On worst-case investment with applications in finance and insurance mathematics

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Summary. We review recent results on the new concept of worst-case portfolio optimization, i.e. we consider the determination of portfolio processes which yield the highest worst-case expected utility bound if the stock price may have uncertain (down) jumps. The optimal portfolios are derived as solutions of non-linear differential equations which itself are consequences of a Bellman principle for worst-case bounds. They are by construction non-constant ones and thus differ from the usual constant optimal portfolios in the classical examples of the Merton problem. A particular application of such strategies is to model crash possibilities where both the number and the height of the crash is uncertain but bounded. We further solve optimal investment problems in the presence of an additional risk process which is the typical situation of an insurer.

1 Introduction

Modelling stock prices at financial markets seems to be a classical field for the use of interacting particle systems. However, the most common stock price models do not contain explicit reference to the market participants, the traders. Even more, modern financial mathematics is based on the "small investor assumption" which requires that the action of the single trader has no impact to prices at all, an assumption which seems to contradict the idea of interaction at all.

The relation to interacting systems lies in a microeconomic modelling of financial markets. An excellent reference for this topic is [FS93]. Here,the authors show in particular how the usual assumption of stock prices following a geometric Brownian motion can be obtained via a limit argument out of a model where only a finite number of traders form the market and the stock prices are determined by supply and demand via the so-called market clearing condition.The geometric Brownian motion model is the limiting model that corresponds to the situation when only uninformed traders ("noise traders") are present.

Looking at the usual stock price models as limits that result from trading activities of many interacting traders, we are in a situation that is similar to limit considerations of particle models in statistical physics or biological applications. The main difference in financial mathematics is that the second step after the stock price modelling, the execution of tasks such as pricing of derivatives or of finding optimal investment straetgies is usually only done in the limit settings (such as the geometric Brownian motion model or other semi-martingale market models).

In this paper two of the main tasks of financial mathematics are touched. One is the modelling of stock prices and the other the determination of optimal investment strategies, the portfolio optimization problem. We will give a survey on the main results of the recently introduced approach of worst-case portfolio optimization (see [KW02] for its first introduction and [KM02], [KO03],[ME03] for generalizations). We specialize on portfolio optimization under the risk of market crashes but applications different from financial mathematics seem to be possible and should be considered in the future (examples could be the optimal control of a production line under the risk of a breakdown, optimal business strategies for food chains under the risk of sudden change of consumer behaviour (such as e.g.during the BSE crisis), evolution of populations/monocultures facing catastrophes). The basic model underlying our approach is worst-case modelling as introduced by Hua and Wilmott [HW97] where upper bounds on both the number of crashes until the time horizon and on the maximum height of a single crash are assumed to be known. Between the crashes the stock price is assumed to move according to a geometric Brownian motion. This makes the setting differ from classical approaches to explain large stock price moves such as e.g. described in [ME76], [EK95], [EKM97] where stock prices are given as Levy processes or other types of processes with heavy tailed distributions. As a second ingredient for our worst-case investment model we are more focused on avoiding large losses in bad situations via trying to put the worst-case bound for the expected utility of terminal wealth as high as possible.

In [KW02] this setup is introduced and the portfolio problem under the threat of a crash is solved in the case of a logarithmic utility function. Deriving systems of non-linear differential equations to characterize the optimal portfolio process for general utility functions and to allow the market parameters to change after each crash are the main achievements of [KM02]. Finally, in [KO03] the optimal investment problem of an insurer is considered who in addition faces a risk process which is non hedgeable in the financial market(a typical example is a life insurer that faces the biometric risk of the population getting older than estimated which seems to be uncorrelated (or at least not perfectly correlated) to the evolution of the financial markets).

The paper is organized as follows: Section 2 describes the set up of the model and contains the main theoretical results in the simple situation where at most one crash can occur. In Section 3 these results will be extended to the situation when the investor faces additional non-hedgeable risk. Finally, Section 4 contains various generalizations and states open problems.

2 The simplest set up of worst-case scenario portfolio optimization

The most basic setup that we consider here consists of a riskless bond and a single risky security with prices during "normal times" given by

$$dP_0\left(t\right) = P_0\left(t\right) rdt,\ P_0\left(0\right) = 1 \tag{1}$$

$$dP_1\left(t\right) = P_1\left(t\right)\left(bdt + \sigma dW\left(t\right)\right),\ P_1\left(0\right) = p_1 \tag{2}$$

for constant market coefficients $b > r$,$\sigma \neq 0$ and a one-dimensional Brownian motion $W\left(t\right)$. At the "crash time" the stock price experiences a sudden relative fall which is assumed to be in the interval $[0, k^*]$ with $0 < k^* < 1$. Otherwise no further assumptions on both the crash size and time are made (we allow for changing market parameters and for multiple crashes in Sections 3 and 4).
We will assume that the investor is able to realize that a crash has happened and therefore introduce a process $N\left(t\right)$ counting the number of jumps (i.e. in our simple setting it is zero before the jump time and one from the jump time onwards). Let $\{f_t\}$ be the P-augmentation of the filtration generated by $W\left(t\right)$ and $N\left(t\right)$. We then define the set of admissible portfolio processes for our investor.

Definition 1. *Let $A\left(x\right)$ be the set of admissible portfolio processes $\pi\left(t\right)$ corresponding to an initial capital of $x > 0$, i.e. $\{f_t\}$ -progressively measurable processes such that*
a) the wealth equation in the usual crash-free setting

$$d\tilde{X}^{\pi}\left(t\right) = \tilde{X}^{\pi}\left(t\right)\left[\left(r + \pi\left(t\right)\left(b - r\right)\right)dt + \pi\left(t\right)\sigma dW\left(t\right)\right], \tag{3}$$

$$\tilde{X}^{\pi}\left(0\right) = x \tag{4}$$

has a unique non-negative solution $\tilde{X}^{\pi}\left(t\right)$ and satisfies

$$\int_0^T \left(\pi\left(t\right)\tilde{X}\left(t\right)\right)^2 dt < \infty \ \ P - a.s. \tag{5}$$

i.e. $\tilde{X}^{\pi}\left(t\right)$ is the wealth process in the crash-free world.
b) the corresponding wealth process $X^{\pi}\left(t\right)$, defined as

$$X^{\pi}\left(t\right) = \begin{cases} \tilde{X}^{\pi}\left(t\right) \ \text{for } t < \tau \\ \left(1 - \pi\left(\tau\right)k\right)\tilde{X}^{\pi}\left(t\right) \ \text{for } t \geq \tau \end{cases}, \tag{6}$$

given the occurrence of a jump of height k at time τ, is strictly positive.
c) $\pi\left(t\right)$ has left-continuous paths with right limits.

This definition allows us to set up the worst-case portfolio problem we want to study:

Definition 2. *a) Let $U(x)$ be a utility function (i.e. a strictly concave, monotonously increasing and differentiable function). Then the problem to solve*

$$\sup_{\pi(.)\in A(x)} \inf_{0\le\tau\le T, 0\le k\le k^*} E\left(U\left(X^\pi\left(T\right)\right)\right) \tag{7}$$

(where the final wealth $X^\pi(T)$ in the case of a crash of size k at the (stopping) time τ is given by

$$X^\pi\left(T\right) = \left(1-\pi\left(\tau\right)k\right)\tilde{X}^\pi\left(T\right) \tag{8}$$

with $\tilde{X}^\pi(\tau)$ as above) is called the worst-case scenario portfolio problem.
b) The value function to the above problem if one crash can still happen is defined as

$$v_1\left(t,x\right) = \sup_{\pi(.)\in A(t,x)} \inf_{t\le\tau\le T, 0\le k\le k^*} E\left(U\left(X^\pi\left(T\right)\right)\right). \tag{9}$$

c) Let $v_0(t,x)$ be the value function for the usual optimisation problem in the crash-free Black-Scholes setting, i.e

$$v_0\left(t,x\right) = \sup_{\pi(.)\in A(t,x)} E\left(U\left(\tilde{X}^\pi\left(T\right)\right)\right). \tag{10}$$

Under the assumption of $b > r$ a first fact which is very usefull and intuitively clear (note the requirement of left-continuity of the strategy !) is that it is optimal - with respect to the worst-case bound - to have all money invested in the bond at the final time (for a formal proof see [KW02]):

Proposition 1. *If $U(x)$ is strictly increasing then an optimal portfolio process $\pi(t)$ for the worst-case problem has to satisfy*

$$\pi\left(T\right) = 0. \tag{11}$$

We further require that the worst possible jump should not lead to a negative wealth process. Therefore, without loss of generality we can restrict to portfolio processes satisfying

$$1/_{k^*} \ge \pi\left(t\right) \ge 0 \text{ for all } t\in[0,T] \text{ a.s..} \tag{12}$$

which in particular implies that we only have to consider bounded portfolio processes. As after a crash it is optimal to follow the optimal portfolio of the crash-free setting, having a wealth of z just after the crash at time s leads to an optimal utility of $v_0(s,z)$. As $v_0(s,.)$ is strictly increasing in the second variable, a crash of maximum size k^* would be the worst thing to happen for an investor following a positive portfolio process at time s. As we only have to consider non-negative portfolio processes, and as by Proposition 1 we have

$$E\left(v_0\left(T,\tilde{X}^\pi(T)(1-\pi(T)k^*)\right)\right)=E\left(v_0\left(T,\tilde{X}^\pi(T)\right)\right)=E\left(U\left(\tilde{X}^\pi(T)\right)\right),$$

it is enough to consider only the effect of the worst possible jump. We have thus shown:

Theorem 1. *"Dynamic programming principle"*
If $U(x)$ and $v_0(t,x)$ are strictly increasing in x then we have

$$v_1(t,x)=\sup_{\pi(.)\in A(t,x)}\inf_{t\le\tau\le T}E\left(v_0\left(\tau,\tilde{X}^\pi(\tau)(1-\pi(\tau)k^*)\right)\right). \tag{13}$$

The dynamic programming principle will be used to derive a dynamic programming equation. A formal proof of the following result is again given in [KM02]. We will only sketch it.

Theorem 2. *"Dynamic programming equation"*
Let the assumptions of Theorem 1 be satisfied, let $v_0(t,x)$ be strictly concave in x, and let there exist a continuously differentiable (with respect to time) solution $\hat{\pi}(t)$ of

$$\begin{aligned}(v_0)_t(t,x)+(v_0)_x(t,x)(r+\hat{\pi}(t)(b-r))x+{}^1\!/\!_2\,(v_0)_{xx}(t,x)\sigma^2\hat{\pi}(t)^2x^2\\ -(v_0)_x(t,x)x\frac{\hat{\pi}'(t)}{(1-\hat{\pi}(t)k^*)}k^*=0\ for\,(t,x)\in[0,T[\times(0,\infty),\end{aligned} \tag{14}$$

$$\hat{\pi}(T)=0. \tag{15}$$

Assume further that we have:
(A) $f(x,y;t)$
$$:=(v_0)_x(t,x)((y-\hat{\pi}(t))(b-r))x+{}^1\!/\!_2\,(v_0)_{xx}(t,x)\sigma^2\left(y^2-\hat{\pi}(t)^2\right)x^2$$
is a concave fuction in (x,y) for all $t\in[0,T)$.
(B) $E^{0,x}\left(\hat{v}\left(t,\tilde{X}^\pi(t)\right)\right)\le E^{0,x}\left(\hat{v}\left(t,\tilde{X}^{\hat{\pi}}(t)\right)\right)$ and $E^{0,x}(\pi(t))\ge\hat{\pi}(t)$ for some $t\in[0,T)$, $\pi\in A(x)$ imply

$$E^{0,x}\left(v_0\left(t,\tilde{X}^\pi(t)(1-\pi(t)k^*)\right)\right)\le E^{0,x}\left(\hat{v}\left(t,\tilde{X}^{\hat{\pi}}(t)\right)\right).$$

Then, $\hat{\pi}(t)$ is indeed the optimal portfolio process before the crash in our portfolio problem with at most one crash. The optimal portfolio process after the crash has happened coincides with the optimal one in the crash free setting. The corresponding value function before the crash is given by :

$$v_1(t,x)=v_0(t,x(1-\hat{\pi}(t)k^*))=E\left[v_0\left(s,\tilde{X}^{\hat{\pi}}(s)(1-\hat{\pi}(s)k^*)\right)\right] \quad for\ 0\le t\le s\le T. \tag{16}$$

Sketch of the proof:
Step 1: Derivation of (14)

The martingale optimality principle of stochastic control (see [KO03b] for a description of the martingale optimality principle) indicates that we obtain a martingale if we plug in the wealth process corresponding to the optimal control into the value function. By using the Bellman principle (13), applying It's formula to the function inside the expectation of the right hand side and leaving aside the sup-opetator we obtain as a sufficient condition for the martingale property of the resulting process $v_0\left(s, \tilde{X}^{\hat{\pi}}(s)(1-\hat{\pi}(s)k^*)\right)$ that the portfolio process $\hat{\pi}(t)$ should satisfy the differential equation (14) with boundary condition

$$\hat{\pi}(T) = 0.$$

In particular, it should be differentiable.
Step 2: Optimality of $\hat{\pi}(t)$
The optimality proof for $\hat{\pi}(t)$ is motivated by the martingale optimality principle of stochastic control (see Korn (2003b)). We therefore introduce $\hat{v}(t,x) := E^{t,x}\left[U\left(\tilde{X}^{\hat{\pi}}(T)\right)\right]$. By considering $\hat{v}\left(t, \tilde{X}^{\pi}(t)\right)$ it will then be shown that under assumptions (A) and (B) all candidate processes $\pi(.)$ that could provide a higher worst case bound than $\hat{\pi}(t)$ do not deliver a higher one.

By verifying the requirements of Theorem 2 we obtain the central result of [KW02] as a special case:

Corollary 1. *There exists a strategy $\hat{\pi}(.)$ such that the corresponding expected log-utility after an immediate crash equals the expected log-utility given no crash occurs at all. It is given as the unique solution $\hat{\pi}(.) \in \left[0, {}^{1}\!/_{k^*}\right)$ of the differential equation*

$$\dot{\pi}(t) = \frac{1}{k^*}(1-\pi(t)k^*)\left(\pi(t)(b-r) - {}^{1}\!/_{2}\pi(t)^2\sigma^2 + {}^{1}\!/_{2}\left(\frac{b-r}{\sigma}\right)^2\right) \quad (17)$$

with

$$\pi(T) = 0.$$

Further, this strategy yields the highest worst-case bound for problem (7). In particular, this bound is active at each future time point ("uniformly optimal balancing"). After the crash has happened the optimal strategy is given by

$$\pi(t) \equiv \pi^* := \frac{b-r}{\sigma^2}. \quad (18)$$

For numerical examples enlightening the performance of $\hat{\pi}(.)$ see [KW02] or [KM02].

Remark: a) The form of the differential equation for the optimal portfolio process in the above corollary in particular underlines that the differential equation in Theorem 2 is only an ordinary differential equation for $\hat{\pi}(.)$ and

not for the value function $v_0(t,x)$ of the crash-free setting. This value function is assumed to be known ! Further, the form of the differential equation (17) also implies that the fraction of wealth invested in the risky stock is continuously reduced over time if there is still the possibility of a crash to happen. This is in line with practitioners' behaviour.
b) In [ME03] the above situation is generalized to the case when the market coefficients after the crash depend on the crash size and crash time. This will introduce new cases that result in different optimal strategies. We will sketch one such situation in Section 4 below.

3 Optimal worst-case investment with non-hedgeable risk

By introducing a non-hedgeable risk process into our scenario we arrive at a worts-case investment problem faced by an insurance company. This company invests at the stock market of the previous section (where for ease of notation we have set $r=0$. The uncertainty of the insurance business is modelled via a risk process of diffusion type,

$$dR(t) = \alpha dt + \beta d\tilde{W}(t). \tag{19}$$

The additional one-dimensional Brownian motion $\tilde{W}(t)$ satisfies

$$\rho = Corr\left(W(t), \tilde{W}(t)\right). \tag{20}$$

The form of the above risk process is justified by a standard diffusion approximation argument (see [BR95]). The presence of this process however also introduces the possibility of bankruptcy. It is therefore convenient to consider the total amount of money $A(t)$ that the investor invests in the stock at time t instead of the portfolio process to describe the investor's activities. The corresponding wealth process $X^A(t)$ is then given by

$$dX^A(t) = A(t)(bdt + \sigma dW(t)) + \alpha dt + \beta d\tilde{W}(t) \tag{21}$$

in normal times. At the crash time it satisfies

$$X^A(\tau) = X^A(\tau-) - kA(\tau). \tag{22}$$

We now consider the worst-case problem of the form

$$\sup_{A(.)\in S(x)} \inf_{0\le\tau\le T, 0\le k\le k^*} E\left(-e^{-\lambda X^A(T)}\right) \tag{23}$$

where $S(x)$ consists of all deterministic strategies $A(t)$ which are left-continuous with right hand limits and almost surely square integrable with respect to time. The positive constant λ measures the investor's attitude towards risk. In the crash-free situation the optimal strategy is known from [BR95] as

$$A(t) \equiv A^* = \frac{b}{\lambda\sigma^2} - \rho\frac{\beta}{\sigma}. \tag{24}$$

As in the setting of Section 2 this also forms the basis for the solution in the crash setting, a result proved in [KO03]:

Theorem 3. *"Optimal deterministic strategy with crash and risk process"*
If A^* is positive then the optimal deterministic amount of money invested in the stock before the crash is given by

$$A(t) = \frac{2k^*}{\lambda\sigma^2(t-T) - {}^{2k^*}\!/_{A^*}} + A^*. \tag{25}$$

The optimal amount of money invested into the stock after a crash equals A^*.

Remark: a) Theorem 3 differs from Corollary 1 by the fact that we now have an explicit expression for the optimal strategy. The reason for this is that the corresponding differential equation - obtained from the indifference argument mentioned in the sketch of the proof of Theorem 2 - can be solved explicitly. Indeed, this is the main difference in the proof of Theorem 3 which otherwise is very similar to the one of Theorem 2.
b) Note that one always invests less money in the stock than in the crash free model. The corresponding optimal wealth process is still a Brownian motion with drift (as in [BR95]) but now with a non-constant one. Figure 1 below shows the typical form of the optimal strategy before and after a crash. Note that the more negative the risk process is correlated with the stock price process the closer the optimal crash strategy approaches the one in the crash free setting.

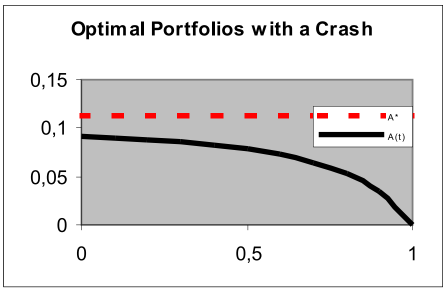

Fig. 1. Optimal investment for insurers with exponential utility and $b = 0.2, r = 0, \sigma = 0.4, k^* = 0.2, T = 1, \alpha = 0.3, \beta = 0.4, \lambda = 100, \rho = -0.1$

4 Generalizations and open problems

a) *Changing market conditions after a crash*
Typically after a crash the market price of risk or some market coefficients change as the expectations on the future perfomance of the stock price is then seen differently by the market participants. This feature is addressed in [KM02] in the stock market setting. It is extended to the insurer‘s case in [KO03]. The new aspect entering the scene is the fact that a crash need not necessarily be extremely disadvantageous if it happens, it can even be advantageous if it happens early when the market situation is better after the crash. To make things more precise, we assume that in normal times after the crash the stock price and the risk process follow

$$dP_1(t) = P_1(t)(b_1 dt + \sigma_1 dW(t)) \tag{26}$$

$$dR(t) = \alpha dt + \beta d\tilde{W}(t) \tag{27}$$

with $\rho_1 = Corr\left(W(t), \tilde{W}(t)\right)$. This leads to an optimal strategy after the crash of

$$A_1{*} = \frac{b_1}{\lambda\sigma_1^2} - \rho_1 \frac{\beta}{\sigma_1}. \tag{28}$$

This new aspect of the possibly advantageous crash leads to the following new optimality result given in [KO03]:

Theorem 4. *"Optimal deterministic strategy with crash, risk process, and changing market"*
Let A^* be positive.
a) If A_1^* is smaller than A^* then the results of Theorem 3 stay valid with A^* replaced by A_1^*.
b) If A_1^* is positive and bigger than A^* then the optimal strategy before the crash is given by

$$A(t) = \min\left(A^*, \frac{2k^*}{\lambda\sigma_1^2(t-T) - {2k^*}/{A_1{*}}} + A_1{*}\right). \tag{29}$$

The optimal amount of money invested into the stock after a crash equals A_1^*.

An example illustrating Theorem 4 is given in Figure 2 where we have used the parameters $b = 0.2, r = 0, \sigma = 0.4, k^* = 0.2, T = 1, \alpha = 0.3, \beta = 0.4, \lambda = 100, \rho = -0.1, b_1 = 0.25, r_1 = 0, \sigma_1 = 0.3$. Note that due to the attractiveness of the crash we are allowed to follow the optimal strategy in the crash-free setting until $t = 0,6$.

b)*n possible crashes*
Further aspects of the model such as the case of at most n possible crashes or more than one stock are considered in [KW02] and in [KM02]. As we

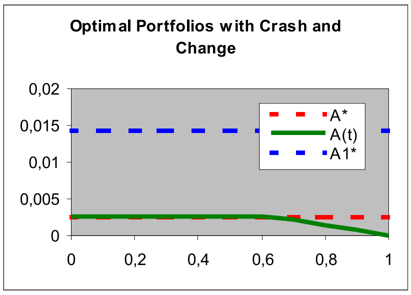

Fig. 2. Optimal investment with crash, changing coefficients, risk process

now have to face n different crash scenarios we have to solve a system of n differential equations which however can be solved in an inductive fashion. Also it is shown in [ME03] that the above results are not changed if there is a probability distribution on the number of crashes that can still happen. The worst case criterion is thus independent on the personal view of the probability for the worst case to appear as long as this probability is positive.

c)*Further aspects*
Interesting topics for future can be (among others):

- *including consumption to the portfolio problem*
- *use of options or option pricing under the threat of a crash*
- *of standard Hamilton-Jacobi-Bellman techniques that do not make use of the indifferernce argument but result in a Hamilton-Jacobi-Bellman equation (or more precisely into a variational inequality) for the value function before the crash.*

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