

# THE LAPICQUE EQUATION IN NEUROELECTRICAL STIMULATION

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## Abstract

The Lapicque equation for electrical stimulation of nerve, proposed in 1907 and further discussed in 1909, is applied to the Hodgkin-Huxley neuron, proposed in 1952 and universally recognised nowadays. We demonstrate how an analytical model of the relationship described by Lapicque can be derived through consideration of this mathematical model of the nerve membrane potential when subject to pulsed current input.

## Keywords

Lapicque, Hodgkin-Huxley, electrical stimulation, electrophysiology

## Introduction

By the beginning of the twentieth century, technology had advanced to the point that electrophysiologists could establish the precise requirements for stimulation of nerve and muscle by pulses of current, which could be controlled in both magnitude and duration. This led to the recognition that the current intensity must exceed a certain minimum value and that it must act for a certain minimum length of time in order for contractions to occur. We are not sure to whom the first clear statement of a quantitative law should be credited, but there is no doubt that two papers [1, 2] by the French electrophysiologist Louis Lapicque (1866–1951), published in 1907 and 1909, based on work which he performed in cooperation with his wife, Marcelle, presented the equation which has stood the test of time. In this paper, we demonstrate how the Lapicque equation naturally arises from an analytical model of electrical nerve stimulation. This theoretical development complements the well-known empirical result of Lapicque's.

## Analysis

Lapicque introduced two terms which have become standard descriptors. In his 1907 paper [1] he fitted experimental data with a curve of the form

$$I_{th} = \frac{I_R}{\left[1 - e^{-\left(\frac{PD}{\tau}\right)}\right]} \quad (1).$$

In this equation, PD is pulse duration and  $I_{th}$  is the corresponding threshold current for excitation. This curve is sketched in Fig. 1. It is seen that as  $PD \rightarrow \infty$ ,  $I_{th} \rightarrow I_R$ . In his 1909 paper [2], Lapicque proposed the term rheobase [“base de courant”] for  $I_R$ . For the pulse duration which gives  $I_{th} = 2I_R$  he coined the name chronaxie [“valeur du temps”]. It is readily derived that chronaxie,  $t_c$ , is related to time constant  $\tau$  by

$$t_c = \tau \ln 2 \quad (2).$$

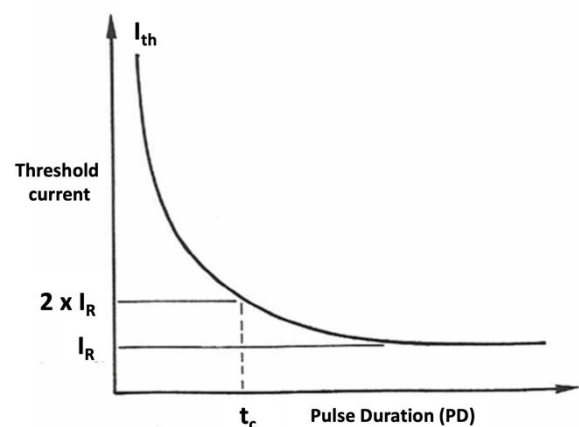


Fig. 1: Lapicque's curve for threshold excitation current vs. pulse duration.

Lapicque was led to the form of his equation by the response of a simple parallel RC circuit to a step function of applied voltage. He thanked his students Chatanay and Levy for help with the analysis. We now propose to show that the Lapicque equation applies to the equivalent circuit of nerve membrane. This equivalent circuit, Fig. 2(a), was established definitively by

the Nobel Prize-winning work of A.L. Hodgkin and A.F. Huxley [3, 4], published in 1952. The nonlinear dynamic controller's conductances are governed by the equations

$$g_{Na} = \hat{g}_{Na} m^3 h \quad (3)$$

$$g_K = \hat{g}_K n^4 \quad (4)$$

where  $h$ ,  $m$  and  $n$  are probabilities governed by rate equations of the typical form

$$\frac{dh}{dt} = \alpha_h(1-h) - \beta_h \cdot h \quad (5)$$

with the  $\alpha$ 's and  $\beta$ 's being nonlinear functions of transmembrane potential difference  $v$ . As per the original publication [3] potentials are measured in  $mV$ , current density in  $\mu A/cm^2$ , conductances in  $mS/cm^2$ , capacitance in  $\mu F/cm^2$  and time in  $ms$ .

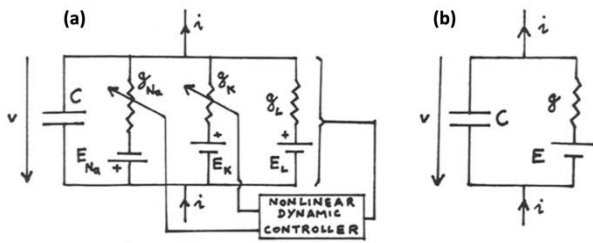


Fig. 2: (a) Hodgkin-Huxley model of neuron and (b) simplification of resting Hodgkin-Huxley neuron using Thévenin's theorem.

The injected current  $i$  is assumed to be a rectangular pulse of threshold amplitude,

$$\begin{aligned} i &= I_{th} \text{ for } 0 < t < PD \\ &= 0 \text{ for } t > PD \end{aligned} \quad (6)$$

with  $PD$  sufficiently short, so that while the current is flowing, the controller has not had time to change  $g_{Na}$  and  $g_K$  significantly from their resting values. With this assumption, the circuit is simplified, via the eponymous theorem due to Leon Thévenin (1857–1926) [5], which is part of the stock-in-trade of all electrical engineers, to that shown on Fig. 2(b), where

$$g = g_{Na} + g_K + g_L \text{ (resting values)} \quad (7)$$

and

$$E = \frac{g_L E_L + g_K E_K - g_{Na} E_{Na}}{g_L + g_K + g_{Na}} \quad (8)$$

The differential equation governing  $v$  in Fig. 2(b) is

$$\frac{dv}{dt} = \left[ \frac{g}{C} \right] \cdot \left[ \frac{i}{g} - E - v \right] \quad (9)$$

using the well-known equation that the current through a capacitor is given by the capacitance times the rate of change of voltage and Kirchoff's current law applied to the circuit shown in Fig. 2(b) [5].

Starting from the rest condition  $v(0) = -E$ , the well-known solution during the time interval  $0 < t < PD$  is

$$v(t) = -E + \frac{I_{th}}{g} \left[ 1 - e^{-\frac{t}{\tau}} \right] \quad (10)$$

With the additional assumption that  $v$  has to reach a threshold value  $v_{th}$  at  $t = PD$  in order to excite the nonlinear controller to enter the regenerative mode in which it causes an action potential, there results

$$I_{th} = \frac{(v_{th} + E)g}{\left[ 1 - e^{-\left(\frac{PD}{\tau}\right)} \right]} \quad (11)$$

This is exactly of the form given by Lapicque.

## Discussion and Conclusion

Up to the time of Lapicque, electrical stimulation in medicine was very subjective. Those who employed it used their intuition, often with great success, to set levels and duration of stimuli. After Lapicque that climate changed, and stimulators could be designed on a rational basis. The 20th century revolution in electronics also changed the technology of electrical stimulation beyond recognition, making possible the convenient, portable and versatile machines which are now widely used for so many diverse applications. A particularly useful application of the Lapicque equation is the relative measurement of excitability of tissue, a topic of great importance in the appropriate use of functional electrical stimulation in rehabilitation settings. It is known for example that flexor muscles have shorter chronaxies than extensors and consequently stimulation parameters must take this into account for effective use. The model derived here may form the basis for relating changes in chronaxie to underlying changes in the physiologically interpretable parameters of the Hodgkin-Huxley model or its variations for different tissue types.

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