ then

$$e^{-\Delta_3(1-\beta)} - 2 = -2(1-\beta)\Delta_3 - (1-\Delta_3)^{1-\beta}.$$

Substituting this into the previous equation implies

$$\begin{aligned} \frac{\partial}{\partial \beta} c(\Delta_3(\beta), \beta) &= -(1-\Delta_3)^{1-\beta}\log(1-\Delta_3) + \Delta_3\left(-2(1-\beta)\Delta_3 - (1-\Delta_3)^{1-\beta}\right) \\ &= -(1-\Delta_3)^{1-\beta}\log(1-\Delta_3) - 2(1-\beta)\Delta_3^2 - \Delta_3(1-\Delta_3)^{1-\beta} \\ &= \left(-\log(1-\Delta_3) - \Delta_3\right)(1-\Delta_3)^{1-\beta} - 2(1-\beta)\Delta_3^2 \\ &= A(\Delta_3, 1-\beta), \end{aligned}$$

where $A(\Delta,\gamma) := \left(-\log(1-\Delta)-\Delta\right)(1-\Delta)^{\gamma} - 2\gamma\Delta^2$. Note

$$\frac{\partial}{\partial\gamma}A(\Delta,\gamma) = \left(-\log(1-\Delta)-\Delta\right)\log(1-\Delta)\,(1-\Delta)^{\gamma} - 2\Delta^2 < 0.$$

Since $\beta \in (\beta_1^*, 1/2)$, $\gamma \in (1/2, 1-\beta_1^*)$ so as $\beta_1^* > 1/5$, then $\gamma \in (1/2, 4/5)$. Therefore for $\gamma > 1/2$ and all $\Delta \in (0,1)$

$$A(\Delta,\gamma) < A(\Delta,1/2) = \left(-\log(1-\Delta)-\Delta\right)(1-\Delta)^{1/2} - \Delta^2 =: a(\Delta).$$

By Lemma 21, $\Delta \mapsto a(\Delta)$ obeys $a(\Delta) < 0$ for all $\Delta \in (0,1)$. Hence, as $\Delta_3 \in (0,1)$ then

$$\frac{\partial}{\partial\beta}c(\Delta_3(\beta),\beta) = A(\Delta_3, 1-\beta) < A(\Delta_3, 1/2) = a(\Delta) < 0,$$

as required. □

Lemma 21. *Let $\Delta \in (0,1)$ and define*

$$a(\Delta) := \left(-\log(1-\Delta)-\Delta\right)(1-\Delta)^{1/2} - \Delta^2.$$

Then $a(\Delta) < 0$ for all $\Delta \in (0,1)$.

Proof. $a(\Delta) < 0$ for all $\Delta \in (0,1)$ is equivalent to

$$\frac{\Delta^2}{(1-\Delta)^{1/2}} > -\log(1-\Delta) - \Delta, \quad \forall\,\Delta \in (0,1). \tag{6.8}$$

Let $x = (1-\Delta)^{1/2}$. Then $x \in (0,1)$, $\Delta = 1-x^2$ and (6.8) is equivalent to

$$\frac{(1-x^2)^2}{x} > -\log x^2 - (1-x^2), \quad x \in (0,1).$$

Define for $x \in (0,1)$

$$b(x) \;:=\; \frac{(1-x^2)^2}{x} + 1 - x^2 + 2\log x = x^{-1} - 2x + x^3 + 1 - x^2 + 2\log x.$$

Then

$$b'(x) \;=\; \frac{3\,(x^2-1)\left(\left(x-\frac{1}{3}\right)^2 + \frac{2}{9}\right)}{x^2},$$

so $b'(x) < 0$ for all $x \in (0,1)$. Thus for all $x \in (0,1)$ then $b(x) > b(1) = 0$, as required. □

6.4 Explicit and Implicit Euler Schemes

In the next theorem we show the Implicit scheme outperforms the Explicit scheme for all $\beta \in (0,1)$ and all $\Delta \in (0,1)$.

Theorem 36. *Let $\beta \in (0,1)$. Then*

$$Error_E(\Delta) > Error_I(\Delta), \quad \forall\, \Delta \in (0,1). \tag{6.9}$$

Proof. Define

$$k(\Delta) \;:=\; Error_E(\Delta) - Error_I(\Delta) = \frac{\tilde{k}(\Delta)}{(1-\beta)\Delta},$$

where $\tilde{k}(\Delta) = (1+\Delta)^{1-\beta} - (1-\Delta)^{1-\beta} - 2(1-\beta)\Delta$ with $\tilde{k}(0) = 0$ and $\tilde{k}(1) = 2^{1-\beta} - 2(1-\beta) = 2(2^{-\beta} - (1-\beta))$. Thus $k(\Delta) > 0$ if $\tilde{k}(\Delta) > 0$ since $(1-\beta)\Delta > 0$. Define $k_2(\beta) := 2^{1-\beta} - 2(1-\beta) = 2(2^{-\beta} - (1-\beta))$ with $k_2(0) = 0$ and $k_2'(\beta) = 2(1 - \log 2e^{-\beta \log 2}) > 0$. Hence $k_2(\beta) > 0$ for all $\beta \in (0,1)$, so $\tilde{k}(1) > 0$ for all $\beta \in (0,1)$. Thus

$$\tilde{k}'(\Delta) = (1-\beta)(1+\Delta)^{-\beta} + (1-\beta)(1-\Delta)^{-\beta} - 2(1-\beta),$$

with $\tilde{k}'(0) = 0$ and $\lim_{\Delta \to 1^-} \tilde{k}'(\Delta) = \infty$ and

$$\tilde{k}''(\Delta) = \beta(1-\beta)\left(\frac{1}{(1-\Delta)^{1+\beta}} - \frac{1}{(1+\Delta)^{1+\beta}}\right) > 0.$$

Hence $\tilde{k}'(\Delta) > 0$ for all $\Delta \in (0,1)$. Since $\tilde{k}(0) = 0$, then $\tilde{k}(\Delta) > 0$ for all $\Delta \in (0,1)$. Hence $Error_E(\Delta) > Error_I(\Delta)\, \forall\, \Delta \in (0,1)$ which is (6.9). □

6.5 Implicit and Transformed Euler Schemes

The following theorem ranks the Implicit and Transformed schemes. Very roughly if Δ is sufficiently small the Implicit scheme outperforms the Transformed scheme if $\beta < 1/2$ while if $\beta > 1/2$ the opposite is true.

Theorem 37. *Let $\beta = \beta_1^*$ be the unique number in $(0,1)$ such that $2^{1-\beta} + e^{-(1-\beta)} = 0$. Then $\beta_1^* \in (1/2, 1/(1+\log 2))$ and the following hold:*

(i) If $\beta \in (0,1/2)$ then

$$Error_T(\Delta) > Error_I(\Delta), \quad \forall\, \Delta \in (0,1).$$

(ii) If $\beta \in (1/2, \beta_1^*)$ *then*

$$\begin{aligned} Error_T(\Delta) &< Error_I(\Delta), \quad \forall\, \Delta < \Delta_3, \\ Error_T(\Delta) &> Error_I(\Delta), \quad \forall\, \Delta > \Delta_3, \end{aligned}$$

where $\Delta = \Delta_3$ *obeys* $(1+\Delta)^{1-\beta} + e^{-\Delta(1-\beta)} = 2$.

(iii) If $\beta \in (\beta_1^*, 1)$ *then*

$$Error_T(\Delta) < Error_I(\Delta), \quad \forall\, \Delta \in (0,1).$$

Proof. Define

$$k(\Delta) \;\; := \;\; Error_T(\Delta) - Error_I(\Delta) = \frac{\tilde{k}(\Delta)}{(1-\beta)\Delta}$$

where $\tilde{k}(\Delta) := (1+\Delta)^{1-\beta} + e^{-\Delta(1-\beta)} - 2$ with $\tilde{k}(0) = 0$ and $\tilde{k}(1) = 2^{1-\beta} + e^{-(1-\beta)} - 2$. Define $k_2(\beta) := 2^{1-\beta} + e^{-(1-\beta)} - 2$ with $k_2(0) = 1/e$, $k_2(1/2) \simeq 0.0207 > 0$ and $k_2(1) = 0$. Then $k_2'(\beta) = e^{\beta-1} - \log 2 e^{\log 2(1-\beta)}$ with $k_2'(1) = 1 - \log 2 > 0$, where $k_2'(\beta) = 0$ if $\beta_1 \simeq 0.78353$ thus $k_2'(\beta) > 0$ if $\beta > \beta_1$ and $k_2'(\beta) < 0$ if $\beta < \beta_1$. Note that

$$k_2(\beta_1) = 2^{1-\beta_1} + e^{-(1-\beta_1)} - 2 = \frac{e^{-(1-\beta_1)}}{\log 2} + e^{-(1-\beta_1)} - 2 < 0.$$

There exists a $\beta_1^* \in (0, \beta_1)$ such that $k_2(\beta_1^*) = \tilde{k}(1) = 0$ with $k_2(\beta_1^*) = \tilde{k}(1) > 0$ if $\beta \in (0, \beta_1^*)$ and $k_2(\beta_1^*) = \tilde{k}(1) < 0$ if $\beta \in (\beta_1^*, 1)$. In fact $\beta_1^* \simeq 0.5906$. Note that

$$\tilde{k}'(\Delta) = (1-\beta)\left((1+\Delta)^{-\beta} - e^{-\Delta(1-\beta)}\right),$$

with $\tilde{k}'(0) = 0$. Then $\tilde{k}'(\Delta) > 0$ if $k_4(\Delta) > 0$ since $1 - \beta > 0$ where $k_4(\Delta) := \Delta(1-\beta) - \beta\log(1+\Delta)$ with $k_4(0) = 0$ and $k_4(1) = 1 - \beta(1+\log 2)$. There exists a β_2^* such that $k_4(1) = 0$ with $k_4(1) > 0$ if $\beta < \beta_2^*$ and $k_4(1) < 0$ if $\beta > \beta_2^*$. In fact $\beta_2^* \simeq 0.562173$. Thus

$$k_4'(\Delta) = 1 - \beta - \frac{\beta}{1+\Delta} \quad \text{and} \quad k_4''(\Delta) = \frac{\beta}{(1+\Delta)^2} > 0.$$

Thus $k_4'(\Delta) = 0$ if $\Delta = \Delta_1 = (2\beta-1)/(1-\beta)$, $k_4'(\Delta) > 0$ if $\Delta > \Delta_1$ and $k_4'(\Delta) < 0$ if $\Delta < \Delta_1$ where $\Delta_1 \in (0,1)$ if $\beta < 2/3$. The second derivative of $\tilde{k}(\Delta)$ is

$$k''(\Delta) = (1-\beta)\left(e^{-\Delta(1-\beta)} - \beta(1+\Delta)^{-(\beta+1)}\right),$$

with $k''(0) = (1-\beta)(1-2\beta)$. Thus $k''(0) > 0$ if $\beta < 1/2$ and $k''(0) < 0$ if $\beta > 1/2$.

Note that $k''(\Delta) > 0$ is equivalent to $k_3(\Delta) > 0$ where

$$k_3(\Delta) := (\beta+1)\log(1+\Delta) - (1-\beta)\Delta + \log\left(\frac{1-\beta}{\beta}\right),$$

where

$$k_3(0) = \log\left(\frac{1-\beta}{\beta}\right) \quad \text{and} \quad k_3(1) = \log\left(\frac{1-\beta}{\beta}\right) + (1+\beta)\log 2 - (1-\beta),$$

where $k_3(0) > 0$ if $\beta < 1/2$ and $k_3(0) < 0$ if $\beta > 1/2$ and

$$k_3'(\Delta) = \frac{1+\beta}{1+\Delta} - (1-\beta),$$

where $k_3'(\Delta) > 0$ if $\Delta_1 < 2\beta/(1-\beta)$ and $k_3'(\Delta) < 0$ if $\Delta_1 > 2\beta/(1-\beta)$. Note that $\Delta_1 > 1$ if $\beta \in (1/3, 1/2)$ and $\Delta_1 \in (0,1)$ if $\beta \in (0, 1/3)$.

When $\beta \in (0, 1/3)$ then $k_3(\Delta) > 0$, $k_3'(\Delta) > 0$ for all $\Delta > \Delta_1 \in (0,1)$ and $k_3(1) > 0$. Thus $k''(\Delta) > 0$ for all $\Delta \in (0,1)$. Since $\tilde{k}'(0) = 0$, $\tilde{k}'(\Delta) > 0$ for all $\Delta \in (0,1)$ and as $\tilde{k}(0) = 0$ then $\tilde{k}(\Delta) > 0$ for all $\Delta \in (0,1)$.

When $\beta \in (1/3, 1/2)$ then $\tilde{k}'(\Delta) > 0$ for all $\Delta \in (0,1)$ since $\Delta_1 > 1$. Therefore combining both results, when $\beta \in (0, 1/2)$, therefore

$$Error_T(\Delta) > Error_I(\Delta), \quad \forall\, \Delta \in (0,1). \tag{6.10}$$

When $\beta \in (1/3, \beta_1^*)$ then $k_4(0) < 0$, $k_4(1) < 1$ and $k_4'(\Delta) < 0$ for $\Delta < \Delta_1$ and $k_4'(\Delta) > 0$ for $\Delta > \Delta_1$. Thus there is a unique $\Delta_2 \in (\Delta_1, 1)$ such that $k_4(\Delta) < 0$ for all $\Delta \in (0, \Delta_2)$ and $k_4(\Delta) > 0$ for all $\Delta \in (\Delta_2, 1)$. Thus $\tilde{k}'(0) = 0$, $\tilde{k}(\Delta)$ is decreasing over $(0, \Delta_2)$ and increasing over $(\Delta_2, 1)$ with $k_2(1) = \tilde{k}(1) > 0$. Therefore there is a unique Δ_3 such that $k_3(\Delta_3) = 0$ with $k_3(\Delta) < 0$ for all $\Delta \in (0, \Delta_3)$ and $k_3(\Delta) > 0$ for all $\Delta \in (\Delta_3, 1)$. Therefore

$$Error_T(\Delta) < Error_I(\Delta), \quad \forall\, \Delta < \Delta_3, \tag{6.11}$$

$$Error_T(\Delta) > Error_I(\Delta), \quad \forall\, \Delta > \Delta_3. \tag{6.12}$$

When $\beta \in (\beta_1^*, \beta_2^*)$ then $k_4(0) = 0$, $k_4(1) > 0$, $k_4'(\Delta) < 0$ if $\Delta \in (0, \Delta_1)$ and $k_4'(\Delta) > 0$ if $\Delta \in (\Delta_1, 1)$. Thus there is a unique Δ_2 such that $k_4(\Delta_2) = 0$ with $k_4(\Delta) < 0$ for $\Delta \in (0, \Delta_2)$ and $k_4(\Delta) > 0$ for $\Delta \in (\Delta_2, 1)$. Thus $\tilde{k}'(\Delta) < 0$ for all $\Delta \in (0, \Delta_2)$ and $\tilde{k}'(\Delta) > 0$ for all $\Delta \in (\Delta_2, 1)$. Thus $\tilde{k}(\Delta) > 0$ for all $\Delta \in (0,1)$ since $\tilde{k}(0) = 0$, $\tilde{k}$ decreasing over $(0, \Delta_2)$, increasing over $(\Delta_2, 1)$ with $\tilde{k}(1) < 0$.

When $\beta > \beta_2^*$ then $k_4(1) < 0$ with $k_4(0) = 0$, $k_4'(\Delta) < 0$ for all $\Delta \in (0,1)$ since $\beta < 2/3$.

Thus $k_4(\Delta) < 0$ for all $\Delta \in (0,1)$. Combining both results yields for $\beta \in (\beta_1^*, 1)$

$$Error_T(\Delta) < Error_I(\Delta), \quad \forall\, \Delta \in (0,1). \tag{6.13}$$

□

Equations (6.11) and (6.12) identify that the Implicit Euler scheme outperforms the Transformed Explicit scheme when $\Delta < \Delta_3$, and vice versa, where Δ_3 is the root of

$$(1+\Delta)^{1-\beta} + e^{-\Delta(1-\beta)} = 2.$$

We now show that $\Delta_3 = \Delta_3(\beta)$ is an increasing function of β where $\beta \in (1/2, \beta_2^*)$. Let $\beta = 1/2$, then we may define $\Delta_3(1/2) := 0$ because for $\beta = 1/2$

$$(1+\Delta_3(\beta))^{1-\beta} + e^{-\Delta_3(\beta)(1-\beta)} = (1+0)^{1-1/2} + e^{-0(1-1/2)} = 2.$$

Similarly, we can define $\Delta_3(\beta_2^*) := 1$ because

$$(1+\Delta_3(\beta_2^*))^{1-\beta_2^*} + e^{-\Delta_3(\beta_2^*)(1-\beta_2^*)} = 2,$$

by definition of β_2^*. The following lemma shows that $\beta \mapsto \Delta_3(\beta)$ is an increasing function.

Lemma 22. *Let $\beta = \beta_2^*$ be the unique number in $(0,1)$ such that $2^{1-\beta} + e^{-(1-\beta)} = 2$ and $\Delta = \Delta_3$ obeys*

$$2 = (1+\Delta)^{1-\beta} + e^{-\Delta(1-\beta)},$$

then $\beta \mapsto \Delta_3(\beta)$ is increasing.

Proof. Define

$$c(\Delta, \beta) := (1+\Delta)^{1-\beta} + e^{-\Delta(1-\beta)} - 2.$$

Then $(\Delta, \beta) \mapsto c(\Delta, \beta)$ is in C^2 in both Δ and β for $\beta \in (1/2, \beta_2^*)$ and $\Delta \in (0,1)$. By the Implicit Function Theorem $\beta \mapsto \Delta_3(\beta)$ is C^2. Since $c(\Delta_3(\beta), \beta) = 0$ and $c(\Delta, \beta) \neq 0$ for $\Delta \neq \Delta_3(\beta)$ hence

$$\frac{d}{d\beta} c(\Delta_3(\beta), \beta) = \frac{\partial}{\partial \Delta} c(\Delta_3(\beta), \beta) \cdot \Delta_3'(\beta) + \frac{\partial}{\partial \beta} c(\Delta_3(\beta), \beta) = 0.$$

If $\frac{\partial}{\partial \Delta} c(\Delta_3(\beta), \beta) > 0$ and $\frac{\partial}{\partial \beta} c(\Delta_3(\beta), \beta) < 0$ then $\Delta_3'(\beta) = \frac{-\partial c/\partial \beta}{\partial c/\partial \Delta} > 0$. Now

$$\frac{\partial c}{\partial \Delta} = (1-\beta)\left((1+\Delta)^{-\beta} - e^{-\Delta(1-\beta)}\right) = \tilde{k}'(\Delta).$$

From Theorem 37 part (ii), for $\beta \in (1/2, \beta_2^*)$, $\tilde{k}'(\Delta) > 0$ for $\Delta \in (\Delta_2, 1)$. But $\Delta_3(\beta) \in$

$(\Delta_2, 1)$, so $\frac{\partial c}{\partial \Delta}(\Delta_3(\beta), \beta) > 0$. Writing $c(\Delta, \beta) = (1+\Delta)^{-\beta \log(1+\Delta)} + e^{-\Delta} e^{\beta\Delta} - 2$ then

$$\begin{aligned} \frac{\partial c}{\partial \beta} &= -(1+\Delta)\log(1+\Delta)e^{-\beta\log(1+\Delta)} + e^{-\Delta}\Delta e^{\beta\Delta} \\ &= -\log(1+\Delta)(1+\Delta)^{1-\beta} + \Delta e^{-\Delta(1-\beta)} := -k_5(\Delta). \end{aligned}$$

Lemma 23 shows that $k_5(\Delta) > 0$ and hence $\frac{\partial}{\partial\beta}\Delta_3(\beta) > 0$, as claimed. □

Lemma 23. *Suppose $\beta \in (0.5, 0.6)$ and define $\gamma := 1 - \beta$ and*

$$k_5(\Delta) = \log(1+\Delta)(1+\Delta)^{\gamma} - \Delta e^{-\Delta\gamma}, \quad \Delta \in (0,1).$$

Then $k_5(\Delta) > 0$ for all $\Delta \in (0, 1)$.

Proof. $k_5(\Delta) > 0$ is equivalent to

$$\left((1+\Delta)e^{\Delta}\right)^{\gamma} > \frac{\Delta}{\log(1+\Delta)},$$

and this is in turn equivalent to

$$\gamma\left(\log(1+\Delta) + \Delta\right) > \log\left(\frac{\Delta}{\log(1+\Delta)}\right) = -\log\left(\frac{\log(1+\Delta)}{\Delta}\right).$$

Thus $k_5(\Delta) > 0$ for all $\Delta \in (0,1)$ is equivalent to $k_6(\Delta) > 0$ for all $\Delta \in (0,1)$ where

$$k_6(\Delta) := \gamma\left(\log(1+\Delta) + \Delta\right) + \log\left(\frac{\log(1+\Delta)}{\Delta}\right).$$

We see that $\lim_{\Delta\to 0^+} k_6(\Delta) = 0$. Also, as $\gamma > 1/4$,

$$\begin{aligned} k_6'(\Delta) &= \gamma\left(1 + \frac{1}{1+\Delta}\right) + \left(\frac{1}{1+\Delta}\right)\left(\frac{1}{\log(1+\Delta)}\right) - \frac{1}{\Delta} \\ &> \frac{1}{4}\left(1 + \frac{1}{1+\Delta}\right) + \left(\frac{1}{1+\Delta}\right)\left(\frac{1}{\log(1+\Delta)}\right) - \frac{1}{\Delta} =: k_7(\Delta). \end{aligned}$$

If $k_7(\Delta) > 0$ for all $\Delta \in (0, 1)$, then $k_6(\Delta) > 0$ for all $\Delta \in (0, 1)$ and the result holds. Thus $k_7(\Delta) > 0$ is equivalent to

$$\frac{1}{4}\left(1 + \frac{1}{1+\Delta}\right) - \frac{1}{\Delta} > -\left(\frac{1}{1+\Delta}\right)\left(\frac{1}{\log(1+\Delta)}\right),$$

which in turn is equivalent to

$$\frac{1}{\log(1+\Delta)} > \frac{1+\Delta}{\Delta} - \frac{2+\Delta}{4} = \frac{1}{\Delta} + \frac{1}{2} - \frac{\Delta}{4}.$$

The right-hand side is positive for $\Delta \in (0,1)$. Thus $k_7(\Delta) > 0$ is equivalent to

$$\left(\frac{1}{\Delta} + \frac{1}{2} - \frac{\Delta}{4}\right)^{-1} > \log\left(1+\Delta\right).$$

Define

$$k_8(\Delta) := \left(\frac{1}{\Delta} + \frac{1}{2} - \frac{\Delta}{4}\right)^{-1} - \log\left(1+\Delta\right).$$

Notice that $k_7(\Delta)$ for all $\Delta \in (0,1)$ if $k_8(\Delta) > 0$ for all $\Delta \in (0,1)$. Clearly $k_8(\Delta) \to 0$ as $\Delta \to 0^+$. If $k_8'(\Delta) > 0$ for all $\Delta \in (0,1)$ then $k_8(\Delta) > 0$ and $k_7(\Delta) > 0$ for all $\Delta \in (0,1)$. Hence $k_6'(\Delta) > 0$ for all $\Delta \in (0,1)$ and so $k_6(\Delta) > 0$ for all $\Delta \in (0,1)$. This shows $k_5(\Delta) > 0$ for all $\Delta \in (0,1)$. Define

$$\rho(\Delta) := \left(1 + \frac{\Delta}{2} - \frac{\Delta^2}{4}\right)^2,$$

for all $\Delta \in (0,1)$. Then

$$k_8'(\Delta) = \left(\frac{1}{\Delta} + \frac{1}{2} - \frac{\Delta}{4}\right)^{-2}\left(\frac{1}{\Delta^2} + \frac{1}{4}\right) - \frac{1}{1+\Delta} = \frac{\Delta^2\left(8\Delta + 8 - \Delta^2\right)}{16(1+\Delta)\rho(\Delta)}.$$

Clearly $8\Delta + 8 - \Delta^2 > 0$ for all $\Delta \in (0,1)$, so $k_8'(\Delta) > 0$ for all $\Delta \in (0,1)$. As indicated above, this shows that $k_5(\Delta) > 0$ for all $\Delta \in (0,1)$, as required. □

Chapter 7

Modified Implicit and Multi-Step Methods

7.1 Introduction

Our analysis to date has considered three one-step numerical methods, all of which use state-dependent adaptive time-stepping. We have discussed in some detail the relative advantages of each, but we make a brief synopsis once again in order to motivate the study of some further schemes which we conduct in this chapter.

The most elementary scheme is the Explicit Euler, with an explicit time-step (i.e. a time-step which depends explicitly on the current state). This has the advantage of being simple to implement but has problems for values of the control parameter $\Delta \geq 1$, in that positivity and asymptotic behaviour of the computed solution gives spurious results. Nevertheless for small Δ, the performance is not significantly inferior to more sophisticated schemes.

We have also considered an Implicit one-step method. It has the advantage of recovering all important qualitative features of the solution and correct asymptotic behaviour without restriction on the control parameter. However, it has two drawbacks. First, like implicit methods in general, it is usually necessary to perform non-linear solving at each time-step in order to proceed. Given that the time-steps must be taken smaller and smaller as the computed solution tends to the equilibrium, this is computationally expensive. Second, the time-step is determined "implicitly", in the sense that the scheme must know the level to which it will move at the next step in order to determine the length of the time-step (in contrast to the Explicit scheme). Whilst this does not present a particular computational problem, as the non-linear solving is still feasible, it is philosophically concerning that we are in effect choosing at the same moment how far we should jump and to where we should jump. In addition, such a method would be questionable in a stochastic setting, because the time-step would depend on the future value of the solution, thereby creating difficulties with the adaptedness in the process.

Our third method is a Transformed Explicit scheme which operates in a new co-ordinate system; then the computed solution of an auxiliary ODE in the new co-ordinate system is pulled back to the original system. In this scheme the time-stepping is again adaptive and explicit. Despite the explicit character of the scheme, the stability, positivity, monotonicity and recovery of finite-stability (or its absence) is achieved unconditionally for all values of the control parameter $\Delta > 0$. This method shares this advantage with the Implicit scheme but being explicit does not require non-linear solving and can be implemented almost as easily as the conventional Explicit scheme. Furthermore, quantitative measures of the asymptotic behaviour are recovered and the scheme is competitive with the Implicit scheme. Indeed, in the case that the solution of the ODE converges super-exponentially and does not hit zero in finite-time, the asymptotic decay rate is recovered exactly. Finally, the explicitness of the method and automatic and certain preservation of positivity make it an attractive choice for simulating SDEs and will show in the second half of this thesis that the performance of this scheme is remarkably reliable for SDEs. Indeed all the desirable properties enumerated above for ODEs are still true then the method is applied to SDEs.

In this chapter, we ask whether we can consider variants of these three methods with a view to establishing better performance. One potential place we could improve matters is to devise an implicit scheme in which the time-step was chosen explicitly. The details of such a scheme are presented here. Very roughly speaking, we show that its performance is comparable to the "double" Implicit scheme for equations with regularly varying non-linearity, since its recovery of qualitative and quantitative features are unconditional on the control parameter. However, in the case when we assume only monotonicity hypotheses on the non-linearity, our results are inconclusive and suggest restrictions on the control parameter may be necessary. This contrasts with the situation for the fully double Implicit scheme, where our theoretical results show there is no such potential restriction. Furthermore, the scheme does not significantly out-perform the Transformed scheme, which has the advantage of avoiding non-linear solving.

We also consider whether linear multi-step methods with adaptive time-stepping might give improved performance, but the results for the two-step schemes we have considered do not point to any significant improvement. Indeed, if anything the inclusion of "out dated" information about the solution in a situation in which the gradient of the solution can change relatively quickly (due to fast or finite-time convergence) tends to make these schemes less attractive. In particular, for the Implicit method, the two-step scheme introduces conditions on the control parameter which were not present in the single-step Implicit scheme. Nevertheless, we will consider another multi-step in Chapter 8 (namely Collocation methods) and find that the midpoint method with adaptive time-stepping appears to give improved performance, as it estimates the simulated generalised Liapunov exponent to $O(\Delta^2)$ as $\Delta \to 0^+$.

7.2 Implicit Euler Scheme with Explicit Adaptive Step Size

In this section we investigate the performance of an Implicit Euler scheme but with an explicit time step. We approximate $x(t_n)$ by x_n, where $x(t_n)$ is the solution x of (1.1) at time t_n. The sequences (x_n), (t_n) and $(h(x_n))$ are defined by

$$x_{n+1} = x_n - h(x_n)f(x_{n+1}), \quad n \geq 0, \quad x_0 = \xi > 0, \tag{7.1}$$

where (t_n) is defined by (1.42) viz.,

$$t_{n+1} = \sum_{j=0}^{n} h(x_j), \quad n \geq 0, \quad t_0 = 0.$$

and

$$h(x) = \frac{\Delta(x)x}{f(x)}, \quad x > 0, \tag{7.2}$$

with $\Delta : [0,\infty) \mapsto [0,\infty)$ continuous and $\Delta(x) \to \Delta \in [0,\infty)$ as $x \to 0^+$. This is equivalent to (3.17) where we have chosen to impose properties on Δ rather than on h directly. Note h obeys (3.2) as a result.

We will see that when only monotonicity assumptions are imposed on f we will restrict $\Delta \in [0, 1)$ while no such restrictions will be required when f is assumed to be regularly varying.

7.2.1 Asymptotic Behaviour with Monotone Assumptions on the Non-Linearity

We suppose that f obeys (3.1) and impose the following monotonicity assumptions on f:

$$f \text{ is an increasing function;} \tag{7.3}$$

$$x \mapsto x/f(x) \text{ is an increasing function.} \tag{7.4}$$

The following results guarantee the existence, positivity and convergence of the solutions of (7.1).

Lemma 24. *Suppose f obeys* (3.1) *and h obeys* (3.2)*. If $x > 0$, the equation*

$$y + h(x)f(y) = x, \tag{7.5}$$

has at least one solution in $(0, x)$ *and no solutions in* $[x, \infty)$*. If* $y(x)$ *is a solution of (7.5),* $y(x) \to 0$ *as* $x \to 0^+$ *and there is a unique solution* $y(x)$ *for each* x *if* f *is increasing.*

Proof. Define for each $x > 0$

$$K(y) := y + h(x)f(y) - x \quad y \in [x, \infty).$$

Then $K(0) = -x < 0$ and $K(x) = h(x)f(x) > 0$. Since $K : [0, \infty) \to \mathbb{R}$ is continuous, $K(y) = 0$ has at least one solution in $(0, x)$. For $y > x$

$$K(y) = y + h(x)f(y) - x > y - x > 0,$$

and $K(x) > 0$. Thus $K(y) > 0$ for all $y \geq x$, and (7.5) has no solutions in $[x, \infty)$. Since any solution $y(x)$ of (7.5) obeys $0 < y(x) < x$, it follows that $y(x) \to 0$ as $x \to 0^+$. Moreover if f is increasing there is a unique $y(x) \in (0, x)$ such that $K(y(x)) = 0$ due to the monotonicity of K. □

Remark 28. If f is increasing and continuous the ODE (1.1) has a unique solution and this property is preserved by the numerical scheme. □

Proposition 13. *Suppose* f *obeys* (3.1) *and* h *obeys* (3.2)*. There exists at least one positive sequence* (x_n) *which obeys* (7.1) *and any such sequence is decreasing and obeys* $x_n \to 0$ *as* $n \to \infty$*. Moreover, if* f *is increasing then the solution is unique.*

Proof. The existence of the sequence is implied by the root of (7.5). Since the solution $y(x) \in (0, x)$ then $x_n > 0 \, \forall \, n \geq 0$ implies $0 < x_{n+1} < x_n$. Since x_n is decreasing, we have $x_n \to L \in [0, \infty)$ as $n \to \infty$. Therefore if $L > 0$ then

$$L = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} \{x_n - h(x_n)f(x_{n+1})\} = L - h(L)f(L),$$

by (3.1), (3.2) and $h(L)f(L) = 0$ which is impossible by (3.1) and (3.2). Hence $x_n \to 0$ as $n \to \infty$. The uniqueness under f being monotone follows from Lemma 24. □

Since (x_n) is positive and decreasing and f obeys (7.3) then $x_{n+1} < u < x_n$ implies $f(x_{n+1}) < f(u) < f(x_n)$. Thus

$$\frac{1}{f(x_n)} < \frac{1}{f(u)} < \frac{1}{f(x_{n+1})}.$$

By (7.1), integrating over $[x_{n+1}, x_n]$ yields

$$\frac{x_n - x_{n+1}}{f(x_n)} \leq \int_{x_{n+1}}^{x_n} \frac{1}{f(u)} \, du \leq \frac{x_n - x_{n+1}}{f(x_{n+1})} = h(x_n). \tag{7.6}$$

Similarly since f obeys (7.4) then

$$\frac{x_{n+1}}{f(x_{n+1})} \cdot \frac{1}{u} < \frac{1}{f(u)} < \frac{x_n}{f(x_n)} \cdot \frac{1}{u},$$

and integrating over $[x_{n+1}, x_n]$ yields

$$\frac{x_{n+1}}{f(x_{n+1})} \log\left(\frac{x_n}{x_{n+1}}\right) \leq \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\, du \leq \frac{x_n}{f(x_n)} \log\left(\frac{x_n}{x_{n+1}}\right).$$

Taking the lower inequality from above and the upper inequality from (7.6) yields

$$\frac{x_{n+1}}{f(x_{n+1})} \log\left(\frac{x_n}{x_{n+1}}\right) \leq \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\, du \leq h(x_n). \tag{7.7}$$

Lemma 25. *Suppose (x_n) is a positive decreasing sequence and the solution of (7.1). If f obeys (3.1), (3.32), (7.3) and (7.4) while h obeys (3.2) and (7.2) with $\Delta \in [0, 1)$ then (7.7) holds and implies for all n sufficiently large*

$$a\left(\frac{1}{1-\Delta(x_n)}\right) h(x_n) \leq \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\, du \leq h(x_n), \tag{7.8}$$

where $a(x) := \log x/(x-1)$, $x > 1$.

Proof. The lower estimate of (7.7) implies

$$\int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\, du \geq \frac{x_{n+1}}{f(x_{n+1})} \log\left(\frac{x_n}{x_{n+1}}\right) = \frac{\log(1/\lambda_n)}{1/\lambda_n - 1} \cdot h(x_n) =: a\,(1/\lambda_n)\, h(x_n),$$

where $\lambda_n := x_{n+1}/x_n$. If f obeys (7.3) then $f(x_{n+1}) < f(x_n)$ and

$$1 - \lambda_n = \Delta(x_n)\frac{f(x_{n+1})}{f(x_n)} < \Delta(x_n),$$

and thus $\lambda_n > 1 - \Delta(x_n)$. Since $\Delta(x_n) \to \Delta \in [0, 1)$ as $n \to \infty$ and a is decreasing we have that

$$a\left(\frac{1}{\lambda_n}\right) > a\left(\frac{1}{1-\Delta(x_n)}\right).$$

Thus

$$a\left(\frac{1}{1-\Delta(x_n)}\right) h(x_n) \leq \int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\, du \leq h(x_n),$$

as claimed. □

The restriction that $\Delta < 1$ is surprising in the light of the unconditional positivity and monotonicity on Δ in Proposition 13 and in view of the unconditional recovery of finite-time stability and global positivity in the Implicit scheme with an implicit step-size. This result is also unexpected in light of analysis later in this chapter, which states in the case when f is regularly varying, that the finite-time stability, global positivity and asymptotic behaviour are recovered without restrictions on Δ.

Theorem 38. *Suppose f obeys* (3.1), (3.32), (7.3) *and* (7.4) *while h obeys* (3.2) *and* (7.2) *with $\Delta \in [0,1)$. Let (t_n) and $\hat{T}_h$ be defined* (1.42) *and* (3.16)*.*

(i) If f obeys (1.7)*, then $\hat{T}_h < \infty$.*

(ii) If f obeys (1.9)*, then $\hat{T}_h = \infty$.*

Proof. By (1.7), $\int_{0^+}^1 1/f(u)\,du < \infty$ then $T_\xi < \infty$ from (3.12). The Comparison Test applied to (7.8) shows the summability of $(\int_{x_{n+1}}^{x_n} 1/f(u)\,du)$ implies that of $(a\left(\frac{1}{1-\Delta(x_n)}\right)h(x_n))$. Thus $(h(x_n))$ is summable since

$$\lim_{n\to\infty} a\left(\frac{1}{1-\Delta(x_n)}\right) = a\left(\frac{1}{1-\Delta}\right) = \frac{(1-\Delta)}{\Delta}\log\left(\frac{1}{1-\Delta}\right),$$

when $\Delta > 0$ and when $\Delta = 0$ the limit is unity. Thus $t_n = \sum_{j=0}^{n-1} h(x_j)$ for $n \geq 1$ obeys $t_n \to \hat{T}_h := \sum_{j=0}^{\infty} h(x_j) < \infty$ as $n \to \infty$.

By (1.9), $\int_{0^+}^1 1/f(u)\,du = \infty$ then $T_\xi = \infty$ from (3.11). The Comparison Test applied to (7.8) shows that $(h(x_n))$ is not summable and thus $(h(x_n))$ obeys $t_n = \sum_{j=0}^{n-1} h(x_j) \to \infty$ as $n \to \infty$. Therefore $t_n \to \infty$ as $n \to \infty$. □

Theorem 39. *Suppose f obeys* (3.1), (3.32), (7.3) *and* (7.4) *while h obeys* (3.2) *and* (7.2) *where $\Delta(x) \to \Delta \in [0,1)$ as $x \to 0^+$. Let F, $\bar{F}$, (t_n) and $\hat{T}_h$ be defined by* (1.11), (1.10), (1.42) *and* (3.16)*.*

(a) Suppose f obeys (1.7)*.*

(i) If $\Delta = 0$, then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and

$$1 \leq \liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq \limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq 1.$$

(ii) If $\Delta \in (0,1)$, then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and

$$\frac{(1-\Delta)}{\Delta}\log\left(\frac{1}{1-\Delta}\right) \leq \liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq \limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq 1.$$

(b) Suppose f obeys (1.9)*.*

(i) If $\Delta = 0$, then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \infty$ as $n \to \infty$ and

$$1 \leq \liminf_{n\to\infty} \frac{F(x_n)}{t_n} \leq \limsup_{n\to\infty} \frac{F(x_n)}{t_n} \leq 1.$$

(ii) If $\Delta \in (0,1)$, then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \infty$ as $n \to \infty$ and

$$\frac{(1-\Delta)}{\Delta}\log\left(\frac{1}{1-\Delta}\right) \leq \liminf_{n\to\infty}\frac{F(x_n)}{t_n} \leq \limsup_{n\to\infty}\frac{F(x_n)}{t_n} \leq 1.$$

Proof. The positivity, monotonicity and convergence of (x_n) have been addressed in Lemma 24 and Proposition 13. Since f obeys (1.7) then $\int_{0+}^{1} 1/f(u)\,du < \infty$ and $t_n \to \hat{T}_h := \sum_{j=0}^{\infty} h(x_j) < \infty$ by Theorem 38. Hence $\hat{T}_h - t_n = \sum_{j=n}^{\infty} h(x_n) \to 0$ as $n \to \infty$. By the second inequality of (7.8)

$$\bar{F}(x_n) = \sum_{j=n}^{\infty}\int_{x_{j+1}}^{x_j}\frac{1}{f(u)}\,du \leq \sum_{j=n}^{\infty} h(x_j) = \hat{T}_h - t_n.$$

Therefore, dividing by $\hat{T}_h - t_n$ and letting $n \to \infty$ yields

$$\limsup_{n\to\infty}\frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq 1.$$

By the first inequality of (7.8) for n all sufficiently large

$$\bar{F}(x_n) = \sum_{j=n}^{\infty}\int_{x_{j+1}}^{x_j}\frac{1}{f(u)}\,du \geq \sum_{j=n}^{\infty} a\left(\frac{1}{1-\Delta(x_n)}\right)h(x_n).$$

Therefore dividing by $\hat{T}_h - t_n$ and letting $n \to \infty$ yields

$$\liminf_{n\to\infty}\frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \quad\geq\quad \liminf_{n\to\infty}\frac{\sum_{j=n}^{\infty} a\left(\frac{1}{1-\Delta(x_n)}\right)h(x_n)}{\sum_{j=n}^{\infty} h(x_n)} = \frac{(1-\Delta)}{\Delta}\log\left(\frac{1}{1-\Delta}\right),$$

by Toeplitz's Lemma with the limit being unity when $\Delta(x_n) \to \Delta = 0$ as $n \to \infty$. Combining both inequalities yields part (i)

$$\frac{(1-\Delta)}{\Delta}\log\left(\frac{1}{1-\Delta}\right) \leq \liminf_{n\to\infty}\frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq \limsup_{n\to\infty}\frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq 1,$$

with the limit on the left-hand side being unity in the case when $\Delta = 0$. Since f obeys (1.9) then $\int_{0+}^{1} 1/f(u)\,du = \infty$ and $t_n \to \infty$ by Theorem 38. By the second ineqaulity of (7.8), for $n \geq 1$

$$F(x_n) - F(x_0) = \sum_{j=0}^{n-1}\int_{x_{j+1}}^{x_j}\frac{1}{f(u)}\,du \leq \sum_{j=0}^{n-1} h(x_n) = t_n.$$

Therefore, dividing by t_n and letting $n \to \infty$ yields

$$\limsup_{n\to\infty} \frac{F(x_n)}{t_n} \leq 1. \tag{7.9}$$

By the first inequality of (7.8), for $n \geq 1$

$$F(x_n) - F(x_0) = \sum_{j=0}^{n-1} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\,du \geq \sum_{j=0}^{n-1} a\left(\frac{1}{1-\Delta(x_n)}\right) h(x_n). \tag{7.10}$$

Therefore, dividing by t_n and letting $n \to \infty$ yields

$$\liminf_{n\to\infty} \frac{F(x_n)}{t_n} \quad \geq \quad \liminf_{n\to\infty} \frac{\sum_{j=0}^{n-1} a\left(\frac{1}{1-\Delta(x_n)}\right) h(x_n)}{\sum_{j=0}^{n-1} h(x_n)} = \frac{(1-\Delta)}{\Delta}\log\left(\frac{1}{1-\Delta}\right),$$

by Toeplitz's Lemma with the limit being unity when $\Delta(x_n) \to \Delta = 0$ as $n \to \infty$. Combining both inequalities yields part (ii)

$$\frac{(1-\Delta)}{\Delta}\log\left(\frac{1}{1-\Delta}\right) \leq \liminf_{n\to\infty} \frac{F(x_n)}{t_n} \leq \limsup_{n\to\infty} \frac{F(x_n)}{t_n} \leq 1,$$

with the limit on the left-hand side being unity in the case when $\Delta = 0$, as claimed. □

Remark 29. An alternative lower bound in Theorem 39 is

$$\int_{x_{n+1}}^{x_n} \frac{1}{f(u)}\,du \geq \frac{x_n - x_{n+1}}{f(x_n)} \quad = \quad h(x_n)\frac{f(x_{n+1})}{f(x_n)} \geq h(x_n)\lambda_n > h(x_n)(1-\Delta(x_n)).$$

This leads to

$$\liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \geq 1 - \Delta,$$

in case (i) and to

$$\liminf_{n\to\infty} \frac{F(x_n)}{t_n} \geq 1 - \Delta,$$

in case (ii). □

7.2.2 Asymptotic Behaviour with Regularly Varying Non-Linearity

In this section we investigate the performance of an Implicit Euler scheme with an Explicit time step by imposing assumptions of regular variation on f. We approximate $x(t_n)$ by x_n, where $x(t_n)$ is the solution x of (1.1) at time t_n. The sequences (x_n), (t_n) and $(h(x_n))$ are defined as before by

$$x_{n+1} = x_n - h(x_n)f(x_{n+1}), \quad n \geq 0, \quad x_0 = \xi > 0,$$

where

$$t_{n+1} = \sum_{j=0}^{n} h(x_j), \quad n \geq 0, \quad t_0 = 0,$$

and

$$h(x) = \frac{\Delta(x)x}{f(x)}, \quad x > 0,$$

with $\Delta : [0, \infty) \mapsto (0, \infty)$ continuous and $\Delta(x) \to \Delta \in (0, \infty)$ as $x \to 0^+$. For brevity we have omitted the case when $\Delta = 0$ because this has been dealt with satisfactorily in Theorem 39 using only monotonicity assumptions on f. Therefore, we have

$$x_{n+1} = x_n - \frac{x_n \Delta(x_n)}{f(x_n)} f(x_{n+1}),$$

so defining $\lambda_n := x_{n+1}/x_n$, then λ_n obeys for $n \geq 0$

$$\lambda_n = 1 - \Delta(x_n) \frac{f(\lambda_n x_n)}{f(x_n)}.$$

We start by determining the asymptotic behaviour when $\beta > 0$ leaving the case when $\beta = 0$ until later.

Lemma 26. *Let* (x_n) *be a positive decreasing solution of* (7.1)*. Suppose* f *obeys* (3.1), (3.32), (7.3) *and* $f \in RV_0(\beta)$, $\beta \in (0, 1]$ *while* h *obeys* (3.2) *and* (7.2) *then*

$$\lim_{n \to \infty} \frac{x_{n+1}}{x_n} = \lambda_*(\Delta),$$

where $\lambda_*(\Delta)$ *is the unique solution of* $\phi(\lambda) = 0$ *where* $\phi(\lambda) := \lambda + \Delta\lambda^\beta - 1$*. Moreover,* $\lambda_*(\Delta) \to 1$ *as* $\Delta \to 0^+$ *and* $1 - \lambda_*(\Delta) \sim \Delta$ *as* $\Delta \to 0^+$*.*

Proof. Note that

$$\begin{aligned}
\lambda_n = 1 - \Delta(x_n)\frac{f(\lambda_n x_n)}{f(x_n)} &= 1 - \Delta(x_n)\left(\frac{f(\lambda_n x_n)}{f(x_n)} - \lambda_n^\beta + \lambda_n^\beta\right) + \Delta\lambda_n^\beta - \Delta\lambda_n^\beta \\
&= 1 - \Delta\lambda_n^\beta + (\Delta - \Delta(x_n))\,\lambda_n^\beta - \Delta(x_n)\left(\frac{f(\lambda_n x_n)}{f(x_n)} - \lambda_n^\beta\right) \\
&= 1 - \Delta\lambda_n^\beta + \epsilon_n,
\end{aligned}$$

where

$$\epsilon_n := (\Delta - \Delta(x_n))\,\lambda_n^\beta - \Delta(x_n)\left(\frac{f(\lambda_n x_n)}{f(x_n)} - \lambda_n^\beta\right).$$

Since (x_n) is positive and decreasing then $0 < x_{n+1} < x_n$ so $0 < \lambda_n = x_{n+1}/x_n < 1$. Thus, for all $n \geq N_1(\epsilon)$, $0 < \lambda_n < 1$. Hence for $n \geq N_1(\epsilon)$

$$\left|\frac{f(\lambda_n x_n)}{f(x_n)} - \lambda_n^\beta\right| \leq \sup_{0<\lambda\leq 1}\left|\frac{f(\lambda x_n)}{f(x_n)} - \lambda^\beta\right|.$$

Since $x_n \to 0$ as $n \to \infty$, by the uniform convergence theorem for $RV_0(\beta)$ functions, the fact that $\Delta(x_n) \to \Delta$ as $n \to \infty$ and that $\lambda_n \in (0,1)\,\forall\, n \geq N_1(\epsilon)$, we have $\epsilon_n \to 0$ as $n \to \infty$. Therefore, $\forall\, \epsilon \in (0, \min(\Delta, 1))$ there is $N_2(\Delta, \epsilon) > 0$ such that $n \geq N_2(\Delta, \epsilon)$ implies $|\epsilon_n| < \epsilon$. Hence for $n \geq \max(N_1(\epsilon), N_2(\Delta, \epsilon))$

$$1 - \Delta\lambda_n^\beta - \epsilon < \lambda_n < 1 - \Delta\lambda_n^\beta + \epsilon.$$

Define $\phi_+(\lambda) := \lambda + \Delta\lambda^\beta - (1-\epsilon)$. Then $\phi_+(0) = -(1-\epsilon) < 0$ and $\phi_+(1) = 1 - (1-\epsilon) + \Delta = \epsilon + \Delta > 0$. Also $\phi_+'(\lambda) = 1 + \Delta\beta\lambda^{\beta-1} > 1 > 0$. Thus by the Intermediate Value Theorem ϕ_+ has a unique zero $\lambda_+(\epsilon) \in (0,1)$. Notice that

$$\phi_+(\lambda_n) \;=\; \lambda_n + \Delta\lambda_n^\beta - (1-\epsilon) = 1 - \Delta\lambda_n^\beta + \epsilon_n + \Delta\lambda_n^\beta - (1-\epsilon) = \epsilon_n + \epsilon > 0,$$

and so $\lambda_n > \lambda_+(\epsilon)$. Similarly defining $\phi_-(\lambda) := \lambda - (1+\epsilon) + \Delta\lambda^\beta$. Then $\phi_-(0) = -(1+\epsilon) < 0$, $\phi_-(1) = \Delta - \epsilon > 0$ with $\phi_-'(\lambda) = 1 + \beta\Delta\lambda^{\beta-1} > 1 > 0$. Thus by the Intermediate Value Theorem ϕ_- has a unique zero $\lambda_-(\epsilon) \in (0,1)$. Notice that

$$\phi_-(\lambda_n) \;=\; \lambda_n + \Delta\lambda_n^\beta - (1+\epsilon) = 1 - \Delta\lambda_n^\beta + \epsilon_n + \Delta\lambda_n^\beta - (1+\epsilon) = \epsilon_n - \epsilon < 0,$$

and so $\lambda_n < \lambda_-(\epsilon)$. Hence for $n \geq \max(N_1(\epsilon), N_2(\Delta, \epsilon))$

$$\lambda_+(\epsilon) < \lambda_n < \lambda_-(\epsilon).$$

Clearly, $\lambda_\pm(\epsilon) \to \lambda_* \in (0,1)$ as $\epsilon \to 0^+$ where $\lambda_* = \lambda_*(\Delta)$ is the unique zero in $(0,1)$ of $\phi(\lambda) = 0$ where $\phi(\lambda) := \lambda + \Delta\lambda^\beta - 1$. Hence by The Squeeze Theorem $\lambda_n \to \lambda_*$ as $n \to \infty$. Therefore

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda_*(\Delta).$$

Since $\lambda_*(\Delta) + \Delta\lambda_*^\beta(\Delta) - 1 = 0$, we see that $\lambda_*(\Delta) \to 1$ as $\Delta \to 0^+$. Moreover, $1 - \lambda_*(\Delta) \sim \Delta$ as $\Delta \to 0^+$. □

Theorem 40. *Suppose f obeys (3.1), (3.32), (7.3) and $f \in RV_0(\beta)$, $\beta \in (0,1]$ while h obeys (3.2) and (7.2) where $\Delta(x) \to \Delta > 0$ as $x \to 0^+$. Let F, $\bar{F}$, (t_n) and $\hat{T}_h$ be defined by (1.11), (1.10), (1.42), (3.16) and $\lambda_*(\Delta)$ be given by*

$$\lambda_*(\Delta) = 1 - \Delta\lambda_*^\beta(\Delta).$$

(i) If f obeys (1.7), then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \frac{1}{\Delta}\int_{\lambda_*(\Delta)}^1 \lambda^{-\beta}\, d\lambda.$$

(ii) If f obeys (1.9), then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = \frac{1}{\Delta} \int_{\lambda_*(\Delta)}^{1} \lambda^{-1}\, d\lambda = \frac{\log(1+\Delta)}{\Delta}.$$

Proof. Note

$$\begin{aligned}
\bar{F}(x_n) = \sum_{j=n}^{\infty} \int_{x_{j+1}}^{x_j} \frac{1}{f(u)}\, du \quad &= \sum_{j=n}^{\infty} \int_{x_{j+1}/x_j}^{1} \frac{x_j}{f(x_j v)}\, dv \\
&= \sum_{j=n}^{\infty} \frac{\Delta(x_j) x_j}{f(x_j)} \cdot \frac{1}{\Delta(x_j)} \int_{x_{j+1}/x_j}^{1} \frac{f(x_j)}{f(v x_j)}\, dv \\
&= \sum_{j=n}^{\infty} h(x_j) \cdot \frac{1}{\Delta(x_j)} \int_{x_{j+1}/x_j}^{1} \frac{\tilde{f}(v x_j)}{\tilde{f}(x_j)}\, dv,
\end{aligned}$$

where $\tilde{f} \in RV_0(-\beta)$. Note that

$$\lim_{j\to\infty} \frac{1}{\Delta(x_j)} \int_{x_{j+1}/x_j}^{1} \frac{\tilde{f}(v x_j)}{\tilde{f}(x_j)}\, dv = \frac{1}{\Delta} \int_{\lambda_*(\Delta)}^{1} v^{-\beta}\, dv,$$

by the uniform convergence theorem for regularly varying functions and Lemma 26. If f obeys (1.7), then $\int_{0^+}^{1} 1/f(u)\, du < \infty$ hence $(\int_{x_{j+1}}^{x_j} 1/f(u)\, du)$ is a convergent series. Hence by Toeplitz's Lemma

$$\begin{aligned}
\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} &= \lim_{n\to\infty} \frac{\sum_{j=n}^{\infty} h(x_j) \cdot 1/\Delta(x_j) \int_{x_{j+1}/x_j}^{1} \tilde{f}(v x_j)/\tilde{f}(x_j)\, dv}{\sum_{j=n}^{\infty} h(x_j)} \\
&= \lim_{j\to\infty} \frac{1}{\Delta(x_j)} \int_{x_{j+1}/x_j}^{1} \frac{\tilde{f}(v x_j)}{\tilde{f}(x_j)}\, dv = \frac{1}{\Delta} \int_{\lambda_*(\Delta)}^{1} v^{-\beta}\, dv.
\end{aligned}$$

If f obeys (1.9), then $\int_{0^+}^{1} 1/f(u)\, du = \infty$ and we must have $\beta = 1$ and so $(\int_{x_{j+1}}^{x_j} 1/f(u)\, du)$ is a divergent series. Hence by Toeplitz's Lemma

$$\begin{aligned}
\lim_{n\to\infty} \frac{F(x_n)}{t_n} &= \lim_{n\to\infty} \frac{F(x_0) + \sum_{j=0}^{n-1} h(x_j) \cdot 1/\Delta(x_j) \int_{x_{j+1}/x_j}^{1} \tilde{f}(v x_j)/\tilde{f}(x_j)\, dv}{\sum_{j=0}^{n-1} h(x_j)} \\
&= \lim_{j\to\infty} \frac{1}{\Delta(x_j)} \int_{x_{j+1}/x_j}^{1} \frac{\tilde{f}(v x_j)}{\tilde{f}(x_j)}\, dv \\
&= \frac{1}{\Delta} \int_{(1+\Delta)^{-1}}^{1} v^{-1}\, dv = \frac{\log(1+\Delta)}{\Delta},
\end{aligned}$$

as claimed. □

Remark 30. With $\hat{T}_h = \sum_{j=0}^{\infty} h(x_j)$, we have

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \frac{1}{\Delta}\int_{\lambda_*(\Delta)}^{1} \lambda^{-\beta}\, d\lambda =: \lambda_{EI}(\Delta).$$

By the Mean Value Theorem for integrals

$$\lambda_{EI}(\Delta) = \frac{1-\lambda_*(\Delta)}{\Delta}\xi^*(\Delta),$$

where $\xi^*(\Delta) \in (1, \lambda_*^{-\beta}(\Delta))$. Since $1 - \lambda_*(\Delta) \sim \Delta$ as $\Delta \to 0^+$ and $\xi^*(\Delta) \to 1$ as $\Delta \to 0^+$ then we have that $\lambda_{EI}(\Delta) \to 1$ as $\Delta \to 0^+$. □

We now tackle the case when f is Slowly Varying. In most instances in this thesis the generalised Liapunov exponent changes as Δ changes. However, Theorems 41 and 42 show the exponent is unity for $\Delta \in (0,1)$ while the exponent is given by the non-constant function $\Delta \mapsto 1/\Delta$ for $\Delta > 1$.

Theorem 41. *Suppose f obeys* (3.1), (3.32), (7.3) *and $f \in RV_0(0)$ while h obeys* (3.2) *and* (7.2) *where $\Delta(x) \to \Delta \in (0,1)$ as $x \to 0^+$. Let $\bar{F}$, (t_n) and $\hat{T}_h$ be defined by* (1.10), (1.42), (3.16) *and $\lambda_*(\Delta)$ be given by*

$$\lambda_*(\Delta) = 1 - \Delta.$$

If f obeys (1.7), *then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and*

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = 1.$$

Proof. Since f is increasing then $f(x_{n+1}) < f(x_n)$ and

$$\Delta(x_n)\frac{f(x_{n+1})}{f(x_n)} < \Delta(x_n),$$

and

$$\lambda_n = 1 - \Delta(x_n)\frac{f(x_{n+1})}{f(x_n)} > 1 - \Delta(x_n).$$

Thus $\lambda_n > 1 - \Delta(x_n)$. Hence $\liminf_{n\to\infty} \lambda_n \geq 1 - \Delta > 0$ for $\Delta \in (0,1)$. By using the calculation of Lemma 26 we have $\lambda_n = 1 - \Delta + \epsilon_n$ and $\epsilon_n \to 0$ as $n \to \infty$ by the uniform convergence theorem for $RV_0(0)$ functions. Hence

$$\lim_{n\to\infty} \lambda_n = 1 - \Delta =: \lambda_*(\Delta).$$

Applying a similar argument as in Theorem 40 part (i) then

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \frac{1}{\Delta}\int_{1-\Delta}^{1} \lambda^{-0}\, d\lambda = 1,$$

as required. □

Theorem 42. *Suppose f obeys (3.1), (3.32), (7.3) and $f \in RV_0(0)$ while h obeys (3.2) and (7.2) where $\Delta(x) \to \Delta \geq 1$ as $x \to 0^+$. Let $\bar{F}$, (t_n) and $\hat{T}_h$ be defined by (1.10), (1.42), (3.16) and $\lambda_*(\Delta)$ be given by*

$$\lambda_*(\Delta) = 1 - \Delta.$$

If f obeys (1.7), then $x_n > 0$ for all $n \geq 0$, (x_n) is decreasing, $x_n \to 0$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \frac{1}{\Delta}.$$

Proof. Recall the definition of the scheme as

$$x_{n+1} = x_n - \frac{x_n \Delta(x_n)}{f(x_n)} f(x_{n+1}).$$

Suppose $\limsup_{n\to\infty} x_{n+1}/x_n =: \underline{\lambda} > 0$ then for every $\epsilon \in (0, \underline{\lambda})$ there is a sequence $n_j \nearrow \infty$ such that $x_{n_j+1}/x_{n_j} \geq \underline{\lambda} - \epsilon$ and

$$x_{n_j+1} = x_{n_j} - x_{n_j}\Delta(x_{n_j})\frac{f(x_{n_j+1})}{f(x_{n_j})}.$$

Now as $\Delta(x_n) \to \Delta$ as $n \to \infty$, we can go far enough in the sequence n_j such that

$$(1+\epsilon)\cdot\Delta > \Delta(x_{n_j}) > (1-\epsilon)\cdot\Delta, \quad \forall\, j > J_1(\epsilon).$$

Thus

$$\Delta(x_{n_j})\frac{f(x_{n_j+1})}{f(x_{n_j})} \;>\; \Delta(x_{n_j})\frac{f((\underline{\lambda}-\epsilon)x_{n_j})}{f(x_{n_j})} > (1-\epsilon)\cdot\Delta\frac{f((\underline{\lambda}-\epsilon)x_{n_j})}{f(x_{n_j})}.$$

Therefore for $j > J_1(\epsilon)$

$$\underline{\lambda} - \epsilon \leq \frac{x_{n_j+1}}{x_{n_j}} \;=\; 1 - \Delta(x_{n_j})\frac{f(x_{n_j+1})}{f(x_{n_j})} < 1 - (1-\epsilon)\cdot\Delta\frac{f((\underline{\lambda}-\epsilon)x_{n_j})}{f(x_{n_j})}.$$

Hence

$$\underline{\lambda} - \epsilon < 1 - (1-\epsilon)\cdot\Delta\frac{f((\underline{\lambda}-\epsilon)x_{n_j})}{f(x_{n_j})}.$$

Since $f \in RV_0(0)$ we may let $n_j \to \infty$ to get $\underline{\lambda} - \epsilon \leq 1 - (1-\epsilon)\cdot\Delta$. Letting $\epsilon \to 0^+$, then by supposition $0 < \underline{\lambda} \leq 1 - \Delta \leq 0$, a contradiction. Hence we cannot have $\underline{\lambda} > 0$. Thus $\underline{\lambda} = 0$. Therefore $x_{n+1}/x_n \to 0$ as $n \to \infty$ if $\Delta \geq 1$. For every $\epsilon \in (0,1)$ there is a $J_2(\epsilon) > 0$ such that for $j > J_2(\epsilon)$, $x_{j+1}/x_j < \epsilon$. Let $j > J_2(\epsilon)$ and define

$\tilde{f} := 1/f \in RV_0(0)$. Then $\tilde{f}(x) \to \infty$ as $x \to 0^+$. Define for $j > J_2(\epsilon)$

$$a_3(j) := \int_{x_{j+1}/x_j}^{1} \frac{\tilde{f}(vx_j)}{\tilde{f}(x_j)}\, dv \quad = \quad \int_{\epsilon}^{1} \frac{\tilde{f}(vx_j)}{\tilde{f}(x_j)}\, dv + \int_{x_{j+1}/x_j}^{\epsilon} \frac{f(x_j)}{f(vx_j)}\, dv =: a_1(j) + a_2(j).$$

Next $\lim_{j\to\infty} \tilde{f}(x_j v)/\tilde{f}(x_j) = 1$ uniformly for all $v \in [\epsilon, 1]$ by the uniform convergence theorem for $RV_0(0)$ functions then $a_1(j) \to 1 - \epsilon$ as $j \to \infty$ and letting $\epsilon \to 0^+$ we arrive at $\liminf_{j\to\infty} a_3(j) \geq 1$. If $x_{j+1}/x_j < v < \epsilon$, then $x_{j+1} < vx_j < \epsilon x_j$, so $f(x_{j+1}) < f(vx_j) < f(\epsilon x_j)$ and

$$\frac{f(x_j)}{f(\epsilon x_j)} < \frac{f(x_j)}{f(vx_j)} < \frac{f(x_j)}{f(x_{j+1})}.$$

Thus

$$a_2(j) \quad \leq \quad \int_{x_{j+1}/x_j}^{\epsilon} \frac{f(x_j)}{f(x_{j+1})}\, dv = \left(\epsilon - \frac{x_{j+1}}{x_j}\right) \frac{f(x_j)}{f(x_{j+1})} \leq \epsilon \cdot \frac{f(x_j)}{f(x_{j+1})}.$$

Since $x_{n+1}/x_n \to 0$ as $n \to \infty$, then

$$0 = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1 - \Delta \lim_{n\to\infty} \frac{f(x_{n+1})}{f(x_n)}.$$

Thus

$$\lim_{n\to\infty} \frac{f(x_{n+1})}{f(x_n)} = \frac{1}{\Delta} \leq 1.$$

Since $a_1(j) \to 1 - \epsilon$ as $j \to \infty$, $\limsup_{j\to\infty} a_2(j) \leq \epsilon\Delta$ then

$$\limsup_{j\to\infty} a_3(j) = \limsup_{j\to\infty} (a_1(j) + a_2(j)) \leq 1 - \epsilon + \epsilon\Delta.$$

Letting $\epsilon \to 0^+$ we get $\lim_{j\to\infty} a_3(j) = 1$. Therefore by Toeplitz's Lemma as $n \to \infty$

$$\bar{F}(x_n) = \sum_{j=n}^{\infty} h(x_j) \cdot \frac{1}{\Delta(x_j)} \int_{x_{j+1}/x_j}^{1} \frac{\tilde{f}(vx_j)}{\tilde{f}(x_j)}\, dv \quad = \quad \sum_{j=n}^{\infty} h(x_j) \cdot \frac{a_3(j)}{\Delta(x_j)} \sim \frac{\hat{T}_h - t_n}{\Delta}.$$

Hence

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \frac{1}{\Delta},$$

when $\Delta \geq 1$, as claimed. □

From the previous result the exponent $1/\Delta \to 0$ as $\Delta \to \infty$. This suggests that if $\Delta(x) \to \infty$ as $x \to 0^+$ we may not recover all aspects of the asymptotic behaviour. The next theorem shows that for any β we either fail to recover the finite hitting time of mispecify the asymptotic behaviour at the finite hitting time.

Theorem 43. *Suppose f obeys* (3.1), (3.32), (7.3) *and $f \in RV_0(0)$ while h obeys* (3.2) *and* (7.2) *where* $\Delta(x) \to \Delta \to \infty$ *as* $x \to 0^+$*. Let* $\bar{F}$*,* (t_n) *and* $\hat{T}_h$ *be defined by* (1.10)*,* (1.42)*,* (3.16)*. If* f *obeys* (1.7)*, then either:* $t_n \to \infty$ *as* $n \to \infty$*; or* $\bar{F}(x_n) = o(\hat{T}_h - t_n)$ *as* $n \to \infty$*.*

Proof. Suppose $\limsup_{n\to\infty} x_{n+1}/x_n =: \underline{\lambda} > 0$. Since $\Delta(x) \to \infty$ as $x \to 0^+$ then $\Delta(x_n) > 1/\epsilon \,\forall\, n \geq N_1(\epsilon)$. Also there is a sequence $n_j \nearrow \infty$ such that $x_{n_j+1}/x_{n_j} > \underline{\lambda} - \epsilon \,\forall\, j \geq J_1(\epsilon)$. Define $N_2(\epsilon) := n_{J_1(\epsilon)}$ and $N_3(\epsilon) := \max(N_1(\epsilon), N_2(\epsilon))$. Then for $n_j > N_3(\epsilon)$

$$\underline{\lambda} - \epsilon < \frac{x_{n_j+1}}{x_{n_j}} = 1 - \Delta(x_{n_j})\frac{f(x_{n_j+1})}{f(x_{n_j})}.$$

Also

$$\Delta(x_{n_j})\frac{f(x_{n_j+1})}{f(x_{n_j})} > \frac{f((\underline{\lambda}-\epsilon)x_{n_j})}{f(x_{n_j})} \cdot \frac{1}{\epsilon}.$$

Hence for $n_j > N_3(\epsilon)$

$$\underline{\lambda} - \epsilon < 1 - \frac{1}{\epsilon} \cdot \frac{f((\underline{\lambda}-\epsilon)x_{n_j})}{f(x_{n_j})}.$$

Letting $j \to \infty$ yields

$$\underline{\lambda} - \epsilon \leq 1 - \frac{(\underline{\lambda}-\epsilon)^\beta}{\epsilon}.$$

If $\beta = 0$, rearranging yields

$$\underline{\lambda} \leq \epsilon + 1 - \frac{1}{\epsilon}.$$

Letting $\epsilon \to 0^+$, $\underline{\lambda} \leq -\infty$, or $\underline{\lambda} = -\infty$ if $\beta = 0$, a contradiction. If $\beta \in (0,1]$ then again $\underline{\lambda} \leq -\infty$ by taking $\epsilon \to 0^+$, a contradiction. Hence $\underline{\lambda} = 0$ in this case also. In the case $t_n \to \hat{T}_h < \infty$ as $n \to \infty$, then

$$\bar{F}(x_n) = \sum_{j=n}^{\infty} h(x_j) \cdot \frac{1}{\Delta(x_j)} \int_{x_{j+1}/x_j}^{1} \frac{\tilde{f}(vx_j)}{\tilde{f}(x_j)}\, dv,$$

where $\tilde{f} \in RV_0(-\beta)$. We show that the integral term is null. Define

$$a_3(j) := \int_{x_{j+1}/x_j}^{1} \frac{\tilde{f}(vx_j)}{\tilde{f}(x_j)}\, dv.$$

Since $x_{j+1}/x_j \to 0$ as $j \to \infty$, there is $J_2(\epsilon) \in \mathbb{N}$ such that $x_{j+1}/x_j < \epsilon \;\forall\, j \geq J_2(\epsilon)$. Take $j \geq J_2(\epsilon)$. Then

$$a_3(j) \;=\; \int_{\epsilon}^{1} \frac{\tilde{f}(x_j v)}{\tilde{f}(x_j)}\, dv + \int_{x_{j+1}/x_j}^{\epsilon} \frac{\tilde{f}(x_j v)}{\tilde{f}(x_j)}\, dv =: a_1(j) + a_2(j).$$

Then $a_1(j) \to \int_\epsilon^1 v^{-\beta}\, dv$ as $j \to \infty$ by the uniform convergence theorem for $RV_0(\beta)$ functions. Thus

$$\liminf_{j\to\infty} a_3(j) \geq \int_\epsilon^1 v^{-\beta}\, dv.$$

If $x_{j+1}/x_j < v < \epsilon$, then $x_{j+1} < vx_j < \epsilon x_j$ so $f(x_{j+1}) < f(vx_j) < f(\epsilon x_j)$ and

$$\frac{f(x_j)}{f(\epsilon x_j)} < \frac{f(x_j)}{f(vx_j)} < \frac{f(x_j)}{f(x_{j+1})}.$$

Integrating implies

$$a_2(j) \quad \leq \quad \int_{x_{j+1}/x_j}^{\epsilon} \frac{f(x_j)}{f(x_{j+1})}\, dv \leq \epsilon \cdot \frac{f(x_j)}{f(x_{j+1})}.$$

Thus

$$\frac{1}{\Delta(x_j)} \int_{x_{j+1}/x_j}^{1} \frac{\tilde{f}(x_j v)}{\tilde{f}(x_j)}\, dv \leq \frac{a_1(j)}{\Delta(x_j)} + \epsilon \cdot \frac{f(x_j)}{\Delta(x_j) f(x_{j+1})}.$$

Since $a_1(j) \to \int_\epsilon^1 v^{-\beta}\, dv$ as $j \to \infty$ and $\Delta(x_j) \to \infty$ as $j \to \infty$, the first term on the right-hand side tends to zero as $j \to \infty$. Since $x_{n+1}/x_n \to 0$ as $n \to \infty$ then

$$0 = \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1 - \lim_{n\to\infty} \frac{\Delta(x_n) f(x_{n+1})}{f(x_n)},$$

or

$$\lim_{n\to\infty} \frac{\Delta(x_n) f(x_{n+1})}{f(x_n)} = 1.$$

Hence

$$\limsup_{j\to\infty} \frac{1}{\Delta(x_j)} \int_{x_{j+1}/x_j}^{1} \frac{\tilde{f}(x_j v)}{\tilde{f}(x_j)}\, dv \leq \epsilon.$$

Letting $\epsilon \to 0^+$ yields

$$\limsup_{j\to\infty} \frac{1}{\Delta(x_j)} \int_{x_{j+1}/x_j}^{1} \frac{\tilde{f}(x_j v)}{\tilde{f}(x_j)}\, dv = 0.$$

Therefore

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = 0,$$

in the case that $t_n \to \hat{T}_h$ as $n \to \infty$; otherwise $t_n \to \infty$ as $n \to \infty$. This exhausts the two claimed properties. □

7.3 Multi-Step Numerical Schemes

In this section we investigate the qualitative properties of multi-step schemes. Our examination is confined to two-step schemes for the sake of brevity.

7.3.1 Two-Step Implicit Euler Scheme

In this section we investigate the performance of a two-period Implicit Euler scheme with an adaptive mesh. We approximate $x(t_n)$ by x_n, where $x(t_n)$ is the solution x of

(1.1) at time t_n. The sequences (x_n), (t_n) and $(h(x_n))$ are defined by $x_0 = \xi > 0$ and

$$x_{n+1} = x_n - \{\alpha h(x_n) f(x_n) + (1-\alpha) h(x_{n+1}) f(x_{n+1})\}, \tag{7.11}$$

where $\alpha \in \mathbb{R}$

$$t_{n+1} = \sum_{j=0}^{n} h(x_{j+1}), \quad n \geq 0, \quad t_0 = 0,$$

and

$$h(x) = \frac{\Delta x}{f(x)}, \quad \text{for some } \Delta > 0. \tag{7.12}$$

Substituting this choice of $h(x)$ into (7.11) implies

$$x_{n+1} = (1 - \alpha\Delta)x_n - (1-\alpha)\Delta x_{n+1}.$$

Concentrating momentarily on explicit schemes, we have already seen that the convergence to a non-trivial limit of

$$\frac{1}{h(x_n)} \int_{x_n}^{x_{n+1}} \frac{1}{f(u)}\, du$$

guarantees the convergence of

$$\frac{\bar{F}(x_n)}{\hat{T}_h - t_n}$$

to the same non-trivial limit. However, note that

$$\frac{1}{h(x_n)} \int_{x_n}^{x_{n+1}} \frac{1}{f(u)}\, du \sim \frac{1}{\Delta} \int_{x_{n+1}/x_n}^{1} \frac{\tilde{f}(\lambda x_n)}{\tilde{f}(x_n)}\, d\lambda, \quad \text{as } n \to \infty,$$

where $\tilde{f} = 1/f$ and $h(x) \sim \Delta x / f(x)$ as $x \to 0^+$. Therefore due to the uniform convergence theorem for regularly varying functions if $x_{n+1}/x_n \to \mu \in (0,1)$ as $n \to \infty$, then

$$\lim_{n\to\infty} \frac{1}{h(x_n)} \int_{x_n}^{x_{n+1}} \frac{1}{f(u)}\, du = \frac{1}{\Delta} \int_{\mu}^{1} \lambda^{-\beta}\, d\lambda.$$

In other words a key ingredient in the success of this approach is the existence of an asymptotic common ratio in $(0,1)$ of the sequence (x_n).

Theorem 44. *Suppose (x_n) is the solution of (7.11) with h given by (7.12). Then*

$$x_{n+1} = \left(\frac{1-\alpha\Delta}{1-\alpha\Delta+\Delta}\right) x_n =: \lambda(\Delta) x_n,$$

and the following case distinction applies:

(i) If $\Delta = 1/\alpha$, then $x_n = 0 \,\forall\, n \geq 1$.

(ii) If $\Delta < 1/\alpha$, then $\lambda(\Delta) \in (0,1)$ and (x_n) is positive for all $n \geq 0$ and decreasing

with $x_n \to 0$ as $n \to \infty$.

(iii) If $\Delta > 1/\alpha$, then either:

(a) (x_n) is not defined for all $n \geq 1$.

(b) $\lambda(\Delta) < 0$ and (x_n) oscillates in sign.

(c) $\lambda(\Delta) > 1$ and (x_n) is increasing with $x_n \to \infty$ as $n \to \infty$.

(d) If $\alpha < 0$, then $\lambda(\Delta) \in (0, 1)$ is decreasing with $x_n \to 0$ as $n \to \infty$.

Proof. If $\Delta = 1/\alpha$, then $1 - \alpha\Delta = 0$, and thus $\lambda(\Delta) = 0$ with $x_n = 0 \, \forall \, n \geq 1$.

If $\Delta < 1/\alpha$ then $1 - \alpha\Delta > 0$ and when $\Delta > 0$ then $0 < 1 - \alpha\Delta < 1 - \alpha\Delta + \Delta$. Hence $\lambda(\Delta) \in (0, 1)$ and (x_n) is positive $\forall \, n \geq 0$, decreasing and $x_n \to 0$ as $n \to \infty$.

If $\Delta > 1/\alpha$, then $1 - \alpha\Delta < 0$. If in addition $1 - \alpha\Delta + \Delta = 0$ then $\lambda(\Delta) = \infty$ and hence $x_n = \infty \, \forall \, n \geq 1$. If $1 - \alpha\Delta + \Delta > 0$ then $\lambda(\Delta) < 0$ and (x_n) oscillates in sign. If $1 - \alpha\Delta < 0$ and $1 - \alpha\Delta + \Delta < 0$ then

$$\lambda(\Delta) := \frac{1 - \alpha\Delta}{1 - \alpha\Delta + \Delta} = \frac{\alpha\Delta - 1}{\alpha\Delta - 1 - \Delta} > 1,$$

and thus (x_n) is increasing.

In the case when $\alpha > 0$ and $\Delta < 1/\alpha$ we have that $\lambda(\Delta) \in (0, 1)$ and hence (x_n) is a positive decreasing sequence with $x_n \to 0$ as $n \to \infty$. This deals with part (ii). Similarly in part (iv), when $\alpha < 0$ and Δ is unrestricted $\lambda(\Delta) \in (0, 1)$. □

7.3.2 Two-Step Explicit Euler Scheme

In this section we investigate the performance of a two-period Explicit Euler scheme with an adaptive mesh. We approximate $x(t_n)$ by x_n, where $x(t_n)$ is the solution x of (1.1) at time t_n. The sequences (x_n), (t_n) and $(h(x_n))$ are defined by

$$x_{n+1} = x_n - \{\alpha h(x_n) f(x_n) + (1 - \alpha) h(x_{n-1}) f(x_{n-1})\}, \tag{7.13}$$

where x_0, $x_{-1} > 0$, $\alpha > 0$ and

$$t_{n+1} = \sum_{j=0}^{n} h(x_j), \quad n \geq 0, \quad t_0 = 0,$$

and

$$h(x) = \frac{\Delta x}{f(x)}, \quad \text{for some } \Delta > 0.$$

Substituting this choice of $h(x)$ into (7.13) implies

$$x_{n+1} = (1-\alpha\Delta)x_n - (1-\alpha)\Delta x_{n-1}. \tag{7.14}$$

In the following theorem, which concerns the Explicit scheme, we are interested in establishing necessary and sufficient conditions for when $x_n > 0$; $0 < x_{n+1}/x_{n+1} \to \lambda(\Delta) \in (0,1)$. The reason for this is that we know when $h(x) = \Delta x/f(x)$ and $f \in RV_0(\beta)$ that we obtain the following case distinctions: when $\int_{0+}^1 1/f(u)\,du = \infty$ then

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = \frac{1}{\Delta}\int_{\lambda(\Delta)}^1 \lambda^{-\beta}\,d\lambda,$$

and when $\int_{0+}^1 1/f(u)\,du < \infty$ then

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \frac{1}{\Delta}\int_{\lambda(\Delta)}^1 \lambda^{-\beta}\,d\lambda.$$

As we mentioned a moment ago, these remarks are also valid for the Explicit scheme studied in Theorem 44. We may use the arguments of this chapter to show that under monotonicity or regular variation hypotheses on f that global positivity, finite-time stability and asymptotic behaviour is faithfully recovered.

Theorem 45. *Suppose*

$$\lambda^2 - (1-\alpha\Delta)\lambda + (1-\alpha)\Delta = 0,$$

is the characteristic equation of (7.14) *with roots* λ_1 *and* λ_2*. Assume* x_0*,* $x_{-1} > 0$*.*

(a) Suppose $\alpha < 1$ *and define*

$$\Delta_- := \frac{2-\alpha-2\sqrt{1-\alpha}}{\alpha^2} \quad \text{and} \quad \Delta_+ := \frac{2-\alpha+2\sqrt{1-\alpha}}{\alpha^2}.$$

(i) Let $\Delta < \Delta_- < 1$*.*

(1) If $x_0 > \lambda_2 x_{-1}$*, then* λ_1*,* $\lambda_2 \in (0,1)$*,* $x_n > 0 \,\forall\, n \geq 0$ *and*

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \max(\lambda_1, \lambda_2) \in (0,1).$$

(2) If $x_0 < \lambda_2 x_{-1}$*, then* (x_n) *is ultimately negative.*

(3) If $x_0 = \lambda_2 x_{-1}$*, then* $x_n > 0 \,\forall\, n \geq 0$ *and*

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda_2.$$

(ii) Let $\Delta > \Delta_-$*. Then* $\lambda_1, \lambda_2 \in \mathbb{C}$ *and* (x_n) *is oscillatory.*

(iii) Let $\Delta > \Delta_+$. Then $\lambda_1, \lambda_2 < 0$ and (x_n) ultimately alternates in sign.

(b) Suppose $\alpha > 1$.

(i) Let $\Delta < 1/\alpha < 1$. Then $\lambda_2 < 0 < \lambda_1 < 1$, $x_n > 0 \, \forall \, n \geq 0$ and

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda_1 < 1.$$

(ii) Let $\Delta > 1/\alpha$.

(1) If $x_0 \neq \lambda_1 x_{-1}$, then

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda_2 < 0.$$

(2) If $x_0 = \lambda_1 x_{-1}$, then $x_n > 0 \, \forall \, n \geq 0$ and

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda_1 \in (0,1).$$

Proof. Consider first the case when $\alpha < 1$ and define the discriminant of the characteristic equation as

$$\delta(\Delta) \;:=\; (1-\alpha\Delta)^2 - 4(1-\alpha)\Delta = (1+\alpha\Delta)^2 - 4\Delta = \alpha^2\Delta^2 + (2\alpha-4)\Delta + 1.$$

The solutions of $\delta(\Delta) = 0$ are

$$\Delta_{-,+} = \frac{-(2\alpha-4) \pm \sqrt{(2\alpha-4)^2 - 4\alpha^2}}{2\alpha^2} = \frac{2-\alpha \pm 2\sqrt{1-\alpha}}{\alpha^2}.$$

We prove part (i) first. If $\alpha < 1$, then $\Delta_-, \Delta_+ \in \mathbb{R}$. Note that $\delta(0) = 1$, $\delta(\Delta) > 0$ when $\Delta < \Delta_-$, $\delta(\Delta) < 0$ when $\Delta \in (\Delta_-, \Delta_+)$ and $\delta(\Delta) > 0$ when $\Delta > \Delta_+$. Furthermore,

$$\delta(1) = \alpha^2 + 2\alpha - 3 = (\alpha-1)(\alpha+3).$$

Thus $\delta(1) < 0$ when $\alpha < 1$. Therefore $0 < \Delta_- < 1 < \Delta_+$. The roots of the characteristic equation (7.14) obey $\lambda_1\lambda_2 = (1-\alpha)\Delta$ and $\lambda_1 + \lambda_2 = 1 - \alpha\Delta$.

If $\alpha < 1$ and $\Delta < \Delta_- < 1$ then $\lambda_1, \lambda_2 \in \mathbb{R}$. Furthermore $\lambda_1\lambda_2 = (1-\alpha)\Delta > 0$ and since $\alpha\Delta < 1$ then $\lambda_1 + \lambda_2 = 1 - \alpha\Delta > 0$. Hence $\lambda_1 > 0$ and $\lambda_2 > 0$ and since $\lambda_1 + \lambda_2 = 1 - \alpha\Delta < 1$ then $\lambda_1, \lambda_2 \in (0,1)$ proving part (a)(i)(1). If $\alpha < 1$ and $\Delta < \Delta_- < 1$ it can be shown that the zeros of the characteristic equation λ_1, λ_2 obey $0 < \lambda_2 < \lambda_1 < 1$. Therefore x_n can be represented by

$$x_n = c_1\lambda_1^n + c_2\lambda_2^n, \quad n \geq -1,$$

where $x(0) = x_0$, $x(-1) = x_{-1}$ are known. This leads to c_1 and c_2 being given by

$$c_1 = \frac{\lambda_1/\lambda_2 \cdot (x_0 - \lambda_2 x_{-1})}{\lambda_1/\lambda_2 - 1} \quad \text{and} \quad c_2 = \frac{\lambda_1 x_{-1} - x_0}{\lambda_1/\lambda_2 - 1}.$$

In the case (2) where $x_0 - \lambda_2 x_{-1} < 0$ then $c_1 < 0$ and so

$$\lim_{n\to\infty} \frac{x_n}{\lambda_1^n} = c_1, \tag{7.15}$$

so (x_n) is ultimately negative in this case. In case (1) where $x_0 - \lambda_2 x_{-1} > 0$ then $c_1 > 0$ and once again (7.15) holds; moreover as $c_1 \neq 0$, we have that $x_{n+1}/x_n \to \lambda_1$ as $n \to \infty$. In the case that $c_2 \geq 0$ it follows that $x_n > 0$ for all $n \geq 0$. If $c_2 < 0$, define $r := \lambda_2/\lambda_1 \in (0,1)$ and $a_n := x_n/\lambda_1^n$, $n \geq 0$. Then $a_n = c_1 + c_2 r^n$. Thus $a_0 = c_1 + c_2 = x_0 > 0$ and for $n \geq 0$

$$a_{n+1} - a_n \;=\; c_1 + c_2 r^{n+1} - c_1 - c_2 r^n = c_2 r^n (r-1) > 0.$$

Hence (a_n) is an increasing sequence as $a_0 > 0$ and $a_n > 0 \,\forall\, n \geq 0$. Hence in this case we again have $x_n > 0$, $n \geq 0$.

In case (3) where, $x_0 - \lambda_2 x_{-1} = 0$, then $c_1 = 0$ and $x_n = c_2 \lambda_2^n$. But $\lambda_1 x_{-1} - x_0 = \lambda_1 x_{-1} - \lambda_2 x_{-1} = (\lambda_1 - \lambda_2) x_{-1} > 0$, so $c_2 > 0$. Therefore $x_n > 0 \,\forall\, n \geq 0$ and $x_{n+1}/x_n \to \lambda_2$ as $n \to \infty$.

To prove part (ii), let $\alpha < 1$ and $\Delta_- < \Delta < \Delta_+$. Then $\lambda_1, \lambda_2 \in \mathbb{C}$ and this generates oscillatory solutions for (x_n).

We now prove part (iii). Let $\alpha < 1$ and $\Delta > \Delta_+$. Then $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \lambda_2 = (1-\alpha)\Delta > 0$ and $\lambda_1 + \lambda_2 = 1 - \alpha\Delta$. Hence either $\lambda_1 > 0$, $\lambda_2 > 0$ or $\lambda_1 < 0$, $\lambda_2 < 0$. Note that for $\alpha < 1$, $\sqrt{1-\alpha} > 0 > \alpha - 1$. Therefore $2 - \alpha + 2\sqrt{1-\alpha} > \alpha$ so $\Delta_+(\alpha) > 1/\alpha$. Thus $\Delta > \Delta_+ > 1/\alpha$, so $\alpha\Delta > 1$. Therefore $\lambda_1 + \lambda_2 < 0$ and so $\lambda_1 < 0$, $\lambda_2 < 0$.

If $\alpha > 1$ and $\Delta < 1/\alpha < 1$, then $\lambda_1\lambda_2 = (1-\alpha)\Delta < 0$ and so $\lambda_2 < 0 < \lambda_1$ and $\lambda_1 + \lambda_2 = 1 - \alpha\Delta \in (0,1)$. Hence $|\lambda_1| = \lambda_1 > |\lambda_2|$ and thus $|\lambda_1| = \max_{i=1,2} |\lambda_i|$. Moreover as

$$\lambda_1 = \frac{1 - \alpha\Delta + \sqrt{(1+\alpha\Delta)^2 - 4\Delta}}{2},$$

it can be checked by hand that $\lambda_1 < 1$ for all $\Delta > 0$. Since $x_{-1} > 0$, $x_0 > 0$, $\alpha > 1$, $1 - \alpha\Delta > 0$ and (x_n) obeys

$$x_{n+1} = (1 - \alpha\Delta) x_n + \Delta(\alpha - 1) x_{n-1}, \quad n \geq 0,$$

it follows that $x_n > 0 \, \forall \, n \geq 0$. Also as $\lambda_1 \neq \lambda_2$ there exists $c_1, c_2 \in \mathbb{R}$ such that

$$x_n = c_1 \lambda_1^n + c_2 \lambda_2^n,$$

and moreover

$$c_1 = \frac{\lambda_1/\lambda_2 \cdot (x_0 - \lambda_2 x_{-1})}{\lambda_1/\lambda_2 - 1} \quad \text{and} \quad c_2 = \frac{\lambda_1 x_{-1} - x_0}{\lambda_1/\lambda_2 - 1}.$$

Since $x_n > 0 \, \forall \, n \geq 0$ it cannot be the case that $c_1 < 0$, for this would imply that (x_n) is ultimately negative. Therefore $c_1 \geq 0$. On the other hand suppose $c_1 = 0$. Then $x_n = c_2 \lambda_2^n \, \forall \, n \geq 0$. Suppose now $c_1 \neq 0$. Since $\lambda_2 < 0$, (x_n) alternates in sign, contradicting the positivity of (x_n). Therefore we cannot have $c_1 = 0$, $c_2 \neq 0$. Next suppose $c_2 = 0$; this implies $x_n \equiv 0 \, \forall \, n \geq 0$, again a contradiction. Therefore it is impossible for $c_1 = 0$. Therefore $c_1 > 0$ is impossible. Therefore it follows from this that $x_n/\lambda_1^n \to c_1$ as $n \to \infty$ and so

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \left(\frac{x_{n+1}}{\lambda_1^{n+1}} \cdot \frac{\lambda_1^n}{x_n} \cdot \lambda_1 \right) = \lambda_1 \in (0,1).$$

When $\alpha > 1$ and $\Delta > 1/\alpha$, $\lambda_1 \lambda_2 < 0$ and $\lambda_1 + \lambda_2 < 0$. Thus $|\lambda_2| > |\lambda_1|$, $\lambda_2 < 0$ and $\lambda_1 > 0$. Once again we have that

$$x_n = c_1 \lambda_1^n + c_2 \lambda_2^n.$$

If $c_2 \neq 0$, then $x_n/\lambda_2^n \to c_2$ as $n \to \infty$ and as $\lambda_2 < 0$, (x_n) ultimately alternates in sign. If $c_2 = 0$, we have $x_n = c_1 \lambda_1^n$; $c_2 = 0$ occurs only when $\lambda_1 x_{-1} = x_0$. This forces

$$c_1 = \frac{\lambda_1/\lambda_2 \cdot (x_0 - \lambda_2 x_{-1})}{\lambda_1/\lambda_2 - 1} = \frac{\lambda_1 x_{-1}(\lambda_1 - \lambda_2)}{\lambda_1 - \lambda_2} = \lambda_1 x_{-1}.$$

Hence $x_n = x_{-1} \lambda_1^{n+1}$, $n \geq -1$ and we have $x_n > 0$ and $x_{n+1}/x_n \to \lambda_1$ as $n \to \infty$. It can again be checked directly that $\lambda_1 \in (0,1)$. □

Remark 31. The case $\alpha = 1$ is not considered because this is the one-step Explicit method. □

The critical values of Δ for which the two-step Explicit scheme produces acceptable qualitative behaviour are summarised below in the case when $\alpha > 0$:

$$\Delta < \Delta(\alpha) = \begin{cases} \frac{2-\alpha-2\sqrt{1-\alpha}}{\alpha^2}, & \alpha < 1, \\ 1, & \alpha = 1, \\ 1/\alpha, & \alpha > 1. \end{cases}$$

Lemma 27. *Let $\alpha \in (0,1)$ and Δ_- be the smallest root of*

$$\delta(\Delta) := \alpha^2\Delta^2 + 2(\alpha-2)\Delta + 1,$$

where

$$\Delta_-(\alpha) = \frac{2-\alpha-2\sqrt{1-\alpha}}{\alpha^2}.$$

Then $\alpha \mapsto \Delta_-(\alpha)$ is increasing and

$$\lim_{\alpha\to 0} \Delta_-(\alpha) = \frac{1}{4}.$$

Proof. Define $k(\alpha) := \sqrt{1-\alpha} = (1-\alpha)^{1/2}$. Then the Taylor Series of $k(\alpha)$ about zero to three terms is

$$k(\alpha) \;=\; k(0) + k'(0)\alpha + \frac{k''(0)\alpha^2}{2} + \frac{k'''(\xi_\alpha)\alpha^3}{6} = 1 - \frac{\alpha}{2} - \frac{\alpha^2}{8} + \frac{k'''(\xi_\alpha)\alpha^3}{6},$$

for some $\xi_\alpha \in (0,\alpha)$. Hence

$$2-\alpha-2\sqrt{1-\alpha} \;=\; 2-\alpha-2\left(1-\frac{\alpha}{2}-\frac{\alpha^2}{8}+\frac{k'''(\xi_\alpha)\alpha^3}{6}\right) = \frac{\alpha^2}{4} - \frac{k'''(\xi_\alpha)\alpha^3}{3}.$$

Hence

$$\lim_{\alpha\to 0^+} \frac{2-\alpha-2\sqrt{1-\alpha}}{\alpha^2} = \frac{1}{4}.$$

Next we have that

$$\Delta_-'(\alpha) \;=\; \frac{\alpha + \alpha(1-\alpha)^{-1/2} + 4(1-\alpha)^{1/2} - 4}{\alpha^3} =: \frac{\tilde{\Delta}(\alpha)}{\alpha^3}.$$

Thus $\Delta_-'(\alpha) > 0$ if $\tilde{\Delta}(\alpha) > 0$ where

$$\tilde{\Delta}(\alpha) := \alpha + \frac{\alpha}{\sqrt{1-\alpha}} + 4\sqrt{1-\alpha} - 4, \quad \forall\, \alpha \in (0,1).$$

To prove $\tilde{\Delta}(\alpha) > 0$, let $\gamma = \sqrt{1-\alpha}$. Then $\gamma^2 = 1-\alpha$ and $\alpha = 1-\gamma^2$. Also $\alpha \in (0,1)$ implies $\gamma \in (0,1)$. Hence $\tilde{\Delta}(\alpha) > 0$ is equivalent to

$$1-\gamma^2 + \frac{1-\gamma^2}{\gamma} + 4\gamma - 4 > 0, \quad \forall\, \gamma \in (0,1),$$

which in turn is equivalent to the valid inequality $(1-\gamma)(\gamma-1)^2 > 0$, $\forall\, \gamma \in (0,1)$. □

Remark 32. By a similar proof to the lemma above, it can be shown that $\Delta_-(\alpha)$ is increasing over $(-\infty, 0)$. □

Remark 33. The Adams-Bashforth method, with an adaptive step-size, is

$$x_{n+1} = x_n - \left\{ \frac{3}{2} h(x_n) f(x_n) - \frac{1}{2} h(x_{n-1}) f(x_{n-1}) \right\},$$

which corresponds to (7.11) with $\alpha = 3/2$. The analysis above identifies $2/3$ as a critical value of Δ. This gives equivalent qualitative performance when $\alpha = \alpha^* \simeq 0.95$ where α^* is such that $\Delta(\alpha^*) = 2/3$. $\square$

Chapter 8

Finite-Time Explosions

8.1 Introduction

In this chapter we wish to explore some topics in recovering good quantitative information on explosion asymptotics in ODEs. Our choice of topics is highly selective but is geared towards the development of computationally efficient methods for SDEs. In addition we wish to understand whether collocation methods for deterministic equations (such as an adaptive midpoint method) might produce superior performance.

In in this chapter we also ask whether there is value in attempting to simulate explosive ODEs by mapping the problem on to an ODE whose solution is stable in finite-time. Our analysis shows that while this might be achievable in principle, the issue is that, in general, appropriate closed-form mappings are difficult to obtain in practice. Therefore, for this reason we will simulate the explosions directly instead.

In our results on simulating finite-time stability we showed under reasonable monotonicity assumptions on the non-linearity f that a step-size $h(x) \sim \Delta x/f(x)$ as $x \to 0^+$ is both optimal in the sense that all salient quantitative and qualitative properties of the finite-time stability are recovered while asymptotically larger step-sizes are unreliable. Therefore it is reasonable to try to emulate this type of result for explosive equations. Although our analysis is less comprehensive than in the finite-time stability case, we identify that a step-size of $O(1/f'(x))$ as $x \to \infty$ is effective.

We have also seen in the finite-time stability case that pre-transforming the coordinate system allows us to use explicit methods whose quantitative and qualitative behaviour is faithful to the original ODE without restrictions on the control parameter Δ. This is certainly advantageous for ODEs but we believe is of even greater value for SDEs with positive solutions due to the difficulty when using explicit methods in preserving the positivity of simulated solutions. Therefore, as a precursor to such stochastic analysis we wish to demonstrate the feasibility and computational efficiency of this approach for ODEs with regularly or rapidly varying non-linearity.

To a certain degree our results in this chapter may be thought of as a feasibility study for different numerical methods and to this end we conclude the chapter by studying two multi-step techniques in the presence of regular variation. In the case of a two-step linear multi-step method with adaptive time-stepping, we show once again that a time-step of $O(x/f(x))$ as $x \to \infty$ produces the desired results but that errors in the simulated generalised Liapunov exponent are generically of $O(\Delta)$ as $\Delta \to 0^+$, as we have found for finite-time stability problems. However by a careful choice of the multipliers of the step-sizes which entails knowledge of the index of regular variation, the error can be improved to $O(\Delta^2)$ as $\Delta \to 0^+$.

Another popular choice of multi-step method is the so called "Theta Method". The most commonly implemented such method is the midpoint method. In the last section of the chapter, we show that the midpoint method when applied to regularly varying equations, also approximates the generalised Liapunov exponent to $O(\Delta^2)$ as $\Delta \to 0^+$. This is interesting because the weighting is independent of the index of regular variation. For Theta methods with $\theta \neq 1/2$ the familiar $O(\Delta)$ error in the Liapunov exponent is recovered.

8.2 Notation and Preliminaries

We examine the asymptotic and qualitative behaviour of Euler discretisations of the scalar non-linear ODE:

$$x'(t) = f(x(t)), \quad t > 0, \quad x(0) = \xi > 0. \tag{8.1}$$

We suppose that f has the following properties:

$$f(x) > 0 \text{ for all } x > 0; \tag{8.2}$$

$$f \in C([0,\infty);\mathbb{R}); \text{ and} \tag{8.3}$$

$$f \text{ is locally Lipschitz continuous on } [0,\infty). \tag{8.4}$$

Sometimes we strengthen (8.3) to (8.4) in order to guarantee a unique solution. In previous chapters we considered the stable differential equation $x'(t) = -f(x(t))$. We have removed the minus sign in this chapter so as to continue working with a positive f.

It is possible to characterise whether: x explodes in finite-time at time T_ξ; or x approaches infinity as $t \to \infty$. Under condition (8.4) on f the Initial Value Problem (8.1) has a unique continuous solution on a maximal interval of existence $[0, T_\xi)$. On this interval of existence, x is positive and increasing. In the case that

$$\int_1^\infty \frac{1}{f(u)}\,du < \infty, \tag{8.5}$$

it follows that $T_\xi < \infty$. A formula for T_ξ is given by

$$T_\xi = \int_\xi^\infty \frac{1}{f(u)}\,du, \tag{8.6}$$

and $\lim_{t\to T_\xi^-} x(t) = \infty$. In the case that

$$\int_1^\infty \frac{1}{f(u)}\,du = \infty, \tag{8.7}$$

it follows that $T_\xi = \infty$ and that $\lim_{t\to\infty} x(t) = \infty$.

We introduce some auxiliary functions to determine the asymptotic behaviour of x. In the case that f obeys (8.5), the function $\bar{F}$ given by

$$\bar{F}(x) = \int_x^\infty \frac{1}{f(u)}\,du, \quad x > 0, \tag{8.8}$$

is well-defined. In the case that f obeys (8.7), the function F given by

$$F(x) = \int_1^x \frac{1}{f(u)}\,du, \quad x > 0, \tag{8.9}$$

is well-defined.

In the case when f obeys (8.7) we have that $F(x(t)) = F(\xi) + t$ implies

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 1, \tag{8.10}$$

while in the case when f obeys (8.5) we have that $\bar{F}(x(t)) = T_\xi - t\,, t \in [0, T_\xi)$ which implies

$$\lim_{t\to T_\xi^-} \frac{\bar{F}(x(t))}{T_\xi - t} = 1. \tag{8.11}$$

Our goal in this chapter is to recover discrete analogues of (8.10) and (8.11) at minimal computational cost.

8.3 Exploding in Finite-Time with Monotonicity

We make the following observations which will be of use in several of our proofs. Suppose (x_n) is an increasing positive sequence such that $x_n \to \infty$ as $n \to \infty$. Suppose $\int_1^\infty 1/f(u)\,du = \infty$. If F is defined by (8.9) then $F(x) \to \infty$ as $x \to \infty$, so $F(x_n) \to \infty$ as $n \to \infty$. Then for $n \geq 1$

$$F(x_n) = \int_1^{x_n} \frac{1}{f(u)}\,du \;=\; \int_1^{x_0} \frac{1}{f(u)}\,du + \int_{x_0}^{x_n} \frac{1}{f(u)}\,du = F(x_0) + \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} \frac{1}{f(u)}\,du.$$

If $F(x_n) \to \infty$ as $n \to \infty$ then

$$\sum_{j=0}^{\infty} \int_{x_j}^{x_{j+1}} \frac{1}{f(u)}\,du = \infty, \tag{8.12}$$

since $F(x_0)$ is finite. Suppose $\int_1^\infty 1/f(u)\,du < \infty$ then $F(x) \to L \in [0,\infty)$ as $x \to \infty$, so $F(x_n) \to L$ as $n \to \infty$. Hence

$$\sum_{j=0}^{\infty} \int_{x_j}^{x_{j+1}} \frac{1}{f(u)}\,du < \infty. \tag{8.13}$$

If T_ξ is defined by (8.6) then for $n \geq 0$

$$T_\xi = \int_\xi^\infty \frac{1}{f(u)}\,du = \sum_{j=0}^{\infty} \int_{x_j}^{x_{j+1}} \frac{1}{f(u)}\,du.$$

Equations (8.12) and (8.13) show that T_ξ is finite or infinite according to whether $F(x)$ is finite or infinite. If $\bar{F}$ is defined by (8.8) then $\bar{F}(x) \to 0$ as $x \to \infty$ so $\bar{F}(x_n) \to 0$ as $n \to \infty$. Then for $n \geq 0$

$$\bar{F}(x_n) = \int_{x_n}^\infty \frac{1}{f(u)}\,du = \sum_{j=n}^{\infty} \int_{x_j}^{x_{j+1}} \frac{1}{f(u)}\,du.$$

The closed-form expressions for $F(x_n)$, $\bar{F}(x_n)$ and T_ξ identify the summand in the last identity as the key sequence in our analysis.

8.4 Explicit Euler Scheme with Adaptive Step Size

In what follows we attempt to recover the asymptotic behaviour of (8.10) and (8.11) by adaptive time-stepping. We will assume a new monotonicity hypothesis on f namely that f' is increasing. This hypothesis is highly compatible with both super-exponential growth and finite-time explosion. We also assume that $f'(x) \to \infty$ as $x \to \infty$. The case when $f'(x)$ converges to a finite limit, A, means that $f(x)$ is asymptotic to Ax as $x \to \infty$ and we are in the more conventional linear, or asymptotically linear, case. Guided by finite-time stability problems, we will parameterise our step-size by a number, Δ. But also in this explosive case we will introduce a second control in the form of a positive function, c, which we will seek to determine carefully at a later stage. This leads to the following collection of hypotheses. Suppose

$$h(x) = \min\left(\frac{c(x)x}{f(x)}, \frac{\Delta(x)}{f'(x + c(x)x)}\right); \tag{8.14}$$

$$f \text{ and } f' \text{ are increasing;} \tag{8.15}$$

$$\Delta \in C([0,\infty);[0,\infty)) \text{ and } \Delta(x) \to \Delta \in (0,\infty) \text{ as } x \to \infty; \tag{8.16}$$

$$f(x) > 0 \text{ for all } x > 0; \text{ and} \tag{8.17}$$

$$c \in C([0,\infty);[0,\infty)),\ c(x) > 0 \text{ for all } x > 0. \tag{8.18}$$

We approximate $x(t_n)$ by x_n, where $x(t_n)$ is the solution x of (8.1) at time t_n. The sequences (x_n), (t_n) and $(h(x_n))$ are defined by

$$x_{n+1} = x_n + h(x_n)f(x_n), \quad n \geq 0, \quad x_0 = \xi > 0, \tag{8.19}$$

where

$$t_{n+1} = \sum_{j=0}^{n} h(x_j), \quad n \geq 0, \quad t_0 = 0, \tag{8.20}$$

and h is given by (8.14). Notice that $h > 0$ and is continuous. By (8.19) and the positivity of f and h then $x_n > 0$ for all $n \geq 0$. Moreover, (x_n) is increasing. Therefore, (t_n) is an increasing sequence and thus

$$\lim_{n\to\infty} t_n =: \hat{T}_h = \sum_{j=0}^{\infty} h(x_j). \tag{8.21}$$

As we have often done we deduce a careful integral estimate which enables us to recover the asymptotic behaviour of our explosive or non-explosive equations.

Lemma 28. *Suppose* (x_n) *is the solution of* (8.19). *Suppose also* f *obeys* (8.15) *and* (8.17), Δ *obeys* (8.16) *while* h *obeys* (8.14) *where* c *obeys* (8.18). *Then*

$$\frac{1}{1+\Delta(x)} \leq \frac{1}{h(x)} \int_x^{x+h(x)f(x)} \frac{1}{f(u)}\, du \leq 1. \tag{8.22}$$

Proof. If f' is increasing, then either $f'(x) \to L \in (-\infty,\infty)$ as $x \to \infty$ or $f'(x) \to \infty$ as $x \to \infty$. We cannot have $L < 0$ as this would force f to be ultimately negative. If $L = 0$, then $f'(x) < 0$ for all $x \in \mathbb{R}$ and this contradicts that f is increasing. Hence we have either $L \in (0,\infty)$ or $L = \infty$. For $x > 0$, and $x < u < x + h(x)f(x)$, then $f(x) < f(u) < f(x + h(x)f(x))$. Thus

$$\frac{f(x)}{f(x+h(x)f(x))} \leq \frac{1}{h(x)} \int_x^{x+h(x)f(x)} \frac{1}{f(u)}\, du \leq 1.$$

Next by the Mean Value Theorem, for every $x > 0$ there is $\theta_x \in (0,1)$ so that

$$f(x + h(x)f(x)) = f(x) + f'(x + \theta_x h(x)f(x)) \cdot h(x)f(x).$$

Hence

$$\frac{f(x + h(x)f(x))}{f(x)} = 1 + f'(x + \theta_x h(x)f(x)) \cdot h(x).$$

Since f' is increasing and $x + \theta_x h(x)f(x) < x + h(x)f(x)$ then

$$\frac{f(x + h(x)f(x))}{f(x)} \leq 1 + f'(x + h(x)f(x)) \cdot h(x).$$

By (8.14), $h(x) \leq c(x)x/f(x)$, so $h(x)f(x) \leq c(x)x$ and again by the monotonicity of f' and (8.14)

$$\frac{f(x + h(x)f(x))}{f(x)} \leq 1 + f'(x + c(x)x) \cdot h(x) \leq 1 + \Delta(x).$$

Hence for $x > 0$ we have (8.22). □

Theorem 46. *Suppose f obeys* (8.15) *and* (8.17), *Δ obeys* (8.16) *while h obeys* (8.14) *where c obeys* (8.18). *Let (t_n) and $\hat{T}_h$ be defined* (8.20) *and* (8.21).

(i) If f obeys (8.5), *then $\hat{T}_h < \infty$.*

(ii) If f obeys (8.7), *then $\hat{T}_h = \infty$.*

Proof. By (8.5), $\int_1^\infty 1/f(u)\,du < \infty$ then $T_\xi < \infty$ from (8.13) since

$$\sum_{j=0}^{\infty} \int_{x_j}^{x_{j+1}} \frac{1}{f(u)}\,du = \int_{\xi}^{\infty} \frac{1}{f(u)}\,du = T_\xi < \infty.$$

The Comparison Test applied to (8.22) shows the summability of $(\int_{x_n}^{x_{n+1}} 1/f(u)\,du)$ implies that of $(h(x_n)/(1+\Delta(x_n)))$ and hence $(h(x_n))$ is summable since $1/(1+\Delta(x_n)) \to 1/(1+\Delta) \in (0,1]$ as $n \to \infty$ and we have that $t_n = \sum_{j=0}^{n-1} h(x_j)$ for $n \geq 1$ obeys $t_n \to \hat{T}_h := \sum_{j=0}^{\infty} h(x_j) < \infty$ as $n \to \infty$.

By (8.7), $\int_1^\infty 1/f(u)\,du = \infty$ then $T_\xi = \infty$ from (8.12) since

$$\sum_{j=0}^{\infty} \int_{x_j}^{x_{j+1}} \frac{1}{f(u)}\,du = \int_{\xi}^{\infty} \frac{1}{f(u)}\,du = T_\xi = \infty.$$

The Comparison Test applied to (8.22) shows that $(h(x_n))$ is not summable and obeys $t_n = \sum_{j=0}^{n-1} h(x_j) \to \infty$ as $n \to \infty$. Therefore, $t_n \to \infty$ as $n \to \infty$. □

We now show that the asymptotic behaviour at the simulated explosion time is recovered by our method.

Theorem 47. *Suppose f obeys (8.5), (8.15) and (8.17), Δ obeys (8.16) while h obeys (8.14) where c obeys (8.18). Let $\bar{F}$, (t_n) and $\hat{T}_h$ be defined by (8.8), (8.20) and (8.21).*

(i) If $\Delta = 0$, then $x_n > 0$ for all $n \geq 0$, (x_n) is increasing, $x_n \to \infty$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = 1.$$

(ii) If $\Delta \in (0, \infty)$, then $x_n > 0$ for all $n \geq 0$, (x_n) is increasing, $x_n \to \infty$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and

$$\frac{1}{1+\Delta} \leq \liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq \limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq 1.$$

Proof. By (8.22) we have that

$$\frac{1}{1+\Delta(x_n)} \leq \frac{1}{h(x_n)} \int_{x_n}^{x_n + h(x_n)f(x_n)} \frac{1}{f(u)}\, du \leq 1.$$

When $\Delta = 0$ the implies

$$\lim_{n\to\infty} \frac{1}{h(x_n)} \int_{x_n}^{x_{n+1}} \frac{1}{f(u)}\, du = 1.$$

By (8.5), $\int_1^\infty 1/f(u)\, du < \infty$ then $t_n \to \hat{T}_h = \sum_{j=0}^\infty h(x_j) < \infty$ as $n \to \infty$ by Theorem 46. Therefore, by Toeplitz's Lemma

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \lim_{n\to\infty} \frac{\sum_{j=n}^\infty \int_{x_j}^{x_{j+1}} 1/f(u)\, du}{\sum_{j=n}^\infty h(x_j)} = \lim_{j\to\infty} \frac{1}{h(x_j)} \int_{x_j}^{x_{j+1}} \frac{1}{f(u)}\, du = 1.$$

When $\Delta \in (0, \infty)$ by (8.22), we have that

$$\bar{F}(x_n) = \sum_{j=n}^\infty \int_{x_j}^{x_{j+1}} \frac{1}{f(u)}\, du \leq \sum_{j=n}^\infty h(x_j) = \hat{T}_h - t_n.$$

Thus

$$\limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq 1.$$

Similarly for the lower bound we get

$$\bar{F}(x_n) = \sum_{j=n}^\infty \int_{x_j}^{x_{j+1}} \frac{1}{f(u)}\, du \geq \sum_{j=n}^\infty \frac{h(x_j)}{1+\Delta(x_j)}.$$

Thus by Toeplitz's Lemma and the fact that $\Delta(x_j) \to \Delta$ as $j \to \infty$

$$\liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \quad \geq \quad \liminf_{n\to\infty} \frac{\sum_{j=n}^{\infty} h(x_j)/(1+\Delta(x_j))}{\sum_{j=n}^{\infty} h(x_j)} = \frac{1}{1+\Delta}.$$

Combining these limits yields part (ii)

$$\frac{1}{1+\Delta} \leq \liminf_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq \limsup_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} \leq 1,$$

as claimed. □

We now consider the corresponding asymptotic behaviour when solutions do not explode. Once again the growth rate is recovered faithfully.

Theorem 48. *Suppose f obeys* (8.7), (8.15) *and* (8.17), *Δ obeys* (8.16) *while h obeys* (8.14) *where c obeys* (8.18). *Let F and (t_n) be defined by* (8.9) *and* (8.20).

(i) If $\Delta = 0$, then $x_n > 0$ for all $n \geq 0$, x_n is increasing, $x_n \to \infty$ as $n \to \infty$, $t_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} = 1.$$

(ii) If $\Delta \in (0,\infty)$, then $x_n > 0$ for all $n \geq 0$, x_n is increasing, $x_n \to \infty$ as $n \to \infty$, $t_n \to \infty$ as $n \to \infty$ and

$$\frac{1}{1+\Delta} \leq \liminf_{n\to\infty} \frac{F(x_n)}{t_n} \leq \limsup_{n\to\infty} \frac{F(x_n)}{t_n} \leq 1.$$

Proof. By (8.22), we have

$$\frac{1}{1+\Delta(x_n)} \leq \frac{1}{h(x_n)} \int_{x_n}^{x_n+h(x_n)f(x_n)} \frac{1}{f(u)}\,du \leq 1.$$

When $\Delta = 0$ this implies

$$\lim_{n\to\infty} \frac{1}{h(x_n)} \int_{x_n}^{x_{n+1}} \frac{1}{f(u)}\,du = 1.$$

By (8.7), $\int_1^\infty 1/f(u)\,du = \infty$, then $t_n = \sum_{j=0}^{n-1} h(x_j) \to \infty$ as $n \to \infty$ by Theorem 46. Therefore, by Toeplitz's Lemma

$$\lim_{n\to\infty} \frac{F(x_n)}{t_n} \quad = \quad \lim_{n\to\infty} \frac{F(x_0) + \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} 1/f(u)\,du}{\sum_{j=0}^{n-1} h(x_j)} = \lim_{j\to\infty} \frac{1}{h(x_j)} \int_{x_j}^{x_{j+1}} \frac{1}{f(u)}\,du = 1.$$

When $\Delta \in (0, \infty)$ by (8.22), we have

$$F(x_n) = F(x_0) + \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} \frac{1}{f(u)}\,du \le F(x_0) + \sum_{j=0}^{n-1} h(x_j).$$

Thus by Toeplitz's Lemma

$$\limsup_{n\to\infty} \frac{F(x_n)}{t_n} \le \limsup_{n\to\infty} \frac{F(x_0) + \sum_{j=0}^{n-1} h(x_j)}{\sum_{j=0}^{n-1} h(x_j)} = 1.$$

Similarly for the lower bound

$$F(x_n) = F(x_0) + \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} \frac{1}{f(u)}\,du \ge F(x_0) + \sum_{j=0}^{n-1} \frac{h(x_j)}{1+\Delta(x_j)}.$$

Thus by Toeplitz's Lemma and that $\Delta(x_j) \to \Delta$ as $j \to \infty$

$$\liminf_{n\to\infty} \frac{F(x_n)}{t_n} \ge \liminf_{n\to\infty} \frac{F(x_0) + \sum_{j=0}^{n-1} h(x_j)/(1+\Delta(x_j))}{\sum_{j=0}^{n-1} h(x_j)} = \frac{1}{1+\Delta}.$$

Combining the limits gives part (ii), as claimed. □

8.4.1 Appropriate Choice of $c(x)$

Up to this point we have left the function c free. We now seek to choose c so as to maximise the step-size, h, thereby reducing the computational effort. Recall

$$h(x) = \min\left(\frac{c(x)x}{f(x)}, \frac{\Delta(x)}{f'(x + c(x)x)} \right).$$

Since f is increasing, if we take $\eta(x) > 0$ and $x \mapsto \eta(x)$ is continuous the choice of

$$c(x) = \frac{\eta(x) f(x)}{x f'(x)},$$

is consistent with (8.18). Thus h satisfies

$$h(x) = \min\left(\frac{\eta(x)}{f'(x)}, \frac{\Delta(x)}{f'(x + \eta(x)\frac{f(x)}{f'(x)})} \right).$$

Now suppose $\eta(x) = \Delta(x)$. Then

$$h(x) = \min\left(\frac{\Delta(x)}{f'(x)}, \frac{\Delta(x)}{f'(x + \Delta(x)\frac{f(x)}{f'(x)})} \right).$$

Since f' is increasing this implies

$$h(x) = \frac{\Delta(x)}{f'(x + \Delta(x)\frac{f(x)}{f'(x)})}. \tag{8.23}$$

We want to consider two important special classes of rapidly growing functions. The class of regularly varying functions we have already examined in depth and the class of functions Γ. We pause to introduce the second class of functions Γ defined as below (see Section 3.10 in [12]):

Definition 49. The class Γ consists of those functions $\phi : \mathbb{R} \to (0, \infty)$ non-decreasing and right-continuous for which there exists a measurable function $g : \mathbb{R} \to (0, \infty)$, called the auxiliary function of ϕ, such that

$$\lim_{x\to\infty} \frac{\phi(x + ug(x))}{\phi(x)} = e^u, \quad \forall\, u \in \mathbb{R}.$$

We record some important facts about Γ. First if g is an auxiliary function of $\phi \in \Gamma$ we must have that

$$g(x) \sim \frac{\int_0^x \phi(u)\, du}{\phi(u)}, \quad \text{as } x \to \infty.$$

Furthermore we must have that $g(x)/x \to 0$ as $x \to \infty$. Moreover, if $\phi \in \Gamma$ then so is $x \mapsto \phi_1(x) := \int_0^x \phi(u)\, du$ and ϕ_1 has the same auxiliary function as ϕ. Therefore

$$\lim_{x\to\infty} \frac{\phi(x) \int_0^x \int_0^y \phi(z)\, dz dy}{\left(\int_0^x \phi(y)\, dy\right)^2} = 1.$$

This limit also characterises Γ and is easily checked.

Finally, functions in Γ obey a uniform convergence theorem and in fact if u is such that $\lim_{x\to\infty} u(x) = u^* \in [-\infty, \infty]$ and $\lim_{x\to\infty}(x + u(x)g(x)) = \infty$ then

$$\lim_{x\to\infty} \frac{\phi(x + u(x)g(x))}{\phi(x)} = e^u.$$

The function $\phi(x) = e^x$ is in Γ with auxiliary function $g(x) = 1$. All functions in Γ are also rapidly-varying at infinity.

Armed with this definition and properties of the class Γ, we now see how the step-size condition (8.23) simplifies for regularly varying functions and functions in Γ.

(i) If $f' \in \mathrm{RV}_\infty(\beta - 1)$. Then $f \in \mathrm{RV}_\infty(\beta)$ and $xf'(x)/f(x) \to \beta$ as $x \to \infty$. Therefore

$$f'\left(x + \Delta(x)\frac{f(x)}{f'(x)}\right) \sim \left(1 + \frac{\Delta}{\beta}\right)^{\beta-1} f'(x), \quad \text{as } x \to \infty.$$

Thus $h(x)f'(x) \to \Delta$ as $x \to \infty$ and $h(x) \sim \Delta/f'(x)$ as $x \to \infty$. This is known to be optimal for $f \in \mathrm{RV}_\infty(\beta)$. We see later in the chapter that having $h(x)f(x)/x \to L$ as $x \to \infty$ is optimal for explosions with regularly varying coefficients.

(ii) Suppose $f' =: \phi$ is in Γ. Then $\phi_1 = f$ is also in Γ and we have that

$$\lim_{x\to\infty} \frac{f'(x)\int_0^x f(u)\,du}{f^2(x)} = \lim_{x\to\infty} \frac{\phi(x)\int_0^x\int_0^y \phi(z)\,dzdy}{\left(\int_0^x \phi(y)\,dy\right)^2} = 1.$$

Therefore as $\Delta(x) \to \Delta$ as $x \to \infty$ and f' has auxiliary function $g(x) = f(x)/f'(x)$, by the uniform convergence theorem for Γ, we have that

$$\lim_{x\to\infty} \frac{f\left(x + \Delta(x)\frac{f(x)}{f'(x)}\right)}{f(x)} = e^{\Delta}.$$

Hence

$$h(x) = \frac{\Delta(x)}{f\left(x + \Delta(x)\frac{f(x)}{f'(x)}\right)} \sim \frac{\Delta}{e^{\Delta} f'(x)}, \quad \text{as } x \to \infty.$$

We show later in this chapter that $h(x)f'(x) \to L$ as $x \to \infty$ is the optimal step-size when $f \in \Gamma$, a fact that is implied by $f' \in \Gamma$.

Hence, under the assumption that f' is increasing it seems h given by (8.23) is of the smallest order possible to recover asymptotic behaviour, at least for important classes like $f' \in \mathrm{RV}_\infty(\beta)$ and Γ.

8.5 Step-Size for Deterministic ODE

If we apply a logarithmic pre-transformation to the solution of the ODE (8.1) before discretising a step-size of $O(1/f'(x))$ as $x \to \infty$ is still needed to capture the explosion asymptotics for f in the subclass Γ of rapidly varying functions. More precisely, we suppose

$$f' \in \Gamma \text{ and } f' \text{ is increasing.} \tag{8.24}$$

Recall that this implies that $f \in \Gamma$ as well and that f and f' share (up to asymptotic equivalence) the same auxiliary function g. The auxiliary function of f' can be chosen to be

$$g(x) = \frac{f(x)}{f'(x)}, \tag{8.25}$$

and this can also be the auxiliary function for f. Since $g(x)/x \to 0$ as $x \to \infty$, we have

$$\lim_{x\to\infty} \frac{f(x)}{xf'(x)} = 0.$$

Since $f' \in \Gamma$ we have seen that $xf'(x)/f(x) \to \infty$ as $x \to \infty$. Therefore for any $M > 1$ there is $x(M) > 0$ such that $xf'(x)/f(x) > M$, $\forall\, x \geq x(M)$. Thus

$$\log\left(\frac{f(x)}{f(x(M))}\right) = \int_{x(M)}^{x} \frac{f'(u)}{f(u)}\,du \geq \int_{x(M)}^{x} \frac{M}{u}\,du = M\log\left(\frac{x}{x(M)}\right).$$

Hence $f(x) \geq C_M x^M$, $\forall\, x \geq x(M)$. Since $M > 1$ is arbitrary then $\int_1^\infty 1/f(u)\,du < \infty$.

8.5.1 Logarithmic Pre-Transformation

We study the ODE (8.1) viz.,

$$x'(t) = f(x(t)), \quad t > 0, \quad x(0) = \xi > 0.$$

Define $z(t) := \log x(t)$. Then the transformed ODE is $z'(t) =: \eta(z(t))$ where $\eta(z) := f(e^z)/e^z$. We discretise this so that z_n approximates $z(t_n)$ and x_n approximates $x(t_n)$ to get

$$z_{n+1} = z_n + \tilde{h}(z_n)\eta(z_n), \quad n \geq 0, \quad z_0 = \log\xi,$$
$$x_{n+1} = e^{z_{n+1}}, \quad n \geq 0, \quad x_0 = \xi,$$

where $h(x) = \Delta(x)/f'(x)$ and

$$t_{n+1} = \sum_{j=0}^{n} \tilde{h}(z_j) = \sum_{j=0}^{\infty} \frac{\Delta(e^{z_j})}{f'(e^{z_j})}, \quad n \geq 0, \quad t_0 = 0,$$

and $\tilde{h}(z) := h(e^z)$. Thus $t_n = \sum_{j=0}^{n-1} h(x_j)$ and

$$x_{n+1} = x_n \exp\left(\frac{h(x_n)f(x_n)}{x_n}\right), \quad n \geq 0, \quad x_0 = \xi > 0. \tag{8.26}$$

Proposition 14. *There is a unique increasing positive sequence (x_n) with $x_0 = \xi > 0$ which obeys (8.26) which obeys $x_n \to \infty$ as $n \to \infty$.*

Proof. The sequence exists and is unique by construction. Since h, $f > 0$ for all $x > 0$, we have that (x_n) is a positive increasing sequence. Since (x_n) is increasing, we have $x_n \to L \in [0, \infty]$ as $n \to \infty$. If $L \in (0, \infty)$ since h, f are continuous

$$L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\left(x_n \exp\left(\frac{h(x_n)f(x_n)}{x_n}\right)\right) = L\exp\left(\frac{h(L)f(L)}{L}\right),$$

which is impossible. Hence $x_n \to \infty$ as $n \to \infty$. □

Lemma 29. *Suppose f obeys (8.24) and (8.25) while Δ obeys (8.16) and $h(x) = \Delta(x)/f'(x)$.*

(i) If $\Delta = 0$*, then*

$$\lim_{x\to\infty} \frac{1}{h(x)} \int_x^{x\exp(h(x)f(x)/x)} \frac{1}{f(u)}\,du = 1.$$

(ii) If $\Delta \in (0,\infty)$*, then*

$$\lim_{x\to\infty} \frac{1}{h(x)} \int_x^{x\exp(h(x)f(x)/x)} \frac{1}{f(u)}\,du = \frac{1-e^{-\Delta}}{\Delta}.$$

Proof. Define

$$\begin{aligned} I(x) &:= \frac{1}{h(x)} \int_x^{x\exp(h(x)f(x)/x)} \frac{1}{f(u)}\,du = \frac{1}{h(x)} \int_0^{x\exp(h(x)f(x)/x)-x} \frac{1}{f(x+v)}\,dv \\ &= \frac{g(x)}{f(x)h(x)} \int_0^{l(x)} \frac{f(x)}{f(x+ug(x))}\,du = \frac{1}{\Delta(x)} \int_0^{l(x)} \frac{f(x)}{f(x+ug(x))}\,du, \end{aligned} \tag{8.27}$$

since

$$\frac{g(x)}{f(x)h(x)} = \frac{f(x)}{f'(x)} \cdot \frac{1}{f(x)} \cdot \frac{f'(x)}{\Delta(x)} = \frac{1}{\Delta(x)},$$

and where

$$l(x) := \frac{x\left(\exp\left(\frac{h(x)f(x)}{x}\right)-1\right)}{g(x)}.$$

Note that (8.25) implies

$$\lim_{x\to\infty} \frac{h(x)f(x)}{x} = \lim_{x\to\infty} \left(\Delta(x) \cdot \frac{f(x)}{xf'(x)}\right) = \Delta \cdot 0 = 0.$$

Since $e^x - 1 \sim x$ as $x \to 0^+$ then

$$l(x) = \frac{x\left(\exp\left(\frac{h(x)f(x)}{x}\right)-1\right)}{g(x)} \sim \frac{h(x)f(x)}{g(x)} = \Delta(x).$$

We now prove part (i). Since f is increasing, for $0 < u < l(x)$ then $f(x) < f(x + ug(x)) < f(x + l(x)g(x))$ and thus

$$\frac{f(x)}{f(x+l(x)g(x))} < \frac{f(x)}{f(x+ug(x))} < 1.$$

Hence

$$\frac{l(x)\cdot f(x)}{\Delta(x)\cdot f(x+l(x)g(x))} \le I(x) \le \frac{l(x)}{\Delta(x)}.$$

Since $l(x) \sim \Delta(x)$ as $x \to \infty$,

$$\liminf_{x\to\infty} \frac{f(x)}{f(x+l(x)g(x))} \le \liminf_{x\to\infty} I(x) \le \limsup_{x\to\infty} I(x) \le 1.$$

Now $l(x) \to 0$ as $x \to \infty$, so by the uniform convergence theorem for $f \in \Gamma$

$$\lim_{x\to\infty} \frac{f(x + l(x)g(x))}{f(x)} = e^0 = 1.$$

Thus $1 \le \liminf_{x\to\infty} I(x) \le \limsup_{x\to\infty} I(x) \le 1$. Therefore, $I(x) \to 1$ as $x \to \infty$, or

$$\lim_{x\to\infty} \frac{1}{h(x)} \int_x^{x\exp(h(x)f(x)/x)} \frac{1}{f(u)}\, du = 1,$$

proving part (i). We now prove part (ii). As f is monotone and $l(x) \to \Delta \in (0,\infty)$ as $x \to \infty$ then for arbitrary $\epsilon \in (0,1)$ and all x sufficiently large we have

$$\int_0^{\Delta(1-\epsilon)} \frac{f(x)}{f(x+ug(x))}\, du \le \int_0^{l(x)} \frac{f(x)}{f(x+ug(x))}\, du \le \int_0^{\Delta(1+\epsilon)} \frac{f(x)}{f(x+ug(x))}\, du. \quad (8.28)$$

Take $h > 0$ arbitrary and $c > 0$ with $h < c$ then

$$\int_0^c \frac{f(x)}{f(x+ug(x))}\, du = \sum_{j=0}^{\lfloor c/h\rfloor -1} \int_{jh}^{(j+1)h} \frac{f(x)}{f(x+ug(x))}\, du + \int_{\lfloor c/h\rfloor h}^{c} \frac{f(x)}{f(x+ug(x))}\, du.$$

For $jh < u < (j+1)h$, $f(x+jhg(x)) < f(x+ug(x)) < f(x+(j+1)hg(x))$. Thus

$$\frac{f(x)h}{f(x+(j+1)hg(x))} < \int_{jh}^{(j+1)h} \frac{f(x)}{f(x+ug(x))}\, du < \frac{f(x)h}{f(x+jhg(x))}.$$

Therefore letting $x \to \infty$ implies

$$\sum_{j=0}^{\lfloor c/h\rfloor -1} e^{-(j+1)h} h \le \liminf_{x\to\infty} \int_0^c \frac{f(x)}{f(x+ug(x))}\, du \le \limsup_{x\to\infty} \int_0^c \frac{f(x)}{f(x+ug(x))}\, du \le \sum_{j=0}^{\lfloor c/h\rfloor -1} e^{-jh} h + e^{\lfloor c/h\rfloor} h.$$

Since h is chosen arbitrarily, we may let $h \to 0$ to give

$$\int_0^c e^{-u}\, du = \lim_{x\to\infty} \int_0^c \frac{f(x)}{f(x+ug(x))}\, du. \quad (8.29)$$

Using (8.29) in (8.28) yields

$$\int_0^{\Delta(1-\epsilon)} e^{-u}\, du \le \liminf_{x\to\infty} \int_0^{l(x)} \frac{f(x)}{f(x+ug(x))}\, du \le \limsup_{x\to\infty} \int_0^{l(x)} \frac{f(x)}{f(x+ug(x))}\, du \le \int_0^{\Delta(1+\epsilon)} e^{-u}\, du.$$

Letting $\epsilon \to 0$ yields

$$\lim_{x\to\infty} \int_0^{l(x)} \frac{f(x)}{f(x+ug(x))}\,du = \int_0^{\Delta} e^{-u}\,du = 1 - e^{-\Delta}.$$

Putting this into (8.27) yields part (ii)

$$\lim_{x\to\infty} \frac{1}{h(x)} \int_x^{x\exp(h(x)f(x)/x)} \frac{1}{f(u)}\,du = \frac{1-e^{-\Delta}}{\Delta},$$

as claimed. □

Theorem 50. *Suppose f obeys* (8.5), (8.24) *and* (8.25) *while Δ obeys* (8.16) *and $h(x) = \Delta(x)/f'(x)$. Let $\bar{F}$, (t_n) and $\hat{T}_h$ be defined by* (8.8), (8.20) *and* (8.21).

(i) If $\Delta = 0$, then $x_n > 0$ for all $n \geq 0$, (x_n) is increasing, $x_n \to \infty$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = 1.$$

(ii) If $\Delta \in (0,\infty)$, then $x_n > 0$ for all $n \geq 0$, (x_n) is increasing, $x_n \to \infty$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \frac{1-e^{-\Delta}}{\Delta}.$$

Proof. The positivity, monotonicity and divergence of (x_n) have been addressed in Proposition 14. Since f obeys (8.5) then $\int_1^\infty 1/f(u)\,du < \infty$, then $\sum_{j=0}^n \int_{x_j}^{x_{j+1}} 1/f(u)\,du$ tends to a finite limit. Suppose $\Delta = 0$. By Lemma 29 part (i)

$$\frac{1}{h(x_n)} \int_{x_n}^{x_{n+1}} \frac{1}{f(u)}\,du = \frac{1}{h(x_n)} \int_{x_n}^{x_n\exp(h(x_n)f(x_n)/x_n)} \frac{1}{f(u)}\,du \to 1, \quad \text{as } n\to\infty. \tag{8.30}$$

Therefore $(h(x_n))$ is a summable sequence and so $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and $\hat{T}_h - t_n = \sum_{j=n}^\infty h(x_j)$. Hence by Toeplitz's Lemma and (8.30)

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \lim_{n\to\infty} \frac{\sum_{j=n}^\infty \int_{x_j}^{x_{j+1}} 1/f(u)\,du}{\sum_{j=n}^\infty h(x_j)} = \lim_{n\to\infty} \frac{1}{h(x_n)} \int_{x_n}^{x_{n+1}} \frac{1}{f(u)}\,du = 1.$$

Suppose $\Delta \in (0,\infty)$. By Lemma 29 part (ii)

$$\frac{1}{h(x_n)} \int_{x_n}^{x_{n+1}} \frac{1}{f(u)}\,du = \frac{1}{h(x_n)} \int_{x_n}^{x_n\exp(h(x_n)f(x_n)/x_n)} \frac{1}{f(u)}\,du \to \frac{1-e^{-\Delta}}{\Delta}, \quad \text{as } n\to\infty. \tag{8.31}$$

Hence by Toeplitz's Lemma and (8.31)

$$\lim_{n\to\infty}\frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \lim_{n\to\infty}\frac{\sum_{j=n}^{\infty}\int_{x_j}^{x_{j+1}} 1/f(u)\,du}{\sum_{j=n}^{\infty} h(x_j)} = \lim_{n\to\infty}\frac{1}{h(x_n)}\int_{x_n}^{x_{n+1}}\frac{1}{f(u)}\,du = \frac{1-e^{-\Delta}}{\Delta},$$

as claimed. □

This result was anticipated by our general Theorem 47 with $h(x) = \Delta(x)/f'(x)$, $\Delta(x)$ tending to a constant and $c(x) = \Delta(x)$.

8.5.2 Power Pre-Transformation

In this section we show that power transformations also recover the blow-up asymptotics of ODEs in which case f obeys (8.24). Suppose the ODE (8.1) viz.,

$$x'(t) = f(x(t)), \quad t > 0, \quad x(0) = \xi > 0.$$

Define $z(t) := x(t)^\theta$, $\theta \in (0,1)$. Then $z'(t) = \eta(z(t))$ where $\eta(z) := \theta\left(z^{1/\theta}\right)^{\theta-1} f(z^{1/\theta})$. The transformed ODE is

$$z'(t) = \eta(z(t)), \quad t > 0, \quad z(0) = \xi^\theta > 0.$$

We discretise this so that z_n approximates $z(t_n)$ and x_n approximates $x(t_n)$ to get

$$z_{n+1} = z_n + \tilde{h}(z_n)\eta(z_n), \quad n \geq 0, \quad z_0 = \xi^\theta,$$
$$x_{n+1} = z_{n+1}^{1/\theta}, \quad n \geq 0, \quad x_0 = \xi,$$

where $h(x) = \Delta(x)/f'(x)$

$$t_{n+1} = \sum_{j=0}^{n}\tilde{h}(z_j) = \sum_{j=0}^{n}\frac{\Delta(z_j^{1/\theta})}{f'(z_j^{1/\theta})}, \quad n \geq 0, \quad t_0 = 0,$$

and $h(z^{1/\theta}) =: \tilde{h}(z)$. Thus $t_n = \sum_{j=0}^{n-1} h(x_j)$ and

$$x_{n+1} = x_n\left(1 + \theta\cdot\frac{h(x_n)f(x_n)}{x_n}\right)^{1/\theta}, \quad n \geq 0, \quad x_0 = \xi > 0. \tag{8.32}$$

Proposition 15. *There is a unique positive, increasing sequence (x_n) which obeys (8.32) and (x_n) obeys $x_n \to \infty$ as $n \to \infty$.*

Proof. The sequence exists and is unique by construction. Since h, $f > 0$ for all $x > 0$, we have that (x_n) is a positive increasing sequence. Since (x_n) is increasing, we have

$x_n \to L \in (0, \infty)$ as $n \to \infty$. If L is finite since h, f are continuous

$$L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \left(x_n \left(1 + \theta \cdot \frac{h(x_n)f(x_n)}{x_n} \right)^{1/\theta} \right) = L \left(1 + \theta \cdot \frac{h(L)f(L)}{L} \right)^{1/\theta},$$

which is impossible. Hence $x_n \to \infty$ as $n \to \infty$. □

Lemma 30. *Suppose f obeys (8.24) and (8.25) while Δ obeys (8.16) and $h(x) = \Delta(x)/f'(x)$.*

(i) If $\Delta = 0$, then

$$\lim_{x\to\infty} \frac{1}{h(x)} \int_x^{x\left(1+\theta\cdot\frac{h(x)f(x)}{x}\right)^{1/\theta}} \frac{1}{f(u)}\,du = 1.$$

(ii) If $\Delta \in (0, \infty)$, then

$$\lim_{x\to\infty} \frac{1}{h(x)} \int_x^{x\left(1+\theta\cdot\frac{h(x)f(x)}{x}\right)^{1/\theta}} \frac{1}{f(u)}\,du = \frac{1-e^{-\Delta}}{\Delta}.$$

Proof. Define

$$\begin{aligned} I(x) &:= \frac{1}{h(x)} \int_x^{x\left(1+\theta\cdot\frac{h(x)f(x)}{x}\right)^{1/\theta}} \frac{1}{f(u)}\,du = \frac{1}{h(x)} \int_0^{x\left(1+\theta\cdot\frac{h(x)f(x)}{x}\right)^{1/\theta}-x} \frac{1}{f(x+v)}\,dv \\ &= \frac{g(x)}{f(x)h(x)} \int_0^{l(x)} \frac{f(x)}{f(x+ug(x))}\,du = \frac{1}{\Delta(x)} \int_0^{l(x)} \frac{f(x)}{f(x+ug(x))}\,du, \end{aligned}$$

since

$$\frac{g(x)}{f(x)h(x)} = \frac{f(x)}{f'(x)} \cdot \frac{1}{f(x)} \cdot \frac{f'(x)}{\Delta(x)} = \frac{1}{\Delta(x)},$$

and where

$$l(x) := \frac{x\left(1+\theta\cdot\frac{h(x)f(x)}{x}\right)^{1/\theta} - x}{g(x)}.$$

Note that

$$\lim_{x\to\infty} \frac{h(x)f(x)}{x} = \lim_{x\to\infty} \left(\Delta(x) \cdot \frac{f(x)}{xf'(x)} \right) = \Delta \cdot 0 = 0.$$

Therefore by L'Hôpital's rule

$$\lim_{x\to\infty} \frac{\left(1+\theta\cdot\frac{h(x)f(x)}{x}\right)^{1/\theta} - 1}{\frac{h(x)f(x)}{x}} = \lim_{y\to 0} \frac{(1+\theta y)^{1/\theta}-1}{y} = \lim_{y\to 0^+} \frac{\theta\,(1+\theta y)^{1/\theta-1}}{\theta} = 1.$$

Therefore as $x \to \infty$

$$l(x) \sim \frac{x\cdot\frac{h(x)f(x)}{x}}{g(x)} = \frac{h(x)f(x)}{g(x)} = \frac{\Delta(x)/f'(x)\cdot f(x)}{f(x)/f'(x)} = \Delta(x).$$

From the calculation in Lemma 29 part (i)

$$\lim_{x\to\infty} \frac{1}{\Delta(x)} \int_0^{l(x)} \frac{f(x)}{f(x+ug(x))}\, du = 1.$$

Thus

$$\lim_{x\to\infty} \frac{1}{h(x)} \int_x^{x\left(1+\theta\cdot\frac{h(x)f(x)}{x}\right)^{1/\theta}} \frac{1}{f(u)}\, du = 1.$$

From the calculation in Lemma 29 part (ii)

$$\lim_{x\to\infty} \frac{1}{\Delta(x)} \int_0^{l(x)} \frac{f(x)}{f(x+ug(x))}\, du = \frac{1}{\Delta}\int_0^{\Delta} e^{-u}\, du = \frac{1-e^{-\Delta}}{\Delta}.$$

Thus

$$\lim_{x\to\infty} \frac{1}{h(x)} \int_x^{x\left(1+\theta\cdot\frac{h(x)f(x)}{x}\right)^{1/\theta}} \frac{1}{f(u)}\, du = \frac{1-e^{-\Delta}}{\Delta},$$

as required. □

Remark 34. Notice that the limit is unity as $\Delta \to 0^+$ while it is 0 if $\Delta \to \infty$. □

Theorem 51. *Suppose f obeys* (8.5), (8.24) *and* (8.25) *while Δ obeys* (8.16) *and $h(x) = \Delta(x)/f'(x)$. Let $\bar{F}$, (t_n) and $\hat{T}_h$ be defined by* (8.8), (8.20) *and* (8.21).

(i) If $\Delta = 0$, then $x_n > 0$ for all $n \geq 0$, (x_n) is increasing, $x_n \to \infty$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = 1$$

(ii) If $\Delta \in (0,\infty)$, then $x_n > 0$ for all $n \geq 0$, (x_n) is increasing, $x_n \to \infty$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \frac{1-e^{-\Delta}}{\Delta}.$$

Proof. The positivity, monotonicity and divergence of (x_n) have been addressed in Proposition 15. The rest of the proof follows the same line of argument as in Theorem 50 but with Lemma 30 playing the role of Lemma 29 in Theorem 50. □

8.6 Multi-Step Numerical Schemes

8.6.1 Two-Step Explicit Euler Scheme

In this section we investigate the performance of a two-step Explicit Euler scheme with an adaptive mesh. We approximate $x(t_n)$ by x_n, where $x(t_n)$ is the solution x of

(8.1) at time t_n. The sequences (x_n), (t_n) and $(h(x_n))$ are defined by

$$x_{n+1} = x_n + \{\alpha h(x_n) f(x_n) + (1-\alpha) h(x_{n-1}) f(x_{n-1})\}, \tag{8.33}$$

where $\alpha > 0$

$$t_{n+1} = \sum_{j=0}^{n} h(x_j), \quad n \geq 0, \quad t_0 = 0,$$

and where $\Delta > 0$ and $h(x) = \Delta x / f(x)$. Substituting this choice of $h(x)$ into (8.33) implies that

$$x_{n+1} = (1 + \alpha\Delta) x_n + (1-\alpha)\Delta x_{n-1}. \tag{8.34}$$

Theorem 52. *Suppose $\alpha > 0$ and let*

$$\lambda^2 - (1+\alpha\Delta)\lambda - (1-\alpha)\Delta = 0,$$

be the characteristic equation of (8.34)*. Then the characteristic equation has roots λ_1, $\lambda_2 \in \mathbb{R}$ and we write $\lambda_1 \geq \lambda_2$. Assume $x_0 > x_{-1} > 0$.*

(i) If $0 < \alpha < 1$, then $\lambda_2 < 0 < 1 < \lambda_1$, $x_n > 0 \, \forall \, n \geq 0$ and

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda_1 > 1.$$

(ii) If $\alpha > 1$, then $0 < \lambda_2 < 1 < \lambda_1$, $x_n > 0 \, \forall \, n \geq 0$ and

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda_1 > 1.$$

Proof. Equation (8.34) can be written as

$$x_{n+1} - x_n = \alpha\Delta(x_n - x_{n-1}) + \Delta x_{n-1}.$$

If $x_0 > x_{-1}$ then (x_n) is increasing and $x_n \to \infty$ as $n \to \infty$. The zeros of the characteristic equation are

$$\lambda_{1,2} = \frac{(1+\alpha\Delta) \pm \sqrt{(1+\alpha\Delta)^2 - 4(\alpha-1)\Delta}}{2},$$

and its discriminant is $\delta(\Delta) := (1-\alpha\Delta)^2 + 4\Delta$. Thus $\delta(\Delta) > 0 \, \forall \, \Delta > 0$ with λ_1, $\lambda_2 \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$. Also as $\lambda_1 \neq \lambda_2$ there exists c_1, $c_2 \in \mathbb{R}$ such that

$$x_n = c_1 \lambda_1^n + c_2 \lambda_2^n,$$

where

$$c_1 = \frac{x_{-1} - x_0/\lambda_2}{1/\lambda_1 - 1/\lambda_2} = \frac{x_0/\lambda_2 - x_{-1}}{1/\lambda_2 - 1/\lambda_1} \quad \text{and} \quad c_2 = \frac{x_0/\lambda_1 - x_{-1}}{1/\lambda_1 - 1/\lambda_2} = \frac{x_{-1} - x_0/\lambda_1}{1/\lambda_2 - 1/\lambda_1}.$$

If $0 < \alpha < 1$ then $\lambda_1\lambda_2 = -(1-\alpha)\Delta = (\alpha - 1)\Delta < 0$. Since $\lambda_1 + \lambda_2 = 1 + \alpha\Delta > 1 > 0$ then $\lambda_2 < 0 < 1 < \lambda_1$. Thus $|\lambda_1| = \lambda_1 > |\lambda_2|$ and $|\lambda_1| = \max_{i=1,2} |\lambda_i|$. Note that $c_1 > 0$ since $\lambda_2 < 0$. It follows that $x_n/\lambda_1^n \to c_1$ as $n \to \infty$ and so

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \left(\frac{x_{n+1}}{\lambda_1^{n+1}} \cdot \frac{\lambda_1^n}{x_n} \cdot \lambda_1 \right) = \lambda_1 > 1.$$

When $\alpha > 1$ then $\lambda_1\lambda_2 > 0$ and $\lambda_1 + \lambda_2 > 0$. Thus $0 < \lambda_2 < 1 < \lambda_1$. Note that $c_1 > 0$ when $x > 1$ since $1/\lambda_2 > 1$ thus $x_0/\lambda_2 > x_0 > x_{-1}$ by supposition thus $x_0 - \lambda_2 x_{-1} > 0$. By the same argument as when $\alpha < 1$ then

$$\lim_{n\to\infty} \frac{x_n}{\lambda_1^n} = c_1 > 0,$$

and we have part (ii). □

We now show that this lemma enables us to recover the asymptotic behaviour of the explosion in the case that $f \in RV_0(\beta)$ for $\beta \geq 1$ and $\int_1^\infty 1/f(u) < \infty$. We write

$$\begin{aligned} \bar{F}(x_n) = \int_{x_n}^{\infty} \frac{1}{f(u)}\,du = \sum_{j=n}^{\infty} \int_{x_j}^{x_{j+1}} \frac{1}{f(u)}\,du &= \sum_{j=n}^{\infty} \frac{\Delta x_j}{f(x_j)} \cdot \frac{f(x_j)}{\Delta x_j} \int_{x_j}^{x_{j+1}} \frac{1}{f(u)}\,du \\ &= \sum_{j=n}^{\infty} \frac{\Delta x_j}{f(x_j)} \cdot \frac{1}{\Delta} \int_1^{x_{j+1}/x_j} \frac{\tilde{f}(vx_j)}{\tilde{f}(x_j)}\,dv, \end{aligned}$$

where $\tilde{f} = 1/f \in RV_0(-\beta)$. Note that

$$\lim_{j\to\infty} \frac{1}{\Delta} \int_{x_j}^{x_{j+1}} \frac{\tilde{f}(u)}{\tilde{f}(x_j)}\,du = \frac{1}{\Delta} \int_1^{\lambda_1(\alpha,\Delta)} v^{-\beta}\,dv,$$

by the uniform convergence theorem and the fact that $x_{j+1}/x_j \to \lambda_1(\alpha, \Delta)$ as $j \to \infty$. Hence

$$\bar{F}(x_n) \sim \frac{1}{\Delta} \int_1^{\lambda_1(\alpha,\Delta)} v^{-\beta}\,dv \cdot (\hat{T}_h - t_n), \quad \text{as } n \to \infty.$$

Therefore

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \frac{1}{\Delta} \int_1^{\lambda_1(\alpha,\Delta)} v^{-\beta}\,dv =: \lambda(\alpha, \Delta).$$

We notice that the asymptotic behaviour at the explosion has been perfectly captured with a step-size h such that $h(x) \sim \Delta(x)/f'(x)$ as $x \to \infty$. This was anticipated by our general Theorem 47 with $c(x) = \Delta(x) = \Delta$.

Theorem 53. *Define*

$$\lambda_1(\alpha,\Delta) := \frac{1+\alpha\Delta+\sqrt{(1+\alpha\Delta)^2-4(\alpha-1)\Delta}}{2},$$

then the error in asymptotic convergence rate $\alpha \mapsto \delta(\alpha,\Delta) := |\lambda(\alpha,\Delta)-1|$ *is minimised at* $\alpha = 1+\beta/2$ *as* $\Delta \to 0^+$ *and moreover* $\delta(1+\beta/2,\Delta) = O(\Delta^2)$ *as* $\Delta \to 0^+$.

Proof. In this calculation β is fixed. The Taylor Series of $\sqrt{(1+\alpha\Delta)^2-4(\alpha-1)\Delta}$ implies that

$$\lambda_1(\alpha,\Delta) = 1+\Delta+(\alpha-1)\Delta^2+O(\Delta^3), \quad \text{as } \Delta \to 0^+.$$

Then

$$\lim_{\Delta\to 0^+}\frac{\lambda(\alpha,\Delta)-1}{\Delta} = \lim_{\Delta\to 0^+}\frac{\int_1^{1+\Delta+(\alpha-1)\Delta^2+O(\Delta^3)} v^{-\beta}\,dv-\Delta}{\Delta^2} = \alpha-1-\frac{\beta}{2}.$$

Hence

$$\lim_{\Delta\to 0^+}\frac{\delta(\alpha,\Delta)}{\Delta} = \lim_{\Delta\to 0^+}\frac{|\lambda(\alpha,\Delta)-1|}{\Delta} = \left|\alpha-1-\frac{\beta}{2}\right| =: C(\alpha).$$

Clearly $C(\alpha)$ is minimised when $\alpha = 1+\beta/2$. Further Taylor Series analysis confirms that $\delta(1+\beta/2,\Delta)/\Delta^2$ has a finite limit as $\Delta \to 0^+$. □

8.7 Collocation Scheme

In this section we investigate the performance of a Collocation Scheme with an Explicit time step. We approximate $x(t_n)$ by x_n, where $x(t_n)$ is the solution x of (8.1) at time t_n. The sequences (x_n), (t_n) and $(h(x_n))$ are defined by

$$x_{n+1} = x_n + h(x_n)f(\theta x_n+(1-\theta)x_{n+1}), \quad n \geq 0, \quad x_0 = \xi > 0, \tag{8.35}$$

where $\theta \in [0,1]$

$$t_{n+1} = \sum_{j=0}^{n} h(x_j), \quad n \geq 0, \quad t_0 = 0,$$

and

$$h(x) = \frac{\Delta(x)x}{f(x)}, \quad x > 0,$$

with $\Delta : (0,\infty) \mapsto (0,\infty)$ continuous and $\Delta(x) \to \Delta \in (0,\infty)$ as $x \to 0^+$.

Lemma 31. *Let* f *be continuous,* $f(x) > 0$ *and* $f \in RV_\infty(\beta)$, $\beta > 1$ *and define for* $x > 0$

$$\tilde{G}_x(\lambda) := \lambda - \Delta(x)\frac{f((1+\theta\lambda)x)}{f(x)}, \quad \lambda \geq 0.$$

Then there is a $\delta > 0$ such that

$$\sup_{x \geq 0} \Delta(x) < \delta(\theta, \beta),$$

implies there is a solution $\lambda = \lambda_0(x) > 0$ of $\tilde{G}_x(\lambda) = 0$ for each $x > 0$.

Proof. Let $\lambda_0 > 0$ be arbitrary. Then

$$\lim_{x \to \infty} \frac{f(x)}{f((1+\theta\lambda_0)x)} = \frac{1}{(1+\theta\lambda_0)^{\beta}}.$$

Therefore, there is $x_0 = x_0(\theta, \lambda_0, \beta)$ such that for $x \geq x_0$ we have

$$\frac{f(x)}{f((1+\theta\lambda_0)x)} > \frac{1}{2(1+\theta\lambda_0)^{\beta}}.$$

Thus

$$\delta_0 := \inf_{x \geq x_0} \frac{\lambda_0 f(x)}{f((1+\theta\lambda_0)x)} \geq \frac{\lambda_0}{2(1+\theta\lambda_0)^{\beta}}.$$

On the other hand, as f is continuous and $f(x) > 0 \, \forall \, x \geq 0$

$$\delta_1 = \inf_{x \in [0, x_0]} \frac{\lambda_0 f(x)}{f((1+\theta\lambda_0)x)} > 0.$$

Clearly δ_0 and δ_1 depend on θ, x_0 and β. Thus with $\delta := \min(\delta_0, \delta_1)$, then

$$\delta = \inf_{x \geq 0} \frac{\lambda_0 f(x)}{f((1+\theta\lambda_0)x)} > 0.$$

Therefore, for all $x \geq 0$

$$\frac{\lambda_0 f(x)}{f((1+\theta\lambda_0)x)} \geq \delta.$$

Then

$$\tilde{G}_x(\lambda_0) = \lambda_0 - \Delta(x)\frac{f((1+\theta\lambda_0)x)}{f(x)} \quad \geq \quad \lambda_0 \left(1 - \frac{\Delta(x)}{\delta}\right).$$

Now suppose $\sup_{x \geq 0} \Delta(x) < \delta$. Then $\tilde{G}_x(\lambda_0) > 0$ for each $x > 0$. For each $x > 0$, $\tilde{G}_x(0) = -\Delta(x) < 0$. Since $\lambda \mapsto \tilde{G}_x(\lambda)$ is continuous by the Intermediate Value Theorem, there is $\lambda \in (0, \lambda_0)$ such that $\tilde{G}_x(\lambda) = 0$ for each $x > 0$. □

Remark 35. For implementation purposes, it is clearly important to have an explicit bound on δ. The following method will generate such a bound, but undoubtedly this bound will not be optimal. However, since we shall generally be interested in taking Δ to be small, a conservative upper bound on δ is not a practical limitation for implementation; rather the problem lies in its *a priori* identification. Incidentally, if we choose $h(x) = \Delta x / f(x)$, then $\Delta(x) = \Delta$ for all x, and we merely need to choose the constant parameter $\Delta < \delta$ in order to guarantee existence of a solution of $\tilde{G}_x(\lambda) = 0$. Pick $\lambda_0 = 1$, say, and noting that θ is given for a particular problem (typically we would choose $\theta = 1/2$ and do so now), we get

$$\delta := \inf_{x \geq 0} \frac{f(x)}{f(\frac{3}{2}x)}.$$

We would use standard calculus and majorisation methods to estimate δ. □

Lemma 32. *If, for each $x > 0$, there is a solution $\lambda > 0$ to $\tilde{G}_x(\lambda) = 0$, where*

$$\tilde{G}_x(\lambda) := \lambda - \Delta(x)\frac{f((1+\theta\lambda)x)}{f(x)},$$

then there is an increasing sequence (x_n) obeying (8.35).

Proof. Let $\tilde{G}_x$ be as given. Define $\bar{G}_x$ by $\bar{G}_x(\lambda + 1) := \tilde{G}_x(\lambda)$ and

$$G_x(y) := y - x - \frac{\Delta(x)x}{f(x)} f((1-\theta)x + \theta y).$$

Then $G_x(\lambda x) = x\bar{G}_x(\lambda)$. Let $n = n_0 \geq 1$ be arbitrary. We prove the claim by induction. Let $x_n > 0$ and $\bar{\lambda}_n$ be a solution of $\tilde{G}_{x_n}(\bar{\lambda}_n)$ which exists by hypothesis. Put $\lambda_n = 1+\bar{\lambda}_n$ and $x_{n+1} = \lambda_n x_n$. Then

$$0 = x_n\tilde{G}_{x_n}(\bar{\lambda}_n) = x_n\tilde{G}_{x_n}(\lambda_n - 1) = x_n\bar{G}_{x_n}(\lambda_n) = x_n G_{x_n}(\lambda_n x_n) = G_{x_n}(x_{n+1}).$$

Hence

$$x_{n+1} = x_n + \frac{\Delta(x_n)x_n}{f(x_n)} f((1-\theta)x_n + \theta x_{n+1}). \tag{8.36}$$

Therefore with $n = n_0$ we may choose $\bar{\lambda}_{n+1} > 0$ to be a solution of $\tilde{G}_{x_{n+1}}(\bar{\lambda}_{n+1}) = 0$ and proceeding in this manner we may construct a sequence obeying (8.36) for all n. Moreover as $\lambda_n > 1$ for all n this sequence is increasing. □

Lemma 33. *Let $f \in RV_\beta, \beta \geq 1$. For $x > 0$, define*

$$\tilde{G}_x(\lambda) := \lambda - \Delta(x)\frac{f((1+\theta\lambda)x)}{f(x)}, \quad \lambda \geq 0.$$

Then $\tilde{G}_x(0) < 0$. If $f(x)/x \to \infty$ as $x \to \infty$, then $\tilde{G}_x(\lambda) \to -\infty$ as $\lambda \to \infty$.

(i) Let $\Delta < \Delta^*(\theta, \beta)$. Then for all x sufficiently large, there is at least one solution $\lambda > 0$ of $\tilde{G}_x(\lambda) = 0$.

(ii) Let $\Delta < \Delta^*(\theta, \beta)$ and x be sufficiently large that there is at least one solution of $\tilde{G}_x(\lambda) = 0$. If

$$\underline{\lambda}(x) = \inf\left\{\lambda > 0 : \tilde{G}_x(\lambda) = 0\right\}, \tag{8.37}$$

then $\lim_{x\to\infty} \underline{\lambda}(x) = \lambda_-(\Delta)$ *where* $\lambda_-(\Delta) > 0$ *is the minimal positive solution of* $\lambda = \Delta(1+\theta\lambda)^\beta$.

Proof. Note

$$\begin{aligned}
&\tilde{G}_x(\lambda)\\
&:= \lambda - \Delta(x)\frac{f((1+\theta\lambda)x)}{f(x)}\\
&= \lambda - \Delta(x)\left(\frac{f((1+\theta\lambda)x)}{f(x)} - (1+\theta\lambda)^\beta + (1+\theta\lambda)^\beta\right)\\
&\qquad + \Delta\left(1+\theta\lambda\right)^\beta - \Delta\left(1+\theta\lambda\right)^\beta\\
&= \tilde{\phi}(\lambda) + (\Delta - \Delta(x))(1+\theta\lambda)^\beta - \Delta(x)\left(\frac{f((1+\theta\lambda)x)}{f(x)} - (1+\theta\lambda)^\beta\right),
\end{aligned}$$

where $\tilde{\phi}(\lambda) := \lambda - \Delta(1+\theta\lambda)^\beta$. Notice that $\tilde{G}_x(0) = -\Delta(x) < 0$. Let $\theta > 0$, $\Delta > 0$ be fixed and suppose $\Delta < \Delta^*(\theta,\beta)$. Then $\tilde{\phi}$ has two zeros at $\lambda_-(\Delta) < \lambda_+(\Delta)$ by Lemma 34 whose statement and proof follow immediately after this lemma. By the regular variation of f for all $\epsilon \in (0,1)$ there exists $x_1(\epsilon) > 0$ such that $x \geq x_1(\epsilon)$ implies

$$\sup_{\lambda\in(0,2\lambda_+(\Delta)]}\left|\frac{f((1+\theta\lambda)x)}{f(x)} - (1+\theta\beta)^\beta\right| < \min\left(\frac{\epsilon}{8}, \frac{\epsilon}{8\Delta}\right).$$

Since $\Delta(x) \to \Delta$ as $x \to \infty$, there is $x_2(\epsilon) > 0$ such that $x \geq x_2(\epsilon)$ implies

$$|\Delta(x) - \Delta| < \min\left(\frac{\epsilon\Delta}{2^\beta\lambda_+(\Delta)8}, \frac{\epsilon}{8}, \frac{\epsilon\Delta}{8}\right).$$

Let $x_3(\epsilon) = \max(x_1(\epsilon), x_2(\epsilon))$. Then for $\lambda \in (0, 2\lambda_+(\Delta)]$ and $x \geq x_3(\epsilon)$,

$$\begin{aligned}
\left|\tilde{G}_x(\lambda) - \tilde{\phi}(\lambda)\right| &= \left|(\Delta - \Delta(x))(1+\theta\lambda)^\beta - \Delta(x)\left(\frac{f((1+\theta\lambda)x)}{f(x)} - (1+\theta\lambda)^\beta\right)\right|\\
&\leq |\Delta - \Delta(x)|\,(1+\theta\lambda)^\beta + |\Delta(x)|\left|\frac{f((1+\theta\lambda)x)}{f(x)} - (1+\theta\lambda)^\beta\right|.
\end{aligned}$$

As $\lambda \leq 2\lambda_+(\Delta)$ then

$$(1+\theta\lambda)^\beta \leq (1+2\theta\lambda_+(\Delta))^\beta < (2+2\theta\lambda_+(\Delta))^\beta = 2^\beta(1+\theta\lambda_+(\Delta))^\beta = \frac{2^\beta\lambda_+(\Delta)}{\Delta},$$

thus

$$|\Delta - \Delta(x)|\,(1+\theta\lambda)^\beta < \min\left(\frac{\epsilon\Delta}{2^\beta\lambda_+(\Delta)8}, \frac{\epsilon}{8}, \frac{\epsilon\Delta}{8}\right)\cdot\frac{2^\beta\lambda_+(\Delta)}{\Delta} \leq \frac{\epsilon}{8}.$$

Similarly

$$\begin{aligned}|\Delta(x)|\left|\frac{f((1+\theta\lambda)x)}{f(x)}-(1+\theta\lambda)^{\beta}\right| &\leq \left(\Delta+\min\left(\frac{\epsilon\Delta}{2^{\beta}\lambda_{+}(\Delta)8},\frac{\epsilon}{8},\frac{\epsilon\Delta}{8}\right)\right)\cdot\min\left(\frac{\epsilon}{8},\frac{\epsilon}{8\Delta}\right)\\ &\leq \Delta\left(1+\frac{\epsilon}{8}\right)\cdot\frac{\epsilon}{8\Delta}<\frac{9\epsilon}{64}.\end{aligned}$$

Hence for $x \geq x_3(\epsilon)$, $\lambda \in (0, 2\lambda_+(\Delta)]$

$$\left|\tilde{G}_x(\lambda)-\tilde{\phi}(\lambda)\right|<\frac{\epsilon}{8}+\frac{9\epsilon}{64}<\frac{3\epsilon}{8}. \tag{8.38}$$

Define next

$$\begin{aligned}\lambda_-^-(\epsilon) &:= \sup\left\{\lambda<\lambda_-(\Delta):\tilde{\phi}(\lambda)=\frac{-3\epsilon}{8}\right\},\\ \lambda_-^+(\epsilon) &:= \inf\left\{\lambda\in(\lambda_-(\Delta),\lambda_+(\Delta)):\tilde{\phi}(\lambda)=\frac{3\epsilon}{8}\right\},\\ \lambda_+^-(\epsilon) &:= \sup\left\{\lambda\in(\lambda_-(\Delta),\lambda_+(\Delta)):\tilde{\phi}(\lambda)=\frac{3\epsilon}{8}\right\},\\ \lambda_+^+(\epsilon) &:= \inf\left\{\lambda>\lambda_+(\Delta):\tilde{\phi}(\lambda)=\frac{-3\epsilon}{8}\right\}.\end{aligned}$$

Clearly $\lambda_-^-(\epsilon), \lambda_+^-(\epsilon) \to \lambda_-(\Delta)$ as $\epsilon \to 0^+$ and $\lambda_+^-(\epsilon), \lambda_+^+(\epsilon) \to \lambda_+(\Delta)$ as $\epsilon \to 0^+$. If $\lambda \in (\lambda_-^-(\epsilon), \lambda_-^+(\epsilon))$ or $\lambda \in (\lambda_+^-(\epsilon), \lambda_+^+(\epsilon))$, then $|\tilde{\phi}(\lambda)| < 3\epsilon/8$, while otherwise $|\tilde{\phi}(\lambda)| \geq 3\epsilon/8$. Therefore for $\epsilon < \epsilon'$ sufficiently small $0 < \lambda_-^-(\epsilon) < \lambda_-^+(\epsilon) < 2\lambda_+(\Delta)$. Now for $x \geq x_3(\epsilon)$,

$$\left|G_x(\lambda_-^-(\epsilon))-\tilde{\phi}(\lambda_-^-(\epsilon))\right|<\frac{3\epsilon}{8}.$$

Thus

$$\tilde{G}_x(\lambda_-^-(\epsilon))-\tilde{\phi}(\lambda_-^-(\epsilon))<\frac{3\epsilon}{8},$$

so

$$\tilde{G}_x(\lambda_-^-(\epsilon))<\tilde{\phi}(\lambda_-^-(\epsilon))+\frac{3\epsilon}{8}=-\frac{3\epsilon}{8}+\frac{3\epsilon}{8}=0.$$

Also for $x \geq x_3(\epsilon)$

$$\tilde{G}_x(\lambda_-^+(\epsilon))-\tilde{\phi}(\lambda_-^+(\epsilon))>\frac{-3\epsilon}{8}.$$

Thus

$$\tilde{G}_x(\lambda_-^+(\epsilon))>\tilde{\phi}(\lambda_-^+(\epsilon))-\frac{3\epsilon}{8}=\frac{3\epsilon}{8}-\frac{3\epsilon}{8}=0.$$

Therefore for $x \geq x_3(\epsilon)$, $\tilde{G}_x(\lambda_-^-(\epsilon)) < 0 < \tilde{G}_x(\lambda_-^+(\epsilon))$. Hence there is a $\lambda(x) \in (\lambda_-^-(\epsilon), \lambda_-^+(\epsilon))$ such that $\tilde{G}_x(\lambda(x)) = 0$. On the other hand suppose there exists $\lambda \in (0, 2\lambda_+(\Delta)]$ such that $\tilde{G}_x(\lambda) = 0$ where $x > x_3(\epsilon)$. Then $|\tilde{\phi}(\lambda)| < 3\epsilon/8$ by (8.38). Hence $\lambda \in (\lambda_-^-(\epsilon), \lambda_-^+(\epsilon))$ or $\lambda \in (\lambda_+^-(\epsilon), \lambda_+^+(\epsilon))$.

Let $x > x_3(\epsilon)$ and define $\underline{\lambda}(x) := \inf\left\{\lambda > 0 : \tilde{G}_x(\lambda) = 0\right\}$. If $\underline{\lambda}(x) \in (0, 2\lambda_+(\Delta)]$, then $|\tilde{\phi}(\underline{\lambda}(x))| < 3\epsilon/8$, so $\underline{\lambda}(x) \in (\lambda_-^-(\epsilon), \lambda_-^+(\epsilon))$ or $\underline{\lambda}(x) \in (\lambda_+^-(\epsilon), \lambda_+^+(\epsilon))$. But the minimality of $\underline{\lambda}(x)$ precludes the second possibility. Thus $x > x_3(\epsilon)$, and $\underline{\lambda}(x) \in [0, 2\lambda_+(\Delta)]$ implies $\lambda_-^-(\epsilon) < \underline{\lambda}(x) < \lambda_-^+(\epsilon)$. On the other hand, if $\underline{\lambda}(x) > 2\lambda_+(\Delta)$ there is a contradiction because $\underline{\lambda}(x) < \lambda_-^+(\epsilon) < 2\lambda_+(\Delta)$ by part (i). Finally, since $\tilde{G}_x(0) = -\Delta(x) < 0$, then $\underline{\lambda}(x) > 0$. Thus $\forall\,\epsilon \in (0,1)$ sufficiently small, there is $x_3(\epsilon) > 0$ such that for $x \geq x_3(\epsilon)$, $\lambda_-^-(\epsilon) < \underline{\lambda}(x) < \lambda_-^+(\epsilon)$. Therefore, it follows that

$$\lambda_-^-(\epsilon) \leq \liminf_{x\to\infty} \underline{\lambda}(x) \leq \limsup_{x\to\infty} \underline{\lambda}(x) \leq \lambda_-^+(\epsilon).$$

Let $\epsilon \to 0^+$, then $\lambda_-^-(\epsilon) \to \lambda_-(\Delta)$ as $\lambda_-^+(\epsilon) \to \lambda_-(\Delta)$. Therefore

$$\lim_{x\to\infty} \underline{\lambda}(x) = \lambda_-(\Delta),$$

as required. □

We prove the postponed result on the zeros of $\tilde{\phi}$ now.

Lemma 34. *Let*

$$\tilde{\phi}(\lambda) = \lambda - \Delta(1+\theta\lambda)^\beta.$$

If $\Delta\theta\beta < \left(\frac{\beta-1}{\beta}\right)^{\beta-1} < 1$, *then there are two positive solutions of* $\tilde{\phi}(\lambda) = 0$.

Proof. Define $\tilde{\phi}(\lambda) := \lambda - \Delta(1+\theta\lambda)^\beta$. Then $\tilde{\phi}(0) = -\Delta < 0$ and $\tilde{\phi}'(\lambda) = 1 - \Delta\theta\beta(1+\theta\lambda)^{\beta-1}$ with $\tilde{\phi}'(0) = 1 - \Delta\theta\beta > 0$. Note that $\tilde{\phi}'(\lambda_*) = 0$ where

$$\lambda_* = \frac{1}{\theta}\left(\left(\frac{1}{\Delta\beta\theta}\right)^{1/(\beta-1)} - 1\right) > 0,$$

if $\Delta\beta\theta < 1$. This is a local maximum of $\tilde{\phi}$ since for all $\lambda > 0$, $\tilde{\phi}''(\lambda) = -\Delta\theta^2\beta(\beta-1)(1+\theta\lambda)^{\beta-2} < 0$. Note that

$$\tilde{\phi}(\lambda_*) \;=\; \lambda_* - \Delta(1+\theta\lambda_*)^{\beta-1}(1+\theta\lambda_*) = \lambda_* - \frac{\Delta(1+\theta\lambda_*)}{\Delta\beta\theta} = \lambda_*\left(1 - \frac{1}{\beta}\right) - \frac{1}{\beta\theta} > 0,$$

since $\Delta\theta\beta < \left(\frac{\beta-1}{\beta}\right)^{\beta-1} < 1$. Hence there is $\lambda_-(\Delta) \in (0, \lambda_*)$ such that $\tilde{\phi}(\lambda_-(\Delta)) = 0$ and $\lambda_+(\Delta) > \lambda_*$ such that $\tilde{\phi}(\lambda_+(\Delta)) = 0$ where $\lambda_+(\Delta) > \lambda_-(\Delta) > 0$. □

Theorem 54. *Let f is continuous and $f \in RV_\infty(\beta)$, $\beta > 1$ and suppose*

$$\sup_{x\geq 0} \Delta(x) < \delta(\theta, \beta).$$

Let $\underline{\lambda}$ *be defined by* (8.37) *and define the sequence* (x_n) *by*

$$x_{n+1} := (1+\underline{\lambda}(x_n))\, x_n, \quad n \geq 0, \quad x_0 = \xi.$$

Then (x_n) *is a positive increasing sequence which satisfies* (8.35) *and*

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1 + \lambda_-(\Delta),$$

where $\lambda_-(\Delta)$ *is the smallest root of* $\tilde{\phi}(\lambda) = 0$ *where* $\tilde{\phi}(\lambda) = \lambda - \Delta(1+\theta\lambda)^\beta$ *and*

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{T_h - t_n} = \frac{1}{\Delta}\int_1^{1+\lambda_-(\Delta)} v^{-\beta}\, dv.$$

Proof. By Lemma 31 there is a solution $\lambda(x)$ of $\tilde{G}_x(\lambda) = 0$ for all $x > 0$. Then by Lemma 32 there is an increasing sequence (x_n) such that (8.35) holds. In particular given an x_n, we find a $\lambda_n = \underline{\lambda}(x_n)$ and define $x_{n+1} = (1+\underline{\lambda}(x_n))x_n$. Then x_n and x_{n+1} obey (8.35) and (x_n) is increasing with $x_n \to \infty$ as $n \to \infty$. By Lemma 33

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} 1 + \underline{\lambda}(x_n) = 1 + \lambda_-(\Delta).$$

Since $f \in RV_\infty(\beta)$ for $\beta > 1$, then

$$\begin{aligned}\bar{F}(x_n) = \int_{x_n}^\infty \frac{1}{f(u)}\, du = \sum_{j=n}^\infty \int_{x_j}^{x_{j+1}} \frac{1}{f(u)}\, du &= \sum_{j=n}^\infty \frac{\Delta x_j}{f(x_j)} \cdot \frac{f(x_j)}{\Delta x_j} \int_{x_j}^{x_{j+1}} \frac{1}{f(u)}\, du \\ &= \sum_{j=n}^\infty \frac{\Delta x_j}{f(x_j)} \cdot \frac{1}{\Delta} \int_1^{x_{j+1}/x_j} \frac{\tilde{f}(vx_j)}{\tilde{f}(x_j)}\, dv,\end{aligned}$$

where $\tilde{f} = 1/f \in RV_\infty(-\beta)$. Note that

$$\lim_{j\to\infty} \frac{1}{\Delta}\int_{x_j}^{x_{j+1}} \frac{\tilde{f}(u)}{\tilde{f}(x_j)}\, du = \frac{1}{\Delta}\int_1^{1+\lambda_-(\Delta)} v^{-\beta}\, dv,$$

by the uniform convergence theorem for $RV_\infty(\beta)$ functions and the fact that $x_{j+1}/x_j \to 1+\lambda_-(\Delta)$ as $j \to \infty$. Hence

$$\bar{F}(x_n) \sim \frac{1}{\Delta}\int_1^{1+\lambda_-(\Delta)} v^{-\beta}\, dv \cdot (T_h - t_n), \quad \text{as } n \to \infty.$$

Therefore

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \frac{1}{\Delta}\int_1^{1+\lambda_-(\Delta)} v^{-\beta}\, dv =: \lambda(\theta, \Delta),$$

as required. ∎

Theorem 55. *Let* $\lambda_- := \lambda_-(\Delta)$ *be the smallest root of* $\tilde{\phi}(\lambda) = 0$ *where* $\tilde{\phi}(\lambda) = \lambda -$

$\Delta(1+\theta\lambda)^{\beta}$. *Then the asymptotic convergence rate, $\lambda(\theta,\Delta)$, estimated by the Collocation scheme according to*

$$\lim_{n\to\infty}\frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \frac{1}{\Delta}\int_1^{1+\lambda_-(\Delta)} v^{-\beta}\,dv =: \lambda(\theta,\Delta),$$

obeys as $\Delta \to 0^+$

$$\lambda(\theta,\Delta) = 1+\beta\left(\theta-\frac{1}{2}\right)\Delta + \left(\left(\beta^2+\frac{\beta(\beta-1)}{2}\right)\theta^2 - \beta^2\theta + \frac{\beta(\beta+1)}{6}\right)\Delta^2 + O(\Delta^3).$$

Moreover,

$$\lambda(1/2,\Delta) = 1+\frac{\beta(\beta+1)}{24}\Delta^2 + O(\Delta^3), \quad \text{as } \Delta \to 0^+.$$

Proof. Computing $\lambda(\theta,\Delta)$ gives

$$\lambda(\theta,\Delta) = \frac{1-(1+\lambda_-)^{1-\beta}}{\Delta(\beta-1)}.$$

We see that $\lambda_-(\Delta)\to 0$ as $\Delta\to 0^+$. The Taylor Series expansion of

$$a_1(x) := \frac{1-(1+x)^{1-\beta}}{\beta-1},$$

as $x\to 0^+$ is

$$\frac{1-(1+x)^{1-\beta}}{\beta-1} = x - \frac{\beta}{2}x^2 + \frac{\beta(\beta+1)}{6}x^3 + O(x^4).$$

Define $\lambda_2 := \lambda_-/\Delta$. Then as $\lambda_- \sim \Delta$ as $\Delta\to 0^+$, then $\lambda_2 \to 1$ as $\Delta \to 0^+$ and

$$\lambda(\theta,\Delta) = \lambda_2(\Delta) - \frac{\beta}{2}\lambda_2^2\Delta + \frac{\beta(\beta+1)}{6}\lambda_2^3\Delta^2 + O(\Delta^3).$$

The Taylor Series expansion of $a_2(x) = (1+x)^\beta$ as $x\to 0^+$ is

$$(1+x)^\beta = 1+\beta x + \frac{\beta(\beta-1)}{2}x^2 + O(x^3).$$

Therefore as $\lambda_- = \Delta\lambda_2$, then

$$\lambda_2 \;=\; (1+\theta\Delta\lambda_2)^\beta = 1+\beta\Delta\theta\lambda_2 + \frac{\beta(\beta-1)}{2}\Delta^2\theta^2\lambda_2^2 + O(\Delta^3).$$

As $\Delta\to 0^+$, then

$$\lambda_2 = 1+\beta\theta\Delta + \left(\beta^2+\frac{\beta(\beta-1)}{2}\right)\theta^2\Delta^2 + O(\Delta^3).$$

Since

$$\lambda_2^2 = 1+2\beta\theta\Delta + 2\left(\beta^2+\frac{\beta(\beta-1)}{2}\right)\theta^2\Delta^2 + \beta^2\theta^2\Delta^2 + O(\Delta^3),$$

and $\lambda_2^3 = 1 + 3\beta\theta\Delta + O(\Delta^2)$, then

$$\lambda(\theta,\Delta) = 1 + \beta\theta\Delta - \frac{\beta}{2}\Delta + \left(\left(\beta^2 + \frac{\beta(\beta-1)}{2}\right)\alpha^2 - \frac{\beta}{2}2\beta\alpha\right)\Delta^2 + \frac{\beta(\beta+1)}{6}\Delta^2 + O(\Delta^3).$$

Hence

$$\lambda(\theta,\Delta) = 1 + \beta\left(\theta - \frac{1}{2}\right)\Delta + \left(\left(\beta^2 + \frac{\beta(\beta-1)}{2}\right)\theta^2 - \beta^2\theta + \frac{\beta(\beta+1)}{6}\right)\Delta^2 + O(\Delta^3).$$

Taking $\theta = 1/2$ eliminates the $O(\Delta)$ term in which case

$$\begin{aligned}\lambda^*(\Delta) := \lambda(1/2,\Delta) &= 1 + \left(\frac{1}{4}\left(\beta^2 + \frac{\beta(\beta-1)}{2}\right)\right)\Delta^2 + O(\Delta^3)\\ &= 1 + \frac{\beta(\beta+1)}{24}\Delta^2 + O(\Delta^3).\end{aligned}$$

□

8.8 Connection Between Finite-Time Explosion and Finite-Time Stability

In this thesis we have devoted the bulk of our efforts to stability problems, but we believe that this also gives insight into the related problem of finite-time blow-up.

One rather natural question arises: given that we have highly reliable methods for examining the asymptotic behaviour of finite-time stability or super-exponential stability, it would be highly convenient if we could, by means of co-ordinate transforms, translate our results from the finite-time stability case to the blow-up case.

The first question is: can any finite-time explosion problem be mapped onto a finite-time stability problem which has an equilibrium at zero (or in other words for which there is a soft landing)? Roughly speaking, the answer to this question is "yes". The second question, which is of practical importance to us, is: can we recast the blow-up problem as a finite-time stability problem, recover information about the blow-up by simulating the stability problem directly and compute the corresponding value of the explosive solution? The answer to this question is "it depends", but there are significant practical challenges in many cases. This likely means that we cannot always circumvent the need to simulate directly the blow-up problem by considering an auxiliary finite-time stability problem.

We formalise this discussion in the theorem below.

Theorem 56. *Suppose $f : (0,\infty) \to (0,\infty)$ is continuous with $\int_1^\infty 1/f(u)\,du < \infty$ and* *let x be the solution of* (8.1) *viz.,*

$$x'(t) = f(x(t)), \quad t \in [0, T), \quad x(0) = \xi > 0.$$

Then x blows-up in the finite-time at

$$T = \int_{\xi}^{\infty} \frac{1}{f(u)}\, du < \infty,$$

i.e. $\lim_{t \to T^-} x(t) = \infty$. *Moreover, suppose that $\epsilon : (0, \infty) \to (0, \infty)$ is continuous with $\epsilon(x) \to 0$ as $x \to \infty$. Then the function Φ given by*

$$\Phi(x) = \int_{x}^{\infty} \frac{\epsilon(u)}{f(u)}\, du, \quad x > 0, \tag{8.39}$$

is well-defined and decreasing. Moreover, if $\eta : [0, \infty) \to [0, \infty)$ is defined by

$$\eta(x) = \begin{cases} \epsilon(\Phi^{-1}(x)), & x > 0, \\ 0, & x = 0, \end{cases} \tag{8.40}$$

then η is continuous on $[0, \infty)$ and the solution of the ODE

$$y'(t) = -\eta(y(t)), \quad t > 0, \quad y(0) = \zeta := \Phi(\xi), \tag{8.41}$$

is such that $y(t) = \Phi(x(t))$, $\forall\, t \in [0, T)$ and

$$\lim_{t \to T^-} y(t) = 0.$$

Proof. Since $\epsilon(x) > 0$ for all $x > 0$ and $\epsilon(x) \to 0$ as $x \to \infty$, it follows from the fact that $\int_1^\infty 1/f(u)\, du < \infty$ that $\int_1^\infty \epsilon(u)/f(u)\, du < \infty$ for every $x > 0$ and therefore that Φ is well-defined. Since ϵ is continuous, Φ is in C^1, positive and obeys

$$\Phi'(x) = \frac{-\epsilon(x)}{f(x)},$$

so clearly Φ is decreasing and hence invertible. Moreover $\Phi(x) \to 0$ as $x \to \infty$ so $\Phi^{-1}(x) \to \infty$ as $x \to 0^+$. Therefore the function η in (8.40) is continuous on $[0, \infty)$ because

$$\lim_{x \to 0^+} \eta(x) = \lim_{x \to 0^+} \epsilon(\Phi^{-1}(x)) = 0 = \eta(0).$$

Moreover $\eta(x) > 0$ for all $x > 0$. Then for any $a > 0$

$$\int_{0^+}^{a} \frac{1}{\eta(u)}\, du = \int_{0^+}^{a} \frac{1}{\epsilon(\Phi^{-1}(u))}\, du = \int_{\infty}^{\Phi^{-1}(a)} \frac{1}{\epsilon(v)} \cdot \frac{-\epsilon(v)}{f(v)}\, du = \int_{\Phi^{-1}(a)}^{\infty} \frac{1}{f(v)}\, dv.$$

Therefore the solution of (8.41) tends to zero in the finite-time T' given by

$$T' = \int_{0+}^{\xi} \frac{1}{\eta(u)}\,du = \int_{\Phi^{-1}(\zeta)}^{\infty} \frac{1}{f(u)}\,du = \int_{\xi}^{\infty} \frac{1}{f(u)}\,du = T.$$

Finally, we check that $y(t) = \Phi(x(t))$, $t \in [0, T)$. Since

$$\int_{y(0)}^{y(t)} \frac{1}{\eta(u)}\,du = \int_0^t \frac{y'(s)}{\eta(y(s))}\,ds = -t, \quad t \in [0, T),$$

so

$$\int_{0+}^{y(t)} \frac{1}{\eta(u)}\,du - \int_{0+}^{y(0)} \frac{1}{\eta(u)} = -t.$$

Thus with $\bar{F}(x) = \int_x^\infty 1/f(u)\,du$, we have

$$\int_{\Phi^{-1}(y(t))}^{\infty} \frac{1}{f(v)}\,dv - \int_{\xi}^{\infty} \frac{1}{f(v)}\,dv = -t,$$

or $\bar{F}(\Phi^{-1}(y(t))) = T - t$, $t \in [0, T)$. But $\bar{F}(x(t)) = T - t$ so $x(t) = \Phi^{-1}(y(t))$ for $t \in [0, T)$ and therefore $y(t) = \Phi(x(t))$, $t \in [0, T)$. □

Remark 36. The substance of the theorem is evident: given an arbitrary scalar blow-up problem, it is possible, by means of a co-ordinate transformation to reformulate the problem as a finite-time stability problem. The theorem also shows what would be needed in order to implement the numerical scheme for the blow-up problem indirectly. One should find a function ϵ which is continuous, positive and vanishing at infinity such that the function Φ, defined by (8.39), has an inverse, Φ^{-1}, which is computable in closed-form. The constraint that Φ^{-1} be computable in closed-form is potentially formidable but the wide range of feasible choices of ϵ makes this a practical proposition in many cases. This makes the function η in (8.40) computable in closed-form. In this case one is free to implement an adaptive time-stepping algorithm to the solution y of the ODE (8.41), and if, at the time point t_n, we have the approximation y_n for $y(t_n)$, the corresponding approximation for $x(t_n)$ is given by $x_n = \Phi^{-1}(y_n)$, which of course can be found if Φ^{-1} is computable in closed-form. We also note that the explosion time of x and the finite stability time of the ODE (8.41) are identical. □

Remark 37. If $x \mapsto \epsilon(x)$ is decreasing, then

$$\lim_{x \to 0^+} \frac{\eta(x)}{x} = \infty.$$

Proof. We have for $u > x$, $\epsilon(u) < \epsilon(x)$. Thus

$$\Phi(x) = \int_x^\infty \frac{\epsilon(u)}{f(u)}\,du \leq \epsilon(x) \int_x^\infty \frac{1}{f(u)}\,du.$$

Hence as $\bar{F}(x) \to 0$ as $x \to \infty$, we get

$$\liminf_{x\to 0^+} \frac{\eta(x)}{x} = \liminf_{x\to 0^+} \frac{\epsilon(\Phi^{-1}(x))}{x} = \liminf_{y\to\infty} \frac{\epsilon(y)}{\Phi(y)} = \liminf_{y\to\infty} \frac{\epsilon(y)}{\epsilon(y)\bar{F}(y)} = \infty,$$

as claimed □

Remark 38. We note also if ϵ is decreasing, that η is increasing, which is a desirable property. To check that $x \mapsto \eta(x)/x$ is decreasing (a condition under which the finite-time stability numerical algorithms perform well) can be carried out for specific f and ϵ. □

Remark 39. The calculations give some insight as to how step-sizes might be chosen for the explosion problem. We have seen that in the case when f is in the appropriate subclass of rapidly varying functions, that a step-size of the order $O(1/f'(x))$ as $x \to \infty$ is optimal. We can also see how this might be achieved by discretising the finite-time stability problem and transforming co-ordinates.

In the case of finite-time stability, we have seen that a step-size $\tilde{h}(y)$ at state y obeying $\tilde{h}(y) \sim \Delta y/\eta(y)$ as $y \to 0^+$ is optimal. This suggests that the corresponding step-size for the explosive equation, when the state in the explosive equation is $x = \Phi^{-1}(y)$, should be

$$h(x) = \tilde{h}(\Phi(x)) \sim \frac{\Delta \int_x^\infty \epsilon(u)/f(u)\,du}{\epsilon(x)}, \quad \text{as } x \to \infty.$$

In the case where ϵ is smoothly regularly varying but $f'' \in \Gamma$, in which case $(ff'')(x)/(f'(x))^2 \to 1$ and $xf'(x)/f(x) \to \infty$, we have that

$$\int_x^\infty \frac{\epsilon(u)}{f(u)}\,du \sim \frac{1}{\frac{d}{dx}\left(f(x)/\epsilon(x)\right)} = \frac{\epsilon^2(x)}{\epsilon(x)f'(x) - f(x)\epsilon'(x)}, \quad \text{as } x \to \infty,$$

so

$$\begin{aligned} h(x) \sim \frac{\Delta}{f'(x) - f(x)/x \cdot x\epsilon'(x)/\epsilon(x)} &= \frac{\Delta}{f'(x)\left(1 - f(x)/xf'(x) \cdot x\epsilon'(x)/\epsilon(x)\right)} \\ &\sim \frac{\Delta}{f'(x)}, \quad \text{as } x \to \infty, \end{aligned}$$

because $xf'(x)/f(x) \to \infty$ as $x \to \infty$ and $x\epsilon'(x)/\epsilon(x)$ tends to a finite limit. □

Example 57. Clearly we need to perform numerical simulations on the explosive ODE (8.1) only when we are unable to compute the solution exactly in closed-form. To show a case in which a closed-form solution cannot be found but in which transformation to a finite-time stability problem is practicable, consider the ODE in which

$$f(x) = x^2(1 + \tfrac{1}{2}\sin x), \quad x \geq 0.$$

Now choose

$$\epsilon(x) = (1 + \tfrac{1}{2}\sin x)\tfrac{1}{x}.$$

Clearly ϵ is positive, continuous and vanishes at infinity. Moreover, it has been chosen to compensate for the analytically awkward sinusoidal term in f. With this choice of ϵ we get

$$\Phi(x) = \int_x^\infty \frac{\epsilon(u)}{f(u)}\,du = \int_x^\infty \frac{1}{u^3}\,du = \frac{1}{2x^2},$$

and $\Phi^{-1}(x) = 1/\sqrt{2x}$. Thus in this case $\Phi^{-1}(x)$ can be found in closed-form. The finite-time stability problem that results is

$$\eta(y) = \eta(\Phi^{-1}(y)) = \sqrt{2y}\left(1 + \tfrac{1}{2}\sin\left(\tfrac{1}{\sqrt{2y}}\right)\right).$$

Here η is rapidly oscillating between $y \mapsto \sqrt{2y}$ and $y \mapsto 3/2\cdot\sqrt{2y}$ as $y \to 0^+$, suggesting a step-size of the form $h(y) = \Delta\sqrt{y}$. □

Chapter 9

Asymptotic Behaviour of Super-Linear SDEs

9.1 Introduction

In this chapter, we determine conditions under which solutions of the SDE (1.17) viz.,

$$dX(t) = f(X(t))\,dt + g(X(t))\,dB(t),$$

tend to zero super-exponentially fast or to zero in finite-time. Very roughly, if we assume the scale function p of X obeys $p(\infty^-) = \infty$ and the limit (1.25) viz.,

$$\lim_{x\to 0^+} \frac{xf(x)}{g^2(x)} =: L,$$

exists with $L \in [-\infty, 1/2)$, the solution will tend to zero almost surely in the case when f is such that the functions $x \mapsto |f(x)|$, $x \mapsto x/|f(x)|$ are asymptotically increasing at zero and $L = -\infty$. The question as to whether X tends to zero in a finite time, generally denoted by T, or not hinges on the finiteness of $\int_{0^+}^1 1/|f(u)|\,du$; if this integral is infinite the solution remains positive for all time and $X(t)$ tends to zero as $t \to \infty$ almost surely. On the other hand if the integral is finite $X(t) > 0, \forall\, t \in [0,T)$, $\lim_{t\to T^-} X(t) = 0$ and T is a.s. finite. This constitutes the a.s. finite-time stability of the equilibrium $x = 0$. Similarly, if $L \in (-\infty, 1/2)$ then $X(t)$ tends to zero in finite-time, or not, with probability one according to whether the integral $\int_{0^+}^1 u/g^2(u)\,du$ is finite or not.

The speed of convergence to zero, in the case when $X(t)$ tends to zero as $t \to \infty$, can also be found. When $L = -\infty$ we prove that

$$\lim_{t\to\infty} \frac{F(X(t))}{t} = 1, \quad \text{a.s.},$$

and when $L \in (-\infty, 1/2)$ we have

$$\lim_{t\to\infty} \frac{G(X(t))}{t} = \frac{1}{2} - L, \quad \text{a.s.},$$

where F and G are defined by (1.29) and (1.34). In the case when there is an almost surely finite $T > 0$ such that $X(t) > 0$, $\forall\, t \in [0, T)$ and $X(t) \to 0$ as $t \to T^-$ we can prove analogous asymptotic results. For example, when $L \in (-\infty, 1/2)$ and we define $\bar{G}$ by (1.33), we find that

$$\lim_{\lambda\to 1^+} \liminf_{x\to 0^+} \frac{(-\log\circ\bar{G}^{-1})(\lambda x)}{(-\log\circ\bar{G}^{-1})(x)} = \infty,$$

implies

$$\lim_{t\to T^-} \frac{\bar{G}(X(t))}{T-t} = \frac{1}{2} - L, \quad \text{a.s.},$$

and

$$\lim_{\lambda\to 1^-} \limsup_{x\to 0^+} \frac{(-\log\circ\bar{G}^{-1})(\lambda x)}{(-\log\circ\bar{G}^{-1})(x)} = 1,$$

implies

$$\lim_{t\to T^-} \frac{-\log X(t)}{(-\log\circ\bar{G}^{-1})((\frac{1}{2}-L)(T-t))} = 1, \quad \text{a.s.}.$$

Analogous results are available in the case when $L = -\infty$. We also consider examples under which these asymptotic conditions on $(-\log\circ\bar{G}^{-1})$ hold and determine sufficient conditions on g which imply these technical conditions and are easier to check.

We introduce the following notation and assumptions throughout our analysis:

$$P = \{\omega \in \Omega \,:\, X(t,\omega) > 0 \text{ for all } t \geq 0\}; \text{and} \tag{9.1}$$

$$A = \{\omega \in \Omega \,:\, \lim_{t\to\infty} X(t,\omega) = 0\}. \tag{9.2}$$

We will consider the situation where $\mathbb{P}[P] = 1$ or $\mathbb{P}[P] = 0$. If the former is true then $\mathbb{P}[A] = 1$. In this case the solution remains positive for all time and converges to the zero equilibrium solution asymptotically on the a.s. event A because of (9.1) and (9.2). The assumption and probabilities of the events in (9.1) and (9.2) may be dealt with by the functions p and v defined by

$$p(x) := \int_{x^*}^{x} \exp\left\{\int_{x^*}^{y} \frac{-2f(z)}{g^2(z)}\,dz\right\} dy, \quad x > 0, \tag{9.3}$$

$$v(x) := \int_{x^*}^{x} p'(y) \int_{x^*}^{y} \frac{2}{p'(z)g^2(z)}\,dz\,dy, \quad x > 0, \tag{9.4}$$

where $x^* > 0$. The function p is referred to as a "scale function" and describes whether the process is attracted to a boundary or not. A boundary is a.s. attracting if the scale function is finite when evaluated at it, but infinite when evaluated at the other boundary. The function v decides whether the process is attracted to a boundary in finite-time or not. It acts as a stochastic analogue of the function F for the ODE (1.1). The finiteness of v can therefore be thought of as a type of stochastic Osgood condition. The finiteness of p and v (as $x \to 0^+$ and $x \to \infty$) give necessary and sufficient conditions to determine the probabilities of the events A, P and $\{T < \infty\}$ if we impose the assumptions on f and g discussed earlier. The value of the lower limit x^* in the definitions of p and v is arbitrary. Indeed if one replaces x^* by any other $x' > 0$, say, then the finiteness of p and v at the boundaries zero and infinity will be the same as for the functions with the original lower limit x^*. For this reason in proofs we choose values of x^* which are convenient for calculations.

In the case that $\mathbb{P}[P] = \mathbb{P}[A] = 1$, we say that the zero solution of (1.17) is a.s. super-exponentially stable if

$$\lim_{t\to\infty} \frac{\log |X(t,\omega)|}{t} = -\infty,$$

and $X(t)$ is a unique, non-trivial, continuous adapted solution of (1.17).

9.2 Global Positivity and Finite-Time Stability

We now look at the asymptotics of the continuous SDE. The first question to resolve is whether $X(t)$ hits zero in finite-time or not. This is resolved by the following result (see Karatzas and Shreve [34], Proposition 5.5.22).

Theorem 58. *Suppose $X(t)$ is the solution of* (1.17)*. Let T and p be defined by* (1.22) *and* (9.3)*.*

(i) If $p(0^+) = -\infty$ and $p(\infty^-) = \infty$, then $T = \infty$ and X is recurrent on $(0, \infty)$ a.s..

(ii) If $p(0^+) > -\infty$ and $p(\infty^-) = \infty$, then

$$\lim_{t\to T^-} X(t) = 0 \quad \textit{and} \quad \sup_{0\le t<T} X(t) < \infty, \qquad \textit{a.s.}.$$

(iii) If $p(0^+) = -\infty$ and $p(\infty^-) < \infty$, then

$$\lim_{t\to T^-} X(t) = \infty \quad \textit{and} \quad \inf_{0\le t<T} X(t) > 0, \qquad \textit{a.s.}.$$

(iv) If $p(0^+) > -\infty$ and $p(\infty^-) < \infty$, then

$$\{\omega : \lim_{t\to T^-} X(t,\omega) = \infty\} \cup \{\omega : \lim_{t\to T^-} X(t,\omega) = 0\},$$

is an a.s. event, with each event in the union having positive probability.

We focus on when zero and infinity are a.s. attracting, cases (ii) and (iii) respectively. We determine sufficient conditions below for when this is the case. Case (i) does not apply to stability problems.

Lemma 35. *Suppose $X(t)$ is the solution of* (1.17)*. Let T and p be defined by* (1.22) *and* (9.3)*. If*

$$\sup_{x>0} \frac{xf(x)}{g^2(x)} < \frac{1}{2}, \tag{9.5}$$

then $p(0^+) > -\infty$ and $p(\infty^-) = \infty$. Furthermore,

$$\lim_{t\to T^-} X(t) = 0 \quad \textit{and} \quad \sup_{0\le t<T} X(t) < \infty, \quad \textit{a.s.}.$$

Proof. From (9.5)

$$\frac{xf(x)}{g^2(x)} \le \Lambda < \frac{1}{2}, \quad \forall\, x.$$

Thus

$$\frac{2f(x)}{g^2(x)} \le \frac{2\Lambda}{x} =: \frac{\Lambda'}{x}, \quad \forall\, x,$$

where $\Lambda' < 1$. We consider first when $x > 0$ is small. For $x < y < u < c$ we may estimate as follows

$$\int_x^c \exp\left\{\int_y^c \frac{2f(u)}{g^2(u)}\,du\right\} dy \le \int_x^c \exp\left\{\int_y^c \frac{\Lambda'}{u}\,du\right\} dy \;=\; \frac{c^{\Lambda'}\left(c^{1-\Lambda'} - x^{1-\Lambda'}\right)}{1-\Lambda'} < \frac{c}{1-\Lambda'}.$$

Hence

$$p(x) = \int_c^x \exp\left\{\int_c^y \frac{-2f(u)}{g^2(u)}\,du\right\} dy = -\int_x^c \exp\left\{\int_y^c \frac{2f(u)}{g^2(u)}\,du\right\} dy > \frac{-c}{1-\Lambda'}.$$

and $p(0^+) > -\infty$. By the argument above for $x > c$, $-2f(x)/g^2(x) > -\Lambda'/x$. Thus

$$\begin{aligned} p(x) := \int_c^x \exp\left\{\int_c^y \frac{-2f(u)}{g^2(u)}\,du\right\} dy \;&\ge\; \int_c^x \exp\left\{\int_c^y \frac{-\Lambda'}{u}\,du\right\} dy \\ &=\; \int_c^x \exp\left\{\log\left(\frac{y}{c}\right)^{-\Lambda'}\right\} dy \\ &=\; \frac{1}{c^{-\Lambda'}}\int_x^x y^{-\Lambda'}\,dy = \frac{c^{\Lambda'}\left(x^{1-\Lambda'} - c^{1-\Lambda'}\right)}{1-\Lambda'}. \end{aligned}$$

As $\Lambda' < 1$, we have $p(x) \to \infty$ as $x \to \infty$. Thus $\sup_{x>0} xf(x)/g^2(x) < 1/2$ implies $p(0^+) > -\infty$ and $p(\infty^-) = \infty$. By Feller's test (Proposition 5.5.22 part (b) in [34])

$$\lim_{t\to T^-} X(t) = 0 \quad \text{and} \quad \mathbb{P}\left[\sup_{0\le t<T} X(t) < \infty\right] = 1,$$

as claimed. ◻

The next result shows that (9.5) cannot be relaxed too much.

Lemma 36. *Suppose $X(t)$ is the solution of* (1.17)*. Let T and p be defined by* (1.22) *and* (9.3)*. If*

$$\inf_{x>0} \frac{xf(x)}{g^2(x)} > \frac{1}{2}, \tag{9.6}$$

then $p(0^+) = -\infty$ and $p(\infty^-) < \infty$. Furthermore,

$$\lim_{t \to T^-} X(t) = \infty \quad \textit{and} \quad \inf_{0 \le t < T} X(t) > 0, \quad \textit{a.s.}.$$

Proof. The proof is similar to that of Lemma 35. □

It should be mentioned that if (9.5) is suppressed and we suppose that $L > 1/2$, then no solutions of (1.17) will tend to zero and so this case is irrelevant to questions of super-exponential convergence or hitting zero in finite-time. We take (9.5) as a standing assumption throughout the rest of this thesis, unless stated otherwise. We claimed earlier that the function v played the role of F in the ODE when zero is attracting. The following result (see e.g. Theorem 5.5.29 in [34]) indicates how the finiteness of T depends on that of v, and we state it to aid understanding.

Theorem 59. *Let T, p and v be defined by* (1.22), (9.3) *and* (9.4)*.*

(i) Suppose $p(0^+) > -\infty$ and $p(\infty^-) = \infty$. If $v(0^+) < \infty$ (resp. $= \infty$), then

$$\lim_{t \to T^-} X(t) = 0 \quad \textit{and} \quad T < \infty, \textit{(resp. } = \infty\textit{)} \quad \textit{a.s.}.$$

(ii) Suppose $p(0^+) = -\infty$ and $p(\infty^-) < \infty$. If $v(\infty^-) < \infty$ (resp. $= \infty$), then

$$\lim_{t \to T^-} X(t) = \infty \quad \textit{and} \quad T < \infty, \textit{(resp. } = \infty\textit{)} \quad \textit{a.s.}.$$

Conditions (9.5) and (9.6) seem to identify the case when $xf(x)$ and $g^2(x)$ are of comparable size is critical. The next result shows that this is the case. Furthermore, the complicated finiteness conditions in Theorem 59 on v are replaced by simple, Osgood-type conditions.

Theorem 60. *Let L, T and p be defined as* (1.25), (1.22) *and* (9.3)*. Suppose that $p(\infty^-) = \infty$.*

(a) If $L = -\infty$ and

(i) $x \mapsto |f(x)|$ is asymptotically increasing, then $\int_{0^+}^1 1/|f(u)|\,du < \infty$ implies $T < \infty$ a.s..

(ii) $x \mapsto |f(x)|/x$ is asymptotically decreasing, then $\int_{0^+}^1 1/|f(u)|\,du < \infty$ implies $T = \infty$ a.s..

(b) If $-\infty < L < 1/2$, then

(i) $\int_{0+}^1 u/g^2(u)\,du < \infty$ *implies* $T < \infty$ *a.s..*

(ii) $\int_{0+}^1 u/g^2(u)\,du = \infty$ *implies* $T = \infty$ *a.s..*

(c) If $L > 1/2$, then $\mathbb{P}[X(t) \to 0 \, as \, t \to T^-] = 0$.

Proof. We prove part(a)(i) first. Define $M(x) := 2x|f(x)|/g^2(x)$ and $\mu(x) := M(x)/x$. Then there exists $x_1 > 0$ such that $M(x) > 0$ and $\mu(x) > 0$ for all $x < x_1$. Note that $L = -\infty$ implies $M(x) \to \infty$ as $x \to 0^+$. Then with x'' temporarily free, for $x < x'' < x_1$ we have

$$p'(x) = \exp\left(-2\int_{x''}^{x} \frac{f(u)}{g^2(u)}\,du\right) \;=\; \exp\left(-\int_x^{x''} \mu(u)\,du\right).$$

Substituting the expression for p' into v implies

$$\begin{aligned} v(x) &= \int_x^{x''}\int_y^{x''} \frac{\exp\left(\int_y^{x''} -\mu(u)\,du\right)}{\exp\left(\int_z^{x''} -\mu(u)\,du\right)} \cdot \frac{2|f(z)|}{g^2(z)} \cdot \frac{1}{|f(z)|}\,dz\,dy \\ &= \int_x^{x''} \left\{ \int_y^{x''} \mu(z) \exp\left(-\int_y^z \mu(u)\,du\right) \frac{|f(y)|}{|f(z)|}\,dz \right\} \frac{1}{|f(y)|}\,dy = \int_x^{x''} I(y) \cdot \frac{1}{|f(y)|}\,dy, \end{aligned}$$

where

$$I(y) := \int_y^{x''} \mu(z) \exp\left(-\int_y^z \mu(u)\,du\right) \frac{|f(y)|}{|f(z)|}\,dz.$$

Since $x \to |f(x)| \sim \phi(x)$ as $x \to 0^+$ where ϕ is increasing then exists x_2 such that

$$\frac{1}{2} < \frac{|f(x)|}{\phi(x)} < 2, \qquad \forall\, x < x_2.$$

Let $x'' = \min(x_1, x_2)$. Hence for $y \le z \le x_1 < x_2$ then $|f(y)| < 2\phi(y)$, $|f(z)| > \phi(z)/2$ and $\phi(y) < \phi(z)$. Thus

$$\frac{|f(y)|}{|f(z)|} \le \frac{2\phi(y)}{\phi(z)/2} \le 4.$$

Hence as $x'' \le x_2$ then

$$\begin{aligned} I(y) \le 4\int_y^{x''} \mu(z) \exp\left(-\int_y^z \mu(u)\,du\right)\,dz &= 4\int_y^{x''} \frac{d}{dz}\left(-\exp\left(-\int_y^z \mu(u)\,du\right)\right)\,dz \\ &= 4\left(1 - \exp\left(-\int_y^{x''} \mu(u)\,du\right)\right) \le 4. \end{aligned}$$

Now $\mu(u) > 0$ as $u \le x'' \le x_1$, so $I(y) \le 4$. Thus $v(x) \le 4\int_x^{x''} 1/|f(y)|\,dy$. So $v(x) \to L' < \infty$ as $x \to 0^+$ since $\int_{0+}^1 1/|f(x)|\,dx < \infty$, as claimed. We now prove part (a)(ii). Since $y \mapsto |f(y)|/y \sim \phi_1(y)$ as $y \to 0^+$ where ϕ_1 is decreasing. There exists x' such that

$$\frac{1}{2} < \frac{|f(x)|/x}{\phi_1(x)} < 2, \quad x < x'.$$

We may also take x' so small such that $M(x) \geq 4$ for all $x < x'$. Thus for $y \leq z \leq x'$ then $\phi_1(y) \geq \phi_1(z)$ and

$$\frac{1}{2} < \frac{|f(y)|/y}{\phi_1(y)} < 2 \quad \text{and} \quad \frac{1}{2} < \frac{|f(z)|/z}{\phi_1(z)} < 2.$$

Thus $|f(y)| > y\phi_1(y)/2$ and $|f(z)| < 2z\phi_1(z) \leq 2y\phi_1(y)$. Hence $|f(y)|/|f(z)| \geq 1/4 \cdot y/z$. Thus

$$v(x) \geq \frac{1}{4}\int_x^{x'} \left(\int_y^{x'} \mu(z) \exp\left(-\int_y^z \mu(u)\,du\right) \frac{1}{z}\,dz \right) \frac{y}{|f(y)|}\,dy.$$

Next, using integration by parts

$$\begin{aligned}
\int_y^{x'} \frac{\mu(z)}{z} \exp\left(-\int_y^z \mu(u)\,du\right)\,dz &= \int_y^{x'} \frac{-1}{z} \cdot \frac{d}{dz}\left(\exp\left(-\int_y^z \mu(u)\,du\right)\right)\,dz \\
&= \frac{-1}{z} \cdot \exp\left(-\int_y^z \mu(u)\,du\right)\Bigg|_y^{x'} - \int_y^{x'} \frac{1}{z^2} \cdot \exp\left(-\int_y^z \mu(u)\,du\right)\,dz \\
&= \frac{-1}{x'} \cdot \exp\left(-\int_y^{x'} \mu(u)\,du\right) - \frac{-1}{y} - \int_y^{x'} \exp\left(-\int_y^z \mu(u)\,du\right)\frac{1}{z^2}\,dz \\
&= \frac{1}{y} - \frac{1}{x'}\exp\left(-\int_y^{x'} \mu(u)\,du\right) - \int_y^{x'} \exp\left(-\int_y^{x'} \mu(u)\,du\right)\frac{1}{z^2}\,dz.
\end{aligned}$$

Thus

$$v(x) \geq \frac{1}{4}\int_x^{x'} \left(1 - \frac{y}{x'}\exp\left(-\int_y^{x'} \mu(u)\,du\right) - \int_y^{x'} \exp\left(-\int_y^{x'} \mu(u)\,du\right)\frac{y}{z^2}\,dz\right)\frac{1}{|f(y)|}\,dy.$$

Thus for $x < x'/2$, the positivity of the integrand implies

$$v(x) \geq \frac{1}{4}\int_x^{x'/2} \left(1 - \frac{y}{x'}\exp\left(-\int_y^{x'} \mu(u)\,du\right) - \int_y^{x'} \exp\left(-\int_y^{x'} \mu(u)\,du\right)\frac{y}{z^2}\,dz\right)\frac{1}{|f(y)|}\,dy.$$

In the integral $y \leq x'/2$, so $y/x' \leq 1/2$. Also $\int_y^{x'} \mu(u)\,du \geq 0$, so $\exp\left(-\int_y^{x'} \mu(u)\,du\right) \leq 1$. Hence

$$v(x) \geq \frac{1}{4}\int_x^{x'/2} \left(\frac{1}{2} - \int_y^{x'} \exp\left(-\int_y^z \mu(u)\,du\right)\frac{y}{z^2}\,dz\right)\frac{1}{|f(y)|}\,dy.$$

Next $u \in [y,z]$ and $z \le x'$. Thus $\mu(u) = M(u)/u \ge 4/u$ and

$$\int_y^z \mu(u)\,du \ge \int_y^z \frac{4}{u}\,du = 4\log\left(\frac{z}{y}\right) = \log\left(\frac{z}{y}\right)^4.$$

Thus

$$\exp\left(-\int_y^z \mu(u)\,du\right) \le \exp\left(\log\left(\frac{z}{y}\right)^{-4}\right) = \left(\frac{z}{y}\right)^{-4}.$$

Thus

$$\int_y^{x'} \exp\left(-\int_y^z \mu(u)\,du\right)\frac{y}{z^2}\,dz \quad \le \quad \int_y^{x'}\left(\frac{z}{y}\right)^{-4}\frac{y}{z^2}\,dz = \left(\frac{y^{-5}}{5} - \frac{x'^{-5}}{5}\right)y^5 \le \frac{1}{5}.$$

Thus

$$v(x) \ge \frac{1}{4}\left(\frac{1}{2} - \frac{1}{5}\right)\int_x^{x'/2} \frac{1}{|f(y)|}\,dy \to \infty, \quad \text{as } x \to 0^+,$$

since $\int_{0^+}^1 1/|f(x)|\,dx = \infty$. We now prove part (b)(i). Write

$$\frac{p'(y)}{p'(z)} = \exp\left(\int_y^z \frac{1}{u}\cdot\frac{2uf(u)}{g^2(u)}\,du\right).$$

Next for all $0 < \epsilon < 2L$ there is $x(\epsilon) > 0$ such that

$$L - \frac{\epsilon}{2} < \frac{uf(u)}{g^2(u)} < L + \frac{\epsilon}{2}.$$

Hence or $y < z < x(\epsilon)$, and $u \in (y,z)$, so

$$\int_y^z \frac{2L-\epsilon}{u}\,du \le \int_y^z \frac{1}{u}\cdot\frac{2uf(u)}{g^2(u)}\,du \le \int_y^z \frac{2L+\epsilon}{u}\,du.$$

Therefore

$$(2L-\epsilon)\log\left(\frac{z}{y}\right) \le \int_y^z \frac{1}{u}\cdot\frac{2uf(u)}{g^2(u)}\,du \le (2L+\epsilon)\log\left(\frac{z}{y}\right),$$

and so

$$\left(\frac{z}{y}\right)^{2L-\epsilon} \le \frac{p'(y)}{p'(z)} \le \left(\frac{z}{y}\right)^{2L+\epsilon}, \quad y < z < x(\epsilon).$$

Recall

$$v(x) = 2\int_x^{x(\epsilon)}\int_y^{x(\epsilon)} \frac{p'(y)}{p'(z)}\cdot\frac{1}{g^2(z)}\,dz\,dy,$$

thus for $x < x(\epsilon)/2$

$$\begin{aligned} v(x) &\leq 2\int_x^{x(\epsilon)}\int_y^{x(\epsilon)}\left(\frac{z}{y}\right)^{2L+\epsilon}\frac{1}{g^2(z)}\,dz\,dy = 2\int_x^{x(\epsilon)}\left(\int_x^z y^{-(2L+\epsilon)}\,dy\right)\frac{z^{2L+\epsilon}}{g^2(z)}\,dz \\ &= 2\int_x^{x(\epsilon)}\left(\frac{z^{1-2L-\epsilon}}{1-2L-\epsilon}-\frac{x^{1-2L-\epsilon}}{1-2L-\epsilon}\right)\frac{z^{2L+\epsilon}}{g^2(z)}\,dz \\ &= \frac{2}{1-2L-\epsilon}\int_x^{x(\epsilon)}\left(1-\left(\frac{x}{z}\right)^{1-2L-\epsilon}\right)\frac{z}{g^2(z)}\,dz \\ &\leq \frac{2}{1-2L-\epsilon}\int_x^{x(\epsilon)}\frac{z}{g^2(z)}\,dz. \end{aligned}$$

Hence $v(x) \to L' < \infty$ as $x \to 0^+$. We now prove part (b)(ii). For $x < x(\epsilon)/2$ and $z > y > x$ then

$$\begin{aligned} v(x) &\geq 2\int_x^{x(\epsilon)}\int_y^{x(\epsilon)}\left(\frac{z}{y}\right)^{2L-\epsilon}\frac{1}{g^2(z)}\,dz\,dy = 2\int_x^{x(\epsilon)}\left(\int_x^z y^{-(2L-\epsilon)}\,dy\right)\frac{z^{2L-\epsilon}}{g^2(z)}\,dz \\ &= 2\int_x^{x(\epsilon)}\left(\frac{z^{1-2L+\epsilon}}{1-2L+\epsilon}-\frac{x^{1-2L+\epsilon}}{1-2L+\epsilon}\right)\frac{z^{2L-\epsilon}}{g^2(z)}\,dz \\ &\geq \left(\frac{2}{1-2L+\epsilon}\right)\int_{2x}^{x(\epsilon)}\left(1-\left(\frac{x}{z}\right)^{1-2L+\epsilon}\right)\frac{z}{g^2(z)}\,dz. \end{aligned}$$

Now $z \geq 2x$, so $1/2 \geq x/z$. Thus $(1/2)^{1-2L+\epsilon} \geq (x/z)^{1-2L+\epsilon}$. Therefore

$$v(x) \geq \left(\frac{2}{1-2L+\epsilon}\right)\left(1-\left(\frac{1}{2}\right)^{1-2L+\epsilon}\right)\int_{2x}^{x(\epsilon)}\frac{z}{g^2(z)}\,dz.$$

Thus $v(x) \to \infty$ as $x \to 0^+$ since $\int_x^1 u/g^2(u)\,du \to \infty$ as $x \to 0^+$. We now prove part (c). Since

$$p'(x) = \exp\left(-2\int_{x^*}^x \frac{f(u)}{g^2(u)}\,du\right) = \exp\left(2\int_{x^*}^x \frac{1}{u}\cdot\frac{-uf(u)}{g^2(u)}\,du\right),$$

thus

$$\lim_{x\to 0^+}\frac{\log p'(x)}{\log(1/x)} = \lim_{x\to 0^+}\frac{2\int_{x^*}^x 1/u\cdot -uf(u)/g^2(u)\,du}{\log(1/x)} = -2\lim_{x\to 0^+}\frac{-xf(x)}{g^2(x)} = 2L.$$

Therefore

$$\lim_{x\to 0^+} p(x) = \begin{cases} -\infty, & L > \frac{1}{2}, \\ > -\infty, & L < \frac{1}{2}. \end{cases}$$

Since $p(\infty^-) = \infty$ by Theorem 58 we have that $\lim_{t\to T^-} X(t) = 0$ if $L < 1/2$ and $\mathbb{P}\left[\lim_{t\to T^-} X(t) = 0\right] = 0$ if $L > 1/2$. □

These Osgood-type conditions strongly suggest that an appropriate auxiliary ODE will describe well the behaviour of the SDE.

9.3 Super-Exponential and Finite-Time Stability

Our next task is to establish convergence rates. These can be obtained under essentially the conditions of Theorem 60. The first case deals with the case when $T = \infty$.

Theorem 61. *Suppose that $p(\infty^-) = \infty$. Let F, G, L and p be defined by* (1.29), (1.34), (1.25) *and* (9.3).

(i) If $L = -\infty$, f obeys (1.27) *and $x \mapsto |f(x)|/x$ is asymptotically decreasing, then*

$$\lim_{t\to\infty} \frac{F(X(t))}{t} = 1, \quad a.s..$$

(ii) If $L \in (-\infty, 1/2)$, g obeys (1.32) *and $x \mapsto g^2(x)/x^2$ is asymptotically decreasing, then*

$$\lim_{t\to\infty} \frac{G(X(t))}{t} = \frac{1}{2} - L, \quad a.s..$$

Proof. We prove part (i) first. Since $L = -\infty$ and $p(\infty^-) = \infty$, then we have that $p(0^+) > -\infty$ and $p(\infty^-) = \infty$. Therefore

$$\lim_{t\to T^-} X(t) = 0 \quad \text{and} \quad \sup_{0\le t<T} X(t) < \infty, \quad \text{a.s..}$$

Furthermore as $x \mapsto |f(x)|/x$ is asymptotically decreasing and $\int_{0^+}^1 1/|f(u)|\,du = \infty$ we have from Theorem 60 that $T = \infty$ a.s. and that $X(t) > 0$, $\forall\, t \in [0,\infty)$ a.s.. Therefore, by Itô's Lemma, we have a.s. for all $t > 0$

$$\log X(t) = \log X(0) + \int_0^t \left(\frac{f(X(s))}{X(s)} - \frac{1}{2}\cdot\frac{g^2(X(s))}{X^2(s)} \right) ds + \int_0^t \frac{g(X(s))}{X(s)}\, dB(s).$$

Let $M(t) := \int_0^t g(X(s))/X(s)\,dB(s)$ which is a continuous local martingale with quadratic variation $\langle M\rangle(t) := \int_0^t g^2(X(s))/X^2(s)\,ds$. Then we can write

$$\log X(t) = \log X(0) + \int_0^t \frac{f(X(s))}{X(s)}\,ds - \frac{1}{2}\langle M\rangle(t) + M(t).$$

If $D := \{\omega : \langle M\rangle(t,\omega) \to \langle M\rangle(\infty,\omega) < \infty,\ t\to\infty\}$ then a.s. on D, the martingale convergence theorem for continuous local martingales holds viz., $M(t)$ converges to a finite limit as $t \to \infty$ a.s. on D - see Proposition 5.1.8 [47]. Therefore we have that

$$\log X(t) \sim \int_0^t \frac{f(X(s))}{X(s)}\,ds, \quad \text{as } t\to\infty \text{ a.s. on } D. \tag{9.7}$$

Then $D' := \{\omega : \langle M\rangle(t,\omega) \to \langle M\rangle(\infty,\omega) = \infty,\ t\to\infty\}$ then a.s. on D', the strong law of large numbers for continuous local martingales holds viz., $M(t)/\langle M\rangle(t) \to 0$ as $t \to \infty$ a.s. on D' - see Exercise 5.1.16 in [47]. Therefore, as $L = -\infty$ and

$X(t) \to 0$ as $t \to \infty$ for each $\omega \in D'$, $t \mapsto \left(\int_0^t f(X(s))/X(s)\,ds\right)(\omega)$ is decreasing on $[T'(\omega), \infty)$ for some $T'(\omega)$. However, because $\langle M\rangle(t) = \int_0^t g^2(X(s))/X^2(s)\,ds \to \infty$ as $t \to \infty$, $g^2(x)/x^2 = o(|f(x)|/x)$ as $x \to 0^+$ and $X(t) \to 0$ as $t \to \infty$, it must be that $\int_0^t f(X(s))/X(s)\,ds \to -\infty$ as $t \to \infty$. Thus by L'Hôpital's Rule

$$\lim_{t\to\infty} \frac{\langle M\rangle(t)}{\int_0^t f(X(s))/X(s)\,ds} = 0, \quad \text{a.s. on } D',$$

and since $M(t)/\langle M\rangle(t) \to 0$ as $t \to \infty$ a.s. on D', we have

$$\log X(t) \sim \int_0^t \frac{f(X(s))}{X(s)}\,ds, \quad \text{as } t \to \infty \text{ a.s. on } D'. \tag{9.8}$$

Hence by (9.7) and (9.8) since $D \cup D'$ is an a.s. event then

$$\lim_{t\to\infty} \frac{-\log X(t)}{\int_0^t -f(X(s))/X(s)\,ds} = 1, \quad \text{a.s..} \tag{9.9}$$

By hypothesis there is a continuous η such that $\eta(x) \sim |f(x)|/x$ as $x \to 0^+$ and η is decreasing. Therefore

$$\Omega^* = \left\{\omega : \lim_{t\to\infty} \frac{-\log X(t)}{\int_0^t \eta(X(s),\omega)\,ds} = 1\right\},$$

is an a.s. event. Define on Ω^*, $I(t) := \int_0^t \eta(X(s))\,ds$, $t \geq 0$. Then I is in $C^1(0,\infty)$ by the continuity of η and X. Thus $-\log X(t)/I(t) \to 1$ as $t \to \infty$ and $I'(t) = \eta(X(t))$ then $X(t) = \eta^{-1}(I'(t))$ by the monotonicity of η. Therefore, for every $\epsilon \in (0,1)$, $\omega \in \Omega^*$ there is a $T(\omega,\epsilon)$ such that for $t \geq T(\omega,\epsilon)$

$$1-\epsilon < \frac{-\log X(t,\omega)}{I(t)} < 1+\epsilon, \tag{9.10}$$

$$1-\epsilon < \frac{-\log \eta^{-1}(I'(t))}{I(t)} < 1+\epsilon. \tag{9.11}$$

We treat the left-hand side of the inequality in (9.11), the analysis of the right-hand side being similar. The left-hand side of (9.11) yields for $t \geq T(\omega,\epsilon)$

$$e^{-(1-\epsilon)I(t)} > \eta^{-1}(I'(t)),$$

and because η is decreasing $\eta(e^{-(1-\epsilon)I(t)}) < I'(t)$ for $t \geq T(\omega,\epsilon)$. Thus we have for $t \geq T(\omega,\epsilon)$

$$\int_{T(\epsilon)}^t \frac{I'(s)}{\eta(e^{-(1-\epsilon)I(s)})}\,ds \geq T(\epsilon) - t,$$

where we drop the ω-dependence for simplicity. Next for $t \geq T(\epsilon)$

$$\int_{T(\epsilon)}^{t} \frac{I'(s)}{\eta(e^{-(1-\epsilon)I(s)})}\,ds = \frac{1}{1-\epsilon}\int_{(1-\epsilon)I(T(\epsilon))}^{(1-\epsilon)I(t)} \frac{1}{\eta(e^{-u})}\,du = \frac{1}{1-\epsilon}\int_{\exp(-(1-\epsilon)I(t))}^{\exp(-(1-\epsilon)I(T(\epsilon)))} \frac{1}{v\eta(v)}\,dv.$$

Since $I(t) \to \infty$ as $t \to \infty$ and $1/(x\eta(x)) \sim 1/|f(x)|$ as $x \to 0^+$ then

$$\int_{T(\epsilon)}^{t} \frac{I'(s)}{\eta(e^{-(1-\epsilon)I(s)})}\,ds \sim \frac{1}{1-\epsilon}\cdot F(e^{-(1-\epsilon)I(t)}), \quad \text{as } t \to \infty.$$

Therefore,

$$1 \leq \liminf_{t\to\infty} \frac{1}{t}\int_{T(\epsilon)}^{t} \frac{I'(s)}{\eta(e^{-(1-\epsilon)I(s)})}\,ds = \frac{1}{1-\epsilon}\cdot \liminf_{t\to\infty} \frac{F(e^{-(1-\epsilon)I(t)})}{t},$$

or

$$\liminf_{t\to\infty} \frac{F(e^{-(1-\epsilon)I(t)})}{t} \geq 1-\epsilon.$$

However, for $t \geq T(\epsilon)$, then $e^{-(1-\epsilon)I(t)} > X(t)$, so for $t \geq T(\epsilon)$ as F is decreasing $F(X(t)) > F(e^{-(1-\epsilon)I(t)})$. Therefore

$$\liminf_{t\to\infty} \frac{F(X(t))}{t} \geq \liminf_{t\to\infty} \frac{F(e^{-(1-\epsilon)I(t)})}{t} \geq 1-\epsilon.$$

Letting $\epsilon \to 0^+$ yields

$$\liminf_{t\to\infty} \frac{F(X(t))}{t} \geq 1.$$

Proceeding similarly pathwise on the right-hand side of (9.11), we get

$$\limsup_{t\to\infty} \frac{F(X(t))}{t} \leq 1,$$

from which the result immediately follows. The proof of part (ii) is similar but with G in place of F, noting that the convergence of $\langle M\rangle$ in this case leads to a contradiction. We set $\eta(x) \sim (\frac{1}{2} - L)g^2(x)/x^2$ in this case. □

If at least one of the conditions below holds

$$\lim_{x\to 0^+} \frac{|f(x)|}{x} = \infty \quad \text{or} \quad \lim_{x\to 0^+} \frac{g^2(x)}{x^2} = \infty,$$

then we have $\lim_{t\to\infty} \log X(t)/t = -\infty$ a.s. in both parts of the theorem. This is the aforementioned super-exponential stability.

In the case when $T < \infty$ a.s., we can once again determine the asymptotics as $t \to T^-$. We give the result when L if finite, the case where $L = -\infty$ being similar.

Theorem 62. *Suppose* $p(\infty^-) = \infty$, $L \in (-\infty, 1/2)$, g *obeys* (1.31) *and* $x \mapsto g^2(x)/x^2$ *is asymptotic to a continuous non-increasing function. Let* T, $\bar{G}$, L *and* p *be defined* *by* (1.22)*,* (1.33)*,* (1.25) *and* (9.3) *where and* $\bar{G}(x) \to 0$ *as* $x \to 0^+$*.*

(i) If

$$\lim_{\lambda\to 1^+} \liminf_{x\to 0^+} \frac{(-\log\circ\bar{G}^{-1})(\lambda x)}{(-\log\circ\bar{G}^{-1})(x)} = \infty, \tag{9.12}$$

then

$$\lim_{t\to T^-} \frac{\bar{G}(X(t))}{T-t} = \frac{1}{2} - L, \quad a.s.. \tag{9.13}$$

(ii) If

$$\lim_{\lambda\to 1^-} \limsup_{x\to 0^+} \frac{(-\log\circ\bar{G}^{-1})(\lambda x)}{(-\log\circ\bar{G}^{-1})(x)} = 1, \tag{9.14}$$

then

$$\lim_{t\to T^-} \frac{-\log X(t)}{(-\log\circ\bar{G}^{-1})((\frac{1}{2}-L)(T-t))} = 1, \quad a.s.. \tag{9.15}$$

We state the following lemma which is used in the proof of Theorem 62. We defer proving the lemma until after the proof of the theorem.

Lemma 37. *Suppose that there is* $\tau(\epsilon) \in (0,T]$ *such that for* $0 < T-t < \tau(\epsilon)$*. Let* T*,* $\bar{G}$ *and* L *be defined by* (1.22)*,* (1.33)*,* (1.25) *where* $L \in (-\infty, 1/2)$*,* $\lambda_-(\epsilon) = (1-\epsilon)\cdot a/2$ *and* $\lambda_+(\epsilon) = (1+\epsilon)\cdot a/2$*.*

$$\bar{G}^{-1}((1-\epsilon)\lambda_-(\epsilon)(T-t))^{\lambda_+(\epsilon)/\lambda_-(\epsilon)} < X(t) < \bar{G}^{-1}((1+\epsilon)\lambda_+(\epsilon)(T-t))^{\lambda_-(\epsilon)/\lambda_+(\epsilon)}. \tag{9.16}$$

If (9.12) *holds, then*

$$\lim_{t\to T^-} \frac{\bar{G}(X(t))}{T-t} = \frac{a}{2}, \quad a.s.. \tag{9.17}$$

Proof of Theorem 62. Suppose (1.25)

$$\lim_{x\to 0^+} \frac{xf(x)}{g^2(x)} = L \in (-\infty, 1/2).$$

By Itô's Lemma, the SDE for $Z(t) = -\log X(t)$ is

$$d(-\log X(t)) = \left(\frac{-f(X(t))}{X(t)} + \frac{1}{2}\cdot\frac{g^2(X(t))}{X(t)^2}\right) dt + \frac{-g(X(t))}{X(t)}\, dB(t). \tag{9.18}$$

Now

$$\lim_{x\to 0^+} \frac{-f(x)/x + g^2(x)/2x^2}{g^2(x)/x^2} = \frac{1}{2} - \lim_{x\to 0^+} \frac{xf(x)}{2g^2(x)} = \frac{1}{2} - L.$$

The conditions on $\bar{G}$ ensure that $X(t) > 0$ for all $t \in [0,T)$ a.s. and that $X(t) \to 0$ as $t \to T^-$. Therefore $\int_0^t g^2(X(s))/X(s)^2\, ds \to \infty$ as $t \to T^-$ and

$$\lim_{t\to T^-} \frac{-\log X(t)}{\int_0^t g^2(X(s))/X^2(s)\, ds} = \frac{1}{2} - L, \quad \text{a.s..}$$

If the integral in the denominator were convergent we would have from (9.18) that $-\log X(t)$ tended to a finite limit as $t \to T^-$ which is impossible and therefore generates a contradiction. Define $\eta(x) := g^2(x)/x^2 \sim \tilde{\eta}(x)$ as $x \to 0^+$ where $\tilde{\eta}$ is permitted to be continuous and decreasing. Then

$$\lim_{t \to T^-} \frac{-\log X(t)}{\int_0^t \tilde{\eta}(X(s))\,ds} = \frac{1}{2} - L, \quad \text{a.s..}$$

Suppose $I(t) := \int_0^t \tilde{\eta}(X(s))\,ds$. Thus $I'(t) = \tilde{\eta}(X(t))$ or $\tilde{\eta}^{-1}(I'(t)) = X(t)$. Thus

$$\lim_{t \to T^-} \frac{-\log \tilde{\eta}^{-1}(I'(t))}{I(t)} = \frac{1}{2} - L =: \frac{a}{2}, \quad \text{a.s..}$$

Hence, for every $\epsilon \in (0,1)$, there is $\tau_1(\epsilon) > 0$ such that $T - t < \tau_1(\epsilon)$ implies

$$(1-\epsilon) \cdot \frac{a}{2} < \frac{-\log \tilde{\eta}^{-1}(I'(t))}{I(t)} < (1+\epsilon) \cdot \frac{a}{2}.$$

Since $\tilde{\eta}$ is decreasing then $\tilde{\eta}\left(e^{-\lambda_-(\epsilon)I(t)}\right) < I'(t) < \tilde{\eta}\left(e^{-\lambda_+(\epsilon)I(t)}\right)$ where $\lambda_\pm(\epsilon) := a(1 \pm \epsilon)/2$. Thus for $T - t < \tau_1(\epsilon)$

$$T - t \le \int_t^T \frac{I'(s)}{\tilde{\eta}\left(e^{-\lambda_-(\epsilon)I(s)}\right)}\,ds \quad \text{and} \quad \int_t^T \frac{I'(s)}{\tilde{\eta}\left(e^{-\lambda_+(\epsilon)I(s)}\right)}\,ds \le T - t.$$

Since $I(t) \to \infty$ as $t \to T^-$ then

$$T - t \le \int_{I(t)}^\infty \frac{1}{\tilde{\eta}\left(e^{-\lambda_-(\epsilon)u}\right)}\,du \quad \text{and} \quad \int_{I(t)}^\infty \frac{1}{\tilde{\eta}\left(e^{-\lambda_+(\epsilon)u}\right)}\,du \le T - t.$$

For $T - t < \tau_1(\epsilon)$, then

$$\begin{aligned}\int_0^{\exp(-\lambda_-(\epsilon)I(t))} \frac{1}{v\tilde{\eta}(v)}\,dv = \int_\infty^{I(t)} \frac{-\lambda_-(\epsilon)e^{-\lambda_-(\epsilon)u}}{\tilde{\eta}\left(e^{-\lambda_-(\epsilon)u}\right)e^{-\lambda_-(\epsilon)u}}\,du &= \lambda_-(\epsilon)\int_{I(t)}^\infty \frac{1}{\tilde{\eta}(e^{-\lambda_-(\epsilon)u})}\,du \\ &\ge \lambda_-(\epsilon)(T-t).\end{aligned}$$

Define $\tilde{G}(x) := \int_0^x 1/(u\tilde{\eta}(u))\,du$. Then $\tilde{G}(x)/\bar{G}(x) \to 1$ $x \to 0^+$ and for $T - t < \tau_1(\epsilon)$, $\tilde{G}\left(e^{-\lambda_-(\epsilon)I(t)}\right) \ge \lambda_-(\epsilon)(T-t)$. Next for every $\epsilon \in (0,1)$ there is $x_1(\epsilon) > 0$ such that for $x < x_1(\epsilon)$, $(1+\epsilon)\cdot\tilde{G}(x) > \bar{G}(x) > (1-\epsilon)\cdot\tilde{G}(x)$. Now $e^{-\lambda_-(\epsilon)I(t)} < x_1(\epsilon)$ for all $T - t < \tau_2(\epsilon)$. Define $\tau_3(\epsilon) = \min(\tau(\epsilon), \tau_2(\epsilon))$. Then $T - t < \tau_3(\epsilon)$ implies

$$\bar{G}\left(e^{-\lambda_-(\epsilon)I(t)}\right) > (1-\epsilon)\cdot\tilde{G}\left(e^{-\lambda_-(\epsilon)I(t)}\right) > (1-\epsilon)\cdot\lambda_-(\epsilon)(T-t).$$

Thus for $T - t < \tau_3(\epsilon)$, $e^{-\lambda_-(\epsilon)I(t)} > \bar{G}^{-1}\left((1-\epsilon)\lambda_-(\epsilon)(T-t)\right)$. Now $T - t < \tau_1(\epsilon)$ implies

$$\lambda_-(\epsilon) < \frac{-\log X(t)}{I(t)} < \lambda_+(\epsilon),$$

thus $e^{-\lambda_-(\epsilon)I(t)} > X(t) > e^{-\lambda_+(\epsilon)I(t)}$. Thus for $T-t<\tau_3(\epsilon)$

$$X(t)^{\lambda_-(\epsilon)/\lambda_+(\epsilon)} > e^{-\lambda_-(\epsilon)I(t)} > \bar{G}^{-1}\left((1-\epsilon)\lambda_-(\epsilon)(T-t)\right),$$

and thus

$$X(t) > \bar{G}^{-1}\left((1-\epsilon)\lambda_-(\epsilon)(T-t)\right)^{\lambda_+(\epsilon)/\lambda_-(\epsilon)}. \tag{9.19}$$

Similarly $\tilde{G}(e^{-\lambda_+(\epsilon)I(t)}) \le \lambda_+(\epsilon)(T-t)$ for $T-t<\tau_1(\epsilon)$. Now $\lambda_+(\epsilon)>\lambda_-(\epsilon)$ so $-\lambda_+(\epsilon)<-\lambda_-(\epsilon)$ and $e^{-\lambda_+(\epsilon)I(t)} < e^{-\lambda_-(\epsilon)I(t)}$. Hence for $T-t<\tau_2(\epsilon)$, $e^{-\lambda_+(\epsilon)I(t)} < e^{-\lambda_-(\epsilon)I(t)} < x_1(\epsilon)$. Thus $(1+\epsilon)\cdot\tilde{G}(e^{-\lambda_+(\epsilon)I(t)}) > \bar{G}(e^{-\lambda_+(\epsilon)I(t)})$. Hence for $T-t<\tau_3(\epsilon)$

$$\bar{G}(e^{-\lambda_+(\epsilon)I(t)}) < (1+\epsilon)\cdot\tilde{G}(e^{-\lambda_+(\epsilon)I(t)}) < (1+\epsilon)\cdot\lambda_+(\epsilon)(T-t).$$

Thus for $T-t<\tau_3(\epsilon)$ we have $e^{-\lambda_+(\epsilon)I(t)} < \bar{G}^{-1}((1+\epsilon)\lambda_+(\epsilon)(T-t))$. Hence for $T-t<\tau_3(\epsilon)$

$$X(t)^{\lambda_+(\epsilon)/\lambda_-(\epsilon)} < e^{-\lambda_+(\epsilon)I(t)} < \bar{G}^{-1}((1+\epsilon)\lambda_+(\epsilon)(T-t)),$$

so

$$X(t) < \bar{G}^{-1}((1+\epsilon)\lambda_+(\epsilon)(T-t))^{\lambda_-(\epsilon)/\lambda_+(\epsilon)}. \tag{9.20}$$

Combining inequalities (9.19) and (9.20) give (9.16) with $a/2 = 1/2 - L$. If (9.12) holds, by Lemma 37 we have

$$\lim_{t\to T^-}\frac{\bar{G}(X(t))}{T-t} = \frac{a}{2} = \frac{1}{2} - L, \quad \text{a.s.},$$

which proves part (i). We now prove part (ii). Inequalities (9.19) and (9.20) hold for $T-t<\tau_3(\epsilon)$ so

$$\bar{G}^{-1}((1-\epsilon)\lambda_-(\epsilon)(T-t))^{\lambda_+(\epsilon)/\lambda_-(\epsilon)} < X(t) < \bar{G}^{-1}((1+\epsilon)\lambda_+(\epsilon)(T-t))^{\lambda_-(\epsilon)/\lambda_+(\epsilon)}.$$

Hence

$$\frac{\lambda_+(\epsilon)}{\lambda_-(\epsilon)}\cdot\left(-\log\circ\bar{G}^{-1}\right)\left((1-\epsilon)\lambda_-(\epsilon)(T-t)\right) > -\log X(t) > \frac{\lambda_-(\epsilon)}{\lambda_+(\epsilon)}\cdot\left(-\log\circ\bar{G}^{-1}\right)\left((1+\epsilon)\lambda_+(\epsilon)(T-t)\right).$$

As $\lambda_+(\epsilon) = (1+\epsilon)\cdot a/2 = (1+\epsilon)\cdot(1/2-L)$ and $\lambda_-(\epsilon) = (1-\epsilon)\cdot(1/2-L)$ then

$$\frac{-\log X(t)}{\left(-\log\circ\bar{G}^{-1}\right)\left((\frac{1}{2}-L)(T-t)\right)} > \frac{\lambda_-(\epsilon)}{\lambda_+(\epsilon)}\cdot\frac{\left(-\log\circ\bar{G}^{-1}\right)\left((1+\epsilon)\lambda_+(\epsilon)(T-t)\right)}{\left(-\log\circ\bar{G}^{-1}\right)\left((\frac{1}{2}-L)(T-t)\right)}.$$

Hence

$$\begin{aligned}
\liminf_{t\to T^-} \frac{-\log X(t)}{(-\log\circ\bar{G}^{-1})\left((\frac{1}{2}-L)(T-t)\right)} &\geq \frac{\lambda_-(\epsilon)}{\lambda_+(\epsilon)}\cdot\liminf_{t\to T^-}\frac{(-\log\circ\bar{G}^{-1})\left((1+\epsilon)\lambda_+(\epsilon)(T-t)\right)}{(-\log\circ\bar{G}^{-1})\left((\frac{1}{2}-L)(T-t)\right)} \\
&= \left(\frac{1-\epsilon}{1+\epsilon}\right)\cdot\liminf_{x\to 0^+}\frac{(-\log\circ\bar{G}^{-1})\left((1+\epsilon)^2 x\right)}{(-\log\circ\bar{G}^{-1})(x)} \\
&= \left(\frac{1-\epsilon}{1+\epsilon}\right)\cdot\liminf_{y\to 0^+}\frac{(-\log\circ\bar{G}^{-1})(y)}{(-\log\circ\bar{G}^{-1})\left(y/(1+\epsilon)^2\right)} \\
&= \left(\frac{1-\epsilon}{1+\epsilon}\right)\cdot\left(\limsup_{y\to 0^+}\frac{(-\log\circ\bar{G}^{-1})\left(y/(1+\epsilon)^2\right)}{(-\log\circ\bar{G}^{-1})(y)}\right)^{-1}.
\end{aligned}$$

Letting $\epsilon\to 0^+$ by (9.14), we get

$$\liminf_{t\to T^-}\frac{-\log X(t)}{(-\log\circ\bar{G}^{-1})\left((\frac{1}{2}-L)(T-t)\right)}\geq 1.$$

Similarly

$$\begin{aligned}
\frac{-\log X(t)}{(-\log\circ\bar{G}^{-1})\left((\frac{1}{2}-L)(T-t)\right)} &< \frac{\lambda_+(\epsilon)}{\lambda_-(\epsilon)}\cdot\frac{(-\log\circ\bar{G}^{-1})((1-\epsilon)\lambda_-(\epsilon)(T-t))}{(-\log\circ\bar{G}^{-1})\left((\frac{1}{2}-L)(T-t)\right)} \\
&= \left(\frac{1+\epsilon}{1-\epsilon}\right)\cdot\frac{(-\log\circ\bar{G}^{-1})\left((1-\epsilon)^2(\frac{1}{2}-L)(T-t)\right)}{(-\log\circ\bar{G}^{-1})\left((\frac{1}{2}-L)(T-t)\right)}.
\end{aligned}$$

Thus

$$\limsup_{t\to T^-}\frac{-\log X(t)}{(-\log\circ\bar{G}^{-1})\left((\frac{1}{2}-L)(T-t)\right)}\leq\left(\frac{1+\epsilon}{1-\epsilon}\right)\cdot\limsup_{x\to 0^+}\frac{(-\log\circ\bar{G}^{-1})((1-\epsilon)^2 x)}{(-\log\circ\bar{G}^{-1})(x)}.$$

By (9.14), letting $\epsilon\to 0^+$, then

$$\limsup_{t\to T^-}\frac{-\log X(t)}{(-\log\circ\bar{G}^{-1})\left((\frac{1}{2}-L)(T-t)\right)}\leq 1,$$

so (9.15) holds, as claimed. □

There appears to be a gap in Theorem 62, as the conditions in (i) and (ii) are not exhaustive. However, in practice these conditions are comprehensive. Sufficient conditions to guarantee (9.12) or (9.14) are available which are more readily verified.

Lemma 38. *If* (9.12) *viz.,*

$$\lim_{\lambda\to 1^+}\liminf_{x\to 0^+}\frac{(-\log\circ\bar{G}^{-1})(\lambda x)}{(-\log\circ\bar{G}^{-1})(x)}=\infty,$$

holds then

$$\limsup_{x\to 0^+}\frac{\bar{G}(x^\lambda)}{\bar{G}(x)}=1,\quad\forall\,\lambda<1. \tag{9.21}$$

Proof. Let $\lambda < 1$ and define

$$b(\lambda) := \liminf_{x\to 0^+} \frac{(-\log\circ \bar{G}^{-1})(\lambda x)}{(-\log\circ \bar{G}^{-1})(x)}.$$

For $\mu < \lambda < 1$, then $\bar{G}^{-1}(\mu x) < \bar{G}^{-1}(\lambda x)$, so $(-\log\circ\bar{G}^{-1})(\mu x) > (-\log\circ\bar{G}^{-1})(\lambda x)$ and thus $b(\mu) \geq b(\lambda)$. Thus, $\mu \mapsto b(\mu)$ is non-increasing. Since $b(\lambda) \to \infty$ as $\lambda \to 1^+$, $b(\lambda) = \infty \;\forall\, \lambda < 1$, so $\liminf_{x\to 0^+}(-\log\circ\bar{G}^{-1})(\lambda x)/(-\log\circ\bar{G}^{-1})(x) = \infty$ and

$$\lim_{x\to 0^+} \frac{(-\log\circ \bar{G}^{-1})(\lambda x)}{(-\log\circ \bar{G}^{-1})(x)} = \infty.$$

Fix, $\lambda < 1$. Then $\forall\, M > 1$, there is $x(\lambda, M) > 0$ such that $\forall\, x < x(\lambda, M)$

$$\frac{(-\log\circ \bar{G}^{-1})(\lambda x)}{(-\log\circ \bar{G}^{-1})(x)} > M.$$

Thus

$$(-\log\circ\bar{G}^{-1})(\lambda x) > M \cdot (-\log\circ\bar{G}^{-1})(x) = (-\log\circ\bar{G}^{-1})(x)^M,$$

or $\bar{G}^{-1}(x)^M > \bar{G}^{-1}(\lambda x)$ for $x < x(M,\lambda)$. Let $y := \bar{G}^{-1}(x)$. Then $x < x(M,\lambda)$ if and only if $y < \bar{G}^{-1}(x(M,\lambda)) =: y(M,\lambda)$. Thus $y^M > \bar{G}^{-1}(\lambda\bar{G}(y))$ so $\bar{G}(y^M) > \lambda\bar{G}(y)$ for $y < y(M,\lambda)$. Therefore for each fixed $\mu < 1$, and all $\lambda := 1/M < 1$ there is $y(1/\lambda,\mu) > 0$ such that $\bar{G}(y^{1/\lambda}) > \mu\bar{G}(y)$, for $y < y(1/\lambda,\mu) = \bar{G}^{-1}(x(1/\lambda,\mu))$. Let $z = y^{1/\lambda}$, and $z(\lambda,\mu) := y(1/\lambda,\mu)^{1/\lambda} = \bar{G}^{-1}(x(1/\lambda,\mu))^{1/\lambda}$. Then for $z < z(\lambda,\mu)$, $G(z) > \mu\bar{G}(z^\lambda)$. Thus for each fixed $\mu < 1$, and all $\lambda < 1$, there is $z(\lambda,\mu)$ such that $\bar{G}(z^\lambda)/\bar{G}(z) < 1/\mu$ for all $z < z(\lambda,\mu)$. Let $z_2(\lambda,\mu) = \min(z(\lambda,\mu),1)$. Since $\lambda < 1$, $\bar{G}(z^\lambda) > \bar{G}(z)$ for all $z < z_2(\lambda,\mu)$. Hence for each fixed $\mu < 1$ and all $\lambda < 1$, there is $z_2(\lambda,\mu) > 0$ such that for $z < z_2(\lambda,\mu)$

$$1 < \frac{\bar{G}(z^\lambda)}{\bar{G}(z)} < \frac{1}{\mu}.$$

Thus for each fixed $\mu < 1$ and all $\lambda < 1$

$$1 \leq \limsup_{z\to 0^+} \frac{\bar{G}(z^\lambda)}{\bar{G}(z)} \leq \frac{1}{\mu}.$$

Therefore $\forall\, \lambda < 1$, $\limsup_{z\to 0^+} \bar{G}(z^\lambda)/\bar{G}(z) = 1$ as required. □

We now give a proof of Lemma 37.

Proof of Lemma 37. By Lemma 38, (9.12) implies (9.21). For $T-t<\tau(\epsilon)$, then

$$\frac{\bar{G}\left(\bar{G}^{-1}((1-\epsilon)\lambda_-(\epsilon)(T-t))^{\lambda_+(\epsilon)/\lambda_-(\epsilon)}\right)}{T-t} < \frac{\bar{G}(X(t))}{T-t} < \frac{\bar{G}\left(\bar{G}^{-1}((1+\epsilon)\lambda_+(\epsilon)(T-t))^{\lambda_-(\epsilon)/\lambda_+(\epsilon)}\right)}{T-t}. \quad (9.22)$$

Thus by (9.22)

$$\begin{aligned}\liminf_{t\to T^-}\frac{\bar{G}(X(t))}{(1-\epsilon)\lambda_-(\epsilon)(T-t)} &\geq \liminf_{t\to T^-}\frac{\bar{G}\left(\bar{G}^{-1}((1-\epsilon)\lambda_-(\epsilon)(T-t))^{\lambda_+(\epsilon)/\lambda_-(\epsilon)}\right)}{(1-\epsilon)\lambda_-(\epsilon)(T-t)}\\ &= \liminf_{x\to 0^+}\frac{\bar{G}\left(\bar{G}^{-1}(x)^{\lambda_+(\epsilon)/\lambda_-(\epsilon)}\right)}{x} = \liminf_{y\to 0^+}\frac{\bar{G}\left(y^{\lambda_+(\epsilon)/\lambda_-(\epsilon)}\right)}{\bar{G}(y)}.\end{aligned}$$

Thus

$$\begin{aligned}\liminf_{t\to T^-}\frac{\bar{G}(X(t))}{T-t} &\geq (1-\epsilon)\cdot\lambda_-(\epsilon)\liminf_{y\to 0^+}\frac{\bar{G}\left(y^{\lambda_+(\epsilon)/\lambda_-(\epsilon)}\right)}{\bar{G}(y)}\\ &= (1-\epsilon)\cdot\lambda_-(\epsilon)\liminf_{z\to 0^+}\frac{\bar{G}(z)}{\bar{G}\left(z^{\lambda_-(\epsilon)/\lambda_+(\epsilon)}\right)}\\ &= (1-\epsilon)^2\cdot\frac{a}{2}\cdot\liminf_{z\to 0^+}\frac{\bar{G}(z)}{\bar{G}\left(z^{(1-\epsilon)/(1+\epsilon)}\right)}\\ &= (1-\epsilon)^2\cdot\frac{a}{2}\cdot\left(\limsup_{z\to 0^+}\frac{\bar{G}\left(z^{(1-\epsilon)/(1+\epsilon)}\right)}{\bar{G}(z)}\right)^{-1} = (1-\epsilon)^2\cdot\frac{a}{2},\end{aligned}$$

by (9.21). Letting $\epsilon\to 0$ yields

$$\liminf_{t\to T^-}\frac{\bar{G}(X(t))}{T-t}\geq\frac{a}{2}. \quad (9.23)$$

By (9.22)

$$\begin{aligned}\limsup_{t\to T^-}\frac{\bar{G}(X(t))}{(1+\epsilon)\lambda_+(\epsilon)(T-t)} &\geq \limsup_{t\to T^-}\frac{\bar{G}\left(\bar{G}^{-1}((1+\epsilon)\lambda_+(\epsilon)(T-t))^{\lambda_-(\epsilon)/\lambda_+(\epsilon)}\right)}{(1+\epsilon)\lambda_+(\epsilon)(T-t)}\\ &= \limsup_{x\to 0^+}\frac{\bar{G}\left(\bar{G}^{-1}(x)^{\lambda_-(\epsilon)/\lambda_+(\epsilon)}\right)}{x}\\ &= \limsup_{y\to 0^+}\frac{\bar{G}\left(y^{\lambda_-(\epsilon)/\lambda_+(\epsilon)}\right)}{\bar{G}(y)}.\end{aligned}$$

Thus by (9.21)

$$\limsup_{t\to T^-}\frac{\bar{G}(X(t))}{T-t}\leq(1+\epsilon)^2\cdot\frac{a}{2}\cdot\limsup_{y\to 0^+}\frac{\bar{G}\left(y^{(1-\epsilon)/(1+\epsilon)}\right)}{\bar{G}(y)} = (1+\epsilon)^2\cdot\frac{a}{2}.$$

Letting $\epsilon \to 0^+$ yields

$$\limsup_{t\to T^-} \frac{\bar{G}(X(t))}{T-t} \le \frac{a}{2},$$

whence the result (9.17) by combining the above limit with (9.23). □

In principle, Equation (9.24) is weaker than (9.12).

Theorem 63. *Suppose*

$$\lim_{\lambda\to 1^-} \limsup_{x\to 0^+} \frac{\bar{G}(x^\lambda)}{\bar{G}(x)} = 1. \tag{9.24}$$

Then

$$\lim_{t\to T^-} \frac{\bar{G}(X(t))}{T-t} = \frac{1}{2} - L, \quad a.s..$$

Proof. The proof of this result uses a modified version of the proof of Lemma 37. □

Next we come up with a condition which implies (9.14) but is easier to check.

Proposition 16. *If*

$$\limsup_{x\to 0^+} \frac{\int_0^x u/g^2(u)\,du}{x^2/g^2(x)\cdot \log(1/x)} =: L^* < \infty, \tag{9.25}$$

then (9.14) *holds.*

Proof. Define $\gamma(x) := g^2(x)/x$. Let x be the solution of the ODE

$$x'(t) = -\gamma(x(t)), \quad t \in [0,T), \quad x(T) = 0.$$

Then $x(t) = \bar{G}^{-1}(T-t)$. Define $u(t) := x(T-t)$, $t \in [0,T)$. Then $u(0) = 0$ and

$$u'(t) = -x'(T-t) = \gamma(x(T-t)) = \gamma(u(t)), \quad t \in [0,T).$$

Hence $u(t) = \bar{G}^{-1}(t)$. Let $z(t) = -\log u(t) = (-\log\circ \bar{G}^{-1})(t)$. Also

$$z'(t) = \frac{-u'(t)}{u(t)} = \frac{-\gamma(u(t))}{u(t)} = \frac{-\gamma(e^{-z(t)})}{e^{-z(t)}} = \frac{-g^2(e^{-z(t)})}{e^{-2z(t)}}.$$

Define $\eta(x) := g^2(x)/x^2$. Then $z'(t) = -\eta(e^{-z(t)}) = -\eta_1(z(t))$ where $\eta_1(z) = \eta(e^{-z})$. Define $\tilde{\eta}_1(z) := \tilde{\eta}(e^{-z})$. Then $\tilde{\eta}_1$ is increasing as $z \mapsto e^{-z}$ and $z \mapsto \tilde{\eta}(z)$ are decreasing. Now $\eta(x) \sim \tilde{\eta}(x)$ as $x \to 0^+$ then

$$\lim_{z\to\infty} \frac{\eta_1(z)}{\tilde{\eta}_1(z)} = \lim_{z\to\infty} \frac{\eta(e^{-z})}{\tilde{\eta}(e^{-z})} = \lim_{x\to 0^+} \frac{\eta(x)}{\tilde{\eta}(x)} = 1.$$

Let $\lambda > 1$. Then, as $t \mapsto z(t)$ is decreasing

$$0 < z(t) - z(\lambda t) = \int_{\lambda t}^{t} z'(s)\,ds = -\int_t^{\lambda t} z'(s)\,ds = \int_t^{\lambda t} \eta_1(z(s))\,ds.$$

Since $z(t) \to \infty$ as $t \to 0^+$, $\eta_1(\infty^-) = \infty$, then

$$\lim_{t\to 0^+} \frac{\int_t^{\lambda t} \eta_1(z(s))\, ds}{\int_t^{\lambda t} \tilde{\eta}_1(z(s))\, ds} = \lim_{t\to 0^+} \frac{\int_t^{\lambda t} \eta_1(z(s))\, ds/z(t)}{\int_t^{\lambda t} \tilde{\eta}_1(z(s))\, ds/z(t)} = 1.$$

Also

$$1 - \frac{z(\lambda t)}{z(t)} = \frac{\int_t^{\lambda t} \eta_1(z(s))\, ds}{z(t)},$$

so

$$\lim_{t\to 0} \frac{1 - z(\lambda t)/z(t)}{\int_t^{\lambda t} \tilde{\eta}_1(z(s))\, ds/z(t)} = 1.$$

Thus since $z(t) = (-\log \circ \bar{G}^{-1})(t)$

$$\lim_{t\to 0^+} \frac{1 - (-\log \circ \bar{G}^{-1})(\lambda t)/(-\log \circ \bar{G}^{-1})(t)}{(\lambda - 1)a_\lambda(t)} = 1,$$

where

$$(\lambda - 1)a_\lambda(t) := \frac{\int_t^{\lambda t} \tilde{\eta}_1(z(s))\, ds}{z(t)}.$$

Hence

$$\lim_{t\to 0^+} \frac{1 - (-\log \circ \bar{G}^{-1})(\lambda t)/(-\log \circ \bar{G}^{-1})(t)}{a_\lambda(t)} = \lambda - 1. \tag{9.26}$$

For each $\lambda > 1$, there is $T(\lambda) > 0$ such that for $t > T(\lambda)$

$$\frac{1 - (-\log \circ \bar{G}^{-1})(\lambda t)/(-\log \circ \bar{G}^{-1})(t)}{a_\lambda(t)} < \frac{3(\lambda - 1)}{2}.$$

Since $\int_t^{\lambda t} \tilde{\eta}_1(z(s))\, ds < \int_t^{\lambda t} \tilde{\eta}_1(z(t))\, ds = (\lambda - 1)t \cdot \tilde{\eta}_1(z(t))$, we get

$$(\lambda - 1)a_\lambda(t) = \frac{\int_t^{\lambda t} \tilde{\eta}_1(z(s))\, ds}{z(t)} \quad < \quad (\lambda - 1)t \cdot \frac{\tilde{\eta}_1(z(t))}{\eta(z(t))} \cdot \frac{\eta(z(t))}{z(t)}.$$

Thus

$$a_\lambda(t) < t \cdot \frac{\tilde{\eta}_1(z(t))}{z(t)}. \tag{9.27}$$

Hence for each $\lambda > 1$, there is $T(\lambda) > 0$, such that for $t > T(\lambda)$

$$0 < 1 - \frac{(-\log \circ \bar{G}^{-1})(\lambda t)}{(-\log \circ \bar{G}^{-1})(t)} < \frac{3(\lambda - 1)}{2} \cdot t \cdot \frac{\tilde{\eta}_1(z(t))}{z(t)}.$$

Thus

$$0 \le \limsup_{t\to 0^+} \left(1 - \frac{(-\log \circ \bar{G}^{-1})(\lambda t)}{(-\log \circ \bar{G}^{-1})(t)}\right) \le \frac{3(\lambda - 1)}{2} \limsup_{t\to 0^+} \left(t \cdot \frac{\tilde{\eta}_1(z(t))}{z(t)}\right).$$

Since $z(t) \to \infty$ as $t \to 0^+$ and $\tilde{\eta}(x) \sim \eta(x)$ as $x \to 0^+$ then

$$\begin{aligned}\limsup_{t\to 0^+}\left(t \cdot \frac{\tilde{\eta}_1(z(t))}{z(t)}\right) &= \limsup_{u\to\infty}\left(z^{-1}(u)\cdot\frac{\tilde{\eta}_1(u)}{u}\right)\\ &= \limsup_{u\to\infty}\left(z^{-1}(u)\cdot\frac{\tilde{\eta}(e^{-u})}{u}\right)\\ &= \limsup_{x\to 0^+}\left(z^{-1}\left(\log(1/x)\right)\cdot\frac{\eta(x)}{\log(1/x)}\right).\end{aligned}$$

Since $z(u) = (-\log\circ\bar{G}^{-1})(u)$ then $z^{-1}(u) = \bar{G}(e^{-u})$. Hence $z^{-1}\left(\log(1/x)\right) = \bar{G}\left(e^{-\log(1/x)}\right) = \bar{G}(x)$. Thus

$$\begin{aligned}\limsup_{t\to 0^+}\left(t \cdot \frac{\tilde{\eta}_1(z(t))}{z(t)}\right) &= \limsup_{x\to 0^+}\left(z^{-1}\left(\log(1/x)\right)\cdot\frac{\eta(x)}{\log(1/x)}\right)\\ &= \limsup_{x\to 0^+}\frac{\int_0^x u/g^2(u)\,du\cdot g^2(x)/x^2}{\log(1/x)} = L^*,\end{aligned}$$

by (9.25). Thus

$$0 \le \limsup_{t\to 0^+}\left(1 - \frac{(-\log\circ\bar{G}^{-1})(\lambda t)}{(-\log\circ\bar{G}^{-1})(t)}\right) \le \frac{3(\lambda-1)L^*}{2},$$

or for each $\lambda > 1$, we have

$$0 \le 1 - \liminf_{t\to 0^+}\frac{(-\log\circ\bar{G}^{-1})(\lambda t)}{(-\log\circ\bar{G}^{-1})(t)} \le \frac{3(\lambda-1)L^*}{2}.$$

Thus for each $\lambda > 1$

$$0 \le 1 - \liminf_{x\to 0^+}\frac{(-\log\circ\bar{G}^{-1})(x)}{(-\log\circ\bar{G}^{-1})(x/\lambda)} \le \frac{3(\lambda-1)L^*}{2}.$$

Put $\mu = 1/\lambda < 1$. Then for each $\mu < 1$, we have

$$0 \le 1 - \liminf_{x\to 0^+}\frac{(-\log\circ\bar{G}^{-1})(x)}{(-\log\circ\bar{G}^{-1})(\mu x)} \le \frac{3}{2}\left(\frac{1}{\mu} - 1\right)L^*.$$

Thus

$$0 \le 1 - \left(\limsup_{x\to 0^+}\frac{(-\log\circ\bar{G}^{-1})(\mu x)}{(-\log\circ\bar{G}^{-1})(x)}\right)^{-1} \le \frac{3}{2}\left(\frac{1}{\mu} - 1\right)L^*.$$

Define

$$L(\lambda) := \limsup_{x\to 0^+}\frac{(-\log\circ\bar{G}^{-1})(\lambda x)}{(-\log\circ\bar{G}^{-1})(x)}.$$

Then for each $\lambda \in (0,1)$

$$0 \le 1 - \frac{1}{L(\lambda)} \le \frac{3}{2}\left(\frac{1}{\lambda} - 1\right)L^*.$$

Hence

$$1 \le L(\lambda) \le \frac{1}{1 - \frac{3}{2}(\frac{1}{\lambda} - 1)L^*}.$$

The denominator on the right-hand side is positive if $1 > \lambda > (1 + 2/3L^*)^{-1}$. Thus

$$1 \le \liminf_{\lambda \to 1^-} L(\lambda) \le \limsup_{\lambda \to 1^-} L(\lambda) \le \limsup_{\lambda \to 1^-} \frac{1}{1 - \frac{3}{2}(\frac{1}{\lambda} - 1)L^*} = 1.$$

Hence $\lim_{\lambda \to 1^-} L(\lambda) = 1$ and thus

$$\lim_{\lambda \to 1^-} \limsup_{x \to 0^+} \frac{(-\log \circ \bar{G}^{-1})(\lambda x)}{(-\log \circ \bar{G}^{-1})(x)} = 1,$$

which is (9.14). □

Remark 40. Since $\bar{G}(x) \to 0$ as $x \to 0^+$ and $x \mapsto g^2(x)/x^2$ is asymptotically decreasing, then

$$\lim_{x \to 0^+} \frac{x^2 \log(1/x)}{g^2(x)} = 0.$$

Proof. Let $\tilde{\eta}(x) \sim \eta(x) = g^2(x)/x^2$ as $x \to 0^+$ and $\tilde{\eta}$ is decreasing. Thus $x\tilde{\eta}(x) \sim g^2(x)/x$ as $x \to 0^+$ and hence $1/(x\tilde{\eta}(x)) \sim x/g^2(x)$ as $x \to 0^+$. Thus, there is $x(\epsilon) > 0$ such that for $x < x(\epsilon)$

$$\frac{1}{x\tilde{\eta}(x)} < \frac{2x}{g^2(x)}.$$

Hence as $\tilde{\eta}$ is decreasing then

$$\int_0^{x(\epsilon)} \frac{1}{u\tilde{\eta}(u)}\,du \le 2\int_0^{x(\epsilon)} \frac{u}{g^2(u)}\,du =: I_2.$$

Let $a_n = 2^{-n}$ and $N(\epsilon)$ be such that $2^{-N(\epsilon)} < x(\epsilon)$. Then $a_{N(\epsilon)} < x(\epsilon)$, so

$$\int_0^{a_{N(\epsilon)}} \frac{1}{u\tilde{\eta}(u)}\,du \le I_2.$$

Thus

$$I_2 \ge \int_0^{a_{N(\epsilon)}} \frac{1}{u\tilde{\eta}(u)}\,du = \sum_{n=N(\epsilon)}^{\infty} \int_{a_{n+1}}^{a_n} \frac{1}{u\tilde{\eta}(u)}\,du.$$

For $a_{n+1} \le u \le a_n$, $1/a_{n+1} \ge 1/u \ge 1/a_n$ and since $\tilde{\eta}$ is decreasing then $\tilde{\eta}(a_{n+1}) \ge \tilde{\eta}(u) \ge \tilde{\eta}(a_n)$ and

$$\frac{1}{\tilde{\eta}(a_{n+1})} \le \frac{1}{\tilde{\eta}(u)} \le \frac{1}{\tilde{\eta}(a_n)}.$$

Hence for $a_{n+1} \le u \le a_n$.

$$\frac{1}{u\tilde{\eta}(u)} \ge \frac{1}{a_n\tilde{\eta}(a_{n+1})}.$$

So
$$\int_{a_{n+1}}^{a_n} \frac{1}{u\tilde{\eta}(u)}\,du \geq \frac{a_n - a_{n+1}}{a_n\tilde{\eta}(a_{n+1})}.$$
Therefore
$$\sum_{n=N(\epsilon)}^{\infty} \frac{a_n - a_{n+1}}{a_n} \cdot \frac{1}{\tilde{\eta}(a_{n+1})} \leq I_2.$$
Thus
$$\sum_{n=N(\epsilon)}^{\infty} \frac{2^{-n} - 2^{-(n+1)}}{2^{-n}} . \frac{1}{\tilde{\eta}(2^{-(n+1)})} \leq I_2,$$
or equivalently
$$\sum_{n=N(\epsilon)}^{\infty} \frac{1}{\tilde{\eta}(2^{-(n+1)})} \leq 2I_2.$$
Define $b_n := 1/\tilde{\eta}(2^{-(n+1)})$. Then $\sum_{n=N(\epsilon)}^{\infty} b_n \leq 2I_2$ and (b_n) is decreasing, so $nb_n \to 0$ as $n \to \infty$ and $n/\tilde{\eta}(2^{-(n+1)}) \to 0$ as $n \to \infty$. Let $2^{-n-1} \leq u \leq 2^{-n}$; then $\tilde{\eta}(2^{-n}) \leq \tilde{\eta}(u) \leq \tilde{\eta}(2^{-(n+1)})$ so for $2 \leq u \leq 2^{-(n+1)}$ then
$$0 < \frac{\log(1/x)}{\tilde{\eta}(x)} \leq \frac{\log(2\cdot 2^n)}{\tilde{\eta}(2^{-n})} = \frac{(\log 2 + n\log 2)}{\tilde{\eta}(2^{-n})} = \frac{\log 2}{\tilde{\eta}(2^{-n})} + \frac{n\log 2}{\tilde{\eta}(2^{-n})} \to 0, \quad \text{as } n \to \infty.$$
Hence
$$\lim_{x\to 0^+} \frac{\log(1/x)}{\tilde{\eta}(x)} = 0.$$
Since $\eta(x) \sim \tilde{\eta}(x)$ as $x \to 0^+$ then
$$\lim_{x\to 0^+} \frac{\log(1/x)}{\eta(x)} = 0.$$
So
$$\lim_{x\to 0^+} \frac{x^2\log(1/x)}{g^2(x)} = 0,$$
as claimed. □

Remark 41. If $g \in RV_0(\beta)$ and $\beta \in [0,1)$ then (9.25) holds.

Proof. By Karamata's Theorem (see e.g. Theorem 1.5.11 in [12])
$$\int_0^x \frac{u}{g^2(u)}\,du \sim \frac{1}{2-2\beta}\cdot\frac{x^2}{g^2(x)}, \quad \text{as } x \to 0^+,$$
since $x \to x/g^2(x) \in RV_0(1-2\beta)$ and thus $\bar{G}(x) \in RV_0(2-2\beta)$. Hence
$$\lim_{x\to 0^+} \frac{\int_0^x u/g^2(u)\,du}{x^2/g^2(x)\cdot\log(1/x)} = \frac{1}{2(1-\beta)} \lim_{x\to 0^+} \frac{x/g^2(x)}{x^2/g^2(x)\cdot\log(1/x)} = 0.$$
□

Remark 42. If $x \mapsto g^2(x)/x$ is asymptotic to a decreasing function, then (9.25) holds.

Proof. Let $\eta_1(x) = g^2(x)/x$ and $\tilde{\eta}_1(x) \sim \eta_1(x)$ as $x \to 0^+$ where $\tilde{\eta}_1$ is decreasing. Hence

$$\lim_{x\to 0^+} \frac{\int_0^x u/g^2(u)\,du}{\int_0^x 1/\tilde{\eta}_1(u)\,du} = \lim_{x\to 0^+} \frac{\int_0^x 1/\eta_1(u)\,du}{\int_0^x 1/\tilde{\eta}_1(u)\,du} = 1.$$

Consider

$$\begin{aligned}\frac{\int_0^x u/g^2(u)\,du}{x^2/g^2(x)\cdot\log(1/x)} &= \frac{\int_0^x 1/\eta_1(u)\,du}{\int_0^x 1/\tilde{\eta}_1(u)\,du}\cdot\frac{\int_0^x 1/\tilde{\eta}_1(u)\,du}{x\cdot x/g^2(x)\cdot\log(1/x)}\\ &= \frac{\int_0^x 1/\eta_1(u)\,du}{\int_0^x 1/\tilde{\eta}_1(u)\,du}\cdot\frac{\int_0^x 1/\tilde{\eta}_1(u)\,du}{x/\tilde{\eta}_1(x)}\cdot\frac{1}{\tilde{\eta}_1(x)/\eta_1(x)}\cdot\frac{1}{\log(1/x)}.\end{aligned}$$

For $0 < u < x$, $\tilde{\eta}_1(u) > \tilde{\eta}_1(x)$ so $1/\tilde{\eta}_1(u) < 1/\tilde{\eta}_1(x)$. Thus

$$\int_0^x \frac{1}{\tilde{\eta}_1(u)}\,du \le \int_0^x \frac{1}{\tilde{\eta}_1(x)}\,du = \frac{x}{\tilde{\eta}_1(x)}.$$

Therefore

$$\limsup_{x\to 0^+} \frac{\int_0^x 1/\tilde{\eta}_1(u)\,du}{x/\tilde{\eta}_1(x)} \le 1.$$

Thus

$$\lim_{x\to 0^+} \frac{\int_0^x u/g^2(u)\,du}{x^2/g^2(x)\cdot\log(1/x)} = 0,$$

as claimed. □

Remark 43. If $g \in RV_0(1)$, then (9.25) is not always true. In fact, we can have

$$\lim_{x\to 0^+} \frac{\int_0^x u/g^2(u)\,du}{x^2/g^2(x)\cdot\log(1/x)} = \infty.$$

□

Remark 40 identifies a critical rate of decay of the step-size when the diffusion is dominant. This is explored further in the following discussion.

Remark 44. Suppose μ is defined by

$$\mu(x) := \frac{g(x)}{x(\log(1/x))^{1/2}}, \tag{9.28}$$

then $\mu(x) \to \infty$ as $x \to 0^+$. Furthermore if $g \in RV_0(1)$, then $\mu \in RV_0(0)$.

Proof. By the definition of μ $g^2(x) = x^2\log(1/x)\,\mu^2(x)$ or $1/\mu^2(x) = x^2/g^2(x)\cdot\log(1/x)$. Thus $1/\mu^2(x) \to 0$ as $x \to 0^+$ so $\mu(x) \to \infty$ as $x \to 0^+$. If $g \in RV_0(1)$ then $x \mapsto g^2(x)/x^2 \in RV_0(0)$, thus $x \mapsto \mu^2(x) = g^2(x)/x^2 \cdot 1/\log(1/x) \in RV_0(0)$ and $\mu \in RV_0(0)$. □

The following lemma determines a sufficient condition on μ such that (9.25) holds.

Lemma 39. *Define* μ *by* (9.28)*. Suppose* $\mu \in C^1$*. Then*

$$\lim_{x\to 0^+} \frac{-x\mu'(x)\log(1/x)}{\mu(x)} =: M^* \in [0,\infty],$$

implies

$$\lim_{x\to 0^+} \frac{\int_0^x u/g^2(u)\,du}{x^2/g^2(x)\cdot\log(1/x)} = \frac{1}{2M^*} \in (0,\infty].$$

Proof. Since $\mu \in C^1$. Then by L'Hôpital's Rule

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\int_0^x u/g^2(u)\,du}{x^2/g^2(x)\cdot\log(1/x)} = \lim_{x\to 0^+} \frac{\int_0^x u/g^2(u)\,du}{1/\mu^2(x)} &= \lim_{x\to 0^+} \frac{x/(x^2\log(1/x)\,\mu^2(x))}{-2\mu(x)^{-3}\cdot\mu'(x)} \\
&= \lim_{x\to 0^+} \frac{1/(x\log(1/x))\cdot 1/\mu^2(x)}{2\cdot 1/\mu^3(x)\cdot -\mu'(x)} \\
&= \frac{1}{2}\left(\lim_{x\to 0^+} \frac{-x\mu'(x)\log(1/x)}{\mu(x)}\right)^{-1} \\
&= \frac{1}{2M^*},
\end{aligned}$$

as claimed. □

We now explore examples to which Theorem 62 applies. The first example show the theorem discriminates for a parametrised family of g's, critical parameter values for which finite-time or super-exponential stability occurs. We can also identify the appropriate case in Theorem 62 to apply.

Example 64. Suppose

$$g(x) = x\log\left(\tfrac{1}{x}\right)^{1/2}\left(\log\log\left(\tfrac{1}{x}\right)\right)^{\alpha}.$$

Then $\mu(x) = (\log\log(1/x))^\alpha$ and

$$\frac{x\mu'(x)}{\mu(x)} = \frac{-\alpha}{\log(1/x)\log\log(1/x)}.$$

Hence

$$\lim_{x\to 0^+} \frac{-x\mu'(x)\log(1/x)}{\mu(x)} = \lim_{x\to 0^+} \frac{\alpha}{\log\log(1/x)} = 0,$$

and so by Lemma 39

$$\lim_{x\to 0^+} \frac{\int_0^x u/g^2(u)\,du}{x^2/g^2(x)\cdot\log(1/x)} = \infty.$$

Also

$$\bar{G}(x) = \int_0^x \frac{u}{g^2(u)}\,du = \int_0^x \frac{u}{u^2 \log(1/u)\left(\log\log(1/u)\right)^{2\alpha}}\,du = \int_{\log\log(1/x)}^{\infty} \frac{1}{w^{2\alpha}}\,dw = \frac{\left(\log\log(1/x)\right)^{-(2\alpha-1)}}{2\alpha-1}.$$

If $\alpha \le 1/2$, then $\int_x^1 u/g^2(u)\,du \to \infty$ as $x \to 0^+$ and we do not hit zero in finite-time. However, if $\alpha > 1/2$ then $\bar{G}(x) \to 0$ as $x \to 0^+$. Thus, for $\alpha > 1/2$

$$\bar{G}(x) = \frac{1}{(2\alpha-1)\left(\log\log(1/x)\right)^{(2\alpha-1)}}.$$

So we have from Lemma 37 and Theorem 62,

$$\lim_{t\to T^-} \frac{\bar{G}(X(t))}{T-t} = \frac{1}{2} - L, \quad \text{a.s.}.$$

By Theorem 63 we could have also concluded that

$$\lim_{x\to 0^+} \frac{\bar{G}(x^\lambda)}{\bar{G}(x)} = 1, \quad \forall\, \lambda < 1.$$

We now demonstrate that (9.12) holds. Since $x = \bar{G}(\bar{G}^{-1}(x))$ then

$$(-\log\circ \bar{G}^{-1})(x) = \exp\left((2\alpha-1)^{-1/(2\alpha-1)} \cdot x^{-1/(2\alpha-1)}\right) = \exp\left(a_* \cdot x^{-1/(2\alpha-1)}\right),$$

where $a_* := (2\alpha-1)^{-1/(2\alpha-1)}$. Then

$$\frac{(-\log\circ \bar{G}^{-1})(\lambda x)}{(-\log\circ \bar{G}^{-1})(x)} = \frac{\exp\left(a_* \cdot (\lambda x)^{-1/(2\alpha-1)}\right)}{\exp\left(a_* \cdot x^{-1/(2\alpha-1)}\right)} = \exp\left(a_* \cdot x^{-1/(2\alpha-1)} \cdot \left(\lambda^{-1/(2\alpha-1)} - 1\right)\right).$$

If $\lambda < 1$ and $\alpha > 1/2$, then $\lambda^{-1/(2\alpha-1)} - 1 > 0$ and $a_* \cdot x^{-1/(2\alpha-1)} \to \infty$ as $x \to 0^+$. Thus

$$\lim_{x\to 0^+} \frac{(-\log\circ \bar{G}^{-1})(\lambda x)}{(-\log\circ \bar{G}^{-1})(x)} = \infty.$$

□

A cursory glance at the structure to Theorem 62 might suggest that the asymptotic behaviour (9.13) and (9.15) are incompatible. The following examples demonstrate that this is not the case by using Theorem 62 to show the implication (9.14) implies (9.15) holds but using Theorem 63 to show that the asymptotic behaviour in (9.13) also holds.

Example 65. Suppose

$$g(x) = x\left(\log\left(\tfrac{1}{x}\right)\right)^{1/2+c} = x\left(\log\left(\tfrac{1}{x}\right)\right)^{1/2}\left(\log\left(\tfrac{1}{x}\right)\right)^{c}.$$

Then $\mu(x) = (\log(1/x))^c$. We have that

$$\begin{aligned}\bar{G}(x) = \int_0^x \frac{u}{g^2(u)}\,du = \int_0^x \frac{u}{u^2(\log(1/u))^{1+2c}}\,du &= \left.\frac{v^{-1-2c+1}}{-1-2c+1}\right|_{\log(1/x)}^{\infty} \\ &= \frac{1}{2c}\left(\log\left(\frac{1}{x}\right)\right)^{-2c}.\end{aligned}$$

Thus $\bar{G}(x) \to 0$ as $x \to 0^+$. We have $\mu'(x) = -c/x \cdot (\log(1/x))^{c-1}$. Hence

$$\frac{-x\mu'(x)\log(1/x)}{\mu(x)} = \frac{-x \cdot -c\,(\log(1/x))^{c-1}\log(1/x)}{x\,(\log(1/x))^c} = c \in (0,\infty).$$

Thus

$$\lim_{x\to 0^+} \frac{\int_0^x u/g^2(u)\,du}{x^2/g^2(x)\cdot\log(1/x)} = \frac{1}{2c} \in (0,\infty).$$

So (9.25) holds and hence (9.14) holds. Thus by Theorem 62 part (ii)

$$\lim_{t\to T^-} \frac{-\log X(t)}{(-\log\circ\bar{G}^{-1})((\frac{1}{2}-L)(T-t))} = 1, \quad \text{a.s..} \tag{9.29}$$

On the other hand we can use Theorem 63 to determine the asymptotic behaviour. Let $\lambda < 1$

$$\bar{G}(x^\lambda) = \frac{1}{2c}\left(\log\left(\frac{1}{x^\lambda}\right)\right)^{-2c} = \frac{1}{2c}\left(\lambda\log\left(\frac{1}{x}\right)\right)^{-2c} = \frac{\lambda^{-2c}}{2c}\left(\log\left(\frac{1}{x}\right)\right)^{-2c}.$$

Thus $\bar{G}(x^\lambda) = \lambda^{-2c}\bar{G}(x)$ and

$$\lim_{x\to 0^+} \frac{\bar{G}(x^\lambda)}{\bar{G}(x)} = \lambda^{-2c}.$$

By Theorem 63

$$\lim_{t\to T^-} \frac{\bar{G}(X(t))}{T-t} = \frac{1}{2} - L, \quad \text{a.s..} \tag{9.30}$$

We now show that (9.29) and (9.30) are equivalent. Since $x = \bar{G}(\bar{G}^{-1}(x))$ then

$$(-\log\circ\bar{G}^{-1})(x) = \left(\frac{1}{2cx}\right)^{1/2c}.$$

Hence (9.29) reads

$$\lim_{t\to T^-} \frac{-\log X(t)}{\left(2c(\frac{1}{2}-L)(T-t)\right)^{-1/2c}} = 1, \quad \text{a.s.,}$$

and (9.30) reads

$$\lim_{t\to T^-} \frac{1/2c\cdot(-\log X(t))^{-2c}}{T-t} = \frac{1}{2} - L, \quad \text{a.s.,}$$

and both limits are clearly equivalent. □

Example 66. Suppose

$$g(x) = x\left(\log\left(\tfrac{1}{x}\right)\right)^{1/2}\left(\log\log\left(\tfrac{1}{x}\right)\right)^{1/2}\left(\log\log\log\left(\tfrac{1}{x}\right)\right)^{\alpha}.$$

Then $\mu(x) = \left(\log\log\left(\tfrac{1}{x}\right)\right)^{1/2}\left(\log\log\log\left(\tfrac{1}{x}\right)\right)^{\alpha}$ and so $\bar{G}$ is given by

$$\begin{aligned}\bar{G}(x) = \int_0^x \frac{u}{g^2(u)}\,du \;&=\; \int_0^x \frac{u}{u^2\log(1/u)\log\log(1/u)\left(\log\log\log(1/u)\right)^{2\alpha}}\,du \\ &=\; \frac{1}{2\alpha-1}\left(\log\log\log(1/x)\right)^{-(2\alpha-1)}.\end{aligned}$$

If $\alpha \le 1/2$, then $\int_x^1 u/g^2(u)\,du \to \infty$ as $x \to 0^+$ while if $\alpha > 1/2$, then $\bar{G}(x) \to 0$. We now check whether condition (9.12) holds. Since $x = \bar{G}(\bar{G}^{-1}(x))$ then

$$(-\log\circ\bar{G}^{-1})(x) = \exp\left(\exp\left((2\alpha-1)^{-1/(2\alpha-1)}\cdot x^{-1/(2\alpha-1)}\right)\right) = \exp\left(\exp\left(a_*\cdot x^{-1/(2\alpha-1)}\right)\right),$$

where $a_* := (2\alpha-1)^{-1/(2\alpha-1)}$. Then

$$\begin{aligned}\frac{(-\log\circ\bar{G}^{-1})(\lambda x)}{(-\log\circ\bar{G}^{-1})(x)} \;&=\; \frac{\exp\left(\exp\left(a_*\cdot(\lambda x)^{-1/(2\alpha-1)}\right)\right)}{\exp\left(\exp\left(a_*\cdot x^{-1/(2\alpha-1)}\right)\right)} \\ &=\; \exp\left(\exp\left(a_*\cdot(\lambda x)^{-1/(2\alpha-1)}\right) - \exp\left(a_*\cdot x^{-1/(2\alpha-1)}\right)\right) \\ &=\; \exp\left(\exp\left(z_*\cdot\lambda^{-1/(2\alpha-1)}\right) - \exp\left(z_*\right)\right),\end{aligned}$$

where $z_* := a_*\cdot x^{-1/(2\alpha-1)}$. If $\lambda < 1$ and $\alpha > 1/2$, then $\lambda^{1/(2\alpha-1)} < 1$ and $\lambda^{-1/(2\alpha-1)} > 1$. Thus $z_* \to \infty$ as $x \to 0^+$. Thus (9.12) holds:

$$\lim_{x\to 0^+}\frac{(-\log\circ\bar{G}^{-1})(\lambda x)}{(-\log\circ\bar{G}^{-1})(x)} = \infty.$$

Thus by Theorem 62, part (i)

$$\lim_{t\to T^-}\frac{\bar{G}(X(t))}{T-t} = \lim_{t\to T^-}\frac{1/(2\alpha-1)\cdot\left(\log\log\log(1/X(t))\right)^{-(2\alpha-1)}}{T-t} = \frac{1}{2} - L, \quad \text{a.s..}$$

□

Chapter 10

Asymptotic Behaviour of Numerical Schemes for Superlinear SDEs

10.1 Introduction

In this chapter, we show by making an appropriate discretisation of the SDE (1.17) viz.,

$$dX(t) = f(X(t))\,dt + g(X(t))\,dB(t),$$

that all the continuous time results in Chapter 9 for super-exponential stability and finite-time convergence can be recovered. This is achieved via a discretisation of the SDE for the process $Z(t) := -\log X(t)$ and taking a step-size at state x for the simulated value of $X(t)$ given by

$$h(x) := \Delta \min\left(1, \frac{x}{|f(x)|}, \frac{x^2}{g^2(x)}\right).$$

More specifically we recover faithfully the positivity of simulated solutions, the presence or absence of a finite stability time and the asymptotic rates of convergence detailed in the main results in Chapter 9 for all positive values of Δ. We recover analogous convergence results regardless of whether these results refer to super-exponential rates of convergence or the asymptotic behaviour as the finite stability time is approached.

10.2 Discrete-Time Stability and Finite-Time Convergence

The logarithmic transformation is also helpful for understanding the convergence rate and asymptotic behaviour of SDEs in the neighbourhood of the time at which equilibrium is reached. Preserving the positivity of solutions of SDEs by conventional direct discretisation is essentially impossible to achieve for the highly non-linear equations studied here even with state-varying step size. A pre-transformation which preserves positivity, such the logarithmic one, is much more intuitively natural for SDEs.

Let $h(x) > 0$ and let $X(t)$ be the solution of (1.17). We approximate the solution $Z(t) := -\log X(t)$ at the time t_n by Z_n. Discretise $Z(t)$ by defining the sequences (Z_n), (X_n) and (t_n), where $Z_0 = -\log\zeta$, $X_0 = \zeta$ and $t_0 = 0$, by

$$Z_{n+1} = Z_n + h(X_n)\cdot\left(\frac{-f(X_n)}{X_n} + \frac{1}{2}\cdot\frac{g^2(X_n)}{X_n^2}\right) + \sqrt{h(X_n)}\cdot\frac{-g(X_n)}{X_n}\cdot\xi_{n+1}, \quad n \geq 0 \tag{10.1}$$

$$X_{n+1} = e^{-Z_{n+1}}, \quad n \geq 0, \tag{10.2}$$

$$t_{n+1} = t_n + h(X_n), \quad n \geq 0, \tag{10.3}$$

where

$$h(x) = \Delta\cdot\min\left(1, \frac{x}{|f(x)|}, \frac{x^2}{g^2(x)}\right). \tag{10.4}$$

and (ξ_n) is a sequence of independent and identically distributed Standard Normal random variables. We define as before

$$\lim_{n\to\infty} t_n =: \hat{T}_h \tag{10.5}$$

recognising that this limit can be finite or infinite. Under monotonicity conditions on f or g (whichever is dominant asymptotically at the boundary zero or infinity) the scheme correctly predicts in all circumstances whether the boundaries are reached in finite time or not. This is laid out in Theorem 67.

Theorem 67. *Suppose Z_n is the solution of* (10.1). *Let L, (t_n), $\hat{T}_h$ be defined by* (1.25), (10.3) *and* (10.5).

(a) If $L \in (-\infty, 1/2)$ and $x \mapsto x^2/g^2(x)$ is asymptotically increasing, then

  (i) $\int_{0+}^1 u/g^2(u)\,du = \infty$ implies $t_n \to \infty$ as $n \to \infty$ a.s..

  (ii) $\int_{0+}^1 u/g^2(u)\,du < \infty$ implies $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ a.s..

(b) If $L = -\infty$ and $x \mapsto x/|f(x)|$ is asymptotically increasing, then

  (i) $\int_{0+}^1 1/|f(u)|\,du = \infty$ implies $t_n \to \infty$ as $n \to \infty$ a.s..

  (ii) $\int_{0+}^1 1/|f(u)|\,du < \infty$ implies $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ a.s..

For simplicity, we prove Theorem 67 with the step-size

$$h(X_n) = \Delta\cdot\min\left(1, 1\Big/\left|\frac{f(X_n)}{X_n} - \frac{g^2(X_n)}{2X_n^2}\right|\right). \tag{10.6}$$

The proof when the step-size obeys (10.4) is similar. Theorem 60 dealt with finite-time or super-exponential stability. Theorem 67 is the discrete time analogue. As can be seen in Theorem 78 we can prove analogous preliminary asymptotic results for those in Lemma 40 below with the simpler step-size specified in (10.4).

Lemma 40. *Suppose* (9.5) *holds,* Z_n *is the solution of* (10.1) *and* X_n *obeys* (10.2) *where*

$$t_{n+1} = \sum_{j=0}^{n} h(X_n) = \sum_{j=0}^{n} \Delta \cdot \min\left(1, 1 \Big/ \left|\frac{f(X_n)}{X_n} - \frac{g^2(X_n)}{2X_n^2}\right|\right).$$

and L *is defined by* (1.25). *Then*

$$\lim_{n\to\infty} X_n = 0, \qquad \lim_{n\to\infty} \sum_{j=0}^{n-1} \min\left(1, \frac{g^2(X_j)}{2X_j^2} - \frac{f(X_j)}{X_j}\right) = \infty,$$

and

$$\lim_{n\to\infty} \frac{\log X_n}{\sum_{j=0}^{n-1} \min\left(1, g^2(X_j)/2X_j^2 - f(X_j)/X_j\right)} = -\Delta, \quad a.s..$$

Proof. Under (9.5), $xf(x)/g^2(x) \le \Lambda < 1/2, \forall\, x \in \mathbb{R}^+$. Thus $f(x)/x \le \Lambda \cdot g^2(x)/x^2$ and so

$$\frac{f(x)}{x} - \frac{g^2(x)}{2x^2} \le \left(\Lambda - \frac{1}{2}\right)\frac{g^2(x)}{x^2} < 0.$$

Hence

$$\tilde{\mu}(x) := \left|\frac{f(x)}{x} - \frac{g^2(x)}{2x^2}\right| = \frac{g^2(x)}{2x^2} - \frac{f(x)}{x} \ge \left(\frac{1}{2} - \Lambda\right)\frac{g^2(x)}{x^2} > 0.$$

Now defining $\tilde{\xi}_{n+1} = -\xi_{n+1}$ we get

$$\begin{aligned} Z_{n+1} &= Z_n + h(X_n)\cdot\tilde{\mu}(X_n) + \sqrt{h(X_n)}\cdot\frac{g(X_n)}{X_n}\cdot\tilde{\xi}_{n+1} \\ &= Z_n + \Delta\cdot\min\left(\frac{1}{\tilde{\mu}(X_n)}, 1\right)\cdot\tilde{\mu}(X_n) + \frac{\sqrt{\Delta}g(X_n)}{X_n}\cdot\min\cdot\left(1, \frac{1}{\sqrt{\tilde{\mu}(X_n)}}\right)\cdot\tilde{\xi}_{n+1} \\ &= Z_n + \Delta\cdot\min\left(1, \tilde{\mu}(X_n)\right) + \sqrt{\Delta}\cdot\min\left(\frac{g(X_n)}{X_n}, \frac{g(X_n)/X_n}{\sqrt{\tilde{\mu}(X_n)}}\right)\cdot\tilde{\xi}_{n+1} \\ &= Z_n + \Delta\cdot\mu_n + \sqrt{\Delta}\cdot\eta_n\cdot\tilde{\xi}_{n+1}, \end{aligned}$$

where

$$\mu_n := \min\left(1, \tilde{\mu}(X_n)\right) \quad \text{and} \quad \eta_n := \min\left(\frac{g(X_n)}{X_n}, \frac{g(X_n)/X_n}{\sqrt{\tilde{\mu}(X_n)}}\right).$$

Thus for $n \ge 1$

$$Z_n = Z_0 + \sum_{j=0}^{n-1}\Delta\cdot\mu_j + \sum_{j=0}^{n-1}\sqrt{\Delta}\cdot\eta_j\cdot\tilde{\xi}_{j+1} = Z_0 + \sum_{j=0}^{n-1}\Delta\cdot\mu_j + M(n),$$

where $M(n) := \sum_{j=0}^{n-1} \sqrt{\Delta} \cdot \eta_j \cdot \tilde{\xi}_{j+1}$. Note that

$$\mu_n = \min\left(1, \tilde{\mu}(X_n)\right) \geq \min\left(1, \left(\frac{1}{2} - \Lambda\right) \frac{g^2(X_n)}{X_n^2}\right) > 0,$$

and

$$\begin{aligned} \eta_n^2 = \min\left(\frac{g^2(X_n)}{X_n^2}, \frac{g^2(X_n)/X_n^2}{\tilde{\mu}(X_n)}\right) &\leq \min\left(\frac{g^2(X_n)}{X_n^2}, \frac{g^2(X_n)/X_n^2}{(1/2-\Lambda)g^2(X_n)/X_n^2}\right) \\ &= \min\left(\frac{g^2(X_n)}{X_n^2}, \frac{1}{(1/2-\Lambda)}\right) \\ &= \frac{1}{1/2-\Lambda} \min\left(1, \left(\frac{1}{2}-\Lambda\right)\frac{g^2(X_n)}{X_n}\right) \leq \frac{\mu_n}{1/2-\Lambda}. \end{aligned}$$

Thus $(1/2 - \Lambda)\, \eta_n^2 \leq \mu_n$. Then M is an L^2-martingale because $\eta^2 \leq (1/2 - \Lambda)^{-1}$ with quadratic variation $\langle M \rangle (n) = \Delta \sum_{j=0}^{n-1} \eta_j^2$ - we give a careful proof of this later in this proof. Consider the events:

$$A = \{\omega : \langle M \rangle (n, \omega) \to L < \infty,\, n \to \infty\} \quad \text{and} \quad A' = \{\omega : \langle M \rangle (n, \omega) \to \infty,\, n \to \infty\}.$$

Then there is an a.s. subevent of A on which $M(n) \to L'$ by the martingale convergence theorem - see Theorem 12.13 in [59]. Moreover, $\eta_n \to 0$ as $n \to \infty$ on A a.s.. Since $\mu_n > 0$ then we have either $\sum_{j=0}^{\infty} \mu_j = \infty$ or $\sum_{j=0}^{\infty} \mu_j < \infty$. In the former case $Z_n \to \infty$ and $X_n \to 0$ as $n \to \infty$ and therefore

$$\lim_{n\to\infty} \frac{Z_n}{\sum_{j=0}^{n-1} \mu_j} = \Delta.$$

In the latter case $\sum_{j=0}^{\infty} \mu_j < \infty$ in which case $Z_n \to Z^* \in (-\infty, \infty)$ as $n \to \infty$, and therefore $X_n \to X^* \in (0, \infty)$ as $n \to \infty$. But this implies $\mu_n \to \min(1, \tilde{\mu}(X^*)) > 0$ as $n \to \infty$, so $\sum_{j=0}^{\infty} \mu_j = \infty$, a contradiction. Therefore a.s. on A

$$\lim_{n\to\infty} \frac{\log X_n}{\sum_{j=0}^{n-1} \mu_j} = -\Delta \quad \text{and} \quad \lim_{n\to\infty} \sum_{j=0}^{n-1} \mu_j = \infty, \quad \forall\, \omega \in A \text{ a.s.}. \tag{10.7}$$

Suppose next we are on A'. Since $\langle M \rangle (n) = \sum_{j=0}^{n-1} \eta_j^2 \to \infty$ as $n \to \infty$ and $(1/2 - \Lambda)\, \eta_n^2 \leq \mu_n$ then $\sum_{j=0}^{n-1} \mu_j \to \infty$ as $n \to \infty$ on A'. Thus

$$\frac{Z_n}{\sum_{j=0}^{n-1} \mu_j} = \frac{Z_0}{\sum_{j=0}^{n-1} \mu_j} + \Delta + \frac{M(n)}{\langle M \rangle (n)} \cdot \frac{\sum_{j=0}^{n-1} \eta_j^2}{\sum_{j=0}^{n-1} \mu_j}. \tag{10.8}$$

On A', a.s. we have $M(n)/\langle M \rangle (n) \to 0$ as $n \to \infty$ by the strong law of large numbers for martingales - see Theorem 12.14 in [59]. Also

$$0 \leq \frac{\sum_{j=0}^{n-1} \eta_j^2}{\sum_{j=0}^{n-1} \mu_j} \leq \frac{1}{1/2 - \Lambda} < \infty.$$

Therefore the last term on the right-hand side of (10.8) tends to zero as $n \to \infty$ on A'. Clearly, so does the first term. Hence once again a.s. on A', we have

$$\lim_{n\to\infty} \frac{\log X_n}{\sum_{j=0}^{n-1} \mu_j} = -\Delta \quad \text{and} \quad \lim_{n\to\infty} \sum_{j=0}^{n-1} \mu_j = \infty, \quad \forall\, \omega \in A' \text{ a.s..} \tag{10.9}$$

Combining (10.7) and (10.9) implies

(i)

$$\lim_{n\to\infty} \frac{\log X_n}{\sum_{j=0}^{n-1} \min\left(1, g^2(X_j)/2X_j^2 - f(X_j)/X_j\right)} = -\Delta, \quad \text{a.s..}$$

(ii)

$$\lim_{n\to\infty} X_n = 0 \quad \text{and} \quad \lim_{n\to\infty} \sum_{j=0}^{n-1} \min\left(1, \frac{g^2(X_j)}{2X_j^2} - \frac{f(X_j)}{X_j}\right) = \infty,$$

as required.

Note that $M(n)$ is in the form $M_n := M(n) = \sum_{j=0}^{n-1} K_j \xi_{j+1}$ where the $|K_j|$'s are uniformly bounded by a constant $C > 0$. Therefore $\mathbb{E}[M_n^2] \leq Cn^2 < \infty$ for all n. Moreover, because the K_j is $\mathscr{F}_j$-measurable, where $\mathscr{F}_n$ is the filtration generated by the iid Standard Normal random variables $(\xi)_{n\geq 1}$, we have that

$$\begin{aligned} \mathbb{E}[M_{n+1}|\mathscr{F}_n] = \mathbb{E}[M_n + K_n\xi_{n+1}|\mathscr{F}_n] &= \mathbb{E}[M_n|\mathscr{F}_n] + \mathbb{E}[K_n\xi_{n+1}|\mathscr{F}_n] \\ &= M_n + K_n\mathbb{E}[\xi_{n+1}|\mathscr{F}_n] = M_n. \end{aligned}$$

Therefore, as (M_n) is clearly adapted to $(\mathscr{F}_n)$, M_n is an L^2-martingale. □

10.3 Asymptotic Behaviour for Discrete Equations

We show in the case that $\hat{T}_h = \infty$, that the rate of super-exponential convergence exhibited by the solution of the SDE in Theorem 61 is recovered precisely by the numerical scheme (10.1). This holds regardless of whether the drift or diffusion term dominates. We also recover, in the case when f and g have finite non-trivial derivatives at zero, the finite Liapunov exponent a.s. of the original SDE.

Theorem 68. *Suppose $L = -\infty$ and $x \mapsto x/|f(x)|$ is asymptotically non-decreasing. Let L, F, $\hat{T}_h$, (t_n) and h be defined by* (1.25), (1.29), (10.5), (10.3) *and* (10.6). *Define*

$$\lim_{x\to 0^+} \frac{x}{|f(x)|} =: c.$$

(a) *Let* $c \in (0,\infty)$*. Then* $X_n \in (0,\infty)$ *for all* $n \geq 0$ *a.s.,* $X_n \to 0$ *as* $n \to \infty$ *a.s.,* $t_n \to \infty$ *as* $n \to \infty$ *a.s. and*

$$\lim_{n\to\infty} \frac{\log X_n}{t_n} = f'(0) < 0, \quad a.s..$$

(b) *Let* $c = 0$.

(i) *If* f *obeys* (1.27)*, then* $X_n \in (0,\infty)$ *for all* $n \geq 0$ *a.s.,* $X_n \to 0$ *as* $n \to \infty$ *a.s.,* $t_n \to \infty$ *as* $n \to \infty$ *a.s. and*

$$\lim_{n\to\infty} \frac{F(X_n)}{t_n} = 1, \quad a.s..$$

(ii) *If* f *obeys* (1.26)*, then* $X_n \in (0,\infty)$ *for all* $n \geq 0$ *a.s.,* $X_n \to 0$ *as* $n \to \infty$ *a.s.,* $t_n \to \hat{T}_h$ *as* $n \to \infty$ *a.s..*

Notice that Theorem 68 part (b) tackles the case of Theorem 67 (b) when $L = -\infty$.

Proof of Theorem 68. We first prove part (a). Since $\lim_{x\to 0^+} x/|f(x)| =: c \in (0,\infty)$ then $|f(x)| \sim x/c$ as $x \to 0^+$ so $f'(0) = -1/c < 0$. Thus

$$\lim_{x\to 0^+} \left(\frac{g^2(x)}{2x^2} - \frac{f(x)}{x} \right) = \lim_{x\to 0^+} \frac{-f(x)}{x} = -f'(0) > 0.$$

Therefore

$$\lim_{n\to\infty} \min\left(1, \frac{g^2(X_n)}{2X_n^2} - \frac{f(X_n)}{X_n}\right) = \min(1, -f'(0)).$$

As

$$t_n = \sum_{j=0}^{n-1} \Delta \cdot \min\left(1, \frac{1}{g^2(X_j)/2X_j^2 - f(X_j)/X_j}\right),$$

then

$$\lim_{n\to\infty} \frac{t_n}{n} = \Delta \cdot \min\left(1, -1/f'(0)\right),$$

so $t_n \to \infty$ as $n \to \infty$. Furthermore,

$$\lim_{n\to\infty} \frac{\log X_n}{n \cdot \min\left(1, -f'(0)\right)} = -\Delta, \quad \text{a.s..}$$

Hence, as $f'(0) < 0$, then

$$\lim_{n\to\infty} \frac{\log X_n}{t_n} = \lim_{n\to\infty} \frac{\log X_n}{n \cdot \Delta \min\left(1, -1/f'(0)\right)} = f'(0) < 0,$$

as required. We now prove part (b). In this case $\lim_{x\to 0^+} x/|f(x)| = 0$ then

$$\frac{g^2(x)}{2x^2} - \frac{f(x)}{x} = -\frac{f(x)}{x}\left(1 - \frac{g^2(x)}{2xf(x)}\right) \sim \frac{|f(x)|}{x}, \quad \text{as } x \to 0^+,$$

and
$$\lim_{x\to 0^+}\left(\frac{g^2(x)}{2x^2}-\frac{f(x)}{x}\right)=\infty.$$
Therefore there is $N_0(\epsilon)$ such that $\forall\, n\geq N_0(\epsilon)$
$$\min\left(1,\frac{g^2(X_n)}{2X_n^2}-\frac{f(X_n)}{X_n}\right)=1.$$
Hence
$$\lim_{n\to\infty}\frac{\log X_n}{n}=-\Delta,\quad \text{a.s.},$$
and for $n\geq N_0(\epsilon)+1$
$$t_n=t_{N_0(\epsilon)}+\sum_{j=N_0(\epsilon)}^{n-1}\frac{\Delta}{g^2(X_j)/2X_j^2-f(X_j)/X_j}.$$

Suppose $\phi(x)\sim x/|f(x)|$ as $x\to 0^+$ and ϕ is non-decreasing. Thus $g^2(x)/2x^2-f(x)/x\sim|f(x)|/x\sim 1/\phi(x)$ as $x\to 0^+$. Hence for every $\epsilon\in(0,1)$, there is $x_0(\epsilon)>0$ such that for $x<x_0(\epsilon)$
$$(1-\epsilon)\cdot\frac{1}{\phi(x)}<\frac{g^2(x)}{2x^2}-\frac{f(x)}{x}<(1+\epsilon)\cdot\frac{1}{\phi(x)}.$$

Define $\alpha_+:=(1-\epsilon)\cdot\Delta$ and $\alpha_-:=(1+\epsilon)\cdot\Delta$ then there is $N_1(\epsilon)>0$ such that for $n\geq N_1(\epsilon)$, $e^{-\alpha_-n}<X_n<e^{-\alpha_+n}$ and there is $N_2(\epsilon)>0$ such that $e^{-\alpha_+n}<x_0(\epsilon),\forall\, n\geq N_2(\epsilon)$. Define $N_3(\epsilon):=\max(N_0(\epsilon),N_1(\epsilon),N_2(\epsilon))+1$, and $n\geq N_3(\epsilon)+1$. Since $N_3(\epsilon)\geq N_0(\epsilon)+1$, then
$$t_n=t_{N_3(\epsilon)}+\sum_{j=N_3(\epsilon)}^{n-1}\frac{\Delta}{g^2(X_j)/2X_j^2-f(X_j)/X_j}.$$

Since $X_n<e^{-\alpha_+n}<x_0(\epsilon)$ for $n\geq\max(N_1(\epsilon),N_2(\epsilon))$, for $n\geq\max(N_1(\epsilon),N_2(\epsilon))$
$$(1-\epsilon)\cdot\frac{1}{\phi(X_n)}<\frac{g^2(X_n)}{2X_n^2}-\frac{f(X_n)}{X_n}<(1+\epsilon)\cdot\frac{1}{\phi(X_n)},$$
so for $n\geq\max(N_1(\epsilon),N_2(\epsilon))$
$$\frac{1}{1-\epsilon}\cdot\Delta\phi(X_n)>\frac{\Delta}{g^2(X_n)/2X_n^2-f(X_n)/X_n}>\frac{1}{1+\epsilon}\cdot\Delta\phi(X_n).$$

Now for $n\geq N_1(\epsilon)$, then $e^{-\alpha_-n}<X_n<e^{-\alpha_+n}$ so as ϕ is non-decreasing for $n\geq\max(N_1(\epsilon),N_2(\epsilon))$, then $\phi(e^{-\alpha_-n})\leq\phi(X_n)\leq\phi(e^{-\alpha_+n})$ thus
$$\frac{1}{1-\epsilon}\cdot\Delta\phi(e^{-\alpha_+n})\geq\frac{\Delta}{g^2(X_n)/2X_n^2-f(X_n)/X_n}\geq\frac{1}{1+\epsilon}\cdot\Delta\phi(e^{-\alpha_-n}).$$

Therefore, for $n \geq N_3(\epsilon)+1 > \max(N_1(\epsilon), N_2(\epsilon))$ then

$$t_n \leq t_{N_3(\epsilon)} + \frac{1}{1-\epsilon}\sum_{j=N_3(\epsilon)}^{n-1} \Delta\phi(e^{-\alpha_+ j}), \tag{10.10}$$

$$t_n \geq t_{N_3(\epsilon)} + \frac{1}{1+\epsilon}\sum_{j=N_3(\epsilon)}^{n-1} \Delta\phi(e^{-\alpha_- j}). \tag{10.11}$$

Next consider $e^{-\alpha(n+1)} \leq u \leq e^{-\alpha n}$. Then, as ϕ is non-decreasing, we can deduce (see the similar calculations between (10.18) and (10.19) for the details) the inequality

$$\phi(e^{-\alpha(n+1)}) \leq \frac{1}{\alpha}\int_{\exp(-\alpha(n+1))}^{\exp(-\alpha n)} \frac{\phi(u)}{u}\,du \leq \phi(e^{-\alpha n}). \tag{10.12}$$

By (10.11) for $n \geq N_3(\epsilon)+1$

$$\begin{aligned} t_n \geq t_{N_3(\epsilon)} + \frac{1}{1+\epsilon}\sum_{j=N_3(\epsilon)}^{n-1} \Delta\phi(e^{-\alpha_- j}) &\geq t_{N_3(\epsilon)} + \frac{1}{1+\epsilon}\sum_{j=N_3(\epsilon)}^{n-1} \frac{\Delta}{\alpha_-}\int_{\exp(-\alpha_-(j+1))}^{\exp(-\alpha_- j)} \frac{\phi(u)}{u}\,du \\ &= t_{N_3(\epsilon)} + \frac{1}{(1+\epsilon)^2}\sum_{j=N_3(\epsilon)}^{n-1} \int_{\exp(-\alpha_-(j+1))}^{\exp(-\alpha_- j)} \frac{\phi(u)}{u}\,du. \end{aligned}$$

Thus for $n \geq N_3(\epsilon)+1$

$$t_n \geq t_{N_3(\epsilon)} + \frac{1}{(1+\epsilon)^2}\int_{\exp(-\alpha_- n)}^{\exp(-\alpha_- N_3(\epsilon))} \frac{\phi(u)}{u}\,du. \tag{10.13}$$

By (10.10), for $n \geq N_3(\epsilon)+1$

$$\begin{aligned} t_n \leq t_{N_3(\epsilon)} + \frac{1}{1-\epsilon}\sum_{j=N_3(\epsilon)}^{n-1} \Delta\phi(e^{-\alpha_+ j}) &\leq t_{N_3(\epsilon)} + \frac{1}{1-\epsilon}\sum_{j=N_3(\epsilon)}^{n-1} \frac{\Delta}{\alpha_+}\int_{\exp(-\alpha_+ j)}^{\exp(-\alpha_+(j-1))} \frac{\phi(u)}{u}\,du \\ &= t_{N_3(\epsilon)} + \frac{1}{(1-\epsilon)^2}\sum_{j=N_3(\epsilon)}^{n-1} \int_{\exp(-\alpha_+ j)}^{\exp(-\alpha_+(j-1))} \frac{\phi(u)}{u}\,du. \end{aligned}$$

Thus for $n \geq N_3(\epsilon)+1$

$$t_n \leq t_{N_3(\epsilon)} + \frac{1}{(1-\epsilon)^2}\int_{\exp(-\alpha_+(n-1))}^{\exp(-\alpha_+(N_3(\epsilon)-1))} \frac{\phi(u)}{u}\,du. \tag{10.14}$$

We now concentrate on the proof of part (b)(i) and suppose $\int_{0+}^1 1/|f(u)|\,du = \infty$. Then $\int_{0+}^1 \phi(u)/u\,du = \infty$ because $\phi(x) \sim x/|f(x)|$ as $x \to 0^+$. Define $\Phi(x) := \int_x^1 \phi(u)/u\,du$. Then $\Phi(x) \to \infty$ as $x \to 0^+$ and $\Phi(x)/F(x) \to 1$ as $x \to 0^+$. By (10.13), for $n \geq$

$N_3(\epsilon)+1$

$$t_n \geq t_{N_3(\epsilon)} + \frac{1}{(1+\epsilon)^2}\cdot\left(\Phi(e^{-\alpha_- n}) - \Phi(e^{-\alpha_- N_3(\epsilon)})\right).$$

So $t_n \to \infty$ as $n \to \infty$. Also, as $e^{-\alpha_- n} < X_n < e^{-\alpha_+ n}$ for $n \geq N_3(\epsilon)+1$ and Φ is decreasing then $\Phi(e^{-\alpha_- n}) > \Phi(X_n)$. Thus for $n \geq N_3(\epsilon)+1$

$$t_n \geq t_{N_3(\epsilon)} + \frac{1}{(1+\epsilon)^2}\cdot\left(\Phi(X_n) - \Phi(e^{-\alpha_- N_3(\epsilon)})\right).$$

Since $\Phi(X_n) \to \infty$ as $n \to \infty$, then

$$\liminf_{n\to\infty} \frac{t_n}{\Phi(X_n)} \geq \frac{1}{(1+\epsilon)^2}.$$

Letting $\epsilon \to 0^+$ yields

$$\liminf_{n\to\infty} \frac{t_n}{\Phi(X_n)} \geq 1.$$

Since $F(x) \sim \Phi(x)$ as $x \to 0^+$ and $X_n \to 0$ as $n \to \infty$, then

$$\limsup_{n\to\infty} \frac{F(X_n)}{t_n} \leq 1. \tag{10.15}$$

From (10.14) for $n \geq N_3(\epsilon)+1$

$$t_{n-1} < t_n \leq t_{N_3(\epsilon)} + \frac{1}{(1-\epsilon)^2}\cdot\left(\Phi(e^{-\alpha_+(n-1)}) - \Phi(e^{-\alpha_+(N_3(\epsilon)-1)})\right).$$

Since $X_{n-1} < e^{-\alpha_+(n-1)}$ for $n \geq N_3(\epsilon)+1$ then $\Phi(X_{n-1}) > \Phi(e^{-\alpha_+(n-1)})$. Thus for $n \geq N_3(\epsilon)+1$

$$t_{n-1} \leq t_{N_3(\epsilon)} + \frac{1}{(1-\epsilon)^2}\cdot\left(\Phi(X_{n-1}) - \Phi(e^{-\alpha_+(N_3(\epsilon)-1)})\right),$$

Dividing by $\Phi(X_{n-1})$ and letting $n \to \infty$ yields

$$\limsup_{n\to\infty} \frac{t_n}{\Phi(X_n)} \leq \frac{1}{(1-\epsilon)^2}.$$

Letting $\epsilon \to 0^+$, taking reciprocals and noting that $F(X_n) \sim \Phi(X_n)$ as $n \to \infty$, then

$$\liminf_{n\to\infty} \frac{F(X_n)}{t_n} \geq 1.$$

Combining with (10.15) yields

$$\lim_{n\to\infty} \frac{F(X_n)}{t_n} = 1, \quad \text{a.s.},$$

as required. This completes the proof of part (b)(i). To prove part (b)(ii) we now suppose $\int_{0^+}^1 1/|f(u)|\,du < \infty$. Then $\int_{0^+}^1 \phi(u)/u\,du < \infty$. By (10.14), for $n \geq N_3(\epsilon)+1$

$$t_n \leq t_{N_3(\epsilon)} + \frac{1}{(1-\epsilon)^2}\int_{\exp(-\alpha_+(n-1))}^{\exp(-\alpha_+(N_3(\epsilon)-1))} \frac{\phi(u)}{u}\,du.$$

Taking the limit as $n \to \infty$ on the right-hand side yields

$$\limsup_{n\to\infty} t_n \leq t_{N_3(\epsilon)} + \frac{1}{(1-\epsilon)^2}\int_0^{\exp(-\alpha_+(N_3(\epsilon)-1))} \frac{\phi(u)}{u}\,du < \infty.$$

Since (t_n) is increasing, it follows that (t_n) tends to a finite limit. □

The next result deals with the case when the diffusion term dominates and covers Theorem 67 part (b) when L is finite and also part (a). Together with Theorem 68, Theorem 69 covers all the parts of Theorem 67.

Theorem 69. *Suppose* $L \in (-\infty, 1/2)$ *and* $x \mapsto x^2/g^2(x)$ *is asymptotically non-decreasing. Let* L, G, $\hat{T}_h$, (t_n) *and* h *be defined by* (1.25), (1.34), (10.5), (10.3) *and* (10.6). *Define*

$$\lim_{x\to 0^+} \frac{x^2}{g^2(x)} =: c^2.$$

(a) Let $c \in (0,\infty)$. *Then* $X_n \in (0,\infty)$ *for all* $n \geq 0$ *a.s.,* $X_n \to 0$ *as* $n \to \infty$ *a.s.,* $t_n \to \infty$ *as* $n \to \infty$ *a.s. and*

$$\lim_{n\to\infty} \frac{\log X_n}{t_n} = -g'(0)^2\left(\tfrac{1}{2} - L\right), \quad a.s..$$

(b) Let $c = 0$.

(i) If g *obeys* (1.32), *then* $X_n \in (0,\infty)$ *for all* $n \geq 0$ *a.s.,* $X_n \to 0$ *as* $n \to \infty$ *a.s.,* $t_n \to \infty$ *as* $n \to \infty$ *a.s. and*

$$\lim_{n\to\infty} \frac{G(X_n)}{t_n} = \frac{1}{2} - L, \quad a.s..$$

(ii) If g *obeys* (1.31), *then* $X_n \in (0,\infty)$ *for all* $n \geq 0$ *a.s.,* $X_n \to 0$ *as* $n \to \infty$ *a.s.,* $t_n \to \hat{T}_h < \infty$ *as* $n \to \infty$ *a.s..*

Proof. We prove part (a) first. Since $\lim_{x\to 0^+} x/g(x) = c \in (0,\infty)$ then $g(x) \sim g'(0)x$ as $x \to 0^+$ and $c = 1/g'(0)$. Then

$$\frac{g^2(x)}{2x^2} - \frac{f(x)}{x} = \frac{g^2(x)}{x^2}\left(\frac{1}{2} - \frac{xf(x)}{g^2(x)}\right).$$

Thus

$$\lim_{x\to 0^+}\left(\frac{g^2(x)}{2x^2} - \frac{f(x)}{x}\right) = g'(0)^2\left(\frac{1}{2} - L\right).$$

Therefore

$$\lim_{n\to\infty} \min\left(1, \frac{g^2(X_n)}{2X_n^2} - \frac{f(X_n)}{X_n}\right) = \min\left(1, g'(0)^2\left(\tfrac{1}{2} - L\right)\right), \quad \text{a.s..}$$

Thus

$$t_n = \sum_{j=0}^{n-1} \Delta \cdot \min\left(1, \frac{1}{g^2(X_j)/2X_j^2 - f(X_j)/X_j}\right),$$

and

$$\lim_{n\to\infty} \frac{\log X_n}{n \cdot \min\left(1, g'(0)^2\left(\tfrac{1}{2} - L\right)\right)} = -\Delta, \quad \text{a.s..}$$

Thus

$$\lim_{n\to\infty} \frac{t_n}{n} = \Delta \cdot \min\left(1, \frac{1}{g'(0)^2\left(\tfrac{1}{2} - L\right)}\right).$$

Hence

$$\lim_{n\to\infty} \frac{\log X_n}{t_n} = \lim_{n\to\infty} \frac{\log X_n}{n \cdot \Delta \cdot \min\left(1, 1/(g'(0)^2\left(\tfrac{1}{2} - L\right))\right)} = -g'(0)^2\left(\tfrac{1}{2} - L\right).$$

We now prove part (b). If $\lim_{x\to 0^+} x/g(x) = 0$ then

$$\frac{g^2(x)}{2x^2} - \frac{f(x)}{x} = \frac{g^2(x)}{x^2}\left(\frac{1}{2} - \frac{xf(x)}{g^2(x)}\right) \sim \frac{g^2(x)}{x^2}\left(\frac{1}{2} - L\right), \quad \text{as } x \to 0^+.$$

Thus

$$\lim_{x\to 0^+}\left(\frac{g^2(x)}{2x^2} - \frac{f(x)}{x}\right) = \left(\frac{1}{2} - L\right)\lim_{x\to 0^+}\frac{g^2(x)}{x^2} = \infty.$$

Therefore, there is $N_0(\omega, \Delta)$ such that $\min\left(1, g^2(X_j)/2X_j - f(X_j)/X_j\right) = 1$ for all $n \geq N_0(\omega, \Delta)$. Hence

$$\lim_{n\to\infty} \frac{\log X_n}{n} = -\Delta, \quad \text{a.s.,}$$

and for $n \geq N_0(\omega, \Delta) + 1$

$$t_n = t_{N_0(\omega,\Delta)} + \sum_{j=N_0(\omega,\Delta)}^{n-1} \frac{\Delta}{g^2(X_j)/2X_j^2 - f(X_j)/X_j}.$$

Since $X_n \to 0$ as $n \to \infty$ and $g^2(x)/2x^2 - f(x)/x \sim \left(\frac{1}{2} - L\right) \cdot g^2(x)/x^2$ as $x \to 0^+$ the convergence of (t_n) to a finite limit is equivalent to the convergence of the series

$$\sum_{j=0}^{n-1} \frac{\Delta X_j^2}{g^2(X_j)}.$$

Suppose $\gamma(x) \sim x^2/g^2(x)$ as $x \to 0^+$ and γ is non-decreasing. Thus

$$\frac{g^2(x)}{2x^2} - \frac{f(x)}{x} \sim \left(\frac{1}{2} - L\right) \cdot \frac{1}{\gamma(x)}, \quad \text{as } x \to 0^+.$$

Hence for every $\epsilon \in (0,1)$, there is $x_0(\epsilon) > 0$ such that for $x < x_0(\epsilon)$

$$(1-\epsilon) \cdot \left(\tfrac{1}{2} - L\right) \cdot \frac{1}{\gamma(x)} < \frac{g^2(x)}{2x^2} - \frac{f(x)}{x} < (1+\epsilon) \cdot \left(\tfrac{1}{2} - L\right) \cdot \frac{1}{\gamma(x)}.$$

Also for every $\epsilon \in (0,1)$ there is $N_1(\epsilon) \in \mathbb{N}$ such that for all $n \geq N_1(\epsilon)$

$$-\Delta - \Delta\epsilon < \frac{\log X_n}{n} < -\Delta + \Delta\epsilon,$$

or $-(1+\epsilon) \cdot \Delta n < \log X_n < -(1-\epsilon) \cdot \Delta n$ which implies for all $n \geq N_1(\epsilon)$

$$e^{-\alpha_- n} = e^{-(1+\epsilon)\Delta n} < X_n < e^{-(1-\epsilon)\Delta n} = e^{-\alpha_+ n},$$

where $\alpha_+ = (1-\epsilon) \cdot \Delta$ and $\alpha_- = (1+\epsilon) \cdot \Delta$. There is $N_2(\epsilon) > 0$ such that $\forall\, n \geq N_2(\epsilon)$ then $e^{-\Delta(1-\epsilon)n} < x_0(\epsilon)$. Let $N_3(\epsilon) := \max(N_0(\epsilon), N_1(\epsilon), N_2(\epsilon)) + 1$. Let $n \geq N_3(\epsilon) + 1$. Since $N_3(\epsilon) \geq N_0(\epsilon) + 1$ then

$$t_n = t_{N_3(\epsilon)} + \sum_{j=N_3(\epsilon)}^{n-1} \frac{\Delta}{g^2(X_j)/2X_j^2 - f(X_j)/X_j}.$$

Since $X_n < e^{-\Delta(1-\epsilon)n} < x_0(\epsilon)$ for $n \geq \max(N_1(\epsilon), N_2(\epsilon))$ for $n \geq \max(N_1(\epsilon), N_2(\epsilon))$ then

$$(1-\epsilon) \cdot \left(\tfrac{1}{2} - L\right) \cdot \frac{1}{\gamma(X_n)} < \frac{g^2(X_n)}{2X_n^2} - \frac{f(X_n)}{X_n} < (1+\epsilon) \cdot \left(\tfrac{1}{2} - L\right) \cdot \frac{1}{\gamma(X_n)}.$$

Hence for $n \geq \max(N_1(\epsilon), N_2(\epsilon))$, we have

$$\frac{1}{(1-\epsilon)} \cdot \frac{\Delta\gamma(X_n)}{(\frac{1}{2} - L)} > \frac{\Delta}{g^2(X_n)/2X_n^2 - f(X_n)/X_n} > \frac{1}{(1+\epsilon)} \cdot \frac{\Delta\gamma(X_n)}{(\frac{1}{2} - L)}.$$

For $n \geq N_1(\epsilon)$, then $e^{-\alpha_- n} < X_j < e^{-\alpha_+ n}$ so as γ is non-decreasing, for $n \geq \max(N_1(\epsilon), N_2(\epsilon))$ then $\gamma(e^{-\alpha_- n}) \leq \gamma(X_n) \leq \gamma(e^{-\alpha_+ n})$. Thus for $n \geq \max(N_1(\epsilon), N_2(\epsilon))$

$$\frac{1}{(1-\epsilon)} \cdot \frac{\Delta\gamma(e^{-\alpha_+ n})}{(\frac{1}{2} - L)} \geq \frac{\Delta}{g^2(X_n)/2X_n^2 - f(X_n)/X_n} \geq \frac{1}{(1+\epsilon)} \cdot \frac{\Delta\gamma(e^{-\alpha_- n})}{(\frac{1}{2} - L)}. \quad (10.16)$$

Therefore for $n \geq N_3(\epsilon)+1 > \max(N_1(\epsilon), N_2(\epsilon))$ then

$$t_n = t_{N_3(\epsilon)} + \sum_{j=N_3(\epsilon)}^{n-1} \frac{\Delta}{g^2(X_j)/2X_j^2 - f(X_j)/X_j},$$

then

$$t_n \leq t_{N_3(\epsilon)} + \sum_{j=N_3(\epsilon)}^{n-1} \frac{1}{(1-\epsilon)} \cdot \frac{\Delta\gamma(e^{-\alpha_+ j})}{(\frac{1}{2}-L)}, \tag{10.17}$$

and

$$t_n \geq t_{N_3(\epsilon)} + \sum_{j=N_3(\epsilon)}^{n-1} \frac{1}{(1+\epsilon)} \cdot \frac{\Delta\gamma(e^{-\alpha_- j})}{(\frac{1}{2}-L)}. \tag{10.18}$$

Consider $e^{-\alpha(n+1)} \leq u \leq e^{-\alpha n}$. Then as γ is non-decreasing $\gamma(e^{-\alpha(n+1)}) \leq \gamma(u) \leq \gamma(e^{-\alpha n})$ or

$$\frac{\gamma(e^{-\alpha(n+1)})}{u} \leq \frac{\gamma(u)}{u} \leq \frac{\gamma(e^{-\alpha n})}{u}.$$

Therefore

$$\gamma(e^{-\alpha(n+1)}) \int_{\exp(-\alpha(n+1))}^{\exp(-\alpha n)} \frac{1}{u}\,du \leq \int_{\exp(-\alpha(n+1))}^{\exp(-\alpha n)} \frac{\gamma(u)}{u}\,du \leq \gamma(e^{-\alpha n}) \int_{\exp(-\alpha(n+1))}^{\exp(-\alpha n)} \frac{1}{u}\,du.$$

Since $\int_{\exp(-\alpha(n+1))}^{\exp(-\alpha n)} 1/u\,du = \log(e^{-\alpha n}) - \log\left(e^{-\alpha(n+1)}\right) = -\alpha n + \alpha(n+1) = \alpha$, we have

$$\gamma(e^{-\alpha(n+1)}) \leq \frac{1}{\alpha} \int_{\exp(-\alpha(n+1))}^{\exp(-\alpha n)} \frac{\gamma(u)}{u}\,du \leq \gamma(e^{-\alpha n}). \tag{10.19}$$

By (10.18), for $n \geq N_3(\epsilon)+1$

$$\begin{aligned} t_n &\geq t_{N_3(\epsilon)} + \frac{1}{1+\epsilon} \cdot \frac{1}{\frac{1}{2}-L} \sum_{j=N_3(\epsilon)}^{n-1} \Delta\gamma(e^{-\alpha_- j}) \\ &\geq t_{N_3(\epsilon)} + \frac{1}{1+\epsilon} \cdot \frac{1}{\frac{1}{2}-L} \sum_{j=N_3(\epsilon)}^{n-1} \frac{\Delta}{\alpha_-} \int_{\exp(-\alpha_-(j+1))}^{\exp(-\alpha_- j)} \frac{\gamma(u)}{u}\,du \\ &= t_{N_3(\epsilon)} + \frac{1}{(1+\epsilon)^2} \cdot \frac{1}{\frac{1}{2}-L} \sum_{j=N_3(\epsilon)}^{n-1} \int_{\exp(-\alpha_-(j+1))}^{\exp(-\alpha_- j)} \frac{\gamma(u)}{u}\,du. \end{aligned}$$

Thus for $n \geq N_3(\epsilon)+1$

$$t_n \geq t_{N_3(\epsilon)} + \frac{1}{(1+\epsilon)^2} \cdot \frac{1}{\frac{1}{2}-L} \int_{\exp(-\alpha_- n)}^{\exp(-\alpha_- N_3(\epsilon))} \frac{\gamma(u)}{u}\,du. \tag{10.20}$$

By (10.17), for $n \geq N_3(\epsilon)+1$

$$\begin{aligned} t_n &\leq t_{N_3(\epsilon)} + \frac{1}{1-\epsilon}\cdot\frac{1}{\frac{1}{2}-L}\sum_{j=N_3(\epsilon)}^{n-1}\Delta\gamma(e^{-\alpha_+ j}) \\ &\leq t_{N_3(\epsilon)} + \frac{1}{1-\epsilon}\cdot\frac{1}{\frac{1}{2}-L}\sum_{j=N_3(\epsilon)}^{n-1}\frac{\Delta}{\alpha_+}\int_{\exp(-\alpha_+ j)}^{\exp(-\alpha_+(j-1))}\frac{\gamma(u)}{u}\,du \\ &= t_{N_3(\epsilon)} + \frac{1}{(1-\epsilon)^2}\cdot\frac{1}{\frac{1}{2}-L}\sum_{j=N_3(\epsilon)}^{n-1}\int_{\exp(-\alpha_+ j)}^{\exp(-\alpha_+(j-1))}\frac{\gamma(u)}{u}\,du. \end{aligned}$$

Thus for $n \geq N_3(\epsilon)+1$

$$t_n \leq t_{N_3(\epsilon)} + \frac{1}{(1-\epsilon)^2}\cdot\frac{1}{\frac{1}{2}-L}\int_{\exp(-\alpha_+(n-1))}^{\exp(-\alpha_+(N_3(\epsilon)-1))}\frac{\gamma(u)}{u}\,du. \tag{10.21}$$

We now concentrate on the proof of part (b)(i) and suppose now $\int_{0^+}^1 u/g^2(u)\,du = \infty$. Then $\int_{0^+}^1 \gamma(u)/u\,du = \infty$ because $\gamma(x) \sim x^2/g^2(x)$ as $x \to 0^+$. Define $\Gamma(x) := \int_x^1 \gamma(u)/u\,du$. Then $\Gamma(x) \to \infty$ as $x \to 0^+$ and $G(x)/\Gamma(x) \to 1$ as $x \to 0^+$. By (10.20), for $n \geq N_3(\epsilon)+1$

$$t_n \geq t_{N_3(\epsilon)} + \frac{1}{(1+\epsilon)^2}\cdot\frac{1}{\frac{1}{2}-L}\left(\Gamma(e^{-\alpha_- n}) - \Gamma(e^{-\alpha_- N_3(\epsilon)})\right),$$

so $t_n \to \infty$ as $n \to \infty$. Also, as $e^{-\alpha_- n} < X_n < e^{-\alpha_+ n}$ for $n \geq N_3(\epsilon)+1$ and Γ is decreasing $\Gamma(e^{-\alpha_- n}) > \Gamma(X_n)$. Thus for $n \geq N_3(\epsilon)+1$

$$t_n \geq t_{N_3} + \frac{1}{(1+\epsilon)^2}\cdot\frac{1}{\frac{1}{2}-L}\left(\Gamma(X_n) - \Gamma(e^{-\alpha_- N_3(\epsilon)})\right).$$

Hence as $\Gamma(X_n) \to \infty$ as $n \to \infty$, then

$$\liminf_{n\to\infty}\frac{t_n}{\Gamma(X_n)} \geq \frac{1}{(1+\epsilon)^2}\cdot\frac{1}{\frac{1}{2}-L}.$$

Letting $\epsilon \to 0^+$ and taking the reciprocal yields

$$\limsup_{n\to\infty}\frac{\Gamma(X_n)}{t_n} \leq \frac{1}{2}-L,$$

and therefore as $G(x) \sim \Gamma(x)$ as $x \to 0^+$, then

$$\limsup_{n\to\infty}\frac{G(X_n)}{t_n} \leq \frac{1}{2}-L, \quad \text{a.s..} \tag{10.22}$$

By (10.21), for $n \geq N_3(\epsilon)+1$

$$t_n \leq t_{N_3(\epsilon)} + \frac{1}{(1+\epsilon)^2}\cdot\frac{1}{\frac{1}{2}-L}\left(\Gamma(e^{-\alpha_+(n-1)}) - \Gamma(e^{-\alpha_+(N_3(\epsilon)-1)})\right).$$

Since $X_{n-1} < e^{-\alpha_+(n-1)}$ for $n \geq N_3(\epsilon)+1$, $\Gamma\left(X_{n-1}\right) > \Gamma\left(e^{-\alpha_+(n-1)}\right)$. Thus for $n \geq N_3(\epsilon)+1$

$$t_{n-1} < t_n \leq t_{N_3(\epsilon)} + \frac{1}{(1+\epsilon)^2}\cdot\frac{1}{\frac{1}{2}-L}\left(\Gamma(X_{n-1}) - \Gamma(e^{-\alpha_+(N_3(\epsilon)-1)})\right).$$

Hence as $\Gamma(X_{n-1}) \to \infty$ as $n\to\infty$

$$\limsup_{n\to\infty}\frac{t_{n-1}}{\Gamma(X_{n-1})} \leq \frac{1}{(1-\epsilon)^2}\cdot\frac{1}{\frac{1}{2}-L}.$$

Proceeding as above,

$$\liminf_{n\to\infty}\frac{G(X_{n-1})}{t_{n-1}} \geq \frac{1}{2}-L, \quad \text{a.s.}.$$

Combining with (10.22) yields

$$\lim_{n\to\infty}\frac{G(X_n)}{t_n} = \frac{1}{2}-L, \quad \text{a.s.}. \tag{10.23}$$

We now concentrate on the proof of part (b)(ii) and suppose now $\int_{0^+}^1 u/g^2(u)\,du < \infty$. Then $\int_{0^+}^1 \gamma(u)/u\,du < \infty$. By (10.21)

$$t_n \leq t_{N_3(\epsilon)} + \frac{1}{(1-\epsilon)^2}\cdot\frac{1}{\frac{1}{2}-L}\int_{\exp(-\alpha_+(n-1))}^{\exp(-\alpha_+(N_3(\epsilon)-1))}\frac{\gamma(u)}{u}\,du.$$

Taking the limit as $n\to\infty$ the right-hand side yields

$$\limsup_{n\to\infty} t_n \leq t_{N_3(\epsilon)} + \frac{1}{(1-\epsilon)^2}\cdot\frac{1}{\frac{1}{2}-L}\int_0^{\exp(-\alpha_+(N_3(\epsilon)-1))}\frac{\gamma(u)}{u}\,du < \infty.$$

Since (t_n) is an increasing sequence, it follows that (t_n) tends to a finite limit, as claimed. □

We are now going to prove an analogue of Theorem 62 which describes the asymptotic behaviour of the SDE in the vicinity of T_ξ. We note that Theorem 69 part (b)(ii) does not supply such asymptotic behaviour in contrast to part (i).

Theorem 70. *Suppose $L \in (-\infty, 1/2)$, g obeys* (1.31) *and $x \mapsto g^2(x)/x^2$ is asymptotic to a continuous non-increasing function. Let L, $\bar{G}$, (t_n), $\hat{T}_h$ be defined by* (1.25), (1.33), (10.3), (10.5).

(i) If

$$\lim_{\lambda\to 1^+}\liminf_{x\to 0^+}\frac{(-\log\circ\bar{G}^{-1})(\lambda x)}{(-\log\circ\bar{G}^{-1})(x)}=\infty,$$

then

$$\lim_{n\to\infty}\frac{\bar{G}(X_n)}{\hat{T}_h-t_n}=\frac{1}{2}-L,\quad a.s..$$

(ii) If

$$\lim_{\lambda\to 1^-}\limsup_{x\to 0^+}\frac{(-\log\circ\bar{G}^{-1})(\lambda x)}{(-\log\circ\bar{G}^{-1})(x)}=1,$$

then

$$\lim_{n\to\infty}\frac{-\log X_n}{(-\log\circ\bar{G}^{-1})((\frac{1}{2}-L)(\hat{T}_h-t_n))}=1,\quad a.s..$$

Proof. We start by developing some useful estimates connecting $\bar{G}$, $\hat{T}_h-t_n$ and X_n. Define $\bar{\Gamma}(x):=\int_0^x\gamma(u)/u\,du$. If $\int_{0^+}^1 u/g^2(u)\,du<\infty$, then $\bar{\Gamma}(x)$, $\bar{G(x)}\to 0$ as $x\to 0^+$ and $\bar{G}(x)/\bar{\Gamma}(x)\to 1$ as $x\to 0^+$. Then $t_n\to\hat{T}_h<\infty$. For $n\geq N_3(\epsilon)+1$,

$$\begin{aligned}\hat{T}_h &= t_{N_3(\epsilon)}+\sum_{j=N_3(\epsilon)}^{\infty}\frac{\Delta}{g^2(X_j)/2X_j^2-f(X_j)/X_j},\\ t_n &= t_{N_3(\epsilon)}+\sum_{j=N_3(\epsilon)}^{n-1}\frac{\Delta}{g^2(X_j)/2X_j^2-f(X_j)/X_j}.\end{aligned}$$

Thus for $n\geq N_3(\epsilon)+1$

$$\hat{T}_h-t_n=\sum_{j=n}^{\infty}\frac{\Delta}{g^2(X_j)/2X_j^2-f(X_j)/X_j},$$

and for $n\geq\max(N_1(\epsilon),N_2(\epsilon))$

$$\frac{1}{1-\epsilon}\cdot\frac{\Delta\gamma(e^{-\alpha_+ n})}{\frac{1}{2}-L}\geq\frac{\Delta}{g^2(X_n)/2X_n^2-f(X_n)/X_n}\geq\frac{1}{1+\epsilon}\cdot\frac{\Delta\gamma(e^{-\alpha_- n})}{\frac{1}{2}-L}.$$

Hence

$$\frac{1}{1-\epsilon}\cdot\frac{1}{\frac{1}{2}-L}\sum_{j=n}^{\infty}\Delta\gamma(e^{-\alpha_+ j})\geq\hat{T}_h-t_n\geq\frac{1}{1+\epsilon}\cdot\frac{1}{\frac{1}{2}-L}\sum_{j=n}^{\infty}\Delta\gamma(e^{-\alpha_- j}).\qquad(10.24)$$

Define $\alpha_-:=(1+\epsilon)\cdot\Delta$ and $\alpha_+:=(1-\epsilon)\cdot\Delta$. Then by (10.19)

$$\begin{aligned}\sum_{j=n}^{\infty}\Delta\gamma(e^{-\alpha_- j})\geq\sum_{j=n}^{\infty}\frac{\Delta}{\alpha_-}\int_{\exp(-\alpha_-(j+1))}^{\exp(-\alpha_- j)}\frac{\gamma(u)}{u}\,du &= \frac{1}{1+\epsilon}\sum_{j=n}^{\infty}\int_{\exp(-\alpha_-(j+1))}^{\exp(-\alpha_- j)}\frac{\gamma(u)}{u}\,du\\ &= \frac{1}{1+\epsilon}\int_0^{\exp(-\alpha_- n)}\frac{\gamma(u)}{u}\,du.\end{aligned}$$

Hence for $n \geq N_3(\epsilon)+1$

$$\hat{T}_h - t_n \geq \frac{1}{(1+\epsilon)^2} \cdot \frac{\bar{\Gamma}(e^{-\alpha_- n})}{\frac{1}{2}-L}. \tag{10.25}$$

Also by (10.19)

$$\begin{aligned}\sum_{j=n}^{\infty} \Delta\gamma(e^{-\alpha_+ j}) \leq \sum_{j=n}^{\infty} \frac{\Delta}{\alpha_+} \int_{\exp(-\alpha_+ j)}^{\exp(-\alpha_+(j-1))} \frac{\gamma(u)}{u}\,du &= \frac{1}{1-\epsilon} \sum_{j=n}^{\infty} \int_{\exp(-\alpha_+ j)}^{\exp(-\alpha_+(j-1))} \frac{\gamma(u)}{u}\,du \\ &= \frac{1}{1-\epsilon} \int_0^{\exp(-\alpha_+(n-1))} \frac{\gamma(u)}{u}\,du.\end{aligned}$$

Hence for $n \geq N_3(\epsilon)+1$

$$\frac{1}{(1-\epsilon)^2} \cdot \frac{\bar{\Gamma}(e^{-\alpha_+(n-1)})}{\frac{1}{2}-L} \geq \hat{T}_h - t_n. \tag{10.26}$$

Next recall that $e^{-\alpha_- n} < X_n < e^{-\alpha_+ n}$ for $n \geq N_3(\epsilon)$. From (10.25), for $n \geq N_3(\epsilon)+1$

$$\bar{\Gamma}(e^{-\alpha_- n}) \leq (1+\epsilon)^2 \cdot (\tfrac{1}{2}-L)(\hat{T}_h - t_n),$$

since $\bar{\Gamma}(e^{-\alpha_- n})/\bar{G}(e^{-\alpha_- n}) > 1/(1+\epsilon)$ then $\forall\, n \geq N_4(\epsilon)$

$$\bar{G}(e^{-\alpha_- n}) \leq (1+\epsilon)^3 \cdot \left(\tfrac{1}{2}-L\right)(\hat{T}_h - t_n). \tag{10.27}$$

Define $N_5(\epsilon) := \max(N_3(\epsilon), N_4(\epsilon))$. Thus for $n \geq N_5(\epsilon)+1$

$$e^{-\alpha_- n} \leq \bar{G}^{-1}\left((1+\epsilon)^3 \left(\tfrac{1}{2}-L\right)(\hat{T}_h - t_n)\right).$$

Now $X_n^{\alpha_-/\alpha_+} < (e^{-\alpha_+ n})^{\alpha_-/\alpha_+} = e^{-\alpha_- n}$ for $n \geq N_3(\epsilon)$. Thus for $n \geq N_5(\epsilon)+1$, as $\alpha_-/\alpha_+ = (1+\epsilon)/(1-\epsilon)$, then

$$X_n^{(1+\epsilon)/(1-\epsilon)} < \bar{G}^{-1}\left((1+\epsilon)^3 \left(\tfrac{1}{2}-L\right)(\hat{T}_h - t_n)\right). \tag{10.28}$$

This implies

$$\frac{1+\epsilon}{1-\epsilon} \cdot -\log X_n > (-\log \circ \bar{G}^{-1})\left((1+\epsilon)^3 \left(\tfrac{1}{2}-L\right)(\hat{T}_h - t_n)\right). \tag{10.29}$$

From (10.26), for $n \geq N_3(\epsilon)+1$, then

$$\bar{\Gamma}\left(e^{-\alpha_+(n-1)}\right) \geq (1-\epsilon)^2 \cdot \left(\tfrac{1}{2}-L\right)(\hat{T}_h - t_n).$$

Now for $n \geq N_6(\epsilon)$, $\bar{G}\left(e^{-\alpha_+(n-1)}\right) \geq (1-\epsilon)\cdot\bar{\Gamma}\left(e^{-\alpha_+(n-1)}\right)$. Define $N_7(\epsilon) := \max(N_3(\epsilon), N_6(\epsilon))$.

For $n \geq N_7(\epsilon) + 1$

$$\bar{G}\left(e^{-\alpha_+(n-1)}\right) \geq (1-\epsilon)^3 \cdot \left(\tfrac{1}{2} - L\right)(\hat{T}_h - t_n). \tag{10.30}$$

Thus

$$e^{-\alpha_+ n} \cdot e^{\alpha_+} = e^{-\alpha_+(n-1)} \geq \bar{G}^{-1}\left((1-\epsilon)^3 \left(\tfrac{1}{2} - L\right)(\hat{T}_h - t_n)\right).$$

For $n \geq N_3(\epsilon)$, $X_n^{\alpha_+/\alpha_-} > (e^{-\alpha_- n})^{\alpha_+/\alpha_-} = e^{-\alpha_+ n}$. Thus for $n \geq N_7(\epsilon) + 1$

$$X_n^{(1-\epsilon)/(1+\epsilon)} \geq e^{-\alpha_+} \cdot \bar{G}^{-1}\left((1-\epsilon)^3 \left(\tfrac{1}{2} - L\right)(\hat{T}_h - t_n)\right), \tag{10.31}$$

or

$$\frac{1-\epsilon}{1+\epsilon} \cdot -\log X_n \leq \alpha_+ + (-\log \circ \bar{G}^{-1})\left((1-\epsilon)^3 \left(\tfrac{1}{2} - L\right)(\hat{T}_h - t_n)\right). \tag{10.32}$$

We now use the estimates we have derived to establish a direct connection between X_n and $\hat{T}_h - t_n$. Recall from (10.27) that for $n \geq N_7(\epsilon) + 1$

$$\bar{G}\left(e^{-\alpha_- n}\right) \leq (1+\epsilon)^3 \cdot \left(\tfrac{1}{2} - L\right)(\hat{T}_h - t_n).$$

Now $X_n^{\alpha_-/\alpha_+} < e^{-\alpha_- n}$ for $n \geq N_3(\epsilon)$ or $X_n^{(1+\epsilon)/(1-\epsilon)} < e^{-\alpha_- n}$ thus for $n \geq N_5(\epsilon) + 1$

$$\bar{G}\left(X^{(1+\epsilon)/(1-\epsilon)_n}\right) \leq (1+\epsilon)^3 \cdot \left(\tfrac{1}{2} - L\right)(\hat{T}_h - t_n). \tag{10.33}$$

Similarly recall (10.30) is

$$\bar{G}\left(e^{\alpha_+} \cdot e^{-\alpha_+ n}\right) \geq (1+\epsilon)^3 \cdot \left(\tfrac{1}{2} - L\right)(\hat{T}_h - t_n),$$

for $n \geq N_7(\epsilon) + 1$. Then because $X_n^{\alpha_+/\alpha_-} > (e^{-\alpha_- n})^{\alpha_+/\alpha_-} = e^{-\alpha_+ n}$, we get

$$\bar{G}\left(e^{\Delta(1-\epsilon)} X_n^{(1+\epsilon)/(1-\epsilon)}\right) \geq (1+\epsilon)^3 \cdot \left(\tfrac{1}{2} - L\right)(\hat{T}_h - t_n). \tag{10.34}$$

Let $y_n = X_n^{(1+\epsilon)/(1-\epsilon)}$; then $y_n^{(1-\epsilon)/(1+\epsilon)} = X_n$. Hence by (10.33)

$$\frac{\bar{G}(X_n)}{(1+\epsilon)^3 \left(\tfrac{1}{2} - L\right)(\hat{T}_h - t_n)} \leq \frac{\bar{G}\left(y_n^{(1-\epsilon)/(1+\epsilon)}\right)}{\bar{G}(y_n)}.$$

Therefore, as $y_n \to 0$ as $n \to \infty$

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\bar{G}(X_n)}{\hat{T}_h - t_n} &\leq (1+\epsilon)^3 \cdot \left(\tfrac{1}{2} - L\right) \limsup_{n\to\infty} \frac{\bar{G}\left(y_n^{(1-\epsilon)/(1+\epsilon)}\right)}{\bar{G}(y_n)} \\
&\leq (1+\epsilon)^3 \cdot \left(\tfrac{1}{2} - L\right) \limsup_{x\to 0^+} \frac{\bar{G}\left(x^{(1-\epsilon)/(1+\epsilon)}\right)}{\bar{G}(x)}.
\end{aligned}$$

Now because (10.33) and (9.12) holds by Lemma 38 then $\limsup_{x\to 0^+} \bar{G}(x^\lambda)/\bar{G}(x) = 1$ for all $\lambda < 1$. Hence letting $\epsilon \to 0^+$ yields

$$\limsup_{n\to\infty} \frac{\bar{G}(X_n)}{\hat{T}_h - t_n} \le \frac{1}{2} - L. \tag{10.35}$$

It remains to prove a corresponding lower bound. Next for $\epsilon \in (0,1), \Delta > 0$, define $x_2(\epsilon,\Delta) := e^{-\Delta(1+\epsilon)(1+2\epsilon)/\epsilon}$. Suppose $x < x_2(\epsilon,\Delta)$, so $x < e^{-\Delta(1+\epsilon)(1+2\epsilon)/\epsilon}$. Hence $x^{\frac{\epsilon(1-\epsilon)}{(1+\epsilon)(1+2\epsilon)}} < e^{-\Delta(1-\epsilon)}$ or $e^{\Delta(1-\epsilon)} < x^{\frac{-\epsilon(1-\epsilon)}{(1+\epsilon)(1+2\epsilon)}}$. Thus $e^{\Delta(1-\epsilon)} < x^{-\frac{1-\epsilon}{1+\epsilon}+\frac{1-\epsilon}{1+2\epsilon}}$ as $x^{\frac{1-\epsilon}{1+\epsilon}} e^{\Delta(1-\epsilon)} < x^{\frac{1-\epsilon}{1+2\epsilon}}$. Now, as $X_n \to 0$ as $n \to \infty$, there is $N_8(\epsilon,\Delta) > 0$ such that $X_n < x_2(\epsilon,\Delta)\, \forall\, n \ge N_8$. Let $N_9(\epsilon) = \max(N_7(\epsilon), N_8(\epsilon))$. Then $X_n^{(1-\epsilon)/(1+\epsilon)} e^{\Delta(1-\epsilon)} < X_n^{(1-\epsilon)/(1+2\epsilon)}$ and

$$\bar{G}\left(X_n^{(1-\epsilon)/(1+\epsilon)} e^{\Delta(1-\epsilon)}\right) < \bar{G}\left(X_n^{(1-\epsilon)/(1+2\epsilon)}\right).$$

By (10.34) for $n \ge N_9(\epsilon) + 1$

$$\bar{G}\left(X_n^{(1-\epsilon)/(1+2\epsilon)}\right) \ge (1+\epsilon)^3 \cdot \left(\tfrac{1}{2} - L\right)(\hat{T}_h - t_n).$$

Now

$$\frac{(1+\epsilon)^3\left(\frac{1}{2}-L\right)(\hat{T}_h - t_n)}{\bar{G}(X_n)} \le \frac{\bar{G}\left(X_n^{(1-\epsilon)/(1+2\epsilon)}\right)}{\bar{G}(X_n)},$$

so

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\hat{T}_h - t_n}{\bar{G}(X_n)} &\le \limsup_{n\to\infty} \frac{\bar{G}\left(X_n^{(1-\epsilon)/(1+2\epsilon)}\right)}{\bar{G}(X_n)} \Big/ (1+\epsilon)^3\left(\tfrac{1}{2}-L\right) \\
&\le \limsup_{x\to 0^+} \frac{\bar{G}\left(x^{(1-\epsilon)/(1+2\epsilon)}\right)}{\bar{G}(x)} \Big/ (1+\epsilon)^3\left(\tfrac{1}{2}-L\right) \\
&= \frac{1}{(1+\epsilon)^3\left(\frac{1}{2}-L\right)}.
\end{aligned}$$

Thus by (9.12) and Lemma 38

$$\liminf_{n\to\infty} \frac{\bar{G}(X_n)}{\hat{T}_h - t_n} \ge (1+\epsilon)^3 \cdot \left(\frac{1}{2} - L\right).$$

Letting $\epsilon \to 0^+$ yields

$$\liminf_{n\to\infty} \frac{\bar{G}(X_n)}{\hat{T}_h - t_n} \ge \frac{1}{2} - L.$$

Combining this with (10.35) completes the proof of part (i). To move to part (ii) recall (10.29): for $n \ge N(\epsilon) + 1$, we have

$$\frac{1+\epsilon}{1-\epsilon} \cdot -\log X_n > (-\log \circ \bar{G}^{-1})\left((1+\epsilon)^3\left(\tfrac{1}{2}-L\right)(\hat{T}_h - t_n)\right).$$

Therefore with $\tau_n := \left(\frac{1}{2} - L\right)(\hat{T}_h - t_n)$

$$\frac{-\log X_n}{(-\log \circ \bar{G}^{-1})\left(\left(\frac{1}{2} - L\right)(\hat{T}_h - t_n)\right)} > \left(\frac{1-\epsilon}{1+\epsilon}\right) \cdot \frac{(-\log \circ \bar{G}^{-1})\left((1+\epsilon)^3 \tau_n\right)}{(-\log \circ \bar{G}^{-1})(\tau_n)},$$

so as $\tau_n \to 0$ as $n \to \infty$, this yields

$$\begin{aligned}\liminf_{n\to\infty} \frac{-\log X_n}{(-\log \circ \bar{G}^{-1})\left(\left(\frac{1}{2} - L\right)(\hat{T}_h - t_n)\right)} &\geq \left(\frac{1-\epsilon}{1+\epsilon}\right) \cdot \liminf_{n\to\infty} \frac{(-\log \circ \bar{G}^{-1})\left((1+\epsilon)^3 \tau_n\right)}{(-\log \circ \bar{G}^{-1})(\tau_n)} \\ &\geq \left(\frac{1-\epsilon}{1+\epsilon}\right) \cdot \liminf_{x\to 0^+} \frac{(-\log \circ \bar{G}^{-1})\left((1+\epsilon)^3 x\right)}{(-\log \circ \bar{G}^{-1})(x)}.\end{aligned}$$

Letting $\epsilon \to 0^+$ and using (9.14) gives

$$\begin{aligned}\liminf_{n\to\infty} \frac{-\log X_n}{(-\log \circ \bar{G}^{-1})\left(\left(\frac{1}{2} - L\right)(\hat{T}_h - t_n)\right)} &\geq 1 \cdot \lim_{\epsilon\to 0^+} \liminf_{x\to 0^+} \frac{(-\log \circ \bar{G}^{-1})\left((1+\epsilon)^3 x\right)}{(-\log \circ \bar{G}^{-1})(x)} \\ &= 1. \qquad (10.36)\end{aligned}$$

Proceeding in a similar manner with (10.32), we arrive at the estimate

$$\limsup_{n\to\infty} \frac{-\log X_n}{(-\log \circ \bar{G}^{-1})\left(\left(\frac{1}{2} - L\right)(\hat{T}_h - t_n)\right)} \leq 1.$$

Combining this with (10.36) gives the desired conclusion of part (ii), as claimed. □

Chapter 11

Finite-Time Stability with Small Noise

11.1 Introduction

We have already observed that the solution of the ODE

$$x'(t) = f(x(t)), \quad t > 0, \quad x(0) = \xi > 0, \tag{11.1}$$

with $F(x) = \int_x^1 1/|f(u)|\, du \to \infty$ as $x \to 0^+$ and $f(x) < 0 \,\forall\, x < 0$ obeys

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 1,$$

and that this rate of decay is recovered for SDEs with "small" noise. In fact if

$$\lim_{x\to 0^+} \frac{g^2(x)}{x|f(x)|} = 0, \tag{11.2}$$

then the solution of the SDE (1.17) obeys

$$\lim_{t\to\infty} \frac{F(X(t))}{t} = 1, \quad \text{a.s.}.$$

However, Theorem 62 suggests that a diffusion term satisfying the small noise condition (11.2) may not be sufficient to ensure preservation of the finite-time stability hitting asymptotics of the ODE (11.1). More precisely, if $\bar{F}(x) = \int_0^x 1/|f(u)|\, du \to 0$ as $x \to 0^+$ then the solution of (11.1) obeys

$$\lim_{t\to T_\xi^-} \frac{\bar{F}(x(t))}{T_\xi - t} = 1, \tag{11.3}$$

where $T_\xi = \int_0^\xi 1/|f(u)|\,du$. While the solution of the SDE (1.17) obeys

$$\lim_{t\to T^-} \frac{\bar{F}(X(t))}{T-t} = 1, \quad \text{a.s..} \tag{11.4}$$

if

$$\lim_{\lambda\to 1^+} \liminf_{x\to 0^+} \frac{(-\log\circ\bar{F}^{-1})(\lambda x)}{(-\log\circ\bar{F}^{-1})(x)} = \infty, \tag{11.5}$$

and

$$\lim_{t\to T^-} \frac{-\log X(t)}{(-\log\circ\bar{F}^{-1})(T-t)} = 1,$$

if

$$\lim_{\lambda\to 1^-} \limsup_{x\to 0^+} \frac{(-\log\circ\bar{F}^{-1})(\lambda x)}{(-\log\circ\bar{F}^{-1})(x)} = 1.$$

In the latter case we do not have asymptotic behaviour of the type seen in (11.3), while in (11.4), the asymptotic behaviour in (11.3) is recovered.

In what follows next, we impose a stricter condition on the noise term which forces it to be smaller than specified in (11.2). In fact, we ask that (1.55) holds with implies

$$\text{there exists } \theta,\, \delta_1 > 0 \text{ such that} \quad \sup_{0<x\le\delta_1} \frac{g^2(x)}{x^{1+\theta}|f(x)|} =: c < \infty. \tag{11.6}$$

It can easily be seen that this is more restrictive than (11.2), indeed (11.6) implies (11.2) for

$$\lim_{x\to 0^+} \frac{g^2(x)}{x|f(x)|} = \lim_{x\to 0^+} \left(\frac{g^2(x)}{x^{1+\theta}|f(x)|} \cdot x^\theta \right) = 0,$$

so we must have $L = -\infty$.

The rationale for this condition is that it allows the asymptotic behaviour in (11.4) to prevail without placing additional conditions on f, such as (11.5). We assume the following hypotheses on f

$$f(x) < 0,\ 0 < x < \delta_1; \tag{11.7}$$

$$x \mapsto |f(x)|/x \text{ is asymptotically decreasing; and} \tag{11.8}$$

$$x \mapsto |f(x)| \text{ is asymptotically increasing.} \tag{11.9}$$

11.2 Continuous Asymptotic Behaviour

We now prove our main theorem in this direction.

Theorem 71. *Suppose* $p(\infty^-) = \infty$. *Suppose also there exists* θ *such that* (1.55) *holds while* f *obeys* (1.26), (11.7), (11.8), (11.9). *Let* p, $\bar{F}$ *and* T *be defined by* (9.3), (1.29) *and* (1.22). *Then* $X(t) > 0$, $\forall\, t \in [0, T)$ *a.s.*, $X(t) \to 0$ *as* $t \to T^-$ *a.s.*, $T < \infty$ *a.s. and*

$$\lim_{t \to T^-} \frac{\bar{F}(X(t))}{T - t} = 1, \quad a.s.. \tag{11.10}$$

Proof. As pointed out above (1.55) implies (11.6). Let $\theta > 0$ be the number in (11.6). If $\eta_1(x) \sim |f(x)|/x$ as $x \to 0^+$ is decreasing since $x \mapsto x^{-\theta}$ is decreasing so $\eta(x) = \eta_1(x)x^{-\theta}$ is decreasing on $(0, \delta_2)$. Here $\eta(x) \sim |f(x)|/x^{1+\theta}$ as $x \to 0^+$ is decreasing. Since

$$\frac{g^2(x)}{x|f(x)|} = \frac{g^2(x)}{x^{1+\theta}|f(x)|} \cdot x^\theta,$$

then

$$\lim_{x \to 0^+} \frac{g^2(x)}{x|f(x)|} = 0.$$

Thus as $f(x) < 0$, we have $xf(x)/g^2(x) \to -\infty$ as $x \to 0^+$. Since (11.8), (11.9) are true and $\bar{F}(x) \to 0$ as $x \to 0^+$, it follows that $X(t) \to 0$ as $t \to T^-$ a.s. from Theorem 60. Consider $Z(t) := X(t)^{-\theta}$. Then $Z(t) \to \infty$ as $t \to T^-$ a.s. and by Itô's Lemma for $t \in [0, T)$

$$X(t)^{-\theta} = X(0)^{-\theta} + \int_0^t -\theta X(s)^{-(\theta+1)} f(X(s)) \left(1 - \frac{(\theta+1)g^2(X(s))}{2X(s)f(X(s))}\right) ds + \int_0^t \frac{-\theta g(X(s))}{X(s)^{\theta+1}}\, dB(s).$$

Define

$$\pi(x) := -\theta x^{-(\theta+1)} f(x) \left(1 - \frac{(\theta+1)g^2(x)}{2xf(x)}\right) \quad \text{and} \quad \nu(x) := \frac{-\theta g(x)}{x^{1+\theta}}.$$

Then

$$\begin{aligned} X(t)^{-\theta} &= X(0)^{-\theta} + \int_0^t \pi(X(s))\, ds + \int_0^t \nu(X(s))\, dB(s) \\ &= X(0)^{-\theta} + \int_0^t \pi(X(s))\, ds + M(t), \end{aligned} \tag{11.11}$$

where $M(t) := \int_0^t \nu(X(s))\, dB(s)$. Thus

$$\lim_{x \to 0^+} \frac{\pi(x)}{x^{-(\theta+1)}|f(x)|} = \theta. \tag{11.12}$$

Hence

$$\limsup_{x\to 0^+} \frac{\nu^2(x)}{\pi(x)} = \limsup_{x\to 0^+} \frac{\theta^2 g^2(x) x^{-(2+2\theta)}}{\theta x^{-(\theta+1)}|f(x)|} = \limsup_{x\to 0^+} \frac{\theta g^2(x)}{x^{1+\theta}|f(x)|} < \infty. \tag{11.13}$$

Define $A := \{\omega : \langle M\rangle(t,\omega) \to L' < \infty \, \text{as} \, t \to T\}$. Then $M(t) \to L^*$ as $t \to T^-$ a.s. on A. Since $X(t) \to 0$ as $t \to T^-$ a.s., the left-hand side of (11.11) tends to infinity as $t \to T^-$. Therefore $t \mapsto \int_0^t \pi(X(s))\,ds \to \infty$ as $t \to T^-$, a.s. on A. Therefore

$$\lim_{t\to T^-} \frac{X(t)^{-\theta}}{\int_0^t \pi(X(s))\,ds} = 1, \quad \text{a.s. on } A.$$

Finally, (11.12) implies that

$$\lim_{t\to T^-} \frac{X(t)^{-\theta}}{\theta \int_0^t |f(X(s))|/X(s)^{\theta+1}\,ds} = 1, \quad \text{a.s. on } A. \tag{11.14}$$

Define $A' := \{\omega : \langle M\rangle(t,\omega) \to \infty \, \text{as} \, t \to T\}$. Then $\limsup_{t\to T^-} M(t) = \infty$ and the $\liminf_{t\to T^-} M(t) = -\infty$ a.s. on A'. Also, we see that $\pi(x) > 0$, $\forall\, x$ sufficiently small, so there is $T' < T$ such that $\pi(X(t)) > 0 \,\forall\, t \in [T', T)$. Therefore $\int_0^t \pi(X(s))\,ds$ tends to a limit as $t \to T^-$. If it is finite, a contradiction results because

$$\liminf_{t\to T^-} X(t)^{-\theta} = \liminf_{t\to T^-} \left(X(0)^{-\theta} + \int_0^t \pi(X(s))\,ds + M(t) \right) = -\infty.$$

Hence, we must have $\int_0^t \pi(X(s))\,ds \to \infty$ as $t \to T^-$ a.s. on A'. Therefore

$$\begin{aligned} \lim_{t\to T^-} \frac{M(t)}{\int_0^t \pi(X(s))\,ds} &= \lim_{t\to T^-} \left(\frac{M(t)}{\langle M\rangle(t)} \cdot \frac{\langle M\rangle(t)}{\int_0^t \pi(X(s))\,ds} \right) \\ &= \lim_{t\to T^-} \frac{M(t)}{\langle M\rangle(t)} \cdot \lim_{t\to T^-} \frac{\int_0^t \nu^2(X(s))\,ds}{\int_0^t \pi(X(s))\,ds}. \end{aligned} \tag{11.15}$$

Consider the second quotient as $t \to T^-$. First, we have $\nu^2(x)/\pi(x) \le c$ from (11.13) for all $x < \delta_1$ and some $c > 0$. Also $X(t) < \delta_1 \,\forall\, t \in [T'', T)$. Thus for $t \in [T'', T)$, $\nu^2(X(t)) \le c\pi(X(t))$. Thus for $t \in [T'', T)$,

$$\begin{aligned} \int_0^t \nu^2(X(s))\,ds &= \int_0^{T''} \nu^2(X(s))\,ds + \int_{T''}^t \nu^2(X(s))\,ds \\ &\le \int_0^{T''} \nu^2(X(s))\,ds + c\int_{T''}^t \pi(X(s))\,ds. \end{aligned}$$

Thus

$$\limsup_{t\to T^-} \frac{\int_0^t \nu^2(X(s))\,ds}{\int_0^t \pi(X(s))\,ds} \le \limsup_{t\to T^-} \left\{ \frac{\int_0^{T''} \nu^2(X(s))\,ds}{\int_0^t \pi(X(s))\,ds} + c \right\} = c.$$

Returning to (11.15) yields $\lim_{t\to T^-} M(t)/\int_0^t \pi(X(s))\,ds = 0$ a.s. on A'. Hence

$$\lim_{t\to T^-} \frac{X(t)^{-\theta}}{\int_0^t \pi(X(s))\,ds} = 1, \quad \text{a.s. on } A'.$$

Using (11.12) yields

$$\lim_{t\to T^-} \frac{X(t)^{-\theta}}{\theta\int_0^t |f(X(s))|/X(s)^{\theta+1}\,ds} = 1, \quad \text{a.s. on } A'. \tag{11.16}$$

By (11.14), (11.16) and since $A\cup A'$ is an a.s. event then

$$\lim_{t\to T^-} \frac{X(t)^{-\theta}}{\theta\int_0^t |f(X(s))|/X(s)^{\theta+1}\,ds} = 1, \quad \text{a.s..} \tag{11.17}$$

Next we have that there exists a function η with $\eta(x)\sim |f(x)|/x^{1+\theta}$ as $x\to 0^+$ which is decreasing. This follows from the asymptotic monotonicity of $x\mapsto |f(x)|/x$ and $x\mapsto x^{-\theta}$. Thus

$$\lim_{t\to T^-} \frac{X(t)^{-\theta}}{\int_0^t \theta\eta(X(s))\,ds} = 1, \quad \text{a.s..}$$

Define $I(t) := \int_0^t \eta(X(s))\,ds$, $t\in[0,T)$. Then $I\in C^1(0,T)$, $I(t)\to\infty$ as $t\to T^-$ and $\eta(X(t)) = I'(t)$. Since η is invertible then

$$\lim_{t\to T^-} \frac{X(t)^{-\theta}}{\theta I(t)} = \lim_{t\to T^-} \frac{\eta^{-1}(I'(t))^{-\theta}}{\theta I(t)} = 1, \quad \text{a.s..}$$

Thus for every $\epsilon\in(0,1)$ and $t\in[T_2(\epsilon),T)$

$$1-\epsilon < \frac{X(t)^{-\theta}}{\theta I(t)} < 1+\epsilon,$$

or for $T_2(\epsilon) < t < T$

$$1-\epsilon < \frac{\eta^{-1}(I'(t))^{-\theta}}{\theta I(t)} < 1+\epsilon.$$

Thus for $t\in(T_2(\epsilon),T)$, $(1-\epsilon)\cdot\theta I(t) < \eta^{-1}(I'(t))^{-\theta} < (1+\epsilon)\cdot\theta I(t)$ or

$$(1-\epsilon)^{-1/\theta}\cdot\theta^{-1/\theta}\cdot I(t)^{-1/\theta} > \eta^{-1}(I'(t)) > (1+\epsilon)^{-1/\theta}\cdot\theta^{-1/\theta}\cdot I(t)^{-1/\theta}.$$

Since η is decreasing for $t\in(T_2(\epsilon),T)$, we have

$$\eta\left((1-\epsilon)^{-1/\theta}\theta^{-1/\theta}I(t)^{-1/\theta}\right) < I'(t) < \eta\left((1+\epsilon)^{-1/\theta}\theta^{-1/\theta}I(t)^{-1/\theta}\right).$$

Hence for $T \geq t \geq T_2(\epsilon)$

$$\int_t^T \frac{I'(s)}{\eta\left(a_- I(s)^{-1/\theta}\right)}\,ds \geq T - t \quad \text{and} \quad \int_t^T \frac{I'(s)}{\eta\left(a_+ I(s)^{-1/\theta}\right)}\,ds \leq T - t,$$

where $a_- := (1-\epsilon)^{-1/\theta} \cdot \theta^{-1/\theta}$, $a_+ := (1+\epsilon)^{-1/\theta} \cdot \theta^{-1/\theta}$. Now recalling that $\bar{F}(x) = \int_0^x 1/|f(u)|\,du$, then as $t \to T^-$

$$\int_t^T \frac{I'(s)}{\eta\left(aI(s)^{-1/\theta}\right)}\,ds \sim \theta a^\theta \bar{F}\left(aI(t)^{-1/\theta}\right),$$

because $I(t) \to \infty$ as $t \to T^-$

$$\int_t^T \frac{I'(s)}{\eta\left(aI(s)^{-1/\theta}\right)}\,ds = \theta a^\theta \int_0^{aI(t)^{-1/\theta}} \frac{1}{u^{1+\theta}\eta(u)}\,du,$$

and $u^{1+\theta}\eta(u) \sim |f(u)|$ as $u \to 0^+$. Therefore

$$1 \leq \liminf_{t\to T^-} \frac{\int_t^T I'(s)/\eta\left(a_- I(s)^{-1/\theta}\right)\,ds}{T-t} = \liminf_{t\to T^-} \frac{\theta a_-^\theta \bar{F}(a_- I(t)^{-1/\theta})}{T-t},$$

and

$$1 \geq \limsup_{t\to T^-} \frac{\int_t^T I'(s)/\eta\left(a_+ I(s)^{-1/\theta}\right)\,ds}{T-t} = \limsup_{t\to T^-} \frac{\theta a_+^\theta \bar{F}(a_+ I(t)^{-1/\theta})}{T-t}.$$

Thus, as $a_\pm^\theta = (1 \pm \epsilon)^{-1} \cdot \theta^{-1}$, then

$$1 - \epsilon \leq \liminf_{t\to T^-} \frac{\bar{F}\left(a_- I(t)^{-1/\theta}\right)}{T-t} \quad \text{and} \quad \limsup_{t\to T^-} \frac{\bar{F}\left(a_+ I(t)^{-1/\theta}\right)}{T-t} \leq 1 + \epsilon, \quad \text{a.s..} \tag{11.18}$$

Recall also for $T_2(\epsilon) < t < T$, $(1-\epsilon)\cdot\theta I(t) < X(t)^{-\theta} < (1+\epsilon)\cdot\theta I(t)$, then

$$(1-\epsilon)^{-1/\theta} \cdot \theta^{-1/\theta} \cdot I(t)^{-1/\theta} > X(t) > (1+\epsilon)^{-1/\theta} \cdot \theta^{-1/\theta} \cdot I(t)^{-1/\theta}.$$

Thus for $T_2(\epsilon) < t < T$

$$a_- I(t)^{-1/\theta} > X(t) > a_+ I(t)^{-1/\theta}. \tag{11.19}$$

Now by (11.19), $\bar{F}\left(a_-/a_+ \cdot X(t)\right) > \bar{F}\left(a_-/a_+ \cdot a_+ I(t)^{-1/\theta}\right) = \bar{F}\left(a_- I(t)^{-1/\theta}\right)$. Thus by (11.18)

$$\liminf_{t\to T^-} \frac{\bar{F}\left(a_-/a_+ \cdot X(t)\right)}{T-t} \geq \liminf_{t\to T^-} \frac{\bar{F}\left(a_- I(t)^{-1/\theta}\right)}{T-t} \geq 1 - \epsilon.$$

Hence

$$\liminf_{t\to T^-} \frac{\bar{F}\left(\left(\frac{1+\epsilon}{1-\epsilon}\right)^{1/\theta} X(t)\right)}{T-t} \geq 1 - \epsilon. \tag{11.20}$$

Similarly by (11.19) $\bar{F}\left(a_+/a_- \cdot X(t)\right) < \bar{F}\left(a_+/a_- \cdot a_- I(t)^{-1/\theta}\right) = \bar{F}\left(a_+ I(t)^{-1/\theta}\right)$. Thus by (11.18)

$$\limsup_{t\to T^-} \frac{\bar{F}\left(a_+/a_- \cdot X(t)\right)}{T-t} \leq \limsup_{t\to T^-} \frac{\bar{F}\left(a_+ I(t)^{-1/\theta}\right)}{T-t} \leq 1+\epsilon.$$

Hence

$$\limsup_{t\to T^-} \frac{\bar{F}\left(\left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/\theta} X(t)\right)}{T-t} \leq 1+\epsilon \tag{11.21}$$

Now

$$\begin{aligned}
\liminf_{t\to T^-} \frac{\bar{F}\left(X(t)\right)}{T-t} &= \liminf_{t\to T^-} \left(\frac{\bar{F}\left(X(t)\right)}{\bar{F}\left(\left(\frac{1+\epsilon}{1-\epsilon}\right)^{1/\theta} X(t)\right)} \cdot \frac{\bar{F}\left(\left(\frac{1+\epsilon}{1-\epsilon}\right)^{1/\theta} X(t)\right)}{T-t} \right) \\
&\geq (1-\epsilon) \cdot \liminf_{t\to T^-} \frac{\bar{F}\left(X(t)\right)}{\bar{F}\left(\left(\frac{1+\epsilon}{1-\epsilon}\right)^{1/\theta} X(t)\right)} \\
&= (1-\epsilon) \cdot \liminf_{x\to 0^+} \frac{\bar{F}(x)}{\bar{F}\left(\left(\frac{1+\epsilon}{1-\epsilon}\right)^{1/\theta} x\right)}.
\end{aligned}$$

Hence if we temporarily assume that

$$\lim_{\epsilon\to 0^+} \liminf_{x\to 0^+} \frac{\bar{F}(x)}{\bar{F}\left(\left(\frac{1+\epsilon}{1-\epsilon}\right)^{1/\theta} x\right)} = 1, \tag{11.22}$$

then

$$\liminf_{t\to T^-} \frac{\bar{F}\left(X(t)\right)}{T-t} \geq 1, \quad \text{a.s.}. \tag{11.23}$$

Also

$$\begin{aligned}
\limsup_{t\to T^-} \frac{\bar{F}\left(X(t)\right)}{T-t} &= \limsup_{t\to T^-} \left(\frac{\bar{F}\left(X(t)\right)}{\bar{F}\left(\left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/\theta} X(t)\right)} \cdot \frac{\bar{F}\left(\left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/\theta} X(t)\right)}{T-t} \right) \\
&\leq (1+\epsilon) \cdot \limsup_{t\to T^-} \frac{\bar{F}\left(X(t)\right)}{\bar{F}\left(\left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/\theta} X(t)\right)} \\
&= (1+\epsilon) \cdot \limsup_{x\to 0^+} \frac{\bar{F}(x)}{\bar{F}\left(\left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/\theta} x\right)}.
\end{aligned}$$

Hence if we once again momentarily suppose that

$$\lim_{\epsilon\to 0^+} \limsup_{x\to 0^+} \frac{\bar{F}(x)}{\bar{F}\left(\left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/\theta} x\right)} = 1, \tag{11.24}$$

then we have

$$\limsup_{t\to T^-}\frac{\bar{F}\left(X(t)\right)}{T-t}\leq 1,\quad \text{a.s..} \tag{11.25}$$

Combining (11.23) and (11.25) gives the result. It remains to prove (11.22) and (11.24) hold. (11.22) is equivalent to

$$\lim_{a\to 1^+}\liminf_{x\to 0^+}\frac{\bar{F}(x)}{\bar{F}(ax)}=1, \tag{11.26}$$

and (11.24) is equivalent to

$$\lim_{a\to 1^-}\limsup_{x\to 0^+}\frac{\bar{F}(x)}{\bar{F}(ax)}=1. \tag{11.27}$$

Putting $y := ax$ in (11.26), then $x = y/a = \alpha y$ where $\alpha := 1/a$. Then (11.26) is equivalent to

$$\lim_{\alpha\to 1^-}\liminf_{y\to 0^+}\frac{\bar{F}(\alpha y)}{\bar{F}(y)}=1, \tag{11.28}$$

as $a > 1$ in (11.26). Similarly (11.27) is equivalent to

$$\lim_{\alpha\to 1^+}\limsup_{y\to 0^+}\frac{\bar{F}(\alpha y)}{\bar{F}(y)}=1. \tag{11.29}$$

We consider (11.29) first. Firstly, $\bar{F}(\alpha y)/\bar{F}(y) > 1$ for $e > \alpha > 1$. Write

$$\bar{F}(\alpha x)=\bar{F}(x)+\int_x^{\alpha x}\frac{1}{|f(u)|}\,du. \tag{11.30}$$

We have $\eta_1(x) \sim |f(x)|/x$ as $x \to 0^+$ and η_1 is decreasing and $\eta_3(x) \sim |f(x)|$ as $x \to 0^+$ and η_3 is increasing. Then for $x < x_1(\epsilon)$ for $\epsilon \in (0,1)$ so small that $(1+\epsilon)^2/(1-\epsilon)^2 \cdot \log\alpha < 1$ (which implies $0 < \epsilon < (\sqrt{1/\log\alpha}-1)/(\sqrt{1/\log\alpha}+1)$) then

$$(1-\epsilon)\cdot\frac{|f(x)|}{x}<\eta_1(x)<(1+\epsilon)\cdot\frac{|f(x)|}{x},$$

and $(1-\epsilon)\cdot|f(x)| < \eta_3(x) < (1+\epsilon)\cdot|f(x)|$. Let $\alpha x < x_1(\epsilon)$. Then, as $|f(x)| > x\eta_1(x)/(1+\epsilon)$ for $\alpha x < x_1(\epsilon)$,

$$\int_x^{\alpha x}\frac{1}{|f(u)|}\,du\leq\int_x^{\alpha x}\frac{1}{u\eta_1(u)}\cdot\frac{1}{1/(1+\epsilon)}\leq(1+\epsilon)\int_x^{\alpha x}\frac{1}{\eta_1(u)}\cdot\frac{1}{u}\,du.$$

Now $\eta_1(x) \geq \eta_1(u) \geq \eta_1(\alpha x)$ for $u \in [x, \alpha x]$, so $1/\eta_1(u) \leq 1/\eta_1(\alpha x)$. Thus as $\eta_1(\alpha x) > (1-\epsilon)\cdot|f(\alpha x)|/\alpha x$ then for $\alpha x < x_1(\epsilon)$

$$\int_x^{\alpha x}\frac{1}{|f(u)|}\,du\leq(1+\epsilon)\cdot\frac{1}{\eta_1(\alpha x)}\cdot\log\alpha\leq\frac{1+\epsilon}{1-\epsilon}\cdot\frac{\alpha x}{|f(\alpha x)|}\cdot\log\alpha. \tag{11.31}$$

Now for $x < x_1(\epsilon)$

$$\bar{F}(x) = \int_0^x \frac{1}{|f(u)|}\,du \geq \int_0^x \frac{1}{1/(1-\epsilon)\cdot \eta_3(x)}\,du,$$

because $|f(x)| < \eta_3(x)/(1-\epsilon)$. Hence for $x < x_1(\epsilon)$

$$\bar{F}(x) \geq (1-\epsilon)\int_0^x \frac{1}{\eta_3(u)}\,du.$$

For $0 \leq u \leq x$, then $\eta_3(u) < \eta_3(x)$. Hence $1/\eta_3(u) > 1/\eta_3(x)$. Thus

$$\bar{F}(x) \geq (1-\epsilon)\cdot \frac{x}{\eta_3(x)},$$

and as $\eta_3(x) < (1+\epsilon)\cdot|f(x)|$, then $1/\eta_3(x) > 1/(1+\epsilon)\cdot 1/|f(x)|$. Thus for $x < x_1(\epsilon)$

$$\bar{F}(x) > \frac{1-\epsilon}{1+\epsilon}\cdot\frac{x}{|f(x)|}. \tag{11.32}$$

Therefore, by (11.32) for $\alpha x < x_1(\epsilon)$

$$\bar{F}(\alpha x) > \frac{1-\epsilon}{1+\epsilon}\cdot\frac{\alpha x}{|f(\alpha x)|}.$$

Thus

$$\frac{\alpha x}{|f(\alpha x)|} < \frac{1+\epsilon}{1-\epsilon}\cdot\bar{F}(\alpha x).$$

Hence by (11.31), for $\alpha x < x_1(\epsilon)$

$$\int_x^{\alpha x}\frac{1}{|f(u)|}\,du \leq \frac{1+\epsilon}{1-\epsilon}\cdot\frac{1+\epsilon}{1-\epsilon}\cdot\bar{F}(\alpha x)\cdot\log\alpha.$$

Putting this in (11.30) yields for $\alpha x < x_1(\epsilon)$

$$\bar{F}(\alpha x) \leq \bar{F}(x) + \frac{(1+\epsilon)^2}{(1-\epsilon)^2}\cdot\bar{F}(\alpha x)\cdot\log\alpha.$$

Thus as $(1+\epsilon)^2/(1-\epsilon)^2\cdot\log\alpha < 1$, then for $\alpha x < x_1(\epsilon)$

$$\bar{F}(\alpha x) \leq \frac{\bar{F}(x)}{1 - \frac{(1+\epsilon)^2}{(1-\epsilon)^2}\cdot\log\alpha}.$$

Thus

$$\limsup_{x\to 0^+}\frac{\bar{F}(\alpha x)}{\bar{F}(x)} \leq \frac{1}{1-\frac{(1+\epsilon)^2}{(1-\epsilon)^2}\cdot\log\alpha}.$$

Letting $\epsilon \to 0^+$ yields

$$1 \leq \limsup_{x\to 0^+}\frac{\bar{F}(\alpha x)}{\bar{F}(x)} \leq \frac{1}{1-\log\alpha}.$$

Therefore
$$1 \leq \liminf_{\alpha\to 1^+} \limsup_{x\to 0^+} \frac{\bar{F}(\alpha x)}{\bar{F}(x)} \leq \limsup_{\alpha\to 1^+} \limsup_{x\to 0^+} \frac{\bar{F}(\alpha x)}{\bar{F}(x)} \leq 1.$$
Hence (11.29) holds. We consider (11.28) for $1/e < \alpha < 1$. Let $\epsilon \in (0,1)$ be so small that $(1+\epsilon)^2/(1-\epsilon)^2 \cdot \log(1/\alpha) < 1$. It is important to obtain a lower bound on $\bar{F}(\alpha y)/\bar{F}(y)$: write
$$\bar{F}(x) = \bar{F}(\alpha x) + \int_{\alpha x}^{x} \frac{1}{|f(u)|}\, du. \tag{11.33}$$
Now by (11.31), for $x < x_1(\epsilon)$,
$$\int_{\alpha x}^{x} \frac{1}{|f(u)|}\, du \leq (1+\epsilon)\cdot \frac{1}{\eta_1(x)} \int_{\alpha x}^{x} \frac{1}{u}\, du = (1+\epsilon)\cdot \frac{1}{\eta_1(x)} \cdot \log\left(\frac{1}{\alpha}\right).$$
Thus, as $\eta_1(x) > (1-\epsilon)\cdot |f(x)|/x$, we have
$$\int_{\alpha x}^{x} \frac{1}{|f(u)|}\, du \leq (1+\epsilon)\cdot \frac{1}{\eta_1(x)} \cdot \log\left(\frac{1}{\alpha}\right) \leq \frac{1+\epsilon}{1-\epsilon}\cdot \frac{x}{|f(x)|} \cdot \log\left(\frac{1}{\alpha}\right).$$
Hence by (11.33) for $x < x_1(\epsilon)$
$$\bar{F}(x) \leq \bar{F}(\alpha x) + \frac{1+\epsilon}{1-\epsilon}\cdot \frac{x}{f(x)} \cdot \log\left(\frac{1}{\alpha}\right). \tag{11.34}$$
Since $x/|f(x)| < (1+\epsilon)/(1-\epsilon)\cdot \bar{F}(x)$ from (11.32), from (11.34) we get for $x < x_1(\epsilon)$
$$\bar{F}(x) \leq \bar{F}(\alpha x) + \frac{(1+\epsilon)^2}{(1-\epsilon)^2}\cdot \bar{F}(x) \cdot \log\left(\frac{1}{\alpha}\right),$$
which rearranges to give for $x < x_1(\epsilon)$
$$\bar{F}(\alpha x) \geq \bar{F}(x)\left(1 - \frac{(1+\epsilon)^2}{(1-\epsilon)^2}\cdot \log\left(\frac{1}{\alpha}\right)\right).$$
Therefore
$$\liminf_{x\to 0^+} \frac{\bar{F}(\alpha x)}{\bar{F}(x)} \geq 1 - \frac{(1+\epsilon)^2}{(1-\epsilon)^2}\cdot \log\left(\frac{1}{\alpha}\right).$$
Letting $\epsilon \to 0^+$ yields for $\alpha \in (1/e, 1)$
$$1 \geq \liminf_{x\to 0^+} \frac{\bar{F}(\alpha x)}{\bar{F}(x)} \geq 1 - \log\left(\frac{1}{\alpha}\right).$$
Thus
$$1 \leq \liminf_{\alpha\to 1^-} \liminf_{x\to 0^+} \frac{\bar{F}(\alpha x)}{\bar{F}(x)} \leq \limsup_{\alpha\to 1^-} \liminf_{x\to 0^+} \frac{\bar{F}(\alpha x)}{\bar{F}(x)} = 1.$$
Hence (11.28) holds. This completes the proof. □

11.3 Power Pre-Transformed Scheme

Define the sequences

$$Z_{n+1} = Z_n + h(X_n) \cdot \left(-\theta X_n^{-(\theta+1)} f(X_n) \left(1 - \frac{(\theta+1)g^2(X_n)}{2X_n f(X_n)} \right) \right) + \sqrt{h(X_n)} \cdot \frac{-\theta g(X_n)}{X_n^{\theta+1}} \cdot \xi_{n+1}, \quad n \geq 0, \quad Z_0 = \zeta^{-\theta}, \quad (11.35)$$

and (ξ_n) is a sequence of independent and identically distributed Standard Normal random variables and

$$X_n = Z_n^{-1/\theta}, \quad n \geq 0, \quad X_0 = \zeta, \quad (11.36)$$

$$t_{n+1} = t_n + h(X_n), \quad n \geq 0, \quad t_0 = 0, \quad (11.37)$$

where h satisfies

$$h : [0, \infty) \mapsto [0, \infty) \text{ is continuous with } h(0) = 0,\ h(x) > 0, \forall\, x > 0 \quad (11.38)$$

$$\text{there exists } \Delta > 0 \text{ such that } h(x) = \Delta \min \left(1, \frac{x}{|f(x)|}, \frac{x^2}{g^2(x)} \right), \forall\, x > 0. \quad (11.39)$$

Our first main result shows, very roughly, that under the conditions of Theorem 71 the discretised solution remains positive, tends to zero and obeys an asymptotic relationship which is the discrete analogue of (11.16). Recall that (11.16) enabled us to obtain direct asymptotic information about the behaviour of the solution of the SDE near the finite stability time.

Theorem 72. *Let $\zeta > 0$. Suppose there exists θ, $\delta_1 > 0$ such that* (11.6) *and*

$$x \in \mathbb{R}/\,\{0\} \text{ implies } x^{-1/\theta} \in (0, \infty). \quad (11.40)$$

holds. Suppose $g : [0, \infty) \mapsto [0, \infty)$ is continuous, f obeys

$$f(0) = 0,\ f(x) < 0\, \forall\, x > 0,\ f \text{ is continuous on } [0, \infty), \quad (11.41)$$

while h obeys (11.38)*. Then $X_n \in (0, \infty)$ for all $n \geq 0$ a.s., $X_n \to 0$ as $n \to \infty$, a.s. and*

$$\lim_{n\to\infty} \frac{X_n^{-\theta}}{\sum_{j=0}^{n-1} h(X_n)\theta X_j^{-(\theta+1)} |f(X_j)|} = 1, \quad a.s.. \quad (11.42)$$

Remark 45. Suppose $p, q \in \mathbb{N}$ with p, q relatively prime, q odd and p even. If $\theta = q/p$ then (11.40) is true

Proof of Remark 45. Let $x \neq 0$. Then

$$x^{-1/\theta} = \frac{1}{(x^{1/q})^p}, \quad \text{for } x \neq 0$$

and so as $x^{1/q} \in \mathbb{R}$ and p is even thus $x^{-1/\theta} > 0$ for $x \neq 0$. □

Proof of Theorem 72. Define

$$\mu_n := h(X_n)\cdot\left(-\theta X_n^{-(\theta+1)} f(X_n)\left(1 - \frac{(\theta+1)g^2(X_n)}{2X_n f(X_n)}\right)\right) \quad \text{and} \quad \nu_n := \sqrt{h(X_n)}\cdot\frac{-\theta g(X_n)}{X_n^{\theta+1}},$$

so that for $n \geq 0$, $Z_{n+1} = Z_n + \mu_n + \nu_n \xi_{n+1}$. First note that $X_n > 0\, \forall\, n \in \mathbb{N}$, a.s.. We prove this by induction as follows: Clearly $X_0 > 0$. Suppose

$$\mathbb{P}\left[X_n > 0\, \forall n \in \{0, \ldots, N\}\right] = 1. \tag{11.43}$$

Then by (11.43)

$$\begin{aligned}
\mathbb{P}\left[X_n > 0\, \forall n \in \{0, \ldots, N+1\}\right] &= \mathbb{P}\left[X_{n+1} > 0 \cap \{X_n > 0\, \forall\, n \in \{0, \ldots, N\}\}\right] \\
&= \mathbb{P}\left[X_{N+1} > 0 | X_n > 0\, \forall n \in \{0, \ldots, N\}\right] \mathbb{P}\left[X_n > 0\, \forall\, n \in \{0, \ldots, N\}\right] \\
&= \mathbb{P}\left[X_{N+1} > 0 | X_n > 0\, \forall\, n \in \{0, \ldots, N\}\right] \\
&= \mathbb{P}\left[Z_{N+1} \in \mathbb{R} | X_n > 0\, \forall\, n \in \{0, \ldots, N\}\right] \\
&= \mathbb{P}\left[Z_{N+1} \in \mathbb{R} | X_N > 0,\ Z_N \text{ well-defined}\right].
\end{aligned}$$

It is clear by (11.41) that if $X_N > 0$, then ν_N and μ_N are well-defined and finite. Thus, as Z_N is well-defined so is Z_{N+1} and hence $X_{N+1} = Z_{N+1}^{-1/\theta} > 0$ by (11.40). Hence

$$A_N := \{X_n \in (0, \infty)\, \forall\, n \in \{0, \ldots, N\}\} \text{ is an a.s. event for each } N.$$

Thus

$$A_\infty := \bigcup_{N \geq 1} A_N = \{\omega : X_n \in (0, \infty)\, \forall\, n \in \mathbb{N}\},$$

is also an a.s. event. Henceforth, we work on this event. Since $f(x) < 0\, \forall\, x > 0$, then

$$\phi(x) := 1 - \frac{(\theta+1)g^2(x)}{2xf(x)} = 1 + \frac{(\theta+1)g^2(x)}{2x|f(x)|} \geq 1.$$

Thus $\mu_n = h(X_n)\theta X_n^{-(\theta+1)}|f(X_n)|\phi(X_n)$ for $\omega \in A_\infty$ by (11.41). Since $h(X_n) > 0\, \forall\, n \geq 0$ then for $n \geq 0$

$$\mu_n \geq h(X_n)\theta X_n^{-(\theta+1)}|f(X_n)| > 0. \tag{11.44}$$

Note next that $Z_{j+1} - Z_j = \mu_j + \nu_j \xi_{j+1}$, $j \geq 0$. So for $n \geq 1$

$$Z_n = Z_0 + \sum_{j=0}^{n-1} \mu_j + \sum_{j=0}^{n-1} \nu_j \xi_{j+1}. \tag{11.45}$$

Define $M_n := \sum_{j=0}^{n-1} \nu_j \xi_{j+1}$. Suppose temporarily that M_n is an L^2-martingale. Since ξ's are Standard Normal random variables, it can be shown that if

$$B_1 = \left\{ \omega : \sum_{j=0}^{\infty} \nu_j(\omega)^2 < \infty \right\} \quad \text{and} \quad B_2 = \left\{ \omega : \sum_{j=0}^{\infty} \nu_j(\omega)^2 = \infty \right\},$$

then on B_1 a.s. $\lim_{n\to\infty} M_n =: M_\infty \in (-\infty, \infty)$ and on B_2 a.s. $\lim_{n\to\infty} M_n / \langle M \rangle (n) = 0$ where $\langle M \rangle (n) := \sum_{j=0}^{n-1} \nu_j^2$. Clearly $B_1 \cup B_2$ is an a.s. event. On B_1, the right-hand side of (11.45) has a limit, possibly infinite, as $n \to \infty$ because M_n converges to a finite limit and $\mu_n > 0$ by (11.44). Suppose on B_1, there is B_1' with positive probability such that $Z_n \to Z^* \in (-\infty, \infty)$ as $n \to \infty$. Then $X_n \to e^{-Z_*} > 0$, $X_* > 0$ as $n \to \infty$ on B_1'. By (11.44) and the continuity of f and h then

$$\liminf_{n\to\infty} \mu_n \geq h(X_*)\theta X_*^{-(\theta+1)} f(X_*) =: \mu_* > 0.$$

As f and g are continuous, $x \mapsto \phi(x)$ is continuous. Finally, it must follow that $\sum_{j=0}^{n-1} \mu_j \to \infty$ a.s. on B_1'. Hence $Z_n \to \infty$ a.s. on B_1', a contradiction. Therefore, on B_1, we must have that $Z_n \to \infty$ or $X_n \to 0$ as $n \to \infty$ a.s.. Hence on B_1

$$\lim_{n\to\infty} \frac{Z_n}{\sum_{j=0}^{n-1} \mu_j} = \lim_{n\to\infty} \left\{ \frac{Z_0}{\sum_{j=0}^{n-1} \mu_j} + 1 + \frac{M_n}{\sum_{j=0}^{n-1} \mu_j} \right\} = 1,$$

because $\sum_{j=0}^{n-1} \mu_j \to \infty$ and M_n tends to a finite limit as $n \to \infty$. Also, since $Z_n \to \infty$ as $n \to \infty$, it follows that $Z_n > 0$ for all n sufficiently large so Z_n is uniquely defined by $Z_n := X_n^{-\theta}$. Therefore

$$\lim_{n\to\infty} \frac{X_n^{-\theta}}{\sum_{j=0}^{n-1} \mu_j} = 1, \quad \text{a.s. on } B_1. \tag{11.46}$$

Now we can work on B_2. On B_2 we have $\sum_{j=0}^{\infty} \nu_j^2 = \infty$. In Lemma 41, which follows the proof of this theorem, we show that

$$\mu_n \geq K(\theta, \delta_1)\nu_n^2, \quad \forall\, n \in \mathbb{N}. \tag{11.47}$$

This implies $\sum_{j=0}^{\infty} \mu_j = \infty$. Recall also $\mu_j > 0 \,\forall\, j$. Now

$$\frac{M_n}{\sum_{j=0}^{n-1} \mu_j} = \frac{M_n}{\langle M \rangle (n)} \cdot \frac{\langle M \rangle (n)}{\sum_{j=0}^{n-1} \mu_j} = \frac{M_n}{\langle M \rangle (n)} \cdot \frac{\sum_{j=0}^{n-1} \nu_j^2}{\sum_{j=0}^{n-1} \mu_j}.$$

By (11.47), we have

$$0 < \frac{\sum_{j=0}^{n-1} \nu_j^2}{\sum_{j=0}^{n-1} \mu_j} \leq \frac{1}{K}.$$

Therefore, as $M_n / \langle M \rangle (n) \to 0$ as $n \to \infty$ a.s. on B_2, it follows that

$$\lim_{n\to\infty} \frac{M_n}{\sum_{j=0}^{n-1} \mu_j} = 0, \quad \text{a.s. on } B_2.$$

Hence

$$\lim_{n\to\infty} \frac{Z_n}{\sum_{j=0}^{n-1} \mu_j} = \lim_{n\to\infty} \left\{ \frac{Z_0}{\sum_{j=0}^{n-1} \mu_j} + 1 + \frac{M_n}{\sum_{j=0}^{n-1} \mu_j} \right\} = 1, \quad \text{a.s. on } B_2.$$

Therefore $Z_n \to \infty$ as $n \to \infty$. Proceeding as in the case of B_1 above we see that

$$\lim_{n\to\infty} \frac{X_n^{-\theta}}{\sum_{j=0}^{n-1} \mu_j} = 1 \quad \text{a.s. on } B_2. \tag{11.48}$$

Since $B_1 \cup B_2$ is an a.s. event by (11.46) and (11.48) we have

$$\lim_{n\to\infty} \frac{X_n^{-\theta}}{\sum_{j=0}^{n-1} \mu_j} = 1, \quad \text{a.s.}. \tag{11.49}$$

Also on B_1 and B_2 a.s. we have that $Z_n \to \infty$ as $n \to \infty$, so clearly $X_n \to 0$ as $n \to \infty$. Next, we determine asymptotic behaviour of μ_n, using the fact that $X_n \to 0$ as $n \to \infty$. Since by (11.6)

$$\frac{g^2(x)}{x|f(x)|} = \frac{g^2(x)}{x^{1+\theta}|f(x)|} \cdot x^\theta \leq c x^\theta,$$

we have that $g^2(x)/(x|f(x)|) \to 0$ as $x \to 0^+$. Hence $g^2(X_n)/(X_n f(X_n)) \to 0$ as $n \to \infty$. Thus $\mu_n \sim h(X_n)\theta X_n^{-(\theta+1)}|f(X_n)|$ as $n \to \infty$. Hence, as $\sum_{j=0}^{n-1} \mu_j \to \infty$ as $n \to \infty$, by Toeplitz's Lemma and (11.49) if follows that

$$\lim_{n\to\infty} \frac{X_n^{-\theta}}{\sum_{j=0}^{n-1} h(X_n)\theta X_j^{-(\theta+1)}|f(X_j)|} = 1,$$

as claimed. It remains to check that M_n is an L^2-martingale. The property $\mathbb{E}\left[M_{n+1}|\mathscr{F}_n\right] = M_n$ follows once $\mathbb{E}\left[\nu_n^2\right] < \infty$. So to prove M_n is L^2, it follows from $\mathbb{E}\left[\langle M \rangle_n\right] = \mathbb{E}\left[M_n^2\right] < \infty$ that it is sufficient to prove $\mathbb{E}\left[\nu_n^2\right] < \infty$ for all n. It is easy to check that $\nu_n^2 \leq K_1(\Delta, \theta) X_n^{-\theta}$ for some constant $K_1 > 0$. Therefore showing that $\mathbb{E}\left[X_n^{-\theta}\right] < \infty$ for all n suffices. Similarly, one can estimate $0 < \mu_n \leq K_2(\Delta, \theta) X_n^{-\theta}$. Since $X_n = Z_n^{-1/\theta}$

and $X_n > 0$, we have $|Z_n| = X_n^{-\theta}$. Now

$$Z_{n+1} = Z_n + \mu_n + \nu_n \xi_{n+1},$$

so $|Z_{n+1}| \leq |Z| + |\mu_n| + |\nu_n \xi_{n+1}|$ or as $\mu_n > 0$

$$X_{n+1}^{-\theta} \leq X_n^{-\theta} + \mu_n + |\nu_n \xi_{n+1}| \leq X_n^{-\theta} + K_2(\Delta, \theta) X_n^{-\theta} + |\nu_n \xi_{n+1}|.$$

Next suppose that $\mathbb{E}\left[X_n^{-\theta}\right] < \infty$. Clearly this is true for $n = 0$. To deal with the general step, since $\nu_n^2 \leq K_1(\Delta, \theta) X_n^{-\theta}$. Thus by the Cauchy-Schwarz inequality

$$\mathbb{E}\left[|\nu_n \xi_{n+1}|\right]^2 \leq \mathbb{E}\left[\nu_n^2\right] \mathbb{E}\left[\xi_{n+1}^2\right] \leq K_1(\Delta, \theta) \mathbb{E}\left[X_n^{-\theta}\right] < \infty.$$

Hence

$$\mathbb{E}\left[X_{n+1}^{-\theta}\right] < 1 + K_2(\Delta, \theta) \mathbb{E}\left[X_n^{-\theta}\right] + \sqrt{K_1(\Delta, \theta)} \mathbb{E}\left[X_n^{-\theta}\right]^{1/2} < \infty,$$

as claimed. □

Lemma 41. *Suppose there exists $\delta_1 > 0$, $\theta > 0$ such that* (11.6) *holds viz.,*

$$\sup_{0 < x \leq \delta_1} \frac{g^2(x)}{x^{1+\theta} |f(x)|} =: C < \infty$$

If

$$\mu_n := h(X_n) \theta X_n^{-(\theta+1)} |f(X_n)| \left(1 + \frac{(\theta+1) g^2(X_n)}{2 X_n |f(X_n|)}\right) \quad \textit{and} \quad \nu_n := \sqrt{h(X_n)} \cdot \frac{-\theta g(X_n)}{X_n^{\theta+1}},$$

then there exists a constant $K = K(\theta, \delta_1) > 0$ such that (11.47) *holds.*

Proof. Let $X_n \in (0, \delta_1]$. Then $\mu_n \geq h(X_n) \theta X_n^{-(\theta+1)} |f(X_n)|$. Also $g^2(X_n) \leq C X_n^{\theta+1} |f(X_n)|$. Thus $\mu_n \geq h(X_n) g^2(X_n) \cdot \theta / C \cdot X_n^{-(2\theta+2)}$. On the other hand $\nu_n^2 = h(X_n) \theta^2 g^2(X_n) X_n^{-(2\theta+2)}$. Thus

$$\frac{\mu_n}{\nu_n^2} \geq \frac{\theta / C}{\theta^2} = \frac{1}{C\theta}.$$

Hence $X_n \in (0, \delta_1]$ implies

$$\frac{\mu_n}{\nu_n^2} \geq \frac{1}{C\theta}. \tag{11.50}$$

Let $X_n > \delta_1$. Then

$$\mu_n \geq h(X_n) \theta X_n^{-(\theta+1)} |f(X_n)| \cdot \frac{(\theta+1) g^2(X_n)}{2 X_n |f(X_n|)} = \frac{\theta(\theta+1) h(X_n) g^2(X_n) X_n^{-(\theta+2)}}{2}.$$

Thus

$$\frac{\mu_n}{\nu_n^2} \geq \frac{\theta(\theta+1) X_n^{-(\theta+2)}}{2\theta^2 X_n^{-(2\theta+2)}} = \frac{(\theta+1) X_n^{\theta}}{2\theta} \geq \frac{(\theta+1)\delta_1^{\theta}}{2\theta}.$$

Thus $X_n > \delta_1$ implies

$$\frac{\mu_n}{\nu_n^2} \geq \frac{(\theta+1)\delta_1^\theta}{2\theta}. \tag{11.51}$$

Hence by (11.50) and (11.51), for any $n \in \mathbb{N}$

$$\frac{\mu_n}{\nu_n^2} \geq \min\left(\frac{(\theta+1)\delta_1^\theta}{2\theta}, \frac{1}{C(\delta_1)\theta}\right) =: K(\theta, \delta_1),$$

so $\mu_n \geq K(\theta, \delta_1)\nu_n^2$, $\forall\, n \in \mathbb{N}$ which is (11.47). □

The next result is an easy consequence of (11.39) and (11.42). The next two lemmas establish that the sequence (X_n) has an asymptotic and deterministic common ratio and this is used to obtain precise asymptotic information in the forthcoming Theorem 73.

Lemma 42. *Suppose all hypotheses of Theorem 72 hold and in addition h obeys* (11.39)*. Then $X_n \in (0, \infty)$ for all $n \geq 0$ a.s., $X_n \to 0$ as $n \to \infty$, a.s. and*

$$\lim_{n\to\infty} \frac{X_n^{-\theta}}{\sum_{j=0}^{n-1} \Delta\theta X_j^{-\theta}} = 1, \quad a.s.. \tag{11.52}$$

Proof. The above discussion shows that (11.42) holds. Define $\tilde{\mu}_n := h(X_n)\theta X_n^{-(\theta+1)}|f(X_n)|$. Then as $n \to \infty$

$$\frac{\tilde{\mu}_n}{\Delta\theta X_n^{-\theta}} = \frac{h(X_n)\theta X_n^{-(\theta+1)}|f(X_n)|}{\Delta\theta X_n^{-\theta}} = \frac{h(X_n)X_n^{-1}|f(X_n)|}{\Delta} = \frac{h(X_n)}{\Delta X_n/|f(X_n)|} \to 1,$$

by the fact that $0 < X_n \to 0$ as $n \to \infty$ and (11.39). By (11.42) and Toeplitz's Lemma, (11.52) holds. □

Lemma 43. *Suppose all hypotheses of Theorem 72 hold and in addition h obeys* (11.39)*. Then*

$$\lim_{n\to\infty} \frac{X_{n+1}}{X_n} = (1+\Delta\theta)^{-1/\theta}, \quad a.s..$$

Proof. Define $S_n := \sum_{j=0}^{n} \Delta\theta X_j^{-\theta}$. Then $S_{n+1} - S_n = \Delta\theta X_{n+1}^{-\theta}$ or

$$\frac{(S_{n+1} - S_n)}{\Delta\theta} = X_{n+1}^{-\theta}.$$

Hence by Lemma 42, (11.52) holds and

$$1 = \lim_{n\to\infty} \frac{X_n^{-\theta}}{S_{n-1}} = \lim_{n\to\infty} \frac{(S_n - S_{n-1})/\Delta\theta}{S_{n-1}}.$$

Thus

$$1 = \frac{1}{\Delta\theta}\left(\lim_{n\to\infty} \frac{S_n}{S_{n-1}} - 1\right),$$

so $\lim_{n\to\infty} S_n/S_{n-1} = 1 + \Delta\theta$. However, by this limit and $X_n^{-\theta}/S_{n-1} \to 1$ as $n \to \infty$ then

$$\lim_{n\to\infty} \frac{X_{n+1}^{-\theta}}{X_n^{-\theta}} = \lim_{n\to\infty} \left(\frac{X_{n+1}^{-\theta}}{S_n} \cdot \frac{S_n}{S_{n-1}} \cdot \frac{S_{n-1}}{X_n^{-\theta}} \right) = 1 + \Delta\theta.$$

Hence

$$\left(\lim_{n\to\infty} \frac{X_{n+1}}{X_n} \right)^{-\theta} = 1 + \Delta\theta,$$

or $X_{n+1}/X_n \to (1 + \Delta\theta)^{-1/\theta}$ as $n \to \infty$ a.s., as claimed. □

Theorem 73. *Suppose all hypotheses of Theorem 72 hold and in addition f obeys (1.26) and (11.8) while h obeys (11.39). Let $\bar{F}$ and (t_n) be defined by (1.28) and (11.37). Then*

$$\lim_{n\to\infty} t_n =: \hat{T}_h < \infty, \quad a.s.. \tag{11.53}$$

Proof. Since $X_n \to 0$ as $n \to \infty$, the summability of $t_n = \sum_{j=0}^{n-1} h(X_j)$ is equivalent to that of $\tau_n := \sum_{j=0}^{n-1} \Delta X_j/|f(X_j)|$. Moreover, if $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ by Toeplitz's Lemma and (11.39)

$$\lim_{n\to\infty} \frac{\hat{T}_h - t_n}{\sum_{j=n}^{\infty} \Delta X_j/|f(X_j)|} = \lim_{n\to\infty} \frac{\sum_{j=n}^{\infty} h(X_j)}{\sum_{j=n}^{\infty} \Delta X_j/|f(X_j)|} = 1. \tag{11.54}$$

Since $(1 + \Delta\theta)^{-1/\theta} < 1$ for any $\Delta > 0$, X_n is decreasing for $n \geq N_1$ and for some $N \in \mathbb{N}$. Now, $x \mapsto x/|f(x) \sim \eta_1(x)$ as $x \to 0^+$ and η_1 is increasing. Thus for every $\epsilon \in (0,1)$ there is $x_1(\epsilon) > 0$ such that for $x < x_1(\epsilon)$

$$(1 - \epsilon) \cdot \eta_1(x) < \frac{x}{|f(x)|} < (1 + \epsilon) \cdot \eta_1(x).$$

Hence, as $X_n \to 0$ as $n \to \infty$, there is $N_2(\epsilon) \in \mathbb{N}$ such that $X_n < x_1(\epsilon) \, \forall \, n \geq N_2(\epsilon)$. Thus for $n \geq \max(N_1, N_2(\epsilon))$, as $X_{n+1} < X_n < x_1(\epsilon)$ then

$$\begin{aligned}
\int_{X_{n+1}}^{X_n} \frac{1}{|f(u)|} \, du = \int_{X_{n+1}}^{X_n} \frac{1}{u} \cdot \frac{u}{|f(u)|} \, du \quad &\leq \quad \int_{X_{n+1}}^{X_n} \frac{1}{u} \cdot (1 + \epsilon) \cdot \eta_1(u) \, du \\
&\leq \quad (1 + \epsilon) \cdot \eta_1(X_n) \int_{X_{n+1}}^{X_n} \frac{1}{u} \, du \\
&= \quad (1 + \epsilon) \cdot \eta_1(X_n) \cdot \log\left(\frac{X_n}{X_{n+1}} \right) \\
&\leq \quad \frac{1 + \epsilon}{1 - \epsilon} \cdot \frac{X_n}{|f(X_n)|} \cdot \log\left(\frac{X_n}{X_{n+1}} \right).
\end{aligned}$$

Thus

$$\int_{X_{n+1}}^{X_n} \frac{1}{|f(u)|} \, du \leq \frac{1 + \epsilon}{1 - \epsilon} \cdot \frac{X_n}{|f(X_n)|} \cdot \log\left(\frac{X_n}{X_{n+1}} \right). \tag{11.55}$$

Similarly

$$\begin{aligned}\int_{X_{n+1}}^{X_n} \frac{1}{|f(u)|}\,du = \int_{X_{n+1}}^{X_n} \frac{1}{u}\cdot\frac{u}{|f(u)|}\,du &\geq \int_{X_{n+1}}^{X_n} \frac{1}{u}\cdot(1-\epsilon)\cdot\eta_1(u)\,du \\ &\geq (1-\epsilon)\cdot\eta_1(X_{n+1})\int_{X_{n+1}}^{X_n}\frac{1}{u}\,du \\ &= \frac{1-\epsilon}{1+\epsilon}\cdot\frac{X_{n+1}}{|f(X_{n+1})|}\cdot\log\left(\frac{X_n}{X_{n+1}}\right).\end{aligned}$$

Therefore for $n \geq \max(N_1, N_2(\epsilon))$

$$\int_{X_{n+1}}^{X_n} \frac{1}{|f(u)|}\,du \geq \frac{1-\epsilon}{1+\epsilon}\cdot\frac{X_{n+1}}{|f(X_{n+1})|}\cdot\log\left(\frac{X_n}{X_{n+1}}\right). \tag{11.56}$$

Hence

$$\frac{\Delta X_{n+1}}{|f(X_{n+1})|} \leq \frac{1+\epsilon}{1-\epsilon}\cdot\frac{\Delta}{\log\left(X_n/X_{n+1}\right)}\int_{X_{n+1}}^{X_n}\frac{1}{|f(u)|}\,du.$$

Since $X_n/X_{n+1} \to (1+\Delta\theta)^{1/\theta}$, $\log\left(X_n/X_{n+1}\right) \to \log(1+\Delta\theta)/\theta$ as $n \to \infty$. Therefore $n \mapsto \Delta X_{n+1}/|f(X_{n+1})|$ is summable because $\sum_{j=0}^{n-1}\int_{X_{j+1}}^{X_j} 1/|f(u)|\,du = \int_{X_n}^{X_0} 1/|f(u)|\,du$ tends to a finite limit as $n \to \infty$, by (1.26) and the fact that $X_n \to 0$ as $n \to \infty$ a.s.. This means that (t_n) tends to a finite limit as $n \to \infty$, as claimed □

Similar to the deterministic case we obtain discrete analogues of the limit $\lim_{t\to T^-} \bar{F}(X(t))/T - t$. Note as in the deterministic case the time indices in these limits differ by unity.

Theorem 74. *Suppose all hypotheses of Theorem 72 hold and in addition f obeys* (1.26) *and* (11.8) *while h obeys* (11.39)*. Then*

$$\frac{\log(1+\Delta\theta)}{\Delta\theta} \leq \liminf_{n\to\infty}\frac{\bar{F}(X_n)}{\hat{T}_h - t_{n+1}}, \quad \limsup_{n\to\infty}\frac{\bar{F}(X_n)}{\hat{T}_h - t_n} \leq \frac{\log(1+\Delta\theta)}{\Delta\theta}. \tag{11.57}$$

Proof. We employ the notation and constructions of Theorem 73. For $n \geq \max(N_1, N_2(\epsilon))$ the following estimate pertains by (11.55) and (11.56)

$$\frac{1-\epsilon}{1+\epsilon}\cdot\frac{\Delta X_{n+1}}{|f(X_{n+1})|}\cdot\frac{1}{\Delta}\log\left(\frac{X_n}{X_{n+1}}\right) \leq \int_{X_{n+1}}^{X_n}\frac{1}{|f(u)|}\,du \leq \frac{1+\epsilon}{1-\epsilon}\cdot\frac{\Delta X_n}{|f(X_n)|}\cdot\frac{1}{\Delta}\log\left(\frac{X_n}{X_{n+1}}\right).$$

Hence for $n \geq \max(N_1, N_2(\epsilon))$

$$\frac{1-\epsilon}{1+\epsilon}\sum_{j=n}^{\infty}\frac{\Delta X_{j+1}}{|f(X_{j+1})|}\cdot a_j \leq \sum_{j=n}^{\infty}\int_{X_{j+1}}^{X_j}\frac{1}{|f(u)|}\,du \leq \frac{1+\epsilon}{1-\epsilon}\sum_{j=n}^{\infty}\frac{\Delta X_j}{|f(X_j)|}\cdot a_j,$$

where $a_j := \log\left(X_n/X_{n+1}\right)/\Delta$. Thus as $X_n \to 0$ as $n \to \infty$ and (11.9) holds, we have

$$\frac{1-\epsilon}{1+\epsilon}\sum_{j=n}^{\infty}\frac{\Delta X_{j+1}}{|f(X_{j+1})|}\cdot a_j \leq \bar{F}(X_n) \leq \frac{1+\epsilon}{1-\epsilon}\sum_{j=n}^{\infty}\frac{\Delta X_j}{|f(X_j)|}\cdot a_j. \tag{11.58}$$

Note $a_n \to \log(1+\Delta\theta)/(\Delta\theta)$ as $n \to \infty$. Also, as $n \to \infty$ then

$$\sum_{j=n}^{\infty} \frac{\Delta X_j}{|f(X_j)|} \cdot a_j \sim \sum_{j=n}^{\infty} \frac{\Delta X_j}{|f(X_j)|} \cdot \frac{\log(1+\Delta\theta)}{\Delta\theta} \sim \left(\hat{T}_h - t_n\right) \frac{\log(1+\Delta\theta)}{\Delta\theta}, \tag{11.59}$$

by (11.54) and Toeplitz's Lemma. By (11.58) and (11.59)

$$\limsup_{n\to\infty} \frac{\bar{F}(X_n)}{\hat{T}_h - t_n} \leq \frac{1+\epsilon}{1-\epsilon} \cdot \frac{\log(1+\Delta\theta)}{\Delta\theta}.$$

Letting $\epsilon \to 0^+$ yields

$$\limsup_{n\to\infty} \frac{\bar{F}(X_n)}{\hat{T}_h - t_n} \leq \frac{\log(1+\Delta\theta)}{\Delta\theta}. \tag{11.60}$$

By (11.58) and (11.59) we have

$$\liminf_{n\to\infty} \frac{\bar{F}(X_n)}{\hat{T}_h - t_{n+1}} \geq \frac{1-\epsilon}{1+\epsilon} \cdot \liminf_{n\to\infty} \frac{\sum_{j=n+1}^{\infty} \Delta X_j/|f(X_j)|}{\hat{T}_h - t_{n+1}} = \frac{1-\epsilon}{1+\epsilon} \cdot \frac{\log(1+\Delta\theta)}{\Delta\theta}.$$

Letting $\epsilon \to 0$ gives

$$\liminf_{n\to\infty} \frac{\bar{F}(X_n)}{\hat{T}_h - t_{n+1}} \geq \frac{\log(1+\Delta\theta)}{\Delta\theta}. \tag{11.61}$$

Combining (11.60) and (11.61) gives (11.57) as required. □

As in the deterministic case we refine the result of Theorem 74 to align the time indices in the denominator.

Theorem 75. *Suppose all hypotheses of Theorem 72 hold and in addition f obeys* (1.26), (11.8) *and* (11.9) *while h obeys* (11.39). *Let $\Delta < \Delta_0 = (e^\theta - 1)/\theta$ and $\bar{F}$, $\hat{T}_h$ and (t_n) be defined by* (1.28), (11.53) *and* (11.37). *Then*

$$\frac{\log(1+\Delta\theta)}{\Delta\theta}\left(1 - \frac{\log(1+\Delta\theta)}{\theta}\right) \leq \liminf_{n\to\infty} \frac{\bar{F}(X_n)}{\hat{T}_h - t_n} \leq \limsup_{n\to\infty} \frac{\bar{F}(X_n)}{\hat{T}_h - t_n} \leq \frac{\log(1+\Delta\theta)}{\Delta\theta}. \tag{11.62}$$

Proof. By (11.61)

$$\liminf_{n\to\infty} \frac{\bar{F}(X_{n+1})}{\hat{T}_h - t_{n+1}} = \liminf_{n\to\infty} \left(\frac{\bar{F}(X_{n+1})}{\bar{F}(X_n)} \cdot \frac{\bar{F}(X_n)}{\hat{T}_h - t_{n+1}}\right) \geq \frac{\log(1+\Delta\theta)}{\Delta\theta} \cdot \liminf_{n\to\infty} \frac{\bar{F}(X_{n+1})}{\bar{F}(X_n)}.$$

Next, as $X_{n+1}/X_n \to (1+\Delta\theta)^{-1/\theta} =: \lambda(\Delta)$ as $n \to \infty$. Therefore for $n \geq N_3(\epsilon)$

$$(1-\epsilon)\cdot\lambda(\Delta) < \frac{X_{n+1}}{X_n} < (1+\epsilon)\cdot\lambda(\Delta).$$

Thus $X_{n+1} \geq (1-\epsilon)\cdot\lambda(\Delta)X_n$. Thus for $n \geq N_3(\epsilon)$ $\bar{F}(X_{n+1}) \geq \bar{F}((1-\epsilon)\lambda(\Delta)X_n)$.

Hence

$$\begin{aligned}\liminf_{n\to\infty}\frac{\bar{F}(X_{n+1})}{\hat{T}_h - t_{n+1}} &\geq \frac{\log(1+\Delta\theta)}{\Delta\theta}\cdot\liminf_{n\to\infty}\frac{\bar{F}\left((1-\epsilon)\lambda(\Delta)X_n\right)}{\bar{F}\left(X_n\right)}\\ &\geq \frac{\log(1+\Delta\theta)}{\Delta\theta}\cdot\liminf_{x\to 0^+}\frac{\bar{F}\left((1-\epsilon)\lambda(\Delta)x\right)}{\bar{F}(x)}.\end{aligned}$$

Since f obeys (11.9), we have for $\alpha\in(1/e,1)$ that

$$\liminf_{x\to 0^+}\frac{\bar{F}(\alpha x)}{\bar{F}(x)}\geq 1-\log\left(\frac{1}{\alpha}\right). \tag{11.63}$$

Set $\alpha := (1-\epsilon)\cdot\lambda(\Delta)$. Clearly, if $\lambda(\Delta)\in(1/e,1)$, we can choose $\epsilon\in(0,1)$ so small that $\alpha\in(1/e,1)$. Now $\alpha\in(1/e,1)$ if and only if $1/e<(1+\Delta\theta)^{-1/\theta}<1$. Clearly we have $(1+\Delta\theta)^{-1/\theta}<1$ for any choice of $\Delta>0$. Also, as $\lim_{\Delta\to 0}(1+\Delta\theta)^{-1/\theta}=1$ and $\lim_{\Delta\to\infty}(1+\Delta\theta)^{-1/\theta}=0$ and $\Delta\mapsto(1+\Delta\theta)^{-1/\theta}$ is decreasing in Δ, there exists $\Delta_0=\Delta_0(\theta)$ such that $\Delta<\Delta_0(\theta)$ implies $1/e<(1+\Delta\theta)^{-1/\theta}<1$. Moreover, Δ_0 is determined by $e=(1+\Delta_0\theta)^{1/\theta}$ or $1/e<\lambda(\Delta)<1$. Thus by (11.63), for $\epsilon\in(0,1)$ sufficiently small we have

$$\liminf_{x\to 0^+}\frac{\bar{F}\left((1-\epsilon)x\lambda(\Delta)\right)}{\bar{F}(x)}\geq 1-\log\left(\frac{1}{(1-\epsilon)\lambda(\Delta)}\right).$$

Hence by (11.57)

$$\liminf_{n\to\infty}\frac{\bar{F}\left(X_n\right)}{\hat{T}_h-t_n}\geq\frac{\log(1+\Delta\theta)}{\Delta\theta}\cdot\left(1-\log\left(\frac{1}{(1-\epsilon)\lambda(\Delta)}\right)\right).$$

Letting $\epsilon\to 0^+$ yields

$$\liminf_{n\to\infty}\frac{\bar{F}\left(X_n\right)}{\hat{T}_h-t_n}\geq\frac{\log(1+\Delta\theta)}{\Delta\theta}\cdot\left(1-\log\left(\frac{1}{\lambda(\Delta)}\right)\right).$$

Now $\lambda(\Delta)=(1+\Delta\theta)^{-1/\theta}$, so $1/\lambda(\Delta)=(1+\Delta\theta)^{1/\theta}$. Therefore

$$\log\left(\frac{1}{\lambda(\Delta)}\right)=\log\left((1+\Delta\theta)^{1/\theta}\right)=\frac{\log(1+\Delta\theta)}{\theta}.$$

Hence

$$\liminf_{n\to\infty}\frac{\bar{F}\left(X_n\right)}{\hat{T}_h-t_n}\geq\frac{\log(1+\Delta\theta)}{\Delta\theta}\cdot\left(1-\frac{\log(1+\Delta\theta)}{\theta}\right),$$

which is (11.62). □

We notice in Theorem 75, as in earlier deterministic results, that the upper and lower estimates in (11.62) tend to unity as $\Delta\to 0^+$. In order that the numerical scheme is computationally efficient and preserving asymptotic behaviour it would be reassuring to show that these limits are indeed Δ-dependent and unequal to unity for small Δ. In the next result we show as in the deterministic case when $|f|$ is regularly varying we have a non-unit limit which tends to unity as $\Delta \to 0^+$.

Theorem 76. *Let* $|f| \in RV_0(\beta)$ *where* $\beta \in [0,1]$. *Then, with* f *obeying* (1.26), (11.8), (11.9) *and* h *obeying* (11.39), *we have for any* $\Delta > 0$,

$$\lim_{n\to\infty} \frac{\bar{F}(X_n)}{\hat{T}_h - t_n} = \frac{1}{\Delta}\int_{(1+\Delta\theta)^{-1/\theta}}^{1} v^{-\beta}\,dv,$$

Remark 46. If $\beta = 0$, then (1.26) and (11.8) hold. If $\beta = 1$, then (11.9) holds. If $\beta \in (0,1)$, then f obeys (1.26), (11.8) and (11.9). □

Proof. Define $\lambda_n := X_{n+1}/X_n$. Then $\lambda_n \to (1+\Delta\theta)^{-1/\theta}$ as $n \to \infty$ and

$$\begin{aligned}\int_{X_{n+1}}^{X_n} \frac{1}{|f(u)|}\,du = \int_{\lambda_n X_n}^{X_n} \frac{1}{|f(u)|}\,du &= \frac{1}{|f(X_n)|}\int_{\lambda_n X_n}^{X_n} \frac{|f(X_n)|}{|f(u)|}\,du \\ &= \frac{1}{|f(X_n)|}\int_{\lambda_n}^{1} \frac{|f(X_n)|}{|f(vX_n)|} X_n\,dv.\end{aligned}$$

Hence

$$\int_{X_{n+1}}^{X_n} \frac{1}{|f(u)|}\,du = \frac{X_n}{|f(X_n)|}\int_{\lambda_n}^{1} \frac{\tilde{f}(vX_n)}{\tilde{f}(X_n)}\,dv, \tag{11.64}$$

where $\tilde{f} = 1/|f| \in RV_0(-\beta)$. Thus by (11.64)

$$\frac{|f(X_n)|}{X_n}\int_{X_{n+1}}^{X_n} \frac{1}{|f(u)|}\,du = \int_{\lambda_n}^{1}\left(\frac{\tilde{f}(vX_n)}{\tilde{f}(X_n)} - v^{-\beta}\right)dv + \int_{\lambda_n}^{1} v^{-\beta}\,dv.$$

Since $\lambda_n \to (1+\Delta\theta)^{-1/\theta}$ as $n \to \infty$ by the uniform convergence theorem for regularly varying functions, we get from (11.64) and the fact that $X_n \to 0$ as $n \to \infty$,

$$\lim_{n\to\infty} \frac{|f(X_n)|}{X_n}\int_{X_{n+1}}^{X_n} \frac{1}{|f(u)|}\,du = \int_{(1+\Delta\theta)^{-1/\theta}}^{1} v^{-\beta}\,dv.$$

Hence

$$\lim_{n\to\infty} \frac{\int_{X_{n+1}}^{X_n} 1/|f(u)|\,du}{\Delta X_n/|f(X_n)|} = \frac{1}{\Delta}\int_{(1+\Delta\theta)^{-1/\theta}}^{1} v^{-\beta}\,dv. \tag{11.65}$$

Therefore by Toeplitz's Lemma and the fact that both numerator and denominator in (11.65) are summable, we have

$$\lim_{n\to\infty} \frac{\sum_{j=n}^{\infty}\int_{X_j}^{X_{j+1}} 1/|f(u)|\,du}{\sum_{j=n}^{\infty} \Delta X_j/|f(X_j)|} = \lim_{n\to\infty} \frac{\int_{X_j}^{X_{j+1}} 1/|f(u)|\,du}{\Delta X_j/|f(X_j)|} = \frac{1}{\Delta}\int_{(1+\Delta\theta)^{-1/\theta}}^{1} v^{-\beta}\,dv.$$

Since $\hat{T}_h - t_n \sim \sum_{j=n}^{\infty} \Delta X_j/|f(X_j)|$ as $n \to \infty$, this yields the result. □

We have already demonstrated under weaker conditions on the diffusion term, that the log-transformed scheme preserves exactly the super-exponential asymptotic behaviour of the SDE. We now show the power-transformation preserves the asymptotic behaviour but with a Δ-dependent Liapunov exponent. We give details now.

For completeness, the following theorem summarises all our results in this section.

Theorem 77. *Suppose all hypotheses of Theorem 72 hold and in addition f obeys* (11.8) *while h obeys* (11.39). *Let F, $\bar{F}$, $\hat{T}_h$ and (t_n) be defined by* (1.29), (1.28), (11.53) *and* (11.37).

(i) If f obeys (1.27), *then $X_n \in (0,\infty)$ for all $n \geq 0$ a.s., $X_n \to 0$ as $n \to \infty$ a.s., $t_n \to \infty$ as $n \to \infty$ a.s. and*

$$\lim_{n\to\infty} \frac{F(X_n)}{t_n} = \frac{\log(1+\Delta\theta)}{\Delta\theta} := \lambda(\Delta).$$

(ii) If f obeys (1.26), *then $X_n \in (0,\infty)$ for all $n \geq 0$ a.s., $X_n \to 0$ as $n \to \infty$ a.s., $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ a.s. and*

$$\lambda(\Delta) \leq \liminf_{n\to\infty} \frac{\bar{F}(X_n)}{\hat{T}_h - t_{n+1}}, \quad \limsup_{n\to\infty} \frac{\bar{F}(X_n)}{\hat{T}_h - t_n} \leq \lambda(\Delta).$$

If in addition f obeys (11.9) *and $\Delta < \Delta_0 = (e^\theta - 1)/\theta$, then*

$$\lambda(\Delta)\left(1 - \Delta\lambda(\Delta)\right) \leq \liminf_{n\to\infty} \frac{\bar{F}(X_n)}{\hat{T}_h - t_n} \leq \limsup \frac{\bar{F}(X_n)}{\hat{T}_h - t_n} \leq \lambda(\Delta).$$

(iii) If $|f| \in RV_0(\beta)$ with $\beta \in (0,1)$, then for any $\Delta > 0$ then $X_n \in (0,\infty)$ for all $n \geq 0$ a.s., $X_n \to 0$ as $n \to \infty$ a.s., $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ a.s. and

$$\lim_{n\to\infty} \frac{\bar{F}(X_n)}{\hat{T}_h - t_n} = \frac{1}{\Delta} \int_{(1+\Delta\theta)^{-1/\theta}}^{1} v^{-\beta}\, dv.$$

Proof. By (11.55) and (11.56), for $n \geq N_3(\epsilon) := \max(N_1, N_2(\epsilon))$

$$\frac{1-\epsilon}{1+\epsilon} \cdot \frac{X_{n+1}}{|f(X_{n+1})|} \cdot \log\left(\frac{X_n}{X_{n+1}}\right) \leq \int_{X_{n+1}}^{X_n} \frac{1}{|f(u)|}\, du \leq \frac{1+\epsilon}{1-\epsilon} \cdot \frac{X_n}{|f(X_n)|} \cdot \log\left(\frac{X_n}{X_{n+1}}\right). \tag{11.66}$$

Since $\sum_{j=0}^{n} \int_{X_{j+1}}^{X_j} 1/|f(u)|\, du \to \infty$ as $n \to \infty$, it follows that $(\Delta X_n/|f(X_n)|)$ is divergent because $\log(X_n/X_{n+1}) \to \log(1+\Delta\theta)/\theta$ as $n \to \infty$. Hence $t_n \to \infty$ as $n \to \infty$ if

$\int_{0^+}^{1} 1/|f(u)|\, du = \infty$. In this case, for

$$\begin{aligned} F(X_n) = \int_{X_n}^{1} \frac{1}{|f(u)|}\, du &= \int_{X_n}^{X_{N_3(\epsilon)}} \frac{1}{|f(u)|}\, du + \int_{X_{N_3(\epsilon)}}^{1} \frac{1}{|f(u)|}\, du \\ &= F(X_{N_3(\epsilon)}) + \sum_{j=N_3(\epsilon)}^{n-1} \int_{X_{j+1}}^{X_j} \frac{1}{|f(u)|}\, du. \end{aligned}$$

Since $t_n \to \infty$ as $n \to \infty$ and $h(X_n) \sim \Delta X_n / f(X_n)$ as $n \to \infty$ then

$$\lim_{n\to\infty} \frac{\sum_{j=0}^{n-1} \Delta X_j / f(X_j)}{t_n} = \lim_{n\to\infty} \frac{\sum_{j=0}^{n-1} \Delta X_j / f(X_j)}{\sum_{j=0}^{n-1} h(X_j)} = 1,$$

by Toeplitz's Lemma. Thus, as $t_{n+1}/t_n \to 1$ as $n \to \infty$ using (11.66) and Toeplitz's Lemma again

$$\begin{aligned} \liminf_{n\to\infty} \frac{F(X_n)}{t_n} = \liminf_{n\to\infty} \frac{F(X_n)}{t_{n+1}} &= \liminf_{n\to\infty} \frac{\sum_{j=N_3(\epsilon)}^{n-1} \int_{X_{j+1}}^{X_j} 1/|f(u)|\, du}{\sum_{j=N_3(\epsilon)+1}^{n} \Delta X_j / f(X_j)} \\ &= \liminf_{n\to\infty} \frac{\sum_{j=N_3(\epsilon)}^{n-1} \int_{X_{j+1}}^{X_j} 1/|f(u)|\, du}{\sum_{j=N_3(\epsilon)}^{n-1} \Delta X_{j+1} / f(X_{j+1})} \\ &\geq \frac{1-\epsilon}{1+\epsilon} \cdot \frac{\log(1+\Delta\theta)}{\Delta\theta}. \end{aligned}$$

Similarly, as $\log(X_n/X_{n+1}) \to \log(1+\Delta\theta)/\theta$ as $n \to \infty$, we have

$$\begin{aligned} \limsup_{n\to\infty} \frac{F(X_n)}{t_n} &\leq \limsup_{n\to\infty} \frac{\sum_{j=N_3(\epsilon)}^{n-1} \int_{X_{j+1}}^{X_j} 1/|f(u)|\, du}{\sum_{j=N_3(\epsilon)}^{n-1} \Delta X_j / |f(X_j)|} \\ &\leq \frac{1+\epsilon}{1-\epsilon} \cdot \limsup_{n\to\infty} \frac{\sum_{j=N_3(\epsilon)}^{n-1} X_j/|f(X_j)| \cdot \log(X_j/X_{j+1})}{\sum_{j=N_3(\epsilon)}^{n-1} \Delta X_j / |f(X_j)|} \\ &\leq \frac{1+\epsilon}{1-\epsilon} \cdot \frac{\log(1+\Delta\theta)}{\Delta\theta}. \end{aligned}$$

Letting $\epsilon \to 0^+$ in both these inequalities gives

$$\lim_{n\to\infty} \frac{F(X_n)}{t_n} = \frac{\log(1+\Delta\theta)}{\Delta\theta},$$

as needed in part (i). Parts (ii) and (iii) are covered by Theorems 75 and 76. □

Remark 47. The bound $\lambda(\Delta)$ is hard to improve. If $\beta = 1$, then $x \mapsto x/|f(x)|$ is asymptotically increasing and $\int_{0^+}^{1} 1/|f(u)|\, du < \infty$ then

$$\lim_{n\to\infty} \frac{\bar{F}(X_n)}{\hat{T}_h - t_n} = \lambda(\Delta).$$

□

11.4 Logarithmic Pre-Transformed Scheme

We now show that we can use the logarithmic transformation in the presence of small noise and recover the full strength of the results in the last section where a power transformation was used. This is interesting because in the last section the choice of the power is connected to an assumption on the coefficients f and g. It can be argued that the logarithmic transformation has the advantage of not relying on this additional information on the coefficients.

Define the sequences (Z_n), (X_n) and (t_n), where $Z_0 = -\log\zeta$, $X_0 = \zeta$ and $t_0 = 0$, by

$$Z_{n+1} = Z_n + \tilde{h}(Z_n)\left(\frac{-f(X_n)}{X_n} + \frac{g^2(X_n)}{2X_n^2}\right) + \sqrt{\tilde{h}(Z_n)}\cdot\frac{g(X_n)}{X_n}\cdot\tilde{\xi}_{n+1}, \quad n \geq 0, \tag{11.67}$$

$$X_{n+1} = e^{-Z_{n+1}}, \quad n \geq 0, \tag{11.68}$$

$$t_{n+1} = t_n + \tilde{h}(Z_n), \quad n \geq 0, \tag{11.69}$$

where $\tilde{h}(z) := h(e^{-z})$ and

$$h(x) = \min\left(\Delta, \frac{\Delta x}{|f(x)|}, \frac{\Delta x^2}{g^2(x)}\right), \tag{11.70}$$

This section consists of two parts. The first gives a fundamental convergence result which we use several times in the remaining part of the thesis in different contexts. The second section proves an analogue of Theorem 77 for finite-time stability under the small noise condition.

11.4.1 Fundamental Convergence Theorem

Theorem 78. *Let L be defined by* (1.25)*. Suppose* (9.5) *holds, Z_n is the solution of* (11.67) *and X_n, t_n and h obey* (11.68), (11.69), (11.70)*. Then $X_n \in (0,\infty)\,\forall\, n \geq 0$ a.s., $X_n \to 0$ as $n \to \infty$ a.s. and*

$$\lim_{n\to\infty} \frac{-\log X_n}{\sum_{j=0}^{n-1} h(X_n)\left(g^2(X_n)/2X_n^2 - f(X_n)/X_n\right)} = 1, \quad a.s..$$

Proof. Define

$$\mu_n := h(X_n)\left(\frac{g^2(X_n)}{2X_n^2} - \frac{f(X_n)}{X_n}\right) \quad \text{and} \quad \sigma_n := \sqrt{h(X_n)}\cdot\frac{g(X_n)}{X_n}.$$

Then for $n \geq 1$

$$Z_n = Z_0 + \sum_{j=0}^{n-1} \mu_j + \sum_{j=0}^{n-1} \sigma_j \tilde{\xi}_{j+1},$$

Suppose for $n \geq 1$ $M(n) = \sum_{j=0}^{n-1} \sigma_j \tilde{\xi}_{j+1}$. Then $M(n)$ is a L^2-martingale, because

$$\sigma_n^2 = h(X_n) \cdot \frac{g^2(X_n)}{X_n^2} \leq \Delta, \quad \text{for all } n \geq 0.$$

by the same consdierations as in Lemma 40. We claim there exists $\theta > 0$ such that

$$\frac{g^2(x)}{2x^2} - \frac{f(x)}{x} \geq \frac{\theta g^2(x)}{x^2}, \quad \forall\, x > 0. \tag{11.71}$$

This is equivalent to

$$\left(\frac{1}{2} - \theta\right) \frac{g^2(x)}{x^2} \geq \frac{f(x)}{x},$$

or $xf(x)/g^2(x) < 1/2 - \theta \,\forall\, x > 0$ and this is true because of hypotheses. By (11.71),

$$\mu_n \geq h(X_n) \cdot \frac{\theta g^2(X_n)}{X_n^2} = \theta \sigma_n^2 > 0.$$

Hence $\langle M \rangle (n) := \sum_{j=0}^{n-1} \sigma_j^2 \leq 1/\theta \cdot \sum_{j=0}^{n-1} \mu_j$. If $\langle M \rangle (n)$ tends to a finite limit on A then $M(n)$ tends to a finite limit on A a.s. by the martingale convergence theorem - see Theorem 12.13 in [59]. Since $\mu_n > 0 \,\forall\, n$ then $\lim_{n\to\infty} Z_n =: Z_\infty \in [-\infty, \infty]$ on A. Clearly, $Z_\infty \in (-\infty, \infty]$. Suppose $Z_\infty \in (-\infty, \infty)$, then $X_n \to e^{-Z_\infty} =: X_\infty \in (0, \infty)$ as $n \to \infty$ on A a.s.. This implies that

$$h(X_n) \to \Delta \min\left(1, \frac{X_\infty}{|f(X_\infty)|}, \frac{X_\infty^2}{g^2(X_\infty)}\right) =: \Delta h_\infty > 0, \quad \text{as } n \to \infty.$$

Therefore by (11.71)

$$\lim_{n\to\infty} \mu_n = \Delta h_\infty \left(\frac{g^2(X_\infty)}{2X_\infty^2} - \frac{f(X_\infty)}{X_\infty}\right) \in (0, \infty).$$

This implies $\sum_{j=0}^{n-1} \mu_j \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{Z_n}{\sum_{j=0}^{n-1} \mu_j} = \lim_{n\to\infty} \frac{Z_0}{\sum_{j=0}^{n-1} \mu_j} + 1 + \lim_{n\to\infty} \frac{M(n)}{\sum_{j=0}^{n-1} \mu_j} = 1.$$

Hence $Z_n \to \infty$ as $n \to \infty$, a contradiction. Therefore a.s. on A, $Z_n \to \infty$, $X_n \to 0$ as $n \to \infty$, so $\sum_{j=0}^{n-1} \mu_j = Z_n - Z_0 - M(n) \to \infty$ and therefore

$$\lim_{n\to\infty} \frac{Z_n}{\sum_{j=0}^{n-1} \mu_j} = 1, \quad \text{a.s. on } A.$$

On A', we have $\langle M\rangle(n) \to \infty$ as $n \to \infty$. Therefore

$$\lim_{n\to\infty} \frac{M(n)}{\langle M\rangle(n)} = 0, \quad \text{a.s. on } A'.$$

by the strong law of large numbers for martingales. Then as $\langle M\rangle(n) \le 1/\theta \cdot \sum_{j=0}^{n-1} \mu_j$ and $\sum_{j=0}^{n-1} \mu_j \to \infty$, we must have

$$\lim_{n\to\infty} \frac{Z_n}{\sum_{j=0}^{n-1} \mu_j} = 1 + \lim_{n\to\infty} \frac{M(n)}{\sum_{j=0}^{n-1} \mu_j} = 1, \quad \text{a.s. on } A',$$

because

$$\limsup_{n\to\infty} \frac{|M(n)|}{\sum_{j=0}^{n-1} \mu_j} = \limsup_{n\to\infty} \left(\frac{|M(n)|}{\langle M\rangle(n)} \cdot \frac{\langle M\rangle(n)}{\sum_{j=0}^{n-1} \mu_j} \right) = 0.$$

This implies $Z_n \to \infty$ and $X_n \to 0$ as $n \to \infty$ a.s. on A'. Combining the analysis on A and A' yields

$$\lim_{n\to\infty} \frac{Z_n}{\sum_{j=0}^{n-1} \mu_j} = 1, \quad \text{a.s.},$$

as claimed. □

11.4.2 Asymptotic Behaviour for Small Noise and Finite-Time Stability

We give here for completeness and ease of comparison the analogue of Theorem 77 in the logarithmic case.

Theorem 79. *Suppose all hypotheses of Theorem 72 hold and in addition f obeys* (11.8) *while h obeys* (11.39). *Let F, $\bar{F}$, $\hat{T}_h$ and t_n be defined by* (1.29), (1.28), (10.5) *and* (11.69). *Then*

(i) If f obeys (1.27), *then $X_n \in (0,\infty)$ for all $n \ge 0$ a.s., $X_n \to 0$ as $n \to \infty$ a.s., $t_n \to \infty$ as $n \to \infty$ a.s. and*

$$\lim_{n\to\infty} \frac{F(X_n)}{t_n} = 1.$$

(ii) If f obeys (1.26), *then $X_n \in (0,\infty)$ for all $n \ge 0$ a.s., $X_n \to 0$ as $n \to \infty$ a.s., $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ a.s. and*

$$1 \le \liminf_{n\to\infty} \frac{\bar{F}(X_n)}{\hat{T}_h - t_{n+1}} \quad \textit{and} \quad \limsup_{n\to\infty} \frac{\bar{F}(X_n)}{\hat{T}_h - t_n} \le 1.$$

(iii) If f obeys (1.26) *and* (11.9), *then $X_n \in (0,\infty)$ for all $n \ge 0$ a.s., $X_n \to 0$ as*

$n \to \infty$ a.s., $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ a.s. and

$$\frac{1-e^{-\Delta}}{\Delta} \leq \liminf_{n\to\infty} \frac{\bar{F}(X_n)}{\hat{T}_h - t_n} \leq \limsup_{n\to\infty} \frac{\bar{F}(X_n)}{\hat{T}_h - t_n} \leq 1.$$

(iv) If f obeys (1.26) *and $|f| \in RV_0(\beta)$ with $\beta \in [0,1]$, then $X_n \in (0,\infty)$ for all $n \geq 0$ a.s., $X_n \to 0$ as $n \to \infty$ a.s., $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ a.s. and*

$$\lim_{n\to\infty} \frac{\bar{F}(X_n)}{\hat{T}_h - t_n} = \frac{1}{\Delta} \int_{e^{-\Delta}}^{1} \lambda^{-\beta}\, dv.$$

Proof. Define for $j \geq 0$

$$D_j := h(X_j)\left(\frac{-f(X_j)}{X_j} + \frac{g^2(X_j)}{2X_j^2}\right) \quad \text{and} \quad T_{j+1} := \sqrt{h(X_j)} \cdot \frac{g(X_j)}{X_j} \cdot \xi_{j+1},$$

and $M(n) := \sum_{j=0}^{n-1} T_{j+1}$, for $n \geq 1$. Then M is a martingale, by the same argument as in Lemma 40, with quadratic variation for $n \geq 1$

$$\langle M \rangle (n) = \sum_{j=0}^{n-1} h(X_j) \frac{g^2(X_j)}{X_j^2},$$

and for $n \geq 1$, $Z_n = Z_0 + \sum_{j=0}^{n-1} D_j + M(n)$. Then by Theorem 78 we have that $\sum_{j=0}^{n-1} D_j$ diverges, $Z_n \to \infty$, $X_n \to 0$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{Z_n}{\sum_{j=0}^{n-1} D_j} = 1, \quad \text{a.s..}$$

As usual $g^2(x)/(xf(x)) \to 0$ as $x \to 0^+$. Hence there is $N_1 > 0$ such that for $n \geq N_1$, $h(X_n) = \Delta X_n/|f(X_n)|$. Thus for $n \geq N_1$

$$D_n := \frac{\Delta X_n}{|f(X_n)|}\left(\frac{|f(X_n)|}{X_n} + \frac{g^2(X_n)}{2X_n^2}\right) = \Delta + \frac{\Delta g^2(X_n)}{2X_n|f(X_n)|},$$

$$T_{n+1} := \frac{\sqrt{\Delta} X_n^{1/2}}{|f(X_n)|^{1/2}} \cdot \frac{g(X_n)}{X_n} \cdot \xi_{n+1} = \frac{\sqrt{\Delta} g(X_n)}{|f(X_n)|^{1/2} X_n^{1/2}} \cdot \xi_{n+1}.$$

Define $\mu(x) := g^2(x)/(x|f(x)|)$, $x > 0$. Then for $n \geq N_1$

$$Z_{n+1} = Z_n + \Delta + \tfrac{1}{2}\Delta\mu(X_n) + \sqrt{\Delta}\sqrt{\mu(X_n)}\xi_{n+1}.$$

Then $0 < \mu(x) \to 0$ as $x \to 0^+$, so $\mu(X_n) \to 0$ as $n \to \infty$. Now, as $D_n \to \Delta$ as $n \to \infty$, then

$$\lim_{n\to\infty} \frac{\log X_n}{n\Delta} = \lim_{n\to\infty} \left(\frac{-Z_n}{\sum_{j=0}^{n-1} D_j} \cdot \frac{\sum_{j=0}^{n-1} D_j}{n\Delta}\right) = -1.$$

Next, there is $\theta > 0$ and $\delta(\theta), C(\theta) > 0$ such that for $x < \delta(\theta)$, $g^2(x) \leq C(\theta)x^{1+\theta}|f(x)|$. Since $X_n \to 0$ as $n \to \infty, X_n < \delta(\theta) \,\forall\, n \geq N_2(\epsilon)$. Let $n \geq N_3(\epsilon) := \max(N_1, N_2(\epsilon))$ then $g^2(X_n) \leq C(\theta)X_n^{1+\theta}|f(X_n)|$ so $\mu(X_n) \leq C(\theta)X_n^{\theta}$ for $n \geq N_3(\epsilon)$. Since X_n tends to zero exponentially fast, $\mu(X_n)$ tends to zero exponentially fast and so does $\sqrt{\mu(X_n)}\xi_{n+1}$, because $\xi_{n+1} = O(\sqrt{\log n})$ as $n \to \infty$. Therefore

$$\lim_{n\to\infty} (Z_{n+1} - Z_n - \Delta) = 0. \tag{11.72}$$

Therefore from (11.72) we have that $X_{n+1}/X_n \to e^{-\Delta}$ as $n \to \infty$ and $X_n e^{n\Delta} \to e^{-L^*}$ as $n \to \infty$. Notice that X_n is decreasing for $n \geq N_3(\epsilon)$. Assume $\bar{F}(x) = \int_0^x 1/|f(u)|\,du \to 0$ as $x \to 0^+$ and $x \mapsto |f(x)|/x$ is asymptotically decreasing. Suppose that $x_1(\epsilon)$ is such that for $x < x_1(\epsilon)$

$$\frac{1}{(1+\epsilon)} \cdot \phi(x) < \frac{|f(x)|}{x} < (1+\epsilon)\cdot\phi(x),$$

where ϕ is decreasing. Then there exists $N_4(\epsilon)$ such that $X_n < x_1(\epsilon) \,\forall\, n \geq N_4(\epsilon)$. Take $N_5(\epsilon) := \max(N_4(\epsilon), N_3(\epsilon))$. Then for $n \geq N_5(\epsilon) + 1$

$$t_n = t_{N_5(\epsilon)} + \sum_{j=N_5(\epsilon)}^{n-1} h\left(X_j\right) = t_{N_5(\epsilon)} + \sum_{j=N_5(\epsilon)}^{n-1} \frac{\Delta X_j}{|f\left(X_j\right)|}.$$

Now we have $(X_j)_{j=N_5(\epsilon)}^{\infty}$ is decreasing. Let $j \geq N_5(\epsilon)$, $u \in [X_{j+1}, X_j]$. Then $\phi\left(X_{j+1}\right) > \phi\left(u\right) > \phi\left(X_j\right)$. Hence

$$\frac{1}{1+\epsilon}\cdot\frac{X_{j+1}}{|f\left(X_{j+1}\right)|} < \frac{1}{\phi\left(X_{j+1}\right)} < \frac{1}{\phi\left(u\right)} < \frac{1}{\phi\left(X_j\right)} < (1+\epsilon)\cdot\frac{X_j}{|f\left(X_j\right)|}.$$

Now

$$\frac{X_{j+1}}{|f\left(X_{j+1}\right)|} < (1+\epsilon)\cdot\frac{1}{\phi(u)} < (1+\epsilon)^2\cdot\frac{u}{|f(u)|}.$$

Thus

$$\int_{X_{j+1}}^{X_j} \frac{1}{u}\cdot\frac{X_{j+1}}{|f\left(X_{j+1}\right)|}\,du \leq (1+\epsilon)^2\int_{X_{j+1}}^{X_j}\frac{1}{|f(u)|}\,du,$$

so

$$\log\left(\frac{X_j}{X_{j+1}}\right)\cdot\frac{X_{j+1}}{|f\left(X_{j+1}\right)|} \leq (1+\epsilon)^2\int_{X_{j+1}}^{X_j}\frac{1}{|f(u)|}\,du.$$

Since the sequence on the right-hand side is summable because $\sum_{j=0}^{\infty}\int_{X_{j+1}}^{X_j} 1/|f(u)|\,du = \int_0^{X_0} 1/|f(u)|\,du = \bar{F}\left(X_0\right) < \infty$ then $\sum_{j=0}^{\infty} X_{j+1}/|f\left(X_{j+1}\right)| < \infty$ because $\log\left(X_j/X_{j+1}\right) \to \Delta$ as $j \to \infty$. Therefore (t_n) tends to a finite limit. Write $\hat{T}_h := \sum_{j=0}^{\infty} h\left(X_j\right)$. Then $\hat{T}_h - t_n = \sum_{j=n}^{\infty} h\left(X_j\right) = \sum_{j=n}^{\infty}\Delta X_j/|f\left(X_j\right)|$ for $n \geq N_5(\epsilon)+1$. Let $n \geq N_5(\epsilon)+1$. For $u \in [X_{n+1}, X_n]$ we have

$$\frac{1}{1+\epsilon}\cdot\frac{X_{n+1}}{|f\left(X_{n+1}\right)|} < \frac{1}{\phi(u)} < (1+\epsilon)\cdot\frac{u}{|f(u)|},$$

$$\frac{1}{1+\epsilon} \cdot \frac{u}{|f(u)|} < \frac{1}{\phi(u)} < (1+\epsilon) \cdot \frac{X_n}{|f(X_n)|}.$$

Hence for $n \geq N_5(\epsilon) + 1$, $X_{n+1}/|f(X_{n+1})| < (1+\epsilon)^2 \cdot u/|f(u)|$ and $u/|f(u)| < (1+\epsilon)^2 \cdot X_n/|f(X_n)|$. Thus

$$\frac{\Delta X_{n+1}}{|f(X_{n+1})|} \cdot \frac{1}{\Delta} \int_{X_{n+1}}^{X_n} \frac{1}{u}\, du \leq (1+\epsilon)^2 \int_{X_{n+1}}^{X_n} \frac{1}{|f(u)|}\, du,$$

and

$$\int_{X_{n+1}}^{X_n} \frac{1}{|f(u)|}\, du \leq (1+\epsilon)^2 \cdot \frac{\Delta X_n}{|f(X_n)|} \cdot \frac{1}{\Delta} \int_{X_{n+1}}^{X_n} \frac{1}{u}\, du.$$

Define

$$a_n := \frac{1}{\Delta} \int_{X_{n+1}}^{X_n} \frac{1}{u}\, du = \frac{1}{\Delta} \log\left(\frac{X_n}{X_{n+1}}\right),$$

so $a_n \to 1$ as $n \to \infty$. Hence for $n \geq N_6(\epsilon)$, $1/(1+\epsilon) < a_n < 1+\epsilon$. Let $n \geq N_7(\epsilon)+1 := \max((N_5(\epsilon)+1), (N_6(\epsilon)+1))$. Then

$$\frac{\Delta X_{n+1}}{|f(X_{n+1})|} \leq (1+\epsilon)^3 \int_{X_{n+1}}^{X_n} \frac{1}{|f(u)|}\, du \quad \text{and} \quad \int_{X_{n+1}}^{X_n} \frac{1}{|f(u)|}\, du \leq (1+\epsilon)^3 \cdot \frac{\Delta X_n}{|f(X_n)|}.$$

Thus for $n \geq N_7(\epsilon) + 1$

$$\begin{aligned} \hat{T}_h - t_n = \sum_{j=n}^{\infty} \frac{\Delta X_j}{|f(X_n)|} \geq \frac{1}{(1+\epsilon)^3} \sum_{j=n}^{\infty} \int_{X_{j+1}}^{X_j} \frac{1}{|f(u)|}\, du &= \frac{1}{(1+\epsilon)^3} \int_0^{X_n} \frac{1}{|f(u)|}\, du \\ &= \frac{1}{(1+\epsilon)^3} \cdot \bar{F}(X_n), \end{aligned}$$

and

$$\begin{aligned} \hat{T}_h - t_{n+1} = \sum_{j=n+1}^{\infty} \frac{\Delta X_j}{|f(X_j)|} = \sum_{j=n}^{\infty} \frac{\Delta X_{j+1}}{|f(X_{j+1})|} &\leq (1+\epsilon)^3 \sum_{j=n}^{\infty} \int_{X_{j+1}}^{X_j} \frac{1}{|f(u)|}\, du \\ &= (1+\epsilon)^3 \cdot \bar{F}(X_n), \end{aligned}$$

Therefore

$$1 \leq \liminf_{n\to\infty} \frac{\bar{F}(X_n)}{\hat{T}_h - t_{n+1}} \quad \text{and} \quad \limsup_{n\to\infty} \frac{\bar{F}(X_n)}{\hat{T}_h - t_n} \leq 1.$$

We now prove part (iii). By the monotonicity assumption on $|f|$ we have for every $\epsilon > 0$ that for $x < x_2(\epsilon)$

$$\frac{1}{1+\epsilon} \cdot \psi(x) < |f(x)| < (1+\epsilon) \cdot \psi(x).$$

Let $N_8(\epsilon)$ be so big that $X_n < x_2(\epsilon)\, \forall\, n \geq N_8$. Then for $n \geq N_9(\epsilon) := \max(N_3(\epsilon), N_8(\epsilon))$, and $u \in [X_{n+1}, X_n]$ then $1/(1+\epsilon) \cdot |f(X_n)| < \psi(u) < (1+\epsilon) \cdot |f(u)|$ and $f(u) <$

$(1+\epsilon)\cdot\psi(u) < (1+\epsilon)^2\cdot|f(X_n)|$. Therefore

$$\frac{1}{(1+\epsilon)^2}\cdot|f(X_{n+1})| < |f(u)| < (1+\epsilon)^2\cdot|f(X_n)|,$$

so

$$\frac{1}{(1+\epsilon)^2}\int_{X_{n+1}}^{X_n}\frac{1}{|f\left(X_n\right)|}\,du \le \int_{X_{n+1}}^{X_n}\frac{1}{|f(u)|}\,du \le (1+\epsilon)^2\int_{X_{n+1}}^{X_n}\frac{1}{|f\left(X_{n+1}\right)|}\,du.$$

Hence

$$\frac{1}{(1+\epsilon)^2}\cdot\frac{X_n-X_{n+1}}{|f\left(X_n\right)|} \le \int_{X_{n+1}}^{X_n}\frac{1}{|f(u)|}\,du \le (1+\epsilon)^2\cdot\frac{X_n-X_{n+1}}{|f\left(X_{n+1}\right)|}.$$

Now

$$\int_{X_{n+1}}^{X_n}\frac{1}{|f(u)|}\,du \ge \frac{1}{(1+\epsilon)^2}\cdot\frac{1}{\Delta}\left(1-\frac{X_{n+1}}{X_n}\right)\frac{\Delta X_n}{|f\left(X_n\right)|}.$$

Since

$$b_n := \frac{1}{\Delta}\left(1-\frac{X_{n+1}}{X_n}\right) \to \frac{1}{\Delta}\left(1-e^{-\Delta}\right) \quad \text{as } n\to\infty,$$

for $n \ge N_{10}(\epsilon)$ then $b_n > 1/(1+\epsilon)\cdot\left(1-e^{-\Delta}\right)/\Delta$. Let $n \ge \max(N_9(\epsilon), N_{10}(\epsilon))$. Then

$$\int_{X_{n+1}}^{X_n}\frac{1}{|f(u)|}\,du \ge \frac{1}{(1+\epsilon)^3}\cdot\frac{\left(1-e^{-\Delta}\right)}{\Delta}\cdot\frac{\Delta X_n}{|f\left(X_n\right)|}.$$

Thus for $n \ge \max(N_9(\epsilon), N_{10}(\epsilon)+1)$

$$\begin{aligned}\hat{T}_h - t_n = \sum_{j=n}^{\infty}\frac{\Delta X_j}{|f\left(X_j\right)|} &\le (1+\epsilon)^3\cdot\frac{\Delta}{\left(1-e^{-\Delta}\right)}\sum_{j=n}^{\infty}\int_{X_{j+1}}^{X_j}\frac{1}{|f(u)|}\,du \\ &= (1+\epsilon)^3\cdot\frac{\Delta}{\left(1-e^{-\Delta}\right)}\cdot\bar{F}\left(X_n\right)\end{aligned}$$

Therefore

$$\liminf_{n\to\infty}\frac{\bar{F}\left(X_n\right)}{\hat{T}_h - t_n} \ge \frac{\left(1-e^{-\Delta}\right)}{\Delta}.$$

If we assume $|f| \in RV_0(\beta), \beta\in[0,1]$, we have that $\tilde{f} = 1/|f| \in RV_0(-\beta)$ and

$$\lim_{j\to\infty}\frac{\int_{X_{j+1}}^{X_j} 1|f(u)|\,du}{\Delta X_j/|f(X_j)|} = \frac{1}{\Delta}\int_{X_{j+1}/X_j}^{1}\frac{\tilde{f}(\lambda X_j)}{\tilde{f}(X_j)}\,d\lambda = \frac{1}{\Delta}\int_{e^{-\Delta}}^{1}\lambda^{-\beta}\,d\lambda,$$

by the fact that $X_{j+1}/X_j \to e^{-\Delta}$ as $j\to\infty$ and the uniform convergence theorem for regularly varying functions. Therefore by Toeplitz's Lemma

$$\lim_{n\to\infty}\frac{\bar{F}\left(X_n\right)}{\hat{T}_h - t_n} = \lim_{n\to\infty}\frac{\sum_{j=n}^{\infty}\int_{X_{j+1}}^{X_j} 1/|f(u)|\,du}{\sum_{j=n}^{\infty}\Delta X_j/|f\left(X_j\right)|} = \frac{1}{\Delta}\int_{e^{-\Delta}}^{1}\lambda^{-\beta}\,d\lambda,$$

which completes the proof. □

Chapter 12

Sub-Exponential Stability

12.1 Introduction

In this section of the thesis, we explore whether it is possible to determine precise asymptotic behaviour for solutions of autonomous SDEs with positive solutions and "weakly attracting" equilibria. Furthermore, we determine whether it is possible to recover numerically this asymptotic behaviour. We make precise later what is meant by "weakly attracting" but for now we note that it excludes the type of drift and diffusion studied in the super-exponential and finite-time stability cases in Chapters 9 and 10. Putting aside for now continuous time results, it would appear superficially that the numerical analysis problem is significantly easier and even perhaps already solved within the literature. We wish to indicate briefly now that this superficial appraisal is incomplete.

The first point at issue is whether positivity of simulations is preserved with certainty without recourse to pre-transformation or adaptive time-stepping. Even in the case where the drift and diffusion obey global linear bounds, this cannot be guaranteed. Furthermore, in the weakly attracting case, it can still be the situation that the drift and diffusion violate linear bounds for large x even though they are well-behaved close to the equilibrium at zero. This suggests that we will have problems preserving positivity with constant step-sizes, however small they are taken. It is then tempting to ask whether adaptive time-stepping without a positivity preserving pre-transformation would suffice as it does for ODEs.

A moment's consideration, however, shows that this cannot be successful if we assume that the increments of Brownian Motion in our simulation are replaced by Standard Normal random variables which can be unbounded. Taking a step-size $h(x)$ at state x leads to the direct discretisation of the SDE given by:

$$X_{n+1} = X_n + h(X_n)f(X_n) + \sqrt{h(X_n)}g(X_n)\xi_{n+1},$$

where (ξ_n) is a sequence of independent and identically distributed Standard Normal random variables. In particular, ξ_{n+1} is independent of X_n which is a function of $\xi_1, \ldots, \xi_n$ only. Therefore, no matter how small h is chosen to control the size of the drift and diffusion terms, the probability of the process X_n changing from positive to negative at any time step is positive. This is not acceptable if we wish to ensure that the simulated process remains positive for all time with probability one.

A related problem is that the solution might change sign but nevertheless converge to the equilibrium from the negative side for large n because asymptotically $h(X_n)f(X_n)$ dominates $\sqrt{h(X_n)}g(X_n)\xi_{n+1}$. This is possible in the case where $h(x) = \Delta\,\forall\, x$. A good bound on the pathwise rate of convergence of X_n is known and $g(x)$ is appropriately small in relation to $f(x)$ as $x \to 0^+$. Then because the ξ_n's are Normal and $\xi_n = O(\sqrt{\log n})$ as $n \to \infty$ if $g(X_n)\sqrt{\log n} = o(f(X_n))$ as $n \to \infty$, there will be no change of sign beyond a certain ω-dependent n.

While it is certainly the case that sub-exponential convergence rates can be recovered by constant step-size discretisations of SDEs, these results rely on global linear bounds on the drift and diffusion and require the imposition of symmetry hypotheses on the drift and diffusion to counteract spurious negative solutions. Rather than make these restrictive assumptions or be forced to devise different methods for weakly and strongly attracting equilibria we instead seek to employ a numerical method which will recover the important qualitative and asymptotic information for the largest possible class of problems. We have already seen that the combination of adaptive time-stepping and logarithmic (or power) pre-transformation performs this task very well for SDEs whose solutions can tend to zero exponentially or super-exponentially fast or in finite time, regardless of whether the drift or diffusion was inducing this stability. Therefore, it seems a natural step to ask whether this will also work if we have sub-exponential stability. Furthermore, we would wish that this performance can be achieved at a reasonable computational cost. We should certainly request that our method uses constant step-sizes asymptotically, bearing in mind that conventional constant step-size methods can recover the right asymptotic behaviour with positive probability.

In this chapter, we show that these requirements can be met. In order to know that we do indeed have the appropriate asymptotic behaviour for the numerical methods, we must also establish new continuous-time asymptotic results under irrestrictive conditions on the drift and diffusion, which yield sub-exponential decay.

Although we do not present our results here, the work in this chapter can easily be adapted to deal with the case where solutions of the SDE (both in continuous-time and discretisaton) grow to infinity sub-exponentially. In fact, we can use once again the very same numerical scheme as outlined in this chapter in the sub-exponential decay case, and in the previous chapters in which super-exponential decay or finite-time stability are covered.

12.2 Asymptotic Behaviour for SDE

Consider the SDE (1.17). Define F and G by (1.29) and (1.34). We suppose that f and g obey the hypotheses (1.19), (1.20) and (1.25) namely:

$$, \quad f, g \in C([0,\infty);\mathbb{R}) \text{ with } f(0) = g(0) = 0;$$

$$g^2(x) > 0 \text{ for all } x > 0; \text{ and}$$

$$\lim_{x\to 0^+} \frac{xf(x)}{g^2(x)} =: L.$$

For simplicity we assume that f and g are locally Lipschitz continuous, (1.21), to guarantee the existence of a unique continuous adapted solution to (1.17). As before let p be the scale function, (9.3), of X and recall by Theorem 58 that $p(\infty^-) = \infty$ and $L < 1/2$, (9.5), implies $X(t) \to 0$ as $t \to T^-$ a.s. where T, (1.22), is the first exit time of X from $(0,\infty)$. Since we are interested in sub-exponential convergence, we will make assumptions on f and g at zero which ensure not only that $T = \infty$ a.s. but also that $\lim_{t\to\infty} \log X(t)/t = 0$ a.s.. We now state and prove our main result concerning sub-exponential rates of convergence of $X(t) \to 0$ as $t \to \infty$.

Theorem 80. *Suppose f and g obey* (1.19), (1.20) *and* (1.25)*. Let F, G and L be defined by* (1.29), (1.34) *and* (1.25)*.*

(i) If $L \in (-\infty, 1/2)$ and $x \mapsto g^2(x)/x^2$ is asymptotic to an increasing function, then

$$\lim_{t\to\infty} \frac{-\log X(t)}{(-\log \circ G^{-1})\left(\left(\frac{1}{2} - L\right)t\right)} = 1, \quad a.s..$$

(ii) If $L = -\infty$ and $x \mapsto |f(x)|/x$ is asymptotic to an increasing function, then

$$\lim_{t\to\infty} \frac{-\log X(t)}{(-\log \circ F^{-1})(t)} = 1, \quad a.s..$$

Proof. We now prove part (i). The proof of part (ii) is similar. Define $Z(t) := -\log X(t), t \geq 0$. Then by Itô's Lemma

$$dZ(t) = \left(\frac{-f(X(t))}{X(t)} + \frac{g^2(X(t))}{2X^2(t)}\right) dt + \frac{-g(X(t))}{X(t)} dB(t).$$

Then $\mathbb{P}[A] > 0$ where $A = \{\omega : X(t,\omega) \to 0 \,\text{as}\, t \to \infty\}$ and

$$\lim_{t\to\infty} \frac{-\log X(t)}{\int_0^t -f(X(s))/X(s) + g^2(X(s))/2X^2(s)\, ds} = 1, \quad \text{a.s. on } A.$$

If $L \in (-\infty, 1/2)$ then

$$\lim_{x\to 0^+} \frac{-f(x)/x + g^2(x)/2x^2}{g^2(x)/x^2} = \frac{1}{2} - \lim_{x\to 0} \frac{xf(x)}{g^2(x)} = \frac{1}{2} - L > 0.$$

Hence

$$\lim_{t\to\infty} \frac{-\log X(t)}{\int_0^t g^2(X(s))/X^2(s)\,ds} = \frac{1}{2} - L.$$

Suppose $\bar{\eta}(x) \sim g^2(x)/x^2$ as $x \to 0^+$ where $\bar{\eta}$ is increasing and continuous. Define $I(t) := \int_0^t \bar{\eta}(X(s))\,ds$, $t \geq 0$. Then $I'(t) = \bar{\eta}(X(t))$, and $\bar{\eta}^{-1}(I'(t)) = X(t)$. Thus

$$\lim_{t\to\infty} \frac{\log \bar{\eta}^{-1}(I'(t))}{I(t)} = -\left(\frac{1}{2} - L\right), \quad \text{a.s. on } A.$$

Therefore for every $\epsilon \in (0,1)$ there is $T_1(\epsilon) > 0$ such that for $t > T_1(\epsilon)$

$$-(1+\epsilon)\cdot\left(\tfrac{1}{2} - L\right) < \frac{\log X(t)}{I(t)} < -(1-\epsilon)\cdot\left(\tfrac{1}{2} - L\right),$$

and

$$-(1+\epsilon)\cdot\left(\tfrac{1}{2} - L\right) < \frac{\log \bar{\eta}^{-1}(I'(t))}{I(t)} < -(1-\epsilon)\cdot\left(\tfrac{1}{2} - L\right).$$

Thus for $t > T_1(\epsilon)$, $e^{-(1+\epsilon)\left(\frac{1}{2}-L\right)I(t)} < X(t) < e^{-(1-\epsilon)\left(\frac{1}{2}-L\right)I(t)}$ and since $\bar{\eta}$ is increasing

$$\bar{\eta}\left(e^{-(1+\epsilon)\left(\frac{1}{2}-L\right)I(t)}\right) < I'(t) < \bar{\eta}\left(e^{-(1-\epsilon)\left(\frac{1}{2}-L\right)I(t)}\right).$$

Hence for $t > T_1(\epsilon)$

$$\frac{I'(t)}{\bar{\eta}\left(e^{-(1+\epsilon)\left(\frac{1}{2}-L\right)I(t)}\right)} > 1 \quad \text{and} \quad \frac{I'(t)}{\bar{\eta}\left(e^{-(1-\epsilon)\left(\frac{1}{2}-L\right)I(t)}\right)} < 1.$$

We now seek to integrate across these inequalities. To this end we prepare the following calculation:

$$\begin{aligned}
\int_{T_1(\epsilon)}^t \frac{I'(s)}{\bar{\eta}\left(e^{-aI(s)}\right)}\,ds = \frac{1}{a}\int_{-aI(t)}^{-aI(T_1(\epsilon))} \frac{1}{\bar{\eta}\left(e^{-u}\right)}\,du \quad &= \quad \frac{1}{a}\int_{\exp(-aI(T_1(\epsilon)))}^{\exp(-aI(t))} \frac{1}{v\bar{\eta}\left(v\right)}\,dv \\
&= \quad \frac{1}{a}\left(N\left(e^{-aI(t)}\right) - N\left(e^{-aI(T_1(\epsilon))}\right)\right).
\end{aligned}$$

where $N(x) := \int_x^1 1/(v\bar{\eta}(v))\,dv \sim \int_x^1 u/g^2(u)\,du = G(x)$ as $x \to 0^+$. Thus for $t \geq T_1(\epsilon)$, we have

$$\frac{1}{1-\epsilon}\cdot\frac{N\left(e^{-(1-\epsilon)\left(\frac{1}{2}-L\right)I(t)}\right) - N_-(\epsilon)}{\left(\frac{1}{2} - L\right)} \leq t - T_1(\epsilon) \leq \frac{1}{1+\epsilon}\cdot\frac{N\left(e^{-(1+\epsilon)\left(\frac{1}{2}-L\right)I(t)}\right) - N_+(\epsilon)}{\left(\frac{1}{2} - L\right)},$$

where

$$N_-(\epsilon) := N\left(e^{-(1-\epsilon)\left(\frac{1}{2}-L\right)I(T_1(\epsilon))}\right) \quad \text{and} \quad N_+(\epsilon) := N\left(e^{-(1+\epsilon)\left(\frac{1}{2}-L\right)I(T_1(\epsilon))}\right).$$

Hence for $t \geq T_1(\epsilon)$

$$N\left(e^{-(1+\epsilon)\left(\frac{1}{2}-L\right)I(t)}\right) \geq (1+\epsilon)\cdot\left(\tfrac{1}{2}-L\right)\cdot(t-T_1(\epsilon)) + N_+(\epsilon),$$
$$N\left(e^{-(1-\epsilon)\left(\frac{1}{2}-L\right)I(t)}\right) \leq (1-\epsilon)\cdot\left(\tfrac{1}{2}-L\right)\cdot(t-T_1(\epsilon)) + N_-(\epsilon).$$

Therefore

$$(1+\epsilon)\cdot\left(\tfrac{1}{2}-L\right) \leq \liminf_{t\to\infty} \frac{N\left(e^{-(1+\epsilon)\left(\frac{1}{2}-L\right)I(t)}\right)}{t}, \quad \limsup_{t\to\infty} \frac{N\left(e^{-(1-\epsilon)\left(\frac{1}{2}-L\right)I(t)}\right)}{t} \leq (1-\epsilon)\cdot\left(\tfrac{1}{2}-L\right).$$

Hence as $I(t) \to \infty$ as $t \to \infty$ and $N(x) \sim G(x)$ as $x \to 0^+$ we have

$$(1+\epsilon)\cdot\left(\tfrac{1}{2}-L\right) \leq \liminf_{t\to\infty} \frac{G\left(e^{-(1+\epsilon)\left(\frac{1}{2}-L\right)I(t)}\right)}{t}, \quad \limsup_{t\to\infty} \frac{G\left(e^{-(1-\epsilon)\left(\frac{1}{2}-L\right)I(t)}\right)}{t} \leq (1-\epsilon)\cdot\left(\tfrac{1}{2}-L\right).$$

For $t \geq T_1(\epsilon)$

$$e^{-(1+\epsilon)\left(\frac{1}{2}-L\right)I(t)} = \left(e^{-(1-\epsilon)\left(\frac{1}{2}-L\right)I(t)}\right)^{(1+\epsilon)/(1-\epsilon)} > X(t)^{(1+\epsilon)/(1-\epsilon)}.$$

Hence for $t \geq T_1(\epsilon)$, $G\left(e^{-(1+\epsilon)\left(\frac{1}{2}-L\right)I(t)}\right) < G\left(X(t)^{(1+\epsilon)/(1-\epsilon)}\right)$. Similarly for $t \geq T_1(\epsilon)$, $G\left(e^{-(1-\epsilon)\left(\frac{1}{2}-L\right)I(t)}\right) > G\left(X(t)^{(1-\epsilon)/(1+\epsilon)}\right)$. Therefore

$$(1+\epsilon)\cdot\left(\tfrac{1}{2}-L\right) \leq \liminf_{t\to\infty} \frac{G\left(X(t)^{(1+\epsilon)/(1-\epsilon)}\right)}{t}, \quad \limsup_{t\to\infty} \frac{G\left(X(t)^{(1-\epsilon)/(1+\epsilon)}\right)}{t} \leq (1-\epsilon)\cdot\left(\tfrac{1}{2}-L\right).$$

Considering the liminf for every $\eta \in (0,1)$, there is $\tilde{T}_2(\eta,\epsilon) > 0$ such that for all $t \geq \tilde{T}_2(\eta,\epsilon)$

$$\frac{G\left(X(t)^{(1+\epsilon)/(1-\epsilon)}\right)}{t} > (1-\eta)\cdot(1+\epsilon)\cdot\left(\tfrac{1}{2}-L\right).$$

Pick $\eta = \epsilon/(1+\epsilon)$ and $T_2(\epsilon) := \tilde{T}_2(\epsilon/(1+\epsilon),\epsilon)$. Then $t \geq T_2(\epsilon)$, $G\left(X(t)^{(1+\epsilon)/(1-\epsilon)}\right) > \left(\frac{1}{2}-L\right)t$. Thus for $t \geq T_2(\epsilon)$, $X(t)^{(1+\epsilon)/(1-\epsilon)} < G^{-1}\left(\left(\frac{1}{2}-L\right)t\right)$. Hence

$$\frac{(1+\epsilon)}{(1-\epsilon)}\cdot\log X(t) < (\log\circ G^{-1})\left(\left(\tfrac{1}{2}-L\right)t\right).$$

Therefore

$$-\log X(t) > \frac{(1-\epsilon)}{(1+\epsilon)} \cdot (-\log \circ G^{-1}) \left(\left(\tfrac{1}{2} - L\right) t \right),$$

and so

$$\liminf_{t\to\infty} \frac{-\log X(t)}{(-\log \circ G^{-1}) \left(\left(\frac{1}{2} - L\right) t \right)} \geq \frac{(1-\epsilon)}{(1+\epsilon)}.$$

Letting $\epsilon \to 0^+$ yields

$$\liminf_{t\to\infty} \frac{-\log X(t)}{(-\log \circ G^{-1}) \left(\left(\frac{1}{2} - L\right) t \right)} \geq 1.$$

Similar consideration with the lim sup yields

$$\limsup_{t\to\infty} \frac{-\log X(t)}{(-\log \circ G^{-1}) \left(\left(\frac{1}{2} - L\right) t \right)} \leq 1,$$

as required. □

We have frequently alluded to the fact that solutions of the SDE are sub-exponential under the hypotheses in Theorem 80. By sub-exponential, we mean that $X(t) \to 0$ as $t \to \infty$ at a rate slower than any negative exponential function or equivalently

$$\lim_{t\to\infty} e^{\epsilon t} X(t) = \infty, \quad \text{a.s. } \forall\, \epsilon > 0. \tag{12.1}$$

If $X(t) \to 0$ as $t \to \infty$ and

$$\lim_{t\to\infty} \frac{\log X(t)}{t} = 0, \quad \text{a.s.}, \tag{12.2}$$

then (12.1) holds. This is easily seen. From (12.2) we have for every $\epsilon > 0$ that there is $T(\epsilon) > 0$ such that for $t \geq T(\epsilon)$

$$\frac{-\epsilon}{2} < \frac{\log X(t)}{t} < \frac{\epsilon}{2},$$

and so $X(t)\exp(\epsilon/2 \cdot t) > 1$ for all $t \geq T(\epsilon)$. Hence $e^{\epsilon t} X(t) \to \infty$ as $t \to \infty$, for each $\epsilon > 0$ as claimed.

Proposition 17. *Under the conditions of Theorem 80, the solution $X(t)$ of* (1.17) *obeys*

$$\lim_{t\to\infty} \frac{\log X(t)}{t} = 0, \quad \textit{a.s.}. \tag{12.3}$$

Proof. We consider only the case where $L = -\infty$ and $|f(x)|/x \to 0$ as $x \to 0^+$. Then

$$\lim_{t\to\infty} \frac{-\log X(t)}{(-\log \circ F^{-1})(t)} = 1, \quad \text{a.s.}. \tag{12.4}$$

Since $F(x) \to \infty$ as $x \to 0^+$ and $\log x \to -\infty$ as $x \to 0^+$, we may use L'Hôpital's Rule to determine the following limit in indeterminate form:

$$\lim_{x\to 0^+} \frac{F(x)}{\log x} = \lim_{x\to 0^+} \frac{\int_x^1 1/|f(u)|\,du}{\log x} = \lim_{x\to 0^+} \frac{-1/|f(x)|}{1/x} = -\infty.$$

Therefore

$$\lim_{t\to\infty} \frac{(-\log\circ F^{-1})(t)}{t} = \lim_{x\to 0^+} \frac{-\log x}{F(x)} = 0.$$

Combining (12.4) and (12.2) gives (12.3), as claimed. □

12.3 Asymptotic Behaviour for Logarithmically Pre-Transformed Scheme

We now show that the asymptotic behaviour of the SDE under sub-exponential hypotheses can be recovered. We make the same assumptions on f and g as the previous section.Our first result proves, when the drift is dominant and sub-linear, that the asymptotic rate of decay of the solution of the SDE is recovered by the difference scheme.

Theorem 81. *Let F and L be defined by* (1.29) *and* (1.25)*. Suppose f and g obey* (1.19), (1.20) *and* (1.25)*. If $L = -\infty$ and $x \mapsto |f(x)|/x$ is asymptotic to an increasing C^1 function at 0, then $X_n \in (0,\infty)$ for all $n \geq 0$ a.s., $X_n \to 0$ as $n \to \infty$ a.s., $t_n \to \infty$ as $n \to \infty$ a.s. and*

$$\lim_{n\to\infty} \frac{-\log X_n}{(-\log\circ F^{-1})(t_n)} = 1, \quad a.s..$$

Proof. We have from the proof of Theorem 78 that $X_n \to 0$ as $n \to \infty$, $Z_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{Z_n}{\sum_{j=0}^{n-1} \tilde{h}(Z_j)\left(-f(X_j)/X_j + g^2(X_j)/2X_j^2\right)} = 1, \quad \text{a.s..}$$

Thus as $L = -\infty$ then

$$\lim_{n\to\infty} \frac{-\log X_n}{\sum_{j=0}^{n-1} h(X_j)\cdot -f(X_j)/X_j} = 1, \quad \text{a.s..}$$

Since $X_n \to 0$ as $n \to \infty$, $f(X_n)/X_n \to 0$ as $n \to \infty$. Because $g^2(X_n)/(X_n f(X_n)) \to 0$ as $n \to \infty$, then $h(X_n) = \Delta$ for all $n \geq N$. Thus for $n \geq N$

$$t_n = \sum_{j=0}^{n-1} h(X_j) = \sum_{j=0}^{N-1} h(X_j) + \sum_{j=N}^{n-1} h(X_j) = t_N + (n-N)\Delta.$$

So $t_n/(n\Delta) \to 1$ as $n \to \infty$ and indeed

$$\lim_{n\to\infty} \frac{-\log X_n}{\sum_{j=0}^{n-1} f(X_j)/X_j} = -\Delta.$$

Define $\eta(x) := \bar{\eta}(e^{-x})$ where $\bar{\eta}$ is the increasing function asymptotic to $x \mapsto |f(x)|/x$. We get

$$\lim_{n\to\infty} \frac{\log X_n}{\sum_{j=0}^{n-1} \bar{\eta}(X_j)} = -\Delta^* := -\Delta. \tag{12.5}$$

Define $S_n := \sum_{j=0}^{n} \bar{\eta}(X_j)$. Then $S_n - S_{n-1} = \bar{\eta}\,(X_n)$ so $X_n = \bar{\eta}^{-1}\,(S_n - S_{n-1})$. Hence

$$\lim_{n\to\infty} \frac{\log \bar{\eta}^{-1}\,(S_n - S_{n-1})}{S_{n-1}} = -\Delta^*.$$

Thus there is $N_1(\epsilon) \in \mathbb{N}$ such that for all $n \geq N_1(\epsilon)$

$$-\Delta^* - \Delta^*\epsilon < \frac{\log X_{n+1}}{S_n} < -\Delta^* + \Delta^*\epsilon, \tag{12.6}$$

or $e^{(-\Delta^*(1+\epsilon)S_n)} < X_{n+1} < e^{(-\Delta^*(1-\epsilon)S_n)}$ and

$$-\Delta^* - \Delta^*\epsilon < \frac{\log \bar{\eta}^{-1}\,(S_{n+1} - S_n)}{S_n} < -\Delta^* + \Delta^*\epsilon,$$

or $e^{(-\Delta^*(1+\epsilon)S_n)} < \bar{\eta}^{-1}\,(S_{n+1} - S_n) < e^{(-\Delta^*(1-\epsilon)S_n)}$ or for $n \geq N_1(\epsilon)$

$$S_n + \bar{\eta}\left(e^{-\Delta^*(1+\epsilon)S_n}\right) < S_{n+1} < S_n + \bar{\eta}\left(e^{-\Delta^*(1-\epsilon)S_n}\right).$$

Define $\Phi_a(x) := M\,(e^{-ax})$ where $M(x) := \int_x^1 1/(v\bar{\eta}(v))\,dv$. Then

$$\Phi_a'(x) = M'\left(e^{-ax}\right) \cdot -ae^{-ax} = \frac{ae^{-ax}}{e^{-ax}\bar{\eta}(e^{-ax})} = \frac{a}{\bar{\eta}\,(e^{-ax})} > 0.$$

Thus by the Mean Value Theorem there is $\theta_n \in (0,1)$ such that

$$\begin{aligned}
\Phi_{\Delta^*(1+\epsilon)}\,(S_{n+1}) &> \Phi_{\Delta^*(1+\epsilon)}\left(S_n + \bar{\eta}\left(e^{(-\Delta^*(1+\epsilon)S_n)}\right)\right) \\
&= \Phi_{\Delta^*(1+\epsilon)}\,(S_n) + \Phi'_{\Delta^*(1+\epsilon)}\left(S_n + \theta_n\bar{\eta}\left(e^{-\Delta^*(1+\epsilon)S_n}\right)\right) \cdot \bar{\eta}\left(e^{-\Delta^*(1+\epsilon)S_n}\right).
\end{aligned}$$

Hence for $n \geq N_1(\epsilon)$, with $y_n := \exp\left(-\Delta^*(1+\epsilon)S_n\right)$, then

$$\begin{aligned}
\Phi_{\Delta^*(1+\epsilon)}\,(S_{n+1}) - \Phi_{\Delta^*(1+\epsilon)}\,(S_n) &> \Delta^*(1+\epsilon) \cdot \frac{\bar{\eta}\left(e^{-\Delta^*(1+\epsilon)S_n}\right)}{\bar{\eta}\left(e^{-\Delta^*(1+\epsilon)\left(S_n+\theta_n\bar{\eta}\left(e^{-\Delta^*(1+\epsilon)S_n}\right)\right)}\right)} \\
&= \Delta^*(1+\epsilon) \cdot \frac{\bar{\eta}(y_n)}{\bar{\eta}(y_n e^{-\Delta^*(1+\epsilon)\theta_n\bar{\eta}(y_n)})} > \Delta^*(1+\epsilon).
\end{aligned}$$

Now $\theta_n \in (0,1)$, so $1 \geq e^{-\Delta^*(1+\epsilon)\theta_n\bar{\eta}(y_n)} \geq e^{-\Delta^*(1+\epsilon)\bar{\eta}(y_n)}$. Hence $y_n \geq y_n e^{-\Delta^*(1+\epsilon)\theta_n\bar{\eta}(y_n)} \geq y_n e^{-\Delta^*(1+\epsilon)\bar{\eta}(y_n)}$ and $\bar{\eta}(y_n) \geq \bar{\eta}\left(y_n e^{-\theta_n\Delta^*(1+\epsilon)\bar{\eta}(y_n)}\right) \geq \bar{\eta}\left(y_n e^{-\Delta^*(1+\epsilon)\bar{\eta}(y_n)}\right)$. Hence

$$\frac{1}{\bar{\eta}(y_n)} \leq \frac{1}{\bar{\eta}\left(y_n e^{-\theta_n\Delta^*(1+\epsilon)\bar{\eta}(y_n)}\right)} \leq \frac{1}{\bar{\eta}\left(y_n e^{-\Delta^*(1+\epsilon)\bar{\eta}(y_n)}\right)}.$$

Thus for $n \geq N_1(\epsilon)$, $\Phi_{\Delta^*(1+\epsilon)}(S_{n+1}) - \Phi_{\Delta^*(1+\epsilon)}(S_n) > \Delta^*(1+\epsilon)$. Therefore

$$\liminf_{n\to\infty} \frac{1}{n}\int_{e^{-\Delta^*(1+\epsilon)S_n}}^{1} \frac{1}{v\bar{\eta}(v)}\,dv = \liminf_{n\to\infty} \frac{\Phi_{\Delta^*(1+\epsilon)}(S_n)}{n} \geq \Delta^*(1+\epsilon). \tag{12.7}$$

Similarly there is $\theta_n \in (0,1)$ such that

$$\begin{aligned}
\Phi_{\Delta^*(1-\epsilon)}(S_{n+1}) &< \Phi_{\Delta^*(1-\epsilon)}\left(S_n + \bar{\eta}\left(e^{-\Delta^*(1-\epsilon)S_n}\right)\right) \\
&= \Phi_{\Delta^*(1-\epsilon)}(S_n) + \Phi'_{\Delta^*(1-\epsilon)}\left(S_n + \theta_n\bar{\eta}\left(e^{-\Delta^*(1-\epsilon)S_n}\right)\right)\cdot\bar{\eta}\left(e^{-\Delta^*(1-\epsilon)S_n}\right).
\end{aligned}$$

Thus

$$\Phi_{\Delta^*(1-\epsilon)}(S_{n+1}) - \Phi_{\Delta^*(1-\epsilon)}(S_n) < \Delta^*(1-\epsilon)\cdot\frac{\bar{\eta}\left(e^{-\Delta^*(1-\epsilon)S_n}\right)}{\bar{\eta}\left(e^{-\Delta^*(1-\epsilon)\left(S_n+\theta_n\bar{\eta}\left(e^{-\Delta^*(1-\epsilon)S_n}\right)\right)}\right)}.$$

Define $y_n := \exp(-\Delta^*(1-\epsilon)S_n)$. Then

$$\Phi_{\Delta^*(1-\epsilon)}(S_{n+1}) - \Phi_{\Delta^*(1-\epsilon)}(S_n) < \Delta^*(1-\epsilon)\cdot\frac{\bar{\eta}(y_n)}{\bar{\eta}\left(y_n e^{-\Delta^*(1-\epsilon)\theta_n\bar{\eta}(y_n)}\right)}.$$

Since $\theta_n \in (0,1)$,

$$\frac{1}{\bar{\eta}\left(y_n e^{-\theta_n\Delta^*(1-\epsilon)\bar{\eta}(y_n)}\right)} \leq \frac{1}{\bar{\eta}\left(y_n e^{-\Delta^*(1-\epsilon)\bar{\eta}(y_n)}\right)}.$$

Hence for $n \geq N_1(\epsilon)$

$$\Phi_{\Delta^*(1-\epsilon)}(S_{n+1}) - \Phi_{\Delta^*(1-\epsilon)}(S_n) < \Delta^*(1-\epsilon)\cdot\frac{\bar{\eta}(y_n)}{\bar{\eta}\left(y_n e^{-\Delta^*(1-\epsilon)\bar{\eta}(y_n)}\right)}. \tag{12.8}$$

Next for $x > 0$, as $\bar{\eta} \in C^1$,

$$0 < \bar{\eta}(x) - \bar{\eta}\left(xe^{-\Delta^*(1-\epsilon)\bar{\eta}(x)}\right) = \bar{\eta}'(x\theta(x))\,x\left(1 - e^{-\Delta^*(1-\epsilon)\bar{\eta}(x)}\right),$$

where $\theta(x) \in (\exp(-\Delta^*(1-\epsilon)\bar{\eta}(x)), 1)$. Thus

$$0 < 1 - \frac{\bar{\eta}(xe^{-\Delta^*(1-\epsilon)\bar{\eta}(x)})}{\bar{\eta}(x)} = \bar{\eta}'(x\theta(x))\cdot x\theta(x)\cdot\frac{1 - e^{-\Delta^*(1-\epsilon)\bar{\eta}(x)}}{\bar{\eta}(x)}\cdot\frac{1}{\theta(x)}.$$

Since $\bar{\eta}(x) \to 0$ as $x \to 0^+$ by L'Hôpital's Rule

$$\lim_{x \to 0^+} \frac{1 - e^{-\Delta^*(1-\epsilon)\bar{\eta}(x)}}{\bar{\eta}(x)} = \lim_{y \to 0^+} \frac{1 - e^{-\Delta^*(1-\epsilon)y}}{y} = \Delta^*(1-\epsilon).$$

Note that $\theta(x) \to 1$ as $x \to 0^+$ by The Squeeze Theorem because $\bar{\eta}(x) \to 0$ as $x \to 0^+$. We have that $\phi(x) \sim f(x)$ as $x \to 0^+$ and $\phi \in C^1$, so $f(0) = 0$ implies $\phi(0) = 0, \bar{\eta}(x) = \phi(x)/x$, so $\bar{\eta}'(x) = (x\phi'(x) - \phi(x))/x^2$ so $x\bar{\eta}'(x) = \phi'(x) - \phi(x)/x = \phi'(x) - \bar{\eta}(x)$. Now

$$\lim_{x \to 0^+} \phi'(x) = \phi'(0^+) = \lim_{x \to 0^+} \frac{\phi(x) - \phi(0)}{x - 0} = \lim_{x \to 0^+} \bar{\eta}(x) = 0.$$

Hence $x\bar{\eta}'(x) \to 0$ as $x \to 0^+$. Therefore

$$\lim_{x \to 0^+} \left(1 - \frac{\bar{\eta}\left(xe^{-\Delta^*(1-\epsilon)\bar{\eta}(x)}\right)}{\bar{\eta}(x)} \right) = 0,$$

or $\lim_{x \to 0^+} \bar{\eta}\left(xe^{-\Delta^*(1-\epsilon)\bar{\eta}(x)}\right)/\bar{\eta}(x) = 1$. Therefore

$$\lim_{n \to \infty} \frac{\bar{\eta}(y_n)}{\bar{\eta}\left(y_n e^{-\Delta^*(1-\epsilon)\bar{\eta}(y_n)}\right)} = 1.$$

Thus from (12.8)

$$\limsup_{n \to \infty} \frac{1}{n} \int_{e^{-\Delta^*(1-\epsilon)S_n}}^{1} \frac{1}{v\bar{\eta}(v)}\, dv = \limsup_{n \to \infty} \frac{\Phi_{\Delta^*(1-\epsilon)}(S_n)}{n} \leq \Delta^*(1-\epsilon). \tag{12.9}$$

Now $\int_x^1 1/(v\bar{\eta}(v))\, dv \sim F(x)$ as $x \to 0^+$. So by (12.7) and (12.9) then

$$\Delta^*(1+\epsilon) \leq \liminf_{n \to \infty} \frac{F\left(e^{-\Delta^*(1+\epsilon)S_n}\right)}{n}, \quad \limsup_{n \to \infty} \frac{F\left(e^{-\Delta^*(1-\epsilon)S_n}\right)}{n} \leq \Delta^*(1-\epsilon)$$

As $t_n \sim n\Delta$ as $n \to \infty$ then

$$\limsup_{n \to \infty} \frac{F\left(e^{-\Delta^*(1-\epsilon)S_n}\right)}{t_n} = \limsup_{n \to \infty} \left(\frac{F\left(e^{-\Delta^*(1-\epsilon)S_n}\right)}{n\Delta} \cdot \frac{n\Delta}{t_n} \right) \leq \frac{\Delta^*}{\Delta} \cdot (1-\epsilon) = (1-\epsilon).$$

Similarly

$$\liminf_{n \to \infty} \frac{F\left(e^{-\Delta^*(1+\epsilon)S_n}\right)}{t_n} \geq \frac{\Delta^*}{\Delta} \cdot (1+\epsilon) = (1+\epsilon).$$

Define $l_0 := 1$. Then for every $\epsilon \in (0, 1)$

$$(1+\epsilon) \cdot l_0 \leq \liminf_{n \to \infty} \frac{F\left(e^{-\Delta^*(1+\epsilon)S_n}\right)}{t_n}, \quad \limsup_{n \to \infty} \frac{F\left(e^{-\Delta^*(1-\epsilon)S_n}\right)}{t_n} \leq (1-\epsilon) \cdot l_0. \tag{12.10}$$

By (12.6), for every $\epsilon \in (0,1)$ there is $N_1(\epsilon) \in \mathbb{N}$ such that for $n \geq N_1(\epsilon)$, $e^{-\Delta^*(1+\epsilon)S_n} < X_{n+1} < e^{-\Delta^*(1-\epsilon)S_n}$. Thus $e^{-\Delta^*(1-\epsilon)S_n} = \left(e^{-\Delta^*(1+\epsilon)S_n}\right)^{(1-\epsilon)/(1+\epsilon)} < X_{n+1}^{(1-\epsilon)/(1+\epsilon)}$ so as F is decreasing then for $n \geq N_1(\epsilon)$

$$F\left(e^{-\Delta^*(1-\epsilon)S_n}\right) > F\left(X_{n+1}^{(1-\epsilon)/(1+\epsilon)}\right). \tag{12.11}$$

Similarly $e^{-\Delta^*(1+\epsilon)S_n} = \left(e^{-\Delta^*(1-\epsilon)S_n}\right)^{(1+\epsilon)/(1-\epsilon)} > X_{n+1}^{(1+\epsilon)/(1-\epsilon)}$. Thus for $n \geq N_1(\epsilon)$

$$F\left(e^{-\Delta^*(1+\epsilon)S_n}\right) < F\left(X_{n+1}^{(1+\epsilon)/(1-\epsilon)}\right). \tag{12.12}$$

By (12.10), (12.11), (12.12)

$$(1+\epsilon)\cdot l_0 \leq \liminf_{n\to\infty} \frac{F\left(X_{n+1}^{(1+\epsilon)/(1-\epsilon)}\right)}{t_n}, \quad \limsup_{n\to\infty} \frac{F\left(X_{n+1}^{(1-\epsilon)/(1+\epsilon)}\right)}{t_n} \leq (1-\epsilon)\cdot l_0.$$

Since $t_n \sim n\Delta$ as $n \to \infty$, $t_{n+1}/t_n \to 1$ as $n \to \infty$. Hence

$$(1+\epsilon)\cdot l_0 \leq \liminf_{n\to\infty} \frac{F\left(X_{n+1}^{(1+\epsilon)/(1-\epsilon)}\right)}{t_{n+1}}, \quad \limsup_{n\to\infty} \frac{F\left(X_{n+1}^{(1-\epsilon)/(1+\epsilon)}\right)}{t_{n+1}} \leq (1-\epsilon)\cdot l_0. \tag{12.13}$$

Hence for every $\eta \in (0,1)$, there is $\tilde{N}_2(\eta,\epsilon) \in \mathbb{N}$ such that for $n \geq \tilde{N}_2(\eta,\epsilon)$

$$\frac{F\left(X_n^{(1-\epsilon)/(1+\epsilon)}\right)}{t_n} \leq (1-\epsilon)\cdot(1+\eta)\cdot l_0.$$

Now fix $\eta = \epsilon/(1-\epsilon) < 1$ if $\eta < 1, \epsilon \leq 1/2$. Then, with $N_2(\epsilon) := \tilde{N}_2(\epsilon/(1-\epsilon),\epsilon)$ we have for $n \geq N_2(\epsilon)$

$$\frac{F\left(X_n^{(1-\epsilon)/(1+\epsilon)}\right)}{t_n} \leq (1-\epsilon)\cdot\left(1+\tfrac{\epsilon}{1-\epsilon}\right)\cdot l_0 = l_0\cdot(1-\epsilon+\epsilon) = l_0.$$

Hence $F\left(X_n^{(1-\epsilon)/(1+\epsilon)}\right) \leq l_0 t_n$ or $X_n^{(1-\epsilon)/(1+\epsilon)} > F^{-1}(l_0 t_n)$ for $n \geq N_2(\epsilon)$. Thus for $n \geq N_2(\epsilon)$

$$\frac{1-\epsilon}{1+\epsilon}\cdot \log X_n > \left(\log\circ F^{-1}\right)(l_0 t_n).$$

Thus for $n \geq N_2(\epsilon)$

$$-\log X_n < \frac{1+\epsilon}{1-\epsilon}\cdot\left(-\log\circ F^{-1}\right)(l_0 t_n),$$

so because $\left(-\log\circ F^{-1}\right)(l_0 t_n) \to \infty$ as $n \to \infty$, then for $n \geq N_2(\epsilon)$

$$\frac{-\log X_n}{\left(-\log\circ F^{-1}\right)(l_0 t_n)} \leq \frac{1+\epsilon}{1-\epsilon}.$$

Therefore

$$\limsup_{n\to\infty} \frac{-\log X_n}{(-\log\circ F^{-1})(l_0 t_n)} \le \frac{1+\epsilon}{1-\epsilon}.$$

Letting $\epsilon \to 0^+$ yields

$$\limsup_{n\to\infty} \frac{-\log X_n}{(-\log\circ F^{-1})(l_0 t_n)} \le 1. \tag{12.14}$$

By (12.13) for every $\eta \in (0,1)$, there is $\tilde{N}_3(\eta,\epsilon) \in \mathbb{N}$ such that for $n \ge \tilde{N}_3(\eta,\epsilon)$,

$$\frac{F\left(X_n^{(1+\epsilon)/(1-\epsilon)}\right)}{t_n} \ge (1+\epsilon)\cdot(1-\eta)\cdot l_0.$$

Now, fix $\eta = \epsilon/(1+\epsilon)$. Then with $N_3(\epsilon) := \tilde{N}_3(\epsilon/(1+\epsilon),\epsilon)$, for $n \ge N_3(\epsilon)$

$$\frac{F\left(X_n^{(1+\epsilon)/(1-\epsilon)}\right)}{t_n} \ge (1+\epsilon)\cdot\left(1-\tfrac{\epsilon}{1+\epsilon}\right)\cdot l_0 = l_0\cdot(1+\epsilon-\epsilon) = l_0.$$

Hence for $n \ge N_3(\epsilon)$ then $F\left(X_n^{(1+\epsilon)/(1-\epsilon)}\right) > l_0 t_n$ or $X_n^{(1+\epsilon)/(1-\epsilon)} < F^{-1}(l_0 t_n)$. Thus for $n \ge N_3(\epsilon)$

$$\frac{1+\epsilon}{1-\epsilon}\cdot\log X_n < (\log\circ F^{-1})(l_0 t_n).$$

So for $n \ge N_3(\epsilon)$

$$-\log X_n > \frac{1-\epsilon}{1+\epsilon}\cdot(-\log\circ F^{-1})(l_0 t_n).$$

Therefore for $n \ge N_3(\epsilon)$

$$\frac{-\log X_n}{(-\log\circ F^{-1})(l_0 t_n)} > \frac{1-\epsilon}{1+\epsilon},$$

so

$$\liminf_{n\to\infty} \frac{-\log X_n}{(-\log\circ F^{-1})(l_0 t_n)} \ge \frac{1-\epsilon}{1+\epsilon}.$$

Letting $\epsilon \to 0^+$ yields

$$\liminf_{n\to\infty} \frac{-\log X_n}{(-\log\circ F^{-1})(l_0 t_n)} \ge 1. \tag{12.15}$$

Combining this with (12.14) yields and noting $l_0 = 1$ yields

$$\lim_{n\to\infty} \frac{-\log X_n}{(-\log\circ F^{-1})(t_n)} = 1, \text{a.s.}, \tag{12.16}$$

as claimed. □

We now deal with the case where the diffusion is of comparable order to the drift or dominates it in the sense that $L \in (-\infty, 1/2)$. We note for $L \in (0, 1/2)$ that the underlying ODE is unstable so we are particularly interested in this case to recover the stabilisation by the noise term. In our main result, Theorem 82 below, we show that the rate of convergence of the SDE recorded in Theorem 80 is preserved by the numerical scheme.

Let Z_n, X_n and t_n be given by (11.67), (11.68) and (11.69) where $\tilde{h}(z) = h(e^{-z})$ given by (10.4) viz.,

$$h(x) = \min\left(\Delta, \frac{\Delta x}{|f(x)|}, \frac{\Delta x^2}{g^2(x)}\right).$$

Theorem 82. *Let G and L be defined by* (1.34) *and* (1.25)*. Suppose f and g obey* (1.19)*,* (1.20) *and* (1.25)*. If $L = (-\infty, 1/2)$ and $x \mapsto g^2(x)/x^2$ is asymptotic to an increasing C^1 function at 0, then $X_n \in (0,\infty)$ for all $n \geq 0$ a.s., $X_n \to 0$ as $n \to \infty$ a.s., $t_n \to \infty$ as $n \to \infty$ a.s. and*

$$\lim_{n\to\infty} \frac{-\log X_n}{(-\log \circ G^{-1})\left(\left(\frac{1}{2} - L\right) t_n\right)} = 1, \quad a.s..$$

Proof. By Theorem 78 we have that $\lim_{n\to\infty} Z_n / \sum_{j=0}^{n-1} \mu_j = 1$. Now as $x \to 0^+$, even when $L = 0$, we have

$$\frac{x}{|f(x)|} = \frac{x^2}{g^2(x)} \cdot \frac{g^2(x)}{x^2} \cdot \frac{x}{|f(x)|} = \frac{x^2}{g^2(x)} \cdot \frac{g^2(x)}{x|f(x)|} \to \infty, \quad \text{as } x \to 0^+,$$

since $x^2/g^2(x) \to \infty$ and $g^2(x)/x|f(x)| \to 1/|L|$ as $x \to 0^+$. Therefore, as $X_n \to 0$ as $n \to \infty$, a.s. then $h(X_n) = \Delta$ for all n sufficiently large because $\Delta X_n/|f(X_n)| \to \infty$ and $\Delta X_n^2/g^2(X_n) \to \infty$ as $n \to \infty$. Therefore $t_n \sim n\Delta$ as $n \to \infty$, and

$$\lim_{n\to\infty} \frac{-\log X_n}{\sum_{j=0}^{n-1} h(X_j)\left(g^2(X_j)/2X_j^2 - f(X_j)/X_j\right)} = 1.$$

Since $h(X_n) = \Delta$ for all n sufficiently large, $X_n \to 0$ as $n \to \infty$ and $xf(x)/g^2(x) \to L \in (-\infty, 1/2)$ we get

$$\lim_{n\to\infty} \frac{\log X_n}{\sum_{j=0}^{n-1} g^2(X_j)/X_j^2} = -\Delta \cdot \left(\frac{1}{2} - L\right) =: -\Delta^*.$$

Let $\bar{\eta}$ be the continuous monotone function such that $\bar{\eta}(x) \sim g^2(x)/x^2$ as $x \to 0^+$. Then

$$\lim_{n\to\infty} \frac{\log X_n}{\sum_{j=0}^{n-1} \bar{\eta}(X_j)} = -\Delta^*,$$

where $\Delta^* = (1/2 - L)\Delta$. Then by following the proof of Theorem 81 from (12.5) and letting $S_n := \sum_{j=0}^{n-1} \bar{\eta}(X_j) \to \infty$ then we obtain the limits (12.7) and (12.9) viz.,

$$\Delta^* \cdot (1+\epsilon) \leq \liminf_{n\to\infty} \frac{1}{n} \int_{e^{-\Delta^*(1+\epsilon)S_n}}^{1} \frac{1}{v\bar{\eta}(v)}\,dv, \quad \limsup_{n\to\infty} \frac{1}{n} \int_{e^{-\Delta^*(1-\epsilon)S_n}}^{1} \frac{1}{v\bar{\eta}(v)}\,dv \leq \Delta^* \cdot (1-\epsilon),$$

and there is $N_1(\epsilon) \in \mathbb{N}$ such that for $n \geq N_1(\epsilon)$ then $e^{-\Delta^*(1+\epsilon)S_n} < X_{n+1} < e^{-\Delta^*(1-\epsilon)S_n}$, which is (12.6) in Theorem 81. Since $t_n \sim n\Delta$ as $n \to \infty$ and $G(x) \sim \int_x^1 1/(v\bar{\eta}(v))\,dv$

as $x \to 0^+$, then

$$\Delta^* \cdot (1+\epsilon) \le \liminf_{n\to\infty} \frac{G\left(e^{-\Delta^*(1+\epsilon)S_n}\right)}{n}, \quad \limsup_{n\to\infty} \frac{G\left(e^{-\Delta^*(1-\epsilon)S_n}\right)}{n} \le \Delta^* \cdot (1-\epsilon),$$

and with $l_0 = 1/2 - L$ we have

$$(1+\epsilon) \cdot l_0 \le \liminf_{n\to\infty} \frac{G\left(e^{-\Delta^*(1+\epsilon)S_n}\right)}{n}, \quad \limsup_{n\to\infty} \frac{G\left(e^{-\Delta^*(1-\epsilon)S_n}\right)}{n} \le (1-\epsilon) \cdot l_0.$$

The rest of the proof mimics that of Theorem 81 from (12.10) to the end with G in the role of F and we get

$$\lim_{n\to\infty} \frac{-\log X_n}{\left(-\log \circ G^{-1}\right)(l_0 t_n)} = 1, \quad \text{a.s.},$$

or

$$\lim_{n\to\infty} \frac{-\log X_n}{\left(-\log \circ G^{-1}\right)\left(\left(\frac{1}{2} - L\right) t_n\right)} = 1, \quad \text{a.s.},$$

as claimed. □

Chapter 13

Sub-Exponential Stability with Small Noise

13.1 Introduction

In the previous chapter we saw that the existence of the limit (1.25) viz.,

$$\lim_{x\to 0^+} \frac{xf(x)}{g^2(x)} =: L,$$

enables us to classify the sub-exponential convergence of solutions of the SDE (1.17) viz.,

$$dX(t) = f(X(t))\,dt + g(X(t))\,dB(t).$$

Roughly, if $L \in (\infty, 1/2)$ and $x \mapsto g^2(x)/x^2$ is increasing then the asymptotic behaviour is given by

$$\lim_{t\to\infty} \frac{-\log X(t)}{(-\log\circ G^{-1})((\frac{1}{2}-L)t)} = 1, \quad \text{a.s.},$$

while in the case when $L = -\infty$ and $x \mapsto |f(x)|/x$ is increasing, we get

$$\lim_{t\to\infty} \frac{-\log X(t)}{(-\log\circ F^{-1})(t)} = 1, \quad \text{a.s.}, \tag{13.1}$$

where F and G are defined by (1.29) and (1.34) viz,.

$$F(x) = \int_x^1 \frac{1}{|f(u)|}\,du \quad \text{and} \quad G(x) = \int_x^1 \frac{u}{g^2(u)}\,du.$$

In the latter case, when $L = -\infty$ and hence $g^2(x) = o(x|f(x)|)$ as $x \to 0^+$, it is impossible to draw stronger conclusions about the rate of decay of solutions such as

$$\lim_{t\to\infty} \frac{F(X(t))}{t} = 1 \quad \text{or} \quad \lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} = 1, \quad \text{a.s.}, \tag{13.2}$$

which prevail for the solution of the ODE

$$x'(t) = f(x(t)), \quad t > 0, \quad x(0) = \zeta > 0. \tag{13.3}$$

Based on the evidence of Chapter 11, a more stringent restriction on the size of the diffusion term should enable decay results of the type listed in (13.2) to be proven.

Recall in the case of finite-time stability in solutions of (1.17), that the "small noise" assumption (1.55) viz.,

$$\text{there exists } \theta > 0 \text{ such that } \quad \limsup_{x \to 0^+} \frac{g^2(x)}{x^{1+\theta}|f(x)|} < \infty,$$

enables us to prove that

$$\lim_{t \to T^-} \frac{\bar{F}(X(t))}{T-t} = 1, \quad \text{a.s..}$$

This asymptotic result improves on that obtained when it is known only that $L = -\infty$. The result also matches asymptotic results for finite-time stability in solutions of the ODE (13.3).

Therefore, it seems reasonable to once again improve the "small noise" condition (1.55), which implies $L = -\infty$ but is stronger since $g^2(x) = O(x^{1+\theta}|f(x)|)$ as $x \to 0^+$. Therefore, our goal under (1.55), is to establish the desired refined asymptotic results in (13.2) and in the first part of this chapter we show that this indeed can be achieved, at the small expense of additional monotonicity hypotheses. Very roughly, if $\theta > 0$ is the number in (1.55) and furthermore $x \mapsto |f(x)|/x^{1+\theta}$ is decreasing then

$$\lim_{t \to \infty} \frac{F(X(t))}{t} = 1, \quad \text{a.s.,} \tag{13.4}$$

while if $x \mapsto |f(x)|/x^{1+\theta}$ is increasing

$$\lim_{t \to \infty} \frac{X(t)}{F^{-1}(t)} = 1, \quad \text{a.s.,} \tag{13.5}$$

The intuition behind this classification is that (13.4) deals with faster than power-law decay rates in X while (13.5) deals with power-law decay.

The second half of this chapter discusses whether these results can be reproduced by the numerical schemes we have presented. In general terms, the precise asymptotic behaviour is recovered but we observe that in the case of power transformations fewer side conditions are needed on f in order to establish the desired rate of decay. Nevertheless, the logarithmic pre-transformation performs equally well for all reasonable functions f and has the advantage that the value of the parameter θ in (1.55) need not be known in order to construct the scheme.

13.2 Polynomial or Sub-Polynomial Decay in SDEs

We are now going to prove our first result.

Theorem 83. *Suppose f, g are continuous, $f(0) = g(0) = 0$, $f(x) < 0$, $g^2(x) > 0$ for all $x > 0$. Suppose further there is $\theta > 0$ and a continuous function $\phi : (0,\infty) \to (0,\infty)$ such that*

$$\limsup_{x\to 0^+} \frac{g^2(x)}{x^{1+\theta}|f(x)|} < \infty$$

$|f(x)| \sim \phi(x)$ as $x \to 0^+$, $x \mapsto \phi(x)/x^{1+\theta}$ is increasing and $\phi(x)/x^{1+\theta} \to 0$ as $x \to 0^+$.

Then the solution of (1.17) obeys

$$\lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} = 1, \quad a.s..$$

Proof. First, since $f(x) < 0$ for all $x > 0$, $g^2(x) > 0$ and $f(0) = 0$, it follows that $p(\infty^-) = \infty$. Also

$$\lim_{x\to 0^+} \frac{g^2(x)}{x|f(x)|} = \lim_{x\to 0^+} \left(\frac{g^2(x)}{x^{1+\theta}|f(x)|} \cdot x^\theta \right) = 0,$$

and as $f(x) < 0$, this implies $L = -\infty$. Furthermore, by assumptions on ϕ, we have

$$\lim_{x\to 0^+} \frac{|f(x)|}{x} = \lim_{x\to 0^+} \frac{\phi(x)}{x} = \lim_{x\to 0^+} \left(\frac{\phi(x)}{x^{1+\theta}} \cdot x^\theta \right) = 0,$$

and

$$\lim_{x\to 0^+} \frac{g^2(x)}{x^2} = \lim_{x\to 0^+} \left(\frac{g^2(x)}{x|f(x)|} \cdot \frac{|f(x)|}{x} \right) = 0,$$

so $T = \inf\{t > 0 : X(t) \notin (0,\infty)\} = \emptyset$ and we have $X(t) > 0$, $\forall\, t \ge 0$, $\lim_{t\to\infty} X(t) = 0$ a.s.. Hence by Itô's Lemma,

$$X(t)^{-\theta} = X(0)^{-\theta} + \int_0^t -\theta X(s)^{-(\theta+1)} f(X(s)) \left(1 - \frac{(\theta+1)g^2(X(s))}{2X(s)f(X(s))} \right) ds + \int_0^t \frac{-\theta g(X(s))}{X(s)^{1+\theta}}\, dB(s).$$

Since $X(t) \to 0$ as $t \to \infty$ and $g^2(x) = o(x|f(x)|)$ as $x \to 0^+$ the integrand in the drift is asymptotic to $\theta X(t)^{-(\theta+1)}|f(X(t))|$ as $t \to \infty$. Define as usual

$$M(t) := \int_0^t \frac{-\theta g(X(s))}{X(s)^{1+\theta}}\, dB(s) \quad \text{and} \quad \langle M \rangle (t) := \int_0^t \frac{\theta^2 g^2(X(s))}{X(s)^{2+2\theta}}\, ds.$$

Suppose $\langle M \rangle (t)$ tends to a finite limit on the event C. Then $M(t)$ converges on C. Moreover, the drift has a limit as $t \to \infty$, which can be finite or infinite because the drift integrand is asymptotically positive. If the drift has a finite limit, then $X(t)^{-\theta}$ tends to a finite limit on C which contradicts the fact that $X(t) \to 0$ as $t \to \infty$ a.s.. Thus the drift tends to infinity as $t \to \infty$ on C and

$$\lim_{t\to\infty} \frac{X(t)^{-\theta}}{\int_0^t \theta X(s)^{-(\theta+1)}|f(X(s))|\,ds} = 1, \quad \text{a.s. on } C. \tag{13.6}$$

On C', $\langle M\rangle(t) \to \infty$ so $M(t)/\langle M\rangle(t) \to 0$ as $t \to \infty$ and $\liminf_{t\to\infty} M(t) = -\infty$. Once again the drift has a limit at infinity. If the drift tends to a finite limit, then $\liminf_{t\to\infty} X(t)^{-\theta} = -\infty$ which is impossible. Thus the drift tends to infinity a.s. on C'. Moreover,

$$\frac{\langle M\rangle(t)}{\int_0^t \theta|f(X(s))|/X(s)^{1+\theta}\,ds} = \frac{\int_0^t \theta^2 g^2(X(s))/X(s)^{2+2\theta}\,ds}{\int_0^t \theta|f(X(s))|/X(s)^{1+\theta}\,ds}.$$

For each $x_0 > 0$ and some $c = c(x_0) > 0$ we have $g^2(x) < cx^{1+\theta}|f(x)|$, $\forall\, x < x_0$. Since $X(t) \to 0$ as $t \to \infty$ there exists $T > 0$ such that for $t > T$, $X(t) < x_0$. Thus

$$\begin{aligned}\frac{\langle M\rangle(t)}{\int_0^t \theta|f(X(s))|/X(s)^{1+\theta}\,ds} &= \frac{\langle M\rangle(T)}{\int_0^t \theta|f(X(s))|/X(s)^{1+\theta}\,ds} + \frac{\int_T^t \theta g^2(X(s))/X(s)^{2+2\theta}\,ds}{\int_0^t |f(X(s))|/X(s)^{1+\theta}\,ds}\\ &\le \frac{\langle M\rangle(T)}{\int_0^t \theta|f(X(s))|/X(s)^{1+\theta}\,ds} + \frac{\int_T^t \theta c X(s)^{1+\theta}|f(X(s))|/X(s)^{2+2\theta}\,ds}{\int_0^t |f(X(s))|/X(s)^{1+\theta}\,ds}\\ &= \frac{\langle M\rangle(T)}{\int_0^t \theta|f(X(s))|/X(s)^{1+\theta}\,ds} + c\theta.\end{aligned}$$

Therefore

$$\limsup_{t\to\infty} \frac{\langle M\rangle(t)}{\int_0^t \theta|f(X(s))|/X(s)^{1+\theta}\,ds} \le c\theta.$$

Hence

$$\lim_{t\to\infty} \frac{M(t)}{\int_0^t \theta|f(X(s))|/X(s)^{1+\theta}\,ds} = 0.$$

Thus

$$\lim_{t\to\infty} \frac{X(t)^{-\theta}}{\int_0^t \theta|f(X(s))|/X(s)^{1+\theta}\,ds} = 1, \quad \text{a.s. on } C'.$$

Combining (13.6) and the last limit gives

$$\lim_{t\to\infty} \frac{X(t)^{-\theta}}{\int_0^t \theta|f(X(s))|/X(s)^{1+\theta}\,ds} = 1, \quad \text{a.s..}$$

Define $\eta(x) := \theta\phi(x)/x^{1+\theta}$ then η is increasing. Moreover

$$\lim_{t\to\infty} \frac{X(t)^{-\theta}}{\int_0^t \eta(X(s))\,ds} = 1. \tag{13.7}$$

Then with $I(t) := \int_0^t \eta(X(s))\,ds$, $t \geq 0$ for every $\epsilon \in (0,1)$ there is $T(\epsilon)$ such that

$$1-\epsilon < \frac{X(t)^{-\theta}}{I(t)} < 1+\epsilon,$$

and $1-\epsilon < \eta^{-1}(I'(t))^{-\theta}/I(t) < 1+\epsilon$ with both inequalities holding for $t \geq T(\epsilon)$. Since η is increasing

$$\eta\left((1-\epsilon)^{-1/\theta} I(t)^{-1/\theta}\right) > I'(t) > \eta\left((1+\epsilon)^{-1/\theta} I(t)^{-1/\theta}\right), \quad t \geq T(\epsilon).$$

Next

$$\int_{T(\epsilon)}^t \frac{I'(s)}{\eta\left((aI(s))^{-1/\theta}\right)}\,ds = \frac{1}{a}\int_{(aI(t))^{-1/\theta}}^{(aI(T(\epsilon)))^{-1/\theta}} \frac{\theta}{v^{1+\theta}\eta(v)}\,dv.$$

Thus for $t \geq T(\epsilon)$,

$$\frac{1}{1-\epsilon}\int_{(1-\epsilon)^{-1/\theta}I(t)^{-1/\theta}}^{I_\epsilon^+} \frac{\theta}{v^{1+\theta}\eta(v)}\,dv \leq t - T(\epsilon).$$

Thus

$$\limsup_{t\to\infty} \frac{1}{t}\int_{(1-\epsilon)^{-1/\theta}I(t)^{-1/\theta}}^{I_\epsilon^+} \frac{\theta}{v^{1+\theta}\eta(v)}\,dv \leq 1-\epsilon.$$

Now $F(x) = \int_x^1 1/|f(u)|\,du \sim \int_x^1 1/\phi(u)\,du = \int_x^1 \theta/(u^{1+\theta}\eta(u))\,du$ as $x \to 0^+$. Hence

$$\limsup_{t\to\infty} \frac{F\left((1-\epsilon)^{-1/\theta}I(t)^{-1/\theta}\right)}{t} \leq 1-\epsilon.$$

Similarly for $t \geq T(\epsilon)$,

$$\frac{1}{1+\epsilon}\int_{(1+\epsilon)^{-1/\theta}I(t)^{-1/\theta}}^{I_\epsilon^-} \frac{\theta}{v^{1+\theta}\eta(v)}\,dv \geq t - T(\epsilon),$$

leading to

$$\liminf_{t\to\infty} \frac{F\left((1+\epsilon)^{-1/\theta}I(t)^{-1/\theta}\right)}{t} \geq 1+\epsilon.$$

Now for $t \geq T(\epsilon)$, $(1-\epsilon)\cdot I(t) < X(t)^{-\theta} < (1+\epsilon)\cdot I(t)$. So $(1-\epsilon)^{-1/\theta}\cdot I(t)^{-1/\theta} > X(t) > (1+\epsilon)^{-1/\theta}\cdot I(t)^{-1/\theta}$. Thus

$$\left(\tfrac{1+\epsilon}{1-\epsilon}\right)^{1/\theta}\cdot X(t) = \tfrac{(1-\epsilon)^{-1/\theta}}{(1+\epsilon)^{-1/\theta}}\cdot X(t) > \tfrac{(1-\epsilon)^{1/\theta}}{(1+\epsilon)^{1/\theta}}\cdot(1+\epsilon)^{-1/\theta}\cdot I(t)^{-1/\theta} = (1-\epsilon)^{-1/\theta}\cdot I(t)^{-1/\theta},$$

so for $t \geq T(\epsilon)$, $F\left(\left(\frac{1+\epsilon}{1-\epsilon}\right)^{1/\theta}X(t)\right) < F\left((1-\epsilon)^{-1/\theta}I(t)^{-1/\theta}\right)$. Hence

$$\limsup_{t\to\infty} \frac{F\left(\left(\frac{1+\epsilon}{1-\epsilon}\right)^{1/\theta}X(t)\right)}{t} \leq 1-\epsilon. \tag{13.8}$$

Similarly

$$\left(\tfrac{1-\epsilon}{1+\epsilon}\right)^{1/\theta} \cdot X(t) < \tfrac{(1-\epsilon)^{1/\theta}}{(1+\epsilon)^{1/\theta}} \cdot (1-\epsilon)^{-1/\theta} \cdot I(t)^{-1/\theta} = (1+\epsilon)^{-1/\theta} \cdot I(t)^{-1/\theta},$$

so

$$\liminf_{t\to\infty} \frac{F\left(\left(\tfrac{1-\epsilon}{1+\epsilon}\right)^{1/\theta} X(t)\right)}{t} \geq 1+\epsilon. \tag{13.9}$$

By (13.8), for every $\nu > 0$ there is $\tilde{T}_2(\nu, \epsilon)$ such that for all $t \geq \tilde{T}_2(\nu, \epsilon)$

$$\frac{F\left(\left(\tfrac{1+\epsilon}{1-\epsilon}\right)^{1/\theta} X(t)\right)}{t} < (1-\epsilon)\cdot(1+\nu).$$

Now fix $\nu = \nu(\epsilon)$ such that $(1-\epsilon)\cdot(1+\nu) = 1$ and write $T_2(\epsilon) = \tilde{T}_2(\nu(\epsilon), \epsilon)$. Then for $t \geq T_2(\epsilon)$

$$\frac{F\left(\left(\tfrac{1+\epsilon}{1-\epsilon}\right)^{1/\theta} X(t)\right)}{t} < 1.$$

Hence $\left(\tfrac{1+\epsilon}{1-\epsilon}\right)^{1/\theta} \cdot X(t) > F^{-1}(t)$ for all $t \geq T_2(\epsilon)$. Therefore, for $t \geq T_2(\epsilon)$

$$\liminf_{t\to\infty} \frac{X(t)}{F^{-1}(t)} \geq \left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/\theta}.$$

Letting $\epsilon \to 0^+$ gives

$$\liminf_{t\to\infty} \frac{X(t)}{F^{-1}(t)} \geq 1.$$

Proceeding similarly with (13.9) yields

$$\limsup_{t\to\infty} \frac{X(t)}{F^{-1}(t)} \leq 1,$$

as claimed. □

In the following example we verify that the results of Theorem 83 hold.

Example 84. Let $\beta > 1$, $2\gamma > 1+\beta$. Suppose $f(x) = -x^\beta$ and $g(x) = x^\gamma$. Then

$$x \mapsto \frac{|f(x)|}{x^{1+\theta}} = x^{\beta-1-\theta},$$

so $x \mapsto |f(x)|/x^{1+\theta}$ is increasing for $0 < \theta < \beta - 1$. Also

$$\limsup_{x\to 0^+} \frac{g^2(x)}{x^{1+\theta}|f(x)|} = \limsup_{x\to 0^+} x^{2\gamma-1-\theta-\beta} < \infty,$$

once $2\gamma - 1 - \theta - \beta \geq 0$ or $0 < \theta \leq 2\gamma - 1 - \beta$. Hence if we pick any $\theta \in (0, \min(\beta -$

$1, 2\gamma - 1 - \beta)$), all the conditions of Theorem 83 are fulfilled and we have that

$$\lim_{t\to\infty} \frac{X(t)}{((\beta-1)t)^{-1/(\beta-1)}} = \limsup_{t\to\infty} \frac{X(t)}{F^{-1}(t)} = 1, \text{a.s.},$$

because $F(x) \sim x^{1-\beta}/(\beta-1)$ as $x \to 0^+$ and so $F^{-1}(t) \sim ((\beta-1)t)^{-1/(\beta-1)}$ as $t \to \infty$.

13.3 Faster-than-Polynomial Decay in SDEs

Theorem 85. *Suppose f, g are continuous, $f(0) = g(0) = 0$, $f(x) < 0$, $g^2(x) > 0$ for all $x > 0$. Suppose further there is $\theta > 0$ and a continuous function $\phi : (0,\infty) \to (0,\infty)$ such that*

$$\limsup_{x\to 0^+} \frac{g^2(x)}{x^{1+\theta}|f(x)|} < \infty,$$

$|f(x)| \sim \phi(x)$ as $x \to 0^+$, $x \mapsto \phi(x)/x^{1+\theta}$ is decreasing and $\phi(x)/x \to 0$ as $x \to 0^+$.

Then the solution of (1.17) *obeys*

$$\lim_{t\to\infty} \frac{F(X(t))}{t} = 1, \quad a.s..$$

Proof. Define $\eta(x) := \theta\phi(x)/x^{1+\theta}$ as the decreasing function. Arguing as in the proof of Theorem 83 and using the hypotheses that $f(x)/x \to 0$ as $x \to 0^+$ we once again have that $X(t) \to 0$ as $t \to \infty$, $X(t) > 0\,\forall\, t > 0$ and that the relationship (13.7) holds, viz.,

$$\lim_{t\to\infty} \frac{X(t)^{-\theta}}{\int_0^t \eta(X(s))\, ds} = 1.$$

Define $I(t) := \int_0^t \eta(X(s))\, ds$. For every $\epsilon \in (0,1)$ there is a $T_1(\epsilon) > 0$ such that for $t > T_1(\epsilon)$

$$1 - \epsilon < \frac{X(t)^{-\theta}}{I(t)} < 1 + \epsilon,$$

or

$$1 - \epsilon < \frac{\eta^{-1}(I'(t))^{-\theta}}{I(t)} < 1 + \epsilon,$$

Since η is decreasing, for $t \ge T_1(\epsilon)$, $\eta\left(((1-\epsilon)I(t))^{-1/\theta}\right) < I'(t) < \eta\left(((1+\epsilon)I(t))^{-1/\theta}\right)$ and

$$((1-\epsilon)I(t))^{-1/\theta} > X(t) > ((1+\epsilon)I(t))^{-1/\theta}. \tag{13.10}$$

Therefore for $t \ge T_1(\epsilon)$

$$\frac{I'(t)}{\eta\left(((1-\epsilon)I(t))^{-1/\theta}\right)} > 1 \quad \text{and} \quad \frac{I'(t)}{\eta\left(((1+\epsilon)I(t))^{-1/\theta}\right)} < 1.$$

Now let $a > 0$ and compute

$$\int_{T_1(\epsilon)}^{t} \frac{I'(s)}{\eta\left((aI(s))^{-1/\theta}\right)}\,ds = \frac{1}{a}\int_{T_1(\epsilon)}^{t} \frac{aI'(s)}{\eta\left((aI(s))^{-1/\theta}\right)}\,ds = \frac{1}{a}\int_{aI(T_1(\epsilon))}^{aI(t)} \frac{1}{\eta\left(u^{-1/\theta}\right)}\,du$$
$$= \frac{1}{a}\int_{(aI(t))^{-1/\theta}}^{(aI(T_1(\epsilon)))^{-1/\theta}} \frac{\theta}{v^{1+\theta}\eta(v)}\,dv.$$

Therefore using the above identity with $a = 1 \pm \epsilon$ we obtain for $t \geq T_1(\epsilon)$

$$\frac{1}{1+\epsilon}\int_{((1+\epsilon)I(t))^{-1/\theta}}^{((1+\epsilon)I(T_1(\epsilon)))^{-1/\theta}} \frac{\theta}{v^{1+\theta}\eta(v)}\,dv \leq t - T_1(\epsilon) \leq \frac{1}{1-\epsilon}\int_{((1-\epsilon)I(t))^{-1/\theta}}^{((1-\epsilon)I(T_1(\epsilon)))^{-1/\theta}} \frac{\theta}{v^{1+\theta}\eta(v)}\,dv.$$

Thus calling the upper limits of integration $I_\epsilon^\pm$ for brevity we get:

$$1-\epsilon \leq \liminf_{t\to\infty} \frac{1}{t}\int_{((1-\epsilon)I(t))^{-1/\theta}}^{I_\epsilon^-} \frac{\theta}{v^{1+\theta}\eta(v)}\,dv, \quad \limsup_{t\to\infty} \frac{1}{t}\int_{((1+\epsilon)I(t))^{-1/\theta}}^{I_\epsilon^+} \frac{\theta}{v^{1+\theta}\eta(v)}\,dv \leq 1+\epsilon.$$

Now $F(x) = \int_x^1 1/|f(u)|\,du \sim \int_x^1 \theta/(u^{1+\theta}\eta(u))\,du$ as $x \to 0^+$. Thus

$$1-\epsilon \leq \liminf_{t\to\infty} \frac{F\left(((1-\epsilon)I(t))^{-1/\theta}\right)}{t}, \quad \limsup_{t\to\infty} \frac{F\left(((1+\epsilon)I(t))^{-1/\theta}\right)}{t} \leq 1+\epsilon.$$

Next by (13.10), $F\left(((1-\epsilon)I(t))^{-1/\theta}\right) < F(X(t)) < F\left(((1+\epsilon)I(t))^{-1/\theta}\right)$. Hence

$$\liminf_{t\to\infty} \frac{F(X(t))}{t} \geq \liminf_{t\to\infty} \frac{F\left(((1-\epsilon)I(t))^{-1/\theta}\right)}{t} \geq 1-\epsilon,$$

and

$$\limsup_{t\to\infty} \frac{F(X(t))}{t} \leq \limsup_{t\to\infty} \frac{F\left(((1+\epsilon)I(t))^{-1/\theta}\right)}{t} \leq 1+\epsilon.$$

Thus letting $\epsilon \to 0^+$ yields $F(X(t))/t \to 1$ as $t \to \infty$ a.s., as claimed. □

Example 86. Suppose $\gamma > 1$, $\beta > 0$ and

$$f(x) = \frac{-x}{\log^\beta(1/x)} \quad \text{and} \quad g(x) = x^\gamma,$$

for all x sufficiently small. Then for $\theta > 0$, $x \mapsto |f(x)|/x$, $x \mapsto |f(x)|/x^{1+\theta}$ are increasing and decreasing functions respectively on an open interval to the right of zero. Consequently, we cannot apply Theorem 83 to the SDE with this drift and diffusion coefficient. However, Theorem 85 can be employed because for $0 < \theta < 2\gamma - 2$ we have

$$\limsup_{x\to0^+} \frac{g^2(x)}{x^{1+\theta}|f(x)|} = 0.$$

Then as $F(x) \sim 1/(\beta+1) \cdot \log^{\beta+1}(1/x)$ as $x \to 0^+$ we have that

$$\lim_{t\to\infty}\left(\frac{1}{\beta+1}\cdot\frac{\log^{\beta+1}(1/X(t))}{t}\right) = \lim_{t\to\infty}\frac{F(X(t))}{t} = 1, \text{a.s.},$$

or

$$\lim_{t\to\infty}\frac{\log X(t)}{t^{1/(\beta+1)}} = -(\beta+1)^{1/(\beta)}, \text{a.s.}.$$

13.4 Refined and Consolidated Results for Subexponential SDEs

In the case that the hypotheses of Theorem 83 hold, we have that $X(t) \sim F^{-1}(t)$ as $t \to \infty$ and therefore it is essentially impossible to obtain more refined asymptotic information concerning X. On the other hand in Theorem 85, we prove only that $F(X(t))/t \to 1$ as $t \to \infty$ which leaves the question as to whether we can prove the stronger limit $X(t) \sim F^{-1}(t)$ as $t \to \infty$. In the following theorem (which also consolidates Theorems 83 and 85) we show that this can be achieved with a small additional cost by imposing a smoothness hypothesis on f. In fact the following hypotheses are employed.

$$\limsup_{x\to 0^+}\frac{g^2(x)}{x^{1+\theta}|f(x)|} < \infty \quad \forall\,\theta \in (0,\theta_0) \tag{13.11}$$

$$\forall\,\theta > 0 \quad x \mapsto |f(x)|/x^{1+\theta} \text{ is asymptotic to a continuous decreasing function} \tag{13.12}$$

$$\exists\,\theta > 0 \quad x \mapsto |f(x)|/x^{1+\theta} \text{ is asymptotic to a continuous increasing function} \tag{13.13}$$

$$|f(x)| \in C^1((0,\infty);(0,\infty)), \quad x \mapsto |f(x)|/x \text{ is asymptotically increasing} \tag{13.14}$$

We start by proving under rather general sub-exponential hypotheses that

$$\lim_{t\to\infty}\frac{X(t)}{F^{-1}(t)} = 1 \quad \Longrightarrow \lim_{t\to\infty}\frac{F(X(t))}{t} = 1 \Longrightarrow \lim_{t\to\infty}\frac{-\log X(t)}{(-\log\circ F^{-1})(t)} = 1.$$

In order to achieve this we start by stating a lemma by Appleby and Patterson [7, 8] concerning the preservation of asymptotic behaviour under transformation.

Lemma 44. *Suppose ϕ is such that $\phi(x) \to \infty$ as $x \to \infty$, $\phi'(x) > 0\,\forall\, x > 0$ and $x \mapsto \phi'(x)$ is decreasing with $\phi'(x) \to 0$ as $x \to \infty$. If b, $c \in C(\mathbb{R}^+;\mathbb{R}^+)$ obey $\lim_{t\to\infty} b(t) = \lim_{t\to\infty} c(t) = \infty$ and $b(t) \sim c(t)$ as $t \to \infty$, then $\phi(b(t)) \sim \phi(c(t))$ as $t \to \infty$.*

Proposition 18. *Suppose $x \in C((0,\infty);(0,\infty))$ is such that $x(t) \to 0$ as $t \to \infty$ and*

$$\lim_{t\to\infty}\frac{F(x(t))}{t} = 1.$$

If $|f|$ is such that $x \mapsto |f(x)|/x$ is increasing and $|f(x)|/x \to 0$ as $x \to 0^+$, then

$$\lim_{t\to\infty} \frac{-\log x(t)}{(-\log \circ F^{-1})(t)} = 1.$$

Proof. Define $y(t) := F^{-1}(t)$, $t \geq 0$, so that $y'(t) = f(y(t))$, $t \geq 0$, $y(0) = 1$. Define $\bar{f}(x) := |f(x)|/x$ so $\bar{f}$ is increasing and set $\phi(x) = (-\log \circ F^{-1})(x)$, $x > 0$. Then as $-\log$ and F^{-1} are decreasing, ϕ is increasing and $\phi(x) \to \infty$ as $x \to \infty$. Also

$$\phi'(x) = \frac{-y'(x)}{y(x)} = \frac{|f(y(x))|}{y(x)} = \bar{f}(y(x)),$$

and since $\bar{f}$ is increasing and y is decreasing, ϕ' is decreasing. Moreover, as $\bar{f}(x) \to 0$ as $x \to 0^+$ and $y(x) \to 0$ as $x \to \infty$, $\phi'(x) \to 0$ as $x \to \infty$. Now set $b(t) = F(x(t))$, $c(t) = t$. Clearly $b(t) \sim c(t)$ as $t \to \infty$ by hypothesis. Therefore all the conditions of Lemma 44 are satisfied and we have as $t \to \infty$:

$$-\log x(t) = (-\log \circ F^{-1})(F(x(t))) = \phi(b(t)) \sim \phi(c(t)) = (-\log \circ F^{-1})(t),$$

as claimed. □

The implication that $x(t) \sim F^{-1}(t)$ as $t \to \infty$ implies $-\log x(t) \sim (-\log \circ F^{-1})(t)$ as $t \to \infty$ is an immediate consequence of the slow variation of $-\log$.

Proposition 19. *Suppose $x \in C((0,\infty);(0,\infty))$ is such that $x(t) \to 0$ as $t \to \infty$ and*

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = 1.$$

If $|f|$ is such that $x \mapsto |f(x)|/x^{1+\theta}$ is decreasing for some $\theta > 0$ with $|f(x)|/x^{1+\theta} \to \infty$ as $x \to 0^+$, then

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 1.$$

Proof. Define $\phi(x) := F(x^{-1/\theta})$, $b(t) := 1/x(t)^\theta$ and $c(t) := 1/F^{-1}(t)^\theta$. Then $b(t) \sim c(t) \to \infty$ as $t \to \infty$. Since $x \mapsto x^{-1/\theta} \to \infty$ as $x \to 0$ and $F(x) \to \infty$ as $x \to 0$, $\phi(x) \to \infty$ as $x \to \infty$ and since

$$\phi'(x) = \frac{x^{-(1+1/\theta)}}{\theta|f(x^{-1/\theta})|} > 0,$$

thus ϕ is increasing. Let $\psi(x) = |f(x)|/x^{1+\theta}$. Then ψ is decreasing, so $x \mapsto 1/(\theta\psi(x))$ is increasing. Since

$$\frac{1}{\theta\psi(x^{-1/\theta})} = \frac{(x^{-1/\theta})^{1+\theta}}{\theta|f(x^{-1/\theta})|}, \tag{13.15}$$

then

$$\phi'(x) = \frac{1}{\theta\psi(x^{-1/\theta})},$$

which must be decreasing. Also as $\psi(x) \to \infty$ as $x \to 0^+$, $\phi'(x) \to 0$ as $x \to \infty$. Hence we can apply Lemma 44 to get as $t \to \infty$:

$$F(x(t)) = F(b(t)^{-1/\theta}) = \phi(b(t)) \sim \phi(c(t)) = F(c(t)^{-1/\theta}) = F(F^{-1}(t)) = t,$$

as claimed. □

We are now in a position to state the main result of this section.

Theorem 87. *Suppose f, g are continuous, $f(0) = g(0) = 0$, $f(x) < 0$, $g^2(x) > 0$ for all $x > 0$. Let $|f(x)|/x \to 0$ as $x \to 0^+$.*

(i) If (13.11) *and* (13.12) *hold, then*

$$\lim_{t\to\infty} \frac{F(X(t))}{t} = 1, \quad a.s..$$

(ii) If (13.11), (13.12) *and* (13.14) *hold, then*

$$\lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} = 1, \quad a.s..$$

(iii) If (13.11) *and* (13.13) *hold, then*

$$\lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} = 1, \quad a.s..$$

To see how part (ii) of the Theorem can improve existing results, we refer to the previous Example 86, in which $\gamma > 1$, $\beta > 0$ and

$$f(x) = \frac{-x}{\log^{\beta}(1/x)} \quad \text{and} \quad g(x) = x^{\gamma}.$$

We can use the fact that $f \in C^1((0,\delta);(0,\infty))$ for some $\delta > 0$, to employ part (ii) of Theorem 87 to prove that

$$\lim_{t\to\infty} \frac{X(t)}{\exp\left(-\left((\beta+1)t\right)^{1/(\beta+1)}\right)} = 1,$$

because $F^{-1}(t) \sim \exp\left(-\left((\beta+1)t\right)^{1/(\beta+1)}\right)$ as $t \to \infty$. This is a stronger statement than we are able to establish using Theorem 85.

Theorem 87 part (ii) requires the following lemmas which we state now to aid understanding. The following result shows how growth rates of ODEs are preserved with respect to changes in the time argument. We state the result with asymptotic monotonicity of functions required, but prove it under the simplifying assumption that such functions are monotone.

Lemma 45. *Suppose $\phi : (0,\infty) \to (0,\infty)$ is continuous with $z \mapsto \phi(z)$ asymptotically increasing and $z \mapsto \phi(z)/z \to 0$ as $z \to \infty$ is asymptotically decreasing. Let $I : (0,\infty) \to \mathbb{R}$ be continuous such that*

$$\lim_{z\to\infty} \frac{\phi(z)}{z} I(\Phi(z)) = 0,$$

where $\Phi(x) = \int_1^x 1/\phi(u)\,du$. Then the solution of the ODE

$$z'(t) = \phi(z(t)), \quad t > 0 \quad , z(0) = \zeta > 0,$$

obeys

$$\lim_{t\to\infty} \frac{z(t+I(t))}{z(t)} = 1.$$

Proof. Define

$$\alpha(x) := \frac{\phi(x)}{x} I(\Phi(x)), \quad x \geq 0.$$

Then $\alpha(x) \to 0$ as $x \to \infty$. Since we may take $z(0) = 1$ without loss of generality then $z(t) = \Phi^{-1}(t) \to \infty$ as $t \to \infty$. Thus for every $\epsilon \in (0,1)$, there is $T(\epsilon) > 0$ such that $\tilde{\alpha}(t) := \alpha(\Phi^{-1}(t))$ obeys $|\tilde{\alpha}(t)| < \epsilon$, $\forall\, t \geq T(\epsilon)$. Let $t \geq T(\epsilon)$ and suppose $I(t) > 0$. By the Mean Value Theorem, there is $\theta_t \in (0,1)$ such that

$$\begin{aligned} z(t) < z(t+I(t)) = z(t) + z'(t+\theta_t I(t)) \cdot I(t) \quad &= \quad z(t) + \phi(z(t+\theta_t I(t))) \cdot I(t) \\ &< \quad z(t) + \phi(z(t+I(t))) \cdot I(t), \end{aligned}$$

since ϕ is increasing. Also, as $z(t) < z(t+I(t))$ as $x \mapsto \phi(x)/x$ is decreasing and $I(t) > 0$

$$1 < \frac{z(t)}{z(t+I(t))} + \frac{\phi(z(t+I(t))) \cdot I(t)}{z(t+I(t))} < \frac{z(t)}{z(t+I(t))} + \frac{\phi(z(t)) \cdot I(t)}{z(t)}.$$

Hence

$$1 - \tilde{\alpha}(t) < \frac{z(t)}{z(t+I(t))}.$$

Thus as $-\epsilon < \tilde{\alpha}(t) < \epsilon, \forall\, t \geq T(\epsilon)$,

$$1 < \frac{z(t+I(t))}{z(t)} < \frac{1}{1-\tilde{\alpha}(t)} < \frac{1}{1-\epsilon}.$$

Thus $t \geq T(\epsilon)$, $I(t) > 0$ implies

$$\left| \frac{z(t+I(t))}{z(t)} - 1 \right| \leq \frac{\epsilon}{1-\epsilon}. \tag{13.16}$$

We now tackle the case when $I(t) < 0$. Then there is $\theta_t \in (0,1)$ such that

$$\begin{aligned} z(t) > z(t+I(t)) = z(t) + z'(t+\theta_t I(t)) \cdot I(t) &= z(t) - z'(t+\theta_t I(t)) \cdot |I(t)| \\ &= z(t) - \phi(z(t+\theta_t I(t))) \cdot |I(t)| \\ &< z(t) - \phi(z(t)) \cdot |I(t)| . \end{aligned}$$

Thus

$$1 \geq \frac{z(t+I(t))}{z(t)} \geq 1 - \frac{\phi(z(t))}{z(t)} \cdot |I(t)| = 1 - |\tilde{\alpha}(t)| > 1 - \epsilon,$$

and so $\forall\, t \geq T(\epsilon)$

$$0 \geq \frac{z(t+I(t))}{z(t)} - 1 \geq 1 - \epsilon.$$

Thus for $t \geq T(\epsilon)$, $I(t) < 0$ implies

$$\left| \frac{z(t+I(t))}{z(t)} - 1 \right| \leq \epsilon. \tag{13.17}$$

Hence by (13.16) and (13.17), for all $t \geq T(\epsilon)$

$$\left| \frac{z(t+I(t))}{z(t)} - 1 \right| \leq \frac{\epsilon}{1-\epsilon}.$$

Thus $z(t+I(t))/z(t) \to 1$ as $t \to \infty$, as claimed. □

Lemma 46. *Let $I : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose $x \mapsto |f(x)|/x^{1+\theta}$ is asymptotically decreasing and $x \mapsto |f(x)|/x$ is asymptotically increasing and*

$$\lim_{x \to 0^+} \frac{f(x)}{x} I\left(F(x)\right) = 0. \tag{13.18}$$

Then

$$\lim_{t \to \infty} \frac{F^{-1}\left(t + I(t)\right)}{F^{-1}(t)} = 1.$$

Proof. Note that $y(t) = F^{-1}(t)$, where $F(x) := \int_x^1 1/|f(u)|\, du$ and

$$y'(t) = f(y(t)), \quad t > 0 : \quad y(0) = 1$$

Define $z(t) := y(t)^{-\theta}$. Then

$$z'(t) = -\theta y(t)^{-\theta-1} y'(t) = \frac{-\theta f(y(t))}{y(t)^{1+\theta}} = \frac{\theta |f(y(t))|}{y(t)^{1+\theta}},$$

or $z'(t) = \phi(z(t))$ where $\phi(z) = \theta|f(z^{-1/\theta})|/\left[z^{-1/\theta}\right]^{1+\theta}$. Then

$$\frac{\phi(z)}{z} = \frac{\theta|f(z^{-1/\theta})|}{z \cdot z^{-1/\theta} \cdot z^{-1}} = \frac{\theta|f(z^{-1/\theta})|}{z^{-1/\theta}} = \theta\phi_1\left(z^{-1/\theta}\right),$$

where $\phi_1(x) := |f(x)|/x$ is asymptotically increasing on $(0,\delta)$, since $z \mapsto z^{-1/\theta}$ is decreasing, $z \mapsto \phi(z)/z$ is asymptotically decreasing. With $\phi_2(x) := |f(x)|/x^{1+\theta}$ we have $\phi(z) = \theta\phi_2\left(z^{-1/\theta}\right)$, so as ϕ_2 is asymptotically decreasing and $z \mapsto z^{-1/\theta}$ is decreasing then ϕ is asymptotically increasing. Therefore by Lemma 46

$$\lim_{t\to\infty} \frac{z(t+I(t))}{z(t)} = 1, \tag{13.19}$$

provided

$$\lim_{z\to\infty} \frac{\phi(z)}{z} I\left(\Phi(z)\right) = 0, \tag{13.20}$$

where

$$\Phi(x) = \int_1^x \frac{1}{\phi(u)}\,du. \tag{13.21}$$

Note that (13.19) implies $y(t+I(t))/y(t) \to 1$ as $t \to \infty$, and hence the result. It remains to show that (13.18) implies (13.20). Now with $x = z^{-1/\theta}$

$$\begin{aligned}\frac{\phi(z)}{z} \cdot I\left(\Phi(z)\right) = \frac{\theta|f(z^{-1/\theta})|}{z^{-1/\theta}} \cdot I\left(\int_1^z \frac{1}{\phi(u)}\,du\right) &= \frac{\theta|f(z^{-1/\theta})|}{z^{-1/\theta}} \cdot I\left(\int_1^z \frac{u^{-(1+\theta)/\theta}}{\theta|f(u^{-1/\theta})|}\,du\right) \\ &= \frac{\theta|f(x)|}{x} \cdot I\left(\int_1^x \frac{v^{1+\theta}}{\theta|f(v)|} \cdot -\theta v^{-(\theta+1)}\,dv\right),\end{aligned}$$

so with $x = z^{-1/\theta}$

$$\frac{\phi(z)}{z} I\left(\Phi(z)\right) = \frac{\theta|f(x)|}{x} \cdot I\left(F(x)\right).$$

Since (13.18) holds, so does (13.20) and therefore the claim. □

Proof of Theorem 87 part (ii). Since $f \in C^1$ by Itô's Lemma we have

$$F(X(t)) = F(X(0)) + t + \int_0^t \frac{g(X(s))}{f(X(s))}\,dB(s) - \frac{1}{2}\int_0^t f'(X(s))\frac{g^2(X(s))}{f^2(X(s))}\,ds.$$

Define

$$M(t) := \int_0^t \frac{g(X(s))}{f(X(s))}\,dB(s) \quad \text{and} \quad I_1(t) := \int_0^t f'(X(s))\frac{g^2(X(s))}{f^2(X(s))}\,ds.$$

Then

$$\langle M\rangle(t) = \int_0^t \frac{g^2(X(s))}{f^2(X(s))}\,ds =: J(t).$$

If $C = \{\omega : \lim_{t\to\infty} \langle M\rangle(t,\omega) < \infty\}$ then $M(t)$ converges to a finite limit on C. This in turn implies $I_1(t)$ tends to a finite limit on C, because $f \in C^1$, $f'(0^+) = 0$ and $X(t) \to 0$

as $t \to \infty$. On C, since both M and I_1 converge, so $F(X(t)) - t \to L^* \in (-\infty, \infty)$ as $t \to \infty$. This implies $X(t) \sim F^{-1}(t)$ as $t \to \infty$ on C. Suppose we are on C' where $\langle M \rangle (t) \to \infty$ as $t \to \infty$. Clearly, as $f'(X(t)) \to 0$ as $t \to \infty$, then $I_1(t)/\langle M \rangle (t) \to 0$ as $t \to \infty$ by L'Hôpital's Rule. Defining

$$I(t) := F(X(0)) + M(t) - \tfrac{1}{2} I_1(t),$$

we see that $I_1(t)/\langle M \rangle (t) \to 0$ as $t \to \infty$, because $M(t)/\langle M \rangle (t) \to 0$ as $t \to \infty$ by the Strong Law of Large Numbers for Martingales. Thus on C', $X(t) = F^{-1}(t + I(t))$ and

$$I_1(t) = o(J(t)) = o\left(\int_0^t \frac{g^2(X(s))}{f^2(X(s))}\, ds \right), \quad \text{as } t \to \infty.$$

We now wish to verify condition (13.18) in Lemma 46 to conclude. Next, as $X(t) \to 0$ as $t \to \infty$, and $g^2(x)/(x^{1+\theta}|f(x)|) \le C^*$ for all $x \in (0, \delta)$ and $X(t)$ is bounded a.s., we have that $g^2(X(t)) \le C^*(\omega) X(t)^{1+\theta} |f(X(t))|$, $t \ge 0$ for some bounded random variable C^*. Hence

$$J(t) \le C^* \int_0^t \frac{X(s)^{1+\theta}}{|f(X(s))|}\, ds, \quad t \ge 0.$$

Now by hypothesis there is a decreasing function η such that $\eta(x) \sim |f(x)|/x^{1+\theta}$ as $x \to 0^+$. Therefore, with

$$\bar{J}_1(t) := \int_0^t \frac{X(s)^{1+\theta}}{|f(X(s))|}\, ds,$$

we either have $\bar{J}_1(t) \to \infty$ as $t \to \infty$ or $\bar{J}_1(t) \to \bar{J}_1(\infty) < \infty$ on C' as $t \to \infty$. In the latter case $\limsup_{t \to \infty} J(t) \le \infty$, a contradiction. Hence $\bar{J}_1(t) \to \infty$ as $t \to \infty$ and so if

$$\bar{J}_2(t) := \int_0^t \frac{1}{\eta(X(s))}\, ds,$$

we have $J(t) \le C^* \bar{J}_1(t)$, so $J(t) \le C^{**} \bar{J}_2(t)$ for all $t \ge 0$ and some bounded random variable C^{**}. Next by Theorem 85, $F(X(t)) \sim t$ as $t \to \infty$, so we have for $t > T(\epsilon)$, $(1-\epsilon) \cdot t < F(X(t)) < (1+\epsilon) \cdot t$. Then $F^{-1}((1-\epsilon)t) > X(t) > F^{-1}((1+\epsilon)t)$. Now for $t \ge T(\epsilon)$, since

$$\frac{1}{\eta\left(F^{-1}\left((1-\epsilon)t\right)\right)} > \frac{1}{\eta\left(X(t)\right)} > \frac{1}{\eta\left(F^{-1}\left((1+\epsilon)t\right)\right)},$$

then for $t > T(\epsilon)$ we have

$$\bar{J}_2(t) \le \bar{J}_2(T(\epsilon)) + \int_{T(\epsilon)}^t \frac{1}{\eta\left(F^{-1}\left((1-\epsilon)s\right)\right)}\, ds.$$

Thus

$$\bar{J}_2(t) \le \bar{J}_2(T(\epsilon)) + \frac{1}{1-\epsilon} \int_{F^{-1}((1-\epsilon)t)}^{F^{-1}((1-\epsilon)T(\epsilon))} \frac{1}{\eta\,(u)} \cdot \frac{1}{|f(u)|}\, du,$$

and so for $t \geq T(\epsilon)$

$$J(t) \leq C^{**} \bar{J}_2(T) + \frac{C^{**}}{1-\epsilon} \int_{F^{-1}((1-\epsilon)t)}^{F^{-1}((1-\epsilon)T(\epsilon))} \frac{1}{\eta(u)} \cdot \frac{1}{|f(u)|} \, du,$$

and $I(t) = o(J(t))$ as $t \to \infty$. Next, as $\eta(x) \sim |f(x)|/x^{1+\theta}$ as $x \to 0^+$, we have for some $C'(\epsilon)$ and all $u < F^{-1}((1-\epsilon)T(\epsilon))$

$$\frac{1}{\eta(u)} < \frac{C'(\epsilon) u^{1+\theta}}{|f(u)|}.$$

Hence for $t \geq T(\epsilon)$, since $F^{-1}((1-\epsilon)t) > F^{-1}(t)$,

$$J(t) \;\leq\; C_1(\epsilon) + C_2(\epsilon) \int_{F^{-1}((1-\epsilon)t)}^{C_3(\epsilon)} \frac{u^{1+\theta}}{f^2(u)} \, du \leq C_1(\epsilon) + C_2(\epsilon) \int_{F^{-1}(t)}^{C_3(\epsilon)} \frac{u^{1+\theta}}{f^2(u)} \, du =: \bar{J}_3(t).$$

Now $I(t) = o(J(t))$ as $t \to \infty$, so if

$$\limsup_{t\to\infty} \left(\frac{f(F^{-1}(t))}{F^{-1}(t)} \bar{J}_3(t) \right) < \infty, \tag{13.22}$$

then

$$\lim_{x\to 0^+} \left(\frac{f(x)}{x} I(F(x)) \right) = 0,$$

and so by Lemma 46

$$\lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} = \lim_{t\to\infty} \frac{F^{-1}(t+I(t))}{F^{-1}(t)} = 1, \quad \text{a.s.},$$

as required. In fact it is clear that it suffices to prove

$$\limsup_{x\to 0^+} \left(\frac{|f(x)|}{x} \int_x^1 \frac{u^{1+\theta}}{|f(u)|^2} \, du \right) < \infty, \tag{13.23}$$

in order to prove (13.22). Finally since $x \mapsto |f(x)|/x$ is asymptotically increasing, we have that $|f(x)|/x \sim \phi_1(x)$ as $x \to 0^+$ where ϕ_1 is increasing. Thus for $x < x_1(\epsilon) < 1$,

$$\frac{1}{1+\epsilon} \cdot \phi_1(x) < \frac{|f(x)|}{x} < (1+\epsilon) \cdot \phi_1(x).$$

Let $x < x_1(\epsilon)$ and write

$$\frac{|f(x)|}{x} \int_x^1 \frac{u^{1+\theta}}{|f(u)|^2} \, du = \frac{|f(x)|}{x} \int_x^{x_1(\epsilon)} \frac{u^{1+\theta}}{|f(u)|^2} \, du + \frac{|f(x)|}{x} \int_{x_1(\epsilon)}^1 \frac{u^{1+\theta}}{|f(u)|^2} \, du.$$

The second term on the right has zero limit as $x \to 0^+$. As to the first, as $\phi_1(u) > \phi_1(x)$

for $u \in [x_1(\epsilon), x]$, then

$$\begin{aligned}\frac{|f(x)|}{x}\int_x^{x_1(\epsilon)} \frac{u^{1+\theta}}{|f(u)|^2}\,du &= \frac{|f(x)|}{x}\int_x^{x_1(\epsilon)} \frac{u^{\theta}}{|f(u)|}\cdot\frac{u}{|f(u)|}\,du \\ &\le \frac{|f(x)|}{x}\int_x^{x_1(\epsilon)} \frac{u^{\theta}}{|f(u)|}\cdot\frac{1+\epsilon}{\phi_1(u)}\,du \\ &\le (1+\epsilon)^2\int_x^{x_1(\epsilon)} \frac{u^{\theta}}{|f(u)|}\cdot\frac{\phi_1(x)}{\phi_1(u)}\,du \\ &\le (1+\epsilon)^2\int_x^{x_1(\epsilon)} \frac{u^{\theta}}{|f(u)|}\,du.\end{aligned}$$

Since $x \mapsto |f(x)|/x^{1+\theta'}$ is asymptotically decreasing for all $\theta' > 0$ sufficiently small, we have for each $\theta' > 0$ that there is $a_{\theta'}^+$, $b_{\theta'}^+$ such that $\forall\, x < b_{\theta'}^+$

$$\frac{|f(x)|}{x^{1+\theta'}} > a_{\theta'}^+.$$

Also as $f(x)/x \to 0$ as $x \to 0^+$ there exists $c_{\theta'}$ such that $\forall\, x < b_{\theta'}^+$, $|f(x)|/x < c_{\theta'}$. Thus for $x < b_{\theta'}^+$

$$a_{\theta'}^+ x^{1+\theta'} < |f(x)| < c_{\theta'} x.$$

Hence

$$\log\left(\frac{1}{a_{\theta'}^+}\right) + (1+\theta')\cdot\log\left(\frac{1}{x}\right) > \log\left(\frac{1}{|f(x)|}\right) > \log\left(\frac{1}{c_{\theta'}}\right) + \log\left(\frac{1}{x}\right),$$

so

$$1 \le \liminf_{x\to 0^+} \frac{\log(1/|f(x)|)}{\log(1/x)} \le \limsup_{x\to 0^+} \frac{\log(1/|f(x)|)}{\log(1/x)} \le 1+\theta'.$$

Then $\log\left(x^{\theta}/|f(x)|\right) \sim \log(1/x)(\theta - 1)$ as $x \to 0^+$ and since θ' can be chosen arbitrarily small

$$\lim_{x\to 0^+} \frac{\log(1/|f(x)|)}{\log(1/x)} = 1.$$

Thus $\lim_{x\to 0^+}\int_x^{x_1(\epsilon)} u^{\theta}/|f(u)|\,du < \infty$. Hence

$$\limsup_{x\to 0^+}\left(\frac{|f(x)|}{x}\int_x^1 \frac{u^{1+\theta}}{|f(u)|^2}\,du\right) < \infty,$$

as required of (13.23). Thus

$$\lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} = 1, \quad \text{a.s.},$$

as claimed. □

13.5 Asymptotic Behaviour of Power Pre-Transformed Scheme

In this section, our main goal is to show that using a power transformation enables us to recover the refined asymptotics for sub-exponential SDEs with small noise given in Theorem 87. This can largely be achieved, but in order to do so we generally impose some additional smoothness on f close to the equilibrium.

Recalling Theorem 87, we have in part (ii) that the small noise condition in conjunction with the assumption that $x \mapsto |f(x)|/x^{1+\theta}$ is increasing close to zero yields $X(t) \sim F^{-1}(t)$ as $t \to \infty$. This deals with SDEs where the non-linearity at the equilibrium is of "power type" or is exceptionally flat. Our first main result (Theorem 88) shows that this prevails for the discretisation of the pre-transformed SDE where we make the transformation $Z(t) = X(t)^{-\theta}$, discretise the SDE for Z and recover X in the discretisation by making the inversion, $X_n = Z_n^{-1/\theta}$, remembering to choose $X_n > 0$ by construction.

In the case when the small noise condition continues to hold but $x \mapsto |f(x)|/x^{1+\theta}$ is decreasing close to the equilibrium we have in Theorem 87 part (i) that $F(X(t)) \sim t$ as $t \to \infty$. This monotonicity assumption covers the case when the non-linearity at the equilibrium promotes sub-exponential and faster than polynomial decay. This result can be recovered for the power pre-transformed numerical method and a sketch of the main points is presented in Theorem 89. In fact, much of the machinery of Theorem 88 can be reused with the change in monotonicity assumption causing the reversal in sense of certain difference inequalities used in Theorem 88.

The second part of Theorem 87 shows that we can improve the result $F(X(t))/t \to 1$ as $t \to \infty$ under the sub-exponential hypotheses in part (i). This works if we are prepared to also request that $f \in C^1(0,\infty)$. We have not presented here an analogous theorem for the power numerical method but have shown in a forthcoming section that such an asymptotic numerical result can be established for the logarithmic pre-transformation with additional conditions which ensure $f \in RV_0(1)$. We anticipate that by following in broad terms the methods of this later result it should be possible to obtain an analogue of part (ii) of Theorem 87 in the power pre-transformed case.

Define $Z(t) := X(t)^{-\theta}$ where

$$dX(t) = f(X(t))dt + g(X(t))dB(t), \quad X(0) = \zeta.$$

Then by Itô's Lemma the SDE for Z is given by

$$\begin{aligned} dZ(t) &= -\theta X(t)^{-\theta-1}\left(f(X(t))dt + g(X(t))\,dB(t)\right) + \frac{\theta(\theta-1)X(t)^{-\theta-2}g^2(X(t))}{2}\,dt \\ &= \left(\frac{-\theta f(X(t))}{X(t)^{\theta+1}} + \frac{\theta(\theta+1)g^2(X(t))}{2X(t)^{\theta+2}}\right)dt - \frac{\theta g(X(t))}{X(t)^{\theta+1}}\,dB(t). \end{aligned}$$

Define the sequences (Z_n), (X_n) and (t_n) by $Z_0 = X_0^{-\theta}$, $X_0 = \zeta$, $t_0 = 0$ and

$$Z_{n+1} = Z_n + h(X_n)\left(\frac{-\theta f(X_n)}{X_n^{\theta+1}} + \frac{\theta(\theta+1)g^2(X_n)}{2X_n^{\theta+2}}\right) + \sqrt{h(X_n)}\cdot\frac{\theta g(X_n)}{X_n^{\theta+1}}\cdot\xi_{n+1}, \quad n \geq 0,$$
$$X_{n+1} = Z_{n+1}^{-1/\theta}, \quad n \geq 0,$$
$$t_{n+1} = t_n + \tilde{h}(X_n), \quad n \geq 0,$$

where
$$h(x) = \min\left(\Delta, \frac{\Delta x}{|f(x)|}, \frac{\Delta x^2}{g^2(x)}\right).$$

Assume

$$f(x) < 0, \forall\, x > 0,\; f(0) = 0,\; g(x) > 0,\; \forall\, x > 0,\; g(0) = 0; \tag{13.24}$$
$$\theta \text{ is chosen so that } x^{-1/\theta} \in [0,\infty)\, \forall\, x \in \mathbb{R}; \tag{13.25}$$
$$\limsup_{x\to 0^+} \frac{g^2(x)}{x^{1+\theta}|f(x)|} < \infty; \tag{13.26}$$

and

$$|f(x)| \sim \phi(x) \text{ as } x \to 0^+,\; x \mapsto \phi(x)/x^{1+\theta} \text{ is increasing,}$$
$$\phi(x)/x^{1+\theta} \to 0 \text{ as } x \to 0^+,\; \phi \in C((0,\infty),(0,\infty)), \tag{13.27}$$

and

$$|f(x)| \sim \phi(x) \text{ as } x \to 0^+,\; \phi(0) = 0,\; \phi(x)/x \to 0 \text{ as } x \to 0^+,$$
$$\phi(x)/x^{1+\theta} \text{ is decreasing, } \phi \in C^1((0,\infty);(0,\infty)). \tag{13.28}$$

Theorem 88. *Assume* (13.24), (13.25), (13.26) *and* (13.27). *Then*

$$\lim_{n\to\infty} \frac{X_n}{F^{-1}(t_n)} = \lim_{n\to\infty} \frac{X_n}{F^{-1}(n\Delta)} = 1, \quad a.s, \tag{13.29}$$

holds.

Proof. Define for $n \geq 0$

$$D_n := h(X_n)\left(\frac{\theta|f(X_n)|}{X_n^{\theta+1}} + \frac{\theta(\theta+1)g^2(X_n)}{2X_n^{\theta+2}}\right) \quad \text{and} \quad T_{n+1} := \frac{\theta\sqrt{h(X_n)}g(X_n)}{X_n^{\theta+1}}\xi_{n+1} = \eta_n\,\xi_{n+1}.$$

Then for $n \geq 1$

$$Z_n = Z_0 + \sum_{j=0}^{n-1} D_j + \sum_{j=0}^{n-1} T_{j+1} = Z_0 + \sum_{j=0}^{n-1} D_j + M(n),$$

where $M(n) := \sum_{j=0}^{n-1} T_{j+1}$ is a martingale with quadratic variation

$$\langle M \rangle (n) = \sum_{j=0}^{n-1} \frac{\theta^2 h(X_j) g^2(X_j)}{X_j^{2\theta+2}}.$$

M is a martingale by the consdierations of Lemma 40, provided the increment of the quadratic variation is bounded which we now show. Note if $x \geq 1$, $h(x) \leq \Delta x^2/g^2(x)$ so for $X_j \geq 1$

$$\frac{\theta^2 h(X_j) g^2(X_j)}{X_j^{2\theta+2}} \leq \theta^2 \cdot \frac{\Delta X_j^2}{g^2(X_j)} \cdot \frac{g^2(X_j)}{X_j^{2\theta+2}} \leq \theta^2 X_j^{-2\theta} \leq \Delta\theta^2.$$

For $x \leq 1$, $|f(x)|/x^{1+\theta} < K_3$, $\phi(x)/x^{1+\theta} \leq K_4$ and $g^2(x) \leq K_2 x^{1+\theta}|f(x)|$. Thus for $X_j < 1$

$$\begin{aligned} \frac{\theta^2 h(X_j) g^2(X_j)}{X_j^{2\theta+2}} \leq \frac{\theta^2 h(X_j)}{X_j^{2\theta+2}} \cdot K_2 X_j^{1+\theta}|f(X_j)| &= \theta^2 K_2 h(X_j) \cdot \frac{|f(X_j)|}{X_j^{1+\theta}} \\ &\leq \theta^2 K_2 K_3 h(X_j) \leq \theta^2 K_2 K_3 \Delta. \end{aligned}$$

Thus, there is $K_5 := K_2 K_3 > 0$ independent of Δ such that $h(X_j)g^2(X_j)/X_j^{2\theta+2} \leq \theta^2 K_5 \Delta$. For $X_n \geq 1$,

$$D_n \geq \frac{\theta(\theta+1) g^2(X_n) h(X_n)}{2X_n^{\theta+2}} \geq \frac{\theta(\theta+1) g^2(X_n) h(X_n)}{2X_n^{2\theta+2}} = \frac{(\theta+1)}{2\theta}\eta_n^2,$$

and for $X_n < 1$, $g^2(X_n) \leq K_2 X_n^{1+\theta}|f(X_n)|$. Hence

$$\eta_n^2 = \frac{\theta^2 h(X_n) g^2(X_n)}{X_n^{2\theta+2}} \leq \frac{K_2 \theta^2 h(X_n) f(X_n)}{X_n^{1+\theta}} \quad \text{and} \quad D_n \geq \frac{\theta h(X_n)|f(X_n)|}{X_n^{\theta+1}},$$

so $\eta_n^2 \leq K_2 \theta D_n$. Therefore for all $n \in \mathbb{N}$

$$\eta_n^2 \leq \max\left(K_2\theta, \frac{2\theta}{\theta+1}\right) D_n =: K_6 D_n.$$

Thus if

(i) $\sum_{j=0}^{n-1} D_j$ diverges, then $\langle M \rangle (n) \leq K_6 \sum_{j=0}^{n-1} D_j$ and either $\langle M \rangle (n)$ is convergent in which case $M(n)$ converges and hence $M(n)/\sum_{j=0}^{n-1} D_j \to 0$ or $\langle M \rangle (n)$ diverges in which case $M(n)/\langle M \rangle (n) \to 0$ as $n \to \infty$ and $M(n)/\sum_{j=0}^{n-1} D_j \to 0$ as $n \to \infty$. Hence

$$\lim_{n\to\infty} \frac{Z_n}{\sum_{j=0}^{n-1} D_j} = 1.$$

(ii) If $\sum_{j=0}^{n-1} D_j$ converges, then $\langle M \rangle (n) \leq K_6 \sum_{j=0}^{n-1} D_j$ converges and so $M(n)$ tends to a finite limit. Hence Z_n tends to a finite limit and so $X_n \to X_\infty \in (0, \infty)$.

Then

$$\lim_{n\to\infty} D_n = h(X_\infty)\left(\frac{\theta|f(X_\infty)|}{X_\infty^{\theta+1}} + \frac{\theta(\theta+1)g^2(X_\infty)}{2X_\infty^{\theta+2}}\right) =: D_\infty > 0,$$

and this means that $\sum_{j=0}^{n-1} D_j \to \infty$ as $n \to \infty$, a contradiction.

Therefore $\sum_{j=0}^{n-1} D_j \to \infty$ as $n \to \infty$, $Z_n \to \infty$ as $n \to \infty$ and $X_n \to 0$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{Z_n}{\sum_{j=0}^{n-1} D_j} = 1, \quad \text{a.s..}$$

Since $X_n \to 0$ and $g^2(x)/(x^{1+\theta}|f(x)|) \le K_2$, $\forall\, x \le 1$ then $x/|f(x)| \to \infty$ as $x \to 0^+$ and

$$\lim_{x\to 0^+} \frac{g^2(x)}{x|f(x)|} = \lim_{x\to 0^+}\left(\frac{g^2(x)}{x^{1+\theta}|f(x)|}\cdot x^\theta\right) = 0,$$

so $x^2/g^2(x) \to \infty$ as $x \to 0^+$. Therefore $h(X_n) = \Delta$ for all n sufficiently large and

$$D_n = \Delta\left(\frac{\theta|f(X_n)|}{X_n^{\theta+1}} + \frac{\theta(\theta+1)g^2(X_n)}{2X_n^{\theta+2}}\right) \sim \frac{\Delta\theta|f(X_n)|}{X_n^{\theta+1}}, \quad \text{as } n \to \infty.$$

Thus

$$\lim_{n\to\infty} \frac{X_n^{-\theta}}{\sum_{j=0}^{n-1} \Delta\theta|f(X_j)|/X_j^{1+\theta}} = 1, \quad \text{a.s.,} \tag{13.30}$$

and for $n \ge N$

$$t_n = t_N + (n-N)\Delta. \tag{13.31}$$

By hypothesis $\tilde{\eta}(x)$ is the increasing function where $\tilde{\eta}(x) \sim \Delta\theta|f(x)|/x^{1+\theta}$ as $x \to 0^+$. Then

$$\lim_{n\to\infty} \frac{X_n^{-\theta}}{\sum_{j=0}^{n-1} \tilde{\eta}(X_j)} = 1, \quad \text{a.s..}$$

Define $S_n := \sum_{j=0}^{n} \tilde{\eta}(X_j)$, $n \ge 1$. Thus $S_n - S_{n-1} = \tilde{\eta}(X_n)$ so $\tilde{\eta}^{-1}(S_n - S_{n-1}) = X_n$. Hence

$$\lim_{n\to\infty} \frac{\left(\tilde{\eta}^{-1}(S_n - S_{n-1})\right)^{-\theta}}{S_{n-1}} = 1 \quad \text{or} \quad \lim_{n\to\infty} \frac{\left(\tilde{\eta}^{-1}(S_{n+1} - S_n)\right)^{-\theta}}{S_n} = 1.$$

Hence for every $\epsilon \in (0,1)$ there is an $N(\epsilon)$ such that for all $n \ge N(\epsilon)$

$$1-\epsilon < \frac{\left(\tilde{\eta}^{-1}(S_{n+1} - S_n)\right)^{-\theta}}{S_n} < 1+\epsilon, \tag{13.32}$$

or $\tilde{\eta}\left((1-\epsilon)^{-1/\theta} S_n^{-1/\theta}\right) > S_{n+1} - S_n > \tilde{\eta}\left((1+\epsilon)^{-1/\theta} S_n^{-1/\theta}\right)$. Thus for $n \ge N(\epsilon)$

$$S_n + \tilde{\eta}\left((1+\epsilon)^{-1/\theta} S_n^{-1/\theta}\right) < S_{n+1} < S_n + \tilde{\eta}\left((1-\epsilon)^{-1/\theta} S_n^{-1/\theta}\right). \tag{13.33}$$

Set $\phi_a(x) := \tilde{\eta}(ax^{-1/\theta})$. Let $\Phi_a(x) := \int_1^x 1/\phi_a(u)\,du$. Then by the Mean Value Theorem there is $\theta_n \in (0,1)$ such that

$$\begin{aligned}\Phi_a(S_{n+1}) &> \Phi_a\left(S_n + \tilde{\eta}\left((1+\epsilon)^{-1/\theta} S_n^{-1/\theta}\right)\right)\\ &= \Phi_a\left(S_n\right) + \Phi_a'\left(S_n + \theta_n\tilde{\eta}\left((1+\epsilon)^{-1/\theta} S_n^{-1/\theta}\right)\right)\cdot\tilde{\eta}\left((1+\epsilon)^{-1/\theta} S_n^{-1/\theta}\right).\end{aligned}$$

Thus

$$\Phi_a(S_{n+1}) - \Phi_a(S_n) > \frac{\tilde{\eta}\left((1+\epsilon)^{-1/\theta} S_n^{-1/\theta}\right)}{\phi_a\left(S_n + \theta_n\tilde{\eta}\left((1+\epsilon)^{-1/\theta} S_n^{-1/\theta}\right)\right)}.$$

Let $a := a_\epsilon := (1+\epsilon)^{-1/\theta}$. Then

$$\Phi_{a_\epsilon}(S_{n+1}) - \Phi_{a_\epsilon}(S_n) > \frac{\tilde{\eta}\left(a_\epsilon S_n^{-1/\theta}\right)}{\tilde{\eta}\left(a_\epsilon\left(S_n + \theta_n\tilde{\eta}\left(a_\epsilon S_n^{-1/\theta}\right)\right)^{-1/\theta}\right)}.$$

Since $S_n + \theta_n\tilde{\eta}\left(a_\epsilon S_n^{-1/\theta}\right) > S_n$ then $\tilde{\eta}\left(a_\epsilon\left(S_n + \theta_n\tilde{\eta}\left(a_\epsilon S_n^{-1/\theta}\right)\right)^{-1/\theta}\right) < \tilde{\eta}\left(a_\epsilon S_n^{-1/\theta}\right)$. Therefore $\Phi_{a_\epsilon}\left(S_{n+1}\right) - \Phi_{a_\epsilon}\left(S_n\right) > 1$ so

$$\liminf_{n\to\infty}\frac{\Phi_{a_\epsilon}\left(S_n\right)}{n} \geq 1.$$

Thus

$$\liminf_{n\to\infty}\frac{1}{n}\int_1^{S_n}\frac{1}{\tilde{\eta}\left(a_\epsilon u^{-1/\theta}\right)}\,du \geq 1.$$

Now $\tilde{\eta}\left(a_\epsilon u^{-1/\theta}\right) \sim \Delta\theta|f(a_\epsilon u^{-1/\theta})|/\left(a_\epsilon u^{-1/\theta}\right)^{1+\theta} = \Delta\theta|f(a_\epsilon u^{-1/\theta})|/(a_\epsilon^{1+\theta}u^{-(1+\theta)/\theta})$ as $x \to 0^+$. Hence

$$\liminf_{n\to\infty}\frac{1}{n}\int_1^{S_n}\frac{a_\epsilon^{1+\theta}u^{-(1+\theta)/\theta}}{\Delta\theta|f(a_\epsilon u^{-1/\theta})|}\,du \geq 1.$$

Let $v = a_\epsilon u^{-1/\theta}$. Hence $u = \left(a_\epsilon/v\right)^\theta$ and so

$$\liminf_{n\to\infty}\frac{1}{n\Delta}\int_{a_\epsilon S_n^{-1/\theta}}^{a_\epsilon}\frac{a_\epsilon^\theta}{|f(v)|}\,dv \geq 1,$$

or

$$\liminf_{n\to\infty}\frac{a_\epsilon^\theta}{t_n}\int_{a_\epsilon S_n^{-1/\theta}}^{a_\epsilon}\frac{1}{|f(v)|}\,dv \geq 1.$$

Hence

$$\liminf_{n\to\infty}\frac{F\left((1+\epsilon)^{-1/\theta} S_n^{-1/\theta}\right)}{t_n} = \liminf_{n\to\infty}\frac{F\left(a_\epsilon S_n^{-1/\theta}\right)}{t_n} \geq a_\epsilon^{-\theta} = 1+\epsilon.$$

Also from (13.32), $(1-\epsilon) \cdot S_n < X_{n+1}^{-\theta} < (1+\epsilon) \cdot S_n$ so

$$(1-\epsilon)^{-1/\theta} \cdot S_n^{-1/\theta} > X_{n+1} > (1+\epsilon)^{-1/\theta} \cdot S_n^{-1/\theta}.$$

Suppose $CX_{n+1} < S_n^{-1/\theta} \cdot (1+\epsilon)^{-1/\theta}$; then $F(CX_{n+1}) > F\left(S_n^{-1/\theta}(1+\epsilon)^{-1/\theta}\right)$. Since $X_{n+1} < (1-\epsilon)^{-1/\theta} S_n^{-1/\theta}$, so $(1-\epsilon)^{1/\theta} X_{n+1} < S_n^{-1/\theta}$ and $\left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/\theta} X_{n+1} < (1+\epsilon)^{-1/\theta} \cdot S_n^{-1/\theta}$. Hence we may take $C = \left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/\theta}$ so

$$\liminf_{n\to\infty} \frac{F\left(\left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/\theta} X_{n+1}\right)}{t_n} \geq 1+\epsilon.$$

Since $t_{n+1} \sim t_n$ as $n \to \infty$ then

$$\liminf_{n\to\infty} \frac{F\left(\left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/\theta} X_{n}\right)}{t_n} \geq 1+\epsilon.$$

Therefore for $\nu = \nu(\epsilon)$ such that $(1+\epsilon)\cdot(1-\nu) = 1$. Then for n sufficiently large we have

$$\frac{F\left(\left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/\theta} X_{n}\right)}{t_n} > (1+\epsilon)\cdot(1-\nu) = 1.$$

Hence $F\left(\left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/\theta} X_n\right) > t_n$. Therefore $\left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/\theta} X_n < F^{-1}(t_n)$. Thus for n sufficiently large, $X_n/F^{-1}(t_n) < \left(\frac{1+\epsilon}{1-\epsilon}\right)^{1/\theta}$. Therefore

$$\limsup_{n\to\infty} \frac{X_n}{F^{-1}(t_n)} \leq 1.$$

We now obtain the corresponding lower bound. Let $a_{-\epsilon} := (1-\epsilon)^{-1/\theta}$. Using the Mean Value Theorem applied to the right-hand side of (13.33)

$$\Phi_{a_{-\epsilon}}(S_{n+1}) < \Phi_{a_{-\epsilon}}(S_n) + \Phi'_{a_{-\epsilon}}\left(S_n + \theta_n \tilde{\eta}\left(a_{-\epsilon} S_n^{-1/\theta}\right)\right) \cdot \tilde{\eta}\left(a_{-\epsilon} S_n^{-1/\theta}\right),$$

then

$$\Phi_{a_{-\epsilon}}(S_{n+1}) - \Phi_{a_{-\epsilon}}(S_n) \quad < \quad \frac{\tilde{\eta}\left(a_{-\epsilon} S_n^{-1/\theta}\right)}{\tilde{\eta}\left(a_{-\epsilon}\left(S_n + \theta_n \tilde{\eta}\left(a_{-\epsilon} S_n^{-1/\theta}\right)\right)^{-1/\theta}\right)}.$$

Since $\theta_n \in (0,1)$ then $S_n + \theta_n \tilde{\eta}\left(a_{-\epsilon} S_n^{-1/\theta}\right) \leq S_n + \tilde{\eta}\left(a_{-\epsilon} S_n^{-1/\theta}\right)$ and

$$\tilde{\eta}\left(a_{-\epsilon}\left(S_n + \theta_n \tilde{\eta}\left(a_{-\epsilon} S_n^{-1/\theta}\right)\right)^{-1/\theta}\right) \geq \tilde{\eta}\left(a_{-\epsilon}\left(S_n + \tilde{\eta}\left(a_{-\epsilon} S_n^{-1/\theta}\right)\right)^{-1/\theta}\right).$$

Thus with $y_n := a_{-\epsilon} S_n^{-1/\theta}$,

$$\Phi_{a_{-\epsilon}}(S_{n+1}) - \Phi_{a_{-\epsilon}}(S_n) \leq \frac{\tilde{\eta}\left(a_{-\epsilon} S_n^{-1/\theta}\right)}{\tilde{\eta}\left(a_{-\epsilon}\left(S_n + \tilde{\eta}\left(a_{-\epsilon} S_n^{-1/\theta}\right)\right)^{-1/\theta}\right)} = \frac{\tilde{\eta}(y_n)}{\tilde{\eta}\left(a_{-\epsilon}\left(S_n + \tilde{\eta}\left(y_n\right)\right)^{-1/\theta}\right)}.$$

Thus

$$\Phi_{a_{-\epsilon}}(S_{n+1}) - \Phi_{a_{-\epsilon}}(S_n) \leq \frac{\tilde{\eta}(y_n)}{\tilde{\eta}\left(a_{-\epsilon}\left(a_{-\epsilon}^{\theta}/y_n^{\theta} + \tilde{\eta}(y_n)\right)^{-1/\theta}\right)}.$$

We now seek to prove for $a = a_{-\epsilon} > 0$

$$\lim_{x \to 0^+} \frac{\tilde{\eta}(x)}{\tilde{\eta}\left(a\left(a^{\theta}/y_n^{\theta} + \tilde{\eta}\left(x\right)\right)^{-1/\theta}\right)} = 1. \tag{13.34}$$

Thus

$$a \cdot \left(\frac{a^{\theta}}{x^{\theta}} + \tilde{\eta}\left(x\right)\right)^{-1/\theta} = a \cdot \left(\frac{a^{\theta}}{x^{\theta}}\left(1 + \frac{x^{\theta}\tilde{\eta}(x)}{a^{\theta}}\right)\right)^{-1/\theta} = x\left(1 + \frac{x^{\theta}\tilde{\eta}(x)}{a^{\theta}}\right)^{-1/\theta},$$

(13.34) is equivalent to

$$\lim_{x \to 0} \frac{\tilde{\eta}(x)}{\tilde{\eta}\left(x\left(1 + x^{\theta}\tilde{\eta}(x)/a^{\theta}\right)^{-1/\theta}\right)} = 1. \tag{13.35}$$

Recall $\tilde{\eta}(x) \sim \Delta\theta|f(x)|/x^{1+\theta}$ as $x \to 0^+$ so $x^{\theta}\tilde{\eta}(x) \sim \Delta\theta|f(x)|/x \to 0$ as $x \to 0^+$. Define $c(x) := x^{\theta}\tilde{\eta}(x)/a^{\theta}$. Then for $\xi(x) \in [(1 + c(x))^{-1/\theta}, 1]$ then

$$\begin{aligned}
\frac{\tilde{\eta}(x) - \tilde{\eta}(x(1 + c(x))^{-1/\theta})}{\tilde{\eta}(x)} &= \frac{\tilde{\eta}'\left(\xi(x)x\right) \cdot x(1 - (1 + c(x))^{-1/\theta})}{\tilde{\eta}(x)} \\
&= \tilde{\eta}'\left(\xi(x)x\right) \cdot \xi(x)x \cdot \frac{1}{\xi(x)} \cdot \frac{(1 - (1 + c(x))^{-1/\theta})}{\tilde{\eta}(x)}.
\end{aligned}$$

Since $c(x) \to 0$ as $x \to 0^+$ and $1 - (1 + h)^{-1/\theta} \sim h/\theta$ as $h \to 0^+$. Hence as $x \to 0^+$

$$1 - (1 + c(x))^{-1/\theta} \sim \frac{c(x)}{\theta} = \frac{1}{\theta} \cdot \frac{x^{\theta}\tilde{\eta}(x)}{a^{\theta}}.$$

Thus $(1 - (1 + c(x))^{-1/\theta})/\tilde{\eta}(x) \sim x^{\theta}/(\theta a^{\theta})$ as $x \to 0^+$. Therefore

$$\begin{aligned}
1 - \frac{\tilde{\eta}\left(x(1 + c(x))^{-1/\theta}\right)}{\tilde{\eta}\left(x\right)} &\sim \tilde{\eta}'\left(\xi(x)x\right) \cdot \xi(x)x \cdot \frac{1}{\xi(x)^{1+\theta}} \cdot \frac{x^{\theta}\xi(x)^{\theta}}{a^{\theta}\theta} \\
&= \tilde{\eta}'\left(\xi(x)x\right) \cdot \left(\xi(x)x\right)^{1+\theta} \cdot \frac{1}{a^{\theta}\theta} \cdot \frac{1}{\xi(x)^{1+\theta}}.
\end{aligned}$$

Finally we show that $y^{1+\theta}\tilde{\eta}'(y) \to 0$ as $y \to 0^+$, which will yield (13.35) with

$$\tilde{\eta}(x) = \theta\Delta\frac{\phi(x)}{x^{1+\theta}}, \quad x > 0,$$

where $\tilde{\eta}$ is increasing C^1 away from zero. By construction $\phi(x) = \theta\Delta x^{1+\theta}\tilde{\eta}(x)$, $\tilde{\eta}(x) \to 0$ as $x \to 0$. Also since $f(x)/x \to 0$ then $\phi(x)/x \to 0$ and $\phi'(0^+) = 0$. Thus $\phi'(x)$ exists for all x and $\phi'(0^+) = 0$. We have for $x > 0$:

$$\tilde{\eta}'(x) = \theta\Delta\left(\frac{\phi'(x)}{x^{1+\theta}} - (1+\theta)\frac{\phi(x)}{x^{2+\theta}}\right).$$

Hence $x^{1+\theta}\tilde{\eta}'(x) = \theta\Delta(\phi'(x) - (1+\theta)\phi(x)/x) \to 0$ as $x \to 0^+$. Now $\phi'(0^+) = 0$ so $\phi'(x) \to 0$ as $x \to 0$ because $\phi \in C^1$. Thus $x^{1+\theta}\tilde{\eta}'(x) \to 0$ as $x \to 0^+$ as needed. Therefore returning to (13.34) with $a = a_{-\epsilon}$, we see that

$$\limsup_{n\to\infty} \frac{\Phi_{a_{-\epsilon}}(S_n)}{n} \le 1 \quad \text{or} \quad \limsup_{n\to\infty} \frac{1}{n}\int_1^{S_n} \frac{1}{\tilde{\eta}\left(a_{-\epsilon}u^{-1/\theta}\right)}\, du \le 1.$$

As for the lim inf we get

$$\limsup_{n\to\infty} \frac{a_{-\epsilon}^{\theta}}{t_n}\int_{a_{-\epsilon}S_n^{-1/\theta}}^{a_{-\epsilon}} \frac{1}{|f(v)|}\, dv \le 1.$$

Thus as $a_{-\epsilon} = (1-\epsilon)^{-1/\theta}$ then

$$\limsup_{n\to\infty} \frac{F\left((1-\epsilon)^{-1/\theta}S_n^{-1/\theta}\right)}{t_n} \le 1-\epsilon.$$

Also from (13.32), $(1-\epsilon)\cdot S_n < X_{n+1}^{-\theta} < (1+\epsilon)\cdot S_n$ so $(1-\epsilon)^{-1/\theta}\cdot S_n^{-1/\theta} > X_{n+1} > (1+\epsilon)^{-1/\theta}\cdot S_n^{-1/\theta}$. Then $(1-\epsilon)^{-1/\theta}\cdot(1+\epsilon)^{1/\theta}\cdot X_{n+1} > (1-\epsilon)^{-1/\theta}\cdot S_n^{-1/\theta}$ or $\left(\frac{1+\epsilon}{1-\epsilon}\right)^{1/\theta} X_{n+1} > (1-\epsilon)^{-1/\theta}S_n^{-1/\theta}$. Therefore $F\left(\left(\frac{1+\epsilon}{1-\epsilon}\right)^{1/\theta} X_{n+1}\right) < F\left((1-\epsilon)^{-1/\theta}S_n^{-1/\theta}\right)$ so

$$\limsup_{n\to\infty} \frac{F\left(\left(\frac{1+\epsilon}{1-\epsilon}\right)^{1/\theta} X_{n+1}\right)}{t_n} \le 1-\epsilon.$$

Since $t_n \sim t_{n+1}$ as $n \to \infty$ then

$$\limsup_{n\to\infty} \frac{F\left(\left(\frac{1+\epsilon}{1-\epsilon}\right)^{1/\theta} X_n\right)}{t_n} \le 1-\epsilon.$$

Thus for any $\nu > 0$ and $n \ge N(\epsilon, \nu)$

$$\frac{F\left(\left(\frac{1+\epsilon}{1-\epsilon}\right)^{1/\theta} X_n\right)}{t_n} \le (1-\epsilon)\cdot(1+\nu).$$

Choose ν such that $(1-\epsilon)\cdot(1+\nu) = 1$. Then $\forall n > N(\epsilon, \nu(\epsilon)) =: N_2(\epsilon)$ then $F\left(\left(\frac{1+\epsilon}{1-\epsilon}\right)^{1/\theta} X_n\right) < t_n$. Hence $\left(\frac{1+\epsilon}{1-\epsilon}\right)^{1/\theta} X_n > F^{-1}(t_n)$. Therefore

$$\liminf_{n\to\infty} \frac{X_n}{F^{-1}(t_n)} \geq \left(\frac{1-\epsilon}{1+\epsilon}\right)^{1/\theta}.$$

Letting $\epsilon \to 0^+$ yields

$$\liminf_{n\to\infty} \frac{X_n}{F^{-1}(t_n)} \geq 1.$$

Combining with the lim sup gives the result, as claimed. □

Theorem 89. *Assume* (13.24), (13.25), (13.26) *and* (13.28). *Then*

$$\lim_{n\to\infty} \frac{F(X_n)}{t_n} = 1, \quad a.s.,$$

holds.

Proof. We can repeat the calculations at the start of Theorem 88 as far as the inequality (13.32) viz, for every $\epsilon \in (0,1)$, there is $N(\epsilon) := N(\epsilon, \omega) \in \mathbb{N}$ such that $\forall n \geq N(\epsilon)$

$$1-\epsilon < \frac{\left(\tilde{\eta}^{-1}\left(S_{n+1}-S_n\right)\right)^{-\theta}}{S_n} < 1+\epsilon. \tag{13.36}$$

Note here that $\tilde{\eta}$ is monotone but in contrast to the situation in Theorem 88, $\tilde{\eta}$ is decreasing. Furthermore, $X_n = \tilde{\eta}^{-1}(S_{n+1}-S_n)$ just as in Theorem 88. The point of departure of this proof from Theorem 88 is at (13.33), where now the fact $\tilde{\eta}$ is decreasing forces

$$S_n + \tilde{\eta}\left((1-\epsilon)^{-1/\theta} S_n^{-1/\theta}\right) < S_{n+1} < S_n + \tilde{\eta}\left((1+\epsilon)^{-1/\theta} S_n^{-1/\theta}\right). \tag{13.37}$$

Putting $a_\epsilon := (1+\epsilon)^{-1/\theta}$ and taking Taylor expansions in the upper inequality in (13.37) as in the argument immediately following (13.33), we see that

$$\Phi_{a_\epsilon}(S_{n+1}) - \Phi_{a_\epsilon}(S_n) \leq 1,$$

where the fact that $\tilde{\eta}$ is decreasing has been explained. This leads to

$$\limsup_{n\to\infty} \frac{\Phi_{a_\epsilon}(S_n)}{n} \leq 1$$

which following the method of calculation in Theorem 88, leads to

$$\limsup_{n\to\infty} \frac{F\left((1+\epsilon)^{-1/\theta} S_n^{-1/\theta}\right)}{t_n} \leq 1+\epsilon.$$

From (13.36) we have that $(1+\epsilon)^{-1/\theta} \cdot S_n^{-1/\theta} < X_{n+1} < (1-\epsilon)^{-1/\theta} \cdot S_n^{-1/\theta}$ from which we get directly

$$\limsup_{n\to\infty} \frac{F\left(X_{n+1}\right)}{t_{n+1}} \leq 1+\epsilon,$$

and letting $\epsilon \to 0^+$ gives the upper bound

$$\limsup_{n\to\infty} \frac{F\left(X_{n}\right)}{t_{n}} \leq 1. \tag{13.38}$$

To get the corresponding lower bound, put $a_\epsilon = (1-\epsilon)^{-1/\theta}$ and take Taylor expansions in the lower inequality in (13.37). This gives

$$\Phi_{a_\epsilon}(S_{n+1}) - \Phi_{a_\epsilon}(S_n) > \frac{\tilde{\eta}(y_n)}{\tilde{\eta}\left(a_\epsilon\left(a_\epsilon^\theta/y_n^\theta + \tilde{\eta}(y_n)\right)^{-1/\theta}\right)},$$

where $y_n = a_\epsilon S_n^{-1/\theta} \to 0$ as $n \to \infty$. The quantity on the right-hand side tends to unity as $n \to \infty$, because the limit in (13.34) still holds. Therefore, we arrive at

$$\liminf_{n\to\infty} \frac{\Phi_{a_\epsilon}(S_n)}{n} \geq 1.$$

This implies, after the usual calculations

$$\liminf_{n\to\infty} \frac{F\left((1-\epsilon)^{-1/\theta} S_n^{-1/\theta}\right)}{t_n} \geq 1-\epsilon.$$

From which we proceed as before to get

$$\liminf_{n\to\infty} \frac{F\left(X_{n+1}\right)}{t_{n+1}} \geq 1-\epsilon,$$

and letting $\epsilon \to 0^+$ yields the lower bound

$$\liminf_{n\to\infty} \frac{F\left(X_{n}\right)}{t_{n}} \geq 1.$$

Combining this with (13.38) yields the result. □

13.6 Polynomial and Sub-Polynomial Decay in Logarithmically Pre-Transformed Scheme

In the previous section it was shown that a power pre-transformation and subsequent discretisation preserved the asymptotic behaviour of sub-exponentially stable SDEs, subject to the small noise condition. However, the choice of power transformation depends on a parameter θ whose value must be inferred from the behaviour of f and g in the neighbourhood of the equilibrium. Therefore, it is pertinent to ask whether we can choose a pre-transformation which does not depend so directly on the structure of the SDE. In this section, we show that a logarithmic pre-transformation performs acceptably, recovering the desired asymptotic rates of decay under the small noise condition. However, in order for this to happen we find it necessary to impose some further control condition on the drift. These conditions are more restrictive than those needed in the case of power transformations. Nevertheless for most drift coefficients that promote sub-exponential decay and possess nice regularity properties, such as regular or rapid variation, these conditions do not represent a practical limitation.

Our first main result is an analogue of Theorem 88 part (iii). However, the desired asymptotic behaviour is recorded under a condition on the solution and noise term which is impossible to verify *a priori*. We will shortly establish sufficient deterministic conditions which can be readily checked in advance of simulations which imply this technical condition.

Theorem 90. *Let f, g be continuous with $f(0) = g(0) = 0$. Suppose $f(x) < 0$, $g^2(x) > 0\,\forall\, x > 0$ and there exists $\theta > 0$ such that*

$$f(x)/x \to 0 \text{ as } x \to 0^+, \quad \sup_{x\in[0,\delta]} \frac{g^2(x)}{x^{1+\theta}|f(x)|} < \infty,$$
$$|f(x)| \sim \phi(x) \text{ as } x \to 0^+,\ \phi \in C^1,\ \phi(0) = 0,$$
$$x \mapsto \phi(x)/x^{1+\theta} \text{ is increasing and } \phi(x)/x^{1+\theta} \to 0 \text{ as } x \to 0^+.$$

Then

$$\lim_{n\to\infty} \left(\frac{g^3(X_n)}{|f(X_n)|X_n^2} \cdot \xi_{n+1}^3 \right) = 0, \tag{13.39}$$

implies

$$\lim_{n\to\infty} \frac{X_n}{F^{-1}(t_n)} = 1, \quad a.s..$$

Remark 48. Under the conditions of Theorem 90 we see that

$$\lim_{x\to 0^+} \frac{g^2(x)}{x|f(x)|} = \lim_{x\to 0^+} \left(\frac{g^2(x)}{x^{1+\theta}|f(x)|} \cdot x^\theta \right) = 0,$$

and

$$x \mapsto \frac{|f(x)|}{x} = \frac{|f(x)|}{x^{1+\theta}} \cdot x^\theta \quad \text{is asymptotically increasing,}$$

so by Theorem 90

$$\lim_{t\to\infty} \frac{-\log X_n}{(-\log\circ F^{-1})(t_n)} = 1, \quad \text{a.s..}$$

□

We now give a deterministic, sufficient condition which implies (13.39) and thereby enables Theorem 90 to be used reliably.

Theorem 91. *Suppose that the conditions of Theorem 90 hold and that*

$$\lim_{y\to 0^+}\left(\left(\frac{f(y)}{y}\right)^{1/3} y^{\theta}(\log\circ F)\left(y^{1+\epsilon}\right)\right)=0,\quad \forall\,\epsilon>0. \tag{13.40}$$

Then

$$\lim_{t\to\infty}\frac{X_n}{F^{-1}(t_n)}=1,\quad a.s..$$

Proof. By virtue of Theorem 90, it suffices to show that condition (13.40) and the deterministic conditions on f and g imply (13.39). We start by observing that for every $\epsilon\in(0,1)$

$$1-\epsilon<\frac{-\log X_n}{(-\log\circ F^{-1})(n\Delta)}<1+\epsilon,\quad \forall\, n\geq N_1(\epsilon).$$

Therefore $X_n<F^{-1}(n\Delta)^{1-\epsilon}$, $\forall\, n\geq N_1(\epsilon)$. Notice also that the iid sequence of Standard Normal random variables (ξ_n) obeys

$$\limsup_{n\to\infty}\frac{|\xi_{n+1}|}{\sqrt{2\log n}}=1,\quad \text{a.s.},$$

so we have for $n\geq N_2(\omega)$ that $|\xi_{n+1}|<2\sqrt{\log(n\Delta)}$. Also by the small noise condition it follows that $g^2(x)\leq Cx^{1+\theta}|f(x)|\,\forall\, x\leq x_1$ and that $X_n<x_1$ for all $n\geq N_3(\omega)$. Also since $x\mapsto|f(x)|/x$ is asymptotically increasing there is $\psi(x)\sim|f(x)|/x$ as $x\to 0^+$ which is increasing so

$$\frac{\psi(x)}{4}<\frac{|f(x)|}{x}<4\psi(x),\quad x<x_2,$$

and $x<x_2$ for $n\geq N_4(\omega)$. Let $N_5(\omega):=\max(N_1,N_2,N_3,N_4)$. Then for $n\geq N_5$ as $x\mapsto\psi(x)$ is increasing

$$\begin{aligned}
\left|\frac{g^3(X_n)}{|f(X_n)|X_n^2}\xi_{n+1}^3\right| &\leq \frac{\left(CX_n^{1+\theta}|f(X_n)|\right)^{3/2}}{|f(X_n)|X_n^2}8\left(\log(n\Delta)\right)^{3/2}\\
&= 8C^{3/2}\left(\frac{|f(X_n)|}{X_n}\right)^{1/2}X_n^{3\theta/2}\left(\log(n\Delta)\right)^{3/2}\\
&< 8C^{3/2}\cdot 2\left(\psi(X_n)\right)^{1/2}\cdot X_n^{3\theta/2}\cdot\left(\log(n\Delta)\right)^{3/2}\\
&< 8C^{3/2}\cdot 2\psi^{1/2}\left(F^{-1}(n\Delta)^{1-\epsilon}\right)\cdot\left(F^{-1}(n\Delta)^{1-\epsilon}\right)^{3\theta/2}\cdot\left(\log(n\Delta)\right)^{3/2}\\
&< 8C^{3/2}\cdot 2\cdot 2\left(\frac{|f(F^{-1}(n\Delta)^{1-\epsilon})|}{F^{-1}(n\Delta)^{1-\epsilon}}\right)^{1/2}\cdot\left(F^{-1}(n\Delta)^{1-\epsilon}\right)^{3\theta/2}\cdot\left(\log(n\Delta)\right)^{3/2}.
\end{aligned}$$

Put $y_n = F^{-1}(n\Delta)^{1-\epsilon}$. Then for $n \geq N_5$

$$\begin{aligned}\left|\frac{g^3(X_n)}{|f(X_n)|X_n^2}\xi_{n+1}^3\right| &\leq 32C^{3/2}\cdot\left(\frac{|f(y_n)|}{y_n}\right)^{1/2}\cdot y_n^{3\theta/2}\cdot\left((\log\circ F)(y_n^{1/(1-\epsilon)})\right)^{3/2}\\ &= 32C^{3/2}\left(\left(\frac{|f(y_n)|}{y_n}\right)^{1/3}\cdot y_n^{\theta}\cdot(\log\circ F)(y_n^{1/(1-\epsilon)})\right)^{3/2}.\end{aligned}$$

Since $y_n \to 0$ as $n \to \infty$ then the term in brackets tends to zero as $n \to \infty$ by (13.40) and the deterministic conditions on f and g imply (13.39). □

The condition (13.40) holds for a large class of functions f. For example if $f \in RV_0(\beta)$, for $\beta > 1$, then $F \in RV_0(1-\beta)$ so $y \mapsto (\log\circ F)(y^{1+\epsilon}) \in RV_0(0)$. Hence

$$y \mapsto \left(\frac{f(y)}{y}\right)^{1/3} y^{\theta}(\log\circ F)(y^{1+\epsilon}) \in RV_0(1/3\cdot(\beta-1)+\theta),$$

so it follows that

$$\lim_{y\to 0^+}\left(\frac{f(y)}{y}\right)^{1/3} y^{\theta}(\log\circ F)(y^{1+\epsilon}) = 0, \quad \forall\,\epsilon > 0,$$

which is (13.40). Hence if the other conditions in Theorem 90 hold we get that $X_n \sim F^{-1}(t_n)$ as $n \to \infty$ where $f \in RV_0(\beta)$ for $\beta > 1$.

If $f \in RV_0(1)$, Theorem 90 is inapplicable owing to the monotonicity hypothesis that $x \mapsto |f(x)|/x^{1+\theta}$ is asymptotically increasing. Therefore, we need to prepare another result to deal with that case. We shortly present Theorem 93 to that end.

The other large class of functions which might attract our attention are those which are "arbitrarily flat" at zero, such as the function $f(x) = e^{-1/x}$, $x > 0$, $f(0) = 0$ which is infinitely differentiable in a right neighbourhood of zero but for which $f^{(n)}(0) = 0$, $\forall\, n \geq 1$. The following conditions, which are easy to check, characterise nicely many such "superflat" functions:

$$\lim_{x\to 0^+}\frac{f(x)f''(x)}{f'(x)^2} = 1 \quad\text{and}\quad \lim_{x\to 0^+}\frac{f(x)}{x} = 0. \tag{13.41}$$

We now prove a result which captures some of the important properties of functions f that obey (13.41). In the lemma below the function f is a *positive, increasing* C^2 function. It is not the function f which stands for the drift in our SDE. However, the salient results for negative f can be read off from the result below.

Lemma 47. *Suppose f is in C^2 and*

$$\lim_{x\to 0^+}\frac{(ff'')(x)}{f'(x)^2} = 1 \quad \textit{and} \quad \lim_{x\to 0^+}\frac{f(x)}{x} = 0.$$

Then

(i) $F(x) = \int_x^1 1/f(u)\, du \sim 1/f'(x)$ *as* $x \to 0^+$.

(ii) $\lim_{x\to 0^+} xf'(x)/f(x) = \infty$,

(iii) $\log(1/f'(x)) \sim \log(1/f(x))$ *as* $x \to 0^+$.

(iv)

$$\lim_{x\to 0^+} \frac{\log(1/f(x))}{\log(1/x)} = \infty.$$

Proof. Define $\psi(x) := \log(1/f(x))$ so $\psi \in C^2$. Then $f(x) = e^{-\psi(x)}$ and $f'(x) = -\psi'(x)f(x)$, $f''(x) = -\psi''(x)f(x) + \psi'(x)^2 f(x)$. Hence $\psi'(x)^2 - \psi''(x) > 0$ and indeed

$$\lim_{x\to 0^+} \frac{(\psi'(x)^2 - \psi''(x))f(x)}{\psi'(x)^2} = \lim_{x\to 0^+} \frac{(ff'')(x)}{f'(x)^2} = 1.$$

Since $f''(x) > 0$ and $f'(0) = 0$, then $f'(x) > 0$ so $\psi''(x)/\psi'(x)^2 \to 0$ as $x \to 0^+$ for $x \in (0, \delta)$. Define $\gamma(x) := -\psi'(x)$. Then $\gamma(x) > 0$ and $M(x) := \gamma'(x)/\gamma^2(x) \to 0$ as $x \to 0^+$. Clearly, as $f(x)/x \to 0$ as $x \to 0^+$ and thus $f'(x) \to 0$ as $x \to 0^+$, we have that $F(x) \to \infty$ and $1/f'(x) \to \infty$ as $x \to 0^+$. By L'Hôpital's rule

$$\lim_{x\to 0^+} \frac{F(x)}{1/f'(x)} = \lim_{x\to 0^+} \frac{-1/f'(x)}{-f''(x)/f'(x)^2} = 1,$$

completing the proof of part (i). For part (ii), we start by noting that $f''(x) > 0$ so $f'(x) > 0$ and f' is increasing on $(0, \delta)$. Thus for $x \in (0, \delta)$, there is $\xi_x \in (0, x)$ such that

$$f(x) = f(0) + f'(\xi_x)x = f'(\xi_x)x < f'(x)x.$$

Since $f' = -\psi' f$, $-\psi'(x)x > 1$ for all $x \in (0, \delta)$. Hence $\gamma(x) > 1/x$ for all $x \in (0, \delta)$. Hence $\gamma(x) \to \infty$ as $x \to 0^+$. Now

$$\int_{\gamma(x)}^{\infty} \frac{1}{v^2}\, dv = \int_{0^+}^{x} \frac{-\gamma'(u)}{\gamma^2(u)}\, du = \int_{0^+}^{x} -M(u)\, du.$$

Thus

$$\gamma(x)^{-1} = \int_{0^+}^{x} -M(u)\, du.$$

Now $1/(x\gamma(x)) = 1/x \cdot \int_{0^+}^{x} -M(u)\, du \to 0$ as $x \to 0^+$. Hence $x\psi'(x) \to -\infty$ as $x \to 0^+$. Thus

$$\lim_{x\to 0^+} \frac{xf'(x)}{f(x)} = \lim_{x\to 0^+} -x\gamma'(x) = \infty.$$

This completes the proof of part (ii). To prove part (iii) notice first from part (ii) that

$$\lim_{x\to 0^+} \frac{f'(x)}{f(x)} = \lim_{x\to 0^+} \left(\frac{1}{x} \cdot \frac{xf'(x)}{f(x)}\right) = \infty.$$

Now define $\eta_f(x) := f'(x)/f(x)$, $\eta_{f'}(x) := f''(x)/f'(x)$. Since $(ff''/f'^2)(x) \to 1$ as $x \to 0^+$, $\eta_f(x) \sim \eta_{f'}(x)$ as $x \to 0^+$, and $\eta_f(x), \eta_{f'}(x) \to \infty$ as $x \to 0^+$. Thus as $x\eta_f(x) \to \infty$ as $x \to 0^+$ for any $K > 1$, we have that $\eta_f(x) > K/x$ for all $x < x(K)$. Hence

$$\int_x^{x(K)} \eta_f(u)\, du > K \int_x^{x(K)} \frac{1}{u}\, du = K \log\left(\frac{x(K)}{x}\right) \to \infty, \quad \text{as } x \to 0^+.$$

Thus $\int_x^\delta \eta_f(u)\, du \to \infty$ as $x \to 0^+$ and likewise $\int_x^\delta \eta_{f'}(u)\, du \to \infty$ as $x \to 0^+$. Moreover, $\int_x^\delta \eta_f(u)\, du \sim \int_x^\delta \eta_{f'}(u)\, du$ as $x \to 0^+$. Also for $x < \delta$

$$\log\left(\frac{f(\delta)}{f(x)}\right) = \int_x^\delta \frac{f'(u)}{f(u)}\, du = \int_x^\delta \eta_f(u)\, du,$$

and as $x \to 0^+, \log(1/f'(x)) \sim \log(1/f(x))$ because as $x \to 0^+$

$$\log\left(\frac{f'(\delta)}{f'(x)}\right) = \int_x^\delta \eta_{f'}(u)\, du \sim \int_x^\delta \eta_f(u)\, du = \log\left(\frac{f(\delta)}{f(x)}\right),$$

completing the proof of part (iii). Finally, to prove part (iv), notice that $xf'(x)/f(x) > K, \forall\, x < x(K)$ and any $K > 0$. Thus $f'(x)/f(x) > K/x$ for $x < x(K)$ and so for $x < x(K)$

$$\log\left(\frac{f(x(K))}{f(x)}\right) = \int_x^{x(K)} \frac{f'(u)}{f(u)}\, du > K \int_x^{x(K)} \frac{1}{u}\, du = K \log\left(\frac{x(K)}{x}\right),$$

so

$$\frac{\log(1/f(x))}{\log(1/x)} > \frac{K \log x(K)}{\log(1/x)} + K + \frac{\log(1/f(x(K)))}{\log(1/x)}.$$

Hence

$$\liminf_{x \to 0^+} \frac{\log(1/f(x))}{\log(1/x)} \geq K,$$

and letting $K \to \infty$ yields the proof of part (iv), as claimed. □

Armed with Lemma 47 we can simplify appreciably the condition (13.40) for functions f obeying (13.41).

Lemma 48. *Suppose* $\psi(x) = \log(1/|f(x)|)$ *and*

$$\limsup_{x \to 0^+} \frac{\log \psi(x^{1+\epsilon})}{\psi(x)} < \frac{1}{3},$$

and let $|f|$ *satisfy* (13.41). *Then* (13.40) *holds.*

Proof. By Lemma 47 parts (i) and (iii), as $x \to 0^+$:

$$\left(\frac{|f(x)|}{x}\right)^{1/3} x^\theta (\log \circ F)(x^{1+\epsilon}) \sim \left(\frac{|f(x)|}{x}\right)^{1/3} x^\theta \log\left(\frac{1}{|f(x^{1+\epsilon})|}\right).$$

Hence with $\psi(x) := \log(1/|f(x)|)$, taking limits as $x \to 0^+$, we get

$$\begin{aligned}
&\log\left(\left(\frac{|f(x)|}{x}\right)^{1/3} x^\theta (\log\circ F)(x^{1+\epsilon})\right)\\
&\sim\ \frac{1}{3}\log|f(x)| + \left(\theta - \frac{1}{3}\right)\log x + \log\log\left(\frac{1}{|f(x^{1+\epsilon})|}\right)\\
&=\ \frac{-1}{3}\log\left(\frac{1}{|f(x)|}\right) + \left(\frac{1}{3} - \theta\right)\log\left(\frac{1}{x}\right) + \log\log\left(\frac{1}{|f(x^{1+\epsilon})|}\right)\\
&=\ \frac{-1}{3}\log\left(\frac{1}{f(x)}\right)\left(1 + \left(-3\left(\frac{1}{3} - \theta\right)\frac{\log(1/x)}{\log(1/f(x))}\right) - 3\frac{\log\log(1/f(x^{1+\epsilon}))}{\log\log(1/f(x))}\right)\\
&=\ \frac{-\psi(x)}{3}\left(1 + \epsilon_1(x) - \frac{3\log\psi(x^{1+\epsilon})}{\psi(x)}\right) \to -\infty,
\end{aligned}$$

since $-1/3 \cdot \log(1/|f(x)|) \to -\infty$ as $x \to 0^+$ and

$$\epsilon_1(x) = -3\left(\frac{1}{3} - \theta\right)\frac{\log(1/x)}{\log(1/|f(x)|)} \to 0, \quad \text{as } x \to 0^+,$$

by Lemma 47, part (iv). □

We now provide an example.

Example 92. Let $f(x) = -e^{-1/x^\alpha}$, $\alpha > 0$. Then it can easily be checked that

$$\lim_{x\to 0^+} \frac{f(x)f''(x)}{f'(x)^2} = 1.$$

Define $\psi(x) := \log(1/|f(x)|) = \log(e^{1/x^\alpha}) = 1/x^\alpha$. Then

$$\lim_{x\to 0^+} \frac{\log\psi(x^{1+\epsilon})}{\psi(x)} = \lim_{x\to 0^+} \frac{\log(1/|f(x)|)}{(x^{1+\epsilon})^\alpha} = \lim_{x\to 0^+} \left(x^\alpha \cdot \log\left(\tfrac{1}{x}\right) \cdot \alpha(1+\epsilon)\right) = 0.$$

Therefore, the condition in Lemma 48 holds, and with this choice of f by Theorem 89 we have that

$$\lim_{n\to\infty} \frac{X_n}{F^{-1}(t_n)} = 1, \quad \text{a.s..}$$

□

Proof of Theorem 90. We have $Z_n = -\log X_n$, $t_n \sim n\Delta$ as $n \to \infty$, $X_n \to 0$ as $n \to \infty$, $X_n > 0$ for all $n \geq 0$, $Z_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{-\log X_n}{(-\log\circ F^{-1})(n\Delta)} = 1.$$

Also for all n sufficiently large $h(X_n) = \Delta$, and therefore

$$Z_{n+1} = Z_n + \Delta\left(\frac{|f(X_n)|}{X_n} + \frac{g^2(X_n)}{2X_n^2}\right) + \sqrt{\Delta} \cdot \frac{g(X_n)}{X_n} \cdot \xi_{n+1}.$$

Define $A(x) := e^{\theta x}$. Then $A(Z_n) = e^{\theta Z_n} = \left(e^{Z_n}\right)^\theta = X_n^{-\theta}$. Define

$$F_n := \Delta \left(\frac{|f(X_n)|}{X_n} + \frac{g^2(X_n)}{2X_n^2} \right) \quad \text{and} \quad R_{n+1} := \sqrt{\Delta} \cdot \frac{g(X_n)}{X_n} \cdot \xi_{n+1}.$$

Then there is η_{n+1} in the interval with endpoints Z_n and Z_{n+1} such that

$$\begin{aligned} A(Z_{n+1}) &= A(Z_n + F_n + R_{n+1}) \\ &= A(Z_n) + A'(Z_n)(F_n + R_{n+1}) + \tfrac{1}{2}A''(\eta_{n+1})(F_n + R_{n+1})^2. \end{aligned}$$

Thus

$$\begin{aligned} &A(Z_{n+1}) - A(Z_n) \\ &= \theta A(Z_n)(F_n + R_{n+1}) + \tfrac{1}{2}\theta^2 A(\eta_{n+1})(F_n^2 + 2F_n R_{n+1} + R_{n+1}^2) \\ &= \theta A(Z_n)F_n + \tfrac{1}{2}\theta^2 A(\eta_{n+1})F_n^2 + \tfrac{1}{2}\theta^2 A(\eta_{n+1})\Delta\frac{g^2(X_n)}{X_n^2}(\xi_{n+1}^2 - 1) + \theta A(Z_n)R_{n+1} \\ &\qquad + \theta^2 A(\eta_{n+1})F_n R_{n+1} + \tfrac{1}{2}\theta^2 A(\eta_{n+1})\Delta\frac{g^2(X_n)}{X_n^2}. \end{aligned}$$

Define $\zeta_{n+1} := \xi_{n+1}^2 - 1$ and

$$D_n := \theta A(Z_n)F_n + \tfrac{1}{2}\theta^2 A(\eta_{n+1})F_n^2 + \tfrac{1}{2}\theta^2 A(\eta_{n+1})\Delta\frac{g^2(X_n)}{X_n^2},$$

$$\begin{aligned} T_{n+1} = \tfrac{1}{2}\theta^2 A(\eta_{n+1})\Delta\frac{g^2(X_n)}{X_n^2}\zeta_{n+1} + \theta A(Z_n)\sqrt{\Delta}\frac{g(X_n)}{X_n}\xi_{n+1} + \\ \theta^2 A(\eta_{n+1})\Delta\left(\frac{|f(X_n)|}{X_n} + \frac{g^2(X_n)}{2X_n^2}\right)\sqrt{\Delta}\frac{g(X_n)}{X_n}\xi_{n+1}. \end{aligned}$$

Then for n sufficiently large $A(Z_{n+1}) = A(Z_n) + D_n + T_{n+1}$. Fix $n \in \mathbb{N}$. If $Z_n > Z_{n+1}$ then $\eta_{n+1} \in [Z_{n+1}, Z_n]$ and $A(\eta_{n+1}) < A(Z_n)$. Then

$$\frac{D_n}{\theta A(Z_n)F_n} - 1 = \frac{1}{2} \cdot \frac{\theta A(\eta_{n+1})}{A(Z_n)} \cdot F_n + \frac{1}{2} \cdot \frac{\theta A(\eta_{n+1})}{A(Z_n)} \cdot \frac{\Delta g^2(X_n)/X_n^2}{F_n} = o(1), \quad \text{as } n \to \infty.$$

If $Z_n < Z_{n+1}$ then $\eta_{n+1} \in [Z_n, Z_{n+1}]$ and $A(\eta_{n+1}) < A(Z_{n+1})$. Then

$$\begin{aligned} 0 < \frac{D_n}{\theta A(Z_n)F_n} - 1 &\leq \frac{1}{2} \cdot \frac{\theta A(Z_{n+1})}{A(Z_n)} \cdot F_n + \frac{1}{2} \cdot \frac{\theta A(Z_{n+1})}{A(Z_n)} \cdot \frac{\Delta g^2(X_n)/X_n^2}{F_n} \\ &= \frac{1}{2} \cdot \theta \left(e^{\theta(Z_{n+1} - Z_n)}F_n + e^{\theta(Z_{n+1} - Z_n)}\frac{\Delta g^2(X_n)/X_n^2}{F_n} \right). \end{aligned}$$

Thus

$$0 < \frac{D_n}{\theta A(Z_n)F_n} - 1 \leq \frac{1}{2}\theta\left(e^{\theta(F_n - R_{n+1})}F_n + e^{\theta(F_n-R_{n+1})}\frac{\Delta g^2(X_n)/X_n^2}{F_n}\right). \quad (13.42)$$

Now $F_n \to 0$ as $n \to \infty$ and $(\Delta g^2(X_n)/X_n^2)/F_n \to 0$ as $n \to \infty$. Moreover since (13.39) holds and

$$R_{n+1}^3 = \Delta^{3/2} \cdot \frac{|f(X_n)|}{X_n} \cdot \frac{g^3(X_n)\xi_{n+1}^3}{X_n^2|f(X_n)|},$$

we have $R_{n+1} \to 0$ as $n \to \infty$. Therefore we have

$$\lim_{n\to\infty} \frac{D_n}{\theta A(Z_n)F_n} = 1.$$

In fact

$$\lim_{n\to\infty} \frac{D_n}{\theta X_n^{-\theta}\Delta|f(X_n)|/X_n} = 1. \quad (13.43)$$

Let

$$T_{n+1}^{(1)} := \frac{1}{2}\theta^2 A(Z_n)\Delta\frac{g^2(X_n)}{X_n^2}\zeta_{n+1} + \theta A(Z_n)\sqrt{\Delta}\frac{g(X_n)}{X_n}\xi_{n+1} + \theta^2 A(Z_n)\Delta\left(\frac{|f(X_n)|}{X_n} + \frac{g^2(X_n)}{2X_n^2}\right)\sqrt{\Delta}\frac{g(X_n)}{X_n}\xi_{n+1}.$$

By applying the induction agrument in Theorem 72, it can be shown that $T_{n+1}^{(1)}$ is a martingale difference and we also define

$$\begin{aligned} T_{n+1}^{(2)} &:= T_{n+1} - T_{n+1}^{(1)} \\ &= A(Z_n)\left(\tfrac{1}{2}\theta^2 \cdot \frac{A(\eta_{n+1}) - A(Z_n)}{A(Z_n)} \cdot \frac{\Delta g^2(X_n)}{X_n^2}\zeta_{n+1} + \theta^2 \cdot \frac{A(\eta_{n+1}) - A(Z_n)}{A(Z_n)} \cdot F_n\frac{\sqrt{\Delta}g(X_n)}{X_n} \cdot \xi_{n+1}\right). \end{aligned}$$

If $Z_{n+1} > Z_n$ then $\eta_{n+1} \in [Z_n, Z_{n+1}]$ and $A(\eta_{n+1}) < A(Z_{n+1})$ so

$$0 < \frac{|A(\eta_{n+1}) - A(Z_{n+1})|}{A(Z_n)} \leq e^{\theta(Z_{n+1}-Z_n)} - 1 = e^{\theta(F_n + R_{n+1})} - 1 = o(1), \quad \text{as } n \to \infty,$$

because F_n, $R_{n+1} \to 0$ as $n \to \infty$. If $Z_n > Z_{n+1}$ then $\eta_{n+1} \in [Z_{n+1}, Z_n]$ and $A(\eta_{n+1}) < A(Z_n)$ so

$$|A(\eta_{n+1}) - A(Z_n)| = A(Z_n) - A(\eta_{n+1}) < A(Z_n) - A(Z_{n+1}).$$

Hence

$$\frac{|A(\eta_{n+1}) - A(Z_n)|}{A(Z_n)} \leq 1 - e^{\theta(Z_{n+1}-Z_n)} = |e^{\theta(Z_{n+1}-Z_n)} - 1| = o(1), \quad \text{as } n \to \infty,$$

because F_n, $R_{n+1} \to 0$ as $n \to \infty$. Hence

$$0 < \frac{|A(\eta_{n+1}) - A(Z_{n+1})|}{A(Z_n)} = o(1), \quad \text{as } n \to \infty,$$

in all cases. Thus

$$T^{(2)}_{n+1} = A(Z_n)\left(o(1) \cdot \frac{\Delta g^2(X_n)}{X_n^2} \cdot \zeta_{n+1} + o(1) \cdot F_n \cdot R_{n+1}\right),$$

where the $o(1)$ terms behave according to

$$o(1) \leq |e^{\theta(Z_{n+1}-Z_n)} - 1| \sim \theta\left(F_n + R_{n+1}\right), \quad \text{as } n \to \infty.$$

Recall $D_n \sim \theta A(Z_n) F_n$ as $n \to \infty$. Then the second term in $T^{(2)}_{n+1}$ is $A(Z_n)o(1)F_nR_{n+1} = o(D_n)$. The first term is

$$A(Z_n) \cdot o(1) \cdot \frac{\Delta g^2(X_n)}{X_n^2} \cdot \zeta_{n+1} \quad = \quad O\left(A(Z_n)(F_n + R_{n+1}) \cdot \frac{\Delta g^2(X_n)}{X_n^2} \cdot \zeta_{n+1}\right), \quad \text{as } n \to \infty.$$

Therefore, if

$$\lim_{n\to\infty}\left(\frac{R_{n+1}}{F_n} \cdot \frac{g^2(X_n)}{X_n^2}\left(\xi^2_{n+1} - 1\right)\right) = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{g^2(X_n)}{X_n^2}\zeta_{n+1} = 0, \tag{13.44}$$

then $T^{(2)}_{n+1} = o(D_n)$ as $n \to \infty$. The second condition in (13.44) holds if $g(X_n)/X_n \cdot \xi_{n+1} \to 0$ as $n \to \infty$. To see this write

$$\left(\frac{g(X_n)}{X_n}\xi_{n+1}\right)^3 = \frac{g^3(X_n)}{|f(X_n)|X_n^2} \cdot \xi^3_{n+1} \cdot \frac{|f(X_n)|}{X_n},$$

and apply (13.39). The first condition in (13.44) is equivalent to

$$\left(\frac{g^2(X_n)}{X_n^2}\xi^2_{n+1} - \frac{g^2(X_n)}{X_n^2}\right) \cdot \frac{g(X_n)\xi_{n+1}/X_n}{|f(X_n)|/X_n} \to 0, \quad \text{as } n \to \infty,$$

or

$$\frac{g(X_n)}{|f(X_n)|}\xi_{n+1}\left(\frac{g^2(X_n)}{X_n^2} \cdot \xi^2_{n+1} - \frac{g^2(X_n)}{X_n^2}\right) \to 0, \quad \text{as } n \to \infty. \tag{13.45}$$

For the second term, recall that $g(X_n)/X_n \cdot \xi_{n+1} \to 0$ as $n \to \infty$ so

$$\frac{g^3(X_n)}{|f(X_n)|X_n^2} \cdot \xi_{n+1} = \frac{g(X_n)}{X_n}\xi_{n+1} \cdot \frac{g^2(X_n)}{|f(Xn)|X_n} \to 0, \quad \text{as } n \to \infty.$$

Hence the second term in (13.45) tends to zero. The first term is

$$R^{(2)}_{n+1} = \frac{g^3(X_n)}{|f(X_n)|X_n^2} \cdot \xi^3_{n+1},$$

and tends to zero by (13.39). Therefore $T^{(2)}_{n+1} = o(D_n)$ as $n \to \infty$, as required. Define now $M(n) := \sum_{j=0}^{n-1} T^{(1)}_{j+1}$. Then

$$A(Z_{j+1}) - A(Z_j) = D(Z_j) + T^{(1)}_{j+1} + T^{(2)}_{j+1},$$

so

$$A(Z_n) = A(Z_0) + \sum_{j=0}^{n-1} D(Z_j) + \sum_{j=0}^{n-1} T^{(2)}_{j+1} + M(n).$$

Since $T^{(2)}_{n+1} = o(D(Z_n))$ as $n \to \infty$ if we can show that $M(n)$ is $o(\sum D(Z_n))$ as $n \to \infty$ we will be able to reduce the remaining asymptotic analysis to that covered in the proof of Theorem 88. Define $\langle M \rangle (n) := \sum_{j=1}^{n} \mathbb{E}\left[(M(j) - M(j-1))^2 | \mathscr{F}_{j-1}\right]$. Hence $\langle M \rangle (n) = \sum_{j=1}^{n} \mathbb{E}\left[T^{(1)^2}_j | \mathscr{F}_{j-1}\right]$. Now

$$\begin{aligned} T^{(1)^2}_j &= A(Z_{j-1})^2 \left(\frac{\theta^2 \Delta g^2(X_{j-1})}{2X^2_{j-1}} \cdot \zeta_j + \frac{\theta\sqrt{\Delta} g(X_{j-1})}{X_{j-1}} \cdot \xi_j + \frac{\theta^2 F_{j-1} \sqrt{\Delta} g(X_{j-1})}{X_{j-1}} \cdot \xi_j \right)^2 \\ &= A(Z_{j-1})^2 \cdot \frac{\theta^2 \Delta g^2(X_{j-1})}{X^2_{j-1}} \left(\frac{\theta\sqrt{\Delta} g(X_{j-1})}{2X_{j-1}} \cdot \zeta_j + \xi_j + \theta F_{j-1} \xi_j \right)^2 . \end{aligned}$$

Thus with $E_j := A(Z_j)^2 \theta^2 \Delta g^2(X_j)/X_j^2$, we have

$$\begin{aligned} &\mathbb{E}\left[T^{(1)^2}_j | \mathscr{F}_{j-1}\right] \\ &= A(Z_{j-1})^2 \cdot \frac{\theta^2 \Delta g^2(X_{j-1})}{X^2_{j-1}} \cdot \mathbb{E}\left[\left(\frac{\theta\sqrt{\Delta} g(X_{j-1})}{2X_{j-1}} \cdot \zeta_j + (1 + \theta F_{j-1})\xi_j \right)^2 \middle| \mathscr{F}_{j-1} \right] \\ &= E_{j-1} \cdot \mathbb{E}\left[\frac{\theta^2 \Delta g^2(X_{j-1})}{4X^2_{j-1}} \cdot \zeta_j^2 + \frac{2\theta\sqrt{\Delta} g(X_{j-1})(1 + \theta F_{j-1})}{2X_{j-1}} \cdot \zeta_j \xi_j + (1 + \theta F_{j-1})^2 \cdot \xi_j^2 \middle| \mathscr{F}_{j-1} \right] \\ &= E_{j-1} \left(\frac{\theta^2 \Delta g^2(X_{j-1})}{4X^2_{j-1}} \cdot \mathbb{E}\left[\zeta_j^2\right] + \frac{\theta\sqrt{\Delta} g(X_{j-1})(1 + \theta F_{j-1})}{X_{j-1}} \cdot \mathbb{E}\left[\zeta_j \xi_j\right] + (1 + \theta F_{j-1})^2 \cdot \mathbb{E}\left[\xi_j^2\right] \right). \end{aligned}$$

Now $\zeta_j = \xi_j^2 - 1$. Then $\zeta_j^2 = \xi_j^4 - 2\xi_j^2 + 1$. Thus $\mathbb{E}\left[\zeta_j^2\right] = 3 - 2.1 + 1 = 2$ and $\mathbb{E}\left[\zeta_j \xi_j\right] = \mathbb{E}\left[\zeta_j(\xi_j^2 - 1)\right] = 0$. Thus

$$\begin{aligned} \mathbb{E}\left[T^{(1)^2}_j \middle| \mathscr{F}_{j-1}\right] &= E_{j-1} \left(\frac{\theta^2 \Delta g^2(X_{j-1})}{2X^2_{j-1}} + (1 + \theta F_{j-1})^2 \right) \\ &= A(Z_{j-1})^2 \cdot \frac{\theta^2 \Delta g^2(X_{j-1})}{X^2_{j-1}} \left(1 + 2\theta F_{j-1} + \theta^2 F^2_{j-1} + \frac{\theta^2 \Delta g^2(X_{j-1})}{2X^2_{j-1}} \right) \\ &\sim A(Z_{j-1})^2 \cdot \frac{\theta^2 \Delta g^2(X_{j-1})}{X^2_{j-1}}, \quad \text{as } j \to \infty. \end{aligned}$$

Hence as $n \to \infty$

$$\langle M \rangle (n) \sim \sum_{j=1}^{n} A(Z_{j-1})^2 \cdot \frac{\theta^2 \Delta g^2(X_{j-1})}{X_{j-1}^2} = \sum_{j=0}^{n-1} A(Z_j)^2 \cdot \frac{\theta^2 \Delta g^2(X_j)}{X_j^2} = \sum_{j=0}^{n-1} E_j.$$

Now $D(Z_j) \sim \theta A(Z_j)\Delta|f(X_j)|/X_j$ as $j \to \infty$. Thus as $j \to \infty$ we have

$$\frac{E_j}{D(Z_j)} \sim \frac{A(Z_j)^2 \cdot \theta^2 \Delta g^2(X_j)/X_j^2}{\theta A(Z_j) \cdot \Delta|f(X_j)|/X_j} = \theta A(Z_j) \cdot \frac{g^2(X_j)}{X_j|f(X_j)|} = \theta X_j^{-\theta} \cdot \frac{g^2(X_j)}{X_j|f(X_j)|} = \frac{\theta g^2(X_j)}{X_j^{1+\theta}|f(X_j)|} \leq E^*.$$

Thus for some a.s. bounded random variable E^*

$$\limsup_{n\to\infty} \frac{\langle M \rangle (n)}{\sum_{j=0}^{n-1} D(Z_j)} \leq E^*.$$

If $\langle M \rangle (n)$ diverges, then $M(n)/\sum_{j=0}^{n-1} D(Z_j) \to 0$ as $n \to \infty$. Since $T_{j+1}^{(2)} = o(D(Z_j))$ as $j \to \infty$ then $A(Z_n) \sim \sum_{j=0}^{n-1} D(Z_j)$ as $n \to \infty$. Thus

$$\lim_{n\to\infty} \frac{X_n^{-\theta}}{\sum_{j=0}^{n-1} \theta A(Z_j)\Delta|f(X_j)|/X_j} = 1,$$

or

$$\lim_{n\to\infty} \frac{X_n^{-\theta}}{\sum_{j=0}^{n-1} \theta \Delta|f(X_j)|/X_j^{1+\theta}} = 1, \quad \text{a.s. on } A_1,$$

where $A_1 := \{\omega : \langle M \rangle (n) \to \infty, n \to \infty\}$. Now consider the event

$$A_1' = \{\omega : \langle M \rangle (n) \to < \infty, n \to \infty\}.$$

On A_1' $M(n)$ converges. We have that $T_{j+1}^{(2)} = o(D(Z_j))$ as $j \to \infty$. Since $D(Z_j) > 0$, $\forall j$ so $\sum_{j=0}^{n-1} D(Z_j) \to \infty$ as $n \to \infty$ or $\sum_{j=0}^{n-1} D(Z_j) \to D^* < \infty$ as $n \to \infty$. In the former case, we have

$$\frac{A(Z_n)}{\sum_{j=0}^{n-1} D(Z_j)} = \frac{A(Z_0)}{\sum_{j=0}^{n-1} D(Z_j)} + 1 + \frac{\sum_{j=0}^{n-1} T_{j+1}^{(2)}}{\sum_{j=0}^{n-1} D(Z_j)} + \frac{M(n)}{\sum_{j=0}^{n-1} D(Z_j)} \to 1, \quad \text{as } n \to \infty.$$

In the latter case, as $T_{j+1}^{(2)} = o(D(Z_j))$ as $j \to \infty$, then $\sum_{j=0}^{n-1} T_{j+1}^{(2)} \to T^*$. Then as $n \to \infty$

$$X_n^{-\theta} = A(Z_n) \to A(Z_0) + D^* + T^* + M^* := X^* \in (-\infty, \infty).$$

If $X^* \neq 0$, then $X_n \to X_\infty \in (-\infty, \infty)$ a contradiction. If $X^* = 0$, then $X_n \to \infty$, also a contradiction. Therefore if $\langle M\rangle(n)$ converges, then

$$\lim_{n\to\infty}\frac{A(Z_n)}{\sum_{j=0}^{n-1}D(Z_j)}=1,\quad \text{on } A_1'.$$

Thus

$$\lim_{n\to\infty}\frac{X_n^{-\theta}}{\sum_{j=0}^{n-1}\theta\Delta|f(X_j)|/X_j^{1+\theta}}=1,\quad \text{a.s.}.$$

Following the proof for the power pre-transformation, we have that (13.30) and (13.31) implies

$$\lim_{n\to\infty}\frac{X_n}{F^{-1}(t_n)}=\lim_{n\to\infty}\frac{X_n}{F^{-1}(n\Delta)}=1,\quad \text{a.s.},$$

as required. □

13.7 Super-Polynomial Decay in Logarithmically Pre-Transformed Scheme

We showed in Theorem 87 part (ii) that when $x\mapsto|f(x)|/x^{1+\theta}$ is decreasing, $x\mapsto|f(x)|/x$ is increasing, $|f|\in C^1((0,\delta);(0,\infty))$, then under the small noise condition, we can strengthen the conclusion of Theorem 85 from

$$\lim_{t\to\infty}\frac{F(X(t))}{t}=1,\quad \text{a.s.},$$

to

$$\lim_{t\to\infty}\frac{X(t)}{F^{-1}(t)}=1,\quad \text{a.s.}.$$

In this section we show that the improved asymptotic result is satisfied by the logarithmically pre-transformed scheme under strengthened hypotheses which ensure that $|f|\in RV_0(1)$. We state now the main result.

Theorem 93. *Suppose that*

(i) $\forall\,\theta>0$

$$\limsup_{x\to0^+}\frac{g^2(x)}{x^{1+\theta}|f(x)|}<\infty$$

(ii)

$$|f|\in C^1,\quad l(x):=|f(x)|/x \text{ is increasing},\quad l'\in RV_0(-1)$$

(iii) $x\mapsto xl'(x)$ *is increasing.*

Then

$$\lim_{n\to\infty}\frac{X_n}{F^{-1}(n\Delta)}=\lim_{n\to\infty}\frac{X_n}{F^{-1}(t_n)}=1,\quad a.s..$$

We now wish to show that the conditions imposed in Theorem 93 imply that of Theorem 87 part (ii). In doing so we show the discrete-time dynamics are preserving the asymptotic behaviour specified in Theorem 87 part (ii) for an important subclass of relevant drift coefficients.

Proposition 20. *Under the conditions of Theorem 93, the solution of the SDE obeys*

$$\lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} = 1, \quad a.s..$$

Proof. Since $|f| \in C((0,\infty);(0,\infty))$, we have automatically the first part of (13.14). We have $l(x) := |f(x)|/x$ increasing so $x \mapsto |f(x)|/x$ certainly fulfills the second part of (13.14). Next as $l' \in RV_0(-1)$ we have that $l \in RV_0(0)$ and therefore as $|f(x)| = xl(x)$, we have $|f| \in RV_0(1)$. Hence $x \mapsto |f(x)|/x^{1+\theta} \in RV_0(-\theta)$ for $\theta > 0$ is asymptotically decreasing at zero, which is condition (13.12). Condition (13.11) is nothing more than the small noise condition we impose in Theorem 93. Therefore all the conditions of Theorem 87 part (ii) hold and it follows that

$$\lim_{t\to\infty} \frac{X(t)}{F^{-1}(t)} = 1, \quad \text{a.s.,}$$

as claimed. □

Example 94. Reconsider Example 86 where $\beta > 0$, $\gamma > 1$ and

$$f(x) = \frac{-x}{\log^{\beta}(1/x)} \quad \text{and} \quad g(x) = x^{\gamma},$$

for $x > 0$ sufficiently small. Then for $0 < \theta < 2\gamma - 2$ the small noise condition in (i) holds, $|f| \in C^1((0,\delta);(0,\infty))$ for some $\delta > 0$ and

$$l(x) = \frac{|f(x)|}{x} = \frac{1}{\log^{\beta}(1/x)} = \log^{-\beta}\left(\frac{1}{x}\right).$$

Clearly l is increasing close to zero, as required in hypothesis (ii) and

$$l'(x) = \frac{\beta}{x} \log^{-(\beta+1)}\left(\frac{1}{x}\right),$$

so $l' \in RV_0(-1)$. Finally $x \mapsto xl'(x) = \beta/\log^{\beta+1}(1/x)$ is an increasing function close to zero as needed in part (ii). Therefore we may conclude from Theorem 93 that

$$\lim_{n\to\infty} \frac{X_n}{\exp\left(-\left((\beta+1)t_n\right)^{1/(\beta+1)}\right)} = 1.$$

In the proof of Theorem 93, we will frequently need to use the fact that solutions of the numerical scheme decay *a priori* faster than any negative power of n. We prove the veracity of this observation now.

Proposition 21. *Let $|f| \in RV_0(1)$, $x \mapsto |f(x)|/x$ be asymptotic to an increasing C^1 function and $f(x) < 0$ for all $x > 0$. If $L = -\infty$, then*

$$\lim_{n\to\infty} \frac{\log X_n}{\log n} = -\infty.$$

Proof. By Theorem 81 we have that

$$\lim_{n\to\infty} \frac{-\log X_n}{(-\log \circ F^{-1})(t_n)} = 1,$$

and because $t_n = n\Delta + t^*$ for all n sufficiently large and $F^{-1}(t+c) \sim F^{-1}(t)$ as $t \to \infty$ for any $c \in \mathbb{R}$, we have that

$$\lim_{n\to\infty} \frac{-\log X_n}{(-\log \circ F^{-1})(n\Delta)} = 1.$$

Since $|f| \in RV_0(1)$, it follows that $F \in RV_0(0)$ and so

$$\lim_{x\to 0^+} \frac{\log F(x)}{\log x} = 0,$$

with the limit being approached from below. Hence

$$\lim_{x\to 0^+} \frac{-\log x}{(-\log \circ F^{-1})(x)} = -\infty.$$

Therefore as $F^{-1}(n\Delta) \to 0$ as $n \to \infty$, this limit implies

$$\lim_{n\to\infty} \frac{(-\log \circ F^{-1})(n\Delta)}{\log n\Delta} = -\infty,$$

and as $\log(n\Delta) \sim \log n$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{-\log X_n}{\log n} = \lim_{n\to\infty} \left(\frac{-\log X_n}{(-\log \circ F^{-1})(n\Delta)} \cdot \frac{(-\log \circ F^{-1})(n\Delta)}{\log n} \right) = \infty,$$

proving the claim. □

Proof of Theorem 93. Let $X_n = e^{-Z_n}$, where for all n sufficiently large, we have

$$Z_{n+1} = Z_n + F_n + R_{n+1},$$

where as before

$$F_n := \Delta \left(\frac{|f(X_n)|}{X_n} + \frac{g^2(X_n)}{2X_n^2} \right) \quad \text{and} \quad R_{n+1} := \frac{\sqrt{\Delta} g(X_n)}{X_n} \xi_{n+1}.$$

Define $A(z) := F(e^{-z})$. Since $F(x) = \int_x^1 1/|f(u)|\,du$ then $F'(x) = -1/|f(x)| = 1/f(x)$ and $F''(x) = -f'(x)/f^2(x)$. Thus $A'(z) = -e^{-z}/f(e^{-z}) = e^{-z}/|f(e^{-z})|$ and $A''(z) = e^{-z}/f(e^{-z}) - e^{-2z}f'(e^{-z})/f^2(e^{-z}) = -e^{-z}/|f(e^{-z})| - e^{-2z}f'(e^{-z})/f^2(e^{-z})$. Furthermore, by Taylor's Theorem there is $\eta_{n+1} \in [\min(Z_n, Z_{n+1}), \max(Z_n, Z_{n+1})]$ such that

$$A(Z_{n+1}) = A(Z_n) + A'(Z_n)(F_n + R_{n+1}) + \tfrac{1}{2}A''(\eta_{n+1})(F_n + R_{n+1})^2.$$

This expansion gives

$$F(X_{n+1}) - F(X_n) = \Delta + \frac{\Delta g^2(X_n)}{2X_n|f(X_n)|} + \frac{\sqrt{\Delta}g(X_n)}{|f(X_n)|}\xi_{n+1} + \frac{1}{2}A''(\eta_{n+1})(F_n^2 + 2F_nR_{n+1} + R_{n+1}^2).$$

Define $\tilde{\eta}_{n+1} := e^{-\eta_{n+1}} \in (\min(X_n, X_{n+1}), \max(X_n, X_{n+1}))$ and

$$B(x) := \frac{x}{f(x)} - \frac{x^2 f'(x)}{f^2(x)},$$

so that $A''(z) = B(e^{-z})$. Hence $A''(\eta_{n+1}) = B(\tilde{\eta}_{n+1})$. Define $\zeta_{n+1} := \xi_{n+1}^2 - 1$. Then

$$\begin{aligned}
&F(X_{n+1}) - F(X_n) - \Delta \\
&= \frac{\Delta g^2(X_n)}{2X_n|f(X_n)|} + \frac{B(\tilde{\eta}_{n+1})F_n^2}{2} + \frac{\sqrt{\Delta}g(X_n)}{|f(X_n)|}\xi_{n+1} + \frac{2B(\tilde{\eta}_{n+1})F_nR_{n+1}}{2} + \frac{B(\tilde{\eta}_{n+1})R_{n+1}^2}{2} \\
&= \frac{\Delta g^2(X_n)}{2X_n|f(X_n)|} + \frac{B(\tilde{\eta}_{n+1})F_n^2}{2} + \frac{2\left(B(\tilde{\eta}_{n+1}) - B(X_n)\right)F_nR_{n+1}}{2} + \frac{\sqrt{\Delta}g(X_n)}{|f(X_n)|}\xi_{n+1} \\
&\quad + \frac{2B(X_n)F_n\sqrt{\Delta}g(X_n)}{2X_n}\xi_{n+1} + \frac{\Delta B(X_n)g^2(X_n)}{2X_n^2}(\xi_{n+1}^2 - 1) + \frac{\Delta B(X_n)g^2(X_n)}{2X_n^2} \\
&\qquad + \frac{\Delta\left(B(\tilde{\eta}_{n+1}) - B(X_n)\right)g^2(X_n)}{2X_n^2}\xi_{n+1}^2 \\
&= \frac{\Delta g^2(X_n)}{2X_n|f(X_n)|} + \frac{\Delta B(X_n)g^2(X_n)}{2X_n^2} + \left(B(\tilde{\eta}_{n+1}) - B(X_n)\right)\left(F_nR_{n+1} + \frac{1}{2}R_{n+1}^2\right) \\
&\quad + \frac{B(\tilde{\eta}_{n+1})F_n^2}{2} + \frac{\sqrt{\Delta}\sqrt{\Delta}g(X_n)}{|f(X_n)|}\xi_{n+1} + \frac{\sqrt{\Delta}B(X_n)F_ng(X_n)}{X_n}\xi_{n+1} \\
&\qquad + \frac{\Delta B(X_n)g^2(X_n)}{2X_n^2}\zeta_{n+1}.
\end{aligned}$$

Let $\Delta B := B(\tilde{\eta}_{n+1}) - B(X_n)$. Then for $n \geq N(\omega)$,

$$D_n = \frac{-\Delta}{2} \cdot \frac{g^2(X_n)}{f^2(X_n)} \cdot f'(X_n) + \Delta B(F_nR_{n+1} + \frac{1}{2}R_{n+1}^2) + \frac{1}{2}B(\tilde{\eta}_{n+1})F_n^2,$$

and for $n \geq 0$

$$\begin{aligned}
T_{n+1} &= \frac{\sqrt{\Delta}g(X_n)}{|f(X_n)|}\left(1 + \frac{|f(X_n)|B(X_n)F_n}{X_n}\right)\xi_{n+1} + \frac{B(X_n)\Delta g^2(X_n)}{2X_n^2} \cdot \zeta_{n+1} \\
&=: K_n\xi_{n+1} + L_n\zeta_{n+1},
\end{aligned}$$

so that for some $N = N(\omega)$ and all $n \geq N(\omega)$

$$F(X_{n+1}) - F(X_n) = \Delta + D_n + T_{n+1}.$$

For $0 \leq n \leq N(\omega)$ set $D_n := F(X_{n+1}) - F(X_n) - \Delta - T_{n+1}$. Then $\forall\, n \geq 0$

$$F(X_{n+1}) - F(X_n) = \Delta + D_n + T_{n+1}, \quad n \geq 0.$$

Then $F(X_n) = F(X_0) + n\Delta + \sum_{j=0}^{n-1} D_j + M(n)$, $n \geq 0$ where $M(n) = \sum_{j=0}^{n-1} T_{j+1}$. Clearly M is a martingale. Moreover

$$\begin{aligned}\mathbb{E}\left[T_{j+1}^2|\mathscr{F}_n\right] &= \mathbb{E}\left[(K_n\xi_{n+1} + L_n\zeta_{n+1})^2|\mathscr{F}_n\right] \\ &= \mathbb{E}\left[K_n^2\xi_{n+1}^2 + 2K_nL_n\xi_{n+1}\zeta_{n+1} + L_n^2\zeta_{n+1}^2|\mathscr{F}_n\right] \\ &= K_n^2 + 2K_nL_n\mathbb{E}\left[\zeta_{n+1}\xi_{n+1}\right] + L_n^2\mathbb{E}\left[\zeta_{n+1}^2\right] = K_n^2 + 2L_n^2.\end{aligned}$$

The last line follows because in the second term $\zeta_{n+1} = \xi_{n+1}^2 - 1$ and $\zeta_{n+1}\xi_{n+1} = \xi_{n+1}^3 - \xi_{n+1}$ so $\mathbb{E}\left[\zeta_{n+1}\xi_{n+1}\right] = \mathbb{E}\left[\xi_{n+1}^3\right] - \mathbb{E}\left[\xi_{n+1}\right] = 0$ and in the third term $\zeta_{n+1}^2 = \xi_{n+1}^4 - 2\xi_{n+1}^2 + 1$ and $\mathbb{E}\left[\zeta_{n+1}^2\right] = \mathbb{E}\left[\xi_{n+1}^4\right] - 2\mathbb{E}\left[\xi_{n+1}^2\right] + 1 = 2$. Thus

$$\mathbb{E}\left[T_{n+1}^2|\mathscr{F}_n\right] = \frac{\Delta g^2(X_n)}{f^2(X_n)}\left(1 + B(X_n)F_n\frac{|f(X_n)|}{X_n}\right)^2 + \tfrac{1}{2}B^2(X_n)\cdot\frac{\Delta^2 g^4(X_n)}{X_n^4}.$$

Then with $\tilde{f}(x) = -f(x) = |f(x)|$, we see that as $n \to \infty$

$$\begin{aligned}B(X_n)F_n\frac{|f(X_n)|}{X_n} \sim \Delta B(X_n)\frac{|f(X_n)|^2}{X_n^2} &= \Delta\left(\frac{-X_n}{\tilde{f}(X_n)} + \frac{\tilde{f}'(X_n)X_n^2}{\tilde{f}^2(X_n)}\right)\frac{\tilde{f}^2(X_n)}{X_n^2} \\ &= \Delta\left(\frac{-\tilde{f}(X_n)}{X_n} + \tilde{f}'(X_n)\right).\end{aligned}$$

By definition then $\tilde{f}(x) = xl(x)$ and $\tilde{f}'(x) = xl'(x) + l(x)$. Hence as l is increasing

$$\tilde{f}'(x) - \frac{\tilde{f}(x)}{x} = xl'(x) =: \lambda(x) > 0,$$

and λ is increasing. Thus $B(X_n)F_n|f(X_n)|/X_n \sim \Delta\lambda(X_n)$ as $n \to \infty$. Also $\lambda(X_n) \to \lambda(0^+) \in [0, \infty)$ as $n \to \infty$, as λ is increasing. Suppose $\lambda(0^+) > 0$. Then $xl'(x) \to \lambda(0^+)$. But $\lambda(0^+) = \lim_{x\to 0}\tilde{f}(x)/x = 0$ forcing a contradiction. Hence $\lambda(0^+) = 0$. Thus $B(X_n)F_n|f(X_n)|/X_n \sim \Delta\lambda(X_n) \to 0$ as $n \to \infty$. Thus the first term in $\mathbb{E}\left[T_{n+1}^2|\mathscr{F}_n\right]$ is $\Delta g^2(X_n)(1 + o(1))/f^2(X_n)$ as $n \to \infty$. Also

$$B(X_n) = \frac{X_n^2}{\tilde{f}(X_n)^2}\left(\frac{-\tilde{f}(X_n)}{X_n} + \tilde{f}'(X_n)\right) = \frac{\lambda(X_n)X_n^2}{\tilde{f}^2(X_n)},$$

so

$$B^2(X_n)\frac{g^4(X_n)}{X_n^4} = \lambda^2(X_n)\cdot\frac{X_n^4}{\tilde{f}^4(X_n)}\cdot\frac{g^4(X_n)}{X_n^4} \quad = \quad \lambda^2(X_n)\cdot\frac{g^4(X_n)}{\tilde{f}^4(X_n)}.$$

Thus

$$\mathbb{E}\left[T_{n+1}^2|\mathscr{F}_n\right] = \Delta\frac{g^2(X_n)}{f^2(X_n)}\left(1+o(1)+\frac{1}{2}\cdot\Delta\lambda^2(X_n)\cdot\frac{g^2(X_n)}{\tilde{f}^2(X_n)}\right), \quad \text{as } n\to\infty,$$

Now

$$\frac{g^2(X_n)}{\tilde{f}^2(X_n)} \leq \frac{CX_n^{1+\theta}|f(X_n)|}{\tilde{f}^2(X_n)} = \frac{CX_n^{1+\theta}}{|f(X_n)|}.$$

Since $l'\in RV_0(-1), l\in RV_0(0)$ and so $\tilde{f}\in RV_0(1)$. Therefore $x\mapsto x^{1+\theta}/\tilde{f}(x)\in RV_0(\theta)$. Since $X_n\to 0$ faster than any power as $n\to\infty$, the sequence $n\mapsto X_n^{1+\theta}/\tilde{f}'(X_n)$ tends to zero faster than any negative power of n. Since $\lambda(X_n)\to 0$ as $n\to\infty$ and $g^2(X_n)/\tilde{f}(X_n)\to 0$ as $n\to\infty$ we have that

$$\mathbb{E}\left[T_{n+1}^2|\mathscr{F}_n\right] = \frac{\Delta^2 g^2(X_n)}{\tilde{f}(X_n)}(1+o(1)).$$

Thus $g^2(X_n)/\tilde{f}^2(X_n)$ is a summable sequence a.s.. Hence $\langle M\rangle(n) = \sum_{j=0}^{n-1}\mathbb{E}\left[T_{j+1}^2|\mathscr{F}_j\right]$ tends to a finite limit as $n\to\infty$ and so $M(n)$ tends to a finite limit as $n\to\infty$, contingent on M being a martingale. Since the projective property for martingales holds it suffices to prove that $\mathbb{E}\left[X_n^\theta\right]<\infty$ for all n and this can be done using an inductive argument in Theorem 72. The first term on the right hand side of $D_n, D_n^{(1)}$ is also summable because $\tilde{f}(X_n)\to 0$ as $n\to\infty$ and $g^2(X_n)/f^2(X_n)$ is summable. Next, as $Z_{n+1} = Z_n + F_n + R_{n+1}$, then

$$\frac{X_{n+1}}{X_n} = e^{-(F_n+R_{n+1})}.$$

Since $F_n \sim \Delta\tilde{f}(X_n)/X_n \to 0$ as $n\to\infty$, showing $R_{n+1}\to 0$ as $n\to\infty$ implies $X_{n+1}/X_n\to 1$ and so $\tilde{\eta}_{n+1}\sim X_n$ as $n\to\infty$, since $\tilde{\eta}_{n+1}$ is contained in the interval $[\min(X_n,X_{n+1}),\max(X_n,X_{n+1})]$. To show $R_{n+1}\to 0$ we bound according to

$$\begin{aligned}|R_{n+1}| = \sqrt{\Delta}\frac{|g(X_n)|}{X_n}|\xi_{n+1}| \quad &\leq \quad \sqrt{\Delta}\frac{CX_n^{1/2+\theta/2}|f(X_n)|^{1/2}}{X_n}|\xi_{n+1}| \\ &= \quad C\sqrt{\Delta}X_n^{\theta/2-1/2}|f(X_n)|^{1/2}|\xi_{n+1}|.\end{aligned}$$

Since $\tilde{f}\in RV_0(1), x\mapsto\tilde{f}(x)^{1/2}x^{\theta/2-1/2}\in RV_0(\theta/2)$. Therefore, as $X_n\to 0$ faster than any negative power of n, and $|\xi_{n+1}| = O(\sqrt{\log n})$ as $n\to\infty$, $R_{n+1}\to 0$ as $n\to\infty$. Therefore as $B(x) = \lambda(x)x^2/\tilde{f}(x)^2$, $\lambda(x) = xl'(x)$ and $\lambda\in RV_0(0)$, we have that $B\in RV_0(0)$. Hence, as $\tilde{\eta}_{n+1}\sim X_n$ as $n\to\infty$, we have that $B(\tilde{\eta}_{n+1})\sim B(X_n)$. Thus the third term in D_n obeys

$$D_n^{(3)} := \frac{B(\tilde{\eta}_{n+1})F_n^2}{2} \sim \frac{B(X_n)F_n^2}{2} \sim \frac{B(X_n)\Delta^2\tilde{f}^2(X_n)}{2X_n^2}.$$

Thus

$$D_n^{(3)} \sim \frac{\Delta^2\lambda(X_n)}{2} \cdot \frac{\tilde{f}^2(X_n)}{X_n^2} \cdot \frac{X_n^2}{\tilde{f}^2(X_n)} = \frac{\Delta^2\lambda(X_n)}{2}.$$

Define the second term in D_n to be

$$D_n^{(2)} := (B(\tilde{\eta}_{n+1}) - B(X_n))\left(F_nR_{n+1} + \tfrac{1}{2}R_{n+1}^2\right) = (\Delta B)F_n\left(1 + \frac{R_{n+1}}{2F_n}\right)\frac{\sqrt{\Delta}g(X_n)}{X_n}\cdot\xi_{n+1}.$$

Now

$$\frac{R_{n+1}}{F_n} = \frac{\sqrt{\Delta}g(X_n)/X_n \cdot \xi_{n+1}}{\Delta\left(\tilde{f}(X_n)/X_n + g^2(X_n)/2X_n^2\right)} = \frac{\Delta^{-1/2}g(X_n)}{\tilde{f}(X_n)}\left(1 + \frac{g^2(X_n)}{2X_n\tilde{f}(X_n)}\right)^{-1} \cdot \xi_{n+1}.$$

The last factor tends to 1 because $g^2(x) \le Cx^{1+\theta}\tilde{f}(x)$ implies $g^2(x)/(x\tilde{f}(x)) \to 0$ as $x \to 0^+$. Also

$$\left|\frac{g(X_n)}{\tilde{f}(X_n)}\xi_{n+1}\right| \le \frac{CX_n^{1/2+\theta/2}\tilde{f}(X_n)^{1/2}}{\tilde{f}(X_n)}|\xi_{n+1}| = CX_n^{1/2+\theta/2}\tilde{f}(X_n)^{-1/2}|\xi_{n+1}|,$$

since $\tilde{f} \in RV_0(1), x \mapsto x^{1/2+\theta/2}\tilde{f}(x)^{-1/2} \in RV_0(\theta/2)$ so $n \mapsto X_n^{1/2+\theta/2}\tilde{f}(X_n)^{-1/2}$ tends to zero faster than any negative power of n, due to the faster than polynomial decay of $X_n \to 0$. Since $|\xi_{n+1}| = O(\sqrt{\log n})$, we have $g(X_n)\xi_{n+1}/\tilde{f}(X_n) \to 0$. Hence $R_{n+1}/F_n \to 0$ as $n \to \infty$. Thus as $n \to \infty$

$$D_n^{(2)} = (\Delta B) \cdot \frac{\Delta\tilde{f}(X_n)}{X_n} \cdot \frac{\sqrt{\Delta}g(X_n)}{X_n} \cdot (1 + o(1)) \cdot \xi_{n+1}.$$

We have that $B(\tilde{\eta}_{n+1}) \sim B(X_n)$. Thus $\Delta B = B(\tilde{\eta}_{n+1}) - B(X_n) = o(B(X_n))$ and so

$$\begin{aligned} D_n^{(2)} = o\left(B(X_n) \cdot \frac{\Delta^{3/2}\tilde{f}(X_n)g(X_n)}{X_n^2} \cdot \xi_{n+1}\right) &= o\left(\frac{\lambda(X_n)X_n^2}{\tilde{f}(X_n)^2} \cdot \frac{\tilde{f}(X_n)g(X_n)}{X_n^2} \cdot \xi_{n+1}\right) \\ &= o\left(\lambda(X_n) \cdot \frac{g(X_n)}{\tilde{f}(X_n)} \cdot \xi_{n+1}\right). \end{aligned}$$

Hence $D_n^{(3)} \sim \frac{1}{2}\Delta^2\lambda(X_n)$ and since $g(X_n)\xi_{n+1}/\tilde{f}(X_n) \to 0$ we see that $D_n^{(2)} = o(\lambda(X_n))$. Hence we have that

$$D_n^{(4)} := D_n^{(2)} + D_n^{(3)} \sim \frac{1}{2}\Delta^2\lambda(X_n) \quad \text{as } n \to \infty,$$

and $D_n = D_n^{(1)} + D_n^{(4)}$ with $D_n^{(1)}$ summable. Hence

$$F(X_n) = F(X_0) + n\Delta + \sum_{j=0}^{n-1} D_j^{(1)} + M(n) + \sum_{j=0}^{n-1} D_j^{(4)} =: c(n) + n\Delta + \sum_{j=0}^{n-1} D_j^{(4)},$$

where $c(n)$ tends to a finite limit and $D_j^{(4)} \sim \Delta^2\lambda(X_j)/2$ as $j \to \infty$. Note that $\lambda(x) = xl'(x) > 0$ for all x sufficiently small. Thus there exists $j^* \in \mathbb{N}$ such that $D_j^{(4)} > 0 \,\forall\, j \geq j^*$. We have that $n \geq j^* + 1$ implies

$$\begin{aligned} F(X_n) &= c(n) + n\Delta + \sum_{j=0}^{j^*-1} D_j^{(4)} + \sum_{j=j^*}^{n-1} D_j^{(4)} \\ &\geq c(n) + \sum_{j=0}^{j^*-1} D_j^{(4)} + n\Delta, \end{aligned}$$

and clearly as $c(n) \to c(\infty)$ as $n \to \infty, c(n) > c^*$ for all $n^* \geq j^* + 1$. Thus there is $F^* \in \mathbb{R}$ such that $F(X_n) > F^* + n\Delta$, $\forall\, n \geq j^* + 1$. Hence $X_n < F^{-1}(F^* + n\Delta)$, $n \geq j^* + 1$. Define $S_n := \sum_{n=0}^{j-1} D_j^{(4)}$. Clearly for $\epsilon > 0$ there is an $N_1(\epsilon) \in \mathbb{N}$ such that for $n > N_1(\epsilon)$

$$D_n^{(4)} < (1+\epsilon) \cdot \frac{\Delta^2}{2}\lambda(X_n),$$

and indeed there is a finite K^* such that $D_n^{(4)} < K^* \cdot \Delta^2/2 \cdot \lambda(X_n)$ for $n \geq j^* + 1$. Thus $n \geq j^* + 1$ the monotonicity of λ implies that

$$S_{j^*+1} \leq S_n \leq S_{j^*+} + K^* \frac{\Delta}{2} \sum_{j=j^*+1}^{n-1} \Delta\lambda\left(F^{-1}\left(F^* + j\Delta\right)\right).$$

Let $F^* + j\Delta \leq x \leq F^* + (j+1)\Delta$. Then

$$F^{-1}\left(F^* + j\Delta\right) > F^{-1}\left(x\right) > F^{-1}\left(F^* + (j+1)\Delta\right),$$

so $\lambda\left(F^{-1}\left(F^* + j\Delta\right)\right) > \lambda\left(F^{-1}\left(x\right)\right) > \lambda\left(F^{-1}\left(F^* + (j+1)\Delta\right)\right)$. Hence for $n \geq j^* + 2$

$$\begin{aligned} S_{j^*+1} \leq S_n &\leq S_{j^*+1} + \frac{\Delta K^*}{2} \sum_{j=j^*+1}^{n-1} \int_{F^*+(j-1)\Delta}^{F^*+j\Delta} \lambda\left(F^{-1}\left(x\right)\right)\, dx \\ &= S_{j^*+1} + \frac{\Delta K^*}{2} \int_{F^*+j^*\Delta}^{F^*+(n-1)\Delta} \lambda\left(F^{-1}\left(x\right)\right)\, dx. \end{aligned}$$

Now integrating by substitution

$$\begin{aligned} S_{j^*+1} \leq S_n &\leq S_{j^*+1} + \frac{\Delta K^*}{2} \int_{F^{-1}(F^*+j^*\Delta)}^{F^{-1}(F^*+(n-1)\Delta)} \lambda(u) \cdot \frac{-1}{\tilde{f}(u)}\, du \\ &= S_{j^*+1} + \frac{\Delta K^*}{2} \int_{F^{-1}(F^*+(n-1)\Delta)}^{F^{**}} \frac{ul'(u)}{\tilde{f}(u)}\, du \\ &= S_{j^*+1} + \frac{\Delta K^*}{2} \int_{F^{-1}(F^*+(n-1)\Delta)}^{F^{**}} \frac{l'(u)}{l(u)}\, du \\ &= S_{j^*+1} + \frac{\Delta K^*}{2} \left(\log l(F^{**}) - \log l\left(F^{-1}(F^* + (n-1)\Delta)\right)\right). \end{aligned}$$

Hence for $n \geq j^* + 2$, there are $\underline{S}, \bar{S}, \bar{K}, \bar{F}$ such that

$$\underline{S} \leq S_n \leq \bar{S} + \bar{K} \log\left(\frac{1}{l\left(F^{-1}(\bar{F} + n\Delta)\right)}\right).$$

Using this estimate, $X_n < F^{-1}\left(F^* + n\Delta\right)$ for $n \geq j^*+1$ and $X_n = F^{-1}(c(n)+n\Delta+S_n)$ we will now obtain the claimed asymptotic behaviour of X_n. First $h(X_n) = \Delta$ for all n sufficiently large, so we may write $t_n = t_N + (n-N)\Delta$. This implies

$$\limsup_{n\to\infty} \frac{X_n}{F^{-1}(n\Delta)} \leq 1 \quad \text{and} \quad \limsup_{n\to\infty} \frac{X_n}{F^{-1}(t_n)} \leq 1,$$

due to the sublinearity of f at zero. On the other hand we have $X_n = F^{-1}\left(c(n) + n\Delta + S_n\right)$ so

$$X_n \geq F^{-1}\left(n\Delta + S^* + \bar{K}\log\left(\frac{1}{l(F^{-1}(\bar{F} + n\Delta))}\right)\right) = F^{-1}\left(n\Delta + I(n\Delta)\right),$$

where

$$I(t) = S^* + \bar{K}\log\left(\frac{1}{l(F^{-1}(\bar{F}+t))}\right).$$

We want to show that

$$\lim_{n\to\infty} \frac{X_n}{F^{-1}(n\Delta)} \geq 1.$$

However

$$\liminf_{n\to\infty} \frac{X_n}{F^{-1}(n\Delta)} \geq \liminf_{n\to\infty} \frac{F^{-1}(n\Delta + I(n\Delta))}{F^{-1}(n\Delta)} \geq \liminf_{t\to\infty} \frac{F^{-1}(t+I(t))}{F^{-1}(t)}.$$

Now by Lemma 46, we have $F^{-1}(t+I(t))/F^{-1}(t) \to 1$ as $t \to \infty$ once

$$\lim_{x\to 0^+} \frac{\tilde{f}(x)}{x} I(F(x)) = 0. \tag{13.46}$$

To get (13.46) it is enough to show that

$$\lim_{t\to\infty} \frac{\tilde{f}(F^{-1}(t))}{F^{-1}(t)} \log\left(\frac{1}{l\left(F^{-1}(\bar{F}+t)\right)}\right) = 0.$$

Next as $l \in RV_0(0)$ and $F^{-1}(\bar{F}+t) \sim F^{-1}(t), t \to \infty$ we have that $l(F^{-1}(\bar{F}+t)) \sim l(F^{-1}(t)), t \to \infty$ and as $\log \in RV_0(0)$ we have

$$\log\left(\frac{1}{l(F^{-1}(\bar{F}+t))}\right) \sim \log\left(\frac{1}{l(F^{-1}(t))}\right), \quad t \to \infty.$$

Hence as $\tilde{f}(x)/x = l(x) \to 0$ as $x \to 0^+$, we have

$$\begin{aligned}
\lim_{t\to\infty} \frac{\tilde{f}(F^{-1}(t))}{F^{-1}(t)} \log\left(\frac{1}{l(F^{-1}(\bar{F}+t))}\right) &= \lim_{t\to\infty} \frac{\tilde{f}(F^{-1}(t))}{F^{-1}(t)} \log\left(\frac{1}{l(F^{-1}(t))}\right) \\
&= \lim_{x\to 0^+} \frac{\tilde{f}(x)}{x} \log\left(\frac{1}{l(x)}\right) \\
&= \lim_{x\to 0^+} \frac{\tilde{f}(x)}{x} \log\left(\frac{x}{\tilde{f}(x)}\right) = \lim_{y\to 0^+} y \log\left(\frac{1}{y}\right) = 0.
\end{aligned}$$

Thus (13.46) holds and $\lim_{t\to\infty} F^{-1}(t+I(t))/F^{-1}(t) = 1$. Hence

$$\liminf_{n\to\infty} \frac{X_n}{F^{-1}(n\Delta)} \geq 1,$$

and the claim holds. □

Chapter 14

Future Work

In this short chapter, we indicate some other aspects of the work which we have started to investigate, or whose study forms a natural sequel to the results in this thesis.

14.1 Strong Approximation

In this work, we have chosen to discretise the SDE

$$dX(t) = f(X(t))\,dt + g(X(t))\,dB(t) \tag{14.1}$$

by discretising the transformed SDE

$$dZ(t) = \tilde{f}(Z(t))\,dt + \tilde{g}(Z(t))\,dB(t)$$

where $Z(t) = -\log X(t)$ with adaptive stepsize according to

$$Z_{n+1} = Z_n + h(X_n)\tilde{f}(Z_n) + \sqrt{h(X_n)}\tilde{g}(Z_n)\xi_{n+1}, \quad n \geq 0 \tag{14.2}$$

where $X_{n+1} = e^{-Z_{n+1}}$ and $t_{n+1} = t_n + h(X_n)$ and $(\xi_n)_{n\geq 1}$ is a sequence of independent standard normal random variables. Clearly, X_n, Z_n and t_n are adapted to the natural filtration $\mathscr{G}_n$ generated by the ξ's. However, a strong approximation of e.g. Z would read

$$Z_{n+1} = Z_n + h_n\tilde{f}(Z_n) + \tilde{g}(Z_n)(B(t_{n+1}) - B(t_n)), \quad n \geq 0 \tag{14.3}$$

where $h_n = t_{n+1} - t_n$. It can be seen from this discretisation (14.3) that replacing $B(t_{n+1}) - B(t_n)$ by $\sqrt{h(X_n)}\xi_{n+1}$, as in (14.2), will not preserve the strong approximation, since the increment ΔB depends on t_n, whereas in (14.2) the ξ's are independent of t_n and other quantities that are $\mathscr{G}_n$-measurable.

The way in which this is tackled is to ensure that the sequence $(t_n)_{n\geq 0}$ is a sequence of stopping times adapted to the natural filtration of B. This can be achieved along the lines of Mao and Liu and Kelly, Rodkina and Rapoo.

To this end, we would modify the sequence defined by writing h_n instead of $h(X_n)$ where h_n is given by

$$h_n = \Delta \min\left(1, \frac{1}{\lfloor f(X_n)/X_n \rfloor}, \frac{1}{\lfloor g^2(X_n)/X_n^2 \rfloor}\right)$$

where $\lfloor x \rfloor$ denotes the integer part of $x \in [0, \infty)$ so that $\lfloor x \rfloor + 1 > x$ and $\Delta > 0$ is a small convergence parameter. This will make the sequence $(t_n)_{n\geq 0}$ a sequence of stopping times adapted to the filtration of B.

In order to recover the desired asymptotic results it is necessary to reconsider the proof of asymptotic stability, but only up to the point we obtain an estimate of the form

$$\lim_{n\to\infty} \frac{\psi(X_{n+1})}{\sum_{j=1}^{n} \phi(X_j)} = 1$$

A key ingredient of the new proofs would be to exploit the fact that ΔB is $\mathscr{F}(t_n)$-conditionally normally distributed, with conditional distribution

$$\Phi_{n+1}(t) = \frac{1}{\sqrt{2\pi(t_{n+1} - t_n)}} \int_{-\infty}^{t} e^{-x^2/(2(t_{n+1}-t_n))} dx,$$

noting in particular properties of the condition moments of ΔB_{n+1} e.g.

$$\mathbb{E}[\Delta B_{n+1} | \mathscr{F}(t_n)] = 0, \quad \text{a.s.},$$
$$\mathbb{E}[\Delta B_{n+1}^2 | \mathscr{F}(t_n)] = t_{n+1} - t_n, \quad \text{a.s.}$$

Consideration of strong convergence also motivates the choice of a power pre-transformation which we have presented here, since strong convergence in the new co-ordinate system would only guarantee control of quantities of the form

$$\mathbb{E}[\sup_{0\leq t\leq T} |\log X(t) - \log \hat{X}(t)|^p]$$

where $\hat{Z} = -\log \hat{X}$ is a suitable continuous time extension of the discrete skeleton generated by the numerical method.

14.2 Growth

The results on SDEs in this thesis, both in continuous and in discrete time, have been confined to stability problems. However, in the case that the solution of (14.1) obeys

$$\lim_{t\to T} X(t) = \infty, \quad \text{a.s.}$$

and $T := \inf\{t > 0 : X(t) \notin (0,\infty)\}$ is such that $T = \infty$ a.s., so that solutions grow but do not exhibit finite-time explosion, we can proceed as above with the identical continuous time analysis and numerical methods. The only change that is needed is to ask that monotonicity conditions or sub- or super-linearity conditions on f and g are satisfied at infinity. In the case, for example, that

$$x \mapsto f(x), \frac{f(x)}{x} \text{ are asymptotic to increasing functions as } x \to \infty,$$

$p(0+) = -\infty$ and

$$\lim_{x\to\infty} \frac{xf(x)}{g^2(x)} = L_\infty \in (1/2, \infty], \tag{14.4}$$

and $\int_1^\infty 1/f(x)\,dx = \infty$ we can show for F defined by

$$F(x) = \int_1^x \frac{1}{f(u)}\,du$$

that $X(t) > 0$ for all $t \geq 0$, $\lim_{t\to\infty} X(t) = \infty$ a.s. and

$$\lim_{t\to\infty} \frac{F(X(t))}{t} = 1 - \frac{1}{2L_\infty}, \quad \text{a.s.}$$

For the numerical scheme, we recover the same sort of result, contingent on the condition

$$\inf_{x>0} \frac{xf(x)}{g^2(x)} > \frac{1}{2} \tag{14.5}$$

holding. More specifically, if the scheme is generated by (14.2), then $X_n > 0$ for all $n \geq 0$, $\lim_{n\to\infty} X_n = \infty$ and

$$\lim_{n\to\infty} \frac{F(X_n)}{t_n} = 1 - \frac{1}{2L_\infty}, \quad \text{a.s.}$$

Analogues of subexponential growth results are also available, with both moderate noise conditions, such as (14.4), and small noise conditions such as

$$\text{There exists } \theta > 0 \text{ such that } \limsup_{x\to\infty} \frac{g^2(x)}{x^{1+\theta} f(x)} < \infty. \tag{14.6}$$

14.3 Explosion

We showed that the asymptotic behaviour at the blow-up time in the deterministic differential equation

$$x'(t) = f(x(t)), \quad t \in [0, T); \quad x(0) = \xi > 0$$

could be recovered by means of adaptive time-stepping when the time step is of size $h(x)$ for state x and h is given by

$$h(x) = \frac{\Delta(x)}{f'(x + \Delta(x) f'(x)/f(x))}$$

in the case that f and f' are increasing, as $\Delta(x) \to \Delta \in [0, \infty)$. In the case where f' is a rapidly growing function in the class Γ, and we take logarithmic pre-transformations, we have further shown that the approximations obey $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{\bar{F}(x_n)}{\hat{T}_h - t_n} = \frac{1 - e^{-\Delta}}{\Delta}$$

and

$$\bar{F}(x) = \int_x^\infty \frac{1}{f(u)}\, du,$$

where the step-size obeys $h(x) f'(x) \to \Delta$ as $x \to \infty$.

This approach can also be applied to determine the blow-up asymptotics for the SDE (14.1). Using the approach we developed in the small noise case (i.e., under condition (14.6)), we can show, provided that $p(0^+) > -\infty$ that there is an a.s. finite T such that $X(t) > 0$ for all $t \in [0, T)$, $\lim_{t\to T^-} X(t) = \infty$ a.s. and

$$\lim_{t\to T^-} \frac{\bar{F}(X(t))}{T - t} = 1, \quad \text{a.s.}$$

so the rate of explosion in the deterministic case is preserved under small noise.

Moreover, if one takes the adaptive time step

$$h(x) = \Delta \min\left(1, \frac{x}{f(x)}, \frac{1}{f'(x)}, \frac{x^2}{g^2(x)}\right)$$

and the condition (14.5) prevails, then for the logarithmically pre-transformed scheme we have $X_n > 0$ for all $n \geq 0$, $X_n \to \infty$ as $n \to \infty$, $t_n \to \hat{T}_h < \infty$ as $n \to \infty$ a.s. and

$$\lim_{n\to\infty} \frac{\bar{F}(X(t))}{\hat{T}_h - t_n} = \frac{1 - e^{-\Delta}}{\Delta} \tag{14.7}$$

Once again, this demonstrates that we have identified the critical order of magnitude of the step-size at which the asymptotic rate of growth is preserved at the singularity.

It should be mentioned that if, for example, $f' \in \mathrm{RV}_\infty(\beta - 1)$ for some $\beta > 1$, the same results hold; however, in this case (or the more general situations in which $f \in \mathrm{RV}_\infty(\beta)$ for $\beta > 1$, or $x \mapsto f(x)/x^{1+\theta}$ is asymptotically decreasing, and $x \mapsto f(x)/x$ is asymptotically increasing) it would suffice to take a step-size of order

$$h(x) = \Delta \min\left(1, \frac{x}{f(x)}, \frac{x^2}{g^2(x)}\right)$$

This gives $O(\Delta)$ estimates on

$$\liminf_{n\to\infty} \frac{\bar{F}(X_n)}{\hat{T}_h - t_n}, \quad \limsup_{n\to\infty} \frac{\bar{F}(X_n)}{\hat{T}_h - t_n},$$

and one recovers the limit (14.7) in the case when $f \in \mathrm{RV}_\infty(\beta)$.

14.4 Recurrence

Our results in this thesis, as well as those discussed in this last section, centre on discretised solutions of the SDE (14.1) in which $X(t)$ tends to 0 or ∞ with probability one.

However, it is also interesting to ask whether the logarithmically transformed scheme can also recover the dynamics when the solution is recurrent on $(0,\infty)$. This arises for the solution of (14.1), for instance, when

$$\lim_{x\to 0^+} \frac{xf(x)}{g^2(x)} =: L_0 > \frac{1}{2}, \quad \lim_{x\to\infty} \frac{xf(x)}{g^2(x)} =: L_\infty < \frac{1}{2}.$$

In particular, we have $T = \infty$ a.s.,

$$\mathbb{P}[\{\lim_{t\to\infty} X(t) = 0\} \cup \{\lim_{t\to\infty} X(t) = \infty\}] = 0$$

and

$$\mathbb{P}[\{\liminf_{t\to\infty} X(t) = 0\} \cap \{\limsup_{t\to\infty} X(t) = \infty\}] = 1 \tag{14.8}$$

If we consider the discretisation (14.2) with the usual step-size

$$h(x) = \Delta \min\left(1, \frac{x}{|f(x)|}, \frac{x^2}{g^2(x)}\right)$$

we can show that $L_0 > 1/2$ and $L_\infty < 1/2$ implies $t_n \to \hat{T}_h = \infty$ as $n \to \infty$ a.s.

$$\mathbb{P}[\{\lim_{n\to\infty} X_n = 0\} \cup \{\lim_{n\to\infty} X_n = \infty\}] = 0,$$

that $\liminf_{n\to\infty} X_n < \limsup_{n\to\infty} X_n$ a.s. and

$$\mathbb{P}[\{\liminf_{n\to\infty} X_n = 0\} \cup \{\limsup_{n\to\infty} X_n = \infty] = 1$$

Therefore, this shows the scheme does not settle down to a limit, and "fills out" the state space at least for very large or very small values. This does not preclude the better result

$$\mathbb{P}[\{\liminf_{n\to\infty} X_n = 0\} \cup \{\limsup_{n\to\infty} X_n = \infty\}] = 1$$

which is the appropriate discrete analogue of (14.8), but which we have not yet been able to prove.

14.5 Non-positive Processes

The logarithmic pre-transformation is of course designed to deal with discretisations of the solution of (14.1) in which $X(t) > 0$ for all $t \in [0, T)$ and the natural state space of the process is $(0, \infty)$.

It is rather natural to ask whether the numerical method can, in some sense, detect when solutions of (14.1) would have natural state space $S \supset (0, \infty)$ in the case when the solution starts at $X(0) = \zeta \in (0, \infty)$. Take as an example the case when $g(0) \neq 0$ and $xf(x) \to 0$ as $x \to 0^+$ with $f(x) < 0$ for $x > 0$, and we define f and g on the interval $(-\infty, \infty)$. Then clearly

$$T = \inf\{t > 0 : X(t) = 0\} < \infty, \quad \text{a.s.}$$

Moreover, using the modulus of continuity of standard Brownian motion and the martingale time-change theorem, one can show that

$$\limsup_{t \to T^-} \frac{\log X(t)}{\log(T - t)} \leq \frac{1}{2}, \quad \text{a.s.} \tag{14.9}$$

It should be noted in the case that f and g are regularly varying at zero, and finite-time stability results we have

$$\lim_{t \to T^-} \frac{\log X(t)}{\log(T - t)} = \lambda \neq \frac{1}{2} \tag{14.10}$$

(except in the case that the index of regular variation of g at zero is zero).

For the numerical scheme, we can show that $t_n \to \hat{T}_h < \infty$ a.s. and

$$\limsup_{n \to \infty} \frac{X_{n+1} - X_n}{\sqrt{h(X_n)}} = \infty, \quad \liminf_{n \to \infty} \frac{X_{n+1} - X_n}{\sqrt{h(X_n)}} = -\frac{|g(0)|}{\Delta}$$

as well as

$$\lim_{n \to \infty} \frac{\log X_n}{\log(\hat{T}_h - t_n)} = \frac{1}{2}. \tag{14.11}$$

In other words, the limits here are representative of the typical behaviour of the solution of an SDE at a point away from the natural boundary, whereas one sees limits of the form (14.10) with limit not equal to 1/2 at a finite-time hitting of a boundary, when that boundary is an equilibrium solution. It should also be noted that (14.11) is consistent with (14.9)

14.6 Preserving Dynamics in an Interval

In this thesis, we have considered SDEs in which the solution stays in $(0,\infty)$. Our approach has been to take transformations of the state space which are not especially reliant on the structure of the drift or diffusion coefficients. This motivates taking logarithmic pre-transformations.

It is reasonable to ask whether the important long-time dynamics can be preserved by discretisation for SDEs in the state space is another subinterval of $\mathbb{R}$. A semi-infinite interval I can be tackled in the same way as $(0,\infty)$ by simply making an affine transformation of I onto $(0,\infty)$, and then proceeding as before. For this to work properly it is however necessary that the finite end point of I is known explicitly.

In the case that the interval is finite and we have $I=(a,b)$ and $-\infty<a<b<\infty$, it is presumably general enough to work on the interval $I'=(0,1)$, once again supposing that a and b are known explicitly. Therefore, we consider a solution of (14.1) such that $X(t)\in(0,1)$ for all $t\in[0,T)$. This suggests that

$$f(0)=g(0)=0, \quad f(1)=g(1)=0, \quad g^2(x)>0 \quad \text{for all } x\in(0,1).$$

A simple C^2 and one-one mapping from $I'=(0,1)$ to $(0,\infty)$ is $T_1:(0,1)\to(0,\infty): x\mapsto T_1(x):=x/(1-x)$. Once again, this transformation is independent of the structure of the SDE being analysed. Then we define the one-one mapping and C^2 mapping $T_2:(0,\infty)\to(-\infty,\infty): x\mapsto T_2(x)=\log x$.

A possible programme for the simulation of the processs X is now as follows: consider the process

$$Z(t)=T_2(T_1(X(t)))=:T_3(X(t)), \quad t\in[0,T)$$

Clearly T_3 is in $C^2((0,1);(-\infty,\infty))$ is increasing, is known in closed-form i.e.,

$$T_3(x)=\log(x/(1-x)), \quad x\in(0,1)$$

and has closed-form inverse $T_3^{-1}:(-\infty,\infty)\to(0,1)$

$$T_3^{-1}(x)=\frac{1}{1-e^{-x}}$$

Moreover, the derivatives of T_3 are also known in closed-form. Therefore, given that we know f and g, using Itô's Lemma, Z obeys the SDE

$$dZ(t)=\left\{T_3'(X(t))f(X(t))+\frac{1}{2}T_3''(X(t))g^2(X(t))\right\}dt+T_3'(X(t))g(X(t))\,dB(t)$$

and as $Z(t) = T_3^{-1}(X(t))$, by defining

$$\tilde{f}(z) = T_3'(T_3^{-1}(z))f(T_3^{-1}(z)) + \frac{1}{2}T_3''(T_3^{-1}(z))g^2(T_3^{-1}(z)), \quad \tilde{g}(z) = T_3'(T_3^{-1}(z))g(T_3^{-1}(z))$$

we have

$$dZ(t) = \tilde{f}(Z(t))\,dt + \tilde{g}(Z(t))\,dB(t)$$

We would now seek to study the dynamics of Z by discretising it as before, and requesting that the step-size when the original SDE is at x is given by

$$h(x) = \Delta \min\left(1, \frac{x}{|f(x)|}, \frac{1-x}{|f(x)|}, \frac{x^2}{g^2(x)}, \frac{(1-x)^2}{g^2(x)}\right)$$

and with $X_0 = \zeta \in (0,1)$, $t_0 = 0$ and $Z_0 = T_3(\zeta)$ we have for $n \geq 0$:

$$\begin{aligned} Z_{n+1} &= Z_n + h(X_n)\tilde{f}(Z_n) + \sqrt{h(X_n)}\tilde{g}(X_n)\xi_{n+1}, \\ X_{n+1} &= T_3^{-1}(Z_{n+1}), \quad t_{n+1} = t_n + h(X_n). \end{aligned}$$

Therefore, the step-size at state z is $\tilde{h}(z) := h(T_3^{-1}(z))$.

14.7 Numerical simulations

This thesis has set out how we might perform numerical simulation of diverse ODEs or SDEs, but we have confined ourselves here to theoretical analysis. Clearly, an important part of future research is to demonstrate that the computer simulations conform broadly to the theory given here, and to investigate the sharpness of the theoretical results.

A related question is to ask how well the scheme with small parameter Δ approximates the true explosion time T. Work of Davila et al suggest for SDEs with $g = \mathrm{o}(f)$ and $h(x) = \Delta/f(x)$ that direct discretision of the SDE leads to $\hat{T}_h$ converging in distribution to T as $\Delta \to 0^+$. It would clearly be of interest to establish similar results for finite-time stability and explosion in our methods, which allow for asymptotically larger step-sizes.

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