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DERIVATIONS OF RELATIVISTIC FORCE TRANSFORMATION EQUATIONS

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Abstract—The correct general form of relativistic transformation equations for the three-vector force is derived without using four-vectors, via the relativistic Newton's second law. The four-vector approach to the problem is also presented. The derivations extend or rectify previous derivations.

1. INTRODUCTION

Jefimenko [1] shows that the Lorentz force expression $\mathbf{f}_L = q(\mathbf{E}+\mathbf{u}\times\mathbf{B})$ transforms with respect to the standard Lorentz transformation

$$x' = \gamma_V(x - Vt), \quad y' = y, \quad z' = z, \quad t' = \gamma_V\left(t - Vx/c^2\right), \tag{1}$$

where $\gamma_V = (1 - V^2/c^2)^{-1/2}$, according to equations

$$\begin{aligned} f'_{Lx} &= \frac{f_{Lx} - (V/c^2)\mathbf{f}_L \cdot \mathbf{u}}{1 - Vu_x/c^2}, \\ f'_{Ly} &= \frac{f_{Ly}}{\gamma_V(1 - Vu_x/c^2)}, \quad f'_{Lz} = \frac{f_{Lz}}{\gamma_V\left(1 - Vu_x/c^2\right)}, \end{aligned} \tag{2}$$

where $\mathbf{f}'_L = q(\mathbf{E}' + \mathbf{u}' \times \mathbf{B}')$. (In this paper the unprimed and primed

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quantities refer to inertial frames Σ and Σ', respectively; the frames are in the standard configuration, Σ' moves relative to Σ with speed V in the positive direction of their common x, x'-axes.) Since the time derivative of the relativistic momentum of a particle with *time-independent* mass m transforms in the same way [2],

$$\begin{aligned} dp'_x/dt' &= \frac{dp_x/dt - (V/c^2)(d\mathbf{p}/dt)\cdot\mathbf{u}}{1 - Vu_x/c^2}, \\ dp'_y/dt' &= \frac{dp_y/dt}{\gamma_V\,(1 - Vu_x/c^2)}, \quad dp'_z/dt' = \frac{dp_z/dt}{\gamma_V\,(1 - Vu_x/c^2)}, \end{aligned} \tag{3}$$

where $\mathbf{p} = m\mathbf{u}/\sqrt{1 - u^2/c^2}$, $\mathbf{p}' = m\mathbf{u}'/\sqrt{1 - u'^2/c^2}$, the author thus shows that the well known relativistic equation of motion of a charged particle in the presence of electric and magnetic fields in the laboratory frame (reference frame Σ)

$$\frac{d}{dt}\left(\frac{m\mathbf{u}}{\sqrt{1 - u^2/c^2}}\right) = q(\mathbf{E} + \mathbf{u}\times\mathbf{B}), \tag{4}$$

is covariant with respect to the Lorentz transformation (1), i.e., as a purely mathematical fact, Equation (4) implies Equation (5)

$$\frac{d}{dt'}\left(\frac{m\mathbf{u}'}{\sqrt{1 - u'^2/c^2}}\right) = q(\mathbf{E}' + \mathbf{u}'\times\mathbf{B}'), \tag{5}$$

and *vice versa*. (Planck [3] was the first to derive and recognize the relativistic equation of motion (4), introducing tacitly the assumption that m is time-independent in his argument. The same tacit assumption is found in the corresponding Einstein's 1905 argument [7].)

It seems that Reference [1] bridges a gap in teaching relativistic electrodynamics. Namely, in many textbooks dealing with the special theory of relativity, the fact that $\mathbf{f}_L = q(\mathbf{E} + \mathbf{u}\times\mathbf{B})$ transforms in the same way as $d\mathbf{p}/dt$ with $m = \text{const}$ is taken (almost always tacitly) as *axiomatic*, and used, e.g., for deriving electric and magnetic fields of a uniformly moving charge from Coulomb's law. (Deriving electric and magnetic fields of a point charge in uniform motion appears to be a recurrent topic (cf, e.g., [4–6]). However, the validity of Equation (2) is far from being self-evident. Another didactic virtue of [1] is that it clearly reveals that the transformation law of the Lorentz force $\mathbf{f}_L$ is determined by the laws of transformation of the quantities composing it, q, $\mathbf{E}$, $\mathbf{u}$ and $\mathbf{B}$. True, deriving the transformations of $\mathbf{E}$ and $\mathbf{B}$ without use of four-tensors, following Einstein's original crooked path [7, 8], from the condition that the so-called source-free Maxwell's equations $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$, $\nabla\cdot\mathbf{B} = 0$ obey the principle of special relativity, is a real *tour de force*.

Jefimenko's argument in [1] implies that any relativistic three-vector force transforms according to equations of the form (2). It appears, however, that the concept of relativistic force has a broader connotation, including forces that involve a time-dependent mass of a particle on which the forces act [9–15]. Thus equations of the form (2) do not express transformation formulae for the relativistic three-vector force (henceforth, "relativistic force") in the general case. On the other hand, formulae that are occasionally presented in the literature as the general form of relativistic force transformation equations [11, 14] are not correct, since the transformation law of a relativistic force generally is *not* determined by time dependence of the mass of a particle.

In this paper the correct general form of relativistic force transformation equations, apparently not found in the literature, is derived in simple and direct ways. In Section 2, a derivation of the transformation formulae without use of four-vectors, via the relativistic Newton's second law, is given. In Section 3 the four-vector approach to the problem is presented, expressing the transformation formulae in the language of four-vectors. Since relativistic force transformation equations are closely related with a variety of basic concepts, we believe that the present paper could be helpful for teaching special relativity and relativistic electrodynamics. Despite ramified applications of relativistic electrodynamics (cf, e.g., [16–20]), it seems that some of its basic concepts still need clarification.

2. GENERAL FORCE TRANSFORMATIONS WITHOUT FOUR-VECTORS

Our derivation follows, *mutatis mutandis*, a simple argument presented by French [2].

The relativistic momentum $\mathbf{p}$ and energy E of a particle relative to the Σ frame, are defined by

$$\mathbf{p} = m\mathbf{u}\gamma_u, \quad E = mc^2\gamma_u, \tag{6}$$

where m is the mass of the particle, $\mathbf{u}$ is its instantaneous velocity, and $\gamma_u = (1 - u^2/c^2)^{-1/2}$; the corresponding Σ' quantities are defined in the same way

$$\mathbf{p}' = m\mathbf{u}'\gamma_{u'}, \quad E' = mc^2\gamma_{u'}. \tag{7}$$

Since the x component of $\mathbf{p}'$ transforms as [2]

$$p'_x = \gamma_V(p_x - VE/c^2), \tag{8}$$

a simple calculation reveals that the time derivative of p'_x transforms as

$$dp'_x/dt' = \frac{dp_x/dt - (V/c^2)dE/dt}{1 - Vu_x/c^2}, \tag{9}$$

all with respect to the Lorentz transformation (1). From the "right-angled triangle identity"

$$E^2 = c^2p^2 + m^2c^4, \tag{10}$$

using equation $\mathbf{p}/E = \mathbf{u}/c^2$, it follows that

$$dE/dt = (d\mathbf{p}/dt) \cdot \mathbf{u} + (c^2/\gamma_u)dm/dt. \tag{11}$$

(Contrary to the tacit assumption made in [2], we assume that m is in the general case a time dependent Lorentz invariant quantity.) Introducing the "proper" time τ elapsed at the particle, $d\tau = dt/\gamma_u$, Equation (11) can be recast as

$$dE/dt = (d\mathbf{p}/dt) \cdot \mathbf{u} + (1/\gamma_u^2)c^2dm/d\tau. \tag{12}$$

Equations (9) and (12) imply

$$dp'_x/dt' = \frac{dp_x/dt - (V/c^2)[(d\mathbf{p}/dt) \cdot \mathbf{u} + (1/\gamma_u^2)c^2dm/d\tau]}{1 - Vu_x/c^2}. \tag{13}$$

Now the correct relativistic generalization of Newton's second law reads

$$d\mathbf{p}/dt = \mathbf{f}, \tag{14}$$

where three-vector $\mathbf{f}$ is a relativistic (Newtonian) force, representing a physical agent which determines the time derivative of the momentum of a particle on which it acts according to Equation (14). On the other hand, the principle of relativity requires that equation of the same form and content

$$d\mathbf{p}'/dt' = \mathbf{f}', \tag{15}$$

applies in the Σ' frame. From the transformation law (13) and Equations (14) and (15) it follows that

$$f'_x = \frac{f_x - (V/c^2)[\mathbf{f} \cdot \mathbf{u} + (1/\gamma_u^2)c^2dm/d\tau]}{1 - Vu_x/c^2}. \tag{16}$$

While Equation (16) is occasionally presented in the literature as transformation equation for the x component of $\mathbf{f}'$ [11, 14], actually this is not so. Namely, despite appearances, transformation equations for a relativistic force exist on their own, regardless of the validity of Equations (14) or (15). True, Equations (14) and (15) are used as scaffolding in the present derivation, but when the construction is over the scaffolding should be removed (in the same way as the "rest", "relativistic", "longitudinal" and "transverse" masses should be removed from special relativity [21]). A little reflection reveals, taking into account that expression $c^2dm/d\tau$ appearing in Equation (16) is Lorentz invariant, that the correct transformation equation for f'_x must have the form

$$f'_x = \frac{f_x - (V/c^2)(\mathbf{f} \cdot \mathbf{u} + \chi/\gamma_u^2)}{1 - Vu_x/c^2}, \tag{17}$$

where χ is a Lorentz invariant quantity. For a given relativistic force $\mathbf{f}$, according to the principle of relativity, $\mathbf{f}'$ depends on Σ'-quantities in the same way as $\mathbf{f}$ depends on the corresponding Σ-quantities, so χ can be ascertained by directly transforming the quantities composing the force $\mathbf{f}'$. Thus, somewhat surprisingly, expression for $\mathbf{f}$ carries information on both $\mathbf{f}'$ and χ.

Transformation equations for the y and z components of $\mathbf{f}'$ are obtained in the same way but much more simply. Since $p'_y = p_y$, $p'_z = p_z$, one gets

$$dp'_y/dt' = \frac{dp_y/dt}{\gamma_V(1 - Vu_x/c^2)}, \quad dp'_z/dt' = \frac{dp_z/dt}{\gamma_V(1 - Vu_x/c^2)}, \tag{18}$$

and thus, using Equations (14) and (15),

$$f'_y = \frac{f_y}{\gamma_V(1 - Vu_x/c^2)}, \quad f'_z = \frac{f_z}{\gamma_V(1 - Vu_x/c^2)}. \tag{19}$$

Equations (17) and (19) are the correct relativistic force transformation equations valid in the general case.

As a by-product of the preceding analysis it follows that Equation (14) *and* equation that determines the "proper" time derivative of the mass of a particle on which the relativistic force $\mathbf{f}$ acts

$$c^2 dm/d\tau = \chi, \tag{20}$$

imply Equation (15), since transformation Equations (13), (18), (17) and (19) apply. Thus only Equations (14) and (20) *taken together* are Lorentz covariant equations of motion of a particle under the action of the force $\mathbf{f}$.

It is not difficult to find transformation equation for $\mathbf{f}'$ in the case when the Σ' frame moves with an arbitrary constant velocity $\mathbf{V} = (V_x, V_y, V_z)$ relative to the Σ frame, and the corresponding coordinate axes are still parallel. (The problem of how to define parallel axes in that case is discussed, e.g., in [9, 22].) The transformation equation reads

$$\mathbf{f}' = \frac{\mathbf{f}/\gamma_V - (\mathbf{V}/c^2)\{\mathbf{f}\cdot\mathbf{u} + \chi/\gamma_u^2 - [1 - (1/\gamma_V)](\mathbf{f}\cdot\mathbf{V})c^2/V^2\}}{1 - \mathbf{V}\cdot\mathbf{u}/c^2}, \tag{21}$$

As an example, consider the Lorentz force $\mathbf{f}_L = q(\mathbf{E} + \mathbf{u}\times\mathbf{B})$. Comparing the first Equation (2) and Equation (17), it follows that for the Lorentz force the corresponding scalar χ vanishes identically. Then, supposing that the Lorentz force is the only force acting on a charged particle in the electromagnetic field, Equation (20) implies that m is constant, consistent with Planck's assumption tacitly made in [3]. As another example, consider a three-force $\mathbf{f} = -(g/\gamma_u)\nabla\Phi$, where a Lorentz scalar g plays the same role as does charge in electromagnetism, and $\Phi = \Phi(\mathbf{r}, t)$ is a Lorentz scalar field; $\mathbf{f}$ represents the relativistic force arising in the scalar meson theory of the nucleus [10, 12–14]. A simple calculation reveals that $\mathbf{f}'$ transforms according to Equations (17) and (19), with $\chi = gd\Phi/d\tau$. (In the calculation, equations $\gamma_{u'} = \gamma_u \gamma_V (1 - Vu_x/c^2)$ [2] and $d\Phi/dt = \partial\Phi/\partial t + (\nabla\Phi) \cdot \mathbf{u}$ are used.) Then Equations (14) and (20) imply that the mass of a particle which moves in the field Φ is given by

$$m = m_{FF} + g\Phi/c^2, \tag{22}$$

where m_{FF} is the field-free mass of the particle [12, 13].

3. THE FOUR-VECTOR APPROACH

For the sake of completeness, in this Section a formal derivation of transformation Equations (17) and (19) via four-vectors is presented, revealing their four-vector content.

Contravariant components of the four-momentum of a particle P^α and its four-velocity U^α are given by equations

$$P^\alpha = mU^\alpha, \quad U^\alpha = \gamma_u(c, \mathbf{u}), \tag{23}$$

which implies

$$P^\alpha = (E/c, \mathbf{p}). \tag{24}$$

(We use the diagonal metric tensor with elements $g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -1$.)

The four-vector equation of motion of a particle under the action of a four-force with contravariant components $F^\alpha = (F^0, \mathbf{F})$ is

$$dP^\alpha/d\tau = F^\alpha. \tag{25}$$

Since $d\tau = dt/\gamma_u$, introducing $\mathbf{f} \equiv \mathbf{F}/\gamma_u$, Equation (25) can be written in the form

$$dE/dt = F^0 c/\gamma_u, \quad d\mathbf{p}/dt = \mathbf{f}. \tag{26}$$

Thus $\mathbf{f}$ is a relativistic force. Transformation equations for the Cartesian components of $\mathbf{f}' \equiv \mathbf{F}'/\gamma_{u'}$ then follow from the well known transformation equations for F'^α

$$F'^x = \gamma_V(F^x - \beta F^0), \;\; F'^y = F^y, \;\; F'^z = F^z, \;\; F'^0 = \gamma_V(F^0 - \beta F^x), \tag{27}$$

where $\beta = V/c$, and for $\gamma_{u'}$

$$\gamma_{u'} = \gamma_u \gamma_V (1 - Vu_x/c^2). \tag{28}$$

(Equation (28) follows directly from $U'^0 = \gamma_V(U^0 - \beta U^x)$.) Thus one gets that f'_x transforms according to equation

$$f'_x = \frac{f_x - (V/c^2)F^0 c/\gamma_u}{1 - Vu_x/c^2}, \tag{29}$$

whereas f'_y and f'_z transform according to Equation (19).

By using identity $F^0 \equiv \mathbf{F}\cdot\mathbf{u}/c + F^\alpha U_\alpha/\gamma_u c$, Equation (29) can be recast as

$$f'_x = \frac{f_x - (V/c^2)(\mathbf{f}\cdot\mathbf{u} + F^\alpha U_\alpha/\gamma_u^2)}{1 - Vu_x/c^2}, \tag{30}$$

which coincides with Equation (17), revealing the four-vector content of the Lorentz invariant χ.

(By the way, the preceding considerations imply that if any three-vector $\mathbf{g}$ transforms with respect to the Lorentz transformation (1) according to equations

$$g'_x = \frac{g_x - (V/c^2)(\mathbf{g}\cdot\mathbf{u} + \xi/\gamma_u^2)}{1 - Vu_x/c^2},$$
$$g'_y = \frac{g_y}{\gamma_V(1 - Vu_x/c^2)}, \quad g'_z = \frac{g_z}{\gamma_V(1 - Vu_x/c^2)},$$

where ξ is a Lorentz invariant, then $\mathbf{g}$ uniquely determines a four-vector with contravariant components $G^\alpha = (\gamma_u\mathbf{g}\cdot\mathbf{u}/c + \xi/\gamma_u c, \gamma_u\mathbf{g})$.)

Note that Equation (25), using identities $U^\alpha U_\alpha \equiv c^2$ and $(dU^\alpha/d\tau)U_\alpha \equiv 0$, implies

$$c^2 dm/d\tau = F^\alpha U_\alpha, \tag{31}$$

and consequently

$$m dU^\alpha/d\tau = F^\alpha - U^\alpha F^\delta U_\delta/c^2, \tag{32}$$

and *vice versa*: Equations (31) and (32) imply Equation (25). Thus Equation (25) is tantamount to Equations (31) and (32).

From the preceding considerations it is clear that two kinds of four-forces are possible, depending on whether $F^\alpha U_\alpha$ vanishes identically or not. Simple four-force for which $F^\alpha U_\alpha \equiv 0$ appears in the literature as the *Minkowski* force [23], or *driving* force [9], or *pure* force [14].

For the convenience of the reader, recall that contravariant components of the four-force F_L^α corresponding to the Lorentz force $\mathbf{f}_L = q(\mathbf{E} + \mathbf{u}\times\mathbf{B})$ can be expressed as

$$F_L^\alpha = qF^{\alpha\beta}U_\beta, \tag{33}$$

where $F^{\alpha\beta}$ are contravariant components of the electromagnetic field tensor defined by equations $F^{\alpha\beta} = -F^{\beta\alpha}$, $(F^{01}, F^{02}, F^{03}) =$

$(-1/c)(E_x,\, E_y,\, E_z)$, $(F^{32},\, F^{13},\, F^{21}) = (B_x,\, B_y,\, B_z)$. Thus identity $F_L^{\alpha}U_{\alpha} \equiv 0$ is a direct consequence of the fact that $F^{\alpha\beta}$ is antisymmetric. (The four-force corresponding to the other relativistic force $\mathbf{f} = -(g/\gamma_u)\nabla\Phi$ discussed above (the "impure" one) has covariant components $F_{\alpha} = \partial\Phi/\partial x^{\alpha}$.)

As can be seen, our above argument reveals that Planck's tacit assumption that the mass of a charged particle that moves in the electromagnetic field is constant [3] is tantamount to assuming that the Minkowski force (33) is the only four-force acting on the particle in the field, i.e., there is no additional "impure" one term (involving also the radiation reaction force).

4. CONCLUSIONS

We have presented simple and direct derivations of relativistic transformation equations for the three-vector force, which generalize or correct previous derivations [1, 2, 14]. Our argument is free from unnecessary assumptions or confusion between distinct concepts, providing another illustration of the importance of Ockham's razor in relativistic considerations [24, 25]. As such it might be helpful for teaching relativistic electrodynamics and special relativity.

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