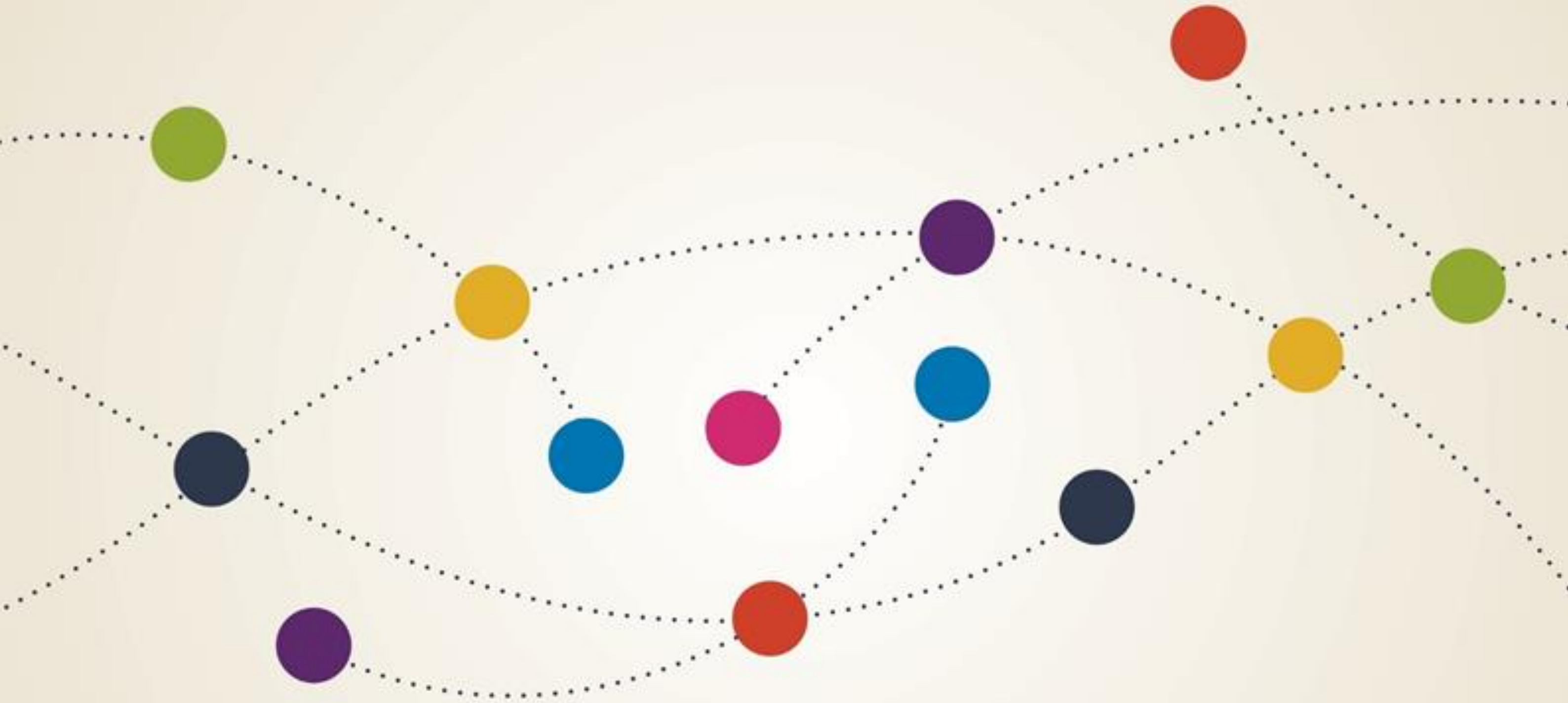

Shape and Space in the senior primary classes

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Introduction

Spatial perception is vitally connected to how we make sense of the world. Geometry, a pillar of mathematics, is very important even for content that is not strictly geometrical (Atiyah, 2001). Research highlights the importance of visuospatial reasoning within and beyond mathematics (Stieff & Uttal, 2015). Unfortunately, the findings of the Trends in International Mathematics and Science Study (Clerkin, Perkins, & Cunningham, 2016) show a relative weakness in geometric shapes and measure compared to number, and a significant gender achievement gap in favour of boys for Irish primary children. This issue must be addressed particularly in light of ongoing technological developments which strengthen the need for high-quality geometry teaching (Watson et al., 2013).

Key concepts

The shift from embodied to symbolic thinking across primary education deserves attention (Dooley, 2019). This is particularly pertinent to Shape and Space where research often addresses 'embodied knowing' and how children can communicate visual, dynamic ideas through gestures (Sinclair et al., 2016). As with other areas of mathematics, Piaget's research has been very influential but also critiqued (Sarama & Clements, 2009). Researchers agree that Euclidean geometry, where deductive logic is used to derive theorems (Watson et al., 2013), has dominated school curricula. Sinclair and Bruce (2015) critique the relatively narrow focus of early geometry tasks, and suggest instead that exploration of multidimensional dynamic spaces is possible, and necessary, at primary level.

Despite substantial research, there is no clear consensus on the definition of spatial ability and its subcomponents (van den Heuvel-Panhuizen, Elia & Robitzsch, 2015). Newcombe et al. (2013) propose that spatial development can be considered to have two strands (a) the development of intra-object (intrinsic) representations along with the ability to transform them and (b) the development of inter-object (extrinsic) representations and the ability to use them to navigate. This links with Sarama and Clements (2009), who draw a fundamental distinction between shape and location, noting different brain systems involved in recognising what an object is versus where an object is. These ideas are further interrogated below, beginning with a section on the overarching idea of visuospatial reasoning.

Visuospatial reasoning

Visuospatial reasoning involves "imagining static or dynamic objects and acting on them" (Sinclair et al., 2016, p. 696), for example, through mental rotation. This topic has received much recent research attention and different terminology is sometimes used to address similar ideas (Mix &

Battista, 2018). Visuospatial reasoning is central to the learning of geometry but visual representations, like graphs or diagrams are helpful in understanding a variety of content areas within and beyond mathematics (Gutierrez, 2018). It has been identified as a gatekeeper for Science, Technology, Engineering and Mathematics (STEM) education and careers and developing visuospatial reasoning is promoted as a means for increasing STEM engagement (Stieff & Uttal, 2015).

Individuals who perform better on spatial tasks also perform better on tests of mathematical ability (Sarama & Clements, 2009). Importantly, research shows that visuospatial reasoning can be improved through teaching (Uttal et al., 2013). Research by Cheng and Mix (2014) goes further. Their research showed that an intervention targeting mental rotation improved children's ability to solve missing-term number sentences of the form 6 + __ = 14. Cheng and Mix suggest that children's increased visuospatial reasoning enabled them to manipulate the terms of the equation. More generally, it has been found that spatial thinking supports the learning of number as the ability to represent magnitude appears to be dependent on visuospatial systems in various regions of the brain (Sarama & Clements, 2009).

Uttal et al. (2013) propose a classification of spatial skills along two dimensions: intrinsic–extrinsic and static–dynamic. Intrinsic information is information related to the object itself, "the specification of the parts, and the relation between the parts" (Uttal et al., 2013, p. 353). Extrinsic information describes the relationship among objects in a group, relative to one another. Considering static and dynamic aspects of objects is important as moving objects may change their intrinsic specification through folding, cutting or rotation for example (Uttal et al., 2013). Other movements may result in changes to extrinsic relations. Representing these two dimensions on a Carroll diagram (figure 1) allows consideration of different types of visuospatial thinking.

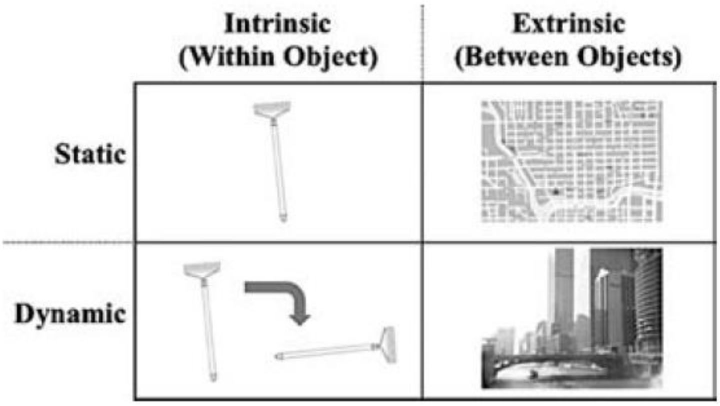


Figure 1. A classification of spatial skills and examples of each process. From Uttal et al., (2013)

Shape

Sarama and Clements' (2009) early learning trajectories for shape attend to identifying and analysing, and composing and decomposing shapes. Composing and decomposing is also emphasized by Bryant (2009) in relation to developing approaches to area measurement for parallelograms, the type of task which may be of relevance in senior primary. Moving beyond early education, a number of theories of geometrical learning have been identified (see Sinclair et al., 2016). Perhaps the most well-known comes from the work of Dina van Hiele-Geldof and Pierre van Hiele (Fuys et al., 1984). The van Hiele model (table 1) details a progression in geometrical thinking starting from a visual level, where thinking is dominated by visual imagery through increasingly sophisticated levels of description, analysis, abstraction and proof (Clements & Battistsa, 1992). Progression between levels is not considered to be related to age but is dependent on the provision of quality learning experiences. Critics argue that the model should not be interpreted as a hierarchical framework, suggesting instead that research demonstrates that children may think in different ways across tasks and may even think in different ways simultaneously (Papademetri-Kachrimani, 2012). The van Hiele levels have been used to consider curricular progression (Sinclair et al., 2016). Such analysis is necessary to ensure that children access a spiral rather than a 'circular' curriculum where topics are repeated at different class levels with no increase in the levels of geometrical reasoning required (Clements & Battistsa, 1992). Researchers agree that developing level 2 and 3 thinking should be an important goal of primary education (ibid.)

Table 1. van Hiele model of geometrical thinking.

0. Pre-recognition	Children may attend to only a subset of a shape's visual characteristics and may be unable to identify many common shapes.
1. Visual	Children recognize shapes solely by their appearance, often by comparison with a known prototype. Limited/no awareness of shape properties.
2. Descriptive/ Analytic	Children characterise shapes by their properties but do not perceive relationships between properties. The child may be unable to identify which properties are necessary and/or sufficient to describe the object.
3. Abstract/ Relational	Children can perceive relationships between properties and between figures. They can form meaningful definitions, classify shapes and give informal justifications for their classifications.

4. Formal Deduction	The individual can reason formally by logically interpreting geometric statements such as axioms, definitions, and theorems.
5. Rigor/ Mathematical	The individual is able to reason formally about mathematical systems by manipulating geometric statements. The objects of this reasoning are relationships between formal constructs.
This overview draws on Clements and Battista (1992) where level 0 was added due to a perceived lack in the original model. This version, rather than van Hiele (1986), is presented as it occurs most frequently in the literature. While 'children' is used to level 3 to indicate common expectations for primary education, the levels are considered to be dependent on learning experiences rather than age.	

Space

Hershkowitz, Parzysz and Van Dormolen (1996) note that the notion of 'space' lacks clarity. Here, children's study of space is conceived as being concerned with place (Rubel et al., 2017), learning to navigate the environment (Newcombe et al., 2013) and representations of location (Bryant, 2009) though it is acknowledged that other perspectives are possible (Hershkowitz et al., 1996).

Children's own environments are an important starting point for this work and navigation and mapping activities create many opportunities for meaningful integration with geography. Spatial orientation involves "understanding and operating on relationships between different positions in space, at first with respect to one's own position and your movement through it, and eventually from a more abstract perspective that includes maps and coordinates at various scales" (Sarama & Clements, 2009, p. 161). Space is often perceived to be centred around the individual and directions like 'right' or 'left' are sometimes considered absolute. Understanding the relative position of directions may require mental rotation. Kotsopoulos, Cordy and Langemeyer (2015) emphasise the importance of attention to dynamic aspects of navigation. In their large-scale mapping activity with 8 - 10 year olds, they found low achievers drew very few objects, produced fewer verbalisations, and less dynamic motions than higher achievers. 'Large scale' mapping refers to situations where the whole area cannot be seen (ibid.). They note that low achieving children tended to use connected smaller-scale spaces (bounded spaces where the whole area can be seen) to make sense of the larger-scale space. They suggest that low achievers may need additional pedagogical approaches to support complex representational thinking and further support in verbalising their thinking.

Tools that have been shown to be effective include oversized floor maps, geographic information systems (GIS), and other digital technologies. Oversized floor maps have been used, even with second-level students, to address observed difficulties with conventional scale map representations and support body-centric understanding and coordination across representations (Rubel et al., 2017). Digital technologies, including location tracking on digital maps, offer rich potential for mapping activities (c.f., Wijer, Jonkers & Driver, 2010). Research also details teaching interventions which have productively explored social inequality through spatial tools such as geographic information systems (GIS), software which facilitates representation of geographically referenced data (Rubel et al., 2017). Ordnance Survey Ireland provides such spatial data, that is, data that can be mapped, about Ireland in its Geohive website (Debruyne et al., 2017). Digital technologies also facilitate participatory mapping where individuals can use mobile devices to record and map data relevant to their topic of study as they navigate the real world (Rubel et al., 2017). These tools provide potential opportunities for meaningful linkage with data, and integration with geography or other subjects, including social justice or environmental issues. Scale and perspective are also central to understanding of maps and many links with measurement, ratio and proportion can be made (NZME, n.d.). The use of multiple representations, including children's own drawings, and making links across representations is recommended (Kotsopoulos et al., 2015).

Maps and graphs may require use of the Cartesian coordinate system. In the Cartesian system, it is possible to plot positions by representing them in terms of their position along horizontal and vertical axes in a two-dimensional plane. Bryant (2009) suggests that while children generally find it easy to locate positions using coordinates or graphs, working out the relation between two or more positions plotted in this way appears harder for them.

Shape and space in senior primary

This section presents further detail of concepts which are of particular relevance to senior primary or which are novel relative to the Irish Primary School Mathematics Curriculum (Government of Ireland, 1999).

Properties, Definitions and Classifications

Van Hiele theory suggests that an emphasis on relationships between properties and classes of shapes is appropriate in senior primary. This involves dimensional deconstruction, i.e., breaking down an object into parts of the same or lower dimensions (Duval, 2005). This is highly complex for 3D shapes since it deals with a number of different kinds of objects, for example, vertices, edges, faces, or 3D subcomponents of the original shape (Laborde, 2008). Duval (2005) proposes that constructions with rulers, compasses or stencils are a key way to help individuals attend to objects from lower dimensions. Construction and deconstruction of 3D solids are also effective for this purpose and digital geometry environments (DGEs) can also play a role in analysis of 3D shape (Laborde, 2008).

Properties are closely related to definitions and classifications (Fujita & Jones, 2007). The formal understanding of definitions is considered to be located at van Hiele level 3 (Sinclair et al., 2016). Although a child may recite the definition of a parallelogram, he/she may still not consider squares or rhombi as parallelograms, since the child's concept image is one in which not all sides are allowed to be equal (De Villier, 1994). Concept image describes the cognitive structure associated with the concept including all the mental pictures and associated properties and processes (Tall & Vinner, 1981). It is generally recommended, that rich and varied concept images are developed prior to formulating definitions (Sinclair et al., 2016).

In developing concept images and definitions, it is necessary to attend to critical aspects, those aspects that can be found in any example (Erez & Yerushalmy, 2006). These aspects are also central to developing hierarchical classifications. Such classifications have an inclusive character, for example, if a statement is true for parallelograms, this means that it is also true for squares, rectangles and rhombuses (Fujita & Jones, 2007). Erez and Yerushalmey (2006) draw on the work of Markman (1989) to propose the following necessary components to understanding hierarchical relations between classes of shapes. These components should be considered in the design of tasks and developed in the resulting interactions with children.

- Recognising that shapes can be labelled with different names and classified in different ways (*a square can also be called a polygon, quadrilateral, parallelogram*)
- Recognising transitive relationships between concepts of shapes (*a square is a rhombus and a rhombus is a parallelogram, so a square is also a parallelogram*.)
- Recognising asymmetry of relationships (*every square is a rectangle, but not every rectangle is a square*)
- Recognising the opposite asymmetry and transitive relations of the critical attributes of shape concepts, for example, critical attributes of the rectangle are included in the critical attributes of the square, but the critical attributes of the square are not included in those of the rectangle.

Transformations

Transformations are of fundamental importance to how we understand shape and recognition of the importance of learning transformational geometry in primary education is a growing trend in recent research (Sinclair & Bruce, 2015). It is theorised that unconscious visual transformations are one of the ways that individuals engage in spatial structuring (Battista, 2007). Transformations involve the movement of objects across two- or three-dimensional space in ways that change their initial location and/or orientation (Kotsopoulos et al., 2015). They include *rotations*, where an of object is turned around an axis or point; *reflections*, where an object is projected across a line of symmetry; *translations,* which preserve the size of an object but move it in some direction; and *dilations*, which transform size or scale rather than location (Kotsopoulos et al., 2015, p. 453). In play, children often construct symmetrical objects and turn, flip or superimpose one object on another to show how objects fit or match, i.e., they explore reflectional and rotational symmetry and test and 'prove' shapes to be congruent (Clements & Sarama, 2009). Early teaching should involve making the mathematics of these situations explicit (Bryant, 2009) by using words such as flip, turn or slide to describe the transformations involved (van de Walle et al., 2013). Later work might involve exploring the connections between rotational and reflectional symmetry and exploration of transformations within tessellations (Davis et al., 2017). DGEs also present rich opportunities for exploration of shape transformations (Sinclair et al., 2016).

Angle

Angles are essential to navigation of space but also highly relevant in the analysis of shape. They are fundamental to the development of geometric understanding (Clements & Battista, 1992). Bryant (2009) suggests that measurement of angle may play a significant, possibly even an essential, part in developing children's understanding. His research synthesis suggests that teaching children about angles in terms of movements or turning is successful, with some evidence to suggest that children can transfer their knowledge to static angle contexts. These ideas are echoed in Kaur's (2020) recent work with 5- 6 year olds, which shows how a DGE might be used to develop understanding of both dynamic and static representations of angle. As with much research in geometry education, Kaur notes that gestures and motion played an important role in children's developing conceptions.

Considerations for teaching

Van Hiele theory posits a central role for the teacher in providing quality learning experiences and proposes a five-phase model of instruction (Clements & Sarama, 2009). This model moves from *introductory* activities where children explore materials freely through to *guided orientation* where children engage with key geometrical ideas. The middle stage is *explication*, where children are supported to describe what they have learned in their own words. The teacher introduces relevant mathematical terminology and orchestrates discussion so that children become explicitly aware of the target geometrical ideas. The final stages involve a child-led *problem solving* phase and an *integration* phase where children reflect on, and consolidate their knowledge. This model has not been widely researched (Watson et al., 2013) but there are some indications of better conceptual understanding where learning experiences more closely align with the van Hiele phases of learning (Sinclair et al., 2016). Research supports the idea of working with children's own language and thinking as suggested in the *explication* phase. For example, verbal reasoning has been shown to be particularly important to middle school children's solving of geometrical problems (Anderon et al., 2008). Also, Clements and Battista (1992) warn that geometrical terminology can be used without mathematical understanding. They recommended that teachers build on children's own language and ideas as they introduce and support the use of important vocabulary.

Research emphasises the importance of visual representations of geometric ideas. It is recommended that children are presented with a wide variety of non-prototypical examples (and non-examples) in different positions or orientations (Clements & Sarama, 2009). DGEs present opportunities for a greater diversity of representations, and the continuous transformation of shapes (Sinclair et al., 2016). Battista (2001) shows how carefully designed problem-solving activities with such resources can provide rich mental models and facilitate the development of powerful geometric reasoning. The availability of mobile touch screen technology offers further possibilities (Sinclair et al., 2016). The research survey of Sinclair et al. concludes that dragging is the most important feature of DGEs, possibly because attention to variance and invariance under dragging may contribute to development of ideas (Battista, 2007). Some research suggests that particular digital games can increase spatial ability and reduce the achievement gap between girls and boys (Yang & Chen; 2010). Technology also exists to support the teaching of location and navigation, for example, programmable devices or mapping tools (Bartolini Bussi & Baccaglini-Frank, 2015; Wijers et al., 2010). Some research on virtual manipulatives, which are digital versions of existing resources such as pattern blocks or tangram pieces, has shown children to construct more creative and complex patterns when using virtual rather than concrete materials (Sinclair & Bruce, 2015). In some cases, this successful use of virtual manipulatives was preceded by work with concrete manipulatives and explicit attention to making connections across representations (Moyer, Niezgoda, & Stanley, 2005). As with tools in mathematics learning more generally, it is cautioned that the mathematics does not reside in the DGE and tools only become meaningful through extended, joint engagement in which the teacher plays an important role (Askew, 2016; Dooley, 2019).

Recent research draws attention to the use of drawing and diagrams to support spatial structuring (Cullen et al., 2018). Thom and McGarvey (2015) propose that the process of drawing can help children attend to significant mathematical aspects and therefore contribute in important ways to building their geometric understanding (see also Kotsopoulos et al., 2015). For example, see figure 2, below which shows a child's drawing of shapes that can be made with triangles. Sinclair et al. (2016) describe a range of studies which show the interplay between gestures and diagrams in building geometrical understanding for learners of all ages. Their research survey indicates the potential of encouraging learners to engage in more gesturing and diagramming. Research suggests that teachers' increased use of gesture prompts children's use of gesture. Supporting children's use of diagrams involves creating contexts in which diagramming is valued and productive.

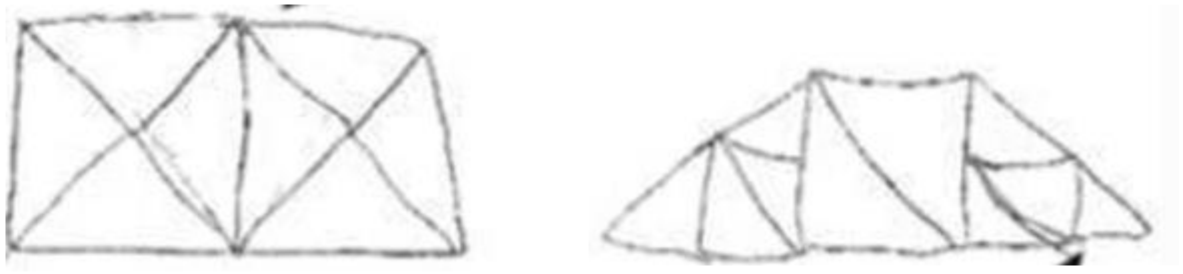


Figure 2. A second grade child's (c. 7 years old) drawings of shapes that can be made with triangles. From Thom and McGarvey (2015).

Key elements in shape and space

Key ideas relevant to *Understanding and Connecting* have been outlined above. It is vital to attend to both static and dynamic aspects of spatial reasoning. Newcombe et al. (2013) propose 'spatializing' the existing curriculum, by attending to the spatial aspects of different curricular areas. Within mathematics, this might mean attending to the spatial structuring involved in length, area or volume measurement as well as making explicit the spatial aspects of visualizations in data, number or algebra as appropriate. Beyond mathematics, there is scope for meaningful connections to real life, to mapping in geography, technology education, constructions in art and science as well as many other subjects.

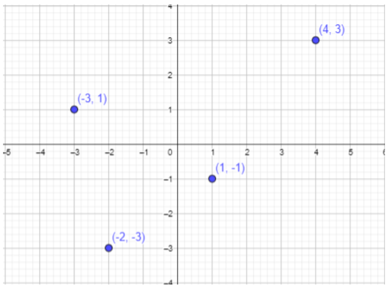
Proof is a key component of geometry and as in other areas of mathematics, empirical and deductive methods should interact and reinforce each other (Clements & Battistsa, 1992). While formal deductive methods may not be appropriate in primary education, exploration and experimentation, accompanied by reasoned conjectures and justifications should be expected and supported (Sinclair et al., 2016). *Reasoning* should be at the heart of many Shape and Space activities, guided by the van Hiele levels as appropriate. *Communicating* should include justifications and argumentation and may involve empirical examples, for example, paper-based or DGE constructions from which children can be prompted to generalise. These activities will be made meaningful by situating them within a problem-solving context, linked with real life contexts where appropriate, so as to provide opportunities for children to engage in *applying and problem-solving*.

Key messages

Primary children are capable of much more than the traditional emphasis on naming shapes. Composing/decomposing, classifying, comparing and mentally manipulating both two- and three-dimensional figures is appropriate. Progression in geometrical reasoning should be visible across the primary curriculum using the van Hiele levels to judge the suitability of learning experiences as appropriate. Visuospatial reasoning should be promoted within and beyond mathematics with attention to dynamic as well as static aspects of geometry. The use of a variety of different visual representations, including non-prototypical examples is recommended with thoughtful use of DGEs or other technology as appropriate. Gesture, motion, construction and drawing have been shown to have the potential to contribute to children's developing understanding. Finally, in line with van Hiele's theory, the teacher is considered to play a pivotal role, with progression in children's understanding of shape and space considered to be dependent on the provision of high-quality learning experiences.

Glossary

Cartesian coordinate system: This allows us to locate points in space using a set of numerical coordinates. On a two dimensional plane, positions are plotted by representing them in terms of their position along two perpendicular axes.



Congruent: Figures are congruent if some distance-preserving transformation (isometry) or combination of transformations exists that takes one shape to another, for example a combination of reflection and rotation. In effect, figures are congruent if they are the same shape and size.

Dimensional deconstruction: Dimensional deconstruction involves breaking down an object into parts of the same or lower dimensions.

Euclidean geometry: This refers to the study of plane and solid figures on the basis of axioms and theorems outlined by the Greek mathematician Euclid. It is associated with using deductive logic to derive theorems. Other geometries exist, for example, spherical geometry, the study of geometric objects surface of a sphere

Hierarchical classification: The classification of a set of concepts so that the more particular concepts form subsets of the more general concepts (De Villier, 1994).

Prototypical examples: Examples of shapes shown to children in commercial resources are often regular (all sides and angles equal). For example, children are much more likely to see examples of equilateral triangles that scalene triangles. Furthermore, these examples are often presented in a similar orientation, with a horizontal 'base'.

Similar: Figures are said to be similar if they have the same shape, i.e., the same set of angles, but are not the same size.

Spatial Structuring: This considers ways in which space or shapes might be structured or ordered. It involves mentally identifying spatial components and associated relationships. In the context of measurement, this structuring generally involves analysing space to consider how a unit might be iterated to find a measurement.

Visuospatial reasoning: Visuospatial reasoning involves imagining static or dynamic objects and acting on them, for example, by mentally rotating them.

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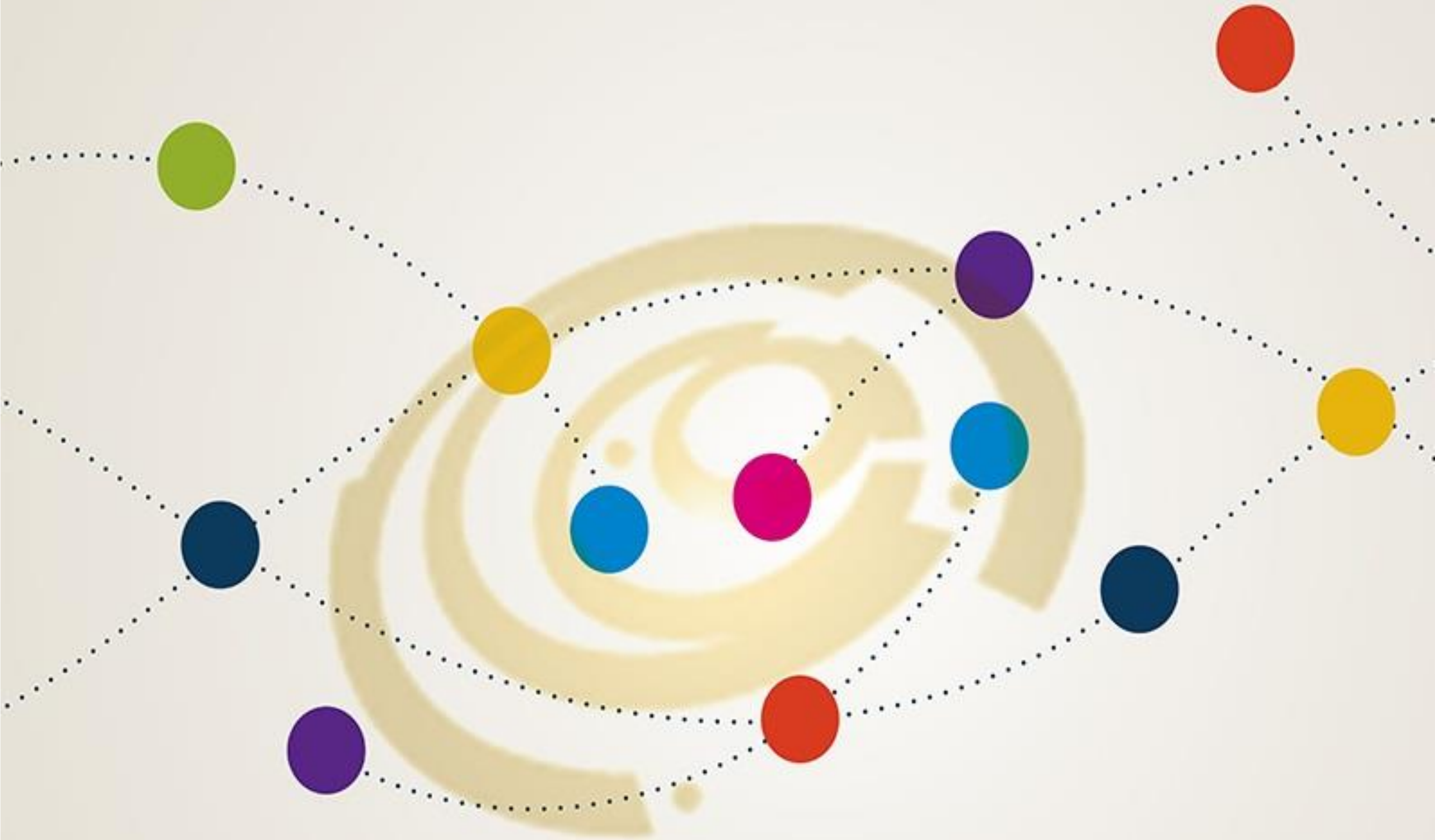

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