Investigating students' learning of differential equations in physics

Author

Diarmaid Hyland B.Sc. (Hons)

Thesis submitted to Dublin City University for the degree of

Doctor of Philosophy

School of Mathematical Sciences

Centre for the Advancement of STEM Teaching and Learning

Dublin City University

September 2018

Research Supervisors

Dr Brien Nolan

Dr Paul van Kampen

Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Doctor of Philosophy is entirely my own work, and that I have exercised reasonable care to ensure that the work is original, and does not to the best of my knowledge breach any law of copyright, and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

Signed: ID No.: Date:

Table of Contents

Table of Abbreviations

Abbreviation	Meaning
PER	Physics Education Research
APOS Theory	Action Process Object Schema Theory
CASE	Cognitive Acceleration through Science Education
IO-DE	Inquiry Oriented Differential Equations
MMPS	Mathematical Modelling and Problem Solving
RME	Realistic Mathematics Education
M(E)R	Multiple (External) Representations
DCU	Dublin City University
ECTS	European Credit Transfer System
RA	Research Aim
RQ	Research Question
MMR	Mixed Methods Research
(O)DE	(Ordinary) Differential Equation
ZPD	Zone of Proximal Development
ADL	Actual Developmental Level
SoV	Separation of Variables
IFM	Integrating Factor Method
DF	Direction Field
CA	Continuous Assessment

List of Tables

List of Figures

Abstract

Diarmaid Hyland: Investigating students' learning of differential equations in physics

There are numerous cases in physics where the value of a quantity and changes in that quantity are related. For example, the speed of an object depends on its acceleration; the radioactivity of a sample depends on the amount of the sample present. Except in highly idealised settings, the analysis of these cases requires students to recognize, set up, and solve an *ordinary differential equation* (ODE). In many universities, ODEs are studied in mathematics before they are applied in physics. However, the aims of mathematicians and physicists can be very different. Mathematics modules tend to emphasise theoretical aspects of ODEs. In contrast, physics modules often emphasise modelling.

This project is a multi-stage investigation that began by identifying the issues experienced by physics students during their study of ODEs before addressing them through the design and implementation of a set of tutorials. Having surveyed a cohort of physics students who completed a typical service module on ODEs, we found that many of them possessed a fragmented concept image of ODEs and insufficient instrumental understanding. Fifteen tutorials were designed to address these issues. Issues with instrumental understanding (primarily the manipulation of exponents and indefinite integration), and broadening the students' concept image were the focuses.

The effectiveness of the tutorials was measured using immediate pre/post-testing, delayed post-tests, and interviews with students. Together, these revealed significant gains in the understanding of ODEs as well as an appreciation of the guided-inquiry approach employed. Although some gaps in instrumental understanding were found to remain, students are hampered more by an incomplete concept image of ODEs. By studying the conceptual difficulties of physics students with ODEs and designing a guided teaching-learning sequence, we have been able to improve students' conceptual understanding of ODEs without impacting negatively on their instrumental understanding.

Conference contributions and publications

Presentations

D. Hyland, P. van Kampen and B.C. Nolan. Investigating students' difficulties with differential equations in physics, **ICTMA**, Nottingham, July 2015.

D. Hyland, P. van Kampen and B.C. Nolan. Investigating students' difficulties with differential equations in physics, **CETL-MSOR Conference**, London, September 2015.

D. Hyland, P. van Kampen and B.C. Nolan. Investigating students' difficulties with differential equations in physics, **SMEC 2016**, Dublin, June 2016.

D. Hyland, P. van Kampen and B.C. Nolan. Investigating students' difficulties with differential equations in physics, **ICME-13**, Hamburg, July 2016.

D. Hyland, P. van Kampen and B.C. Nolan. Investigating students' difficulties with differential equations in physics, **CETL-MSOR Conference**, Loughborough, September 2016.

D. Hyland, P. van Kampen and B.C. Nolan. Investigating students' learning of differential equations in physics, **GIREP-MPTL 2018**, San Sebastian, July 2018.

Workshops

D. Hyland, P. van Kampen and B.C. Nolan. Investigating students' difficulties with differential equations in physics, **GIREP-ICPE-EPEC 2017**, Dublin, July 2017.

D. Hyland, P. van Kampen and B.C. Nolan. Investigating students' learning of differential equations in physics, **SMEC 2018**, Dublin, June 2018.

Posters

D. Hyland, P. van Kampen and B.C. Nolan. Investigating students' learning of differential equations in physics, **CERME-11**, Dublin, February 2017.

Papers

D. Hyland, P. van Kampen and B.C. Nolan. Outcomes of a service teaching module on ODEs for physics students. **International journal of mathematics education in science and technology**, 2017, **49** (5), p.743-758.

Acknowledgements

There are many people who helped tremendously at various stages of my doctoral studies. I want to start by thanking my supervisors, Paul and Brien. I appreciate the patience, belief, and commitment you've shown me during my time at DCU, particularly the last four years. I've seen first-hand the dedication and selflessness you exhibit towards other people and I aspire to replicate this across all facets of my life. You were, and continue to be, mentors and friends.

The Schools of Physical and Mathematical Sciences have also shown me great hospitality during my time here. I would like to thank the members of both schools for their support and guidance. Particular thanks are owed to Prof. Turlough Downes, Dr Eabhnat Ní Fhloinn, and Dr Sinéad Breen for their altruistic approach to my research, and to Dr Mossy Kelly for the rich conversations. Eabhnat also provided great guidance throughout my studies, particularly early on when I was still finding my way. For this I am eternally grateful. I'm especially grateful for the friendship of the other postgraduate students in both schools and their positive attitudes toward me and each other.

I was very fortunate to have my doctoral studies funded by the National Forum for the Enhancement of Teaching and Learning in Higher Education. The Forum is an exceptionally close-knit group of like-minded people who offered me more support than could be expected.

Finally, I would like to thank my family – Sandra, Mick, Emer, Conor, Sinéad, and Cathal. They provide me with the encouragement and support needed to embrace any challenge I encounter.

1. Introduction and background

The study of calculus is a crucial prerequisite for the study of introductory physics at undergraduate level; so much so that research into learning of limits, derivatives, and integration by physics students (among others) has grown substantially in recent years. The premise of these studies is that an improved understanding of how physics students learn calculus can be used to develop better instruction that will benefit them throughout their undergraduate degree.

We act on the same premise at a more advanced stage of students' development: we regard the study of ordinary differential equations (ODEs) as an important pillar in the study of intermediate and advanced physics. It is our hope that by carrying out a detailed investigation of students' learning of ODEs, we can develop and implement a revised curriculum that results in improved outcomes for all participants.

Five educational concepts are central to this research[1]: the characterisation and importance of understanding; an educational paradigm and instructional design; service teaching; mathematics in physics education research (PER); and teaching and learning of ODEs. We will briefly outline research on each of these as it pertains to this project before describing our project in more detail.

1.1. Importance and characterization of understanding

Understanding is the end goal for many educators, but it can carry different meanings depending on the user, the subject matter, and the context in which it is used. To many people, understanding is synonymous with 'knowing', but to education researchers its meaning is multi-faceted.

Certainly, understanding something requires knowledge. It is often (perhaps predominantly, within our education system) limited to knowing how to carry out a procedure, but less often does it include why that procedure is chosen or when it would be an inappropriate choice. When it expands to include these facets, it becomes more difficult to teach and harder to assess accurately [1]. This can cause educators to abandon their attempts in favour of the tried and tested methods of teaching and assessing the 'knowing' version of understanding.

The appreciation for the depth of the meaning of understanding within education research has resulted in theories of how students learn in general and subsequently in theories specific to

[1] The words "research", "project", and "study" are used synonymously throughout this thesis.

mathematics education. Educational psychologists like Jean Piaget and Lev Vygotsky paved the way for other researchers to develop discipline-specific theories like APOS Theory [2] and the three worlds of mathematics [3], tailored to the needs of the mathematics learner and educator. Other influential names in relation to student understanding of mathematics are Richard Skemp, Shlomo Vinner, David Tall, Ed Dubinsky, and Anna Sfard whose contributions to the field will be used extensively throughout this project. Particularly, the manner in which Skemp, Vinner, and Tall described understanding has provided mathematics educators with a useful shared language.

Skemp pioneered the distinction between understanding and knowledge by discussing *instrumental* and *relational understanding* [4]. In this paper, he defined relational understanding as "knowing what to do and why it should be done" and contrasted this with instrumental understanding, which he equated to having "rules without reasons". This dichotomy later expanded to include two more categories (logical [5] and symbolic [6]), though for the purposes of this research it is sufficient to describe only the initial categories. The hierarchy implied is quite similar to the one we outlined in the preceding paragraphs, where instrumental understanding can be seen as more superficial than relational understanding: the 'how to carry out a given procedure' as opposed to the 'when or why the procedure is relevant'.

The clarity of Skemp's distinction has had an impact with practitioners, but begs the question of why we see relational understanding emphasised so rarely in practice. To this he also provides the answer that "Relational understanding would take too long to achieve, and to be able to use a particular technique is all that these pupils are likely to need." [4, p.24]. Skemp explains that "The backwash effect of examinations", "Over-burdened syllabi", and "Difficulty of assessment" [4, p.24] all play roles in its rarity.

Another useful way of looking at understanding of mathematics is that of Tall and Vinner [7] who introduced the ideas of *concept definition* and *concept image* to the lexicon of mathematics educators. They regard the concept definition to be "a form of words used to specify that concept" [7, p.152] whereas they use the term concept image to describe "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" [7, p.152]. A distinction is also made as to how the learner accrues their concept definition and concept image. Concept definition "may be learnt by the individual in a rote fashion or more meaningfully learnt and related to a greater or lesser degree to the concept as a whole." [7, p.152]. Concept image "is built over the years through experiences of all kinds, changing as the individual meets new stimuli and matures." [7, p.152].

They also explain the evolution of a student's concept image over time, highlighting that it may not be coherent at all times:

> We shall call the portion of the concept image which is activated at a particular time the *evoked concept image*. At different times, seemingly conflicting images may be evoked. Only when conflicting aspects are evoked *simultaneously* need there be any actual sense of conflict or confusion. [7, p.152]

Vinner [8] goes on to explain that a concept image, and not a concept definition, is required to respond to concepts. Tall [9] builds on this point, explaining that "Concept definition (where the concept was introduced by means of a definition) will remain inactive or even be forgotten. In thinking, almost always the concept image will be evoked." [9, p.40]. Having provided a rich description of both terms, in addition to the instances where either is elicited through questioning, Tall [9] concludes that a more sophisticated approach to instruction is required in order for students to "form a more coherent concept" [9, p.40], something that the author admits is not easily achieved.

1.2. Educational paradigm and instructional design

In the opening section, the subtlety of the concept of understanding was discussed, as was the acknowledgement that its pursuit and achievement are complex. In order to maximise the chance of achievement, Gresalfi and Lester [10] explain that subscribing to an educational paradigm and allowing it to guide what happens in the classroom is necessary. In the present project, a constructivist approach to teaching and learning is adopted. Constructivism is chosen above other theories of learning primarily because of the scale and usefulness of relevant research that incorporate it in their conceptual frameworks [2, 14, 17, 24, 53, 82]. In particular, the theory has been well developed as it applies to mathematics [2, 3]. We will also draw on the guiding principles of constructivism at several stages throughout this research. This adds a level of consistency and transparency that strengthens the research and its findings. Constructivism will be explained briefly before its influence on our classroom practices will be discussed.

Constructivism says that students construct their own understanding by actively discussing and grappling with concepts, and we subscribe to this point of view. Social constructivism in particular prioritises the role of community in learning. Crotty [11] has highlighted the link that exists between one's worldview, theoretical lens, methodological approach, and methods of data collection, and explained how these elements should be consistent with one another. It follows then that our instructional design must build on constructivism.

This means that our activities must maximise the opportunity for students to construct their own knowledge in a classroom environment that supports communication, especially argumentation and cooperation.

Carpenter and Lehrer [12] explain how understanding impacts instruction. For them, there are five types of activities a classroom must provide students with an opportunity to engage with in order for 'learning with understanding to occur on a widespread basis'. The types of activities are

> a, develop appropriate relationships, b, extend and apply their mathematical knowledge, c, reflect about their own mathematical experiences, d, articulate what they know, and e, make mathematical knowledge their own. [12, p.24]

Further to this, Carpenter and Lehrer [12] identify the necessary dimensions of instruction that need to be considered in order to organise a classroom that enables students to engage in the aforementioned activities

> a, tasks or activities that engage students in the problems that they solve; b, tools that represent mathematical ideas and problem situations; and c, normative practices, which are the standards regulating mathematical activity, agreed on by the students and the teacher [12, p. 24]

Their work aligns clearly with research by McDermott [13], Shaffer and McDermott [14], Redish [15], Hestenes [16], and Doughty et al. [17], who agree that students must be actively engaged in the learning process to gain understanding.

When discussing alternative instructional designs taken by other researchers, we focus on attempts that are rooted in a constructivist paradigm: these place a far larger emphasis on the role of the student in the classroom than the lecturer, what Williams [18] calls a connectionist approach as opposed to a transmissionist approach. Within these different approaches, an emphasis on group work is prevalent, emphasising the social aspect of constructivism championed by Vygotsky. These approaches are in line with the overall goals of increasing understanding and the ability to solve problems (often in context). Numerous attempts at improving on traditional instruction have resulted in several names being used to describe pedagogical approaches that adopt these guiding principles. Active learning [19], cooperative learning [20], problem-based learning (PBL) [21], Cognitive Acceleration through Science Education (CASE) [22], guided-inquiry [23], and inquiry-based learning (IBL) [24] are some of the more prominent approaches that seek to apply the principles discussed above. Though none of these are identical, they all share a constructivist underpinning and as such, can be grouped together in contrast to a traditional lecture format.

Inevitably, a change in emphasis leaves less time for practising solution techniques and can lead to the omission of material which is a common concern of mathematicians. Yoshinobu and Jones [25] question this concern however, showing that students succeeded at least as well as their peers on later courses. McDermott [13] goes further, citing work of the Physics Education Group at the University of Washington, which shows that students can improve their conceptual understanding without impacting negatively on their performance on end-of-chapter problems. Within this body of work, there is a study [14] that uses pre/post-testing to show that guided-inquiry compared favourably to traditional instruction. Rasmussen and Kwon [28] also use testing to conclude of the Inquiry Oriented Differential Equations (IO-DE) project that "all IO-DE students from each of the three institutions, regardless of academic backgrounds and gender differences, outperformed traditionally taught comparison students." [28, p.193]. They also concluded of their inquiry-oriented approach that

> It can help students build the type of conceptual understanding that makes mathematics meaningful to them. It can facilitate students' development of mathematical reasoning ability. It can positively influence their beliefs about knowing and doing mathematics. [28, p.194]

In addition to the effect on student achievement, the pedagogies employed by reform curricula are also appreciated by students. Walker [29] explains that in a study conducted at the University of Manchester on the effectiveness of group work, students reported

> 1. Groupwork helped them to consolidate and further explore material covered in lectures. 2. Peer discussions facilitated their understanding of the lecture material. 3. Groupwork helped them to retain their concentration levels. And 4. They felt more able to ask for clarification or help from their peers than from the lecturer. [29, p.78]

Cui et al. [26] also noted in their own study that the students believed an increased emphasis on conceptual understanding would better facilitate their application of mathematical knowledge in physics.

Despite lecturing still being the dominant teaching approach employed by mathematicians in university courses [30], there is a large body of research that advocates alternative pedagogical approaches be taken in service mathematics [31, 32] and in service teaching of ODEs specifically [32-38]. The studies cited in this section show that successfully introducing such an approach increases performance and is positively received by the students. An intervention that shared these design principles would be of increased benefit to learners of mathematics at university level.

1.3. Service teaching

Service mathematics, defined as “mathematics as taught to non-mathematics specialists and students studying science, engineering, and other technical subjects” [39, p.1031], is widely practiced at universities worldwide. Its popularity results from the perceived benefit of students being taught by experts and is generally considered an effective way of transmitting information to a large cohort of students. That being said, service teaching is not without its drawbacks.

Service teaching of mathematics can often involve students solving exclusively decontextualised problems that place little to no emphasis on how the mathematics itself can help the students with the study of their own discipline [40]. The ability to use the mathematics they have learned in their chosen discipline is not shown by many students, even if they performed well in the service module [26, 40-45]. Beyond this, even students who achieve a pass grade in fundamental courses display ‘limited ability’ in more advanced modules [39].

Nankervis [46] reported several interesting findings after carrying out a review of service teaching at Royal Melbourne Institute of Technology. She stressed that the issues do not arise with service teaching itself (which has many potential benefits) but with several factors related to its delivery that affect students’ levels of satisfaction with service-taught modules. The study mentions the relevance of the content of service-taught courses to the students’ own discipline, the quality of teaching and learning, and assessments specifically as reasons for low satisfaction rates among students.

The issues observed internationally are also seen in Ireland [47], where Mathematics Lecturers reported that students displayed a “lack of fluency in basic arithmetic and algebra skills”, “Gaps (or in some cases absence) in basic prerequisite knowledge in important areas of the school syllabus”, and “An inability to use or apply mathematics except in the simplest or most practised way.” [48, p.3].

These issues may be partly due to course designers over-estimating the capability of students, as reported by Prendergast et al. [49] who made this point in relation to changes in the pre-university curriculum in Ireland. Another area for improvement with respect to course design is highlighted by Gill and O’Donoghue [50] who explain how a lack of industry involvement is a missed opportunity to improve the quality of the course being delivered. This practice of exclusion goes against the recommendations of Bajpai [51] who called for course design to involve this type of external input.

In addition to the issues with design and performance, there can be a disconnect between how students and lecturers view each other’s performance. It has been reported [50] in the Irish context that lecturers hold mathematics students and service students to different standards, and do not attempt to engage them with the concepts as much as they would mathematics students. This lack of engagement hampers the students’ conceptual understanding, which may affect their ability to apply it in their own studies, as has been observed for chemistry students learning mathematics [43]. The previous pages indicate that a change in practice is warranted from both parties.

Studies calling for a need for lecturers to be stimulating [52], to enable students to become independent learners [29], for changes in pedagogic approach [53], and for the immersion of mathematics in real-world problems [54-57] are becoming more widespread and are leading to more thoughtful conversations about the teaching and learning of mathematics, whether it be for mathematics students or students from other disciplines who take mathematics modules as part of their studies. Despite the substantial increase in research in this area, publications still call for more research on teaching and learning as well as the need for mathematicians and mathematics educators alike to become more reflective on their practice [58, 59].

Where service teaching of mathematics to physics students is concerned, Caballero et al. [60] conducted a study that found that students often struggle to link mathematics from previous service-taught modules with physics. A similar phenomenon is reported by Thompson et al. [61], who identified that students can successfully apply and manipulate mathematical equations but are unable to generate or interpret these equations. Stephan and Rasmussen [62] observe that less time is left to highlight the applications and understanding of the mathematics when a course focuses on algorithmic methods to solve DEs, making the transition to using the mathematics in physics more difficult. Redish and Kuo [63] observed that physicists and mathematicians make meaning with mathematics in different ways. This makes using the language of mathematics in physics more complex. Manogue and Dray [64] also used a language metaphor and suggested that mathematics and physics courses use a different dialect of the same language.

It has been observed that the aims of mathematicians can differ significantly from the aims of physicists. In the case of service mathematics modules on ODEs, mathematics lecturers tend to emphasise the theoretical aspects of DEs. Solution techniques are also studied. In physics modules, however, modelling is often emphasised [42, 63, 64] because of its relevance to physics. There are many cases in physics where the value of a quantity is related to changes in that quantity. For example, the change in temperature of a body depends on the difference in temperature between the body and its environment; the radioactivity of a sample depends on the amount of the sample present; the speed of an object depends on its acceleration. Except in highly idealised situations, the study of these and many other models requires students to recognise the need for the use of a DE, to set up the appropriate DE, to solve it, and interpret the solution. The study of ODEs also underpins the study of partial differential equations which have a major role in physics at undergraduate level (Maxwell's equations, wave equation, and heat equation for example). Therefore, students must not only apply mathematical knowledge to solve a DE, but should recognize that a DE is an appropriate tool to describe the mathematical structure of the problem in question. The difference in the use of DEs in physics and mathematics modules is clear, and the issues the students may have with solving them are compounded by the differences in the purpose of mathematics and physics teachers.

The disconnect between the content and approach of service-taught mathematics modules and the use of that content in physics modules leaves students under-prepared. Clearly, this under-preparedness remains an issue as the students progress through their undergraduate degree, causing further difficulty along the way.

1.4. Mathematics in physics education research

Physics Education Research (PER) has seen rapid growth in the recent past, with a significant amount of work focusing on mathematics in PER. Within this, many of the issues raised with service teaching apply specifically to physics students. One of the findings of such studies [26, 65-68] is that it is common for students to struggle to apply their mathematical knowledge in a physics context. Other common themes in PER are identifying difficulties [26], developing instruction [69], or developing teaching materials [70].

In a study of students' difficulties with vector calculus in electrodynamics, Bollen et al. [71] found "strong indications that traditional instruction is not sufficient" for physics students to fully understand mathematics in a physical context, in their case Maxwell's equations in electrodynamics. This study also recommended more emphasis being placed on linking mathematics and physics when designing instruction.

Mathematical modelling and problem solving in real world situations is becoming central to many reform curricula at third level. The idea of mathematical modelling being taught to students is not a new one, but it has seen a resurgence of late that has obvious potential benefits to mathematics in physics education. An exemplar is a module at the University of Leeds [72] called Mathematical Modelling and Problem Solving (MMPS). MMPS, as described by Savage and Grove [73, p.121] aims to "enable students to understand the key role that mathematical modelling plays in the investigation and solution of physical problems". It has been received very well by students whose performance on the assessment was "clearly positive". Of more significance however, was the student feedback which was highly complimentary of the methods of learning experienced during the module [73].

With respect to calculus in PER, there are studies on the concept of a limit [74], of differentials [66], and of integration from several perspectives [75]. Each of these studies provide great insight into student thinking and Jones [75] states that more often than not, integration as summation is the most useful approach for students to take. Less prominent, however, is work viewing integration as anti-differentiation which is perhaps most useful for students studying differential equations.

1.5. The teaching and learning of ODEs

Much of the research on the teaching and learning of differential equations focuses on the following themes: introducing a geometric approach to solving differential equations [79, 80]; students' conception of the solution to a differential equation [76-78]; and the effect of a change in pedagogy on conceptual understanding [32-36] as discussed in a previous section. In addition to these themes are studies focused on single facets of solving ODEs, such as Wittmann and Black's [81] work on separation of variables.

The issues with mathematics mentioned in previous sections are also found in students learning differential equations. Wittmann and Black [81] found that students who are otherwise strong algebraically struggled with algebraic manipulation in the context of separation of variables. This has many similarities to the studies that found students have difficulty using their mathematics in physics courses.

In addition to research on students' ability to solve ODEs analytically, the point is made by Stephan and Rasmussen [62] that graphing plays a larger role in the teaching and learning of differential equations than is typically seen in other maths modules. They mention graphical representations are a third method along with numerical and analytical techniques for the study and application of DEs. The authors talk about the importance of teaching this in a "bottom-up" manner which is in keeping with the Realistic Mathematics Education (RME) approach that influenced the IO-DE project. RME is a theory of instruction in mathematics that was first developed in The Netherlands [82]. The central thesis of the RME approach is emphasising the role of realistic experiences in the learning process [83]. A similar view of combining various approaches is encouraged by Artigue [84] who used

> didactical engineering in an instructional approach to differential equations (for first year university students of mathematics and physics) that, from the beginning coordinated algebraic, numerical, and graphical approaches with the solution of associated differential equations. [84, p.243].

Yerushalmy and Schwartz [85, p.16] mention direction fields specifically when discussing recent approaches to teaching differential equations, stating that "These representations are valuable and can lead to greater insight where applicable". Any change to increase the emphasis on graphical methods such as direction fields within a course would be seen as a step towards a change in focus (from procedural fluency to conceptual understanding) being made which is encouraged by Artigue et al. [39, p.1031]. Direction fields are an example of an external representation as described by Zhang and Norman [86, 87]. The importance of using Multiple External Representations (MER) is described by Ainsworth [88, 89], as is their role in supporting learning and problem solving. The inclusion of MER, while potentially beneficial, must be done carefully. Camacho-Machín et al. 'found evidence that students do not use graphical representations to explore meanings and mathematical relations and they experience difficulties to move back and forth from one type of representation to another' [78, p.1].

An emerging body of research on student learning in differential equations [77, 90] highlights several aspects of student reasoning surrounding the notion that solutions to differential equations are collections of functions. Also involving the concept of solution is the work done by Keene, Glass, and Kim [91] who developed a framework for Relational Understanding of Procedures in ODEs. Inspired by Starr's [92] suggestion that deep and shallow procedural knowledge is possible, and the work of Hassenbrank and Hodson [93], Keene, Glass, and Kim [91, p.2] began wondering 'is it possible to evaluate if students really come to know how to find solutions to ordinary differential equations with deep conceptual knowledge' and developed The Framework for Relational Understanding of Differential Equations, the idea being that if students develop an understanding in six particular areas, they have a relational understanding of a technique (procedure). The six areas, which go beyond simply carrying out a procedure, includes 'making connections within and across representations' [91, p.2] and align themselves with much of the research described in this section [62, 78-80, 85, 88, 89] by emphasizing the role MR plays in student learning.

What is sure to follow the development of this framework is a conversation about assessing conceptual understanding in differential equations. Keene, Glass, and Kim [91] draw attention to the small amount of work done in relation to this to date, and a broadening of what we hope to assess will be a consequence of any reform effort. Similar calls for a change in assessment have already been made [94], and Rowland and Jovanoski [95, p.514] specify that "more qualitative or conceptual type questions [be included] into the curriculum… as these will force students to move away from a purely manipulation focus to more of a focus on understanding".

Changes of this nature will be a welcome addition, given the limitations of 'commonly used achievement tests' that, at best provide 'only indirect and highly limited information on students' understanding of specific conceptual domains' [1].

1.6. Summary

Having reviewed the relevant literature, it is clear that there are several emerging themes that must be considered throughout this project. There is a disconnect between what we consider to be understanding and what is typically prioritised in our teaching. This is being discussed and addressed more frequently of late, and the resulting research on reform efforts and assessment are promising steps in the right direction. The research done on PER, specifically research investigating calculus is similarly clear in its recommendation of pedagogical change, but also emphasises the importance of how students conceive the mathematics, which ties back to our opening section.

There is a clear body of evidence supporting changes in pedagogy and inclusion of modelling where service teaching physics students is concerned that this project must build upon in pursuit of its aims. What is also apparent is that service teaching mathematics can have several undesirable outcomes, particularly for physics students, whose advanced modules rely heavily on the topics they are service-taught. To assess aspects of the problem at Dublin City University (DCU), Ireland, we conducted a survey of academic staff in the School of Physical Sciences.

The survey (Appendix A) asked lecturers about the use of differential equations in their modules and how student ability plays a role in their modular design. The responses from this survey indicated that the role of differential equations in advanced physics modules has diminished in recent years to the point of omission in certain cases. Lecturers feel hamstrung by the students' inability to apply the differential equations they are learning in mathematics modules to the physics they study as part of their degree programme. The issues cited in the survey represent areas for attention in relation to service teaching both locally and further afield [96, p.118].

Our research recognises the magnitude of this issue as it currently stands and so we turned our focus to the service mathematics module taken by physics students at DCU. The module in question, Introduction to Differential Equations (MS225), carries five European Credit Transfer and Accumulation System (ECTS) credits and is taught by mathematicians at DCU. In addition to the physics undergraduates, MS225 is also taken by prospective mathematics teachers, all of whom are in the third year of their undergraduate studies.

Having found a setting in which our research could take place, we decided on our research aims, and subsequently, our research questions which we outline in the next chapter. The initial phase of this research (described in Chapter 3) has been reported previously [97]. The subsequent chapters have not been reported on to date, although they build on the findings of the initial phase.

How voice and tense are used throughout this thesis

The research described in this thesis was conducted by a team of three people. I carried out the research under the supervision of two members of academic staff at DCU. I was the primary creator of materials during the project (worksheets, pre/post-tests, interview scripts etc.) in that I would draft these items before they were finalised at meetings of the research team. There are some exceptions to this standard: the survey of academic staff (Appendix A) was facilitated by Dr van Kampen who is a member of staff in this school, and a question on the Diagnostic Survey was borrowed from a previous study [66]. Though meetings were typically used to shape drafted material, sometimes more significant changes were needed, such as Worksheet 2 and 7 which contain mathematical concepts that warrant particular care when explaining. In these instances, the supervisors' expertise was required to ensure the mathematical components of the materials were at an appropriate standard. These three individuals are referred to collectively as 'the research team'. Two other people are mentioned at certain stages, though they are not members of the research team. The role of each person mentioned in the thesis is outlined in Table 1-1.

The Research Team	Student/Tutor for MS225	Diarmaid Hyland
	Supervisor	Brien Nolan
		Paul van Kampen
Independent	Lecturer for MS225	Member of academic staff 1
	Interviewer	Member of academic staff 2

Table 1-1: Roles of individuals involved in this research

The register changes throughout the thesis, driven by the manner in which the author chose to communicate each chapter. The change in register is most notable with respect to tense and voice. Generally, the passive voice was used to discuss facts and the active voice was reserved for opinion and details of this project specifically. To this, Chapter 1 (Introduction and Background) is almost exclusively written in the passive voice because it reviews the literature in an objective manner. Chapter 3 to 7, on the other hand, deal with results and discussion of the research conducted by the research team. This is more accurately portrayed through the active voice. Chapter 2 (Methodology) uses a combination of active and passive voice but in a manner consistent with the rest of the thesis. Where methodologies in general are discussed the passive voice is used and where references are made to this project the active voice is used.

The use of tense also changes throughout the thesis. Generally, the tenses are consistent with the time of writing (Spring 2018). Exceptions to this occur when the project is spoken of chronologically, most notably in Chapter 3 and 4. These chapters discuss the creation of the intervention at a time before its existence so the future tense is used. The other chapters were written retrospectively and as such, the design and implementation of the intervention has already occurred.

Reference list

1. D. Niemi. Assessing conceptual understanding in mathematics: representations, problem solutions, justifications, and explanations. **The Journal of Educational Research,** 1996, **89** (6), p.351-363.
2. I. Arnon, J. Cottrill, E. Dubinksy, A. Oktac, S. Roa Fuentes, M. Trigueros, and K. Weller. **APOS Theory: A framework for research and curriculum development in mathematics education.** New York: Springer. 2013.
3. D.O. Tall. The three worlds of mathematics. **For the Learning of Mathematics**, 2004, **23** (3), p.29–33.
4. R.R. Skemp. Relational Understanding and Instrumental Understanding. **Mathematics Teaching**, 1976, **77** (1), p.20–26.
5. R.R. Skemp. Goals for learning and quality of understanding. **Mathematics Teaching,** 1979, **88**, p.44–49.
6. R.R. Skemp. Symbolic understanding. **Mathematics Teaching**, 1982, 99, p. 59–61.
7. D.O. Tall and S. Vinner. Concept image and concept definition in mathematics, with special reference to limits and continuity. **Educational Studies in Mathematics**, 1981, **12** (2), p.151-169.
8. S. Vinner. Conflicts between definitions and intuitions: the case of the tangent. In: **Proceedings of the 6th Conference of the International Group for the Psychology of Mathematics Education**, 1982, p. 24-28.
9. D.O. Tall. Concept image and concept definition. **In**: J. De Lange and M. Doorman, (eds), **Senior Secondary Mathematics Education**. Utrecht: OW & OC, 1988, p.37–41. Available from: http://homepages.warwick.ac.uk/staff/David.Tall/pdfs/dot1988e-concept-image-icme.pdf.
10. M.S. Gresalfi and F. Lester. What's worth knowing in mathematics? **Constructivist Instruction**, 2009, p.276-302.
11. M. Crotty. **The foundations of social research: Meaning and perspective in the research process**. Sage, 1998.

12. T.P. Carpenter, and R. Lehrer. Teaching and learning mathematics with understanding. **Mathematics Classrooms that Promote Understanding**, 1999, p.19-32.

13. L.C. McDermott. Guest comment: How we teach and how students learn a mismatch? **American Journal of Physics**, 1993, **61**, p.295-298.

14. P.S. Shaffer and L.C. McDermott. Research as a guide for curriculum development: An example from introductory electricity. part ii: Design of instructional strategies. **American Journal of Physics**, 1992, **60** (11), p.994-1003.

15. E.F. Redish. Implications of cognitive studies for teaching physics. **American Journal of Physics**, 1994, **62** (9), p.796–803.

16. D. Hestenes. Who needs physics education research!? **American Journal of Physics**, 1998, **66**, p.465-467.

17. L. Doughty, E. McLoughlin, P. van Kampen. What integration cues, and what cues integration in intermediate electromagnetism. **American Journal of Physics**, 2014, **82** (11), p.1093–1103.

18. J. Williams. Mathematics education and the transition into higher education – transmaths demands better learning-teaching dialogue. **In**: Grove, M., Croft, T., J. Kyle and D. Lawson (eds), **Transitions in undergraduate mathematics education**. Birmingham: The University of Birmingham, 2015, p. 25-38.

19. Conference Board of Mathematical Sciences. **Active Learning in Post-Secondary Mathematics Education**, 2018. [Online] Available from: https://www.cbmsweb.org/2016/07/active-learning-in-post-secondary-mathematics-education/ [Accessed 1 June 2018].

20. J. Cooper and R. Mueck. Student involvement in learning: Cooperative learning and college instruction. **Journal on Excellence in College Teaching**, **1** (1), 1990, p.68-76.

21. J.R. Savery and T.M. Duffy. Problem based learning: An instructional model and its constructivist framework. **Educational Technology**, 1995, **35** (5), p.31-8.

22. P.S. Adey. **The science of thinking, and science for thinking: a description of cognitive acceleration through science education (CASE). innodata monographs 2**. 1999. [Online] Available from: http://www.ibe.unesco.org. [Accessed 30 May 2018].

23. C.C. Kuhlthau, L.K. Maniotes and A.K. Caspari. **Guided inquiry: Learning in the 21st century**, 2nd edition. Santa Barbara (CA): ABC-CLIO, 2015.
24. C.L. Rasmussen, K. Marrongelle, O.N. Kwon, and A. Hodge. Four goals for instructors using inquiry-based learning. **Notices of the American Mathematical Society**, 2017, **64** (11).
25. S. Yoshinobu and M.G. Jones. The coverage issue. **PRIMUS**, **22**, 2012, p.303-316.
26. L. Cui, N.S. Rebello, P.R. Fletcher, and A.G. Bennett. Transfer of learning from college calculus to physics courses. In: **Proceedings of the NARST 2006 National Meeting**. San Francisco, California. 2006. [Online]. Available from: https://www.researchgate.net/profile/N_Sanjay_Rebello/publication/242297311_TRANSFER_OF_LEARNING_FROM_COLLEGE_CALCULUS_TO_PHYSICS_COURSES/links/0c960532638f5bfd86000000.pdf [Accessed 1 June 2017]
27. L.C. McDermott. Research on conceptual understanding in mechanics. **Physics Today**, 1984, **37**, p.24-32.
28. C.L. Rasmussen, and O.N. Kwon. An inquiry-oriented approach to undergraduate mathematics. **Journal of Mathematical Behaviour**, 2007, **26**, p.189-194.
29. L. Walker. Enabling students to become independent learners **In**: M. Grove, T. Croft, J. Kyle and D. Lawson (eds). **Transitions in undergraduate mathematics education**. Birmingham: The University of Birmingham, 2015, p.71-84.
30. B. Braun, P. Bremser, A.M. Duval, E. Lockwood, and D. White. What does active learning mean for mathematicians? **Notices of the American Mathematical Society**, 2017, **64** (2), p.124-128.
31. M. Carr and E. Ní Fhloinn. A project-based-learning approach to teaching second-order differential equations to engineers. In: **18th SEFI-MWG European Seminar on Mathematics in Engineering Education**. Gothenburg, Sweden. 27-29 June 2016. [Online] Available from: *A project-based-learning approach to teaching second-order differential equations to engineers.* [Accessed 1 Jan 2018]
32. D.G. Mallet and S.W. McCue. Constructive development of the solutions of linear equations in introductory ordinary differential equations. **International Journal of Mathematics Education in Science and Technology**, 2009, **40** (5) p.587–595.

33. K. Afasamaga-Fuata. An undergraduate student's understanding of differential equations through concept maps and vee diagrams. In **Proceedings of the First International Conference on Concept Mapping**. Pamplona, Spain, 2004.
34. M.K. Ju and O.N. Kwon. Ways of talking and ways of positioning: students' beliefs in an inquiry-oriented differential equations class. **Journal of Mathematical Behavior**, 2007, **26**, p.267–280.
35. S. Waddy, J.H. Kim and M. Glass. Tutoring dialogue goals for conceptual understanding of differential equations: preliminary work. In **Proceedings of the 20th Midwest AI and Cognitive Science Conference.** Indiana Purdue Fort Wayne, 2009.
36. C.L. Rasmussen and K.D. King. Locating starting points in differential equations: a realistic mathematics education approach. **International Journal of Mathematical Education in Science and Technology**, 2000, **31** (2), p.161-172, DOI: 10.1080/002073900287219
37. O.N. Kwon, C.L. Rasmussen, and K. Allen. Students' retention of mathematical knowledge and skills in differential equations. **School Science and Mathematics.** 2005, **105** (5), pp.227-239.
38. C.L. Rasmussen, O.N. Kwon, K. Allen, K. Marrongelle, and M. Burtch. Capitalizing on advances in mathematics and K-12 mathematics education in undergraduate mathematics: An inquiry-oriented approach to differential equations. **Asia Pacific Education Review,** 2006, **7** (1), p.85-93.
39. M. Artigue, C. Batanero and P. Kent. Mathematical thinking and learning at post-secondary level. **In**: F.K. Lester Jr (ed.), **Second handbook of research on mathematics teaching and learning vol. 2**. Information age publishing, 2007, p.1011-1049.
40. S. Britton, P.B. New, M.D. Sharma, and D. Yardley. A case study of the transfer of mathematics skills by university students. **International Journal of Mathematics Education in Science and Technology**, 2003, **36** (1), p.1–13.
41. J. Tuminaro. **A cognitive framework for analyzing and describing introductory students' use and understanding of mathematics in physics.** Ph.D. thesis, Maryland, 2004.
42. F.R. Yeatts and J.R. Hundhausen. Calculus and physics: Challenges at the interface.

American Journal of Physics, 1992, **60**, p.716-721.

43. R.A. Hoban, O.E. Finlayson, and B.C. Nolan. Transfer in chemistry: a study of students' abilities in transferring mathematical knowledge to chemistry. **International Journal of Mathematics Education in Science and Technology**, 2013, **44** (1), p.14-35.
44. A. Selden and J. Selden. HOW WE GOT IN. **The Journal of Mathematical Behavior**, 1989, **8**, p.45-50.
45. J.D. Novak. Meaningful learning: the essential factor for conceptual change in limited or inappropriate propositional heirachies leading to empowerment of learners. **Science Education**, 2002, **86**, p.548-571.
46. K. Nankervis. Service Teaching: Student experiences, issues and future directions at RMIT. 2008.
47. F. Faulkner, A. Hannigan, and O. Gill. Trends in the mathematical competency of university entrants in Ireland by leaving certificate mathematics grade. **Teaching Mathematics and its Applications: An International Journal of the Irish Mathematics Association**, 2010, **29**(2), p.76-93.
48. J. O'Donoghue. **An intervention to assist at risk students in service mathematics courses at the University of Limerick.** Limerick: University of Limerick; 1999.
49. M. Prendergast, F. Faulkner, C. Breen, and M. Carr. Mind the gap: an initial analysis of the transition of a second level curriculum reform to higher education. **Teaching Mathematics and its Applications: An International Journal of the Irish Mathematics Association**, January 2017.
50. O. Gill and J. O'Donoghue. Service mathematics in Irish universities: some findings from a recent study. **Adults Learning Mathematics,** 2007, **2** (2), pp.6-19.
51. A.C. Bajpai. The role of mathematics in engineering education: a mathematician's view. **International Journal of Mathematics Education in Science and Technology,** 1985, **16** (3), p.417-430.
52. C. Alsina. 2001. Why the professor must be a stimulating teacher **In**: D. Holton (ed.). **The teaching and learning of mathematics at university level: an ICME study**. Netherlands: Kluwer Academic Publishers, p.3-12.

53. N. Challis. 2015. Group work within undergraduate mathematics **In**: M. Grove, T. Croft, J. Kyle and D. Lawson (eds). **Transitions in undergraduate mathematics education**. Birmingham: The University of Birmingham, p.85-96.
54. M. Savage and M. Grove. Mathematical modelling and problem solving in real-world physical situations. **In**: M. Grove, T. Croft, J. Kyle and D. Lawson (eds). **Transitions in undergraduate mathematics education.** Birmingham: The Higher Education Academy, University of Birmingham, 2015, p.115-127.
55. Institute of Mathematics and its Applications (IM). **Mathematics Matters in Engineering**. Southend-on-Sea: I.M.A. 1999.
56. A. Kelly. **Strategies for teaching maths to scientists and engineers**. Report of CTI Mathematics Workshop 16. University of Birmingham. 23 Mar 1994.
57. J. Ahmad, J. Appleby and P. Edwards. 2001. Engineering mathematics should be taught by engineers. In **Proceedings of UMTC, 2001, the Twenty Seventh Conference of Undergraduate Mathematics Teaching. University of Birmingham**, 3-6 Sept. 2001. [Online]. Available from: http://www.umtc.ac.uk/umtc2001/proceedings01.pdf. [Accessed Jan 2017]
58. M.P. Carlson and C.L. Rasmussen. 2008. **Making the connection: research and teaching in undergraduate mathematics education**, USA: Mathematical Association of America.
59. D. Holton. **The teaching and learning of mathematics at university level: an ICME study.** Netherlands: Kluwer Academic Publishers, 2001.
60. M.D. Caballero, B.R. Wilcox, L. Doughty and S.J. Pollock. Unpacking students' use of mathematics in upper-division physics: where do we go from here? **European Journal of Physics**, 2015, **36** (6), p.1–28.
61. J. Thompson, B. Bucy, D. Mountcastle. Assessing student understanding of partial derivatives in thermodynamics. **In**: P. Heron, L. McCullough and J. Marx. (eds). Physics Education Research Conference, Salt Lake City: Utah. 2005.
62. M. Stephan and C.L. Rasmussen. Classroom mathematical practices in differential equations. **Journal of Mathematical Behaviour**, 2002, **21**, p.459–490.
63. E.F Redish and E. Kuo. Language of Physics, Language of Math: Disciplinary Culture and

Dynamic Epistemology. **Science Education**, 2015, **24,** p.561–590.

64. C.A. Manouge and T. Dray. Bridging the gap between mathematics and the physical sciences. 2004. [Online], available from: www.math.oregonstate.edu/bridge/papers/bridge.pdf. [Accessed 10 June 2018].
65. T. Wemyss and P. van Kampen. Categorization of first-year university students' interpretations of numerical linear distance-time graphs. **Physical Review Special Topics – Physics Education Research,** 2013, **9**, p.1-17.
66. M. Artigue, J. Menigaux, and L. Viennot, Some aspects of students' conceptions and difficulties about differentials. **European Journal of Physics**, 1990, **11**, p. 262.
67. F. R. Yeatts, Calculus and physics: Challenges at the interface. **American Journal of Physics**, 1992, **60**, p.716.
68. E. F. Redish, Problem Solving and the Use of Math in Physics Courses. **The World View on Physics Education.** Delhi: India, August 21-22, 2005, p.1-10. [Online] Available from: http://www.compadre.org/per/items/detail.cfm?ID=3706. [Accessed 15 May 2018].
69. M.R. Wilson. One preservice secondary teacher's understanding of function: The impact of a course integrating mathematical content and pedagogy. **Journal for Research in Mathematics Education**, 1994, **25** (4), p.346-370.
70. D.A. Zollman, N.S. Rebello, and K. Hogg. Quantum mechanics for everyone: Hands-on activities integrated with technology. **American Journal of Physics**, 2002, **70** (3), p.252-259.
71. L. Bollen, P. van Kampen, M. De Cock. Students' difficulties with vector calculus in electrodynamics. **Physical Review Special Topics- Physics Education Research**, 2015, **11**, p.1–14.
72. University of Leeds. **Faculty of engineering**, 2016. [Online] Available from: https://www.engineering.leeds.ac.uk/teaching/electronic/module/MATH1400 [Accessed 1 March 2016].
73. M. Savage and M. Grove. Mathematical modelling and problem solving in real-world physical situations. **In**: M. Grove, T. Croft, J. Kyle, and D. Lawson, (eds). **Transitions in**

undergraduate mathematics education. Birmingham: The Higher Education Academy, University of Birmingham, 2015, p.115-127.

74. M.A. McDonald and A. Brown. Developing notions of infinity. **In**: M.P. Carlson and C.L. Rasmussen (eds). **Making the connection: research and teaching in undergraduate mathematics education**. Washington (DC): The Mathematical Association of America, 2008, p.55-64.
75. S. R. Jones. Understanding the integral: students' symbolic forms. **Journal of Mathematical Behavior**, 2013, **32**, p.122-141.
76. D. Raychaudhuri. Dynamics of a definition: a framework to analyse student construction of the concept of solution to a differential equation. **International Journal of Mathematics Education in Science and Technology**, 2009, **39** (2), p.161–177.
77. C.L. Rasmussen. New directions in differential equations: A framework for interpreting students' understandings and difficulties. **Journal of Mathematical Behavior**, 2001, **20**, p.55–87.
78. M. Camacho-Machín, J. Perdomo-Diaz and M. Santos-Triago. An exploration of students' conceptual knowledge built in a first ordinary differential equations course (part I). **The Teaching of Mathematics**, 2012, **Vol XV** (1), p.1-20.
79. S. Habre. Investigating students' approval of a geometrical approach to differential equations and their solutions. **International Journal of Mathematics Education in Science and Technology,** 2003, **34** (5), p.651–662.
80. S. Habre. Exploring Students' Strategies to Solve Ordinary Differential Equations in a Reformed Setting. **Journal of Mathematical Behaviour**, 2000, **18** (4), p.455–472.
81. M.C. Wittmann and K.C. Black. Mathematical actions as procedural resources: an example from the separation of variables. **Physical Review Special Topics- Physics Education Research**, 2015, **11** (2), p.1-13.
82. M. Van Den Heuvel-Panhuizen. The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. **Educational Studies in Mathematics**, 2003, **54** (1), p.9-35.

83. M. Van den Heuvel-Panhuizen and P. Drijvers. Realistic mathematics education. **In**: S. Lerman (ed.). **Encyclopedia of mathematics education**. Netherlands: Springer, 2014, p.521-525.

84. M. Artigue and M.J. Perrin-Glorian. Didactic engineering, research and development tool: some theoretical problems linked to this duality. **For the Learning of Mathematics**, 1991, 11 (1), p.13-8.

85. M. Yerushalmy and J.L. Schwartz. A procedural approach to exploration in calculus. **International Journal of Mathematics Education in Science and Technology**, 1999, **30** (6), p.903-914.

86. J. Zhang and D.A. Norman. Representations in distributed cognitive tasks. **Cognitive Science**, 1994, **18**, p.87-122.

87. J. Zhang. The nature of external representations in problem solving. **Cognitive Science**, 1997, **21** (2), p.179-217.

88. S. Ainsworth. The functions of multiple representations. **Computers and Education**, 1999, **33**, p.131-152.

89. S. Ainsworth. The educational value of multiple-representations when learning complex scientific concepts. **In**: J.K. Gilbert et al., (eds), **Visualization: theory and practice in science education**, Springer. 2008, p.191-208.

90. M. Zandieh and M. McDonald. Student understanding of equilibrium solutions in differential equations. **In**: F. Hitt & M. Santos (eds), **Proceedings of the 21st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education**, Columbus, OH: ERIC. 1999, p.253–258.

91. K.A. Keene, M. Glass and J.H. Kim. Identifying and Assessing Relational Understanding in Ordinary Differential Equations. In **Proceedings 2011 frontiers in education (FIE 20110)**. Rapid City, SD. October, 2011, p. S4B-1.

92. J.R. Starr. Reconceptualizing procedural knowledge. **Journal for Research in Mathematics Education**, 2005, **36** (5), p.404-415.

93. J. Hassenbrank and T.A. Hodgson. A Framework for Developing Algebraic Understanding & Procedural Skill: An Initial Assessment. In **Proceedings of Research in Undergraduate**

Mathematics Annual Conference. February 2007. [Online] Available from: http://www.sigmaa,maa,org.com/rume/crume2007/epoc.html [Accessed September 2016].

94. P. Davis Asking Good Questions About Differential Equations. **The College Mathematics Journal**, 1994, **25** (5), p. 394-400.
95. D.R. Rowland and Z. Jovanoski. Student interpretations of the terms in first-order ordinary differential equations in modelling contexts. **International Journal of Mathematics Education in Science and Technology**, 2004, **35** (4), p.503-516.
96. IoP. **Mind the Gap: Mathematics and the transition from A-levels to physics and engineering degrees**. London, UK: Institute of Physics, 2011.
97. D. Hyland, P. van Kampen, and B.C. Nolan. Outcomes of a service teaching module on ODEs for physics students. **International Journal of Mathematics Education in Science and Technology**, 2017, **49** (5), p.743-758.

2. Research methodology

This chapter states the research aims (RAs) and questions (RQs) that guide this investigation and describes the methodology employed. A variety of instruments was used to gather the data necessary to answer the research questions. These instruments are described in detail, and their role in the research is illustrated using a design diagram (Figure 2-2). The chapter finishes by outlining some of the methods used to analyse the various forms of data harvested across the project.

2.1. Research aims and research questions

The research aims of this project are as follows:

1. To study physics students' learning of ODEs, with a focus on the obstacles they encounter that hinder success;
2. To develop an intervention that addresses the obstacles described and thereby promotes more successful learning.

The intention of RA1 is to make an inventory of the obstacles students encounter when studying ODEs. The motivation behind this aim arose from the literature on mathematics in PER discussed in Section 1.4. Our review of this literature highlighted several potential barriers our students may experience during their studies, all of which must be identified locally before an intervention (RA2) can be designed. To make this inventory, we look at concept image and instrumental understanding (discussed in detail Section 1.1, recapped in Section 2.2). Concept image has previously been used to investigate students' understanding of mathematics in physics [4, 5, 6] and we believe that it provides an appropriate lens to study students' understanding of ODEs.

RA2 will build on the findings of RA1 by developing an intervention based on the inventory of obstacles identified. The literature on similar reform curricula is discussed throughout Chapter 1, with a particular focus in Section 1.2.

These research aims lead to the following research questions, where Research Questions 1 and 2 address the first research aim and Research Question 3 addresses the second research aim.

1. Do our students have the necessary instrumental understanding in the following areas to succeed in their study of ODEs?

a. Manipulation of exponents in equations;
b. Evaluating indefinite integrals.

2. Do our students have a well-developed concept image of ODEs upon completion of this module?
   a. What is brought to mind when presented with an ODE?
   b. What do students know about ODEs and their applications?
   c. Do students understand what a solution to an ODE is?
3. Has the intervention benefitted our students when learning ODEs?
   a. How does the instrumental understanding of students who experienced the intervention compare to those who completed the module prior to its implementation?
   b. In what way has the concept image of students who experienced the intervention grown?
   c. How is the intervention viewed by the participants?

2.2. Concept image and instrumental understanding

Before describing how the research methodology enabled the answering of the research questions, it is necessary to explain what we mean when we refer to concept image and instrumental understanding. When we refer to concept image in our second research question, we are referring to Tall and Vinner [4] who defined the concept image as ‘the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes’ [4, p.152]. For example, a person’s concept image of ODEs may comprise generic ideas like “a relation between a function and its derivatives” or “an equation that has a derivative in it”, but also specific instances like “$\frac{d^2y}{dx^2} + y = 0$”. The concept image evoked depends on the context. Tall and Vinner note that students are more likely to use the ideas they form from their experience with a concept (i.e., their concept image) than the formal concept definition.

In our first research question, we refer to terminology first used by Skemp [5]. In this paper, Skemp talks about instrumental understanding as having ‘rules without reasons’. We use second order linear

ODEs as an example highlighting the difference between instrumental understanding and relational understanding in the context of ODEs. When solving for the general solution of a second order Euler-Cauchy ODE, a putative solution of $y(x) = x^r$ is used to generate an auxiliary equation (which transpires to be a quadratic equation for the constant r). Then x^r is a solution of the ODE if and only if r is a solution of the auxiliary equation. A student with instrumental understanding knows that this putative solution must be used when solving Euler-Cauchy equations, and requires no other knowledge or explanation in order to successfully obtain the general solution. A student who immediately writes down the associated quadratic auxiliary equation clearly appears to use 'rules without reasons'.

However, a student with relational understanding of the fundamental solution understands the significance of the structure of monomials of the form $y=x^r$ *vis a vis* the terms that arise in an Euler-Cauchy equation. They can explain the role of the exponent in generating the auxiliary equation and can explain why a different putative solution is used for Euler-Cauchy ODEs and for constant coefficient ODEs.

In describing and assessing students' instrumental understanding in relation to general calculus and algebra skills, we are assessing their ability to carry out small-scale tasks independent of context, and independent of any necessity to explain or validate their reasoning: this is instrumental understanding. Our research questions are best described and answered using concept image and instrumental understanding.

2.3. Research design (MMR)

An appropriate methodological approach will be vital to successfully addressing the research questions. Given the varied nature of the data being collected, we used a form of mixed methods research (MMR) referred to by Creswell, Fetters, Plano Clark, and Morales [6] as a *mixed methods intervention design* [6, p.162].

Cohen [7] states that

> a mixed methods paradigm rests on an ontology that recognizes that phenomena are complex to the extent that single methods approaches might result in partial, selective, and incomplete understanding and on an epistemology, that requires pragmatic combinations of methods – in sequence, in parallel or in synthesis – in

> order to fully embrace and comprehend the phenomenon and to do justice to its several facets. [7, p.220]

In short, MMR is a relatively modern solution to the problem of answering increasingly more complex questions, but how do the research aims outlined at the beginning of this chapter warrant an MMR approach, and why is it the best approach for this study?

Creswell [8] discusses scenarios where an MMR approach is required, many of which apply to this research:

- When qualitative research or quantitative research alone is insufficient to fully understand the problem;
- When better contextualized instruments, measures, or interventions are needed to reach certain populations;
- When exploration is required before administering instruments;
- When the gathering of trend data and individual perspectives from community members is needed;
- When qualitative experiences are required to confirm quantitative measures.

Creswell [8] also lays out several characteristics of MMR and criteria to evaluate the suitability of MMR to a piece of research. These criteria have been used to ascertain the suitability of MMR to this project. Taking what Creswell [8] calls "essential" and "core" characteristics of MMR, the following paragraphs detail the specific pieces of this research that warrant the use of an MMR approach.

At its most basic, MMR is required in response to questions where both quantitative data and qualitative data are collected and analysed. This research meets that requirement, given the dual nature of the research aims and subsequent research questions posed. For example, in several instances we require data on the knowledge and understanding of a cohort of students pre-intervention and post-intervention. This requirement suggests the use of surveys, since "… surveys gather data at a particular point in time with the intention of describing the nature of existing conditions…" [7, p.426]. These surveys provide both a quantitative database (closed questions from a Diagnostic Survey, pre-tests, and immediate and delayed post-tests) and a qualitative database (open-ended questions from the Diagnostic Survey, pre-tests, the immediate post-tests, and interviews), through which the research questions will be addressed. The quantitative part serves to highlight broad areas students may struggle with or excel in, while the open-ended parts capture the specificity of a particular situation and allow for a greater authenticity, richness, depth of response, honesty, and candour which, as is argued elsewhere by Creswell [9], are the hallmarks of qualitative data. The question of weighting is often discussed during the design stage of MMR; in this instance, the qualitative data and quantitative data are weighted similarly.

Simply having separate databases is not enough, however, to warrant an MMR approach. Another key characteristic of MMR is the integration (or combination) of the findings from the quantitative and qualitative data [9]. In this project, the integration of qualitative and quantitative data occurs at two stages: during the design of the intervention, where qualitative and quantitative data inform the design of the worksheets used in tutorials (including their revision); and at the end of the project, where the data generated during and after the intervention inform the research findings.

In addition to the integration of data, some data is transformed during the analysis of the interviews. Transforming data involves treating qualitative themes quantitatively to compare the information with previously analysed data [9]. The use of this technique enhances the ability to answer Research Question 3a and 3b, as it allows the interview data to be utilised where appropriate.

An intervention mixed methods design [6] is an appropriate methodology when an intervention is being used to address an issue. It involves gathering data during three different stages in the research: before, during and after the intervention. This project aims to develop an intervention that addresses the obstacles described by physics students when learning ODEs, making this MMR approach the most appropriate choice.

2.3.1. Design diagram

It is customary to use a design diagram to illustrate the specific MMR design employed for the project in question. Figure 2-1 shows a decontextualized design diagram, illustrating the *mixed methods intervention design*, while Figure 2-2 shows a design diagram applied to this project.

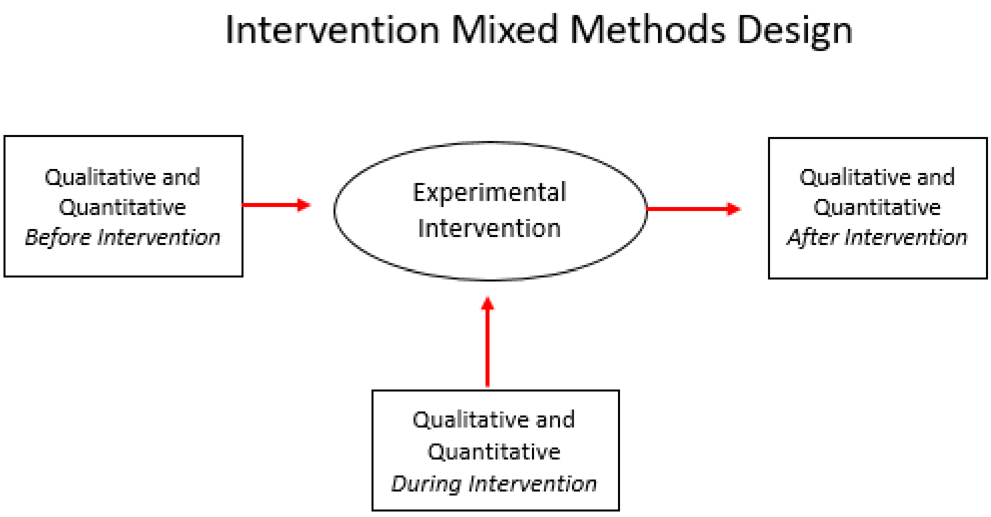


Figure 2-1: Intervention mixed methods design diagram

2.4. Instruments

This research project uses several instruments designed to harvest data at different stages throughout the research. Each of these instruments is described briefly below, and the stage at which they are utilised is illustrated in Figure 2-2.

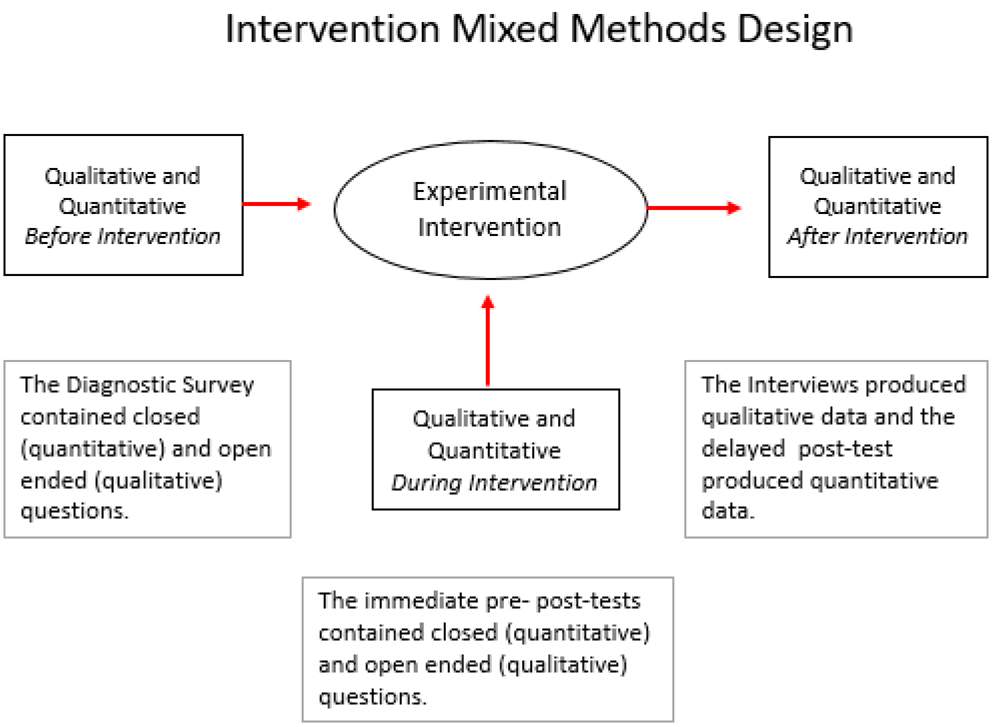


Figure 2-2: Design diagram for this project

2.4.1. Diagnostic Survey

The Diagnostic Survey presented in Appendix B was designed to address RQ1 and 2. It contained a mixture of closed and open-ended questions, resulting in the collection of qualitative and quantitative data. The closed questions assessed students' ability to solve decontextualized mathematical tasks and provided a measure of their instrumental understanding. The open-ended questions sought to elicit the students' concept images of ODEs. In addition to informing RQs, the data gathered from the Diagnostic Survey contributed to the design of the intervention. Given its purpose, the Diagnostic Survey was given to students at the pre-intervention stage as shown in the design diagram above (Figure 2-2).

The Diagnostic Survey was completed by students (*n*=18) enrolled in MS225 in semester one of the 2014-2015 academic year. The results from the first cohort of students informed RQ1 and 2, and are described in Chapter 3. It was also given to the first cohort of students (*n*=34) that completed the intervention the following year as a point of comparison. This data was useful during the revision of the intervention, but does not align directly with the RQs for this project, and is not reported as a result. In total, 52 students who consented to take part in the study (29% of the cohort) completed the Diagnostic Survey. These students were selected from attendees at a lecture for MS225 in the final week of semester in the 2014-2015 academic year and intervention attendees in the 2015-2016 academic year. All students who completed the Diagnostic Survey did so voluntarily.

2.4.2. Pre/post-tests

The pre/post-tests (Appendix C) were designed to inform RQ3. The nature of the questions varied with each pre/post-test, so a combination of qualitative and quantitative data was gathered through closed and open-ended questions. When RQ3a was the focus, the questions were mathematical tasks designed to assess their instrumental understanding. When the focus was on RQ3b and 3c, the questions were designed to evoke students' concept image of certain aspects of ODEs. Using pre/post-tests allows for data to be gathered on the individual and group level simultaneously which provides a large amount of data to inform the research questions. They were also completed by all attendees which resulted in a large response rate that represented every member of the cohort. They were administered immediately before and after their respective worksheets and so are represented in the design diagram by the *during intervention* stage. Administering the pre/post-tests directly before and after the worksheets eliminated any chance of external influences on student responses. The amount of data obtained and the timing of the tests made using immediate pre/post-tests the most appropriate method of data collection. It must be stressed that pre-tests may not be an interference-free instrument; indeed, pre-tests sometimes serve a dual goal of providing the researcher with data and priming the students for the tutorial ahead.

The pre/post-tests were introduced to the intervention during the 2016-2017 academic year and continued for the 2017-2018 academic year. In total, 120 students who consented to take part in the study (100% of the cohort) completed at least one immediate pre/post-test across both years of data collection. There was a 71% response rate for immediate pre/post-testing on average across both cohorts: 68% in 2016-2017 and 74% in 2017-2018. The students who chose to respond self-selected from tutorial attendees.

2.4.3. Interviews

Semi-structured interviews were also conducted as part of this research project. They followed the design guidelines described by Jääskeläinen [10], although slight variations in how they were run existed between both years (described below). They yielded large amounts of qualitative data that informed RQ3.

The first cycle of interviews presented in Appendix D had 3 sections: Section 1 contained several short questions common to each interview designed to assess RQ3a and 3b; Section 2 contained a longer question that assessed one aspect of student understanding related to either 3a or 3b; and the third section sought student feedback on the intervention and other aspects of the students' studies, informing RQ3c.

Interviews were chosen because of the richness of the data obtained. They can facilitate more open-ended questions than pen and paper tests, and allow for spontaneous lines of enquiry to be followed by the interviewer. With respect to RQ3, the interview setting allowed students to express themselves more freely than any other medium. A known disadvantage of using interviews is the possible occurrence of self-selection effects with students who volunteer to participate.

During the first cycle of interviews, nine were carried out involving nineteen different students. The reason for using small group interviews was to replicate the environment in which the students were used to working as much as possible. This was intended to maximise student comfort in the interview generally, and to maximise achievement in students' work on the longer question in Section 2. Seven of the nine interviews were facilitated by a mathematics education researcher and member of academic staff in the School of Mathematical Sciences at DCU. They have no other direct involvement in the research. The remaining two interviews were carried out by the author. The change in interviewer was done to facilitate students who were unavailable during the days the interviews took place. Both the rescheduled interviews took place one day after the other seven and followed the same guidelines and line of questioning. Because we wanted to know about the impact of the tutorials, having a tutor conduct the interview is not ideal from the perspective of neutrality [11]: it may cause the student to respond to a question differently to how they otherwise would with an outside interviewer. The students who were interviewed by the author were made aware of this by the research team and chose to participate.

The interviews that were carried out during the second year were informed by data from the previous year as well as another round of pre/post-test data. This meant that more was known, and so questions became more focused. The interviews were one-to-one, meaning there were twenty-one interviews conducted. The author conducted all of the interviews during this cycle. This was done because the goals of the interviews had changed, and the author was best suited to facilitate the interviews. For the first cycle of interviews, a script was prepared ahead of time with general questions that an academic with knowledge of ODEs and constructivism could negotiate effectively. By the second cycle of interviews, more was known about the identity of the obstacles our students encounter, but not necessarily about the specificities of each obstacle. In order to investigate the nature of the obstacles in depth, the interviewer would need to be informed on the evolution of the project and the students taking part in interviews. This required a member of the research team act as interviewer, which reduced to the author because of the demand on time interviewing 21 individuals across a week entails.

Each interview across both years was conducted one week after the intervention was completed and are represented by the final stage in the design diagram. In total, 40 students who consented to take part in the study (33% of the cohort) took part in interviews. These students self-selected from students participated in the intervention.

2.4.4. Exam question (delayed post-test)

In addition to the interviews, data informing RQ3 was gathered through a written delayed post-test (Appendix E) that was given as a question on the terminal exam for MS225. The exam question resembled the Diagnostic Survey in that it contained a mixture of closed and open-ended questions. The closed questions assessed students' ability to solve decontextualised mathematical tasks and were a measure of their instrumental understanding. The open-ended questions sought to elicit the students' concept image of ODEs.

In addition to providing additional data that informed RQ3, the delayed nature of the test allowed for comparisons to be made with some immediate post-test data on similar concepts. This is achieved by aligning the questions on the delayed post-test with questions from immediate pre/post-tests that were administered earlier in the intervention. In this instance, mathematical tasks and the concept of direction fields were the focus of the delayed post-test.

The exam took place six weeks after the lectures and tutorials finished for the module. Seventy students completed the delayed post-test. In total, 118 students who consented to take part in the study (98% of the cohort) completed the delayed post-test; the remaining students were absent for the exam. The students who completed the delayed post-test as a question on the terminal exam did so out of choice: students were given five questions on their exam, of which four would contribute to their mark.

2.4.5. Follow-up Participant Survey

The Follow-up Participant Survey (Appendix E) was given to a subset of the 2015-2016 participants (n=9, all prospective teachers) one year after they completed the module. It asked students about the effect of the tutorials for MS225 on their learning in a subsequent Mechanics module, and how the tutorials differed to other mathematics tutorials. The questions were entirely open-ended and the answers given informed Research Question 3c. In total, nine students who consented to take part in the study (14% of the cohort) completed the Follow-up Participant Survey.

An illustration of the instruments used and the research questions they address is contained in Table 2-1.

Instrument	1a	1b	2a	2b	2c	3a	3b	3c
The Diagnostic Survey	✓	✓	✓	✓	✓			
Follow-Up Participant Survey							✓	✓
Pre/Post 1						✓		
Pre/Post 2							✓	
Pre/Post 3							✓	
Pre/Post 4						✓	✓	✓
Pre/Post 5						✓	✓	✓
Post 7							✓	
Pre 8							✓	
Post 9							✓	
Pre/Post 11							✓	
Pre/Post 12						✓		
Interview 16-17						✓	✓	✓
Interview 17-18						✓	✓	✓
Delayed Post 17						✓	✓	

Table 2-1: Instrument versus research question addressed

2.5. Data analysis

The nature of the RQs required that data of various forms was collected. The techniques for analysing each type of data are outlined below.

The quantitative data generated during this project consists entirely of student solutions to decontextualized mathematical tasks. The research team developed and agreed upon a rubric prior to data analysis that ensured that each data piece was analysed consistently. An example of this is contained in Appendix G, where a rubric for a question on the delayed post-test is shown. With respect to the immediate pre/post-testing, the author used a similar rubric to categorise each student's response. The research team then met to confirm the categorisation and discuss all liminal cases. The research team always arrived at an agreed upon category for each student response.

The qualitative data, which consists of responses to interview questions and open-ended post-test questions is more complex however, and must be dealt with in a different manner. The qualitative data gathered during this research is in response to open-ended questions on the Diagnostic Survey, pre/post-tests, and interviews. All of this data is analysed as described by Thomas [11], who details a general inductive approach to qualitative data analysis.

Thomas [11] provides an outline of a general inductive approach for qualitative data analysis with the primary purpose of allowing “research findings to emerge from the frequent, dominant or significant themes inherent in raw data, without the restraints imposed by structured methodologies” [11, p.283]. A general inductive analysis uses coding to develop categories. The coding process Thomas [11] describes aims to condense the raw data and to establish transparent and defensible links between the research objectives and the summary findings. Examples of this approach applied to interview transcripts and student responses to open-ended questions on the Diagnostic Survey are included in Appendix G.

Lincoln and Guba [13] report that techniques similar to those used with other types of qualitative data analysis can be used to assess the trustworthiness of findings derived from inductive analysis. In addition to these techniques, procedures to assess the trustworthiness of the category system, such as independent coding, coding consistency check, and stakeholder checks are also described by Thomas [11]. A coding consistency check was carried out on student responses to open-ended questions on the delayed post-test (Appendix G) in a mathematical context. The final stage in the process (reporting the findings) emphasises the top-level categories and uses detailed descriptions and quotations to illustrate the meaning of the findings.

Miles & Huberman [14] and Stemler [15] describe similar approaches to that of Thomas. Stemler delves into more detail about the generation and revision of categories when a team of researchers are analysing the results independently, and Miles and Huberman contribute to the discussion by illustrating the components of data analysis with the following flow model shown in Figure 2-3 [14, p.23].

FIGURE 1. *Components of data analysis: flow model*

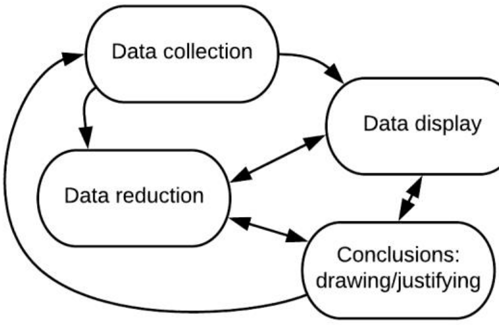


Figure 2-3: Miles and Huberman's components of data collection: flow model

The general inductive approach described by Thomas [11], in addition to the advances mentioned [14,15] will be used to analyse the open-ended questions contained in the Diagnostic Survey, both forms of pre/post-test, and the interview transcriptions.

Given the detailed nature of the data, reporting them in a clear and concise way requires care and creativity. This is done through the integration of the qualitative and quantitative data (cf. Section 2.3). As part of Chapter 5 (Results and discussion), joint displays will be used as visual representations of the interview data and its analysis, specifically for the data that corresponds to RQ3c. A joint display is defined as a way to "integrate the data by bringing the data together through a visual means to draw out new insights beyond the information gained from the separate quantitative and qualitative results." [16] Joint displays are an excellent way to represent large amounts of data of different forms, and are complex by their nature.

With respect to the results in this project, joint displays will be used for two purposes. Firstly, tables that include representative quotes and simple tallies, though some will also contain themes from the interview data. More complex, joint displays will be used to represent all of the data that feeds into RQ3c. The qualitative data from the interview transcriptions will be integrated with a tally of the frequency with which they are mentioned by students. This results in a visual representation that conveys information about themes, their frequency, and their relationship to other themes. An illustrative example is provided in Figure 2-4 along with a description of the information contained within it.

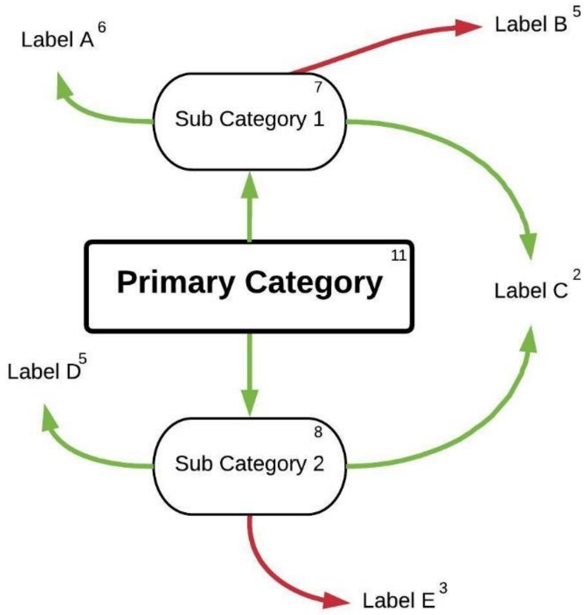


Figure 2-4: Example of visual representation of data analysis

Figure 2-4 is an example of data representation that uses text, numbers, and colour to convey information. These are just some of the ways Kirk [17] mentions for encoding data, all of which derive from combinations of two properties: *marks* and *attributes*, defined by Kirk [17] as follows:

> Marks are visible features like dots, lines and areas. An individual mark can represent a record or instance of data. [17, p.151]

> Attributes are variations applied to the appearance of marks, such as the size, position, or colour. They are used to represent the values held by different quantitative or categorical variables against each record or instance. [17, p.151-152]

Using a general inductive analysis will result in the creation of a category, sub categories, and labels that can overlap. This can be quite complicated, and Figure 2-4 provides a visual representation of the same data that can help the reader understand the complex nature of the data. Table 2-2 describes the marks and attributes used in Figure 2-4 and the information they convey.

Mark	Meaning
Labels	Each label represents the name of a category, sub category, or individual data point.
Numbers	The numbers beside each label represents the frequency with which that label was mentioned.
Attribute	**Meaning**
Arrows	Represent links between categories and sub categories, and sub categories and individual data points.
Borders	The primary category is contained within a square border, the subcategories have an elliptical border, and individual data points have no borders.
Colour	Colour is used to indicate whether the link between labels is considered by students to be positive or negative. Green is used for positive links and red is used for negative links.

Table 2-2: Description of marks and attributes

Thomas' [11] approach is also used to analyse qualitative data in a mathematical context during this research. A question that appeared on the delayed post-test question required students to sketch a solution curve through a point on a direction field. Students' attempts to answer this question (sketches of a solution curve) constitute qualitative data, but the mathematical context of the data required a different approach to their analysis.

Each member of the research team categorised the responses independently before meeting to discuss their categories. Mutual categories and cases were identified and subsequent independent categorisations were carried out until all members of the team were in agreement. The nature of the question and student responses is reported in 5.2.2.2, where the final categories are used in the description of the results.

2.6. Ethics

Prior to the project, ethical approval was sought and received from the research ethics committee at DCU. The application included the administering of a plain language statement and informed consent form (Appendix H). The plain language statement was read aloud to all prospective participants in the study prior to their involvement. Participation was voluntary, and all students who chose to take part at any stage of the project signed an informed consent form before they generated data. Both of these items made clear to students exactly what was required of them were they to participate in the study. They were given the opportunity to ask questions before deciding whether to take part.

Students were reassured that they would not be adversely affected if they chose not to participate. All students chose to participate.

2.7. Summary of methodology

This chapter begins by describing the RAs and RQs and the terms instrumental understanding and concept image and how they relate. It then provides an overview of the MMR approach adopted by the project and justification for its use. The MMR approach is illustrated using a design diagram. The instruments used to harvest the data are explained before the manner in which the data will be analysed is outlined. The RQs and each of the instruments are referred to through each of the following chapters, beginning with Chapter 3 which describes how the Diagnostic Survey was used to answer RQ1 and 2.

2.8. The intervention and its constituent parts

The Diagnostic Survey came before the intervention. It comprised four sections, and was given to students who had just completed MS225, one month prior to their exam. The four sections were: prior mathematical learning; conceptual issues in the study of differential equations; transfer issues; and modelling. Upon analysing the results, several areas of concern with student understanding were highlighted. An intervention was conceived and designed based on these areas and other areas of concern reported in the relevant literature. The timing of each part of the intervention is outlined in Table 2-3.

The intervention comprises a set of 15 tutorials. Tutorials are fifty-minute, timetabled slots. During tutorials, the students complete worksheets in a small group setting. Often, the worksheets are immediately preceded and/or followed by pre/post-test questions. The pre/post-tests are considered part of the intervention because students who attend the tutorials complete them, and they play a role in priming and reflecting on the worksheet activities. Figure 2-5 illustrates the intervention and its constituent parts.

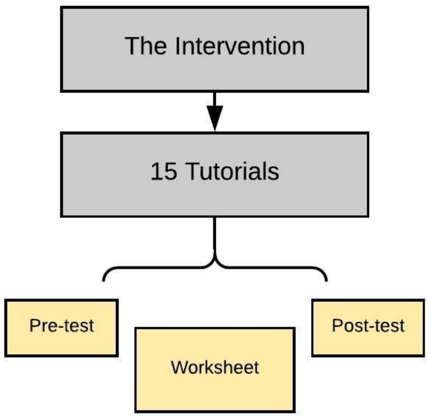


Figure 2-5: The intervention and its constituent parts

It is important to note that tutorials are more than a fifty-minute window of time during which a worksheet (and typically a pre-test and post-test) is completed by students. Tutorials are an environment (developed from a social constructivist paradigm) where interaction and atmosphere are key. This encapsulates far more than just a time slot in which teaching material is completed by students. This is similar to the difference between RME [18] and the materials used in an RME classroom. The form of instruction underpins the learning. The worksheets should be designed to work within the learning environment and not independent of it. An example from the teaching and learning of ODEs would be the Inquiry-Oriented approach [19] and its teaching materials as two intertwined parts.

After the tutorials are completed, students are asked to volunteer to take part in interviews relating to the tutorials. A delayed post-test (in the form of an exam question) is also administered, although the structure of the exam means completing the question is optional for students.

The intervention has been given to students enrolled in MS225 in DCU each year since its development. To date, two complete cycles of the intervention has taken place (Table 2-3). After each iteration, changes are made to each component of the intervention if required. The changes are based on answers to questions on pre/post-tests and in interviews.

Year	Diagnostic S.	Worksheets	Pre/post-tests	Interviews	Exam Q
Pre 2014					
14-15	✓				
15-16	✓	✓			
16-17		✓	✓	✓	✓
17-18		✓	✓	✓	✓

Table 2-3: Instrument used in each year

Our first interaction with students was administering the Diagnostic Survey to the outgoing cohort in first semester of the 2014-2015 academic year. We used this data to draft worksheets for the following cohort. The worksheets were trialled and the Diagnostic Survey was given to this cohort. The comparison between the 2014-2015 cohort and the 2015-2016 cohort reflected favourably on worksheets so they were revised and used with the 2016-2017 cohort in conjunction with pre/post-tests.

After the intervention in 2016-2017, interviews were conducted with students and a delayed post-test was an optional question on the final exam. The worksheets were revised again, and were administered during the 2017-2018 academic year along with pre/post-tests, interviews, and delayed post-test question.

Reference list

1 S. Vinner and T. Dreyfus. Images and definitions for the concept of function. **Journal of Research in Mathematics Education**, 1989, **20** (4), p.356-366.

2 L. Doughty, E. McLoughlin and P. van Kampen. What integration cues, and what cues integration in intermediate electromagnetism. **American Journal of Physics**, 2014, **82,** p.1093-1103.

3 L. Bollen, P. van Kampen and M. DeCock. Students' difficulties with vector calculus in electrodynamics. **Physical Review Special Topic- Physics Education Research**, 2015, **11**, p.1-14.

4. D.O. Tall and S. Vinner. Concept image and concept definition in mathematics, with special reverence to limits and continuity. **Educational Studies in Mathematics**, 1981, **12**, p.151-169.

5. R.R. Skemp. Relational Understanding and Instrumental Understanding. **Mathematics Teaching**, 1976, **77**, p.20–26.

6. J.W. Creswell, M.D. Fetters, V.L. Plano Clark and A. Morales. Mixed methods intervention trials. **Mixed Methods Research for Nursing and the Health Sciences**, 2009, **27**, p.161-80.

7. L. Cohen, L. Mannion and K. Morrison. **Research Methods in Education**, 7th ed. Routledge. 2011.

8. J.W. Creswell. **Steps in Conducting a Scholarly Mixed Methods Study**. *DBER Speaker Series*. 2013. [Online] Available from: http://digitalcommons.unl.edu/dberspeakers/48 [Accessed 10 May 2018].

9. J.W. Creswell. **Research design: qualitative, quantitative, and mixed methods approaches**, 2nd ed. Thousand Oaks (CA): Sage productions. 2003.

10. R. Jääskeläinen. Think-aloud protocol. **In**: Y. Gambier and L. van Doorslaer (eds). **Handbook of translation studies volume 1**. Amsterdam: John Benjamins publishing company. 2010, p.371-374.
11. A.J. Onwuegbuzie and N.L. Leech. Linking research questions to mixed methods data analysis procedures 1. **The Qualitative Report**, 2006, **11**(3), p.474-98.
12. D.R. Thomas. A general inductive approach for analyzing qualitative evaluation data. **American Journal of Evaluation,** 2006, **27**, p.20-26.
13. Y.S. Lincoln and E.G. Guba. **Naturalistic inquiry**. Beverly Hills, CA: Sage. 1985.
14. M.B. Miles and A.M. Huberman. **Qualitative analysis: An expanded sourcebook.** Thousand Oaks, CA: Sage. 1984.
15. S.E. Stemler. Content analysis. **Emerging Trends in the Social and Behavioral Sciences: An Interdisciplinary, Searchable, and Linkable Resource,** 2015.
16. M.D. Fetters, L.A. Curry and J.W. Creswell. Achieving integration in mixed methods designs-principles and practices. **Health Service Research**, 2013, **48** (6 Pt 2), p.2134-2156.
17. A. Kirk. **Data visualisation: a handbook for data driven design.** London: Sage, 2016.
18. M. Van Den Heuvel-Panhuizen. The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. **Educational Studies in Mathematics**, 2003, **54** (1), p.9-35.
19. C.L. Rasmussen, and O.N. Kwon. An inquiry-oriented approach to undergraduate mathematics. **Journal of Mathematical Behavior**, 2007, **26**, p.189-194.

3. Pre-intervention

In this chapter, we report on the initial stage of this project in which we are investigating how a service-taught mathematics module prepares students for using ordinary differential equations (ODEs) in physics. We explore students' instrumental understanding and concept image of ODEs having completed a standard module on this topic. The module carries 5 ECTS credits and is taught by mathematicians at Dublin City University (DCU), Ireland. A paper reporting this stage of the research has previously been published [1].

3.1. Background

In order to gain insight into the degree of student difficulty with ODEs in physics at DCU, a survey of academic staff (Appendix A) in the School of Physical Sciences was conducted. The survey asked lecturers about how they use ODEs in their modules, and whether and to what extent student ability influences their modular design. The responses to these questions detailed how the role of ODEs in advanced physics modules has lessened gradually in recent years (to the point of omission in certain cases). Lecturers feel constrained by the students' inability to use the ODEs they are being service-taught in the physics they are learning as part of their degree programme. The issues cited by lecturers in the survey highlight areas for potential improvement of service teaching in DCU and further afield. The project focused on a group of students who have completed a typical 12-week introductory service module on ODEs (MS225) in DCU. The physics students at DCU complete this module in their third year of study and ODEs begin to appear in advanced physics modules from this time onward. This stage of the study focused on Research Question 1 and 2.

1) Do our students have the necessary instrumental understanding in the following areas to succeed in their study of ODEs?
    a) Manipulation of exponents in equations
    b) Evaluating indefinite integrals
2) Do our students have a well-developed concept image of ODEs upon completion of this module?

a) What is brought to mind when presented with an ODE?

b) What do students know about ODEs and their applications after completing their module?

c) Do students understand what a solution to an ODE is?

Instrumental understanding relates to students' ability to complete the mathematical tasks they encounter during their study of ODEs, making it an integral part of this study. We have investigated our students' conceptual understanding through establishing their concept image of ODEs. When the students' instrumental understanding is investigated in conjunction with their concept image, it will provide a more detailed account of their strengths and weaknesses in relation to their understanding of ODEs. We stress that at this stage of the project there has not yet been a change in our teaching approach; rather, the results from this stage will inform the development of future interventions.

3.2. Methodology

Though the project uses a mixed methods research design [6] (c.f. Section 2.3), this chapter focuses on the initial stage of data collection which comes from a Diagnostic Survey. The Diagnostic Survey is designed to inform RQ1 and 2.

In Section 2.2, we explained what we mean when we refer to instrumental understanding and concept image. Tall and Vinner [3, p.152] defined the concept image as 'the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes'. Tall and Vinner [3] explain that what students write down or otherwise communicate to us is not necessarily their entire concept image, since the question may not have cued them to share everything. What is obtained from students' answers is called their 'evoked concept image' [3 p.152]. In order to assess this in our students, we designed open-ended questions that encouraged them to explain terms to us as they understand them. Other investigations of students' understanding of mathematics in physics [4, 5, 6] have used concept image in this manner and we believe that it provides an appropriate lens through which our students' understanding of ODEs can be studied. The data generated from the questions in the Diagnostic Survey will give us the best opportunity to identify certain aspects of students' strengths and weaknesses in relation to ODEs.

In our first research question, we refer to instrumental understanding, a term first used by Skemp [7]. In this paper, Skemp describes instrumental understanding as possessing "rules without reasons", and contrasts it with relational understanding, which he explains is "knowing both what to do and why" [7, p.20]. In describing and assessing students' instrumental understanding in relation to general calculus and algebra skills, we are investigating their ability to systematically solve decontextualised mathematical tasks independent of any requirement to justify or validate their reasoning: this is instrumental understanding. Our research questions are best described and answered using concept image and instrumental understanding, but the dichotomy presented by Skemp is noteworthy (cf. Section 1.1). We also recognise a connection with Sfard's identification of a duality (as opposed to a dichotomy) in relation to mathematical concepts as having complementary aspects referred to as *operational* and *structural* [8]. We will return to this duality below in our discussion of the results of this research.

3.2.1. Administration and data collection

Prior to being issued to students, the Diagnostic Survey was trialled with a group of postgraduate students from the Schools of Mathematical, Physical, and Chemical Sciences at DCU. When it was finalised, the Diagnostic Survey was administered to students during a lecture slot in the final week of the semester. It comprised two sections, which were given to students one at a time. Students were allowed twenty minutes per section and no student required additional time to complete their work. The students were informed that if they did not know the answer to a question that they should state this on their sheet and move onto the next question. This was done to allow us to differentiate between students who were unable to complete the questions and students who had insufficient time to do so. The students completed the Diagnostic Survey individually without the use notes or other resources.

3.3. Instrument

We designed a Diagnostic Survey (Figure 3-1) to answer the research questions outlined in the Section 3.1. Section 1 of the Diagnostic Survey focused on RQ1 and Section 2 of the Diagnostic Survey focused on RQ2. The term 'differential equation' was used as synonymous with ODE during the Diagnostic Survey and throughout the module.

Section 1

Q1. Find x and y if

- $5=3+x$
- $4=3+xe^{-y}$

Q2. Simplify the following.

$$\frac{(-k)^2}{k^{-1}} + (k^2 + k^1)^2$$

Q3. Integrate each of the following;

- $\int(x^2 + \frac{1}{x^3})\,dx$
- $\int \frac{1}{x}\,dx$
- $\int xe^x\,dx$

Section 2

Q1. Write down everything you can think of when you see each of the following. (note: C is a positive constant in each case)

- $\frac{dN}{dt} = -C$
- $\frac{dN}{dt} = -CN$
- $\frac{dN}{dt} = -Ct$

Q2. What are differential equations, and why are they useful?

Q3. Write down everything you know about the solutions to differential equations.

Figure 3-1: The Diagnostic Survey

Section 1 was divided into three questions; Question 3 comprised three parts. Each question that appeared in this section of the Diagnostic Survey is a closed mathematical task that arises during the course of the module. They are not claimed to be an exhaustive list of problem areas for students but were selected for inclusion by the research team based on prior teaching experience, their relevance to the module, and the related literature [9].

Question 1 contains a system of two equations that students are asked to solve for x and y. This question was included on the Diagnostic Survey because an isomorphic system of equations occurs when students solve a problem based on Newton's Law of Cooling. Newton's Law of Cooling models how the temperature of an object changes with time using a first order ODE. Problems of this form appear in MS225 and in many modules on ODEs. In order to successfully complete this question, an instrumental understanding of both manipulating exponential functions and isolating an unknown when it is an exponent is required.

The purpose of Question 2 is to assess students' instrumental understanding of exponents. Classroom observations had signalled to us that manipulating exponents (specifically negative ones) cause some students difficulties. Students' ability to use the power rule of integration successfully is also affected by their ability to manipulate exponents and including a question that assesses students' instrumental understanding of exponents will shed light on both of these difficulties and potentially allow us to distinguish between both. We intended to narrow the band of potential mistakes by writing k as k^1. However, we note that this inadvertently led us to use non-standard notation, and thus taint the diagnostic element of the item.

Question 3 was a three-part integration question involving the calculation of indefinite integrals. Integration is a key procedure when solving ODEs and students will need to be able to calculate a range of anti-derivatives during MS225 and in their future studies.

The first part of Question 3 required students to integrate two terms for which the power rule of integration works. The second part of Question 3 asked students to evaluate $\int \frac{1}{x} dx$, a common calculation that students will face repeatedly during their study of ODEs. The final part asked students to integrate xe^x. Students have been exposed to integration by parts and one would expect students to apply this approach in this canonical integration-by-parts task, but other methods, such as the method of undetermined coefficients, are also possible. Integration by parts frequently appears when solving first order linear ODEs. It is clearly more complex operationally than the other two integrals and could conceivably cause students issues, warranting its inclusion.

Section 2 of the Diagnostic Survey contained five questions in total, three of which are described and reported on in this chapter. The purpose of this section of the Diagnostic Survey was to identify how students view and understand ODEs upon completion of MS225. This was achieved by evoking their concept image using open-ended questions, where students could answer as they saw appropriate. The phrasing of the questions was designed to evaluate the students' evoked concept image in the same manner used by Vinner and Dreyfus [4], Doughty et al. [5], and Bollen et al. [6] to study function, integration, and divergence, gradient, and curl respectively.

Question 1 of Section 2 of the Diagnostic Survey asked students to write down everything they can think of when they see each of the following equations:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\mathrm{C} \qquad \frac{\mathrm{d}N}{\mathrm{d}t} = -\mathrm{C}N \qquad \frac{\mathrm{d}N}{\mathrm{d}t} = -\mathrm{C}t$$

where C is a positive constant in each case. Question 2 asked students what differential equations are, and why are they useful, and Question 3 asked students to write down everything they know about solutions to differential equations.

3.4. Results and discussion

In this section, we present the results of the Diagnostic Survey. We will begin with the students' responses to the mathematical tasks asked in Section 1 of the Diagnostic Survey before describing the answers given to Section 2. Each section of the Diagnostic Survey was completed by eighteen students.

3.4.1. Research Question 1: Do our students have the necessary instrumental understanding in the following areas to succeed in their study of ODEs?

Research Question 1a: manipulating exponents in equations

As mentioned in the previous section of this chapter, the questions in Section 1 evaluate instrumental understanding. Our analysis focused on how many students got each question correct and what errors occurred in the incorrect answers. Question 1 required students to solve a system of non-linear algebraic equations for x and y. Ten students (56%) obtained the correct values for x and y. The eight (44%) students who were unsuccessful all obtained a correct value for x and reduced the problem to solving $\frac{1}{2} = e^{-y}$, but failed to correctly isolate y.

We conclude that just over half of our students are capable of manipulating exponents sufficiently well for the purposes of this module on ODEs, but a significant minority have not yet mastered the use of the natural log function *vis a vis* its relationship with exponentials.

Research Question 1b: evaluating indefinite integrals

Question 3 comprised a three-part integration question that was answered with varying levels of success. The second part of the question was answered best by students, with seventeen of the eighteen students (94%) integrating $\frac{1}{x}$ correctly. With respect to part one of the question, nine students (50%) integrated $x^2 + \frac{1}{x^3}$ correctly. One student did not give an answer and the remaining eight students failed to integrate the $\frac{1}{x^3}$ term correctly. Of these eight students, one student had the correct term with an incorrect sign, another student left the $\frac{1}{x^3}$ term unaltered, three students arrived at $\ln x^3$

as their answer, and the remaining three students made errors when manipulating exponents during the integration process.

We attribute the issues encountered by students when integrating the $\frac{1}{x^3}$ term to a large extent to difficulties with negative exponents. This viewpoint is corroborated by the answers to Question 2. Only seven students (39%) answered the question correctly; of the eleven students who were incorrect, ten students made an error manipulating the expression $\frac{(-k)^2}{k^{-1}}$. The responses to both Questions 2 and 3 suggest that our students find manipulating negative exponents more difficult than manipulating positive exponents. Table 3-1 shows how the students' answers to Question 2 and 3 are linked. All 7 students who answered the exponents question correctly also answered the integration question correctly; of the 9 students who answered the exponents question incorrectly, 7 answered the integration question incorrectly, and 2 were correct. The students who answered the exponents question with 'do not know' and left the question unanswered also integrated incorrectly.

Exponentials \ Integration	Correct	Incorrect
Correct	7	-
Incorrect	2	7
Don't know	-	1
Blank	-	1

Table 3-1: Correlation between answers to Q2 and Q3

The final part of Question 3, the integration of xe^x, was answered correctly by nine students (50%). Of the nine students who were incorrect, two attempted to integrate by parts but did not complete their calculation; four students used the product rule for differentiation, suggesting a potential confusion between differentiation techniques and integration techniques; and three students combined various incorrect integration and differentiation algorithms.

We conclude that our students' ability to evaluate indefinite integrals is probably not at the level required. These results were obtained on completion of the module, and about half of the students were unable to integrate correctly two integrals they would have encountered repeatedly during the module, and beforehand in other modules. Unsatisfactory as this is, we note that this need not impede students' progress with acquiring broader or deeper knowledge of ODEs, since these difficulties occur towards the end of the process of solving ODEs (and may even go undetected).

3.4.2. Research Question 2: Do our students have a well-developed concept image of ODEs upon completion of this module?

Questions 1, 2, and 3 of Section 2 assessed students' concept image of ODEs upon completion of the module and allowed us to answer Research Questions 2a, b, and c respectively. The open-ended nature of the questions implied that the answers given by students could take several different forms. Using a data analysis technique described by Thomas [10] (c.f. Section 2.5), we were able to reduce the data to the form given in Table 3-2 and Table 3-3. These tables contain information about the form and content of answers given for each question in this section. A tallied display was used because it best suited the form of response given by students, which typically consisted of isolated examples of the uses of differential equations.

Research Question 2a: what is brought to mind when presented with an ODE?

The most common approach used by students when answering Question 1 was to attempt to solve the three ODEs analytically. For the first ODE, ten students (56%) did this. In each case, students attempted to solve the ODE by separating variables, which they succeeded in; eight of these completed the solution but provided no commentary, explanation, or validation for their approach. Another analytical approach, taken by two students (11%), was to differentiate with respect to t (Figure 3-2 and Figure 3-3). They may have been prompted to differentiate when they saw a derivative. The other answers given by students included four students (22%) explaining that N decreases at a rate C with respect to time. Two students (11%) graphed N vs t to show that the slope was constant.

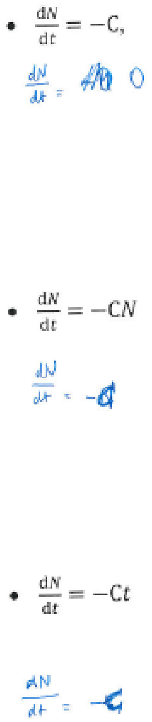


Figure 3-2: Student that differentiated the RHS with respect to t

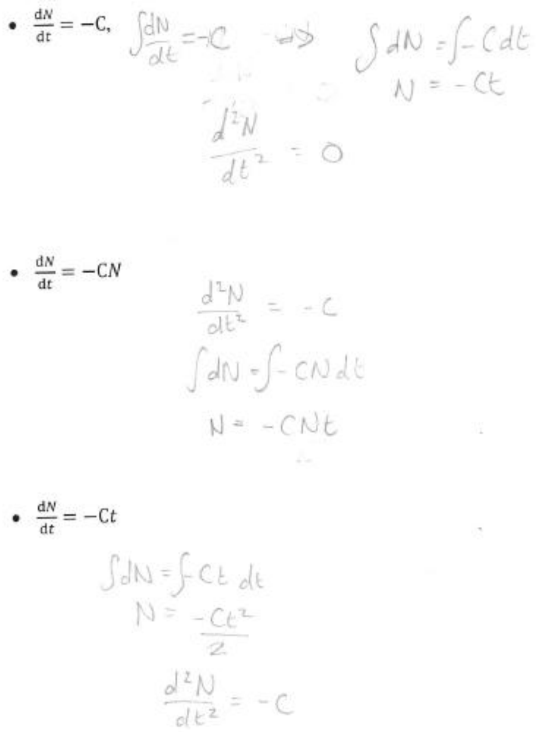


Figure 3-3: Student that differentiated both sides with respect to t

The students approached the other equations in Question 1 in a similar manner, with answers occurring with similar frequency. Of note is that the number of students who attempted to separate and solve the given ODE varies across the equations: Students attempted to separate the first equation (n=10; 56%) less frequently than the second equation (n=13; (72%)) or the third equation (n=11; (61%)). This finding suggests that students may identify an equation as a differential equation more easily when either the dependent or independent variable is explicitly present (rather than just implicitly in the form of the derivative), though the sample is too small to make a definitive statement. In this case, students identified the equation that contained the dependent variable (the second) as a differential equation most frequently. Fewer students identified the equation that contained the independent variable (the third equation) as a differential equation, and fewer still identified the equation that contained neither variable (the first equation) as a differential equation. This suggests that structure may play a part in a students' ability to recognise ODEs. Also of note was the frequency with which these ODEs caused students to mention differentiation, or in one case to differentiate the right-hand side with respect to t. Mentioning differentiation or derivatives was most frequent in the first ODE, where talk of differential equations was least frequent. It is understandable that these terms do not always appear together. In fact, in order for a student to respond in both ways, it would require them to think of a derivative as representing a process that has taken place (a function has been differentiated) at the same time as thinking of an equation where the derivative is simply another term. This can be understood in terms of Sfard's notion of duality [8].

Research Question 2b: what do students know about ODEs and their applications after completing the module?

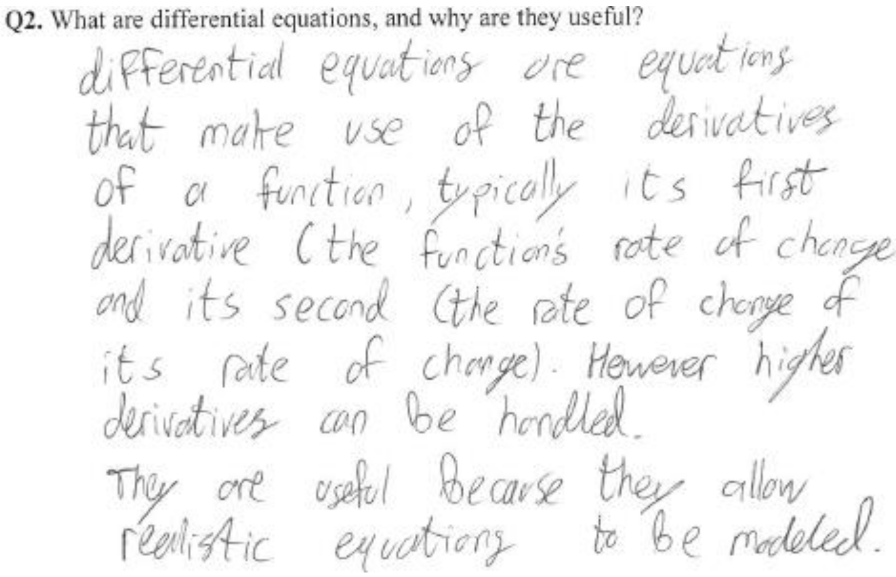
Question 2 in this section yielded varying levels of responses from students regarding their understanding of differential equations and their uses. While eight students (44%) described differential equations as pertaining to rate of change, only one student (6%) gave a formal definition in their answer. Arguably more significant is the number of students (*n*=6; 33%) who, at the end of a twelve-week module on ODEs, left their survey blank or wrote 'I don't know' in response to what a differential equation is. This implies that these students had a purely instrumental view of ODEs.

Applications for differential equations were offered more readily by students. The frequency with which some terms were mentioned indicated in Table 3-2.

Use	**Frequency**
Population	5
Predict	5
Used to mathematically model complex systems	3
Stock market	2
Weather (chaos)	1
Optics	1
Semiconductors	1
Physical systems	1
Acceleration	1
Chemical reactions	1

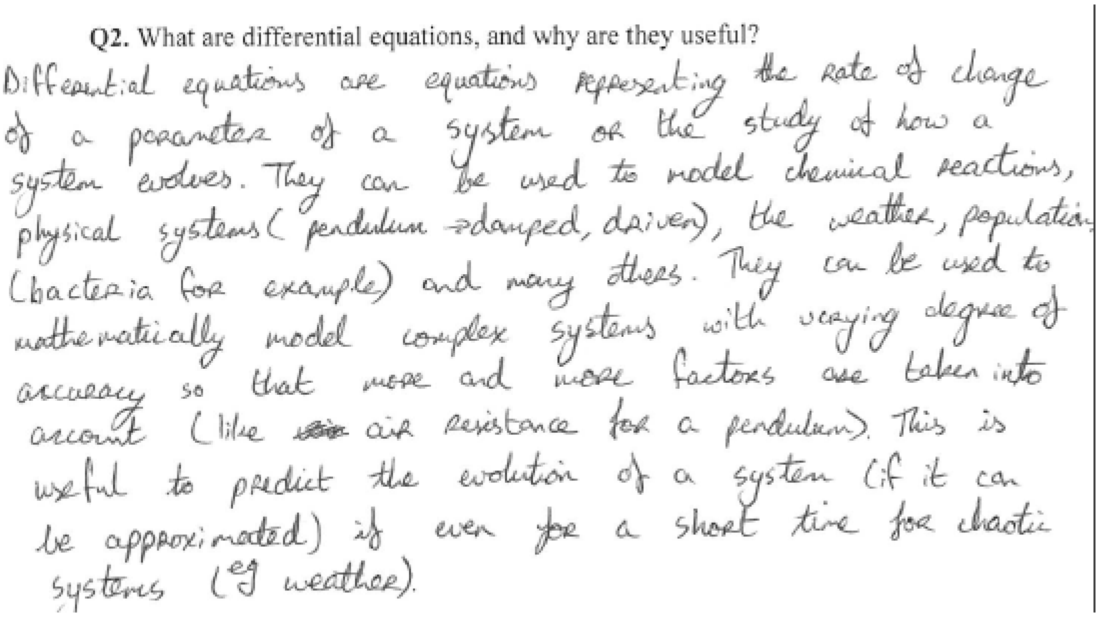
Table 3-2: Students' concept images of what ODEs are for

Five of the students (28%) did not explain why differential equations may be useful or where they may be applied; two of these did not give a response to the question as to what a differential equation is. Even among the thirteen (72%) students who we considered to have answered the question, only two students (11%) offered satisfactory definitions (Figure 3-4). The answers shown in Figure 3-4 were both considered to be correct but represented two different ways of understanding differential equations. Answer (a) is more succinct and uses more formal language to explain what differential equations are. This is close to the definition given to students in lectures (an ODE is an equation involving one or more derivative of a function, say *y*, which depends on a single variable, say *x*). Answer (b) is more descriptive and describes differential equations with reference to many examples. Both are valid answers, but the second answer could be considered more desirable for a physics student.



Q2. What are differential equations, and why are they useful?

differential equations are equations that make use of the derivatives of a function, typically its first derivative (the function's rate of change) and its second (the rate of change of its rate of change). However higher derivatives can be handled.
They are useful because they allow realistic equations to be modeled.



Q2. What are differential equations, and why are they useful?

Differential equations are equations representing the rate of change of a parameter of a system or the study of how a system evolves. They can be used to model chemical reactions, physical systems (pendulum damped, driven), the weather, population (bacteria for example) and many others. They can be used to mathematically model complex systems with varying degree of accuracy so that more and more factors are taken into account (like air resistance for a pendulum). This is useful to predict the evolution of a system (if it can be approximated) if even for a short time for chaotic systems (eg weather).

Figure 3-4: Satisfactory definitions for ODE given by students

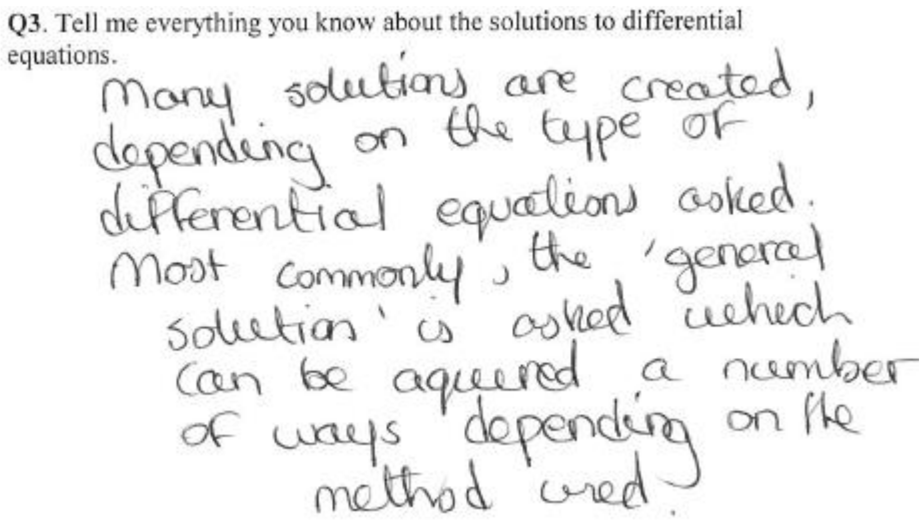
Research Question 2c: do students understand what a solution to an ODE is?

Question 3 required students to write down everything they know about solutions to differential equations. An example of how student responses were analysed is included in Appendix G. The breakdown of terms and the frequency with which they were mentioned is shown in Table 3-3.

Answer	Frequency
General solution	7
Initial condition	7
Contain a constant	4
Differentiation	3
Integration	3
Numerically/analytically	3
Particular solution	3
Homogenous/non-homogeneous	3
Not all differential equations have solutions	2
Different methods e.g. Euler/Runge-Kutta	2
Rate of change of something	1
Tangent to a curve	1

Table 3-3: Students' concept images of solutions of ODEs

The students' propensity toward procedures and solution techniques is clearly evident from Table 3-3 where terms associated with answering questions dominate the top of the table and can be seen throughout (e.g. mentioning numerical and analytical as two ways of obtaining solutions and mentioning various methods for solving DEs). Figure 3-5 shows a typical student response (the fact that particular solutions are not always asked for in questions explains its position further down the table). It is also notable that some students give isolated terms in a disjointed manner. An example is the discrepancy in frequency between related terms: for example, to an expert the term 'initial condition', mentioned seven times, would go hand in hand with 'particular solution', but particular solution as an answer is only seen three times; and none of the students linked the two explicitly.


Q3. Tell me everything you know about the solutions to differential equations.

Many solutions are created, depending on the type of differential equations asked. Most commonly, the 'general solution' is asked which can be aquired a number of ways depending on the method used.

Figure 3-5: Typical student response to what they know about solutions to ODEs

A fundamental aspect of the solution of an ODE was mentioned by only one student (6%): that they are functions. Difficulty grasping that a function can be a solution to an equation is noted by Mallet and McCue [11] and agrees with the findings of Rasmussen [12, p.66] who posited that 'this switch from conceptualising solutions as numbers to conceptualising them as functions is non-trivial for students'. This contrasts with the frequency with which 'contain a constant' is mentioned – again, to an expert these ideas are linked. This further suggests a focus on algorithms and procedures: awareness of the fact that solving an ODE (where including a constant in an indefinite integral may be required) produces a function is not central to the relevant procedure.

A useful way to interpret the results of Section 2 of the Diagnostic Survey is in terms of Sfard's identification of the dual process-object nature of mathematical concepts [8]. Almost all of the students' responses refer to processes related to ODEs and their solutions. Only one student mentioned that the solution of an ODE is a function - thus alluding to the concept of 'solution of an ODE' as a mathematical object. It seems that for many of our students, the presence of the derivative symbol evokes the process of differentiation, or even further, is taken as a prompt to differentiate something. They have difficulty understanding the symbol $\frac{dy}{dx}$ as an object that can enter into a mathematical equation the way it does in an ODE. Their poor conception of ODEs and their solutions as mathematical objects (or structures) is in keeping with Sfard's remark that 'in the process of concept formation, operational conceptions would precede the structural' [8, p.10]. Attaining understanding of this nature through the processes of interiorisation, condensation and reification is 'a long and inherently difficult process' [8, p.1]. But, as Sfard further notes, a structural understanding and approach is advantageous, allowing as it does 'more room… for more information' [8, p.28]. From this point of view, it is perhaps not surprising that our students struggle to apply ODEs in physics: understanding ODEs as tools (objects) that can be applied in different situations is a prerequisite of this application.

3.5. Conclusions

The Diagnostic Survey was designed to give us an overall picture of students' instrumental understanding and concept image of ODEs on completion of a standard service-taught module. RQ1 looked at students' instrumental understanding of manipulation of exponents and indefinite integrals.

The analyses of the results to Section 1 of the Diagnostic Survey led us to conclude that there are areas where students' instrumental understanding is not strong enough to successfully complete the standard of questions on ODEs they encounter during MS225. Although the majority of students were able to complete Question 1, a large minority (44%) were unable to use the natural log function as the inverse of the exponential function which was necessary in order to successfully complete the task. A similar level of success was observed in Question 3, where only half of the students successfully integrated $\frac{1}{x^3}$. We believe the requirement to manipulate negative exponents caused the high level of difficulty experienced by students. This is corroborated by the results of Question 2 of Section 1 of the Diagnostic Survey which also required students to manipulate negative exponents as part of the correct solution. All the students who could manipulate negative exponents in Question 2 were able to integrate $\frac{1}{x^3}$. Question 2 was the only question where less than half of the students were correct, but the results of Section 1 as a whole suggest students' instrumental understanding needs significant improvement if students are going to excel in their use of ODEs throughout their studies.

RQ2 investigated students' concept images of ODEs upon completion of MS225. The results show that students' concept images are poorly developed, and their understanding is at the level of processes rather than concepts. This is in line with Sfard's notion of the process-object duality of mathematical concepts [8]. First-year students' poor conceptual understanding in introductory calculus modules has been reported on previously [13], but the problem is also evident for third-year students in this module on ODEs. If a vast improvement does not occur in the conceptual understanding of the students completing this module, using differential equations to model situations in physics effectively will likely continue to be a challenge for them. We will now discuss the implication of these findings for teaching ODEs.

3.6. Implications for teaching

In order to maximise students' ability to correctly apply ODEs in their future studies, the issues discussed in the results section need to be addressed. While students who completed MS225 showed adequate instrumental understanding in their assessment, there was still a lack of understanding of

ODEs preventing their physics lecturers from including material on future courses, as described in the Survey of Academic Staff (Appendix A). This lack of understanding, illustrated by fragmented concept images, was evident in the results from the Diagnostic Survey. The identification of shortcomings in student understanding begins the next stage in this research project: designing an intervention that seeks to address the obstacles encountered by physics students when studying ODEs.

The intervention, which we describe in detail in Chapter 4, has taken the form of a set of fifteen tutorials. The tutorials replaced the tutorials that were given previously and were delivered to students in conjunction with the lectures for MS225, but did not affect the module in any other way. Results from the Diagnostic Survey and findings from relevant studies were used to decide on the content of the tutorials. This set of tutorials focused on the following findings from the Diagnostic Survey:

1. Section 1 of the Diagnostic Survey identified some areas for improvement relating to exponents, algebraic systems, and integration. While not all students had difficulties with each of these concepts, an instrumental understanding of each is required to solve the ODEs encountered by these students. The intervention addressed these areas when dealing with students' mathematical ability.
2. The results of the Diagnostic Survey highlighted students' conceptual understanding of ODEs as an area in need of attention. Specifically, ODEs, their solutions, and the function nature of their solutions were areas in relation to which we felt that students may be able to develop a more comprehensive concept image. We have therefore devoted time in the intervention to further developing students' concept image. Redish and Kuo [14] talked about the need for physics instructors to foster components of conceptual understanding (mathematical fluency in a physical context, mathematical modelling of physical systems) in their students. While this is certainly the case, there is no reason why mathematics instructors cannot work towards the same goal which will allow students to succeed in using mathematics in physics.
3. Prior to the intervention being implemented, the tutorials delivered as part of MS225 took a traditional format. A problem sheet was made available to students and they attended a fifty-minute recitation session each week. We believe that a change in pedagogical approach

might foster better learning for our students. Of the research on the teaching and learning of differential equations, Rasmussen and Kwon [15] address the manner in which content is delivered most directly. Their paper concentrates on improving the method of teaching, resulting in The Inquiry Orientated Differential Equations (IO-DE) project which was described in the opening chapter as adapting the instructional design of RME. RME also placed an emphasis on embedding real-world problems in teaching. A similar emphasis on real-world problems in the mathematics classroom is being attempted at the University of Leeds [16] who have changed the structure of their MATH 1400 module to include more mathematical modelling and problem-solving. This is particularly useful to students studying physics who would benefit from the relevance of the problem settings. These projects are prime examples of successfully changing an instructional design at tertiary level. We aim to do the same. The tutorials will use guided-inquiry worksheets and adopt a cooperative learning pedagogical approach. This is influenced by previous studies that reported success with similar goals to ours [5, 18]. This approach is also supported by McDermott and Shaffer [19], who compared the effectiveness of guided-inquiry tutorials to traditional instruction using a control study.

4. Introducing graphical representations was considered when designing the tutorials. Stephan and Rasmussen [20] make the point that graphing plays a bigger role in the teaching and learning of differential equations than is usually seen in other maths modules. They mention graphical representations as a third method (along with numerical and analytical techniques) for the study and application of differential equations. They talk about the importance of teaching this in a 'bottom-up' manner which aligns with the RME approach that influenced the IO-DE project. This prompted us to include direction fields in the intervention to help develop the third method for students, who until now have focused almost exclusively on analytical techniques to solve problems.

In this chapter, we have described the first stage of a multiphase project that aims to identify and address the obstacles encountered by physics students during their study of ODEs. We designed a Diagnostic Survey which was given to students who completed a service mathematics module on ODEs. The results of the Diagnostic Survey identified students' strengths and weaknesses in relation to ODEs. These findings will inform the next stage of the research: the design of an intervention that seeks to address the obstacles encountered by students. A comprehensive description of the design and implementation of the intervention will be explained in detail in Chapter 4.

Reference list

1. D. Hyland. P. van Kampen and B.C. Nolan. Outcomes of a service teaching module on ODEs for physics students. **International Journal of Mathematics Education in Science and Technology**, 2017, **49** (5), p.743-758.
2. J.W. Creswell and V.L. Plano Clark**. Designing and conducting mixed methods research.** Thousand Oaks (CA): SAG Publications Inc., 2007.
3. D.O. Tall and S. Vinner. Concept image and concept definition in mathematics with particular reference to limits and continuity. **Educational Studies in Mathematics**, 1981, **12** (2), p.151-169
4. S. Vinner and T. Dreyfus. Images and definitions for the concept of function. **Journal of Research in Mathematics Education**, 1989, **20** (4), p.356-366.
5. L. Doughty, E. McLoughlin, and P. van Kampen. What integration cues, and what cues integration in intermediate electromagnetism. **American Journal of Physics**, 2014, **82,** p.1093-1103.
6. L. Bollen, P. van Kampen and M. De Cock. Students' difficulties with vector calculus in electrodynamics. **Physical Review Special Topic- Physics Education Research**, 2015, **11**, p.1-14.
7. R.R. Skemp. Relational understanding and instrumental understanding. **Mathematics Teaching**, 1976, **77**, p.20-26.
8. A. Sfard. On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. **Educational Studies in Mathematics**, 1991, **22** (1), p.1-36.
9. J. Confrey and E. Smith. Splitting, covariation, and their role in the development of exponential functions. **Journal of Research in Mathematics Education**, 1995, **26**, p.66-86.
10. D.R. Thomas. A general inductive approach for analysing qualitative evaluation data. **American Journal of Evaluation**, 2006, **27**, p.20-26.
11. D.G. Mallet and S.W. McCue. Constructive development of the solutions of linear

equations in introductory ordinary differential equations. **International Journal of Mathematics Education in Science and Technology**, 2009, **40** (5), p.587-595.

12. C.L. Rasmussen. New directions in differential equations: A framework for interpreting students' understandings and difficulties. **Journal of Mathematical Behavior**, 2001, **20**, p.55–87.

13. J. Bezuidenhout. Limits and continuity: some conceptions of first-year students. **International Journal of Mathematics Education in Science and Technology**, 2001, **32** (4), p. 487-500.

14. E.F. Redish and E. Kuo. Language of physics, language of math: disciplinary culture and dynamic epistemology. **Science Education**, 2015, **24**, p.561-590.

15. C.L. Rasmussen and O.N. Kwon. An inquiry-oriented approach to undergraduate mathematics. **Journal of Mathematical Behaviour**, 2007, **26**, p.189-194.

16. University of Leeds. **Faculty of engineering**, 2016. [Online] Available from: https://www.engineering.leeds.ac.uk/teaching/electronic/module/MATH1400 [Accessed 1 March 2016]

17. M. Savage and M. Grove. Mathematical modelling and problem solving in real-world physical situations. **In**: M. Grove, T. Croft, J. Kyle and D. Lawson, (eds). **Transitions in undergraduate mathematics education**. Birmingham: The Higher Education Academy, University of Birmingham, 2015, p.115-127.

18. L.C. McDermott, P.S Shaffer PS, the Physics Education Group at University of Washington. **Tutorials in introductory physics.** Upper Saddle River (NJ): Prentice-Hall; 2002.

19. L.C. McDermott and P.S. Shaffer. Research as a guide for curriculum development: an example from introductory electricity. Part II: design of instructional strategies. **American Journal of Physics**, 1992, **60** (11), p.1003-1013.

20. M. Stephan and C.L. Rasmussen. Classroom mathematical practices in differential equations. **Journal of Mathematical Behavior**, 2002, **21**, p.459-490.

4. The intervention

The second research aim of this project was to develop an intervention that addresses the obstacles described in Chapter 3, with the intention of promoting more successful learning that would broaden the students' concept image and make the module more useful for future physics modules. The intervention comprises several components: worksheets, pre/post-tests, and interviews, as well as a small-group setting.

The design of the worksheets is the primary focus of this chapter. This process involved several key elements and was carried out in a manner similar to that described by McDermott [1]. McDermott describes three key elements for developing instructional materials: research on student understanding; use of the findings to guide the development of curriculum; and assessment of student learning [1, p.1134]. These elements are also emphasised in this work, where the use of the Diagnostic Survey, design of the worksheets, and the subsequent data collection through pre/post-testing and interviews maximises the capabilities of our Mixed Methods Research (MMR) approach. Before the worksheets could be developed, it was important to learn about student understanding of ODEs. This was accomplished using the Diagnostic Survey as outlined in Chapter 3. The Diagnostic Survey gave rise to several implications for teaching that were central to the design of the worksheets. It was also important to review the literature for other areas the Diagnostic Survey may not have captured. The results of the Diagnostic Survey in conjunction with other relevant research from the literature encompass the first and second key elements described by McDermott. The assessment of student learning is achieved through a variety of methods (pre/post-tests and interviews described in Chapter 2), whose results will be discussed in Chapter 5. An additional element to consider when designing instructional materials is the theory of learning that will underpin the intervention. A large emphasis is placed on the educational theory in this research as described in Section 1.2. The implications of adopting a social constructivist approach will be explained in Section 4.2.

This chapter describes all of the aspects of the intervention, culminating in the set of worksheets presented in full. In addition to discussing these, an outline of the design parameters imposed on the intervention and a description of how the intervention was revised in successive years is included.

4.1.Design restrictions

The intervention was designed to replace the existing tutorials for MS225. In order for this to occur, the tutorials needed to work within the same constraints as the previous tutorials.

The previous tutorials could be described as recitation sessions: A problem sheet was made available for students online at the beginning of the week that contained a number of mathematical tasks that allowed students to practice the procedures they saw during lectures. The students then had the choice to attend the tutorial session which was allocated one fifty-minute slot per week on their timetable. In line with custom and practice, attendance or completion of tutorial work was neither compulsory nor graded.

Working within this format imposes several restrictions on the intervention. The module has three fifty-minute sessions timetabled per week, within a twelve-week semester. Typically, this allows for two lectures and one tutorial (and is timetabled as such) although this is subject to change within reason. Each session was timetabled individually, so there was no possibility for a two-hour session. Further to this time constraint was our desire to use immediate pre/post-testing, meaning that each tutorial that used pre/post-testing would need to contain a worksheet which would take no more than forty-five minutes to complete (allowing for fifty-minute sessions).

4.2.Underlying theory of learning

Section 1.2 indicated that constructivism is the educational paradigm adopted by this project and outlined some key elements of this approach. In the present section, the link between constructivism and worksheet design will be made in more detail. First, we consider the distinctions between different types of constructivism [2], specifically cognitive constructivism and social constructivism. In order to discuss cognitive constructivism, it is necessary to mention briefly the concept of cognitive psychology.

Cognitive psychology is described by Blake and Pope as

> A branch of psychology that focuses on studies (*sic*) mental processes, which include how people think, perceive, remember, and learn. Its core focus is on how people acquire, process, and store information. [3, p.59].

Cognitive constructivism places the same emphasis on the mental processes of the student within a constructivist paradigm. This is often distinguished from social constructivism, which is the focus of our work. Social constructivism is a term used to describe constructivism where particular emphasis is placed on the importance of community for the learning process. It is associated with the work of Vygotsky, who wrote about the *zone of proximal development* (ZPD) [4].

4.2.1. The Zone of Proximal Development

Vygotsky [4] describes *the zone of proximal development* (ZPD) as

> the distance between the *actual developmental level* as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers. [4, p.86]

This was a novel way to conceptualise intelligence, in that it accounted for functions that had 'not yet matured but are in the process of maturation' [4, p.86]. Using the actual developmental level (ADL) and ZPD, Vygotsky introduced terms that characterise mental development retrospectively (ADL) and prospectively (ZPD).

Vygotsky states that the creation of the ZPD is an essential feature of learning [4], explaining that

> learning awakens a variety of internal development processes that are able to operate only when the child is interacting with people in his environment and in cooperation with his peers. Once these processes are internalized, they become part of the child's independent developmental achievement. [4, p.90]

It follows from Vygotsky's description of the ZPD that maximising the opportunity for learners to interact with their peers or others in their environment is desirable. The manner in which we chose to incorporate this in the worksheets is through facilitated group work. Our decision agrees with Dubinsky [5] who explains how subscribing to a constructivist approach to knowledge construction means that learning will happen best through conversation with others. The students' interactions with the facilitator also feeds back to Vygotsky's [4] ZPD. A detailed overview of how the groups were constructed and facilitated in coherence with Vygotsky's work is given in the following sections.

4.2.2. Group work

The potential benefits of students working in groups in education, both in general and in undergraduate mathematics are outlined in Section 1.2. A study of students' attitudes on group work in undergraduate mathematics by Walker [6] reported several favourable outcomes. Group work helped them to consolidate and further explore material covered in lectures; peer discussions facilitated their understanding of the lecture material; group work helped them to retain their concentration levels; and they felt more able to ask for clarification or help from their peers than from the lecturer [6, p.78]. The research team should also be aware of potential drawbacks to group work (for example, it is common for certain students to prefer to work individually).

In the present chapter, we relate this to this project. By subscribing to a social constructivist theory of learning, we are accepting that group work will play a significant role in student learning. Although the results reported in Chapter 1 are positive, they are not an automatic consequence of grouping students. There are several considerations to be made when planning group work to maximise the effectiveness of the approach: principal among these are group size and composition. Assigning roles to individual group members is a common practice, although this is not the case in this intervention. Many of the issues that assigning roles can help counteract (such as preventing students from becoming passive and increasing communication between members) are not issues with our students, given their familiarity and comfort with their peers. We say this knowing the background of our students and the degree programme they are pursuing. The students are all in the third year of their degree programme, meaning they have developed a close working relationship with their peers. This is a consequence of the time in classes together, but is amplified by a number of courses they take prior to MS225 that require them to work collaboratively. In addition to the interaction between peers, the number of physics students and prospective mathematics teachers per intake is low, ranging between 11 and 34. The combination of these factors results in groups of students that are comfortable with one another within a group setting and have a level of familiarity that is beyond what may be typical of other groups.

Group size is an important aspect to consider when designing the worksheets. We have decided to use groups of no more than four students that are of mixed-ability and gender balanced where possible. This is line with the findings of Gillies [7], who adds that the facilitators need to be trained in how to implement this pedagogical strategy. Gillies [7, p.47] adds that ‘children in structured groups give more detailed and explanatory help to each other, ask deeper and more comprehensive questions, and achieve higher learning outcomes’ when these criteria are met.

Structuring groups in such a manner is similar to that how Lotan describes complex instruction

> Complex instruction is a pedagogical approach that enables teachers to teach at a high intellectual level in academically, linguistically, racially, ethnically as well as socially heterogeneous classrooms. [8, p.15]

In addition to how groups are structured, complex instruction also emphasises the importance of *lateral relations* [9, p.33] and *delegation of authority* [9, p.34], both of which are recognised by this research team for their value in the learning process.

4.2.3. Activities and questions on worksheets

In a study examining physics students' use of integration in the context of electromagnetism, Doughty [10] describes the manner in which their worksheets are structured as follows:

> The worksheets are structured in a way that guides students through a set of prescribed tasks towards the construction of their own definition of a concept, identification of the relationship between concepts, or solving a complex problem. Each task requires an explanation of reasoning and the emphasis is on qualitative understanding. [10, p.3]

The worksheets designed as part of the present research are designed in a similar manner. Given the difference in domain, our worksheets also include a significant number of mathematical tasks, intended to allow students to develop their instrumental understanding in relation to mathematical concepts that are relevant to the study of ODEs.

Each activity comprises several questions. The purpose of each activity, as Doughty [10] explains, is to build towards a student-constructed understanding of a given concept. Krainer [12] describes this as seeing learners as producers of knowledge, not just consumers. By extension, each question must act as a stepping stone [12, 13] for students on their way to the destination of understanding the given concept. The relative difficulty of successive questions now becomes very important, and a major focus of the design team. The optimum 'step size' is that through which the group can navigate using a combination of argumentation and justification. An unnecessarily small 'jump' between questions, and students do not need to interact with each other (which is required to evoke the ZPD and for learning to occur). An oversized gap between questions is one that falls outside the ZPD of the individual with their peers (and even the tutor), resulting in no learning taking place, and subsequently, the destination not being reached. An increase in scaffolded questions was requested by students in a physics context who believed that questions of this nature would help them solve problems in physics [14].

Accepting the ‘activity as a set of scaffolded questions’ model results in worksheets that typically comprise three related activities for students to grapple with. This can cover several concepts, though some concepts may require multiple activities to be explored fully. Each activity should end with some form of reflection to allow students to consolidate their learning [13, 15] or to highlight any remaining misconceptions at the earliest opportunity [13]. In the case of our worksheets, this is achieved through built-in instances where students must discuss their answers with tutors before they progress. These conversations also offer students the opportunity to ‘articulate what they know’, something that Carpenter and Lehrer [15] deem a necessary provision for students to engage with in order for ‘learning with understanding to occur on a widespread basis’ [15, p. 24]

When the overall structure of the worksheets is decided, individual tasks must be designed that fit cohesively with the structure of the worksheet, the pedagogy being employed, and the concept being constructed. Several studies on task design [12, 13, 16-19] were studied during the design process of the worksheets. The purpose of this investigation was to identify a variety of tasks that would foster an understanding of concepts that were also conducive to the pedagogical approach being employed in this research. Several task types were identified and are explained below along with instances where they were employed in the worksheets.

Swan [16], and Pointon and Sangwin [17] recognise the richness of example generation, including it in their mathematical question taxonomy. The taxonomy included eight separate task types shown in Table 4-1. In addition to example generation, Swan [16] mentions ‘classifying mathematical objects’, ‘interpreting multiple representations’, ‘evaluating mathematical statements’, and ‘analysing reasoning and solutions’ as types of tasks that he believes foster conceptual understanding. Examples of each of Swan’s five task framework can be seen throughout the worksheets where they are designed to be ‘accessible, extendable, encourage ‘what if’ and ‘what if not?’ questions’ [16, p.8].

Number	Question type
1	Factual recall
2	Carry out a routine calculation or algorithm
3	Classify some mathematical object
4	Interpret situation or answer
5	Prove, show, justify – (general argument)
6	Extend a concept
7	Construct an instance
8	Criticize a fallacy

Table 4-1: Pointon and Sangwin's taxonomy of mathematical questions

Pointon and Sangwin [17] list "Construct an instance" as task seven of eight, where tasks one to four represent 'adoptive learning' and five to eight require students to behave like 'experts' and are characterised as 'adaptive learning' [18, p.815]. Curley and Meehan [19] also demonstrated the potential benefit of example generation activities, writing about an introductory analysis course in an Irish context. Figure 4-1 shows an example generation activity from Worksheet 11 (Appendix I-12).

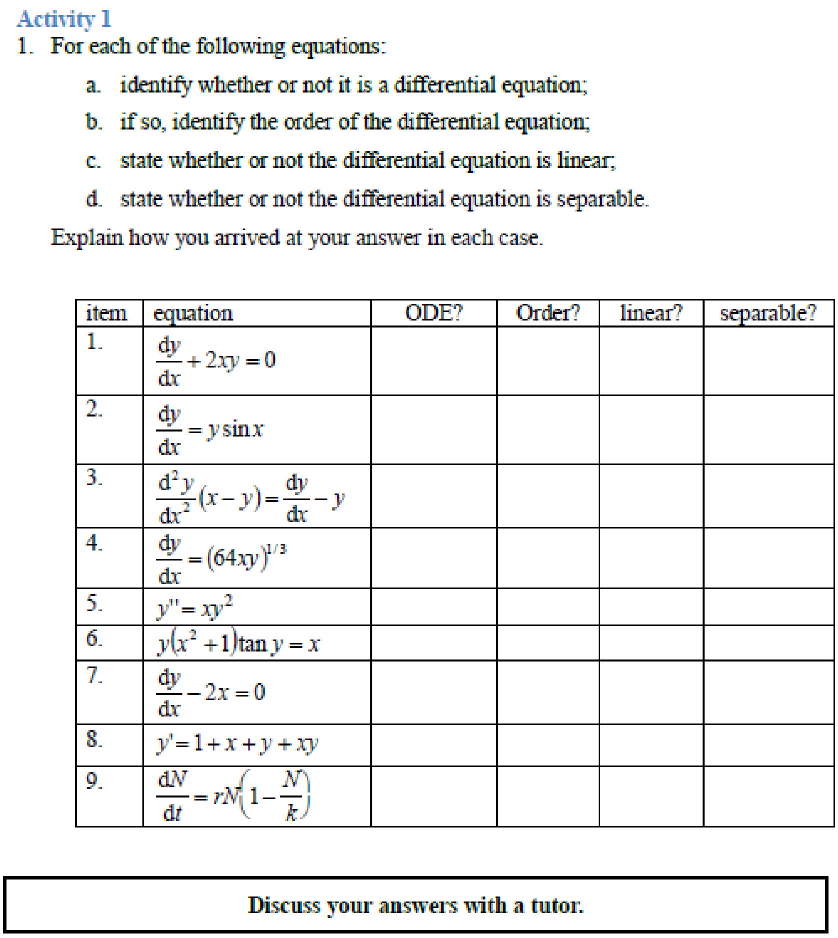
Activity 2

1. Work in your groups to generate examples of second order ODEs with the following characteristics in each case:
    a. ODE 1 is non-linear;
    b. ODE 2 is linear and homogeneous;
    c. ODE 3 is linear and non-homogeneous;
    d. ODE 4 is linear, homogeneous and has constant coefficients;
    e. ODE 5 is linear, non-homogeneous, and the coefficients are not constant.

Figure 4-1: Example generation exercise

Example generation activities are used at several stages of the intervention, particularly at the end of worksheets that deal with classification of ODEs. When learning ODEs, our students encounter terms such as linear, separable, and homogeneous. When these properties are correctly identified, they give the student information about the ODE and inform the solution techniques available to them. Many of these properties are not mutually exclusive, that is, a first order ODE can be both separable and linear. This is where a simple example generation exercise can be extended to require students to grapple with several properties at once. The example shown in Figure 4-1 focuses on the properties of linearity and homogeneity, and on the nature of the coefficients which impact the students' choice of solution technique.

The 'classifying mathematical objects' task type is used repeatedly throughout the intervention. In addition to those described by Swan [16], it provides several ancillary benefits for the learner. The task makes up the primary activity in worksheets that are completed early during the intervention (Figure 4-2). The task promotes conversation at an early stage which can help build relationships within each group. From a mathematical perspective, the students are repeatedly grappling with the concept definition of ODE and other related terms through increasingly difficult stages. Given that the concept of an ODE is novel to most (if not all) of the students, allowing their initial experiences to include the concept definition minimises any discrepancy between it and their 'evoked concept image' [20, p.152] when answering a question similar to those in this activity. A coherent concept image for students is desirable because it will avoid the potential for conflict or confusion in the future [20, p.152]. For this reason, the task is used during the introduction of second order ODEs in Worksheet (Appendix I-12).



Activity 1

1. For each of the following equations:
   a. identify whether or not it is a differential equation;
   b. if so, identify the order of the differential equation;
   c. state whether or not the differential equation is linear;
   d. state whether or not the differential equation is separable.

   Explain how you arrived at your answer in each case.

item	equation	ODE?	Order?	linear?	separable?
1.	$\frac{dy}{dx}+2xy=0$				
2.	$\frac{dy}{dx}=y\sin x$				
3.	$\frac{d^2y}{dx^2}(x-y)=\frac{dy}{dx}-y$				
4.	$\frac{dy}{dx}=(64xy)^{1/3}$				
5.	$y''=xy^2$				
6.	$y(x^2+1)\tan y=x$				
7.	$\frac{dy}{dx}-2x=0$				
8.	$y'=1+x+y+xy$				
9.	$\frac{dN}{dt}=rN\left(1-\frac{N}{k}\right)$				

Discuss your answers with a tutor.

Figure 4-2: Classifying mathematical objects

A study by Stein [21, p.351] investigating the processes associated with the maintenance of high-level cognitive demands also recommends in-built reflection for students to self-monitor, 'scaffolding of student thinking', and using feedback and questioning to press students for explanation and justification. Furthermore, Stein [21, p.351] highlights the importance of 'selecting tasks that build on students' prior knowledge'. Failure to include tasks that go beyond the correct answer will result in a decline of high-level tasks [21].

4.2.4. Role of the tutor

The role of the tutor can often take a back seat to the role of the student when reform curricula are discussed in research, but the tutor's impact should not go entirely undiscussed. When a social constructivist viewpoint is adopted, the notion of the ZPD must be considered. Inherent in the definition of the ZPD is the difference between the *actual developmental level* (ADL) and the level of the tutor or more capable peers in the group [4]. This means that the potential of the student is implicitly defined by the ability of the tutor in certain instances. Given that so much learning takes place in groups and the questions are designed in the manner described, the role of the tutor is very important.

In Chapter 1, we detailed how social constructivism has influenced several reform curricula at undergraduate level, one of which [22] took place in the context of ODEs. Rasmussen et al. [22] describe four research-based goals in Inquiry-Based Learning (IBL) classrooms based on previous research [23-25] they conducted. These studies, one of which focused on differential equations, helped identify the following four goals [22, p.1]:

1. get students to share their thinking,
2. help students to orient to and engage in others' thinking,
3. help students deepen their thinking, and
4. build on and extend student ideas.

Though these goals are not exhaustive, the authors claim they 'serve as a solid foundation for successful teaching in IBL classrooms' for a tutor who incorporates them into their teaching.

In a study of integration in physics that used a similar instructional approach, Doughty [10] describes the role of the tutor as "facilitator, promoting and monitoring discussions and supporting students to reach their own answers." [10, p. 3]. This will be adopted and extended for this intervention. It is critical that the tutor is involved in (or has a deep understanding of) the development of the instructional materials. In order to elicit the optimal response from a student, the tutor must have a clear destination in mind. They must understand what the intended outcome of each activity is, and, through building a relationship with each student, understand the appropriate step size for each question that results in the student constructing an appropriate understanding of a concept with as little help as possible. Such a task is very difficult to achieve, but made less difficult if a tutor possesses these traits.

The focus of the tutor for MS225 has changed significantly since our involvement began. Before the intervention, the tutor ran recitation sessions where their priority was on students obtaining correct solutions to preassigned tasks. A change in priority in the direction of the four goals described by Rasmussen et al. [22] has occurred since the tutorials were designed and implemented. This sees the focus of the tutor shift to ensuring groups function in a cohesive manner, in an environment that supports learning through communication and experience.

We also note the dual role our tutor serves: that of tutor and interviewer outlined in Section 2.4.3. It is important for the tutor to understand the demands (and potential conflicts) of both of these roles. In the present section, we describe their role as tutor, which has some overlap with their role as an interviewer. Research carried out in the context of ODEs [22, p.1] listed 'get students to share their thinking' and 'help students deepen their thinking' as goals of the tutor. These goals translate to the role of an interviewer in a semi-structured format also. The potential conflict this presents is in how it may influence students' answers. Students are most likely to answer questions pertaining to RQ3c differently when asked by an active participant in the intervention as opposed to an outside researcher. Given that data informing RQ3c has already been gathered during the first round of interviews, data from this round can be evaluated against this in order to check if any changes may be attributable to the change in interviewer. In addition to this, the author made sure to reiterate to students that participating in the interview would not harm their performance in MS225 in any way.

This section explains how social constructivism can influence a classroom environment, but it is also studied in the context of undergraduate mathematics. Research using APOS Theory [27] has shown that group work similar to that described above has been successful in the undergraduate mathematics context. Research specific to differential equations has also shown that students can work in groups [22, 28], as have studies in Irish [29] and British [6,30] settings.

4.3. Tutorial content

The next step in the design of the intervention was to choose central concepts and themes around which to build individual or groups of worksheets. A combination of the Diagnostic Survey, the professional experience of the research team, and relevant literature were used to choose the following six focus points: instrumental understanding, separation of variables, the derivative, the solution of an ODE, mathematical modelling, and multiple external representations (direction fields). Before going into detail on each of these themes, an explanation of the set of worksheets is provided. Table 4-2 shows the name of each worksheet, the sequence in which they are completed by students, and primary theme they concern.

Number	Name	Primary theme
1	Calculus review	Instrumental understanding
2	Meaning of the derivative	The derivative
3	ODEs and their solutions	Solution to ODE
4	Separable equations	Instrumental understanding
5	First order ODEs	Instrumental understanding
6	Practice solving first order ODEs	Instrumental understanding
7	Direction fields	Direction fields
8	Modelling with first order ODEs I	Modelling
9	Modelling with first order ODEs II	Modelling
10	Modelling with first order ODEs III	Modelling
11	Second order ODEs	Instrumental understanding
12	More second order ODEs	Instrumental understanding
13	Solutions to second order ODEs	Solution to ODE
14	Practice solving second order ODEs	Instrumental understanding
15	Modelling with second order ODEs	Modelling

Table 4-2: Worksheet order, title, and primary theme

Although the content of the worksheets follows the progress of lectures, there are several other factors that influence their sequencing. The necessity for a calculus review worksheet is greater because this module is run in semester one of the academic year and students are typically less well versed at this time of year than at the beginning of the second semester. The tutorial on direction fields is placed directly after the final tutorial on first order equations because it is intended to be an extension and an alternative to the analytical solution techniques practiced by students until then. It is necessarily before the modelling tutorials because they culminate in an activity that requires students to have seen direction fields. The modelling worksheets are the only worksheets that are not strictly stand-alone worksheets. Worksheet 8 and 9 are best completed in order and are a prerequisite for Worksheet 10.

Generating the questions contained within the intervention was done in a number of ways. The vast majority of mathematical tasks and qualitative questions were created by the author and finalised by the research team. This process was guided by the direction in which the research team wanted the intervention to progress. Some examples, such as those contained in Activity 2 of Worksheet 4 were taken from textbooks [31]. Textbooks were used as sources of initial value problems. Other sources used to create the intervention were lecture notes (Worksheet 3, 11, and 12) and mathematical tasks from the previous tutorials for MS225 (Worksheet 13). These were used to tie the lectures to the intervention where possible.

4.3.1. Instrumental understanding of basic mathematical skills

There are several recurrent mathematical techniques that are required to carry out some of the solution techniques of ODEs. For example, the ability to manipulate algebraic equations, isolate unknowns in logarithmic expressions, and anti-differentiate a range of functions are all required for first order ODEs. Similarly, finding the roots of a quadratic equation and solving a system of simultaneous equations are skills required when finding the general solution of certain second order ODEs. We saw in Chapter 1 and Chapter 3 that assuming that all students possess these skills would be incorrect. It is clear, therefore, that the students' instrumental understanding must be prioritised in the intervention.

Each of the mathematical tasks mentioned above must be grappled with by students repeatedly throughout the intervention if the issues are to be resolved. To this end, there are three worksheets (Appendix I-1, I-6, and I-14) whose primary focus is on instrumental understanding of calculus, first order, and second order ODEs respectively. The worksheet on calculus focuses on differentiation and integration, with tasks on manipulating exponents built into each question. Instrumental understanding also plays a central role in many of the other worksheets, where students are afforded time within a group setting, and individually, to practice mathematical tasks. This strategy aligns well with Vygotsky's theory [4] which posits that learning occurs initially through peer-to-peer interaction and subsequently though a process of internalisation, leading to deep understanding.

4.3.2. Separation of variables

A solution technique that students studying MS225 will encounter is Separation of Variables (SoV). This technique is used to solve ODEs of the form $\frac{dy}{dx} = A(y)B(x)$. Two areas to consider in relation to SoV are the procedure as a decontextualised mathematical task, and how SoV impact students studying physics. Wittmann and Black [32, p.1] found that 'students who are otherwise strong in their algebraic skills (as observed in classroom activities or on examinations) have difficulties when separating variables'. They explain that the student's success in viewing the $\frac{dy}{dx}$ in an ODE as a fraction renders the task of SoV relatively simple, but the potential for difficulty is noteworthy.

Beyond issues with instrumental understanding, Wilcox and Pollock [33] conducted a study of upper-division physics students' difficulties with SoV and noted several challenges. Among the challenges was students' difficulty with recognition of SoV as the appropriate tool and "spontaneously reflecting on their solutions" [33, p.1]. Classification tasks and in-built points of reflection (as described above) were included in several worksheets (most notably in Appendix I-4) to confront these potential issues. In addition to these findings, Wilcox and Pollock report student difficulty with identifying implicit boundary conditions [33]. Although the students studying MS225 may not display this issue during the intervention, it is a potential issue in future modules (particularly for the physics students). For this reason, there are several tasks throughout the intervention where initial conditions are deliberately embedded in a word problem (Figure 4-3).

Activity 2

1. Each of the following paragraphs describes a differential equation. Write down this differential equation and separate and solve where possible.
    a. The rate of change of a population is directly proportional to the size of the population in question.
    b. According to Newton's law of cooling, the time rate of change of the temperature $T(t)$ of a body immersed in a medium of constant temperature T_0 is proportional to the difference $T_0 - T$.
    c. The amount $A(t)$ of a certain drug in the bloodstream, measured by the excess over the naturally occurring level of the drug, will decline at a rate proportional to the current excess amount.
2. Using the following information, find an exact solution to each specific problem below outlined in general in Question 1 above.
    a. A certain city had a population of 25,000 in 1960 and a population of 30,000 in 1970. Assume that its population is growing exponentially. What population can its town planners expect the city to have in 2020?
    b. A cake is removed from an oven at 180 °C and left to cool at room temperature (20 °C). After 30 minutes the temperature of the cake is 70 °C. When will its temperature be 50 °C?
    c. Suppose that sodium pentobarbital is used to anaesthetize a dog. The dog is anesthetized when its bloodstream contains at least 45 milligrams (mg) of sodium pentobarbital per kilogram of the dog's body mass. Suppose also that sodium pentobarbital is eliminated exponentially from the dog's bloodstream with a half-life of 5 hours. What single dose should be administered in order to anaesthetize a 25 kg dog for 1 hour?

Figure 4-3: Word problems with implicit boundary conditions

4.3.3. The derivative

Concern among educators that students studying calculus 'may not be developing a clear conceptual understanding' is not new [34]. Given that introductory calculus is a prerequisite to the study of ODEs, this issue requires attention during the intervention. A concept that is central to the study of calculus [48], and ODEs as a result, is the derivative.

Not only has unsatisfactory conceptual understanding of the derivative been reported elsewhere [35], the Diagnostic Survey results highlighted a lack of connection made by students between the derivative and ODEs. In response to both of these issues, Worksheet 2 (Appendix I-2) was designed. This worksheet focuses on (but is not limited to) a geometric approach to the meaning of the derivative. This approach was chosen for several reasons: it emphasises the derivative as a rate of change; it exposes students to graphical representations early in the intervention; and it is an approach that is accessible to the entire cohort of students.

4.3.4. Solutions of ODEs

The concept of the solution of a differential equation has been the subject of several studies in recent years [49-51]. The focus of each study has been slightly different, but each motivated by the premise that an increase in the conceptual understanding of students should be worked towards. This attention alone is cause for ensuring our students are allowed time to develop an understanding of the structure and significance of a solution to an ODE, but the Diagnostic Survey also highlighted the concept of solution as an area to focus on. In Chapter 3, it was reported that only one of the eighteen students who completed the Diagnostic Survey understood that the solution to an ODE is a function. This suggests that student difficulties reported internationally [50] may also be prevalent locally.

To address the issues students have with the concept of a solution to an ODE, two tutorials were designed as part of the intervention. The first worksheet (Appendix I-3) focused on ODEs and their solutions, and was designed for students to grapple with the concept definitions of ODE and a solution to an ODE in a meaningful way.

It was clear from the Diagnostic Survey that some students have problems with these definitions. This may be caused by a lack of awareness of the importance of definitions in advanced mathematics and their role, something that is highlighted by Alcock and Simpson [40]. An issue such as this may explain the results reported in the Diagnostic Survey on the question of what an ODE is and what constitutes a solution to an ODE. Furthermore, we recall from Tall and Vinner [20] that concept definition "may be learnt by the individual in a rote fashion or more meaningfully learnt and related to a greater or lesser degree to the concept as a whole"; our focus on the concept of ODE and its solution is an attempt to ensure the latter, more meaningful learning occurs. Figure 4-4 is an excerpt from the activity on the solution of an ODE, which culminates in a definition being provided to students.

Tutorial 3: Differential Equations and their Solutions

Activity 3

1. Given that $y = e^x + 2x$, show that:
   a. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = e^x + 2$;
   b. $\dfrac{\mathrm{d}y}{\mathrm{d}x} - y = 2(1 - x)$.
2. Are either of the above differential equations? Justify your answer.
3. What does $y = e^x + 2x$ represent in this context?

Discuss your answers with a tutor.

Definition

A solution to a differential equation is any function $y(x)$ that satisfies the differential equation in question. That is, when we calculate the derivatives of y, and substitute for y and its derivatives, the left hand side is equal to the right hand side for all values of x in the domain of y.

Figure 4-4: Activity on solution of an ODE

The second worksheet that focuses on the concept of solution of an ODE is Worksheet 14 (Appendix I-15). This worksheet presents students with five second order linear ODEs, five exact solutions, and a portion of the graph for five functions. Students must work to correctly match the elements. This worksheet is designed to strengthen the link between graphical and analytical solution techniques and to emphasise the function nature of solutions. Physics students have linked the ODEs and their solutions to simple and damped harmonic oscillation which is a positive sign of beneficial links forming.

A final note on the concept of solution relates to an activity from Worksheet 12 (Appendix I-13). This activity (Figure 4-5) gets students to consider the relationship between the general solution to an inhomogeneous second order linear ODE and a particular solution of the equation.

Activity 2

1. Consider the second order linear inhomogeneous equation,

$$a(x)y''+b(x)y'+c(x)y = d(x), \tag{1}$$

and its homogeneous counterpart,

$$a(x)y''+b(x)y'+c(x)y = 0. \tag{2}$$

Let y_g be the general solution of (1), and let y_h be the general solution of (2). Let y_p be a particular solution of (1).

Explain why it must be the case that

$$y_g = y_h + y_p. \tag{3}$$

Hint: Discuss what it means to be a general solution of the homogeneous problem and what it means to be a particular solution.

Figure 4-5: Activity on the general solution to an inhomogeneous second order ODE

We have found that it is necessary for our students to have this conversation during tutorial time. Understanding this content is important for what comes in Worksheet 14, and in our experience the students do not develop sufficient understanding from seeing this done. This thread is developed in Worksheet 14 with two questions that link back to Worksheet 12 and build on the implications of this on the graphs of exact solutions (Figure 4-6). Students who before mention oscillators here might see, or can be brought to see, a driven element to the system.

2. What similarities exist between equations (4) and (5)?

3. What similarities exist between the graphs of the solutions to equations (4) and (5)? How is the inhomogeneous element of equation (5) reflected in the graph of its solution?

Figure 4-6: Activity on the graphical link between solutions of inhomogeneous second order linear ODEs and their homogeneous equivalent

4.3.5. Modelling

The inclusion of modelling in undergraduate mathematics courses has been recommended by several studies in recent years [29, 41]. There are many benefits associated with modelling that have led to these calls, many of which are magnified by the fact that the students in this study are studying physics.

It is reported [42] that mathematical modelling and mathematical problem-solving help to form skills that are essential for the workplace. This point resonates with one made in relation to the need to involve industry in the design of service courses [43] in Chapter 1. Other skills often listed as desirable in employees [44] are effective communication and an ability to work within a group setting, something that is fostered in our social constructivist setting. Effective communication can also result in improved relationships being developed. In a study conducted by Hernandez-Martinez et al., [41] the inclusion of modelling caused 'a critical shift in relations between students and teacher' which, the authors argue 'allowed for a more dialogical relationship' [29, p.31].

Research on mathematical modelling is also of interest in PER. According to Redish and Kuo [46], four components of 'expert physics practice' are required to succeed in physics: fluency with mathematical processing in the context with physics; fluency with the mathematical modelling of physical systems; an ability to blend physical meaning with mathematical structures; and an ability to evaluate and interpret results. Redish and Kuo believe that "we, as physics instructors must explicitly foster these components of expert physics practice to help students succeed in using math in physics." [46, p.583]. Within modelling tasks, it is important that the context in which the task is based is "experientially real" to the learner [47, p.111].

We believe that the cyclic nature of modelling needs to be emphasised for students to replicate the process themselves. While modelling is mentioned in all the textbooks on ODEs the research team studied, not every textbook described modelling as a process explicitly and fewer still illustrated modelling as a cyclical process. A similar distinction has been made with the definition of problem-solving, where Lesh and Doerr [44] chose the requirement for extension and refinement as "the most important criteria that distinguishes 'non-routine problems' from 'exercises'" [44, p.9] as opposed to definitions based on the processing of information [52]. Zawojeski and Lesh [52] explain that the model-eliciting definition is preferable to the data processing definition because students operate ‘primarily on their own interpretations’ [52, p.318]

Our modelling worksheets (Appendices I-8 to I-11) are designed around the modelling cycle presented by Savage and Grove [42] (Figure 4-7). This cycle incorporates each point made above, and emphasises the cyclical nature of modelling. Worksheet 8 and 9 (Appendices I-8 and I-9) guide the students through two complete cycles of modelling and Worksheet 10 (Appendices I-10 and I-11) provides them with the opportunity to practice their newly acquired skill themselves. A question at the end of Activity 1 of Worksheet 8 (Appendix I-8) asks students to consider the appropriate unit of measurement for a term in an equation. Rowland and Jovanoski [54] reported an unawareness of certain students of the necessity of unit analysis.

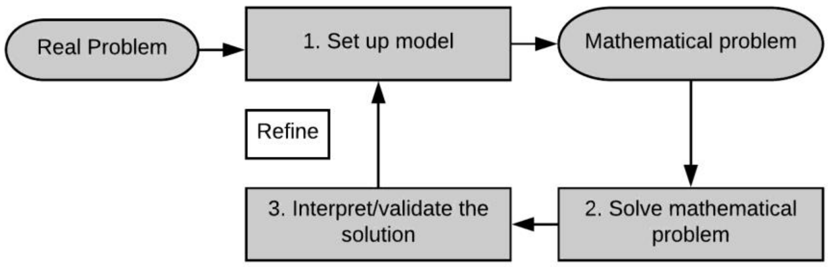


Figure 4-7: Modelling cycle used in the intervention

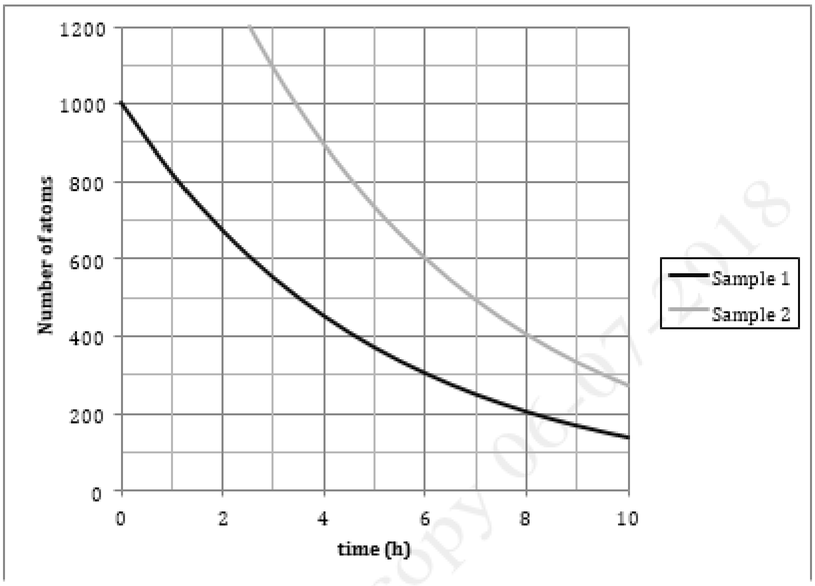
4.3.6. Direction fields

The intervention contains a worksheet on direction fields that has strong links to the work students do on modelling (Activity 3, Appendix I-9) and on the concept of solution. Prior to the intervention, direction fields were not a part of the course but there is a strong argument to be made for their inclusion. As mentioned previously, Swan [16] advocated the inclusion of tasks requiring students to interpret multiple representations, which can be incorporated in the study of first order ODEs through direction fields.

Direction fields are an excellent way to incorporate graphical representations into the study of first order ODEs and are an important tool for motivating students to think about the visual component of differential equations [55, p.150]. Graphical representations are one of three solution methods (along with numerical and analytical) mentioned by Stephan and Rasmussen [56], and as a form of visualisation are considered by Borrelli and Coleman [57, p.1] to be vital to understand the dynamical aspect of an introductory ODE course. In addition, the ability to interpret graphical representations is considered 'critical in physics' by Mc Dermott [1].

Any improvement in graphing ability brought about through the inclusion of direction fields would be of great benefit to our students. A question given in Section 2 of the Diagnostic Survey (Figure 4-8) provided an insight into the abilities of students studying MS225 before the intervention. The question, designed to assess the students' ability to compare individual points and rates of change of an exponential function resulted in a significant majority of students answering incorrectly, with only 11% of students answering the question correctly. Although not the sole cause of incorrect answers, noticing that the curves had the same decay constant played the largest part in such a small number of students answering correctly.

The number of atoms N in a radioactive sample is given by $\frac{dN(t)}{dt} = -\lambda N(t)$, where $\lambda > 0$ is the decay constant.



The number of atoms in two radioactive samples is shown in the diagram.
Do the two samples have:

- a different number of atoms at the start with the same decay constant?
- the same number of atoms at the start with different decay constants?
- a different number of atoms at the start and different decay constants?
- or is it impossible to tell?

Explain your answer.

Figure 4-8: Question 4 from Section 2 of The Diagnostic Survey

Direction fields differ from each of the other points of emphasis because it is the only concept that is being added to the curriculum. With the other concepts and themes, students will have encountered them in various degrees of detail in lectures prior to completing the respective tutorial. This means the direction field tutorial must stand alone as an exhaustive instruction for students where other tutorials can act in a supplementary role.

4.4. The nature of the revision

The opening of this chapter explains how the design of the intervention was influenced in part by three components outlined by McDermott [1]. Although not explicitly mentioned, the importance of revision between successive iterations is accounted for within McDermott's [1] work. Revision is a critical part of the development of all instructional material [58-61]. Research Aim 2 sought to develop an intervention that addresses the obstacles encountered by our students, thereby promoting more successful learning. Inherent in this aim is an iterative cycle of revision. At the end of each year of the intervention, a large amount of data was collected that, when analysed, highlighted areas of improvement for the intervention. Revision of instructional materials is also considered key by the physics education research group at The University of Washington. McDermott [1] explains that 'use of the findings to guide the development of curriculum' is a component of the development of their instructional design. The worksheets as presented in Section 4.5 underwent three cycles of revision. Each cycle resulted in several changes, all of which are detailed in Table 4-3.

2015-2016 Revisions	
Change	**Explanation**
Addition	Immediate pre/post-testing was introduced
Addition	Different versions of Worksheet 10 were created for each cohort
2016-2017 Revisions	
Addition	Delayed post-testing was introduced
Change	Standardize formatting of the worksheets and immediate pre/post-tests
Change	Reassign CA to tutorial attendance and participation
2017-2018 Revisions	
Addition	A task on evaluating mathematical statements was added to post-test 5
Addition	Task on Superposition Principle was added to Worksheet 11
Change	Different versions of Worksheet 15 were created for each cohort
Change	Post-test 7, Question 1: The nature of one of the incorrect options was changed, requiring students to use multiple forms of reasoning in order to choose the correct answer

Table 4-3: Changes implemented during each cycle of revision

An example of a significant change was the tailoring of Worksheet 10 (Appendix I-10 and I-11). During the first iteration of the intervention, the difference in abilities between students with respect to different skills became apparent. Most obvious among the differences was an ability of physics students to model physical situations more independently than the student teachers. Rather than subject one cohort to an inappropriate challenge, the decision was made by the research team to design two versions of the worksheet for the following year. The consequence of this is that two versions of Worksheet 10 (Appendix I-10 and I-11) are contained within the intervention: a structured version (for student teachers) and an unstructured version (for physics students).

The unstructured version offered students six problems in word form that they needed to describe mathematically using the modelling cycle they had practiced. The problems range in context and difficulty. The structured version of the worksheet contained two problems, given in the same form as before, but one also contained additional information that students could use as required during the tutorial. The additional help is provided to ensure that difficulty or context are not barriers that impede the students' practice of the modelling cycle. Had the students been unable to focus on the structural and systematic characteristics of the situation to be modelled, the activity would fail to be 'model-eliciting' [52]. Figure 4-9 and Figure 4-10 show both versions of the worksheet.

Activity 1

Complete the project assigned to your group. The projects are as follows:

1. **The policewoman:** The body of a homicide victim has been found and police want to estimate a time of death. To do this they need to know how the temperature of a body cools over time.

   Discuss the situation within your group and describing what you think would happen to the temperature T over time t.

   You may want to consider the following questions when modelling this situation:

   - What assumptions can we make to simplify the situation?
   - Will the body temperature, $T(t)$, increase or decrease after the person dies?
   - Will it increase or decrease at a constant rate?
   - What factor affects the increase or decrease?
   - Is your mathematical model based on the relationship between the change in temperature (dT/dt) and the above factor?
   - When discussing populations, we had a constant r that we considered to be the difference between birth rate and death rate. What could k be in this model?
   - Classify and solve the equation you have constructed. Can you evaluate the constant in front of the exponential?
   - How can you test your model to see if it is accurate? How accurate is it?
   - What does the graph of T v t look like?
   -

2. **The football:** A football is released at rest from the top of a building and falls towards the ground. Create a model that describes the velocity of this ball.

Figure 4-9: Worksheet 10 Activity 1 structured

Activity 1

Complete the project assigned to your group. The projects are as follows:

1. **The policewoman:** The body of a homicide victim has been found and police want to estimate a time of death. To do this they need to know how the temperature of a body cools over time.
2. **Hell's kitchen:** To increase efficiency in their restaurant, the chef wants to minimise the amount of time food needs to be in the oven for. To do this they need to know how the temperature of an object increases over time when placed in an environment that is warmer than itself.
3. **Historic cooking:** A historic method of cooking involved heating stones in a fire before placing them in water to raise its temperature. This works on the principle that the temperature of the stone will decrease until it reaches thermal equilibrium with the water. Imagine a single stone being placed in a vast amount of water, such that the temperature of the stone decreases to the initial temperature of the water. Model this change in temperature over time.
4. **Rabbits:** A population of rabbits survives by eating grass in a field. The grass is limited, but sustains a certain number of rabbits. Foxes get introduced to the environment such that they survive by hunting the rabbits. Create a model that describes the population of rabbits.
5. **The football:** A football is released at rest from the top of a building and falls towards the ground. Create a model that describes the velocity of this ball.
6. **Skydiving:** Skydivers use parachutes to ensure they reach the ground at a speed that won't harm them. Create a model that describes the velocity of a skydiver from the moment they begin their jump until they effectively stop accelerating.

Figure 4-10: Worksheet 10 Activity 1 unstructured

Another example, this time coming from interview feedback, and through cooperation with the module lecturer concerns how the module is assessed. The revision was to assign a portion of the continuous assessment (10% of 20%) to attendance and participation for tutorials. This was proposed after the first set of interviews where student feedback suggested that this would reduce the number of students whose attendance tapered off for no significant reason. Using extrinsic motivators like this to encourage non-attendees is not the most elegant solution, but the research team agreed that if it resulted in students maintaining attendance it would be the best outcome for students. At least three cases of this were observed in an interview from the following year, where students explained that the CA motivated them to attend tutorials initially, but that, as they saw how beneficial the tutorials were for their learning, it became an afterthought.

A final example of revision has come from the most recent revision cycle, and involves including a new task type in the worksheets, Swan's [16] 'evaluating mathematical statements'. This task requires students to rate a given statement as being always, sometimes, or never true, and where justification must be provided. We use this task type to attempt to resolve an issue of classification observed in tutorials and recorded during interviews. Some students have difficulty with the notion of a first order ODE being linear and separable. It seems that, for some, a false dichotomy exists that does not allow for an ODE to possess both properties. A possible reason for this may be the following. Students learn the SoV technique to solve separable equations, and learn the IFM technique to solve linear equations. They may incorrectly 'reverse engineer' to conclude that these two quite different techniques apply to two mutually exclusive categories of equation. Given the nature of the difficulty and the preliminary feedback obtained, we decided to include the new question (Figure 4-11) on the post-test for Worksheet 5 (Appendix I-5). A revision will need to be considered should the issue remain beyond Worksheet 5 because this tutorial marks the end of activities on theoretical aspects of first order ODEs.

New task: Evaluating mathematical statements (always/sometimes/never)

For each of the following statements, state whether they are true always, sometimes, or never. Explain your choice in each case.

1. Separable ODEs are linear.
2. Separable ODEs are first order.
3. Separable ODEs are homogeneous.

Figure 4-11: Activity on evaluating mathematical statements

4.5. The worksheets and pre/post-tests

In designing the intervention, we worked within various design parameters, and with a focus on a social constructivist approach to the teaching and learning of ODEs. Although the intervention includes pre/post-tests and interviews, the focus of this chapter was on the worksheets that were completed by students during the tutorials. The worksheets were designed in accordance with the criteria outlined by McDermott and implemented by Doughty [10]. The tasks were chosen to address key difficulties identified in the literature or in the Diagnostic Survey. Each worksheet dealt with at least one, and often several, of the overarching themes described in Section 4.4, and has undergone three revision cycles.

The revisions ranged in magnitude and typically stemmed from student responses to other elements of the intervention (pre/post-test questions or interview questions). The most recent version of the intervention contains a set of fifteen forty-five-minute worksheets (Appendix I), ten of which are accompanied by pre-tests and immediate post-tests (Appendix C). The intervention also includes a delayed post-test in the form of a question on the terminal exam and some students opt to take part in interviews.

Reference list

1. L. C. McDermott. Oersted Medal Lecture 2001: ‘‘Physics Education Research—The Key to Student Learning’’. **American Journal of Physics**, 2001, **69** (11), p.1129-1137.
2. J.D. Raskin. Constructivism in psychology: Personal construct psychology, radical constructivism, and social constructionism. **American Communication Journal**, 2002, **5** (3), p.1-25.
3. B. Blake and T. Pope. Developmental Psychology: Incorporating Piaget’s and Vygotsky’s Theories in Classrooms. **Journal of Cross-Disciplinary Perspectives in Education**, 2008, **1** (1), p.59-67.
4. L. Vygotsky. **Mind in society: the development of higher mental process.** Cambridge (MA): Harvard University Press, 1978.
5. E. Dubinsky. A radical constructivist does cooperative learning. [Online]. Available from: https://www.google.com/url?sa=t&rct=j&q=&esrc=s&source=web&cd=1&ved=0ahUKEwjZktyDqOfbAhWCchQKHdL1DFAQFggnMAA&url=http%3A%2F%2Fwww.math.kent.edu%2F~edd%2FRadConstrCL.pdf&usg=AOvVaw2zxLPloMi8CYW1J4L05_EM [Accessed 30 July 2016].
6. L. Walker. Enabling students to become independent learners **In**: M. Grove, T. Croft, J. Kyle, and D. Lawson (eds). **Transitions in undergraduate mathematics education.** Birmingham: The University of Birmingham, 2015, p. 71-84.
7. R.M. Gillies. Structuring cooperative group work in classrooms. **International Journal of Educational Research**, 2003, **39**, p.35-49.
8. R.A. Lotan. Complex instruction: An overview. **In**: E.G. Cohen and R.A. Lotan (eds), **Working for equity in heterogeneous classrooms**. New York: Teachers College Press. 1997, p. 15-27.
9. E.G. Cohen, R.A. Lotan and N.C. Holthuis. Organising the classroom for learning. **In**: E.G. Cohen and R.A. Lotan (eds), **Working for equity in heterogeneous classrooms**. New York: Teachers College Press. 1997, p.31-43.

10. L. Doughty. 2013. **Designing, implementing and assessing guided-inquiry based tutorials in introductory physics**. PhD thesis. Dublin City University.

11. L.C. McDermott and P.S. Shaffer. Research as a guide for curriculum development: An example from introductory electricity. Part I: Investigation of student understanding. **American Journal of Physics,** 1992, **60** (11), pp.994-1003.

12. K. Krainer. Powerful tasks: A contribution to a high level of acting and reflecting in mathematics instruction. **Educational Studies in Mathematics**, 1993, **24**, p. 65-93.

13. A. Bell. Principles for the design of teaching. **Educational Studies in Mathematics**, 1993, **24** (1), p.5-34.

14. L. Cui, N.S. Rebello, P.R. Fletcher, and A.G. Bennett. Transfer of learning from college calculus to physics courses. In: **Proceedings of the NARST 2006 National Meeting**. San Francisco, California. 2006. [Online]. Available from: https://www.researchgate.net/profile/N_Sanjay_Rebello/publication/242297311_TRANSFER_OF_LEARNING_FROM_COLLEGE_CALCULUS_TO_PHYSICS_COURSES/links/0c960532638f5bfd86000000.pdf [Accessed 1 June 2017]

15. T.P. Carpenter and R. Lehrer. Teaching and learning mathematics with understanding. **In**: E. Fennema and T.A. Romberg (eds). **Mathematics classrooms that promote understanding**. New Jersey: Lawrence Erlbaum Associates. 1999, p.19-32.

16. M. Swan. The impact of task-based professional development on teachers' practices and beliefs: A design research study. **Journal of Mathematics Teacher Education**, 2007, **10** (4-6), p.217-37.

17. A. Pointon and C. Sangwin. An analysis of undergraduate core material in the light of hand-held computer algebra systems. **International Journal of Mathematics Education in Science and Technology**, 2003, **34** (5), p.671-686.

18. C. Sangwin. New opportunities for encouraging higher level mathematical learning by creative use of emerging computer assessment. **International Journal of Mathematics Education in Science and Technology**, 2003, **34** (6), p.813-829.

19. N. Curley and M. Meehan. Example generation exercises in an introductory analysis course. **MSOR Connections**, 2010, **10** (1), p.48-51.

20. D.O. Tall and S. Vinner. Concept image and concept definition in mathematics with particular reference to limits and continuity. **Educational Studies in Mathematics**, 1981, **12** (2), p.151-169.

21. M.K. Stein, J. Remillard and M.S. Smith. How curriculum influences student learning. **In**: F.K. Lester Jr (ed.), **Second handbook of research on mathematics teaching and learning vol. 1.** Information age publishing, 2007, p.319-369.

22. C.L. Rasmussen, K. Marrongelle, O.N. Kwon and A. Hodge. Four goals for instructors using inquiry-based learning. **Notices of the American Mathematical Society**, 2017, **64** (11).

23. K. Marrongelle and C.L. Rasmussen. Meeting new teaching challenges: Teaching strategies that mediate between all lecture and all student discovery. **In**: M. Carlson and C.L. Rasmussen (eds). **Making the connection: Research and teaching in undergraduate mathematics education**. Washington, DC: Mathematical Association of America, 2008, p. 167-178.

24. C.L. Rasmussen. and O.N. Kwon. An inquiry-orientated approach to undergraduate mathematics. **Journal of Mathematical Behavior**, 2007, **26**, p.189-194.

25. M. Wawro, M. Zandieh, C.L. Rasmussen and C. Andrews-Larson. **Inquiry oriented linear algebra: Course materials.** 2013. [Online] Available from: http://iola.math.vt.edu. [Accessed 15 March 2018].

26. L. Doughty. 2013. **Designing, implementing and assessing guided-inquiry based tutorials in introductory physics**. PhD thesis. Dublin City University.

27. I. Arnon, J. Cottrill, E. Dubinksy, A. Oktac, S. Roa Fuentes, M. Trigueros, and Weller, K. **APOS Theory: A framework for research and curriculum development in mathematics education.** New York: Springer. 2013.

28. E. Yackel, C.L. Rasmussen, and K. King. Social and sociomathematical norms in an advanced undergraduate mathematics course. **Journal of Mathematical Behavior**, 2000, **19** (3), pp.275-287.

29. B.C. Nolan and E. Ni Fhloinn. Notes on group-work based tutorials in a large service teaching module. In: T. Dooley, S. NicMhuirí, M OReilly and R. Ward (eds). **Fifth**

Conference on Research in Mathematics Education (MEI5) Dublin, September 2013, p.248-259.

30. J. Williams. Mathematics education and the transition into higher education – transmaths demands better learning-teaching dialogue **In**: *M.* Grove, T. Croft, J. Kyle and D. Lawson (eds). **Transitions in undergraduate mathematics education**. Birmingham: The University of Birmingham, 2015, p.25-38.
31. W.E Kohler and L.W. Johnson. **Elementary Differential Equations**. Pearson New International Edition, Pearson Higher Ed. 2013.
32. M.C. Wittmann and K.C. Black. Mathematical actions as procedural resources: an example from the separation of variables. **Physical Review Special Topics- Physics Education Research**, 2015, **11** (2), p.1-13.
33. B.R. Wilcox and S.J. Pollock. Upper-division student difficulties with separation of variables. **Physical Review Special Topics- Physics Education Research**, 2015, **11** (2), p.1-13.
34. T.A. Grundmeier, J. Hansen and E. Sousa. An exploration of definition and procedural fluency in integral calculus. **Problems, Resources, and Issues in Mathematics Undergraduate Studies**, 2006, **16** (2), p.178-91.
35. A. Orton. Understanding rate of change. **Mathematics in School**, 1984, **13** (5) p.23-26.
36. D. G. Mallet and S.W. McCue. Constructive development of the solutions of linear equations in introductory ordinary differential equations. **International Journal of Mathematics Education in Science and Technology**, 2009, **40** (5), p.587–595.
37. D. Raychaudhuri. Dynamics of a definition: a framework to analyse student construction of the concept of solution to a differential equation. **International Journal of Mathematics Education in Science and Technology**, 2009, **39** (2), p.161–177.
38. C.L. Rasmussen. New directions in differential equations: A framework for interpreting students' understandings and difficulties. **Journal of Mathematical Behavior**, 2001, **20**, p.55–87.

39. S. Habre. Investigating students' approval of a geometrical approach to differential equations and their solutions. **International Journal Mathematics Education in Science and Technology**, 2003, **34** (5), p.651-662.
40. L. Alcock and A. Simpson. **Ideas from mathematics education: An introduction for mathematicians.** The Higher Education Academy. Maths Stats and OR Network (MSOR Network). 2009.
41. P. Hernandez-Martinez, J.S. Williams, L. Black, P. Davis, M. Pampuka and G. Wake. Mathematics coursework as facilitator of formative assessment, student-centred activity and understanding. **Research in Mathematics Education**, 2011, **13** (2), p 197-212.
42. M. Savage and M. Grove. Mathematical modelling and problem solving in real-world physical situations. **In**: M. Grove, T. Croft, J. Kyle and D. Lawson, (eds). **Transitions in undergraduate mathematics education**. Birmingham: The University of Birmingham, 2015, p.115-128.
43. A.C. Bajpai. The role of mathematics in engineering education: a mathematician's view. **International Journal of Mathematics Education in Science and Technology**, 1985, **16** (3), p.417-430.
44. R.A. Lesh, and H.M. Doerr. Symbolizing, communicating, and mathematizing: Key components of models and modelling. **In**: P. Cobb and E. Yackel (eds). **Symbolizing and communicating in mathematics classrooms**. Mahwah, NJ: Lawrence Erlbaum Associates. 1998.
45. W. Oaks and A.G. Rud Jr. The EPICS model in engineering education: Perspectives on problem-solving abilities needed for success beyond schools. **In:** R.A. Lesh and H.M. Doerr (eds), **Beyond constructivism: Models and modelling perspectives on mathematics problem solving, learning, and teaching.** New Jersey: Lawrence Erlbaum Associates, Inc., 2003.
46. E.F. Redish and E. Kuo. Language of physics, language of math: disciplinary culture and dynamic epistemology. **Science Education**, 2015, **24**, p.561-590.
47. K. Gravemeijer and M. Doorman. Context problems in realistic mathematics education: A calculus course as an example. **Educational Studies in Mathematics**, 2009, **39**, p.111–129.

48. J. Ferrini-Mundy and K.G. Graham. An overview of the calculus curriculum reform effort: Issues for learning, teaching, and curriculum development. **The American Mathematical Monthly**, 1991, **98** (7), p.627-35.

49. D. Raychaudhuri. Dynamics of a definition: a framework to analyse student construction of the concept of solution to a differential equation. **International Journal of Mathematics Education in Science and Technology**, 2009, **39** (2), p.161–177.

50. C.L. Rasmussen. New directions in differential equations: A framework for interpreting students' understandings and difficulties. **Journal of Mathematical Behavior**, 2001, **20**, p.55–87.

51. M. Camacho-Machín, J. Perdomo-Diaz and M. Santos-Triago. An exploration of students' conceptual knowledge built in a first ordinary differential equations course (part I). **The Teaching of Mathematics**, 2012, **Vol XV** (1), p.1-20.

52. W.A. Wickelgren. **How to solve problems: Elements of a theory of problems and problem solving.** San Francisco (CA). 1974.

53. J.S. Zawojeski and R.A. Lesh. A models and modelling perspective on problem solving. **In**: R.A. Lesh and H.M. Doerr (eds). **Beyond constructivism: models and modeling perspectives on mathematics problem solving, learning and teaching**. New Jersey: Lawrence Erlbaum Associates. 2003, p.317-336.

54. D.R. Rowland and Z. Jovanoski. Student interpretations of the terms in first-order ordinary differential equations in modelling contexts. **International Journal of Mathematics Education in Science and Technology**, 2004, **35** (40), p.503-516.

55. H. Gollwitzer. Visualisation in differential equations. **In**: W. Zimmermann and S. Cunningham (eds), **Visualization in teaching and learning mathematics**. Washington, DC: Mathematical Association of America, 1991, p.149-155.

56. M. Stephan and C.L. Rasmussen. Classroom mathematical practices in differential equations. **Journal of Mathematical Behavior**, 2002, **21**, p.459–490.

57. R.L. Borrelli and C.S. Coleman. **Differential equations, a modelling perspective**. New York: John Wiley and Sons. 1998.

58. T. Plomp. Educational research design: an introduction. **In**: T. Plomp and N. Nieveen (eds). **Educational design research - part A: an introduction.** The Netherlands: Netherlands Institute for Curriculum Design, 2013, p.10-51.

59. T.C. Reeves. Design research from a technology perspective. **In**: J. van den Akker, K. Gravemeijer, S. McKenney and N. Nieveen (eds), **Educational design research**. London: Routledge, 2006, p.52-66.

60. S. McKenney. Computer-based support for science education materials developers in Africa: Exploring potentials. **PhD thesis**. Enschede, the Netherlands: University of Twente, 2001.

61. J. Van den Akker, K. Gravemeijer, S. McKenney and N. Nieveen. (eds). **Educational design research**. London: Routledge. 2006.

62. R. Lesh, M. Hoover, B. Hole, E. Kelly and T. Post. Principles for developing thought-revealing activities for students and teachers. **In**: A. Kelly and R. Lesh (eds), **Handbook of research design in mathematics and science education.** Mahwah (NJ): Erlbaum, 2000.

5. Results and discussion

In this chapter, we discuss to what extent the intervention has benefitted our students and the data gathered to answer that question. Chapter 3 described our students' instrumental understanding (RQ1) and their concept image of ODEs in a standard setting (RQ2), which resulted in the development of the intervention described in Chapter 4. RQ3 focused on this intervention, paying particular attention to how the intervention benefitted our students. RQ3 has three sub-questions, all of which focus on the students who participated in the intervention. Answering these sub-questions will provide insight into students' instrumental understanding, concept image, and personal experience of the intervention. We include a reminder of RQ3 before discussing each sub-question in turn.

RQ3 - Has the intervention benefitted our students when learning ODEs?

a. How does the instrumental understanding of students who experienced the intervention compare to those who completed the module prior to its implementation?
b. In what way has the concept image of students who experienced the intervention grown?
c. How is the intervention viewed by the participants?

The focus of part a and part b of RQ3 is on student learning. These sub-questions were designed to align with RQ1 and RQ2 respectively. The manner in which they are worded allows for baselines to be used and comparisons to be drawn. The data reported in Chapter 3 relates to students who completed MS225 prior to the intervention. This means the results generated by them provide a point of comparison for students who completed the intervention. Their instrumental understanding is well known, as are elements of their concept image. When the data generated to answer RQ3 is analysed, it can be compared to the results from RQ1 and RQ2 to highlight how students' instrumental understanding and concept image develop. RQ3c is more straightforward. Its goal is to elicit student experiences to inform cycles of revision for the intervention and similar reform efforts.

Data was obtained through a combination of student interviews and pre/post-testing (both immediate and delayed). In total, 18 different immediate pre/post-tests, three delayed post-tests, and 30 interviews were conducted across two cycles of the intervention. This amounted to over 1,500 items of written data and over 11.5 hours of audio recordings. Rather than discussing the data in its entirety and relating it to the research questions, we will report on the research questions in turn, drawing on all the pertinent data. Appendix G includes examples of how the qualitative data generated in response to the delayed post-test and interviews was analysed.

Throughout this chapter, we report on data from the Diagnostic Survey, interviews, and the pre/post-tests. We remind the reader that the Diagnostic Survey was completed by students prior to the creation of the intervention. The interviews and pre/post-tests, on the other hand, were completed by students who participated in the intervention. This makes the cohorts non-identical, albeit similar: the link between these cohorts is that they were enrolled in MS225 in successive years, they are pursuing the same degrees, meaning they have completed the same modules prior to, and alongside MS225, and will have satisfied basic entry requirements in order to begin their studies at DCU. We will be explicit in identifying the cohort that generated a given data set when we present them. We begin by discussing RQ3a which looked at the instrumental understanding of students after completing the intervention.

5.1 Research Question 3a: How does the instrumental understanding of students who experienced the intervention compare to those who completed the module prior to its implementation?

Research Question 3a sought to investigate how the intervention impacts the instrumental understanding of its participants. In doing so, we drew on data from questions on the pre/post-tests to Tutorial 1, 4, and 5, from the delayed post-test from 2016-2017, and from interviews in 2017-2018. These tutorials were chosen because the primary theme they addressed was instrumental understanding. The results will be discussed under two themes: procedural competence and classification. 'Procedural competence' is the ability of a student to complete all of the steps in a solution technique correctly. Classification of ODEs is the ability to correctly identify the properties of an equation, such as whether or not the equation is an ODE, whether or not it is linear, separable or homogeneous, and what its order is. It is necessary to introduce procedural competence in this section to distinguish instrumental understanding at different levels of granularity. We will continue to use instrumental understanding to describe the students' ability to carry out small-scale tasks independent of context, and independent of any necessity to explain or validate their reasoning. Procedural competence on the other hand describes instrumental understanding at a higher level, such as the ability to carry out separation of variables without giving a reason why.

5.1.1. Instrumental understanding of basic mathematical skills

On the basis of the Diagnostic Survey we concluded that 'There are areas where students' instrumental understanding is not strong enough to successfully complete the level of questions on differential equations they encounter during this module.' [1]. Specifically, we saw that (44%) showed an inability to use the natural log function as the inverse of the exponential function, only half of the students successfully integrated $\frac{1}{x^3}$, and that negative exponents caused students difficulty. The intervention tackled each of these issues through a combination of activities within worksheets and pre-emptive actions of the tutor. Data was gathered through immediate and delayed pre/post-testing to inform this question. We will discuss the immediate pre/post-test data before discussing the delayed pre/post-test data.

The first worksheet of the intervention is a review of calculus. The focus of the worksheet is on computing a range of derivatives and indefinite integrals with the need for students to correctly manipulate exponents recurring throughout the tasks. Indefinite integration and manipulating negative exponents are two of the areas (using the natural log function as the inverse of the exponential function was the third) highlighted by the Diagnostic Survey. Given the instrumental focus of this tutorial we decided to pair it with the pre/post-test questions shown in Table 5-1. Using them in the pre/post-test allowed us to gather data on the prevalence of these issues with the new cohort of students. The table also contains the combined success rate of both cohorts, which showed slight to medium gains from pre-test to post-test across each question, albeit across non-identical tasks. The post-test results were similar to those recorded during the Diagnostic Survey, though these students had a slightly higher success rate in each question. Knowing that our current students performed similarly on these tasks to students who completed the Diagnostic Survey gave us a baseline from which to work.

Pre-test	Post-test	Percentage of correct responses (Pre-test; *N*=84)	Percentage of correct responses (Post-test; *N*=84)
1. Simplify the expression $\frac{(-x)^2}{x^{-1}}$.	1. Simplify the expression $\frac{(-y)^2}{y^{-3}}$	71% (60)	75% (63)
2. Compute the indefinite integral $\int \frac{1}{x^3}\mathrm{d}x$.	2. Compute the indefinite integral $\int 4x^{-2}\mathrm{d}x$.	62% (52)	73% (61)
3. Solve for *y* the equation: $1 = 2e^{-y}$.	3. Solve for *x* the equation: $e^{x+2} = 1$.	57% (48)	87% (73)

Table 5-1: Mathematical tasks on pre/post-test 1

A discrepancy in difficulty may be present in the tasks on indefinite integration and using the natural log function. In an attempt to design non-identical questions for post-test Question 2 and 3, the research team unintentionally altered the structure of the problem. Question 2 on the post-test contained a negative exponent where it did not on the pre-test, which could be considered more difficult for students to complete. Similarly, the exponents in Question 3 differ in sign, with the post-test also containing a constant. This discrepancy should be taken into account in our discussion of the results.

5.1.2. Procedural competence with separation of variables

The measure of growth across the intervention comes from analysing student responses to an item on the delayed post-test and comparing them with their answers on pre/post-test 1. The type of tasks contained on the pre/post-test were embedded in the delayed post-test. The delayed post-test task required students to find the general solution of $\frac{dy}{dx} = \frac{0.5+y}{2}$. This first order ODE is both linear and separable, meaning that it can be solved using two techniques available to the students: separation of variables (SoV) and the integrating factor method (IFM). The choice of which method to use was left to the students with 17 of the 70 students choosing the use the IFM. Successfully acquiring the general solution requires the ability to execute the type of tasks that the Diagnostic Survey and pre/post-test 1 identified as problematic (Algebraic manipulations, indefinite integration, and using the natural log function). The rubric used to analyse responses to this question is contained in Appendix G. Table 5-2 is a breakdown of the outcome of students attempting to find the general solution to the following ODE: $\frac{dy}{dx} = \frac{0.5+y}{2}$.

Outcome	Separation of Variables (SoV) (*N*=53)	Integrating Factor Method (IFM) (*N*=17)
1 – Correct	40% (21)	29% (5)
2 – Algebra errors	32% (17)	41% (7)
3a – Integration errors 3b –Integration constant omitted	2% (1) 8% (4)	18% (3) 12% (2)
4 – Natural log/exponential	8% (4)	0% (0)
5 – no error, incomplete	11% (6)	0% (0)

Table 5-2: Outcomes of Student attempts at delayed post-test with focus on instrumental understanding within two procedures

Our focus when analysing the data relating to RQ3a is the nature of the unsuccessful attempts. These are of interest because they allow us to investigate the prevalence of the issues identified by the Diagnostic Survey in students' work. Algebraic errors (improperly manipulating terms in the ODE) are now by far the most prevalent type of error. Of the issues identified by the Diagnostic Survey, manipulating negative exponents was not the source of any error, and the other two issues (errors with the natural log/exponential and with integration) combined for eight errors out of seventy students, or 14 errors if we consider the +*c* to be an error in indefinite integration.

Though this situation is unsatisfactory, issues of this nature have been reported by Wittmann and Black [1]. They found that 'students who are otherwise strong in their algebraic skills (as observed in classroom activities or on examinations) have difficulties when separating variables.' [2, p.1]. We have observed this also, and include Figure 5-1 as an exemplar. Figure 5-1 shows a student attempt to separate the ODE $\frac{dy}{dx} = \frac{0.5+y}{2}$. The correct procedure in Wittmann and Black's [1] language is to use *grouping* (view $(0.5 + y)$ as one term), *divide* (both sides of the ODE by the term $(0.5 + y)$), and *multiply* (both sides by dx).

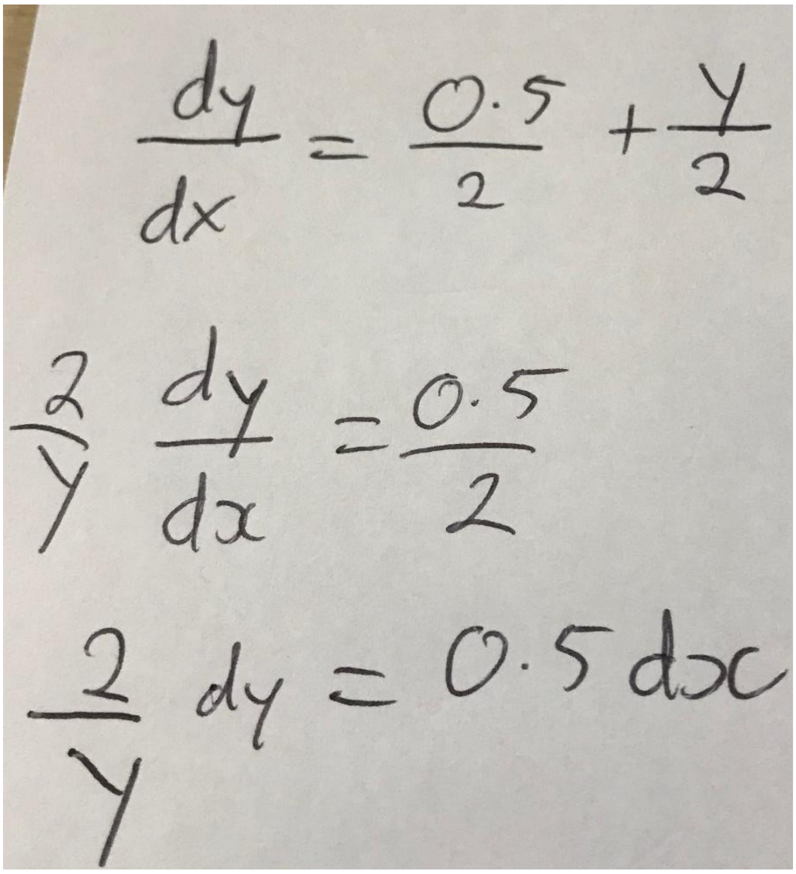


Figure 5-1: Algebraic error when separating variables

The error in Figure 5-1 is failing to *group*, instead separating the fraction into two terms. A subsequent error occurs with the $\frac{y}{2}$ term, before successfully *multiplying* the dx. In addition to the insights on grouping we can take from this answer, we note the student's success in viewing the $\frac{dy}{dx}$ as a fraction, which Wittmann and Black [1] explain renders the task of SoV relatively simple. Finally, some success was observed during later interviews in which this issue occurred. It seems that if the students are asked to explain the purpose of their initial step (expanding the RHS) in relation to their goal, they can self-correct. They can explain that expanding the RHS to introduce another term is not bringing them any closer to separating the variables, and they return to the ODE as presented in the question. Eventually, each student that initially made an error saw that *grouping* allows them to proceed toward a correct solution. The three points made in relation to the example in Figure 5-1 provide valuable insight that will inform future cycles of revision of the intervention.

We conclude that the cumulative effect of the intervention reduces the issues with respect to instrumental understanding identified in the Diagnostic Survey. This is based on delayed post-test and interview data. We also note that reducing the frequency with which these errors occur uncovered algebraic manipulations as the most problematic area for students post-intervention which will become a focus of future interventions. With respect to SoV, a major focus of Worksheet 4, 5, and 6 is on solving first order ODEs. An increased emphasis should be placed on correct procedures (grouping, multiplying, dividing etc) and on treating the derivative as a fraction when solving separable equations.

5.1.3. Classification

Analysing the responses to Section B of the Diagnostic Survey highlighted students' inability to define a differential equation (33% of students left this question blank or wrote they did not know), which we considered to be a gap in their concept image. It suggests a shortcoming in an element of the instrumental understanding that can often fall behind procedural competence in order priority: classification. By classification in the context of ODEs, we refer to the students' ability to identify an ODE, and subsequently its order, and whether or not it is linear, separable, and homogeneous. The data we present is initially from interviews conducted with the 2017-2018 cohort. This data is supplemented with pre/post-test data from Tutorial 4.

We consider classification to be a form of instrumental understanding with respect to this data because we did not ask students to elaborate on their answers; students needed only to evaluate the given equation against a rule in a decontextualised setting. This is in line with Skemp's [3, p.20] definition of instrumental understanding as having "rules without reasons". An example of this was observed repeatedly during interviews when students were asked a variety of questions that required them to classify equations. We stress that we are not claiming that our students do not possess relational understanding relevant to these properties of ODEs; we are only arguing that an instrumental understanding is sufficient to succeed with the questions posed to them in interviews. Table 5-3 shows the frequency with which responses were given to questions during student interviews. Each question required students to decide whether equations carried the aforementioned properties.

Property	Student response	Frequency
ODE	Contains a derivative and an equals sign.	15/17
Order	Power of the highest derivative present	14/14
Linear	Fits in a certain structure $a(x)\frac{dy}{dx} + b(x)y = c(x)$	8/15
Separable	Fits in a certain structure $m(y)dy = n(x)dx$	18/18
Homogeneous	When written in linear form $c(x) = 0$	4/4

Table 5-3: Frequency of responses to properties of ODEs

The frequency with which students give the same response to these questions is the result of an emphasis being placed on classification across the intervention. While we believe that the intervention may have also fostered relational understanding relevant to this part of their studies, this is certainly a case of 'rules without reasons' being sufficient to engage with the question. This is also seen when some of the incorrect answers are examined.

Every student responded correctly, and with the same reasoning, to questions relating to order, separability, and homogeneity. There were two students who did not use 'derivative and an equals' in response to whether the equation was an ODE. One of the students answered the question correctly, but only mentioned the derivative as being required in their explanation. The other student exhibited reasoning discussed in Chapter 3 that 'students may identify an equation as a differential equation more easily when the dependent or independent variable is present.' When shown the three ODEs:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\mathrm{C} \qquad \frac{\mathrm{d}N}{\mathrm{d}t} = -\mathrm{C}N \qquad \frac{\mathrm{d}N}{\mathrm{d}t} = -\mathrm{C}t$$

they identified the second and third equation as ODEs, explaining that 'there has to be an *N* or a *t* in it'. Though only one of the 17 students presented with these equations made this point explicitly, classroom observations suggest it is more widespread.

The property of linearity was clearly the most difficult for students to identify, with seven of 15 students providing incorrect answers. Of this seven, six students offered correct information but in an incomplete form. Three of these students described linear as a form of equation such that, if the

ODE can be written in this form it must be linear. The following excerpt is representative of this group of students

> D: That's linear because it's $a(x)\mathrm{d}y$, plus $b(x)y$ equals something. (17-16)

Another three students mentioned the coefficients of the function and its derivative in their answers. They spoke about conditions that must be met by the coefficients in order for the ODE to be linear. An example of this is

> B: Linear is when you have no y^2s in it and you can write the function of y by itself, plus the other thing. (17-13)

The final student did not display partially correct reasoning, mentioning straight lines in their answer. Emphasising the importance of classification of ODEs throughout the intervention resulted in a dramatic improvement in students' instrumental understanding. This is shown with data from pre/post-tests from Tutorial 4. Questions on the immediate post-test asked students to mention something they learned and were still unclear about. The responses to these questions were dominated by terms associated with classification (Table 5-4).

<table>
<tr><td colspan="3">Post-test 4</td></tr>
<tr><td>Cohort</td><td>2016-2017 (55 responses)</td><td>2017-2018 (45 responses)</td></tr>
<tr><td rowspan="2">Learned</td><td colspan="2">What linear - separable - order means</td></tr>
<tr><td>28 (51%) - 15 (27%) - 4 (7%)</td><td>26 (58%) - 11 (24%) - 5 (11%)</td></tr>
<tr><td rowspan="2">Still unclear
(N/A most popular with 23 and 25 each year)</td><td colspan="2">Separating the 'harder' ones (phrasing taken from student responses)</td></tr>
<tr><td>11 (20%)</td><td>1 (2%)</td></tr>
</table>

Table 5-4: Post-test 4 results

The post-test was dominated by positive responses from students with respect to classification of ODEs. The majority of students (n=23, 25 respectively) reported no areas in which they were still uncertain in the post-test. Separation of variables was the most commonly mentioned area that students were still unclear about. The dramatic reduction in this response (from n=11 to n=1) across successive years is a positive sign, however when coupled with our previous point that algebraic manipulation emerges as the largest issue post-intervention, this is an area that warrants further investigation.

We conclude that the intervention is successful in reducing the issues cited in Chapter 3 with procedural competence. Delayed post-test data shows a reduction in the number of errors made by students when using the natural log and exponential function as inverses of each other, manipulating exponents, and carrying out indefinite integration. Reducing the prevalence of these issues has exposed students' ability to manipulate equations as an area of concern. This issue has been identified by Wittmann and Black [1], who found that students who are otherwise strong algebraically struggled with algebraic manipulation in the context of separation of variables. These issues require further investigation and the intervention will change in accordance with the findings.

In relation to classification, the interview data was overwhelmingly positive in relation to students' ability to identify ODEs, their order, and whether they are separable and homogeneous. It highlighted the property of linearity as a relative weakness however. This must be addressed in future iterations of the intervention and the potential relationship between it and the students' difficulty manipulating equations must also be investigated.

5.2. Research Question 3b: In what way has the concept image of students who experienced the intervention grown?

The most prominent takeaway from Chapter 3 related to the concept image of students. We reported that 'ODEs, their solutions, and the function nature of their solutions were areas in relation to which we felt that students may be able to develop a more comprehensive concept image'. The results presented in this section focus on direction fields and solutions to ODEs. By introducing direction fields into the curriculum, we are expanding the concept image of the students. Yerushalmy [4, p.16] mentions direction fields specifically when discussing recent approaches to teaching differential equations, stating that "These representations are valuable and can lead to greater insight where applicable". We measure the growth in concept image after the inclusion of direction fields using data from the immediate post-test and the delayed post-test. We also look at the function nature of the solution and an understanding of what it means to be a solution to an equation. Here, we draw on interview data to support our claims.

5.2.1. Solutions of ODEs

A focus of the 2017-2018 interviews was to learn more about what students understand about solutions to ODEs. The notion that solutions to ODEs are collections of functions and how students reason about this has previously been the focus of research [5, 6] and we wanted to investigate this in relation to the students in this study. Rasmussen [5] details the solution-as-function dilemma having carried out task-based interviews with students. Where Rasmussen [5] asked specific questions about the nature and behaviour of different solutions, we looked at the concept of solution more generally. We focused on two notions: solutions are functions (of the correctly identified independent and dependent variables), and solutions can be verified by substituting them into the ODE (and obtaining an identity). Table 5-5 shows the success rate of students in response to both of these properties of solutions to ODEs.

Category	Solution is a function	Function of correct variables	Sub in to ODE to verify	0=0 iff solution
Correct	11	12	10	9
Incorrect	1	0	1	2

Table 5-5: Student responses to questions on solutions of ODEs

In relation to the function nature of solutions to ODEs we identified two separate elements in responses that described student understanding. We looked at to what extent students understood that the solution to an ODE is a function, and that the independent and dependent variables of the solution are encapsulated in the derivative term(s) in the ODE. Eleven of the twelve students who were asked knew that the solution would be a function, and that the function would have the form $y = f(x)$ (or equivalently, $y(x)$). In response to the same question, the twelfth student stated that '$y(x) =$ something', indicating that they understood the importance of the variables in the derivative but not stating explicitly that $y(x)$ is a function. As we will see below, the likelihood that this student is simply recalling something they remember from repeatedly solving problems cannot be ignored. Therefore, their response was considered to be incorrect in the first category but correct in the second. There were also two aspects identified with respect to the meaning of a solution. Firstly, students must explain that substituting the solution function into the ODE would be how the solution is verified. Ten of the eleven students questioned explained this using their own words. The remaining student stated that they were not sure how to verify a solution. The second aspect of verifying the solution is the application of the first. However, when asked to verify a solution to an ODE, one of the students who understood that the equality holds, was unsure how to show this. They understood that if they differentiated their solution they would get $\frac{dy}{dx} = \frac{(0.5+y)}{2}$, but did not recognise the $y(x)$ on the RHS of the ODE as the solution. Thus, this student failed to appreciate what we will call the dual role of $y(x)$ in verifying a solution: it is both the function whose derivative appears in the equation, and the function which appears in the equation in un-differentiated form. The final student, who did not know how to verify the solution was the 'twelfth student' referred to above.

Question 3 in Section B of the Diagnostic Survey asked students to write down everything they knew about the solutions to differential equations. Although it is an open-ended question, the fact that only one student out of eighteen (6%) mentioned that solutions are functions is in stark contrast to the post-intervention results, where 11 of the 12 students that were interviewed on completion of the entire module (92%) stated this when asked to share everything they knew about solutions to differential equations. Further, all 12 students understood the variables of the function. A similar majority of students (90%, *n*=11) interviewed demonstrated an understanding of what it means to be a solution to an equation, with all except one of these students understanding the dual role of $y(x)$ in verifying the solution.

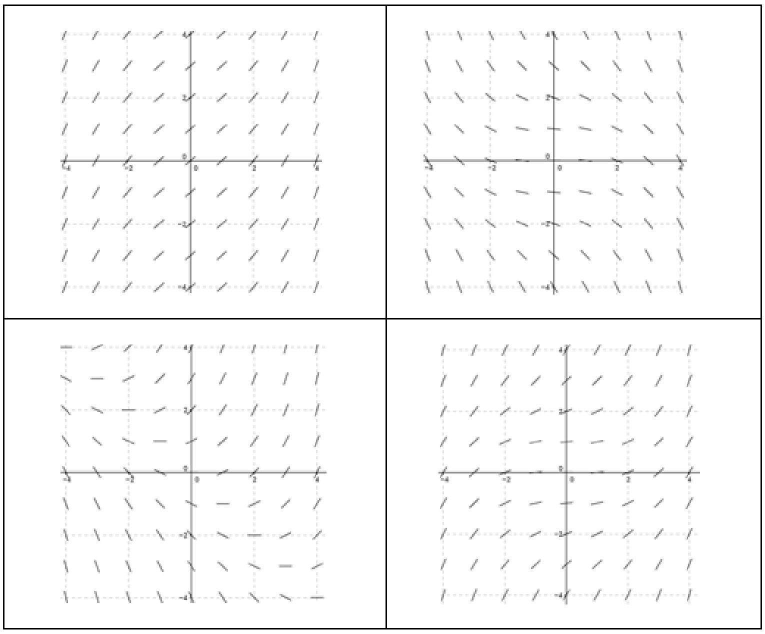
The Diagnostic Survey highlighted significant shortcomings in this area of students' conceptual understanding, which resulted in it being a primary theme during the design of the intervention (cf. Section 4.3.4). Interview data from 2017-2018 was used to highlight the improved understanding of the concept of a solution to an ODE as a result of the intervention. Specifically, the data showed that notions of 'solution as a function' and 'verification of a solution' are known by the vast majority of students. The next section reports on the growth in students' concept image that resulted from the introduction of direction fields in the intervention.

5.2.2. Direction fields

Direction fields were added to MS225 through the intervention (cf. Section 4.3.6) and were not mentioned during lectures. Nor did students meet this concept in any other module. Thus, we deem any understanding of direction fields acquired by students as an addition, and so an expansion of their concept image. The post-test data on direction fields will therefore contribute toward RQ3b. We used immediate and delayed post-testing to (i) measure students' ability to identify direction fields when given the ODE in analytical form and (ii) match a solution curve to the appropriate direction field.

We begin by reporting the students' answers to the immediate post-test questions before describing the answers given to questions on the delayed post-test. 42 students completed the immediate post-test (Figure 5-3) and 46 students completed delayed post-test (Figure 5-4) in the 2016 academic year. 34 students completed the immediate post-test in the 2017 academic year.

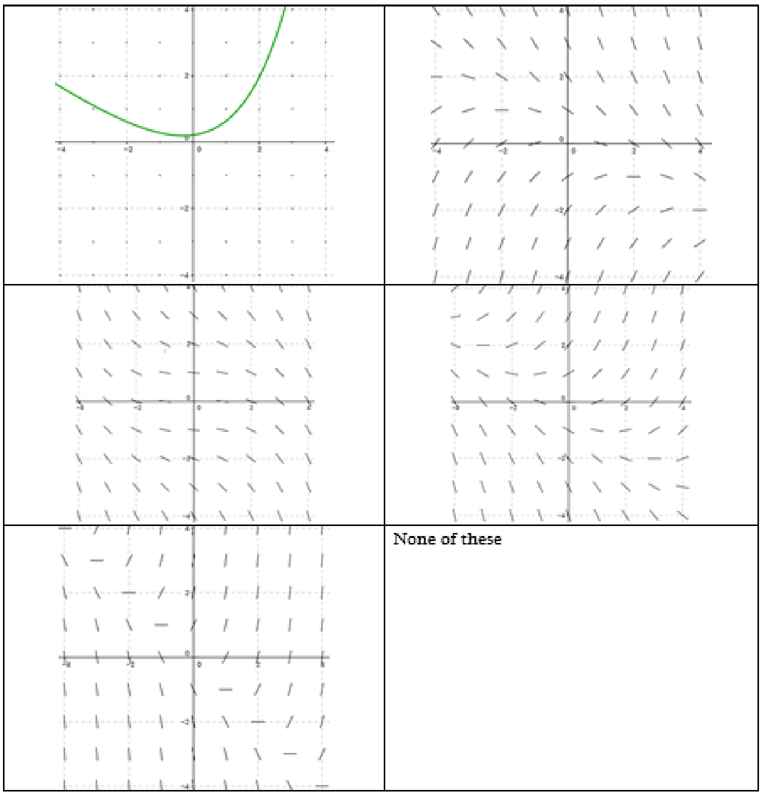
1. Which of the following pictures shows the direction field for the ODE $\frac{dy}{dx} = \frac{1}{10}\left(x^2 + y^2\right)$? Tick the appropriate box.



Explain how you arrived at your answer.

Figure 5-2: Direction field immediate post-test Question 1

2. Match the particular solution shown to the appropriate picture of one of the direction fields below.



Explain how you arrived at your answer.

Figure 5-3: Direction field immediate post-test Question 2

Identify the direction field that corresponds to the equation $\frac{dy}{dx} = \frac{0.5+y}{2}$ from the choices below. Explain your answer.

[5 marks]

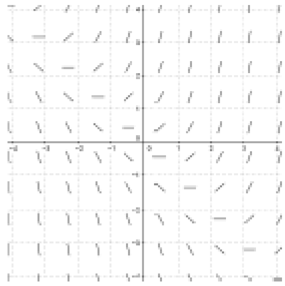
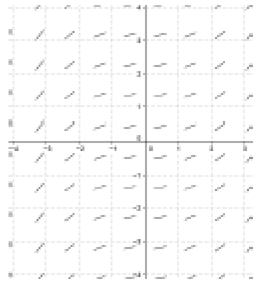
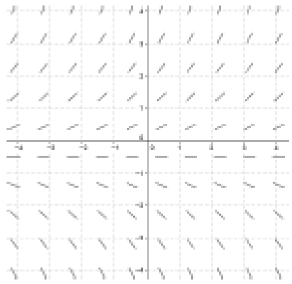
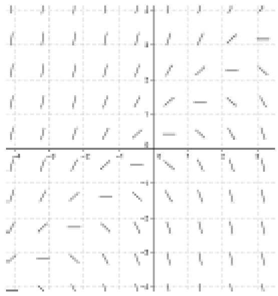
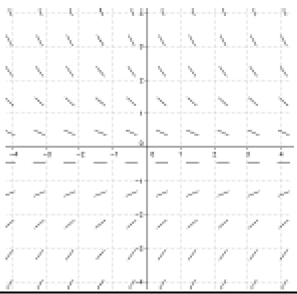
	
	
	None of these

Figure 5-4: Direction field delayed post-test

The post-test questions evaluate the students' ability to correctly match an ODE with a direction field that represents it, and when given a solution curve, the students' ability to correctly match it to the appropriate direction field. Our analysis of the data focused on how many students got each question right and what errors occurred in the incorrect answers. The direction field worksheet underwent only very minor formatting revisions, so it was anticipated that the results from the following year would not differ greatly from that of the previous year. However, we report the cohorts separately as a precaution.

5.2.2.1. Immediate post-test

Can students match an ODE to a direction field?

(2016 cohort, n=42)

Question 1 required students to match the ODE to a picture of its direction field. 35 students (83%) did so successfully, with the remaining seven students (17%) choosing an incorrect direction field (*n*=6) or leaving the question blank (*n*=1). The question also required students to explain their reasoning. The results of this are summarised in in Table 5-6. The table also outlines the frequency with which the reasoning was given.

Explanation	Frequency (*N*=35)
The slope at all points must be positive	66% (23)
Tested various points to see if slope matched the diagram	26% (9)
Blank	3% (1)
Other (each mentioned by one student)	6% (2)

Table 5-6: Frequency of response to direction field post-test question of Figure 5-2

Requiring an explanation allowed us to see what reasoning led to students' successful and unsuccessful answers. Using the ODE (either to substitute in points and calculate slopes or purely to identify sign) was common to almost every student who gave a correct answer. Something of note among the incorrect responses was the prevalence of the same reasoning. All six of the students who gave an incorrect solution used the same reasoning as students who were correct. The salient difference was that they either did not use this information to select the correct answer, or (in the case of those who tested points) did not exhaust this method to rule out all other options. An example of incorrectly using information is a student who reasoned that the slope will be positive at every point going on to choose an option where negative slopes are present. An example of a student who tested points in a non-exhaustive manner is explained below (Figure 5-5).

(2017 cohort, n=34)

As observed with the previous cohort, students who were correct used appropriate reasoning to arrive at their answer, but students who chose incorrectly also displayed correct reasoning. 31 students (91%) chose the correct direction field. All three of the students who chose incorrectly used correct, but incomplete reasoning to arrive at their answer. An example is shown (Figure 5-5) where a student correctly evaluated the slope at a variety of points but arrived at the incorrect answer because their method was not exhaustive.

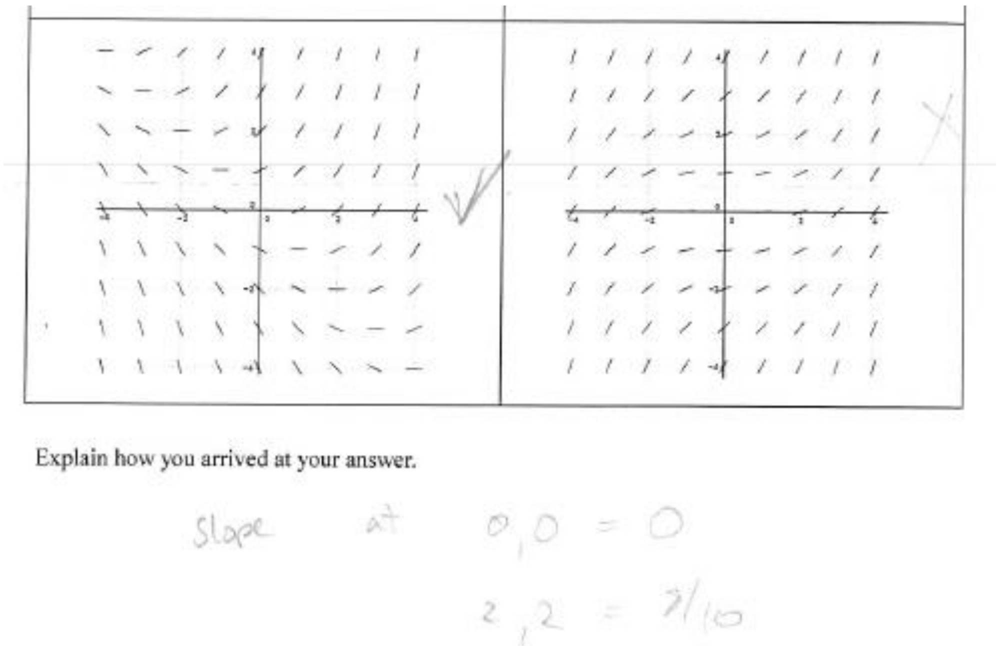


Figure 5-5: Incomplete 'evaluating points' approach

Across both cohorts, 66 students (87%) correctly identified the direction field of a given ODE. Furthermore, the ten students who chose incorrectly used correct reasoning techniques but did not apply them extensively enough to rule out all of the incorrect options. We conclude that the tutorial is effective at instructing students on constructing and reading direction fields.

Can students match a solution curve of an ODE to a direction field?

Question 2 on the immediate post-test required students to match the solution curve of an ODE to the correct direction field (Figure 5-3). The students were given five options from which to choose: one correct direction field; three incorrect direction fields; and the option that none of the above are correct. The results to this question are described below, separated by year.

(2016 cohort, n=42)

Students performed well on this task with 39 of the students (93%) answering correctly. The reasons given for their selection were also very consistent. The most popular explanation (given by 25 students) was that the direction field chosen offered the 'best fit' for the solution curve. The students took a visual approach, superimposing the curve onto each potential direction field before picking the one where the line segments on the field acted as tangents to the solution curve. An exemplar for this technique is shown in Figure 5-6. The second most popular justification (given by four students) was to pick points that the curve passed through, calculate them or investigate them on each field, and choose the closest match. Two students described the direction fields qualitatively and then examined the solution curve to see if they were consistent.

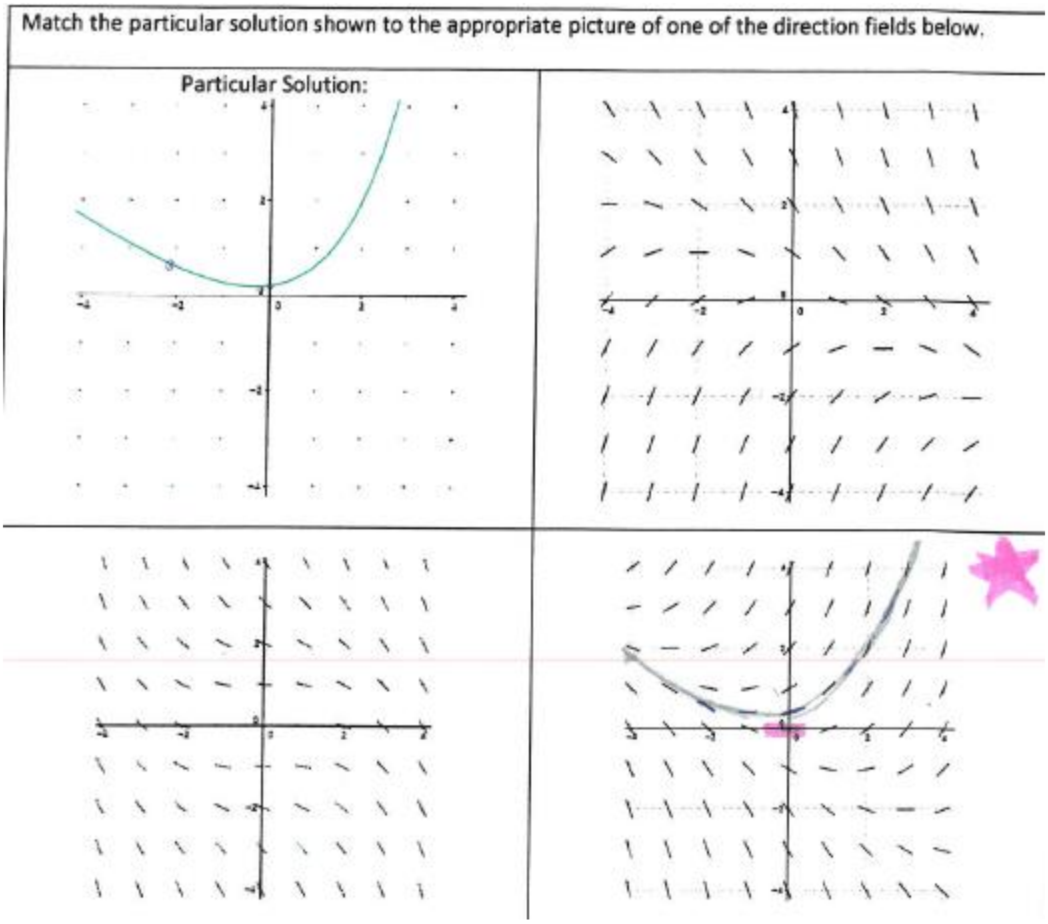


Figure 5-6: 'Superimposing the curve' approach

Of the three students who chose incorrectly, only one picked the incorrect field, opting for the direction field in the bottom left corner where they superimposed the curve onto the field. This student showed similar reasoning to the vast majority of correct answers but executed incorrectly in this instance. The remaining two chose "none of the above". In one case the student said that the answer will be similar to the correct answer but will not have $\frac{dy}{dx} = o\ at\ (0,0)$, which, to them, ruled out the correct answer. In the second case the student drew in what they deemed to be the correct answer which is shown in Figure 5-7. Their direction field appears to be based on a quadrant by quadrant qualitative approach that shares some similarities with the correct option, but is an oversimplification that will not work in the case of this, or similarly complex first order ODEs.

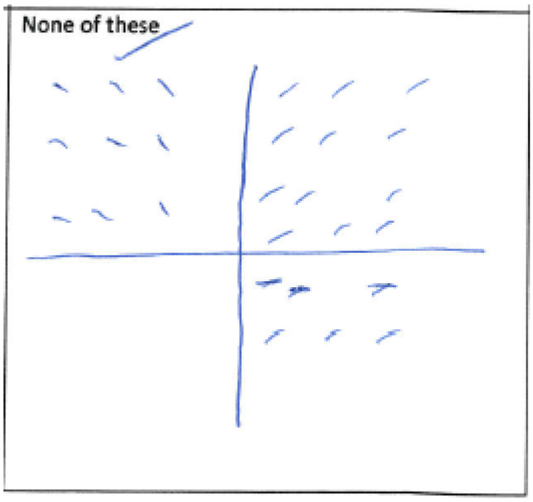


Figure 5-7: Student-constructed answer

(2017 cohort, n=34)

The results for the 2017 cohort are in line with those of the previous year. 28 students (85%) chose the correct direction field, with four of the remaining five students choosing incorrectly and one student leaving the post-test blank. Also consistent with the previous cohort were the explanations given and the frequency with which they were cited. The most common explanation that accompanied a correct answer (given by nine students) was that the solution curve and the chosen direction field had matching slopes. Six students evaluated the slope at certain points and used this to select their answer. Four students used qualitative arguments to justify their choice, and the remaining nine students gave no explanation with their correct answer.

An interesting answer (Figure 5-8) appears to have used a combination of the two most popular techniques in a unique way. It appears that the student drew tangent lines to the solution curve at various points and also extended the tangent lines at the same point on the direction field to highlight the solution curve to arrive at their answer. This unique combination is indicative of a student who could reason with this task in both an analytical and graphical manner.

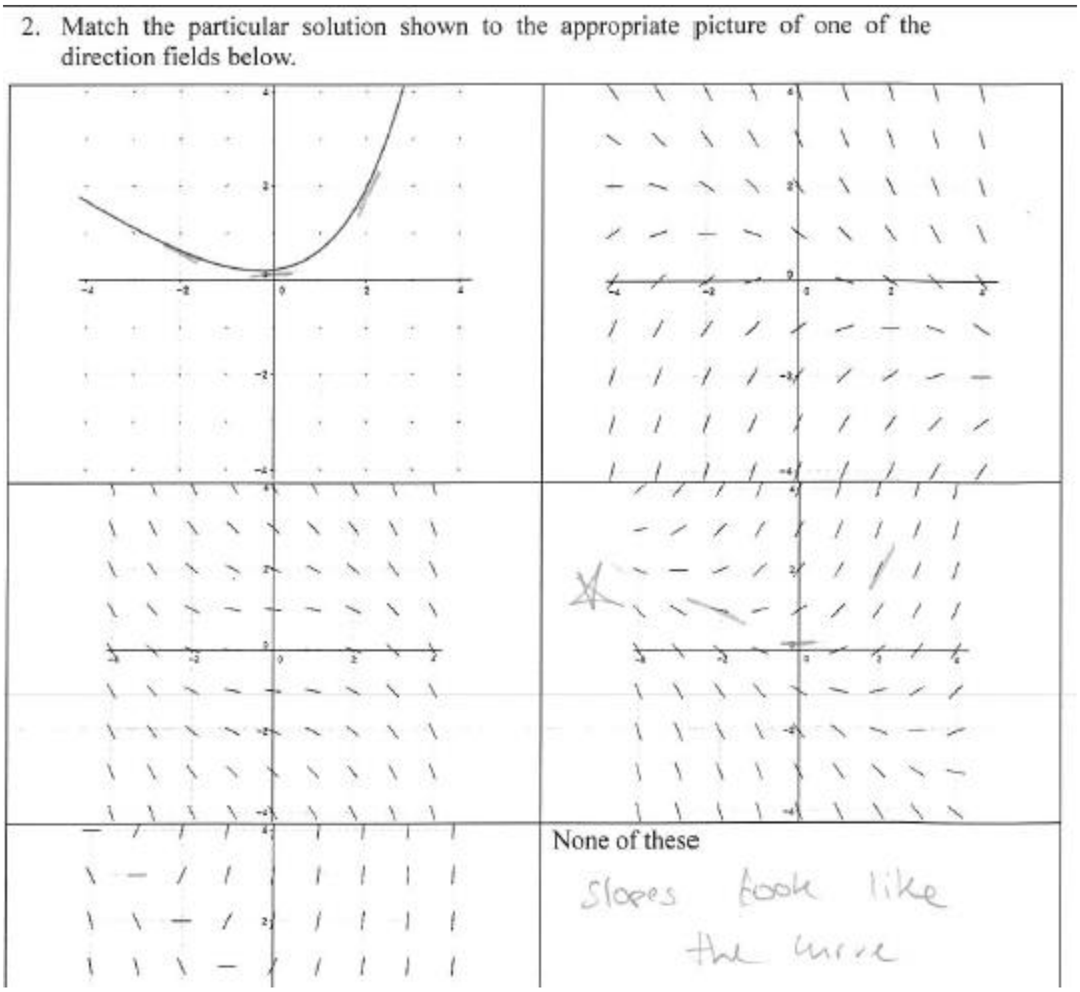


Figure 5-8: Answer containing a combination of two approaches

There were four incorrect answers provided to this question. Three of these four students gave an explanation. One student chose “none of these” but reasoned correctly by attempting to join slope lines. They were unsuccessful because the lines they drew did not match the solution curve. One student’s qualitative approach resulted in them selecting the direction field on the bottom left of the page, writing “negative slope for negative *x*-values, positive slope for positive *x*-values”. The final student left the question unanswered.

We conclude that the tutorial is effective at instructing students on the relationship between solution curves and direction fields. Across both cohorts, 67 students (88%) correctly matched a solution curve to its direction field. Similar to the previous question, a large portion of the incorrect solutions carried correct reasoning. All of the students who chose incorrectly used reasoning that when used correctly, would yield the correct answer.

5.2.2.2. Delayed post-test

(2016 cohort only, n=46)

To measure the long-term effectiveness of the direction field worksheet, a delayed post-test was administered as part of the final exam for MS225 11 weeks after the tutorial was completed. The delayed post-test contained an isomorphic equivalent to Question 1 in the immediate post-test and a question similar to Question 2 in the immediate post-test. It was similar in that it dealt with a solution curve on a differential equation, but differed by not providing students with the curve. Instead, it asked students to draw the solution curve themselves, providing them with a point through which their curve must pass. Extending the task from the immediate post-test in this manner allowed us to assess additional elements of students' understanding of direction fields.

Of the 46 students who completed the delayed post-test, 39 students (85%) identified the correct direction field. This reflects strongly on the effectiveness of the tutorial over time when compared to the immediate post-test success rate of 88%: the interval between the tutorial and delayed post-test was 11 weeks and students had not worked with direction fields during this time.

15 students (33%) successfully answered the second question on the delayed post-test which required them to draw the solution of the ODE that passed through the point (1,1) on the direction field they chose in Question 1. The nature of the incorrect answers are outlined in the Table 5-7.

Description of attempt	Number of students
Drew the correct curve beginning at (1,1) and continuing in the positive direction	9
Drew a small segment of the correct tangent line at the point (1,1)	6
Did not submit the answer sheet with their exam	4
Drew inaccurate curves through (1,1)	3
Drew multiple curves on the direction field, one of which was the correct solution curve through (1,1)	2
Drew a tangent line through (1,1)	2
Drew several line segments at (x,1) with the same slope as the correct tangent at (1,1)	1
Drew a horizontal line through (1,1)	1
Drew a straight line through (1,1) that was perpendicular to the tangent at (1,1)	1
Blank	1

Table 5-7: Categorisation of incorrect responses to delayed post-test question

While it is clear that the students were less successful with the second delayed post-test question, we believe this is due (at least in part) to the difference in task between the immediate and delayed post-test. Specifically, we believe that drawing in a solution curve is more difficult than deciding which direction field a curve belongs to. Implicit in seventeen of the incorrect answers is an ability to evaluate the slope at (1,1) and draw a correct curve at that point. We wonder if these students would have been successful had they been given a question that was structurally similar to Question 2 on the immediate post-test, but additional research needs to be carried out to investigate this.

Even at 48% success (15 and 17 students), this question highlights an area of improvement for the tutorial. In particular, 18 of the 30 incorrect responses described in the table do not correspond to functions. Thus, establishing a stronger link between analytic properties of solutions ("solutions are functions") and their geometric properties (in the form of graphs – satisfying the vertical line test) is warranted.

5.2.3. Conclusions

The purpose of the post-testing was to give us an indication of the effectiveness of our worksheet on direction fields and provide insight into RQ3b. This was measured by students' performance on post-test questions. There was an immediate post-test and a delayed post-test. Both contained two questions that corresponded to our research question in terms of expanding the concept image of students who experienced the intervention. We looked at students' ability to match an ODE to a direction field and to match a solution curve of an ODE to a direction field. Having analysed the results to the immediate post-test, we concluded that the tutorial is effective at instructing students in identifying direction fields when given the ODE in analytical form. This was based on the success rates of 87% and 88% for each question respectively.

The results of the delayed post-test were more mixed. While the success rate in response to Question 1 remained largely unchanged (from 88% to 85%), Question 2 saw a reduction, albeit on non-identical tasks. Nonetheless, this points to an opportunity to improve the worksheet on direction fields, which would further strengthen the concept image of participants.

5.3. Research Question 3c: How is the intervention viewed by the participants?

When instructional materials are designed as part of research, they should be considered live documents that must adapt and evolve as their role changes. Our worksheets do this through cycles of revision after each cohort of students have experienced them and given feedback. RQ3c was designed to gather and analyse the opinions of the students who took part in the intervention. It was answered with interview data from 2016-2017 and 2017-2018. In total, 30 interviews were conducted with 40 students.

5.3.1. Interview data

All of the data used to answer RQ3c was subjected to a general inductive analysis approach [7] as described in Chapter 2 (cf. Section 2.5, Appendix G). The vast majority of this data was generated through semi-structured interviews (cf. Section 2.4.3). These interviews took place across two years. In the first year they were done with students in small groups whereas in the second year they were done individually. When interview excerpts are used, different letters are used at the beginning of each line to differentiate between speakers. Each interview is identifiable by the code contained at the end of the excerpt. The code details the year in which the interview was conducted and the interview number separated by a hyphen, so the interviews from the 2016-2017 academic year are labelled 16-1 to 16-9 and the interviews from the 2017-2018 academic year are labelled 17-1 to 17-21.

In the case of the interview data, the general inductive analysis approach was used to identify a primary category, subcategories, and labels that are discussed in the following paragraphs. The primary category, or theme that emerged from the analysis was *tutorials in the intervention are different to lectures and other tutorials*. This means that all of the data from interviews had this theme in common.

The 'difference' referred to in the primary category has multiple layers. Primarily it speaks to the experience students had during tutorials and how they contrast that to their experience in lectures. Beneath our primary category, we identified three sub categories: differences with respect to pedagogy (specifically *groupwork*), *questions*, and the working environment (*interaction pattern*). A visual representation of the general inductive analysis for both years of interviews is shown below (Figure 5-9, Figure 5-10). These are joint displays [8], that combine qualitative data (categories and their relations) with quantitative data (the frequency with which each term is mentioned).

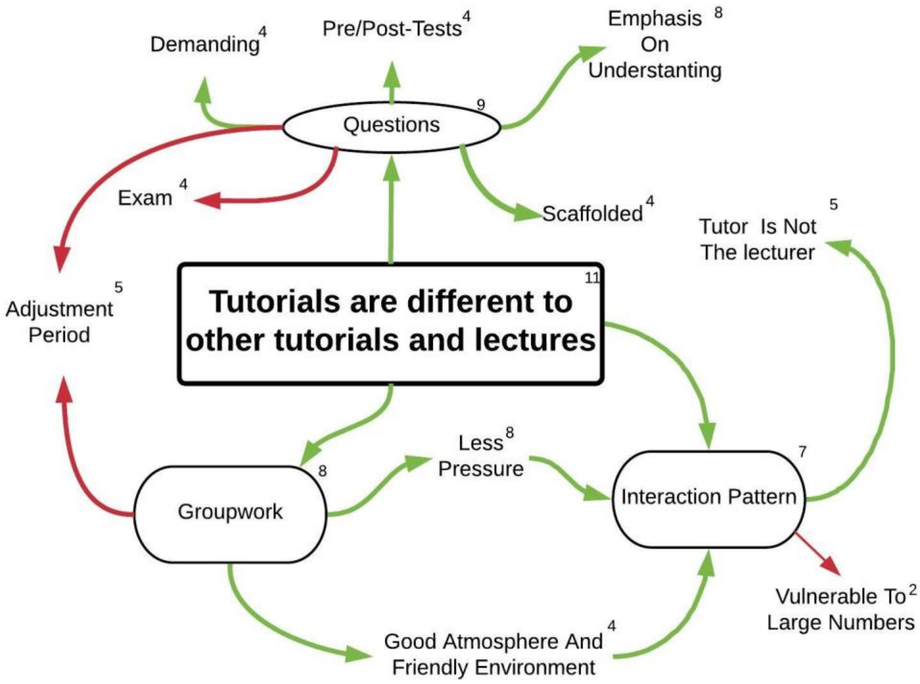


Figure 5-9: Visualization of 2016-2017 data analysis

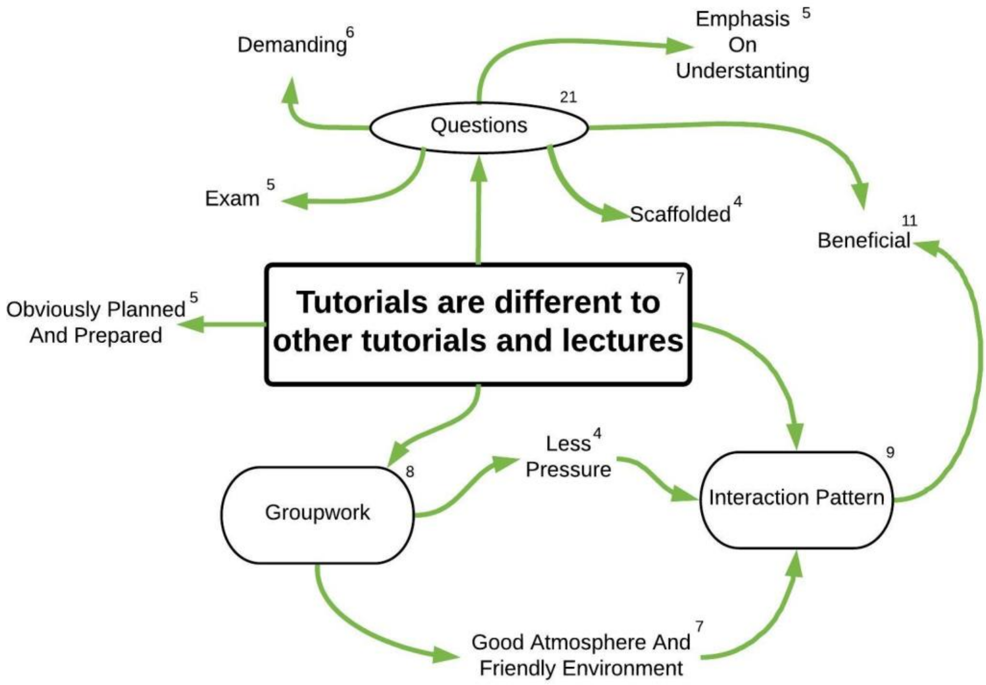


Figure 5-10: Visualization of 2017-2018 data analysis

In Section 2.5, we explained how all visual representations of data contain '*marks*' and '*attributes*'. These are ways of encoding data where marks 'represent or record an instance of data' and attributes 'are variations applied to the appearance of marks' [9, p.151-152]. Table 5-8 outlines the marks and attributes used in Figure 5-9 and Figure 5-10.

Mark	**Meaning**
Labels	Each label represents the name of a category, sub category, or individual data point.
Numbers	The numbers beside each label represents the frequency with which that label was mentioned.
Attribute	**Meaning**
Arrows	Represent links between categories and sub categories, and sub categories and individual data points.
Borders	The primary category is contained within a square border, the subcategories have an elliptical border, and individual data points have no borders.
Colour	Colour is used to indicate whether the link between labels is positive or negative. Green is used for positive links and red is used for negative links.

Table 5-8: Description of marks and attributes used in Figure 5-8 and Figure 5-9

The joint displays show a high level of similarity between the successive cohorts. On both occasions students expressed an appreciation for the prevalence of group work which led to a pleasant working atmosphere. This point was mentioned in 16 of the interviews, with the following excerpt from an interview with two students (R and A) from 2016-2017 being archetypical

> Interviewer: Did you enjoy the MS225 tutorials?
> R: Like I did.
> A: Yes definitely.
> R: They were interactive.
> A: Yes, they take a different approach than what an ordinary lecture would be, instead of just everybody sitting in silence you kind of interact with people and get their thoughts on different things and you get to discuss some just freely instead of just being silent and listening to somebody in a lecture I think it's far more helpful.
> R: Yes, you're actually learning by doing it and talking to someone beside you or if you're [the tutor] walking around obviously you're going to help as well. (16-9)

These students also reported that the tutorials were far more interactive and engaging than they are used to (as did 15 others), and while five interviewees in 2016-2017 said this took adjusting to, they all said this was a positive aspect of their experience. The students felt less pressure in relation to tutorials; this is due in part to the relaxed atmosphere that developed but also because there was no preparatory work required in order to attend. Nine students also noted that they found it easier to ask questions at these tutorials than other mathematics tutorials which is in line with the findings of Walker [10]. This point is made clearly when a student is comparing the intervention to other tutorials

> H: …in tutorials the lecturer will just sort of stand up on the board and just write out solutions but you wouldn't really have a chance to like ask a lot of questions or say 'I can't understand this can you go back over it again?' because you'd be holding up like 34 people. (16-4)

A difference in the type of questions asked of the participants during the intervention was obvious, and was mentioned in some form in every interview. They saw a clear change in emphasis toward understanding the concepts (14 interviews) and they appreciated that they were scaffolded or 'built up from basics' (14 interviews). While the idea of a change in emphasis toward understanding was appreciated by students, it caused a small number to feel anxious (five interviews). This point is well articulated by a student as follows

> C: There was a little bit of anxiety on my side from seeing that I hadn't been doing exam questions or I hadn't been doing more concrete problems up until a certain point. Now, I know that's how it's structured and it did work well, but it's probably no harm to have a direction of, these are tutorial sheets you could be doing in your own time in the MLC [Maths Learning Centre] taking over, and to just kind of keep you working. Just, I'm sure I'll fly the exam in two days, but it

> would help when… just set your mind at ease knowing that you have done the work. (17-5)

In this instance, the anxiety is related to their performance on the terminal exam, and how it may be adversely impacted by not spending all their time on 'exam style questions'. We believe this to mean questions that assess a band of their instrumental understanding (limited to ability to complete algorithms) which they, rightly or wrongly, have grown accustomed to dealing with almost exclusively. The point is well made by the student, they appreciate the manner in which they have learned ODEs during the intervention but cannot disregard the study skills that they have relied on until now.

A positive finding relating to the subcategory of *questions* is that students reported, as a good thing, the questions being 'more demanding'. In addition to this, they saw the guided-inquiry as a way of learning that has a longer lasting impact and will stand them in better stead in their future studies and careers. When asked whether they found the guided-inquiry approach frustrating, one student said the following:

> C: Oh no, that's key. It's whenever… whenever you're just given or presented with the answer, that's absolutely no use to you. You don't… it's never going to stick in my head if I'm just given the answer, this is how you do it. Whereas, you know that kind of thing where if you eventually stumble across the answer yourself, you're more likely to remember it, more likely to… (17-5)

Their desire to learn in a more meaningful manner is in line with the design and intentions of the module and contrasted by the behaviours of other students described by Skemp [11, p.10] "whose goal is to understand instrumentally". Evidence of the existence of this type of student is provided during our comparison of both cohorts, but the fact that the former is seen in greater numbers is reassuring.

5.3.2. Conclusion

We used a general inductive approach to analyse the interview data from each year they were conducted. Both times, *The tutorials are different to other tutorials and lectures* emerged as the primary category and *Questions, Groupwork*, and *Interaction Pattern* became subcategories. With respect to questions, students appreciated the switch in emphasis towards understanding, and scaffolding questions. Some students grew frustrated at the apparent disconnect between questions of this nature and their terminal exam, but conceded that it would be more beneficial to their long-term learning. The sub categories of *Questions* and *Interaction Pattern* were closely linked. A friendly environment and good atmosphere that lead to a feeling of reduced pressure applied to both categories, whereas the interaction pattern also encompassed the tutor and lecturer being different people, which students described as positive. The overall nature of the feedback given by students was overwhelmingly positive. They are eager to learn and are more than willing to apply themselves when they see a benefit attached to the work. A direct comparison between the 2016-2017 and 2017-2108 cohort is made in Chapter 6.

Reference list

1. D. Hyland. P. van Kampen and B.C. Nolan. Outcomes of a service teaching module on ODEs for physics students. **International Journal of Mathematics Education in Science and Technology**, 2017, **49** (5), p.743-758.
2. M.C. Wittmann and K.C. Black. Mathematical actions as procedural resources: an example from the separation of variables. **Physical Review Special Topics- Physics Education Research**, 2015, **11** (2), p.1-13.
3. R.R. Skemp. Relational understanding and instrumental understanding. **Mathematics Teaching**, 1976, **77** (1), p. 20-26.
4. M. Yerushalmy and J.L. Schwartz. A procedural approach to exploration in calculus. **International Journal of Mathematics Education in Science and Technology**, 1999, **30** (6), p.903-914.
5. C.L. Rasmussen. New directions in differential equations: A framework for interpreting students' understandings and difficulties. **Journal of Mathematical Behavior**, 2001, **20**, p.55–87.
6. M. Zandieh and M. McDonald. Student understanding of equilibrium solutions in differential equations. In: F. Hitt and M. Santos (eds), **Proceedings of the 21st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education**, Columbus (OH): ERIC. 1999, p.253–258
7. D.R. Thomas. A general inductive approach for analyzing qualitative evaluation data. **American Journal of Evaluation**, 2006, **27**, p.20-26
8. M.D. Fetters, L.A. Curry and J.W. Creswell. Achieving integration in mixed methods designs-principles and practices. **Health Service Research**, 2013, **48**, p.2134-2156.
9. A. Kirk. **Data visualisation: a handbook for data driven design.** London: Sage, 2016.
10. L. Walker. Enabling students to become independent learners **In**: M. Grove, T. Croft, J. Kyle and D. Lawson (eds). **Transitions in undergraduate mathematics education**. Birmingham: The University of Birmingham, 2015, p.71-84.
11. R.R. Skemp. Faux Amis. **The Arithmetic Teacher**, 1978, **26** (3), p.9-15.

6. Post-intervention

In this chapter we will discuss MS225 and its students after two complete cycles of the intervention. We will do this in three ways: by comparing the feedback given by both cohorts during interviews, by profiling typical students who participated in the intervention, and by describing the changes the module has undergone since our involvement (Table 4-3). Each of these themes highlight areas in which the intervention has benefitted the participants since its introduction. We begin by comparing the feedback given by students in each cohort.

6.1. Interview data comparison of 2016-2017 and 2017-2018 with respect to RQ3c

Drawing comparisons between data sets allows us to identify relative changes in feedback. Much can be learned through comparing successive cohorts and any differences identified may be evidence of the impact of the cycles of revision the intervention has undergone (cf. Section 4.4). We note that other factors can also contribute to changes in feedback (for example a change in interviewer and differences between cohorts). There were seven notable differences between the data sets after analysis. Three terms that were new in 2017-2018 and four appeared in 2016-2107 that did not reappear in 2017-2018. Table 6-1 provides a summary of the changes between both cohorts.

We discuss each of these changes in turn, beginning with the ones that only appeared in 2016-2017 before discussing the ones that emerged for the first time in 2017-2018. In each case, we describe the label before explaining why we think it appeared in one year and not the other.

Table 6-1: Differences in feedback between both cohorts who have participated in the intervention

No longer appearing		
No.	**Label**	**Description**
1	*Adjustment period*	It took students time to adjust to the change in pedagogy
2	*Pre/post-tests*	The pre/post-tests were cited as a way to track progress
3	*Tutor and lecturer are different*	The tutor was a different person to the lecturer
4	*Vulnerable to large numbers*	Interaction time with the tutor is dependent on the number of students attending the tutorial.
Additions		
No.	**Label**	**Description**
1	*Beneficial*	The tutorials were beneficial to the learners which drove attendance. Guided-inquiry was also recognized as beneficial for longer term learning.
2	*Obviously planned and prepared*	Students noticed and appreciated that the tutorials were well prepared and planned with their backgrounds in mind.
Changes		
No.	**Label**	**Description**
1	*Exam*	The relationship between the intervention and the exam was viewed negatively and positively by students in each cohort. The balance changed from 3-4 to 5-1 (positive to negative).

6.1.1. Labels that appeared in 2016-2017 only

No *adjustment period for the change in pedagogy* was reported by the 2017-2018 cohort. This was in contrast to the 2016-2017 cohort, where a minority of students expressed difficulty adjusting to the change in pedagogical approach. An excerpt from Interview 7 is representative of this point, which was mentioned in five interviews in total.

> Z: It's a bit frustrating when you're being asked do you understand that, and every other maths module we've ever done is like, just show me how to do it and that's it. I don't really care after that. (16-7)

Braun et al. [1] caution practitioners of active learning methods to expect resistance from students, particularly at the beginning. They give the example of students who have had success with traditional methods potentially feeling threatened by the new environment. We believe there is some overlap between the students described by Braun et al. [1] and the students represented by the excerpt from Interview 7 although we regret that this is the case among the prospective teachers (as well as the physics students) given their training. This concern was not raised in the 2017-2018 interviews, most likely due to a portion of the continuous assessment for MS225 being assigned to tutorial attendance and participation. It is possible, however, that this concern was not raised because students did not want to raise it with their tutor, who acted as interviewer.

While the reallocation of CA was successful in incentivising high achieving students to attend, it undoubtedly motivated students who are driven primarily by their final grade to attend, with one student from 2017-2018 stating simply that

> D: Well if they contribute to CA I'll go. (17-16)

The *vulnerability of the pedagogical approach to large numbers* did not emerge as a label from the analysis of the 2017-2018 data**.** In 2016-2017, student feedback highlighted a reduction in student-tutor interaction time when there are a high number of students at the tutorial. Two of the nine groups of students interviewed cited this when asked about areas of potential improvement, with one of the groups acknowledging

> Z: But that's the type of tutorial you need a smaller group. (16-7)

Certainly, a large student-tutor ratio has an effect on classrooms that adopt an inquiry-based approach more than ones that use a lecture format but we believe this shortcoming is more than made up for in other areas, as illustrated in Figure 5-9 and Figure 5-10. Also, both of the groups who mentioned this referred to the same tutorial. The reason this issue was not cited by students during the 2017-2018 is a consequence of better planning (to avoid a repeat occurrence) and a smaller student cohort in 2017-2018.

The observation *The tutor and the lecturer are different* was not made in 2017-2018**.** This point is largely self-explanatory and was mentioned in 2016-2017. The fact remained the same across both years but details of the interviews changed, as mentioned previously. In this case, we believe that the change in interviewer is the primary reason this point was omitted by students. In 2016-2017, a member of academic staff unattached to the project conducted the interviews whereas in 2017-2018 the tutor conducted the interviews. We believe that the students who did mention this, felt it necessary to do so to the 'outside' interviewer during the first set of interviews but not to the tutor in the following year.

The final point that appeared in 2016-2017 but not the following year was that the pre/post-tests allowed students to observe their progress across tutorials. This point was cited by four of the nine groups during 2016-2017 and described by one group as follows.

> X: Post-test and pre-test, we would have never done that in a tutorial before. If you actually look at them and before you know something, like if you actually know something, and it's kind of good then, and it's actually kind of useful. (16-7)

The reason for this change is not understood by the research group. The pre/post-tests were administered identically across both years. Students answered the same questions attached to the same worksheets and at the same stage in their studies. At best, we offer the following that may clarify the situation. More time was spent on student feedback during the 2016-2017 interviews, and as such, more data was created. The pre/post-tests were mentioned by students in response to a question asking them about activities in the intervention. In answering the questions, student may have thought of activities that were unique to the intervention, which reminded them of the pre/post-tests. This is consistent with the above quote which explains that they 'never done that in a tutorial before'.

The four labels that appeared in 2016-2017 but not in 2017-2108 were identified as: *no adjustment period for the change in pedagogy*; *the vulnerability of the pedagogical approach to large numbers*; *the tutor and the lecturer are different*; and *the pre/post-tests* allowed students to observe their progress across tutorials. With the exception of the point on pre/post-testing, a change in interviewer or a revision to the intervention can be attributed to the change between cohorts. In the case of the points on pedagogy, we consider the nature of the revision and the outcome of its implementation to be an improvement to the intervention. We will now discuss the emergence of four subcategories during the analysis of the 2017-2018 interview data.

6.1.2. Terms that emerged or evolved in 2017-2018

In 2017-2018, a positive aspect of the link between the intervention and exams was mentioned during student interviews**.** Analysis from the 2016-2017 interviews revealed an apparent disconnect between the tutorials and the exam that caused anxiety for some students. While they appreciated the content and delivery of the tutorials, and even preferred them to other tutorials, they struggled to see how they related to the exam as exemplified by the excerpt from interview 17-5 on page 125.

Analysis of data from 2017-2108 revealed a reduction of this concern (from four of nine groups to one of twenty-one students) and also yielded a positive link between tutorials and exams (mentioned by five of twenty-one students). An example of this is the following response, given by a student

> E: Yeah, no definitely, it was towards an understanding, that you look at what's in the exam eventually and you know what you're doing not reaming off a method that you just learned off. (17-8)

Though contradictory evidence needs to be clarified, some students explained how the tutorials have left them in an ideal position heading in to their exam and that they would even use them as a study aid. Possibilities for the reduction in student dissatisfaction at the disconnect between the exam and the tutorials include the reallocation of CA (10% of the final grade was attached to tutorial attendance and participation) and the delayed post-test question appearing on the previous years exam. As described in Chapter 5, the reallocation of CA acts as motivation for some students to participate in tutorials. This, coupled with students seeing a 'typical' tutorial question on the terminal exam from the previous year, may have shown students how the intervention relates to their final grade.

The idea of the tutorials being beneficial appeared repeatedly in 2017-2018, resulting in its appearance in Figure 5-10. It conveys two meanings here. Firstly, it relates to *Interaction Pattern* because students felt like their interactions with peers and with the tutor were of benefit to their learning, which factored into their decision to attend as outlined in this comment

> Q: But I felt like this tutorial, it felt more like kind of even going back into secondary school, like he was really like, very engaging. And I felt like I should go because I know I'm going to actually do something in this. Important yeah. And very engaging, and like I knew like when I'd go in there I knew I'd get an hour's work done. Everytime I was in. (16-3)

Secondly, students saw the type of questioning used throughout the intervention as beneficial. This is closely related to how the manner in which students learn may affect how long their understanding lasts. The point is made in Chapter 5 that six of the twenty-one students believe learning ODEs through guided-inquiry better equips them for any subsequent ODEs they meet in their studies. The long-term benefits associated with learning through guided-inquiry [2] were supported on a small scale using the Follow-up Participant Survey (Appendix E). All of the nine respondents said that the tutorials were beneficial to them in Mechanics and seven of the nine students referenced a different approach as the key behind this.

The same point can be made specifically of physics students. Each physics student that was questioned about how the approach affects their long-term learning agreed that both the content of tutorials and the manner in which they are delivered is of more benefit to them moving forward than traditional instruction. This was in reference to both subsequent modules and eventual careers, highlighted by this student response

> M: Yeah, I think so, I suppose going back to the Vibrations and Waves, I think had I known what I know now about them, it would have been easier to understand where all the equations where coming from. I think last year we kind of just said, this is a formula, we're going to use it, but I can't remember exactly them saying, this is a differential equation. And say even just a quick introduction as to why it's cosine, sine, whatever. And then we'll say the exponentials for damping and stuff like that. So yeah, I think it's good. (17-14)

In fact, one student even suggested increasing the amount of modelling in tutorials when asked for potential improvements. The quote below was made by one of the students from the 2017-2018 cohort in response to a call for potential improvements

> D: I think just have more examples to do with, like give them a sheet of like scenarios give them like do them in a way like an experiment was conducted or something like these things and that the differential equation involved or find the differential equation like we did that in one of the tutorials I remember it was kind of like [00:03:19] I think it was baking a cake or something and the temperature going down, and that kind of stuff but I think just having something like that and just like if this was the situation that happened, because we did labs for the and we understand like things kept constant other stuff, like the whole like tutorials based on that one. (17-16)

The final label to emerge from the 2017-2018 interview data was an acknowledgement from students that the worksheets were *obviously prepared and personalised*. This is closely linked to our previous point on how beneficial the intervention was to students. In one instance, students describe the tutorials as follows

> J: We definitely look forward – well, not look forward – but it's one of our favourites to go to where you'd be, not as if you'd be sitting there. You'd be actually doing something, you'd be trying to encourage and-
> K: You'd be concentrating the whole time.
> J: Yeah. And the 45 minutes you're like, yeah I learned something today. (16-5)

There were a variety of reasons attributed to the students' perception that the worksheets were obviously planned. They refer to the questions and structuring of the worksheets, the pre/post-tests, and the recognition of the prior learning of the students. In the following exchanges, four students (eight in total) express an appreciation for the intervention incorporating their primary subject

> K: Because we're physics, population. Necessarily, it mightn't be on our exam but it's beneficial to learning rather than just, here are the numbers, here are the formulas, do it. You don't know what you're doing it kind of for, so... (16-5)
>
> F: Yes, and like a lot of the stuff that we were doing is like relevant to us as physicists as opposed to last year where it was like linear mathematics and I still don't have a clue about any of that stuff. (16-4)
>
> J: Rather than just telling you, he spent of lot of time on modelling. (16-5)
>
> Y: A way of explaining things and bringing – like to the population or certainly the first question you did there, he was really relating it to us. (16-7)

Whether tutorials will be of benefit to students appeared repeatedly during interviews and is the strongest intrinsic motivator we found driving tutorial attendance.

When we compared the data that informed RQ3a between both cohorts we saw strong agreement. The primary category (*Tutorials are different to lectures and other tutorials*) remained the same, as did the three subcategories (*Groupwork*, *Questions*, *Interaction Pattern*). There were changes at the lowest level however. In total, four labels did not reappear (*adjustment period*, *pre/post-test*, *tutor is not lecturer*, and *vulnerable to large numbers*), two new labels (*beneficial* and *obviously prepared and planned*) emerged, and one label (*exam*) took on a new meaning. We described each of these changes in this section, which are also summarised in Table 6-1. With the exception of the disappearance of the label *pre/post-tests* from the 2017-2018 data, each of these changes are explained by elements of revision or circumstances surrounding the interviews.

We will now describe the characteristics of a typical student who engages with the intervention. This profile is based on the results described in Chapter 5 and in this section.

6.2. Student profile

Analysing the interview data was very informative. RQ3c was designed to inform the research team on the opinions and experiences of students who completed the intervention, with the expectation that this feedback could improve aspects of its content and delivery. We believe that it has been successful in doing so and will continue to provide insight that will improve the intervention as a product. We found the candour of the students during the interviews particularly useful and note that their contributions have influenced most, if not all, the revisions since we began.

We reported on many aspects of the intervention in Chapters 4, 5 and 6 that emerged during the analysis of interview and pre/post-test data. Within this, there was a significant amount of data that contributed to our understanding of our students. We have used this data to describe typical students who participated in the intervention. Of these students we suggest the following:

- For most students, attendance is tied to how beneficial they deem a tutorial to be to their learning. They are willing to grapple with 'more demanding' questions in the process.
- Students appreciate when content is tailored to them as learners with regard to difficulty level and area of interest. This point is particularly true in the case of service maths. Alignment of mathematics tasks with core areas of study should go beyond using a word paragraph to hide initial conditions for a given problem. Students consider many end-of-chapter 'word problems' to be situated in unfamiliar contexts. In many instances, they learn to scan the text and extract the necessary information before solving the problem without ever grappling with the context. Our students (both those studying physics and education) reacted very well to the modelling tutorials they completed and some of the physicists even requested a more sophisticated version of the modelling tutorials to be developed for second order ODEs.
- All of our students noticed the shift in emphasis toward understanding and saw the role of scaffolded questions in this. Some students did take time to adjust to the change in approach however.
- Students recognised the benefit in partaking in guided-inquiry. In this case, they agreed that the understanding they have of ODEs (which used guided-inquiry during tutorials) will last far longer, and is far more accessible than their other mathematics modules (which used traditional instruction during tutorials).
- Students react to the environment. The nature of these tutorials requires students to work closely with their peers and the tutor. This requires relationships to form and grow. This takes time for some students but once it does it results in an excellent learning environment, one in which students ask and respond to questions within their group and with the tutor; one where they gradually gain control and eventually autonomy over their learning (as it pertains to *delegation of authority* [2]). Interaction patterns and environment were major themes of the interviews, as was the notion of pressure and anxiety. These positives are amplified and

negatives are mitigated to an extent when students develop stronger relationships with the other tutorial participants.

- Despite everybody's best intentions, the focus of learning will always fall on the final exam. We saw this happen far less frequently as a result of the intervention, but it was still apparent in the 2017-2018 data. There are several reasons for this: students that 'buy in' to inquiry learning can see the long-term effect it has on their understanding which gives them confidence and reduces their focus on the solitary objective being to pass the module; reallocating CA required students whose primary motivation is achievement to engage with the intervention in a meaningful way; the appearance of the delayed post-test on the past exam paper gave students a tangible link between the intervention and succeeding in the terminal exam. Despite this, one student out of twenty-one who were interviewed in 2017-2018 wanted a more visible link between the worksheets and the exam. At this level, we believe only a cultural shift will eliminate this type of thinking; certainly, a twelve-week intervention for third-year university students will not make enough of a difference in isolation.

The above points are intended to describe the outcomes of typical students who participated in the tutorials we designed when studying MS225. These points are supported by a representative quote and tally from each year (Table 6-2).

Point	Quote	2016-2017 tally (9)	2017-2018 tally (21)
Tutorials are beneficial (mentioned in previous section) and questions are more demanding.			
More demanding	Interviewer: Would you… are they more challenging because you're working for the full hour in that way? C: Oh, a hundred per cent. You can't… you know yourself, the only way I ever learn in maths is by making mistakes and by trying to work my way through things. If somebody just pops up the board and says, oh you can't do Q3, here's exactly how it's done, that's no benefit to me. Whereas, when you are scattered around the room and you're just given just enough information to steer us in the right direction, or to get us back and focused towards a more linear…. that was brilliant. (17-5)	4	2
Tutorials are personalised, within this modelling (for example) is mentioned.			
Well prepared and personalised	Y: A way of explaining things and bringing – like to the population or certainly the first question you did there, he was really relating it to us. (16-7) Z: It was well-prepared. (16-7)	4	4
Modelling	J: Rather than just telling you, he spent of lot of time on modelling. Because we're physics, population. Necessarily, it mightn't be on our exam but it's beneficial to learning rather than just, here are the numbers, here are the formulas, do it. You don't know what you're doing it kind of for, so... (16-5)	3	3
Shift towards understanding, scaffolded questions, and adjusting.			
Emphasis on understanding	Interviewer: So is it that you know the material now, or do you understand it? Do you think you understand it as well as just knowing how to do them? H: I think – I understand. I don't understand it completely, but I – yeah, I don't understand it completely but it's a lot better than most modules because 95% of modules to be honest we don't understand. A lot of it, do you know. To be honest, I need to say it. O: It's pure – for a lot of them, it's pure rote learning. It's just layers and layers of notes, learn this, do it this way, don't ask why, just do it. H: But there's like – compared to other modules, definitely, yeah. (16-3) M: There was a lot of thinking. Interview: So you don't think maths is thinking? M: No, but a lot of it is procedure, a lot of this is a lot of deep thinking. (16-1)	8	5
Scaffolded Qs	H: I'd say the types, because they're very building. They're very like scaffolding questions. They won't just	5	9

	throw you into the deep end. And then like, if you kind of half knew it and then your friend didn't have a clue then you'd be like, well I know this. (16-3)		
negative adjustment period	Z: It's a bit frustrating when you're being asked do you understand that, and every other maths module we've ever done is like, just show me how to do it and that's it. I don't really care after that. (16-7)	5	0
The long-lasting effect of guided-inquiry			
Guided-inquiry results in longer lasting understanding	V: When asked whether the manner in which they learned ODEs will better prepare them for using ODEs in the future "Oh yeah, honestly I would, I'm not saying that the question has tempted to be set up, but you know that, because say for...like I was saying, calculus or something, I'd my exam the other day and I thought it went really well. If I was to go back...if I was to do calculus even in May, the exam, I'd have to put in just as much study as I did for this one, maybe a tiny bit less, whereas if I'm...after I used differential, if I was to do it again in May, the amount of study I'd have to do would be definitely a lot less than what I have to do for this, because I'd remember things more. (17-11) Interviewer: Is the 'just enough information' approach does that get frustrating over a while or…? C: Oh no, that's key. It's whenever… whenever you're just given or presented with the answer, that's absolutely no used to you. You don't… it's never going to stick in my head if I'm just given the answer, this is how you do it. Whereas, you know that kind of thing where if you eventually stumble across the answer yourself, you're more likely to remember it, more likely to. (17-5)	4	6
The effects of the learning environment.			
An appreciation for the prevalence of group work, leading to a pleasant working atmosphere	Interviewer: Did you enjoy the MS225 tutorials? R: Like I did. A: Yes, definitely. R: They were interactive. A: Yes, they take a different approach than what an ordinary lecture would be instead of just everybody sitting in silence you kind of interact with people and get their thoughts on different things and you get to discuss some just freely instead of just being silent and listening to somebody in a lecture I think it's far more helpful. R: Yes you're actually learning by doing it and talking to someone beside you or if you're walking around obviously you're going to help as well. (16-9)	8	8
Less pressure: atmosphere	Interviewer: And did you always know more after? M: Yeah, we wanted more.	4	2

	T: It actually worked in the class, then if you needed a hand, he was there to help. But at the same time, if you wanted to bop on by yourself. (16-1)		
Less pressure so easier to ask questions	Comparing the intervention to other tutorials H: …tutorials the lecturer will just sort of stand up on the board and just write out solutions but you wouldn't really have a chance to like ask a lot of questions or say 'I can't understand this can you go back over it again?' because you'd be holding up like 34 people. (16-4) Q: You weren't afraid to ask a question, or help with something that maybe we'd moved on from, but coming back to it. Or whatever, yeah. (16-3)	4	2
Less pressure: no prep work	Z: You went with a pen and a piece of paper. (16-7)	4	1

Table 6-2: Additional interview excerpts with tally

6.3.MS225

Our influence on MS225 across four years has resulted in several major changes to its content, delivery, and assessment. This would not have been possible without the support and commitment with the lecturer for MS225, whose interest in education and willingness to facilitate our study was crucial. Not only were they able to maintain alignment between the lectures and tutorials through change, they also adapted their own lectures to ensure the module remained self-contained. We greatly appreciate their effort and will discuss the changes MS225 has undergone.

Several changes have occurred to the manner in which MS225 is assessed since our involvement began. The first was a reallocation of continuous assessment. Previously, 20% of the student's grade was assigned to two in-class tests carried out during semester. Data gathered during interviews and the cooperation of the lecturer allowed for half of this to be reallocated to the tutorials. The continuous assessment was awarded to students for attendance and participation in tutorials (5% for each). In addition to this, a delayed post-test question appeared on the terminal exam for MS225 for the 2016-2017 and 2017-2018 cohorts. This opened another channel through which data could be obtained. The data from both cohorts has been used to inform RQ3. In Section 6.2 we explained the influence this change had on student attitudes which we believe is a critical step towards a change in culture.

Reference list

1. B. Braun, P. Bremser, A.M. Duval, E. Lockwood and D. White. What does active learning mean for mathematicians? **Notices of the American Mathematical Society**, 2017, **64** (2), p. 124-128.
2. M.R. Blanchard, S.A. Southerland, J.W. Osborne, V.D. Sampson, L.A. Annetta and E.M. Granger. Is inquiry possible in light of accountability? A quantitative comparison of the relative effectiveness of guided inquiry and verification laboratory instruction. **Science Education**, 2010, **94** (4), p.577-616.
3. E.G. Cohen, R.A. Lotan and N.C. Holthuis. Organising the classroom for learning. **In**: E.G. Cohen and R.A. Lotan (eds), **Working for equity in heterogeneous classrooms**. New York: Teachers College Press. 1997, p.31-43.

7. Conclusions

In this chapter we summarise our findings for each of the research questions posed at the beginning of this project. We then explore the potential for our work to be expanded upon and improved, before ending with a summary of the project.

7.1. Evaluation of student learning

In the initial stage of this project we outlined two research aims and three research questions to investigate. We will recap on these before describing our findings for each research question in order.

Research Aims

1. To study physics students' learning of ODEs, with a focus on the obstacles they encounter that hinder success;
2. Develop an intervention that addresses the obstacles described and thereby promotes more successful learning.

Research Questions

1. Do our students have the necessary instrumental understanding in the following areas to succeed in their study of ODEs?
    a. Manipulation of exponents in equations;
    b. Evaluating indefinite integrals.
2. Do our students have a well-developed concept image of ODEs upon completion of this module?
    a. What is brought to mind when presented with an ODE?
    b. What do students know about ODEs and their applications?
    c. Do they understand what a solution to an ODE is?
3. Has the intervention benefitted our students when learning ODEs?
    a. How does the instrumental understanding of students who experienced the intervention compare to those who completed the module prior to its implementation?

b. In what way has the concept image of students who experienced the intervention grown?

c. How is the intervention viewed by the participants?

Research Question 1 and 2 correspond to Research Aim 1. Their purpose was to understand in detail what students excelled at and struggled with having completed an introductory module on ODEs. Once this was known, Research Aim 2 sought to develop an intervention whose purpose was to address the issues in a manner that also saw students develop their understanding of the other areas of their study of ODEs. Research Question 3 was an evaluation of the intervention on three fronts: instrumental understanding of students, concept image of students, and student feedback. In this way, Research Question 3a was aligned with RQ1, RQ3b aligned with RQ2, and RQ3c informed each cycle of revision the intervention would undergo.

7.1.1. Findings of Research Question 1

Three mathematical tasks were given to students as part of the Diagnostic Survey (Figure 7-1) to inform RQ1.

Section 1

Q1. Find *x* and *y* if

- $5=3+x$
- $4=3+xe^{-y}$

Q2. Simplify the following.

$$\frac{(-k)^2}{k^{-1}} + (k^2 + k^1)^2$$

Q3. Integrate each of the following;

- $\int (x^2 + \frac{1}{x^8})\,dx$
- $\int \frac{1}{x}dx$
- $\int xe^x\,dx$

Section 2

Q1. Write down everything you can think of when you see each of the following. (note: C is a positive constant in each case)

- $\frac{dN}{dt} = -C$
- $\frac{dN}{dt} = -CN$
- $\frac{dN}{dt} = -Ct$

Q2. What are differential equations, and why are they useful?

Q3. Write down everything you know about the solutions to differential equations.

Figure 7-1: The Diagnostic Survey

The analysis of the results to Section 1 of the Diagnostic Survey led us to conclude that there are areas where students' instrumental understanding is not strong enough to successfully complete the standard of questions on ODEs they encounter during MS225. Although the majority of students were able to complete Question 1, a large minority (44%) were unable to use the natural log function as the inverse of the exponential function which was necessary in order to successfully complete the task. A similar level of success was observed in Question 3, where only half of the students successfully integrated $\frac{1}{x^3}$. We believe the requirement to manipulate negative exponents caused the high level of difficulty experienced by students. This is corroborated by the results of Question 2 of Section 1 of the Diagnostic Survey which also required students to manipulate negative exponents as part of the correct solution. All the students who could manipulate negative exponents in Question 2 were able to integrate $\frac{1}{x^3}$. Question 2 was the only question where less than half of the students were correct, but the results of Section 1 as a whole suggest students' instrumental understanding needs significant improvement if students are going to excel in their use of ODEs throughout their studies.

7.1.2. Findings of Research Question 2

RQ2 investigated students' concept images of ODEs having completed MS225. The results show that students possess poorly developed concept images and that their understanding is at the level of processes rather than concepts. This is in keeping with Sfard's notion of the process-object duality of mathematical concepts [1]. The fact that first-year students' poor conceptual understanding in introductory calculus modules has been reported on previously [2], but the problem is also evident for third-year students in this module on ODEs. If a vast improvement does not occur in the conceptual understanding of the students completing this module, using differential equations to model situations in physics effectively will likely continue to be a challenge for them.

7.1.3. Findings of Research Question 3

RQ3 was designed to harvest data relating to the intervention. The findings can be best understood when compared with those of RQ1 and RQ2. Since our involvement, the instrumental understanding of students has improved, especially in relation to the areas identified by the Diagnostic Survey. In Chapter 5, we saw a vast reduction in the frequency with which these errors occur in students who participated in the intervention. The reduction of issues with these areas helped identify algebraic manipulations in the context of separation of variables (SoV) as the most frequent difficulty students were having with respect to their instrumental understanding. Difficulty with algebra in the context of SoV was previously identified by Wittman and Black [3], and will now become a point of emphasis for the intervention. The instrumental understanding of the students also saw growth through an improvement in classification skills.

Excellent progress was reported in Chapter 5 with respect to students' concept image. We saw a sharp improvement in students' conception of a solution to an ODE and its nature as a function. The introduction of direction fields through the intervention also contributed to an improved concept image. Prior to the intervention, direction fields were not part of the curriculum for MS225 but post-testing revealed significant achievement by students after just one hour was devoted to their inclusion. The immediate post-test showed that students were able to identify direction fields when given the ODE in analytical form (88%) and to match a solution curve to the appropriate direction field (93%) with great success. The delayed post-test also showed that the worksheet had a long-lasting impact, with the success of students' ability to identify direction fields when given the ODE in analytical form remaining similar (85%). The delayed post-test contained a more difficult, albeit similar question that students were less successful at answering, with 33% of students completing the task correctly. This highlighted an area in which the worksheet may improve for future cohorts.

RQ3c was answered using data from the interviews, where students were asked about their experiences during the intervention and areas they believed could be improved. A wide variety of data was analysed when answering this question with a high level of consistency emerging between both cohorts. The students noticed the change in emphasis toward understanding the intervention was trying to instil. They cited the change in questioning most often as where they saw this change, but also noted the prevalence of group work and change in interaction pattern as keys to its success. Together, these three themes formed the basis of the analysis, with all other points falling under these headings and under the overarching theme of *The tutorials in the intervention are different to lectures and other tutorials*.

7.2. Avenues to extend the research

We began this chapter by stating our research aims and questions, and our findings across its four-year timeline. In detailing this progression, we describe many positive outcomes but we also identified other aspects of the research that could be investigated in their own right. We will discuss these in relation to their potential to build on the research we have reported.

There is an excellent opportunity to expand the findings and impact of this research through generalisability. In this study we focused on the physics students in DCU, but many other universities both domestically and further afield have similar cohorts of students for whom the same issues may be relevant. Were we to restart our efforts, offering the survey of academic staff to other institutes and the Diagnostic Survey to their students would be useful: in addition to an increase in responses which may unearth new findings, the ability to compare different cohorts of students for relative strengths could yield useful information about the presence (or otherwise) and severity of each issue in each location.

A similar extension could be achieved by looking beyond physics students. ODEs appear in many different subjects (e.g. Engineering, Chemistry, Economics, Biology) and so, service teaching of ODEs is carried out with students studying these subjects. It is reasonable to assume that difficulties may exist for these students, and an equivalent intervention may be able to improve their learning. An intervention for students studying subjects other than physics may look considerably different to the one developed here, but carrying out a survey of academics in that subject and an appropriate Diagnostic Survey would be an ideal starting point. All of the decontextualised activities in the intervention can used for students studying any discipline, but the subject-specific material would require changes. An example of this would be the final modelling worksheet, where students practice modelling a real-life situation. The situations given to students need to be taken from their discipline. A less obvious example is the manner in which the intervention approaches the meaning of the derivative. This tutorial uses a geometric approach that is within the grasp of physics students but may be beyond students with different backgrounds.

The opportunity to continue our work with the students who participate in the intervention is also exciting. The most obvious way of doing this is by working with members of the academic staff in the School of Physical Sciences to inform them on the learning of their students with respect to ODEs. An improved understanding of their students' abilities will enhance the design of their modules and the manner in which they teach them. A collaboration of this nature would involve making an informed decision on how best to include (or reintroduce) ODEs in the module.

This collaboration could work both ways. An example of where the physics lecturers might improve the intervention would be improving the worksheet on modelling with second order ODEs for physics students. When asked for ways they would improve the intervention, a physics student expressed a desire for increasingly difficult modelling problems to be utilised. They suggested a realistic problem from their studies in which ODEs were relevant. Developing problems of this nature for each of the different physics programmes would be enhanced by the involvement of the physics lecturers, whose subject insight and programme knowledge is unmatched.

In relation to mathematics education more generally, we see two areas of immediate relevance. The first comes from the analysis of the interview data pertaining to RQ3c. Though we did not highlight this above, members of the research team were reminded of the work of Lee and Johnson-Wilder [4] on mathematical resilience. It is our believe that mathematical resilience may be a factor in the success of students participating in the intervention. A cursory look at the answers given by students who engaged and succeeded during the intervention show evidence of the four aspects of mathematical resilience [5]: growth, value, knowing how to work at mathematics, and knowing how to recruit support. In addition to these traits being displayed by the successful students, absence of these traits is also evident from students who were less successful. Establishing mathematical resilience in our students would also benefit them outside the classroom as it was found in [6] the students who have developed resilience in the domain of mathematics carry their skills into their wider lives. Unfortunately, the researchers identified this issue only after the interviews were completed and so did not have the opportunity to pursue it with appropriate follow-up questions. Further interviews would be required in order to investigate the issue thoroughly

Another area for further investigation relates to threshold concepts and troublesome knowledge [7, 8], and a concept inventory [9]. A vast amount of data was obtained across the duration of this project that has given useful insights into the learning of students studying ODEs. This is a necessary part of any research hoping to contribute to the identification of threshold concepts for a given topic, but there are other elements that must also be present. Our primary impediment to creating an inventory of questions or a list of concepts that are integral to the study of ODEs is volume of students. Though work on concept inventories began in physics [9], they have translated to mathematics through Epstein’s [10] calculus concept inventory and O’Shea et al.’s [11] work on developing a function concept inventory. Should we wish to explore this avenue further, the appropriate action would be to use our data to develop an initial ODE concept inventory. This would need to be trialled locally before being proposed as a widely applicable ODE concept inventory.

7.3. Summary

Prior to this project beginning, the service teaching of differential equations to physics students took place in the third year of their undergraduate programme, when they took MS225. MS225 is a five-credit module taken during first semester. It consisted of two lectures and one tutorial each week. The tutorials amounted to recitation sessions, where student were given a problem sheet in advance and could choose to attend a session where the tutor would place solutions on the board and answer questions posed to them by the students. The tutorials were not well attended, although students seemed to perform well on their end-of-term exam. Students need to demonstrate competence with several solution techniques in order to perform well in this assessment.

The suggestion that an issue may exist was brought to light by the survey of academic staff in the School of Physical Science in DCU (Appendix A). Respondents to this survey explained how the use of differential equations in their modules has diminished over time, to the point of omission in certain cases. They cited student difficulties as the primary cause of the reduction. This prompted our investigation which initially sought to identify the exact nature of the difficulties encountered by physics students when learning ODEs. When these issues were understood, we then sought to address them by designing and implementing an intervention.

We designed and administered a Diagnostic Survey to the outgoing MS225 students that investigated their understanding in several areas, primarily instrumental understanding and concept image. We identified three major issues with their instrumental understanding: manipulating exponents, carrying out indefinite integration, and using the natural log and exponential functions as inverses of each other. We also identified significant shortcomings in their concept image, particularly with respect to what an ODE is and what constitutes a solution of an ODE.

These issues were used as central themes around which our intervention would be designed. The intervention would replace the tutorials for MS225 and no other changes would be made. Our review of the literature also suggested that students' concept image would benefit greatly from the inclusion of multiple representations (direction fields) and modelling. These were also included in the intervention.

Trialling the intervention and administering the Diagnostic Survey with the following cohort of students led to a marked improvement across each of the major themes. We then extended the intervention to include pre/post-testing and interviews. This resulted in more detailed data being obtained that further informed iterative cycles of revision. The intervention has been used in full for the 2016-2017 and 2017-2018 cohorts and has generated a wide variety of data.

In the Results and discussion section, we showed how the intervention has led to a significant reduction in the issues in instrumental understanding initially seen in the Diagnostic Survey. This reduction uncovered an issue students have with algebraic manipulation. We also showed evidence of an increase in students' ability to classify ODEs. With respect to their concept image, the addition of direction fields has been successful in instructing students on the identification of direction fields when given an ODE and on matching solution curves to their appropriate ODE. It requires further work in instructing students on how to draw solution curves through specific points on ODEs, which we suggest may be supported by a strengthening of the relationship between analytical and graphical solution techniques. Interview data also shows that student understanding of solutions to an ODE has grown. In the Diagnostic Survey only one student mentioned "function" when asked about the solution to an ODE, whereas interview data across both cohorts shows a vast majority of students have a multifaceted understanding of the concept of the solution to an ODE. Students know that it is a function (the variables of which are encapsulated in the derivative) and they understand that it can be verified by substituting it back into the ODE.

Research Question 3c, which sought the opinions of the students about the intervention was the driving force behind almost all of the revisions to date. All students noticed the difference between the tutorials to MS225 and other tutorials or lectures. Some students required a period of transition to adjust to the change in pedagogy, but all of the students interviewed found it more beneficial to their learning. They showed an awareness of and appreciation for the emphasis on understanding and reported feeling less pressure in tutorials which increased their activity within their group and with the tutor. When asked if the manner in which the tutorials were conducted has resulted in them performing better with ODEs in future studies, each student said yes. This is supported by the Follow-

Up Participant Survey (Appendix E), where eight of nine students reiterated this belief. It is hoped that these changes provide the academic staff in the School of Physical Sciences with the opportunity to increase or even reintroduce ODEs into their courses in a meaningful way. This, in turn, may lead to students feeling comfortable using their understanding of ODEs and other mathematics in the remainder of their degree and in their future studies or careers.

Our work is an example of how scratching beneath the surface can unearth a host of interesting information about students. We have used this information to improve the outcomes for students and have done so in a manner they found enjoyable, engaging, and worthwhile. We hope this experience shows them the applicability of mathematics to their study, and enhances (or rekindles) positive opinions about the subject. We believe that the issues we have looked at (in some depth) are not unique to DCU, and that mathematicians and mathematics educators will recognise them from their own institutions. We have designed an intervention that has had a major impact on students in a relatively short window of their education, but that also has the potential to expand rapidly and help students in different locations and in adjacent fields of interest.

7.4. Our contribution to mathematics and physics education research

This project has made several original contributions to the fields of physics education research and mathematics education research.

By designing and carrying out the Diagnostic Survey, we were the first research team to assess Irish physics students' concept image and instrumental understanding of ODEs. We discovered areas for improvement in the students' instrumental understanding and that they possess fragmented concept images. Some of our findings confirmed the presence of difficulties reported internationally, most notably with students' conception of solutions of ODEs. We also saw things in our cohort that, to the best of our knowledge, have not be described internationally such as how the nature of the terms in an ODE impact how readily students recognise it to be an ODE. These findings were combined with the relevant literature to inform our most significant contribution: an intervention.

The intervention comprises fifteen stand-alone worksheets, seventeen immediate pre/post-tests, and one delayed post-test. The worksheets are an appropriate accompaniment to an introductory module on ODEs and are self-contained. When we implemented the intervention as part of an introductory module on ODEs in DCU, we saw a significant improvement in students' concept image of ODEs and a reduction in errors related to instrumental understanding. Interviews carried out with students described the intervention in a positive manner.

The intention of the research team moving forward is to broaden the impact of the intervention by implementing it in other universities and with students from other disciplines. There is also potential to use the data harvested via interview and pre/post-testing to investigate other phenomena described in popular literature, such as concept inventories and the notion of mathematical resilience.

Reference list

1. A. Sfard. On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. **Educational Studies in Mathematics**, 1991, **22** (1), p.1-36
2. J. Bezuidenhout. Limits and continuity: some conceptions of first-year students. **International Journal of Mathematics Education in Science and Technology**, 2001, **32** (4), p.487-500.
3. M.C. Wittmann and K.C. Black. Mathematical actions as procedural resources: an example from the separation of variables. **Physical Review Special Topics- Physics Education Research**, 2015, **11** (2), p.1-13.
4. S. Johnston-Wilder and C. Lee. Mathematical Resilience. **Mathematics Teaching**, 2010, **218**, p.38-41.
5. C. Lee and S. Johnston-Wilder. The Construct of Mathematical Resilience. In: **Understanding Emotions in Mathematical Thinking and Learning**, 2017, p. 269-291.
6. S. Johnston-Wilder, C. Lee, L. Garton, S. Goodlad and J. Brindley. (2013). Developing coaches for mathematical resilience. In **ICERI 2013: 6th International Conference on Education, Research and Innovation**, Seville, Spain. Online, Available from: www.mathematicalresilience.org/papers.htm [Accessed 10 May 2018]
7. J.H. Meyer and R. Land. **Threshold concepts and troublesome knowledge: Linkages to ways of thinking and practising within the disciplines**. Edinburgh: University of Edinburgh, 2003.
8. J.H. Meyer and R. Land. **Overcoming barriers to student understanding: Threshold concepts and troublesome knowledge.** Routledge, 2006.
9. D. Hestenes, M. Wells and G. Swackhammer, Force concept inventory. **The Physics Teacher**, 1992, **30**, p.141-158.
10. J. Epstein. The calculus concept inventory – measurement of the effect of teaching methodology in mathematics. **Notices of the American Mathematical Society**, 2013, **60** (8), p.1018-1026.

11. A. O'Shea, S. Breen and B. Jaworski. The development of a function concept inventory. **International Journal of Research in Undergraduate Mathematics Education**, 2016, **2**, p. 279-296.

Appendices

Appendix A: Survey of academic staff in the School of Physical Sciences, DCU

Questions posed to lecturers in the School of Physical Sciences, DCU.

We'd like to know what you do with Differential Equations in your module:

- students set them up and solve them?

- you give the DE and students solve them?

- you give the DE and its solution?

- you feel DEs should be part of your module but avoid it because you feel students can't do them and/or it would take too much time?

- etc...

Lecturer	**Module**	**Comment**
Lecturer 1	Module 1	derivation validates the equation. Equation given on exam-sheet. (Heat conduction) use of differential equation to derive formulae examinable (thermal expansion for linear, area, volume.)
Lecturer 1	Module 2	derivation is examinable (Lambert Beers law, grating equation, LiDAR,)
Lecturer 2	Module 3 Module 4	I don't use them in either Module 3 (Light and Optics) or Module 4 (Geophysics).
Lecturer 3	Module 5	I refer to my lectures Module 5 Quantum Physics I and Microfluidics: * students set them up and solve them? No * you give the DE and students solve them? No * you give the DE and its solution? Yes * you feel DEs should be part of your module but avoid it because you feel students can't do them and/or it would take too much time? Yes

Lecturer 4	Module 6	I am dealing with the simplest forms of differential equations, like dx/dt=x**3, and while during the 3rd, and especially 4th year students should be used to solve it without any problems, they are still feel rather uncomfortable with it :((.
Lecturer 5	Module 7	Use existing ones, Set up some, they solve, make dimensionless, some simple, some not.
Lecturer 6	Module 8	I'm not sure if this is what you're looking for, but the only place I really use them in solid state is for the Debye heat capacity (http://hyperphysics.phy-astr.gsu.edu/hbase/solids/phonon.html#c2) Basically all I ask is that the students understand how to go from the internal energy to the heat capacity and understand the substitutions that need to be made. So basically everything is handed to them.
Lecturer 7	Module 9 Module 10	I use Diff. Equations in Quantum Physics 2 and the 3rd year labs! In the good old days i.e. 10 years ago I would actually solve the Harmonic Oscillator and the Hydrogen atom using the 'series solution' method. I would work through the full gory solution. In the past few years I have been reduced to giving the DEs and stating their solutions ... I would then give tutorial/homework problems where I would ask them to show a particular function was a solution to a given DE ... but this concept seems to be beyond most of them. The real problem at present is that they do not see DEs in the Math modules until they are in 3rd year! so they are covering them

		at the same time or even after they are introduced in the Quantum ... I have no idea how [redacted] handles this problem with the 2nd year Quantum module. The computational project for the 3rd year AP is effectively 5 weeks of learning how to solve DEs numerically, starting with simple discrete time-step models moving on to solving the damped driven H.O using the Euler method and then solving other DE using modifications of the simple Euler method. Finally I get them to solve some coupled DEs and chaotic models. Last year one or two of the better students actually implemented a Runge Kutta algorithm. I know that most of the AP students don't really like this project ... primarily because they never really took the time to learn how to program in 1st and 2nd year! but I think it is a really good exercise for them to go through and it is the only module, as far as I know, where they learn a little about Chaotic systems. Maybe we should get together and discuss [redacted] project? I would very much welcome any input on how to improve the AP3 comp. projects.

Appendix B: The Diagnostic Survey

Section 1

Class: (e.g. AP2, SE3, BT2,...)________________________

Q1. Find *x* and *y* if

- 5=3+x
- 4=3+xe^{-y}

Q2. Simplify the following.

$$\frac{(-\mathrm{k})^2}{\mathrm{k}^{-1}} + (\,\mathrm{k}^2 + \mathrm{k}^1)^{2}$$

Q3. Integrate each of the following;

- $\int (x^2 + \frac{1}{x^3})\,\mathrm{d}x$

- $\int \frac{1}{x}\,\mathrm{d}x$

- $\int xe^x\,\mathrm{d}x$

- $\int \frac{1}{A+Bx}\,dx$

Section 2

Part 1.

Q1. Write down everything you can think of when you see each of the following. (note: C is a positive constant in each case)

- $\frac{\mathrm{d}N}{\mathrm{d}t} = -\mathrm{C}$,

- $\frac{\mathrm{d}N}{\mathrm{d}t} = -\mathrm{C}N$

- $\frac{\mathrm{d}N}{\mathrm{d}t} = -\mathrm{C}t$

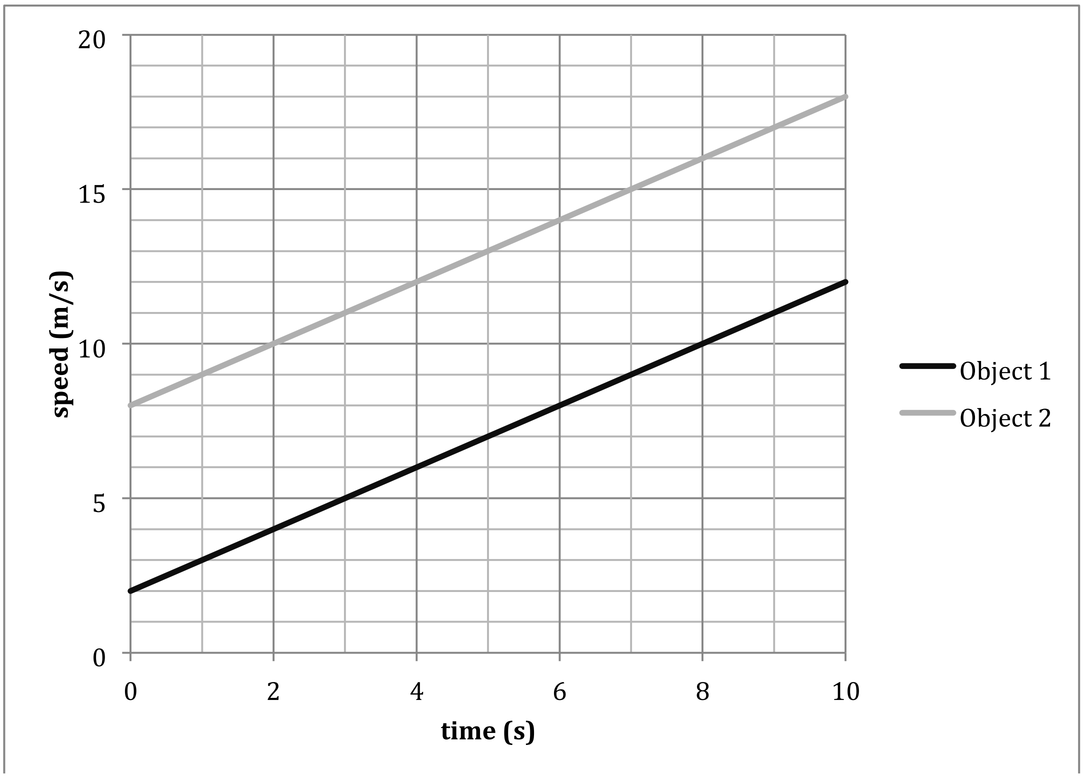
Q2. What are differential equations, and why are they useful?

Q3. Write down everything you know about the solutions to differential equations.

Part 2.

Q4

a) Let *v(t)* be the speed of an object moving with constant acceleration. Then $\frac{dv(t)}{dt} = a$, where a is the acceleration.

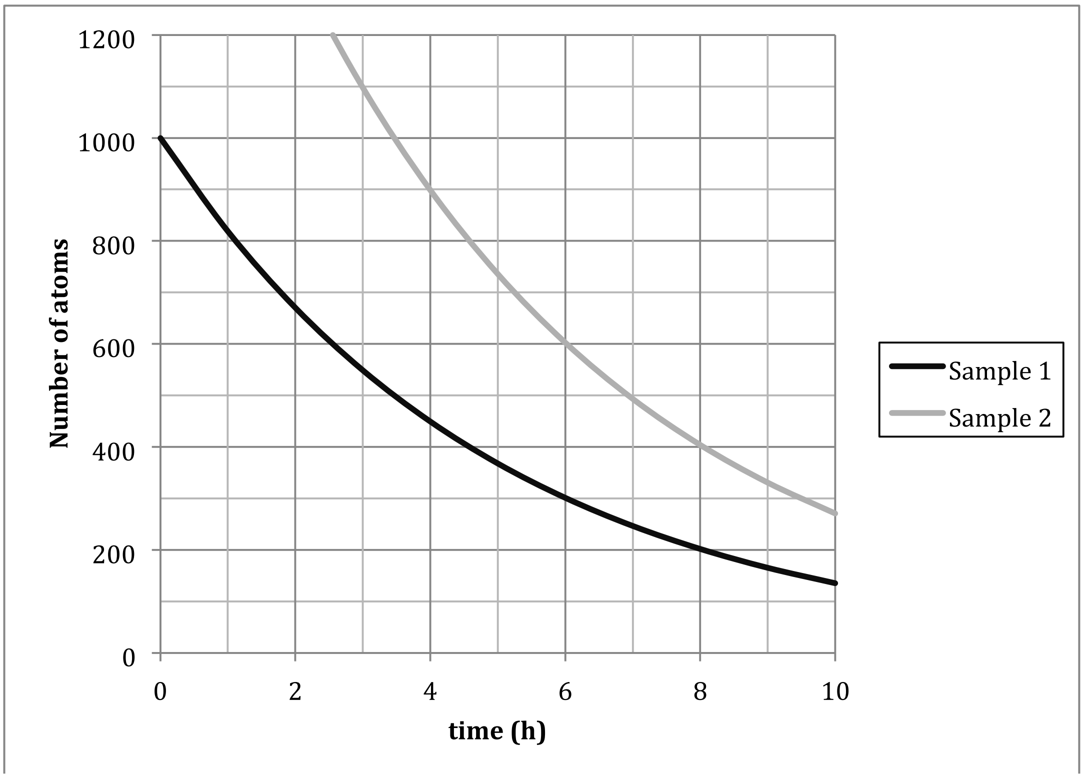


The speed of two objects moving with constant acceleration is shown in the diagram. Do the two objects move:

- with different initial speeds but the same accelerations?
- with the same initial speeds but different accelerations?
- with different initial speeds and different accelerations?
- or is it impossible to tell?

Explain your answer.

b) The number of atoms N in a radioactive sample is given by $\frac{dN(t)}{dt} = -\lambda N(t)$, where $\lambda > 0$ is the decay constant.



The number of atoms in two radioactive samples is shown in the diagram.
Do the two samples have:

- a different number of atoms at the start with the same decay constant?
- the same number of atoms at the start with different decay constants?
- a different number of atoms at the start and different decay constants?
- or is it impossible to tell?

Explain your answer.

Appendix C: Immediate pre/post-tests

1. C-1: Pre-test 1
2. C-2: Post-test 1
3. C-3: Pre-test 2
4. C-4: Post-test 2
5. C-5: Pre-test 3
6. C-6: Post-test 3
7. C-7: Pre-test 4
8. C-8: Post-test 4
9. C-9: Pre-test 5
10. C-10: Post-test 5
11. C-11: Post-test 7
12. C-12: Pre-test 8
13. C-13: Post-test 9 (V1 and V2)
14. C-14: Pre-test 11
15. C-15: Post-test 11
16. C-16: Pre-test 12
17. C-17: Post-test 12

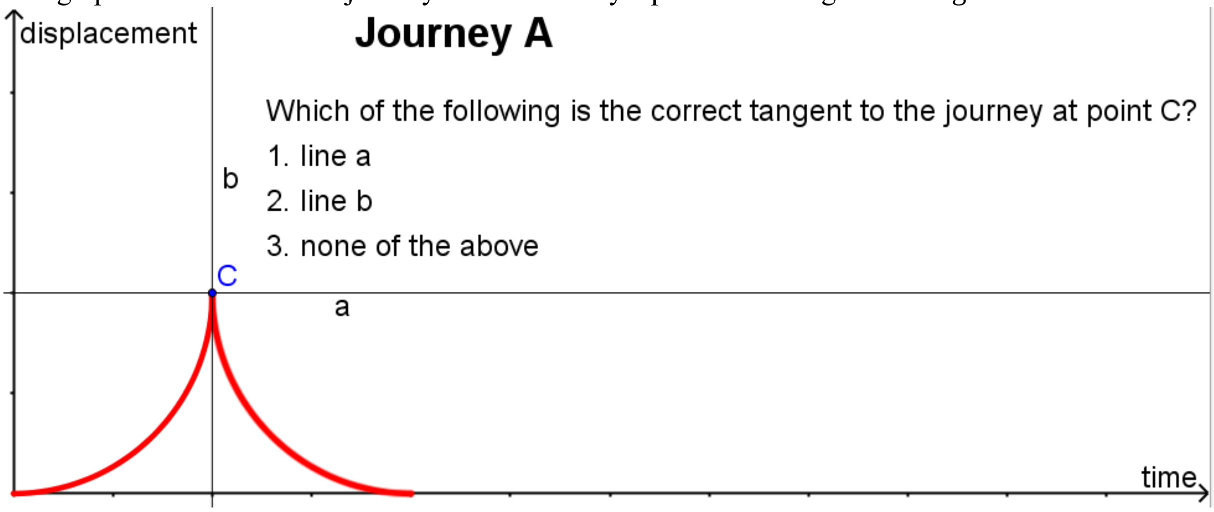
C-1: Pre-Test 1

1. Simplify the expression $\frac{(-x)^2}{x^{-1}}$.
2. Compute the indefinite integral $\int \frac{1}{x^3}\,dx$.
3. Solve for *y* the equation: $1 = 2e^{-y}$.

2. Simplify the expression $\dfrac{(-x)^2}{x^{-1}}$.
3. Compute the indefinite integral $\int 4x^{-2}\mathrm{d}x$.
4. Solve for x the equation: $e^{x+2} = 1$.

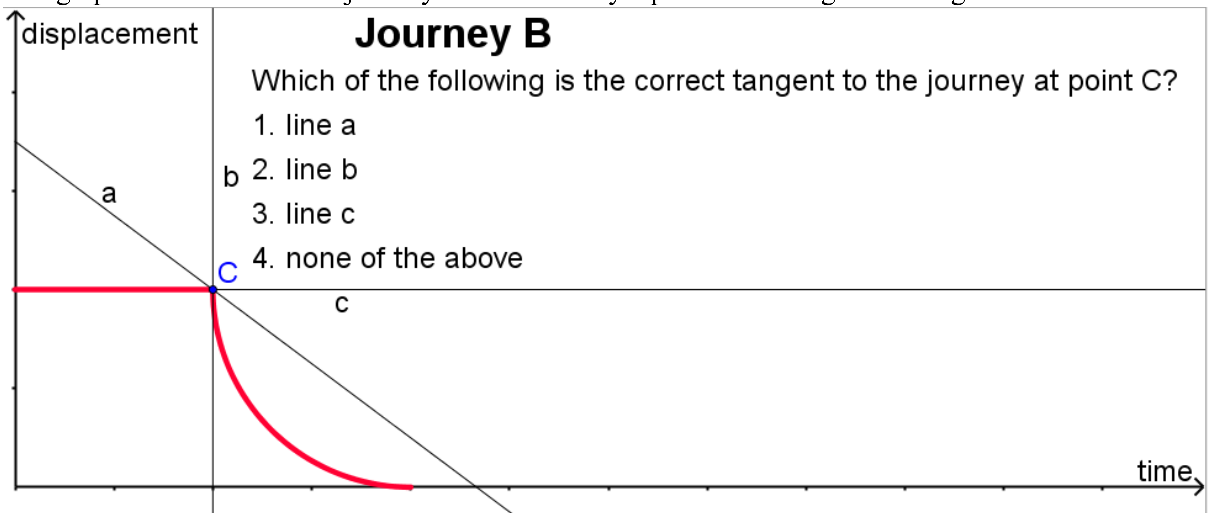
C-3: Pre-Test 2

The graph below shows the "journey" undertaken by a particle moving on a straight line.



C-4: Post-Test 2

The graph below shows the “journey” undertaken by a particle moving on a straight line.



C-5: Pre-Test 3

1. Which of the following are ODEs for $y(x)$? Place one tick in each row.

item	equation	yes	no	don't know
a.	$\frac{\mathrm{d}y}{\mathrm{d}x} = 4y + 2$			
b.	$4y + \sin x = \frac{\mathrm{d}y}{\mathrm{d}x}$			
c.	$\frac{\mathrm{d}y}{\mathrm{d}x} = x + 2$			
d.	$x = \cos y^2$			

2. Which of the following are possible solutions to an ODE for $y(x)$? Place one tick in each row.

item	equation	yes	no	don't know
a.	$y = 4$			
b.	$y = \sin x$			
c.	$x = 1,\ y = 3$			
d.	$x^2 + y^2 = 5$			
e.	$y = x^2 + t$			

C-6: Post-Test 3

1. What influenced your decisions in your pretest when deciding whether or not something was an ODE and a possible solution to an ODE?

2. Would you change any of your answers now? Why (not)?

3. Has this tutorial clarified what an ODE is and what constitutes a solution to an ODE for you?

4. Which of the following are possible solutions to the ODE $\frac{\mathrm{d}y}{\mathrm{d}x} = -ky$? Place one tick in each row.

possible solution	yes	no	unsure
$-\frac{ky^2}{2}$			
0			
Ae^{-kx}			

C-7: Pre-Test 4

1. Rate the following statements on a scale of 1-5, where 1 is strongly agree and 5 is strongly disagree:

Statement	**1**	**2**	**3**	**4**	**5**
I am able to solve most problems involving separable ODEs.					
I am unsure which method to use when I attempt a problem involving separable ODEs.					
The maths involved in solving problems on separable ODEs is very hard.					

C-8: Post-Test 4

1. Rate the following statements on a scale of 1-5, where 1 is strongly agree and 5 is strongly disagree:

Having done the tutorial…	**1**	**2**	**3**	**4**	**5**
I have increased my ability to solve most problems involving separable ODEs.					
I am more sure which method to use when I attempt a problem involving separable ODEs.					
The maths involved in solving problems on separable ODEs does not seem as hard as before.					

2. Write down one thing you learned from this tutorial.

3. Write down one thing you are still unclear about having completed the tutorial.

4. Write down one thing you liked about the tutorial.

5. Write down one thing you disliked about the tutorial.

C-9: Pre-Test 5

1. Rate the following statements on a scale of 1-5, where 1 is strongly agree and 5 is strongly disagree:

Statement	**1**	**2**	**3**	**4**	**5**
I am able to solve most problems involving first order linear equations.					
I am unsure which method to use when I attempt a problem involving first order linear equations.					
The maths involved in solving problems on first order linear equations is very hard.					

C-10: Post-Test 5

1. Rate the following statements on a scale of 1-5, where 1 is strongly agree and 5 is strongly disagree:

Having done the tutorial…	**1**	**2**	**3**	**4**	**5**
I have increased my ability to solve most problems involving first order linear equations.					
I am more sure which method to use when I attempt a problem involving first order linear equations.					
The maths involved in solving problems on first order linear equations does not seem as hard as before.					

2. Write down one thing you learned from this tutorial.

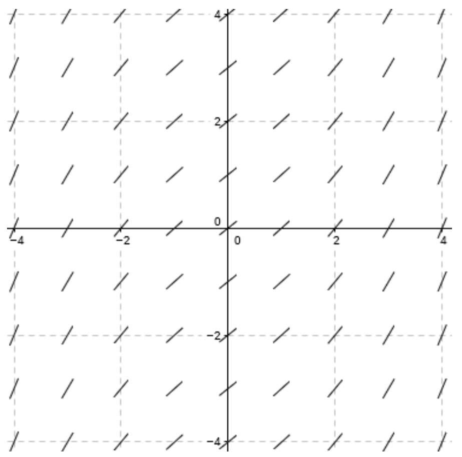
3. Write down one thing you are still unclear about having completed the tutorial.

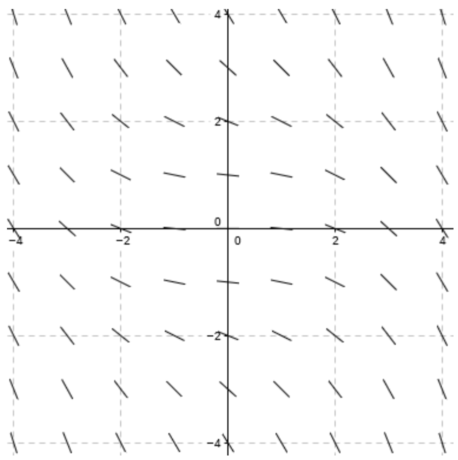
4. Write down one thing you liked about the tutorial.

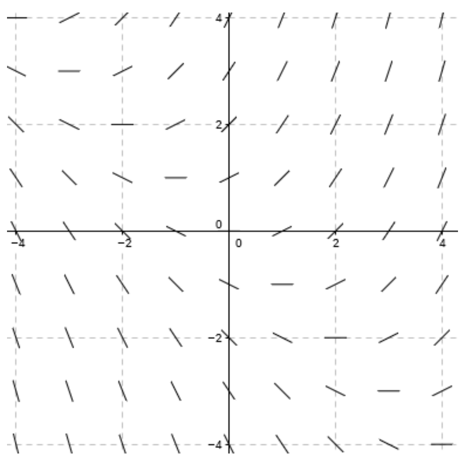
5. Write down one thing you disliked about the tutorial.

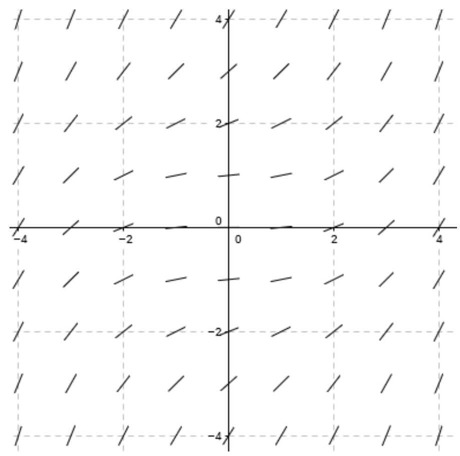
C-11: Post-Test 7

1. Which of the following pictures shows the direction field for the ODE $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{10}\left(x^2 + y^2\right)$? Tick the appropriate box.



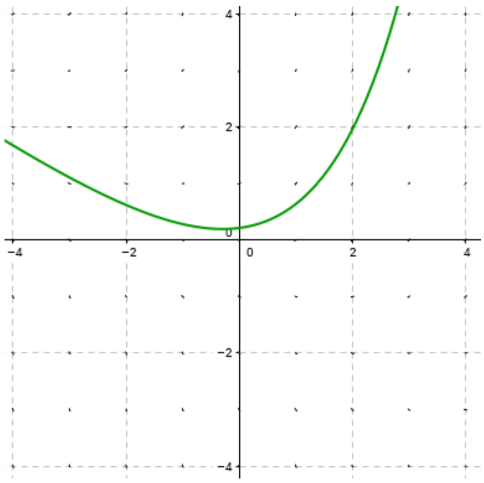
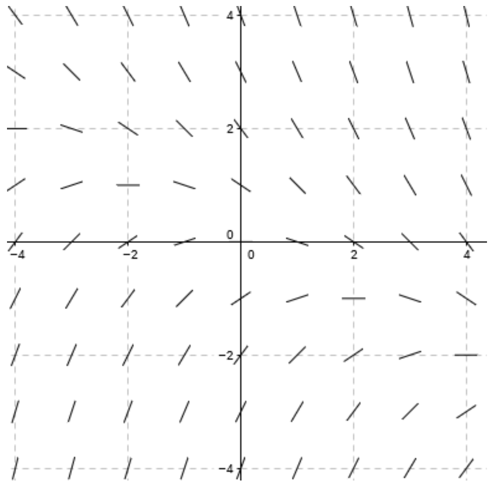
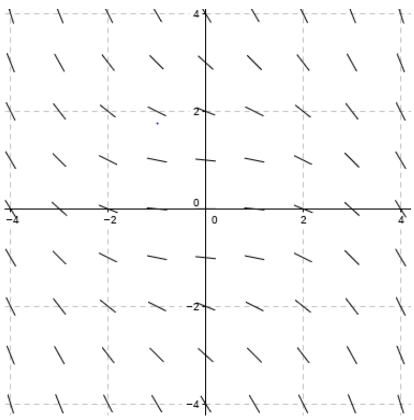
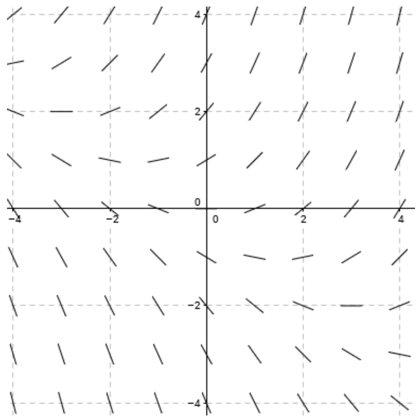
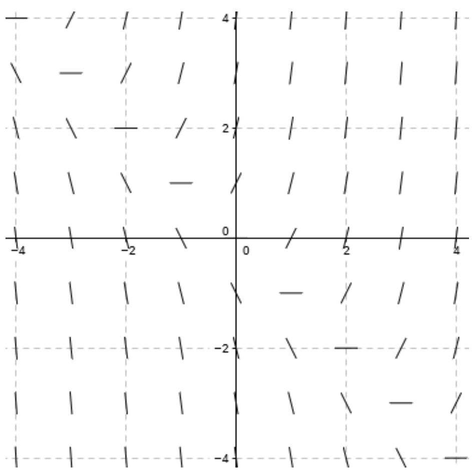






Explain how you arrived at your answer.

2. Match the particular solution shown to the appropriate picture of one of the direction fields below.

	
	
	None of these

Explain how you arrived at your answer.

C-12: Pre-Test 8

1. A cake removed from a hot oven cools over time until it reaches room temperature. In an attempt to describe this cooling mathematically, a student proposes the following equation where T is the temperature of the cake, t is time, and k is a positive constant.

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -k$$

   a. Do you think this student's equation models the cooling of the cake well? Please explain why/why not.

   b. Are there any improvements you would make or situations where this model will not work? If so, explain why would you make each change or why no changes are necessary.

C-13: Post-Test 9 (V1 and V2)

1. To summarise the last two tutorials, our current model of population dynamics is as follows:
   $\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{k}\right)$.
   a. What do the following terms stand for in this model?
      - $\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{k}\right)$;
      - N;
      - r;
      - k;
   b. On a scale of 1-10, how accurately do you believe this model portrays a change in population over time? Explain in detail why you picked this number.

(version 2)

1. A scientist proposes the following alternate model of population dynamics:
   $\frac{\mathrm{d}N}{\mathrm{d}t} = rN - E$. In this model,

- $\frac{\mathrm{d}N}{\mathrm{d}t}$ is the rate of change of the population (t measured in years).
- N is the number of individuals in the population
- r is the constant birth rate of the population minus the death rate of the population (per year).
- E is the net inward migration rate (the number of immigrants entering the country minus the number of emigrants leaving the country per year).

   a. Outline at least two good points about this model.

   b. Outline at least two bad points about this model.

   c. Compare this model to the model we developed: $\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{k}\right)$.

   Is the new model more or less accurate than the model we developed? Why do you think so?

C-14: Pre-Test 11

There exists a 2nd order linear ODE with $y(t) = C_1 e^{-t/2} + C_2 e^{-2t}$ and $y(t) = 0$ as solutions. Generate 2 other solutions to this ODE.

C-15: Post-Test 11

In the pretest, you were asked the following:

> There exists a 2nd order linear ODE with $y(t) = C_1 e^{-t/2} + C_2 e^{-2t}$ and $y(t) = 0$ as solutions.
>
> Generate 2 other solutions to this ODE.

Revisit the pretest question and explain how the information given enables you to generate many solutions.

C-16: Pre-Test 12

1. Rate the following statements on a scale of 1-5, where 1 is strongly agree and 5 is strongly disagree:

Statement	**1**	**2**	**3**	**4**	**5**
I am confident in my ability to solve second order, linear, constant coefficient homogeneous ODEs.					
I am confident in my ability to solve second order, linear, Euler-Cauchy homogeneous ODEs.					
I am confident in my ability to solve second order, linear, constant coefficient inhomogeneous ODEs.					
I am confident in my ability to solve second order, linear, Euler-Cauchy inhomogeneous ODEs.					

C-17: Post-Test 12

1. Rate the following statements on a scale of 1-5, where 1 is strongly agree and 5 is strongly disagree:

Having done the tutorial…	**1**	**2**	**3**	**4**	**5**
I am more confident in my ability to solve second order, linear, constant coefficient homogeneous ODEs.					
I am more confident in my ability to solve second order, linear, Euler-Cauchy homogeneous ODEs.					
I am more confident in my ability to solve second order, linear, constant coefficient inhomogeneous ODEs.					
I am more confident in my ability to solve second order, linear, Euler-Cauchy inhomogeneous ODEs.					

2. Consider the following second order linear inhomogeneous ODEs. For which one of these is the function $y_g(x) = C_1 e^{3x} + C_2 e^{-2x} + 3\sin x$ the general solution?

   a. $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - 6y = 3\sin x$

   b. $x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 6y = 3\sin x$

   c. $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - 6y = -3\cos x - 21\sin x$

   d. $x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 6y = -3\cos x - 21\sin x$

Appendix D: Interview questions

Section A: Introductory questions (2016-2017)

Q1. Write down everything you can think of when you see each of the following. (note: C is a positive constant in each case)

- $\frac{\mathrm{d}N}{\mathrm{d}t} = -\mathrm{C}$

- $\frac{\mathrm{d}N}{\mathrm{d}t} = -\mathrm{C}N$

- $\frac{\mathrm{d}N}{\mathrm{d}t} = -\mathrm{C}t$

Q2. What are differential equations, and why are they useful?

Q3. What do you know about the solutions to differential equations?

Section B: Main Question

1. Consider the differential equation

$$\frac{dy}{dx} = -k(y - b)$$

$$Where\ k\ and\ b\ are\ positive\ constants.$$

This is a first order linear ODE with y(x) as its solution.

Is this problem separable? Can you separate it?

Once you separate, how do you solve for y(x)? Can you solve it?

Great, now I am going to give you a similar problem. In this one there is a $'\sin x$' instead of b.

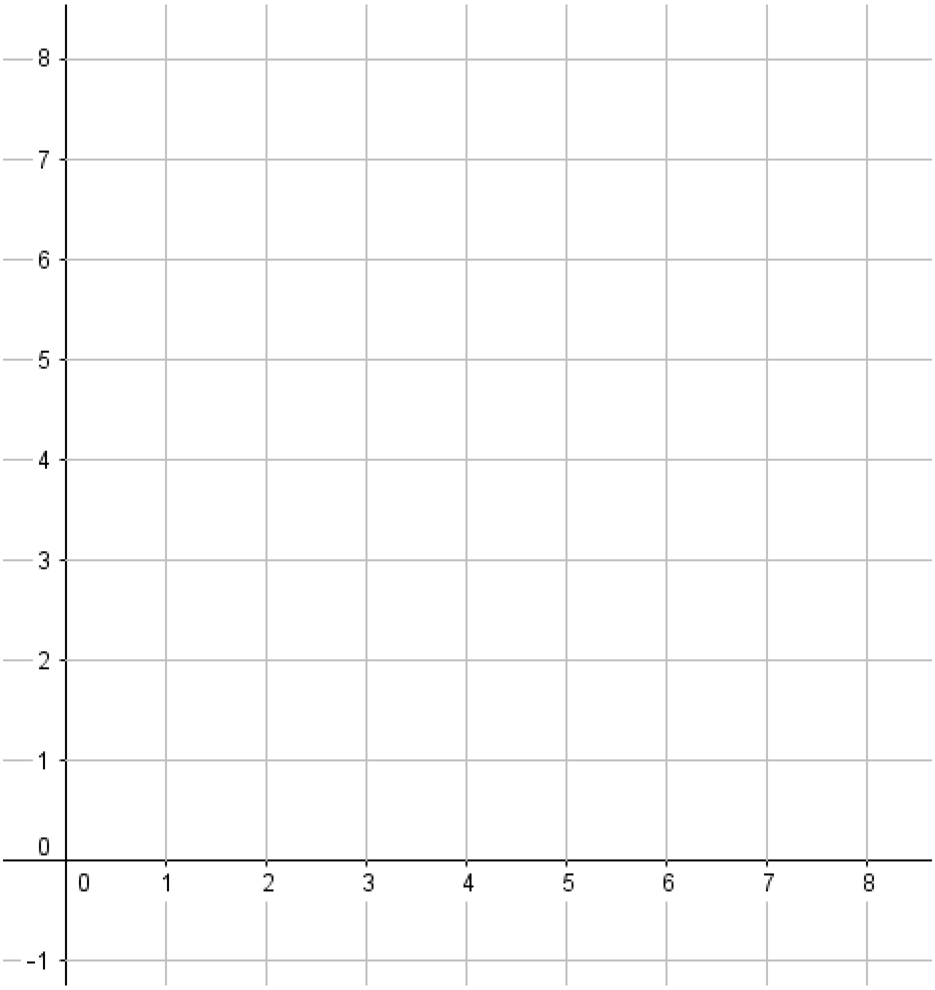
$$\frac{dy}{dx} = -k(y - \sin x)$$

2. Consider the differential equation

$$\frac{dy}{dx} = -(y-5)$$

Fill in the following points on the slope field for this DE

(2,3) (5,5.5) (4,6) (3,5) (2,4.5)



Then show the students a completed direction field and ask them to draw in the curves that pass through (1,1), (1,5), (3,6).

What do these curves represent in relation to this DE?

What can you tell me about the solution to the DE

$$\frac{dy}{dx} = -(y-5)$$

3. Read the following and discuss.

Student A and Student B are trying to model heat transfer. To begin, they make the following assumptions

- Room temperature is constant.
- Objects above room temperature will cool until they reach room temperature.
- Objects below room temperature will heat until the reach room temperature.
- The rate of heating or cooling depends on the difference between the temperature of the object and the temperature of the room. That is, the change in temperature of an object is proportional to the temperature difference between it and the room.

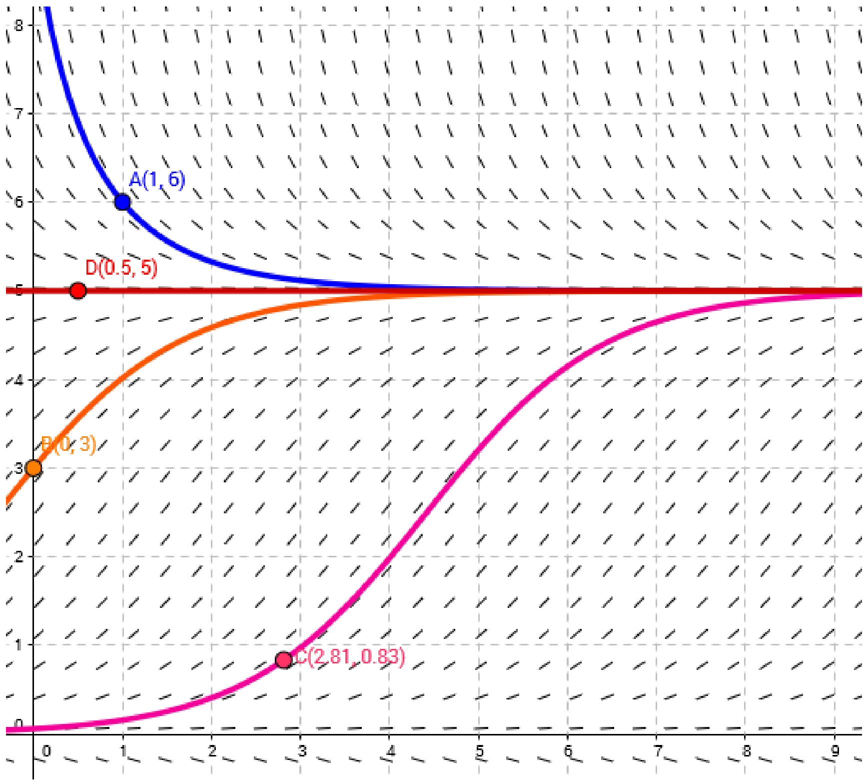
The students agree that the following DE is consistent with the assumptions they have made.

$$\frac{dT}{dt} = -k(T - t_0)$$

Where

- T is the temperature of the object.
- T_0 is room temperature.
- K is a positive constant.

The students attempt to graph the behaviour of T(t) and come up with the following.



Student A is happy that this graph represents the solution to the differential equation they began with whereas Student B is unsure. Do you think this graph represents the solution to the differential equation they began with? Why? How would you convince the students that you are correct?

Section C: Tutorial feedback

1. Did you enjoy the MS225 tutorials?
2. How did you feel about working in groups during the tutorials? Did it benefit your learning? Was it conducive to your learning? Did it fit well with the activities in the tutorials?
3. Do you feel that tutorials like these better prepare you for differential equations you will encounter in later modules?
4. Some students grew frustrated after the first few tutorials because they were not solving enough problems. They later felt that the tutorials where less problem solving occurs are necessary and that they allow problems to be better understood and, as a result, solved more easily. What do you think of this assessment? Did you feel like this at the time?
5. Looking back at the tutorials now, do you feel like an appropriate amount of time was spent solving problems to ensure students could complete all the problems they are required to complete?
6. Have you any additional feedback related to the tutorials for MS225, their design, content, or delivery?

2017-2018

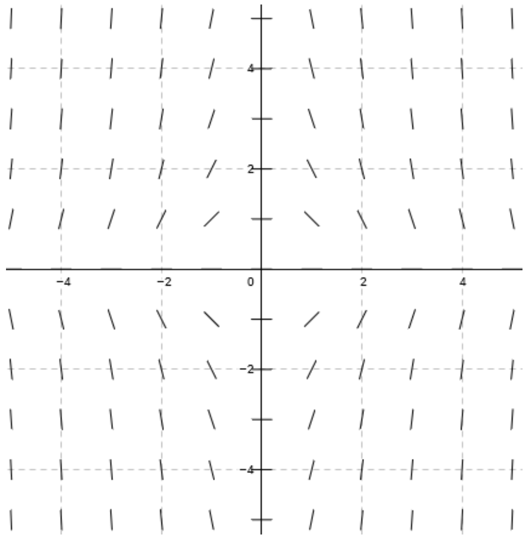
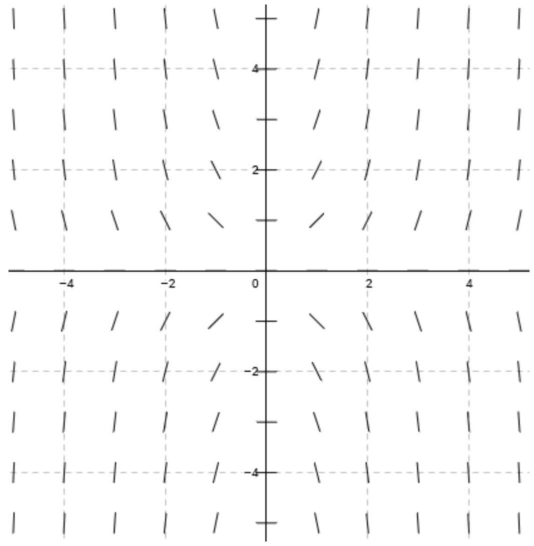
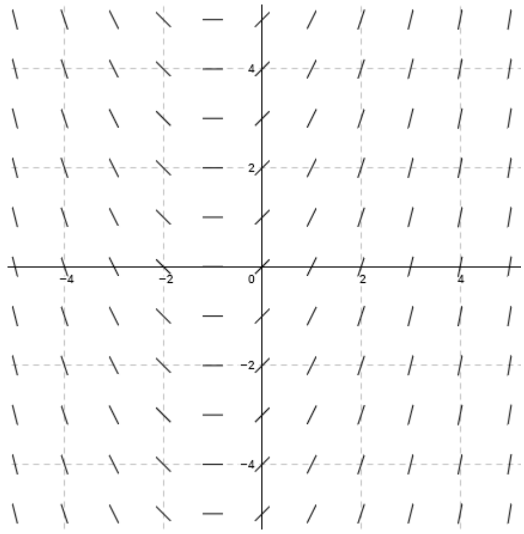
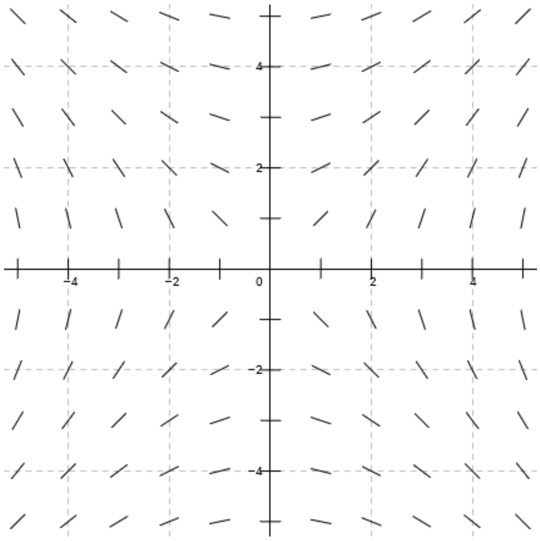
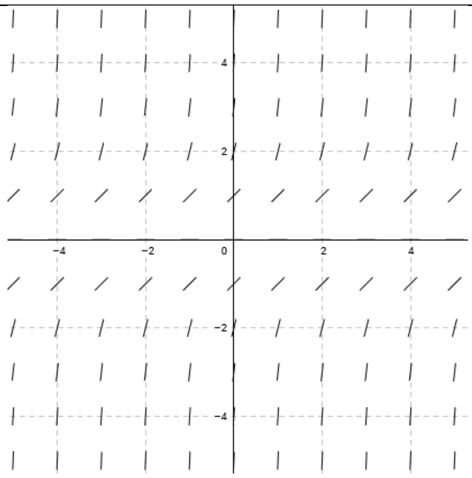
The interviews in 2017-2018 centred on the delayed post-test from the previous year (Appendix E). Where students have seen this question before (on a past exam paper) and progressed through this question quickly, a second ODE (below) was used with the same questions. The interviews also asked students to tell the interviewer everything they knew about the solutions to an ODE, and what a differential equation was.

a) When categorizing a *differential equation* it is useful to know what *order* the equation is and whether or not the equation is *linear*, *separable*, or *homogeneous*. Define each of the italicised words in this sentence and consider the equation $\frac{dy}{dx} = xy$ when answering the following questions:
   1. Is this equation a *differential equation*?
   2. What is the *order* of this equation?
   3. Is this equation *linear*?
   4. Is this equation *separable*?
   5. Is this equation *homogeneous*?

b) Find the general solution of the equation $\frac{dy}{dx} = xy$.

c) Identify the direction field that corresponds to the equation $\frac{dy}{dx} = xy$ from the choices overleaf. Explain your answer.

d) Sketch the particular solution that contains the point (1,1) on the direction field you chose in part c.

	
	
	None of these

Appendix E: Delayed post-test 2016-2017

a) When categorizing a *differential equation* it is useful to know what *order* the equation is and whether or not the equation is *linear*, *separable*, or *homogeneous*. Define each of the italicised words in this sentence and consider the equation $\frac{dy}{dx} = \frac{0.5+y}{2}$ when answering the following questions:

6. Is this equation a *differential equation*?
7. What is the *order* of this equation?
8. Is this equation *linear*?
9. Is this equation *separable*?
10. Is this equation *homogeneous*?

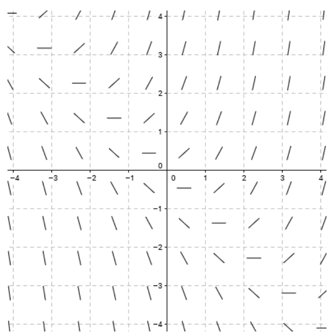
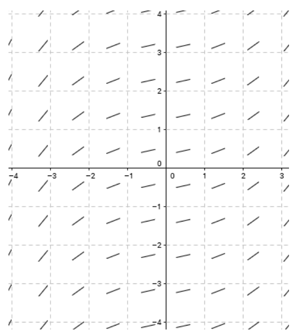
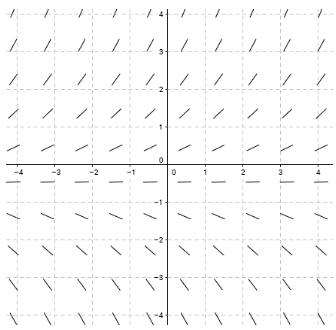
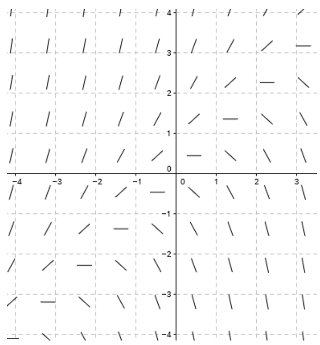
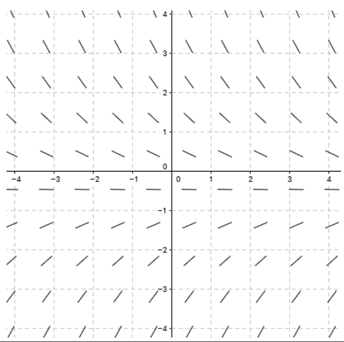
[5 marks]

b) Find the general solution of the equation $\frac{dy}{dx} = \frac{0.5+y}{2}$.

[10 marks]

c) Identify the direction field that corresponds to the equation $\frac{dy}{dx} = \frac{0.5+y}{2}$ from the choices below. Explain your answer.

[5 marks]

	
	
	None of these

d) Sketch the particular solution that contains the point (1,1) on the direction field you chose in part c.

[2 marks]

e) A student proposes that the value of the *x* coordinate is less significant than the value of the y coordinate when considering the behaviour of the solutions to the equation $\frac{dy}{dx} = \frac{0.5+y}{2}$ that pass through the point (*x*,*y*). Do you agree with the student's proposition? Justify your answer.

[3 marks]

Appendix F: Follow-up Participant Survey

Questions for SE4 students who completed MS225 Introduction to Differential Equations.

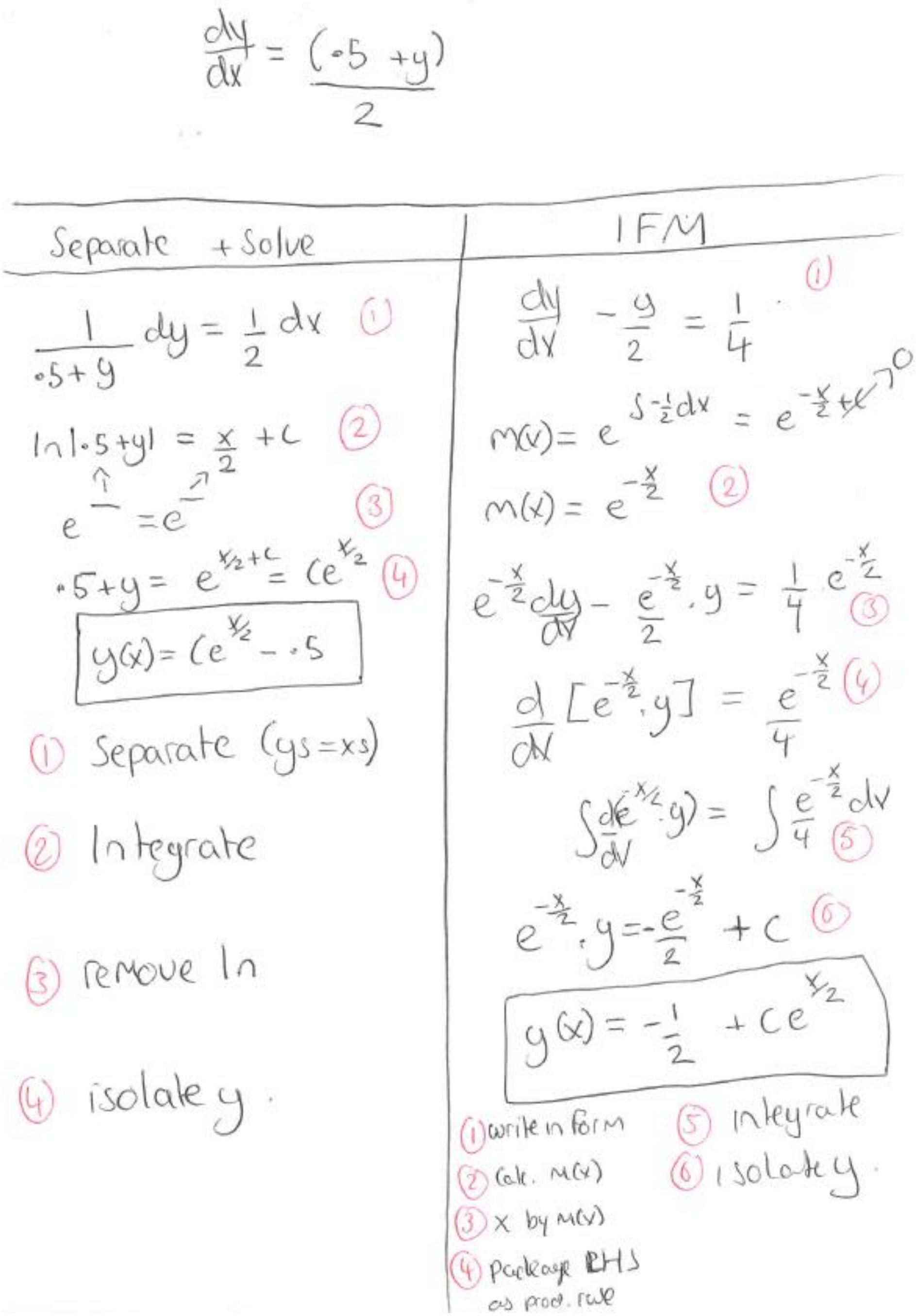
1. Have the tutorials for MS225 helped you understand differential equations that appear in MS339 (Mechanics)? How?

2. How did the tutorials for MS225 differ from the tutorials you attend or have attended for other modules? Were these differences good or bad? Please explain.

3. What is a differential equation?

Appendix G: Data analysis

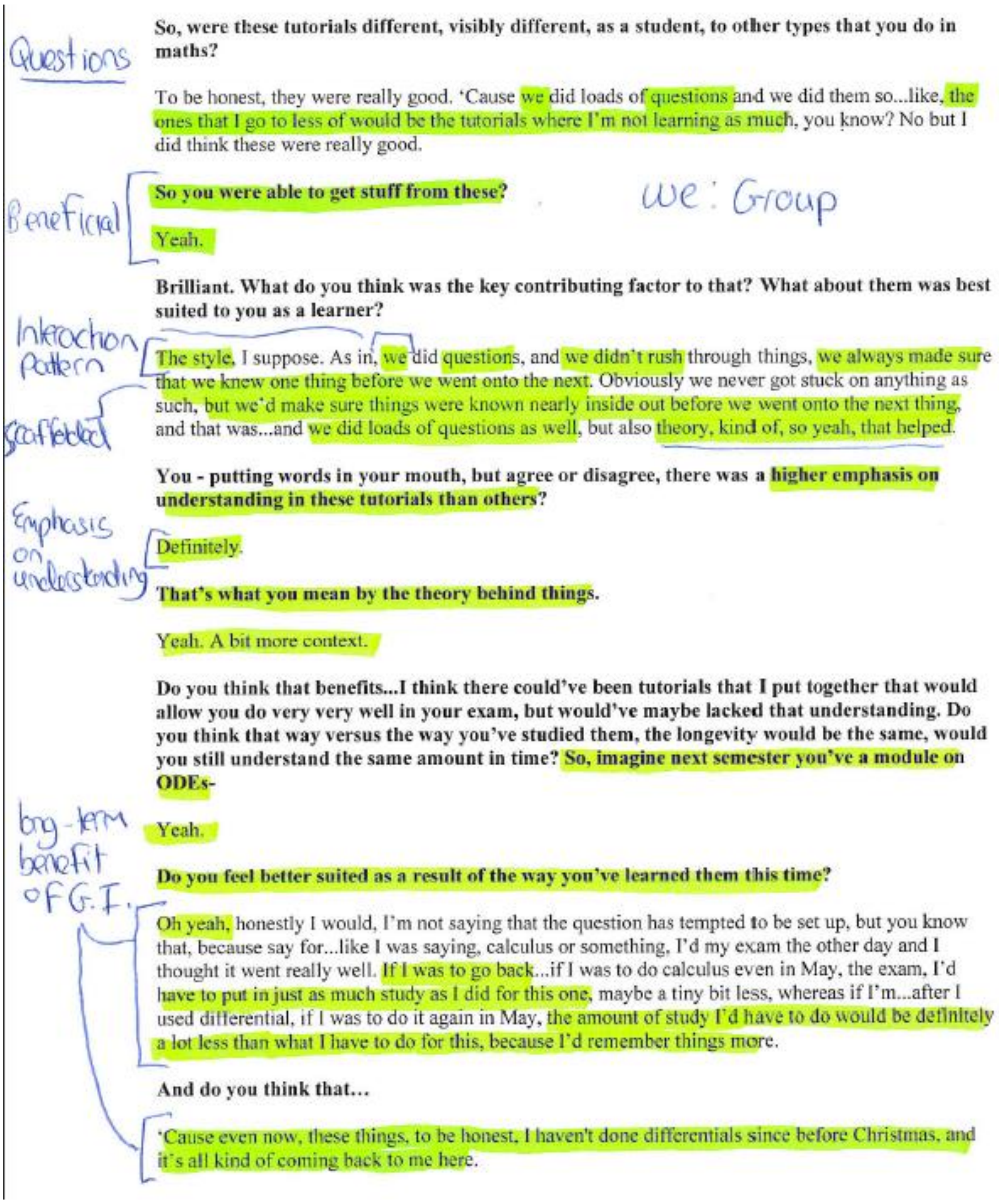
Prior to analysing the responses to Question 1b on the delayed post-test, a rubric was developed and agreed upon by the research team. The rubric contains an exhaustive list of solution techniques and details elements of instrumental understanding required to successfully navigate each step. This was used to analyse the data generated in response to this question, resulting in Table 5-2.



Below is the final categorisation for answers to Q1c and d on the delayed post-test. This final categorisation was agreed upon by the research team following independent categorisation by two members of the team followed by a meeting in which categories were agreed upon and liminal cases were discussed. This was used to analyse the data generated in response to this question, resulting in Table 5-7. All answers were categorised by the team during this meeting. The tally separates attendees and non-attendees of Tutorial 7.

Total	**Attended**	**did not attend**	**1c**
7% (5)	9% (4)	5% (1)	c, calc point(s) + global
16% (11)	13% (6)	23% (5)	c, calc points
9% (6)	7% (3)	14% (3)	c, global
49% (33)	57% (26)	32% (7)	c, no explanation
9% (6)	9% (4)	9% (2)	no answer
10% (7)	7% (3)	18% (4)	wrong, no explanation
100% (68)	**100% (46)**	**100% (22)**	all
Total	**Attended**	**did not attend**	**1d**
22% (15)	24% (11)	18% (4)	good
15% (10)	13% (6)	18% (4)	good but no asymptote
21% (14)	17% (8)	27% (6)	incorrect, straight line
16% (11)	13% (6)	23% (5)	no answer
18% (12)	22% (10)	9% (2)	other
9% (6)	11% (5)	5% (1)	poor, just slope at (1,1)
100% (68)	**100% (46)**	**100% (22)**	all

A general inductive analysis was conducted on all interview data. This analysis generates categories from large amounts of qualitative data. Coding is used on the raw text during the process. An excerpt from Interview 17-11 that was subjected to this analysis is shown below.



So, were these tutorials different, visibly different, as a student, to other types that you do in maths?

To be honest, they were really good. 'Cause we did loads of questions and we did them so...like, the ones that I go to less of would be the tutorials where I'm not learning as much, you know? No but I did think these were really good.

So you were able to get stuff from these?

Yeah.

Brilliant. What do you think was the key contributing factor to that? What about them was best suited to you as a learner?

The style, I suppose. As in, we did questions, and we didn't rush through things, we always made sure that we knew one thing before we went onto the next. Obviously we never got stuck on anything as such, but we'd make sure things were known nearly inside out before we went onto the next thing, and that was...and we did loads of questions as well, but also theory, kind of, so yeah, that helped.

You - putting words in your mouth, but agree or disagree, there was a higher emphasis on understanding in these tutorials than others?

Definitely.

That's what you mean by the theory behind things.

Yeah. A bit more context.

Do you think that benefits...I think there could've been tutorials that I put together that would allow you do very very well in your exam, but would've maybe lacked that understanding. Do you think that way versus the way you've studied them, the longevity would be the same, would you still understand the same amount in time? So, imagine next semester you've a module on ODEs-

Yeah.

Do you feel better suited as a result of the way you've learned them this time?

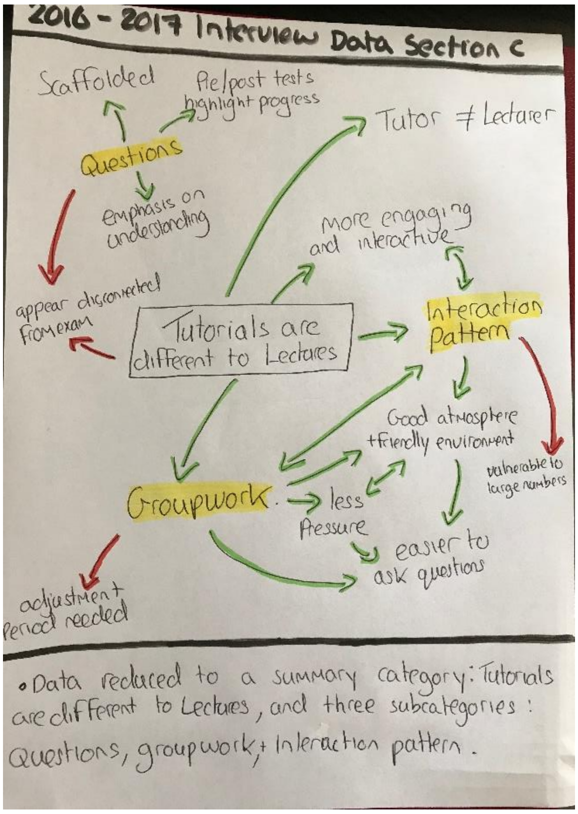
Oh yeah, honestly I would, I'm not saying that the question has tempted to be set up, but you know that, because say for...like I was saying, calculus or something, I'd my exam the other day and I thought it went really well. If I was to go back...if I was to do calculus even in May, the exam, I'd have to put in just as much study as I did for this one, maybe a tiny bit less, whereas if I'm...after I used differential, if I was to do it again in May, the amount of study I'd have to do would be definitely a lot less than what I have to do for this, because I'd remember things more.

And do you think that…

'Cause even now, these things, to be honest, I haven't done differentials since before Christmas, and it's all kind of coming back to me here.

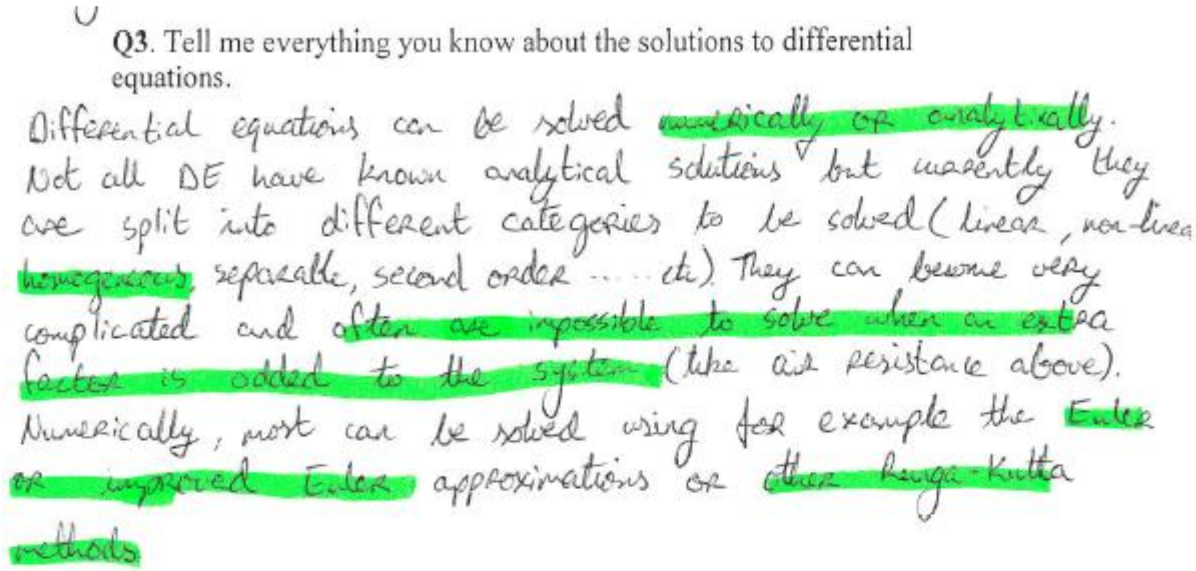
This analysis began by preparation of the data files and close reading of the text. The author then identified frequent themes, such as those in the margin above. Each of these themes are seen in Figure 5.10, with the exception of *long-term benefit of guided-inquiry*, which appears in Table 6.2.

With respect to the data from 2016-2017, the initial categorisation resulted in 6 primary categories (or themes): different to lectures, questions, interaction pattern, groupwork, exam, atmosphere. The next step in the procedure was to revise and refine categories. When this was done, a hierarchical system for the categories was adopted, resulting in a primary category and three sub categories. One of the initial categories (*tutorials are different to lectures*) emerged as common to all coded data, and the remaining five categories were reduced to three subcategories (*groupwork, questions, interaction pattern*). One of the advanced revisions of the visualisations in shown below.



From this stage, other revisions were carried out that eventually resulted in Figure 5.9. An example of this would be the repositioning of the label *tutor is not the lecturer,* relabelling *tutorials are different to lectures,* and an omission of the label *easier to ask questions.* This was done because the number of times these labels were mentioned was below a threshold, and the labels were deemed less significant than the others in the visualisation.

Data from the Diagnostic Survey was also analysed using this approach. Table 3-3 was generated when this approach was applied to student responses to Q3. A student answer that has undergone coding shown below. This answer contributed four terms to the table: Numerically/analytically; Homogeneous/non-homogeneous; Not all differential equations have solutions; Different methods e.g. Euler/Runge-Kutta.



Q3. Tell me everything you know about the solutions to differential equations.

Differential equations can be solved numerically or analytically. Not all DE have known analytical solutions but currently they are split into different categories to be solved (linear, non-linea homogeneous, separable, second order ..... etc) They can become very complicated and often are impossible to solve when an extra factor is added to the system (like air resistance above). Numerically, most can be solved using for example the Euler or improved Euler approximations or other Runge-Kutta methods

Appendix H: Research ethics

PLAIN LANGUAGE STATEMENT

This project aims to investigate the difficulties students have with differential equations in physics. The project is being carried out by Diarmaid Hyland, Dr Paul van Kampen, and Dr Brien Nolan. The objectives of the project are to understand and address the difficulties that physics students have in applying their knowledge and skills relating to differential equations.

If you decide to participate in the study, we will ask you to complete pre- and post-tests for the tutorials you attend. You may also be asked to take part in group interviews. It is designed to help us identify areas of interest for our study and will in no way affect your grade within this or any other module. Please understand that taking part in this study is completely voluntary and that complete anonymity is maintained throughout the process. All raw data collected from the participants will be coded to ensure confidentiality and protected according to national (Data Protection Acts 1998 & 2003) and international legislation (EU Directive 95/46/EC). All students who have participated in the project will receive access to the reports produced.

The data will be collected and analyzed by the researchers only. When we complete our work with the data, it will be disposed of in the correct manner ensuring the participants' privacy remains intact.

If you decide to participate in the study, you will ensure that all students who undertake modules dealing with differential equations in the future will have benefited from your feedback. Your data will form a central part of this research project and will hopefully lead to an improvement in how differential equations are taught at third level.

Finally, we would like to highlight that there are no risks of taking part in this study. Also, participants can change their mind at any stage and withdraw from the study.

Contact details for further information (including REC contact details)

For more information about the project: email Diarmaid.hyland3@mail.dcu.ie

If participants have concerns about this study and wish to contact an independent person, please contact The Secretary, Dublin City University Research Ethics Committee, c/o Office of the Vice-President for Research, Dublin City University, Dublin 9. Tel 01-7008000

DUBLIN CITY UNIVERSITY

Informed Consent Form

Research Study Title: Investigating students' difficulties with differential equations in physics.

The purpose of the research is to determine the degree to which different difficulties physics students have with differential equations contribute to the problems that arise in using these in physics.

Participant – please complete the following (Circle Yes or No for each question)

I have read the Plain Language Statement (or had it read to me) *Yes/No*

I understand the information provided *Yes/No*

I have had an opportunity to ask questions and discuss this study *Yes/No*

I have received satisfactory answers to all my questions *Yes/No*

I am aware that involvement requires me to complete pre- and post-test and/or interviews *Yes/No*

Please understand that taking part in this study is completely voluntary and that complete anonymity is maintained throughout the process. All raw data collected from the participants will be coded to ensure confidentiality and protected according to national (Data Protection Acts 1998 & 2003) and international legislation (EU Directive 95/46/EC). All students who have participated in the project will receive access to the reports produced.

I have read and understood the information in this form. My questions and concerns have been answered by the researchers, and I have a copy of this consent form. Therefore, I consent to take part in this research project

Participants Signature: ______________________

Name in Block Capitals: ______________________

Witness: ______________________

Date: ______________________

Appendix I: The worksheets

1. I-1: Worksheet 1 – Calculus review
2. I-2: Worksheet 2 – Meaning of the derivative
3. I-3: Worksheet 3 – DEs and their solutions
4. I-4: Worksheet 4 – Separable ODEs
5. I-5: Worksheet 5 – First order linear ODEs
6. I-6: Worksheet 6 – Practicing solving first order ODEs
7. I-7: Worksheet 7 – Direction fields
8. I-8: Worksheet 8 – Modelling 1
9. I-9: Worksheet 9 – Modelling 2
10. I-10: Worksheet 10 – Modelling 3 (structured version)
11. I-11: Worksheet 10 – Modelling 3 (unstructured version)
12. I-12: Worksheet 11 – Second Order ODEs with constant coefficients
13. I-13: Worksheet 12 – More second order ODEs
14. I-14: Worksheet 13 – Practicing solving second order ODEs
15. I-15: Worksheet 14 – Solutions to second order ODEs
16. I-16: Worksheet 15 – Modelling with second order ODEs

I-1: Worksheet 1 – Calculus review

Introduction

Today you will review calculus problems that you will meet later in the course. Studying differential equations requires a fluency with differentiation, integration, and other mathematics that you will look at in this tutorial.

Activity 1

Discuss each of the following problems in your group. Describe how you would attempt to solve each question and do so if possible. Spend no more than 10 minutes on each question. There are 4 questions in total.

1. Differentiate each of the following functions with respect to x:

   a. $f(x) = \dfrac{x^3 + 7x + 4}{2x - 3}$;

   b. $f(x) = -\cos\left(x^{5/2}\right)$.

2. Calculate each of the following indefinite integrals:

a. $\int \left(x^2 - 5x^{-5}\right) \mathrm{d}x$;

b. $\int \cos(3x) \mathrm{d}x$;

c. $-\int e^{-2x} \mathrm{d}x$.

3. Calculate each of the following indefinite integrals:

a. $\int \cos^3(3x)\mathrm{d}x$;

b. $\int (3 + 3\tan^2 x)\mathrm{d}x$.

4. Calculate each of the following indefinite integrals:

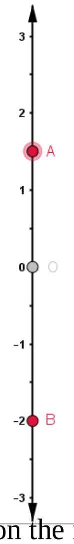
a. $\int \frac{2x+5}{\sqrt{9-x^2}} \mathrm{d}x$;

b. $\int \frac{1}{\sqrt{1-x^3}} \mathrm{d}x$.

I-2: Worksheet 2 – Meaning of the derivative

Introduction

This tutorial is about the meaning of a derivative. You will use the graphs of several journeys to improve your understanding of derivatives which will help you with your study of differential equations.



You will be working with displacement-time graphs. The *displacement* of a point is the signed distance of the point from the origin O of the line on which the point lies. *Signed* means that the displacement may be positive or negative, depending on which side of the origin the point lies. Thus in the diagram, the point A has displacement +1.5, and the point B has displacement -2.

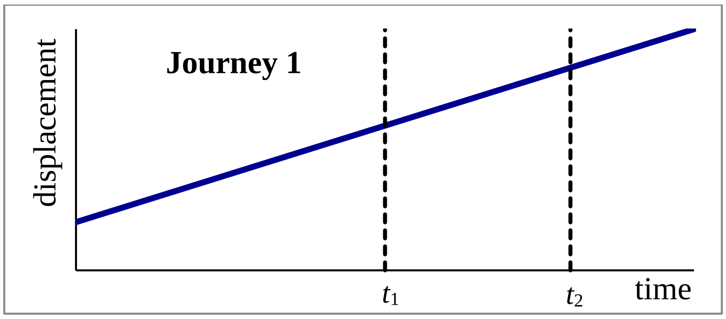
Activity 1

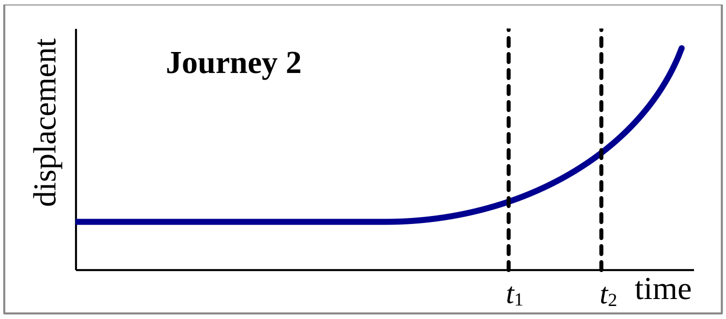
1. In your groups, discuss each of the displacement-time graphs (Journeys 1-4) shown on the next page, and describe a journey that could be represented by each of these graphs.
2. Answer each of the following questions individually before discussing your answers with your group:
    a. Do any of these journeys represent motion at constant velocity? If so, which do/don't and why?
    b. Do any of these journeys represent accelerated motion? If so, which journeys do/don't and why?
    c. Are all of these journeys plausible? Defend your opinion.
    d. Is it possible to cross either axis? Defend your opinion.
    e. Who travelled fastest?
    f. How fast are the objects represented in Journey 3 and Journey 4 travelling at $t = t_1$?

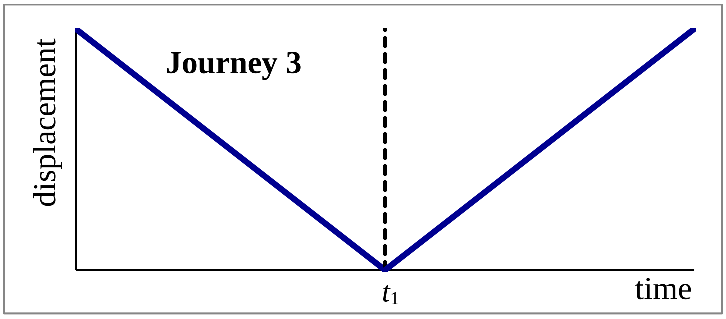
Discuss your answers with a tutor.

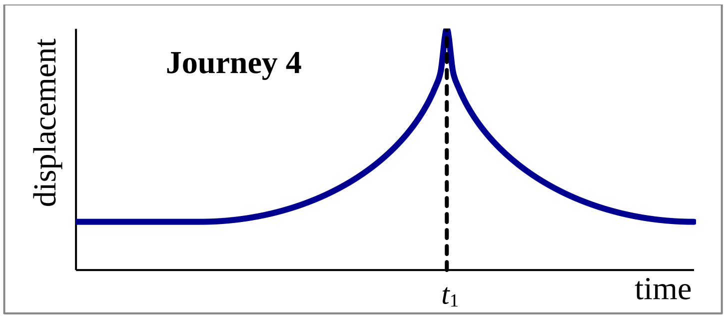
3. Revisit Journey 1 and discuss the velocity:
    a. over the entire journey;
    b. over the time interval from t_1 to t_2;
    c. at the instant $t = t_1$.

Discuss your answers with a tutor.


Journey 1
displacement
time
t1
t2


Journey 2
displacement
time
t1
t2


Journey 3
displacement
time
t1


Journey 4
displacement
time
t1

4. Revisit Journey 2 and discuss the velocity:
   a. over the entire journey;
   b. over the time interval from t_1 to t_2;
   c. at the instant $t = t_1$. What is the velocity at $t = t_1$? How did you arrive at your answer? How accurate do you think your answer is?

Discuss your answers with a tutor.

Defining the derivative

The slope of the tangent line at $P = (t, f(t))$ is the limit of the change in the function divided by the corresponding change in the independent variable as that change approaches 0:

$$\text{slope of tangent at } (t, f(t)) = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}. \qquad (1)$$

When this limit exists, it is called it the derivative of f at t, and we write this limit as $f'(t)$. We say that f is *differentiable* at t, or (equivalently) that f has a *derivative* at t.

5. Discuss the derivative at $t = t_1$ for Journey 3 and Journey 4.

I-3: Worksheet 3 – DEs and their solutions

Introduction

Todays tutorial is about differential equations: what they are and what constitutes a solution to a differential equation.

Given a function *y* of a variable *x*, a differential equation is an equation that involves one or more of the derivatives of *y*.

Activity 1

1. Using the definition of a differential equation, examine the following equations and decide in your groups whether or not they are differential equations. Justify your decision in each case.

   a. $\frac{\mathrm{d}y}{\mathrm{d}x} = xy + 15;$

   b. $y^2 + x^2 = 5;$

   c. $x\frac{\mathrm{d}^3y}{\mathrm{d}t^3} - \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 0;$

   d. $m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - \gamma v\,.$

Discuss your answers with a tutor.

Activity 2

1. For the following questions, discuss whether or not the text given can be described using a differential equation and defend your answer.

   a. Scientists have observed that radioactive materials have an instantaneous rate of decay (that is, a rate of decrease) that is proportional to the amount of material present.

   b. The growth rate of a population of bacteria is constant.

   c. Torricelli's law implies that the *time rate of change* (the rate of change with respect to time *t*) of the volume *V* of water in a draining tank is proportional to the square root of the depth *y* of the water in the tank.

Discuss your answers with a tutor.

Activity 3

1. Given that $y = e^x + 2x$, show that:

   a. $\frac{dy}{dx} = e^x + 2$;

   b. $\frac{dy}{dx} - y = 2(1 - x)$.

2. Are either of the above differential equations? Justify your answer.

3. What does $y = e^x + 2x$ represent in this context?

Discuss your answers with a tutor.

Definition

A solution to a differential equation is any function $y(x)$ that satisfies the differential equation in question. That is, when we calculate the derivatives of y, and substitute for y and its derivatives, the left-hand side is equal to the right-hand side for all values of x in the domain of y.

Activity 4

1. Examine the following functions. Decide in your groups whether or not they are solutions to the differential equations that accompany them, and justify your decision. Throughout these problems, primes denote derivatives with respect to x.

   a. Is the function $y = x^3 + 7$ a solution to the differential equation $y' = 3x^2$?

   b. Is the function $x = 7t^2 + 3t$ a solution to the differential equation $\frac{dx}{dt} - 4t = x - 1$?

   c. Is the function $y = -e^{-3x}$ a solution to the differential equation $y'' = 3y'$?

   d. Is the function $y = (1 + x^2)^{-1}$ a solution to the differential equation $y' + 2xy^2 = 0$?

2. For what values of r are the following statements true?

   a. The function $y = x^2 - rx$ is a solution to the differential equation $y' = 2x + 10$.

   b. The function $y = Ce^{x^r}$ is a solution to the differential equation $\frac{dy}{dx} = 2xy$.

Discuss your answers with a tutor.

I-4: Worksheet 4 – Separable ODEs

Introduction

Separable equations are a form of first order differential equation that can be written in the form $N(y)\mathrm{d}y = M(x)\mathrm{d}x$. That is, you can isolate the terms of each variable on opposite sides of the equation. Today you will examine their structure in further detail. You will also practice recognizing and solving this type of diffential equation.

Activity 1

2. For each of the following equations:
   a. identify whether or not it is a differential equation;
   b. if so, identify the order of the differential equation;
   c. state whether or not the differential equation is linear;
   d. state whether or not the differential equation is separable.

   Explain how you arrived at your answer in each case.

item	equation	ODE?	Order?	linear?	separable?
1.	$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 0$				
2.	$\frac{\mathrm{d}y}{\mathrm{d}x} = y \sin x$				
3.	$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}(x-y) = \frac{\mathrm{d}y}{\mathrm{d}x} - y$				
4.	$\frac{\mathrm{d}y}{\mathrm{d}x} = (64xy)^{1/3}$				
5.	$y'' = xy^2$				
6.	$y(x^2+1)\tan y = x$				
7.	$\frac{\mathrm{d}y}{\mathrm{d}x} - 2x = 0$				
8.	$y' = 1 + x + y + xy$				
9.	$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{k}\right)$				

Discuss your answers with a tutor.

3. Separate and solve (differential) equations 1, 2, 4, and 7.

Activity 2

1. Each of the following paragraphs describes a differential equation. Write down this differential equation and separate and solve where possible.

    a. The rate of change of a population is directly proportional to the size of the population in question.

    b. According to Newton's law of cooling, the time rate of change of the temperature $T(t)$ of a body immersed in a medium of constant temperature T_0 is proportional to the difference $T_0 - T$.

    c. The amount $A(t)$ of a certain drug in the bloodstream, measured by the excess over the naturally occurring level of the drug, will decline at a rate proportional to the current excess amount.

2. Using the following information, find an exact solution to each specific problem below outlined in general in Question 1 above.

    a. A certain city had a population of 25,000 in 1960 and a population of 30,000 in 1970. Assume that its population is growing exponentially. What population can its town planners expect the city to have in 2020?

    b. A cake is removed from an oven at 180 °C and left to cool at room temperature (20 °C). After 30 minutes the temperature of the cake is 70 °C. When will its temperature be 50 °C?

    c. Suppose that sodium pentobarbital is used to anaesthetize a dog. The dog is anesthetized when its bloodstream contains at least 45 milligrams (mg) of sodium pentobarbital per kilogram of the dog's body mass. Suppose also that sodium pentobarbital is eliminated exponentially from the dog's bloodstream with a half-life of 5 hours. What single dose should be administered in order to anaesthetize a 25 kg dog for 1 hour?

I-5: Worksheet 5 – First order linear ODEs

Introduction

Today you will work in your groups to consolidate everything you have learned about first order linear ODEs to date.

Activity 1

Go to the expert group relating to your group letter and complete the task. You have 12 minutes to complete this activity. When the 12 minutes elapse return to your original group for Activity 2.

The task for expert group A is as follows:

1. Explain what "order" means in relation to differential equations and how to identify the order of a differential equation. Create a list of examples for which your explanation works. Include some straightforward examples for other members of your home group to practice with, and some more complex examples.
2. Explain what is meant by "linear" and "non-linear" in relation to differential equations and how to identify whether a differential equation is linear or non-linear. Create a list of examples for which your explanation works. Again include some straightforward examples for other members of your home group to practice with, and some more complex examples.

The task for expert group B is as follows:

1. Explain what is meant by "separable" and "non-separable" in relation to differential equations and how to identify whether a differential equation is separable or non-separable. Create a list of examples for which your explanation works. Include some straightforward examples for other members of your home group to practice with, and some more complex examples.
2. Explain what is meant by "linear" and "non-linear" in relation to differential equations and how to identify whether a differential equation is linear or non-linear. Create a list of examples for which your explanation works. Again include some straightforward examples for other members of your home group to practice with, and some more complex examples.

The task for expert group C is as follows:

1. Create an algorithm that can be used to solve any first order linear ordinary differential equation. Ideally the other members of your home group should be able to use your algorithm without any further instruction.

Activity 2

3. Return to your home group.

   In turn, explain to your group what your task was, and how you answered it. Use the A1 sheet provided to display the answers to all expert group tasks and ensure that each member understands each solution. You have 8 minutes to complete this activity.

Activity 3

1. Identify which of the following equations are first order ordinary differential equations and solve to obtain the general solution.

   a. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = xy$

   b. $\dfrac{\mathrm{d}y}{\mathrm{d}x} + y\cos x = x^2$

   c. $x\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2y = -4\dfrac{\mathrm{d}y}{\mathrm{d}x}$

   d. $\sqrt{a^2 + b^2} = 10\sin\phi$

   e. $y' + ay = be^{-\omega t}$

   f. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -2xy^2$

   g. $\dfrac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \dfrac{N}{k}\right)$

   h. $\dfrac{\mathrm{d}T}{\mathrm{d}t} = -k(T - A)$

   i. $m\dfrac{\mathrm{d}v}{\mathrm{d}t} = mg - \gamma v$

I-6: Worksheet 6 – Practicing solving first order ODEs

Introduction

Today you will practice categorizing, separating and solving various first order linear ordinary differential equations. You are not expected to finish the entire worksheet within the hour, but are advised to complete it in your independent study time.

Activity 1

1. For each of the following equations:
   a. identify whether or not it is a differential equation;
   b. if so, identify the order of the differential equation;
   c. state whether or not the differential equation is linear;
   d. state whether or not the differential equation is separable.

   Explain how you arrived at your answer in each case.

item	equation	ODE?	Order?	linear?	separable?
1.	$\frac{dy}{dx} + 2xy = 0$				
2.	$\frac{dy}{dx} = y \sin x$				
3.	$\frac{d^2y}{dx^2}(x - y) = \frac{dy}{dx} - y$				
4.	$\frac{dy}{dx} = (64xy)^{1/3}$				
5.	$y'' = xy^2$				
6.	$y(x^2 + 1)\tan y = x$				
7.	$\frac{dy}{dx} - 2x = 0$				
8.	$y' = 1 + x + y + xy$				
9.	$\frac{dN}{dt} = rN\left(1 - \frac{N}{k}\right)$				

Activity 2

1. Separate the following ODEs:

   a. $\frac{dy}{dx} = xy$;

   b. $x^2 \frac{dy}{dx} = y^2$;

   c. $x \frac{dx}{dt} = (x^2 + 1)(t^2 + 1)$;

   d. $x \frac{dy}{dx} = \frac{1}{x}$;

   e. $x \frac{dy}{dx} + \frac{y}{\ln|x|} = 0$.

Activity 3

1. Find the general solution to the following differential equations (some are separable and some are not):

a.	$\frac{dy}{dx} = xy$	i.	$x \frac{dy}{dx} = \frac{1}{x}$
b.	$\frac{dy}{dx} - 2y = x$	j.	$z^2 \frac{dz}{dx} = xz^3 + x$
c.	$\frac{dy}{dx} + \frac{y}{x} = 1$	k.	$y' = -3y + e^{-2x}$
d.	$x^2 \frac{dy}{dx} = y^2$	l.	$xy' + 2y = 2x^2 - 3x$
e.	$\frac{dx}{dt} = tx$	m.	$xy' + \frac{1}{2}y = x$
f.	$\frac{dx}{dt} = -x + t$	n.	$\frac{dy}{dx} = y \tan x + \sec x$
g.	$x \frac{dx}{dt} = (x^2 + 1)(t^2 + 1)$	o.	$x \frac{dy}{dx} + \frac{y}{\ln|x|} = 0$
h.	$\frac{dx}{dt} + \frac{e^t}{e^t + 1} x = e^{2t}$	p.	$x \frac{dy}{dx} = -xy - y + 6$

2. Find the particular solution to the differential equations from Question 1 using the information provided below:

   a. $y(0) = 2$;

   b. $y(0) = 0$;

   c. $y(1) = 3$;

   d. $y(1) = 1$;

   e. $x(0) = 2$;

   f. $x(0) = 0$;

   g. $x(0) = 2$;

   h. $x(0) = \frac{1}{6}$;

   i. $y(2) = 4$;

   j. $z(0) = 0$;

Further reading

Useful links when revising first order linear ordinary differential equations:

- https://ocw.mit.edu/index.htm
- http://tutorial.math.lamar.edu/Classes/DE/Linear.aspx
- http://www.sosmath.com/diffeq/first/lineareq/lineareq.html
- https://www.khanacademy.org/math/differential-equations/first-order-differential-equations

Useful books available in DCU library:

- http://www.sciencedirect.com/science/handbooks/18745717/3 (E-book)
- E. Kreyszig, "Advanced Engineering Mathematics" (recommended reading on course descriptor)

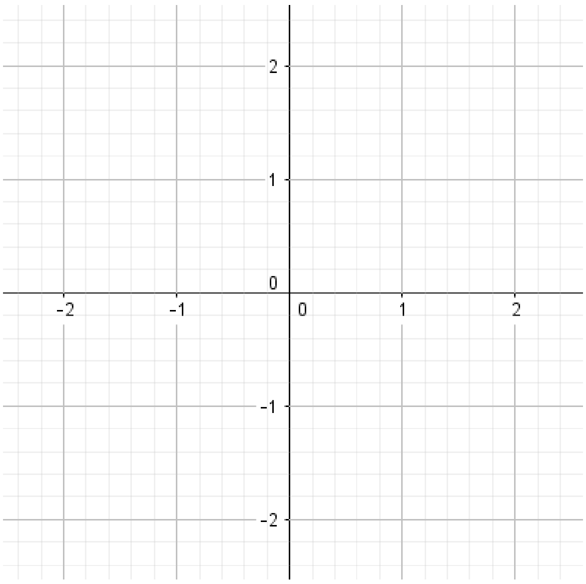
I-7: Worksheet 7 – Direction fields

Introduction

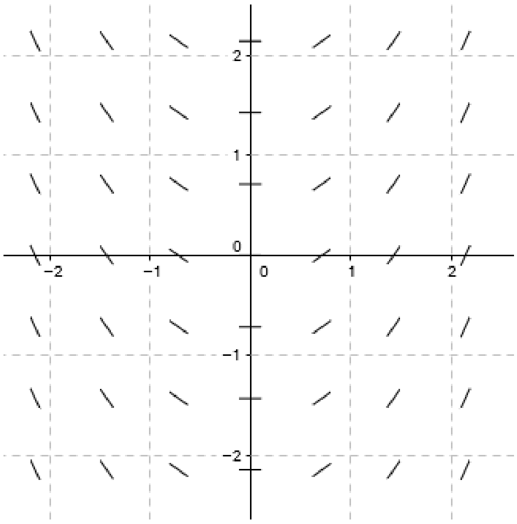
This tutorial focuses on *direction fields* and graphing solutions to differential equations. A direction field (also called *slope field*) is a graphical representation of the solutions of a first order ODE. It is useful because it can be created without solving the ODE analytically.

Constructing a direction field

Consider the first order ODE $\frac{dy}{dx} = x$, and imagine that you do not know how to solve it. To get a feel for the behaviour of the solution, you may evaluate $\frac{dy}{dx}$ at a variety of points (x, y) and use each value as the slope of the tangent to the solution curve at (x, y). Starting with a blank pair of axes, you may populate the graph by evaluating the slope at different points, and plotting the slopes as short line segments centred on each point.

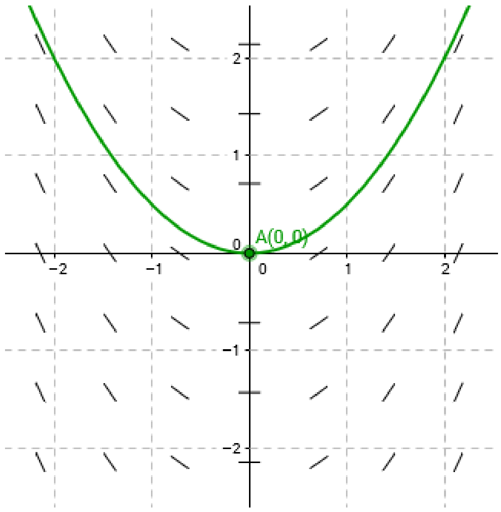


For example, let's evaluate $\frac{dy}{dx}$ at point $(0,0)$. At this point $\left.\frac{dy}{dx}\right|_{x=0} = \left.x\right|_{x=0} = 0$, so we draw a line segment with slope 0 at the point $(0,0)$. At the point (1,1), $\left.\frac{dy}{dx}\right|_{x=1} = \left.x\right|_{x=1} = 1$, so draw a line segment with slope 1 at the point (1,1). When you evaluate each of the points and draw in the line segments of appropriate slope you begin to see a picture of the behaviour of the solutions.

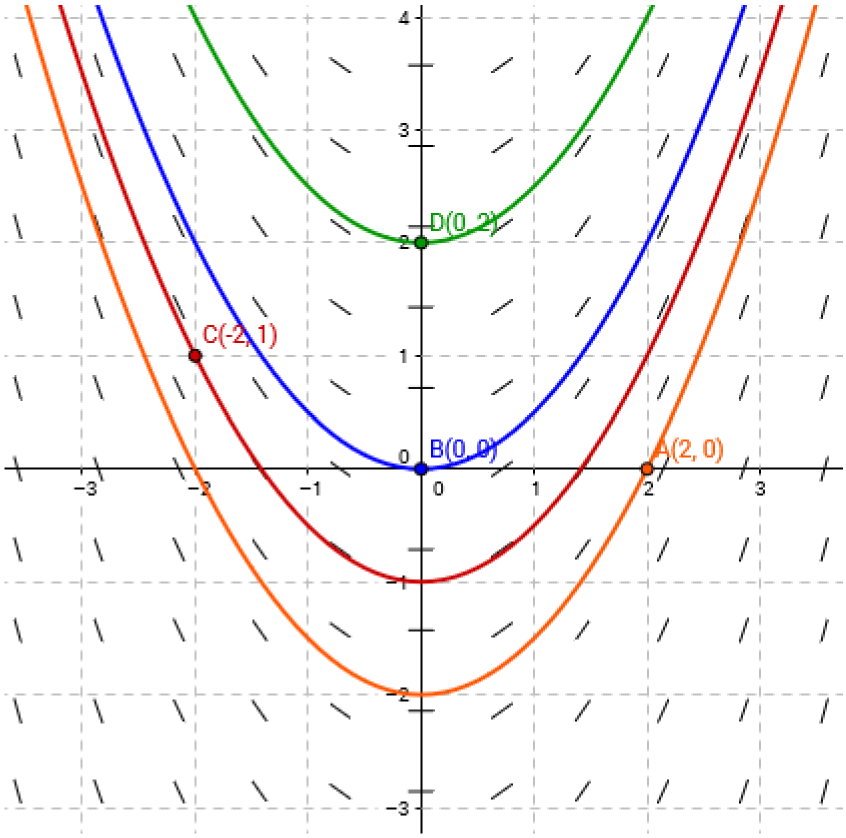


A large number of such line segments are shown at left. If you draw curves tangent to these segments, each curve satisfies the relation between the derivative and the function given by the ODE. Each curve therefore represents a particular solution to the ODE, and all curves together represent a family of functions that are the general solution to the ODE.

As an example, starting at the point $(0,0)$ you can work forwards and backwards to construct the graph of the solution to the differential equation $\frac{dy}{dx} = x$ that passes through the point $(0,0)$, as shown at right.



Four particular solutions to the first order differential equation $\frac{dy}{dx} = x$ are shown in the direction field below.

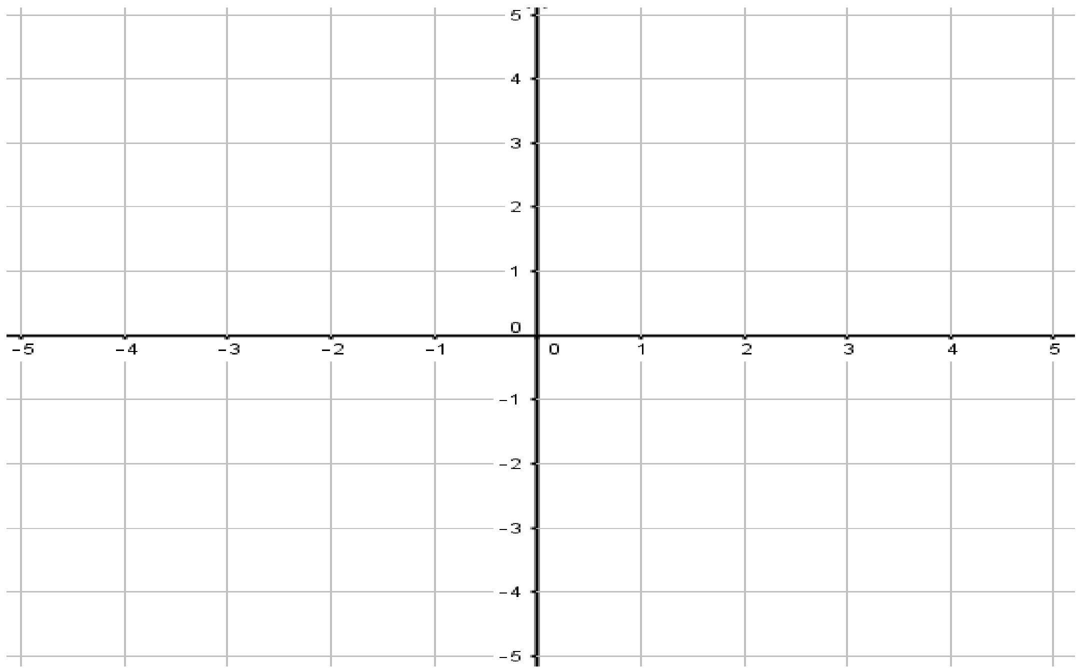


In this case, you could easily have solved the ODE, and obtained $y(x) = \frac{1}{2}x^2 + C$ as the general solution. The four graphs above show solutions for four different values of *C*. The direction field method is at its most useful when it is difficult or even impossible to find analytical solutions to a given ODE. In the remainder of the tutorial, you will work with direction fields in small groups.

Activity 1

1. Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = y + 2$.

    a. Use the table and axes provided to draw a direction field for this ODE. You may divide the work among the members of your group but every student must complete the direction field on their tutorial. There are 121 points to consider (but it's not as nasty as it looks!)



x	y	$\frac{dy}{dx}$

x	y	$\frac{dy}{dx}$

x	y	$\frac{dy}{dx}$

x	y	$\frac{dy}{dx}$

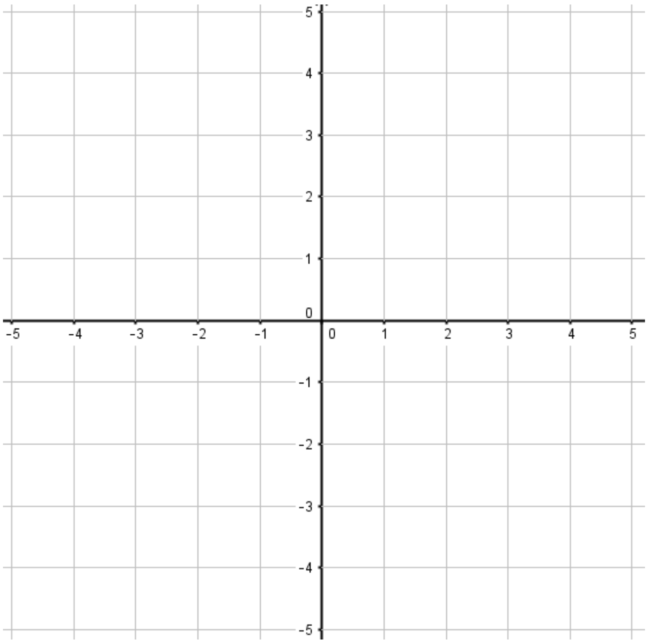
    b. How does the value of x affect the slope at each point?

    c. Draw the solution to the differential equation that passes through each of the following points: (0,0); (1,-3); and (-2,1) on the same diagram as your direction field.

2. Now consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.

    a. Use the table and axes provided to draw a direction field for this ODE. When you have finished your direction field compare it to that of others within your group and resolve any discrepancies.

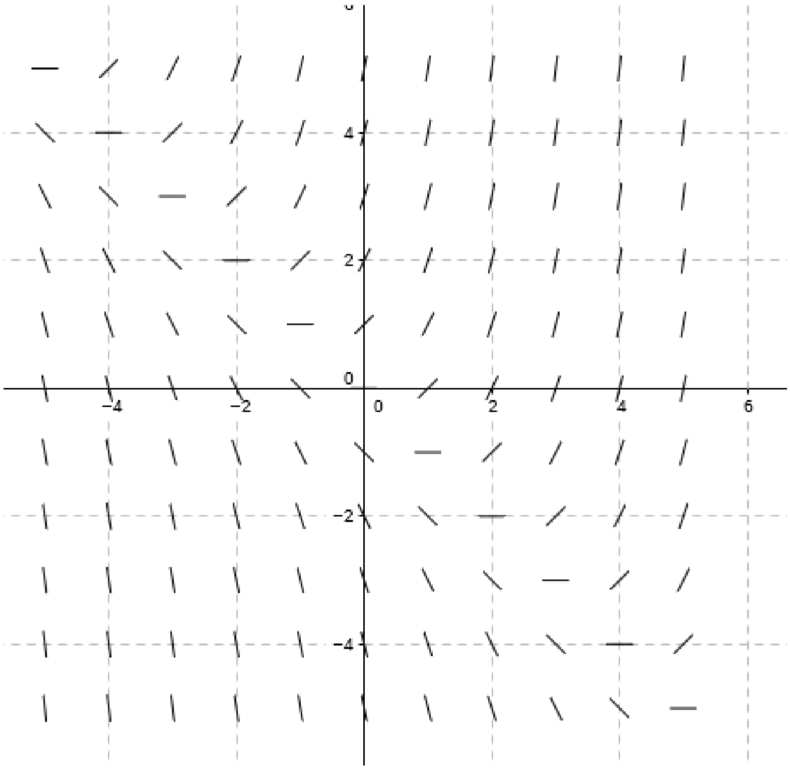
x	y	$\frac{dy}{dx}$



b. Describe the differences between this direction field and the direction field constructed in Question 1.

3. Which of these differential equations is described in the direction field below?

- $\frac{dy}{dx} = -\frac{x}{y}$;
- $\frac{dy}{dx} = x - y$;
- $\frac{dy}{dx} = y - x$;
- $\frac{dy}{dx} = x + y$;
- $\frac{dy}{dx} = \frac{x}{y}$.



Activity 2

1. Using the Geogebra file on the MS225 Loop page, construct a direction field with the following properties: the solution curves whose tangents are represented by the direction field
   - depend on both *x* and *y*;
   - are increasing for all $x > 5$;
   - have a vertical asymptote at $x = 2$.

Extension Question

1. Given that $y' = \frac{-y}{x^2 + y^2}$:
   a. Sketch the direction field for $-5 < x < 5,\ -5 < y < 5$.
2. Now consider the solution *y*(*x*) through (*x*,*y*)=(0,1) (i.e. the solution with initial value y(0)=1).
   a. Use the direction field plot to show that $y(x) > 0$ for $x > 0$.
   b. Use the analytical form of the ODE to explain that $y(x)$ is decreasing for all *x*.

I-8: Worksheet 8 – Modelling 1

Introduction

In previous tutorials you considered population models in which the rate of change of a population (i.e., the growth rate) is directly proportional to the size of the population in question. This may be expressed as follows:

$$\frac{dN}{dt} = rN, \qquad (1)$$

where r is a constant.

Activity 1

Classify and solve differential equation (1).

Activity 2

1. Consider a population with an initial total of 1,000,000 people. The number of people in the population was recorded over a period of time. After a certain amount of time had passed the population had risen to 2,000,000 people. Assuming that the population follows the model represented by (1), which of the curves shown in the graph below best describes how the number of people in a population changed over time? (Assume that the death rate is 0.)

Discuss your answers with a tutor.

2. What are the units of r in equation (1)?

3. Use the Excel worksheet provided to investigate how changing the value of *r* affects the population (*N*) and its growth rate (d*N*/d*t*) over time (*t*). The Excel file contains graphs of population vs time, growth rate vs time, and growth rate vs ppopulation. Cell B2 contains a value for *r* that you will vary during this investigation.

    a. Increase the value of *r* gradually from 0.2 to 2.0. Observe what happens to each graph across this range.

    b. **Without changing *r***, predict what would happen to each graph if you were to increase *r* above 2.0. Justify your prediction. Only then use Excel to check whether or not your prediction is correct.

    c. Make a statement about the behaviour of the graphs when *r* is greater than 2.0 and increasing.

    d. Decrease the value of *r* gradually from 0.2 to 0.02. Observe what happens to each graph across this range.

    e. **Without changing *r***, predict what would happen to each graph if you were to decrease *r* below 0.02 but greater than 0. Justify your prediction. Only then use Excel to check whether or not your prediction is correct.

    f. Make a statement about the behaviour of the graphs when $0 < r < 0.02$.

4. You have investigated the behaviour of the population for positive values of *r* and can predict how increasing or decreasing its value affects the three graphs. Your next task is to continue this line of investigation for negative values of *r*.

    a. Make a statement about the behaviour of the graphs when *r* is negative and decreasing.

    b. Make a statement about the behaviour of the graphs when *r* is negative and increasing.

5. Use the Excel file to investigate what would happen to the population over a long time for each of the following values of *r*:

    - $r = 5.0$;
    - $r = -5.0$;
    - $r = 0.0000000001$;
    - $r = 0$.

    Summarize the effect of the value of *r* on each graph having completed your investigation.

6. Given that $\frac{dN}{dt} = rN$, how could we have predicted that the graph of growth rate vs population would have been linear? What can we say about the value of its slope?

Extending the model

- The birth rate of a given population is the ratio of the number of live births to the total population size over a specified period of time.
- The death rate of a given population is the ratio of the number of deaths to the total population size over a specified period of time.

Activity 3

1. Many models of population dynamics use

$$r = r_b - r_d, \tag{2}$$

where r_b and r_d are the birth rate and death rate respectively. Is this a realistic way of viewing r?

2. Suppose an initial population of 1,000,000 people can be modelled by equation (1) with an r value of 0.01. This results in a population that grows slowly over time. Write down what would happen to the population in the following instances:
    a. A disease results in the death rate of the population getting closer to, but not surpassing the birth rate.

    b. The birth rate increases as the death rate remains constant.

    c. The birth rate decreases below the death rate.

Discuss your answer with a tutor before progressing.

Activity 4

Write down your latest model for population dynamics.

I-9: Worksheet 9 – Modelling 2

Introduction

In the model used in last weeks tutorial, $r > 0$ will cause a population to grow exponentially, which becomes unrealistic on long time scales. At a certain point, a population reaches a limit at which it can survive sustainably before other factors such as limitations in food supply cannot sustain them. For this reason, a model for population change over time is often written as follows:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{k}\right), \tag{1}$$

Activity 1

1. Use equation (1) to describe how the population (N) changes when the population is much smaller than k.

2. Use equation (1) to describe how the population (N) changes when the population is close to, but still smaller than k.

3. Use equation (1) to describe how the population (N) changes when the population is equal to k.

4. Use equation (1) to describe how the population (N) changes when the population is greater than k.

5. Use your answers to questions 1-4 to give an interpretation of k.

6. Discuss how each of the following situations affect N and k:
   a. A better than expected crop yield results in an abundance of food.

   b. A severe drought causes a shortage of water within a population.

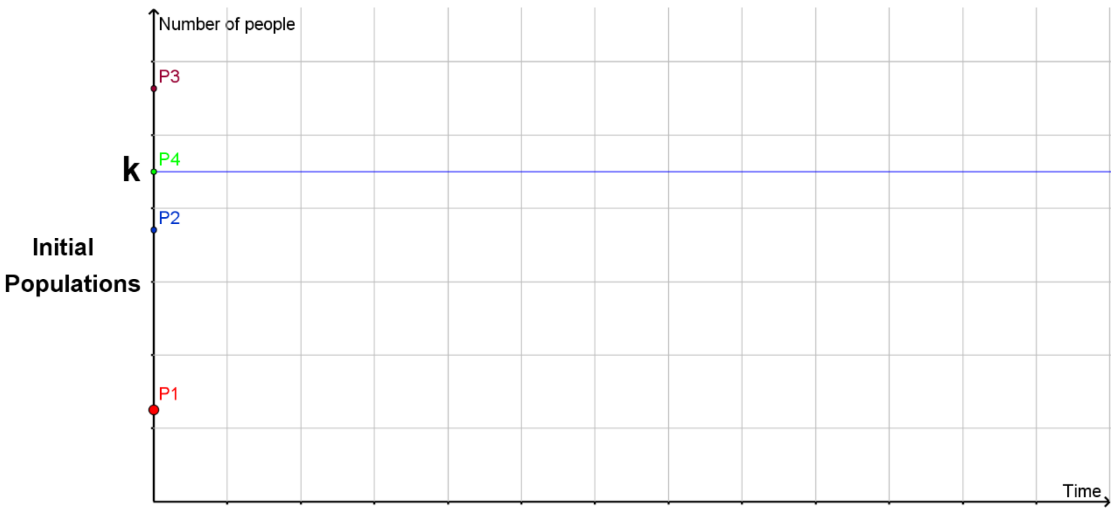
Activity 2

Complete the following questions, using the Excel file provided to support your answers:

1. Briefly describe what each graph represents.
2. Identify the initial population (N_0) and k on each graph if possible.
3. At what time is the population growth largest? Use evidence from the graphs to defend your answer.
4. How does changing r affect these graphs? Compare and contrast to how changing r affected the graphs in last week's tutorial.
5. Draw each of the following curves on the graph paper provided. Using the Excel file, keep k and r constant and adjust N_0 as follows:
   - $N_0 = 0.1k$;
   - $N_0 = 0.25k$;
   - $N_0 = 0.5k$;
   - $N_0 = 0.75k$;
   - $N_0 = k$;
6. Draw a horizontal line at $0.5k$. This splits the graph into two sections, from 0 to $0.5k$ and from $0.5k$ to k.
   a. How does dN/dt behave in each of these two sections?
   b. What can you infer about dN/dt at $0.5k$?
7. Will the graphs change if N_0 is greater than k? If so, which ones and how?

Activity 3

Sketch the evolution of populations beginning at the initial populations P1, P2, P3, P4 indicated on the graph below. Use the Excel file provided if necessary. The value of k has been drawn already.



I-10: Worksheet 10 – Modelling 3 (structured version)

Introduction

Today each group will model a real life situation as accurately as possible. It will be important to draw on everything you learned in the two previous tutorials.

Activity 1

Complete the project assigned to your group. The projects are as follows:

1. **The policewoman:** The body of a homicide victim has been found and police want to estimate a time of death. To do this they need to know how the temperature of a body cools over time.

   Discuss the situation within your group and describing what you think would happen to the temperature T over time t.

   You may want to consider the following questions when modelling this situation:

   - What assumptions can we make to simplify the situation?
   - Will the body temperature, $T(t)$, increase or decrease after the person dies?
   - Will it increase or decrease at a constant rate?
   - What factor affects the increase or decrease?
   - Is your mathematical model based on the relationship between the change in temperature (dT/dt) and the above factor?
   - When discussing populations, we had a constant r that we considered to be the difference between birth rate and death rate. What could k be in this model?
   - Classify and solve the equation you have constructed. Can you evaluate the constant in front of the exponential?
   - How can you test your model to see if it is accurate? How accurate is it?
   - What does the graph of T v t look like?
   -

2. **The football:** A football is released at rest from the top of a building and falls towards the ground. Create a model that describes the velocity of this ball.

I-11: Worksheet 10 – Modelling 3 (unstructured version)

Introduction

Today each group will model a real life situation as accurately as possible. It will be important to draw on everything you learned in the two previous tutorials.

Activity 1

Complete the project assigned to your group. The projects are as follows:

3. **The policewoman:** The body of a homicide victim has been found and police want to estimate a time of death. To do this they need to know how the temperature of a body cools over time.
4. **Hell's kitchen:** To increase efficiency in their restaurant, the chef wants to minimise the amount of time food needs to be in the oven for. To do this they need to know how the temperature of an object increases over time when placed in an environment that is warmer than itself.
5. **Historic cooking:** A historic method of cooking involved heating stones in a fire before placing them in water to raise its temperature. This works on the principle that the temperature of the stone will decrease until it reaches thermal equilibrium with the water. Imagine a single stone being placed in a vast amount of water, such that the temperature of the stone decreases to the initial temperature of the water. Model this change in temperature over time.
6. **Rabbits:** A population of rabbits survives by eating grass in a field. The grass is limited, but sustains a certain number of rabbits. Foxes get introduced to the environment such that they survive by hunting the rabbits. Create a model that describes the population of rabbits.
7. **The football:** A football is released at rest from the top of a building and falls towards the ground. Create a model that describes the velocity of this ball.
8. **Skydiving:** Skydivers use parachutes to ensure they reach the ground at a speed that won't harm them. Create a model that describes the velocity of a skydiver from the moment they begin their jump until they effectively stop accelerating.

I-12: Worksheet 11 – Second order ODEs with constant coefficients

Introduction

This tutorial focuses on second order linear ODEs. We will recap on their structure, the various types that we will encounter, and how to solve them. Below is an extract from your lecture notes.

3. Second order linear ODEs

Recall. A second order ODE is linear if it can be written

$$y'' + p(x)y' + q(x)y = r(x)$$

If $r = 0$, this equation is homogeneous.

Superposition Principle. If y_1 and y_2 are solutions of

$$y'' + p(x)y' + q(x)y = 0,$$

so is $c_1y_1 + c_2y_2$ for any constants c_1, c_2.

Activity 1

1. In your groups, identify which of the following equations are second order linear ODEs and state whether or not they are homogeneous in each case. Throughout these problems, primes denote derivatives with respect to *t*.

equation	order?	linear?	homogeneous?	
a) $25y''+20y'+4y=0$				
b) $\frac{d^2y}{dt^2}+\frac{dy}{dt}+y=2ty^2$				
c) $\sin x+5y=2x\frac{dy}{dx}-x^2\frac{d^2y}{dx^2}$				
d) $y=4t\left(t\frac{d^2y}{dt^2}+\frac{dy}{dt}\right)-2y$				
e) $x^2\frac{d^2y}{dx^2}-\sin x\frac{dy}{dx}+y^{1/2}=0$				
f) $a(x)\frac{d^2y}{dx^2}+b(x)\frac{dy}{dx}+c(x)y=d(x)$				

2. Outline the difference between a Constant Coefficient and Euler-Cauchy type problem and revisit questions 1-6 above and identify which equations are constant coefficient, and which are Euler-Cauchy.

Activity 2

1. Work in your groups to generate examples of second order ODEs with the following characteristics in each case:
    a. ODE 1 is non-linear;

    b. ODE 2 is linear and homogeneous;

    c. ODE 3 is linear and non-homogeneous;

    d. ODE 4 is linear, homogeneous and has constant coefficients;

    e. ODE 5 is linear, non-homogeneous, and the coefficients are not constant.

Activity 3

1. Classify and find the general solution of the following equations:
    a) $y'' - 3y' + 4y = 0$
    b) $25y'' - 100y' + 100y = 0$
    c) $y = 4t\left(t\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{\mathrm{d}y}{\mathrm{d}t}\right) - 2y$
    d) $\frac{\mathrm{d}}{\mathrm{d}x}\left(4h^2\frac{\mathrm{d}h}{\mathrm{d}x} + 7h\right) - 18h = 0$
2. Find a second order linear ODE which has $y(t) = e^{-t}(\sin 3t + \cos 3t)$ as a solution.

I-13: Worksheet 12 – More second order ODEs

Introduction

In last week's tutorial, you looked at second order linear homogeneous ODEs and how to solve constant coefficient problems. Today you will practice solving Euler-Cauchy problems and extend your solution techniques to cater for inhomogeneous problems.

Activity 1

1. Solve the following second order linear ODEs. As before, primes denote derivatives with respect to *x*.

    a) $x^2y'' - 6xy' + 9y = 0$

    b) $x^2y'' - 5xy' + 4y = 0$

    c) $x^2y'' + 8xy' + 25 = 0$

2. When solving 2nd order linear constant coefficient equations, we look for solutions of the form $y(x) = e^{/x}$, and when solving 2nd order linear Euler-Cauchy equations, we look for solutions of the form $y(x) = x^{/}$. Why do we use these different forms for the different types of equation?

3. Complete the following table and use the initial conditions provided below to solve the corresponding questions.

equation	order?	linear?	homogeneous?	CC/EC?
a) $y'' + y = x + 2$ $y(0) = 4,\ y'(0) = 2$				
b) $2x^2y'' + 3xy' - 15y = 0$ $y(1) = 0,\ y'(1) = 1$				
c) $t^2\dfrac{d^2y}{dt^2} - 5t\dfrac{dy}{dt} + 5y = 10$ $y(1) = 4,\ \dfrac{dy}{dt}(0) = 6$				

4. Why are 2 initial conditions needed to solve second order problems when only one is needed to solve first order problems?

Discuss your answers with a tutor.

Activity 2

1. Consider the second order linear inhomogeneous equation,

$$a(x)y'' + b(x)y' + c(x)y = d(x), \tag{1}$$

and its homogeneous counterpart,

$$a(x)y'' + b(x)y' + c(x)y = 0. \tag{2}$$

Let y_g be the general solution of (1), and let y_h be the general solution of (2). Let y_p be a particular solution of (1).

Explain why it must be the case that

$$y_g = y_h + y_p. \tag{3}$$

Hint: Discuss what it means to be a general solution of the homogeneous problem and what it means to be a particular solution.

I-14: Worksheet 13 – Practicing solving second order ODEs

Introduction

Today you will practice solving various second order linear ordinary differential equations. Doing so involves techniques you have learned in lectures and tutorials over the last 6 weeks. You are not expected to finish the entire worksheet within the hour, but are advised to use it in your independent study time.

Activity 1

1. For each of the following equations, find the general solution, and use the initial conditions given in questions g-i to find a particular solution.
   a) $2y'' - y' - 3y = 0$
   b) $x^2 y'' - 7xy' + 16y = 0$
   c) $\dfrac{d^2y}{dt^2} - 6\dfrac{dy}{dt} + 13y = 0$
   d) $6x^2 y'' - 5xy' + 4y = 2x + 1$
   e) $\dfrac{d^2x}{dt^2} + 8\dfrac{dx}{dt} + 16x = 3t - 2$
   f) $t^2\dfrac{d^2x}{dt^2} + 5t\dfrac{dx}{dt} + 5x = 9$
   g) $6\dfrac{d^2x}{dt^2} - \dfrac{dx}{dt} - x = 2t + 2$ with $x(0) = 2$ and $x'(0) = -1$
   h) $9x^2\dfrac{d^2x}{dt^2} + 3x\dfrac{dx}{dt} + t = -2$ with $x(1) = -2$ and $x'(1) = 6$
   i) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 13y = 0$ with $y(0) = 0$ and $y'(0) = 3$

I-15: Worksheet 14 – Solutions to second order ODEs

Introduction

In this tutorial we will investigate some of the properties of homogeneous second order linear ODEs with constant coefficients.

Activity 1

1. Your task is to complete the table by associating the correct solution and graph with each of the following differential equations:

differential equation	solution	graph of solution
1) $y''+10y'+9y=0$		
2) $y''+4y'+9y=0$		
3) $y''+9y=0$		
4) $y''-6y'+9y=0$		
5) $y''-6y'+9y=3x$		

Solutions:

a) $y(x)=xe^{3x}$.

b) $y(x)=-\frac{1}{3}\sin 3x$.

c) $y(x)=-\frac{1}{8}e^{-9x}+\frac{1}{8}e^{-x}$.

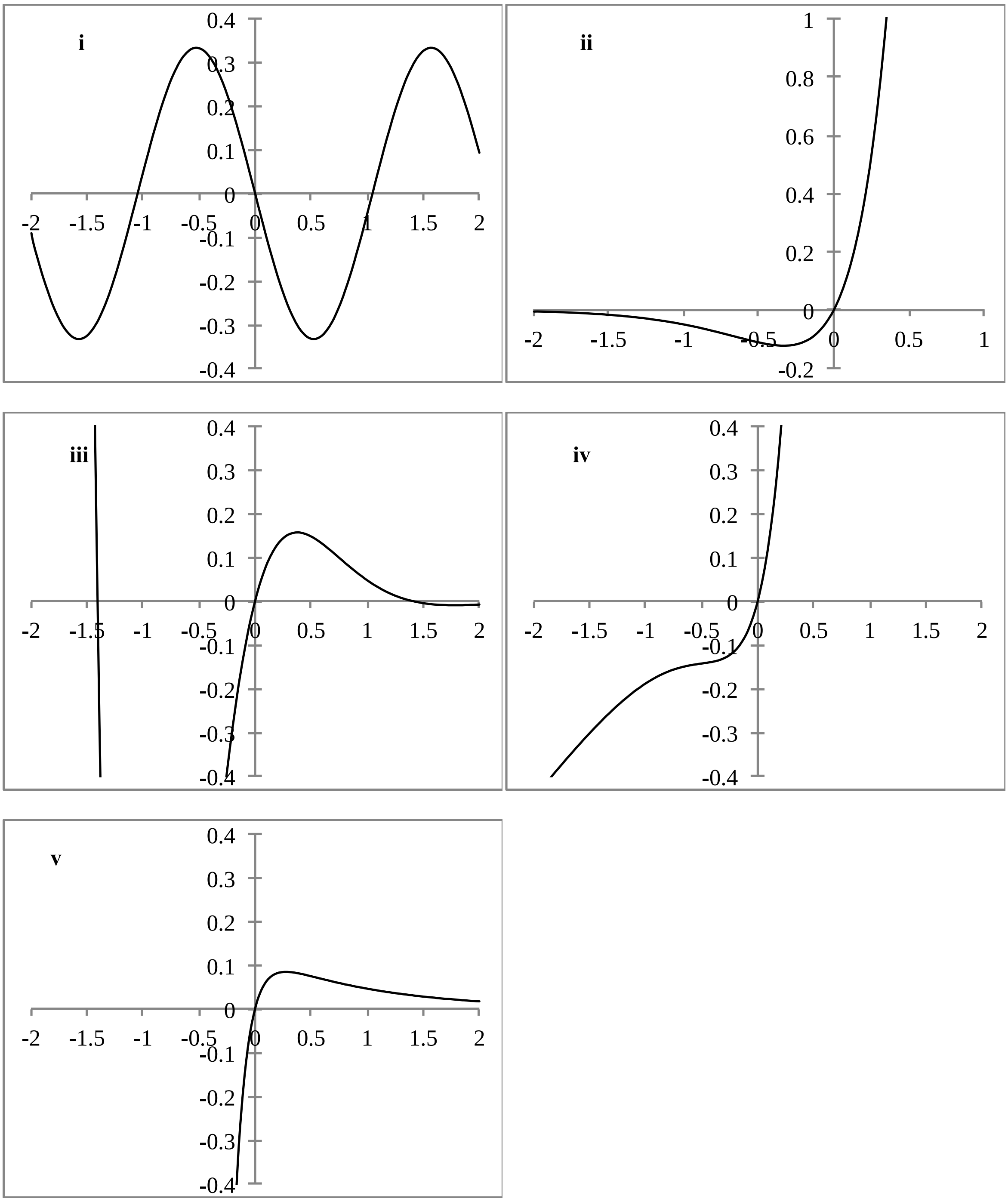
d) $y(x)=\dfrac{e^{-2x}}{\sqrt{5}}\sin\left(x\sqrt{5}\right)$.

e) $y(x)=-\frac{2}{9}e^{3x}+\frac{4}{3}xe^{3x}+\frac{1}{3}x+\frac{2}{9}$.

Graphs: see next page

2. What similarities exist between equations (4) and (5)?

3. What similarities exist between the graphs of the solutions to equations (4) and (5)? How is the inhomogeneous element of equation (5) reflected in the graph of its solution?



i
0.4 0.3 0.2 0.1 0 -0.1 -0.2 -0.3 -0.4
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
ii
1 0.8 0.6 0.4 0.2 0 -0.2
-2 -1.5 -1 -0.5 0 0.5 1
iii
0.4 0.3 0.2 0.1 0 -0.1 -0.2 -0.3 -0.4
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
iv
0.4 0.3 0.2 0.1 0 -0.1 -0.2 -0.3 -0.4
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2
v
0.4 0.3 0.2 0.1 0 -0.1 -0.2 -0.3 -0.4
-2 -1.5 -1 -0.5 0 0.5 1 1.5 2

I-16: Worksheet 15 – Modelling with second order ODE

Introduction

In this tutorial you will look at one example of where second order linear ODEs model a real life phenomenon: harmonic oscillation.

Activity 1

1. In this tutorial you will investigate how the coefficients in a second order linear, constant coefficient ODE affect the behaviour of the solution. Using the two Geogebra files provided, give a detailed explanation of how the coefficients of the ODE affect the solution.