On the Pathwise Large Deviations of Stochastic Differential and Functional Differential Equations with Applications to Finance

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Declaration

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To My Parents

Contents

Abstract

The thesis deals with the asymptotic behaviour of highly nonlinear stochastic differential equations, as well as linear and nonlinear functional differential equations. Both ordinary functional and neutral equations are analysed. In the first chapter, a class of nonlinear stochastic differential equations which satisfy the Law of the Iterated Logarithm is studied, and the results applied to a financial market model. Mainly scalar equations are considered in the first chapter. The second chapter deals with a more general class of finite-dimensional nonlinear SDEs and SFDEs, employing comparison and time change methods, as well as martingale inequalities, to determine the almost sure rate of growth of the running maximum of functionals of the solution. The third chapter examines the exact almost sure rate of growth of the large deviations for affine stochastic functional differential equations, and for equations with additive noise which are subject to relatively weak nonlinearities at infinity. The fourth chapter extends conventional conditons for existence and uniqueness of neutral functional differential equations to the stochastic case. The final chapter deals with large fluctuations of stochastic neutral functional differential equations.

Introduction and Preliminaries

0.1 Introduction

The classical Efficient Market Hypothesis by Fama in the 1960's (cf. eg. [34]) asserts that current prices of assets truly reflect the information available to all investors and the their collective beliefs about future. This implies that no investors can outperform the market by using any public information. In particular, the weakest form of efficiency refers to that historical price information can not be used to generate any profit. Stochastic differential equations (SDEs) are common tools in the modelling of financial objects in efficient markets. The famous stock pricing model Geometric Brownian Motion (GBM) is a good example. However, the presence of price bubbles and crashes shows that markets are not always efficient, especially when the prices deviate significantly from their fundamental value. These phenomena are thought to be caused by widely–used feedback trading strategies. In order to reflect the occasional price persistency, it is reasonable to use stochastic functional differential equations (SFDEs) with delays to model price evolution.

SFDEs are commonly used in modelling systems which evolve in a random environment and whose evolution depends on the past states of the system through either memory or time delay. Examples include population biology (Mao [59], Mao and Rassias [61, 62]), neural networks (cf. e.g. Blythe et al. [20]), viscoelastic materials subjected to heat or mechanical stress Drozdov and Kolmanovskii [32], Caraballo et al. [26], Mizel and Trutzer [64, 65]), or financial mathematics (Ahn et al. [1, 2], Arrojas et al. [14], Hobson and Rogers [46]).

To date there is comparatively little literature regarding the size of large fluctuations of the solution of SDEs and SFDEs. In this thesis, we mainly study the rates at which large fluctuations of solutions of both SDEs and SFDEs tend to infinity. More precisely, if X is the solution of the stochastic equation, we try to find two constants C_1 and C_2, and a deterministic and continuous function ϱ with $\varrho(t) \to \infty$ as $t \to \infty$ such that

$$C_1 \le \limsup_{t\to\infty} \frac{|X(t)|}{\varrho(t)} \le C_2, \quad \text{a.s. conditionally on some non–null event } A,$$

or in some cases

$$\limsup_{t\to\infty} \frac{|X(t)|}{\varrho(t)} = 1, \quad \text{a.s. conditionally on some non-null event } A.$$

We call such a function the *essential growth rate of the running maxima* of X. In applications this is important, as the size of the large fluctuations may represent the largest bubble or crash in a financial market (or the largest epidemic in a disease model,or a population explosion in an ecological model). By comparing results of both SDEs and SFDEs, we investigate how feedback trading strategies affect the size of the largest fluctuations in stock prices or returns.

The *Law of the Iterated Logarithm* (LIL) is one of the most important characteristics of finite–dimensional standard Brownian motions. In Chapter 1, we classify a family of SDEs which has the form

$$dX(t) = f(X(t), t)\,dt + g(X(t), t)\,dB(t),$$

and whose solutions obey the LIL. We give sufficient conditions on f and g which ensure LIL–type results. Moreover, we investigate the relation between the drift coefficient f and the ergodicity of the process. The results are used in the modelling of market inefficiency: The usual source of randomness in the SDE (namely Brownian motion) which governs the evolution of a Geometric Brownian Motion, is replaced by a semimartingale which obeys the LIL and whose increments (changes in the logarithm of prices) are no longer Gaussian and independent. This semimartingale is constructed in such a way that it reflects the risk–averse behaviour of investors, and it shows how bias can effect the long-run average value of log-returns. The technique used in this chapter is a combination of stochastic comparison principle and Motoo's theorem.

In Chapter 2, we compare this Motoo–Comparison technique with the existing EMI–GI (Exponential Martingale Inequality and Gronwall Inequality) technique developed by Mao. We extend SDEs in Chapter 1 to some highly non-linear SDEs using the Motoo–

Comparison technique. Moreover, we show that the technique also works well on some SFDEs with point delay which have recurrent solutions.

In Chapter 3, we study the essential growth rate of the partial maxima growth rate of solutions of finite–dimensional affine SFDEs with additive noise. The general idea is that the solution of linear SFDEs can be written in terms of the fundamental solution (or the resolvent). The roots of the characteristic equation determine the asymptotic behaviour of the resolvent, which in turn determine the asymptotic behaviour of the corresponding stochastic solution. Moreover, if the resolvent decays exponentially, then the stochastic process is Gaussian and asymptotically stationary, therefore the partial maxima growth rate has order $\sqrt{\log t}$. The results can even be extended to some SFDEs with maximum functionals, provided that the non-linear term grows slower than linear order at infinity.

In Chapter 4, we study the existence and uniqueness of solutions of stochastic neutral functional differential equations (SNFDEs) of the form

$$d(X(t) - D(X_t)) = f(X_t)\,dt + g(X_t)\,dB(t).$$

The existing result on SNFDEs which was developed by Mao in the 1990's requires that the neutral functional D to satisfy a *global* contraction condition, that is, D satisfies

$$|D(\phi) - D(\varphi)| \le \kappa||\phi - \varphi||_{\sup}, \quad \text{for all } \phi, \varphi \in C([-\tau, 0]; \mathbb{R}^d).$$

where $\kappa < 1$. One the other hand, in the 1970's, Hale developed a *local* contraction condition on the deterministic neutral functional differential equations (NFDEs) of the form

$$\frac{d}{dt}(x(t) - D(x_t)) = f(x_t).$$

The "local" condition is much weaker than the "global" condition, enabling us to remove the condition $\kappa < 1$ in most cases. We adapt Mao's technique for the stochastic case and extend Hale's theorem to SNFDEs. By giving some equations which do not have solutions, we show that Hale's condition is an optimal one, and in the case of a maximal neutral functional D, that Mao's condition can not be relaxed.

In the final chapter, we again study the essential growth rate of the running maxima of the solutions of SNFDEs. As in Chapter 3, the characteristic question of the under lying deterministic resolvent is crucial in determining the asymptotic behaviour of the stochastic process. Many elements in the results and method of proof can be extended from those in Chapter 3. Since the equations are affine, we concentrate on solutions which are Gaussian and asymptotically stationary. For simplicity, we deal with scalar and affine equations only, believing that extensions to finite–dimensional and weakly nonlinear equations are relatively routine. In comparison with the non-neutral resolvent, the neutral resolvent also decays exponentially. However, unlike the non-neutral resolvent which is everywhere differentiable, the differentiability of neutral resolvent is uncertain. Therefore the technique used in the neutral case is distinct from that in Chapter 3.

0.2 Preliminaries

Notations The following notations are used in this thesis:

$\mathbb{R}$: set of real numbers.

$\mathbb{R}^+$: set of non-negative real numbers.

$\mathbb{R}^d$: d-dimensional Euclidean space.

$\mathbb{C}$: set of complex numbers.

$\mathbb{R}^{d\times r}$: set of d by r matrices.

A^T : the transpose of $A \in \mathbb{R}^{d\times r}$.

$\det A$: the determinate of a square matrix A.

$\mathrm{Re}(z)$: the real part of $z \in \mathbb{C}$.

$\mathrm{Im}(z)$: the imaginary part of $z \in \mathbb{C}$.

$x \vee y$: the maximum value between x and y.

$x \wedge y$: the minimum value between x and y.

$f * g$: the convolution of two functions f and g.

$\langle \cdot, \cdot \rangle$: the standard inner product on $\mathbb{R}^d$.

D^+ : the upper Dini derivative, i.e. if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then

$$D^+ f(t) := \limsup_{h\to 0^+} \frac{f(t+h) - f(t)}{h}.$$

$|\cdot|$: the Euclidean norm on a row or column vector.

$||\cdot||$: the Frobenius norm of a matrix $A \in \mathbb{R}^{d\times r}$.

$||\cdot||_{\mathrm{op}}$: the operator norm of a matrix $A \in \mathbb{R}^{d\times r}$, i.e. $||A||_{\mathrm{op}} = \sup_{x\in\mathbb{R}^r, |x|=1} |Ax| = \sqrt{\lambda_{max}(A^T A)}$, where $\lambda_{max}(A^T A)$ stands for the largest eigenvalue of the square matrix $A^T A$. Note that $||A||_{\mathrm{op}} \le ||A|| \le \sqrt{r}||A||_{\mathrm{op}}$.

$|\cdot|_\infty$: the maximum norm of a row or column vector.

$||\cdot||_{\mathrm{sup}}$: the supremum norm.

$\mathbf{e}_i$: the i-th standard basis vector in $\mathbb{R}^d$.

$\mathcal{N}(a, b)$: normal distribution with mean a and standard distribution b.

C^p: set of functions whose p-th derivative are continuous.

$\mathcal{RV}_\infty(\beta)$: the family of functions which are regularly varying at infinity with index β. A measurable function $f : [l, \infty) \to (0, \infty)$ for some $l \in (0, \infty)$, is called *regularly varying of index* $\beta \in \mathbb{R}$ if and only if $f(\lambda x)/f(x) \to \lambda^\beta$ as $x \to \infty$, for all $\lambda > 0$.

$\mathcal{SRV}_\infty(\beta)$: the family of functions which are smoothly varying at infinity with index β (cf. [19, Section 1.8]). A function $f \in \mathcal{RV}_\infty(\beta)$ *varies smoothly* with index β, if and only if $h(x) := \log f(e^x)$ is C^∞, and $h'(x) \to \beta$, $h^{(n)}(x) \to 0$ $(n = 2, 3, ...)$ as $x \to \infty$. One consequence is that $xf'(x)/f(x) \to \beta$ as $x \to \infty$.

$M([a, b]; \mathbb{R}^{d\times d})$: the space of finite Borel measures on $[a, b]$ with values in $\mathbb{R}^{d\times d}$.

$L^p([a, b]; \mathbb{R}^d)$: the family of Borel measurable functions $h : [a, b] \to \mathbb{R}^d$ such that $\int_a^b |h(x)|^p\,dx < \infty$.

$\mathcal{M}^p([a, b]; \mathbb{R}^d)$: the family of processes $\{h(t)\}_{a\le t\le b}$ in $L^p([a, b]; \mathbb{R}^d)$ such that $\mathbb{E}[\int_a^b |h(x)|^p\,dx] < \infty$.

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P})$: a complete probability spaces with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions, i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets.

Definitions and Technical Issues The major relevant definitions and theorems on technical issues are given here:

Scale function and speed measure: let $I := (l, r)$ with $-\infty \le l < r \le \infty$, and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be the drift and diffusion coefficients of a scalar autonomous stochastic differential equation respectively. Moreover, f and g satisfy the *non-degeneracy* and *local integrability* conditions:

$$g^2(x) > 0, \quad \forall\, x \in I; \tag{0.2.1}$$

$$\forall\, x \in I, \quad \exists \epsilon > 0 \quad \text{such that} \int_{x-\epsilon}^{x+\epsilon} \frac{1 + |f(y)|}{g^2(y)} dy < \infty. \tag{0.2.2}$$

Under the above conditions, a scale function and speed measure of solution of this SDE are defined as

$$s_c(x) = \int_c^x e^{-2\int_c^y \frac{f(z)}{g^2(z)}\,dz}\,dy, \quad m(dx) = \frac{2dx}{s_c'(x)g^2(x)}, \;\; c,\, x \in I, \tag{0.2.3}$$

where I is the state space of the process. These functions help us to determine the recurrence and stationary of a process on I by Feller's test for explosions (cf. [49]). Moreover, Feller's test allows us to examine whether a process will escape from its space in finite time. This in turn relies on the v-function.

v-function: if s_c is a scale function, then the v-function is defined as

$$v_c(x) = \int_c^x s_c'(y) \int_c^y \frac{2dz}{s_c'(z)g^2(z)}\, dy, \; c \in \mathbb{R}, \; x \in \mathbb{R}. \tag{0.2.4}$$

A process will reach the boundary of its state space within finite time if and only if $v_c(l+) = v_c(r-) = \infty$. Note that the real number $c \in (l, r)$ appeared in the definitions of both scale function and v-function does not affect whether or not s and v are finite at the boundaries l and r.

Doob's continuous martingale representation theorem: suppose M is a continuous local martingale defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and the square variation $\langle M \rangle$ is an absolutely continuous function of t for $\mathbb{P}$-almost every ω. Then there is an extended space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ of $(\Omega, \mathcal{F}, \mathbb{P})$ on which is defined a one-dimensional Brownian motion $W = \{W(t), \tilde{\mathcal{F}}_t; 0 \le t < \infty\}$ and a $\tilde{\mathcal{F}}_t$-adapted process X with $\tilde{\mathbb{P}}$-a.s.

$$\int_0^t X^2(s) ds < \infty, \quad 0 \le t < \infty,$$

such that we have the representations $\tilde{\mathbb{P}}$-a.s.

$$M(t) = \int_0^t X(s)\, dW(s), \quad \langle M \rangle(t) = \int_0^t X^2(s)\, ds, \quad 0 \le t < \infty.$$

In the proof of the above martingale representation theorem (which can be found in [49, Theorem 3.4.2]), the new Brownian motion W was constructed by a continuous local martingale with respect to the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a another Brownian motion, say $\widehat{B}$, which was defined on the extended part of $(\Omega, \mathcal{F}, \mathbb{P})$ in $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Moreover, $\widehat{B}$ is independent of M. Therefore in this report, any conclusion made with respect to the extended measure $\tilde{\mathbb{P}}$ about the underlying process with diffusion M defined on $(\Omega, \mathcal{F}, \mathbb{P})$ coincides with that with measure $\mathbb{P}$. Therefore we do not make explicit reference to the probability spaces when stating results.

Properties of measures: The total variation of a measure ν in $M([-\tau, 0]; \mathbb{R}^{d \times r})$ on a Borel set $B \subseteq [-\tau, 0]$ is defined by

$$|\nu|_m(B) := \sup \sum_{i=1}^N ||\nu(E_i)||,$$

where $(E_i)_{i=1}^N$ is a partition of B and the supremum is taken over all partitions. The total variation defines a positive scalar measure $|\nu|_m$ in $M([-\tau, 0]; \mathbb{R})$. One can easily establish for the measure $\nu = (\nu_{i,j})_{i,j=1}^d$ the inequality

$$|\nu|_m(B) \le C \sum_{i=1}^d \sum_{j=1}^d |\nu_{i,j}|(B) \qquad \text{for every Borel set } B \subseteq [-\tau, 0] \tag{0.2.5}$$

with $C = 1$. Then, by the equivalence of every norm on finite-dimensional spaces, the inequality (0.2.5) holds true for the arbitrary norms and some constant $C > 0$. Moreover, as in the scalar case we have the fundamental estimate

$$\left| \int_{[-\tau,0]} \nu(ds)\, f(s) \right| \le \int_{[-\tau,0]} |f(s)|\; |\nu|_m(du)$$

for every function $f : [-\tau, 0] \to \mathbb{R}^{d\times r}$ which is $|\nu|_m$-integrable.

Convolution: The convolution of a function f and a measure ν is defined by

$$\nu * f : \mathbb{R}^+ \to \mathbb{R}^{d\times r}, \qquad (\nu * f)(t) := \int_{[0,t]} \nu(ds)\, f(t-s).$$

The convolution of two functions is defined analogously.

Stochastic Fubini's Theorem (cf., e.g., [68, Ch. IV.6, Theorem 64]): Let X be a semimartingale and $(A, \mathcal{A})$ be a measurable space, $H_t^a = H(a, t, \omega)$ be a bounded $\mathcal{A} \otimes \mathcal{P}$ measurable function ($\mathcal{P}$ denotes the predictable σ-algebra), and let μ be a finite measure on $\mathcal{A}$. Let $Z_t^a = \int_0^t H_s^a\, dX_s$ be $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$ measurable such that for each $a \in A$, Z^a is a càdlàg (i.e.,stochastic process which a.s. has sample paths that are left continuous with right limits.) version of $H^a \cdot X$. Then $Y_t = \int_A Z_t^a \mu(da)$ is a càdlàg version of $H \cdot X$, where $H_t = \int_A H_t^a \mu(da)$.

Chapter 1

Solutions of Stochastic Differential Equations obeying the Law of the Iterated Logarithm

1.1 Introduction

The following Law of the Iterated Logarithm (LIL) is one of the most important results on asymptotic behaviour of finite-dimensional standard Brownian motions,

$$\limsup_{t\to\infty} \frac{|B(t)|}{\sqrt{2t\log\log t}} = 1, \quad \text{a.s.} \tag{1.1.1}$$

Classical work on iterated logarithm–type results, as well as associated lower bounds on the growth of transient processes date back to Dvoretzky and Erdős [33]. There is an interesting literature on iterated logarithm results and the growth of lower envelopes for self-similar Markov processes (cf. e.g., Rivero [72], Chaumont and Pardo [27]) which exploit a Lamperti representation [53], processes conditioned to remain positive (cf. Hambly et al. [45]), and diffusion processes with special structure (cf. e.g. Bass and Kumagi [18]).

In contrast to these papers the analysis here is inspired by work of Motoo [67] on iterated logarithm results for Brownian motions in finite dimensions, in which the asymptotic behaviour is determined by means of time change arguments to reduce the process under study to a stationary one. Our paper concentrates mainly on iterated logarithm upper bounds of solutions of stochastic differential equations, as well as obtaining lower envelopes for the growth rate. Our goal has been to establish these results under the minimum continuity and asymptotic conditions on the drift and diffusion coefficients. An advantage of this approach is that it enables us to analyse a class of equations of the form

$$dX(t) = f(X(t))\,dt + g(X(t))\,dB(t)$$

for which $xf(x)/g^2(x)$ tends to a finite limit as $x \to \infty$ in the case when f and g are regularly varying at infinity. Ergodic type–theorems are also presented. We also show

how results can be extended to certain classes of non-autonomous and finite-dimensional equations. We employ extensively comparison arguments of various kinds throughout.

In this chapter, we give the sufficient conditions ensuring that these processes obey the LIL in the sense of (1.1.1). In particular, for a parameterized family of autonomous SDEs, we observed that solutions can change from being recurrent to transient when a critical value of the bifurcation parameter $L = \sigma^2/2$ (where $\lim_{x\to\pm} xf(x) = L$ and $g(x) = \sigma$ for all $x \in \mathbb{R}$) is exceeded while preserving the properties of the LIL. Among the results, we examine the extent to which the drift can be perturbed so that in the long-run the size of the large deviations remains the same as for Brownian motion.

In [57], Mao shows that if X is the solution of the d–dimensional equation

$$dX(t) = f(X(t), t)\,dt + g(X(t), t)\,dB(t), \quad t \geq 0$$

and if there exist positive real numbers ρ, K such that for all $x \in \mathbb{R}^d$ and $t \geq 0$, $x^T f(x,t) \leq \rho$, and $||g(x,t)||_{op} \leq K$, then

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \leq K\sqrt{e}, \quad \text{a.s.} \tag{1.1.2}$$

The main steps of the Mao's proof are as follows: first, make a suitable Itô transformation; then estimate the size of the Itô integral term by a Riemann integral by means of the *exponential martingale inequality* (EMI); and finally apply *Gronwall's inequality* (GI) to determine the asymptotic rate of growth.

In contrast, the results in this chapter are established through a combination of *comparison principles* and *Motoo's theorem.* Motoo's theorem (cf. [67]) determines the *exact* asymptotic growth rate of the partial maxima of a *stationary* or *asymptotically stationary* process governed by an autonomous SDE. Since we will frequently refer this theorem, it is stated here for convenience.

Theorem 1.1.1. *Let $f : (l,\infty) \to \mathbb{R}$ and $g : (l,\infty) \to \mathbb{R}$ satisfy 0.2.1 and 0.2.2, and X be the unique continuous adapted process satisfying*

$$dX(t) = f(X(t))\,dt + g(X(t))\,dB(t), \qquad t \geq 0.$$

If a scale function s and the speed measure m, as defined in the preliminaries, satisfy

$$s(l) = -\infty, \quad s(\infty) = \infty \quad \textit{and} \quad m(l,\infty) < \infty,$$

then X is asymptotically stationarily recurrent on its state space (l,∞). Moreover, for some $t_0 > 0$, if $\varrho : (t_0,\infty) \to (0,\infty)$ is an increasing function with $\varrho(t) \to \infty$ as $t \to \infty$, then

$$\mathbb{P}\left[\limsup_{t\to\infty} \frac{X(t)}{\varrho(t)} \geq 1\right] = 1 \quad \textit{or} \quad 0$$

depending on whether

$$\int_{t_0}^{\infty} \frac{1}{s(\varrho(t))}\, dt = \infty \quad \textit{or} \quad \int_{t_0}^{\infty} \frac{1}{s(\varrho(t))}\, dt < \infty.$$

for some $t_0 > 0$.

In [67], Motoo also gave a proof of the Law of the Iterated Logarithm for a finite-dimensional Brownian motion. This proof is crucially reliant on applying a change in both space and scale. He considers an autonomous non–stationary δ–dimensional Bessel process R_δ, which is governed by the following scalar equation

$$dR_\delta(t) = \frac{\delta - 1}{2R_\delta(t)}\, dt + dB(t) \tag{1.1.3}$$

with $R_\delta(0) = r_0 \geq 0$. The Bessel process R_δ is turned into an autonomous process with finite speed measure (i.e., solutions that possess limiting distributions), to which the Motoo's theorem can be applied. More precisely, if we let

$$S_\delta(t) = e^{-t} R_\delta^2(e^t - 1), \tag{1.1.4}$$

then

$$dS_\delta(t) = (\delta - S_\delta(t))\, dt + 2\sqrt{S_\delta(t)}\, d\tilde{B}(t). \tag{1.1.5}$$

It is reasonable to ask whether a combination of space and scale transformation of this classic type could reduce general non-stationary autonomous SDEs to those with finite speed measure to which Motoo's theorem could then be applied. If we consider general transformations of the form

$$Y(t) = \lambda(t) P(X(\gamma(t)))$$

where $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is increasing, $P \in C^2(\mathbb{R};\mathbb{R})$ and $\lambda \in C^1(\mathbb{R}^+;\mathbb{R}^+)$ which is related to γ, the resulting SDE for Y will be non–autonomous, and in particular, will have non-autonomous diffusion coefficient. Adapting the proof of Motoo's theorem to cope with SDEs with non-autonomous diffusion coefficients introduces formidable difficulties. Because the independence of excursions, on which the proof stands can no longer be assured.

However, in this chapter, with the well–known stochastic comparison principle on the monotonicity of the drift coefficients, we are able to investigate a much wider class of SDEs which are related to (1.1.3) through (1.1.4)—or similar rescalings— that give equations of the type (1.1.5). In addition, with ordinary Itô transformations, we could map an even wider class of nonlinear equations onto those of known nature as shown in the next chapter. A detailed discussion on the relative advantages and disadvantages of this comparison-Motoo technique with the existing EMI-Gronwall approach can also be found in the next chapter.

In [3], Appleby et al. applied processes obeying the Law of the Iterated Logarithm to inefficient financial market models. In this chapter, we further investigate the ergodic–like properties of these processes, and interpret the results in financial market.

This chapter considers a number of closely related equations, and proves a number of diverse asymptotic results. In order to understand the relationships between these results and to ease the readers' path through the chapter, we give a synopsis and discussion about the main results, and their applications in Section 1.2. Full statements of the theorems and detailed proofs are found in succeeding sections.

The work in this chapter appears mainly in a paper, joint with John Appleby [12].

1.2 Synopsis and Discussion of Main Results

In this section, we give a discussion of the results proven in this chapter. First, we prove the LIL and other asymptotic growth bounds for transient processes for autonomous SDEs. Second, we discuss general non-autonomous equations for which the LIL holds, under some unified estimates on the drift. Third, we prove comprehensive results for a parameterized family of autonomous SDEs with constant diffusion coefficient, which do not require uniform estimates on the drift. Finally, we discuss some extension of these results to multidimensional SDEs and the applications of the results in this chapter to inefficient financial markets.

Transient processes Our first main result, Theorem 1.3.1, concerns transient solutions of the scalar autonomous stochastic differential equation

$$dX(t) = f(X(t))\,dt + g(X(t))\,dB(t) \tag{1.2.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ satisfies 0.2.1, $g(x) = \sigma$ for $x \in \mathbb{R}$, and

$$\lim_{x\to\infty} xf(x) = L_\infty > \frac{\sigma^2}{2}. \tag{1.2.2}$$

If we define $A := \{\omega : \lim_{t\to\infty} X(t,\omega) = \infty\}$, then $\mathbb{P}[A] > 0$, and we show that the solution X obeys

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = |\sigma|, \quad \text{a.s. conditionally on } A \tag{1.2.3}$$

and

$$\liminf_{t\to\infty} \frac{\log \frac{X(t)}{\sqrt{t}}}{\log\log t} = -\frac{1}{\frac{2L_\infty}{\sigma^2} - 1}, \quad \text{a.s. conditionally on } A.$$

X exhibits similar transient behaviour at minus infinity if

$$\lim_{x\to-\infty} xf(x) = L_{-\infty} > \frac{\sigma^2}{2}. \tag{1.2.4}$$

These results were established through comparison with a generalized Bessel process (Lemma 1.3.1) which has similar behaviour to X. The modulus of a finite-dimensional Brownian motion (i.e., a Bessel process) with dimension greater than two is known to be transient, and when the dimension is less than or equal to two, the process is recurrent on the positive real line. However, for general Bessel processes, the critical dimension altering its behaviour does not have to be an integer. This fact is eventually captured in Theorem 1.3.1 by condition (1.2.2) (or (1.2.4)). More precisely, if exactly one of the parameters L_∞ and $L_{-\infty}$ is greater than the critical value $\sigma^2/2$, then the process tends to infinity or minus infinity almost surely while still obeying the Law of the Iterated Logarithm. If on the other hand L_∞ and $L_{-\infty}$ are both greater than $\sigma^2/2$, and we denote the event $\{\omega : \lim_{t\to\infty} X(t,\omega) = -\infty\}$ by $\tilde{A}$, we have that $\mathbb{P}[\tilde{A}] = 1 - \mathbb{P}[A]$ and both probabilities are positive and can be computed explicitly in terms of the scale function and the deterministic initial value of the process (cf. [49, Proposition 5.5.22]). Motoo's theorem also aids us to find an exact pathwise lower bound on the growth rate of the process. This result could also be very useful in determining the pathwise decay rates of asymptotically stable SDEs. In Theorem 1.3.2, the constant diffusion coefficient σ is replaced by a state-dependent coefficient $g(\cdot)$ tending to σ as x tends to infinity, and similar results are obtained by means of a random time-change argument. Theorem 1.3.1 lays the foundation for further results concerning generalized transient problems with unbounded diffusion coefficients. For example, suppose X obeys (1.2.1), where g is strictly positive and regularly varying at infinity with index β ($0 < \beta < 1$), and f and g are related via,

$$\lim_{x\to\infty} \frac{xf(x)}{g^2(x)} = L_\infty > \frac{1}{2}.$$

Then by Itô's rule, if A is as previously defined, it is easy to show that

$$\limsup_{t\to\infty} \frac{X(t)}{G^{-1}(\sqrt{2t\log\log t})} = 1, \quad \text{a.s. conditionally on } A$$

and

$$\liminf_{t\to\infty} \frac{\log \frac{G(X(t))}{\sqrt{t}}}{\log\log t} = -\frac{1-\beta}{2L_\infty - 1}, \quad \text{a.s. conditionally on } A.$$

where G is defined as

$$G(x) = \int_c^x \frac{1}{g(y)}\,dy, \quad c \in \mathbb{R}.$$

Example 1.2.1. Suppose f and g are locally Lipschitz continuous, and obeys conditions 0.2.1 and 0.2.2. Moreover, $\lim_{x\to\infty} f(x)/x^{-1/3} = 1$ and $\lim_{\to\infty} g(x)/x^{1/3} = 1$. Then $P[A] > 0$ where A is as previously defined, and

$$\limsup_{x\to\infty} \frac{X(t)}{(2t\log\log t)^{\frac{3}{2}}} = 3^{-3}, \quad \text{a.s. conditionally on } A$$

$$\liminf_{x\to\infty} \frac{\log \frac{X^{\frac{1}{3}}(t)}{\sqrt{t}}}{\log\log t} = -\frac{2}{3}, \quad \text{a.s. conditionally on } A.$$

The probability of A also depends on $L_\infty := \lim_{x\to l} xf(x)/g^2(x)$ where l is the lower bound on the state space of X. Appleby et al. (cf. [13] and [10]) studied the stability problem with f and g satisfying similar conditions. The techniques can be adapted to this problem by considering the reciprocal of the stable process (in fact, it even allows $\beta = 1$), which produce less sharper results than the one obtained in this example.

Another application of these results is given in the next subsection: we make use of the upper envelope of the growth rate (1.2.3) to determine upper bounds for a more general type of equation which obeys the Law of the Iterated Logarithm.

General conditions and ergodicity In Section 1.4 , we state and prove three theorems which give sufficient conditions ensuring Law of the Iterated Logarithm-type results, and which support results in following sections of the chapter. We will study the one–dimensional non–autonomous equation

$$dX(t) = f(X(t), t)\, dt + \sigma\, dB(t), \quad t \geq 0, \tag{1.2.5}$$

with $X(0) = x_0$.

From the results in Section 1.3 , in Theorem 1.4.1, it can be easily shown that if

$$\sup_{(x,t)\in\mathbb{R}\times\mathbb{R}^+} xf(x,t) = \rho \in (0, \infty), \tag{1.2.6}$$

then

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t \log\log t}} \leq |\sigma|, \quad \text{a.s.} \tag{1.2.7}$$

Furthermore, in Theorem 1.4.2, we prove that

$$\inf_{(x,t)\in\mathbb{R}\times\mathbb{R}^+} xf(x,t) = \mu > -\frac{\sigma^2}{2}, \tag{1.2.8}$$

implies

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t \log\log t}} \geq |\sigma|, \quad \text{a.s.} \tag{1.2.9}$$

Hence if both (1.2.6) and (1.2.8) are satisfied, we can determine the exact growth rate of the partial maxima. Moreover, we can establish an ergodic-type theorem on the average value of the process, as described by the following two inequalities which can be deduced from the known result [71, Exercise XI.1.32]:

$$\limsup_{t\to\infty} \frac{\int_0^t \frac{X^2(s)}{(1+s)^2} ds}{\log t} \le 2\rho + \sigma^2, \quad \text{a.s.} \tag{1.2.10}$$

$$\liminf_{t\to\infty} \frac{\int_0^t \frac{X^2(s)}{(1+s)^2} ds}{\log t} \ge 2\mu + \sigma^2 > 0, \quad \text{a.s.} \tag{1.2.11}$$

(1.2.7) was obtained by the construction of two transient processes as described in Section 1.3. It appears that a condition of the form (1.2.6) is necessary to ensure that the solution obeys the LIL. Suppose, for instance in equation (1.2.1) that there is $\alpha \in (0,1)$ such that $x^\alpha f(x) \to C > 0$ as $x \to \infty$. Then $X(t) \to \infty$ on some event Ω' with positive probability and

$$\lim_{t\to\infty} \frac{X(t)}{t^{\frac{1}{1+\alpha}}} = [C(1+\alpha)]^{\frac{1}{1+\alpha}}, \quad \text{a.s. on } \Omega',$$

which obviously violates the Law of the Iterated Logarithm (cf. [37, Theorem 4.17.5]). It is worth noticing that ρ does not appear in the estimate in (1.2.7). This fact is used in Theorem 1.6.3 which deals with multidimensional systems. However ρ does affect the average value of X in the long-run, as seen in (1.2.10). As mentioned in the introduction, by the Motoo-comparison approach, the estimate on the constant on the righthand side of (1.2.7) has been reduced by a factor of $\sqrt{e}$. In addition, this approach enables us to find the lower estimate (1.2.9), which is the same size as the upper estimate. This has been unachievable to date by the exponential martingale inequality approach. Condition (1.2.8) is sufficient but unnecessary for getting a LIL-type of lower bound, as will be seen in Theorem 1.4.3.

We noted already that the parameters ρ and μ in the drift do not affect the growth of the partial maxima as given by (1.2.7) and (1.2.9). However, (1.2.10) and (1.2.11) show that these parameters are important in determining the “average” size of the process, with larger contributions from the drift leading to larger average values. To cast further light on this we consider the related deterministic differential equation

$$x'(t) = f(x(t)), \quad t \ge 0, \tag{1.2.12}$$

where $xf(x) \to C > 0$ as $x \to \infty$, with the initial condition $x(0) > 0$ and is sufficiently large. Then it is easy to verify that $x^2(t)/t \to 2C$ as $t \to \infty$. Moreover, the solution satisfies

$$\lim_{t\to\infty} \frac{\int_0^t \frac{x^2(s)}{(1+s)^2} ds}{\log t} = 2C. \tag{1.2.13}$$

Comparing this with (1.2.10) and (1.2.11), suggests that, on average, the absolute value of the solution of stochastic equation (1.2.5) under condition (1.2.6) and (1.2.8) captures the basic growth rate $\sqrt{t}$ of the corresponding deterministic solution (1.2.12). It is known that the Brownian motion $X(t) := \sigma B(t)$ obeys $\mathbb{E}[X^2(t)] = \sigma t$, and using (1.2.10) and (1.2.11) with $\rho = \mu = 0$, it must also obey

$$\lim_{t\to\infty} \frac{\int_0^t \frac{X^2(s)}{(1+s)^2} ds}{\log t} = \sigma^2, \quad \text{a.s.} \tag{1.2.14}$$

We notice how this is also consistent with the behaviour of the ODE (1.2.12). (1.2.14) indicates how the Brownian motion excursions in the solution of (1.2.5) contributes the σ^2 term in (1.2.10) and (1.2.11). These two extreme cases (where there is no diffusion in the first, and no drift in the second) indicate that the contributions of drift and diffusion are of similar magnitude, and this is reflected in (1.2.10) and (1.2.11).

Theorem 1.4.3 deals with processes with integrable drift coefficients. For an autonomous equation with drift coefficient $f \in L^1(\mathbb{R};\mathbb{R})$ and constant diffusion coefficient, there exist positive constants $\{C_i\}_{i=1,2,3,4}$ such that

$$C_1 \le \limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \le C_2, \quad \text{a.s.}$$
$$-C_3 \le \liminf_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \le -C_4, \quad \text{a.s.}$$

The definitions of the estimates can be found in Section 1.4. These processes are recurrent and can be transformed to some other processes which are drift-free with bounded diffusion coefficient, which preserve the largest fluctuation size. This result is consistent with those in [37, Chapter 4], which essentially say that if the drift coefficient is zero on average along the real line and the diffusion coefficient has a positive limit σ for large values, then process has a limiting distribution of $\mathcal{N}(0, \sigma\sqrt{t})$, which exactly characterizes the Brownian motion $\{\sigma B(t)\}_{t\ge 0}$.

Recurrent processes In Section 1.5, we investigate scalar autonomous equation

$$dX(t) = f(X(t))\,dt + \sigma\,dB(t) \tag{1.2.15}$$

where the drift coefficient satisfies

$$\lim_{x\to\infty} xf(x) = L_\infty \le \sigma^2/2 \quad \text{and} \quad \lim_{x\to-\infty} xf(x) = L_{-\infty} \le \sigma^2/2. \tag{1.2.16}$$

These hypothesis are complementary to those in Section 1.3. Simple calculations on Feller's test [49] show that under condition (1.2.16), processes are no longer transient but are recurrent on the real line. However results in Section 1.3 together with Theorem 1.4.3 (which deals with integrable drift) suggest that solutions should still have asymptotic behaviour similar to the LIL. The upper bound given by Theorem 1.4.1 automatically applies, while difficulties arise in finding the lower bound on the limsup without condition (1.2.8), particularly when L_∞ and $L_{-\infty}$ are of the same sign. The subdivision of the main result into various theorems is necessitated by slight distinctions in proofs, which in turn depends on the value of both L_∞ and $L_{-\infty}$. The results are summarized with $\sigma = 1$ in Figure 1.

Theorem 1.5.1 is a direct result of Motoo's theorem: it shows that $-\sigma^2/2$ is another critical value at which the behaviour of the process changes from being stationary or asymptotically stationary to non-stationary. The LIL is no longer valid when $L_{\pm\infty} < -\sigma^2/2$. By constructing another asymptotically stationary process as a lower bound for X^2 and X in Theorem 1.5.2 and 1.5.3 respectively, we obtain the following exact estimate on the polynomial Liapunov exponent $|X|$:

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{\log t} = \frac{1}{2}, \quad \text{a.s.} \tag{1.2.17}$$

(1.2.17) is a less precise result than the LIL. It shows that the partial maxima of the solution grows at least as fast as $K_\varepsilon t^{(1-\varepsilon)/2}$ for $\varepsilon \in (0,1)$ and some positive K_ε, which is still consistent with the LIL and supports our conjecture. Using the same construction (see Lemma 1.5.2) and comparison technique, together with Theorem 1.4.3, we obtain Theorem 1.5.4 which gives upper and lower estimates on the growth rate of the partial maxima.

Note that we have excluded zero from Figure 1 for the purpose of stating consistent results on pairs of intervals for L_∞ and $L_{-\infty}$. Theorem 1.5.4 covers the case that at least one of the limits is zero and the drift coefficient f changes sign for an even number of

$L_{-\infty}$ \ L_∞	$(-\infty,-\frac{1}{2})$	$[-\frac{1}{2},0)$	$(0,\frac{1}{2}]$	$(\frac{1}{2},\infty)$
$(-\infty,-\frac{1}{2})$	asymptotically stationary violates LIL Theorem 1.5.1	recurrent **C**, **D** Theorem 1.5.3	recurrent **B** Theorem 1.5.4 Part (i)	$\lim_{t\to\infty} X(t)=\infty$ **A** Theorem 1.3.1
$[-\frac{1}{2},0)$	recurrent **C**, **D** Theorem 1.5.3	recurrent **C**, **D** Theorem 1.5.2	recurrent **B** Theorem 1.5.4 Part (i)	$\lim_{t\to\infty} X(t)=\infty$ **A** Theorem 1.3.1
$(0,\frac{1}{2}]$	recurrent **B** Theorem 1.5.4 Part (ii)	recurrent **B** Theorem 1.5.4 Part (ii)	recurrent **C**, **D** Theorem 1.5.2	$\lim_{t\to\infty} X(t)=\infty$ **A** Theorem 1.3.1
$(\frac{1}{2},\infty)$	$\lim_{t\to\infty} X(t)=-\infty$ **A** Corollary 1.3.2	$\lim_{t\to\infty} X(t)=-\infty$ **A** Corollary 1.3.2	$\lim_{t\to\infty} X(t)=-\infty$ **A** Corollary 1.3.2	$\lim_{t\to\infty} X(t)=\pm\infty$ **A** Theorem 1.3.1, Corollary 1.3.2

Table 1.1: Asymptotic behaviour of X obeying (1.2.1) where $\lim_{x\to\infty} xf(x) = L_\infty$ and $\lim_{x\to-\infty} xf(x) = L_{-\infty}$ and $g(x) = 1$. **A** signifies that X obeys the Law of the Iterated Logarithm exactly; **B** that $|X(t)|$ is bounded above and below by $\sqrt{2t\log_2 t}$ as $t\to\infty$; **C** that X has a polynomial upper Liapunov exponent equal to 1/2; and **D** that the asymptotic behaviour is consistent with the Law of the Iterated Logarithm.

times. In particular, if f remains non–negative or non–positive on the real line, X can be pathwise compared with the Brownian motion $\{\sigma B(t)\}_{t\geq 0}$ directly, so an exact estimate can be obtained (Corollary 1.5.1). Otherwise, Theorem 1.5.2 and 1.5.3 are sufficient to cover the rest of the possible cases (Remark 1.5.1).

Multidimensional processes In Section 1.6, we generalize results from Section 1.4 to the following d-dimensional equation driven by an m–dimensional Brownian motion

$$dX(t) = f(X(t),t)\,dt + g(X(t),t)\,dB(t). \tag{1.2.18}$$

Theorem 1.6.1 extends the result of Theorem 1.4.1 to SDEs with bounded diffusion coefficients under condition similar to (1.2.6). Through a random time-change to the process, we prove that

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \leq C_a, \quad \text{a.s.}$$

where $C_a := \sup_{(x,t)\in\mathbb{R}^d\times\mathbb{R}^+} ||g(x,t||_{op}$. In a like manner, Theorem 1.6.2 complements Theorem 1.4.2 in $\mathbb{R}^d$. The generalisation of these results to unbounded diffusion coefficients can be found in the next chapter. Finally, Theorem 1.6.3 shows that if the Euclidean norm of a multidimensional process generally grows at the rate of the iterated logarithm, then the order of the actual size of the largest fluctuations of the norm is given by the coordinate process with the largest fluctuations. This result is an extension of the LIL for a d-dimensional Brownian motion (1.1.1). Mao (cf. [57]) pointed out the fact that the independent individual components of the multidimensional Brownian motion are not simultaneously of the order $\sqrt{2t\log\log t}$, for otherwise we would have $\sqrt{d}$ rather than unity on the right-hand side of (1.1.1). We establish this fact for drift–perturbed finite–dimensional Brownian motions. To simplify the analysis, we look at the following equation in $\mathbb{R}^d$:

$$dX(t) = f(X(t),t)\,dt + \Gamma\,dB(t), \quad t\geq 0 \tag{1.2.19}$$

where Γ is a $d\times d$ diagonal invertible matrix with diagonal entries $\{\gamma_i\}_{1\leq i\leq d}$. If $\langle x, f(x,t)\rangle \leq \rho$, then

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \leq \max_{1\leq i\leq d} |\gamma_i|, \quad \text{a.s.}$$

Furthermore if there exists one coordinate process X_i with drift coefficient f_i satisfying (1.2.8), then we have

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \geq |\gamma_i|, \quad \text{a.s.}$$

In the more general case that Γ is any invertible matrix, with the same conditions as above, the proof of this result can be easily adapted to show that with respect to the norm $|x|_\Gamma := |\Gamma^{-1}x|$, the solution of (1.2.19) satisfies

$$\limsup_{t\to\infty} \frac{|X(t)|_\Gamma}{\sqrt{2t\log\log t}} = 1, \quad \text{a.s.}$$

Applications to inefficient financial markets According to Fama [34], when efficiency refers only to historical information which is contained in every private trading agent's information set, the market is said to be *weakly efficient* (cf.[35, Definition 10.17]. Weak efficiency implies that successive price changes (or returns) are independently distributed. Formally, let the market model be described by a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that trading takes place in continuous time, and that there is one risky security. Let $h > 0$, $t \geq 0$ and let $r_h(t+h)$ denote the return of the security from t to $t+h$, and let $S(t)$ be the price of the risky security at time t. Also let $\mathcal{F}(t)$ be the collection of historical information available to every market participant at time t. Then the market is weakly efficient if

$$\mathbb{P}[r_h(t+h) \leq x|\mathcal{F}(t)] = \mathbb{P}[r_h(t+h) \leq x], \quad \forall\, x \in \mathbb{R},\ \ h > 0,\ \ t \geq 0.$$

Here the information $\mathcal{F}(t)$ which is publicly available at time t is nothing other than the generated σ-algebra of the price $\mathcal{F}^S(t) = \sigma\{S(u) : 0 \leq u \leq t\}$. An equivalent definition of weak efficiency in this setting is that

$$r_h(t+h) \text{ is } \mathcal{F}^S(t)\text{-independent, for all } h > 0 \text{ and } t \geq 0. \tag{1.2.20}$$

Geometric Brownian Motion is the classical stochastic process that is used to describe stock price dynamics in a weakly efficient market. More concretely, it obeys the linear SDE

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dB(t), \quad t \geq 0 \tag{1.2.21}$$

with $S(0) > 0$. Here $S(t)$ is the price of the risky security at time t, μ is the appreciation rate of the price, and σ is the volatility. It is well-known that the logarithm of S grows linearly in the long-run. The increments of $\log S$ are stationary and Gaussian, which is a consequence of the driving Brownian motion. That is, for a fixed time lag h,

$$r_h(t+h) := \log \frac{S(t+h)}{S(t)} = (\mu - \frac{1}{2}\sigma^2)h + \sigma(B(t+h) - B(t))$$

is Gaussian distributed. Clearly $r_h(t+h)$ is $\mathcal{F}^B(t)$-independent, because B has independent increments. Therefore if $\mathcal{F}^B(t) = \mathcal{F}^S(t)$, it follows that the market is weakly efficient. To see this, note that S being a strong solution of (1.2.21) implies that $\mathcal{F}^S(t) \subseteq \mathcal{F}^B(t)$. On the other hand, since

$$\log S(t) = \log S(0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma B(t), \quad t \geq 0,$$

we can rearrange for B in terms of S to get that $\mathcal{F}^B(t) \subseteq \mathcal{F}^S(t)$, and hence $\mathcal{F}^B(t) = \mathcal{F}^S(t)$. Due to this reason, equation (1.2.21) has been used to model stock price evolution under the classic Efficient Market Hypothesis.

In order to reflect the phenomenon of occasional weak inefficiency resulting from feedback strategies widely applied by investors, in [3] SDEs whose solutions obey the Law of the Iterated Logarithm are applied to inefficient financial market models. More precisely, a semi-martingale X, which is slightly drift-perturbed and obeys the Law of the Iterated Logarithm, is introduced into equation (1.2.21) as the driving semimartingale instead of Brownian motion. It is shown that if a process S_* satisfies

$$dS_*(t) = \mu S_*(t)\,dt + S_*(t)\,dX(t), \quad t \geq 0, \quad S_*(0) > 0, \tag{1.2.22}$$

then S_* preserves some of the main characteristics of the standard Geometric Brownian Motion S. More precisely, the size of the long-run large deviations from the linear trend of the cumulative returns is preserved, along with the exponential growth of S. This is despite the fact that the increments of $\log S_*$ are now correlated and non-Gaussian.

In this paper, we further investigate the effect of this drift perturbation on the cumulative returns in (1.2.22) with the process X satisfying (1.2.5) or (1.2.15), say. We do not wish to provide a complicated and empirically precise model, but rather a simple and tractable model, and to interpret the mathematical results.

With a modest bias in the trend (e.g. captured by condition (1.2.6) and (1.2.8)), the excursions in prices from the linear trend are no longer independent. The largest possible sizes of these excursions coincide with those under no bias (as seen in (1.2.7) and (1.2.9)). However, by ergodic–type results (e.g. (1.2.10) and (1.2.11)), the stronger the positive bias that the investors have, the larger the average values of price excursions, and consequently the smaller the volatility that arises around the average values. This causes the price to persist on average further from the long-run growth trend that the GBM model would allow. This is made precisely in (1.2.24) below. This persistence could make investors believe that the cumulative returns are close to their true values and are unbiased, which might cause a more dramatic fall in cumulative returns later on. Moreover, if the market is even more pessimistic after a relatively large drop in returns, the bias tends to have a longer negative impact on the market.

In the model presented below, we presume that the returns evolve according to the strength of the various agents trading in the market. At a given time, each agent determines a threshold which signals whether the market is overbought or oversold. The agents become more risk-cautious in their trading strategies when these overbought or oversold thresholds are breached. If we make the simplifying assumption that one agent is representative of all, then the threshold level is simply the weighted average of the threshold for all the individuals.

Using these ideas, we are led to study the equation

$$dX(t) = f(X(t))[1 - \alpha I_{\{|X(t)|>k\sigma\sqrt{t}\}}]\,dt + \sigma\,dB(t). \tag{1.2.23}$$

Here f is assumed continuous and odd on $\mathbb{R}$ so that the positive and negative returns are treated symmetrically. Moreover, in order that the bias be modest, we require $\lim_{|x|\to\infty} xf(x) = L \in (0, \sigma^2/2]$. In (1.2.23), I is the indicator function, and $\alpha \in (0,1]$ measures the extent of short-selling or "going long" in the market. Here an increased α is associated with an increased tendency to sell short or go long. We presume that investors believe that the de–trended security returns are given by Brownian motion without drift, and the returns obey the Law of the Iterated Logarithm. Moreover, we assume that the investors can estimate the value of σ by tracking the size of the largest deviations.

We briefly indicate how the threshold level is arrived at. The standard Brownian motion (which the investors believe models the security return) is scaled by σ, and therefore, at time t, has standard deviation $\sigma\sqrt{t}$. If each agent i chooses a multiple k_i of this standard deviation as his/her threshold level, and assuming that all agents are representative, there exists a weighted coefficient k, such that $k\sigma\sqrt{t}$ measures the overall market threshold level. In practice, the value of k might be different for price increases and falls. We treat two situations with one fixed k here for simplicity.

Given these assumptions, we prove the following. First, X is recurrent on $\mathbb{R}$ and obeys the Law of the Iterated Logarithm by the results in Section 1.4 and 1.5. Second, we determine the long-run average value of the de–trended cumulative returns by proving the following ergodic–type theorem:

$$\lim_{t\to\infty} \frac{\int_0^t \frac{X^2(s)}{(1+s)^2}\,ds}{\log t} = \Lambda_{L,\sigma,\alpha,k} > \sigma^2, \quad \text{a.s.} \tag{1.2.24}$$

Here, $\Lambda_{L,\sigma,\alpha,k}$ measures the market bias from the unbiased value of σ^2. It can be computed and is given in Section 1.7. Our assumptions on parameters ensure that $\Lambda_{L,\sigma,\alpha,k} > \sigma^2$. This means that the presence of bias increases the "average size" of the departures of the returns from the trend growth rate. Therefore, in theory, the long-run "average size" Λ computed from observing the largest size of the fluctuation of the log-returns is too much different from σ, then it's an indication that there exists bias in the drift, and by the formula of $\Lambda_{L,\sigma,\alpha,k}$, we can compute the size of the bias L.

To establish (1.2.24), we first transform the solution X of (1.2.23) into a process Y by a change in both time and scale; second, we construct two equations with continuous and time-homogenous drift coefficients and with finite speed measures, such that Y is trapped between the solutions of these equations; third, by adjusting certain auxiliary parameters, we obtain an ergodic–type theorem for Y, which in turn implies (1.2.24). From a mathematical point of view, we have proved an ergodic–type theorem for a non–autonomous equation using the stochastic comparison principle.

Finally, we confirm that equation (1.2.22) with X satisfying (1.2.23) does represent an inefficient market in the *weak* sense, i.e., we want to show that

$$r_{*,h}(t+h) \text{ is } \mathcal{F}^{S_*}(t)\text{-dependent, for all } h>0 \text{ and } t\geq 0, \tag{1.2.25}$$

where r_* is the return. It is easy to verify that

$$S_*(t) = S_*(0)e^{(\mu-\frac{1}{2}\sigma^2)t+X(t)}, \quad X(t) = \log\frac{S_*(t)}{S_*(0)} - (\mu - \frac{1}{2}\sigma^2)t, \quad t \geq 0.$$

Therefore $\mathcal{F}^{S_*}(t) = \mathcal{F}^X(t)$. In the proof of the main result of this section, we establish the strong existence and uniqueness of the solution of equation (1.2.23) (this requires a little care because of the discontinuity of the drift coefficient). Since $X(0) = 0$ is deterministic, and X is a strong solution, we have $\mathcal{F}^X(t) \subseteq \mathcal{F}^B(t)$ for $t \geq 0$. On the other hand, by writing $F(t, x) := f(x)[1 - \alpha I_{\{|x|>k\sigma\sqrt{t}\}}]$, we get

$$B(t) = \frac{1}{\sigma}\left(X(t) - \int_0^t F(s, X(s))ds\right), \quad t \geq 0.$$

Hence $\mathcal{F}^B(t) \subseteq \mathcal{F}^X(t)$ for $t \geq 0$. Consequently $\mathcal{F}^{S_*}(t) = \mathcal{F}^B(t) = \mathcal{F}^X(t)$ for $t \geq 0$. So we may replace $\mathcal{F}^{S_*}(t)$ by $\mathcal{F}^B(t)$ in (1.2.25). Next, the increments $r_{*,h}$ of $\log S_*$ obey

$$\begin{aligned} r_{*,h}(t+h) &:= \log\frac{S_*(t+h)}{S_*(t)} \\ &= (\mu - \frac{1}{2}\sigma^2)h + \sigma(B(t+h) - B(t)) + \int_t^{t+h} F(s, X(s))ds \\ &= (\mu - \frac{1}{2}\sigma^2)h + (X(t+h) - X(t)). \end{aligned}$$

Now suppose for some $t \geq 0$, that $r_{*,h}(t+h)$ is $\mathcal{F}^B(t)$-independent. Since $[(\mu - \frac{1}{2}\sigma^2)h + \sigma(B(t+h)-B(t))]$ is $\mathcal{F}^B(t)$-independent, $\int_t^{t+h} F(s, X(s))ds$ must also be $\mathcal{F}^B(t)$-independent. However, by the Markov property of X, $\int_t^{t+h} F(s, X(s))ds$ is a functional of $X(t)$ and the increments of B. Hence, $\int_t^{t+h} F(s, X(s))ds$ is $\mathcal{F}^X(t)$-dependent, and since $\mathcal{F}^X(t) = \mathcal{F}^B(t)$, this gives a contradiction. Therefore (1.2.25) is proved.

1.3 Asymptotic Behaviour of Transient Processes

In this section, we study processes which obey (1.2.1) and are transient as time goes to infinity. To do this, introduce an auxiliary process: let $\delta > 2$ and consider

$$dY(t) = \sigma^2 \frac{\delta - 1}{2Y(t)}\, dt + \sigma\, dB(t) \quad \text{for } t \geq 0, \tag{1.3.1a}$$

$$Y(0) = y_0 > 0, \tag{1.3.1b}$$

where y_0 is deterministic. The solution of the above equation is a generalized Bessel process of dimension higher than 2; $\delta > 2$ does not have to be an integer. If $\delta > 2$ is an integer, then $Y(t) = \sigma|W(t)|$ where W is a δ–dimensional Brownian motion. Therefore, in the general case, we expect Y to grow to infinity like e.g. a three-dimensional Bessel process. This can be confirmed by [49, Chapter 3.3 Section C]. In fact, as proven in the following lemma, Y should also obey the Law of the Iterated Logarithm. The proof is the same in spirit as that in Motoo [67], but is briefly given here in the language of stochastic differential equations for the reasons of consistency with the technique of this chapter. We moreover employ Motoo's techniques to establish a lower bound on the growth rate.

Lemma 1.3.1. *Let $\delta > 2$ and Y be the unique continuous adapted process which obeys* (1.3.1)*. Then Y is a positive process a.s., and satisfies*

$$\limsup_{t\to\infty} \frac{Y(t)}{\sqrt{2t\log\log t}} = |\sigma| \quad a.s. \tag{1.3.2}$$

and

$$\liminf_{t\to\infty} \frac{\log \frac{Y(t)}{\sqrt{t}}}{\log\log t} = -\frac{1}{\delta-2}, \quad a.s. \tag{1.3.3}$$

Proof. Let $Z(t) = Y(t)^2$. By Itô's rule, we get

$$dZ(t) = \sigma^2\,\delta\,dt + 2\sqrt{Z(t)}\sigma\,d\widehat{B}(t), \quad t \geq 0$$

with $Z(0) = y_0^2$, where by Doob's martingale representation theorem, we have replaced the original Brownian motion B by $\widehat{B}$ in an extended probability space. Therefore

$$\begin{aligned} Z(e^t-1) &= y_0^2 + \int_0^{e^t-1} \sigma^2\delta\,ds + \int_0^{e^t-1} 2\sqrt{Z(s)}\sigma\,d\widehat{B}(s) \\ &= y_0^2 + \int_0^t \sigma^2\delta e^s\,ds + \int_0^t 2\sigma\sqrt{Z(e^s-1)}e^{\frac{s}{2}}\,dW(s), \end{aligned}$$

where W is again another Brownian motion. If $\widetilde{Z}(t) = Z(e^t-1)$, then

$$d\widetilde{Z}(t) = \sigma^2\delta e^t\,dt + 2\sigma\sqrt{\widetilde{Z}(t)}e^{\frac{t}{2}}\,dW(t), \quad t \geq 0.$$

If $H(t) := e^{-t}\widetilde{Z}(t)$, then $H(0) > 0$ and H obeys

$$dH(t) = (\sigma^2\delta - H(t))\,dt + 2\sigma\sqrt{H(t)}\,dW(t), \quad t \geq 0. \tag{1.3.4}$$

Therefore by Lemma 2.3.1, we have

$$\limsup_{t\to\infty} \frac{H(t)}{2\log t} = \sigma^2, \quad \text{a.s.} \tag{1.3.5}$$

Using the definition of Y in terms of H and Z we obtain (1.3.2).

To prove (1.3.3), consider the transformation $H_*(t) := 1/H(t)$. H_* is well-defined, a.s. positive, and by Itô's rule obeys

$$dH_*(t) = [(4\sigma^2 - \sigma^2\delta)H_*^2(t) + H_*(t)]\,dt - 2\sigma \frac{H_*^2(t)}{\sqrt{H_*(t)}}\,dW(t), \quad t \geq 0.$$

It is easy to show that the scale function satisfies

$$s_{H_*}(x) = K_1 \int_1^x y^{\frac{\delta-4}{2}} e^{\frac{1}{2\sigma^2 y}}\,dy, \quad x \in \mathbb{R},$$

for some positive constant K_1, and H_* obeys all the conditions of Motoo's theorem. By L'Hôpital's rule, for some positive constant K_2, we have

$$\lim_{x\to\infty} \frac{s_{H_*}(x)}{x^{\frac{\delta-2}{2}}} = K_2.$$

Let $\varrho_1(t) = t^{2/(\delta-2)}$ then for some $t_1 > 0$,

$$\int_{t_1}^{\infty} \frac{1}{s_{H_*}(\varrho_1(t))}\,dt = \int_{t_1}^{\infty} \frac{1}{K_2 t}\,dt = \infty.$$

Hence by Motoo's theorem,

$$\limsup_{t\to\infty} \frac{H_*(t)}{\varrho_1(t)} = \limsup_{t\to\infty} \frac{H_*(t)}{t^{\frac{2}{\delta-2}}} \geq 1, \quad \text{a.s.}$$

On the other hand, for $\epsilon \in (0, \delta-2)$,

$$\lim_{x\to\infty} \frac{s_{H_*}(x)}{x^{\frac{\delta-2-\epsilon}{2}}} = \infty.$$

Let $\varrho_2(t) = t^{2/(\delta-2-\epsilon-\theta)}$, where $\theta \in (0, \delta-2-\epsilon)$. Then for some $t_2 > 0$, we get

$$\int_{t_2}^{\infty} \frac{1}{s_{H_*}(\varrho_2(t))}\,dt \leq \int_{t_2}^{\infty} \frac{1}{t^{\frac{\delta-2-\epsilon}{\delta-2-\epsilon-\theta}}}\,dt < \infty,$$

a.s. on an a.s. event $\Omega_{\epsilon,\theta} := \Omega_\epsilon \cap \Omega_\theta$, where Ω_ϵ and Ω_θ are both a.s. events. From this by letting $\epsilon \downarrow 0$ and $\delta \downarrow 0$ through rational numbers, it can be deduced that

$$\limsup_{t\to\infty} \frac{\log H_*(t)}{\log t} = \frac{2}{\delta - 2}, \quad \text{a.s. on } \cap_{\epsilon,\theta\in\mathbb{Q}}\Omega_{\epsilon,\theta}.$$

Using the relation between H_* and Y, we get the desired result (1.3.3). □

Corollary 1.3.1. *Let $\delta > 2$ and Y be the unique continuous adapted process which obeys* (1.3.1a)*, but with $Y(0) = y_0 < 0$. Then Y obeys*

$$\liminf_{t\to\infty} \frac{Y(t)}{\sqrt{2t\log\log t}} = -|\sigma|, \quad a.s. \tag{1.3.6}$$

and

$$\liminf_{t\to\infty} \frac{\log \frac{|Y(t)|}{\sqrt{t}}}{\log\log t} = -\frac{1}{\delta - 2}, \quad a.s. \tag{1.3.7}$$

Proof. Letting $Y_*(t) = -Y(t)$ and applying the same analysis as Lemma 1.3.1 to Y_*, the results can be easily shown. The details are omitted. □

We are now in a position to determine the asymptotic behaviour of (1.2.1) when the diffusion coefficient is constant.

Theorem 1.3.1. *Let X be the unique continuous adapted process which obeys* (1.2.1)*. Let $A := \{\omega : \lim_{t\to\infty} X(t,\omega) = \infty\}$. If*

$$\lim_{x\to\infty} xf(x) = L_\infty; \tag{1.3.8}$$

$$g(x) = \sigma, \;\; x \in \mathbb{R},$$

where $\sigma \neq 0$ and $L_\infty > \sigma^2/2$, then $\mathbb{P}[A] > 0$ and X satisfies

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = |\sigma| \quad a.s. \text{ conditionally on } A, \tag{1.3.9}$$

and

$$\liminf_{t\to\infty} \frac{\log \frac{X(t)}{\sqrt{t}}}{\log\log t} = -\frac{1}{\frac{2L_\infty}{\sigma^2} - 1}, \quad a.s. \text{ conditionally on } A. \tag{1.3.10}$$

Proof. First note that given $L_\infty > \sigma^2/2$, the existence of such a non-null event A in the sample space is guaranteed by Feller's test [49, Proposition 5.5.22]. From now on, we assume that we are working in A, and will frequently suppress ω-dependence and A a.s. qualifications accordingly. We compare X with $Y_{+\epsilon}$, where $Y_{+\epsilon}$ is given by

$$dY_{+\epsilon}(t) = \frac{L_\infty + \epsilon}{Y_{+\epsilon}(t)}\,dt + \sigma\,dB(t), \quad t \geq 0$$

with $Y_{+\epsilon}(0) > 0$ and $(L_\infty + \epsilon) > (L_\infty - \epsilon) > \sigma^2/2$, so that L_∞ takes the same role as δ in (1.3.1) as we let $\epsilon \downarrow 0$. Since $\lim_{x\to\infty} xf(x) = L_\infty$ and $\lim_{t\to\infty} X(t) = \infty$, there exists $T_1(\epsilon,\omega) > 0$, such that for all $t \geq T_1(\epsilon,\omega)$, $L_\infty - \epsilon < X(t)f(X(t)) < L_\infty + \epsilon$ and $X(t) > 0$. Hence $(L_\infty - \epsilon)/X(t) < f(X(t)) < (L_\infty + \epsilon)/X(t)$, $t \geq T_1(\epsilon,\omega)$. Let $\Delta(t) = Y_{+\epsilon}(t) - X(t)$. We now consider three cases:

Case 1: if $X(T_1) < Y_{+\epsilon}(T_1)$, i.e., $\Delta(T_1) > 0$, we claim that

$$\text{for all} \quad t > T_1(\epsilon,\omega), \quad X(t) < Y_{+\epsilon}(t).$$

Suppose to the contrary there exists a minimal $t^* > T_1(\epsilon,\omega)$ such that $X(t^*) = Y_{+\epsilon}(t^*)$. Then $\Delta(t^*) = 0$ and $\Delta'(t^*) \leq 0$. But

$$\Delta'(t) = \frac{L_\infty + \epsilon}{Y_{+\epsilon}(t)} - f(X(t)) > \frac{L_\infty + \epsilon}{Y_{+\epsilon}(t)} - \frac{L_\infty + \epsilon}{X(t)}, \quad \text{for all } t \geq T_1(\epsilon,\omega),$$

so

$$\Delta'(t^*) > \frac{L_\infty + \epsilon}{Y_{+\epsilon}(t^*)} - \frac{L_\infty + \epsilon}{X(t^*)} = 0,$$

which gives a contradiction.

Case 2: if $X(T_1) > Y_{+\epsilon}(T_1) > 0$, i.e., $\Delta(T_1) < 0$, we show that

$$\text{for all} \quad t \geq T_1(\epsilon,\omega), \quad X(t) \leq Y_{+\epsilon}(t) - \Delta(T_1).$$

Now for all $t \geq T_1(\epsilon,\omega)$,

$$\Delta'(t) = \frac{L_\infty + \epsilon}{Y_{+\epsilon}(t)} - f(X(t)) > \frac{L_\infty + \epsilon}{Y_{+\epsilon}(t)} - \frac{L_\infty + \epsilon}{X(t)} = \frac{-\Delta(t)(L_\infty + \epsilon)}{Y_{+\epsilon}(t)X(t)}. \tag{1.3.11}$$

In particular

$$\Delta'(T_1) > \frac{-\Delta(T_1)(L_\infty + \epsilon)}{Y_{+\epsilon}(T_1)X(T_1)} > 0. \tag{1.3.12}$$

There are now two possibilities: either $X(t) > Y(t)$ for all $t > T_1(\epsilon, \omega)$ or there is $T_2(\omega) > T_1(\epsilon, \omega)$, such that $X(T_2) = Y_{+\epsilon}(T_2)$. If $X(t) > Y_{+\epsilon}(t)$, $\forall\, t > T_1(\epsilon, \omega)$, then $\Delta'(t) > 0$, so Δ is increasing on $[T_1(\epsilon, \omega), \infty)$. Therefore $Y_{+\epsilon}(t) - X(t) = \Delta(t) > \Delta(T_1)$, we are done. The analysis of the situation where there exists $T_2(\omega) > T_1(\epsilon, \omega)$ such that $X(T_2) = Y_{+\epsilon}(T_2)$ is dealt with by case 3.

Case 3: if $X(T_1) = Y_{+\epsilon}(T_1)$, i.e., $\Delta(T_1) = 0$, we claim that

$$\text{for all} \quad t > T_1(\epsilon, \omega), \quad X(t) < Y_{+\epsilon}(t).$$

We note first from (1.3.12) that $\Delta'(T_1) > 0$. Hence, there exists $T_3(\omega) > T_1(\epsilon, \omega)$ such that $\Delta(t) > 0$ for $t \in (T_1, T_3)$. Suppose in contradiction to the claim, that $T_3(\omega)$ is such that $\Delta(T_3) = 0$. Then $\Delta'(T_3) \leq 0$, which is impossible by (1.3.11).

Combining the above results, for almost all ω in A, we have

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t \log\log t}} \leq \limsup_{t\to\infty} \frac{Y_{+\epsilon}(t)}{\sqrt{2t \log\log t}}. \tag{1.3.13}$$

A lower estimate on X can be deduced by a similar argument. For the same ϵ, define $Y_{-\epsilon}$ by

$$dY_{-\epsilon}(t) = \frac{L_\infty - \epsilon}{Y_{-\epsilon}(t)}\, dt + \sigma\, dB(t), \quad t \geq 0$$

with $Y_{-\epsilon}(0) > 0$. Note that $L_\infty - \epsilon > \sigma^2/2$, so $Y_{-\epsilon}$ is guaranteed to be positive. Then, by arguing as above, we obtain an analogous result to (1.3.13), namely

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t \log\log t}} \geq \limsup_{t\to\infty} \frac{Y_{-\epsilon}(t)}{\sqrt{2t \log\log t}}. \tag{1.3.14}$$

We are now in a position to prove (1.3.9). Using (1.3.13), and letting Ω^*_ϵ be the a.s. event on which

$$\limsup_{t\to\infty} \frac{Y_{+\epsilon}(t)}{\sqrt{2t \log\log t}} = |\sigma|,$$

we have

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \le |\sigma|, \quad \text{a.s. on } \Omega^*_\epsilon \cap A.$$

Letting $\Omega_* = \cap_{\epsilon\in\mathbb{Q}^+\cap(0,1)}\Omega^*_\epsilon$, it follows that

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \le |\sigma|, \quad \text{a.s. on } \Omega_* \cap A, \tag{1.3.15}$$

as required. Similarly using (1.3.14), and letting $\Omega^*_{-\epsilon}$ be the a.s. event on which

$$\limsup_{t\to\infty} \frac{Y_{-\epsilon}(t)}{\sqrt{2t\log\log t}} = |\sigma|,$$

we have

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \ge |\sigma|, \quad \text{a.s. on } A \cap \Omega^*_{-\epsilon}.$$

With $\Omega_{**} = \cap_{\epsilon\in\mathbb{Q}\cap(0,1)}\Omega^*_{-\epsilon}$, it follows that

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \ge |\sigma|, \quad \text{a.s. on } A \cap \Omega_{**} \tag{1.3.16}$$

as required. Combining (1.3.15) and (1.3.16) gives (1.3.9).

To prove (1.3.10), notice that $Y_{+\epsilon}$ obeys (1.3.1) with $\delta = \delta_\epsilon = 1 + 2(L_\infty + \epsilon)/\sigma^2$. Then, by (1.3.3) we have

$$\liminf_{t\to\infty} \frac{\log \frac{Y_{+\epsilon}(t)}{\sqrt{t}}}{\log\log t} = -\frac{1}{\delta_\epsilon - 2} = -\frac{1}{2(L_\infty+\epsilon)/\sigma^2 - 1}, \quad \text{a.s. on } \Omega^+_\epsilon \tag{1.3.17}$$

where Ω^+_ϵ is an almost sure event. Therefore by (1.3.13), a.s. on $A \cap \Omega^+_\epsilon$ we have

$$\liminf_{t\to\infty} \frac{\log \frac{X(t)}{\sqrt{t}}}{\log\log t} \le -\frac{1}{2(L_\infty+\epsilon)/\sigma^2 - 1}.$$

If $A^* = A \cap \{\cap_{\epsilon\in\mathbb{Q}\cap(0,1)}\Omega^+_\epsilon\}$, then A^* is an a.s. subset of A and

$$\liminf_{t\to\infty} \frac{\log \frac{X(t)}{\sqrt{t}}}{\log\log t} \le -\frac{1}{2L_\infty/\sigma^2 - 1}, \quad \text{a.s. on } A^*. \tag{1.3.18}$$

Proceeding similarly with $Y_{-\epsilon}$ and using (1.3.14) we can prove that

$$\liminf_{t\to\infty} \frac{\log \frac{X(t)}{\sqrt{t}}}{\log\log t} \ge -\frac{1}{2L_\infty/\sigma^2 - 1}, \quad \text{a.s. on } A^{**}, \tag{1.3.19}$$

where A^{**} is an a.s. subset of A. Combining (1.3.18) and (1.3.19) now yields (1.3.10). $\square$

By Feller's test, depending on the value of $L_{-\infty}$, we can compute the probability of the event A defined in the previous theorem. Suppose that $L_\infty > \sigma^2/2$. If $L_{-\infty} \le \sigma^2/2$, then $\mathbb{P}[A] = 1$. If $L_{-\infty} > \sigma^2/2$, and we define $\tilde{A} := \{\omega : \lim_{t\to\infty} X(t,\omega) = -\infty\}$, then $A \cup \tilde{A}$ is an a.s. event, and $\mathbb{P}[A]$, $\mathbb{P}[\tilde{A}] \in (0,1)$. The exact values of $\mathbb{P}[A]$ and $\mathbb{P}[\tilde{A}]$ depend on the deterministic initial value of X. In a like manner, we can prove similar results when the roles of L_∞ and $L_{-\infty}$ are interchanged. By Corollary 1.3.1, it is not difficult to show the following result. The details of the proof are omitted.

Corollary 1.3.2. *Let X be the unique continuous adapted process which obeys* (1.2.1). *Let $\tilde{A} := \{\omega : \lim_{t\to\infty} X(t,\omega) = -\infty\}$. If*

$$\lim_{x\to\infty} xf(x) = L_{-\infty}, \quad g(x) = \sigma, \quad x \in \mathbb{R}$$

where $\sigma \neq 0$ and $L_{-\infty} > \sigma^2/2$, then $\mathbb{P}[\tilde{A}] > 0$ and X satisfies

$$\liminf_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = -|\sigma| \quad a.s. \text{ on } \tilde{A},$$

and

$$\liminf_{t\to\infty} \frac{\log\frac{|X(t)|}{\sqrt{t}}}{\log\log t} = -\frac{1}{\frac{2L_{-\infty}}{\sigma^2} - 1}, \quad a.s. \text{ on } \tilde{A}.$$

Theorem 1.3.1 can now be used to prove a more general result for (1.2.1), where instead of being constant, g now obeys

$$\forall\, x \in \mathbb{R}, \quad g(x) \neq 0, \quad \lim_{x\to\infty} g(x) = \sigma \in \mathbb{R}/\{0\}. \tag{1.3.20}$$

Theorem 1.3.2. *Let X be the unique continuous adapted process which obeys* (1.2.1). *Let $A := \{\omega : \lim_{t\to\infty} X(t,\omega) = \infty\}$. If there exist positive real numbers L_∞ and σ such that $L_\infty > \sigma^2/2$, f obeys* (1.3.8), *and g obeys* (1.3.20), *then X satisfies* (1.3.9) *and* (1.3.10).

Proof. Define the local martingale

$$M(t) = \int_0^t g(X(s))\,dB(s), \quad t \ge 0.$$

Therefore, by (1.3.20) we have

$$\lim_{t\to\infty}\frac{1}{t}\langle M\rangle(t)=\lim_{t\to\infty}\frac{1}{t}\int_0^t g^2(X(s))\,ds=\sigma^2,\quad \text{a.s. conditionally on } A. \tag{1.3.21}$$

For each $0\le s<\infty$, define the stopping time $\nu(s):=\inf\{t\ge 0:\ \langle M\rangle(t)>s\}$. By the time-change theorem for martingales [49, Theorem 3.4.6], the process defined as $W(t):=M(\nu(t))$ is a standard Brownian motion with respect to the filtration $\mathcal{Q}_t:=\mathcal{F}_{\nu(t)}$. If $\widetilde{X}(t):=X(\nu(t))$, then

$$d\widetilde{X}(t)=\frac{f(\widetilde{X}(t))}{g^2(\widetilde{X}(t))}\,dt+dW(t),\quad t\ge 0.$$

Now, since $\lim_{t\to\infty}xf(x)/g^2(x)=L_\infty/\sigma^2>1/2$, by Theorem 1.3.1, for almost all $\omega\in A$,

$$\limsup_{t\to\infty}\frac{\widetilde{X}(t)}{\sqrt{2t\log\log t}}=1,\quad \liminf_{t\to\infty}\frac{\log\frac{\widetilde{X}(t)}{\sqrt{t}}}{\log\log t}=-\frac{1}{\frac{2L_\infty}{\sigma^2}-1}.$$

That is for almost all $\omega\in A$,

$$\limsup_{t\to\infty}\frac{X(t)}{\sqrt{2\langle M\rangle(t)\log\log\langle M\rangle(t)}}=1,\quad \liminf_{t\to\infty}\frac{\log\frac{X(t)}{\sqrt{\langle M\rangle(t)}}}{\log\log\langle M\rangle(t)}=-\frac{1}{\frac{2L_\infty}{\sigma^2}-1}. \tag{1.3.22}$$

Combining (1.3.21) with these limits, the desired assertion can be obtained. □

A similar result can be developed in the case when $X(t)\to-\infty$ under the assumptions that $xf(x)\to L_{-\infty}>\sigma^2/2$ and $g(x)\to\sigma$ as $x\to-\infty$. The proof is essentially the same as that of Theorem 1.3.2, and hence omitted.

The following theorem is a even more generalized result on transient processes and is obtained by Theorem 1.3.1.

Theorem 1.3.3. *Let X be the unique continuous adapted process which obeys* (1.2.1). *Let $A:=\{\omega:\lim_{t\to\infty}X(t,\omega)=\infty\}$. If there exists a positive real numbers $L_\infty>1/2$ such that*

$$\lim_{x\to\infty}\frac{xf(x)}{g^2(x)}=L_\infty. \tag{1.3.23}$$

And g obeys

$$\forall\, x\in\mathbb{R},\quad g(x)>0;\quad g\in\mathcal{RV}_\infty(\beta),\quad 0<\beta<1. \tag{1.3.24}$$

Then

$$\limsup_{t\to\infty} \frac{X(t)}{G^{-1}(\sqrt{2t\log\log t})} = 1, \quad \textit{a.s. conditionally on } A, \tag{1.3.25}$$

and

$$\liminf_{t\to\infty} \frac{\log \frac{G(X(t))}{\sqrt{t}}}{\log\log t} = -\frac{1-\beta}{2L_\infty - 1}, \quad \textit{a.s. conditionally on } A, \tag{1.3.26}$$

where G is defined as

$$G(x) = \int_c^x \frac{1}{g(y)}\,dy, \quad x, c \in \mathbb{R}. \tag{1.3.27}$$

Proof. Again by Feller's test, under condition (1.3.23), the existence of such a non–null event A is guaranteed. Recall that $g \in \mathcal{RV}_\infty(\beta)$ means $\lim_{x\to\infty} g(\lambda x)/g(x) = \lambda^\beta$ for all $\lambda \in \mathbb{R}$. By the smooth variation theorem [19, Theorem 1.8.2], there exists a function $l \in C^1([0,\infty);(0,\infty))$ and $l \in \mathcal{SRV}_\infty(\beta)$ with $\lim_{x\to\infty} g(x)/l(x) = 1$ such that $\lim_{x\to\infty} xl'(x)/l(x) = \beta$. Moreover, we can extend l to $(-\infty, 0)$ such that $l(x) > 0$ for $x \in (-\infty, 0)$ and $l \in C^1(\mathbb{R};(0,\infty))$. Then the function $H : \mathbb{R} \to \mathbb{R}$ given by

$$H(x) := \int_1^x \frac{1}{l(y)}\,dy$$

is well-defined. Moreover, $H'(x) = 1/l(x)$ and $H''(x) = -l'(x)/l^2(x)$. Since $\beta \in (0,1)$, it follows that $\lim_{x\to\infty} H(x) = \infty$ and $\lim_{x\to\infty} G(x)/H(x) = 1$. Since both g and l are strictly positive, G and H are monotone increasing on $\mathbb{R}$. By Itô's rule, we have

$$dH(X(t)) = \Big[\frac{f(X(t))}{l(X(t))} - \frac{1}{2} l'(X(t))\frac{g^2(X(t))}{l^2(X(t))}\Big]\,dt + \frac{g(X(t))}{l(X(t))}dB(t).$$

Let $Y(t) := H(X(t))$ for all $t \ge 0$. Then $X(t) = H^{-1}(Y(t))$. Hence if we could prove

$$\lim_{x\to\infty} H(x)\Big[\frac{f(x)}{l(x)} - \frac{1}{2}l'(x)\frac{g^2(x)}{l^2(x)}\Big] =: I_\infty > \frac{1}{2}, \tag{1.3.28}$$

then by Theorem 1.3.2, we get (1.3.26) and

$$\limsup_{t\to\infty} \frac{Y(t)}{\sqrt{2t\log\log t}} = 1, \quad \text{a.s. conditionally on } A,$$

which implies (1.3.25) since $\lim_{x\to\infty} G^{-1}(x)/H^{-1}(x) = 1$. Now by the definition of H and L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{H(\lambda x)}{H(x)} = \lim_{x\to\infty} \frac{\lambda l(x)}{l(\lambda x)} = \lambda^{1-\beta}$$

Thus $H \in \mathcal{RV}_\infty(1-\beta)$. Hence

$$\lim_{x\to\infty} \frac{H(x)}{x/l(x)} = \lim_{x\to\infty} \frac{1}{l(x)} \cdot \frac{l^2(x)}{l(x) - xl'(x)} = \frac{1}{1-\beta}.$$

Therefore

$$\lim_{x\to\infty} l'(x)H(x) = \lim_{x\to\infty} \frac{xl'(x)}{l(x)} \cdot \frac{H(x)}{x/l(x)} = \frac{\beta}{1-\beta}.$$

Also

$$\lim_{x\to\infty} H(x) \cdot \frac{f(x)}{l(x)} = \lim_{x\to\infty} \frac{H(x)}{x/l(x)} \cdot \frac{xf(x)}{l^2(x)} = \frac{L_\infty}{1-\beta}.$$

Since $L_\infty > 1/2$, the above two equations implies (1.3.28). □

1.4 General Conditions Ensuring the Law of the Iterated Logarithm and Ergodicity

Theorem 1.4.1. *Let X be the unique continuous adapted process satisfying* (1.2.5). *If there exists a positive real number ρ such that*

$$\forall\,(x,t) \in \mathbb{R}\times\mathbb{R}^+, \quad xf(x,t) \le \rho, \tag{1.4.1}$$

then

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \le |\sigma|, \quad a.s. \tag{1.4.2}$$

and

$$\limsup_{t\to\infty} \frac{\int_0^t \frac{X^2(s)}{(1+s)^2}\,ds}{\log t} \le 2\rho + \sigma^2, \quad a.s. \tag{1.4.3}$$

Proof. Without loss of generality, we can choose $\rho > \sigma^2/2$. Consider

$$dX^2(t) = (2X(t)f(X(t),t) + \sigma^2)\,dt + 2X(t)\sigma\,dB(t)$$

and

$$dX_u(t) = (2\rho + \sigma^2)\,dt + 2\sqrt{X_u(t)}\sigma\,dB(t). \tag{1.4.4}$$

with $X_u(0) > X^2(0)$. By the comparison theorem (cf. e.g.Proposition 5.2.18 [49]), $X_u(t) \geq X^2(t)$ for all $t \geq 0$ a.s. From the proof of Lemma 1.3.1, we know that $\mathbb{P}[\lim_{t\to\infty} X_u(t) = \infty] = 1$. Moreover, X_u obeys

$$\limsup_{t\to\infty} \frac{X_u(t)}{2t\log\log t} \leq \sigma^2 \quad \text{a.s.}$$

Hence the assertion (1.4.2) is obtained.

The second part of the theorem can be easily deduced from (1.4.4) by the following known result (cf. e.g.[71, Exercise XI.1.32]). We omit its proof. □

Lemma 1.4.1. *Suppose that Q is the unique continuous adapted process satisfying*

$$dQ(t) = \delta\,dt + 2\sqrt{Q(t)}\,dB(t), \quad t \geq 0$$

with $Q(0) \geq 0$ and $\delta > 0$. Then Q obeys

$$\lim_{t\to\infty} \frac{\int_1^t \frac{Q(s)}{s^2}\,ds}{\log t} = \delta, \quad a.s.$$

We now establish lower bounds corresponding to the upper bounds given in the previous theorem.

Theorem 1.4.2. *Let X be the unique continuous adapted process satisfying* (1.2.5). *If there exists a real number μ such that*

$$\inf_{(x,t)\in\mathbb{R}\times\mathbb{R}^+} xf(x,t) = \mu > -\frac{\sigma^2}{2}, \tag{1.4.5}$$

then

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \geq |\sigma|, \quad a.s. \tag{1.4.6}$$

Moreover,

$$\liminf_{t\to\infty} \frac{\int_0^t \frac{X^2(s)}{(1+s)^2}\,ds}{\log t} \geq 2\mu + \sigma^2, \quad a.s. \tag{1.4.7}$$

Proof. We begin with a change in both time and scale on X to transform it to a stationary process. Let $Y(t) = e^{-t}X(\frac{1}{2}(e^{2t}-1))$. By Itô's rule, it can be shown that for $t \geq 0$

$$dY^2(t) = \Big[-2Y^2(t) + 2Y(t)e^t f\big(Y(t)e^t, \frac{1}{2}(e^{2t}-1)\big) + \sigma^2\Big]\,dt + 2\sigma\sqrt{Y^2(t)}\,dW(t)$$

with $Y^2(0) = x_0^2$, where by Doob's martingale representation theorem given in the preliminaries, we have replaced

$$\int_0^t Y(s)\,dB(s) \quad \text{by} \quad \int_0^t \sqrt{Y^2(s)}\,dW(s).$$

W is another Brownian motion in an extended space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Consider the processes governed by the following two equations,

$$dY_1(t) = (-2Y_1(t) + 2\mu + \sigma^2)\,dt + 2\sigma\sqrt{|Y_1(t)|}\,dW(t), \tag{1.4.8}$$

$$dY_2(t) = (-2Y_2(t))\,dt + 2\sigma\sqrt{|Y_2(t)|}\,dW(t) \tag{1.4.9}$$

with $x_0^2 \geq Y_1(0) \geq Y_2(0) = 0$. Instead of applying Lemma 2.3.1 directly, we give more details on estimating the asymptotic growth rate of Y_1 using Motoo's theorem. By Yamada and Watanabe's uniqueness theorem (cf.[49, Proposition 5.2.13]), $Y_2(t) = 0$ for all $t \geq 0$ a.s. for all $t \geq 0$. Applying the comparison theorem twice, we have $Y^2(t) \geq Y_1(t) \geq Y_2(t) = 0$ for all $t \geq 0$ a.s. So the absolute values in (1.4.8) can be removed. Now it is easy to check that a scale function and the speed measure of Y_1 are

$$s_{Y_1}(x) = e^{-\frac{1}{\sigma^2}}\int_1^x e^{\frac{y}{\sigma^2}} y^{-\frac{2\mu+\sigma^2}{2\sigma^2}}\,dy, \quad m_{Y_1}(dx) = \frac{1}{2}\sigma^2 e^{-\frac{1}{\sigma^2}} e^{\frac{-x}{\sigma^2}} x^{\frac{2\mu+\sigma^2}{2\sigma^2}-1}\,dx$$

respectively. Without loss of generality, we can choose $\mu \in (-\sigma^2/2, \sigma^2/2]$. Then $s_{Y_1}(\infty) = \infty$, $s_{Y_1}(0) > -\infty$ and $m_{Y_1}(0,\infty) < \infty$. In addition, the v function of Y_1 as defined in (0.2.4) satisfies $v(0) < \infty$. So by Feller's test for explosions, Y_1 reaches zero within finite time on some event. A direct calculation confirms that $m_{Y_1}(\{0\}) = 0$. By the definition of an instantaneously reflecting point in [71, Chapter VII, Definition 3.11], we conclude that zero is a reflecting barrier for Y_1, hence for almost all $\omega \in \Omega$, Y_1 is a recurrent process with finite speed measure to which Motoo's theorem as stated in the introduction of this chapter can now be applied. Let $\varrho(t) = \sigma^2 \log t$. Since $\mu \in (-\sigma^2/2, \sigma^2/2]$, by L'Hôpital's rule

$$\lim_{x\to\infty} \frac{s_{Y_1}(x)}{e^{\frac{x}{\sigma^2}}} = \lim_{x\to\infty} x^{-\frac{2\mu+\sigma^2}{2\sigma^2}} = 0.$$

This implies that there exists $x_* > 0$ such that for all $x > x_*$, $s_{Y_1}(x) < e^{x/\sigma^2}$. Since ϱ is an increasing function, there exists $t_0 > 0$ such that for all $t > t_0$, $\varrho(t) > x_*$, so $s_{Y_1}(\varrho(t)) < t$. Hence

$$\int_{t_0}^{\infty} \frac{1}{s_{Y_1}(\varrho(t))}\, dt \geq \int_{t_0}^{\infty} \frac{1}{t}\, dt = \infty.$$

Therefore, by Motoo's theorem

$$\limsup_{t\to\infty} \frac{Y^2(t)}{\log t} \geq \limsup_{t\to\infty} \frac{Y_1(t)}{\log t} \geq \sigma^2, \quad \text{a.s.}$$

Using the relation between X and Y, we get the desired result (1.4.6).

For the second part of the conclusion, consider the following equation

$$dZ(t) = (2\mu + \sigma^2)\, dt + 2\sigma \sqrt{|Z(t)|}\, dW(t), \quad t \geq 0,$$

with $Z(0) \leq x_0^2$. Then $X^2(t) \geq Z(t)$ for $t \geq 0$ a.s. Again, by applying Lemma 1.4.1 to Z, (1.4.7) is proved. □

The following corollary combines Theorem 1.3.2 with Theorem 1.4.1 and Theorem 1.4.2.

Corollary 1.4.1. *Let X be the unique continuous adapted process satisfying the equation*

$$dX(t) = f(X(t), t)\, dt + g(X(t))\, dB(t), \quad t \geq 0,$$

with $X(0) = x_0$. Suppose $g : \mathbb{R} \to \mathbb{R}$ is even and satisfies

$$\forall\, x \in \mathbb{R}, \quad g(x) \neq 0, \quad \lim_{|x|\to\infty} g(x) = \sigma \in \mathbb{R}/\{0\}. \tag{1.4.10}$$

(i) If there exists a positive constant ρ such that f satisfies (1.4.1)*, then X obeys* (1.4.2)*.*

(ii) If there exists a constant μ such that

$$\inf_{(x,t)\in\mathbb{R}\times\mathbb{R}^+} \frac{xf(x,t)}{g^2(x)} = \mu > \frac{1}{2},$$

then X obeys (1.4.6)*.*

Proof. Without loss of generality, we can choose $\rho > \sigma^2/2$. Consider the equation

$$dX_u(t) = \frac{\rho}{X_u(t)}\,dt + g(X_u(t))\,dB(t), \quad t \geq 0$$

with $X_u(0) > |x_0| \vee 0$. It is easy to check that the scale function of X_u satisfies $s_{X_u}(\infty) < \infty$ and $s_{X_u}(0) = -\infty$. Thus $\mathbb{P}[\lim_{t\to\infty} X_u(t) = \infty] = 1$. Moreover $v_{X_u}(\infty) = v_{X_u}(0) = \infty$, which implies that $\mathbb{P}[X_u(t) > 0; \forall\, 0 < t < \infty] = 1$. Hence

$$dX_u^2(t) = [2\rho + g^2(X_u(t))]\,dt + 2X_u(t)g(X_u(t))\,dB(t), \quad t \geq 0.$$

Also by Theorem 1.3.2, X_u obeys

$$\limsup_{t\to\infty} \frac{X_u}{\sqrt{2t\log\log t}} = |\sigma|, \quad \text{a.s.}$$

Now since

$$dX^2(t) = [2X(t)f(X(t),t) + g^2(X(t))]\,dt + 2X(t)g(X(t))\,dB(t), \quad \text{a.s.}$$

Therefore $X_u^2(t) \geq X^2(t)$ for all $t \geq 0$ a.s., which implies (1.4.2). For the second part of the theorem, applying the same random time change to X as in the proof of Theorem 1.3.2, we obtain the first member of (1.3.22). Combining this result with (1.4.10), we get (1.4.6). $\square$

Next corollary applies the ergodic-type theorems (Theorem 1.4.1, 1.4.2 and Lemma 1.4.1) to the growth process with non–constant diffusion coefficient which is dealt in Theorem 1.3.2. We supply the proof here which is similar to that of Lemma 1.4.1.

Corollary 1.4.2. *Let X be the unique continuous adapted process which obeys* (1.2.1)*. Let $A := \{\omega : \lim_{t\to\infty} X(t,\omega) = \infty\}$. If there exist positive real numbers L_∞ and σ such that $L_\infty > \sigma^2/2$, f obeys* (1.3.8)*, and g obeys* (1.3.20)*, then X satisfies*

$$\lim_{t\to\infty} \frac{\int_0^t \frac{X^2(s)}{(1+s)^2}\,ds}{\log t} = 2L_\infty + \sigma^2, \quad \textit{a.s. conditionally on } A. \tag{1.4.11}$$

Proof. Applying the transformation $Y(t) := (e^{-t/2}X(e^t-1))^2$ for $t \ge 0$, we get

$$Y(t) = x_0^2 - \int_0^t Y(s)\,ds + \int_0^t 2\tilde{X}(s)f(\tilde{X}(s))\,ds + \int_0^t g^2(\tilde{X}(s))\,ds \\ + \int_0^t 2\tilde{X}(s)e^{\frac{-s}{2}}g(\tilde{X}(s))\,d\tilde{B}(s), \tag{1.4.12}$$

where $\tilde{X}(t) := X(e^t - 1)$, and as before, $\tilde{B}$ is another standard Brownian motion in an extended probability space. It can be verified that for almost all $\omega \in A$,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \tilde{X}(s)f(\tilde{X}(s))\,ds = L_\infty, \quad \lim_{t\to\infty} \frac{1}{t}\int_0^t g^2(\tilde{X}(s))\,ds = \sigma^2. \tag{1.4.13}$$

Let

$$M(t) := \int_0^t 2\tilde{X}(s)e^{\frac{-s}{2}}g(\tilde{X}(s))\,d\tilde{B}(s),$$

which has the quadratic variation

$$\langle M\rangle(t) := \int_0^t 4\tilde{X}^2(s)e^{-s}g^2(\tilde{X}(s))\,ds.$$

We have

$$\lim_{t\to\infty} \frac{\langle M\rangle(t)}{\int_0^t Y(s)\,ds} = 4\sigma^2, \quad \text{a.s. conditionally on } A. \tag{1.4.14}$$

Suppose $D := \{\omega : \lim_{t\to\infty}\langle M\rangle(t) < \infty\}$ with $\mathbb{P}[D] > 0$. Then $\int_0^\infty Y(s)\,ds < \infty$, a.s. on $A \cap D$. Thus

$$\lim_{t\to\infty} \frac{Y(t)}{t} = 2L_\infty + \sigma^2, \quad \text{a.s. on } A\cap D,$$

which contradicts

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = |\sigma|, \quad \text{a.s. conditionally on } A. \tag{1.4.15}$$

Therefore $\mathbb{P}\left[\lim_{t\to\infty}\langle M\rangle(t)=\infty\right]=1$. Note that (1.4.15) implies $\lim_{t\to\infty} Y(t)/t=0$ a.s. conditionally on A. Also,

$$\lim_{t\to\infty}\frac{M(t)}{\int_0^t Y(s)\,ds}=\lim_{t\to\infty}\frac{M(t)}{\langle M\rangle(t)}\cdot\frac{\langle M\rangle(t)}{\int_0^t Y(s)\,ds}=0,\quad \text{a.s. conditionally on } A.$$

Now since for all $t\geq 0$, $Y(t)\geq 0$ a.s., we have

$$\int_0^t Y(s)\,ds\leq x_0^2+\int_0^t 2\tilde{X}(s)f(\tilde{X}(s))\,ds+\int_0^t g^2(\tilde{X}(s))\,ds+M(t).$$

Dividing both sides by $\int_0^t Y(s)\,ds$, taking limits as $t\to\infty$ using (1.4.13), and rearranging the resulting inequality, we get

$$\liminf_{t\to\infty}\frac{t}{\int_0^t Y(s)\,ds}\geq\frac{1}{2L_\infty+\sigma^2},\quad \text{a.s. conditionally on } A.$$

That is

$$\limsup_{t\to\infty}\frac{\int_0^t Y(s)\,ds}{t}\leq 2L_\infty+\sigma^2,\quad \text{a.s. conditionally on } A.$$

Finally, since

$$\lim_{t\to\infty}\frac{M(t)}{t}=\lim_{t\to\infty}\frac{M(t)}{\int_0^t Y(s)\,ds}\cdot\frac{\int_0^t Y(s)\,ds}{t}=0,\quad \text{a.s. conditionally on } A,$$

by (1.4.12) we get

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t Y(s)\,ds=2L_\infty+\sigma^2,\quad \text{a.s. conditionally on } A,$$

from which the desired result (1.4.11) can be obtained. □

Besides being of independent interest, the following result deals with SDEs wih integrable drift coefficients, and will be used extensively in Section 1.5 to prove comparison results.

Theorem 1.4.3. *Let X be the unique continuous adapted process satisfying* (1.2.15) *with $X(0)=x_0$. If $f\in L^1(\mathbb{R};\mathbb{R})$, then there exist positive real numbers $\{C_i\}_{i=1,2,3,4}$ such that*

$$C_1\leq\limsup_{t\to\infty}\frac{X(t)}{\sqrt{2t\log\log t}}\leq C_2,\quad a.s. \tag{1.4.16}$$

$$-C_3\leq\liminf_{t\to\infty}\frac{X(t)}{\sqrt{2t\log\log t}}\leq -C_4,\quad a.s. \tag{1.4.17}$$

where

$$C_1 = \frac{|\sigma| e^{\frac{-2}{\sigma^2}\sup_{x\in\mathbb{R}}\int_0^x f(z)dz}}{e^{\frac{-2}{\sigma^2}\int_0^\infty f(z)dz}}, \quad C_2 = \frac{|\sigma| e^{\frac{-2}{\sigma^2}\inf_{x\in\mathbb{R}}\int_0^x f(z)dz}}{e^{\frac{-2}{\sigma^2}\int_0^\infty f(z)dz}},$$
$$C_3 = \frac{|\sigma| e^{\frac{-2}{\sigma^2}\inf_{x\in\mathbb{R}}\int_0^x f(z)dz}}{e^{\frac{2}{\sigma^2}\int_{-\infty}^0 f(z)dz}}, \quad C_4 = \frac{|\sigma| e^{\frac{-2}{\sigma^2}\sup_{x\in\mathbb{R}}\int_0^x f(z)dz}}{e^{\frac{2}{\sigma^2}\int_{-\infty}^0 f(z)dz}}.$$

Proof. Consider the scale function of X defined as the following

$$s(x) = \int_0^x e^{-2\int_0^y \frac{f(z)}{\sigma^2}\,dz}\,dy, \quad x \in \mathbb{R}.$$

Then $s \in C^2(\mathbb{R};\mathbb{R})$ and for all $x \in \mathbb{R}$ we have

$$s'(x)f(x) + \frac{1}{2}\sigma^2 s''(x) = 0. \tag{1.4.18}$$

Since $f \in L^1$, there exist real numbers k_1 and k_2, such that $\int_0^\infty f(z)dz = k_1$ and $\int_{-\infty}^0 f(z)\,dz = k_2$, which implies $\lim_{x\to\infty} s'(x) = e^{-2k_1/\sigma^2}$ and $\lim_{x\to-\infty} s'(x) = e^{2k_2/\sigma^2}$. So $s(\infty) = \infty$ and $s(-\infty) = -\infty$. Thus $\limsup_{t\to\infty} X(t) = \infty$ and $\liminf_{t\to\infty} X(t) = -\infty$ a.s. Also by L'Hôpital's rule,

$$\lim_{x\to\infty}\frac{s(x)}{x} = e^{\frac{-2k_1}{\sigma^2}}, \quad \lim_{x\to-\infty}\frac{s(x)}{x} = e^{\frac{2k_2}{\sigma^2}}. \tag{1.4.19}$$

Let $Y(t) = s(X(t))$, by Itô's rule and (1.4.18),

$$dY(t) = \sigma s'(X(t))\,dB(t), \quad t \geq 0,$$

with $Y(0) = s(X(0))$. Now since s is strictly increasing, the above equation can be written as

$$dY(t) = g(Y(t))\,dB(t), \quad t \geq 0,$$

where $g(x) = \sigma s'(s^{-1}(x))$, for all $x \in \mathbb{R}$. Y also is a recurrent process on $\mathbb{R}$. Moreover, (1.4.19) gives

$$\lim_{t\to\infty}\frac{\sup_{0\leq s\leq t} Y(s)}{\sup_{0\leq s\leq t} X(s)} = e^{\frac{-2k_1}{\sigma^2}} \quad \text{and} \quad \lim_{t\to\infty}\frac{\inf_{0\leq s\leq t} Y(s)}{\inf_{0\leq s\leq t} X(s)} = e^{\frac{2k_2}{\sigma^2}}, \quad \text{a.s.} \tag{1.4.20}$$

For $0 \le t < \infty$, define the continuous local martingale

$$M(t) := \int_0^t g(Y(s))\,dB(s)$$

which has the quadratic variation $\langle M\rangle(t) = \int_0^t g^2(Y(s))\,ds$. Thus $\langle M\rangle'(t) > 0$ for $t > 0$ and $\langle M\rangle$ is an increasing function. Now

$$\begin{aligned}\inf_{x\in\mathbb{R}} g^2(x) = \inf_{x\in\mathbb{R}} \sigma^2 s'(s^{-1}(x))^2 = \sigma^2 \inf_{x\in\mathbb{R}} e^{\frac{-4}{\sigma^2}\int_0^{s^{-1}(x)} f(z)\,dz} \\ = \sigma^2 e^{\frac{-4}{\sigma^2}\sup_{x\in\mathbb{R}}\int_0^x f(z)\,dz} > 0.\end{aligned}$$

Similarly

$$\sup_{x\in\mathbb{R}} g^2(x) = \sigma^2 e^{\frac{-4}{\sigma^2}\inf_{x\in\mathbb{R}}\int_0^x f(z)\,dz} < \infty.$$

Let $g_1^2 = \inf_{x\in\mathbb{R}} g^2(x)$ and $g_2^2 = \sup_{x\in\mathbb{R}} g^2(x)$, so for all $t \ge 0$,

$$g_1^2 t \le \langle M\rangle(t) \le g_2^2 t, \quad \text{a.s.} \tag{1.4.21}$$

Thus $\lim_{t\to\infty}\langle M\rangle(t) = \infty$ almost surely. Now Define, for each $0 \le s < \infty$, the stopping time $\lambda(s) = \inf\{t \ge 0; \langle M\rangle(t) > s\}$. It is obvious that λ is continuous and tends to infinity almost surely. So $\langle M\rangle(\lambda(t)) = t$, and $\lambda^{-1}(t) = \langle M\rangle(t)$ for $t \ge 0$. By the time-change theorem for martingales in [49], the time-changed process $W(t) := M(\lambda(t))$ is a standard one-dimensional Brownian motion with respect to the filtration $\mathcal{G}_t := \mathcal{F}_{\lambda(t)}$. Hence we have

$$Z(t) := Y(\lambda(t)) = Y(\lambda(0)) + \int_0^{\lambda(t)} g(Y(s))\,dB(s) = Z(0) + W(t)$$

where Z is $\mathcal{G}_t$-adapted. So the Law of the Iterated Logarithm holds for Z, that is

$$1 = \limsup_{t\to\infty} \frac{Y(\lambda(t))}{\sqrt{2t\log\log t}} = \limsup_{t\to\infty} \frac{Y(t)}{\sqrt{2\langle M\rangle(t)\log\log\langle M\rangle(t)}}, \quad \text{a.s.}$$

Note that by (1.4.21) for all $t \ge 0$,

$$\log g_1^2 + \log t \le \log\langle M\rangle(t) \le \log g_2^2 + \log t, \quad \text{a.s.}$$

We have

$$\lim_{t\to\infty} \frac{\log\log \langle M\rangle(t)}{\log\log t} = 1, \quad \text{a.s.},$$

which implies

$$\limsup_{t\to\infty} \frac{Y(t)}{\sqrt{2\langle M\rangle(t)\log\log t}} = 1, \quad \text{a.s.}$$

Similarly

$$\liminf_{t\to\infty} \frac{Y(t)}{\sqrt{2\langle M\rangle(t)\log\log t}} = -1, \quad \text{a.s.}$$

Now as $\langle M\rangle(t) \le g_2^2 t$, we have

$$\limsup_{t\to\infty} \frac{Y(t)}{\sqrt{2t\log\log t}} = \limsup_{t\to\infty} \sqrt{\frac{\langle M\rangle(t)}{t}} \cdot \frac{Y(t)}{\sqrt{2\langle M\rangle(t)\log\log t}} \le g_2, \quad \text{a.s.}$$

Similarly

$$\limsup_{t\to\infty} \frac{Y(t)}{\sqrt{2t\log\log t}} \ge g_1, \quad \text{a.s.}$$

And

$$-g_2 \le \liminf_{t\to\infty} \frac{Y(t)}{\sqrt{2t\log\log t}} \le -g_1, \quad \text{a.s.}$$

Finally combine the above results with (1.4.20), we get

$$e^{\frac{2k_1}{\sigma^2}} g_1 \le \limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \le e^{\frac{2k_1}{\sigma^2}} g_2, \quad \text{a.s.}$$
$$-e^{\frac{-2k_2}{\sigma^2}} g_2 \le \liminf_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \le -e^{\frac{-2k_2}{\sigma^2}} g_1, \quad \text{a.s.}$$

The proof is complete. □

1.5 Recurrent Processes with Asymptotic Behaviour Close to the Law of the Iterated Logarithm

In this section, we again study solutions of (1.2.15), where the drift coefficient satisfies

$$\lim_{x\to\infty} xf(x) = L_\infty \le \frac{\sigma^2}{2} \quad \text{and} \quad \lim_{x\to-\infty} xf(x) = L_{-\infty} \le \frac{\sigma^2}{2}. \tag{1.5.1}$$

As mentioned previously, the solutions are no longer transient but are now recurrent on the real line. Results vary according to the values of L_∞ and $L_{-\infty}$. We classify these results into four main cases. The first result is a direct and easy application of Motoo's theorem. However, we state as a theorem here for two reasons: first, it shows that $-\sigma^2/2$ is another critical value for the process; second, it provides a way to construct a process of known nature to which we can compare processes in the other three cases.

Theorem 1.5.1. *Let X be the unique continuous adapted process satisfying* (1.2.15). *If f satisfies* (1.5.1) *and $L_\infty \in (-\infty, -\sigma^2/2)$, $L_{-\infty} \in (-\infty, -\sigma^2/2)$, then X is recurrent and has finite speed measure. Moreover X obeys*

$$\limsup_{t\to\infty} \frac{\log X(t) \vee 1}{\log t} = \frac{1}{1 - 2L_\infty/\sigma^2}, \quad \limsup_{t\to\infty} \frac{\log((-X(t) \vee 1)}{\log t} = \frac{1}{1 - 2L_{-\infty}/\sigma^2}, \quad a.s.$$

Hence

$$\limsup_{t\to\infty} \frac{\log |X(t)|}{\log t} = \frac{1}{1 - 2(L_\infty \vee L_{-\infty})/\sigma^2}, \quad a.s.$$

Proof. Condition (1.5.1) implies that for any $\epsilon > 0$, there exists $x_\epsilon > 0$ such that

$$L_\infty - \epsilon < xf(x) < L_\infty + \epsilon < -\frac{\sigma^2}{2}, \quad x > x_\epsilon;$$
$$L_{-\infty} - \epsilon < xf(x) < L_{-\infty} + \epsilon < -\frac{\sigma^2}{2}, \quad x < -x_\epsilon.$$

It can be shown that setting $c = x_\epsilon$ in (0.2.3), for any $x > x_\epsilon$, the scale function satisfies

$$\int_{x_\epsilon}^{x} \left(\frac{y}{x_\epsilon}\right)^{\frac{-2(L_\infty+\epsilon)}{\sigma^2}} dy \le s(x) \le \int_{x_\epsilon}^{x} \left(\frac{y}{x_\epsilon}\right)^{\frac{-2(L_\infty-\epsilon)}{\sigma^2}} dy. \tag{1.5.2}$$

Since $L_\infty \in (-\infty, -\sigma^2/2)$, we have $s(\infty) = \infty$. A similar estimate can be used to get $s(-\infty) = -\infty$. For some constants $K_{1,\epsilon}$ and $K_{2,\epsilon}$, the speed measure is given by

$$m(0, \infty) \le K_{1,\epsilon} + K_{2,\epsilon} \int_{x_\epsilon}^{\infty} x^{\frac{2(L_\infty+\epsilon)}{\sigma^2}} dx < \infty.$$

Similarly $m(-\infty, 0) < \infty$, so $m(-\infty, \infty) < \infty$. Hence X is recurrent on $\mathbb{R}$ and has finite speed measure. We can therefore apply Motoo's theorem to X. By L'Hôpital's rule, we have

$$0 \le \limsup_{x\to\infty} \frac{s(x)}{x^{1-\frac{2(L_\infty-\epsilon)}{\sigma^2}}} \le \lim_{x\to\infty} \frac{e^{-\frac{2}{\sigma^2}\int_0^{x_\epsilon} f(z)\,dz - \frac{2}{\sigma^2}\int_{x_\epsilon}^{x} \frac{L_\infty-\epsilon}{z}\,dz}}{(1-\frac{2(L_\infty-\epsilon)}{\sigma^2})x^{-2(L_\infty-\epsilon)/\sigma^2}}$$
$$= \frac{K_{3,x_\epsilon}}{1-\frac{2(L_\infty-\epsilon)}{\sigma^2}}$$

for some positive real number K_{3,x_ϵ}. So if $\varrho_1(t) = t^{1/[1-2(L_\infty-\epsilon)/\sigma^2]}$, we get

$$\int_1^\infty \frac{1}{s(\varrho_1(t))}\,dt \ge \int_1^\infty \frac{1}{K_{4,\epsilon}t}\,dt = \infty,$$

for some positive real number $K_{4,\epsilon}$. Hence

$$\limsup_{t\to\infty} \frac{X(t)}{t^{\frac{1}{1-2(L_\infty-\epsilon)/\sigma^2}}} \ge 1, \quad \text{a.s. on an a.s. event } \Omega_\epsilon,$$

which implies

$$\limsup_{t\to\infty} \frac{\log(X(t)\vee 1)}{\log t} \ge \frac{1}{1-2(L_\infty-\epsilon)/\sigma^2}, \quad \text{a.s. on } \Omega_\epsilon.$$

By considering the a.s. event $\Omega^* = \cap_{\epsilon\in\mathbb{Q}}\Omega_\epsilon$, we have

$$\limsup_{t\to\infty} \frac{\log(X(t)\vee 1)}{\log t} \ge \frac{1}{1-2L_\infty/\sigma^2}, \quad \text{a.s. on } \Omega^*. \tag{1.5.3}$$

Similarly using (1.5.2) for some positive constant $K_{5,\epsilon}$,

$$\liminf_{x\to\infty} \frac{s(x)}{x^{1-2(L_\infty+\epsilon)/\sigma^2}} \ge \frac{K_{5,\epsilon}}{1-\frac{2(L_\infty+\epsilon)}{\sigma^2}} > 0.$$

If we choose $\varrho_2(t) = t^{\frac{1+\epsilon}{1-2(L_\infty+\epsilon)/\sigma^2}}$, then for some positive constant $K_{6,\epsilon}$,

$$\int_1^\infty \frac{1}{s(\varrho_2(t))}\,dt \le \int_1^\infty \frac{1}{K_{6,\epsilon}t^{1+\epsilon}}\,dt < \infty.$$

Hence

$$\limsup_{t\to\infty} \frac{X(t)}{t^{\frac{1+\epsilon}{1-2(L_\infty+\epsilon)/\sigma^2}}} \le 1, \quad \text{a.s. on } \Omega_\epsilon. \tag{1.5.4}$$

Letting $\epsilon \downarrow 0$ through rational numbers, and combining with (1.5.3) we get

$$\limsup_{t\to\infty} \frac{\log(X(t)\vee 1)}{\log t} = \frac{1}{1-2L_\infty/\sigma^2}, \quad \text{a.s. on } \Omega^*. \tag{1.5.5}$$

Now let $Y(t) = -X(t)$, $g(x) = -f(-x)$ and $\tilde{B}(t) = -B(t)$. Then

$$\lim_{x\to\infty} xg(x) = \lim_{x\to\infty} -xf(-x) = \lim_{y\to-\infty} yf(y) = L_{-\infty}$$

and

$$dY(t) = g(Y(t))\,dt + \sigma\,d\tilde{B}(t).$$

Hence by applying the line of argument above we obtain

$$\limsup_{t\to\infty} \frac{Y(t)}{t^{\frac{1}{1-2(L_{-\infty}-\epsilon)/\sigma^2}}} \geq 1, \quad \text{a.s. on some a.s. event } \tilde{\Omega}_\epsilon,$$

$$\limsup_{t\to\infty} \frac{Y(t)}{t^{\frac{1+\epsilon}{1-2(L_{-\infty}+\epsilon)/\sigma^2}}} \leq 1, \quad \text{a.s. on } \tilde{\Omega}_\epsilon,$$

and so as before, we have

$$\limsup_{t\to\infty} \frac{\log(Y(t)\vee 1)}{\log t} = \frac{1}{1-2L_{-\infty}/\sigma^2}, \quad \text{a.s. on some a.s. event } \tilde{\Omega}^*.$$

Finally combining the above equation with (1.5.5), we get

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{\log t} = \frac{1}{1-2(L_\infty \vee L_{-\infty})/\sigma^2}, \quad \text{a.s.}$$

□

The previous theorem is not part of the main focus of this section. Indeed, it shows that solutions are asymptotically stationary, and do not behave asymptotically in a manner close to the LIL. However, taking the results of Theorem 1.5.1, Theorem 1.3.1 and Theorem 1.3.2 together, we can exclude the necessity to study these regions of $(L_\infty, L_{-\infty}, \sigma^2)$ parameter space further.

The rest of our analysis focusses on the parameter regions not covered by these results. Before moving on to the next theorem, we give a lemma which is a building block for the construction of appropriate comparison processes.

Lemma 1.5.1. *Suppose $f:\mathbb{R}\to\mathbb{R}$ is locally Lipschitz continuous and satisfies* (1.5.1). *If $L_\infty \in [-\sigma^2/2,\infty)$ and $L_{-\infty} \in [-\sigma^2/2,\infty)$ and $f(0)=0$, then for every $\epsilon>0$ there exists* *an odd function* $q_\epsilon : \mathbb{R} \to \mathbb{R}$ *such that*

$$q_\epsilon \textit{ is locally Lipschitz continuous on } \mathbb{R}; \tag{1.5.6a}$$

$$\lim_{x\to\pm\infty} xq_\epsilon(x) = -\frac{\sigma^2}{2} - \epsilon; \tag{1.5.6b}$$

$$f(x) \geq q_\epsilon(x), \quad x \geq 0; \tag{1.5.6c}$$

$$f(x) \leq q_\epsilon(x), \quad x \leq 0. \tag{1.5.6d}$$

Moreover, the function $G_\epsilon : (-\infty, \infty) \to \mathbb{R}$ *defined by* $G_\epsilon(x) = \sqrt{|x|}q_\epsilon(\sqrt{|x|})$ *is globally Lipschitz continuous on* $(-\infty, \infty)$.

Proof. For every $\epsilon > 0$ there exists $x_\epsilon > 1$ such that

$$L_\infty - \frac{\epsilon}{2} < xf(x) < L_\infty + \frac{\epsilon}{2}, \quad x > x_\epsilon, \tag{1.5.7}$$

$$L_{-\infty} - \frac{\epsilon}{2} < xf(x) < L_{-\infty} + \frac{\epsilon}{2}, \quad x < -x_\epsilon. \tag{1.5.8}$$

Since f is locally Lipschitz continuous, there is a constant $K > 0$ such that

$$|f(x) - f(y)| \leq K|x - y|, \quad |x| \vee |y| \leq 1. \tag{1.5.9}$$

Now define $f_\epsilon : [x_\epsilon, \infty) \to \mathbb{R}$ by $f_\epsilon(x) = (L_\infty \wedge L_{-\infty} - \varepsilon/2)x^{-1}$ and

$$C_\epsilon = 1 + K + \left\{(-\min_{x\in[1,x_\epsilon]} f(x)) \vee \max_{x\in[-x_\epsilon,-1]} f(x) \vee 0\right\} + [-f_\epsilon(x_\epsilon)]^+.$$

where

$$[x]^+ := \begin{cases} x, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Then

$$C_\epsilon \geq 1 + K; \quad C_\epsilon + f_\epsilon(x_\epsilon) \geq 1. \tag{1.5.10}$$

Also

$$-C_\epsilon < f(x), \quad x \in [1, x_\epsilon] \tag{1.5.11}$$

and

$$C_\epsilon > f(x), \quad x \in [-x_\epsilon, -1]. \tag{1.5.12}$$

By the second inequality in (1.5.10), and the fact that $L_\infty \wedge L_{-\infty} \geq -\sigma^2/2$, we may define $\delta_\epsilon : [x_\epsilon, \infty) \to [0, \infty)$ by

$$\delta_\epsilon(x) = \frac{\frac{\sigma^2}{2} + L_\infty \wedge L_{-\infty} + \frac{\epsilon}{2}}{\frac{\frac{\sigma^2}{2} + L_\infty \wedge L_{-\infty} + \frac{\epsilon}{2}}{f_\epsilon(x_\epsilon) + C_\epsilon} + x - x_\epsilon}, \quad x \geq x_\epsilon.$$

Now we define the candidate function q_ϵ. It is given for $x \geq 0$ by

$$q_\epsilon(x) = \begin{cases} -C_\epsilon x, & x \in [0,1], \\ -C_\epsilon, & x \in (1, x_\epsilon], \\ f_\epsilon(x) - \delta_\epsilon(x), & x > x_\epsilon, \end{cases}$$

and extended for $x \leq 0$ according to $q_\epsilon(x) = -q_\epsilon(-x)$. Clearly q_ϵ is odd by definition, and is obviously Lipschitz continuous on $(-x_\epsilon, x_\epsilon)$. Since

$$\lim_{x \to x_\epsilon^+} q_\epsilon(x) = f_\epsilon(x_\epsilon) - \delta_\epsilon(x_\epsilon) = f_\epsilon(x_\epsilon) - f_\epsilon(x_\epsilon) - C_\epsilon = -C_\epsilon = q_\epsilon(x_\epsilon),$$

we have that q_ϵ is locally Lipschitz continuous on $\mathbb{R}$. Noting that

$$\lim_{x\to\infty} x f_\epsilon(x) = L_\infty \wedge L_{-\infty} - \frac{\epsilon}{2}, \quad \lim_{x\to\infty} x\delta_\epsilon(x) = \frac{\sigma^2}{2} + L_\infty \wedge L_{-\infty} + \frac{\epsilon}{2},$$

we get

$$\lim_{x\to\infty} x q_\epsilon(x) = L_\infty \wedge L_{-\infty} - \frac{\epsilon}{2} - \left(\frac{\sigma^2}{2} + L_\infty \wedge L_{-\infty} + \frac{\epsilon}{2}\right) = -\frac{\sigma^2}{2} - \epsilon.$$

Since q_ϵ is odd, the same limit pertains as $x \to -\infty$.

Finally, we show that $xf(x) \geq xq_\epsilon(x)$, $x \in \mathbb{R}$. For $x \in [0,1]$, because $f(0) = 0$, and (1.5.9) holds, we have $|f(x)| \leq K|x| = Kx$. Hence

$$f(x) \geq -Kx \geq -Kx - x \geq -C_\epsilon x = q_\epsilon(x).$$

For $x \in [-1, 0]$ we have $|f(x)| \leq K|x| = -Kx$. Hence

$$f(x) \leq -Kx \leq -Kx - x \leq -C_\epsilon x = q_\epsilon(x),$$

where we have used the first inequality of (1.5.10) to deduce the third inequality in each case, and the definition of q_ϵ and the fact that it is an odd function at the last steps.

By (1.5.11), for $x \in [1, x_\epsilon]$ we have $q_\epsilon(x) = -C_\epsilon < f(x)$, and as q_ϵ is odd, for $x \in [-x_\epsilon, -1]$ using (1.5.12) we get $q_\epsilon(x) = C_\epsilon > f(x)$. It remains to establish inequalities on (x_ϵ, ∞) and $(-\infty, -x_\epsilon)$. We noted earlier that $\delta_\epsilon(x) > 0$ for $x > x_\epsilon$. Hence, by the definition of q_ϵ, this fact and (1.5.7) yield

$$q_\epsilon(x) = f_\epsilon(x) - \delta_\epsilon(x) < f_\epsilon(x) = \frac{L_\infty \wedge L_{-\infty} - \epsilon/2}{x} \leq \frac{L_\infty - \epsilon/2}{x} < f(x),$$

for $x > x_\epsilon$, as required. We now consider the case when $x < -x_\epsilon$. Since q_ϵ is odd, we get

$$q_\epsilon(x) = -q_\epsilon(-x) = -f_\epsilon(-x) + \delta_\epsilon(-x) > -f_\epsilon(-x),$$

the last step coming from the fact that $\delta_\epsilon(-x) > 0$ for $-x > x_\epsilon$. By the definition of f_ϵ, we have

$$q_\epsilon(x) > \frac{L_\infty \wedge L_{-\infty} - \epsilon/2}{x}, \quad x < -x_\epsilon.$$

Thus, as $x < 0$, we get

$$xq_\epsilon(x) < L_\infty \wedge L_{-\infty} - \frac{\epsilon}{2} \leq L_{-\infty} - \frac{\epsilon}{2} < xf(x),$$

using (1.5.8) at the last step. Hence $xq_\epsilon(x) < xf(x)$ for $x < -x_\epsilon$.

We conclude by dealing with the continuity of G_ϵ. For $x \in [0, 1]$ we have $G_\epsilon(x) = -C_\epsilon x$, so G_ϵ is Lipschitz continuous on $[0, 1)$. Since for any $M > 1$ the functions $x \mapsto \sqrt{x}$ and $x \mapsto q_\epsilon(x)$ are Lipschitz continuous from $[1, M] \to [1, \sqrt{M}]$ and $[1, \sqrt{M}] \to \mathbb{R}$ respectively, the composition $[1, M] \to \mathbb{R} : x \mapsto q_\varepsilon(\sqrt{x})$ is Lipschitz continuous. Thus, as $[1, M] \to [1, \sqrt{M}] : x \mapsto \sqrt{x}$ is Lipschitz continuous, the product $G_\epsilon : [1, M] \to \mathbb{R} : x \mapsto G_\epsilon(x) = \sqrt{x} q_\epsilon(\sqrt{x})$ is Lipschitz continuous. Since $M > 1$ is arbitrary, recalling that G_ϵ is Lipschitz continuous on $[0, 1)$ and continuous at $x = 1$, we have that G_ϵ is locally Lipschitz continuous on $[0, \infty)$. Moreover, as $\sqrt{\cdot}$ and $q_\epsilon(\cdot)$ are actually globally Lipschitz continuous on $[1, \infty)$, and G_ϵ is

Lipschitz continuous on $[0,1]$, it follows that G_ϵ is globally Lipschitz continuous on $[0,\infty)$. Finally since G_ϵ is an even function, it is also globally Lipschitz continuous on $\mathbb{R}$. $\square$

Armed with this result, we are now in a position to determine the asymptotic behaviour for X when $L_\infty \in [-\sigma^2/2, \sigma^2/2]$, $L_{-\infty} \in [-\sigma^2/2, \sigma^2/2]$.

Theorem 1.5.2. *Let X be the unique continuous adapted process satisfying* (1.2.15). *Suppose f satisfies* (1.5.1) *and there exists at least one $x_* \in \mathbb{R}$ such that $f(x_*) = 0$. If $L_\infty \in [-\sigma^2/2, \sigma^2/2]$ and $L_{-\infty} \in [-\sigma^2/2, \sigma^2/2]$, then X is recurrent and satisfies*

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t \log\log t}} \le |\sigma|, \quad a.s.$$

Moreover

$$\limsup_{t\to\infty} \frac{\log |X(t)|}{\log t} = \frac{1}{2}, \quad a.s. \tag{1.5.13}$$

Proof. Again, the first part of the conclusion can be obtained immediately by Theorem 1.4.1. Therefore we also have the following upper estimate

$$\limsup_{t\to\infty} \frac{\log |X(t)|}{\log t} \le \frac{1}{2}, \quad \text{a.s.}$$

For the rest of the proof, the main idea is to compare X^2 with a squared stationary process described in Theorem 1.5.1. In what follows we fix $\epsilon \in (0,1)$. By hypothesis, there exists at least one $x_* \in \mathbb{R}$ such that $f(x_*) = 0$. Consider the process $\widetilde{X}$ governed by the following equation,

$$d\widetilde{X}(t) = \tilde{f}(\widetilde{X}(t))\,dt + \sigma\,dB(t), \quad t \ge 0,$$

where $\widetilde{X}(t) = X(t) - x_*$ and $\tilde{f}(x) = f(x + x_*)$. Thus $\tilde{f}(0) = 0$. By Itô's rule, we have

$$\begin{aligned}
d\widetilde{X}^2(t) &= \big(2\widetilde{X}(t)\tilde{f}(\widetilde{X}(t)) + \sigma^2\big)\,dt + 2\widetilde{X}(t)\sigma\,dB(t) \\
&= \big[2\big(\widetilde{X}(t)\tilde{f}(\widetilde{X}(t)) - \widetilde{X}(t)q_\epsilon(\widetilde{X}(t))\big) + 2\widetilde{X}(t)q_\epsilon(\widetilde{X}(t)) + \sigma^2\big]\,dt + 2\widetilde{X}(t)\sigma\,dB(t).
\end{aligned}$$

If q_ϵ is defined as in the previous lemma, then for all $x \in \mathbb{R}$, $\phi(x) := x\tilde{f}(x) - xq_\epsilon(x) \geq 0$, with $\phi(0) = 0$. Since q_ϵ is odd, we can rewrite the above equation governing $\widetilde{X}^2(t) =: Y(t)$ as

$$dY(t) = (2\psi(Y(t)) + 2\sqrt{|Y(t)|}q_\epsilon(\sqrt{|Y(t)|}) + \sigma^2)\,dt + 2\sqrt{|Y(t)|}\sigma\,dW(t)$$

where $Y(0) = (x_0 - x_*)^2$, W is another Brownian motion in an extended space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, and $\psi(x) = \phi(\sqrt{|x|})$. Consider now the processes governed by the following two equations

$$dY_\epsilon(t) = (2\sqrt{|Y_\epsilon(t)|}q_\epsilon(\sqrt{|Y_\epsilon(t)|}) + \sigma^2)\,dt + 2\sqrt{|Y_\epsilon(t)|}\sigma\,dW(t)$$

$$dY_0(t) = (2\sqrt{|Y_0(t)|}q_\epsilon(\sqrt{|Y_0(t)|}))\,dt + 2\sqrt{|Y_0(t)|}\sigma\,dW(t)$$

with $Y(0) \geq Y_\epsilon(0) \geq Y_0(0) = 0$. Since the drift coefficient of Y_0 is globally Lipschitz continuous by the previous lemma, we can use Yamada and Watanabe's uniqueness theorem again to show that for every $\epsilon \in (0,1)$, there exists an a.s. event Ω_ϵ, such that $Y(t) \geq Y_\epsilon(t) \geq Y_0(t) = 0$ for all $t \geq 0$ a.s. on Ω_ϵ. Therefore all the absolute values can be removed. Now by the definition and properties of q_ϵ, it is easy to check that the scale function and the speed measure of Y_ϵ satisfy

$$s(\infty) = \infty, \quad s(0) > -\infty, \quad \text{and } m(0,\infty) < \infty$$

respectively. A similar argument to that used in Theorem 1.4.2 shows that zero is a reflecting barrier for Y_ϵ. Therefore Y_ϵ is a recurrent process on $\mathbb{R}^+$ with finite speed measure to which we can apply Motoo's theorem in order to determine the growth rate of its largest deviations. Now since $\lim_{x\to\infty} \sqrt{x}q(\sqrt{x}) = -\sigma^2/2 - \epsilon$, for the same ϵ, there exists x_ϵ such that for all $x > x_\epsilon$,

$$-\frac{\sigma^2}{2} - \epsilon(1+\epsilon) < \sqrt{x}q_\epsilon(\sqrt{x}) < -\frac{\sigma^2}{2} - \epsilon(1-\epsilon).$$

Let s be the scale function of Y_ϵ, then for some real positive constants $K_{1,\epsilon}$,

$$0 \leq \limsup_{x\to\infty} \frac{s(x)}{x^{1+\epsilon(1+\epsilon)/\sigma^2}} \leq \lim_{x\to\infty} \frac{\int_{x_\epsilon}^x \left(\frac{y}{x_\epsilon}\right)^{\frac{\epsilon(1+\epsilon)}{\sigma^2}} dy}{x^{1+\epsilon(1+\epsilon)/\sigma^2}} = \frac{K_{1,\epsilon}}{1+\epsilon(1+\epsilon)/\sigma^2}.$$

If we choose $\varrho(t) = t^{\frac{1}{1+\epsilon(1+\epsilon)/\sigma^2}}$, then

$$\int_1^\infty \frac{1}{s(\varrho(t))}\,dt \geq \int_1^\infty \frac{1}{t}\,dt = \infty.$$

Again by Motoo's theorem we have

$$\limsup_{t\to\infty} \frac{Y_\epsilon(t)}{t^{1/(1+\epsilon(1+\epsilon)/\sigma^2)}} \geq 1, \quad \text{a.s. on an a.s. event } \Omega_\epsilon^*,$$

which implies

$$\limsup_{t\to\infty} \frac{\log Y_\epsilon(t)}{\log t} \geq \frac{1}{1+\epsilon(1+\epsilon)/\sigma^2}, \quad \text{a.s. on } \Omega_\epsilon^*.$$

Hence on the a.s. event $\Omega_\epsilon^{**} = \Omega_\epsilon \cap \Omega_\epsilon^*$,

$$\limsup_{t\to\infty} \frac{\log Y(t)}{\log t} \geq \frac{1}{1+\epsilon(1+\epsilon)/\sigma^2} \quad \text{a.s.}$$

Considering the a.s. event $\Omega^* = \cap_{\epsilon\in\mathbb{Q}}\Omega_\epsilon^{**}$, we have

$$\limsup_{t\to\infty} \frac{\log Y(t)}{\log t} \geq 1, \quad \text{a.s.}$$

which implies

$$\limsup_{t\to\infty} \frac{\log|\widetilde{X}(t)|}{\log t} \geq \frac{1}{2}, \quad \text{a.s.},$$

and hence the result. □

Using the same technique as was employed to prove Theorem 1.5.2, we may construct a locally Lipschitz continuous function q_ϵ such that for all $x \in \mathbb{R}$, $f(x) \geq q_\epsilon(x)$, and $\lim_{|x|\to\infty} xq_\epsilon(x) = -\sigma^2/2 - \epsilon$. Instead of comparing pathwise with X^2, we manufacture a solution with drift coefficient q_ϵ and directly compare it with X. The proof is left to the reader.

Theorem 1.5.3. *Let X be the unique continuous adapted process satisfying* (1.2.15). *Suppose f satisfies* (1.5.1) *and there exists at least one $x_* \in \mathbb{R}$ such that $f(x_*) = 0$. If $L_{-\infty} \in (-\infty, -\sigma^2/2)$ and $L_\infty \in [-\sigma^2/2, 0]$, or $L_\infty \in (-\infty, -\sigma^2/2)$ and $L_{-\infty} \in [-\sigma^2/2, 0]$,* *then X is recurrent and obeys*

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \le |\sigma|, \quad a.s.$$

Moreover,

$$\limsup_{t\to\infty} \frac{\log|X(t)|}{\log t} = \frac{1}{2}, \quad a.s.$$

Remark 1.5.1.

Even though zeros are not included on the intervals for $L_{\pm\infty}$ in Figure 1 in Section 3, the construction of q_ϵ in either Theorem 1.5.2 or Theorem 1.5.3 covers the case when one or both of L_∞ and $L_{-\infty}$ is zero. Therefore we can always get the result (1.5.13) if the drift coefficient f reaches zero along the real line at least once. However, if f changes its sign an even number of times, more precise estimates on the growth rate can be obtained, despite the fact that at least one of L_∞ and $L_{-\infty}$ is zero. Lemma 1.5.2 and Theorem 1.5.4 deal with this case. In particular, if f remains non-negative (or non-positive) on the real line, we could compare X with the Brownian motion $\{\sigma B(t)\}_{t\ge 0}$ directly. This fact is stated in Corollary 1.5.1 without proof.

In order to apply a comparison argument to the next category of parameter values, we need to construct an appropriate drift coefficient, just as was done in Lemma 1.5.1 and Theorem 1.5.2.

Lemma 1.5.2. *Suppose $f:\mathbb{R}\to\mathbb{R}$ is locally Lipschitz continuous and satisfies* (1.5.1)*.*

(i) If $L_{-\infty}\in(-\infty,0]$ and $L_\infty\in[0,\infty)$, and there exists $x_*>0$ such that for all $|x|>x_*$, $f(x)\ge 0$, then there exists an even function $q_{x_*}:\mathbb{R}\to\mathbb{R}$ such that for all $x\in\mathbb{R}$, $f(x)\ge q_{x_*}(x)$.

(ii) If $L_\infty\in(-\infty,0]$ and $L_{-\infty}\in[0,\infty)$, and there exists $x_*>0$ such that for all $|x|>x_*$, $f(x)\le 0$, then there exists an even function $q_{x_*}:\mathbb{R}\to\mathbb{R}$ such that for all $x\in\mathbb{R}$, $f(x)\le q_{x_*}(x)$.

Moreover, q_{x_*} in either case is globally Lipschitz continuous.

Proof. Under the conditions in Part (i), define $C := \min_{x\in[-x_*,x_*]} f(x) \wedge 0$ and construct q_{x_*} according to:

$$q_{x_*}(x) = \begin{cases} C, & |x| < x_*, \\ -Cx + C + Cx_*, & x_* \le x \le x_* + 1, \\ Cx + C + Cx_*, & -x_* - 1 \le x \le -x_*, \\ 0, & |x| > x_* + 1. \end{cases}$$

It is obvious that q_{x_*} is even, globally Lipschitz continuous, and $f(x) \ge q_{x_*}(x)$ for all $x \in \mathbb{R}$. By a similar argument, we get the second part of the assertion. $\square$

Theorem 1.5.4. *Let X be the unique continuous adapted process satisfying* (1.2.15)*, and suppose f satisfies* (1.5.1)*.*

(i) If $L_{-\infty} \in (-\infty, 0]$ and $L_\infty \in [0, \sigma^2/2]$, and there exists $x_* > 0$ such that for all $|x| > x_*$, $f(x) \ge 0$, then X is recurrent and there exists a deterministic $\varsigma > 0$ such that

$$\varsigma \le \limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \le |\sigma|, \quad a.s.$$

(ii) If $L_\infty \in (-\infty, 0]$ and $L_{-\infty} \in [0, \sigma^2/2]$, and there exists $x_* > 0$ such that for all $|x| > x_*$, $f(x) \le 0$, then X is recurrent and there exists a deterministic $\varsigma > 0$ such that

$$-|\sigma| \le \liminf_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \le -\varsigma, \quad a.s.$$

Proof. We show assertion (i) first. Consider another process Y governed by the equation

$$dY(t) = q_{x_*}(Y(t))\,dt + \sigma\,dB(t), \quad t \ge 0,$$

with $Y(0) \le X(0)$, where q_{x_*} is the function defined in Lemma 1.5.2. Note that $q_{x_*} \in L^1(\mathbb{R};\mathbb{R})$, so by Theorem 1.4.3, we have

$$\varsigma \le \limsup_{t\to\infty} \frac{Y(t)}{\sqrt{2t\log\log t}}, \quad \text{a.s.}$$

where

$$\varsigma = \frac{|\sigma| e^{\frac{-2}{\sigma^2} \sup_{x\in\mathbb{R}} \int_0^x q_{x*}(z)\,dz}}{e^{\frac{-2}{\sigma^2}\int_0^\infty q_{x*}(z)\,dz}}.$$

By Lemma 1.5.2 part (i), $f(x) \geq q_{x_*}(x)$ for all $x \in \mathbb{R}$, so a comparison argument gives

$$\varsigma \leq \limsup_{t\to\infty} \frac{Y(t)}{\sqrt{2t\log\log t}} \leq \limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}}, \quad \text{a.s.}$$

Combining this with the result of Theorem 1.4.1, we get the first part of the theorem. For part (ii), let $\bar{X}(t) = -X(t)$, $\bar{f}(x) = -f(-x)$ and $\bar{B}(t) = B(t)$. Then $\bar{X}$ obeys

$$d\bar{X}(t) = \bar{f}(\bar{X}(t))\,dt + \sigma\,d\bar{B}(t).$$

Now

$$\lim_{x\to\infty} x\bar{f}(x) = \lim_{y\to-\infty}(-y)(-f(y)) = \lim_{y\to-\infty} yf(y) = L_{-\infty} > 0.$$

Similarly $\lim_{y\to-\infty} y\bar{f}(y) = L_\infty < 0$. Therefore by the first part of the proof we get

$$\varsigma \leq \limsup_{t\to\infty} \frac{\bar{X}(t)}{\sqrt{2t\log\log t}}, \quad \text{a.s.}$$

which implies

$$\liminf_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} \leq -\varsigma, \quad \text{a.s.}$$

Combining this limit with the result of Theorem 1.4.1, the second assertion is proved. □

Corollary 1.5.1. *Let X be the unique continuous adapted process satisfying* (1.2.15).

(i) Suppose f remains non-negative on the real line. If $L_{-\infty} \in (-\infty, 0]$ and $L_\infty \in [0, \sigma^2/2]$, then X is recurrent and satisfies

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = \limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} = |\sigma|, \quad a.s.$$

(ii) Suppose f remains non-positive on the real line. If $L_\infty \in (-\infty, 0]$ and $L_{-\infty} \in [0, \sigma^2/2]$, then X is recurrent and satisfies

$$\liminf_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = \limsup_{t\to\infty} \frac{-|X(t)|}{\sqrt{2t\log\log t}} = -|\sigma|, \quad a.s.$$

The lower estimate on the asymptotic growth rate of partial maxima of $|X|$ in this section can also be obtained when the limit in condition (1.5.1) is replaced by a limit superior or limit inferior in the appropriate way. For example, in Theorem 1.5.2, we can alter (1.5.1) to $\liminf_{x\to-\infty} xf(x) = L_{-\infty}$ and $\limsup_{x\to\infty} xf(x) = L_\infty$. Hence we are able to estimate the growth rate of the partial maxima (or minima) of solutions in this section in terms of either the Law of the Iterated Logarithm or the polynomial Liapunov exponent for all real values of L_∞ and $L_{-\infty}$.

1.6 Generalization to Multidimensional Systems

In this section, we generalize some of the main results in the scalar case to finite-dimensional processes. We show that analogous results can be obtained by using the same technique under adjusted conditions.

Theorem 1.6.1. *Let X be the unique continuous adapted process satisfying the d-dimensional equation* (1.2.18)*, where $X(0) = x_0 \in \mathbb{R}^d$, $f : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$, $g : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^{d\times m}$ and B is a m-dimensional Brownian motion. If there exist positive real numbers ρ, C_a and C_b such that*

$$\forall\,(x,t) \in \mathbb{R}^d \times \mathbb{R}^+, \quad x^T f(x,t) \le \rho; \tag{1.6.1a}$$

$$\forall\,(x,t) \in \mathbb{R}^d \times \mathbb{R}^+, \quad ||g(x,t)||_{op} \le C_a, \quad |x^T g(x,t)| \ge C_b|x|. \tag{1.6.1b}$$

then

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \le C_a, \quad a.s. \tag{1.6.2}$$

Proof. By Itô's rule,

$$d|X(t)|^2 = [2X^T(t)f(X(t),t) + ||g(X(t),t)||^2]\,dt + 2X^T(t)g(X(t),t)\,dB(t). \tag{1.6.3}$$

Let N be the martingale $N(t) = \int_0^t X^T(s)g(X(s),s)\,dB(t)$, which has quadratic variation $\langle N\rangle(t) = \int_0^t |g^T(X(s),s)X(s)|^2\,ds$. Then by the martingale representation theorem, there exists a scalar Brownian motion $\widetilde{B}$ on an extended probability space with measure $\tilde{\mathbb{P}}$ such that

$$N(t) = \int_0^t |g^T(X(s), s)X(s)| \, d\widetilde{B}(t), \quad \tilde{\mathbb{P}} - \text{a.s.}$$

We can therefore rewrite (1.6.3) as

$$d|X(t)|^2 \quad = \quad (2X^T(t)f(X(t), t) \; + \; ||g(X(t), t)||^2) \, dt \; + \; 2|X(t)|\Phi(X(t), t) \, d\widetilde{B}(t),$$

where

$$\Phi(x, t) = \begin{cases} \sigma \in [C_b, C_a], & x = 0, \\ \frac{|x^T g(x,t)|}{|x|}, & x \neq 0. \end{cases} \tag{1.6.4}$$

Note by (1.6.1b) that

$$C_b \leq \Phi(x, t) \leq C_a, \quad \text{for all } (x, t) \in \mathbb{R}^d \times \mathbb{R}^+. \tag{1.6.5}$$

If $Y(t) := |X(t)|^2$, then

$$dY(t) = [2X^T(t)f(X(t), t) + ||g(X(t), t)||^2] \, dt + 2\sqrt{Y(t)}\Phi(X(t), t) \, d\widetilde{B}(t).$$

Now define the martingale $M(t) = \int_0^t \Phi(X(s), s) \, d\widetilde{B}(s)$ which has the quadratic variation $\langle M \rangle(t) = \int_0^t \Phi^2(X(s), s) \, ds$. For each $0 \leq s < \infty$, define the stopping time $\eta(s) := \inf\{t \geq 0 : \langle M \rangle(t) > s\}$. Again by the time-change theorem for martingales, the process defined by $W(t) := M(\eta(t))$ is a standard Brownian motion with respect to the filtration $\mathcal{K}_t := \mathcal{F}_{\eta(t)}$. By Proposition 3.4.8 in [49], we have, almost surely

$$\int_0^{\eta(t)} 2\sqrt{Y(s)} \, dM(s) = \int_0^t \sqrt{Y(\eta(s))} \, dW(s) \quad \text{for each } 0 \leq t < \infty.$$

Hence it can be shown that

$$Z(t) = x_0^2 + \int_0^t \frac{2X^T(\eta(s))f(X(\eta(s)), \eta(s)) + ||g(X(\eta(s)), \eta(s))||^2}{\Phi^2(X(\eta(s)), \eta(s))} \, ds + \int_0^t 2\sqrt{Z(s)} \, dW(s), \tag{1.6.6}$$

where $Z(t) := Y(\eta(t))$. Now it is easy to see that the drift coefficient of (1.6.6) is bounded above by $K_u := (2\rho + mC_a^2)/C_b^2$ due to (1.6.1). Consider the process governed by the equation

$$dZ_u(t) = K_u \, dt + 2\sqrt{|Z_u(t)|} \, dW(t), \quad t \geq 0,$$

with $Z_u(0) \geq x_0^2$. A similar argument as given in the proof of Theorem 1.4.2 shows that Z_u is non-negative. Applying the comparison theorem again, we have, for almost all $\omega \in \Omega$, $0 \leq Z(t) \leq Z_u(t)$ for all $t \geq 0$. Let $V_u(t) := e^{-t} Z_u(e^t - 1)$. By Itô's rule, it can be shown that

$$dV_u(t) = (-V_u(t) + K_u) \, dt + 2\sqrt{|V_u(t)|} \, d\widetilde{W}(t), \quad t \geq 0,$$

where $\widetilde{W}$ is another one-dimensional Brownian motion. Applying Lemma 2.3.1, we obtain

$$\limsup_{t \to \infty} \frac{V_u(t)}{2 \log t} = 1, \quad \text{a.s.}$$

Using the relation between V_u and Z_u, and then comparing Z_u with Z, we get

$$\limsup_{t \to \infty} \frac{Z(t)}{2t \log \log t} \leq \limsup_{t \to \infty} \frac{Z_u(t)}{2t \log \log t} \leq 1, \quad \text{a.s.}$$

Since $\eta^{-1}(t) = \langle M \rangle(t)$ for $t \geq 0$, and $Z(t) = Y(\eta(t))$, we have

$$\limsup_{t \to \infty} \frac{Y(t)}{2\langle M \rangle(t) \log \log \langle M \rangle(t)} \leq 1, \quad \text{a.s.}$$

By (1.6.5), $C_b^2 \, t \leq \langle M \rangle(t) \leq C_a^2 \, t$ for all $t \geq 0$ a.s. Thus

$$\limsup_{t \to \infty} \frac{Y(t)}{2t \log \log t} \leq C_a^2, \quad \text{a.s.}$$

Since $Y(t) = |X(t)|^2$, the assertion (1.6.2) is therefore proved. □

We now establish the corresponding lower bound.

Theorem 1.6.2. *Let X be the unique continuous adapted process satisfying the d-dimensional equation* (1.2.18), *where B is a m-dimensional Brownian motion. If* (1.6.1b) *holds and* *there exists a positive real number* μ *such that*

$$\inf_{(x,t)\in\mathbb{R}^d\times\mathbb{R}^+}\left(2x^T f(x,t)+||g(x,t)||^2\right)=\mu,$$

then

$$\limsup_{t\to\infty}\frac{|X(t)|}{\sqrt{2t\log\log t}}\geq C_b, \quad a.s. \tag{1.6.8}$$

Proof. Proceeding in the same way as in the previous theorem, we arrive at the process Z governed by (1.6.6), i.e.,

$$dZ(t) \quad = \quad \frac{2X^T(\eta(t))f(X(\eta(t)),\eta(t))+||g(X(\eta(t)),\eta(t))||^2}{\Phi^2(X(\eta(t)),\eta(t))}\,dt \ + \ 2\sqrt{Z(t)}\,dW(t),$$

where Φ is as defined in (1.6.4). By condition (1.6.7), it is obvious that the drift coefficient is bounded below by $K_l := \mu/(mC_a^2)$. Let Z_l be the non-negative process with $Z(0) \geq Z_l(0) \geq 0$ which satisfies the SDE

$$dZ_l(t) = K_l\,dt + 2\sqrt{Z_l(t)}\,dW(t), \quad t \geq 0.$$

Then $Z(t) \geq Z_l(t)$, for all $t \geq 0$ a.s. Applying the same change in time and scale to Z_l as in the previous proof, and defining $V_l(t) := e^{-t}Z_l(e^t-1)$, we get

$$dV_l(t) = (-V_l(t)+K_l)\,dt + 2\sqrt{|V_l(t)|}\,d\widetilde{W}(t), \quad t \geq 0.$$

Applying Lemma 2.3.1 again yields

$$\limsup_{t\to\infty}\frac{V_l(t)}{2\log t}=1, \quad \text{a.s.}$$

Following a similar argument as in Theorem 1.6.1, we get the desired result (1.6.8). $\square$

Our last theorem covers the special case where the diffusion coefficient is constant, diagonal and invertible. In this result, we use the notation $\langle x,y\rangle$ to denote the standard inner product of x and y in $\mathbb{R}^d$, and e_i as the i–th standard basis vector.

Theorem 1.6.3. *Let X be the unique continuous adapted process satisfying the d-dimensional equation*

$$dX(t) = f(X(t),t)\,dt + \Gamma\,dB(t), \quad t \geq 0 \tag{1.6.9}$$

with $X(0) = x_0 \in \mathbb{R}^d$, $f : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ and Γ is a $d \times d$ diagonal and invertible matrix with diagonal entries $\{\gamma_i\}_{1\leq i\leq d}$. B is a d-dimensional Brownian motion.

(i) If there exists a positive real number ρ such that

$$\forall\,(x,t) \in \mathbb{R}^d \times \mathbb{R}^+, \quad x^T f(x,t) \leq \rho, \tag{1.6.10}$$

then

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \leq \max_{1\leq i\leq d} |\gamma_i|, \quad a.s. \tag{1.6.11}$$

(ii) If there exists $i \in \{1, 2...d\}$ such that

$$\inf_{(x,t)\in\mathbb{R}^d\times\mathbb{R}^+} \langle x, e_i\rangle\langle f(x,t), e_i\rangle = \mu > -\frac{\gamma_i^2}{2}, \tag{1.6.12}$$

then

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \geq |\gamma_i|, \quad a.s. \tag{1.6.13}$$

(iii) Moreover, if (1.6.10) holds, and there exists $i \in \{1, 2...d\}$ such that (1.6.12) holds and $|\gamma_i| = \max_{1\leq j\leq d} |\gamma_j|$, then

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} = |\gamma_i|, \quad a.s.$$

Proof. It is obvious that part (iii) of the conclusion is a consequence of part (i) and (ii). To prove part (i), let $Y(t) := \Gamma^{-1}X(t)$, $\tilde{f}(x,t) = \Gamma^{-1}f(\Gamma x, t)$, so that

$$dY(t) = \tilde{f}(Y(t),t)\,dt + I_d\,dB(t), \quad t \geq 0.$$

Therefore

$$d|Y(t)|^2 = (2Y^T(t)\tilde{f}(Y(t),t) + d)\,dt + 2Y^T(t)\,dB(t), \quad t \geq 0.$$

Define $Z(t) := |Y(t)|^2$. Then the above equation can be written as

$$dZ(t) = (2Y^T(t)\tilde{f}(Y(t),t) + d)\,dt + 2\sqrt{Z(t)}\,dW(t), \quad t \geq 0.$$

where W is another one–dimensional Brownian motion. If we can show that

$$\forall\,(y,t) \in \mathbb{R}^d \times \mathbb{R}^+, \quad y^T\tilde{f}(y,t) \leq K, \tag{1.6.14}$$

for some positive K, then the non-negative process governed by

$$dZ_u(t) = (2K + d)\,dt + 2\sqrt{Z_u(t)}\,dW(t), \quad t \geq 0,$$

with $Z_u(0) \geq x_0^2$ satisfies $Z_u(t) \geq Z(t)$ for all $t \geq 0$ almost surely. As in the proof of the previous theorem, we have

$$\limsup_{t\to\infty} \frac{Z(t)}{2t\log\log t} \leq \limsup_{t\to\infty} \frac{Z_u(t)}{2t\log\log t} \leq 1, \quad \text{a.s.}$$

Thus

$$\limsup_{t\to\infty} \frac{\sqrt{\frac{X_1^2(t)}{\gamma_1^2} + \frac{X_2^2(t)}{\gamma_2^2} + ... + \frac{X_d^2(t)}{\gamma_d^2}}}{\sqrt{2t\log\log t}} \leq 1, \quad \text{a.s.}$$

Since

$$\frac{1}{\max_{1\leq i\leq d}|\gamma_i|}\sqrt{X_1^2(t) + \cdots + X_d^2(t)} \leq \sqrt{\frac{X_1^2(t)}{\gamma_1^2} + \frac{X_2^2(t)}{\gamma_2^2} + ... + \frac{X_d^2(t)}{\gamma_d^2}},$$

assertion (1.6.11) is proved. Now it is left to show (1.6.14). Let $y := \Gamma^{-1}x$, so that for $1 \leq i \leq d$, the i-th components are related by $y_i = x_i/\gamma_i$. Hence condition (1.6.10) gives

$$\begin{aligned} y^T\tilde{f}(y,t) &= y^T\Gamma^{-1}f(\Gamma y,t) = \Sigma_{i=1}^d \frac{y_i}{\gamma_i} f_i(\Gamma y,t) \\ &= \Sigma_{i=1}^d \frac{x_i}{\gamma_i^2} f_i(x,t) \leq \frac{1}{\min_{1\leq i\leq d}\gamma_i^2}\Sigma_{i=1}^d x_i f_i(x,t) \leq \frac{\rho}{\min_{1\leq i\leq d}\gamma_i^2}. \end{aligned}$$

The proof of part (i) is complete. For part (ii), note for each $1 \leq i \leq d$ and all $t \geq 0$, that $|X(t)| \geq |X_i(t)|$. Consider a particular X_i which is governed by

$$dX_i(t) = f_i(X(t),t)\,dt + \gamma_i\,dB_i(t), \quad t \geq 0.$$

Here by (1.6.12) and Theorem 1.4.2, we have

$$\limsup_{t\to\infty} \frac{|X_i(t)|}{\sqrt{2t\log\log t}} \geq |\gamma_i|, \quad \text{a.s.}$$

and so the inequality (1.6.13) is obvious. □

1.7 Application to a Financial Market Model

In this section, for the purposes mentioned in Section 1.2, we present an ergodic–type theorem for the solution of the equation

$$dX(t) = f(X(t))[1 - \alpha I_{\{|X(t)|>k\sigma\sqrt{t}\}}]\, dt + \sigma\, dB(t). \tag{1.7.1}$$

A detailed discussion can be found in the end of Section 1.2.

Theorem 1.7.1. *Suppose f is locally Lipschitz continuous and odd on $\mathbb{R}$, and satisfies,*

$$\lim_{|x|\to\infty} xf(x) = L \in (0, \sigma^2/2], \quad f(x) \geq 0 \quad \text{for all } x \geq 0. \tag{1.7.2}$$

Let x_0 be deterministic, $0 < \alpha \leq 1$, $\sigma > 0$, $k > 0$ and I be the indicator function. Then there is a unique strong continuous solution X of (1.7.1) with $X(0) = x_0$. Moreover, X obeys

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} = \sigma, \quad a.s.$$

and

$$\lim_{t\to\infty} \frac{\int_0^t \frac{X^2(s)}{(1+s)^2}ds}{\log t} = \Lambda_{L,\sigma,\alpha,k} \quad a.s., \tag{1.7.3}$$

where

$$\Lambda_{L,\sigma,\alpha,k} := \frac{\int_0^{k^2\sigma^2} e^{\frac{-x}{2\sigma^2}} x^{\frac{\sigma^2+2L}{2\sigma^2}}\, dx + (k^2\sigma^2)^{\frac{L\alpha}{\sigma^2}} \int_{k^2\sigma^2}^\infty e^{\frac{-x}{2\sigma^2}} x^{\frac{\sigma^2+2L(1-\alpha)}{2\sigma^2}}\, dx}{\int_0^{k^2\sigma^2} e^{\frac{-x}{2\sigma^2}} x^{\frac{2L-\sigma^2}{2\sigma^2}}\, dx + (k^2\sigma^2)^{\frac{L\alpha}{\sigma^2}} \int_{k^2\sigma^2}^\infty e^{\frac{-x}{2\sigma^2}} x^{\frac{2L(1-\alpha)-\sigma^2}{2\sigma^2}}\, dx} > \sigma^2. \tag{1.7.4}$$

Remark 1.7.1.

In the case when $f(x) = 0$, then $L = 0$, and we can independently prove (1.2.14), which is consistent with (1.7.3) ($\Lambda_{L,\sigma,\alpha,k} = \sigma^2$). On the other hand, letting $L \to 0$ in (1.7.4) yields $\lim_{L\to 0+} \Lambda_{L,\sigma,\alpha,k} = \sigma^2$.

Remark 1.7.2.

As claimed earlier, we have $\Lambda_{L,\sigma,\alpha,k} > \sigma^2$ under the hypotheses of Theorem 1.7.1. To see this, for $L \in (0, \sigma^2/2]$, let

$$I := \int_0^{k^2\sigma^2} e^{\frac{-x}{2\sigma^2}} x^{\frac{2L-\sigma^2}{2\sigma^2}}\, dx$$

and

$$J := (k^2\sigma^2)^{\frac{L\alpha}{\sigma^2}} \int_{k^2\sigma^2}^{\infty} e^{\frac{-x}{2\sigma^2}} x^{\frac{2L(1-\alpha)-\sigma^2}{2\sigma^2}}\, dx.$$

Integration by parts gives

$$\int_0^{k^2\sigma^2} e^{\frac{-x}{2\sigma^2}} x^{\frac{\sigma^2+2L}{2\sigma^2}}\, dx = -2e^{\frac{-k^2}{2}} k^{1+\frac{2L}{\sigma^2}} \sigma^{3+\frac{2L}{\sigma^2}} + (\sigma^2+2L)I$$

and

$$(k^2\sigma^2)^{\frac{L\alpha}{\sigma^2}} \int_{k^2\sigma^2}^{\infty} e^{\frac{-x}{2\sigma^2}} x^{\frac{\sigma^2+2L(1-\alpha)}{2\sigma^2}}\, dx = 2e^{\frac{-k^2}{2}} k^{1+\frac{2L}{\sigma^2}} \sigma^{3+\frac{2L}{\sigma^2}} + (\sigma^2+2L(1-\alpha))J.$$

Then by (1.7.4)

$$\Lambda_{L,\sigma,\alpha,k} = \sigma^2 + \frac{2LI + 2L(1-\alpha)J}{I+J} > \sigma^2,$$

as claimed.

Proof. We first discuss the existence of a strong solution of (1.7.1), which is not directly obvious because the drift coefficient of (1.7.1) is discontinuous. However, by condition (1.7.2) and the continuity of f, the drift coefficient of X is uniformly bounded on $[0,\infty)\times\mathbb{R}$. Therefore, we may apply Proposition 5.3.6 and Remark 5.3.7 in [49] to obtain a weak solution. Moreover, by Corollary 5.3.11 in [49], the weak solution of (1.7.1) is unique in the sense of probability law. On the other hand, Theorem V.41.1 in [73] by Nakao and Le Gall gives us the pathwise uniqueness of the solution. This, together with the weak existence implies the existence of a strong solution by Corollary 5.3.23 in [49]. For a given initial value x_0, and a fixed Brownian motion B, this strong solution is unique.

By the Ikeda–Watanabe comparison theorem [73, Theorem V.43] which only requires the continuity of one of the drift coefficients in the two equations being compared, the first part of the theorem can easily be obtained by Theorem 1.4.1 and 1.4.2.

Now consider the transformation $Y(t) := e^{-t}X^2(e^t-1)$. By Itô's rule, and the fact that f is odd, there exists a standard Brownian motion W such that

$$dY(t) = \big(-Y(t)+\sigma^2+2\sqrt{Y(t)}e^{\frac{t}{2}}f(\sqrt{Y(t)}e^{\frac{t}{2}})[1-\alpha I_{\{Y(t)>k^2\sigma^2(1-e^{-t})\}}]\big)\,dt \\ + 2\sigma\sqrt{Y(t)}\,dW(t). \quad (1.7.5)$$

For any $0<\varepsilon<1/2$, there exists a deterministic $T_{1,\varepsilon}>0$ such that for all $t>T_{1,\varepsilon}$, $e^{-t}<\varepsilon$, so $k^2\sigma^2(1-\varepsilon)<k^2\sigma^2(1-e^{-t})<k^2\sigma^2$. Due to (1.7.2) and continuity of f, there exists a $K>L(1+\varepsilon)$ such that for all $x\in\mathbb{R}$, $xf(x)<K$, and there exists a deterministic $x_\varepsilon>0$ such that for all $x>x_\varepsilon$, $L(1-\varepsilon)<xf(x)<L(1+\varepsilon)$. For any $0<\eta<1\wedge k^2\sigma^2(1-\varepsilon)$, there exists a deterministic $T_{2,\varepsilon,\eta}>T_{1,\varepsilon}$ such that $e^{T_{2,\varepsilon,\eta}/2}\sqrt{\eta}=x_\varepsilon$. Thus for all $t>T_{2,\varepsilon,\eta}$ and $Y(t)>\eta$, $L(1-\varepsilon)<\sqrt{Y(t)}e^{t/2}f(\sqrt{Y(t)}e^{t/2})<L(1+\varepsilon)$. Choose θ_1, $\theta_2>0$ so small that $\theta_1<2L$, $\theta_1\vee\theta_2\vee\eta<k^2\sigma^2/6$, which implies $\eta+\theta_1<k^2\sigma^2(1-\varepsilon)-\theta_2$. Now consider $Y_u:=Y_{u,\varepsilon,\eta,\theta_1,\theta_2}$ and $Y_l:=Y_{l,\varepsilon,\eta,\theta_1,\theta_2}$ governed by the following two equations respectively: for $t\ge T_{2,\varepsilon,\eta}$,

$$dY_u(t) = [-Y_u(t)+\sigma^2+2G_u(Y_u(t))]\,dt + 2\sigma\sqrt{Y_u(t)}\,dW(t), \quad (1.7.6)$$

$$dY_l(t) = [-Y_l(t)+\sigma^2+2G_l(Y_l(t))]\,dt + 2\sigma\sqrt{Y_l(t)}\,dW(t) \quad (1.7.7)$$

with Y_l and Y_u chosen so that $0\le Y_l(T_{2,\varepsilon,\eta})<Y(T_{2,\varepsilon,\eta})<Y_u(T_{2,\varepsilon,\eta})$ a.s., where $G_u:\mathbb{R}^+\to\mathbb{R}^+/\{0\}$ is defined by

$$G_u(x)=\begin{cases} K, & 0\le x<\eta,\\ -\frac{K-L(1+\varepsilon)}{\theta_1}x+(K+\frac{K-L(1+\varepsilon)}{\theta_1}\eta), & \eta\le x<\eta+\theta_1,\\ L(1+\varepsilon), & \eta+\theta_1\le x<k^2\sigma^2,\\ -\frac{L\alpha(1+\varepsilon)}{\theta_2}x+L(1+\varepsilon)(1+\frac{\alpha k^2\sigma^2}{\theta_2}), & k^2\sigma^2\le x<k^2\sigma^2+\theta,\\ L(1-\alpha)(1+\varepsilon), & k^2\sigma^2+\theta_2\le x.\end{cases}$$

$G_l : \mathbb{R}^+ \to \mathbb{R}^+$ is defined by

$$G_l(x) = \begin{cases} 0, & 0 \le x < \eta, \\ \frac{L(1-\varepsilon)}{\theta_1}x - \frac{L(1-\varepsilon)\eta}{\theta_1}, & \eta \le x < \eta + \theta_1, \\ L(1-\varepsilon), & \eta + \theta_1 \le x < k_\varepsilon - \theta_2, \\ -\frac{L\alpha(1-\varepsilon)}{\theta_2}x + L(1-\alpha)(1-\varepsilon) + \frac{L\alpha k_\varepsilon(1-\varepsilon)}{\theta_2}, & k_\varepsilon - \theta_2 \le x < k_\varepsilon, \\ L(1-\alpha)(1-\varepsilon), & k_\varepsilon \le x, \end{cases}$$

where $k_\varepsilon := k^2\sigma^2(1-\varepsilon)$. Note that G_u and G_l are globally Lipschitz continuous on $\mathbb{R}^+$. Again by Ikeda–Watanabe's comparison theorem, it can be verified that $Y_l(t) \le Y(t) \le Y_u(t)$ for all $t \ge T_{2,\varepsilon,\eta}$ a.s. on an a.s. event $\Omega_* := \Omega_{\varepsilon,\eta,\theta_1,\theta_2}$. Choose $c \in (\eta+\theta_1, k^2\sigma^2(1-\varepsilon)-\theta_2)$ in definition (0.2.3). Then direct calculations on a scale function and speed measure of Y_l give that

$$\begin{aligned}
\zeta_{1,\varepsilon,\eta,\theta_1,\theta_2} &:= \int_0^\infty x\, m_{Y_l}(dx) \\
&= \frac{1}{2\sigma^2}\Bigg[\int_0^\eta e^{\frac{c-2L(1-\varepsilon)}{2\sigma^2}}\Big(\frac{\eta+\theta_1}{c}\Big)^{\frac{\sigma^2+2L(1-\varepsilon)}{2\sigma^2}}\Big(\frac{\eta}{\eta+\theta_1}\Big)^{\frac{-2L(1-\varepsilon)\eta/\theta_1+\sigma^2}{2\sigma^2}} e^{\frac{-x}{2\sigma^2}}\Big(\frac{x}{\eta}\Big)^{\frac{1}{2}}\,dx \\
&+ \int_\eta^{\eta+\theta_1} e^{\frac{c-2L(1-\varepsilon)(\eta+\theta_1)/\theta_1}{2\sigma^2}}\Big(\frac{\eta+\theta_1}{c}\Big)^{\frac{\sigma^2+2L(1-\varepsilon)}{2\sigma^2}} e^{\frac{2L(1-\varepsilon)/\theta_1-1}{2\sigma^2}x}\Big(\frac{x}{\eta+\theta_1}\Big)^{\frac{\sigma^2-2L(1-\varepsilon)\eta/\theta_1}{2\sigma^2}}\,dx \\
&+ \int_{\eta+\theta_1}^{k^2\sigma^2(1-\varepsilon)-\theta_2} e^{\frac{c-x}{2\sigma^2}}\Big(\frac{x}{c}\Big)^{\frac{\sigma^2+2L(1-\varepsilon)}{2\sigma^2}}\,dx \\
&+ \int_{k^2\sigma^2(1-\varepsilon)-\theta_2}^{k^2\sigma^2(1-\varepsilon)} \Big(\frac{k^2\sigma^2(1-\varepsilon)-\theta_2}{c}\Big)^{\frac{\sigma^2+2L(1-\varepsilon)}{2\sigma^2}} e^{\frac{c-x-2L\alpha(1-\varepsilon)(x-k^2\sigma^2(1-\varepsilon)+\theta_2)/\theta_2}{2\sigma^2}} \\
&\qquad \Big(\frac{x}{k^2\sigma^2(1-\varepsilon)-\theta_2}\Big)^{\frac{\sigma^2+2L(1-\alpha)(1-\varepsilon)+2L\alpha k^2\sigma^2(1-\varepsilon)^2/\theta_2}{2\sigma^2}}\,dx \\
&+ \int_{k^2\sigma^2(1-\varepsilon)}^\infty c^{\frac{-\sigma^2-2L(1-\varepsilon)}{2\sigma^2}} e^{\frac{c-2L\alpha(1-\varepsilon)}{2\sigma^2}}(k^2\sigma^2(1-\varepsilon)-\theta_2)^{\frac{2L\alpha(1-\varepsilon)-2L\alpha k^2\sigma^2(1-\varepsilon)^2/\theta_2}{2\sigma^2}} \\
&\qquad (k^2\sigma^2(1-\varepsilon))^{\frac{2L\alpha k^2(1-\varepsilon)^2}{2\theta_2}} e^{\frac{-x}{2\sigma^2}} x^{\frac{\sigma^2+2L(1-\alpha)(1-\varepsilon)}{2\sigma^2}}\,dx\Bigg] < \infty. \qquad (1.7.8)
\end{aligned}$$

Similar calculations give $\int_0^\infty m_{Y_l}(dx) =: \zeta_{2,\varepsilon,\eta,\theta_1,\theta_2} < \infty$. Hence by the ergodic theorem

[73, Theorem V.53.1], for almost all $\omega \in \Omega_*$,

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t Y(s)\,ds = \liminf_{t\to\infty} \frac{1}{t}\int_{T_{2,\varepsilon,\eta}}^t Y(s)\,ds$$

$$\geq \lim_{t\to\infty} \frac{1}{t}\int_{T_{2,\varepsilon,\eta}}^t Y_l(s)\,ds = \frac{\zeta_{1,\varepsilon,\eta,\theta_1,\theta_2}}{\zeta_{2,\varepsilon,\eta,\theta_1,\theta_2}}. \quad (1.7.9)$$

Now we let the parameters tend to zero through rational numbers in the order ε, θ_1, θ_2 and η. We consider each term in the square brackets in (1.7.8) in turn. As $\varepsilon \downarrow 0$, the first integral on the interval $(0,\eta)$ becomes

$$J_1 := e^{\frac{c-2L}{2\sigma^2}} c^{\frac{-\sigma^2-2L}{2\sigma^2}} (\eta+\theta_1)^{\frac{L}{\sigma^2}} \Big(\frac{\eta+\theta_1}{\eta}\Big)^{\frac{L\eta}{\sigma^2\theta_1}} \int_0^\eta e^{\frac{-x}{2\sigma^2}} x^{\frac{1}{2}}\,dx.$$

Hence

$$\lim_{\eta\to 0}\big(\lim_{\theta_1\to 0} J_1\big) = \lim_{\eta\to 0} e^{\frac{c-2L}{2\sigma^2}} c^{\frac{-\sigma^2-2L}{2\sigma^2}} \eta^{\frac{L}{\sigma^2}} e^{\frac{L}{\sigma^2}} \int_0^\eta e^{\frac{-x}{2\sigma^2}} x^{\frac{1}{2}}\,dx = 0.$$

Similarly, as $\varepsilon \downarrow 0$, the second integral becomes

$$J_2 := e^{\frac{c-2L(\eta+\theta_1)/\theta_1}{2\sigma^2}} c^{\frac{-\sigma^2-2L}{2\sigma^2}} (\eta+\theta_1)^{\frac{L+L\eta/\theta_1}{\sigma^2}} \int_\eta^{\eta+\theta_1} e^{\frac{2L/\theta_1-1}{2\sigma^2}x} x^{\frac{\sigma^2-2L\eta/\theta_1}{2\sigma^2}}\,dx.$$

Since $\theta_1 < 2L$, we have

$$J_2 \leq e^{\frac{c}{2\sigma^2}-\frac{L\eta}{\sigma^2\theta_1}-\frac{L}{\sigma^2}} c^{\frac{-\sigma^2-2L}{2\sigma^2}} (\eta+\theta_1)^{\frac{L+L\eta/\theta_1}{\sigma^2}} e^{\frac{L(\eta+\theta_1)}{\theta_1\sigma^2}} (\eta+\theta_1)^{\frac{1}{2}}\theta_1.$$

Hence $\lim_{\theta_1\to 0} J_2 = 0$. For the third integral, as $\varepsilon, \theta_1, \theta_2$ and η tend to zero, it tends to

$$\int_0^{k^2\sigma^2} e^{\frac{c-x}{2\sigma^2}} \Big(\frac{x}{c}\Big)^{\frac{\sigma^2+2L}{2\sigma^2}}\,dx.$$

Also as $\varepsilon \downarrow 0$, the fourth integral becomes

$$J_4 := e^{\frac{c}{2\sigma^2}+\frac{L\alpha k^2}{\theta_2}-\frac{L\alpha}{\sigma^2}} c^{\frac{-\sigma^2-2L}{2\sigma^2}} (k^2\sigma^2-\theta_2)^{\frac{L\alpha}{\sigma^2}-\frac{L\alpha k^2}{\theta_2}} \int_{k^2\sigma^2-\theta_2}^{k^2\sigma^2} e^{\frac{-(1+2L\alpha/\theta_2)x}{2\sigma^2}} x^{\frac{\sigma^2+2L(1-\alpha)+2L\alpha k^2\sigma^2/\theta_2}{2\sigma^2}}\,dx.$$

It can be verified that

$$J_4 \leq c^{\frac{-\sigma^2-2L}{2\sigma^2}} e^{\frac{c-k^2\sigma^2+\theta_2}{2\sigma^2}} (k^2\sigma^2-\theta_2)^{\frac{L\alpha}{\sigma^2}} (k^2\sigma^2)^{\frac{1}{2}+\frac{L(1-\alpha)}{\sigma^2}} \Big(\frac{k^2\sigma^2}{k^2\sigma^2-\theta_2}\Big)^{\frac{L\alpha k^2}{\theta_2}}\theta_2.$$

Letting $\theta_2 \downarrow 0$, since $\lim_{\theta_2\to 0}\big(\frac{k^2\sigma^2}{k^2\sigma^2-\theta_2}\big)^{\frac{L\alpha k^2}{\theta_2}} = e^{\frac{L\alpha}{\sigma^2}}$, we have $\lim_{\theta_2\to 0} J_4 = 0$. Finally, as $\varepsilon \downarrow 0$, the last integral becomes

$$J_5 := c^{\frac{-\sigma^2-2L}{2\sigma^2}} e^{\frac{c}{2\sigma^2}} e^{\frac{-L\alpha}{\sigma^2}} (k^2\sigma^2-\theta_2)^{\frac{L\alpha}{\sigma^2}} \Big(\frac{k^2\sigma^2}{k^2\sigma^2-\theta_2}\Big)^{\frac{L\alpha k^2}{\theta_2}} \int_{k^2\sigma^2}^{\infty} e^{\frac{-x}{2\sigma^2}} x^{\frac{\sigma^2+2L(1-\alpha)}{2\sigma^2}}\, dx.$$

Letting $\theta_2 \downarrow 0$, we have

$$\lim_{\theta_2\to 0} J_5 = c^{\frac{-\sigma^2-2L}{2\sigma^2}} e^{\frac{c}{2\sigma^2}} (k^2\sigma^2)^{\frac{L\alpha}{\sigma^2}} \int_{k^2\sigma^2}^{\infty} e^{\frac{-x}{2\sigma^2}} x^{\frac{\sigma^2+2L(1-\alpha)}{2\sigma^2}}\, dx.$$

Hence

$$\begin{aligned}\lim_{\varepsilon,\theta_1,\theta_2,\eta\to 0} \zeta_{1,\varepsilon,\eta,\theta_1,\theta_2} = \frac{1}{2\sigma^2} c^{\frac{-\sigma^2-2L}{2\sigma^2}} e^{\frac{c}{2\sigma^2}} \Bigg(&\int_0^{k^2\sigma^2} e^{\frac{-x}{2\sigma^2}} x^{\frac{\sigma^2+2L}{2\sigma^2}}\, dx \\ &+ (k^2\sigma^2)^{\frac{L\alpha}{\sigma^2}} \int_{k^2\sigma^2}^{\infty} e^{\frac{-x}{2\sigma^2}} x^{\frac{\sigma^2+2L(1-\alpha)}{2\sigma^2}}\, dx\Bigg).\end{aligned}$$

In a similar fashion, it is easy to check that as $\varepsilon \downarrow 0$, $\theta_1 \downarrow 0$, $\theta_2 \downarrow 0$ and $\eta \downarrow 0$, $\zeta_{2,\varepsilon,\eta,\theta_1,\theta_2}$ also tends to a finite limit. Indeed,

$$\begin{aligned}\lim_{\varepsilon,\theta_1,\theta_2,\eta\to 0} \zeta_{2,\varepsilon,\eta,\theta_1,\theta_2} = \frac{1}{2\sigma^2} c^{\frac{-\sigma^2-2L}{2\sigma^2}} e^{\frac{c}{2\sigma^2}} \Bigg(&\int_0^{k^2\sigma^2} e^{\frac{-x}{2\sigma^2}} x^{\frac{2L-\sigma^2}{2\sigma^2}}\, dx \\ &+ (k^2\sigma^2)^{\frac{L\alpha}{\sigma^2}} \int_{k^2\sigma^2}^{\infty} e^{\frac{-x}{2\sigma^2}} x^{\frac{2L(1-\alpha)-\sigma^2}{2\sigma^2}}\, dx\Bigg).\end{aligned}$$

This implies that

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t Y(s)\, ds \ge \Lambda_{L,\sigma,\alpha,k}, \quad \text{a.s. on } \Omega_{**} := \cap_{\{\varepsilon,\eta,\theta_1,\theta_2\in\mathbb{Q}\}}\Omega_{*}. \tag{1.7.10}$$

where $\Lambda_{L,\sigma,\alpha,k}$ is given by (1.7.4) and Ω_{**} is an a.s. event. In an analogous manner, by the definition of G_u, we have

$$\begin{aligned}
\kappa_{1,\varepsilon,\eta,\theta_1,\theta_2} &:= \int_0^\infty x\, m_{Y_u}(dx) \\
&= \frac{1}{2\sigma^2}\Bigg[\int_0^\eta e^{\frac{c+2K-2L(1+\varepsilon)}{2\sigma^2}}\Big(\frac{\eta+\theta_1}{c}\Big)^{\frac{\sigma^2+2L(1+\varepsilon)}{2\sigma^2}}\Big(\frac{\eta}{\eta+\theta_1}\Big)^{\frac{\sigma^2+2K+2(K-L(1+\varepsilon))\eta/\theta_1}{2\sigma^2}} \\
&\qquad e^{\frac{-x}{2\sigma^2}}\Big(\frac{x}{\eta}\Big)^{\frac{\sigma^2+2K}{2\sigma^2}}\,dx \\
&\quad + \int_\eta^{\eta+\theta_1} e^{\frac{c+2(K-L(1+\varepsilon))(\eta+\theta_1)/\theta_1}{2\sigma^2}}\Big(\frac{\eta+\theta_1}{c}\Big)^{\frac{\sigma^2+2L(1+\varepsilon)}{2\sigma^2}} e^{-\frac{1+2(K-L(1+\varepsilon))/\theta_1}{2\sigma^2}x} \\
&\qquad \Big(\frac{x}{\eta+\theta_1}\Big)^{\frac{\sigma^2+2K+2(K-L(1+\varepsilon))\eta/\theta_1}{2\sigma^2}}\,dx \\
&\qquad + \int_{\eta+\theta_1}^{k^2\sigma^2} e^{\frac{c-x}{2\sigma^2}}\Big(\frac{x}{c}\Big)^{\frac{\sigma^2+2L(1+\varepsilon)}{2\sigma^2}}\,dx \\
&\quad + \int_{k^2\sigma^2}^{k^2\sigma^2+\theta_2} e^{\frac{c+2L(1+\varepsilon)\alpha k^2\sigma^2/\theta_2}{2\sigma^2}}\Big(\frac{k^2\sigma^2}{c}\Big)^{\frac{\sigma^2+2L(1+\varepsilon)}{2\sigma^2}} e^{-\frac{1+2L(1+\varepsilon)\alpha/\theta_2}{2\sigma^2}x} \\
&\qquad \Big(\frac{x}{k^2\sigma^2}\Big)^{\frac{\sigma^2+2L(1+\varepsilon)(1+\alpha k^2\sigma^2/\theta_2)}{2\sigma^2}}\,dx \\
&\quad + \int_{k^2\sigma^2+\theta_2}^\infty e^{\frac{c-2L(1+\varepsilon)\alpha}{2\sigma^2}}\Big(\frac{k^2\sigma^2}{c}\Big)^{\frac{\sigma^2+2L(1+\varepsilon)}{2\sigma^2}}\Big(\frac{k^2\sigma^2+\theta_2}{k^2\sigma^2}\Big)^{\frac{\sigma^2+2L(1+\varepsilon)(1+\alpha k^2\sigma^2/\theta_2)}{2\sigma^2}} e^{\frac{-x}{2\sigma^2}} \\
&\qquad \Big(\frac{x}{k^2\sigma^2+\theta_2}\Big)^{\frac{\sigma^2+2L(1-\alpha)(1+\varepsilon)}{2\sigma^2}}\,dx\Bigg] < \infty
\end{aligned}$$

Similar calculations give $\int_0^\infty m_{Y_u}(dx) =: \kappa_{2,\varepsilon,\eta,\theta_1,\theta_2} < \infty$. Also by the ergodic theorem,

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t Y(s)\,ds \le \lim_{t\to\infty}\frac{1}{t}\int_0^t Y_u(s)\,ds = \frac{\kappa_{1,\varepsilon,\eta,\theta_1,\theta_2}}{\kappa_{2,\varepsilon,\eta,\theta_1,\theta_2}}, \quad \text{a.s. on } \Omega_*. \tag{1.7.11}$$

Again, let $\varepsilon \downarrow 0$, $\theta_1 \downarrow 0$, $\theta_2 \downarrow 0$ and $\eta \downarrow 0$ through rational numbers and proceeding as for Y_l, we get the same limit $\Lambda_{L,\sigma,\alpha,k}$ as obtained the lower bound. Combining this with (1.7.11) and (1.7.10), we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t Y(s)\,ds = \Lambda_{L,\sigma,\alpha,k}, \quad \text{a.s. on } \Omega_{**}.$$

Using the relation $Y(t) = e^{-t}X^2(e^t - 1)$, the desired result (1.7.3) is obtained. $\square$

Chapter 2

Extension Results on Non-Linear SDEs using the Motoo-Comparison Techniques

2.1 Introduction

In this chapter, we study the almost sure asymptotic growth rate of the *partial maxima* $t \mapsto \sup_{0\le s\le t} |X(s)|$, where $\{X(t)\}_{t\ge 0}$ is the solution of the following d-dimensional SDE.

$$dX(t) = f(X(t), t)dt + g(X(t), t)\, dB(t), \quad t \ge 0 \tag{2.1.1}$$

with initial value $X(0) = x_0 \in \mathbb{R}^d$. Here $f : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ and $g : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^{d\times m}$, both f and g satisfy the local Lipschitz condition. We attempt to find deterministic upper and lower estimates on the rate of growth of the partial maxima by finding constants C_1 and C_2, and a function $\varrho : (0, \infty) \to (0, \infty)$ such that

$$0 < C_2 \le \limsup_{t\to\infty} \frac{\sup_{0\le s\le t} |X(s)|}{\varrho(t)} \le C_1, \quad \text{a.s.} \tag{2.1.2}$$

We often refer to such a function ϱ as an *essential rate of growth.*

In this work, we do not attempt to give a comprehensive theory about large deviations, but rather to demonstrate for particular classes of problems, three different, general and complementary methods for determining growth estimates. Two of the methods are variants of existing estimation techniques, with which we can even find the large deviations of certain stochastic functional differential equations (SFDEs); one is, to our knowledge, a new method. These methods, and the basic ideas behind them, are indicated in the introduction.

In [57, Chapter 2] and [55], Mao considered some classes of SDEs whose solutions are closely related to the Ornstein–Uhlenbeck processes, or which obeys iterated logarithm-type growth bounds. The results in these works are achieved mainly through the combination of the *exponential martingale inequality* (EMI) and *Gronwall's inequality* (GI) (see also [54, 56]). More precisely, the process is transformed by Itô's formula to a one–dimensional process. The transformation is determined by hypotheses on the drift and diffusion coefficients, and also guided by the conjectured rate of growth. With the exponential martingale inequality, the size of the fluctuations of the Itô integral term can then be estimated in terms of its square variation. If the transformation is well–chosen, the square variation is a Riemann integral with an integrand which does not grow faster than linearly in the new scalar state variable. This results in a Riemann integral inequality, which depends on random times, to which Gronwall's inequality can be applied.

This general approach is quite powerful, because it allows us to reduce a *stochastic* differential equation to a integral inequality that can be treated by *deterministic* techniques. This method has proved effective not only in the estimation of the growth rates of large deviations, but also in estimating moments of solutions. Furthermore, as illustrated in [57], it can play an important role in numerical approximations of solutions, such as Caratheodory's or Cauchy–Maruyama's methods. Mao adapts and generalizes this EMI-GI technique to a variety of non-linear SDEs not covered in [55, 57]. The results are stated in Section 3 without proofs.

As mentioned in Chapter 1, in [67], Motoo gave a proof of the Law of the Iterated Logarithm (LIL) for a finite-dimensional Brownian motion. In chapter 1, this technique was generalized to a class of SDEs whose solutions obey the LIL, mainly by means of the *stochastic comparison principle*. This method produces an upper estimate on the growth rate ϱ which is consistent with that obtained by the EMI-GI technique. Moreover, it supplies a shaper upper estimate on C_1 in (2.1.2). There is another advantage associated with this comparison approach: it allows us to obtain a lower estimate in (2.1.2), which we have been unable to establish to date using the exponential martingale inequality. In fact, in certain cases we can even show that the constants C_1 and C_2 in (2.1.2) coincide. These results are interesting because they show that the general exponential martingale approach correctly predicts the essential rate of growth ϱ. Also, the gains made using the comparison approach come at a cost, requiring more restrictive conditions, especially when dealing with multi-dimensional cases.

In Section 2.3.1, we present generalisations of results in Chapter 1 using the stochastic comparison technique. We consider the same nonlinearities in the drift and diffusion coefficients covered by results in Section 2.2 , which eases comparison between hypotheses and conclusions. The proofs are postponed to Section 2.5.

In Section 2.3.2, we also state a result which improves upon the EMI-GI technique to produce a more accurate upper bound on the growth rate for a one-dimensional process obeying the LIL. As in the results in [55, 57] we start by applying Itô's rule, but instead of using the exponential martingale inequality, we apply the LIL for martingales to the Itô integral term. An estimate on the size of the fluctuations of this integral can be furnished by means of the upper estimates already found in Section 2.2. This leads to a Riemann integral inequality involving random times, in common with those found in EMI-GI–type proofs. This integral inequality can be used to formulate an equivalent differential inequality, just as is used in the proof of the classical Gronwall inequality. The next step, which is entirely novel, involves the *construction* of a random, but differentiable process which satisfies a related differential inequality. By applying standard theorems on deterministic differential inequalities, we can improve the estimates established using the EMI-GI method, without requiring any additional conditions. Like the EMI-GI results, this result gives upper estimates only. The proof of this theorem is also postponed to Section 2.5.

All the techniques in this chapter are quite general and exhibit distinct advantages and disadvantages. Both techniques and results have the potential for extension. For example, one could use an alternative Itô transformation, an alternative Riemann integral inequality, or even a different differential inequality. Transformation techniques can be used to map other SDEs onto those studied in this paper. Moreover, it seems fruitful to apply Motoo's theorem together with appropriate comparison arguments to certain stochastic functional differential equations exhibiting monotonicity in the delay, and the start of such a programme of work is indicated in this chapter.

Since 1960's, a number of papers has emerged concerning deterministic differential equations with maximum delay functionals on the righthand side. Halanay [40], as well as Baker and Tang [16], studied the stability theory of solutions of linear differential equations with a maximum delay which is taken on a time interval with a fixed length. To date, there has been comparatively little literature in the corresponding SDEs with maxima delay in either the linear or nonlinear cases. In a recent paper (cf. [11]), Appleby and Wu studied the following scalar equation

$$dX(t) = [-g(X(t)) + \sup_{-\tau \le s \le t} f(X(s))]\,dt + \sigma\,dB(t), \quad t \ge 0, \tag{2.1.3}$$

where g and f are of linear order. Both recurrent and transient solutions were investigated, with the results applied to inefficient financial markets.

In Section 2.4, we again study (2.1.3), where g and f are now asymptotically polynomial functions. For reasons of consistency in this paper (particularly with respect to techniques used), we do not concern ourselves with the case when the solution is transient. Instead we focus on the case when the solution is recurrent. This happens when the reinforcing historical term f is dominated by the mean–reverting instantaneous term g, in the sense that g grows at least as fast as f when $|x|$ tends to infinity. The results are proved using a combination of Motoo's theorem and the type of stochastic comparison principles described in Section 2.3.1. Moreover, we also use another type of comparison argument which involves the construction of a random but differentiable process which satisfies a differential inequality, as described in Section 2.3.2. This technique has been exploited in [11] and in Appleby and Rodkina [9] for highly nonlinear SFDEs with a fading noise intensity. The results show that the presence of the delay does not affect the essential growth rate $\{\varrho(t)\}_{t>0}$ in (2.1.2), but that it does affect the estimates C_1 and C_2. However, it is the degree of non-linearity of g determines ϱ. The proof is postponed to Section 2.5.

The work in this chapter appears in a paper joint with John Appleby and Xuerong Mao [7].

2.2 Results Obtained by the Exponential Martingale Inequality

The following theorem is given in Mao [57]. It generalises a similar result proven in [55]. Note that $X^*(t) := \sup_{0 \le s \le t} X(s)$, for all $t \ge 0$.

Theorem 2.2.1. *Assume that there is a pair of constant $\rho \geq 0$ and $\sigma > 0$ such that*

$$\langle x, f(x,t)\rangle \leq \rho \quad \textit{and} \quad \|g^T(x,t)\|_{op}^2 \leq \sigma \tag{2.2.1}$$

for all $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$, then the solution of equation (2.1.1) obeys

$$\limsup_{t\to\infty} \frac{X^*(t)}{\sqrt{2t\log\log t}} \leq \sqrt{\sigma e} \quad a.s. \tag{2.2.2}$$

We remark that this result gives the correct iterated logarithm rate of growth modulo the constant on the righthand side in the scalar case where $f \equiv 0$ and g is constant. See [57, Theorem 5.4] and following remarks [57, pages 69 and 70].

The following is a generalisation of Theorem 2.2.1.

Theorem 2.2.2. *Let $\theta \in (0,1)$, $\rho > 0$ and $\sigma > 0$ be three constants such that*

$$\langle x, f(x,t)\rangle \leq \rho(1+|x|^{2(1-\theta)}) \quad \textit{and} \quad \|g^T(x,t)\|_{op}^2 \leq \rho + \sigma|x|^{2(1-\theta)} \tag{2.2.3}$$

for all $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$. Then equation (2.1.1) *has a unique global solution which obeys*

$$\limsup_{t\to\infty} \frac{X^*(t)}{(2t\log\log t)^{\frac{1}{2\theta}}} \leq (\theta^2\sigma e)^{\frac{1}{2\theta}} \quad a.s. \tag{2.2.4}$$

In comparison with Theorem 2.2.1, Theorem 2.2.2 may allow both f and g to grow sub–linearly. The following corollary describes this situation more precisely.

Corollary 2.2.1. *Assume that there are positive constants α, β, K_1 and K_2 such that $\alpha \in [0,1)$, $0 < 2\beta \leq 1+\alpha$,*

$$|f(x,t)| \leq K_1(1+|x|^\alpha) \quad \textit{and} \quad \|g^T(x,t)\|_{op}^2 \leq K_1 + K_2|x|^{2\beta} \tag{2.2.5}$$

for all $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$. Let X be the solution of equation (2.1.1).

(i) If $2\beta < 1+\alpha$, then

$$\lim_{t\to\infty} \frac{X^*(t)}{(2t\log\log t)^{\frac{1}{1-\alpha}}} = 0 \quad a.s. \tag{2.2.6}$$

(ii) If $2\beta = 1+\alpha$*, then*

$$\limsup_{t\to\infty} \frac{X^*(t)}{(2t\log\log t)^{\frac{1}{1-\alpha}}} \leq \left(\frac{1}{4}(1-\alpha)^2 K_2 e\right)^{\frac{1}{1-\alpha}} \quad a.s. \tag{2.2.7}$$

The next corollary covers the situation where f decays like $|x|^{-\alpha}$ as $|x|\to\infty$ for some $\alpha\in(0,1)$ while g may still grow sub-linearly.

Corollary 2.2.2. *Assume that there are positive constants* α, β, K_1 *and* K_2 *such that* $\alpha\in(0,1)$*,* $0<2\beta\leq 1-\alpha$*,*

$$|x|^{\alpha}|f(x,t)| \leq K_1 \quad \textit{and} \quad \|g^T(x,t)\|_{op}^2 \leq K_1 + K_2|x|^{2\beta} \tag{2.2.8}$$

for all $(x,t)\in\mathbb{R}^d\times\mathbb{R}^+$*. Let* X *be the unique solution of* (2.1.1)*.*

(i) If $2\beta < 1-\alpha$*, then*

$$\lim_{t\to\infty} \frac{X^*(t)}{(2t\log\log t)^{\frac{1}{1+\alpha}}} = 0 \quad a.s. \tag{2.2.9}$$

(ii) If $2\beta = 1-\alpha$*, then*

$$\limsup_{t\to\infty} \frac{X^*(t)}{(2t\log\log t)^{\frac{1}{1+\alpha}}} \leq \left(\frac{1}{4}(1+\alpha)^2 K_2 e\right)^{\frac{1}{1+\alpha}} \quad a.s. \tag{2.2.10}$$

Roughly speaking, these new results show that when f obeys a polynomial growth or decay condition with exponent $\alpha\in(-1,1)$, and $\langle x, f(x)\rangle$ dominates $\|g(x,t)\|_{\mathrm{op}}^2$ for large $|x|$, then the a.s. partial maxima of the solution still exhibits an iterated logarithm–type of growth bound.

We now turn to consider asymptotic behaviour in the cases when the linear growth bound on f is sharp. Since the results above cover the case when the drift coefficient behaves according to $|x|^{\alpha}$ for $\alpha\in(-1,1)$, and $\alpha>1$ corresponds to cases where f does not obey a linear growth bound, by covering the case $\alpha=1$, we have a reasonably complete picture of the asymptotic behaviour when the drift exhibits polynomial behaviour in $|x|$. More precisely, we build on work in Mao [54, 57] in which it is assumed that

$$\langle x, f(x,t)\rangle \leq \pm\gamma|x|^2+\rho \quad \text{and} \quad ||g(x,t)|| \leq K.$$

Our main aim here is to show that we can remove the condition that the diffusion coefficient g be bounded. The following theorem is a generalization of [57, Theorem 5.3 on page 66], and deals with the case when f can *grow* linearly and $\langle x, f(x,t)\rangle$ slightly dominates $\|g(x,t)\|_{\text{op}}^2$.

Theorem 2.2.3. *Assume that there are positive constants ρ, σ, γ and θ such that $\theta \in (0,1]$,*

$$\langle x, f(x,t)\rangle \leq \gamma |x|^2 + \rho \quad \textit{and} \quad \|g^T(x,t)\|_{op}^2 \leq \rho + \sigma |x|^{2(1-\theta)} \tag{2.2.11}$$

for all $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$. Then the solution of equation (2.1.1) obeys

$$\lim_{t\to\infty} \frac{X^*(t)}{e^{\gamma t}(\log\log t)^{\frac{1}{2\theta}}} = 0 \quad \textit{a.s.} \tag{2.2.12}$$

We note that the rate of growth is essentially $e^{\gamma t}$ modulo an iterated logarithmic factor. This exponential rate of growth is the best estimate one can expect in the deterministic case when $g \equiv 0$, suggesting that the estimate is quite sharp in some cases at least.

We now consider the case where f is linear, but tends to push the solution of the related deterministic system $x'(t) = f(x(t),t)$ towards a bounded domain. Once again we assume that $|\langle x, f(x,t)\rangle|$ dominates $\|g^T(x,t)\|_{\text{op}}^2$. The following theorem is an extension of [57, Theorem 5.5 on page 69].

Theorem 2.2.4. *Assume that there are positive constants ρ, σ, γ and θ such that $\theta \in (0,1)$,*

$$\langle x, f(x,t)\rangle \leq -\gamma |x|^2 + \rho \quad \textit{and} \quad \|g^T(x,t)\|_{op}^2 \leq \rho + \sigma |x|^{2(1-\theta)} \tag{2.2.13}$$

for all $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$. Then the solution of equation (2.1.1) obeys

$$\limsup_{t\to\infty} \frac{X^*(t)}{(\log t)^{\frac{1}{2\theta}}} \leq \left(\frac{\theta\sigma e}{\gamma}\right)^{\frac{1}{2\theta}} \quad \textit{a.s.} \tag{2.2.14}$$

In the scalar and autonomous case, the condition (2.2.13) implies that X is a recurrent process with a finite speed measure. In the case that $f(x) \sim -\gamma x$ and $g^2(x) \sim \sigma |x|^{2(1-\theta)}$ as $x \to \infty$, we may use Motoo's theorem to show that the solution obeys

$$\limsup_{t\to\infty} \frac{X^*(t)}{(\log t)^{\frac{1}{2\theta}}} = \left(\frac{\theta\sigma}{\gamma}\right)^{\frac{1}{2\theta}}, \quad \text{a.s.}$$

and so the estimate for the essential rate of growth obtained in Theorem 2.2.4, which covers finite–dimensional and non–autonomous equations, is sharp.

2.3 Results Obtained by Comparison Principles

In Chapter 1, we gave a different approach to finding both upper and lower bounds on the asymptotic growth rates of solutions of scalar autonomous SDEs based on comparison arguments and Motoo's theorem.

The following lemma is a direct application of the above theorem and it plays an important role in this section. The details of the proof are omitted.

Lemma 2.3.1. *Let U be the unique continuous adapted solution of the following equation*

$$dU(t) = (-aU(t) + b)dt + c\sqrt{|U(t)|}dB(t), \quad t \geq 0,$$

with $U(0) = u$, where a, b and c are positive real numbers. Then for all $t \geq 0$, $U(t) \geq 0$ a.s. Moreover U is stationary and obeys

$$\limsup_{t\to\infty} \frac{U(t)}{\log t} = \frac{c^2}{2a} \quad a.s.$$

2.3.1 Comparison principle results

Our first results are analogues of Theorem 2.2.2 and Corollaries 2.2.1 and 2.2.2 which give iterated logarithm–type estimates on the growth rate of the partial maximum of the solution of (2.1.1) when the drift and diffusion coefficients obey polynomial growth conditions. We supply both upper and lower estimates on the rate of growth of the partial maxima.

Firstly, note that the following Lemma from Chapter 1 is an analogue of Theorem 2.2.1.

Lemma 2.3.2. *If there exist real positive numbers ρ, σ_1 and σ_2 such that for all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$,*

$$xf(x,t) \leq \rho \quad and \quad \sigma_2 \leq |g(x,t)|^2 \leq \sigma_1, \tag{2.3.1}$$

then the solution of the one-dimensional equation (2.1.1) obeys

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \leq \sqrt{\sigma_1} \quad a.s. \tag{2.3.2}$$

A lower bound on the solution is also given in Chapter 1:

Lemma 2.3.3. *If there exist real positive numbers σ_1 and σ_2 such that for all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$,*

$$\inf \frac{xf(x,t)}{g^2(x,t)} = L > -\frac{1}{2} \quad \textit{and} \quad \sigma_2 \leq |g(x,t)|^2 \leq \sigma_1, \tag{2.3.3}$$

for some real number L, then the solution of the one-dimensional equation (2.1.1) *obeys*

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \geq \sqrt{\sigma_2} \quad a.s. \tag{2.3.4}$$

We now state an analogue of Theorem 2.2.2. The following result uses Lemma 2.3.2, and is an extension of Lemma 2.3.2 to the multi-dimensional case.

Theorem 2.3.1. *Let $\theta \in (0,1)$, and suppose there exist positive real numbers ρ, σ_1 and σ_2 such that for all $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$,*

$$x^T f(x,t) \leq \rho |x|^{2(1-\theta)}; \tag{2.3.5a}$$

$$||g(x,t)||_{op}^2 \leq \sigma_1 |x|^{2(1-\theta)} \quad \textit{and} \quad |x^T g(x,t)|^2 \geq \sigma_2 |x|^{2(2-\theta)}; \tag{2.3.5b}$$

$$|x^T g(x,t)| = 0 \quad \textit{iff} \quad x = 0 \in \mathbb{R}^d. \tag{2.3.5c}$$

If in addition

$$f(0,t) = 0 \quad \textit{and} \quad g(0,t) = 0, \quad \textit{for all } t \geq 0, \tag{2.3.6}$$

then the solution of the finite-dimensional equation (2.1.1) *obeys*

$$\limsup_{t\to\infty} \frac{|X(t)|}{(2t\log\log t)^{\frac{1}{2\theta}}} \leq (\theta^2\sigma_1)^{\frac{1}{2\theta}} \quad a.s. \tag{2.3.7}$$

In [57], if a equation with drift and diffusion coefficients satisfies the above assumption (2.3.6), then the solution does not reach zero almost surely, provided that it starts from a non–zero point. The condition is technical here; it is not needed to establish a comparable upper bound in Theorem 2.2.2.

Note that the estimate on the righthand side of (2.3.7) is smaller than that obtained in (2.2.4) in Theorem 2.2.2 by a factor of $e^{1/(2\theta)}$. This is a common feature of the technique combining Motoo's theorem and comparison principle: all results in this section have shaper estimates than those stated in the last section. However, (2.3.5c) and the lower bound on $|x^T g(x,t)|$ in (2.3.5b) are not needed in Theorem 2.2.2, whose proof uses the exponential martingale inequality. Such extra technical conditions are simply needed to complete the proof using the comparison principle approach. The presence of additional conditions of this type are another common feature and a disadvantage of the results stated in this section.

By a similar argument, we have the following theorem on the lower estimate from Lemma 2.3.3. There is no comparable theorem available using the exponential martingale inequality.

Theorem 2.3.2. *Let* $\theta \in (0,1)$*, if there exist* $L \in \mathbb{R}$*,* $\sigma_1 > 0$ *and* $\sigma_2 > 0$ *such that for all* $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$

$$\inf_{x \in \mathbb{R}^d} \frac{|x|^2\big(2x^T f(x,t) + ||g(x,t)||^2\big)}{|x^T g(x,t)|^2} = L > 2 - 2\theta; \tag{2.3.8}$$

and (2.3.5b) *and* (2.3.5c) *also hold. If in addition* f *and* g *obey* (2.3.6)*, then the solution of the finite–dimensional equation* (2.1.1) *obeys*

$$\limsup_{t\to\infty} \frac{|X(t)|}{(2t\log\log t)^{\frac{1}{2\theta}}} \geq (\theta^2\sigma_2)^{\frac{1}{2\theta}} \quad a.s. \tag{2.3.9}$$

The results of Theorem 2.3.1 and 2.3.2 together show that the partial maximum has an identifiable deterministic essential rate of growth given by $\varrho(t) = (2t\log\log t)^{\frac{1}{2\theta}}$. As indicated in the introduction, this shows that both the exponential martingale approach and the upper bound identified by the comparison argument produce sharp bounds on the growth rate.

We now consider results which parallel Theorem 5.5 in [57, page 69]. In addition, we provide results regarding the lower estimates.

Theorem 2.3.3. *Suppose there exist positive real numbers* γ*,* ρ*,* σ_1 *and* σ_2 *such that for all* $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$

$$||g(x,t)||_{op}^2 \leq \sigma_1 \quad and \quad |x^T g(x,t)|^2 \geq \sigma_2|x|, \tag{2.3.10}$$

and g obeys (2.3.5c)*.*

(i) If $x^T f(x,t) \le -\gamma|x|^2 + \rho$, then the solution of equation (2.1.1) *obeys*

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{\log t}} \le \left(\frac{\sigma_1}{\gamma}\right)^{\frac{1}{2}} \quad a.s. \tag{2.3.11}$$

(ii) If $x^T f(x,t) \ge -\gamma|x|^2$, then the solution of equation (2.1.1) *obeys*

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{\log t}} \ge \left(\frac{\sigma_2}{\gamma}\right)^{\frac{1}{2}} \quad a.s.$$

Note once again that the estimate on the righthand side of (2.3.11) is smaller than that obtained in [57, Theorem 5.5] by a factor of $\sqrt{e}$. However, as before, the extra technical conditions (2.3.5c) and the lower bound on $|x^T g(x,t)|$ in (2.3.10) are required to obtain the upper estimate.

The following result may be compared directly with Theorem 2.2.4. It is a generalisation of Theorem 2.3.3.

Theorem 2.3.4. *Suppose $\theta \in (0,1)$, and there exist positive real numbers γ, σ_1 and σ_2 such that for all $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$,*

$$||g(x,t)||_{op}^2 \le \sigma_1 |x|^{2(1-\theta)} \quad and \quad |x^T g(x,t)|^2 \ge \sigma_2 |x|^{4-2\theta} \tag{2.3.12}$$

Suppose moreover that g obeys (2.3.5c)*, and f and g obey* (2.3.6)*.*

(i) If $x^T f(x,t) \le -\gamma|x|^2$, the solution of equation (2.1.1) *obeys*

$$\limsup_{t\to\infty} \frac{|X(t)|}{(\log t)^{\frac{1}{2\theta}}} \le \left(\frac{\theta\sigma_1}{\gamma}\right)^{\frac{1}{2\theta}} \quad a.s.$$

(ii) If $x^T f(x,t) \ge -\gamma|x|^2$, the solution of equation (2.1.1) *obeys*

$$\limsup_{t\to\infty} \frac{|X(t)|}{(\log t)^{\frac{1}{2\theta}}} \ge \left(\frac{\theta\sigma_2}{\gamma}\right)^{\frac{1}{2\theta}} \quad a.s.$$

Again, in the above theorem, as a trade off for getting a shaper estimate, we sacrifice the positive constants in (2.2.13) for technical reasons.

2.3.2 A comparison result using a priori estimates

In this subsection, we state a result which further improves a theorem given in Section 2.2, by reusing the idea of estimation on the Itô integral when constructing a Riemann integral inequality.

Theorem 2.3.5. *Let $f, g : \mathbb{R} \to \mathbb{R}$ and B be a one–dimensional standard Brownian motion. Suppose that $X = \{X(t); t \geq 0\}$ is the unique adapted continuous solution of*

$$dX(t) = f(X(t))\,dt + g(X(t))\,dB(t), \quad t \geq 0,$$

with $X(0) = x_0$. If there exist positive real numbers ρ and σ such that for all $x \in \mathbb{R}$,

$$xf(x) \leq \rho; \tag{2.3.13}$$

$$\limsup_{|x|\to\infty} |g(x)| = \sigma \quad \textit{and} \quad g^2(x) > 0, \tag{2.3.14}$$

then

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} \leq \sqrt{2\sigma} \quad \textit{a.s.} \tag{2.3.15}$$

In this theorem, the global bound on g which appeared in the upper estimates in all previous sections has been reduced to an ultimate bound σ for large values of $|x|$. No extra technical conditions are imposed on the lower bound of $|g|$, as are needed in the comparison arguments, while the factor independent of the diffusion bound on the righthand side of (2.2.2) in Theorem 2.2.1 is reduced from $\sqrt{e}$ to $\sqrt{2}$ in (2.3.15). Based on Lemma 2.3.2, we conjecture that the optimal factor is unity, and that the size of the large deviations of the process will depend on the behaviour of the diffusion coefficient as $|x| \to \infty$.

2.4 Recurrent Solutions of Stochastic Functional Differential Equations with Maximum Delay

In this section, we investigate the large deviations of SFDEs of the following type:

$$dX(t) = [-g(X(t)) + \sup_{-\tau \le s \le t} f(X(s))]\, dt + \sigma\, dB(t), \quad t \ge 0, \tag{2.4.1}$$
$$X(t) = \psi(t), \quad t \in [-\tau, 0],$$

where g and f are asymptotically polynomial functions.

The first theorem in this section concerns SDEs without delay. It provides the fundamental essential growth rate of the partial extrema of the solutions despite the presence of delay. The result is obtained by a direct application of Motoo's theorem; however we state both the result and give details in order to make the paper more self-contained.

Theorem 2.4.1. *Let V be the unique continuous adapted process obeying the following equation*

$$dV(t) = -g(V(t))\, dt + \sigma\, dB(t), \quad t \ge 0, \tag{2.4.2}$$

with $V(0) = v_0$. If there exist positive real numbers θ and a such that

$$\lim_{|x|\to\infty} \frac{\mathrm{sgn}\,(x) g(x)}{|x|^\theta} = a, \tag{2.4.3}$$

then V is recurrent on $\mathbb{R}$. Moreover,

$$\limsup_{t\to\infty} \frac{V(t)}{(\log t)^{\frac{1}{1+\theta}}} = \left[\frac{\sigma^2(1+\theta)}{2a}\right]^{\frac{1}{1+\theta}}, \quad a.s. \tag{2.4.4}$$

$$\liminf_{t\to\infty} \frac{V(t)}{(\log t)^{\frac{1}{1+\theta}}} = -\left[\frac{\sigma^2(1+\theta)}{2a}\right]^{\frac{1}{1+\theta}}, \quad a.s. \tag{2.4.5}$$

Theorem 2.4.1 shows that for SDEs with polynomial drift coefficients of degree θ and additive noise, the growth rate of the partial maxima is logarithmic with the degree of logarithmic growth increasing as the strength of mean–reversion decreases. The result can certainly be generalized to equations with non-constant diffusion coefficient as shown in Chapter 1. To ease later analysis on delay equations using comparison arguments, we retain throughout the condition of a constant diffusion coefficient.

Before moving on to delay equations, we state a lemma which will prove to be convenient in the proofs of later theorems.

Lemma 2.4.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous, odd and non-decreasing function. If there exists a real number $C \geq 1$ such that*

$$\forall\, x, y \in \mathbb{R}, \quad g(|x| + |y|) \leq C\,(g(|x|) + g(|y|))\,, \tag{2.4.6}$$

then

$$\forall\, x \geq 0,\ y \in \mathbb{R}, \quad -Cg(x+y) \leq -g(|x|) + (C+1)g(|y|); \tag{2.4.7}$$

$$\forall\, x < 0,\ y \in \mathbb{R}, \quad Cg(x+y) \leq -g(|x|) + (C+1)g(|y|). \tag{2.4.8}$$

The following theorem deals with the situation when $\theta \in (0, 1)$.

Theorem 2.4.2. *Let X be the unique continuous adapted process obeying* (2.4.1)*. Suppose that g is an odd function, both g and f are non-decreasing on $\mathbb{R}$, and*

$$\forall\, x, y \in \mathbb{R}^+, \quad g(x+y) \leq g(x) + g(y); \tag{2.4.9a}$$

$$\forall\, x, y \in \mathbb{R}, \quad |f(x+y)| \leq f(|x|) + f(|y|). \tag{2.4.9b}$$

Furthermore

$$\lim_{|x|\to\infty} \frac{\operatorname{sgn}(x) g(x)}{|x|^\theta} = a > 0 \tag{2.4.10}$$

where $0 < \theta < 1$.

(i) If

$$\lim_{|x|\to\infty} \frac{\operatorname{sgn}(x) f(x)}{|x|^\theta} = b > 0 \tag{2.4.11}$$

with $a > b$, then

$$C_1 \leq \limsup_{t\to\infty} \frac{|X(t)|}{(\log t)^{\frac{1}{1+\theta}}} \leq C_2, \quad a.s. \tag{2.4.12}$$

where

$$C_1 := \left[\frac{\sigma^2(1+\theta)}{2(a-b)}\right]^{\frac{1}{1+\theta}}, \quad C_2 := \left[1+\left(\frac{3a+b}{a-b}\right)^{\frac{1}{\theta}}\right]\left[\frac{\sigma^2(1+\theta)}{2a}\right]^{\frac{1}{1+\theta}}. \tag{2.4.13}$$

(ii) If

$$\lim_{|x|\to\infty}\frac{g(x)}{f(x)} = \infty, \tag{2.4.14}$$

then

$$\left[\frac{\sigma^2(1+\theta)}{2a}\right]^{\frac{1}{1+\theta}} \le \limsup_{t\to\infty}\frac{|X(t)|}{(\log t)^{\frac{1}{1+\theta}}} \le (3^{\frac{1}{\theta}}+1)\left[\frac{\sigma^2(1+\theta)}{2a}\right]^{\frac{1}{1+\theta}}, \quad a.s. \tag{2.4.15}$$

It is obvious that the second part of the theorem is a special case of the first part and can be obtained by letting $b \downarrow 0$. It can be seen that because the function g in the instantaneous term dominates the function f in the delay , the essential growth rate of the solution of the delay equation is the same as that of the equation without delay. In fact, in case (ii) where f is negligible relative to g, we can obtain sharper estimates on the growth rate, which are moreover close to those seen in (2.4.4) and (2.4.5) for the non-delay equation. Clearly, if f dominates g, we cannot expect solutions to be recurrent, so an analysis of large deviations using Motoo's theorem cannot be applied. If $\theta \in (1,\infty)$, we have the following theorem.

Theorem 2.4.3. *Let X be the unique continuous adapted process obeying* (2.4.1)*. Suppose that g is an odd function, both g and f are non-decreasing on $\mathbb{R}$, and there exists $C > 1$ such that g obeys* (2.4.6)*. Furthermore, suppose g obeys* (2.4.10)*, where $\theta > 1$.*

(i) If f obeys (2.4.11)*, with $b < a2^{1-\theta}C^{-1} < a$, then*

$$C_1 \le \limsup_{t\to\infty}\frac{|X(t)|}{(\log t)^{\frac{1}{1+\theta}}} \le C_3, \quad a.s. \tag{2.4.16}$$

where C_1 is defined as in Theorem 2.4.2 and

$$C_3 := \left[\left(\frac{b2^{\theta-1}+(2+\frac{1}{C})a}{\frac{1}{C}a-b2^{\theta-1}}\right)^{\frac{1}{\theta}}+1\right]\left[\frac{\sigma^2(1+\theta)}{2a}\right]^{\frac{1}{1+\theta}}.$$

(ii) If f and g obey (2.4.14)*, then*

$$\left[\frac{\sigma^2(1+\theta)}{2a}\right]^{\frac{1}{1+\theta}} \leq \limsup_{t\to\infty} \frac{|X(t)|}{(\log t)^{\frac{1}{1+\theta}}} \leq \left[(2C+1)^{\frac{1}{\theta}}+1\right]\left[\frac{\sigma^2(1+\theta)}{2a}\right]^{\frac{1}{1+\theta}} \tag{2.4.17}$$

As in Theorem 2.4.2, the second part of Theorem 2.4.3 is also a special case of the first part. Theorem 2.4.2, together with Theorem 2.4.3 suggests that when the historical delay term is dominated by the mean-reverting instantaneous term, the solution is recurrent. Also because of the autocorrelation provoked by the delay term, solutions tend to experience slightly larger extreme fluctuations. Therefore we would expect the exact growth rate to be greater than that seen when the non-linear term involving f is instantaneous.

It is worth noticing that τ does not appear in the estimates on either side of inequalities (2.4.12) and (2.4.16). These estimates are global. If we replace "$\sup_{-\tau\leq s\leq t}$" with "$\sup_{t-\tau\leq s\leq t}$", by the stochastic comparison principle, results remain the same. This means that the essential growth rate of the long-run large fluctuations is insensitive to the length of the time interval on which the maximum value is taken. However, this does not necessarily mean that the size of the fluctuations is *independent* of the delay. An example which shows that the delay can matter is given in the next chapter. However, in the next chapter, we only deal with SFDEs which are linear, or which have negligible nonlinearities at infinity.

2.5 Proofs of Section 2.3 and Section 2.4

Proof of Theorem 2.3.1 Let $Y(t) := |X(t)|^\theta$. By the Itô formula, we compute

$$\begin{aligned} dY(t) &= \frac{1}{Y(t)}\Bigg[\frac{1}{2}\theta|X(t)|^{2\theta-2}\big(2X^T(t)f(X(t),t)+||g(X(t),t)||^2\big) \\ &\quad -(\theta-\frac{1}{2}\theta^2)|X(t)|^{2\theta-4}|X^T(t)g(X(t),t)|^2\Bigg]dt \\ &+ \theta|X(t)|^{\theta-2}X^T(t)g(X(t),t)dB(t). \end{aligned}$$

Let $M(t)=\int_0^t \theta|X(s)|^{\theta-2}X^T(s)g(X(s),s)dB(s)$ which has the quadratic variation

$$\langle M\rangle(t) = \int_0^t \theta^2|X(s)|^{2(\theta-2)}|X^T(s)g(X(s),s)|^2\,ds.$$

Then by Doob's martingale representation theorem (see e.g., [49, Theorem 3.4.2]), there is a one–dimensional Brownian motion $\widetilde{B}$ in an extended probability space with measure $\widetilde{\mathbb{P}}$ such that

$$M(t) = \int_0^t \theta|X(s)|^{\theta-2}|X^T(s)g(X(s),s)|d\widetilde{B}(s), \quad \widetilde{\mathbb{P}} - \text{a.s.}$$

Hence

$$\begin{aligned} dY(t) &= \frac{1}{Y(t)}\bigg[\frac{1}{2}\theta|X(t)|^{2\theta-2}\big(2X^T(t)f(X(t),t)+||g(X(t),t)||^2\big) \\ &\quad -(\theta-\frac{1}{2}\theta^2)|X(t)|^{2\theta-4}|X^T(t)g(X(t),t)|^2\bigg]dt \\ &+ \theta|X(t)|^{\theta-2}|X^T(t)g(X(t),t)|d\widetilde{B}(t). \end{aligned} \tag{2.5.1}$$

We now show that the drift and diffusion coefficients of above equation are bounded by some positive real numbers as (2.3.1). Since $\theta - \frac{1}{2}\theta^2 > 0$ and (2.3.5), we have

$$\begin{aligned} &\frac{1}{2}\theta|x|^{2\theta-2}\big(2x^Tf(x,t)+||g(x,t)||^2\big) - (\theta-\frac{1}{2}\theta^2)|x|^{2\theta-4}|x^Tg(x,t)|^2 \\ &\le \frac{1}{2}\theta|x|^{2\theta-2}\big(2x^Tf(x,t)+||g(x,t)||^2\big) \\ &\le \theta\rho + \frac{1}{2}\theta\sigma_1 m. \end{aligned}$$

Also

$$\theta|x|^{\theta-2}|x^Tg(x,t)| \le \theta|x|^{\theta-2}|x|||g(x,t)||_{\text{op}} \le \theta|x|^{\theta-1}\sqrt{\sigma_1}|x|^{1-\theta} = \theta\sqrt{\sigma_1},$$

and

$$\theta|x|^{\theta-2}|x^Tg(x,t)| \ge \theta|x|^{\theta-2}\sqrt{\sigma_2}|x|^{2-\theta} = \theta\sqrt{\sigma_2}.$$

Hence by Lemma 2.3.2, we get the desired result (2.3.7).

Proof of Theorem 2.3.2 By (2.5.1) and Lemma 2.3.3, we see that the conclusion (2.3.9) is obvious if we can show that for all $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$,

$$\inf_{(x,t)\in\mathbb{R}^d\times\mathbb{R}^+}\left\{\frac{\frac{1}{2}\theta|x|^{2\theta-2}\big(2x^Tf(x,t)+||g(x,t)||^2\big)}{\theta^2|x|^{2\theta-4}|x^Tg(x,t)|^2} - \frac{(\theta-\frac{1}{2}\theta^2)|x|^{2\theta-4}|x^Tg(x,t)|^2}{\theta^2|x|^{2\theta-4}|x^Tg(x,t)|^2}\right\} > -\frac{1}{2}.$$

But the above is equivalent to part (a) of condition (2.3.8), therefore the proof is complete.

Proof of Theorem 2.3.3 (i) By Itô's formula, for all $t \ge 0$,

$$d|X(t)|^2 = [2X^T(t)f(X(t),t)+||g(X(t),t)||^2]\,dt + 2X^T(t)g(X(t),t)\,dB(t).$$

Again, by the martingale representation theorem, we can replace the martingale defined as $M(t) := \int_0^t X^T(s)g(X(s),s)\,dB(s)$ by $M(t) = \int_0^t |X^T(s)g(X(s),s)|\,d\widetilde{B}(s)$ in an extended probability space which has measure $\widetilde{\mathbb{P}}$ and supports the one–dimensional Brownian motion $\widetilde{B}$. So

$$d|X(t)|^2 = [2X^T(t)f(X(t),t) + ||g(X(t),t)||^2]dt + 2|X(t)|\Phi(|X(t)|,t)d\widetilde{B}(t)$$

where

$$\Phi(x,t) = \begin{cases} \sigma \in [\sqrt{\sigma_2}, \sqrt{\sigma_1}], & x = 0, \\ |x^T g(x,t)|/|x|, & x \neq 0. \end{cases}$$

Due to the above definition of Φ, $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$, $\sqrt{\sigma_2} \leq \Phi(x,t) \leq \sqrt{\sigma_1}$. Now define $N(t) := \int_0^t \Phi(X(s),s)d\widetilde{B}(s)$, which has the quadratic variation $\langle N\rangle(t) = \int_0^t \Phi^2(X(s),s)\,ds$. Thus

$$\forall\, t \geq 0, \quad \sigma_2 t \leq \langle N\rangle(t) \leq \sigma_1 t. \tag{2.5.2}$$

$\lim_{t\to\infty}\langle N\rangle(t) = \infty$ due to (2.3.10). For each $0 \leq s < \infty$, define the stopping time $\lambda(s) := \inf\{t \geq 0; \langle N\rangle(t) > s\}$. Hence for all $t \geq 0$, $\langle N\rangle(\lambda(t)) = t$ and $\lambda(t) = \langle N\rangle^{-1}(t)$. By martingale time-change theorem (see e.g., [49, Theorem 3.4.6]), the process W defined by

$$W(t) := \int_0^{\lambda(t)} \Phi(X(s),s)d\widetilde{B}(s) \quad \forall\, t \geq 0$$

is a standard Brownian motion with respect to the filtration $\mathcal{G}_t := \mathcal{F}_{\lambda(t)}$. Proposition 3.4.8 in [49]gives us, almost surely

$$\int_0^{\lambda(t)} |X(s)|dN(s) = \int_0^t |X(\lambda(s))|dW(s) \quad \text{for each } 0 \leq t < \infty.$$

Thus if $Y(t) := |X(\lambda(t))|^2$, from (2.3.10), we have

$$\begin{aligned} dY(t) &= \frac{2X^T(\lambda(t))f(X(\lambda(t)),\lambda(t)) + ||g(X(\lambda(t)),\lambda(t))||^2}{\Phi^2(\sqrt{Y(t)},\lambda(t))}dt + 2\sqrt{Y(t)}dW(t) \\ &\leq \left(\frac{-2\gamma Y(t)}{\sigma_1} + \frac{\rho}{\sigma_2} + \frac{m\sigma_1}{\sigma_2}\right)dt + 2\sqrt{Y(t)}dW(t). \end{aligned}$$

Consider the process governed by

$$dZ(t) = \left(\frac{-2\gamma Z(t)}{\sigma_1} + \frac{\rho}{\sigma_2} + \frac{m\sigma_1}{\sigma_2}\right)dt + 2\sqrt{|Z(t)|}dW(t),$$

with $Z(0) \geq |x_0|^2$. By Lemma 2.3.1, we have $\forall t \geq 0$, $Z(t) \geq 0$. Using the comparison theorem (Proposition 5.2.18 in [49]) $\forall t \geq 0$, $Z(t) \geq Y(t)$ a.s. Also by Lemma 2.3.1, we get

$$\limsup_{t\to\infty} \frac{Y(t)}{\log t} \leq \limsup_{t\to\infty} \frac{Z(t)}{\log t} = \frac{\sigma_1}{\gamma}, \quad \text{a.s.}$$

That is

$$\limsup_{t\to\infty} \frac{|X(\lambda(t))|}{(\log t)^{\frac{1}{2}}} \leq \frac{\sqrt{\sigma_1}}{\sqrt{\gamma}}, \quad \text{a.s.}$$

which implies

$$\limsup_{t\to\infty} \frac{|X(t)|}{(\log \lambda^{-1}(t))^{\frac{1}{2}}} \leq \frac{\sqrt{\sigma_1}}{\sqrt{\gamma}}, \quad \text{a.s.}$$

Combining the above inequality with (2.5.2), the desired result is obtained. The proof for part (ii) is essentially the same as part (i), except that the process Z is constructed to go below Y pathwise using the condition $x^T f(x,t) \geq -\gamma|x|^2$. We omit the details.

Proof of Theorem 2.3.4 As in the proof of Theorem 2.3.1, we set $Y(t) := |X(t)|^\theta$ $t \geq 0$ which obeys (2.5.1). Now since $(\theta - \frac{1}{2}\theta^2) > 0$ and (2.3.12) it is easy to see that for all $(x,t) \in \mathbb{R}^d \times \mathbb{R}^+$, the drift coefficient of (2.5.1) satisfies

$$\frac{\theta}{2}|x|^{2\theta-2}(2x^T f(x,t) + ||g(x,t)||^2) - (\theta - \frac{\theta^2}{2})|x|^{2\theta-4}|x^T g(x,t)|^2 \leq -\theta\gamma|x|^{2\theta} + \frac{1}{2}\theta\sigma_1 m,$$

and the diffusion coefficient satisfies

$$\theta\sqrt{\sigma_2} \leq \theta|x|^{\theta-2}|x^T g(x,t)| \leq \theta|x|^{\theta-2}|x|||g(x,t)||_{\text{op}} \leq \theta\sqrt{\sigma_1}.$$

Therefore by applying Theorem 2.3.3, we get the desired result in part (i). Part (ii) follows a similar argument.

Proof of Theorem 2.3.5 Given (2.3.14), for any $\varepsilon \in (0,1)$, there exists $x_\varepsilon > 0$ such that for all $x > x_\varepsilon$, $g^2(x) < \sigma^2(1+2\varepsilon)^{\frac{1}{3}}$. Moreover, there exists a real number C such that for all $x \in \mathbb{R}$, $g^2(x) \le C^2$. Hence

$$
\begin{aligned}
&\int_0^t X^2(s)g^2(X(s))\,ds \\
&= \int_0^t X^2(s)g^2(X(s))1_{\{X^2(s)>x_\varepsilon^2\}}\,ds + \int_0^t X^2(s)g^2(X(s))1_{\{X^2(s)\le x_\varepsilon^2\}}\,ds \\
&\le \sigma^2(1+2\varepsilon)^{\frac{1}{3}}\int_0^t X^2(s)1_{\{X^2(s)>x_\varepsilon^2\}}ds + C^2\int_0^t X^2(s)1_{\{X^2(s)\le x_\varepsilon^2\}}\,ds \\
&\le \sigma^2(1+2\varepsilon)^{\frac{1}{3}}\int_0^t X^2(s)ds + C^2x_\varepsilon^2 t. \qquad (2.5.3)
\end{aligned}
$$

Let $M(t) = \int_0^t 2X(s)g(X(s))\,dB(s)$ for $t \ge 0$, which has quadratic variation

$$\langle M\rangle(t) = 4\int_0^t X^2(s)g^2(X(s))\,ds. \qquad (2.5.4)$$

Define $A = \{\omega : \lim_{t\to\infty}\langle M\rangle(t) < \infty\}$. Then there exists a real number L such that $\lim_{t\to\infty} M(t) = L$ a.s. conditionally on A. Then for almost all $\omega \in A$, the result is obviously true. Consider the complement A^c of A. For almost all $\omega \in A^c$ and $\varepsilon \in (0,1)$, there exists a random time $T_{\varepsilon,1} > 0$ such that for all $t > T_{\varepsilon,1}$,

$$M(t) < \sqrt{2\langle M\rangle(t)\log\log\langle M\rangle(t)}(1+2\varepsilon)^{\frac{1}{3}}. \qquad (2.5.5)$$

Now by Lemma 2.3.2 (or Theorem 2.2.1),

$$\limsup_{t\to\infty}\frac{X^2(t)}{2t\log\log t} \le C^2, \quad \text{a.s.} \qquad (2.5.6)$$

For the same $\varepsilon \in (0,1)$, there exists $T_{\varepsilon,2} > 0$ such that for all $t > T_{\varepsilon,2}$,

$$X^2(t) < (1+2\varepsilon)^{\frac{1}{3}}C^2 2t\log\log t, \quad \text{a.s.}$$

By L'Hôpital's Rule

$$\lim_{t\to\infty}\frac{\int_{T_{\varepsilon,2}}^t 2s\log\log s\,ds}{t^2\log\log t} = \lim_{t\to\infty}\frac{2t\log\log t}{2t\log\log t + t^2\frac{1}{t\log t}} = 1.$$

Hence

$$
\begin{aligned}
\limsup_{t\to\infty}\frac{\int_0^t X^2(s)\,ds}{t^2\log\log t} &= \limsup_{t\to\infty}\frac{\int_{T_{\varepsilon,2}}^t X^2(s)\,ds}{t^2\log\log t} \\
&\le \limsup_{t\to\infty} C^2(1+2\varepsilon)^{\frac{1}{3}}\frac{\int_{T_{\varepsilon,2}}^t 2s\log\log s\,ds}{t^2\log\log t} \le C^2(1+2\varepsilon)^{\frac{1}{3}}, \quad \text{a.s.}
\end{aligned}
$$

Letting $\varepsilon \downarrow 0$ through rational numbers, we get

$$\limsup_{t\to\infty} \frac{\int_0^t X^2(s)\,ds}{t^2 \log\log t} \le C^2, \quad \text{a.s.}$$

Thus by (2.5.3) and (2.5.4)

$$\limsup_{t\to\infty} \frac{\langle M\rangle(t)}{t^2 \log\log t} \le 4\sigma^2 C^2, \quad \text{a.s. conditionally on } A^c.$$

Therefore

$$\limsup_{t\to\infty} \frac{\log\log \langle M\rangle(t)}{\log\log t} \le 1, \quad \text{a.s. conditionally on } A^c.$$

So there exists $T_{\varepsilon,3} > 0$ such that for all $t > T_{\varepsilon,3}$,

$$\log\log \langle M\rangle(t) < (1+2\varepsilon)^{\frac{2}{3}} \log\log t, \quad \text{a.s. conditionally on } A^c.$$

Let $T_{\varepsilon,4} = T_{\varepsilon,1} \vee T_{\varepsilon,3}$, then by (2.5.5), we have

$$\forall\, t > T_{\varepsilon,4}, \quad M(t) < \sqrt{2\langle M\rangle(t) \log\log t}(1+2\varepsilon)^{\frac{2}{3}}, \quad \text{a.s.} \tag{2.5.7}$$

Now for all $t > T_{\varepsilon,4}$,

$$\begin{aligned} X^2(t) &\le x_0^2 + (2\rho + C^2)t + M(t) \\ &\le x_0^2 + (2\rho + C^2)t + (1+2\varepsilon)^{\frac{2}{3}} \sqrt{2\langle M\rangle(t)\log\log t}, \quad \text{a.s.} \end{aligned}$$

Define $Y(t) = \int_0^t X^2(s)\,ds$ for $t \ge 0$. Then for any $\tau > x_0^2/C^2 > 0$ and $t > T_{\varepsilon,4}$,

$$\begin{aligned} Y'(t) &\le 2(\rho + C^2)(t+\tau) \\ &\quad + (1+2\varepsilon)^{\frac{2}{3}} \sqrt{8(\log\log(t+\tau))[\sigma^2(1+2\varepsilon)^{\frac{1}{3}} Y(t) + C^2 x_\varepsilon^2 (t+\tau)]}, \end{aligned} \tag{2.5.8}$$

with $Y(T_{\varepsilon,4}) = y_\varepsilon > 0$. Now suppose the following

$$\tau > e^e; \tag{2.5.9a}$$

$$\tau \log\log \tau > \frac{C^2 x_\varepsilon^2}{C_\varepsilon \sigma^2 2\varepsilon (1+2\varepsilon)^{\frac{1}{3}}}; \tag{2.5.9b}$$

$$\log\log \tau > \frac{2(\rho + C^2)}{\varepsilon\sigma\sqrt{8C_\varepsilon}}; \tag{2.5.9c}$$

$$\tau^2 \log\log \tau > \frac{y_\varepsilon}{C_\varepsilon}; \tag{2.5.9d}$$

where

$$C_\varepsilon = 2(1+2\varepsilon)^4\sigma^2. \tag{2.5.9e}$$

Define

$$y_+(t) = C_\varepsilon(t+\tau)^2 \log\log(t+\tau), \quad t \geq T_{\varepsilon,4}. \tag{2.5.10}$$

By (2.5.9b),

$$\begin{aligned}
&\sigma^2(1+2\varepsilon)^{\frac{1}{3}} y_+(t) + C^2 x_\varepsilon^2(t+\tau) \\
&= \sigma^2(1+2\varepsilon)^{\frac{1}{3}} C_\varepsilon(t+\tau)^2 \log\log(t+\tau) + C^2 x_\varepsilon^2(t+\tau) \\
&< \sigma^2(1+2\varepsilon)^{\frac{1}{3}} C_\varepsilon(t+\tau)^2 \log\log(t+\tau) + \sigma^2 C_\varepsilon 2\varepsilon(1+2\varepsilon)^{\frac{1}{3}}(t+\tau)^2 \log\log(t+\tau) \\
&= (1+2\varepsilon)^{\frac{4}{3}}\sigma^2 C_\varepsilon(t+\tau)^2 \log\log(t+\tau).
\end{aligned}$$

Hence

$$\begin{aligned}
&(1+2\varepsilon)^{\frac{2}{3}}\sqrt{8(\log\log(t+\tau))[\sigma^2(1+2\varepsilon)^{\frac{1}{3}} y_+(t) + C^2 x_\varepsilon^2(t+\tau)]} \\
&\qquad\qquad \leq (1+2\varepsilon)^{\frac{4}{3}}\sigma\sqrt{8C_\varepsilon}(t+\tau)\log\log(t+\tau).
\end{aligned}$$

Next since (2.5.9c) holds, we have

$$\begin{aligned}
&(1+2\varepsilon)^{\frac{2}{3}}\sqrt{8(\log\log(t+\tau))[\sigma^2(1+2\varepsilon)^{\frac{1}{3}} y_+(t) + C^2 x_\varepsilon^2(t+\tau)]} \\
&\qquad + 2(\rho+C^2)(t+\tau) \\
&< \varepsilon\sigma\sqrt{8C_\varepsilon}(t+\tau)\log\log(t+\tau) + (1+2\varepsilon)^{\frac{4}{3}}\sigma\sqrt{8C_\varepsilon}(t+\tau)\log\log(t+\tau) \\
&= (1+2\varepsilon)^2\sigma\sqrt{8C_\varepsilon}(t+\tau)\log\log(t+\tau).
\end{aligned} \tag{2.5.11}$$

Now by (2.5.9d)

$$y_+(T_{\varepsilon,4}) = C_\varepsilon(T_{\varepsilon,4}+\tau)^2\log\log(T_{\varepsilon,4}+\tau) \geq C_\varepsilon\tau^2\log\log\tau > y_\varepsilon = Y(T_{\varepsilon,4}). \tag{2.5.12}$$

(2.5.9e) together with (2.5.11) gives

$$\begin{aligned}
y_+'(t) &= 2C_\varepsilon(t+\tau)\log\log(t+\tau) + \frac{C_\varepsilon(t+\tau)^2}{(t+\tau)\log(t+\tau)} \\
&> 2C_\varepsilon(t+\tau)\log\log(t+\tau) \\
&= (1+2\varepsilon)^2\sigma\sqrt{8C_\varepsilon}(t+\tau)\log\log(t+\tau) \\
&> 2(\rho+C^2)(t+\tau) \\
&\qquad + (1+2\varepsilon)^{\frac{2}{3}}\sqrt{8(\log\log(t+\tau))[\sigma^2(1+2\varepsilon)^{\frac{1}{3}} y_+(t) + C^2 x_\varepsilon^2(t+\tau)]}.
\end{aligned} \tag{2.5.13}$$

Thus by (2.5.8), (2.5.12) and (2.5.13), we have

$$\forall\, t > T_{\varepsilon,4}, \quad Y(t) < y_+(t) = 2(1+2\varepsilon)^4\sigma^2(t+\tau)^2\log\log(t+\tau), \quad \text{a.s. conditionally on } A^c.$$

Combining above with (2.5.12), for almost all $\omega \in A^c$ and any $t > T_{\varepsilon,4}$,

$$X^2(t) \le x_0^2 + (2\rho + C^2)(t+\tau) + (1+2\varepsilon)^{\frac{2}{3}}\sqrt{(t+\tau)\log\log(t+\tau)} \times \sqrt{8\left[2\sigma^4(1+2\varepsilon)^{\frac{13}{3}}(t+\tau)(\log\log(t+\tau)) + C^2x_\varepsilon^2\right]}.$$

This implies that for almost all $\omega \in A^c$ and any $t > T_{\varepsilon,4}$

$$\frac{X^2(t)}{(t+\tau)\log\log(t+\tau)} \le \frac{x_0^2}{(t+\tau)\log\log(t+\tau)} + \frac{2\rho + C^2}{\log\log(t+\tau)} + (1+2\varepsilon)^{\frac{2}{3}}\sqrt{8\left[2\sigma^4(1+2\varepsilon)^{\frac{13}{3}} + \frac{C^2x_\varepsilon^2}{(t+\tau)\log\log(t+\tau)}\right]}.$$

Letting $t \to \infty$, we get

$$\limsup_{t\to\infty}\frac{X^2(t)}{t\log\log t} \le 4(1+2\varepsilon)^{\frac{13}{9}}(1+3\varepsilon)^{\frac{1}{2}}\sigma^2, \quad \text{a.s. conditionally on } A^c.$$

Finally letting $\varepsilon \downarrow 0$ through rational numbers, the desired result is obtained.

Proof of Theorem 2.4.1 (2.4.3) implies that for any fixed $0 < \varepsilon < 1$, there exists $x_\varepsilon > 1$ such that

$$\forall\, x > x_\varepsilon, \quad x^\theta a(1-\varepsilon) \le g(x) \le x^\theta a(1+\varepsilon); \tag{2.5.14}$$

$$\forall\, x < -x_\varepsilon, \quad -|x|^\theta a(1+\varepsilon) \le g(x) \le -|x|^\theta a(1-\varepsilon). \tag{2.5.15}$$

Consider the scale function s_v of V defined as

$$s_v(x) = \int_1^x e^{-2\int_1^y \frac{-g(z)}{\sigma^2}dz}dy, \quad x \in \mathbb{R}.$$

Due to (2.5.14), it is easy to verify that for $y > x_\varepsilon$,

$$s_v(x) \ge K_1\int_1^x e^{\frac{2a(1-\varepsilon)}{\sigma^2(1+\theta)}y^{1+\theta}}dy,$$

where

$$K_1 := e^{-\frac{2a(1-\varepsilon)}{\sigma^2(1+\theta)}x_\varepsilon^{1+\theta} + \frac{2}{\sigma^2}\int_1^{x_\varepsilon} g(z)dz}.$$

Hence $s_v(\infty) = \infty$. Similarly, $s_v(-\infty) = -\infty$. The speed measure m_v of V is defined as

$$m_v(dx) = \frac{2}{\sigma^2} e^{-\frac{2}{\sigma^2}\int_1^x g(z)dz} dx.$$

Again, using both (2.5.14) and (2.5.15), it can be shown that $m_v(-\infty, \infty) < \infty$. Hence V is asymptotically stationary on $\mathbb{R}$, and therefore we can apply Motoo's theorem to determine the growth rate of its large deviations. Now, for $y > x_\varepsilon$,

$$K_2 \int_1^x e^{\frac{2a(1-\varepsilon)}{\sigma^2}\int_{x_\varepsilon}^y z^\theta dz} dy \le s_v(x) \le K_2 \int_1^x e^{\frac{2a(1+\varepsilon)}{\sigma^2}\int_{x_\varepsilon}^y z^\theta dz} dy,$$

where $K_2 := e^{2/\sigma^2 \int_1^{x_\varepsilon} g(z)dz}$. Dividing both sides of this inequality by the quantity $e^{(2a(1+\varepsilon)/(\sigma^2(1+\theta)))x^{1+\theta}}$, where $\theta > 0$, and letting $x \to \infty$, we get

$$\lim_{x\to\infty} \frac{s_v(x)}{e^{\frac{2a(1+\varepsilon)}{\sigma^2(1+\theta)}x^{1+\theta}}} = 0.$$

Thus there exists $x_0 > 0$ such that

$$\forall\, x > x_0, \quad s_v(x) \le e^{\frac{2a(1+\varepsilon)}{\sigma^2(1+\theta)}x^{1+\theta}}.$$

For $t > 0$, define

$$\varrho_1(t) := \left[\frac{\sigma^2(1+\theta)}{2a(1+\varepsilon)} \log t\right]^{\frac{1}{1+\theta}}.$$

There exists t_{x_0} such that for all $t > t_{x_0}$, $\varrho_1(t) > x_0$, which in turn implies $s_v(\varrho_1(t)) < t$. Hence

$$\int_{t_{x_0}}^\infty \frac{1}{s_v(\varrho_1(t))} dt \ge \int_{t_{x_0}}^\infty \frac{1}{t} dt = \infty.$$

By Motoo's theorem,

$$\limsup_{t\to\infty} \frac{V(t)}{(\log t)^{\frac{1}{1+\theta}}} \ge \left[\frac{\sigma^2(1+\theta)}{2a(1+\varepsilon)}\right]^{\frac{1}{1+\theta}}, \quad \text{a.s.}$$

Letting $\varepsilon \downarrow 0$ through the rational numbers, we have

$$\limsup_{t\to\infty} \frac{V(t)}{(\log t)^{\frac{1}{1+\theta}}} \ge \left[\frac{\sigma^2(1+\theta)}{2a}\right]^{\frac{1}{1+\theta}}, \quad \text{a.s.} \tag{2.5.16}$$

For $t > 0$, define

$$\varrho_2(t) := \left[\frac{\lambda\sigma^2(1+\theta)}{2a(1-\varepsilon)} \log t\right]^{\frac{1}{1+\theta}} + 1,$$

where $\lambda > 1$. Since $s_v(x)$ is increasing, for $\varrho_2(t) > x_\varepsilon$, we get

$$K_1 e^{\frac{2a(1-\varepsilon)}{\sigma^2(1+\theta)}(\varrho_2(t)-1)^{1+\theta}} \le s_v(\varrho_2(t)),$$

that is

$$\frac{1}{s_v(\varrho_2(t))} \le K_1^{-1} e^{-\frac{2a(1-\varepsilon)}{\sigma^2(1+\theta)}(\varrho_2(t)-1)^{1+\theta}}.$$

Hence

$$\lim_{t\to\infty} \frac{\log \frac{1}{s_v(\varrho_2(t))}}{\log t} \le -\lambda.$$

For any fixed $0 < \epsilon < \theta - 1$, there exists $t_\epsilon > 0$ such that

$$\forall\, t > t_\epsilon, \quad \log \frac{1}{s_v(\varrho_2(t))} \le (-\lambda + \epsilon) \log t,$$

which implies

$$\int_{t_\epsilon}^{\infty} \frac{1}{s_v(\varrho_2(t))}\, dt \le \int_{t_\epsilon}^{\infty} \frac{1}{t^{\lambda-\epsilon}}\, dt < \infty.$$

Applying Motoo's theorem again, we have

$$\limsup_{t\to\infty} \frac{V(t)}{(\log t)^{\frac{1}{1+\theta}}} \le \left[\frac{\lambda\sigma^2(1+\theta)}{2a(1+\varepsilon)}\right]^{\frac{1}{1+\theta}}, \quad \text{a.s.}$$

Letting $\lambda \downarrow 1$ and $\varepsilon \downarrow 0$ through the rational numbers, it follows that

$$\limsup_{t\to\infty} \frac{V(t)}{(\log t)^{\frac{1}{1+\theta}}} \le \left[\frac{\sigma^2(1+\theta)}{2a}\right]^{\frac{1}{1+\theta}}, \quad \text{a.s.}$$

Combining this inequality with (2.5.16), we get the first part of the theorem. For the second part of the theorem, for $t \ge 0$, let $\widehat{V}(t) := -V(t)$, $\hat{g}(x) := -g(-x)$ and $\widehat{B}(t) := B(t)$. Then we have

$$d\widehat{V}(t) = -\hat{g}(\widehat{V}(t))\, dt + \sigma\, d\widehat{B}(t), \quad t \ge 0.$$

where $\hat{g}$ also satisfies

$$\lim_{|x|\to\infty} \frac{\operatorname{sgn}(x)\hat{g}(x)}{|x|^\theta} = a.$$

Hence by (2.4.4),

$$\limsup_{t\to\infty} \frac{\widehat{V}(t)}{(\log t)^{\frac{1}{1+\theta}}} = \left[\frac{\sigma^2(1+\theta)}{2a}\right]^{\frac{1}{1+\theta}}, \quad \text{a.s.}$$

which in turn implies (2.4.5).

Proof of Lemma 2.4.1 We first prove (2.4.7). For $x \geq 0$, we first consider the case when $y < 0$.If $x + y \geq 0$,

$$\begin{aligned}
Cg(x+y) - g(|x|) + (C+1)g(|y|) &= Cg(x+y) - g(x+y+(-y)) + (C+1)g(|y|) \\
&\geq Cg(x+y) - C(g(x+y) + g(-y)) + (C+1)g(|y|) \\
&= Cg(x+y) - C(g(x+y)) + Cg(y) + (C+1)g(|y|) \\
&\geq Cg(y) + (C+1)g(|y|) \geq 0,
\end{aligned}$$

where we have used (2.4.6) in the second line. If $x + y < 0$, then $y < -x$. Since g is non-decreasing and odd, $g(y) < g(-x) = -g(x)$. Also as $g(x + y) \geq g(y)$, we have

$$\begin{aligned}
Cg(x+y) - g(|x|) + (C+1)g(|y|) &\geq Cg(x+y) + g(y) + (C+1)g(|y|) \\
&\geq Cg(y) + g(y) + (C+1)g(|y|) = 0.
\end{aligned}$$

When $y \geq 0$, since $C \geq 1$,

$$-Cg(x+y) \leq -Cg(x) \leq -g(|x|) + (C+1)g(|y|).$$

Therefore we have proved (2.4.7).

Now we prove (2.4.8). For $x < 0$, we also consider $y < 0$ first. Note $x + y < x$ and $g(x) \leq 0 \leq g(|y|)$, so

$$\begin{aligned}
-Cg(x+y) - g(|x|) + (C+1)g(|y|) &\geq -Cg(x) + g(x) + (C+1)g(|y|) \\
&= g(x)(1-C) + (C+1)g(|y|) \geq 0.
\end{aligned}$$

When $y \geq 0$, if $x + y \geq 0$, then $g(|y|) \geq g(x + y) \geq g(x)$, thus

$$\begin{aligned}
-Cg(x+y) - g(|x|) + (C+1)g(|y|) &= -Cg(x+y) + g(x) + Cg(|y|) + g(|y|) \\
&= Cg(|y|) - Cg(x+y) + g(|y|) + g(x) \geq 0.
\end{aligned}$$

Finally when $y \geq 0$ and $x + y < 0$,

$$\begin{aligned}
-Cg(x+y) - g(|x|) + (C+1)g(|y|) &= Cg(-x-y) + Cg(|y|) - g(|x|) + g(|y|) \\
&\geq g(-x-y+y) - g(|x|) + g(y) \\
&= g(-x) - g(|x|) + g(y) \geq 0.
\end{aligned}$$

Hence (2.4.8) is also proven.

Proof of Theorem 2.4.2 (i) Consider the process Y governed by the following SDE:

$$dY(t) = [-g(Y(t)) + f(Y(t))]\,dt + \sigma\,dB(t), \quad t \geq 0,$$

where $Y(t) = \phi(t) \leq \psi(t)$ for $t \in [-\tau, 0]$. Then by the comparison principle, for all $t \geq 0$, $X(t) \geq Y(t)$ a.s. We notice by (2.4.10) and (2.4.11), and the fact that $a > b$, that Y obeys all the properties of V in Theorem 2.4.1. Therefore,

$$\limsup_{t\to\infty} \frac{Y(t)}{(\log t)^{\frac{1}{1+\theta}}} = C_1, \quad \text{a.s.}$$

where C_1 is given by the formula in (2.4.13). For the upper estimate, consider the process Z governed by the following equation

$$dZ(t) = -g(Z(t))\,dt + \sigma\,dB(t), \quad t \geq 0,$$

with $Z(t) = \psi(t)$ for $t \in [-\tau, 0]$. For all $t \geq -\tau$, let $Q(t) := X(t) - Z(t)$, then

$$Q'(t) = -g(Q(t) + Z(t)) + g(Z(t)) + \sup_{-\tau\leq s\leq t} f(Q(s) + Z(s)), \quad t \geq 0,$$

with $Q(t) = 0$ for $t \in [-\tau, 0]$. Now, if $Q(t) = 0$, then $D^+|Q(t)| = |Q'(t)|$. If $Q(t) > 0$, by (2.4.9), Lemma 2.4.1 and the fact that both g and f are non-decreasing,

$$\begin{aligned}
D^+|Q(t)| &= Q'(t) \\
&\leq -g(|Q(t)|) + 2g(|Z(t)|) + g(|Z(t)|) + \sup_{-\tau\leq s\leq t} f(|Q(s)| + |Z(s)|) \\
&\leq -g(|Q(t)|) + 3g(|Z(t)|) + f(\sup_{-\tau\leq s\leq t} |Q(s)|) + f(\sup_{-\tau\leq s\leq t} |Z(s)|).
\end{aligned}$$

If $Q(t) < 0$,

$$\begin{aligned}
D^+|Q(t)| &= -Q'(t) \\
&= g(Q(t) + Z(t)) - g(Z(t)) - \sup_{-\tau\leq s\leq t} f(Q(s) + Z(s)) \\
&\leq g(|Q(t)| + |Z(t)|) + |g(Z(t))| + \sup_{-\tau\leq s\leq t} |f(Q(s) + Z(s))| \\
&\leq -g(|Q(t)|) + 3g(|Z(t)|) + f(\sup_{-\tau\leq s\leq t} |Q(s)|) + f(\sup_{-\tau\leq s\leq t} |Z(s)|),
\end{aligned}$$

where we have chosen $C = 1$ in Lemma 2.4.1. Now by Theorem 2.4.1, Z obeys

$$\limsup_{t\to\infty} \frac{|Z(t)|}{(\log t)^{\frac{1}{1+\theta}}} = \left[\frac{\sigma^2(1+\theta)}{2a}\right]^{\frac{1}{1+\theta}}, \quad \text{a.s.} \tag{2.5.17}$$

Let (2.5.17) be true on the a.s. event Ω^*. Then for every $\omega \in \Omega^*$, for any fixed $\varepsilon \in (0,(a-b)/(a+b))$, there exists $T_1(\varepsilon,\omega) > 0$ such that

$$\forall\, t > T_1(\varepsilon,\omega), \qquad \sup_{-\tau\le s\le t} |Z(s)| \le \left[\frac{\sigma^2(1+\theta)}{2a}\right]^{\frac{1}{1+\theta}} (\log t)^{\frac{1}{1+\theta}}(1+\varepsilon), \quad \text{on } \Omega^*. \tag{2.5.18}$$

Since f is non-decreasing, (2.5.18) implies on Ω^* that

$$\forall\, t > T_1(\varepsilon,\omega), \quad f(\sup_{-\tau\le s\le t} |Z(s)|) \le f\left(\left[\frac{\sigma^2(1+\theta)}{2a}\right]^{\frac{1}{1+\theta}} (\log t)^{\frac{1}{1+\theta}}(1+\varepsilon)\right). \tag{2.5.19}$$

Also (2.4.11) implies that for the same ε, there exists $x_\varepsilon > 0$ such that

$$\forall\, x > x_\varepsilon, \quad bx^\theta(1-\varepsilon) \le f(x) \le bx^\theta(1+\varepsilon).$$

Now there exists $T_{x_\varepsilon} > 0$ such that

$$\forall\, t > T_{x_\varepsilon}, \quad \left[\frac{\sigma^2(1+\theta)}{2a}\right]^{\frac{1}{1+\theta}} (\log t)^{\frac{1}{1+\theta}}(1+\varepsilon) > x_\varepsilon.$$

Choosing $T_2(\varepsilon,\omega) := T_1(\varepsilon,\omega) \vee T_{x_\varepsilon}$, we see that

$$\forall\, t > T_2(\varepsilon,\omega), \quad f(\sup_{-\tau\le s\le t} |Z(s)|) \le b\left[\frac{\sigma^2(1+\theta)}{2a}\right]^{\frac{\theta}{1+\theta}} (\log t)^{\frac{\theta}{1+\theta}}(1+\varepsilon)^{1+\theta}, \quad \text{on } \Omega^*.$$

Similarly for some $T_3(\varepsilon,\omega) > 0$,

$$\forall\, t > T_3(\varepsilon,\omega), \quad g(|Z(t)|) \le a\left[\frac{\sigma^2(1+\theta)}{2a}\right]^{\frac{\theta}{1+\theta}} (\log t)^{\frac{\theta}{1+\theta}}(1+\varepsilon)^{1+\theta}, \quad \text{on } \Omega^*.$$

Hence if $T_4(\varepsilon,\omega) := T_2(\varepsilon,\omega) \vee T_3(\varepsilon,\omega)$, then for all $t > T_4(\varepsilon,\omega)$,

$$\begin{aligned} D_+|Q(t)| \le -g(|Q(t)|) &+ f(\sup_{-\tau\le s\le t} |Q(s)|) \\ &+ (3a+b)\left[\frac{\sigma^2(1+\theta)}{2a}\right]^{\frac{\theta}{1+\theta}} (\log t)^{\frac{\theta}{1+\theta}}(1+\varepsilon)^{1+\theta}, \quad \text{on } \Omega^*. \end{aligned}$$

Let t_+ be a positive real number such that $\log t_+ > 0$. Consider the randomly parameterised function U_ε given by

$$U_\varepsilon(t) := \begin{cases} K_\varepsilon(\log t)^{\frac{1}{1+\theta}} + \rho(\varepsilon,\omega), & t \in [t_+,\infty), \\ K_\varepsilon(\log t_+)^{\frac{1}{1+\theta}} + \rho(\varepsilon,\omega), & t \in [-\tau,t_+), \end{cases} \tag{2.5.20}$$

where K_ε, $\rho(\varepsilon,\omega) > 0$. Hence U_ε is a continuous, positive and non-decreasing function on its domain. By (2.4.10) and (2.4.11), for the same ε, there exists $T_5(\varepsilon,\omega) > 0$ such that for all $t > T_5(\varepsilon,\omega)$, $-g(U_\varepsilon(t)) \le -aU_\varepsilon^\theta(t)(1-\varepsilon)$ and $f(\sup_{-\tau\le s\le t} U_\varepsilon(s)) = f(U_\varepsilon(t)) \le bU_\varepsilon^\theta(t)(1+\varepsilon)$. Let $T_6(\varepsilon,\omega) := T_5(\varepsilon,\omega)\vee t_+ \vee T_4(\varepsilon,\omega)$. Choose

$$K_\varepsilon := \left[\frac{(3a+b)\left(\frac{\sigma^2(1+\theta)}{2a}\right)^{\frac{\theta}{1+\theta}}(1+\varepsilon)^{1+\theta}}{a(1-\varepsilon)-b(1+\varepsilon)}\right]^{\frac{1}{\theta}}$$

and let $\rho_{\varepsilon,\omega} > 0$ be such that $U_\varepsilon(-\tau) > \max_{t\in[-\tau,T_6]}|Q(t)|$. For all $t > T_6(\varepsilon,\omega)$,

$$\begin{aligned}
U_\varepsilon'(t) &+ g(U_\varepsilon(t)) - f(\sup_{-\tau\le s\le t} U_\varepsilon(s)) - (2a+b)\left[\frac{\sigma^2(1+\theta)}{2a}\right]^{\frac{\theta}{1+\theta}}(\log t)^{\frac{\theta}{1+\theta}}(1+\varepsilon)^{1+\theta}\\
&\ge \frac{K_\varepsilon}{1+\theta}(\log t)^{\frac{-\theta}{1+\theta}}\frac{1}{t} + (a(1-\varepsilon)-b(1+\varepsilon))\,U_\varepsilon^\theta(t)\\
&\qquad - (2a+b)\left[\frac{\sigma^2(1+\theta)}{2a}\right]^{\frac{\theta}{1+\theta}}(\log t)^{\frac{\theta}{1+\theta}}(1+\varepsilon)^{1+\theta}\\
&> (a(1-\varepsilon)-b(1+\varepsilon))\,K_\varepsilon^\theta(\log t)^{\frac{\theta}{1+\theta}}\\
&\qquad - (2a+b)\left[\frac{\sigma^2(1+\theta)}{2a}\right]^{\frac{\theta}{1+\theta}}(\log t)^{\frac{\theta}{1+\theta}}(1+\varepsilon)^{1+\theta} = 0.
\end{aligned}$$

Therefore by [52, Theorem 8.1.4, volume II], for all $t\in[T_6,\infty)$, $U_\varepsilon(t) > |Q(t)|$. Hence

$$\limsup_{t\to\infty}\frac{|Q(t,\omega)|}{(\log t)^{\frac{1}{1+\theta}}} \le \lim_{t\to\infty}\frac{U_\varepsilon(t,\omega)}{(\log t)^{\frac{1}{1+\theta}}} = K_\varepsilon, \quad \text{on } \Omega^*.$$

Letting $\varepsilon\downarrow 0$, we have

$$\limsup_{t\to\infty}\frac{|Q(t,\omega)|}{(\log t)^{\frac{1}{1+\theta}}} \le \left[\frac{(3a+b)}{a-b}\right]^{\frac{1}{\theta}}\left[\frac{\sigma^2(1+\theta)}{2a}\right]^{\frac{1}{1+\theta}}. \tag{2.5.21}$$

Because $\omega\in\Omega^*$ and $\mathbb{P}[\Omega^*]=1$, (2.5.21) holds a.s. Now for all $t\in[-\tau,\infty)$, $|X(t)| \le |Q(t)|+|Z(t)|$. Therefore combining with (2.5.17) and (2.5.21), we get the desired upper estimate for C_2 in (2.4.13).

(ii) Let $b=\varepsilon>0$ and ε be so small that $\varepsilon < (a-\varepsilon)/(a+\varepsilon)$. Then we can reprise the proof of part (i) with $b=\varepsilon$, from which we obtain

$$\limsup_{t\to\infty}\frac{|Q(t,\omega)|}{(\log t)^{\frac{1}{1+\theta}}} \le K_\varepsilon = \left[\frac{(3a+\varepsilon)\left(\frac{\sigma^2(1+\theta)}{2a}\right)^{\frac{\theta}{1+\theta}}(1+\varepsilon)^{1+\theta}}{a(1-\varepsilon)-\varepsilon(1+\varepsilon)}\right]^{\frac{1}{\theta}}, \quad \text{on } \Omega^*.$$

Let $\varepsilon\downarrow 0$, we get the upper bound in (2.4.15).

Proof of Theorem 2.4.3 Proceeding in the same way as in the proof of the previous theorem, we arrive at

$$Q'(t) = -g(Q(t) + Z(t)) + g(Z(t)) + \sup_{-\tau \le s \le t} f(Q(s) + Z(s)), \quad t \ge 0.$$

Due to the fact that $\theta > 1$ and g satisfies (2.4.10), C is guaranteed to be greater than 1. Then in a similar manner as in the proof of the previous theorem, it is not difficult to show that for all $t \ge 0$,

$$D_+|Q(t)| \le -\frac{1}{C} g(|Q(t)|) + \frac{2C+1}{C} g(|Z(t)|) + f(\sup_{-\tau \le s \le t} |Q(s)| + \sup_{-\tau \le s \le t} |Z(s)|).$$

By an analogous argument as in the previous proof and the conditions (2.4.10) and (2.4.11), as well as the inequality $(x+y)^\theta \le 2^{\theta-1}(x^\theta + y^\theta)$ for $x, y \ge 0$ and $\theta > 1$, it can be shown that there exists $T_7(\varepsilon, \omega) > 0$ such that for all $t > T_7(\varepsilon, \omega)$,

$$\begin{aligned} D_+|Q(t)| \le -\frac{1}{C} g(|Q(t)|) &+ b(1+\varepsilon)2^{\theta-1}(\sup_{-\tau \le s \le t} |Q(s)|)^\theta \\ &+ (b2^{\theta-1} + (2 + \frac{1}{C})a)\left[\frac{\sigma^2(1+\theta)}{2a}\right]^{\frac{\theta}{1+\theta}} (1+\varepsilon)^{1+\theta} (\log t)^{\frac{\theta}{1+\theta}}. \end{aligned}$$

Here we require $\varepsilon \in (0, (\frac{a}{c} - b2^{\theta-1})/(\frac{a}{c} + b2^{\theta-1}))$. Now again consider the function U_ε defined as (2.5.20), there exists a $T_8(\varepsilon, \omega) > 0$ such that for all $t > T_8(\varepsilon, \omega)$, $g(U_\varepsilon) > a(1-\varepsilon)U_\varepsilon^\theta$. Let $T_9(\varepsilon, \omega) := T_7(\varepsilon, \omega) \vee t_+ \vee T_8(\varepsilon, \omega)$. This time we choose

$$K_\varepsilon := \left[\frac{(1+\varepsilon)^{1+\theta}(b2^{\theta-1} + (2 + \frac{1}{C})a)\left(\frac{\sigma^2(1+\theta)}{2a}\right)^{\frac{\theta}{1+\theta}}}{\frac{1}{C}a(1-\varepsilon) - b(1+\varepsilon)2^{\theta-1}}\right]^{\frac{1}{\theta}}$$

and $\rho(\varepsilon, \omega) > 0$ large enough such that $U_\varepsilon(-\tau) \ge |Q(T_9)|$. Then by a similar calculation as before, we get the desired results in both part (i) and (ii) of the theorem.

Chapter 3

Stochastic Affine Functional Differential Equations

3.1 Introduction

Increasingly real–world systems are modelled using stochastic differential equations with delay, as they represent systems which evolve in a random environment and whose evolution depends on the past states of the system through either memory or time delay. Examples include population biology (Mao [59], Mao and Rassias [61, 62],), neural networks (cf. e.g. Blythe et al. [20]), viscoelastic materials subjected to heat or mechanical stress Drozdov and Kolmanovskii [32], Caraballo et al. [26], Mizel and Trutzer [64, 65]), or financial mathematics Anh et al. [1, 2], Arriojas et al. [14], Hobson and Rogers [46].

In such stochastic models of phenomena in engineering and physics it is often of great importance to know that the system is *stable*, in the sense that the solution of the mathematical model converges in some sense to equilibrium. Consequently, a great deal of mathematical activity has been devoted to the question of stability of point equilibria of stochastic functional differential equations and also to the rate at which solutions converge. The literature is extensive, but a flavour of the work can be found in the monographs of Mao [56, 57], Mohammed [66], and Kolmanovskii and Myskhis [50].

However, in disciplines such as mathematical biology or finance, it is less usual for systems to converge to an equilibrium; more typically, the solutions may be stable in the sense that there is a stationary distribution to which the solution converges (see e.g. Reiß et al. [69], Küchler and Mensch [51], Mao [58]).

Mao and Rassias [62] have established *upper bounds* on the partial maxima growth rate of solutions some special stochastic delay differential equations (SDDEs) with fixed delays, with their results having particular application to population biology. Their methods enable them to recover results for highly nonlinear systems which are moreover sharp in the sense that the rate of growth of the corresponding non-delay systems are recovered when the fixed delay is set equal to zero. However, their methods do not automatically extend to differential equations with more general delay functionals, nor can they obtain lower bounds on the rate of growth of the partial maxima.

This chapter deals with a simpler class of stochastic functional differential equations (SFDEs) than [62] (in the sense that the equations are essentially linear) but with a more general type of delay functional, covering both point and distributed delay by using measures in the delay. In common with [62], but by different methods, we obtain an upper bound on the rate of growth of the partial maxima. However, in contrast to [62], we are also able to establish a lower bound on rate of growth of the partial maxima; indeed, as these bounds are equal, we can determine the *exact* a.s. rate of growth of the partial maxima. The results exploit the fact that given an exponentially decaying resolvent, the finite delay in the equation forces the limiting autocovariance function to decay exponentially fast, so that the solution of the linear equation is an asymptotically stationary Gaussian process. The results apply to both scalar and finite– dimensional equations and can moreover be extended to equations with a weak nonlinearity at infinity.

More precisely, we study the asymptotic behaviour of the finite–dimensional process which satisfies

$$X(t) = \psi(0) + \int_0^t L(X_s)\,ds + \int_0^t \Sigma\,dB(s), \quad t \geq 0, \tag{3.1.1a}$$

$$X(t) = \phi(t), \quad t \in [-\tau, 0]. \tag{3.1.1b}$$

where B is an m–dimensional standard Brownian motion, Σ is a $d \times m$–matrix with real entries, and $L : C[-\tau, 0] \to \mathbb{R}^d$ is a linear functional with $\tau \geq 0$ and

$$L(\phi) = \int_{[-\tau,0]} \nu(ds)\phi(s), \quad \phi \in C([-\tau, 0]; \mathbb{R}^d).$$

The asymptotic behaviour of (3.1.1) is determined in the case when the resolvent r of the deterministic equation $x'(t) = L(x_t)$, $t \geq 0$ obeys $r \in L^1([0,\infty); \mathbb{R}^{d\times d})$. In particular, we show that the partial maxima of each component grows according to

$$\limsup_{t\to\infty} \frac{\langle X(t), \mathbf{e}_i\rangle}{\sqrt{2\log t}} = \sigma_i, \quad \liminf_{t\to\infty} \frac{\langle X(t), \mathbf{e}_i\rangle}{\sqrt{2\log t}} = -\sigma_i, \quad \text{a.s.} \tag{3.1.2}$$

where $\sigma_i > 0$ depends on Σ and the resolvent r. Moreover

$$\limsup_{t\to\infty} \frac{|X(t)|_\infty}{\sqrt{2\log t}} = \max_{i=1,\dots,d} \sigma_i, \quad \text{a.s.} \tag{3.1.3}$$

Linear stochastic delay *difference* equations are commonly seen in the time series modelling of interest rates and volatilities in inefficient markets, in which historical information is incorporated in the dynamical system at any given time. An autoregressive (AR) model can be seen as a discretised version of the linear SFDE (3.1.1) when the measure ν is purely discrete. More precisely, if the continuous-time equation has only an instantaneous term and p point delays equally spaced in time, an AR(p) process results from the discretisation. If the mesh size of the discretisation is chosen sufficiently small, properties such as stationarity of the continuous equation can be preserved by the AR model. Conversely, an appropriately parameterised AR(p) model can converge weakly to the solution of (3.1.1) with a discrete measure as the parameter tends to a limit.

An extension and application in which the conditional variance obeys an autoregressive equation is given by the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model developed by Bollerslev (cf. e.g.,[21, 30]); such models are often used to model stock volatilities. There is an extensive literature on GARCH and AR models applied to finance, with nice recent introductions provided in e.g., [35]. A wealth of basic results on linear time series models is also contained in the classic text [23]. The results in this chapter concerning Gaussian stationary solutions of linear SFDEs provide the basic framework for estimating the large deviations of interest rates or volatilities simulated by continuous time semimartingale analogues of both scalar and vector autoregressive processes. An interesting and related literature on continuous time linear stochastic models also exists in the time series literature (see e.g., [22, 24, 63]), but the emphasis in those works does not overlap with the thrust of this chapter.

The non-linear problem (3.3.18) illustrated in this chapter deals only non-linearity that is lower than linear order at infinity in a sense made precise by (3.3.16). It is therefore interesting to ask how the results here could be developed to deal with other forms of non-linearity in the presence of additive noise. In Chapter 2, the asymptotic behaviour of scalar SFDEs of the form

$$dX(t) = (aX(t) + b \sup_{t-\tau \le s \le t} X(s))\, dt + \sigma\, dB(t), \quad t \ge 0 \tag{3.1.4}$$

is considered. Note that (3.1.4) is not in the form of either (3.1.1) or (3.3.18) with the condition (3.3.16). In Chapter 2, it is shown that if the solution is *recurrent* on the real line, then the presence of the maximum functional does not significantly change the essential growth rate of the solution of the related non-delay linear equation $dY(t) = \alpha Y(t)\, dt + \sigma\, dB(t)$ where $\alpha < 0$. More specifically, it is shown that there exist deterministic c_1, c_2 such that

$$0 < c_1 \le \limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2\log t}} \le c_2 < +\infty, \quad \text{a.s.}$$

which recovers the exact square root logarithmic growth rate of Y

$$\limsup_{t\to\infty} \frac{|Y(t)|}{\sqrt{2\log t}} = \frac{|\sigma|}{\sqrt{2|\alpha|}}, \quad \text{a.s.}$$

Since we illustrate in the present chapter that equations of the form (3.3.18) have exact square root logarithmic growth rate, this suggests that it is linearity, or "near linearity" that generates Gaussian-like large fluctuations.

For a scalar autonomous SDE which has no delay and whose solution is *stationary* we can apply Motoo's theorem to estimate the growth rate of the partial maximum, even when the drift coefficient is not of linear leading order at infinity (in contrast to (3.1.4) and (3.3.18) with the condition (3.3.16)). These techniques can even be extended to finite–dimensional and non-stationary processes as seen in Chapter 1. Similarly, if we add some delay factor into a stationary non-linear SDE, provided the order of this delay term is smaller than that of the instantaneous term at infinity, we show in forthcoming work that the size of the large fluctuations of the non-delay process are preserved, with the growth rate depending on the degree of non-linearity of the instantaneous term.

The work in this chapter appears in a paper joint with John Appleby and Xuerong Mao [6].

3.2 A Recapitulation on the Fundamentals of Stochastic Functional Differential Equations

We first turn our attention to the deterministic delay equation underlying the SDE (3.1.1). For a fixed constant $\tau \geq 0$ we consider the deterministic linear delay differential equation

$$\begin{aligned} x'(t) &= \int_{[-\tau,0]} \nu(du)\, x(t+u), \quad \text{for } t \geq 0, \\ x(t) &= \phi(t) \quad \text{for } t \in [-\tau, 0], \end{aligned} \tag{3.2.1}$$

for a measure $\nu \in M([-\tau,0];\mathbb{R}^{d\times d})$. The initial function ϕ is assumed to be in the space $C[-\tau,0] := \{\phi : [-\tau,0] \to \mathbb{R}^d : \text{continuous}\}$. A function $x : [-\tau,\infty) \to \mathbb{R}^d$ is called a *solution* of (3.2.1) if x is continuous on $[-\tau,\infty)$, its restriction to $[0,\infty)$ is continuously differentiable, and x satisfies the first and second identity of (3.2.1) for all $t \geq 0$ and $t \in [-\tau,0]$, respectively. It is well–known that for every $\phi \in C[-\tau,0]$ the problem (3.2.1) admits a unique solution $x = x(\cdot,\phi)$.

The *fundamental solution* or *resolvent* of (3.2.1) is the unique locally absolutely continuous function $r : [0,\infty) \to \mathbb{R}^{d\times d}$ which satisfies

$$r(t) = I_d + \int_0^t \int_{[\max\{-\tau,-s\},0]} \nu(du) r(s+u)\, ds \quad \text{for } t \geq 0, \tag{3.2.2}$$

where I_d is the $d \times d$ identity matrix. It plays a role which is analogous to the fundamental system in linear ordinary differential equations and the Green function in partial differential equations. For later convenience we set $r(t) = 0$ for $t \in [-\tau, 0)$.

The solution $x(\cdot,\phi)$ of (3.2.1) for an arbitrary initial segment ϕ exists, is unique, and can be represented as

$$x(t,\phi) = r(t)\phi(0) + \int_{-\tau}^0 \int_{[-\tau,u]} r(t+s-u)\nu(ds)\, \phi(u)\, du, \quad \text{for } t \geq 0, \tag{3.2.3}$$

cf. Diekmann et al [31, Chapter I].

Define the function $h_\nu : \mathbb{C} \to \mathbb{C}$ by

$$h_\nu(\lambda) = \det\left(\lambda I_d - \int_{[-\tau,0]} e^{\lambda s}\, \nu(ds)\right),$$

Define also the set

$$\Lambda = \{\lambda \in \mathbb{C} : h_\nu(\lambda) = 0\}.$$

The function h is analytic, and so the elements of Λ are isolated. Define

$$v_0(\nu) := \sup\{\mathrm{Re}\,(\lambda) :\ h_\nu(\lambda) = 0\}, \tag{3.2.4}$$

where $\mathrm{Re}\,(z)$ denotes the real part of a complex number z. Furthermore, the cardinality of $\Lambda' := \Lambda \cap \{\mathrm{Re}\,(\lambda) = v_0(\nu)\}$ is finite. Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ we have

$$e^{-v_0(\nu)t} r(t) = \sum_{\lambda_j \in \Lambda'} \{p_j(t)\cos(\mathrm{Im}\,(\lambda_j)t) + q_j(t)\sin(\mathrm{Im}\,(\lambda_j)t)\} + o(e^{-\varepsilon t}), \quad t \to \infty,$$

where p_j and q_j are matrix–valued polynomials of degree $m_j - 1$, with m_j being the multiplicity of the zero $\lambda_j \in \Lambda'$ of h, and $\mathrm{Im}\,(z)$ denoting the imaginary part of a complex number z. Hence, for every $\epsilon > 0$ there exists a $C(\epsilon) > 0$ such that

$$|r(t)| \leq C(\epsilon) e^{(v_0(\nu)-\epsilon)t}, \quad t \geq 0. \tag{3.2.5}$$

Therefore if $v_0(\nu) < 0$, then r decays to zero exponentially. This is a simple restatement of Diekmann et al [31, Theorem 1.5.4 and Corollary 1.5.5]. Furthermore, the following lemma regarding r is given in [4]:

Lemma 3.2.1. *Let r satisfy* (3.2.2), *and $v_0(\nu)$ be defined as* (3.2.4). *Then the following statements are equivalent:*

(a) $v_0(\nu) < 0$.

(b) r decays exponentially as $t \to \infty$.

(c) $r(t) \to 0$ as $t \to \infty$.

(d) $r \in L^1(\mathbb{R}^+; \mathbb{R}^{d\times d})$.

(e) $r \in L^2(\mathbb{R}^+; \mathbb{R}^{d\times d})$.

Let us introduce some notation for (3.2.1). For a function $x : [-\tau, \infty) \to \mathbb{R}^d$ we define the *segment of* x at time $t \geq 0$ by the function

$$x_t : [-\tau, 0] \to \mathbb{R}^d, \qquad x_t(u) := x(t+u).$$

If we equip the space $C[-\tau,0]$ of continuous functions with the supremum norm, Riesz' representation theorem guarantees that every continuous functional $L : C[-\tau,0] \to \mathbb{R}^d$ is of the form

$$L(\psi) = \int_{[-\tau,0]} \nu(du)\,\psi(u),$$

for a $d \times d$ matrix–valued measure $\nu \in M([-\tau,0];\mathbb{R}^{d\times d})$. Hence, we will write (3.2.1) in the form

$$x'(t) = L(x_t) \quad \text{for } t \geq 0, \qquad x_0 = \phi$$

and assume L to be a continuous and linear functional on $C([-\tau,0];\mathbb{R}^d)$.

We study the following stochastic differential equation with time delay:

$$\begin{aligned} dX(t) &= L(X_t)\,dt + \Sigma\,dB(t) \quad \text{for } t \geq 0, \\ X(t) &= \phi(t) \quad \text{for } t \in [-\tau,0], \end{aligned} \tag{3.2.6}$$

where L is a continuous and linear functional on $C([-\tau,0];\mathbb{R}^d)$ for a constant $\tau \geq 0$, and Σ is a $d \times m$ matrix with real entries.

For every $\phi \in C([-\tau,0];\mathbb{R}^d)$ there exists a unique, adapted strong solution $(X(t,\phi) : t \geq -\tau)$ with finite second moments of (3.2.6) (cf., e.g., Mao [57]). The dependence of the solutions on the initial condition ϕ is neglected in our notation in what follows; that is, we will write $x(t) = x(t,\phi)$ and $X(t) = X(t,\phi)$ for the solutions of (3.2.1) and (3.2.6) respectively.

By Reiß et al [70, Lemma 6.1] the solution $(X(t) : t \geq -\tau)$ of (3.2.6) obeys a variation-of-constants formula

$$X(t) = \begin{cases} x(t) + \int_0^t r(t-s)\Sigma\,dB(s), & t \geq 0, \\ \phi(t), & t \in [-\tau,0], \end{cases} \tag{3.2.7}$$

where r is the fundamental solution of (3.2.1).

3.3 Statement and Discussion of Main Results

3.3.1 One-dimensional SFDEs

We start with some preparatory lemmata, used to establish the almost sure rate of growth of the partial maxima of the solution of a scalar version of (3.2.6).

Lemma 3.3.1. *Suppose* $(a_n)_{n=1}^{\infty}$ *is a real sequence with* $\limsup_{n\to\infty} a_n \geq 0$, γ *is a non-negative and non-decreasing sequence, with* $\gamma(n) \to \infty$ *as* $n \to \infty$. *Then*

$$\limsup_{n\to\infty} \frac{\max_{1\leq j\leq n} a_j}{\gamma(n)} = \limsup_{n\to\infty} \frac{a_n}{\gamma(n)}.$$

We also need the following continuous analogue of Lemma 3.3.1, which appeared as Lemma 2.6.3 in [57].

Lemma 3.3.2. *Suppose* $y : [0,\infty) \to [0,\infty)$ *and* $\vartheta : [0,\infty) \to (0,\infty)$ *be a non-decreasing function with* $\vartheta(t) \to \infty$ *as* $t \to \infty$. *Then*

$$\limsup_{t\to\infty} \frac{\max_{0\leq s\leq t} y(s)}{\vartheta(t)} = \limsup_{t\to\infty} \frac{y(t)}{\vartheta(t)}.$$

We require the following results about sequences of identically distributed normal random variables.

Lemma 3.3.3. *If* $(X_n)_{n=1}^{\infty}$ *is a sequence of jointly normal standard random variables, then*

$$\limsup_{n\to\infty} \frac{|X_n|}{\sqrt{2\log n}} \leq 1, \quad a.s. \tag{3.3.1}$$

Moreover

$$\limsup_{n\to\infty} \frac{\max_{1\leq j\leq n} X_j}{\sqrt{2\log n}} \leq 1, \quad a.s. \tag{3.3.2}$$

The next result gives precise information on the growth of the partial maxima of a sequence of normal random variables which have an exponentially decaying autocovariance function. The proof was an early work of Appleby which can be found in [6].

Lemma 3.3.4. *Suppose* $(X_n)_{n=1}^{\infty}$ *is a sequence of jointly normal standard random variables satisfying*

$$|\operatorname{Cov}(X_i, X_j)| \leq \alpha^{|i-j|}$$

for some $\alpha \in (0,1)$. *Then*

$$\lim_{n\to\infty} \frac{\max_{1\leq j\leq n} X_j}{\sqrt{2\log n}} = 1, \quad a.s. \tag{3.3.3}$$

These lemmata are used to determine the size of the large fluctuations of the solution of (3.2.6) in the scalar case, i.e., the case in which $d = 1$ and the solution X of (3.2.6) is a one–dimensional process. If $m > 1$ and $\Sigma = (\Sigma_1, \Sigma_2, \ldots, \Sigma_m)$ is a $1 \times m$–matrix we note that the martingale

$$M(t) = \sum_{j=1}^{m} \int_0^t \Sigma_j \, dB_j(s), \quad t \geq 0,$$

can be rewritten as

$$M(t) = \int_0^t \sigma \, dW(s), \quad t \geq 0,$$

where $\sigma = (\sum_{j=1}^m \Sigma_j^2)^{1/2}$ and W is a one–dimensional Brownian motion. Therefore, in the scalar case it suffices to study the equation

$$\begin{aligned} dX(t) &= L(X_t)\, dt + \sigma \, dW(t) \quad \text{for } t \geq 0, \\ X(t) &= \phi(t) \quad \text{for } t \in [-\tau, 0], \end{aligned} \tag{3.3.4}$$

where $\phi \in C([-\tau, 0]; \mathbb{R})$.

Theorem 3.3.1. *Suppose that r is the solution of* (3.2.2) *with $d = 1$, and that $v_0(\nu) < 0$, where $v_0(\nu)$ is defined as* (3.2.4). *Let X be the unique continuous adapted process which obeys* (3.3.4). *Then*

$$\limsup_{t \to \infty} \frac{|X(t)|}{\sqrt{2 \log t}} = |\sigma| \sqrt{\int_0^\infty r^2(s)\, ds} =: \Gamma, \quad a.s. \tag{3.3.5}$$

Moreover,

$$\limsup_{t \to \infty} \frac{X(t)}{\sqrt{2 \log t}} = |\sigma| \sqrt{\int_0^\infty r^2(s)\, ds}, \quad a.s. \tag{3.3.6}$$

$$\liminf_{t \to \infty} \frac{X(t)}{\sqrt{2 \log t}} = -|\sigma| \sqrt{\int_0^\infty r^2(s)\, ds}, \quad a.s. \tag{3.3.7}$$

Theorem 3.3.1 can be applied in the case where X is a mean-reverting Ornstein-Uhlenbeck process. Consider the OU process governed by the following equation

$$dU(t) = -\alpha U(t)\, dt + \sigma \, dB(t), \quad t \geq 0 \tag{3.3.8}$$

with $U(0) = u_0$ and $\alpha > 0$. Then U is a Gaussian process and has a limiting distribution $N(0, \sigma^2/2\alpha)$. It can easily be shown that $e^{\alpha t} U(t) = u_0 + M(t)$, where $M(t) = \sigma \int_0^t e^{\alpha s} dB(s)$

is a continuous martingale with quadratic variation $\gamma(t) := \sigma^2(e^{2\alpha t}-1)/2\alpha$. By the time-change theorem for martingales [49, Theorem 3.4.6], $M(\gamma^{-1}(t))$ is a standard Brownian motion. Hence by the Law of the Iterated Logarithm for standard one-dimensional Brownian motion,

$$\limsup_{t\to\infty} \frac{|M(\gamma^{-1}(t))|}{\sqrt{2t\log\log t}} = 1, \quad \text{a.s.}$$

which implies

$$\limsup_{t\to\infty} \frac{|U(t)|}{\sqrt{2\log t}} = \frac{|\sigma|}{\sqrt{2\alpha}}, \quad \text{a.s.} \tag{3.3.9}$$

Thus it can be seen in this simple case that a short and independent proof of (3.3.5) can be given. In the general case with linear distributed delay, the solution of (3.3.4) can be represented by (3.2.7). Moreover, under the condition $v_0(\nu) < 0$, the solution is asymptotically Gaussian distributed with mean zero and variance Γ^2. However, since the characteristic equation of r in general has infinitely many roots, it is difficult to write an explicit solution for r, and hence for X. Consequently the value of Γ is not easily computed. Moreover, since the process given by the stochastic integral in (3.2.7) is not in general a martingale, the martingale time-change approach given above for the OU process is not available. We therefore use Mill's estimate together with Lemma 3.3.4 (both on Gaussian random variables) to prove (3.3.5) on a sequence of mesh points a_n. Then we investigate the behaviour of the solution in continuous time by choosing a_n so that the distance between the mesh points tends to zero as $n \to \infty$. This enables us to closely control the behaviour of X on the interval $[a_n, a_{n+1}]$.

The condition $v_0(\nu) < 0$ is essential in Theorem 3.3.1. If $v_0(\nu) \geq 0$, then asymptotic stationarity of the stochastic solution not assured. The case of $v_0(\nu) \geq 0$ has not been studied in the thesis mainly for two reasons. Firstly, the emphasis of this thesis is on the large deviations of *recurrent* rather than transient solutions of stochastic functional differential equations. Transient solutions are expected in general in the case of $v_0(\nu) > 0$. Secondly, although the results in the deterministic case, the asymptotic hehaviour of the unstable part is relatively straightforward because it is equivalent to a finite-dimensional differential equation, the analysis for the stochastic case is more complicated. Appleby et al. (cf.[8]) studied the case when $v_0(\nu) \geq 0$ in the case of a simple root of the characteristic equation. Their results can be summarized as the following:

(a) If $v_0(\nu) = 0$, then

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log t}} = L_1;$$

(b) If $v_0(\nu) > 0$, then

$$\lim_{t\to\infty} e^{-v_0(\nu)t} X(t) = L_2(\omega), \quad \text{a.s.}$$

where L_1 is deterministic and L_2 is a random variable. Their theorem requires new results on the asymptotic behaviour of stochastic convolution integrals. However, these convolution results are simplified by virtue of the fact the leading root of the characteristic equation is real and simple. In the case where the roots have multiplicity great than one, or the roots are complex, new convolution results are needed. Moreover, these results cannot easily use martingale techniques, because the exponential contribution cannot be entirely factor outside of the stochastic integrals. Problems of this type are particular to stochastic convolution integrals. Such analysis is of genuine interest, and worthy of study in its own right. However, we do not address this question in this thesis.

Theorem 3.3.1, together with these two results, connects the location of the roots of the characteristic equation to the asymptotic behaviour of the resolvent r, and hence to the asymptotic behaviour of the stochastic process X. If the underlying deterministic equation is stable in such a way that the resolvent tends to zero ($v_0(\nu) < 0$), then the process is Gaussian and asymptotically stationary. If mean-reverting force is just compensated by reinforcement ($v_0(\nu) = 0$), then the process obeys the law the iterated logarithm, and behaves like a Brownian motion. Finally, if the resolvent is exponentially unstable ($v_0(\nu) > 0$), then the process is exponentially transient.

The generalized Langevin equation mentioned at the end of Chapter 2 is an example of a process to which Theorem 3.3.1 can be applied, provided that $v_0(\nu) < 0$. We now characterise when this deterministic condition is satisfied in terms of the parameters (a, b, τ). The discussion summarises the analysis in e.g., Chapter XI.3 in [31].

Example 3.3.1. Let $a, b, \tau > 0$. Consider

$$r'(t) = ar(t) + br(t-\tau), \quad t \geq 0; r(t) = 0, \quad t \in [-\tau, 0); \quad r(0) = 1.$$

Then the characteristic equation is $h(z) = z - a - be^{-z\tau}$ and $\nu = a\delta_{\{0\}} + b\delta_{\{-\tau\}}$. Let $\rho(t) = r(t\tau)$, for $t \geq -1$. Then we have

$$\rho'(t) = \tau a\rho(t) + \tau b\rho(t-1), \quad t \geq 0; \rho(t) = 0, \quad t \in [-1, 0); \quad \rho(0) = 1.$$

Let $\alpha := \tau a$ and $\beta := \tau b$. Note that $v_0(\nu) < 0$ if and only if $\rho \in L^1(0, \infty)$ if and only if $r \in L^1(0, \infty)$. Define the smooth parameterised curve C_0 in $\mathbb{R}^2$ by

$$C_0 := \{\, (\alpha, \beta) = \left(\frac{\nu \cos \nu}{\sin \nu}, -\frac{\nu}{\sin \nu}\right), \nu \in (0, \pi) \,\}.$$

Then $v_0(\nu) < 0$ if and only if $(\alpha, \beta) \in S$ where

$$S := \{\, (\alpha, \beta) : \ \alpha < -\beta, \ \alpha > C(\beta) \,\},$$

and $C : (-\infty, -1) \to (-\infty, 1]$ is the strictly increasing function which is implicitly defined by $(C(\beta), \beta) \in C_0$. In fact C is asymptotic to the identity transform as $\beta \to -\infty$, and we have $C(-1) = 1$ (which defines the point of intersection of C_0 with the line $\alpha = -\beta$), and $C(-\pi/2) = 0$. This condition and the definition of S shows that solutions of the equation

$$\rho'(t) = \beta\rho(t-1), \quad t > 0$$

obey $\rho \in L^1(0, \infty)$ if and only if $-\pi/2 < \beta < 0$. The stability in this case and the dependence on the delay τ is discussed further at the end of the chapter.

3.3.2 Finite-dimensional SFDEs

We can extend the result of Theorem 3.3.1 to the solution of the general finite–dimensional equation (3.2.6). First, we state a lemma which gives the lower estimate on the limsup of the absolute value of an asymptotic Gaussian stationary process. The proof of the lemma is due to Appleby, and it can be found in [5].

Lemma 3.3.5. *Let B be an m–dimensional standard Brownian motion. Suppose that for each $j = 1, \ldots, m$, γ_j is a deterministic function such that $\gamma_j \in C([0, \infty); \mathbb{R}) \cap$*

$L^2([0,\infty);\mathbb{R}^{d\times d})$. *Define*

$$U(t) = \sum_{j=1}^{m} \int_0^t \gamma_j(t-s)\,dB_j(s), \quad t \geq 0.$$

Then

(a) For every $\theta \in (0,1)$, there is an a.s. event Ω_θ such that

$$\limsup_{n\to\infty} \frac{|U(n^\theta)|}{\sqrt{2\log n}} \leq \left(\sum_{j=1}^{m} \int_0^\infty \gamma_j^2(s)\,ds\right)^{1/2}, \quad \textit{a.s. conditionally on } \Omega_\theta.$$

(b) If there exists $c > 0$ and $\alpha > 0$ such that $|\gamma_j(t)| \leq ce^{-\alpha t}$ for all $t \geq 0$ and $j = 1,\ldots,m$ then

$$\limsup_{t\to\infty} \frac{|U(t)|}{\sqrt{2\log t}} \geq \left(\sum_{j=1}^{m} \int_0^\infty \gamma_j^2(s)\,ds\right)^{1/2}, \quad \textit{a.s.}$$

Furthermore we have

$$\limsup_{t\to\infty} \frac{U(t)}{\sqrt{2\log t}} \geq \left(\sum_{j=1}^{m} \int_0^\infty \gamma_j^2(s)\,ds\right)^{1/2}, \quad \textit{a.s.} \tag{3.3.10}$$

$$\liminf_{t\to\infty} \frac{U(t)}{\sqrt{2\log t}} \leq -\left(\sum_{j=1}^{m} \int_0^\infty \gamma_j^2(s)\,ds\right)^{1/2}, \quad \textit{a.s.} \tag{3.3.11}$$

Theorem 3.3.2. *Suppose that r is the solution of* (3.2.2) *and that $v_0(\nu) < 0$, where $v_0(\nu)$ is defined as* (3.2.4)*. Let X be the unique continuous adapted d-dimensional process which obeys* (3.2.6)*. Then for each $1 \leq i \leq d$,*

$$\limsup_{t\to\infty} \frac{X_i(t)}{\sqrt{2\log t}} = \sigma_i \quad \textit{and} \quad \liminf_{t\to\infty} \frac{X_i(t)}{\sqrt{2\log t}} = -\sigma_i, \quad \textit{a.s.} \tag{3.3.12}$$

where

$$\sigma_i = \sqrt{\sum_{k=1}^{m} \int_0^\infty \rho_{ik}^2(s)\,ds} \tag{3.3.13}$$

and $\rho(t) = r(t)\Sigma \in \mathbb{R}^{d\times m}$. Moreover

$$\limsup_{t\to\infty} \frac{|X(t)|_\infty}{\sqrt{2\log t}} = \max_{i=1,\ldots,d} \sigma_i, \quad \textit{a.s.} \tag{3.3.14}$$

The next result shows that (3.2.6) can be perturbed by a nonlinear functional N in the drift (which is of lower than linear order at infinity) without changing the asymptotic behaviour of the underlying affine stochastic functional differential equation. To make this claim more precise, we characterise the perturbing nonlinear functional N as follows: suppose $N : [0,\infty) \times C[-\tau, 0] \to \mathbb{R}^d$ obeys

$$\begin{gathered}\text{For all } n \in \mathbb{N} \text{ there exists a } K_n > 0 \text{ such that if } \varphi,\, \psi \in C([-\tau,0];\mathbb{R}^d) \\ \text{obey } \|\varphi\|_{\sup} \vee \|\psi\|_{\sup} \le n, \text{ then } |N(t,\varphi) - N(t,\psi)| \le K_n \|\varphi - \psi\|_{\sup}, \\ \text{and } N \text{ is continuous in its first argument;}\end{gathered} \tag{3.3.15}$$

$$\lim_{\|\varphi\|_{\sup}\to\infty} \frac{|N(t,\varphi)|}{\|\varphi\|_{\sup}} = 0, \quad \text{uniformly in } t; \tag{3.3.16}$$

$$t \to |N(t,0)| \text{ is bounded on } [0,\infty). \tag{3.3.17}$$

Consider the following nonlinear stochastic differential equation with time delay:

$$\begin{aligned} dX(t) &= (L(X_t) + N(t, X_t))\, dt + \Sigma\, dB(t) \quad \text{for } t \ge 0, \\ X(t) &= \phi(t) \quad \text{for } t \in [-\tau, 0], \end{aligned} \tag{3.3.18}$$

where L is a continuous and linear functional on $C([-\tau,0];\mathbb{R}^d)$ for a constant $\tau \ge 0$, and Σ is a $d \times m$ matrix with real entries.

Since L is linear and N obeys (3.3.15) and (3.3.16), for every $\phi \in C([-\tau,0];\mathbb{R}^d)$ there exists a unique, adapted strong solution $(X(t,\phi) : t \ge -\tau)$ with finite second moments of (3.3.18) (cf., e.g., Mao [57]).

Theorem 3.3.3. *Suppose that N obeys* (3.3.15) *and* (3.3.16)*. Also suppose that r is the solution of* (3.2.2) *and $v_0(\nu) < 0$, where $v_0(\nu)$ is defined as* (3.2.4)*. Let X be the unique continuous adapted d-dimensional process which obeys* (3.3.18)*. Then for each $1 \le i \le d$,*

$$\limsup_{t\to\infty} \frac{X_i(t)}{\sqrt{2\log t}} = \sigma_i, \quad \textit{and} \quad \liminf_{t\to\infty} \frac{X_i(t)}{\sqrt{2\log t}} = -\sigma_i, \quad \textit{a.s.} \tag{3.3.19}$$

where σ_i is given by (3.3.13)*. Moreover*

$$\limsup_{t\to\infty} \frac{|X(t)|_\infty}{\sqrt{2\log t}} = \max_{1\le i\le d} \sigma_i, \quad \textit{a.s.} \tag{3.3.20}$$

The above theorem was due to Appleby, the proof can be found in [6].

3.4 Proofs of Section 3.3

3.4.1 Proof of Section 3.3.1

Define $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\,du$. Mill's estimate tells us that

$$1 - \Phi(x) \le \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-\frac{x^2}{2}}, \quad x > 0.$$

Indeed, we also have

$$\lim_{x\to\infty} \frac{1-\Phi(x)}{\frac{1}{\sqrt{2\pi}}\frac{1}{x}e^{-\frac{x^2}{2}}} = 1. \tag{3.4.1}$$

Proof of Lemma 3.3.1 Let

$$\limsup_{n\to\infty} \frac{\max_{1\le j\le n} a_j}{\gamma(n)} = L_1, \quad \limsup_{n\to\infty} \frac{a_n}{\gamma(n)} = L_2$$

Clearly $L_1 \ge L_2$. Since $\limsup_{n\to\infty} a_n \ge 0$ and γ is positive, $L_2 \ge 0$. If $L_2 = \infty$, then $L_1 = \infty$ and the result holds. It remains to prove $L_1 \le L_2$ when $L_2 \in [0,\infty)$. Note for all $\epsilon > 0$ that there exists $N = N(\epsilon) \in \mathbb{N}$ such that for all $n > N$, $a_n < L_2(1+\epsilon)\gamma(n)$. Therefore

$$\begin{aligned} L_1 &= \limsup_{n\to\infty} \frac{\max_{1\le j\le n} a_j}{\gamma(n)} \\ &= \limsup_{n\to\infty} \max\Big(\frac{\max_{1\le j\le N} a_j}{\gamma(n)}, \frac{\max_{N\le j\le n} a_j}{\gamma(n)}\Big). \end{aligned}$$

If $\max_{1\le j\le N} a_j > \max_{N\le j\le n} a_j$ for all $n \ge N$, then $L_1 = 0 \le L_2$, and the proof is complete. If $\max_{1\le j\le N} a_j \le \max_{N\le j\le n} a_j$ for some $n \ge N$, we have that there is $N_1 \ge N$ such that

$$\max_{N\le j\le n} a_j \ge \max_{N\le j\le N_1} a_j \quad \text{for all } n \ge N_1.$$

Therefore

$$\begin{aligned} L_1 &= \limsup_{n\to\infty} \frac{\max_{N\le j\le n} a_j}{\gamma(n)} \\ &\le \limsup_{n\to\infty} \frac{\max_{N\le j\le n} L_2(1+\epsilon)\gamma(j)}{\gamma(n)} \\ &= L_2(1+\epsilon) \limsup_{n\to\infty} \frac{\max_{N\le j\le n} \gamma(j)}{\gamma(n)} \\ &= L_2(1+\epsilon). \end{aligned}$$

Letting $\epsilon \to 0$, we get $L_1 \le L_2$. The proof is complete.

Proof of Lemma 3.3.3 For every $\epsilon > 0$, Mill's estimate gives

$$\mathbb{P}[|X_n| > \sqrt{2(1+\epsilon)\log n}] \le \frac{2}{\sqrt{2\pi}} \frac{1}{\sqrt{2(1+\epsilon)\log n}} \frac{1}{n^{1+\epsilon}},$$

so by Borel-Cantelli lemma, for each $\epsilon > 0$, we have

$$\limsup_{n\to\infty} \frac{|X_n|}{\sqrt{2(1+\epsilon)\log n}} \le 1, \quad \text{a.s.}$$

By letting $\epsilon \to 0$ through rational numbers we get (3.3.1). Moreover

$$\limsup_{n\to\infty} \frac{\max_{1\le j\le n} X_j}{\sqrt{2\log n}} \le \limsup_{n\to\infty} \frac{\max_{1\le j\le n} |X_j|}{\sqrt{2\log n}} = \limsup_{n\to\infty} \frac{|X_n|}{\sqrt{2\log n}} \le 1 \quad \text{a.s.},$$

where we have used Lemma 3.3.1 at the penultimate step.

Proof of Theorem 3.3.1 Since $v_0(\nu) < 0$, we have that $r(t) \to 0$ as $t \to \infty$, so the first term on the righthand side of (3.2.7) tends to zero as $t \to \infty$. We analyse the behaviour of the second term. We first establish

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2\log t}} \le |\sigma| \sqrt{\int_0^\infty r^2(s)\,ds}, \quad \text{a.s.} \tag{3.4.2}$$

Define

$$\tilde{X}(t) := \sigma \int_0^t r(t-s)\,dB(s), \quad \tilde{X}(n^\epsilon) := \sigma \int_0^{n^\epsilon} r(n^\epsilon - s)\,dB(s), \quad \text{for some } \epsilon \in (0,1).$$

It is helpful to define

$$v(t) = \sigma^2 \int_0^t r^2(s)\,ds, \quad t \ge 0, \tag{3.4.3}$$

and so

$$v(n^\epsilon) = \sigma^2 \int_0^{n^\epsilon} r^2(s)\,ds.$$

Then both $\tilde{X}(n^\epsilon)$ and $\tilde{X}(t)$ are normally distributed with mean 0 and variances $v(n^\epsilon)$ and $v(t)$ respectively, where v is given by (3.4.3). Since $r \in L^1([0,\infty);\mathbb{R})$ and $r(t) \to 0$ as $t \to \infty$, we have $r \in L^2([0,\infty);\mathbb{R})$ and so

$$v(t) = \sigma^2 \int_0^t r^2(s)\,ds \le \sigma^2 \int_0^\infty r^2(s)\,ds =: \Gamma^2.$$

Clearly $\lim_{t\to\infty} v(t) = \Gamma^2$ and $\lim_{n\to\infty} v(n^\epsilon) = \Gamma^2$. If $Z(n^\epsilon) := \tilde{X}(n^\epsilon)/\sqrt{v(n^\epsilon)}$, by using a similar proof as in Lemma 3.3.3, we obtain

$$\limsup_{n\to\infty} \frac{|Z(n^\epsilon)|}{\sqrt{2\log n}} \le 1, \quad \text{a.s.}$$

Therefore

$$\limsup_{n\to\infty} \frac{|\tilde{X}(n^\epsilon)|}{\sqrt{2\log n}} \leq \Gamma, \quad \text{a.s.} \tag{3.4.4}$$

Now, by a stochastic Fubini theorem (which is stated in the preliminary), we get

$$\begin{aligned}\tilde{X}(t) &= \sigma \int_0^t \left(1 + \int_0^{t-s} r'(u)du\right) dB(s) \\ &= \sigma B(t) + \sigma \int_0^t \int_s^t r'(u-s)du\, dB(s) \\ &= \sigma B(t) + \sigma \int_0^t \int_0^u r'(u-s)\, dB(s)\, du.\end{aligned} \tag{3.4.5}$$

Therefore

$$|\tilde{X}(t)| \leq \sigma|B(t) - B(n^\epsilon)| + \sigma \left|\int_{n^\epsilon}^t \int_0^u r'(u-s)\, dB(s)\, du\right| + |\tilde{X}(n^\epsilon)|. \tag{3.4.6}$$

We now consider each of the three terms on the lefthand side of (3.4.6). By the properties of a standard Brownian motion, we have

$$\begin{aligned}\mathbb{P}\left[\sup_{n^\epsilon \leq t \leq (n+1)^\epsilon} |B(t) - B(n^\epsilon)| > 1\right] &\leq 2\mathbb{P}\left[\sup_{0\leq t\leq (n+1)^\epsilon - n^\epsilon} B(t) > 1\right] \\ &= 2\mathbb{P}[|B((n+1)^\epsilon - n^\epsilon)| > 1] \\ &= 4\mathbb{P}\left[Z > \frac{1}{\sqrt{(n+1)^\epsilon - n^\epsilon}}\right],\end{aligned}$$

where Z is a standard normal random variable. Since $\{(n+1)^\epsilon - n^\epsilon\}/n^{\epsilon-1} \to \epsilon$ as $n \to \infty$, by Mill's estimate and the Borel–Cantelli lemma, there exists $N(\omega) \in \mathbb{N}$, such that for all $n > N$

$$\sup_{n^\epsilon \leq t \leq (n+1)^\epsilon} |B(t) - B(n^\epsilon)| \leq 1, \quad \text{a.s.}$$

That is

$$\limsup_{n\to\infty} \sup_{n^\epsilon \leq t \leq (n+1)^\epsilon} |B(t) - B(n^\epsilon)| \leq 1, \quad \text{a.s.} \tag{3.4.7}$$

For the double integral term in (3.4.6), define

$$U_n := \sup_{n^\epsilon \leq t \leq (n+1)^\epsilon} \left|\int_{n^\epsilon}^t \int_0^u r'(u-s)\, dB(s)\, du\right|.$$

Then, by Hölder's inequality

$$\begin{aligned}
\mathbb{E}[U_n^{2k}] &\le \mathbb{E}\left[\sup_{n^\epsilon \le t \le (n+1)^\epsilon} \left(\int_{n^\epsilon}^t \left|\int_0^u r'(u-s)\,dB(s)\right|\,du\right)^{2k}\right] \\
&\le \mathbb{E}\left[\sup_{n^\epsilon \le t \le (n+1)^\epsilon} (t-n^\epsilon)^{2k-1} \int_{n^\epsilon}^t \left|\int_0^u r'(u-s)\,dB(s)\right|^{2k} du\right] \\
&= \mathbb{E}\left[\big((n+1)^\epsilon - n^\epsilon\big)^{2k-1} \int_{n^\epsilon}^{(n+1)^\epsilon} \left|\int_0^u r'(u-s)\,dB(s)\right|^{2k} du\right] \\
&= \big((n+1)^\epsilon - n^\epsilon\big)^{2k-1} \int_{n^\epsilon}^{(n+1)^\epsilon} \mathbb{E}\left|\int_0^u r'(u-s)\,dB(s)\right|^{2k} du.
\end{aligned}$$

Now, for $u \ge 0$, $\int_0^u r'(u-s)\,dB(s)$ is a Gaussian process with mean 0, variance $\int_0^u r'(s)^2\,ds$. Since r decays exponentially by Lemma 3.2.1, the variance is bounded above by $\int_0^\infty r'(s)^2\,ds =: L$. Hence there exists $C_k > 0$ such that

$$\int_{n^\epsilon}^{(n+1)^\epsilon} \mathbb{E}\left|\int_0^u r'(u-s)\,dB(s)\right|^{2k} du \le \int_{n^\epsilon}^{(n+1)^\epsilon} C_k L^k\,du = C_k L^k\big((n+1)^\epsilon - n^\epsilon\big).$$

By Chebyshev's inequality, we therefore get

$$\mathbb{P}(|U_n| \ge 1) \le \mathbb{E}[U_n^{2k}] \le C_k L^k\big((n+1)^\epsilon - n^\epsilon\big)^{2k}.$$

If we choose an integer $k \ge (1-\epsilon)^{-1}$, as $\{(n+1)^\epsilon - n^\epsilon\}/n^{\epsilon-1} \to \epsilon$ as $n \to \infty$, by the Borel–Cantelli lemma we obtain

$$\limsup_{n\to\infty} \sup_{n^\epsilon \le t \le (n+1)^\epsilon} \left|\int_{n^\epsilon}^t \int_0^u r'(u-s) dB(s)\,du\right| \le 1, \quad \text{a.s.} \tag{3.4.8}$$

Gathering the results from (3.4.4) to (3.4.8), we see that

$$\limsup_{n\to\infty} \sup_{n^\epsilon \le t \le (n+1)^\epsilon} \frac{|\tilde{X}(t)|}{\sqrt{2\log t}} \le \frac{\Gamma}{\sqrt{\epsilon}} \quad \text{a.s.}$$

which implies

$$\limsup_{t\to\infty} \frac{|\tilde{X}(t)|}{\sqrt{2\log t}} \le \frac{\Gamma}{\sqrt{\epsilon}} \quad \text{a.s.}$$

Finally, letting $\epsilon \to 1$, we obtain

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2\log t}} = \limsup_{t\to\infty} \frac{|\tilde{X}(t)|}{\sqrt{2\log t}} \le \Gamma \quad \text{a.s.,}$$

which is (3.4.2). We next show that

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2\log t}} \ge \Gamma \quad \text{a.s.} \tag{3.4.9}$$

Define the discrete Gaussian process $(\tilde{X}(n))_{n\geq 1}$ where $\tilde{X}(n) := \sigma \int_0^n r(n-s)\, dB(s)$. $\tilde{X}(n)$ has variance $v^2(n) := \sigma^2 \int_0^n r^2(s) ds$, so $(Z_n)_{n=1}^{\infty}$ is a sequence of standard normal random variables where $Z_n := \tilde{X}(n)/v(n)$.

We next prove that there exists a constant $\alpha \in (0,1)$, such that $|\operatorname{Cov}(Z_i, Z_j)| \leq \alpha^{|i-j|}$. To find this constant α, let $h \geq 0$ and $n = m+h$. Then

$$|\operatorname{Cov}(Z_{m+h}, Z_m)| = \frac{|\int_0^m r(s+h) r(s)\, ds|}{\sqrt{\int_0^{m+h} r^2(s) ds \int_0^m r^2(s) ds}}.$$

By the Cauchy-Schwarz inequality

$$|\operatorname{Cov}(Z_{m+h}, Z_m)|^2 \leq \frac{\int_0^m r^2(s+h)\, ds}{\int_0^{m+h} r^2(s)\, ds} = 1 - \frac{\int_0^h r^2(s)\, ds}{\int_0^{m+h} r^2(s)\, ds}.$$

Next define $\Gamma_1 = \int_0^\infty r^2(s)\, ds$. Then $\int_0^{m+h} r^2(s)\, ds \leq \Gamma_1$, so

$$|\operatorname{Cov}(Z_{m+h}, Z_m)|^2 \leq 1 - \frac{\int_0^h r^2(s)\, ds}{\int_0^{m+h} r^2(s)\, ds} \leq 1 - \frac{\int_0^h r^2(s)\, ds}{\Gamma_1} = \frac{\int_h^\infty r^2(s)\, ds}{\Gamma_1}. \qquad (3.4.10)$$

Now define

$$\alpha := \sup_{h\in\mathbb{N}} \alpha(h), \text{ where } \alpha(h) := \exp\left[\frac{1}{2h} \log \frac{\int_h^\infty r^2(s)\, ds}{\Gamma_1}\right].$$

We show that $\alpha \in (0,1)$. Since $r \in L^1(0,\infty)$, by (3.2.5) there exists $C > 0$ and $\lambda > 0$ such that $|r(t)| \leq Ce^{-\lambda t}$ for all $t \geq 0$. Hence

$$\frac{\int_h^\infty r^2(s)\, ds}{\Gamma_1} \leq \frac{C^2}{\Gamma_1} \int_h^\infty e^{-2\lambda s}\, ds = \frac{C^2 e^{-2\lambda h}}{2\lambda \Gamma_1},$$

so

$$\frac{1}{2h} \log \frac{\int_h^\infty r^2(s)\, ds}{\Gamma_1} \leq -\lambda + \frac{1}{2h} \log \frac{C^2}{2\lambda\Gamma_1}. \qquad (3.4.11)$$

Let $\lceil x \rceil$ denote the minimum integer which is greater than $x \in \mathbb{R}$. If $h' := 1 + \lceil (1/\lambda) \log(C^2/2\lambda\Gamma_1) \rceil$, then for all $h > h'$

$$\frac{\lambda}{2} > \frac{1}{2h} \log \frac{C^2}{2\lambda\Gamma_1}. \qquad (3.4.12)$$

Substituting (3.4.12) into (3.4.11), we obtain $0 < \alpha(h) \leq e^{-\lambda/2}$ for all $h > h'$. For $h < h'$, since r is continuous and $r(0) = 1$, $\int_0^h r^2(s)\, ds > 0$ for all $h > 0$, and therefore we have that $\int_h^\infty r^2(s)\, ds < \int_0^\infty r^2(s)\, ds$ for all $h > 0$. This implies $\alpha(h) \in (0,1)$ for all integers h such that $0 < h \leq h'$, and so $\alpha \in (0,1)$. Therefore

$$\alpha \geq \exp \frac{1}{2h} \log \frac{\int_h^\infty r^2(s)\, ds}{\Gamma_1}, \quad h \in \mathbb{N},$$

which gives

$$\frac{\int_h^\infty r^2(s)\,ds}{\Gamma_1} \le \alpha^{2h}, \quad h \in \{0\} \cup \mathbb{N}. \tag{3.4.13}$$

Combining (3.4.10) and (3.4.13), we get $|\operatorname{Cov}(Z_n, Z_m)| \le \alpha^{|n-m|}$. Thus by Lemma 3.3.4,

$$\lim_{n\to\infty} \frac{\max_{1\le j\le n} \tilde{X}(n)/v(n)}{\sqrt{2\log n}} = 1, \quad \text{a.s.}$$

Since Lemma 3.3.1 implies

$$\limsup_{n\to\infty} \frac{|\tilde{X}(n)|/v(n)}{\sqrt{2\log n}} = \limsup_{n\to\infty} \frac{\max_{1\le j\le n} |\tilde{X}(n)|/v(n)}{\sqrt{2\log n}},$$

combining these relations gives

$$\limsup_{n\to\infty} \frac{|\tilde{X}(n)|/v(n)}{\sqrt{2\log n}} = 1, \quad \text{a.s.}$$

Therefore

$$\begin{aligned}\limsup_{t\to\infty} \frac{|X(t)|}{\Gamma\sqrt{2\log t}} &= \limsup_{t\to\infty} \frac{|\tilde{X}(t)|}{\Gamma\sqrt{2\log t}} \\ &= \limsup_{t\to\infty} \frac{|\tilde{X}(t)|/v(t)}{\sqrt{2\log t}} \ge \limsup_{n\to\infty} \frac{|\tilde{X}(n)|/v(n)}{\sqrt{2\log n}},\end{aligned}$$

which implies (3.4.9). Since (3.4.2) also holds, we have established (3.3.5).

It remains to prove (3.3.6) and (3.3.7). We prove (3.3.6). First, note by (3.3.5) that

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2\log t}} \le \limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2\log t}} = \Gamma, \quad \text{a.s.}$$

By the definitions of $\tilde{X}$, Z and v, we deduce that

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2\log t}} = \limsup_{t\to\infty} \frac{\tilde{X}(t)}{\sqrt{2\log t}} \ge \limsup_{n\to\infty} \frac{\tilde{X}(n)}{\sqrt{2\log n}} = \limsup_{n\to\infty} \frac{Z_n v(n)}{\sqrt{2\log n}}.$$

Using the fact that $v(n) \to \Gamma$ as $n \to \infty$, Lemma 3.3.1, and then Lemma 3.3.4, we obtain

$$\begin{aligned}\limsup_{n\to\infty} \frac{Z_n v(n)}{\sqrt{2\log n}} &= \limsup_{n\to\infty} \frac{Z_n}{\sqrt{2\log n}} \cdot \Gamma = \limsup_{n\to\infty} \frac{\max_{1\le j\le n} Z_j}{\sqrt{2\log n}} \cdot \Gamma \\ &= \lim_{n\to\infty} \frac{\max_{1\le j\le n} Z_j}{\sqrt{2\log n}} \cdot \Gamma = \Gamma,\end{aligned}$$

and so (3.3.6) holds. (3.3.7) may be obtained by a symmetric argument.

3.4.2 Proof of Section 3.3.2

Proof of Theorem 3.3.2 Let x be the solution of (3.2.1). Then $x(t) \to 0$ as $t \to \infty$, because $v_0(\nu) < 0$. Then $\tilde{X}(t) = X(t) - x(t)$ where

$$\tilde{X}(t) := \int_0^t r(t-s)\Sigma\, dB(s), \quad t \geq 0.$$

Notice that $\tilde{X}(t) \in \mathbb{R}^d$ for each $t \geq 0$. Also $\tilde{X}(t) = \int_0^t \rho(t-s)\, dB(s)$, $t \geq 0$, where $\rho(t) = r(t)\Sigma$ is a $d \times m$–matrix valued function in which each entry must obey $|\rho_{ij}(t)| \leq Ce^{-v_0(\nu)t/2}$, $t \geq 0$ for some $C > 0$. Hence $\tilde{X}_i(t) := \langle X(t), \mathbf{e}_i \rangle$ obeys

$$\tilde{X}_i(t) = \sum_{j=1}^m \int_0^t \rho_{ij}(t-s)\, dB_j(s), \quad t \geq 0.$$

Define $\rho_i(t) \geq 0$ with $\rho_i^2(t) = \sum_{j=1}^m \rho_{ij}^2(t)$, $t \geq 0$. Then $\tilde{X}_i(t)$ is normally distributed with mean zero and variance $v_i(t) = \int_0^t \rho_i^2(s)\, ds$. Since $\rho_i \in L^2(0, \infty)$, we have that $v_i(t) \to \int_0^\infty \rho_i^2(s)\, ds = \int_0^\infty \sum_{j=1}^m \rho_{ij}^2(t)\, dt =: \sigma_i^2$ as $t \to \infty$. Moreover $|\rho_i(t)| \leq Cme^{-v_0(\nu)t/2}$, $t \geq 0$. Then by part (b) of Lemma 3.3.5, we have

$$\limsup_{t\to\infty} \frac{|\tilde{X}_i(t)|}{\sqrt{2\log t}} \geq \sigma_i, \quad \limsup_{t\to\infty} \frac{\tilde{X}_i(t)}{\sqrt{2\log t}} \geq \sigma_i, \quad \liminf_{t\to\infty} \frac{\tilde{X}_i(t)}{\sqrt{2\log t}} \leq \sigma_i, \quad \text{a.s.} \tag{3.4.14}$$

We now wish to prove

$$\limsup_{t\to\infty} \frac{|\tilde{X}_i(t)|}{\sqrt{2\log t}} \leq \sigma_i, \quad \text{a.s.} \tag{3.4.15}$$

Also by part (a) of Lemma 3.3.5, for each $0 < \epsilon < 1$, we have

$$\limsup_{n\to\infty} \frac{|\tilde{X}_i(n^\epsilon)|}{\sqrt{2\log(n^\epsilon)}} \leq \sqrt{\frac{\sigma_i^2}{\epsilon}}, \quad \text{a.s.} \tag{3.4.16}$$

In a similar manner to (3.4.5), we can rewrite $\tilde{X}$ according to

$$\begin{aligned}\tilde{X}_i(t) &= \sum_{j=1}^m \int_0^t \left(\rho_{ij}(0) + \int_0^{t-s} \rho_{ij}'(u)\, du\right) dB_j(s) \\ &= \sum_{j=1}^m \rho_{ij}(0)B_j(t) + \sum_{j=1}^m \int_0^t \int_0^u \rho_{ij}'(u-s)\, dB_j(s)\, du.\end{aligned}$$

Hence for $t \in [n^\epsilon, (n+1)^\epsilon]$, we get

$$\tilde{X}_i(t) - \tilde{X}_i(n^\epsilon) = \sum_{j=1}^m \rho_{ij}(0)\left(B_j(t) - B_j(n^\epsilon)\right) + \sum_{j=1}^m \int_{n^\epsilon}^t \int_0^u \rho_{ij}'(u-s)\, dB_j(s)\, du,$$

which implies

$$\sup_{n^\epsilon \le t \le (n+1)^\epsilon} |\tilde{X}_i(t) - \tilde{X}_i(n^\epsilon)| \le \sum_{j=1}^m |\rho_{ij}(0)| \sup_{n^\epsilon \le t \le (n+1)^\epsilon} |B_j(t) - B_j(n^\epsilon)| + \sum_{j=1}^m U_n^{(i,j)},$$

where we have defined

$$U_n^{(i,j)} = \sup_{n^\epsilon \le t \le (n+1)^\epsilon} \left| \int_{n^\epsilon}^t \int_0^u \rho'_{ij}(u-s)\, dB_j(s)\, du \right|.$$

Then using the technique used to prove (3.4.8), we can show that

$$\limsup_{n\to\infty} U_n^{(i,j)} \le 1, \quad \text{a.s.}$$

By (3.4.7), we have

$$\limsup_{n\to\infty} \sup_{n^\epsilon \le t \le (n+1)^\epsilon} |B_j(t) - B_j(n^\epsilon)| \le 1, \quad \text{a.s.}$$

Therefore,

$$\limsup_{n\to\infty} \frac{\sup_{n^\epsilon \le t \le (n+1)^\epsilon} |\tilde{X}_i(t) - \tilde{X}_i(n^\epsilon)|}{\sqrt{2\log n^\epsilon}} = 0, \quad \text{a.s.} \tag{3.4.17}$$

Using this estimate and (3.4.16), we obtain

$$\limsup_{n\to\infty} \sup_{n^\epsilon \le t \le (n+1)^\epsilon} \frac{|\tilde{X}_i(t)|}{\sqrt{2\log t}} \le \sqrt{\frac{\sigma_i^2}{\epsilon}}, \quad \text{a.s.,}$$

which implies

$$\limsup_{t\to\infty} \frac{|\tilde{X}_i(t)|}{\sqrt{2\log t}} \le \sqrt{\frac{\sigma_i^2}{\epsilon}}, \quad \text{a.s.}$$

Letting $\epsilon \to 1$ through the rational numbers implies (3.4.15). Combining (3.4.14) and (3.4.15) yields

$$\limsup_{t\to\infty} \frac{|\tilde{X}_i(t)|}{\sqrt{2\log t}} \le \sigma_i, \quad \text{a.s.}$$

Proceeding as at the end of Theorem 3.3.1, we can also establish (3.3.12).

To prove (3.3.14), note that there is an $i^* \in \{1,\ldots,d\}$ such that $\sigma_{i^*} = \max_{1\le i\le d} \sigma_i$. Then $\max_{1\le i\le d} |X_i(t)| \ge |X_{i^*}(t)|$. Hence

$$\limsup_{t\to\infty} \frac{\max_{1\le i\le d} |X_i(t)|}{\sqrt{2\log t}} \ge \limsup_{t\to\infty} \frac{|X_{i^*}(t)|}{\sqrt{2\log t}} = \sigma_{i^*} = \max_{i=1,\ldots,d} \sigma_i, \quad \text{a.s.} \tag{3.4.18}$$

Let p be an integer greater than unity. Note that $\max_{1\le i\le d}|x_i| \le (\sum_{i=1}^d |x_i|^p)^{1/p}$, so we have

$$\begin{aligned}\left(\limsup_{t\to\infty}\frac{\max_{1\le i\le d}|X_i(t)|}{\sqrt{2\log t}}\right)^p &= \limsup_{t\to\infty}\frac{(\max_{1\le i\le d}|X_i(t)|)^p}{(\sqrt{2\log t})^p}\\ &\le \limsup_{t\to\infty}\frac{\sum_{i=1}^d|X_i(t)|^p}{(\sqrt{2\log t})^p}\\ &\le \sum_{i=1}^d \limsup_{t\to\infty}\frac{|X_i(t)|^p}{(\sqrt{2\log t})^p}\\ &= \sum_{i=1}^d\left(\limsup_{t\to\infty}\frac{|X_i(t)|}{\sqrt{2\log t}}\right)^p = \sum_{i=1}^d \sigma_i^p.\end{aligned}$$

Hence

$$\limsup_{t\to\infty}\frac{\max_{1\le i\le d}|X_i(t)|}{\sqrt{2\log t}} \le \left(\sum_{i=1}^d \sigma_i^p\right)^{1/p}, \quad \text{a.s.}$$

Letting $p\to\infty$ through the natural numbers, it yields

$$\limsup_{t\to\infty}\frac{\max_{1\le i\le d}|X_i(t)|}{\sqrt{2\log t}} \le \max_{1\le i\le d}\sigma_i, \quad \text{a.s.}, \tag{3.4.19}$$

since $\left(\sum_{i=1}^d \sigma_i^p\right)^{1/p} \to \max_{1\le i\le d}\sigma_i$ as $p\to\infty$. Combining (3.4.18) and (3.4.19) yields (3.3.14).

3.5 A Note on the Generalized Langevin Delay Equations

As mentioned in the comments of Theorem 3.3.1, the Ornstein-Uhlenbeck process governed by the Langevin equation (3.3.8) obeys (3.3.9). We now take the following special case of the linear scalar SFDE (3.3.4), namely the *generalized Langevin delay equation*, as an example

$$dX(t) = (-aX(t) + bX(t-\tau))\,dt + \sigma\,dB(t), \quad t\ge 0, \tag{3.5.1}$$

with $X(t)=\phi(t)\in C([-\tau,0];\mathbb{R})$ and $a>b>0$. Küchler and Mensch [51] studied the stationarity and the covariance function of this equation in great detail. It can be shown that the solution of (3.5.1) has the explicit form

$$X(t) = r(t)\psi(0) + b\int_{-\tau}^0 r(t-s-\tau)\psi(s)\,ds + \sigma\int_0^t r(t-s)\,dB(s), \quad t\ge 0, \tag{3.5.2}$$

where r is the fundamental solution of the corresponding deterministic differential equation and satisfies

$$r'(t) = -ar(t) + br(t-\tau), \quad t \geq 0 \tag{3.5.3}$$

with $r(0) = 1$, $r(t) = 0$ for $t \in [-\tau, 0)$. For X be stationary, it is necessary and sufficient that $r \in L^1([0,\infty);\mathbb{R})$. Then by Theorem 3.3.1, X satisfies

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2\log t}} = |\sigma|\sqrt{\int_0^\infty r^2(s)\,ds}, \quad \text{a.s.} \tag{3.5.4}$$

In common with the non-linear delay SFDEs studied in Chapter 3, when the historical term is dominated by a mean reverting polynomial instantaneous term, the solution is recurrent on the real line. Moreover, the growth rate of the partial maxima of the corresponding non-delay equations are recovered when the fixed delay τ is set equal to zero. In other words, the general growth rate is determined by the polynomial degree of the instantaneous term.

However, the distribution of the solution of affine SFDEs is *Gaussian*. In this case, the large fluctuations given in (3.5.4) are not unexpected. For a general linear SFDE, it is possible to write an explicit solution in terms of the resolvent. However, because the characteristic equation of the resolvent has in general infinitely many roots, it is difficult to write down a useable explicit formula for r which satisfies (3.2.2). This makes the computation for the constant on the right–hand side of (3.3.5) much less straightforward in comparison with the non-delay case. Even in the special case of (3.5.4), where X and r obey (3.5.1) and (3.5.3) respectively, the exact value of the constant K may not be easily computed. However, by obtaining an explicit solution for r on each time interval $[n\tau, (n+1)\tau]$ (cf. Appendix A), we can approximate the size of the large deviations of X at any given time t in the long-run.

Despite the inconvenience of computing the exact value of K, we at least know that $K > |\sigma|\sqrt{2a}$. This is due to the autocorrelation provoked by the delay term, which causes the process to fluctuate at greater amplitudes, especially at extreme values. This feature of linear SFDEs is shared with non-linear SFDEs, and it could be used to capture a phenomena present in financial markets, namely that feedback trading tends to induce more extreme events. Since prices or returns are correlated in some way due to feedback trading strategy used by traders, the size of the large fluctuation of prices and returns in the presence of this feedback tends to be greater than when there is no memory.

We also note that the growth rate of the partial maxima of the solution depends on the length of the delay τ, because the resolvent depends itself on τ. If we look at the following SFDE

$$dX(t) = bX(t-\tau)\,dt + \sigma\,dB(t), \quad t \geq 0$$

where $b > 0$, then the solution X is still stationary as long as $b\tau < \pi/2$, but the growth rate of the partial maxima of X is very sensitive to τ as τ approaches $\pi/2b$ from below. Indeed, when $\tau = \pi/2b$, the resolvent is no longer square integrable (cf. [38]).

Chapter 4

Existence and Uniqueness of Stochastic Neutral Functional Differential Equations

4.1 Introduction

Over the last ten years, a body of work has emerged concerning the properties of stochastic neutral equations of Itô type. Of course, one of the most fundamental questions is whether solutions of such equations exist and are unique. A great many of these results have been established by Mao and co-workers.

In this chapter, for simplicity we concentrate on autonomous stochastic neutral functional differential equations (SNFDEs), and establish existence and uniqueness of solutions under weaker conditions than currently extant in the literature. The solutions will be unique within the class of continuous adapted processes, and will also exist on $[0,\infty)$. Also for simplicity, we assume that all functionals are globally linearly bounded and globally Lipschitz continuous (with respect to the sup–norm topology). The most general finite–dimensional neutral equation of this type is

$$d(X(t) - D(X_t)) = f(X_t)\,dt + g(X_t)\,dB(t), \quad 0 \le t \le T; \tag{4.1.1}$$

$$X(t) = \psi(t), \quad t \in [-\tau, 0]. \tag{4.1.2}$$

where $\tau > 0$, $\psi \in C([-\tau,0];\mathbb{R}^d)$ (i.e., the space of continuous functions from $[-\tau,0] \to \mathbb{R}^d$ with sup norm), B is an m–dimensional standard Brownian motion, D and f are functionals from $C([-\tau,0];\mathbb{R}^d)$ to $\mathbb{R}^d$ and $g : C([-\tau,0];\mathbb{R}^d \times \mathbb{R}^m) \to \mathbb{R}^d$. It is our belief that the results presented in this chapter can be extended to non–autonomous equations, to equations which obey only local Lipschitz continuity conditions, and to equations with local linear growth bounds. Naturally, in these circumstances, we cannot expect solutions to necessarily be global; instead, one can talk only about the existence of local solutions.

To the best of the authors' knowledge, all existing existence results concerning stochastic neutral equations in general, and (4.1.1) in particular, involve a "contraction condition" on the operator D on the righthand side. We term the operator D the *neutral functional* throughout this chapter, and the functional $E : C([-\tau, 0]; \mathbb{R}^d) \to \mathbb{R}^d$ defined by $E(\phi) := \phi(0) - D(\phi)$ the *neutral term*. The contraction condition on D is that there exists a $\kappa \in (0, 1)$ such that

$$|D(\phi) - D(\varphi)| \leq \kappa \|\phi - \varphi\|_{\sup}, \quad \text{for all } \phi,\, \varphi \in C([-\tau, 0]; \mathbb{R}^d), \tag{4.1.3}$$

where $\|\phi\|_{\sup} := \sup_{-\tau \leq s \leq 0} |\phi(s)|$ and $\phi \in C([-\tau, 0]; \mathbb{R}^d)$. Under this condition, as well as conventional Lipschitz conditions on f and g, it can be shown that (4.1.1) has a unique continuous adapted solution on $[0, T]$ for every $T > 0$.

While the condition (4.1.3) is certainly sufficient to ensure existence and uniqueness of solutions, until now it has not been understood whether this condition is necessary. However, comparison with the existence theory for the deterministic neutral equation corresponding to (4.1.1) viz.,

$$\frac{d}{dt}(x(t) - D(x_t)) = f(x_t), \quad 0 \leq t \leq T; \tag{4.1.4}$$

$$x(t) = \psi(t), \quad t \in [-\tau, 0]. \tag{4.1.5}$$

would lead one to suspect that the condition (4.1.3) is too strong, at least in some circumstances. To take a simple scalar example, suppose that $f : C([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$ is globally Lipschitz continuous, and that $w \in C([-\tau, 0]; \mathbb{R}^+)$ is such that

$$\int_{-\tau}^{0} w(s)\, ds > 1. \tag{4.1.6}$$

Then the solution of

$$\frac{d}{dt}(x(t) - \int_{-\tau}^{0} w(s) x(t+s)\, ds = f(x_t), \quad 0 \leq t \leq T;$$
$$x(t) = \psi(t), \quad t \in [-\tau, 0].$$

exists and is unique in the class of continuous functions. On the other hand, extant results do not enable us to make a definite conclusion concerning the existence and uniqueness of solutions of

$$d(X(t) - \int_{-\tau}^{0} w(s)X(t+s)\,ds) = f(X_t)\,dt + g(X_t)\,dB(t), \quad 0 \le t \le T; \tag{4.1.7}$$

$$X(t) = \psi(t), \quad t \in [-\tau, 0]. \tag{4.1.8}$$

when $g : C([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$ is also globally Lipschitz continuous, because the functional D defined by

$$D(\phi) = \int_{-\tau}^{0} w(s)\phi(s)\,ds \tag{4.1.9}$$

does not obey (4.1.3) if w obeys (4.1.6).

It transpires that the condition of uniform non–atomicity at zero of the functional D, which was introduced by Hale in the deterministic theory, and ensures the existence and uniqueness of a solution of the equation (4.1.4), also ensures the existence of a unique solution of (4.1.1), under Lipschitz continuity conditions on f and g. We discuss this non–atomicity condition presently, but note that it entails the existence of a number $s_0 \in (0, \tau)$ and a non–decreasing function $\kappa : [0, s_0] \to \mathbb{R}$ such that $\kappa(s_0) < 1$ and

$$|D(\phi) - D(\varphi)| \le \kappa(s)\|\phi - \varphi\|_{\sup} \quad \text{for all } \phi,\, \varphi \in C([-\tau, 0]; \mathbb{R}^d),$$

$$\text{such that } \phi = \varphi \text{ on } [-\tau, -s] \text{ and } s \in [0, s_0]. \tag{4.1.10}$$

Roughly speaking, it can be seen that (4.1.10) relaxes (4.1.3) by allowing the functions ϕ and φ to be equal on a subinterval of $[-\tau, 0]$, thereby effectively reducing the Lipschitz constant in (4.1.3) from a number greater than unity to a number less than unity. As an example, the functional in (4.1.9) obeys (4.1.10) even under the condition (4.1.6) on w. Therefore, we can conclude that (4.1.7) has a unique solution; existing results would however require w to obey $\int_{-\tau}^{0} w(s)\,ds < 1$.

The condition (4.1.3) has to date played a very important role in the analysis of properties of solutions of (4.1.1). It is a key assumption in proofs of estimates on the almost sure and p-th mean rate of growth of solutions of (4.1.1). It is also required in results which deal with the almost sure and p–th mean asymptotic stability of solutions. Results on the L^p continuity of solutions, and even results on numerical methods to simulate the solution of (4.1.1), rely on the condition (4.1.3). However, corresponding results for the underlying deterministic equation (4.1.4) regarding asymptotic behaviour, regularity of solutions, and numerical methods can be established under the weaker condition (4.1.10).

It is therefore reasonable to ask whether fundamental results on e.g., asymptotic behaviour, can still be established for solutions of (4.1.1) under the weaker condition (4.1.10), which is shown in this chapter to be sufficient to ensure solutions exist. Towards this end, in this chapter we prove results on p–th mean exponential estimates on the growth of the solution of (4.1.1) using the condition (4.1.10) in place of (4.1.3). Although we confine our attention here to the study of these exponential estimates, it is of obvious interest to investigate further the properties of solutions of stochastic neutral equations under the weaker non–atomicity condition (4.1.10) which have, owing to the absence of existence results, remained unconsidered until now.

It is worthy mentioning that Turi et al. (cf. [25], [48] and [47]) studied the existence of solutions of NFDEs with weakly singular kernels. Their results show that Hale's condition is sufficient but not necessary. We do not consider the measures which are weakly singular in this work.

Neutral functional differential equations (NFDEs) have been used to describe various processes in physics and engineering sciences [44, 75]. For example, transmission lines involving nonlinear boundary conditions [42], cell growth dynamics [15], propagating pulses in cardiac tissue [29] and drillstring vibrations [17] have been described by means of NFDEs.

4.2 Preliminaries

In this section, we give some definitions of the notation, state and comment on known results on the existence of solutions of the SNFDEs, and introduce in precise terms the weaker conditions used here on the neutral functional D which will still guarantee existence and uniqueness of solutions of (4.1.1).

Let ϕ be a function from $[-\tau, t_1] \to \mathbb{R}^d$. Let $t \in [0, t_1] \subset \mathbb{R}$. We use ϕ_t to denote the function on $[-\tau, 0]$ defined by $\phi_t(s) = \phi(t + s)$ for $-\tau \leq s \leq 0$.

4.2.1 Existing Results for Stochastic Neutral Equations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with the filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ satisfying the usual conditions. Let $\tau > 0$ and $0 < T < \infty$. Let the functionals D, f and g defined by

$$D : C([-\tau, 0]; \mathbb{R}^d) \to \mathbb{R}^d, \quad f : C([-\tau, 0]; \mathbb{R}^d) \to \mathbb{R}^d, \quad g : C([-\tau, 0]; \mathbb{R}^d) \to \mathbb{R}^{d\times m}$$

all be Borel–measurable.

Consider the d–dimensional neutral stochastic functional differential equation

$$d(X(t) - D(X_t)) = f(X_t)\,dt + g(X_t)\,dB(t), \quad 0 \leq t \leq T. \tag{4.2.1}$$

This should be interpreted as the integral equation

$$X(t)-D(X_t) = X(0)-D(X_0)+\int_0^t f(X_s)\,ds+\int_0^t g(X_s)\,dB(s), \quad \text{for all } t \in [0, T]. \tag{4.2.2}$$

For the initial value problem we must specify the initial data on the interval $[-\tau, 0]$ and hence we impose the initial condition

$$X_0 = \psi = \{\psi(\theta) : -\tau \leq \theta \leq 0\} \in L^2_{\mathcal{F}(0)}([-\tau, 0]; \mathbb{R}^d), \tag{4.2.3}$$

that is ψ is an $\mathcal{F}(0)$–measurable $C([-\tau, 0]; \mathbb{R}^d)$–valued random variable such that $\mathbb{E}[|\psi|^2] < +\infty$. The initial value problem for equation (4.2.1) is to find the solution of (4.2.1) satisfying the initial data (4.2.3). We give the definition of the solution in this context

Definition 4.2.1. An $\mathbb{R}^d$–valued stochastic process $X = \{X(t) : -\tau \leq t \leq T\}$ is called a solution to equation (4.2.1) with initial data (4.2.3) if it has the following properties:

(i) $t \mapsto X(t, \omega)$ is continuous for almost all $\omega \in \Omega$ and X is $\{\mathcal{F}(t)\}_{t\geq 0}$–adapted;

(ii) $\{f(X_t)\} \in L^1([0, T]; \mathbb{R}^d)$ and $\{g(X_t)\} \in L^2([0, T]; \mathbb{R}^{d\times m})$;

(iii) $X_0 = \psi$ and (4.2.2) holds for every $t \in [0, T]$.

A solution X is said to be unique if any other solution $\bar{X}$ is indistinguishable from it i.e.,

$$\mathbb{P}[X(t) = \bar{X}(t) \text{ for all } -\tau \leq t \leq T] = 1.$$

We now make the following assumptions on the functionals f and g in order to ensure the existence and uniqueness of solutions of (4.2.1). They will hold throughout the chapter.

Assumption 4.2.1. *There exists $K > 0$ such that for all $\phi, \varphi \in C([-\tau, 0]; \mathbb{R}^d)$*

$$|f(\varphi) - f(\phi)| \leq K\|\varphi - \phi\|_{sup}, \quad ||g(\varphi) - g(\phi)|| \leq K\|\varphi - \phi\|_{sup}. \tag{4.2.4}$$

There exists $\bar{K} > 0$ such that for all $\phi, \varphi \in C([-\tau, 0]; \mathbb{R}^d)$

$$|f(\varphi)| \leq \bar{K}(1 + \|\varphi\|_{sup}), \quad ||g(\varphi)|| \leq \bar{K}(1 + \|\varphi\|_{sup}). \tag{4.2.5}$$

The following result is Theorem 6.2.2 in [57]; it concerns the existence and uniqueness of solutions of the stochastic neutral functional differential equation (4.2.1).

Theorem 4.2.1. *Suppose that the functionals f and g obey* (4.2.4) *and* (4.2.5) *and that the functional D obeys*

There exists $\kappa \in (0, 1)$ such that for all $\phi, \varphi \in C([-\tau, 0]; \mathbb{R}^d)$

$$|D(\varphi) - D(\phi)| \leq \kappa\|\varphi - \phi\|_{sup}. \tag{4.2.6}$$

Then there exists a unique solution X to (4.2.1) *with initial data* (4.2.3). *Moreover the solution belongs to $\mathcal{M}^2([-\tau, T]; \mathbb{R}^d)$.*

On the other hand, a restriction of this type on the neutral functional D such as (4.2.6) is not needed in the case when it depends purely on delayed arguments. See [57, Theorem 6.3.1].

4.2.2 Assumptions on the Neutral Functional

In order to orient the reader to the question of existence which is addressed in this chapter, we must first introduce some results and notation from the theory of deterministic neutral differential equations. Consider systems of nonlinear functional differential equations of neutral type having the form

$$\frac{d}{dt}E(x_t) = f(x_t), \tag{4.2.7}$$

where the operator $E : C \to \mathbb{R}^d$ is *atomic* at 0 and *uniformly atomic* at 0 in the sense of Hale [41, pp 170–173], and where $f : C \to \mathbb{R}^d$ is continuous and uniformly Lipschitzian in the last argument. In (4.2.7), instead of the atomicity assumption on E, we may assume that E is of the form

$$E(\phi) = \phi(0) - D(\phi)$$

where $D : C \to \mathbb{R}^d$ is continuous and is uniformly nonatomic at zero on C in the following sense.

Definition 4.2.2. For any $\phi \in C$, and $s \geq 0$, let

$$Q(\phi, s) = \{\varphi \in C : \varphi(\theta) = \phi(\theta), \theta < -s, \theta \in [-\tau, 0]\}.$$

We say that a continuous function $D : C \to \mathbb{R}^d$ is *uniformly nonatomic at zero* on C if, for any $\phi \in C$, there exist T_1 such that $0 < T_1 < \tau$, independent of ϕ, and a positive scalar function $\rho(\phi, s)$, defined for $\phi \in C$, $0 \leq s \leq T_1$, nondecreasing in s such that

$$\rho_0(s) := \sup_{\phi \in C} \rho(\phi, s), \quad \rho_0(T_1) =: k < 1, \tag{4.2.8}$$

and

$$|D(\varphi_1) - D(\varphi_2)| \leq \rho_0(s) \|\varphi_1 - \varphi_2\|_{\text{sup}}, \text{ for } \varphi_1,\, \varphi_2 \in Q(\phi, s) \text{ and all } 0 \leq s \leq T_1. \tag{4.2.9}$$

We note that the definition implies both that ρ_0 is non–decreasing and that ρ_0 is independent of ϕ. Therefore a consequence of (4.2.9) is

$$|D(\varphi_1) - D(\varphi_2)| \leq \rho_0(s) \|\varphi_1 - \varphi_2\|_{\text{sup}}, \text{ for } \varphi_1,\, \varphi_2 \in Q(\phi, s),$$

$$\text{and all } 0 \leq s \leq T_1 \text{ and all } \phi \in C. \tag{4.2.10}$$

We tend to use this consequence of the definition in practice.

It is instructive to compare the conditions (4.2.8) and (4.2.9) with Mao's condition (4.2.6) on the neutral functional D. We first note that (4.2.6) implies both (4.2.8) and (4.2.9) and so implies that D is uniformly nonatomic at 0 in $C([-\tau, 0]; \mathbb{R}^d)$, so that (4.2.6) is not a weaker condition that uniform nonatomicity. Indeed, as shown by the functional given in (4.1.9), the condition (4.2.6) is a strictly stronger condition.

It is known ([28, 41, 43]) that under these assumptions on D, and f for each $\phi \in C$ there is a unique solution of (4.2.7) with initial value ϕ at 0. The solution is continuous with respect to initial data. For definition of solutions see [43]. In the sequel T_1 is fixed and is in the interval of definition $[0, T]$, of solutions of (4.2.7).

We make the following related assumption on the functional.

Assumption 4.2.2. *Let $\phi \in C([-\tau, 0]; \mathbb{R}^d)$ and assume D_0, $D_1 : C \to \mathbb{R}^d$ such that*

$$D(\phi) = D_0(\phi) + D_1(\phi). \tag{4.2.11}$$

Suppose there exists $\delta > 0$ and $H : C([-\tau, 0]; \mathbb{R}^d) \to \mathbb{R}^d$ such that

$$D_0(\phi) := H(\{\phi(s) : -\tau \le s \le -\delta < 0\}), \textit{for all } \phi \in C([-\tau, 0]; \mathbb{R}^d).$$

Suppose further that D_1 is uniformly non–atomic at zero on C, so that there exists $0 < T_1 \le \delta$ and $k \in (0, 1)$ as given in definition 4.2.2 such that (4.2.8) *and* (4.2.10) *hold.*

We can choose $T_1 < \delta$ without loss of generality in order to ensure that the pure delay functional D_0 which depends on $\phi \in C([-\tau, 0]; \mathbb{R}^d)$ only on $[-\tau, -\delta]$ does not interact with the functional D_1 which can depend on ϕ on all $[-\tau, 0]$. One consequence of the decomposition of D in (4.2.11) is that the continuity condition on k required in Hale's definition of uniform non–atomicity can be dropped for D_0.

We make a linear growth assumption on D which is slightly non–standard also.

Assumption 4.2.3. *For all $\phi \in C([-\tau, 0]; \mathbb{R}^d)$, there exist $k \in (0, 1)$ and $K_D > 0$ such that*

$$|D(\phi)| \le K_D(1 + \sup_{-\tau \le s \le -T_1} |\phi(s)|) + k \sup_{-T_1 \le s \le 0} |\phi(s)|. \tag{4.2.12}$$

The numbers k and T_1 can be chosen to be the same as those in Assumption 4.2.2 without loss of generality, and we choose to do so. One reason for this is that the choice that $T_1 < \delta$ in Assumption 4.2.2 ensures that the pure delay functional D_0 does not make a contribution to the constant k in the second term on the right hand side of (4.2.12) which might force $k > 1$. The linear growth bound on $D(\phi)$ arising from the dependence on ϕ over the interval $[-\tau, -T_1]$ guarantees the existence of second moments of the solution of (4.1.1). Notice that no restriction is made on the size of the constant K_D, while we require $k \in (0, 1)$.

4.3 Discussion of Main Results

In this section we state and discuss the main results of the chapter. We state our main existence result, and give examples of functionals to which it applies. We then show, under the condition that D is uniformly non–atomic at zero in $C([-\tau, 0]; \mathbb{R}^d)$, that the solution X of (4.2.1) enjoys exponential growth bounds in a p–th mean sense. Finally, we give examples of equations for which the neutral functional D is not uniformly non–atomic at zero, and for which solutions of (4.2.1) do not exist.

4.3.1 Existence result

The main result of this chapter relaxes the contraction constant in (4.2.6) in the case when the functional D is composed of a mixture of pure delay and instantaneously interacted functional. For any $T > 0$ and $\tau \geq 0$ we define $\mathcal{M}^2([-\tau, T]; \mathbb{R}^d)$ to be the space of all $\mathbb{R}^d$–valued adapted process $U = \{U(t) : -\tau \leq t \leq T\}$ such that

$$\mathbb{E}\left[\sup_{-\tau \leq s \leq T} |U(s)|^2\right] < +\infty.$$

Theorem 4.3.1. *Suppose that the functionals D obeys Assumption 4.2.2 and Assumption 4.2.3, f and g obey Assumption 4.2.1. Then there exists a unique solution to equation* (4.1.1). *Moreover, the solution is in $\mathcal{M}^2([-\tau, T]; \mathbb{R}^d)$.*

We now give two examples to which Theorem 4.3.1 can be applied.

Example 4.3.1. Consider the neutral functional D defined by

$$D(\varphi) = h_0(\varphi(0)) + \sum_{i=1}^{N} h_i(\varphi(-\tau_i)) + \int_{[-\tau_0,0]} w(s)h(\varphi(s))\,ds, \tag{4.3.1}$$

where $\varphi \in C([-\tau_*,0];\mathbb{R}^d)$ where $\tau_* := \max_{i\geq 1}\{\tau_i\} \vee \tau_0$; h is global Lipschitz continuous and linearly bounded; w is continuous; For each $i \in \mathbb{N}$, $\tau_i > 0$, h_i is continuous and globally linearly bounded. Hence, it is easy to see that under either of the following two conditions, a unique solution exists:

(i) If h_0 is also globally Lipschitz continuous and linearly bounded, moreover, for any $x, y \in \mathbb{R}^d$, $|h_0(x) - h_0(y)| \leq k|x-y|$ with $0 < k < 1$.

(ii) If $h_0(x) = Ax$, $A \in \mathbb{R}^{d\times d}$ and $\det(I-A) \neq 0$. In this case, equation (4.1.1) can be rearranged by dividing both sides by $(I-A)^{-1}$ to obtain a unique solution regardless the value of k.

This is because for some $L > 0$,

$$\begin{aligned}\int_{[-\tau_0,0]} w(s)(h(\varphi_1(s)) - h(\varphi_2(s)))\,ds &\leq L\int_{[-\tau_0,0]} w(s)|\varphi_1(s) - \varphi_2(s)|\,ds\\ &\leq L \sup_{-\tau_*\leq s\leq 0} |\varphi_1 - \varphi_2| \int_{[-\tau_0,0]} w(s)\,ds,\end{aligned}$$

we can choose $T_1 \in [0,\tau_0]$ such that

$$L\int_{[-\tau_0,-T_1]} w(s)\,ds + k < 1,$$

which ensures that D satisfies the condition of being uniformly non-atomic at zero. The two cases illustrate the importance of both invertibility and non-atomicity in ensuring a unique solution of equation (4.1.1).

Example 4.3.2. Consider $D(\varphi) = K\max_{-\tau\leq s\leq -\tau'} |\varphi(s)|$ where $0 \leq \tau' < \tau$. If $\tau' > 0$, then for all $K \in \mathbb{R}$, a unique solution exists. In this case, D plays the role of D_0 in (4.1.1). However, if $\tau' = 0$, then we require that $|K| < 1$.

4.3.2 Exponential estimates on the solution

In this subsection we state our results on the existence of moment of the solution of (4.1.1). Results of this kind have been proven by Mao in [60, Chapter 6] under the condition (4.2.6). However, in this chapter we establish similar estimates under the weaker assumption that D is uniformly non–atomic at zero. In our proof, this relaxation of the condition comes at the expense of a strengthening of our hypotheses on the functionals D, f and g. The new hypotheses, which tend to preclude the functionals being closely related to maximum functionals, are nonetheless very natural for equations with point or distributed delay. The proofs rely on differential and integral inequalities, in contrast to those in [60, Chapter 6].

Theorem 4.3.2. *Suppose that f and g are globally Lipschitz continuous and that D is uniformly non–atomic at zero. Then there exists a unique continuous solution X of equation* (4.1.1)*. Suppose further that there exist positive real numbers C_f, C_g and C_D such that*

$$|f(\varphi)| \le C_f + \int_{[-\tau,0]} \nu(ds)|\varphi(s)|; \tag{4.3.2}$$

$$||g(\varphi)|| \le C_g + \int_{[-\tau,0]} \eta(ds)|\varphi(s)|; \tag{4.3.3}$$

$$|D(\varphi)| \le C_D + \int_{[-\tau,0]} \mu(ds)|\varphi(s)|, \tag{4.3.4}$$

where ν, η and $\mu \in M([-\tau,0];\mathbb{R}^+)$. Let $p \ge 2$, $\varepsilon > 0$ and define

$$\beta_1 = \beta_1(p,\varepsilon) := \frac{\varepsilon p(p-1)}{2}, \quad \lambda(du) = \lambda_{p,\varepsilon}(du) := \nu(du)\cdot\frac{1}{\varepsilon^{p-1}} + \eta(du)\cdot\frac{p-1}{\varepsilon^{\frac{p-2}{2}}}.$$

Then there exists a positive real number $\theta = \theta(p,\varepsilon)$ such that X obeys

$$\limsup_{t\to\infty} \frac{1}{t}\log \mathbb{E}[|X(t)|^p] \le \theta + \frac{\varepsilon p(p-1)}{2}, \tag{4.3.5}$$

where θ satisfies

$$\int_{[-\tau,0]} e^{(\theta+\beta_1)s}\mu(ds) + \int_0^\tau e^{-\theta s}\int_{[-s,0]} e^{\beta_1 u}\lambda(du)\,ds + \frac{e^{-\theta\tau}}{\theta}\int_{[-\tau,0]} e^{\beta_1 u}\lambda(du) = 1.$$

We make no claims about the optimality of the exponent in (4.3.5), although $\varepsilon > 0$ could be chosen so as to minimise $\varepsilon \mapsto \theta(p,\varepsilon) + \beta_1(p,\varepsilon)$ for a given value of $p \geq 2$. In a later work we show that an exact exponent can be determined in the case $p = 2$ for a scalar linear stochastic neutral equation.

Remark 4.3.1. We notice that a functional of a form similar to (4.3.1) satisfies the conditions (4.3.2), (4.3.3) or (4.3.4). Suppose for $i = 1, \dots, N$ that $h_i : \mathbb{R}^d \to \mathbb{R}^{d'}$ is globally linearly bounded, and satisfies the bound $|h_i(x)| \leq K_i(1 + |x|)$ for $x \in \mathbb{R}^d$, and that $\nu_i \in M([-\tau, 0]; \mathbb{R}^{d\times d'})$, and let

$$f(\varphi) = \sum_{i=1}^{N} \int_{[-\tau_i,0]} \nu_i(ds)h_i(\varphi(s)), \quad \varphi \in C([-\tau, 0]; \mathbb{R}^d),$$

where $\tau = \max_{i=1,\dots,N} \tau_i \in (0, \infty)$. Then

$$|f(\varphi)| \leq \sum_{i=1}^{N} \int_{[-\tau_i,0]} K_i|\nu_i|(ds) + \sum_{i=1}^{N} \int_{[-\tau_i,0]} K_i|\nu_i|(ds)|\varphi(s)|.$$

Now set $C_f = \sum_{i=1}^{N} \int_{[-\tau_i,0]} K_i|\nu_i|(ds)$ and $\nu(ds) := \sum_{i=1}^{N} K_i|\nu_i|(ds)$ where we define $\nu_i(E) = 0$ for every Borel set $E \subset [-\tau, -\tau_i)$, so that f obeys (4.3.2).

Remark 4.3.2. First, we note that the conditions (4.3.2), (4.3.3) and (4.3.4) imply Assumption 4.2.1 and Assumption 4.2.3, with which Lemma 4.4.1 can be applied. Second, for any $p \geq 2$, the conditions (4.3.2), (4.3.3) and (4.3.4) imply

$$|f(\varphi)|^p \leq C_f + \int_{[-\tau,0]} \nu(ds)|\varphi(s)|^p; \tag{4.3.6}$$

$$||g(\varphi)||^p \leq C_g + \int_{[-\tau,0]} \eta(ds)|\varphi(s)|^p; \tag{4.3.7}$$

$$|D(\varphi)|^p \leq C_D + \int_{[-\tau,0]} \mu(ds)|\varphi(s)|^p, \tag{4.3.8}$$

respectively for a different set of C_f, C_g and C_D, and rescaled measures ν, η and μ. Therefore, for the reason of convenience, we will be using conditions (4.3.6), (4.3.7) and (4.3.8) in the proof of Theorem 4.3.2.

4.3.3 Non-existence of Solutions of SNFDEs

In this section, we give examples of scalar stochastic neutral equations which do not have a solution. To the best of our knowledge, examples of stochastic neutral equations which do not have solutions have not appeared in the literature to date. Our purpose in constructing such examples is to demonstrate the importance of the existence conditions (4.2.10) and (4.2.6) in ensuring the existence of solutions. We show that both these sufficient conditions are in some sense sharp in two ways. First, we show that if either condition (4.2.10) or (4.2.6) is slightly relaxed, then solutions to our examples do not exist. Second, by considering the equations for which solutions do not exist as members of parameterised families of equations, we can show that small changes in the parameters lead to equations which have unique solutions. We emphasise in each case that the underlying deterministic equation is also ill–posed. Therefore in the examples we consider, the presence of a stochastic perturbation does not make the stochastic NFDE well–posed. In fact, our theory in this chapter shows that the addition of a well–behaved stochastic term (e.g., Lipschitz continuous) does not modify the existence or uniqueness of solutions. It is an open and interesting problem as to whether there are a class of NFDEs or of reasonable stochastic perturbations which can give differing existence and uniqueness results. However, as this question is not required for the analysis of the pathwise large deviations of affine equations in the following chapter, we do not pursue it here.

Regarding ill–posed equations, we consider both equations with continuously distributed functionals and with maximum type functionals. The first class of equation shows the condition (4.2.10) cannot readily be improved for such equations. On the other hand, the more conservative condition (4.2.6) is shown to be quite sharp for equations with max–type functionals.

Equation with continuously distributed delay

Let the functional f defined by $f : C([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$ be Borel–measurable. Let $h \in C(\mathbb{R}; \mathbb{R})$, $w \in C^1([-\tau, 0]; \mathbb{R})$ and $\sigma \neq 0$. Consider the one–dimensional stochastic neutral functional differential equation

$$d\left(\epsilon X(t) + \int_{-\tau}^{0} w(s)h(X(t+s))\,ds\right) = f(X_t)\,dt + \sigma\,dB(t), \quad 0 \le t \le T. \tag{4.3.9}$$

where $\epsilon \in \mathbb{R}$. For the initial value problem we must specify the initial data on the interval $[-\tau, 0]$ and hence we impose the initial condition

$$X_0 = \psi = \{\psi(\theta) : -\tau \le \theta \le 0\} \in L^2_{\mathcal{F}(0)}([-\tau, 0]; \mathbb{R}), \tag{4.3.10}$$

that is ψ is an $\mathcal{F}(0)$–measurable $C([-\tau, 0]; \mathbb{R})$–valued random variable such that $\mathbb{E}[|\psi|^2] < +\infty$. (4.3.9) should be interpreted as the integral equation

$$\begin{aligned}\epsilon X(t) + \int_{-\tau}^{0} w(s)h(X(t+s))\,ds &= \epsilon X(0) + \int_{-\tau}^{0} w(s)h(\psi(s))\,ds \\ &\quad + \int_0^t f(X_s)\,ds + \int_0^t \sigma\,dB(s), \quad \text{for all } t \in [0, T], \text{ a.s.}. \end{aligned} \tag{4.3.11}$$

The initial value problem for equation (4.3.9) is to find the solution of (4.3.9) satisfying the initial data (4.3.10). In this context a solution is an $\mathbb{R}$–valued stochastic process $X = \{X(t) : -\tau \le t \le T\}$ to equation (4.3.9) with initial data (4.3.10) if it has the following properties:

(i) $t \mapsto X(t, \omega)$ is continuous for almost all $\omega \in \Omega$ and X is $\{\mathcal{F}(t)\}_{t\ge 0}$–adapted;

(ii) $\{f(X_t)\} \in L^1([0, T]; \mathbb{R})$;

(iii) $X_0 = \psi$ and (4.3.11) holds.

Proposition 4.3.1. *Let $\tau > 0$. Let $h \in C(\mathbb{R}; \mathbb{R})$, $w \in C^1([-\tau, 0]; \mathbb{R})$, $\psi \in C([-\tau, 0]; \mathbb{R})$ and $\sigma \neq 0$. Suppose also that $t \mapsto f(x_t)$ is in $C([0, \infty); \mathbb{R})$ for each $x \in C([-\tau, \infty); \mathbb{R})$. Let $T > 0$ and $\epsilon = 0$. Then there is no process $X = \{X(t) : -\tau \le t \le T\}$ which is a solution of* (4.3.9)*,* (4.3.10)*.*

We note that a solution does not exist *for any* $T > 0$.

It is the hypotheses $\epsilon = 0$ that is crucial in ensuring the non–existence of a solution. In (4.3.9) we may define the neutral functional D by

$$D(\varphi) := (1-\epsilon)\varphi(0) - \int_{-\tau}^{0} w(s)h(\varphi(s))\,ds, \quad \varphi \in C([-\tau, 0]; \mathbb{R}).$$

Suppose that h is globally Lipschitz continuous with Lipschitz constant k_h. Let $\phi \in C([-\tau,0];\mathbb{R})$ and suppose that $\varphi_1, \varphi_2 \in Q(\phi, s)$ for $s < \tau$. Clearly D cannot be uniformly non–atomic at 0 on $C([-\tau,0];\mathbb{R})$ for otherwise (4.3.9) would have a solution.

We now show, however, for $\epsilon \in (0,2)$ that D is uniformly non–atomic at 0 on $C([-\tau,0];\mathbb{R})$, and so (4.3.9) does have a solution. First note that

$$D(\varphi_1) - D(\varphi_2) = (1-\epsilon)(\varphi_1(0) - \varphi_2(0)) - \int_{-\tau}^{0} w(u)\,(h(\varphi_1(u)) - h(\varphi_2(u)))\,du.$$

Since $\varphi_1(u) = \varphi_2(u) = \phi(u)$ for $u \in [-\tau, -s)$, we have

$$D(\varphi_1) - D(\varphi_2) = (1-\epsilon)(\varphi_1(0) - \varphi_2(0)) - \int_{-s}^{0} w(u)\,(h(\varphi_1(u)) - h(\varphi_2(u)))\,du. \quad (4.3.12)$$

Therefore by (4.3.12) we have

$$\begin{aligned}|D(\varphi_1) - D(\varphi_2)| &\le |1-\epsilon||\varphi_1(0) - \varphi_2(0)| + \int_{-s}^{0} |w(u)||h(\varphi_1(u)) - h(\varphi_2(u))|\,du \\ &\le |1-\epsilon||\varphi_1(0) - \varphi_2(0)| + k_h \int_{-s}^{0} |w(u)||\varphi_1(u) - \varphi_2(u)|\,du \\ &\le |1-\epsilon|\|\varphi_1 - \varphi_2\|_{\sup} + k_h\|\varphi_1 - \varphi_2\|_{\sup} \int_{-s}^{0} |w(u)|\,du \\ &= \rho_0(s)\|\varphi_1 - \varphi_2\|_{\sup},\end{aligned}$$

where we define

$$\rho_0(s) := |1-\epsilon| + k_h \int_{-s}^{0} |w(u)|\,du, \quad s \in [-\tau, 0].$$

Clearly ρ_0 is non–decreasing. For every $\epsilon \in (0,2)$ we have $|1-\epsilon| < 1$, so because w is continuous, there exists a $0 < T_1 < \tau$ such that $\rho_0(T_1) < 1$. In this case, D is uniformly non–atomic at 0 on $C([-\tau,0];\mathbb{R})$. Therefore for $\epsilon \in (0,2)$ we see that (4.3.9) has a unique solution by Theorem 4.3.1. In the case when $\epsilon > 2$ or $\epsilon < 0$, simply divide (4.3.9) by ϵ. The properties on f, w and h etc. guarantee the existence and uniqueness by Theorem 4.3.1 using the above arguments in the case $\epsilon = 1$.

Proposition 4.3.2. *Let $\tau > 0$ and $\epsilon \neq 0$. Suppose $h \in C(\mathbb{R};\mathbb{R})$ is globally Lipschitz continuous, $w \in C([-\tau,0];\mathbb{R})$, $\psi \in C([-\tau,0];\mathbb{R})$ and $\sigma \neq 0$. Suppose also that there is $K > 0$*

$$|f(\phi) - f(\varphi)| \le K \sup_{-\tau \le s \le 0} |\phi(s) - \varphi(s)|, \quad \text{for all } \phi, \varphi \in C([-\tau,0];\mathbb{R})$$

Let $T > 0$. Then there is a unique solution $X = \{X(t) : -\tau \le t \le T\}$ of (4.3.9), (4.3.10).

Equations with maximum functionals

Let $\kappa > 0$ and suppose that $g : C([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$ is globally Lipschitz continuous. Consider the SNFDE

$$d(X(t) + \kappa \max_{-\tau \le s \le 0} |X(t+s)|) = g(X_t)\, dB(t), \quad 0 \le t \le T, \text{ a.s.} \tag{4.3.13}$$

In the case when $\kappa \in (0, 1)$, (4.2.6) holds for the functional D defined by

$$D(\varphi) = \kappa \max_{s \in [-\tau, 0]} |\varphi(s)|, \quad \varphi \in C([-\tau, 0]; \mathbb{R}), \tag{4.3.14}$$

and for any given $T > 0$, (4.3.13) has a solution by Mao [57, Theorem 6.2.2]. This could also be concluded from the fact that D is uniformly non–atomic at 0 on $C([-\tau, 0]; \mathbb{R})$, in which case Theorem 4.3.1 applies.

We suppose now that $\kappa \ge 1$. We note that (4.2.6) does not apply to the functional D in (4.3.14). To see this consider $\varphi_2 \in C([-\tau, 0], \mathbb{R})$ and let $\varphi_1 = \alpha\varphi_2$ for some $\alpha > 0$. Then

$$\begin{aligned} |D(\varphi_2) - D(\varphi_1)| &= |\kappa\|\varphi_2\|_{\sup} - \kappa\|\varphi_1\|_{\sup}| \\ &= \kappa|\|\varphi_2\|_{\sup} - \alpha\|\varphi_2\|_{\sup}| = \kappa|1 - \alpha|\|\varphi_2\|_{\sup}. \end{aligned}$$

On the other hand $\kappa\|\varphi_2 - \varphi_1\|_{\sup} = \kappa\|\varphi_2 - \alpha\varphi_2\|_{\sup} = \kappa|1 - \alpha|\|\varphi_2\|_{\sup}$, so

$$|D(\varphi_2) - D(\varphi_1)| = \kappa\|\varphi_2 - \varphi_1\|_{\sup},$$

which violates (4.2.6), as $\kappa \ge 1$.

Also, we see that D in (4.3.14) does not satisfy (4.2.9). To see this suppose that $\varphi_1, \varphi_2 \in Q(s, 0)$ is such that $\varphi_2(0) > 0$, φ_2 is non–decreasing, and $\varphi_1 = \alpha\varphi_2$ for $\alpha > 0$. Then

$$D(\varphi_2) = \kappa \max_{u \in [-\tau, 0]} |\varphi_2(u)| = \kappa \max_{u \in [-s, 0]} |\varphi_2(u)| = \kappa \max_{u \in [-s, 0]} \varphi_2(u) = \kappa\varphi_2(0).$$

Similarly

$$D(\varphi_1) = \kappa \max_{u \in [-s, 0]} |\varphi_1(u)| = \kappa \max_{u \in [-s, 0]} \alpha\varphi_2(u) = \kappa\alpha\varphi_2(0).$$

Hence $|D(\varphi_2) - D(\varphi_1)| = \kappa|1 - \alpha|\varphi_2(0)$. On the other hand

$$\|\varphi_2 - \varphi_1\|_{\sup} = \max_{u \in [-s, 0]} |\varphi_2(u) - \varphi_1(u)| = \max_{u \in [-s, 0]} |1 - \alpha||\varphi_2(u)| = |1 - \alpha|\varphi_2(0).$$

Thus $|D(\varphi_2) - D(\varphi_1)| = \kappa\|\varphi_2 - \varphi_1\|_{\sup}$, so (4.2.8) and (4.2.9) cannot both be satisfied, because $\kappa \geq 1$.

We now prove that (4.3.13) does not have a solution.

Proposition 4.3.3. *Let $\tau > 0$. Let $\psi \in C([-\tau, 0]; \mathbb{R})$. Suppose also that*

$$\text{There exists } \delta > 0 \text{ such that } \delta := \inf_{\varphi \in C([-\tau,0];\mathbb{R})} g^2(\varphi). \tag{4.3.15}$$

Let $T > 0$ and $\kappa \geq 1$. Then there is no process $X = \{X(t) : -\tau \leq t \leq T\}$ which is a solution of (4.3.13).

4.4 Auxiliary Results

The proofs of the main results are facilitated by a number of supporting lemmata. We state and discuss these here.

We first give a lemma which is necessary in proving the uniqueness and existence of the solution.

Lemma 4.4.1. *Let X be the unique continuous solution of equation* (4.2.1) *with initial condition* (4.2.3)*. If both* (4.2.5) *and* (4.2.12) *hold, then for any $p \geq 2$, there exist positive constants K_1 and K_2 depending on T such that*

$$\mathbb{E}[\sup_{-\tau \leq s \leq T} |X(s)|^p] \leq K_1 e^{K_2 T}. \tag{4.4.1}$$

In our proofs of moment estimates, we will need to use the fact that the p–th moment of the solution is a continuous function. Although the continuity of the moments is known for solutions of SNDEs, the contraction condition (4.2.6) is used in proving this continuity. Therefore, under our weaker assumptions, we need to prove this result afresh. To prove the continuity, we first need an elementary inequality.

Lemma 4.4.2. *Let $p \geq 1$. Suppose that $U, V \in \mathbb{R}^d$ are random variables in $L^{2(p-1)}$. If $c_p > 0$ is the number such that*

$$(a + b)^{2(p-1)} \leq c_p(a^{2(p-1)} + b^{2(p-1)}), \quad \textit{for all } a, b \geq 0,$$

then

$$|\mathbb{E}[|U|^p] - \mathbb{E}[|V|^p]| \le p\left(c_p\mathbb{E}[(|U|^{2(p-1)}] + c_p\mathbb{E}[|V|^{2(p-1)}]\right)^{1/2}\mathbb{E}[|U-V|^2]^{1/2}.$$

The continuity of the moments applies to general processes; since we will also employ it for an important auxiliary process, we do not confine the scope of the result to the solution of (4.2.1).

Lemma 4.4.3. *Let $p \ge 1$. Let $\tau, T > 0$. Let $X = \{X(t) : t \in [-\tau, T]\}$ be a $\mathbb{R}^d$–valued stochastic process with a.s. continuous paths, such that*

$$\mathbb{E}[\max_{-\tau\le s\le T}|X(s)|^2] < +\infty, \quad \mathbb{E}[\max_{-\tau\le s\le T}|X(s)|^{2(p-1)}] < +\infty. \tag{4.4.2}$$

Then

$$\lim_{t\to s}\mathbb{E}[|X(t)-X(s)|^2] = 0, \quad \text{for all } s \in [0,T], \tag{4.4.3}$$

and so

$$\lim_{t\to s}\mathbb{E}[|X(t)|^p] = \mathbb{E}[|X(s)|^p] \quad \text{for all } s \in [0,T]. \tag{4.4.4}$$

We find it useful to prove a variant of Gronwall's lemma. The argument is a slight modification of arguments given in Gripenberg, Londen and Staffans [39, Theorems 9.8.2 and 10.2.15]. The result gives us the freedom to *construct* an upper bound via an integral *inequality*, rather than relying on precise knowledge of the asymptotic behaviour of a solution of an *equation*. We avail of this freedom in proving a.s. and p-th mean exponential estimates on the solution of the neutral SFDE.

Lemma 4.4.4. *Suppose that $\kappa \in M([0,\infty);\mathbb{R}^+)$ is such that $(-\kappa)$ has non–positive resolvent ρ given by*

$$\rho + (-\kappa) * \rho = -\kappa.$$

Let f be in $L^1_{loc}(\mathbb{R}^+)$ and $x \in L^1_{loc}(\mathbb{R}^+)$ obey

$$x(t) \le (\kappa * x)(t) + f(t), \quad t \ge 0. \tag{4.4.5}$$

If $y \in L^1_{loc}(\mathbb{R}^+)$ *obeys*

$$y(t) \geq (\kappa * y)(t) + f(t), \quad t \geq 0; \quad y(0) \geq x(0), \tag{4.4.6}$$

then $x(t) \leq y(t)$ *for all* $t \geq 0$.

4.5 Proof of Section 4.4 and Section 4.3

Proof of Lemma 4.4.1 First, consider $t \in [0, T_1]$. Define

$$\xi_m := T_1 \wedge \inf\{t \in [0, T_1] \quad | \quad |X(t)| \geq m\}, \quad m \in \mathbb{N}.$$

Set $X^m(t) = X(t \wedge \xi_m)$. Hence

$$X^m(t) = \psi(0) - D(\psi) + D(X^m_t) + \int_0^t f(X^m_s)\, ds + \int_0^t g(X^m_s)\, dB(s).$$

By the inequality (cf. [57, Lemma 6.4.1]),

$$|a+b|^p \leq (1 + \varepsilon^{\frac{1}{p-1}})^{p-1}(|a|^p + \frac{|b|^p}{\varepsilon}), \quad \forall\, p > 1,\ \varepsilon > 0, \text{ and } a, b \in \mathbb{R}, \tag{4.5.1}$$

it is easy to show that

$$|X^m(t)|^p \leq (1 + \varepsilon^{\frac{1}{p-1}})^{p-1}\left(|D(X^m_t) - D(\psi)|^p + \frac{1}{\varepsilon}|J^m_1(t)|^p\right),$$

where

$$0 < \varepsilon < \left(\frac{1}{k^{\frac{p}{3p-3}}} - 1\right)^{p-1} \tag{4.5.2}$$

k is defined in (4.2.12), and

$$J^m_1(t) := \psi(0) + \int_0^t f(X^m_s)\, ds + \int_0^t g(X^m_s)\, dB(s).$$

Given (4.2.12),and using (4.5.1), for any $\varepsilon > 1$, we have

$$
\begin{aligned}
&|X^m(t)|^p \\
&\le \frac{(1+\varepsilon^{\frac{1}{p-1}})^{2p-2}}{\varepsilon}|D(\psi)|^p + (1+\varepsilon^{\frac{1}{p-1}})^{2p-2}|D(X_t^m)|^p + \frac{(1+\varepsilon^{\frac{1}{p-1}})^{p-1}}{\varepsilon}|J_1^m(t)|^p \\
&\le \frac{(1+\varepsilon^{\frac{1}{p-1}})^{2p-2}}{\varepsilon}|D(\psi)|^p + \frac{(1+\varepsilon^{\frac{1}{p-1}})^{p-1}}{\varepsilon}|J_1^m(t)|^p \\
&\quad + (1+\varepsilon^{\frac{1}{p-1}})^{2p-2}\left[K_D\left(1+\sup_{-\tau\le s\le -T_1}|X^m(t+s)|\right) + k\sup_{-T_1\le s\le 0}|X^m(t+s)|\right]^p \\
&\le \frac{(1+\varepsilon^{\frac{1}{p-1}})^{2p-2}}{\varepsilon}|D(\psi)|^p + \frac{(1+\varepsilon^{\frac{1}{p-1}})^{p-1}}{\varepsilon}|J_1^m(t)|^p \\
&\quad + (1+\varepsilon^{\frac{1}{p-1}})^{2p-2}[K_D + (K_D+k)\sup_{-\tau\le s\le 0}|\psi(s)| + k\sup_{0\le s\le t}|X^m(s)|]^p \\
&\le \frac{(1+\varepsilon^{\frac{1}{p-1}})^{2p-2}}{\varepsilon}|D(\psi)|^p + \frac{(1+\varepsilon^{\frac{1}{p-1}})^{p-1}}{\varepsilon}|J_1^m(t)|^p \\
&\quad + (1+\varepsilon^{\frac{1}{p-1}})^{3p-3}k^p\sup_{0\le s\le t}|X^m(s)|^p \\
&\quad + \frac{(1+\varepsilon^{\frac{1}{p-1}})^{3p-3}}{\varepsilon}[K_D + (K_D+k)\sup_{-\tau\le s\le 0}|\psi(s)|]^p
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\sup_{0\le s\le t}|X^m(s)|^p \\
&\le \frac{(1+\varepsilon^{\frac{1}{p-1}})^{2p-2}}{\varepsilon}|D(\psi)|^p + \frac{(1+\varepsilon^{\frac{1}{p-1}})^{3p-3}}{\varepsilon}[K_D + (K_D+k)\sup_{-\tau\le s\le 0}|\psi(s)|]^p \\
&+ (1+\varepsilon^{\frac{1}{p-1}})^{3p-3}k^p\sup_{0\le s\le t}|X^m(s)|^p + \frac{(1+\varepsilon^{\frac{1}{p-1}})^{p-1}}{\varepsilon}\sup_{0\le s\le t}|J_1^m(t)|^p.
\end{aligned}
$$

Due to (4.5.2), $(1+\varepsilon^{\frac{1}{p-1}})^{3p-3}k^p < 1$, the above inequality implies

$$
\begin{aligned}
\sup_{0\le s\le t}|X^m(s)|^p \le{}& \frac{1}{1-(1+\varepsilon^{\frac{1}{p-1}})^{3p-3}k^p}\left\{\frac{(1+\varepsilon^{\frac{1}{p-1}})^{2p-2}}{\varepsilon}|D(\psi)|^p\right. \\
&\left. + \frac{(1+\varepsilon^{\frac{1}{p-1}})^{3p-3}}{\varepsilon}[K_D + (K_D+k)\sup_{-\tau\le s\le 0}|\psi(s)|]^p\right\} \\
&+ \frac{(1+\varepsilon^{\frac{1}{p-1}})^{p-1}}{\varepsilon[1-(1+\varepsilon^{\frac{1}{p-1}})^{3p-3}k^p]}\sup_{0\le s\le t}|J_1^m(t)|^p.
\end{aligned}
$$

Since

$$
\sup_{-\tau\le s\le t}|X^m(s)|^p \le \sup_{-\tau\le s\le 0}|\psi(s)|^p + \sup_{0\le s\le t}|X^m(s)|^p,
$$

we get

$$\sup_{-\tau\le s\le t}|X^m(s)|^p \le \Bigg\{\frac{1}{1-(1+\varepsilon^{\frac{1}{p-1}})^{3p-3}k^p}\Bigg[\frac{(1+\varepsilon^{\frac{1}{p-1}})^{2p-2}}{\varepsilon}|D(\psi)|^p$$
$$+\frac{(1+\varepsilon^{\frac{1}{p-1}})^{3p-3}}{\varepsilon}[K_D+(K_D+k)\sup_{-\tau\le s\le 0}|\psi(s)|]^p\Bigg]+\sup_{-\tau\le s\le 0}|\psi(s)|^p\Bigg\}$$
$$+\frac{(1+\varepsilon^{\frac{1}{p-1}})^{p-1}}{\varepsilon[1-(1+\varepsilon^{\frac{1}{p-1}})^{3p-3}k^p]}\sup_{0\le s\le t}|J_1^m(t)|^p. \quad (4.5.3)$$

Now

$$\begin{aligned}
&\sup_{0\le s\le t}|J_1^m(s)|^p\\
&=\sup_{0\le s\le t}\left|\psi(0)+\int_0^s f(X_u^m)\,du+\int_0^s g(X_u^m)\,dB(u)\right|^p\\
&\le \frac{(1+\varepsilon^{\frac{1}{p-1}})^{p-1}}{\varepsilon}\sup_{-\tau\le s\le 0}|\psi(s)|^p\\
&\qquad+(1+\varepsilon^{\frac{1}{p-1}})^{p-1}\sup_{0\le s\le t}\left|\int_0^s f(X_u^m)\,du+\int_0^s g(X_u^m)\,dB(u)\right|^p\\
&\le \frac{(1+\varepsilon^{\frac{1}{p-1}})^{p-1}}{\varepsilon}\sup_{-\tau\le s\le 0}|\psi(s)|^p+\frac{(1+\varepsilon^{\frac{1}{p-1}})^{2p-2}}{\varepsilon}\sup_{0\le s\le t}\left(\int_0^s\left|f(X_u^m)\right|du\right)^p+\\
&\qquad+(1+\varepsilon^{\frac{1}{p-1}})^{2p-2}\sup_{0\le s\le t}\left|\int_0^s g(X_u^m)\,dB(u)\right|^p
\end{aligned}$$

Taking expectations on both sides of the inequality, and let $\alpha=\varepsilon^{1/(p-1)}$, by Assumption 4.2.1, we have

$$\begin{aligned}
\mathbb{E}[\sup_{0\le s\le t}|J_1^m(s)|^p] \le \left(\frac{1+\alpha}{\alpha}\right)^{p-1}&\sup_{-\tau\le s\le 0}|\psi(s)|^p\\
&+\frac{(1+\alpha)^{2p-2}}{\alpha^{p-1}}\mathbb{E}\left[\sup_{0\le s\le t}\left(\int_0^s \bar{K}(1+||X_u^m||_{\sup})\,du\right)^p\right]\\
&+(1+\alpha)^{2p-2}\mathbb{E}\left[\sup_{0\le s\le t}\left|\int_0^s g(X_u^m)\,dB(u)\right|^p\right].
\end{aligned}$$

By the Burkholder-Davis-Gundy inequality, let $C_p := [p^{p+1}/(2(p-1)^{p-1})]^{p/2}$, the above inequality implies that

$$\begin{aligned}
\mathbb{E}[\sup_{0\le s\le t}|J_1^m(s)|^p] \le & \left(\frac{1+\alpha}{\alpha}\right)^{p-1}\sup_{-\tau\le s\le 0}|\psi(s)|^p \\
&+\frac{(1+\alpha)^{2p-2}}{\alpha^{p-1}}\bar{K}^p\mathbb{E}\left[\left(\int_0^t(1+||X_u^m||_{\sup})\,du\right)^p\right] \\
&+(1+\alpha)^{2p-2}C_p\mathbb{E}\left[\left(\int_0^t||g(X_s^m)||_s^2\,ds\right)^{\frac{p}{2}}\right] \\
\le & \left(\frac{1+\alpha}{\alpha}\right)^{p-1}\sup_{-\tau\le s\le 0}|\psi(s)|^p \\
&+\frac{(1+\alpha)^{2p-2}}{\alpha^{p-1}}\bar{K}^pT_1^{p-1}\mathbb{E}\left[\int_0^t(1+||X_u^m||_{\sup})^p\,du\right] \\
&+(1+\alpha)^{2p-2}C_p\bar{K}^p\mathbb{E}\left[\left(\int_0^t(1+||X_u^m||_{\sup})^2\,du\right)^{\frac{p}{2}}\right]
\end{aligned}$$

where we have used Hölder's inequality in the second line. Thus

$$(1+||X_u^m||_{\sup})^p \le (1+\alpha)^{p-1}(\alpha^{1-p}+||X_u^m||_{\sup}^p),$$

and

$$\begin{aligned}
\left(\int_0^t(1+||X_u^m||_{\sup})^2\,du\right)^{\frac{p}{2}} &\le T_1^{\frac{(p-2)p}{4}}\int_0^t(1+||X_u^m||_{\sup})^p\,du \\
&\le (1+\alpha)^{p-1}T_1^{\frac{(p-2)p}{4}}\int_0^t(\alpha^{1-p}+||X_u^m||_{\sup}^p)\,du.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{E}[\sup_{0\le s\le t}|J_1^m(s)|^p] \le & \left(\frac{1+\alpha}{\alpha}\right)^{p-1}\sup_{-\tau\le s\le 0}|\psi(s)|^p \\
&+\left[\frac{(1+\alpha)^{3p-3}}{\alpha^{p-1}}\bar{K}^pT_1^{p-1}+(1+\alpha)^{3p-3}C_p\bar{K}^pT_1^{\frac{(p-2)p}{4}}\right]\mathbb{E}\left[\int_0^t(\alpha^{1-p}+||X_u^m||_{\sup}^p)\,du\right].
\end{aligned} \tag{4.5.4}$$

Taking expectations on both sides of (4.5.3), and inserting the above inequality into (4.5.3), we have

$$\begin{aligned}
\frac{1}{\varepsilon}+\mathbb{E}[\sup_{-\tau\le s\le t}|X^m(s)|^p] &\le \kappa_1+\kappa_2\int_0^t\left(\frac{1}{\varepsilon}+\mathbb{E}[||X_u^m||_{\sup}^p]\right)du \\
&\le (\frac{1}{\varepsilon}+\kappa_1)+\kappa_2\int_0^t\left(\frac{1}{\varepsilon}+\mathbb{E}[\sup_{-\tau\le u\le s}|X^m(u)|^p\right)du,
\end{aligned}$$

where

$$\kappa_1 := \frac{1}{1-(1+\varepsilon^{\frac{1}{p-1}})^{3p-3}k^p}\Bigg[\frac{(1+\varepsilon^{\frac{1}{p-1}})^{2p-2}}{\varepsilon}|D(\psi)|^p + \frac{(1+\varepsilon^{\frac{1}{p-1}})^{3p-3}}{\varepsilon}[K_D+(K_D+k)\sup_{-\tau\le s\le 0}|\psi(s)|]^p\Bigg] + \left[1+\frac{(1+\varepsilon^{\frac{1}{p-1}})^{p-1}}{\varepsilon}\right]\sup_{-\tau\le s\le 0}|\psi(s)|^p,$$

and

$$\kappa_2 := \frac{(1+\varepsilon^{\frac{1}{p-1}})^{p-1}}{\varepsilon[1-(1+\varepsilon^{\frac{1}{p-1}})^{3p-3}k^p]}\times \left[\frac{(1+\varepsilon^{\frac{1}{p-1}})^{3p-3}}{\varepsilon}\bar{K}^pT_1^{p-1} + (1+\varepsilon^{\frac{1}{p-1}})^{3p-3}C_p\bar{K}^pT_1^{\frac{(p-2)p}{4}}\right].$$

Now the Gronwall inequality yields that

$$\frac{1}{\varepsilon}+\mathbb{E}[\sup_{-\tau\le s\le T_1}|X^m(s)|^p] \le (\frac{1}{\varepsilon}+\kappa_1)e^{\kappa_2 T_1},$$

Consequently

$$\mathbb{E}[\sup_{-\tau\le s\le T_1}|X^m(s)|^p] \le (\frac{1}{\varepsilon}+\kappa_1)e^{\kappa_2 T_1}.$$

Letting $m\to\infty$ and $\varepsilon\to[1/k^{p/(3p-3)}-1]^{p-1}$, we get

$$\mathbb{E}[\sup_{-\tau\le s\le T_1}|X(s)|^p] \le \left[\left(\frac{1}{k^{\frac{p}{3p-3}}}-1\right)^{p-1}+\kappa_1\right]e^{\kappa_2 T_1}.$$

For $t\in[nT_1,(n+1)T_1]$ $(n\in\mathbb{N})$, assertion (4.4.1) can be shown by applying the same analysis as in the case of $t\in[0,T_1]$.

Proof of Lemma 4.4.2 Let $x,y\ge 0$ and $p\ge 1$. Then there exists $\theta(x,y)\in[0,1]$ such that

$$x^p-y^p = p[\theta x+(1-\theta)y]^{p-1}(x-y).$$

Thus for U, $V\in\mathbb{R}^d$ we have $\theta(U,V)\in[0,1]$ such that

$$|U|^p-|V|^p = p[\theta|U|+(1-\theta)|V|]^{p-1}(|U|-|V|).$$

Therefore

$$\begin{aligned}\mathbb{E}[|U|^p]-\mathbb{E}[|V|^p] &= p\mathbb{E}[[\theta|U|+(1-\theta)|V|]^{p-1}(|U|-|V|)]\\ &\le p\mathbb{E}[[\theta|U|+(1-\theta)|V|]^{2(p-1)}]^{1/2}\mathbb{E}[(|U|-|V|)^2]^{1/2}\\ &\le p\mathbb{E}[[|U|+|V|]^{2(p-1)}]^{1/2}\mathbb{E}[(|U|-|V|)^2]^{1/2}.\end{aligned}$$

Similarly, as $|V|^p - |U|^p = p[\theta|U| + (1-\theta)|V|]^{p-1}(|V| - |U|)$, we have

$$\begin{aligned}
\mathbb{E}[|V|^p] - \mathbb{E}[|U|^p] &= p\mathbb{E}[[\theta|U| + (1-\theta)|V|]^{p-1}(|V| - |U|)] \\
&\le p\mathbb{E}[[\theta|U| + (1-\theta)|V|]^{2(p-1)}]^{1/2}\mathbb{E}[(|V| - |U|)^2]^{1/2} \\
&= p\mathbb{E}[[\theta|U| + (1-\theta)|V|]^{2(p-1)}]^{1/2}\mathbb{E}[(|U| - |V|)^2]^{1/2} \\
&\le p\mathbb{E}[[|U| + |V|]^{2(p-1)}]^{1/2}\mathbb{E}[(|U| - |V|)^2]^{1/2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
|\mathbb{E}[|U|^p] - \mathbb{E}[|V|^p]| &\le p\mathbb{E}[[|U| + |V|]^{2(p-1)}]^{1/2}\mathbb{E}[(|U| - |V|)^2]^{1/2} \\
&= p\mathbb{E}[[|U| + |V|]^{2(p-1)}]^{1/2}\mathbb{E}[||U| - |V||^2]^{1/2}
\end{aligned}$$

Now $||U| - |V|| \le |U - V|$, so $||U| - |V||^2 \le |U - V|^2$. Therefore

$$|\mathbb{E}[|U|^p] - \mathbb{E}[|V|^p]| \le p\mathbb{E}[[|U| + |V|]^{2(p-1)}]^{1/2}\mathbb{E}[|U - V|^2]^{1/2}.$$

Since $(a+b)^{2(p-1)} \le c_p(a^{2(p-1)} + b^{2(p-1)})$ for all $a, b \ge 0$, we have

$$|\mathbb{E}[|U|^p] - \mathbb{E}[|V|^p]| \le p\left(c_p\mathbb{E}[(|U|^{2(p-1)}] + c_p\mathbb{E}[|V|^{2(p-1)}]\right)^{1/2}\mathbb{E}[|U - V|^2]^{1/2},$$

as required.

Proof of Lemma 4.4.3 Let $0 \le s \le t \le T$. We first prove (4.4.3). By the continuity of the sample paths, we have $\lim_{t\to s} X(t) = X(s)$ a.s. for each $s \in [0, T]$. On the other hand, because

$$|X(t)| \le \max_{0\le u\le T} |X(u)|,$$

we have that $|X(t)|$ is dominated by a random variable which is in L^2 by (4.4.2). Then by the Dominated Convergence Theorem, we have that $X(t)$ converges to $X(s)$ in L^2 viz.,

$$\lim_{t\to s} \mathbb{E}[|X(t) - X(s)|^2] = 0,$$

which is (4.4.3). Now we prove (4.4.4). Let $0 \le s \le t \le T$. Define $M_p(T) := \mathbb{E}[\max_{-\tau\le s\le T} |X(s)|^{2(p-1)}]$. Since (4.4.2) holds, by Lemma 4.4.2

$$\begin{aligned}
&|\mathbb{E}[|X(t)|^p] - \mathbb{E}[|X(s)|^p]| \\
&\le p\left(c_p\mathbb{E}[(|X(t)|^{2(p-1)}] + c_p\mathbb{E}[|X(s)|^{2(p-1)}]\right)^{1/2}\mathbb{E}[|X(t) - X(s)|^2]^{1/2} \\
&\le p\left(2c_pM_p(T)\right)^{1/2}\mathbb{E}[|X(t) - X(s)|^2]^{1/2}.
\end{aligned}$$

Now (4.4.3) implies (4.4.4).

Proof of Lemma 4.4.4 By (4.4.5) and (4.4.6), there are $g \geq 0$ and a $h \geq 0$, both in $L^1_{\text{loc}}(\mathbb{R}^+)$ such that

$$x(t) = (\kappa * x)(t) + f(t) - g(t), \quad y(t) = (\kappa * y)(t) + f(t) + h(t), \quad t \geq 0.$$

Since ρ is the resolvent of $-\kappa$, we have the variation of constants formulae:

$$x = f - g - \rho * (f - g), \quad y = f + h - \rho * (f + h).$$

Therefore

$$\kappa * x = \kappa * (f - g) - \kappa * \rho * (f - g) = [\kappa - \kappa * \rho] * f - [\kappa - \kappa * \rho] * g = -\rho * f + \rho * g.$$

Similarly $\kappa * y = -\rho * f - \rho * h$. Hence

$$x(t) \leq (\kappa * x)(t) + f(t) = -(\rho * f)(t) + (\rho * g)(t) + f(t) \leq f(t) - (\rho * f)(t),$$

where we have used the fact that g is non–negative and ρ is non–positive at the last step. Similarly

$$y(t) \geq (\kappa * y)(t) + f(t) = -(\rho * f)(t) - (\rho * h)(t) + f(t) \geq f(t) - (\rho * f)(t),$$

where we have used the fact that h is non–negative and ρ is non–positive at the last step. Therefore $x(t) \leq f(t) - (\rho * f)(t) \leq y(t)$ for all $t \geq 0$, which proves the claim.

Proof of Theorem 4.3.1 We first establish the existence of the solution on $[0, T_1]$, where $T_1 \in (0, \delta)$ as defined in Assumption 4.2.2. Define that for $n = 0, 1, 2, \ldots$, $X^n_{1,0} = \psi$ and $X^0_1(t) = \psi(0)$ for $0 \leq t \leq T_1$. Define the Picard Iteration, for $n \in \mathbb{N}$, $t \in [0, T_1]$,

$$X^n_1(t) - D(X^{n-1}_{1,t}) = \psi(0) - D(\psi) + \int_0^t f(X^{n-1}_{1,s})\, ds + \int_0^t g(X^{n-1}_{1,s})\, dB(s). \tag{4.5.5}$$

Hence

$$X^1_1(t) - X^0_1(t) = D(X^0_{1,t}) - D(\psi) + \int_0^t f(X^0_{1,s})\, ds + \int_0^t g(X^0_{1,s})\, dB(s).$$

By Assumption 4.2.3,

$$
\begin{aligned}
|X_1^1(t) - X_1^0(t)|^2 &\le \frac{1}{\alpha}|D(X_{1,t}^0) - D(\psi)|^2 + \frac{1}{1-\alpha}|I(t)|^2 \\
&\le \frac{1}{\alpha}\Bigg(K_D(1 + \sup_{-\tau \le s \le -T_1} |X_1^0(t+s)|) \\
&\qquad\qquad + k \sup_{-T_1 \le s \le 0} |X_1^0(t+s)| + |D(\psi)| \Bigg)^2 + \frac{1}{1-\alpha}|I(t)|^2
\end{aligned}
$$

where

$$
I(t) := \int_0^t f(X_{1,s}^0)\, ds + \int_0^t g(X_{1,s}^0)\, dB(s).
$$

It follows that

$$
\begin{aligned}
&\sup_{0 \le t \le T_1} |X_1^1(t) - X_1^0(t)|^2 \\
&\le \frac{1}{\alpha}\Bigg(K_D(1 + \sup_{-\tau \le s \le 0} |\psi(s)|) + k \sup_{-T_1 \le s \le T_1} |X_1^0(s)| \\
&\qquad\qquad + |D(\psi)| \Bigg)^2 + \frac{1}{1-\alpha} \sup_{0 \le s \le T_1} |I(t)|^2 \\
&= \frac{1}{\alpha}\Bigg(K_D + (K_D + k) \sup_{-\tau \le s \le 0} |\psi(s)| + |D(\psi)| \Bigg)^2 + \frac{1}{1-\alpha} \sup_{0 \le s \le T_1} |I(t)|^2
\end{aligned}
$$

By Assumption 4.2.1, it can be shown that

$$
\mathbb{E}\Bigg[\sup_{0 \le t \le T_1} |I(t)|^2 \Bigg] \le 2\bar{K}T_1(T_1 + 4)(\sup_{-T_1 \le s \le 0} |\psi(s)|^2 + 1).
$$

This implies that

$$
\begin{aligned}
\mathbb{E}\Bigg[\sup_{0 \le t \le T_1} |X_1^1(t) - X_1^0(t)|^2 \Bigg] &\le \frac{1}{\alpha}\Bigg(K_D + (K_D + k) \sup_{-\tau \le s \le 0} |\psi(s)| + |D(\psi)| \Bigg)^2 \\
&\qquad + \frac{2\bar{K}T_1(T_1 + 4)}{1-\alpha}(\sup_{-T_1 \le s \le 0} |\psi(s)|^2 + 1|) =: C. \quad (4.5.6)
\end{aligned}
$$

Now for all $n \in \mathbb{N}$ and $0 \le t \le T_1 < \delta$ (δ is defined in Assumption 4.2.2), follow the same argument as in the proof of the uniqueness, we have $D_0(X_{1,t}^n) - D_0(X_{1,t}^{n-1}) = 0$. Therefore

$$
\begin{aligned}
X_1^{n+1}(t) - X_1^n(t) &= D_1(X_{1,t}^n) - D_1(X_{1,t}^{n-1}) \\
&\quad + \int_0^t \big(f(X_{1,s}^n) - f(X_{1,s}^{n-1})\big)\, ds + \int_0^t \big(g(X_{1,s}^n) - g(X_{1,s}^{n-1})\big)\, dB(s).
\end{aligned}
$$

Again by (4.2.10), we have

$$
\begin{aligned}
&|D_1(X_{1,t}^n) - D_1(X_{1,t}^{n-1})| \\
&\le k\|X_{1,t}^n - X_{1,t}^{n-1}\|_{\sup} \\
&= k \max\{\sup_{-\tau\le s\le -T_1} |X_1^n(t+s) - X_1^{n-1}(t+s)|, \\
&\qquad\qquad \sup_{-T_1\le s\le 0} |X_1^n(t+s) - X_1^{n-1}(t+s)|\} \\
&= k \sup_{-T_1\le s\le 0} |X_1^n(t+s) - X_1^{n-1}(t+s)| \\
&= k \sup_{0\le s\le t} |X_1^n(s) - X_1^{n-1}(s)|.
\end{aligned}
$$

Apply the same analysis as in the proof of the uniqueness, we get

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le T_1} |X_1^{n+1}(t) - X_1^n(t)|^2\Big] \qquad (4.5.7) \\
&\le \frac{k^2}{\alpha}\mathbb{E}\Big[\sup_{0\le t\le T_1} |X_1^n(t) - X_1^{n-1}(t)|^2\Big] \\
&\qquad + \frac{2K(T_1+4)}{1-\alpha}\int_0^{T_1} \mathbb{E}\Big[\sup_{0\le s\le t} |X_1^n(s) - X_1^{n-1}(s)|^2\Big]\, dt \\
&\le \Big(\frac{k^2}{\alpha} + \frac{2KT_1(T_1+4)}{1-\alpha}\Big)\mathbb{E}\Big[\sup_{0\le t\le T_1} |X_1^n(s) - X_1^{n-1}(s)|^2\Big].
\end{aligned}
$$

Now let

$$
\gamma := \frac{k^2}{\alpha} + \frac{2KT_1(T_1+4)}{1-\alpha}.
$$

We show that there exist such T_1 and α so that $\gamma < 1$. Fix $0 < \mu < 1$. Choose T_1 such that $k = \rho_0(T_1) < \mu$ and $2KT_1(T_1+4) < (1-\mu^2)^2/[2(1+\mu^2)]$. Let $\alpha = (1/2)\mu^2 + (1/2)$, then $k^2 < \mu^2 < \alpha < 1$, which implies $\gamma < 1$. Combining (4.5.7) with (4.5.6), we have

$$
\mathbb{E}\Big[\sup_{0\le t\le T_1} |X_1^{n+1}(t) - X_1^n(t)|^2\Big] \le \gamma^n C. \qquad (4.5.8)
$$

Choose $\epsilon > 0$, so that $(1+\epsilon)\gamma < 1$. Hence by Chebyshev's inequality,

$$
\mathbb{P}\Big\{\sup_{0\le t\le T_1} |X_1^{n+1}(t) - X_1^n(t)| > \frac{1}{(1+\epsilon)^n}\Big\} \le (1+\epsilon)^{2n}\gamma^n C.
$$

Since $\sum_{n=0}^{\infty}(1+\epsilon)^{2n}\gamma^n C < \infty$, by Borel-Cantelli lemma, for almost all $\omega \in \Omega$, there exists $n_0 = n_0(\omega) \in \mathbb{N}$ such that

$$
\sup_{0\le t\le T_1} |X_1^{n+1}(t) - X_1^n(t)| \le \frac{1}{(1+\epsilon)^{2n}}, \quad \text{for } n > n_0.
$$

This implies that

$$X_1^n(t) = X_1^0(t) + \sum_{i=0}^{n-1}[X_1^{i+1}(t) - X_1^i(t)],$$

converge uniformly on $t \in [0, T_1]$ a.s. Let the limit be $X_1(t)$ for $t \in [0, T_1]$ which is continuous and $\mathcal{F}(t)$-adapted. Moreover, by (4.5.8), $\{X_1^n(t)\}_{n\in\mathbb{N}} \to X_1(t)$ in L^2 on $t \in [0, T_1]$. By Lemma 4.4.1, $X_1(\cdot) \in \mathcal{M}^2([-\tau, T_1]; \mathbb{R}^d)$. Note that

$$\begin{aligned}
\mathbb{E}\left[\left|\int_0^t f(X_{1,s}^n)\,ds - \int_0^t f(X_{1,s})\,ds\right|^2\right] &\le \mathbb{E}\left[\left(\int_0^t |f(X_{1,s}^n) - f(X_{1,s})|\,ds\right)^2\right]\\
&\le \mathbb{E}\left[\left(\int_0^t K\|X_{1,s}^n - X_{1,s}\|_{\sup}\,ds\right)^2\right]\\
&\le K^2T_1^2\int_0^{T_1}\mathbb{E}[\|X_{1,s}^n - X_{1,s}\|_{\sup}^2]\,ds,\\
&\to 0, \quad \text{as } n \to \infty,
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t g(X_{1,s}^n)\,dB(s) - \int_0^t g(X_{1,s})\,dB(s)\right|^2\right]\\
&\qquad = \mathbb{E}\left[\left|\int_0^t \Big(g(X_{1,s}^n) - g(X_{1,s})\Big)\,dB(s)\right|^2\right]\\
&\qquad = \mathbb{E}\left[\int_0^t \left|g(X_{1,s}^n) - g(X_{1,s})\right|^2 ds\right]\\
&\qquad \le K^2\int_0^{T_1}\mathbb{E}[\|X_{1,s}^n - X_{1,s}\|_{\sup}^2]\,ds\\
&\qquad \to 0, \quad \text{as } n \to \infty,
\end{aligned}$$

and

$$\mathbb{E}[|D(X_{1,t}^n) - D(X_{1,t})|] \le k\mathbb{E}[\|X_{1,t}^n - X_{1,t}\|] \to 0, \quad \text{as } n \to \infty.$$

Hence let $n \to \infty$ in (4.5.5), almost surely that

$$X_1(t) = \psi(0) - D(\psi) + D(X_{1,t}) + \int_0^t f(X_{1,s})\,ds + \int_0^t g(X_{1,s})\,dB(s).$$

Therefore $\{X_1(t)\}_{t\in[0,T_1]}$ is the solution on $[0, T_1]$ on an almost sure event Ω_{T_1}. We now prove the existence of the solution on the interval $[T_1, 2T_1]$. Define $X_{2,T_1}^n = X_{1,T_1}$ for $n = 0, 1, 2...$, and $X_2^0(t) = X_1(T_1)$ for $t \in [T_1, 2T_1]$. Define the Picard Iteration, for $n \in \mathbb{N}$,

$$X_2^n(t) - D(X_{2,t}^{n-1}) = X_1(T_1) - D(X_{1,T_1}) + \int_{T_1}^t f(X_{2,s}^{n-1})\,ds + \int_{T_1}^t g(X_{2,s}^{n-1})\,dB(s).$$

Following the same argument as in the case of $t \in [0, T_1]$, it can be shown that there exists continuous $\{X_2(t)\}_{t\in[T_1,2T_1]}$ such that $X_2^n(t) \to X_2(t)$ in L^2 for $t \in [T_1, 2T_1]$ almost surely. Moreover, $X_2(\cdot) \in \mathcal{M}^2([T_1, 2T_1]; \mathbb{R}^d)$, and $X_2(\cdot)$ almost surely satisfies the equation

$$X_2(t) = X_1(T_1) - D(X_{1,T_1}) + D(X_{2,t}) + \int_{T_1}^{t} f(X_{2,s})\, ds + \int_{T_1}^{t} g(X_{2,s})\, dB(s).$$

Therefore $\{X_2(t)\}_{t\in[T_1,2T_1]}$ is the solution on $[T_1, 2T_1]$ on an almost sure event Ω_{2T_1}. Let $X(t) := \{X_n(t) \cdot I_{\{t\in[nT_1,(n+1)T_1]\}}\}_{n\in\mathbb{N}\cup\{0\}}$, then $X(\cdot)$ is the solution of (4.1.1) on the entire interval $[0, T]$ which is in $\mathcal{M}^2([0, T]; \mathbb{R})$.

For the *uniqueness*, consider $t \in [0, T_1]$, suppose that both X and Y are solutions to (4.1.1), with initial solution $X(t) = Y(t) = \psi(t)$ for $t \in [-\tau, 0]$. Then

$$\begin{aligned} X(t) - Y(t) = D_0(X_t) - D_0(Y_t) + D_1(X_t) - D_1(Y_t) &+ \int_0^t \big(f(X_s) - f(Y_s)\big)\, ds \\ &+ \int_0^t \big(g(X_s) - g(Y_s)\big)\, dB(s). \end{aligned}$$

Let $s \in [-\tau, -\delta]$, by (4.2.12), we have $t + s \leq T_1 - \delta < 0$, and so $X(t + s) = Y(t + s) = \psi(t + s)$. Then $|D_0(X_t) - D_0(Y_t)| = 0$. Hence

$$\begin{aligned} |X(t) - Y(t)| \leq{}& |D_1(X_t) - D_1(Y_t)| \\ &+ \left| \int_0^t \big(f(X_s) - f(Y_s)\big)\, ds + \int_0^t \big(g(X_s) - g(Y_s)\big)\, dB(s) \right|. \end{aligned}$$

Let $k^2 < \alpha < 1$, where k is given by (4.2.8). Then we get

$$|X(t) - Y(t)|^2 \leq \frac{1}{\alpha}|D_1(X_t) - D_1(Y_t)|^2 + \frac{1}{1-\alpha}|J(t)|^2,$$

where we have used the inequality (cf. [57, Lemma 6.2.3])

$$(a + b)^2 \leq \frac{1}{\alpha}a^2 + \frac{1}{1-\alpha}b^2, \quad 0 < \alpha < 1. \tag{4.5.9}$$

and define

$$J(t) := \int_0^t \big(f(X_s) - f(Y_s)\big)\, ds + \int_0^t \big(g(X_s) - g(Y_s)\big)\, dB(s).$$

Now by (4.2.10), since $0 \le t \le T_1$,

$$
\begin{aligned}
&|D_1(X_t) - D_1(Y_t)| \\
&\le k\|X_t - Y_t\|_{\sup} \\
&= k\{\sup_{-\tau \le s \le -T_1} |X(t+s) - Y(t+s)|, \sup_{-T_1 \le s \le 0} |X(t+s) - Y(t+s)|\} \\
&= k \sup_{-T_1 \le s \le 0} |X(t+s) - Y(t+s)|.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
|X(t) - Y(t)|^2 &\le \frac{k^2}{\alpha} \sup_{-T_1 \le s \le 0} |X(t+s) - Y(t+s)|^2 + \frac{1}{1-\alpha}|J(t)|^2 \\
&= \frac{k^2}{\alpha} \sup_{0 \le s \le t} |X(s) - Y(s)|^2 + \frac{1}{1-\alpha}|J(t)|^2.
\end{aligned}
$$

Moreover,

$$
\sup_{0 \le s \le t} |X(s) - Y(s)|^2 \le \frac{k^2}{\alpha} \sup_{0 \le s \le t} |X(s) - Y(s)|^2 + \frac{1}{1-\alpha} \sup_{0 \le s \le t} |J(t)|^2.
$$

Since α has been chosen such that $0 < k^2 < \alpha < 1$, it follows that

$$
\sup_{0 \le s \le t} |X(s) - Y(s)|^2 \le \frac{1}{(1-\alpha)(1-\frac{k^2}{\alpha})} \sup_{0 \le s \le t} |J(t)|^2.
$$

Now, by (4.2.1) and similar argument as in the proof of Lemma 4.4.1, it is easy to show that

$$
\mathbb{E}\left[\sup_{0 \le s \le t} |J(t)|^2\right] \le 2K(T_1+4) \int_0^t \mathbb{E}\left[\sup_{0 \le u \le s} |X(u) - Y(u)|^2\right] ds.
$$

It follows that

$$
\mathbb{E}\left[\sup_{0 \le s \le t} |X(s) - Y(s)|^2\right] \le \frac{2K(T_1+4)}{(1-\alpha)(1-\frac{k^2}{\alpha})} \int_0^t \mathbb{E}\left[\sup_{0 \le u \le s} |X(u) - Y(u)|^2\right] ds.
$$

Using Gronwall's inequality, we have that

$$
\forall\, 0 \le t \le T_1, \quad \mathbb{E}\left[\sup_{0 \le s \le t} |X(s) - Y(s)|^2\right] = 0,
$$

which implies that

$$
\mathbb{E}\left[\sup_{0 \le t \le T_1} |X(t) - Y(t)|^2\right] = 0.
$$

Therefore we can conclude that on an a.s. event Ω_{T_1}, for all $0 \le t \le T_1$, $X(t) = Y(t)$ a.s. Apply the same argument on the interval $[T_1, 2T_1]$ given $X(t) = Y(t)$ on $[-\tau, T_1]$ a.s., it can be shown that $X(t) = Y(t)$ on the entire interval $[-\tau, T]$ a.s.

Proof of Theorem 4.3.2 Let $Y(t) := X(t) - D(X_t)$, then by the inequality (4.5.1), we have

$$|X(t)|^p \le (1+\varepsilon^{\frac{1}{p-1}})^{p-1}(|Y(t)|^p + \frac{1}{\varepsilon}|D(X_t)|^p). \tag{4.5.10}$$

By Itô's formula,

$$\begin{aligned}|Y(t)|^p = |\psi(0) - D(\psi)|^p + \int_0^t \bigg(p|Y(s)|^{p-2}Y^T(s)f(X_s) \\ + \frac{p(p-1)}{2}|Y(s)|^{p-2}||g(X_s)||^2 \bigg) ds + \int_0^t p|Y(s)|^{p-2}Y^T(s)g(X_s)\, dB(s).\end{aligned}$$

Hence if

$$\mathbb{E}\bigg[\int_0^t |Y(s)|^{2p-2}||g(X_s)||^2\, ds\bigg] < \infty, \tag{4.5.11}$$

we get

$$\begin{aligned}\mathbb{E}[|Y(t)|^p] = |\psi(0) - D(\psi)|^p + \mathbb{E}\bigg[\int_0^t \bigg(p|Y(s)|^{p-2}Y^T(s)f(X_s) \\ + \frac{p(p-1)}{2}|Y(s)|^{p-2}||g(X_s)||^2 \bigg) ds\bigg].\end{aligned}$$

We assume (4.5.11) holds at the moment, and will show that it is true at the end of this proof. Define $x(t) := \mathbb{E}[|X(t)|^p]$, and $y(t) := \mathbb{E}[|Y(t)|^p]$. Then

$$\begin{aligned}y(t+h) - y(t) &= \int_t^{t+h} \mathbb{E}\big[p|Y(s)|^{p-2}Y^T(s)f(X_s) + \frac{p(p-1)}{2}|Y(s)|^{p-2}||g(X_s)||^2\big]\, ds \\ &\le \int_t^{t+h} \mathbb{E}\big[p|Y(s)|^{p-1}|f(X_s)| + \frac{p(p-1)}{2}|Y(s)|^{p-2}||g(X_s)||^2\big]\, ds \\ &\le \int_t^{t+h} \bigg\{ p\mathbb{E}\bigg[\frac{\varepsilon(p-1)}{p}|Y(s)|^p + \frac{|f(X_s)|^p}{p\varepsilon^{p-1}}\bigg] \\ &\qquad + \frac{p(p-1)}{2}\mathbb{E}\bigg[\frac{\varepsilon(p-2)}{p}|Y(s)|^p + \frac{2||g(X_s)||^p}{p\varepsilon^{(p-2)/2}}\bigg]\bigg\}\, ds \\ &= \int_t^{t+h} \bigg\{ \frac{\varepsilon p(p-1)}{2}y(s) + \frac{1}{\varepsilon^{p-1}}\mathbb{E}[|f(X_s)|^p] + \frac{p-1}{\varepsilon^{(p-2)/2}}\mathbb{E}[||g(X_s)||^p]\bigg\}\, ds \\ &\le \int_t^{t+h} \bigg\{ \frac{\varepsilon p(p-1)}{2}y(s) + \frac{1}{\varepsilon^{p-1}}\mathbb{E}\bigg[C_f + \int_{[-\tau,0]} \nu(du)|X(u+s)|^p\bigg] \\ &\qquad + \frac{p-1}{\varepsilon^{(p-2)/2}}\mathbb{E}\bigg[C_g + \int_{[-\tau,0]} \eta(du)|X(u+s)|^p\bigg]\bigg\}\, ds,\end{aligned}$$

where we have used the inequalities (cf. [57, Lemma 6.2.4])

$$\forall\, p \ge 2, \text{ and } \varepsilon, a, b > 0, \quad a^{p-1}b \le \frac{\varepsilon(p-1)a^p}{p} + \frac{b^p}{p\varepsilon^{p-1}}$$

and

$$a^{p-2}b^2 \le \frac{\varepsilon(p-2)a^p}{p} + \frac{2b^p}{p\varepsilon^{(p-2)/2}},$$

in the second inequality, conditions (4.3.6) and (4.3.7) in the last inequality. By the continuity of $t \mapsto \mathbb{E}[|X(t)|^p]$ and $t \mapsto \mathbb{E}[|Y(t)|^p]$, it is then easy to see that

$$D^+y(t) \le \frac{\varepsilon p(p-1)}{2} y(t) + \frac{C_f}{\varepsilon^{p-1}} + \frac{C_g(p-1)}{\varepsilon^{\frac{p-2}{2}}} + \int_{[-\tau,0]} \lambda(ds)x(t+s),$$

where

$$\lambda(ds) := \nu(ds) \cdot \frac{1}{\varepsilon^{p-1}} + \eta(ds) \cdot \frac{p-1}{\varepsilon^{\frac{p-2}{2}}}. \tag{4.5.12}$$

Hence

$$y(t) \le e^{\beta_1 t}y(0) + \int_0^t e^{\beta_1(t-u)}\left(\beta_2 + \beta_3 + \int_{[-\tau,0]} \lambda(ds)x(u+s)\right) du, \tag{4.5.13}$$

where

$$\beta_1 := \frac{\varepsilon p(p-1)}{2}, \quad \beta_2 := \frac{C_f}{\varepsilon^{p-1}}, \quad \beta_3 := \frac{C_g(p-1)}{\varepsilon^{\frac{p-2}{2}}}.$$

Now since

$$|X(t)| \le |X(t) - D(X_t)| + |D(X_t)|,$$

again by (4.5.1),

$$|X(t)|^p \le (1+\varepsilon^{\frac{1}{p-1}})^{p-1}(\frac{1}{\varepsilon}|D(X_t)|^p + |X(t) - D(X_t)|^p),$$

it follows that

$$\begin{aligned} x(t) &\le (1+\varepsilon^{\frac{1}{p-1}})^{p-1}\left(\frac{1}{\varepsilon}\mathbb{E}[|D(X_t)|^p] + y(t)\right) \\ &\le \frac{(1+\varepsilon^{\frac{1}{p-1}})^{p-1}}{\varepsilon} C_D + \frac{(1+\varepsilon^{\frac{1}{p-1}})^{p-1}}{\varepsilon} \int_{[-\tau,0]} \mu(ds)x(t+s) \\ &\qquad + (1+\varepsilon^{\frac{1}{p-1}})^{p-1} y(t), \end{aligned}$$

Combining the above inequality with (4.5.13), we get

$$\begin{aligned} x(t) \le{}& (1+\varepsilon^{\frac{1}{p-1}})^{p-1} e^{\beta_1 t} y(0) + \beta_4 C_D + \beta_4 \int_{[-\tau,0]} \mu(ds)x(t+s) \\ &+ (1+\varepsilon^{\frac{1}{p-1}})^{p-1} \int_0^t e^{\beta_1(t-u)}\left(\beta_2 + \beta_3 + \int_{[-\tau,0]} \lambda(ds)x(u+s)\right) du, \end{aligned}$$

where $\beta_4 := (1 + \varepsilon^{1/(p-1)})^{p-1}/\varepsilon$. Let $x_e(t) = e^{-\beta_1 t}x(t)$ for $t \geq -\tau$. Since $e^{-\beta_1 t} \leq 1$ for $t \geq 0$, then

$$\begin{aligned} x_e(t) &\leq (1 + \varepsilon^{\frac{1}{p-1}})^{p-1}y(0) + \beta_4 C_D e^{-\beta_1 t} + \beta_5(1 - e^{-\beta_1 t}) \\ &\qquad + \beta_4 \int_{[-\tau,0]} \mu(ds)e^{-\beta_1 t}x(t+s) \\ &\qquad\qquad + (1 + \varepsilon^{\frac{1}{p-1}})^{p-1} \int_0^t e^{-\beta_1 u} \int_{[-\tau,0]} \lambda(ds)x(u+s)\, du \\ &\leq \left[(1 + \varepsilon^{\frac{1}{p-1}})^{p-1}y(0) + \beta_4 C_D + \beta_5\right] + \beta_4 \int_{[-\tau,0]} e^{\beta_1 s}\mu(ds)x_e(t+s) \\ &\qquad + \int_0^t \int_{[-\tau,0]} e^{\beta_1 s}\lambda(ds)x_e(u+s)\, du, \end{aligned}$$

where

$$\beta_5 := \frac{1}{\beta_1}(1 + \varepsilon^{\frac{1}{p-1}})^{p-1}(\beta_2 + \beta_3).$$

Let $\beta_6 := (1 + \varepsilon^{\frac{1}{p-1}})^{p-1}y(0) + \beta_4 C_D + \beta_5$, $\mu_e(ds) := e^{\beta_1 s}\mu(ds)$ and $\lambda_e(ds) := e^{\beta_1 s}\lambda(ds)$, thus

$$x_e(t) \leq \beta_6 + \beta_4 \int_{[-\tau,0]} \mu_e(ds)x_e(t+s) + \int_0^t \int_{[-\tau,0]} \lambda_e(ds)x_e(u+s)\, du.$$

Now let $\mu(E) = \lambda(E) = 0$ for $E \subset (-\infty, -\tau)$, so $\mu_e(E) = \lambda_e(E) = 0$ for $E \subset (-\infty, -\tau)$. Define $\mu_e^+(E) := \mu_e(-E)$ and $\lambda_e^+(E) := \lambda_e(-E)$ for $E \subset [0, \infty)$. Hence

$$\begin{aligned} \int_{[-\tau,0]} \mu_e(ds)x_e(t+s) &= \int_{(-\infty,0]} \mu_e(ds)x_e(t+s) \\ &= \int_{[0,\infty)} \mu_e^+(ds)x_e(t-s) \\ &= \int_{[0,t]} \mu_e^+(ds)x_e(t-s) + \int_{(t,\infty)} \mu_e^+(ds)x_e(t-s) \\ &= \int_{[0,t]} \mu_e^+(ds)x_e(t-s) + \int_{(t,t+\tau]} \mu_e^+(ds)\psi_e(t-s), \end{aligned}$$

where $\psi_e(t) := e^{-\beta_1 t}|\psi(t)|^p$ and ψ is the initial condition for X on $[-\tau, 0]$. Similarly,

$$\int_{[-\tau,0]} \lambda_e(ds)x_e(u+s) = \int_{[0,t]} \lambda_e^+(ds)x_e(u-s) + \int_{(t,t+\tau]} \lambda_e^+(ds)\psi_e(u-s).$$

Consequently,

$$\begin{aligned} x_e(t) &\leq \beta_6 + \beta_4 \int_{[0,t]} \mu_e^+(ds)x_e(t-s) + \beta_4 \int_{(t,t+\tau]} \mu_e^+(ds)\psi_e(t-s) \\ &\qquad + \int_0^t \int_{[0,u]} \lambda_e^+(ds)x_e(u-s)\, du + \int_0^t \int_{(u,u+\tau]} \lambda_e^+(ds)\psi_e(u-s)\, du. \qquad (4.5.14) \end{aligned}$$

Let $\Lambda_e^+(t) := \int_{[0,t]} \lambda_e^+(ds)$. By Fubini's theorem and the integration-by-parts formula,

$$\begin{aligned}\int_0^t \int_{[0,u]} \lambda_e^+(ds) x_e(u-s)\,du &= \int_{s=0}^t \lambda_e^+(ds) \int_{u=s}^t x_e(u-s)\,du \\ &= \int_{s=0}^t \lambda_e^+(ds) \int_{v=0}^{t-s} x_e(v)\,dv \\ &= \int_0^{t-s} x_e^+(v)\,dv \cdot \Lambda_e^+(s)\Big|_{s=0}^t + \int_0^t \Lambda_e^+(s) x_e(t-s)\,ds \\ &= \int_0^t \Lambda_e^+(s) x_e(t-s)\,ds.\end{aligned} \tag{4.5.15}$$

Also

$$\begin{aligned}&\int_0^t \int_{(u,u+\tau]} \lambda_e^+(ds)\psi_e(u-s)\,du \\ &= \int_{[0,t+\tau]} \lambda_e^+(ds) \int_{(s-\tau)\vee 0}^{s\wedge t} \psi_e(u-s)\,du \\ &= \int_{[0,t]} \lambda_e^+(ds) \int_{(s-\tau)\vee 0}^{s\wedge t} \psi_e(u-s)\,du + \int_{(t,t+\tau]} \lambda_e^+(ds) \int_{(s-\tau)\vee 0}^{s\wedge t} \psi_e(u-s)\,du \\ &= \int_{[0,t]} \lambda_e^+(ds) \int_{(s-\tau)\vee 0}^{s} \psi_e(u-s)\,du + \int_{(t,t+\tau]} \lambda_e^+(ds) \int_{(s-\tau)\vee 0}^{t} \psi_e(u-s)\,du.\end{aligned} \tag{4.5.16}$$

Now, if $t \geq \tau$, the second integral in (4.5.16) is zero; if $0 \leq t < \tau$, then

$$\begin{aligned}\int_{(t,t+\tau]} \lambda_e^+(ds) \int_{(s-\tau)\vee 0}^{t} \psi_e(u-s)\,du &= \int_{(t,\tau]} \lambda_e^+(ds) \int_{(s-\tau)\vee 0}^{t} \psi_e(u-s)\,du \\ &= \int_{(t,\tau]} \lambda_e^+(ds) \int_0^t \psi_e(u-s)\,du \\ &= \int_{(t,\tau]} \lambda_e^+(ds) \int_{-s}^{t-s} \psi_e(v)\,dv \\ &\leq \int_{(t,\tau]} \lambda_e^+(ds) \tau ||\psi_e||_{\sup} \\ &\leq \tau ||\psi_e||_{\sup} \int_{[0,\tau]} \lambda_e^+(ds).\end{aligned} \tag{4.5.17}$$

For the first integral in (4.5.16),

$$\begin{aligned}\int_{[0,t]} \lambda_e^+(ds) \int_{(s-\tau)\vee 0}^{s} \psi_e(u-s)\,du &= \int_{[0,\tau]} \lambda_e^+(ds) \int_{(s-\tau)\vee 0}^{s} \psi_e(u-s)\,du \\ &= \int_{[0,\tau]} \lambda_e^+(ds) \int_0^s \psi_e(u-s)\,du \\ &= \int_{[0,\tau]} \lambda_e^+(ds) \int_{-s}^0 \psi_e(v)\,dv \\ &\leq \int_{[0,\tau]} \lambda_e^+(ds) \tau ||\psi_e||_{\sup} \\ &\leq \tau ||\psi_e||_{\sup} \int_{[0,\tau]} \lambda_e^+(ds).\end{aligned} \tag{4.5.18}$$

Inserting (4.5.17) and (4.5.18) into (4.5.16), we have

$$\int_0^t \int_{(u,u+\tau]} \lambda_e^+(ds)\psi_e(u-s)\,du \leq 2\tau||\psi_e||_{\sup} \int_{[0,\tau]} \lambda_e^+(ds). \tag{4.5.19}$$

Moreover, if $t \geq \tau$, then

$$\beta_4 \int_{(t,t+\tau]} \mu_e^+(ds)\psi_e(t-s) = 0;$$

if $0 \leq t < \tau$, then

$$\begin{aligned} \beta_4 \int_{(t,t+\tau]} \mu_e^+(ds)\psi_e(t-s) &= \beta_4 \int_{(t,\tau]} \mu_e^+(ds)\psi_e(t-s) \\ &\leq \beta_4 \int_{(t,\tau]} \mu_e^+(ds)||\psi_e||_{\sup} \\ &\leq \beta_4 ||\psi_e||_{\sup} \int_{[0,\tau]} \mu_e^+(ds). \end{aligned} \tag{4.5.20}$$

Therefore combining (4.5.15), (4.5.19) and (4.5.20) with (4.5.14), we have

$$x_e(t) \leq \beta_7 + \int_{[0,t]} \Big(\beta_4\mu_e^+(ds) + \Lambda_e^+(s)\,ds\Big) x_e(t-s), \quad t \geq 0. \tag{4.5.21}$$

where

$$\beta_7 := \beta_6 + \Big(\beta_4 \int_{[0,\tau]} \mu_e^+(ds) + 2\tau \int_{[0,\tau]} \lambda_e^+(ds)\Big)||\psi_e||_{\sup}.$$

Choose small $\rho > 0$ and define

$$z(t) := \beta_7 + \int_{[0,t]} \Big(\beta_4\mu_e^+(ds) + \Lambda_e^+(s)\,ds + \rho\,ds\Big) z(t-s), \quad t \geq 0.$$

Then by Lemma 4.4.4, we get $z(t) \geq x_e(t)$ for $t \geq 0$.

Next we determine the asymptotic behaviour of z. Note that the measure

$$\alpha(ds) := \beta_4\mu_e^+(ds) + \Lambda_e^+(s)\,ds + \rho\,ds \tag{4.5.22}$$

has an absolutely continuous component. Moreover α is a positive measure. Also, we can find a number $\theta > 0$ such that $\int_{[0,\infty)} e^{-\theta s}\alpha(ds) = 1$. Now, define the measure $\alpha_\theta \in M([0,\infty);\mathbb{R})$ by $\alpha_\theta(ds) = e^{-\theta s}\alpha(ds)$. Then α_θ is a positive measure with a nontrivial absolutely continuous component such that $\alpha_\theta(\mathbb{R}^+) = 1$. Also, we have that

$$\begin{aligned} \int_{[0,\infty)} s\alpha_\theta(ds) &= \int_{[0,\infty)} se^{-\theta s}\alpha(ds) \\ &= \int_{[0,\infty)} se^{-\theta s}(\beta_4\mu_e^+(ds) + \Lambda_e^+(s)\,ds + \rho\,ds) \\ &= \beta_4 \int_{[0,\tau]} se^{-\theta s}\mu_e^+(ds) + \int_{[0,\infty)} se^{-\theta s}\Lambda_e^+(s)\,ds + \rho \int_{[0,\infty)} se^{-\theta s}\,ds, \end{aligned}$$

since $\mu_e^+(E) = 0$ for all $E \subset (\tau, \infty)$. Now, we note that because $\Lambda_e^+(t) \leq \Lambda_e^+(\infty) = \int_{[0,\tau]} \lambda_e^+(ds) < +\infty$ for all $t \geq 0$, the second integral on the righthand side is finite, and therefore we have that $\int_{[0,\infty)} t\alpha_\theta(dt) < +\infty$. Next define $z_\theta(t) := e^{-\theta t} z(t)$ for $t \geq 0$ so that

$$z_\theta(t) = \beta_7 e^{-\theta t} + \int_{[0,t]} \alpha_\theta(ds) z_\theta(t-s), \quad t \geq 0.$$

Now, define $-\gamma$ to be the resolvent of $-\alpha_\theta$. Then, by the renewal theorem (cf. [39, Theorem 7.4.1]), the existence of γ is guaranteed. Moreover, γ is a positive measure and is of the form

$$\gamma(dt) = \gamma_1(dt) + \gamma_1([0,t])\, dt$$

where $\gamma_1 \in M(\mathbb{R}^+;\mathbb{R})$ and $\gamma_1(\mathbb{R}^+) = 1/\int_{\mathbb{R}^+} t\alpha_\theta(dt)$, which is finite. Since $(-\gamma) + (-\alpha_\theta) * (-\gamma) = -\alpha_\theta$, let $h(t) := \beta_7 e^{-\theta t}$, we have

$$\begin{aligned} z_\theta = h + \alpha_\theta * z_\theta &= h + \gamma * z_\theta - \alpha_\theta * \gamma * z_\theta \\ &= h + \gamma * (z_\theta - \alpha_\theta * z_\theta) \\ &= h + \gamma * h, \end{aligned}$$

that is

$$\begin{aligned} z_\theta(t) &= \beta_7 e^{-\theta t} + \beta_7 \int_{[0,t]} \gamma(ds) e^{-\theta(t-s)} \\ &= \beta_7 e^{-\theta t} + \beta_7 \int_{[0,t]} \Big(\gamma_1(ds) + \gamma_1([0,s])ds\Big) e^{-\theta(t-s)}. \end{aligned}$$

Thus

$$\limsup_{t\to\infty} \frac{x_e(t)}{e^{\theta t}} \leq \limsup_{t\to\infty} \frac{z(t)}{e^{\theta t}} = \limsup_{t\to\infty} z_\theta(t) \leq \frac{\beta_7}{\int_{\mathbb{R}^+} t\alpha_\theta(dt)} + \frac{\beta_7}{\theta \int_{\mathbb{R}^+} t\alpha_\theta(dt)}.$$

Hence there exists $C > 0$ such that $x_e(t) \leq Ce^{\theta t}$ for $t \geq 0$. Therefore $\mathbb{E}[|X(t)|^p] = x(t) = e^{\beta_1 t} x_e(t) \leq Ce^{(\theta+\beta_1)t}$ for $t \geq 0$, which implies

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}[|X(t)|^p] \leq \theta + \beta_1.$$

Now in (4.5.22), let $\rho \to 0$, then $\theta \to \theta_*$, where

$$\int_{[0,\infty)} e^{-\theta_* s} \alpha(ds) = \int_{[0,\infty]} e^{-\theta_* s} \Big(\mu_e^+(ds) + \Lambda_e^+(s)\, ds\Big) = 1. \tag{4.5.23}$$

Note

$$\int_0^\infty e^{-\theta_* s}\mu_e^+(ds) \;=\; \int_0^\tau e^{-\theta_* s}\mu_e^+(ds) \;=\; \int_{-\tau}^0 e^{-\theta_* s}\mu_e(ds) \;=\; \int_{-\tau}^0 e^{(-\theta_*+\beta_1)s}\mu(ds),$$

and

$$\begin{aligned}\int_0^\infty e^{-\theta_* s}\Lambda_e^+(s)\,ds &= \int_0^\tau e^{-\theta_* s}\Lambda_e^+(s)\,ds + \int_\tau^\infty e^{-\theta_* s}\int_{[0,\tau]}\lambda_e^+(du)\,ds\\ &= \int_0^\tau e^{-\theta_* s}\int_{[0,s]}\lambda_e^+(du)\,ds + \frac{e^{-\theta_*\tau}}{\theta_*}\int_{[0,\tau]}\lambda_e^+(du)\\ &= \int_0^\tau e^{-\theta_* s}\int_{[-s,0]}\lambda_e(du)\,ds + \frac{e^{-\theta_*\tau}}{\theta_*}\int_{[-\tau,0]}\lambda_e(du)\\ &= \int_0^\tau e^{-\theta_* s}\int_{[-s,0]}e^{\beta_1 u}\lambda(du)\,ds + \frac{e^{-\theta_*\tau}}{\theta_*}\int_{[-\tau,0]}e^{\beta_1 u}\lambda(du).\end{aligned}$$

where λ is defined in (4.5.12). Replace θ_* by θ, we get the desired result.

Finally, we show that (4.5.11) holds for $t \geq 0$. By Hölder's inequality, we get

$$\mathbb{E}\left[\int_0^t |Y(s)|^{2p-2}||g(X_s)||^2\right]ds \leq \int_0^t \mathbb{E}[|Y(s)|^{4p-4}]^{\frac{1}{2}}\mathbb{E}[||g(X_s)||^4]^{\frac{1}{2}}\,ds.$$

Given (4.3.7), by Lemma 4.4.1, let $\varepsilon = 1$ in (4.5.1), there exist positive real numbers K_1 and K_2 such that

$$\begin{aligned}\mathbb{E}[||g(X_s)||^4] &\leq \mathbb{E}\left[\left(C_g + \int_{[-\tau,0]}\eta(du)|X(s+u)|\right)^4\right]\\ &\leq 8\mathbb{E}\left[C_g^4 + \left(\int_{[-\tau,0]}\eta(du)|X(s+u)|\right)^4\right]\\ &\leq 8C_g^4 + 8\left(\int_{[-\tau,0]}\eta(du)\right)^3\left(\int_{[-\tau,0]}\eta(du)\mathbb{E}[|X(s+u)|^4]\right)\\ &\leq 8C_g^4 + 8\left(\int_{[-\tau,0]}\eta(du)\right)^4 K_1 e^{K_2 s}.\end{aligned}$$

There also exist positive real numbers K_3 and K_4 such that

$$\begin{aligned}\mathbb{E}[|Y(s)|^{4p-4}] &= \mathbb{E}[|X(s) - D(X_s)|^{4p-4}]\\ &\leq 2^{4p-5}\left(\mathbb{E}[|X(s)|^{4p-4}] + \mathbb{E}[|D(X_s)|^{4p-4}]\right)\\ &\leq 2^{4p-5}\left(K_3 e^{K_4 s} + \mathbb{E}[|D(X_s)|^{4p-4}]\right).\end{aligned}$$

Apply the same analysis to $\mathbb{E}[|D(X_s)|^{4p-4}]$ as $\mathbb{E}[||g(X_s)||^4]$ using (4.3.8), it is easy to see that

$$\int_0^t \mathbb{E}[|Y(s)|^{4p-4}]^{\frac{1}{2}}\mathbb{E}[||g(X_s)||^4]^{\frac{1}{2}}\,ds < \infty.$$

Hence (4.5.11) holds.

Proof of Proposition 4.3.1 Let Ω_1 be an almost sure event such that $t \mapsto B(t,\omega)$ is nowhere differentiable on $(0,\infty)$. Let $T>0$. Suppose that $X = \{X(t) : -\tau \leq t \leq T\}$ is a solution of (4.3.9), (4.3.10). Then X is $\{\mathcal{F}(t)\}_{t\geq 0}$–adapted and is such that $t \mapsto X(t,\omega)$ is continuous on $[-\tau, T]$ for all $\omega \in \Omega_2$, where Ω_2 is an almost sure event. Define $C_T = \{\omega : X(\cdot,\omega) \text{ obeys } (4.3.11)\}$ and

$$A_T = C_T \cap \Omega_1 \cap \Omega_2,$$

Thus $\mathbb{P}[C_T] > 0$ and so $\mathbb{P}[A_T] > 0$. Hence for each $\omega \in A_T$, we have for all $t \in [0,T]$

$$\int_{-\tau}^{0} w(s)h(X(t+s,\omega))\,ds = \int_{-\tau}^{0} w(s)h(\psi(s))\,ds + \int_0^t f(X_s(\omega))\,ds + \sigma B(t,\omega),$$

so

$$\sigma B(t,\omega) = F(t,\omega), \quad t \in [0,T], \tag{4.5.24}$$

where we have defined

$$F(t,\omega) := \int_{-\tau}^{0} w(s)h(X(t+s,\omega))\,ds - \int_{-\tau}^{0} w(s)h(\psi(s))\,ds - \int_0^t f(X_s(\omega))\,ds.$$

It is not difficult to show that the righthand side of (4.5.24) viz., $t \mapsto F(t,\omega)$ is differentiable on $[0,T]$ for each $\omega \in A_T$, while the lefthand side of (4.5.24) is not differentiable anywhere in $[0,T]$ for each $\omega \in A_T$. This contradiction means that $\mathbb{P}[A_T] = 0$; hence with probability zero there are no sample paths of X which satisfy (4.3.9), (4.3.10).

Proof of Proposition 4.3.3 Suppose X is a solution on $[-\tau, T]$. Then with $A := \psi(0) + \kappa \max_{s\in[-\tau,0]} |\psi(s)|$

$$X(t) + \kappa \max_{s\in[t-\tau,t]} |X(s)| = A + \int_0^t g(X_s)\,dB(s), \quad t \in [0,T], \quad \text{a.s.}$$

Clearly $X(t) + \kappa \max_{s\in[t-\tau,t]} |X(s)| \geq -|X(t)| + \kappa|X(t)| = (\kappa-1)|X(t)| \geq 0$. Therefore

$$M(t) := \int_0^t -g(X_s)\,dB(s) \leq A, \quad t \in [0,T], \quad \text{a.s.} \tag{4.5.25}$$

Note that $A \geq 0$. Clearly M is a local martingale with $\langle M\rangle(t) = \int_0^t g^2(X_s)\,ds \geq \delta t$ by (4.3.15). By the martingale time change theorem, there exists a standard Brownian motion $\tilde{B}$ such that $M(t) = \tilde{B}(\langle M\rangle(t))$ for $t \in [0,T]$. Therefore by (4.5.25) we have

$$\max_{0\leq u\leq T} \tilde{B}(\langle M\rangle(u)) \leq A, \quad \text{a.s.}$$

Since $\langle M\rangle(T) \geq \delta T$ and $t \mapsto \langle M\rangle(t)$ is increasing on $[0, T]$ we have

$$\max_{0\leq s\leq \delta T} \tilde{B}(s) \leq \max_{0\leq u\leq T} \tilde{B}(\langle M\rangle(u)) \leq A, \quad \text{a.s.},$$

which is false, because $\tilde{B}$ is a standard Brownian motion $\delta T > 0$ and $A \geq 0$ is finite, recalling that $|W(\delta T)|$ and $\max_{0\leq s\leq \delta T} W(s)$ have the same distribution for any standard Brownian motion W. Hence there is no process X which is a solution on $[-\tau, T]$.

Chapter 5

Large Deviations of Stochastic Neutral Functional Differential Equations

5.1 Introduction

In the previous chapter, we studied the existence and uniqueness of solutions of stochastic neutral functional differential equations (SNFDEs). In this chapter, we continuous our study in the large deviations of solutions of SNFDEs.

We focus on linear SNFDEs with distributed delay and additive noise. Moreover, the solutions of these equations are Gaussian and asymptotically stationary. The main idea of the theory is analogous to that in Chapter 3. The characteristic equation determines the behaviour of the fundamental solution (resolvent), which in turn determines the behaviour of the stochastic solution. As a result, the statements of the theorem is very similar to those in Chapter 3 concerning non-neutral SFDEs. In the proof of Theorem 3.3.1, the differentiability of the underlying resolvent plays a crucial role in controlling the behaviour of the process between mesh points. Due to the uncertainty of the differentiability of the resolvent of the SNFDE, we cannot apply the same analysis as in Theorem 3.3.1.

More precisely, we study the equation

$$d\left(X(t) - \int_{[-\tau,0]} \mu(ds)X(t+s)\right) = \left(\int_{[-\tau,0]} \nu(ds)X(t+s)\right) dt + \sigma\, dB(t), \quad t \geq 0 \quad (5.1.1)$$

with $X(t) = \phi(t)$ for $t \in [-\tau, 0]$, where $\tau > 0$, $\mu, \nu \in M = M([-\tau, 0]; \mathbb{R})$. The initial function ϕ is assumed to be in the space $C[-\tau, 0] := \{\phi : [-\tau, 0] \to \mathbb{R} : \text{continuous}\}$.

We first turn our attention to the deterministic delay equation underlying (5.1.1). For a fixed constant $\tau \geq 0$ we consider the deterministic linear delay differential equation

$$\begin{aligned} \frac{d}{dt}\left(x(t) - \int_{[-\tau,0]} \mu(ds)x(t+s)\right) &= \int_{[-\tau,0]} \nu(ds)\, x(t+s), \quad \text{for } t \geq 0, \\ x(t) &= \phi(t) \quad \text{for } t \in [-\tau, 0], \end{aligned} \quad (5.1.2)$$

A function $x : [-\tau, \infty) \to \mathbb{R}$ is called a *solution* of (5.1.2) if x is continuous on $[-\tau, \infty)$ and x satisfies the first and second identity of (5.1.2) for all $t \geq 0$ and $t \in [-\tau, 0]$, respectively. From the existence result for both stochastic and deterministic neutral equation discussed in Chapter 4, for every $\phi \in C[-\tau, 0]$ the problem (5.1.2) admits a unique solution $x = x(\cdot, \phi)$ provided that $\mu(\{0\}) \neq 1$. This condition on μ is equivalent to the notion of *uniform non–atomicity at* 0 of the functional $D : C[-\tau, 0] \to \mathbb{R}$ given by

$$D(\psi) = \int_{[-\tau,0]} \mu(ds)\psi(s), \quad \psi \in C([-\tau, 0]; \mathbb{R}).$$

For $\mu(\{0\}) \in \mathbb{R}/\{1\}$, (5.1.2) can be rescaled, so that a unique solution exists. Hence without loss of generality, we assume that

$$\mu(\{0\}) = 0. \tag{5.1.3}$$

The *fundamental solution* or *resolvent* of (5.1.2) is the unique locally absolutely continuous function $\rho : [0, \infty) \to \mathbb{R}$ which satisfies

$$\begin{gathered}\frac{d}{dt}\left(\rho(t) - \int_{[-\tau,0]} \mu(ds)\rho(t+s)\right) = \left(\int_{[-\tau,0]} \nu(ds)\rho(t+s)\right), \quad t \geq 0; \\ \rho(t) = 0, \quad t \in [-\tau, 0); \quad \rho(0) = 1.\end{gathered} \tag{5.1.4}$$

Similar to Chapter 3, for a function $x : [-\tau, \infty) \to \mathbb{R}$ we denote the *segment of* x at time $t \geq 0$ by the function

$$x_t : [-\tau, 0] \to \mathbb{R}, \qquad x_t(s) := x(t+s).$$

If we equip the space $C[-\tau, 0]$ of continuous functions with the supremum norm, Riesz' representation theorem guarantees that every continuous functional $D : C[-\tau, 0] \to \mathbb{R}$ is of the form

$$D(\psi) = \int_{[-\tau,0]} \mu(ds)\, \psi(s),$$

for a scalar measure $\mu \in M$. Hence, we will write (5.1.2) in the form

$$\frac{d}{dt}[x(t) - D(x_t)] = L(x_t) \quad \text{for } t \geq 0, \qquad x_0 = \phi$$

where

$$L(\psi) = \int_{[-\tau,0]} \nu(ds)\, \psi(s),$$

and assume D and L to be continuous and linear functionals on $C([-\tau,0];\mathbb{R})$.

Fix a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ with a filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ satisfying the usual conditions and let $(B(t): t\geq 0)$ be a standard m–dimensional Brownian motion on this space. Equation (5.1.1) can be written as

$$\begin{aligned} d[X(t)-D(X_t)] &= L(X_t)\,dt+\sigma\,dB(t) \quad \text{for } t\geq 0,\\ X(t) &= \phi(t) \quad \text{for } t\in[-\tau,0], \end{aligned} \tag{5.1.5}$$

where D and L are as previously defined, and $\sigma\in\mathbb{R}$.

The dependence of the solutions on the initial condition ϕ is neglected in our notation in what follows; that is, we will write $x(t)=x(t,\phi)$ and $X(t)=X(t,\phi)$ for the solutions of (5.1.2) and (5.1.5) respectively.

We also constrain ourselves with the condition

$$\inf_{\mathrm{Re}\,(z)\geq-\alpha}\left|1-\int_{[-\tau,0]}e^{zs}\mu(ds)\right|>0 \quad \text{for some}\,\alpha>0. \tag{5.1.6}$$

It is easy to see that the above condition implies that

$$h_0:=1-\int_{[-\tau,0]}e^{zs}\mu(ds)\neq 0 \quad \text{for every}\, z\in\mathbb{C} \text{ with } \mathrm{Re}\,z\geq 0. \tag{5.1.7}$$

Define the function $h_{\mu,\nu}:\mathbb{C}\to\mathbb{C}$ by

$$h_{\mu,\nu}(\lambda)=\lambda\left(1-\int_{[-\tau,0]}e^{\lambda s}\mu(ds)\right)-\int_{[-\tau,0]}e^{\lambda s}\nu(ds).$$

The asymptotic behaviour of ρ relies on the value of

$$v_0(\mu,\nu):=\sup\left\{\mathrm{Re}\,(\lambda):\lambda\in\mathbb{C},h_{\mu,\nu}(\lambda)=0\right\} \tag{5.1.8}$$

We summarize some conditions on the asymptotic behaviour of ρ in the following lemma:

Lemma 5.1.1. *Let ρ satisfy* (5.1.4), *and $v_0(\mu,\nu)$ be defined as* (5.1.8). *If* (5.1.7) *holds, then the following statements are equivalent:*

(a) $v_0(\mu,\nu)<0$.

(b) ρ decays to zero exponentially.

(c) $\rho(t)\to 0$ *as* $t\to\infty$.

(d) $\rho \in L^1(\mathbb{R}^+;\mathbb{R})$.

(e) $\rho \in L^2(\mathbb{R}^+;\mathbb{R})$.

In Chapter 1 and Chapter 6 of [36], it is shown that a condition on the zeros of $h_{\mu,\nu}$ suffices to determine the asymptotic behaviour of the differential resolvent ρ. In our work, we have found it necessary to also assume a restriction on the zeros of h_0. In Chapter 1 in [36], Frasson analysed the relationship between the zeros of h_0 and $h_{\mu,\nu}$. For μ with jumps and zeros of sufficiently large modulus, there is a one-to-one correspondence between the zeros of h_0 and the zeros of $h_{\mu,\nu}$.

Frasson's asymptotic analysis suggests that condition (5.1.6) and/or (5.1.7) maybe dropped. It is interesting to probe however why we find it useful to retain these conditions. The condition 5.1.6 implies that the neutral operator is "D–stable". Under this condition, Staffans has shown in [74] that a deterministic NFDE in $\mathbb{R}^d$ of the form $dD(x_t)/dt = f(x_t, t)$ (where D is a linear operator from the space $C[-\tau, 0]$ to $\mathbb{R}^d$) into a retarded FDE with infinite delays. We exploit a similar reformulation of the stochastic equation in this chapter in order to derive a representation of the solution and to study large deviations. We do this in order to avail of the variation of constants formula due to Reiss, Riedle and van Gaans for retarded SFDEs, and to make use of our asymptotic analysis of large fluctuations of affine SFDEs studied in Chapter 3. Our philosophy in some sense parallels that of Staffans. But we have a technical reason for our approach also, which necessitates the assumption of D– stability of the neutral operator. In the proof of the result on the large deviations of the stochastic solution, we need to write the differential resolvent ρ of the neutral differential equation in terms of a continuously differentiable function κ and the integral resolvent ρ_0 of $(-\mu_+)$ (which is a reflection version of the measure μ). Condition 5.1.6 ensures that ρ_+ a finite measure on $\mathbb{R}^+$, which is an important fact in the proof.

The solution of the neutral equation can be represented in terms of the deterministic solution and the fundamental solution.

Theorem 5.1.1. *Suppose that L and D are linear functionals and that μ obeys* (5.1.3).

If x is the solution of (5.1.2) *and ρ is the continuous solution of* (5.1.4)*, then the unique continuous adapted process X which satisfies* (5.1.5) *obeys*

$$X(t) = x(t) + \int_0^t \rho(t-s)\sigma\,dB(s), \quad t \geq 0, \tag{5.1.9}$$

and $X(t) = \phi(t)$ for $t \in [-\tau, 0]$.

5.2 Statement and Discussion of Main Results

We start with some preparatory lemmata, used to establish the almost sure rate of growth of the partial maxima of the solution of a scalar version of (5.1.5).

Theorem 5.2.1. *Suppose that ρ is the solution of* (5.1.4) *and that μ satisfies* (5.1.7)*. Moreover, $v_0(\mu,\nu) < 0$, where $v_0(\mu,\nu)$ is defined as* (5.1.8)*. Let X be the unique continuous adapted process which obeys* (5.1.5)*. Then*

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2\log t}} = |\sigma|\sqrt{\int_0^\infty \rho^2(s)\,ds} =: \Gamma, \quad a.s. \tag{5.2.1}$$

Moreover,

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2\log t}} = \Gamma, \quad \textit{and} \quad \liminf_{t\to\infty} \frac{X(t)}{\sqrt{2\log t}} = -\Gamma, \quad a.s.$$

The results of Theorem 5.2.1 is very similar to those of Theorem 3.3.1. The proof of Theorem 3.3.1 depends on two key properties of the differential resolvent r satisfying (3.2.2) with initial condition zero on $[-\tau, 0)$. The first is that r decays exponential fast because $v_0(\nu) < 0$. This is in common with the condition $v_0(\mu,\nu) < 0$ in Theorem 5.2.1. The second is that r is in $C^1((0,\infty);\mathbb{R})$, which plays a crucial role in controlling the behaviour of the process between mesh points. In contrast with the differentiability of r, the neutral differential resolvent ρ may not be differentiable everywhere on $(0,\infty)$. Therefore the proof of Theorem 5.2.1 deviates from Theorem 3.3.1 in controlling the behaviour of the process between mesh points.

One could extend Theorem 5.2.1 to finite-dimensional and non-linear problems in the same way as in Theorem 3.3.2 and Theorem 3.3.3. Since the technique (which involving constructing differential inequalities for Theorem 3.3.3 as seen in Chapter 2) and the result are essentially the same as in Chapter 3, we do not supply theorem here.

5.3 Proofs of Section 5.2

Proof of Lemma 5.1.1 We extend the measures μ and ν to $M((-\infty,0];\mathbb{R})$ by assuming $\mu(E)=\nu(E)=0$ for every Borel set $E\subseteq(-\infty,-\tau)$. Introduce the measures μ_+ and ν_+ in $M([0,\infty);\mathbb{R})$, related to μ and ν in $M((-\infty,0];\mathbb{R})$ by $\mu_+(E):=\mu(-E)$, $\nu_+(E):=\nu(-E)$, where $(-E):=\{x\in\mathbb{R}:-x\in E\}$. Then for $t\geq 0$,

$$\begin{aligned}\int_{[-\tau,0]}\mu(ds)\rho(t+s)&=\int_{[0,\tau]}\mu_+(ds)\rho(t-s)\\&=\int_{[0,\infty)}\mu_+(ds)\rho(t-s)-\int_{(\tau,\infty)}\mu_+(ds)\rho(t-s)\\&=\int_{[0,\infty)}\mu_+(ds)\rho(t-s)\\&=\int_{[0,t]}\mu_+(ds)\rho(t-s)+\int_{(t,\infty)}\mu_+(ds)\rho(t-s)\\&=\int_{[0,t]}\mu_+(ds)\rho(t-s).\end{aligned}\tag{5.3.1}$$

The last step is obtained by the fact that $\rho(t)=0$ for $t\in[-\tau,0)$ and $\mu(\{0\})=0$. Similarly

$$\int_{[-\tau,0]}\nu(ds)\rho(t+s)=\int_{[0,t]}\nu_+(ds)\rho(t-s),\quad t\geq 0.\tag{5.3.2}$$

Define

$$\kappa(t):=\begin{cases}\rho(t)-\int_{[-\tau,0]}\mu(ds)\rho(t+s), & t\geq 0,\\ 0, & t\in[-\tau,0).\end{cases}$$

Also since $\rho(0)=1$ and $\mu(\{0\})=0$, $\kappa(0)=1$. Moreover, by (5.3.1) and (5.3.2), we have

$$\kappa(t)=\rho(t)-\int_{[0,t]}\mu_+(ds)\rho(t-s),\quad t\geq 0,$$

and

$$\kappa'(t)=\int_{[0,t]}\nu_+(ds)\rho(t-s),\quad t\geq 0.\tag{5.3.3}$$

That is $\kappa=\rho-\mu_+*\rho$ and $\kappa'(t)=(\nu_+*\rho)(t)$. Then $\rho=\kappa-\rho_0*\kappa$, where ρ_0 is the integral resolvent of $(-\mu_+)$. Given 5.1.6, by Corollary 4.4.7 in [39], $\rho_0\in M(\mathbb{R}^+;\mathbb{R})$. Moreover,

$\kappa'(t) = (\beta * \kappa)(t)$, where $\beta := \nu_+ - \nu_+ * \rho_0$. Hence, under (5.1.7) and using Theorem 3.6.1 in [39], we have

$$\lim_{t\to\infty} \kappa(t) = 0 \quad \Leftrightarrow \quad \lim_{t\to\infty} \rho(t) = 0; \tag{5.3.4}$$

$$\kappa \text{ decays to zero exponentially} \quad \Leftrightarrow \quad \rho \text{ decays to zero exponentially}; \tag{5.3.5}$$

$$\kappa \in L^1(\mathbb{R}^+;\mathbb{R}) \quad \Leftrightarrow \quad \rho \in L^1(\mathbb{R}^+;\mathbb{R}); \tag{5.3.6}$$

$$\kappa \in L^2(\mathbb{R}^+;\mathbb{R}) \quad \Leftrightarrow \quad \rho \in L^2(\mathbb{R}^+;\mathbb{R}). \tag{5.3.7}$$

Now by Theorem 3.3.17 from [39], if $\beta \in M(\mathbb{R}^+;\mathbb{R})$ has a finite first moment, i.e. $\int_{[0,\infty)} t|\beta|_m(dt) < \infty$, where $\beta := \nu_+ - \nu_+ * \rho_0$, then

$$\lim_{t\to\infty} \kappa(t) = 0 \quad \Leftrightarrow \quad \kappa \in L^1(\mathbb{R}^+;\mathbb{R}). \tag{5.3.8}$$

We now show that β has a finite first moment. Note that

$$\begin{aligned}
\int_{[0,\infty)} t|\beta|_m(dt) &\le \int_{[0,\infty)} t|\nu_+|_m(dt) + \int_{[0,\infty)} t|\nu_+ * \rho_0|_m(dt) \\
&= \int_{[0,\tau]} t|\nu_+|_m(dt) + \int_{[0,\infty)} t|\nu_+ * \rho_0|_m(dt).
\end{aligned}$$

Since ρ_0 decays exponentially, there exists $\alpha > 0$ such that $\int_{[0,\infty)} e^{\alpha t}|\rho_0|_m(dt) < \infty$. Thus by Young's inequality,

$$\begin{aligned}
\int_{[0,\infty)} t|\nu_+ * \rho_0|_m(dt) &\le \frac{1}{\alpha}\int_{[0,\infty)} e^{\alpha t}|\nu_+ * \rho_0|_m(dt) \\
&\le \frac{1}{\alpha}\int_{[0,\infty)} e^{\alpha t}|\nu_+|_m(dt) \int_{[0,\infty)} e^{\alpha t}|\rho_0|_m(dt) \\
&= \frac{1}{\alpha}\int_{[0,\tau]} e^{\alpha t}|\nu_+|_m(dt) \int_{[0,\infty)} e^{\alpha t}|\rho_0|_m(dt) \\
&< \infty.
\end{aligned}$$

So β has finite first moment. Therefore (5.3.8) holds. Moreover,

$$\int_{[0,\infty)} e^{\alpha t}|\beta|_m(dt) < \infty. \tag{5.3.9}$$

So by (5.3.4), (5.3.6) and (5.3.8), statement (c) and (d) are equivalent. Now if $\kappa \in L^1(\mathbb{R}^+;\mathbb{R})$, due to (5.3.9), we have that κ decays to zero exponentially, which by (5.3.5) implies that ρ decays to zero exponentially. Hence (b) and (c) are equivalent. If $\lim_{t\to\infty} \rho(t) = 0$, then ρ decays to zero exponentially, which implies $\rho \in L^2(\mathbb{R}^+;\mathbb{R})$. On the other hand, if $\rho \in L^2(\mathbb{R}^+;\mathbb{R})$, then $\kappa \in L^2(\mathbb{R}^+;\mathbb{R})$. Also $\kappa' \in L^2(\mathbb{R}^+;\mathbb{R})$. Let $f := \kappa^2$. Then $|f'| = 2|\kappa\kappa'| \leq |\kappa|^2 + |\kappa'|^2$, so $f' \in L^1(\mathbb{R}^+;\mathbb{R})$. Therefore as $f \in L^1(\mathbb{R}^+;\mathbb{R})$, we have $\lim_{t\to\infty}\kappa(t) = 0$, consequently $\lim_{t\to\infty}\rho(t) = 0$. Hence (b)–(e) are equivalent. For part (a), suppose $\rho \in L^1$, which holds if and only if $\kappa \in L^1$, which in turn is equivalent to

$$\lambda - \hat{\beta}(\lambda) \neq 0, \quad \mathrm{Re}\,(\lambda) \geq 0. \tag{5.3.10}$$

Now $\hat{\rho}_0(\lambda) = \hat{\mu}_+(\lambda)/(\hat{\mu}_+(\lambda) - 1)$ for all $\mathrm{Re}\,\lambda \geq 0$, because $1 - \hat{\mu}_+(\lambda) \neq 0$ for all $\mathrm{Re}\,\lambda \geq 0$ due to (5.1.7). We have, for $\mathrm{Re}\,\lambda \geq 0$,

$$\begin{aligned}
\lambda - \hat{\beta}(\lambda) &= \lambda - \hat{\nu}_+(\lambda) + \hat{\nu}_+(\lambda)\hat{\rho}_0(\lambda) \\
&= \lambda - \hat{\nu}_+(\lambda) - \hat{\nu}_+(\lambda)\hat{\mu}_+(\lambda)\frac{1}{1-\hat{\mu}_+(\lambda)} \\
&= \frac{1}{1-\hat{\mu}_+(\lambda)}\Big[\lambda(1-\hat{\mu}_+(\lambda)) - \hat{\nu}_+(\lambda)(1-\hat{\mu}_+(\lambda)) - \hat{\nu}_+(\lambda)\hat{\mu}_+(\lambda)\Big] \\
&= \frac{1}{1-\hat{\mu}_+(\lambda)}\Big[\lambda(1-\hat{\mu}_+(\lambda)) - \hat{\nu}_+(\lambda)\Big] \\
&= \frac{1}{1-\hat{\mu}_+(\lambda)}\Big[\lambda(1-\int_{[-\tau,0]} e^{\lambda s}\mu(ds)) - \int_{[-\tau,0]} e^{\lambda s}\nu(ds)\Big]
\end{aligned}$$

Clearly, under (5.1.7), (5.3.10) holds if and only if

$$\lambda(1-\int_{[-\tau,0]} e^{\lambda s}\mu(ds)) - \int_{[-\tau,0]} e^{\lambda s}\nu(ds) \neq 0, \quad \text{for all } \mathrm{Re}\,\lambda \geq 0$$

which is true if and only if $v_0(\mu,\nu) < 0$. Hence statement (a)–(e) are all equivalent.

Proof of Theorem 5.1.1 First, as in the proof of Lemma 5.1.1, we extend the measures μ and ν to $M((-\infty,0];\mathbb{R})$ by assuming

$$\mu(E) = \nu(E) = 0 \quad \text{for every Borel set } E \subseteq (-\infty,-\tau).$$

For any Borel set $E \subseteq \mathbb{R}$ we use the notation

$$-E := \{x \in \mathbb{R} : -x \in E\}$$

to define the reflected Borel set $(-E)$. Now, we introduce the measures μ_+ and ν_+ in $M([0,\infty);\mathbb{R})$, related to μ and ν in $M((-\infty,0];\mathbb{R})$ by

$$\mu_+(E) = \mu(-E), \quad \nu_+(E) = \nu(-E).$$

Therefore for $t \geq 0$, X satisfies

$$d\left(X(t) - \int_{[0,\tau]} \mu_+(ds)X(t-s)\right) = \left(\int_{[0,\tau]} \nu_+(ds)X(t-s)\right) dt + \sigma\, dB(t),$$

with $X(t) = \phi(t)$ for $t \in [-\tau, 0]$. Similarly the deterministic solution x satisfying (5.1.2) satisfies

$$d\left(x(t) - \int_{[0,\tau]} \mu_+(ds)x(t-s)\right) = \left(\int_{[0,\tau]} \nu_+(ds)x(t-s)\right) dt,$$

with $x(t) = \phi(t)$ for $t \in [-\tau, 0]$. In a similar manner as in the proof of Lemma 5.1.1, it can be shown that

$$\int_{[-\tau,0]} \mu(ds)\rho(t+s) = \int_{[0,t]} \mu_+(ds)\rho(t-s) \quad \text{and} \quad \int_{[-\tau,0]} \nu(ds)\rho(t+s) = \int_{[0,t]} \nu_+(ds)\rho(t-s).$$

Hence, for $t \geq 0$, the fundamental solution ρ satisfies

$$\frac{d}{dt}\left(\rho(t) - \int_{[0,t]} \mu_+(ds)\rho(t-s)\right) = \int_{[0,t]} \nu_+(ds)\rho(t-s), \tag{5.3.11}$$

with $\rho(t) = 0$ for $t \in [-\tau, 0)$ and $\rho(0) = 1$. Define $W(t) := X(t) - x(t)$ for $t \geq -\tau$, then W obeys

$$d\left(W(t) - \int_{[0,t]} \mu_+(ds)W(t-s)\right) = \int_{[0,t]} \nu_+(ds)W(t-s)\, dt + \sigma\, dB(t), \quad t \geq 0;$$
$$W(t) = 0, \quad t \in [-\tau, 0],$$

and is the unique solution of the above equation. Now define κ by

$$\kappa(t) := \rho(t) - \int_{[0,t]} \mu_+(ds)\rho(t-s), \quad t \in \mathbb{R}^+.$$

Then $\kappa(0) = 1$ and $\kappa(t) = 0$ for all $t < 0$. Moreover, $\kappa \in C^1((0,\infty);\mathbb{R})$. We may write $\kappa = \rho - \mu_+ * \rho$. Let

$$Z(t) := W(t) - \int_{[0,\tau]} \mu_+(ds)W(t-s), \quad t \in \mathbb{R}.$$

Then $Z(0) = W(0) = 0$, and we may write $Z = W - \mu_+ * W$. Clearly Z is continuous. By definition, $\mu_+ \in M([0,\infty);\mathbb{R})$, then by Theorem 4.1.5 (half line Paley–Wiener theorem) in [39], we may define ρ_0 to be the integral resolvent of $(-\mu_+)$, i.e.,

$$\rho_0 - \mu_+ * \rho_0 = -\mu_+, \tag{5.3.12}$$

where $\rho_0 \in M_{\text{loc}}([0,\infty);\mathbb{R})$. Then by Theorem 4.1.7 in [39],

$$\rho = \kappa - \rho_0 * \kappa, \tag{5.3.13}$$

and $W = Z - \rho_0 * Z$. Therefore

$$
\begin{aligned}
dZ(t) &= (v_+ * W)(t)\,dt + \sigma\,dB(t) \\
&= \Big[v_+ * (Z - \rho_0 * Z)\Big](t)\,dt + \sigma\,dB(t) \\
&= \Big[(v_+ - v_+ * \rho_0) * Z\Big](t)\,dt + \sigma\,dB(t)
\end{aligned}
$$

Now by (5.3.11), $\kappa' = v_+ * \rho = v_+ * (\kappa - \rho_0 * \kappa) = (v_+ - v_+ * \rho_0) * \kappa$. Since $\kappa(0) = 1$ and $Z(0) = 0$, we have that Z obeys

$$Z(t) = \sigma \int_0^t \kappa(t-s)\,dB(s), \quad t \geq 0.$$

Finally, note that

$$W(t) = Z(t) - (\rho_0 * Z)(t) = \sigma \int_0^t \kappa(t-s)\,dB(s) - \sigma \int_{[0,t]} \rho_0(ds) \int_0^{t-s} \kappa(t-s-u)\,dB(u).$$

By Fubini's Theorem, we have for all $t \geq 0$,

$$
\begin{aligned}
W(t) &= \sigma \int_0^t \kappa(t-s)\,dB(s) - \sigma \int_{u=0}^t \int_{s\in[0,t-u]} \rho_0(ds)\kappa(t-s-u)\,dB(u) \\
&= \sigma \int_0^t \kappa(t-s)\,dB(s) - \sigma \int_0^t (\rho_0 * \kappa)(t-u)\,dB(u) \\
&= \sigma \int_0^t [\kappa(t-s) - (\rho_0 * \kappa)(t-s)]\,dB(s) \\
&= \sigma \int_0^t \rho(t-s)\,dB(s).
\end{aligned}
$$

Hence $X(t) = x(t) + W(t) = x(t) + \sigma \int_0^t \rho(t-s)\,dB(s), \quad t \geq 0$.

Proof of Theorem 5.2.1 By Theorem 5.1.1, following the same argument as in the proof of Theorem 3.3.1 in Chapter 3, it can be shown that

$$\limsup_{n\to\infty} \frac{|X(n^\epsilon)|}{\sqrt{2\log n}} \leq |\sigma| \sqrt{\int_0^\infty \rho^2(s)\,ds}, \quad \text{a.s.} \tag{5.3.14}$$

where $0 < \epsilon < 1$. The proof for the above upper estimate in Theorem 3.3.1 does not depend on the differentiability of the resolvent. However, for the lower estimate, Theorem 3.3.1 does depend on the differentiability of the resolvent. Since the differentiability of ρ on $\mathbb{R}^+$ is uncertain in the neutral case, we cannot apply the same argument as in Theorem

3.3.1 which connects the result on the mesh points with that on continuous time. Now $|X(t)| \le |X(t) - X(n^\epsilon)| + |X(n^\epsilon)|$ for $n^\epsilon \le t \le (n+1)^\epsilon$. Since

$$\begin{aligned}
X(t) - X(n^\epsilon) &= x(t) - x(n^\epsilon) + Z(t) - Z(n^\epsilon) \\
&\qquad - \left(\int_{[0,t]} \rho_0(ds) Z(t-s) - \int_{[0,n^\epsilon]} \rho_0(ds) Z(n^\epsilon - s) \right) \\
&= x(t) - x(n^\epsilon) + Z(t) - Z(n^\epsilon) \\
&\qquad - \left(\int_{[0,t]} \rho_0(ds) Z(t-s) - \int_{[0,n^\epsilon]} \rho_0(ds) Z(t-s) \right. \\
&\qquad \left. + \int_{[0,n^\epsilon]} \rho_0(ds) Z(t-s) - \int_{[0,n^\epsilon]} \rho_0(ds) Z(n^\epsilon - s) \right) \\
&= x(t) - x(n^\epsilon) + Z(t) - Z(n^\epsilon) - \int_{[n^\epsilon,t]} \rho_0(ds) Z(t-s) \\
&\qquad - \int_{[0,n^\epsilon]} \rho_0(ds) \Big(Z(t-s) - Z(n^\epsilon - s) \Big).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sup_{n^\epsilon \le t \le (n+1)^\epsilon} |X(t) - X(n^\epsilon)| \\
&\le \sup_{n^\epsilon \le t \le (n+1)^\epsilon} |x(t) - x(n^\epsilon)| + \sup_{n^\epsilon \le t \le (n+1)^\epsilon} |Z(t) - Z(n^\epsilon)| + \sup_{n^\epsilon \le t \le (n+1)^\epsilon} \left| \int_{[n^\epsilon,t]} \rho_0(ds) Z(t-s) \right| \\
&\qquad + \sup_{n^\epsilon \le t \le (n+1)^\epsilon} \left| \int_{[0,n^\epsilon]} \rho_0(ds) \big(Z(t-s) - Z(n^\epsilon - s) \big) \right|. \qquad (5.3.15)
\end{aligned}$$

We now consider each of the four terms on the right-hand side of (5.3.15) in turn. It is easy to see that

$$\lim_{n\to\infty} \sup_{n^\epsilon \le t \le (n+1)^\epsilon} |x(t) - x(n^\epsilon)| = 0.$$

Applying the same argument as in the proof of Theorem 3.3.1 for $\tilde{X}$, it can be shown that

$$\limsup_{n\to\infty} \sup_{n^\epsilon \le t \le (n+1)^\epsilon} |Z(t) - Z(n^\epsilon)| \le 2, \quad \text{a.s.} \qquad (5.3.16)$$

For the third term,

$$\begin{aligned}
\sup_{n^\epsilon \le t \le (n+1)^\epsilon} \left| \int_{[n^\epsilon,t]} \rho_0(ds) Z(t-s) \right| &\le \sup_{n^\epsilon \le t \le (n+1)^\epsilon} \int_{[n^\epsilon,t]} |\rho_0|_m(ds) |Z(t-s)| \\
&\le \sup_{n^\epsilon \le t \le (n+1)^\epsilon} \sup_{n^\epsilon \le s \le t} |Z(t-s)| \cdot \int_{[n^\epsilon,\infty)} |\rho_0|_m(ds) \\
&= \sup_{n^\epsilon \le s \le t \le (n+1)^\epsilon} |Z(t-s)| \cdot \int_{[0,\infty)} |\rho_0|_m(ds) \\
&= \sup_{0 \le u \le (n+1)^\epsilon - n^\epsilon} |Z(u)| \cdot \int_{[0,\infty)} |\rho_0|_m(ds),
\end{aligned}$$

which implies

$$\lim_{n\to\infty} \sup_{0\le u\le (n+1)^\epsilon - n^\epsilon} |Z(u)|\cdot \int_{[0,\infty)} |\rho_0|_m(ds) \le |Z(0)| \int_{[0,\infty)} |\rho_0|_m(ds) = 0, \quad \text{a.s.} \quad (5.3.17)$$

For the last term on the right-hand side of (5.3.15), we note that for $t \ge 0$,

$$\begin{aligned} Z(t) &= \kappa(t)Z(0) + \sigma\int_0^t \kappa(t-s)\,dB(s) \\ &= \sigma\int_0^t (1 + \int_0^{t-s} \kappa'(v)\,dv)\,dB(s) \\ &= \sigma B(t) + \sigma\int_0^t\int_0^{t-s} \kappa'(v)\,dv\,dB(s) \\ &= \sigma B(t) + \sigma\int_0^t\int_s^t \kappa'(u-s)\,du\,dB(s) \\ &= \sigma B(t) + \sigma\int_0^t\int_0^u \kappa'(u-s)\,dB(s)\,du. \end{aligned}$$

So for $n^\epsilon \le t \le (n+1)^\epsilon$,

$$Z(t-s) - Z(n^\epsilon - s) \,=\, \sigma\Big(B(t-s) - B(n^\epsilon - s)\Big) + \sigma\int_{n^\epsilon - s}^{t-s}\int_0^u \kappa'(u-v)\,dB(v)\,du.$$

Hence

$$\begin{aligned} &\sup_{n^\epsilon\le t\le (n+1)^\epsilon}\left|\int_{[0,n^\epsilon]} \rho_0(ds)(Z(t-s) - Z(n^\epsilon - s))\right| \\ &\qquad \le |\sigma| \sup_{n^\epsilon\le t\le (n+1)^\epsilon}\int_{[0,n^\epsilon]} |\rho_0|_m(ds)|B(t-s) - B(n^\epsilon - s)| \\ &\qquad + |\sigma| \sup_{n^\epsilon\le t\le (n+1)^\epsilon}\int_{[0,n^\epsilon]} |\rho_0|_m(ds)\left|\int_{n^\epsilon - s}^{t-s}\int_0^u \kappa'(u-v)\,dB(v)\,du\right|. \quad (5.3.18) \end{aligned}$$

For the first term on the right-hand side of (5.3.18), for some $p_\epsilon > 1$ and $q_\epsilon > 1$ such that $1/p_\epsilon + 1/q_\epsilon = 1$,

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sup_{n^\epsilon \le t \le (n+1)^\epsilon} \int_{[0,n^\epsilon]} |\rho_0|_m(ds)|B(t-s) - B(n^\epsilon - s)|\right)^{p_\epsilon}\right] \\
&\le \mathbb{E}\left[\sup_{n^\epsilon \le t \le (n+1)^\epsilon} \left(\int_{[0,n^\epsilon]} |\rho_0|_m(ds)\right)^{\frac{p_\epsilon}{q_\epsilon}} \left(\int_{[0,n^\epsilon]} |\rho_0|_m(ds)|B(t-s) - B(n^\epsilon - s)|^{p_\epsilon}\right)\right] \\
&\le \left(\int_{[0,\infty)} |\rho_0|_m(ds)\right)^{\frac{p_\epsilon}{q_\epsilon}} \mathbb{E}\left[\sup_{n^\epsilon \le t \le (n+1)^\epsilon} \left(\int_{[0,n^\epsilon]} |\rho_0|_m(ds)|B(t-s) - B(n^\epsilon - s)|^{p_\epsilon}\right)\right] \\
&\le \left(\int_{[0,\infty)} |\rho_0|_m(ds)\right)^{\frac{p_\epsilon}{q_\epsilon}} \mathbb{E}\left[\int_{[0,n^\epsilon]} |\rho_0|_m(ds) \sup_{n^\epsilon \le t \le (n+1)^\epsilon} |B(t-s) - B(n^\epsilon - s)|^{p_\epsilon}\right] \\
&\le \left(\int_{[0,\infty)} |\rho_0|_m(ds)\right)^{\frac{p_\epsilon}{q_\epsilon}} \int_{[0,n^\epsilon]} |\rho_0|_m(ds)\, \mathbb{E}\left[\sup_{n^\epsilon - s \le u \le (n+1)^\epsilon - s} |B(u) - B(n^\epsilon - s)|^{p_\epsilon}\right] \\
&\le \left(\int_{[0,\infty)} |\rho_0|_m(ds)\right)^{\frac{p_\epsilon}{q_\epsilon}} \int_{[0,n^\epsilon]} |\rho_0|_m(ds)\, \mathbb{E}\left[\sup_{n^\epsilon - s \le u \le (n+1)^\epsilon - s} \left|\int_{n^\epsilon - s}^{u} dB(v)\right|^{p_\epsilon}\right] \\
&\le \left(\int_{[0,\infty)} |\rho_0|_m(ds)\right)^{\frac{p_\epsilon}{q_\epsilon}} \int_{[0,n^\epsilon]} |\rho_0|_m(ds) \left(\frac{32}{p_\epsilon}\right)^{\frac{p_\epsilon}{2}} \mathbb{E}\left[((n+1)^\epsilon - n^\epsilon)^{\frac{p_\epsilon}{2}}\right] \\
&\le \left(\frac{32}{p_\epsilon}\right)^{\frac{p_\epsilon}{2}} \left(\int_{[0,\infty)} |\rho_0|_m(ds)\right)^{\frac{p_\epsilon + q_\epsilon}{q_\epsilon}} [(n+1)^\epsilon - n^\epsilon]^{\frac{p_\epsilon}{2}},
\end{aligned}
$$

where we have used the Hölder inequality and Burkholder-Davis-Gundy inequality in the second and penultimate lines respectively. Hence by the Chebyshev inequality

$$
\begin{aligned}
&\mathbb{P}\left[\sup_{n^\epsilon \le t \le (n+1)^\epsilon} \int_{[0,n^\epsilon]} |\rho_0|_m(ds)|B(t-s) - B(n^\epsilon - s)| > 1\right] \\
&\qquad \le \mathbb{E}\left[\left(\sup_{n^\epsilon \le t \le (n+1)^\epsilon} \int_{[0,n^\epsilon]} |\rho_0|_m(ds)|B(t-s) - B(n^\epsilon - s)|\right)^{p_\epsilon}\right] \\
&\qquad \le \left(\frac{32}{p_\epsilon}\right)^{\frac{p_\epsilon}{2}} \left(\int_{[0,\infty)} |\rho_0|_m(ds)\right)^{\frac{p_\epsilon + q_\epsilon}{q_\epsilon}} [(n+1)^\epsilon - n^\epsilon]^{\frac{p_\epsilon}{2}}.
\end{aligned}
$$

Now since $\lim_{n\to\infty}[(n+1)^\epsilon - n^\epsilon]/n^{(\epsilon-1)} = \epsilon$, if we choose $p_\epsilon = 4/(1-\epsilon) > 1$, then by the Borel-Cantelli lemma, we get

$$
\limsup_{n\to\infty} \sup_{n^\epsilon \le t \le (n+1)^\epsilon} \int_{[0,n^\epsilon]} |\rho_0|_m(ds)|B(t-s) - B(n^\epsilon - s)| \le 1 \quad \text{a.s.} \tag{5.3.19}
$$

For the second term on the right-hand side of (5.3.18), define $I(u) := \int_0^u \kappa'(u-v)dB(v)$ and $H_n(s) := \int_{n^\epsilon - s}^{(n+1)^\epsilon - s} |I(u)|\,du$. Then

$$\begin{aligned}
A_n &:= \sup_{n^\epsilon \le t \le (n+1)^\epsilon} \int_{[0,n^\epsilon]} |\rho_0|_m(ds) \left| \int_{n^\epsilon - s}^{t-s} \int_0^u \kappa'(u-v)\,dB(v)\,du \right| \\
&\le \sup_{n^\epsilon \le t \le (n+1)^\epsilon} \int_{[0,n^\epsilon]} |\rho_0|_m(ds) \int_{n^\epsilon - s}^{t-s} \left| \int_0^u \kappa'(u-v)\,dB(v) \right| du \\
&\le \int_{[0,n^\epsilon]} |\rho_0|_m(ds) H_n(s).
\end{aligned}$$

Therefore if $p_\epsilon > 1$ and $q_\epsilon > 1$ are such that $1/p_\epsilon + 1/q_\epsilon = 1$, then by Hölder's inequality we have

$$\begin{aligned}
A_n^{p_\epsilon} &\le \left(\int_{[0,n^\epsilon]} |\rho_0|_m(ds) H_n(s) \right)^{p_\epsilon} \\
&\le \left(\int_{[0,n^\epsilon]} |\rho_0|_m(ds) \right)^{\frac{p_\epsilon}{q_\epsilon}} \int_{[0,n^\epsilon]} |\rho_0|_m(ds) H_n(s)^{p_\epsilon} \\
&\le \left(\int_{[0,n^\epsilon]} |\rho_0|_m(ds) \right)^{\frac{p_\epsilon}{q_\epsilon}} \int_{[0,n^\epsilon]} |\rho_0|_m(ds) \left((n+1)^\epsilon - n^\epsilon\right)^{p_\epsilon - 1} \int_{n^\epsilon - s}^{(n+1)^\epsilon - s} |I(u)|^{p_\epsilon}\,du.
\end{aligned}$$

Since $I(u)$ is normally distributed with zero mean and variance $\int_0^u \kappa'(u)^2\,du$, and $\kappa' \in L^2$, we have that

$$\mathbb{E}[I(u)^2] \le \int_0^\infty \kappa'(s)^2\,ds, \quad u \ge 0.$$

Therefore there exists $K(p) > 0$ such that $\mathbb{E}[|I(u)|^p] \le K(p)$ for all $u \ge 0$. Therefore

$$\begin{aligned}
\mathbb{E}[A_n^{p_\epsilon}] &\le \left((n+1)^\epsilon - n^\epsilon\right)^{p_\epsilon - 1} \left(\int_{[0,\infty)} |\rho_0|_m(ds) \right)^{p_\epsilon/q_\epsilon} \\
&\qquad \times \int_{[0,n^\epsilon]} |\rho_0|_m(ds) \int_{n^\epsilon - s}^{(n+1)^\epsilon - s} \mathbb{E}[|I(u)|^{p_\epsilon}]\,du \\
&\le \left((n+1)^\epsilon - n^\epsilon\right)^{p_\epsilon - 1} \left(\int_{[0,\infty)} |\rho_0|_m(ds) \right)^{p_\epsilon/q_\epsilon} \\
&\qquad \times \int_{[0,n^\epsilon]} |\rho_0|_m(ds)\left((n+1)^\epsilon - n^\epsilon\right) K(p_\epsilon) \\
&\le K(p_\epsilon) \left((n+1)^\epsilon - n^\epsilon\right)^{p_\epsilon} \left(\int_{[0,\infty)} |\rho_0|_m(ds) \right)^{p_\epsilon/q_\epsilon + 1}.
\end{aligned}$$

Therefore we have

$$\mathbb{P}[A_n > 1] \le \mathbb{E}[A_n^{p_\epsilon}] \le K(p_\epsilon) \left((n+1)^\epsilon - n^\epsilon\right)^{p_\epsilon} \left(\int_{[0,\infty)} |\rho_0|_m(ds) \right)^{p_\epsilon/q_\epsilon + 1}. \tag{5.3.20}$$

Let $p_\epsilon = 2/(1-\epsilon)$; then $p_\epsilon > 2$. Then the righthand side of (5.3.20) is summable in n, because $\{(n+1)^\epsilon - n^\epsilon\}/n^{\epsilon-1} \to \epsilon$ as $n \to \infty$, so by the Borel–Cantelli Lemma we have that

$$\limsup_{n\to\infty} \sup_{n^\epsilon \le t \le (n+1)^\epsilon} \int_{[0,n^\epsilon]} |\rho_0|_m(ds) \left| \int_{n^\epsilon - s}^{t-s} \int_0^u \kappa'(u-v)\, dB(v)\, du \right| = \limsup_{n\to\infty} A_n \le 1, \quad \text{a.s.} \tag{5.3.21}$$

Combining (5.3.18), (5.3.19) and (5.3.21), it follows that

$$\limsup_{n\to\infty} \frac{\sup_{n^\epsilon \le t \le (n+1)^\epsilon} \left| \int_{[0,n^\epsilon]} \rho_0(ds)(Z(t-s) - Z(n^\epsilon - s)) \right|}{\sqrt{2\log n}} = 0, \quad \text{a.s.} \tag{5.3.22}$$

Gathering the results (5.3.14), (5.3.15), (5.3.16), (5.3.17) and (5.3.22), it gives

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2\log t}} \le |\sigma| \sqrt{\int_0^\infty \rho^2(s)\, ds}, \quad \text{a.s.}$$

For the lower bound, we can apply the same analysis as in the proof of (3.3.1). Therefore (5.2.1) is proved.

Explicit Formula for the Fundamental Solution of the Deterministic Delay Differential Equation (3.5.3)

Theorem A.0.1. *Suppose r satisfies $r'(t) = ar(t) + br(t-\tau)$ for $t \geq 0$; $r(0) = 1$ and $r(1) = 0$ for $t \in [-\tau, 0)$. Here $a, b \in \mathbb{R}$ and $\tau > 0$. Then*

$$\text{for } t \in [n\tau, (n+1)\tau], \quad r(t) = e^{at} \sum_{j=0}^{n} \frac{(be^{-a\tau})^j}{j!}(t-j\tau)^j, \quad n \geq 0. \tag{A.0.1}$$

Proof. On the interval $[0,\tau]$, $r'(t) = ar(t)$. So for $t \in [0,\tau]$, $r(t) = e^{at}$. Let $x(t) = e^{-at}r(t)$, for $t \geq -\tau$. Then $x(t) = 0$ for $t \in [-\tau, 0)$, $x(0) = 1$ and for $t > 0$ we have

$$x'(t) = e^{-at}r'(t) - ae^{-at}r(t) = be^{-at}r(t-\tau) = be^{-a\tau}x(t-\tau).$$

Let $\alpha = be^{-a\tau}$. Then $x'(t) = \alpha x(t-\tau)$, $t > 0$. Consider $t \in [0,\tau]$, then

$$x(t) = x(0) + \int_0^t x'(s)\,ds = 1 + \int_0^t \alpha x(s-\tau)\,ds = 1.$$

For $t \in [\tau, 2\tau]$,

$$x(t) = x(\tau) + \int_\tau^t \alpha x(s-\tau)\,ds = 1 + \alpha \int_\tau^t 1\,ds = 1 + \alpha(t-\tau).$$

In general for $t \in [n\tau, (n+1)\tau]$, we have

$$x_n(t) = x_{n-1}(n\tau) + \int_{n\tau}^t \alpha x_{n-1}(s-\tau)\,ds = x_{n-1}(n\tau) + \alpha \int_{(n-1)\tau}^{t-\tau} x_{n-1}(s)\,ds, \tag{A.0.2}$$

where $x_n(t) := x(t)$ when $t \in [n\tau, (n+1)\tau]$. We proceed the rest of the proof by *induction*. Suppose

$$x_n(t) = \sum_{j=0}^{n} \frac{\alpha^j}{j!}(t-j\tau)^j, \quad \text{for } t \in [n\tau, (n+1)\tau].$$

If we could should that

$$x_{n+1}(t) = \sum_{j=0}^{n+1} \frac{\alpha^j}{j!}(t - j\tau)^j, \quad \text{for } t \in [(n+1)\tau, (n+2)\tau], \tag{A.0.3}$$

then the proof is complete. Now by (A.0.2),

$$\begin{aligned}
x_{n+1}(t) &= x_n((n+1)\tau) + \alpha \int_{n\tau}^{t-\tau} x_n(s)\, ds \\
&= 1 + \sum_{j=1}^{n} \frac{\alpha^j}{j!}((n+1)\tau - j\tau)^j + \alpha \int_{n\tau}^{t-\tau} 1 + \sum_{j=1}^{n} \frac{\alpha^j}{j!}((s - j\tau)^j\, ds \\
&= 1 + \sum_{j=1}^{n} \frac{\alpha^j}{j!}((n+1)\tau - j\tau)^j + \alpha(t-\tau) - \alpha n\tau + \sum_{j=1}^{n} \frac{\alpha^{j+1}}{j!} \int_{n\tau}^{t-\tau} (s - j\tau)^j\, ds \\
&= 1 + \sum_{j=1}^{n} \frac{\alpha^j}{j!}((n+1)\tau - j\tau)^j + \alpha(t-\tau) - \alpha n\tau \\
&\qquad\qquad + \sum_{j=1}^{n} \frac{\alpha^{j+1}}{(j+1)!}[(t - (1+j)\tau)^{j+1} - (n\tau - j\tau)^{j+1}] \\
&= 1 + \left[\sum_{k=2}^{n+1} \frac{\alpha^k}{k!}(t - k\tau)^k + \alpha(t-\tau)\right] + \left[\sum_{j=1}^{n} \frac{\alpha^j}{j!}((n+1)\tau - j\tau)^j - \alpha n\tau\right] \\
&\qquad\qquad - \sum_{j=1}^{n} \frac{\alpha^{j+1}}{(j+1)!}(n\tau - j\tau)^{j+1} \\
&= 1 + \sum_{k=1}^{n+1} \frac{\alpha^k}{k!}(t - k\tau)^k + \sum_{j=2}^{n} \frac{\alpha^j}{j!}((n+1)\tau - j\tau)^j - \sum_{k=2}^{n+1} \frac{\alpha^k}{k!}((n+1)\tau - k\tau)^k \\
&= \sum_{k=0}^{n+1} \frac{\alpha^k}{k!}(t - k\tau)^k.
\end{aligned}$$

We get the final line in the above equation by the fact that the last two terms in the penultimate line are equal. Since $r(t) = e^{at}x(t)$, we therefore obtain the desired result (A.0.1). $\square$

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