CYLINDRICALLY SYMMETRIC MODELS OF GRAVITATIONAL COLLAPSE

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Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Doctor of Philosophy is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

Signed: Louise Noble ID No.: 52159761

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Contents

Abstract

In this thesis we examine two main problems. Firstly, we attempt to match the most general cylindrically symmetric vacuum spacetime with a Robertson-Walker interior. The matching conditions show that the interior must be dust filled, the boundary must be comoving and the vacuum region must be polarized. We use a result of Thorne's to simplify the line element. We can then prove that the matching is impossible. This demonstrates the impossibility of generalising the Oppenheimer-Snyder model of gravitational collapse to the cylindrically symmetric case The second problem is an analysis of cylindrically symmetric spacetimes with self-similarity modelling gravitational collapse The field equations and regularity conditions are examined firstly for a vacuum spacetime and then for a dust filled spacetime. The vacuum case leads to an explicit solution but no solutions that are of relevance to gravitational collapse. In the dust case, the solution of the field equations reduces to the solution of a non-linear third-order ordinary differential equation. A dynamical systems approach is then adopted, and an autonomous three-dimensional system is obtained. A unique solution is found to emanate from the regular axis $\{r = 0,\ t < 0\}$, where t and r are time and radial coordinates which emerge naturally from the analysis This solution persists up to $\{t = 0,\ r > 0\}$, which we define as Σ_0 The solution coming from Σ_0 has one parameter (a bifurcation has occurred) and propagates up to the future null cone, $\mathcal{F}$, through the scaling origin p_o, where $p_o = \{(r,t) = (0,0)\}$. We describe the physical invariants of the system and discuss the nature of such a spacetime in terms of its global structure.

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Part I

Introduction

Chapter 1

Notation

In Part II we shall use lower-case Roman indices running through $0, 1, 2, 3$ and Greek indices running through $1, 2, 3$ (labelling the hypersurface coordinates) with prime and overdot referring to differentiation with respect to T and t respectively In Part III we shall use lower-case Roman letters to denote spacetime indices $0, 1, 2, 3$ where 0 is the time direction. Lower-case Greek letters will denote space indices $1, 2, 3$. We shall also employ the Einstein summation convention as follows: in any expression containing a superscript and subscript which are identical, a summation over the repeated index is to be understood.

The dynamical systems analysis used in Part III requires several changes of independent variable. The original independent variable is

$$s = \ln|\xi| = \ln\left(\frac{r}{|t|}\right),$$

and we will call our new independent variables $\tau, T, \lambda, \tilde{\lambda}$. There will be some repetition of these terms. Apart from Chapter 5 (in which τ occurs twice) there will be no repetition within a chapter. We hope that this does not cause confusion, it is purely to reduce terminology. This part of the thesis also has numerous equilibria. We shall use the notation E_i, E_i^*, L_i and P_i where i labels the equilibrium point or set. There will be repetition of this notation, but never within a chapter.

Parameters will be denoted as a letter with a hat, e.g. $\hat{y}$ We use units where $c = 8\pi G = 1$.

The Levi-Civita symbol is given by

$$\epsilon^{abcd} = \begin{cases} +1 & \text{if (a,b,c,d) is an even permutation of (0,1,2,3)} \\ -1 & \text{if (a,b,c,d) is an odd permutation of (0,1,2,3)} \\ 0 & \text{if any two labels are repeated.} \end{cases}$$

Chapter 2

Background and summary

One of the outstanding issues of General Relativity (GR) is that of the final state of gravitational collapse. GR predicts that a star of sufficiently large mass, in the last stage of its evolution, will contract without limit due to its huge gravity, and form a spacetime singularity

It was proved by Hawking and Penrose [17] that singularities appear for any spacetime symmetry. However, their singularity theorems do not give information about how the spacetime curvature or energy density diverges to form the singularity. Singularities can occur as a black hole (which cannot be observed) or a naked singularity (which can be observed) A naked singularity represents a point at which all physical laws must break down. Moreover, it has the potential to influence the external universe, threatening predictability in physical laws everywhere.

To avoid this problem Penrose proposed his Cosmic Censorship Conjecture (CCC) [30], which states that all singularities in gravitational collapse are hidden within black holes (weak conjecture). Although examples of naked singularities have been found in theoretical models, they are often considered too unrealistic to be a real counter-example to the CCC Indeed naked singularities are found in the collapse of cylindrically symmetric fluids but such examples do not refute the CCC as they are not asymptotically flat spacetimes (and the strict symmetry is not considered physically realistic). However, an asymptotically flat model could be constructed with a cylindrically symmetric portion, and so studies of cylindrically symmetric spacetimes can be thought of as physically viable.

The self-similarity hypothesis is another important advance in the study of gravitational collapse [9]. The hypothesis asserts that under a variety of physical circumstances solutions will naturally evolve to a self-similar form. Self-similarity is highly relevant to the CCC as many of the putative counter-examples involve self-similarity. Self-similarity also plays an important role in critical collapse behaviour. Critical collapse studies the phase space of isolated gravitating systems, for a variety of matter sources, in terms of basins of attraction: collapse to a black hole, formation of a stable star, or dispersion (leaving empty flat spacetime behind) Any one-parameter (say p) family of initial data was found to have a critical value $p = p^*$ such that for $p > p^*$ a black hole is formed and for $p < p^*$ no black hole is formed. In addition, near the critical value $p \simeq p^*$ a universal scaling relation was found for the black hole mass [16]

$$M_{BH} = C(p)(p - p^*)^\gamma,$$

where $C(p)$ is a constant which depends on the initial data. In the original Choptuik model of spherically symmetric massless scalar field, the critical exponent γ is universal to all families of initial data studied and has been numerically determined as $\gamma \sim 0.37$.

So it is clear that even models restricted to spherical symmetry show that gravitational collapse is a topic of great richness and complexity Indeed, the majority of studies in gravitational collapse are with spherically symmetric spacetimes, which do not contain gravitational radiation, and studies in cylindrically symmetric collapse suggest that gravitational radiation may be a major factor in how collapse proceeds, see §2.2

In order to gain insight into non-spherical gravitational collapse and the nonlinearity of the field equations, we study a cylindrically symmetric spacetime filled with dust. Cylindrical symmetry has the advantage of producing tractable field equations, displaying gravitational radiation and naked singularities while departing from spherical symmetry. So it is with a view to presenting a realistic non-spherical collapse model that we study a cylindrically symmetric self-similar dust spacetime in Part III of this thesis

In the remainder of this chapter we shall sketch the development of the general theory of relativity and introduce the quantities and concepts that are to be used in this thesis. Next we will review the current status of research in cylindrically symmetric gravitational collapse models and the self-similarity hypothesis. We will outline the tools from dynamical systems analysis which are needed in the analysis of Part III, Chapters 4-7 To conclude this chapter, we give a brief summary of the thesis.

2.1 General theory of relativity

2.1.1 Differential geometry

Before describing the formulation of GR it is necessary to describe some of the mathematical concepts involved. We begin with $\mathcal{M}$, a 4-dimensional C^r manifold with a C^r atlas $\{(U_\alpha, \Phi_\alpha)\}$ on $\mathcal{M}$, where each (U_α, Ψ_α) is called a local chart and consists of an open set U_α of $\mathcal{M}$ and a one-to-one mapping $\Psi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$. We also have the property that the U_α cover $\mathcal{M}$, i.e. $\mathcal{M} = \bigcup_\alpha U_\alpha$. A local chart enables us to assign local coordinates $\{x^a\} = (x^0, x^1, x^2, x^3) = (t, x, y, z)$ to points in U_α. Consider a point $p \in \mathcal{M}$. Let $\mathcal{F}_p(\mathcal{M})$ be the set of all C^∞ functions defined on some open neighbourhood of p A tangent vector at p is a real valued function v_p $\mathcal{F}_p(\mathcal{M}) \rightarrow \mathbb{R}$. We denote the set of all tangent vectors at p as $T_p(\mathcal{M})$ and with suitable properties $T_p(\mathcal{M})$ becomes a linear vector space We can express $v_p \in T_p(\mathcal{M})$ in terms of a local basis $\{\frac{\partial}{\partial x^a}\} = \{\partial_a\} = \{e_a\}$ to get

$$v_p = v_p^a \frac{\partial}{\partial x^a}.$$

Elements of $T_p(\mathcal{M})$ are called contravariant vectors We have the dual vector space $T_p^*(\mathcal{M})$ where any $w_p \in T_p^*(\mathcal{M})$ is called a covariant vector and can be written as

$$w_p = w_a^p dx^a,$$

where $\{dx^a\} = \{e^a\}$ is the basis dual to $\{e^a\}$ with $e^a e_b = \delta_b^a$.

Given two coordinate systems $\{x^a\}$ and $\{x^{a'}\}$, both covering a region U, if we define the Jacobian matrices

$$X^{a'}_{b} = \frac{\partial x^{a'}}{\partial x^{b}}, \quad X^{a}_{b'} = \frac{\partial x^{a}}{\partial x^{b'}}, \quad X^{a}_{b'}X^{b'}_{c} = \delta^{a}_{c},$$

then we can also define a contravariant vector v at a point $p \in \mathcal{M}$ as an object having four components v^a which under a change of coordinates about p transform according to

$$v^a = X^a_{b'}v^{b'},$$

where the partial derivatives are evaluated at p. A covariant vector is similarly defined, and transforms according to

$$w_a = X^{b'}_{a} w_{b'}.$$

We go on to define mixed tensors in the usual way, e.g., a tensor $\mathbf{T}$ of type $\binom{1}{2}$ has components T^a_{bc} which satisfy

$$T^{a'}_{b'c'} = \frac{\partial x^{a'}}{\partial x^{i}}\frac{\partial x^{j}}{\partial x^{b'}}\frac{\partial x^{k}}{\partial x^{c'}}T^{i}_{jk}.$$

Thus tensorial equations are coordinate independent or covariant

At each $p \in \mathcal{M}$ we define a symmetric, bilinear and non-degenerate mapping g_p,

$$g_p \cdot T_p(\mathcal{M}) \times T_p(\mathcal{M}) \to \mathbb{R},$$

by symmetric we mean that for all $u, v \in T_p(\mathcal{M})$ we have that $g(u,v) = g(v,u)$ and by non-degenerate we mean that there is no non-zero vector $u \in T_p(\mathcal{M})$ such that $g(u,v) = 0$ for all $v \in T_p(\mathcal{M})$. A metric tensor g on $\mathcal{M}$ is the specification of such a mapping g_p at each $p \in \mathcal{M}$. We can define a unique symmetric tensor of type $\binom{0}{2}$ with components, g_{ab} with respect to the basis $\{e_a\}$, by the relations

$$g^{ab}g_{bc} = \delta^a_c,$$

the tensors g^{ab} and g_{ab} can be used to raise and lower indices e.g

$$T^a_{\ b} = g^{ac}T_{cb}$$

We define the interval between two events as

$$ds^2 = g_{ab}dx^a dx^b.$$

The metric tensor components g_{ab} allows us to write the length of a contravariant nonzero vector v^a as

$$|g_{ab}v^a v^b|^{\frac{1}{2}} = |v^a v_a|^{\frac{1}{2}}.$$

So we describe v^a as follows:

$$\begin{aligned} \text{timelike if} \quad g_{ab}v^a v^b &< 0, \\ \text{spacelike if} \quad g_{ab}v^a v^b &> 0, \\ \text{null if} \quad g_{ab}v^a v^b &= 0, \end{aligned}$$

and if the tangent vector to a curve is everywhere null, we describe the curve as null. Similarly we define a timelike (or spacelike) curve as a curve whose tangent vector is everywhere timelike (or spacelike). A particle with mass follows a timelike path or curve, while a photon follows a null path. A material particle's path through spacetime is called its world line and the interval (or proper time interval) between points on its world line is given by $d\tau^2 = -ds^2$, and we say that the particle's world velocity is $u^a = \frac{dx^a}{d\tau}$.

A Lorentzian metric is one which can be diagonalised at any point to the matrix form diag$(-1, 1, 1, 1)$ and which therefore has trace (or signature) $sgn = +2$. We define spacetime as a (connected, Hausdorff) Riemannian manifold $\mathcal{M}$ on which a Lorentzian metric tensor g is defined at each point $p \in \mathcal{M}$.

It is not straightforward to introduce derivatives acting on vector fields because derivatives involve taking the limit of the difference of vectors at different points, but these vectors will belong to different tangent spaces. The usual partial derivative does not preserve tensor character. Introducing a covariant method of differentiation adds a structure to the manifold, an affine connection, which essentially describes how to parallelly transport a tensor in our curved spacetime The metric connection is chosen

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}\left(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}\right).$$

We can use this to define the covariant derivative of a tensor of type $\binom{k}{l}$ as

$$\begin{aligned}\nabla_c T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} \equiv T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l ; c} &= \partial_c T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} + \Gamma^{a_1}_{dc} T^{d \cdots a_k}{}_{b_1 \cdots b_l} + \cdots + \Gamma^{a_k}_{dc} T^{a_1 \cdots d}{}_{b_1 \cdots b_l} \\ &\quad - \Gamma^{d}_{b_1 c} T^{a_1 \cdots a_k}{}_{d \cdots b_l} - \cdots - \Gamma^{d}_{b_l c} T^{a_1 \cdots a_k}{}_{b_1 \cdots d}.\end{aligned}$$

The metric connection has the property that the covariant derivative of the metric vanishes.

Next we can write out the Riemann tensor, which measures the non-commutativity of the covariant derivative:

$$\nabla_a \nabla_b V^d - \nabla_b \nabla_a V^d = R^d{}_{abc} V^c,$$

and can be written in terms of the connection as follows.

$$R^a{}_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed}.$$

The Ricci tensor and scalar are both contractions of the Riemann tensor which measure the curvature of the manifold

$$R_{cd} = g^b_a R^a{}_{cbd} = \delta^b_a R^a{}_{dbc}, \quad R = g^{ab} R_{ab}.$$

These are used to construct the Einstein tensor

$$G^{ab} = R^{ab} - \frac{1}{2} g^{ab} R.$$

Next we introduce the energy momentum tensor, T^{ab}, which covariantly describes the matter and energy content of spacetime. For perfect fluid we have

$$T^{ab} = \left(\rho + \frac{p}{c^2}\right) u^a u^b - p g^{ab}, \tag{2.1.1}$$

where ρ is the density, p is the pressure, $u^a = \frac{\mathrm{d}x^a}{\mathrm{d}\tau}$ is the world velocity of the fluid and c is the speed of light in a vacuum. In Part II of this thesis we will use an energy momentum tensor of the form (2.1.1). In Part III we will use the energy momentum tensors representing a vacuum $T^{ab} = 0$ and dust $T^{ab} = \rho u^a u^b$

2.1.2 Formulation of the field equations

Following on from his Special Theory of Relativity (SR) (1905), Einstein developed General Relativity (GR), a relativistic theory of gravity in 1915. In SR all inertial frames are equivalent, and the equations of motion of a free particle in an inertial reference frame S', with local coordinates $\{x'^a\}$, are given by

$$\frac{\mathrm{d}^2 x'^a}{\mathrm{d}s^2} = 0$$

But if we move to a non-inertial or general reference frame S, with local coordinates $\{x^a\}$, then the equations of motion become

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}s^2} + \Gamma^a_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}s}\frac{\mathrm{d}x^c}{\mathrm{d}s} = 0. \qquad (2.1.2)$$

The second term in (2.1.2) represents inertial forces, which are locally indistinguishable from gravitational forces (by the principle of equivalence). We infer that the metric is playing the role of gravitational potentials. So in the presence of gravitation, spacetime is described by a 4-dimensional Riemannian manifold with a metric g in some local coordinates system $\{x^a\}$. Guided by the principle of minimal gravitational coupling and the principle of covariance, Einstein then formulated his famous field equations

$$G_{ab} = \kappa T_{ab},$$

where $\kappa = \frac{8\pi G}{c^4}$, (G is Newton's gravitational constant). In Part III of this thesis we will impose self-similarity to reduce the field equations from partial to ordinary differential equations. Dynamical systems analysis will then be used to study possible solutions.

2.1.3 Null geodesics and trapped surfaces

A curve $x^a(s)$ which is a solution to (2 1.2) and which obeys the null condition is a null geodesic, which can represent a light ray. The effect of spacetime curvature would be to focus or distort a small bundle of these rays. To quantify this effect we consider the expansion of a congruence of null geodesics We can think of the congruence of null geodesics as the histories of photons If we place a small circular opaque disk (radius $r = 1$) in the path of the photons so that the rays strike it perpendicularly, then a short distance dr from the disk a plane screen is placed so that the rays strike it perpendicularly. A shadow of the disk will appear on the screen, see Figure 2.1.

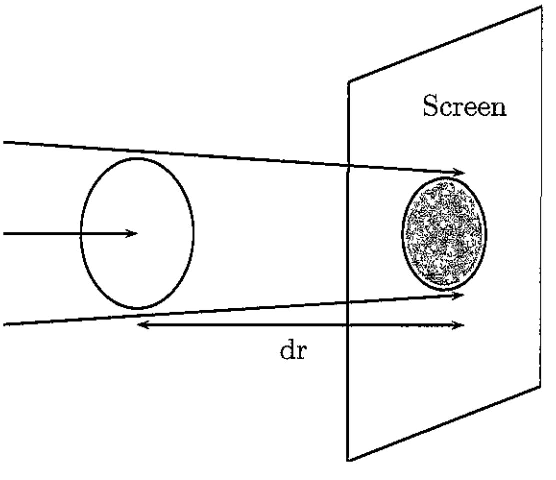

Figure 2.1. Shadow of a disk.

We will not consider the case where the shadow is a rotated or sheared/squeezed but where the radius of the shadow has changed to $r_s = 1 + \theta dr$: if the shadow has expanded then $\theta > 0$ and if the shadow has contracted $\theta < 0$. We can derive the formula for this expansion scalar as $\theta = \nabla_a v^a$ where v^a is the null tangent vector field to the bundle of rays, satisfying

$$v^a \nabla_b v^b = 0.$$

A trapped cylinder is a two-dim spacelike surface, T, (a cylinder of constant t and r) having the property that the expansion scalars $\theta^{\pm}$ of the ingoing and outgoing future directed null geodesics orthogonal to T are everywhere negative A compact trapped surface signals the presence of a singularity.

2.2 Review of research on cylindrical symmetry

Cylindrically symmetric spacetimes have the advantage of producing comparatively manageable field equations while departing from spherical symmetry. They also introduce two topics of great physical interest: gravitational radiation and naked singularities. Cylindrically symmetric spacetimes, unlike spherically symmetric spacetimes can contain gravitational waves. As the LIGO experiment and others are currently trying to actually detect and measure gravitational waves, this topic is of great interest There exist examples of naked singularities in cylindrically symmetric spacetimes and the cosmic censorship hypothesis, which postulates that a naked singularity cannot exist is another area of great interest. Much work has been done on this question but the hypothesis is still unproven.

The majority of recent research on cylindrical symmetry has focussed on these two main areas and their overlap. Einstein and Rosen (1937) made the first study of cylindrically symmetric gravitational waves [13]. Investigating a more general cylindrically symmetric vacuum, Thorne provided a neat argument to show that horizons cannot evolve in the vacuum region surrounding a collapsing infinite cylinder [38] He also introduced C energy, a localizable and locally measurable covariant vector which obeys a conservation law. C energy is shown to be propagated by Einstein-Rosen waves and by cylindrical electromagnetic waves [37]. He went on to propose his hoop conjecture: black holes with horizons form when and only when a mass M gets compacted into a region whose circumference in every direction is $C \leq \frac{4\pi GM}{c^2}$

The hoop conjecture was upheld in studies of spindle gravitational collapse [35] (but the cosmic censorship hypothesis was not). However, more analytical and numerical accuracy can be achieved using an infinitely long cylindrical distribution of matter which is considered an acceptable approximation to the more plausible spindle shaped matter-cloud. Apostolatos and Thorne [1] proved that even an infinitesimal amount of rotation can halt the collapse of an infinite cylindrical null dust shell Piran [32] found numerically that the collapse of an infinite rotating perfect fluid cylinder resulted in the emission of large amounts of gravitational radiation; up to 65% of the rest mass energy is released during the bounce.

Echeverria describes a gravitational wave burst just before the formation of a singularity in the cylindrically symmetric collapse of an infinite null dust shell [12].

Chiba investigated the gravitational collapse of a cylindrically symmetric dust fluid, assuming a form for the density function and performed numerical and analytical calculations. The author found negligible gravitational wave emission during the free fall time [11]. Nakao and Morisawa modelled the collapse of cylindrical dust fluid, not null dust [25], however it is not clear that the perturbation scheme used by the authors is consistent Assuming that, as the gravitational collapse proceeds, the speed of the collapsing matter approaches the speed of light, the deviation of the 4-velocity of the dust fluid from null is treated as a perturbation, and so linear perturbation analysis is applied. The authors found that in the shell approximation they got results which were consistent with Echeverria's - thinner widths led to greater amounts of gravitational radiation. They found that modelling dust collapse led to collapse occurring first on the symmetry axis and then accreting from outer regions The formation of the singularities are not almost simultaneous and thus less gravitational radiation is produced, which is consistent with Chiba's result.

Berger, Chrusciel and Moncrief proved that asymptotic flatness, energy conditions and cylindrical symmetry exclude the existence of compact trapped surfaces [6]. Solutions which describe the collapse of cylindrical shells of null dust are considered in [26] by Nolan, and it is shown that globally naked singularities can arise when the space-times are asymptotically flat for each fixed z.

These and other results furnish us with a clearer picture of non-spherical collapse in which gravitational radiation and angular momentum play an important role. In many instances these results are in sharp contrast to results for the corresponding spherically symmetric model. For example Berger *et al's* strong cosmic censorship result [6], Thorne's [38] result ruling out certain types of horizon. However, the purpose of this study is not specifically geared towards either of these two topics. In this thesis we obtain analytic solutions of the field equations for a cylindrically symmetric spacetime filled with dust. As we are interested in collapse scenarios we add the physically reasonable constraint of self-similarity. Whether gravitational radiation is emitted from our solution will not be discussed. Our main concern will be the global structure of the resulting spacetime. We will explore the existence and uniqueness of solutions, and we will then consider the formation and nature of singularities.

2.3 Motivation for self-similarity

Self-similarity plays an important role in a wide range of relativistic and Newtonian problems. We define a similarity (self-similar) solution of the field equations as one for which the resulting spacetime admits the homothetic Killing vector $\vec{k}$ satisfying

$$\mathcal{L}_{\vec{k}} g_{ab} = 2g_{ab}$$

This is called continuous self-similarity or similarity of the first kind. There is also a generalisation to continuous self-similarity called kinematic self-similarity, also called similarity of the zeroth, second and infinite kind. This form of self-similarity is not considered in this thesis. Much work has been done on continuous and kinematic self-similarity in spherical symmetric models.

Research has shown that solutions will naturally evolve to a self-similar form in many important situations. For a recent review see [9]. We are specifically interested in gravitational collapse, and this has been shown to exhibit critical phenomena in a wide variety of cases [16]. There are two types of critical collapse observed, type II has continuous self-similarity and type I does not. So in order to study critical collapse, solutions may be found by imposing self-similarity.

A detailed study of line elements and perfect fluid solutions with a G_2 isometry and a homothety has been performed in [7]. Recently Sharif *et al* have looked at cylindrically symmetric systems with self-similarity. They investigated cylindrically symmetric systems with perfect fluid and with kinematic self-similarity [36]. Three different equations of state were inserted into the field equations. Without a full solution to the (complicated) field equations the authors classify the solutions using two first order differential equations linking pressure and density terms which could be isolated from the field equations.

Work has also been done on self-similar cylindrically symmetric spacetimes with scalar fields [42]. In this study a class of exact solutions to the massless scalar field equations is found. This class is separated into two cases, one of which leads to a degenerate black hole (with the definition of a black hole (and associated terminology) in a non-asymptotically flat spacetime provided by Hayward [18]).

2.4 Dynamical systems

The third part of this thesis deals with a three dimensional (3-dim) autonomous dynamical system. We give a brief summary of the dynamical systems methods used [41],[31]. To begin we consider the non-linear autonomous system of ordinary differential equations (ODEs)

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = f(\vec{x}), \tag{2.4.1}$$

where $\vec{x} \in \mathbb{R}^n$ and $f . \mathbb{R}^n \to \mathbb{R}^n$.

Theorem 2.4.1 (Fundamental existence and uniqueness theorem). *Consider the initial value problem*

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = f(\vec{x}), \quad \vec{x}(0) = a. \tag{2 4 2}$$

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is of class $C^1(\mathbb{R}^n)$, then for all $a \in \mathbb{R}^n$, there exists an interval $(-\delta, \delta)$ and a unique function $\psi_a : (-\delta, \delta) \to \mathbb{R}^n$ such that

$$\frac{\mathrm{d}\psi_a(t)}{\mathrm{d}t} = f(\psi_a(t)), \quad \psi_a(0) = a,$$

i.e., $\psi_a(t)$ is a unique solution to (2 4.2).

We can extend the interval of definition of the solution $\psi_a(t)$ by successively reapplying the theorem, to obtain the maximal interval of definition

Theorem 2.4.2 (Maximality). *Let $\psi_a(t)$ be the unique solution of* (2.4.2) *and let (α, β) denote the interval of maximal existence on which $\psi_a(t)$ is defined. If β is finite then*

$$\lim_{t \to \beta^-} \|\psi_a(t)\| = +\infty,$$

where $\|\,\|$ denotes the standard norm in $\mathbb{R}^n$.

Therefore, if a solution $\psi_a(t)$ of (2.4.2) is bounded for $t \geq 0$ then the solution is defined for all $t \geq 0$. This result is also valid for the left–hand limit.

Theorem 2.4.3 (Dependence on parameters [31]). *Let E be an open subset of $\mathbb{R}^{n+m}$ containing the point (x_0, c_0) where $x_0 \in \mathbb{R}^n$ and $c_0 \in \mathbb{R}^m$ and assume that $f \in C^1(E)$*

It then follows that there exists $a > 0$ and $\delta > 0$ such that for all $\vec{y} \in N_\delta(x_0)$ and $c \in N_\delta(c_0)$, the initial value problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, c), \quad x(0) = y,$$

has a unique solution $u(t, y, c)$ with $u \in C^1(G)$ where $G = [-a, a] \times N_\delta(x_0) \times N_\delta(c_0)$, where $N_\delta(x_0)$ is the $\delta-$neighbourhood of x_0 i.e. an open ball of positive radius δ,

$$N_\delta(x_0) = \{x \in R^n || x - x_0| < \delta\}.$$

Next we define the flow of (2.4.1) to be the one-parameter family of maps $\{\phi_t\}_{t\in\mathbb{R}}$ such that $\phi_t \cdot \mathbb{R}^n \to \mathbb{R}^n$ and $\phi_t a = \psi_a(t)$ for all $a \in \mathbb{R}^n$.

Theorem 2.4.4 (Global existence theorem). *Let M be a compact manifold and let $f \in C^1(M)$ then for $a \in M$ the initial value problem* (2.4.2) *has a unique solution defined for all $t \in I$, where $I \subset \mathbb{R}$.*

2.4.1 Equilibrium points

The equilibrium points of (2.4.1) are points $\vec{x_0} \in \mathbb{R}^n$ such that

$$f(\vec{x_0}) = 0$$

An equilibrium point $\vec{x_0}$ is called a hyperbolic equilibrium point of (2.4.1) if none of the eigenvalues of the matrix $Df(\vec{x_0})$ have zero real part. We can linearise (2.4.1) about $\vec{x_0}$ to get

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = Df(\vec{x_0}) \cdot \vec{x} = A \cdot \vec{x}. \tag{2.4 3}$$

Suppose A is a diagonalisable $n \times n$ matrix with real eigenvalues λ_j, where $j = 1, \cdots, n$, and corresponding real eigenvectors, $v^1, \cdots, v^{n_s}$ are the n_s eigenvectors whose eigenvalues are negative, $u^1, \cdots, u^{n_u}$ are the n_u eigenvectors whose eigenvalues are positive and $w^1, \cdots, w^{n_c}$ are the n_c eigenvectors whose eigenvalues are zero, then $n_s + n_u + n_c = n$. The subspaces spanned by the eigenvectors can be divided into three classes:

$$\begin{aligned}
\text{the stable subspace } E^S &= \text{Span}\{v^1, \cdots, v^{n_s}\}, \\
\text{the unstable subspace } E^U &= \text{Span}\{u^1, \cdots, u^{n_u}\}, \\
\text{the centre subspace} E^C &= \text{Span}\{w^1, \cdots, w^{n_c}\}.
\end{aligned}$$

Theorem 2.4.5 (Stable Manifold Theorem). *Let E be an open subset of $\mathbb{R}^n$ containing $\vec{x_0}$, let $f \in C^1(E)$ and let ϕ_t be the flow of the non-linear system (2.4.1). Suppose that $f(\vec{x_0}) = 0$ and that $Df(\vec{x_0})$ has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. Then, there exists a k-dimensional differentiable manifold S, tangent to the stable subspace E^S of the linear system (2.4.3) at $\vec{x_0}$, such that for all $t \geq 0$, $\phi_t(S) \subset S$ and for any $\vec{x_1} \in S$ we have*

$$\lim_{t\to\infty} \phi_t(\vec{x_1}) = \vec{x_0},$$

and there exists a $(n-k)$-dimensional differentiable manifold U, tangent to the unstable subspace E^U (2.4 3) at $\vec{x_0}$, such that for all $t \leq 0$, $\phi_t(U) \subset U$ and for any $\vec{x_1} \in U$

$$\lim_{t\to-\infty} \phi_t(\vec{x_1}) = \vec{x_0}.$$

The Hartman-Grobman Theorem shows that near a hyperbolic equilibrium point $\vec{x_0}$ the nonlinear system (2.4.1) has the same qualitative structure as the linear system (2.4.3)

Theorem 2.4.6 (Hartman-Grobman Theorem). *Let E be an open subset of $\mathbb{R}^n$ containing $\vec{x_0}$, let $f \in C^1(E)$ and let ϕ_t be the flow of the nonlinear system (2.4.1). Suppose that $f(\vec{x_0}) = 0$ and that the matrix $A = Df(\vec{x_0})$ has no eigenvalue with zero real part. Then there exists a homeomorphism H of an open set U containing $\vec{x_0}$ onto an open set V containing $\vec{x_0}$ such that for each $\vec{x_1} \in U$, there is an open interval $I_0 \subset \mathbb{R}$ containing zero, such that for all $t \in I_0$*

$$H \circ \phi_t(\vec{x_1}) = e^{At} H(\vec{x_1});$$

i e., H maps trajectories of (2.4.1) near $\vec{x_0}$ onto trajectories of (2 4.3) near $\vec{x_0}$ and preserves the parametrisation.

Theorem 2.4.7 (The Centre Manifold Theorem). *Let $f \in C^r(E)$ where E be an open subset of $\mathbb{R}^n$ containing $\vec{x_0}$ and $r \geq 1$. Suppose that $f(\vec{x_0}) = 0$ and that the matrix $Df(\vec{x_0})$ has n_s eigenvalues with negative real part, n_u eigenvalues with positive real part, and $n_c = n - n_u - n_s$ eigenvalues with zero real part. Then there exists an n_c-dimensional centre manifold C of class C^r tangent to the centre subspace E^C of* (2 4.3) *at $\vec{x_0}$ which is invariant under the flow ϕ_t of* (2.4.1).

2.4.2 Equilibrium sets

We may also find non-isolated equilibria of the dynamical system (2.4.1), e.g., a curve of equilibrium points, which we call an equilibrium set. An equilibrium set is said to be normally hyperbolic if the only eigenvalues with zero real parts are those whose corresponding eigenvectors are tangent to the set

Theorem 2.4.8 (Aulbach [3]). *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be three times continuously differentiable and suppose that* (2.4.1) *has a compact C^1 manifold M of equilibrium points which is normally hyperbolic i.e. for each $\tilde{x} \in M$ the Jacobian $D(f(\tilde{x}))$ has (n-dim(M)) eigenvalues with real part different from zero Then there exists an $\mathbb{R}^n$ neighbourhood N of M such that any solution of* (2.4.1) *having a positive semi-trajectory in N lies in the stable manifold S of some equilibrium point in M.*

This theorem implies that normally hyperbolic equilibrium manifolds admit a transverse C^0-foliation with hyperbolic linear flows on the leaves. Therefore the stability of a normally hyperbolic equilibrium set can be completely classified by considering the signs of the non-zero eigenvalues, the centre manifold dynamics need not be considered.

In order to find some quantitative information about solutions approaching a normally hyperbolic equilibrium set we present a generalisation of the Hartman-Grobman theorem, which was formulated for non-hyperbolic equilibrium points, i.e. when the linearisation included one or more zero eigenvalue:

Theorem 2.4.9 (Kirchgraber-Palmer [20]). *Without loss of generality we can write*

(2.4.1) *in the form*

$$\frac{\mathrm{d}y_c}{\mathrm{d}t} = A_0 \cdot y_c + g_0(y_c, y_s, y_u), \tag{2.4.4a}$$
$$\frac{\mathrm{d}y_s}{\mathrm{d}t} = A_1 \cdot y_s + g_1(y_c, y_s, y_u), \tag{2.4.4b}$$
$$\frac{\mathrm{d}y_u}{\mathrm{d}t} = A_2 \cdot y_u + g_2(y_c, y_s, y_u), \tag{2.4.4c}$$

where the eigenvalues of A_0 have zero real parts, the eigenvalues of A_1 have negative real parts and the eigenvalues of A_2 have positive real parts, and $y_c \in \mathbb{R}^{n_c}$, $y_s \in \mathbb{R}^{n_s}$, $y_u \in \mathbb{R}^{n_u}$ where $n = n_c+n_s+n_u$, and where the functions g_0, g_1 and g_2 are C^1 functions defined for $\vec{y} = (y_c, y_s, y_u)^T$ near the origin (equilibrium set) where they vanish together with their first order partial derivatives. We extend the domains of definition of g_0, g_1 and g_2 to the whole of $\mathbb{R}^{n_c} \times \mathbb{R}^{n_s} \times \mathbb{R}^{n_u}$ using bump functions, this can be done in such a way that g_0, g_1 and g_2 are bounded and satisfy Lipschitz conditions with sufficiently small Lipschitz constants Then there exists a homeomorphism which takes solutions of (2.4.4) to solutions of

$$\frac{\mathrm{d}y_c}{\mathrm{d}t} = A_0 y_c + g_0(y_c, \phi_1(y_c), \phi_2(y_c, \phi_1(y_c))), \tag{2.4.5a}$$
$$\frac{\mathrm{d}y_s}{\mathrm{d}t} = A_1 y_s, \tag{2.4.5b}$$
$$\frac{\mathrm{d}y_u}{\mathrm{d}t} = A_2 y_u, \tag{2.4.5c}$$

where the set $\{(y_c, \phi_1(y_c), \phi_2(y_c, \phi_1(y_c)))|y_c \in \mathbb{R}^{n_c}\}$ is the centre manifold. Therefore the flow corresponding to the non-linear system (2.4.4) is equivalent to the product of the flow on the centre manifold and the linear flow, (2 4.5b), (2.4.5c).

In the case of a normally hyperbolic equilibrium set we have seen that the centre manifold dynamics do not affect the solution. Therefore the above theorem demonstrates that if (2.4.4) had a normally hyperbolic equilibrium set then trajectories in the stable subspace of (2.4.4), S, would be topologically equivalent to trajectories in the stable subspace E^S of (2.4.5) (spanned by eigenvectors of A_1).

2.4.3 Invariant sets and associated properties

A set $S \subset \mathbb{R}^n$ is an invariant set of the flow ϕ_t of (2.4.1) if, for all $\vec{x} \in S$ and for all $t \in \mathbb{R}$, $\phi_t(\vec{x}) \in S$. Invariant sets include equilibrium points, stable, unstable and centre manifolds and α- and ω-limit sets which we will now define. Given an initial point $a \in E$, a point $p \in E$ is an ω-limit point of (2.4.1) if there exists a sequence $t_n \to \infty$ such that

$$\lim_{n\to\infty} \phi_{t_n}(a) = p.$$

The α-limit point is defined similarly by using a sequence $t_n \to -\infty$. The dynamical systems we consider in this thesis are all of dimension three or four. Some tools to simplify the analysis when $n > 2$ are the following.

Definition If S has the property that $\phi_t(\vec{x}) \in S$ for all $\vec{x} \in S$ and for all $t > 0$, then we say that S is positively invariant.

Definition Let ϕ_t be a flow in $\mathbb{R}^n$, let S be an open subset of an invariant set of ϕ_t and let $Z : S \to \mathbb{R}$ be a differentiable function. If

$$\dot{Z} = \nabla Z \cdot f < 0 \quad (\text{or} > 0), \quad \text{on} \quad S,$$

then Z is monotone decreasing (or increasing) on S.

Proposition 2.4.10. *Let $S \subset \mathbb{R}^n$ be an invariant set of a flow ϕ_t. If there exists a monotone function $Z : S \to \mathbb{R}$ on S, then S contains no equilibrium points and periodic orbits.*

Theorem 2.4.11 (LaSalle invariance principle [41]). *Consider the system (2.4.1) with flow ϕ_t. Let S be a closed, bounded and positively invariant set of ϕ_t and let Z be a C^1 monotone function. Then, for all $\vec{x_0} \in S$ we have $\omega(\vec{x_0}) \subset M$ where M is the largest invariant subset $\{x \in S | \dot{Z} = \nabla Z \cdot f = 0\}$.*

Theorem 2.4.12 (Monotonicity principle [41]). *Let ϕ_t be the flow of (2.4.1) with S an invariant set. Let $Z : S \to \mathbb{R}$ be a C^1 function whose range is (a, b), where $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{\infty\}$ and $a < b$. If Z is decreasing on orbits in S then, for* *all* $x \in S$

$$\begin{aligned} \omega(x) &\subseteq \{s \in \bar{S} \setminus S | \lim_{y \to s} Z(y) \neq b\}, \\ \alpha(x) &\subseteq \{s \in \bar{S} \setminus S | \lim_{y \to s} Z(y) \neq a\}, \end{aligned}$$

where $\bar{S}$ *is the closure of* S.

We can also use Liapunov functions to determine the stability of a given equilibrium point. They can be described by the following theorem.

Theorem 2.4.13 (Liapunov stability theorem [31]). *Let* $\vec{x_0}$ *be an equilibrium point of* (2.4.1). *Let* $V : \mathbb{R}^n \to \mathbb{R}$ *be a* C^1 *function such that*

$$V(\vec{x_0}) = 0, \quad V(\vec{x}) > 0, \quad \forall \vec{x} \in U/\{\vec{x_0}\},$$

where U *is a neighbourhood of* $\vec{x_0}$.

1. *If* $\dot{V}(\vec{x}) < 0$ *for all* $\vec{x} \in U/\{\vec{x_0}\}$, *then* $\vec{x_0}$ *is asymptotically stable*

2. *If* $\dot{V}(\vec{x}) \leq 0$ *for all* $\vec{x} \in U/\{\vec{x_0}\}$, *then* $\vec{x_0}$ *is stable*

3. *If* $\dot{V}(\vec{x}) > 0$ *for all* $\vec{x} \in U/\{\vec{x_0}\}$, *then* $\vec{x_0}$ *is unstable,*

where $V(\vec{x}) = \nabla V(\vec{x}) \cdot f(\vec{x})$. *A function* $V : \mathbb{R}^n \to \mathbb{R}$ *which satisfies* $V(\vec{x_0}) = 0$, $V(\vec{x}) > 0$ *for all* $\vec{x} \in U/\{\vec{x_0}\}$ *and* $\dot{V}(\vec{x}) \leq 0$ *(respectively* < 0*) for all* $\dot{x} \in U/\{\vec{x_0}\}$, *is called a Liapunov function (respectively, a strict Liapunov function) for the equilibrium point* $\vec{x_0}$.

In Part III of this thesis we study a third order non-linear ODE which we derived from the field equations for a cylindrically symmetric self-similar spacetime. This can be written as a 3-dim autonomous dynamical system. In order to follow the evolution of our solution from the regular axis it was necessary to transform the dependent and independent variables of this system a number of times. As a result we needed to analyse two 3-dim hyperbolic equilibrium points, one 4-dim hyperbolic equilibrium point, two normally hyperbolic 3-dim equilibrium sets and a non-hyperbolic 3-dim equilibrium set. The hyperbolic equilibrium points are easily described using the

Hartman-Grobman theorem, and the solutions can be written out explicitly This means that we can obtain physical information about the solution, i.e., expansion scalars, density functions. We can describe the stability of the normally hyperbolic equilibrium sets using Theorem 2 4 8. Finally to describe the non-hyperbolic equilibrium set we use a Liapunov function to classify the stability. To obtain quantitative information about this equilibrium set we needed to employ other methods which will be described in detail in Part III, Chapter 6.

2.5 Structure of the thesis

In Part II, Chapter 3 we shall describe the results of matching a cylindrically symmetric perfect fluid to a vacuum. We begin by describing the matching hypersurface - we describe the embedding in detail We then simplify the vacuum line element Finally we inspect the matching equations and obtain a contradiction which rules out the matching We also show that continuity of the metric alone rules out the matching.

In Part III we begin a study of self-similar cylindrically symmetric spacetimes. In Chapter 4 we consider a cylindrically symmetric vacuum spacetime. We impose self-similarity of the first kind on the Einstein-Rosen line element. We can then write our metric functions in terms of a similarity variable $\xi = \frac{r}{t}$. The field equations will be a set of ODEs and a full solution is obtained analytically. No solutions are found that are of relevance to gravitational collapse. But a family of regular solutions is found that can be thought of as a 'ground state' for cylindrically symmetric spacetimes which are self-similar The problem is then formulated for a dust spacetime. We describe and impose regularity conditions at the axis. We will show that a solution to the field equations is obtained by solving a third order non-linear ODE in one of the metric functions We rewrite this third order ODE as a three dimensional (3-dim) autonomous dynamical system

In Chapter 5 we will begin to analyse the 3-dim autonomous dynamical system at the regular axis. A hyperbolic equilibrium point which corresponds to the regular axis is found and we invoke standard theorems (as described in §2.4) to describe the asymptotic behaviour of this solution We will prove that this is a unique solution.

Next we will show that this unique solution must approach a specific normally hyperbolic equilibrium set, which corresponds to the past null cone through the singular or scaling origin p_o, where $p_o = \{(r, t) = (0, 0)\}$. We will denote by $\mathcal{N}$ the past null cone at p_o which will be generated by all past pointing null geodesics from p_o, and we will denote by $\mathcal{F}$ the future null cone at p_o will be generated by all future pointing null geodesics from p_o. The hypersurface $t = 0$, $r > 0$ will be referred to as Σ_0.

In Chapter 6 we will examine the solution evolving from $\mathcal{N}$ into the future. We prove the existence of a unique solution emanating from a hyperbolic equilibrium point We analyse the possible evolution of this solution and obtain two possible cases - we use a numerical procedure to eliminate one of these cases. We then know which equilibrium set our unique solution will approach. This equilibrium set is non-hyperbolic. We provide a Liapunov function which proves the asymptotic stability of this equilibrium set. We then perform a compactification of the phase space and a polar blow up of the equilibrium point in the new variables. We obtain a new dynamical system. After performing some analysis to reduce the number of equilibrium points of the new system, we finally arrive at a 3-dim hyperbolic equilibrium point at Σ_0. We will use the theorems of §2.4 to write the asymptotic behaviour of his solution. We will conclude this chapter by examining the physical properties of this solution

In Chapter 7 we will inspect the solution emanating from Σ_0 into the future. We identify the point Σ_0 with a hyperbolic equilibrium point of the dynamical system. The solution has one parameter, which indicates that a bifurcation has occurred. We prove that these solutions must evolve to a certain equilibrium point, located at $\mathcal{F}$. The existence of a one-parameter solution at $\mathcal{F}$ indicates that the singularity at p_o is naked.

In Chapter 8 we present our conclusions, and some suggestions for further work.

Part II

Isotropic cylindrically symmetric stellar models

Chapter 3

Matching a cylindrically symmetric perfect fluid to a vacuum

To begin our investigation of the effect cylindrical symmetry has on gravitational collapse we investigate a cylindrical version of the "standard" model of spherical collapse, the Oppenheimer-Snyder model [29] However, unlike that model we do not need the *a priori* assumption that the interior comprises pressureless dust. We apply the standard matching techniques, without any conditions of staticity, to the cylindrically symmetric case to obtain a general result about the evolution of cylindrically symmetric objects in a vacuum spacetime. These results were published in [27].

We use the following conventions in this chapter, Latin indices run through $0, 1, 2, 3$ and Greek indices run through $1, 2, 3$ labelling the hypersurface coordinates, with prime and overdot referring to differentiation with respect to T and t respectively.

3.1 Matching formalism

We now outline the formalism used to match or glue together two spacetimes, [5]. In order to match together two separate spacetimes we begin with two oriented C^3 4-dimensional manifolds $\mathcal{V}^{\pm}$, with boundaries $\Sigma^{\pm}$. These manifolds are endowed with C^2 Lorentzian metrics $g^{\pm}_{ab}$. In order to match these boundaries we require an identification of the boundaries. So we define a diffeomorphism from Σ^+ to Σ^- as follows, i.e , there exists an abstract 3-dimensional C^3 manifold Σ and two C^3 embeddings

$$\Psi^+ : \Sigma \to \mathcal{V}^+, \quad \Psi^- . \Sigma \to \mathcal{V}^-,$$

which satisfy $\Psi^+(\Sigma) = \Sigma^+$ and $\Psi^-(\Sigma) = \Sigma^-$. We let $\mathcal{V}^\pm$ have coordinates $x^\alpha_\pm$, and Σ have coordinates ξ^α, where $x^a_\pm = \Psi^{a\pm}(\xi^\alpha)$. This identification means we have glued

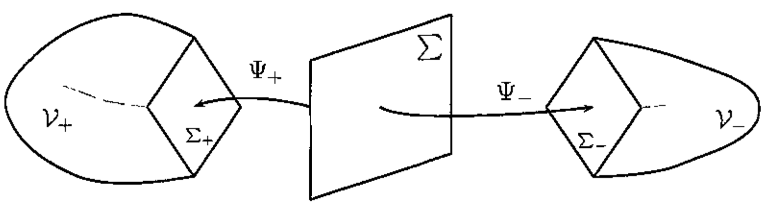

Figure 3 1· Matching of spacetimes.

together $\mathcal{V}^+$ and $\mathcal{V}^-$ at their boundaries to form a single manifold $\mathcal{V} = \mathcal{V}^+ \cup \mathcal{V}^-$. We can obtain the vectors tangent to $\Sigma_\pm$ from the embeddings $\Psi^\pm$ as follows·

$$\mathrm{d}\Psi^\pm \left(\frac{\partial}{\partial \xi^\alpha}\bigg|_\Sigma \right) = \frac{\partial \Psi^{\pm a}}{\partial \xi^\alpha_\pm} \frac{\partial}{\partial x^a_\pm} = e^a_{\alpha\pm} \frac{\partial}{\partial x^a_\mp} = \vec{e}^{\,\pm}_\alpha\big|_{\Sigma_\pm},$$

and we define unit normal vectors $n^a_{\alpha\pm}$ where

$$\vec{n}^- \cdot \vec{n}^-\big|_{\Sigma_-} = \vec{n}^+ \cdot \vec{n}^+\big|_{\Sigma_+}, \quad \vec{n}^\pm \cdot \vec{e}^{\,\pm}_\alpha\big|_{\Sigma_\pm} = 0,$$

where the dot means inner product with the metric g_{ab} in $\mathcal{V}^\pm$. The first and second fundamental forms are given by

$$g^\pm_{\alpha\beta} = e^{\pm a}_\alpha e^{\pm b}_\beta \, g_{ab}|_{\Sigma_\pm}, \quad K^\pm_{\alpha\beta} = -n^\pm_a e^{\pm b}_\beta \nabla_b e^{\pm a}_\alpha$$

Matching conditions require the equality of the first and second fundamental forms

$$g^+_{\alpha\beta} = g_{\alpha\beta}, \quad K^+_{\alpha\beta} = K^-_{\alpha\beta}. \tag{3.1.1}$$

3.2 Description of matching problem

As a preliminary we examine a general matching of a spacetime $(\mathcal{V}^-, g^-)$ to a vacuum spacetime $(\mathcal{V}^+, g^+)$, with the matching condition

$$[T_{ab}]n^b = \left(T^+_{ab} - T^-_{ab}\right) n^b = 0, \tag{3.2.1}$$

where $T^\pm_{ab}$ are the energy momentum tensors in $\mathcal{V}^\pm$ respectively [24]. The conditions (3.2.1) are known as the Israel junction conditions and follow from the standard (Darmois) matching conditions of General Relativity – continuity of the first and second fundamental forms – which are used throughout. For vacuum $\mathcal{V}^+$, (3.2.1) implies that

$$\left(T^-_{ab}\right) n^b = 0, \tag{3.2.2}$$

on the matching hypersurface Σ. Assuming that the energy momentum tensor of $(\mathcal{V}^-, g^-)$ is that of a perfect fluid, we have

$$T^-_{ab} = (p+\rho)u_a u_b + p g_{ab},$$

where u_a is a unit future pointing timelike vector. Then (3.2.2) becomes

$$p n_a + (p+\rho) n^b u_b u_a = 0 \tag{3.2 3}$$

If we invoke the weak energy condition, $\rho \geq 0$ and $\rho + p \geq 0$ and require $\rho \neq 0$ to avoid a trivial case, then (3.2.3) implies that $p = 0$ on Σ and $u_b n^b = 0$. In other words, matching with vacuum can only be done with pressureless dust, and the normal to the matching hypersurface Σ is always spacelike: the matching hypersurface must be timelike everywhere We are considering the case where $(\mathcal{V}^-, g^-)$ is a Robertson-Walker (RW) spacetime. Since the pressure of such a spacetime is homogeneous, this implies that the pressure must vanish everywhere. So we will consider a RW interior with cylindrically symmetric line element, given in coordinates $\{t, \rho, x, \varphi\}$ adapted to the Killing vector fields, $\{\eta_\varphi^- = \frac{\partial}{\partial\varphi}, \eta_x^- = \frac{\partial}{\partial x}\}$, which is given by

$$ds_-^2 = -dt^2 + a^2(t)(d\rho^2 + \Upsilon_{,\rho}^2(\rho,\epsilon)dx^2 + \Upsilon^2(\rho,\epsilon)d\varphi^2), \qquad (3.2.4)$$

where $a(t)$ is the scale factor and for collapsing dust

$$a(t) = \begin{cases} a_1(1-\cosh\varsigma), & t = a_1(\varsigma - \sin\varsigma), & \epsilon = +1, \\ a_0|t|^{2/3}, & & \epsilon = 0, \\ a_1(\cosh\varsigma - 1), & t = a_1(\sinh\varsigma - \varsigma), & \epsilon = -1, \end{cases} \qquad (3.2.5)$$

where a_0 and a_1 are constants, and where $\Upsilon(\rho,\epsilon)$ satisfies

$$\Upsilon(\rho,\epsilon) = \begin{cases} \sinh\rho, & \epsilon = -1, \\ \rho, & \epsilon = 0, \\ \sin\rho, & \epsilon = +1, \end{cases} \qquad (3.2.6)$$

and where ϵ is the curvature index so that $\epsilon = 1, 0, -1$ for closed, flat or open RW models, respectively. We will match to a general cylindrically symmetric unpolarized vacuum exterior spacetime $(\mathcal{V}^+, g^+)$, which has the line element, given in coordinates $\{T, R, Z, \Phi\}$ adapted to the Killing vector fields, $\{\eta_\Phi^+ = \frac{\partial}{\partial\Phi}, \eta_Z^+ = \frac{\partial}{\partial Z}\}$, [21] (see also [8],[4])

$$ds_+^2 = e^{2(\gamma-\psi)}(-dT^2 + dR^2) + e^{2\psi}(dZ + \omega d\Phi)^2 + \alpha^2 e^{-2\psi}d\Phi^2, \qquad (3.2.7)$$

where γ, ψ, ω and α are functions of T and R[1]. This line element admits cylindrical waves with two polarisations (as its Killing vectors are not hypersurface orthogonal) - if we require that $\omega = 0$ then we have cylindrical waves with one polarisation and a line element with this condition is called polarised. Next we will name the intrinsic

[1]The intention is that if the matching were successful we would propose a global picture of the whole matched spacetime with the infinite RW cylinder truncated at two values of z and hemispherical caps inserted at these points $z = z_1$ and $z = z_2$ so that the RW portion corresponds to $z_1 < z < z_2$ [5]. Thus the vacuum region should include the axis beyond the caps. This implies orthogonal transitivity of the isometry group and so (3.2 7) applies [8]

coordinates in the hypersurface Σ as $\{\tau, z, \phi\}$, so our local coordinate systems are

$$\begin{aligned}\text{Interior } \mathcal{V}^- : x_-^a &= \{t, \rho, x, \varphi\}, \\ \text{Exterior } \mathcal{V}^+ : x_+^a &= \{T, R, Z, \Phi\}, \\ \text{Matching hypersurface } \Sigma : \xi^\alpha &= \{\tau, z, \phi\}\end{aligned}$$

3.2.1 The embedding Ψ^-

In order to have a description of $\Sigma^\pm$, the hypersurfaces in $\mathcal{V}^\pm$ respectively ($\Sigma^+ \subset \mathcal{V}^\pm$), in terms of the coordinates $\xi^\alpha = \{\tau, z, \phi\}$ we need to follow established methods [40]. We will describe first the embedding Ψ^-. We can choose ϕ in Σ so that the vector field $\frac{\partial}{\partial \phi}$ tangent to Σ is mapped by $d\Psi$ to the Killing vector η_φ^- in $\mathcal{V}_-$ restricted to Σ_-, i.e.,

$$\begin{aligned}
d\Psi^-\left(\left.\frac{\partial}{\partial \phi}\right|_\Sigma\right) &= \left.\frac{\partial}{\partial \varphi}\right|_{\Sigma_-} = \vec{e}_3^{\,-}, && (3.2.8)\\
\text{but}\quad d\Psi^-\left(\left.\frac{\partial}{\partial \phi}\right|_\Sigma\right) &= \frac{\partial \Psi^{0-}}{\partial \phi}\left.\frac{\partial}{\partial t}\right|_{\Sigma_-} + \frac{\partial \Psi^{1-}}{\partial \phi}\left.\frac{\partial}{\partial \rho}\right|_{\Sigma_-} + \frac{\partial \Psi^{2-}}{\partial \phi}\left.\frac{\partial}{\partial x}\right|_{\Sigma_-} + \frac{\partial \Psi^{3-}}{\partial \phi}\left.\frac{\partial}{\partial \varphi}\right|_{\Sigma_-},\\
\text{therefore}\quad (3.2.8) \Rightarrow\ & \frac{\partial \Psi^{0-}}{\partial \phi} = \frac{\partial \Psi^{1-}}{\partial \phi} = \frac{\partial \Psi^{2-}}{\partial \phi} = 0, \quad \frac{\partial \Psi^{3-}}{\partial \phi} = 1. && (3.2.9)
\end{aligned}$$

Similarly we can choose z in Σ so that

$$\begin{aligned}
d\Psi^-\left(\left.\frac{\partial}{\partial z}\right|_\Sigma\right) &= \left.\frac{\partial}{\partial x}\right|_{\Sigma_-} = \vec{e}_2^{\,-},\\
\text{but}\quad d\Psi^-\left(\left.\frac{\partial}{\partial z}\right|_\Sigma\right) &= \frac{\partial \Psi^{0-}}{\partial z}\left.\frac{\partial}{\partial t}\right|_{\Sigma_-} + \frac{\partial \Psi^{1-}}{\partial z}\left.\frac{\partial}{\partial \rho}\right|_{\Sigma_-} + \frac{\partial \Psi^{2-}}{\partial z}\left.\frac{\partial}{\partial x}\right|_{\Sigma_-} + \frac{\partial \Psi^{3-}}{\partial z}\left.\frac{\partial}{\partial \varphi}\right|_{\Sigma_-},\\
\Rightarrow\ & \frac{\partial \Psi^{0-}}{\partial z} = \frac{\partial \Psi^{1-}}{\partial z} = \frac{\partial \Psi^{3-}}{\partial z} = 0, \quad \frac{\partial \Psi^{2-}}{\partial z} = 1, && (3.2.10)\\
\Rightarrow\ & \Psi^{2-} = z + c_1(\tau), \quad \Psi^{3-} = \phi + c_2(\tau) \quad \text{by} \quad (3.2.9), (3.2.10).
\end{aligned}$$

Next we choose τ in Σ so that $d\Psi^-\left(\frac{\partial}{\partial \tau}\right)$ is orthogonal to $d\Psi^-\left(\frac{\partial}{\partial \phi}\right)$ and $d\Psi^-\left(\frac{\partial}{\partial z}\right)$. This implies that

$$\frac{\partial \Psi^{2-}}{\partial \tau} = 0, \quad \frac{\partial \Psi^{3-}}{\partial \tau} = 0,$$

which implies that $c_1(\tau)$ = constant and $c_2(\tau)$ = constant so that with two inconsequential translations we can get $\Psi^{2-} = z$ and $\Psi^{3-} = \phi$ and we can write

$$\mathrm{d}\Psi^- \left(\left.\frac{\partial}{\partial \tau}\right|_{\Sigma} \right) = \frac{\partial \Psi^{0-}}{\partial \tau} \left.\frac{\partial}{\partial t}\right|_{\Sigma_-} + \frac{\partial \Psi^{1-}}{\partial \tau} \left.\frac{\partial}{\partial \rho}\right|_{\Sigma_-} = \vec{e}_1^{\,-}.$$

The embedding Ψ^- in these coordinates is given by

$$\Sigma^- : \{t = \Psi^{0-}(\tau),\ \rho = \Psi^{1-}(\tau),\ x = z,\ \varphi = \phi\} \tag{3.2.11}$$

We can now simply choose $\Psi^{0-}(\tau) = \tau$ and therefore

$$\vec{e}_1^{\,-} = \frac{\partial}{\partial t} + \frac{\partial \Psi^{1-}}{\partial \tau}\frac{\partial}{\partial \rho}, \quad \vec{e}_2^{\,-} = \frac{\partial}{\partial x}, \quad \vec{e}_3^{\,-} = \frac{\partial}{\partial \varphi}$$

we know that

$$\vec{n}^- \cdot \vec{e}_2^{\,-} = \vec{n}^- \cdot \vec{e}_3^{\,-} = 0, \quad \text{and} \quad \vec{n}^- \cdot \vec{u} = 0 \Rightarrow \vec{n}^- \propto \frac{\partial}{\partial \rho},$$

using the fact that $\vec{u} = \frac{\partial}{\partial t}$ But $\vec{n}^- \cdot \vec{e}_1^{\,-} = 0$ then gives us

$$\frac{\partial \Psi^{1-}}{\partial \tau} = 0$$

and so the boundary occurs at some constant $\rho = \rho_0 > 0$ so that

$$\Sigma^- : \{t = \tau,\ \rho = \rho_0,\ x = z,\ \varphi = \phi\}. \tag{3.2.12}$$

3.2.2 The embedding Ψ^+

Next we will consider the embedding of Σ^+. As the axial Killing vector η_Φ^+ is uniquely defined, the vector field $\frac{\partial}{\partial \phi}$ must be mapped to the axial generator in $\mathcal{V}^+$ by $d\Psi^+$, i.e.

$$\mathrm{d}\Psi^+ \left(\left.\frac{\partial}{\partial \phi}\right|_{\Sigma} \right) = \left.\frac{\partial}{\partial \Phi}\right|_{\Sigma_+} = \vec{e}_3^{\,+}.$$

Next the image of $\frac{\partial}{\partial z}$ by $d\Psi^+$ must complete the basis of Killing vectors generating the G_2 group in $\mathcal{V}^+$ restricted to Σ^+, it will take the general form

$$d\Psi^+\left(\left.\frac{\partial}{\partial z}\right|_{\Sigma}\right) = a\left.\frac{\partial}{\partial Z}\right|_{\Sigma_-} + b\left.\frac{\partial}{\partial \Phi}\right|_{\Sigma_-} = \vec{e}_2^{\,+}, \tag{3.2.13}$$

where a and b are arbitrary constants and $a \neq 0$ We can do a coordinate transformation which preserves the form of the line element (3.2.7)

$$Z \to Z' = \frac{Z}{a}, \quad \Phi \to \Phi' = \Phi - \frac{b}{a}Z,$$

to arrive at

$$d\Psi^+\left(\left.\frac{\partial}{\partial z}\right|_{\Sigma}\right) = \left.\frac{\partial}{\partial Z'}\right|_{\Sigma_+} = \vec{e}_2^{\,+}.$$

Finally we can require without loss of generality that $d\Psi^+\left(\frac{\partial}{\partial \tau}\right)$ is orthogonal to $d\Psi^+\left(\frac{\partial}{\partial \phi}\right)$ and $d\Psi^+\left(\frac{\partial}{\partial x}\right)$. This implies that

$$\frac{\partial \Phi'}{\partial \tau} = 0, \quad \frac{\partial Z'}{\partial \tau} = 0.$$

Now we drop the prime for simplicity, so that we can write

$$d\Psi^+\left(\left.\frac{\partial}{\partial \tau}\right|_{\Sigma}\right) = \frac{\partial \Psi^{0+}}{\partial \tau}\left.\frac{\partial}{\partial T}\right|_{\Sigma_+} + \frac{\partial \Psi^{1+}}{\partial \tau}\left.\frac{\partial}{\partial R}\right|_{\Sigma_+} = \vec{e}_1^{\,+}.$$

The embedding Ψ^+ in these coordinates is given by

$$\Sigma^+ : \{T = \Psi^{0+}(\tau),\ R = \Psi^{1+}(\tau),\ Z = z,\ \Phi = \phi\}. \tag{3.2.14}$$

It is convenient to use t instead of τ from now on, and we will rename $\Psi^{0+}(\tau) = T(t)$ and $\Psi^{1+}(\tau) = R_0(t)$ to get

$$\Sigma^+ : \{T = T(t),\ R = R_0(T(t)),\ Z = z,\ \Phi = \phi\} \tag{3.2.15}$$

3.2.3 Choice of matching

In theory four different matchings of $(\mathcal{V}^-, g^-)$ and $(\mathcal{V}^+, g^+)$ are possible, depending on the choice of continuous normals $\vec{n}^{\pm}$ to $\Sigma^{\pm}$ in $\mathcal{V}^{\pm}$. However, our aim is to describe a spacetime consisting of a RW interior and a vacuum exterior For $\vec{n}^-$, we choose the normal to point toward cylinders of increasing radius. For $\vec{n}^+$, we want to do the same. This requires that $\vec{n}^+$ points towards larger values of α. The coordinate R has yet to be specified and it may happen that either $\alpha_{,R} > 0$ or $\alpha_{,R} < 0$. We assume further that the axis of the vacuum spacetime resides in the region removed to accommodate the RW portion. Thus $\alpha_{,R} < 0$ in $\mathcal{V}^+$ can only come about if R decreases away from Σ^+ Hence the $\alpha_{,R} < 0$ case can be converted to the $\alpha_{,R} > 0$ case by a coordinate transformation of the form

$$R \to \hat{R} = R^* - R.$$

Thus in the coordinates of (3.2.15), we will assume that $\vec{n}^+$ points in the direction of increasing R. We will refer to this arrangement of $\Sigma^{\pm}$ and $\vec{n}^{\pm}$ by saying that $\vec{n}$ points out of $\mathcal{V}^-$ and into $\mathcal{V}^+$

3.3 Reduction to the polarised case

We begin with the line element (3.2 7) The tangent vectors $e^a_\alpha = \frac{\partial x^a}{\partial \xi^\alpha}$ to the hypersurface Σ are

$$e^a_1 = \left(\frac{\partial T}{\partial t}, \frac{\partial R}{\partial t}, 0, 0\right), \quad e^a_2 = (0,0,1,0), \quad e^a_3 = (0,0,0,1),$$

so that

$$\vec{e}_1 \;=\; e^a_1 \frac{\partial}{\partial x^a} = \frac{dT}{dt}\left(\frac{\partial}{\partial T} + (R_0')\frac{\partial}{\partial R}\right).$$

Thus there is a tangential derivative proportional to

$$\frac{\partial}{\partial T} + (R_0')\frac{\partial}{\partial R},$$

which for convenience we will refer to as *the* tangential derivative. (The other tangential derivatives, ∂_ϕ, ∂_z, are trivial in the sense that they play no role in the dynamics.) We will require the following matching equations

$$g^{+}_{z\phi} \overset{\Sigma}{=} g^{-}_{z\phi} \quad \Leftrightarrow \quad \omega \overset{\Sigma}{=} 0, \tag{3.3 1}$$

$$g^{+}_{\tau\tau} \overset{\Sigma}{=} g^{-}_{\tau\tau} \quad \Rightarrow \quad \left(1-(R_0{}')^2\right) > 0, \tag{3.3.2}$$

$$K^{+}_{z\phi} \overset{\Sigma}{=} K^{-}_{z\phi} \quad \Leftrightarrow \quad \left((R_0')\frac{\partial \omega}{\partial T} + \frac{\partial \omega}{\partial R}\right) \overset{\Sigma}{=} 0, \tag{3 3.3}$$

where we use $\overset{\Sigma}{=}$ to indicate equality on Σ. We can take the tangential derivative of ω and then evaluate it on Σ

$$\left(\frac{\partial \omega}{\partial T} + (R_0')\frac{\partial \omega}{\partial R}\right) \overset{\Sigma}{=} 0$$

By our matching condition (3.3.3) this implies

$$\frac{\partial \omega}{\partial T}(1-(R_0')^2) \overset{\Sigma}{=} 0$$

Using the matching conditions (3 3 2) then gives the result

$$\frac{\partial \omega}{\partial T} \overset{\Sigma}{=} 0, \quad \frac{\partial \omega}{\partial R} \overset{\Sigma}{=} 0, \tag{3.3.4}$$

or equivalently

$$\omega^{(1)} \overset{\Sigma}{=} 0, \tag{3.3 5}$$

where $\omega^{(k)}$ denotes all partial derivatives of ω of order k.
It is then straightforward to show that $\omega^{(2)} \overset{\Sigma}{=} 0$ by considering the field equation (A.1), which we can write in the form

$$\frac{\partial^2 \omega}{\partial T^2} - \frac{\partial^2 \omega}{\partial R^2} = f(\omega, \omega^{(1)}), \tag{3 3.6}$$

where f is some polynomial function satisfying $f(0,0) = 0$, which by (3.3.1) and (3.3.5) equals zero when evaluated at Σ. By taking tangential derivatives of both of (3.3.5) and comparing with (3.3.6), we obtain the result $\omega^{(2)} = 0$.

In like manner, we can show that if ω is $C^k, k \leq \infty$ on a neighbourhood of Σ, then $\omega^{(k)} \stackrel{\Sigma}{=} 0$ We use an induction argument If we assume that $\omega^{(j)} \stackrel{\Sigma}{=} 0$ for $0 \leq j \leq k$ is true then by proving $\omega^{(k+1)} \stackrel{\Sigma}{=} 0$ is true and using (3.3.5) we have the desired result.

To show that $\omega^{(k+1)} \stackrel{\Sigma}{=} 0$ is true, we take the tangential derivative of our assumed $\omega^{(k)} \stackrel{\Sigma}{=} 0$. There are $(k+1)$ of these tangential derivative equations, and they are of the form

$$\frac{\partial^{k+1}\omega}{\partial T^{k+1}} + R_0' \frac{\partial^{k+1}\omega}{\partial R \partial T^k} \stackrel{\Sigma}{=} 0, \tag{3.3.7}$$

$$\frac{\partial^{k+1}\omega}{\partial T^k \partial R} + R_0' \frac{\partial^{k+1}\omega}{\partial R^2 \partial T^{k-1}} \stackrel{\Sigma}{=} 0, \tag{3 3.8}$$

$$\frac{\partial^{k+1}\omega}{\partial T^{k-1} \partial R^2} + R_0' \frac{\partial^{k+1}\omega}{\partial R^3 \partial T^{k-2}} \stackrel{\Sigma}{=} 0 \quad \text{etc.} \tag{3.3.9}$$

Then we consider the field equation (3.3 6). Taking successive partial derivatives of (3 3.6) we have

$$\frac{\partial^{k+1}\omega}{\partial T^{k+1}} - \frac{\partial^{k+1}\omega}{\partial T^{k-1} \partial R^2} = F(\omega, \cdots, \omega^{(k)}),$$

and the form of f in (A.1) shows that $F(0, \cdots, 0) = 0$. But evaluated on Σ we know by assumption that

$$\omega^{(j)} \stackrel{\Sigma}{=} 0 \quad \text{for} \quad 0 \leq j \leq k,$$

and therefore

$$\frac{\partial^{k+1}\omega}{\partial T^{k+1}} - \frac{\partial^{k+1}\omega}{\partial T^{k-1} \partial R^2} \stackrel{\Sigma}{=} 0.$$

This equation together with (3.3.7) and (3.3.8) gives the relation

$$\frac{\partial^{k+1}\omega}{\partial T^k \partial R}(1 - (R_0')^2) \stackrel{\Sigma}{=} 0,$$

and so by (3.3.2)

$$\frac{\partial^{k+1}\omega}{\partial T^k \partial R} \stackrel{\Sigma}{=} 0$$

Substituting this equation into the appropriate tangential equation shows, by a cascade effect, each partial derivative of order $(k+1)$ to be zero when evaluated at Σ, proving our assertion We can then write down the following lemma.

Lemma 3.3.1. *If ω is analytic on a neighbourhood Ω of Σ, then $\omega \equiv 0$ on Ω.*

Proof: Let $(R_1, T_1) \in \Omega$. Then we can write $R_1 = R_0(T) + R_*, T_1 = T + T_*$ for some numbers R_*, T_* and where $(R_0(T), T) \in \Sigma$. By analyticity, we can write

$$\omega(R_0(T) + R_*, T + T_*) = \left[\sum_{n=1}^{\infty} \frac{1}{n!} \left(T_* \frac{\partial}{\partial T} + R_* \frac{\partial}{\partial R}\right)^n \omega\right]_\Sigma \tag{3 3.10}$$

The result follows immediately.

Therefore, assuming that $\omega(R, T)$ is an analytic function we see that $\omega = 0$ on a neighbourhood of Σ by using the matching conditions and the vacuum field equations. So we have henceforth that $\omega = 0$ and (3 2.7) becomes

$$ds_+^2 = e^{2(\gamma-\psi)}(-dT^2 + dR^2) + e^{2\psi}dZ^2 + \alpha^2 e^{-2\psi}d\Phi^2. \tag{3 3.11}$$

3.4 Further simplification of the vacuum line element

We note that the general solution of the vacuum field equation

$$\frac{\partial^2 \alpha}{\partial R^2} - \frac{\partial^2 \alpha}{\partial T^2} = 0 \tag{3.4.1}$$

$$\text{is} \quad \alpha(T, R) = F(U) + G(V), \tag{3.4 2}$$

where $U = T - R$ and $V = T + R$. Following Thorne [1] we characterise a spacetime with line element (3.3.11) at any event p as follows: $D^{(+)}$ if $\nabla\alpha$ is spacelike and points away from the symmetry axis, $D^{(-)}$ if $\nabla\alpha$ is spacelike and points toward the symmetry axis, $D^{(0\uparrow)}$ if $\nabla\alpha$ is timelike and points toward the future, and $D^{(0\downarrow)}$ if $\nabla\alpha$

is timelike and points toward the past. So at any event p, the character is

$$\begin{aligned}
D^{(+)} &\Leftrightarrow \frac{\partial F}{\partial U} < 0,\ \frac{\partial G}{\partial V} > 0,\\
D^{(-)} &\Leftrightarrow \frac{\partial F}{\partial U} > 0,\ \frac{\partial G}{\partial V} < 0,\\
D^{(0\uparrow)} &\Leftrightarrow \frac{\partial F}{\partial U} > 0,\ \frac{\partial G}{\partial V} > 0,\\
D^{(0\downarrow)} &\Leftrightarrow \frac{\partial F}{\partial U} < 0,\ \frac{\partial G}{\partial V} < 0.
\end{aligned}$$

It is straightforward to show that $D^{(+)}$ and $D^{(-)}$ at p imply no trapped cylinders at p, while $D^{(0\uparrow)}$ and $D^{(0\downarrow)}$ implies trapping. To prove this we note that the standard line element can be rewritten in terms of null coordinates U and V as follows

$$ds_+^2 \;=\; -e^{2(\gamma-\psi)}dUdV + e^{2\psi}dZ^2 + \alpha^2 e^{-2\psi}d\Phi^2.$$

The condition for a two-cylinder, S, of constant T and R to be untrapped is that

$$\theta_k^+\theta_l^+ < 0,$$

where θ_k^+, θ_l^+ are respectively the expansions of the future pointing outgoing and ingoing null geodesics k_+^a, l_+^a orthogonal to S, given by

$$k_+^a = a(U)e^{-2(\gamma-\psi)}\delta_V^a, \quad l_+^a = b(V)e^{-2(\gamma-\psi)}\delta_U^a,$$

where $a(U) > 0$ and $b(V) > 0$. We find the expansions of these null geodesics to be

$$\theta_k^+ = \nabla_a k_+^a = a(U)e^{-2(\gamma-\psi)}\frac{\alpha_{,V}}{\alpha}, \tag{3.4.3}$$

$$\theta_l^+ := \nabla_a l_+^a = b(V)e^{-2(\gamma-\psi)}\frac{\alpha_{,U}}{\alpha}, \tag{3.4.4}$$

and we can write

$$\theta_k^+\theta_l^+ = a(U)b(V)e^{-4(\gamma-\psi)}\frac{\alpha_{,V}\alpha_{,U}}{\alpha^2} = a(U)b(V)\frac{e^{-4(\gamma-\psi)}}{\alpha^2}\frac{\partial F}{\partial U}\frac{\partial G}{\partial V}.$$

The result follows by inspection.
We require that there are no trapped surfaces initially.

$$D^{(+)} \text{ or } D^{(-)} \text{ at } T = 0.$$

However, we note that in the absence of trapped cylinders

$$D^{(+)} \Leftrightarrow \frac{\partial \alpha}{\partial R} > 0, \quad \text{and} \quad D^{(-)} \Leftrightarrow \frac{\partial \alpha}{\partial R} < 0.$$

These constraints, together with the assumptions of §3.2.3 rule out $D^{(-)}$ initially,

$$D^{(+)} \text{ at } T = 0 \Leftrightarrow \left. \frac{\partial G(V)}{\partial V} \right|_{T=0} > 0 \tag{3 4.5}$$

Thorne [1] showed that in the vacuum region outside a cylindrical shell of matter, with the constraint (3.4.5), the only possible character change is

$$D^{(+)} \to D^{(0\downarrow)}.$$

If $D^{(+)}$ changes to $D^{(0\downarrow)}$ then there exists some point $p = (T_1, R_1)$, $T_1 > 0, R_1 > 0$, where

$$\left. \frac{\partial G(V)}{\partial V} \right|_{V=V_1} < 0,$$

where $V_1 = T_1 - R_1$ But the ingoing null hypersurface $V = V_1$ intersects $T = 0$ at $R = R_1$, so

$$\left. \frac{\partial G(V)}{\partial V} \right|_{T=0, R=R_1} < 0. \tag{3.4.6}$$

The contradiction between equations (3.4.5) and (3.4.6) implies that

$$D^{(+)} \text{ at } T = 0 \Rightarrow D^{(+)} \; \forall \; T \geq 0.$$

The argument also holds in the vacuum region outside our cylindrical star. Furthermore it has been shown, [39], that in a spacetime of character $D^{(+)}$ we can make a coordinate transformation

$$(T, R) \rightarrow \left(\hat{T}(T, R), \hat{R}(T, R)\right) \tag{3.4.7}$$

whereby $\alpha(R, T)$ becomes the new radial variable $\hat{R}$. Therefore if our vacuum spacetime does not contain trapped cylinders initially <u>and</u> is not radially closed ($D^{(+)}$ at $T = 0$) we can use the above results to describe the vacuum exterior spacetime, $(\mathcal{V}^+, g^+)$, by

$$ds_+^2 = -e^{2(\gamma-\psi)}(dT^2 - dR^2) + e^{2\psi}dZ^2 + R^2e^{-2\psi}d\Phi^2, \tag{3.4.8}$$

where we have rewritten $\hat{T}$ and $\hat{R}$ as T and R without confusion.

3.5 Impossibility of the matching

Thus far, we have shown that the most general matching of a non-vacuum RW universe with a vacuum cylindrically symmetric spacetime reduces to the case where the RW universe is dust-filled, the boundary is co-moving, the vacuum region is polarized and has character $D^{(+)}$. In this section, we show that this matching configuration is impossible. More generally, we show that metric matching alone rules out the matching of a collapsing RW universe across a co-moving hypersurface with a polarized cylindrical vacuum spacetime. The interior line element is

$$ds_-^2 = -dt^2 + a^2(t)(d\rho^2 + \Upsilon_{,\rho}^2(\rho, \epsilon)dx^2 + \Upsilon^2(\rho, \epsilon)d\varphi^2), \tag{3.5.1}$$

and the exterior line element is

$$ds_+^2 = -e^{2(\gamma-\psi)}(dT^2 - dR^2) + e^{2\psi}dZ^2 + R^2e^{-2\psi}d\Phi^2. \tag{3.5.2}$$

By a collapsing RW universe, we mean one for which the scale factor $a(t)$ decays to zero in finite time:

$$\lim_{t \to 0^-} a(t) = 0,$$

where by a time translation we have set the time of complete collapse to be at $t = 0$. Of course this includes the dust model considered above. Note that since we have dropped the junction condition $[K_{ab}] = 0$, the matching condition (3.2 1) no longer holds, and so we are not restricted to dust Metric continuity across the comoving hypersurface $\rho = \rho_0$ yields

$$\Upsilon_{,\rho}(\rho, \epsilon)a(t) \stackrel{\Sigma}{=} \exp(\psi(R_0(T), T)),$$

where

$$\Upsilon_{,\rho}(\rho, \epsilon)|_\Sigma = \begin{cases} \cosh\rho_0, & \epsilon = -1, \\ 1, & \epsilon = 0, \\ \cos\rho_0, & \epsilon = +1 \end{cases} \tag{3.5.3}$$

We note that if $\rho_0 = \frac{\pi}{2}$ in the case $\epsilon = +1$, then the matching conditions are violated. So we rule out this case. Noting then that $\Upsilon_{,\rho}(\rho, \epsilon)|_\Sigma \neq 0$, we immediately obtain

$$\lim_{T \to T_*} \psi(R_0(T), T) = -\infty, \tag{3 5.4}$$

where

$$T_* = \lim_{t \to 0^-} T_0(t),$$

where $T_0(t)$ is the solution of the metric matching condition

$$\left(\frac{dT_0}{dt}\right)^2 \stackrel{\Sigma}{=} e^{-2(\gamma-\psi)}(1 - (R_0')^2)^{-1}.$$

Now ψ satisfies the linear wave equation in 3-dimensional Minkowski spacetime (A 2), the solution of which can be written in the integral form

$$\begin{aligned} \psi(T, x, y) &= \frac{1}{2\pi}\frac{\partial}{\partial T}\left\{\int_{S(T)} \frac{\psi_0(x', y')}{[T^2 - (x - x')^2 - (y - y')^2]^{1/2}}\, dx'dy'\right\} \\ &+ \frac{1}{2\pi}\int_{S(T)} \frac{\psi_1(x', y')}{[T^2 - (x - x')^2 - (y - y')^2]^{1/2}}\, dx'dy', \end{aligned} \tag{3.5.5}$$

where

$$S(T) = \{(T, x, y) : T^2 \geq (x - x')^2 + (y - y')^2\},$$

and

$$\psi_0 .= \psi|_{T=0}, \quad \psi_1 = \frac{\partial \psi}{\partial T}|_{T=0}$$

are Cauchy initial data set on an arbitrary initial time slice (which we label as $T = 0$). We assume that these initial data are finite in an appropriate sense. Imposing smoothness and compact support are sufficient, although more general data would also satisfy our requirements [2]. This forms part of our assumption that all initial data for the problem are regular. Then the solution (3.5.5) obeys an *a priori* bound which holds for all finite $T > 0$ [6, 2]. Hence for any $T_1 > 0$

$$|\psi(R, T_1)| < +\infty, \quad R \geq 0.$$

So if $T_* < +\infty$, the limit equation (3 5.4) cannot be satisfied.

A similar conclusion holds in the case that $T_0 = +\infty$. We can expand (3.5.5) in inverse powers of T to obtain a uniformly convergent series representation [2]

$$\psi(R, T) = \sum_{k=1}^{\infty} \frac{\psi_k(R)}{T^k},$$

which yields $\lim_{T \to \infty} \psi(R, T) = 0$ uniformly in R for all $R \geq 0$. Hence (3.5.4) cannot be satisfied in this case, and so metric matching is ruled out

3.6 Null expansions

We can also use and extend to the cylindrical case, a result of Fayos, Senovilla and Torres [15], that if we have two C^3 orientable spacetimes $\mathcal{U}^-$ and $\mathcal{U}^+$ carrying C^2 metrics g^- and g^+ respectively, then every quantity in the resultant matched spacetime $\mathcal{U}^4$ constructed from the metric, its first derivatives and some C^1 tensor fields must be continuous across the boundary In the spherically symmetric case the null geodesic congruences are invariantly defined so the signs of the expansion scalars of these congruences must be continuous across the boundary. The outgoing radial null geodesics of the interior spacetime $\mathcal{V}_-$ are generated by

$$k^a_- = \frac{1}{a(t)}\delta^a_t + \frac{1}{a(t)^2}\delta^a_r,$$

with expansion scalar

$$\theta^-_k = \nabla_a k^a_- = \frac{2\dot{a}}{a^2} + \frac{1}{\rho a^2}. \tag{3.6.1}$$

The outgoing radial null geodesics of the exterior spacetime $\mathcal{V}^+$ have expansion scalar

$$\theta^+_k = \frac{a(U)}{2R}e^{2(\psi-\gamma)} \quad \text{for some} \quad a(U) > 0. \tag{3.6.2}$$

We can conclude that in $\mathcal{V}^-$, due to the t-dependance in the scale factor $a(t) = a_0|t|^{2/3}$ as $-t \to 0$, $\theta^-_k \to -\infty$ whereas $\mathcal{V}^+$ has θ^+_k strictly positive The discontinuity in the sign of θ across the boundary is in agreement with [15]. We could equivalently show that the region $\mathcal{V}^-$ does display the formation of trapped surfaces, $\theta = \theta^-_k\theta^-_l > 0$, whereas the region $\mathcal{V}^+$ does not.

3.7 Conclusions and discussion

We summarize the above as follows:

Proposition 3.7.1. *Let $(\mathcal{V}^+, g^+)$ be a vacuum cylindrically symmetric spacetime with metric described by (3 2.7), and with the following assumptions:*

(i). In $\mathcal{V}^+$ the metric function ω is analytic

(ii). In $\mathcal{V}^+$ the metric function ψ has regular initial data

(iii). $\mathcal{V}^+$ contains no trapped surfaces initially and is not radially closed.

Let $(\mathcal{V}^-, g^-)$ be a Robertson Walker spacetime with the energy conditions $\rho > 0$ and $\rho + p \geq 0$. Let $(\mathcal{V}^+, g^+)$ and $(\mathcal{V}^-, g^-)$ be matched across a C^2 hypersurface Σ with continuous normal $\vec{n}$ pointing out of $\mathcal{V}^-$ and into $\mathcal{V}^+$. Then at some value of the cosmological time and for all subsequent times the matching breaks down

This result demonstrates the impossibility of the existence of an isotropic, cylindrically symmetric star, that evolves from a regular initial state (or even a star with a cylindrically symmetric portion). Matching may be possible up until a trapped surface forms in $\mathcal{V}^-$ at a time $t = t^*$ By rearrangement of the matching conditions we can show *that* t^* *is given by the largest value of* t for which

$$({R_0}')^2 < 1,$$

where

$$({R_0}')^2 \stackrel{\Sigma}{=} 4\dot{a}^2 \left(\frac{\Upsilon}{\Upsilon_{,\rho}}\right)^2 .$$

In the time up until t^*, matching of the two space-times may be possible However, the initial conditions necessarily imply evolution to a state where matching is not possible.

A spherically symmetric static vacuole in a dust RW cosmology was shown to be possible [14] and it was deduced that the observed cosmological expansion would not affect local physics on astrophysical scales. Senovilla and Vera, [33], proved that embedding a cylindrically symmetric *static* region in an expanding RW cosmology is always impossible irrespective of the matter inside the cavity. Mars, [22],[23], investigated the Einstein-Straus model with a general static cavity embedded in a RW cosmology and obtained the result that the boundary of the static region must be a 2-sphere and that for various reasonable energy momentum tensors the interior is also spherically symmetric. These results were extended to stationary axisymmetric cavities in [28]. We can consider the complementary matching of a cylindrically symmetric vacuum interior with a RW exterior and impose regularity on the axis of a vacuum interior without affecting the matching, i.e. the axis is not singular. Matching of these two spacetimes may be possible for a finite amount of time up until a trapped cylinder appears in the RW exterior This leads to a contradiction and prevents the matching from persisting, and so again we do not have a valid physical configuration. Since our results also hold taking the vacuum region to be the interior and the RW the exterior, they complement [33] by also ruling out a dynamical cylindrically symmetric *vacuum* interior.

In light of these results the impossibility of a cylindrical isotropic star is perhaps unsurprising. However, the purpose of this study is to obtain a clearer picture of simple non-spherical, and more specifically, cylindrically symmetric systems in General Relativity.

Part III

Self-similar cylindrical gravitational collapse

Chapter 4

Self-similar cylindrically symmetric spacetime

The aim of this chapter is to examine cylindrically symmetric gravitational collapse. We will look at a vacuum model and a dust model. In order to make the field equations tractable we impose the assumption of continuous self-similarity. As described in the Introduction, self-similarity may be a physically reasonable assumption insofar as it has been seen to evolve naturally in many models and has been observed in critical collapse models.

4.1 Cylindrically symmetric spacetimes

A spacetime with cylindrical symmetry will have a line element

$$ds^2 = g_{ab}dx^a dx^b \tag{4.1.1}$$

which admits two spacelike commuting Killing vector fields, i.e. there exists coordinates $x^2 = z$ and $x^3 = \phi$ such that these Killing vector fields are

$$\begin{aligned} l_{(z)} &= \partial_z \quad \text{translational invariance,} \\ l_{(\phi)} &= \partial_\phi \quad \text{rotational invariance.} \end{aligned}$$

Where lower-case Roman indices run through $0, 1, 2, 3$ If (4.1.1) admits these two Killing vectors then

$$\mathcal{L}_{l_{(z)}} g_{ab} = \mathcal{L}_{l_{(\phi)}} g_{ab} = 0,$$

which implies that g_{ab} are independent of z and ϕ. The azimuthal angle ϕ will be identified at 0 and 2π i e. ϕ is periodic with period 2π and z is not periodic. To obtain a simple form for our line element we impose some further restrictions (whole cylinder symmetry):We require that $l_{(z)}$ and $l_{(\phi)}$ are each hypersurface orthogonal therefore

$$\epsilon^{abcd} l_{(z)b} l_{(z)c,d} = \epsilon^{abcd} l_{(\phi)b} l_{(\phi)c,d} = 0$$

where ϵ^{abcd} is the Levi-Civita symbol. We also require that $l_{(\phi)} \cdot l_{(z)} = 0$, so that $l_{(z)}$ lies in the hypersurface orthogonal to $l_{(\phi)}$ and $l_{(\phi)}$ lies in the hypersurface orthogonal to $l_{(z)}$. We choose x^0 and x^1 as coordinates for the 2-surfaces orthogonal to $l_{(z)}$ and $l_{(\phi)}$ and then (4.1.1) becomes

$$ds^2 = g_{00}\mathrm{d}(x^0)^2 + 2g_{01}\mathrm{d}x^0\mathrm{d}x^1 + g_{11}\mathrm{d}(x^1)^2 + g_{22}\mathrm{d}z^2 + g_{33}\mathrm{d}\phi^2, \tag{4.1.2}$$

where g_{ab} depend only on x^0 and x^1. As we require that our metric is Lorentzian then g_{ab} must have one negative and three positive eigenvalues. If we name $x^0 = t$ (where we specify that t *is the time coordinate*) and $x^1 = r$ and introduce $\gamma, \delta, \kappa, \nu, \sigma$ arbitrary functions of t and r, then we can write the line element as

$$ds^2 = -\gamma^2\mathrm{d}t^2 + 2\gamma\delta\mathrm{d}t\mathrm{d}r + \kappa^2\mathrm{d}r^2 + \nu^2\mathrm{d}z^2 + \sigma^2\mathrm{d}\phi^2 \tag{4.1.3}$$

There is the freedom in the x^0 coordinate to make the transformation

$$\lambda\mathrm{d}t' = \gamma\mathrm{d}t - \delta\mathrm{d}r$$

to obtain

$$ds^2 = -\lambda^2\mathrm{d}t'^2 + (\delta^2 + \kappa^2)\mathrm{d}r^2 + \nu^2\mathrm{d}z^2 + \sigma^2\mathrm{d}\phi^2. \tag{4.1.4}$$

This diagonalises the line element and is equivalent to choosing t' orthogonal to r. We rename $\delta^2 + \kappa^2 = \mu^2$ and $t' = t$ to get

$$ds^2 = -\lambda^2 \mathrm{d}t^2 + \mu^2 \mathrm{d}r^2 + \nu^2 \mathrm{d}z^2 + \sigma^2 \mathrm{d}\phi^2, \tag{4.1.5}$$

where λ, μ, ν and σ are all functions of r and t. We see that the length of a covariant vector pointing in the t-direction, $\nabla_a t < 0$, and thus t is a timelike coordinate, whereas r, z, ϕ are spacelike coordinates

We note the z-axis is now fixed and (4.1.5) has the following symmetries:

1. $z \to z' = z + \Delta z$ translations along the z-axis,
2. $z \to z' = -z$ reflection in all surfaces perpendicular to the z-axis,
3. $\phi \to \phi' = \phi + \Delta\phi$ rotations about the z-axis,
4. $\phi \to \phi' = -\phi$ reflection in all surfaces containing the z-axis.

The norms of the Killing vectors are geometrical invariants, the circumferential radius

$$r_\phi = \sqrt{l_{(\phi)a} l^a_{(\phi)}},$$

and the specific length

$$r_z = \sqrt{l_{(z)a} l^a_{(z)}},$$

can be combined to form a function

$$R(t, r) = r_\varphi r_z, \tag{4.1.6}$$

which plays a similar role to the areal radius in spherical symmetry [18].

4.1.1 Regularity conditions

To have cylindrical symmetry some physical and geometrical conditions are needed. We define 'the axis' to be situated at $r = 0$.

1 There must exist a regular axis:

$$\Upsilon = |l^i_{(\phi)} l^j_{(\phi)} g_{ij}| \longrightarrow 0, \quad \text{as } r \longrightarrow 0, \tag{4.1.7}$$

2. Spacetime is locally flat near the axis:

$$\frac{\Upsilon_{,i}\Upsilon_{,j}g^{ij}}{4\Upsilon} \longrightarrow 1, \quad \text{as } r \longrightarrow 0,$$

3. The density must remain finite and non zero as $r \longrightarrow 0$:

$$0 < \rho < \infty.$$

4.2 Self-similar cylindrically symmetric vacuum spacetime

The cylindrically symmetric self-similar vacuum spacetime may be thought of as a "ground state" for this study of cylindrically symmetric self-similarity

The whole cylindrically symmetric line element for a vacuum spacetime, the Einstein-Rosen metric, is given by

$$\mathrm{d}s^2 = e^{2(\mu-\nu)}(-\mathrm{d}t^2 + \mathrm{d}r^2) + e^{2\nu}\mathrm{d}z^2 + \sigma^2 e^{-2\nu}\mathrm{d}\phi^2,$$

where μ, ν and σ are functions of r and t. As we have seen, self-similarity is a useful assumption We define a similarity (self-similar) solution of the field equations as one for which the resulting spacetime admits the homothetic vector k^i satisfying

$$\mathcal{L}_{\vec{k}} g_{ij} = k_{i,j} + k_{j,i} = 2g_{ij} \tag{4.2.1}$$

We choose a homothetic vector $\vec{k}$ which commutes with the two Killing vectors i.e., the Abelian similarity group H_3, Bianchi type I (tilted case). There are other homothetic vectors, see [7], but we focussed on this one as it is cylindrically symmetric·

$$k^i = \alpha(r,t)\delta^i_0 + \beta(r,t)\delta^i_1, \tag{4.2.2}$$

where as we are interested in the tilted case so $\alpha(r,t) \neq 0$ and $\beta(r,t) \neq 0$.

4.2.1 Case 1.

We will consider the special case where $\alpha - \beta \neq 0$ and $\alpha + \beta \neq 0$ first. By inserting (4.2.2) into (4.2.1) we obtain a set of 5 equations. These can be reduced to the following

$$\begin{aligned}
\dot{\alpha} + (\dot{\mu} - \dot{\nu})\alpha + (\mu' - \nu')\beta &= 1, && (4.2.3a)\\
\beta' + (\dot{\mu} - \dot{\nu})\alpha + (\mu' - \nu')\beta &= 1, && (4.2.3b)\\
\alpha' - \dot{\beta} &= 0, && (4.2.3c)\\
\alpha\dot{\nu} + \beta\nu' &= 1, && (4.2.3d)\\
\alpha(\sigma\dot{\nu} - \dot{\sigma}) + \beta(\sigma\nu' - \sigma') &= -\sigma, && (4.2.3e)
\end{aligned}$$

where overdot represents differentiation with respect to t and prime represents differentiation with respect to r. Clearly we have

$$(4.2.3a) - (4.2.3b) \Rightarrow \dot{\alpha} - \beta' = 0. \tag{4.2.4}$$

Then if we differentiate (4.2.4) and (4.2.3c) we obtain

$$\ddot{\alpha} - \alpha'' = 0, \quad \ddot{\beta} - \beta'' = 0,$$

which has the solution

$$\alpha = f(u) + g(v), \quad \beta = h(u) + k(v),$$

where $u = t - r$ and $v = t + r$. Next we can use (4.2.4) and (4.2.3c) to simplify this to

$$\alpha = f(u) + g(v), \quad \beta = -f(u) + g(v) + C,$$

where C is a constant. It is convenient now to write

$$\alpha + \beta = 2g(v) + C, \quad \alpha - \beta = 2f(u) - C,$$

and our homothetic Killing vector as

$$\begin{aligned} \vec{k} &= \alpha(r,t)\frac{\partial}{\partial t} + \beta(r,t)\frac{\partial}{\partial r} \\ &= (\alpha - \beta)\frac{\partial}{\partial u} + (\alpha + \beta)\frac{\partial}{\partial v} \\ &= (2f(u) - C)\frac{\partial}{\partial u} + (2g(v) + C)\frac{\partial}{\partial v}. \end{aligned}$$

We can define new coordinates $\bar{u}$ and $\bar{v}$ by

$$\frac{\mathrm{d}\bar{u}}{\mathrm{d}u} = \frac{\bar{u}}{(\alpha - \beta)} = \frac{\bar{u}}{(2f(u) - C)}, \quad \frac{\mathrm{d}\bar{v}}{\mathrm{d}v} = \frac{\bar{v}}{(\alpha + \beta)} = \frac{\bar{v}}{(g(v) + C)},$$

to obtain

$$\vec{k} = \bar{u}\frac{\partial}{\partial \bar{u}} + \bar{v}\frac{\partial}{\partial \bar{v}}.$$

If we define

$$2(\bar{\mu} - \bar{\nu}) = 2(\mu - \nu) + \ln(\alpha + \beta) + \ln(\alpha - \beta) - \ln \bar{u} - \ln \bar{v},$$

and insert this into (4.2 3a) and (4.2.3b) we get

$$\bar{u}\frac{\partial(\bar{\mu} - \bar{\nu})}{\partial \bar{u}} + \bar{v}\frac{\partial(\bar{\mu} - \bar{\nu})}{\partial \bar{v}} = 0,$$

which implies that $(\bar{\mu} - \bar{\nu})$ is a function of $\frac{\bar{u}}{\bar{v}}$. With this definition we can write

$$e^{2(\mu - \nu)}\mathrm{d}u\mathrm{d}v = e^{2(\bar{\mu} - \bar{\nu})}\mathrm{d}\bar{u}\mathrm{d}\bar{v}$$

Similarly, we can define

$$\bar{\nu} = \nu - \ln \bar{u},$$

to show that $\bar{\nu}$ is a function of $\frac{\bar{u}}{\bar{v}}$. Finally (4.2.3d) and (4.2.3e) give us

$$\begin{aligned}\alpha\dot{\sigma}+\beta\sigma' &= 2\sigma,\\ \Rightarrow \bar{u}\frac{\partial\sigma}{\partial\bar{u}}+\bar{v}\frac{\partial\sigma}{\partial\bar{v}} &= 2\sigma,\\ \Rightarrow \sigma &= \bar{u}^2\bar{\sigma}\left(\frac{\bar{u}}{\bar{v}}\right).\end{aligned}$$

If we define $\bar{r}$ and $\bar{t}$ by $\bar{u}=\bar{t}-\bar{r}$ and $\bar{v}=\bar{t}+\bar{r}$ then we can rewrite

$$\bar{u}^2=\bar{r}^2\left(1-\left(\frac{\bar{t}}{\bar{r}}\right)+\left(\frac{\bar{t}}{\bar{r}}\right)^2\right)=\bar{r}^2 f(\bar{\zeta}),$$

where $\bar{\zeta}=\frac{\bar{t}}{\bar{r}}$ and

$$\bar{\sigma}=\bar{\sigma}\left(\frac{\bar{u}}{\bar{v}}\right)=\bar{\sigma}\left(\frac{\bar{t}-\bar{r}}{\bar{t}+\bar{r}}\right)=\bar{\sigma}\left(\frac{\bar{\zeta}-1}{\bar{\zeta}+1}\right)\equiv\sigma(\bar{\zeta}).$$

Then we get

$$e^{2\nu}\mathrm{d}z^2+\sigma^2e^{-2\nu}\mathrm{d}\phi^2=\bar{u}^2(e^{2\bar{\nu}}\mathrm{d}z^2+\bar{\sigma}^2e^{-2\bar{\nu}}\mathrm{d}\phi^2)=\bar{r}^2(e^{2\tilde{\nu}}\mathrm{d}z^2+\tilde{\sigma}^2e^{-2\tilde{\nu}}\mathrm{d}\phi^2)$$

where $e^{2\tilde{\nu}}=f(\bar{\zeta})e^{2\bar{\nu}}$, $\tilde{\sigma}=f(\bar{\zeta})\bar{\sigma}$ and $e^{2\tilde{\mu}}=f(\bar{\zeta})^{-1}e^{2\bar{\mu}}$, finally we obtain

$$\mathrm{d}s^2=e^{2(\tilde{\mu}-\tilde{\nu})}(-\mathrm{d}\bar{t}^2+\mathrm{d}\bar{r}^2)+\bar{r}^2\left(e^{2\tilde{\nu}}\mathrm{d}z^2+\tilde{\sigma}^2e^{-2\tilde{\nu}}\mathrm{d}\phi^2\right)$$

We then simply drop the bars and tildes and rename as follows:

$$\mathrm{d}s^2=e^{2(\mu-\nu)}(-\mathrm{d}t^2+\mathrm{d}r^2)+r^2\left(e^{2\nu}\mathrm{d}z^2+\sigma^2e^{-2\nu}\mathrm{d}\phi^2\right),$$

where μ, ν and σ now denote functions of the similarity variable $\zeta=\frac{t}{r}$.

The field equations for a vacuum spacetime with this line element are obtained from

$$G_{ab}=0.$$

We have the following five equations:

$$\begin{aligned}
G_{11} = 0 &\Rightarrow \zeta^2\sigma - \dot{\sigma}((1+\zeta^2)\dot{\mu} + \zeta) + \sigma(1 + 2\zeta(\dot{\mu} - \dot{\nu}) + (1+\zeta^2)\dot{\nu}^2) = 0, \\
G_{12} = 0 &\Rightarrow \zeta\ddot{\sigma} - \dot{\sigma}(2\zeta\mu) + \sigma(2(\dot{\mu} - \dot{\nu}) + 2\zeta\nu^2) = 0, \\
G_{22} = 0 &\Rightarrow \ddot{\sigma} + \dot{\sigma}(\zeta - (1+\zeta^2)\dot{\mu}) + \sigma(-1 + 2\zeta(\dot{\mu} - \dot{\nu}) + (1+\zeta^2)\dot{\nu}^2) = 0, \\
G_{33} = 0 &\Rightarrow \sigma\left(2\zeta(\dot{\mu} - \mu) + (\zeta^2 - 1)(\dot{\nu}^2 + \dot{\mu} - 2\ddot{\nu})\right) + (\zeta^2 - 1)(\ddot{\sigma} - 2\dot{\nu}\dot{\sigma}) = 0, \\
G_{44} = 0 &\Rightarrow 2\zeta(\dot{\mu} - \dot{\mu}) + (\zeta^2 - 1)(\dot{\nu}^2 + \ddot{\mu}) = 0.
\end{aligned}$$

where overdot now represents differentiation with respect to ζ. These can be rearranged as follows

$$G_{11} - G_{22} = 0 \quad \Rightarrow \quad \ddot{\sigma}(1 - \zeta^2) + \dot{\sigma}(2\zeta) - 2\sigma = 0.$$

This can be integrated twice to obtain

$$\sigma = c_1\zeta^2 + c_2\zeta + c_1, \tag{4.2.5}$$

where c_1 and c_2 are the two constants of integration. Next we write

$$G_{33} - G_{44} = 0 \quad \Rightarrow \quad -2\ddot{\nu}\sigma + \ddot{\sigma} - 2\dot{\nu}\dot{\sigma} = 0,$$

and using our expression for (4.2.5) and integrating once we get

$$\dot{\nu} = \frac{c_1\zeta + c_3}{c_1\zeta^2 + c_2\zeta + c_1}, \tag{4.2.6}$$

where c_3 is a constant of integration. Next we use $G_{12} = 0$ and $G_{11} = 0$ to obtain two expressions for $\dot{\mu}$. We equate these to get

$$\frac{-c_1\zeta + \sigma\dot{\nu}(1 - \zeta\dot{\nu})}{\sigma - \zeta\dot{\sigma}} = \frac{c_1(\zeta^2+1) + \sigma\dot{\nu}(\dot{\nu}(\zeta^2+1) - 2\zeta)}{c_2(1-\zeta^2)}.$$

We can use (4.2.6) in this expression and simplify to obtain

$$c_2 = \frac{c_1^2 + c_3^2}{c_3},$$

which we can put back into (4.2.5) to get,

$$\sigma = \frac{1}{c_3}(c_3\zeta + c_1)(c_1\zeta + c_3)$$

Next we examine (4.2.6) which gives us

$$\dot{\nu} = \frac{c_3}{c_3\zeta + c_1},$$

which we integrate to get

$$\nu = \ln(c_3\zeta + c_1) + c_4,$$

where c_4 is the constant of integration. Finally, we can return to $G_{12} = 0$ to get

$$\dot{\mu} = \frac{c_3}{c_3\zeta + c_1},$$

which we integrate to get

$$\mu = \ln(c_3\zeta + c_1) + c_5,$$

where c_5 is the constant of integration. With this solution the line element becomes

$$\begin{aligned} \mathrm{d}s^2 &= e^{2(c_5-c_4)}(-\mathrm{d}t^2 + \mathrm{d}r^2) + r^2e^{2c_4}(c_3\zeta + c_1)^2\mathrm{d}z^2 + r^2e^{-2c_4}(c_1\zeta + c_3)^2\mathrm{d}\phi^2 \\ &= a_1(-\mathrm{d}t^2 + \mathrm{d}r^2) + a_2(c_3t + c_1r)^2\mathrm{d}z^2 + a_3(c_1t + c_3r)^2\mathrm{d}\phi^2, \end{aligned}$$

where we renamed our constants as $a_1, a_2, a_3 > 0$. We categorise this line element as follows.

case 1(a). If $c_3 > c_1$ then we can define new time and radial coordinates as follows. $t \to \tilde{t} = c_3t + c_1r$, $r \to \tilde{r} = c_1t + c_3r$. Then we get

$$\mathrm{d}s^2 = -\mathrm{d}\tilde{t}^2 + \mathrm{d}\tilde{r}^2 + \left(\frac{a_2}{a_1}\right)\tilde{t}^2\mathrm{d}z^2 + \left(\frac{a_3}{a_1}\right)\tilde{r}^2\mathrm{d}\phi^2.$$

If $\frac{a_3}{a_1} \neq 1$ then this spacetime will have a conical singularity (or string along the z-axis), if $\frac{a_3}{a_1} = 1$ we have a flat spacetime which could represent a 'ground state' for our study of cylindrically symmetric self-similar spacetimes.

case 1(b). If $c_3 < c_1$ then we can define new time and radial coordinates as follows:

$t \rightarrow \tilde{t} = c_1 t + c_3 r$, $r \rightarrow \tilde{r} = c_3 t + c_1 r$. Then we get

$$\mathrm{d}s^2 = -\mathrm{d}\tilde{t}^2 + \mathrm{d}\tilde{r}^2 + \left(\frac{a_2}{a_1}\right)\tilde{r}^2\mathrm{d}z^2 + \left(\frac{a_3}{a_1}\right)\tilde{t}^2\mathrm{d}\phi^2.$$

This spacetime does not obey the regularity conditions (see §4.1.1) which are necessary for defining a physical cylindrical geometry

case 1(c). If $c_3 = c_1$ then we can define null coordinates as follows. $u = t - r$, $u = t + r$. Then we get

$$\mathrm{d}s^2 = \mathrm{d}u\mathrm{d}v + \left(\frac{a_2}{a_1}\right)v^2\mathrm{d}z^2 + \left(\frac{a_3}{a_1}\right)v^2\mathrm{d}\phi^2.$$

This spacetime is singular for the null hypersurface $v = 0$. This spacetime does not obey the regularity conditions (see §4.1.1) which are necessary for defining a physical cylindrical geometry.

4.2.2 Case 2.

We will consider the case where $\alpha - \beta = 0$ and $\alpha + \beta \neq 0$. In this case we can show that the homothethic vector becomes

$$\vec{k} = g(v)\left(\partial_v\right), \tag{4 2 7}$$

and applying 4.2.1 and the field equations we can simplify the line element to

$$\mathrm{d}s^2 = A(u)\mathrm{d}u\mathrm{d}v + v^2\left(C_1\mathrm{d}z^2 + C_2\mathrm{d}\phi^2\right),$$

where C_1 and C_2 are constants. This spacetime does not obey the regularity conditions (see §4.1.1) which are necessary for defining a physical cylindrical geometry

4.2.3 Case 3.

We will consider the case where $\alpha - \beta = 0$ and $\alpha + \beta \neq 0$. In this case we can show that the homothethic vector becomes

$$\vec{k} = f(u)\left(\partial_u\right), \tag{4.2.8}$$

and applying 4.2.1 and the field equations we can simplify the line element to

$$\mathrm{d}s^2 = B(v)\mathrm{d}u\mathrm{d}v + u^2\left(C_3\mathrm{d}z^2 + C_4\mathrm{d}\phi^2\right),$$

where C_3 and C_4 are constants. This spacetime does not obey the regularity conditions (see §4.1.1) which are necessary for defining a physical cylindrical geometry

4.3 Self-similar cylindrically symmetric dust spacetime

The matter field we will consider is dust, with energy momentum tensor

$$T^{ab} = \rho u^a u^b,$$

where ρ is the density and u^a is a unit future-pointing timelike vector field which is tangent to the flow lines of the dust We take the coordinates t and r to be co-moving which means that $u^a \propto \delta^a_t$ and $u^a\nabla_a r = 0$ i.e the dust particles are at rest in these coordinates $(u^\mu = 0)$ We take t to increase into the future. We can show that $u^a = \frac{1}{\lambda}\delta^a_t$ and the conservation equation $T^{ab}_{\ ;a} = 0$ implies that

$$\frac{\mathrm{d}\lambda}{\mathrm{d}r} = 0 \Rightarrow \lambda = f(t),$$

thus if we let $\tilde{t}$ be defined by

$$\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t} = f(t)$$

then (4.1.5) becomes

$$ds^2 = -d\tilde{t}^2 + \mu^2 dr^2 + \nu^2 dz^2 + \sigma^2 d\phi^2,$$

and we then rename $\tilde{t}$ as t.

As we have discussed in the Introduction, and based on a large body of evidence, self-similarity is physically reasonable assumption to impose in a study of gravitational collapse. As before, we define a similarity (self-similar) solution of the field equations as one for which the resulting spacetime admits the homothetic vector k^i satisfying

$$\mathcal{L}_{\vec{k}} g_{ij} = k_{i,j} + k_{j,i} = 2g_{ij}. \tag{4.3.1}$$

We again choose a homothetic vector $\vec{k}$ which commutes with the two Killing vectors i.e., the Abelian similarity group H_3, Bianchi type I (tilted case). There are other homothetic vectors, see [7], but we focussed on this one as it is cylindrically symmetric:

$$k^i = \alpha(r,t)\delta^i_0 + \beta(r,t)\delta^i_1, \tag{4.3.2}$$

where as we are interested in the tilted case so $\alpha(r,t) \neq 0$ and $\beta(r,t) \neq 0$. In this model the source of the gravitational field is dust, described by the energy momentum tensor, T^{ij}, and it follows that T^{ij} is also conformally invariant, i.e.,

$$\begin{aligned} \mathcal{L}_k T^{ij} &= 2T^{ij}, \\ \Rightarrow u^i &= k^i_{,j} u^j - u^i_{,j} k^j. \end{aligned} \tag{4.3.3}$$

By inserting (4.3.2) into (4.3.1) we obtain the following equations

$$\frac{\partial \alpha}{\partial t} = 1, \tag{4.3.4a}$$

$$\frac{\alpha}{\mu}\frac{\partial \mu}{\partial t} + \frac{\beta}{\mu}\frac{\partial \mu}{\partial r} + \frac{\mathrm{d}\beta}{\mathrm{d}r} = 1, \tag{4.3.4b}$$

$$\frac{\partial \alpha}{\partial r} = \mu^2 \frac{\partial \beta}{\partial t}, \tag{4.3 4c}$$

$$\alpha\frac{\partial \nu}{\partial t} + \beta\frac{\partial \nu}{\partial r} = \nu, \tag{4.3.4d}$$

$$\alpha\frac{\partial \sigma}{\partial t} + \beta\frac{\partial \sigma}{\partial r} = \sigma. \tag{4 3 4e}$$

Then we insert (4.3.2) into (4.3 3) to get

$$\frac{\partial \alpha}{\partial t} = 1 \quad \text{and} \quad \frac{\partial \beta}{\partial t} = 0$$

So using (4.3.4c) we can show that

$$\frac{\partial \alpha}{\partial r} = 0 \quad \Rightarrow \alpha = \alpha(t), \quad \text{and} \quad \frac{\partial \beta}{\partial t} = 0 \quad \Rightarrow \beta = \beta(r).$$

We can define new independent variables $\bar{t}$ and $\bar{r}$ by

$$\frac{\mathrm{d}\bar{t}}{\mathrm{d}t} = \frac{\bar{t}}{\alpha}, \quad \frac{\mathrm{d}\bar{r}}{\mathrm{d}r} = \frac{\bar{r}}{\beta},$$

and a new dependant variable

$$\ln \bar{\mu} = \ln \mu + \ln \beta - \ln \bar{r},$$

and recast our equations to get

$$\begin{aligned}
(4.3\ 4\mathrm{b}) \Rightarrow \bar{t}\frac{\partial \bar{\mu}}{\partial \bar{t}} + \bar{r}\frac{\partial \sigma}{\partial \bar{r}} &= 0, \\
(4.3\ 4\mathrm{d}) \Rightarrow \bar{t}\frac{\partial \nu}{\partial \bar{t}} + \bar{r}\frac{\partial \nu}{\partial \bar{r}} &= \nu, \\
(4.3\ 4\mathrm{e}) \Rightarrow \bar{t}\frac{\partial \sigma}{\partial \bar{t}} + \bar{r}\frac{\partial \sigma}{\partial \bar{r}} &= \sigma.
\end{aligned}$$

These equations indicate that we can write our metric functions as

$$\mu = \bar{\mu}\left(\frac{\bar{r}}{\bar{t}}\right), \quad \nu = \bar{r}\bar{\nu}\left(\frac{\bar{r}}{\bar{t}}\right), \quad \sigma = \bar{r}\bar{\sigma}\left(\frac{\bar{r}}{\bar{t}}\right).$$

We can then show that our line element becomes

$$\mathrm{d}s^2 = -\mathrm{d}\bar{t}^2 + \bar{\mu}^2\mathrm{d}\bar{r}^2 + \bar{r}^2\left(\bar{\nu}^2\mathrm{d}z^2 + \bar{\sigma}^2\mathrm{d}\phi^2\right).$$

For clarity we rename our variables, by dropping the bars and we define $\zeta = \frac{t}{r}$ and $\xi = \frac{r}{t}$. Finally, we arrive at

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mu(\xi)^2\mathrm{d}r^2 + r^2\left(\nu(\xi)^2\mathrm{d}z^2 + \sigma(\xi)^2\mathrm{d}\phi^2\right).$$

We will show that we can write our density ρ in terms of the similarity variable after we have written out our field equations.

4.4 Einstein's field equations for dust

The field equations

$$G_{ab} = T_{ab}$$

where we have set $8\pi G = c = 1$, gives us the following five equations.

$$G_{11} = \rho \;\Rightarrow\; -\frac{\ddot{\sigma}}{\sigma}\zeta^2 + \frac{\dot{\sigma}}{\sigma}\left(\zeta + \frac{\dot{\nu}}{\nu}(\mu^2 - \zeta^2) + \frac{\dot{\mu}}{\mu}(\mu^2 + \zeta^2)\right) + \\ + \frac{\dot{\mu}}{\mu}\frac{\dot{\nu}}{\nu}(\mu^2 + \zeta^2) - 1 - \zeta\frac{2\dot{\mu}}{\mu} + \frac{\dot{\nu}}{\nu}\zeta - \frac{\dot{\nu}}{\nu}\zeta^2 = \rho r^2\mu^2, \tag{4.4.1a}$$

$$G_{12} = 0 \;\Rightarrow\; \frac{\ddot{\sigma}}{\sigma} - \frac{\dot{\sigma}}{\sigma}\frac{\dot{\mu}}{\mu} + \frac{2\dot{\mu}}{\zeta\mu} - \frac{\dot{\nu}}{\nu}\frac{\dot{\mu}}{\mu} + \frac{\dot{\nu}}{\nu} = 0, \tag{4.4.1b}$$

$$G_{22} = 0 \;\Rightarrow\; \frac{\ddot{\sigma}}{\sigma}\mu^2 + \frac{\sigma}{\sigma}\left(\zeta + \frac{\dot{\nu}}{\nu}(\mu^2 - \zeta^2)\right) - 1 + \frac{\zeta\dot{\nu}}{\nu} + \frac{\mu^2\ddot{\nu}}{\nu} = 0, \tag{4.4.1c}$$

$$G_{33} = 0 \;\Rightarrow\; \frac{\ddot{\sigma}}{\sigma}(\mu^2 - \zeta^2) + \frac{\dot{\sigma}}{\sigma}\frac{\dot{\mu}}{\mu}(\mu^2 + \zeta^2) + \mu\ddot{\mu} - \frac{\dot{\mu}\zeta}{\mu} = 0, \tag{4.4.1d}$$

$$G_{44} = 0 \;\Rightarrow\; \frac{\ddot{\nu}}{\nu}(\mu^2 - \zeta^2) + \frac{\dot{\mu}}{\mu}\frac{\dot{\nu}}{\nu}(\mu^2 + \zeta^2) + \mu\ddot{\mu} - \frac{\dot{\mu}\zeta}{\mu} = 0. \tag{4.4.1e}$$

where overdot represents differentiation with respect to ζ This set of ordinary differential equations is not in a convenient or workable form, so we will rearrange them as follows:

$$(G_{33}+G_{44})-(\mu^2-\zeta^2)G_{12}=0 \quad \Rightarrow \quad \frac{\dot{\sigma}}{\sigma}+\frac{\dot{\nu}}{\nu}-\frac{1}{\zeta}+\frac{\ddot{\mu}}{\dot{\mu}}=0. \tag{4.4.2}$$

Then we notice that $(\mu^2G_{12}-G_{22})=0$ has only first order derivatives

$$\frac{\dot{\sigma}}{\sigma}(-\zeta+\dot{\mu}\mu)+\frac{\dot{\nu}}{\nu}(-\zeta+\dot{\mu}\mu)+1+\frac{2\dot{\nu}}{\nu}\zeta-\frac{\dot{\sigma}}{\sigma}\frac{\dot{\nu}}{\nu}\left(\mu^2-\zeta^2\right)=0. \tag{4.4.3}$$

We can use (4.4.2) to substitute for $\frac{\sigma}{\sigma}$ in (4.4.3) and we get a quadratic equation in $\frac{\nu}{\nu}$. Similarly we can use (4.4.2) to substitute for $\frac{\nu}{\nu}$ in (4.4.3) and we the same quadratic equation in $\frac{\sigma}{\sigma}$

$$Q(\chi) \equiv \chi^2+\left(\frac{\mu}{\mu}-\frac{1}{\zeta}\right)\chi+\left(\frac{\ddot{\mu}}{\dot{\mu}}\zeta+\frac{\dot{\mu}\mu}{\zeta}+\ddot{\mu}\mu\right)\frac{1}{\mu^2-\zeta^2}=0. \tag{4.4.4}$$

This is a quadratic equation in χ where

$$\chi=\frac{\dot{\sigma}}{\sigma} \qquad \text{and} \qquad \chi=\frac{\dot{\nu}}{\nu}.$$

The roots of (4 4.4) are $\chi=\delta_1\pm\delta_2$ where

$$\begin{aligned}\delta_1 &= \frac{1}{2}\left(\frac{1}{\zeta}-\frac{\ddot{\mu}}{\mu}\right),\\ \delta_2 &= \frac{1}{2}\sqrt{\left(\frac{1}{\zeta}-\frac{\ddot{\mu}}{\dot{\mu}}\right)^2-\frac{4\left(\frac{\mu}{\mu}\zeta+\frac{\mu\mu}{\zeta}+\ddot{\mu}\mu\right)}{\mu^2-\zeta^2}}.\end{aligned}$$

Consider the case when $\chi=\frac{\nu}{\nu}$. Our solutions are

$$\frac{\dot{\nu}}{\nu}=\delta_1\pm\delta_2.$$

We will label them as $\left(\frac{\nu}{\nu}\right)_{\pm}$ Then

$$\dot{\nu}_{\pm} = \nu_{\pm}(\delta_1 \pm \delta_2) \quad = \quad \nu_{\pm}\delta_{\pm}. \tag{4.4.5}$$

We can differentiate (4.4.5) to get

$$\ddot{\nu}_{\pm} = \nu_{\pm}\delta_{\pm} + \nu_{\pm}\dot{\delta_{\pm}} \tag{4.4.6}$$

But we can write G_{44} as

$$\nu \quad = \quad \dot{\nu}\alpha_1 + \nu\alpha_2. \tag{4.4.7}$$

Then (4.4.6) and (4.4.7) imply

$$\dot{\nu}_{\pm}(\alpha_1 - \delta_{\pm}) + \nu_{\pm}(\alpha_2 - \dot{\delta_{\pm}}) = 0.$$

We have then

$$\begin{aligned} -\dot{\nu}_{\pm} + \nu_{\pm}\delta_{\pm} &= 0, \\ \dot{\nu}_{\pm}(\alpha_1 - \delta_{\pm}) + \nu_{\pm}(\alpha_2 - \dot{\delta_{\pm}}) &= 0, \end{aligned}$$

which imply

$$\begin{aligned} \delta_{\pm}(\alpha_1 - \delta_{\pm}) + (\alpha_2 - \dot{\delta_{\pm}}) &= 0, \\ \Rightarrow (\delta_1 + \delta_2)(\alpha_1 - \delta_1 - \delta_2) + (\alpha_2 - \dot{\delta_1} - \delta_2) &= 0, \qquad (4.4.8) \\ \Rightarrow (\delta_1 - \delta_2)(\alpha_1 - \delta_1 + \delta_2) + (\alpha_2 - \dot{\delta_1} + \dot{\delta_2}) &= 0, \qquad (4.4.9) \end{aligned}$$

where α_1 and α_2 depend on $\mu, \dot{\mu}, \ddot{\mu}$ and ζ (the same result holds for $\frac{\sigma}{\sigma}$ as well). If we add and subtract (4.4.8) and (4.4.9), we obtain

$$\begin{aligned} 2\dot{\delta_1} + 2(\delta_1^2 + \delta_2^2) - 2\delta_1\alpha_1 - 2\alpha_2 &= 0, \\ 2\dot{\delta_2} + 4\delta_2^2 - 2\delta_2\alpha_1 &= 0. \end{aligned}$$

Both of these equations yield the same third order ODE in $\mu(\zeta)$. So we arrive at

$$\frac{\dddot{\mu}}{\dot{\mu}} - 2\left(\frac{\ddot{\mu}}{\dot{\mu}}\right)^2 + \left(\frac{\dot{\mu}}{\zeta\mu} + \frac{\ddot{\mu}}{\mu}\right)\frac{\mu^2+\zeta^2}{\mu^2-\zeta^2} + \frac{2\mu^2}{\zeta(\mu^2-\zeta^2)}\frac{\ddot{\mu}}{\mu} = 0$$

Finally we can rewrite our density equation

$$G_{11} + \zeta^2 G_{12} + (4.4.2) = 0 \Rightarrow \frac{2\dot{\mu}\mu}{\zeta} = \rho r^2 \mu^2.$$

These calculations show that (4.4.1) are equivalent to the following set of equations, and this is the set we will work with from now on:

$$\begin{aligned}
\frac{\dot{\sigma}}{\sigma} + \frac{\dot{\nu}}{\nu} - \frac{1}{\zeta} + \frac{\dot{\mu}}{\mu} &= 0, && (4.4.10a)\\
\frac{2\dot{\mu}\mu}{\zeta} &= \rho r^2\mu^2, && (4.4.10b)\\
Q(\chi) \equiv \chi^2 + \left(\frac{\ddot{\mu}}{\dot{\mu}} - \frac{1}{\zeta}\right)\chi + \left(\frac{\ddot{\mu}}{\dot{\mu}}\zeta + \frac{\dot{\mu}\mu}{\zeta} + \ddot{\mu}\mu\right)\frac{1}{\mu^2-\zeta^2} &= 0, && (4.4.10c)\\
\frac{\sigma}{\sigma}\mu^2 + \frac{\sigma}{\sigma}\left(\zeta + \frac{\dot{\nu}}{\nu}(\mu^2-\zeta^2)\right) - 1 + \frac{\zeta\dot{\nu}}{\nu} + \frac{\mu^2\ddot{\nu}}{\nu} &= 0, && (4.4.10d)\\
\frac{\dddot{\mu}}{\mu} - 2\left(\frac{\ddot{\mu}}{\dot{\mu}}\right)^2 + \left(\frac{\dot{\mu}}{\zeta\mu} + \frac{\mu}{\mu}\right)\frac{\mu^2+\zeta^2}{\mu^2-\zeta^2} + \frac{2\mu^2}{\zeta(\mu^2-\zeta^2)}\frac{\ddot{\mu}}{\dot{\mu}} &= 0. && (4.4.10e)
\end{aligned}$$

In terms of the similarity variable, ξ, the field equations become,

$$\begin{aligned}
\frac{\sigma'}{\sigma} + \frac{\nu'}{\nu} + \frac{3}{\xi} + \frac{\mu''}{\mu'} &= 0, && (4.4.11a)\\
-2\xi^3\mu'\mu &= \rho r^2\mu^2, && (4.4.11b)\\
-\mu'''\xi^2 + 2\xi\mu'' + 2\mu' + 2\xi^2\frac{(\mu'')^2}{\mu'} + \left(-\frac{\xi(\mu')^2}{\mu} - \frac{\xi^2\mu''\mu'}{\mu}\right)\frac{\mu^2\xi^2+1}{\mu^2\xi^2-1} + & \\
+\frac{2\xi^2\mu^2}{(\mu^2 z^2-1)}(z\mu'' + 2\mu') &= 0, && (4.4.11c)\\
Q(\chi) \equiv \chi^2 + \left(\frac{\mu''}{\mu'} + \frac{3}{\xi}\right)\chi + \left(-\frac{2}{\xi^2} - \frac{\mu''}{\xi\mu'} + \mu'\mu\xi + \xi^2\mu''\mu\right)\frac{1}{\mu^2\xi^2-1} &= 0 && (4.4.11d)
\end{aligned}$$

Where prime now refers to differentiation with respect to ξ. The quadratic equation (4.4.11d) holds for both $\frac{\sigma'}{\sigma}$ and $\frac{\nu'}{\nu}$.

We note that (4.4.11c) is an ODE in terms of only one metric function, i.e. μ We will demonstrate in §4.6 that a solution to this equation will lead to a full solution Therefore the subsequent chapters of this thesis will deal with the analysis of (4.4.11c).

4.5 Regularity conditions at the axis

We define 'the axis' to be situated at $\sqrt{l_{(\phi)}l^{(\phi)}} = r_\phi = 0$, where r_ϕ is a geometric invariant (4.1.6). We will impose *initial* regularity conditions on the axis, i.e , that the axis be regular for $t < 0$. This indicates that a singularity has not formed yet. In order to examine regularity conditions at the axis it is more convenient to use the set (4.4.11). We have introduced the Killing vectors $l_{(z)} = \partial_z$ and $l_{(\phi)} = \partial_\phi$, and we now impose the following conditions:

1. There must exist a regular axis:

$$\begin{aligned} \Upsilon &= |l^i_{(\phi)} l^j_{(\phi)} g_{ij}| \to 0, \qquad \text{as } |\xi| \to 0, \\ &\Rightarrow \lim_{|\xi|\to 0} r\sigma(\xi) = 0. \end{aligned} \tag{4.5.1}$$

   Thus, if we assume the expansion

$$\sigma(\xi) = \xi^n \sum_{i=0}^{\infty} \sigma_i \xi^i$$

   for $\sigma(\xi)$, we must have $n > -1$.

2. Spacetime is locally flat near the axis·

$$\begin{aligned} &\frac{\Upsilon_{,i}\Upsilon_{,j}g^{ij}}{4\Upsilon} \to 1, \qquad \text{as } |\xi| \to 0, \\ \Rightarrow\ &\lim_{|\xi|\to 0} \frac{\sigma(\xi)}{\mu(\xi)} = 1, \end{aligned}$$

   so $\mu(\xi)$ must have the same leading order as $\sigma(\xi)$

3. The density must remain finite and non zero as $|\xi| \to 0$:

$$\Rightarrow 0 < \lim_{|\xi|\to 0} \rho(\xi) < \infty.$$

To gain some intuition about the behaviour of $\mu(\xi)$ near the axis we assume that $\mu(\xi)$ may be written as

$$\mu(\xi) = \xi^n \sum_{i=0}^{\infty} \mu_i \xi^i,$$

where $\mu_0 \neq 0$ and μ_i are our constant coefficients and the series converges in a neighbourhood of $\xi = 0$. When we insert this into (4.4.11c) we obtain

$$|\xi|^{3n-2} \sum_{i=0}^{\infty} a_i \xi^i + |\xi|^{5n} \sum_{i=0}^{\infty} b_i \xi^i = 0, \tag{4.5.2}$$

where $a_0 = \mu_0^3 n^3 (1 + 2n)$ and $b_0 = -\mu_0^5 n^2 (2 + 3n)$. However, we know $n > -1$ which implies that our leading order term is $|\xi|^{3n-2}$. So to ensure (4.5.2) is satisfied as $|\xi| \to 0$ we require that

$$a_0 = 0 \quad \Rightarrow \quad n^3(1 + 2n) = 0 \quad \Rightarrow \quad n = -\frac{1}{2}$$

Subsequent terms in (4 5.2) are eliminated by choice of μ_i, $i \geq 1$. Regularity Condition 3 rules out $n = 0$

$$\Rightarrow \mu(\xi) = \xi^{-\frac{1}{2}} \sum_{i=0}^{\infty} \mu_i \xi^i,$$

which we can write in detail as

$$\mu(\xi) = |\xi|^{-1} \left(\mu_0 |\xi|^{\frac{1}{2}} + \frac{\mu_0^3}{2} |\xi|^{\frac{3}{2}} - \frac{3\mu_0^5}{56} |\xi|^{\frac{5}{2}} + \frac{11\mu_0^7}{1904} |\xi|^{\frac{7}{2}} + \cdots \right). \tag{4.5.3}$$

This will help us to choose the appropriate coordinates for later dynamical systems analysis of (4.4.11c).

4.6 Further analysis of the field equations

Next we return to our examination of (4.4.10c). We can say the following

$$\left(\frac{\dot{\sigma}}{\sigma}, \frac{\nu}{\nu}\right) = \begin{cases} (\delta_1 + \delta_2, \delta_1 + \delta_2) \\ (\delta_1 + \delta_2, \delta_1 - \delta_2) \\ (\delta_1 - \delta_2, \delta_1 + \delta_2) \\ (\delta_1 - \delta_2, \delta_1 - \delta_2) \end{cases}$$

Out of the four possibilities two distinct cases arise:

Case 1.

$$\left(\frac{\dot{\sigma}}{\sigma}, \frac{\dot{\nu}}{\nu}\right) = \begin{cases} (\delta_1 + \delta_2, \delta_1 + \delta_2) \\ (\delta_1 - \delta_2, \delta_1 - \delta_2) \end{cases} .$$

In this case we have

$$\begin{aligned} \frac{\dot{\sigma}}{\sigma} + \frac{\dot{\nu}}{\nu} = 2(\delta_1 \pm \delta_2) \quad \Rightarrow \quad \delta_2 = 0 \quad \text{by} \quad (4.4.10\text{a}), \\ \Rightarrow \frac{\dot{\sigma}}{\sigma} = \delta_1 \quad \text{and} \quad \frac{\dot{\nu}}{\nu} = \delta_1, \\ \Rightarrow \sigma = C_1\sqrt{\frac{\zeta}{\dot{\mu}}} \quad \text{and} \quad \nu = C_2\sqrt{\frac{\zeta}{\dot{\mu}}}, \end{aligned} \tag{4.6.1}$$

where C_1 and C_2 are constants of integration We can then substitute these expressions for $\frac{\sigma}{\sigma}$ and $\frac{\nu}{\nu}$ into (4.4.10d) which gives us another third order equation in μ, which we will rewrite as $\dddot{\mu} \equiv T_2(\mu)$. We can subtract this from (4.4.10e) and it again gives us the relation $\delta_2 = 0$.

Furthermore, we can show that (4.4.10e) is a linear combination of δ_2 and $\dot{\delta}_2$. Thus $\dddot{\mu} = T_2(\mu)$ is also a linear combination of δ_2 and $\dot{\delta}_2$. Therefore the system is fully determined by the equation $\delta_2 = 0$.

Case 2.

$$\left(\frac{\sigma}{\sigma}, \frac{\dot{\nu}}{\nu}\right) = \begin{cases} (\delta_1 + \delta_2, \delta_1 - \delta_2) \\ (\delta_1 - \delta_2, \delta_1 + \delta_2) \end{cases}$$

In this case we have the relation

$$\frac{\dot{\sigma}}{\sigma} + \frac{\dot{\nu}}{\nu} = 2\delta_1 = \frac{1}{\zeta} - \frac{\dot{\mu}}{\dot{\mu}},$$

which agrees with (4 4.10a). Substituting these values for $\frac{\sigma}{\sigma}$ and $\frac{\nu}{\nu}$ into the differential equation (4.4.10d) we obtain (4.4.10e). Thus, this expression for σ and ν satisfies all the other field equations. So to obtain a full solution to the field equations we solve (4.4.10e) to get $\mu(\xi)$, then use (4.4.10c) to get $\frac{\sigma}{\sigma}$ and $\frac{\dot{\nu}}{\nu}$ and integrate these to obtain expression for σ and ν

Proposition 4.6.1. *Regularity conditions at the axis rule out Case 1.*

Proof. If we assume that our solution obeys Case 1 then

$$\sigma(\xi)^2\mu'(\xi)\xi^3 = -C_1^2, \tag{4.6.2}$$

where we have rewritten (4.6.1) in terms of ξ for convenience. If we apply Regularity Conditions 1 and 2 we can write

$$\sigma(\xi) = |\xi|^n \sum_{i=0}^{\infty} \sigma_i \xi^i, \quad \mu(\xi) = |\xi|^n \sum_{i=0}^{\infty} \mu_i \xi^i,$$

where $n > -1$ and $\mu_0 = \sigma_0 \neq 0$, which we can then substitute into (4.6.2) to get

$$n\sigma_0^2\mu_0\xi^{3n+2} + \mathcal{O}(\xi^{3n+3}) = -C_1^2.$$

As $|\xi| \to 0$ this equation is satisfied if and only if $3n + 2 = 0$ which is not consistent with the limiting behaviour of (4.4 11c). Thus, our regularity conditions are not compatible with (4.4 11c) and (4.6.1). Therefore only Case 2 needs to be considered. □

4.7 Null geodesics

At any point or event $p \in \mathcal{M}$ we can define the null cone or light cone as the subset of $\mathcal{M}$ generated by all null geodesics from p. Self-similarity in cylindrical symmetry singles out a point, the singular origin or scaling origin p_o, at which the homothetic vector $\vec{k}$ vanishes identically and in the present coordinates this corresponds to $\{(r,t) = (0,0)\}$. We will denote by $\mathcal{N}$ the past null cone at p_o which will be generated by all past pointing null geodesics from p_o, and we will denote by $\mathcal{F}$ the future null cone at p_o, generated by all future pointing null geodesics from p_o. We note that p_o must be singular in the sense that $\rho|_{p_o}$ is infinite, and that if $\mathcal{F}$ exists as a part of $\mathcal{M}$, then p_o is a naked singularity In fact as p_o is not part of the spacetime the future null generator, γ, of $\mathcal{N}$ is future incomplete i.e. $\gamma : [\tau_0, \tau_*) \rightarrow \mathcal{M}$ where γ is inextendible at $\tau = \tau_*$ and

$$\lim_{\tau \rightarrow \tau_*} (r,t)\Big|_{\gamma} = (0,0) = p_o.$$

To find the radial null geodesics we apply the Euler-Lagrange equations to the Lagrangian

$$\mathcal{L} = -\left(\frac{\mathrm{d}t}{\mathrm{d}u}\right)^2 + \mu^2 \left(\frac{\mathrm{d}r}{\mathrm{d}u}\right)^2 = -\dot{t}^2 + \mu^2 \dot{r}^2,$$

where u is an affine parameter and $\frac{\mathrm{d}t}{\mathrm{d}u} = \dot{t}$. This gives

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\partial \mathcal{L}}{\partial \dot{t}}\right) - \frac{\partial \mathcal{L}}{\partial t} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \frac{\partial \mathcal{L}}{\partial r} = 0.$$

We obtain

$$\begin{aligned} \frac{\mathrm{d}t}{\mathrm{d}u} &= -\mu \dot{r}^2 \frac{\partial \mu}{\partial t}, \\ \frac{\mathrm{d}(\dot{r}\mu^2)}{\mathrm{d}u} &= \mu \dot{r}^2 \frac{\partial \mu}{\partial r}. \end{aligned} \tag{4.7.1}$$

Then we can use the self-similarity of the line element to write

$$\frac{\partial \mu}{\partial t} = \frac{\mathrm{d}\mu}{\mathrm{d}\xi}\left(\frac{-\xi}{t}\right), \quad \text{and} \quad \frac{\partial \mu}{\partial r} = \frac{\mathrm{d}\mu}{\mathrm{d}\xi}\frac{1}{t},$$

and we get

$$\frac{\mathrm{d}t}{\mathrm{d}u}\left(\frac{1}{\mu\dot{r}^2}\right)=\left(\frac{\xi}{t}\right)\frac{\mathrm{d}\mu}{\mathrm{d}\xi},$$
$$\frac{\mathrm{d}(\dot{r}\mu^2)}{\mathrm{d}u}\left(\frac{1}{\mu\dot{r}^2}\right)=\left(\frac{1}{t}\right)\frac{\mathrm{d}\mu}{\mathrm{d}\xi},$$

which gives us

$$\frac{\mathrm{d}\dot{t}}{\mathrm{d}u}=\ddot{t}=\xi\frac{\mathrm{d}(\dot{r}\mu^2)}{\mathrm{d}u}. \tag{4.7.2}$$

To simplify the right hand side of (4.7.2) we consider

$$\begin{aligned}\frac{\mathrm{d}(\xi r\mu^2)}{\mathrm{d}u} &= \xi\frac{\mathrm{d}(\dot{r}\mu^2)}{\mathrm{d}u}+(\dot{r}\mu^2)\frac{\mathrm{d}\xi}{\mathrm{d}u},\\ \Rightarrow \xi\frac{\mathrm{d}(\dot{r}\mu^2)}{\mathrm{d}u} &= \frac{\mathrm{d}(\mu^2 r\xi)}{\mathrm{d}u}-\frac{\mu^2\dot{r}}{t}\left(\dot{r}-\xi\dot{t}\right)\end{aligned}$$

Using (4.7 2) and the fact that our geodesic must be null, $\dot{t}=\pm\mu\dot{r}$ this becomes

$$\frac{\mathrm{d}\dot{t}}{\mathrm{d}u}=\frac{\mathrm{d}(\pm\mu\dot{t}\xi)}{\mathrm{d}u}-\frac{\dot{t}^2}{t}\left(1\pm\xi\mu\right)$$

which we can write as

$$\frac{\mathrm{d}\left(tt(1\pm\mu\xi)\right)}{\mathrm{d}u}=0.$$

By rewriting (4.7.1) in terms of ζ we get

$$\frac{\ddot{t}|t|}{(\dot{t})^2}=\frac{|\zeta|}{\mu}\frac{\partial\mu}{\partial\zeta}=\frac{x-y}{x},$$

where x and y are defined in (5 1.2) in the next chapter, and we will also show in the next chapter that for $|\zeta|=1$, $\frac{x-y}{x}=C_1>0$. So we can integrate along $|\zeta|=1$ to get, $\dot{t}=C_2t^{C_1}$, where C_2 is a constant of integration. Thus

$$\dot{t}\Big|_{p_o}=0.$$

Therefore the only radial null geodesics through p_o are given by solutions to

$$(1 \pm \mu\xi) = 0.$$

To locate the past null cone, $\mathcal{N}$, we look for the first negative value of $\xi = \xi_{\mathcal{N}}$ which satisfies

$$\xi = -\frac{1}{\mu}, \tag{4.7.3}$$

as $\mu > 0$, and to locate the future null cone, $\mathcal{F}$ we look for the first positive value of $\xi = \xi_{\mathcal{F}}$ which satisfies

$$\xi = \frac{1}{\mu}. \tag{4.7.4}$$

4.8 Expansion scalars

We can think of a congruence of null geodesics as the histories of photons. The effect of spacetime curvature would be to focus or distort a small bundle of these rays. To quantify this effect we consider the expansion of a congruence of null geodesics. We define the null vectors $l^a_\pm$ by

$$l^a_\pm = \beta_\pm(t,r)\left(\delta^a_0 \pm \frac{\delta^a_1}{\mu}\right)$$

The integral curves of $l^a_\pm$ are the outgoing and ingoing null surfaces. We require that this vector is future pointing $\beta_\pm(t,r) > 0$. If the outgoing null vector l^a_- is affinely parametrised by u then we have

$$\begin{aligned} \left(\frac{\partial l^a_-}{\partial x^b} + \Gamma^a_{db} l^d_-\right) l^b_- &= 0, \\ \Rightarrow \mu\frac{\partial \beta_-}{\partial t} + \beta_-\frac{\partial \mu}{\partial t} &= \frac{\partial \beta_-}{\partial r}. \end{aligned} \tag{4.8.1}$$

The expansion of the ingoing null geodesic congruence defined by l^a_- is given by

$$\begin{aligned}\theta_- = l^a_{-,a} &= \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^a}\left(\sqrt{-g}l^a_-\right)\\ &= \frac{\beta_-}{r}\left(\frac{\dot{\nu}}{\nu}+\frac{\dot{\sigma}}{\sigma}\right)\left(1+\frac{\zeta}{\mu}\right)-\frac{2\beta_-}{r\mu} \quad \text{by (4.8.1)}\\ &= \frac{\beta_-}{r}\left(\frac{1}{\zeta}-\frac{1}{\mu}-\frac{\ddot{\mu}}{\dot{\mu}}-\frac{\zeta\ddot{\mu}}{\mu\dot{\mu}}\right) \quad \text{by (4.4 10a).}\end{aligned}$$

Similarly, we can find the expansion of the outgoing null geodesic congruence defined by l^a_+

$$\begin{aligned}\theta_+ = l^a_{+,a} &= \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^a}\left(\sqrt{-g}l^a_+\right)\\ &= \frac{\beta_+}{r}\left(\frac{1}{\zeta}+\frac{1}{\mu}-\frac{\ddot{\mu}}{\dot{\mu}}+\frac{\zeta\ddot{\mu}}{\mu\dot{\mu}}\right).\end{aligned}$$

We have

$$\theta_+\theta_- = \frac{\beta_+\beta_-}{r^2}\left(\left(\frac{1}{\zeta}-\frac{\mu}{\mu}\right)^2-\left(\frac{1}{\mu}+\frac{\zeta\ddot{\mu}}{\mu\dot{\mu}}\right)^2\right),$$

where

$$\begin{aligned}\theta_+\theta_- &< 0 \quad \Rightarrow \quad \text{untrapped cylinders,}\\ \theta_+\theta_- &> 0 \quad \Rightarrow \quad \text{trapped cylinders.}\end{aligned}$$

4.9 The autonomous dynamical system

In order to obtain a solution to the field equations, we will begin to look for solutions of (4.4 11c)

$$\begin{aligned}-\mu'''\xi^2+2\xi\mu''+2\mu'+2\xi^2\frac{(\mu'')^2}{\mu'}+\left(-\frac{\xi(\mu')^2}{\mu}-\frac{\xi^2\mu''\mu'}{\mu}\right)\frac{\mu^2\xi^2+1}{\mu^2\xi^2-1}+&\\ +\frac{2\xi^2\mu^2}{(\mu^2\xi^2-1)}\left(\xi\mu''+2\mu'\right) &= 0.\end{aligned}$$

If we rewrite this equation using the follow prescription

$$x = \mu(\xi)|\xi|, \quad y = \frac{dx}{ds}, \quad z = \frac{dy}{ds}, \quad s = \ln|\xi|, \quad \vec{x} = (x, y, z)^T,$$

the result is a 3-dim autonomous dynamical system

$$\begin{aligned} \frac{dx}{ds} &= y, \\ \frac{dy}{ds} &= z, \\ \frac{dz}{ds} &= y - \frac{2(y-z)^2}{(x-y)} + \frac{(x-y)(x-2y+z)(x^2+1)}{x(x^2-1)} \\ &- \frac{2x^2(y-z)}{(x^2-1)} \equiv f(x,y,z). \end{aligned}$$

The regularity conditions at the axis are used to derive the limit

$$\lim_{|\xi|\to 0} \vec{x} = \vec{0}.$$

Then we can construct the initial value problem

$$\frac{d\vec{x}}{ds} = \vec{F}(\vec{x}) = \begin{pmatrix} y \\ z \\ f(\vec{x}) \end{pmatrix}, \quad \vec{x}(s = -\infty) = 0.$$

In the next three chapters we will analyse this dynamical system, starting at the regular axis $r = 0, t < 0, |\xi| = 0$ continuing through $t = 0, r > 0, |\xi| = \infty$, and then examining the one-parameter family of solutions that propagate to $r = t$

Chapter 5

The future of the regular axis

5.1 Proof of the existence of solutions emanating from the regular axis

Motivated by the analysis of the regularity conditions and the field equations in §4.5 we define a solution of (4.4.11c) which is regular at the axis by

$$\mu = \mu_0|\xi|^{-\frac{1}{2}} + \mathcal{O}(|\xi|^{\frac{1}{2}}), \quad \xi \to 0, \tag{5 1 1a}$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}\xi} = \frac{\mu_0}{2}|\xi|^{-\frac{3}{2}} + \mathcal{O}(|\xi|^{-\frac{1}{2}}), \quad \xi \to 0. \tag{5.1.1b}$$

We now define

$$s = \ln|\xi|, \quad x = \mu(\xi)|\xi|, \quad y = \frac{dx}{ds}, \quad z = \frac{dy}{ds}. \tag{5.1.2}$$

These definitions are independent of (4.4.11c).

Proposition 5.1.1. *For every* $\mu_o > 0$ *there exists a unique solution of* (4.4.11c) *of the form* (5.1.1).

In the remainder of this section we construct the proof of this proposition with the following series of lemmata.

Lemma 5.1.2. *The solution* (5.1.1) *is equivalent to the following:*

$$\begin{aligned} x &= \mu_0 \exp\left(\frac{s}{2}\right) + \mathcal{O}\left(\exp\left(\frac{3s}{2}\right)\right), \quad s \to -\infty, && (5.1.3a)\\ y &= \frac{\mu_0}{2} \exp\left(\frac{s}{2}\right) + \mathcal{O}\left(\exp\left(\frac{3s}{2}\right)\right), \quad s \to -\infty && (5.1.3b)\end{aligned}$$

Proof. The result follows immediately from the definitions (5.1.2). □

Lemma 5.1.3. *Existence of the solution* (5.1.1) *is equivalent to existence of the solution* (x, y, z) *of*

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s} &= y, && (5.1.4a)\\ \frac{\mathrm{d}y}{\mathrm{d}s} &= z, && (5.1.4b)\\ \frac{\mathrm{d}z}{\mathrm{d}s} &= y - \frac{2(y-z)^2}{(x-y)} + \frac{(x-y)(x-2y+z)(x^2+1)}{x(x^2-1)} - \frac{2x^2(y-z)}{(x^2-1)}. && (5.1.4c)\end{aligned}$$

where

$$\begin{aligned} x &= \mu_0 \exp\left(\frac{s}{2}\right) + \mathcal{O}\left(\exp\left(\frac{3s}{2}\right)\right), \quad s \to -\infty && (5.1.5a)\\ y &= \frac{\mu_0}{2} \exp\left(\frac{s}{2}\right) + \mathcal{O}\left(\exp\left(\frac{3s}{2}\right)\right), \quad s \to -\infty, && (5.1.5b)\\ z &= \frac{\mu_0}{4} \exp\left(\frac{s}{2}\right) + \mathcal{O}\left(\exp\left(\frac{3s}{2}\right)\right), \quad s \to -\infty. && (5.1.5c)\end{aligned}$$

Proof. First we note that it follows immediately from the definitions (5.1.2) that (5.1.5) implies the solution (5.1.1). Next, to prove the converse we use the system (5.1.4) to write

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{(y-z)}{(x-y)^2}\right) = -\frac{(x^2+1)}{x(x^2-1)} + \frac{(2x-y)(x^2+1)}{x(x^2-1)}\frac{(y-z)}{(x-y)^2}.$$

Let $u = \frac{(y-z)}{(x-y)^2}$, then by Lemma 5.1.2 we have $x - y > 0$ for all $s < s_0$, for some

$s_0 > -\infty$. Therefore, u is a well-defined quantity for $s < s_0$ Then we have

$$\frac{\mathrm{d}u}{\mathrm{d}s} - \frac{(2x-y)(x^2+1)}{x(x^2-1)}u = -\frac{(x^2+1)}{x(x^2-1)}. \qquad (5\,1.6)$$

The general solution of this equation is

$$\begin{aligned} u &= \frac{1}{M(s)}\left\{\int -\frac{(x^2+1)}{x(x^2-1)}M(s)\mathrm{d}s + C\right\}, \\ \text{where} \quad M(s) &= \exp\left\{\int -\frac{(2x-y)(x^2+1)}{x(x^2-1)}\mathrm{d}s\right\} \end{aligned}$$

Suppose we have a regular solution (5.1.1), then by Lemma 5 1.2 we have

$$\begin{aligned} x &= \mu_0 \exp\left(\frac{s}{2}\right) + \mathcal{O}\left(\exp\left(\frac{3s}{2}\right)\right), \quad s \to -\infty, \\ y &= \frac{\mu_0}{2}\exp\left(\frac{s}{2}\right) + \mathcal{O}\left(\exp\left(\frac{3s}{2}\right)\right), \quad s \to -\infty, \end{aligned} \qquad (5.1\,7)$$

and so,

$$\begin{aligned} (2x-y)(x^2+1) &= \frac{3\mu_0}{2}\exp\left(\frac{s}{2}\right) + \mathcal{O}\left(\exp\left(\frac{3s}{2}\right)\right), \quad s \to -\infty, \\ x(x^2-1) &= -\mu_0 \exp\left(\frac{s}{2}\right) + \mathcal{O}\left(\exp\left(\frac{3s}{2}\right)\right), \quad s \to -\infty, \\ \frac{(2x-y)(x^2+1)}{x(x^2-1)} &= -\frac{3}{2} + \mathcal{O}\left(\exp(s)\right), \quad s \to -\infty, \\ \frac{(x^2+1)}{x(x^2-1)} &= \frac{1}{\mu_0}\exp\left(-\frac{s}{2}\right) + \mathcal{O}\left(\exp\left(\frac{s}{2}\right)\right), \quad s \to -\infty. \end{aligned}$$

Let us take, without loss of generality,

$$\begin{aligned} M(s) &= \exp\left\{\int_{-\infty}^{s} -\frac{(2x-y)(x^2+1)}{x(x^2-1)}\mathrm{d}s\right\} \\ &= \exp\left\{\int_{-\infty}^{s}\left(\frac{3}{2} + \mathcal{O}(\exp \bar{s})\right)\mathrm{d}\bar{s}\right\} \\ &= \exp\left(\frac{3s}{2}\right) + \mathcal{O}(\exp 2s), \quad s \to -\infty. \end{aligned}$$

Then we get

$$
\begin{aligned}
u &= C\exp\left(-\frac{3s}{2}\right)+\frac{1}{\mu_0}\exp\left(\frac{3}{2}\right)+\mathcal{O}(1), \quad s\to-\infty,\\
z &= y-(x-y)^2u\\
&= -\frac{\mu_0^2}{4}C\exp\left(-\frac{s}{2}\right)+\mathcal{O}\left(\exp\left(\frac{s}{2}\right)\right), \quad s\to-\infty.
\end{aligned}
$$

Thus, the general solution of (5.1.6) with x and y corresponding to a regular solution has

$$
z=c\exp\left(-\frac{s}{2}\right)+\mathcal{O}\left(\exp\left(\frac{s}{2}\right)\right), \quad s\to-\infty. \tag{5.1.8}
$$

But, if (x,y,z) is a solution of (5.1.4), then we have

$$
y=\int_{-\infty}^{s} z(\bar{s})\mathrm{d}\bar{s}.
$$

Then, comparing (5.1.8) and (5 1.7) gives $c=0$, and returning to (5.1 6), with this in place, gives

$$
z=\frac{\mu_0}{4}\exp\left(\frac{s}{2}\right)+\mathcal{O}\left(\exp\left(\frac{3s}{2}\right)\right), \quad s\to-\infty.
$$

We have the result that if μ is a regular solution and (x,y,z) is the corresponding solution of (5.1.4), then

$$
\begin{aligned}
x &= \mu_0\exp\left(\frac{s}{2}\right)+\mathcal{O}\left(\exp\left(\frac{3s}{2}\right)\right), \quad s\to-\infty,\\
y &= \frac{\mu_0}{2}\exp\left(\frac{s}{2}\right)+\mathcal{O}\left(\exp\left(\frac{3s}{2}\right)\right), \quad s\to-\infty,\\
z &= \frac{\mu_0}{4}\exp\left(\frac{s}{2}\right)+\mathcal{O}\left(\exp\left(\frac{3s}{2}\right)\right), \quad s\to-\infty.
\end{aligned}
$$

□

Lemma 5.1.4. *There exists a new independent variable τ, which is a monotone increasing function of s and has the asymptotic behaviour $\tau\to-\infty$ as $s\to-\infty$. Let (x,y,z) correspond to* (5.1.1). *Then,* *(i). there exists an $s_0 > -\infty$ such that $x > 0$ and $x - y > 0$ for all $s \in (-\infty, s_0]$*

(ii). the function

$$\tau = -\int_s^{s_0} \frac{d\bar{s}}{x(x-y)}$$

is defined on $s \in (-\infty, s_0]$ and satisfies

$$\tau(s_0) = 0, \quad \tau \sim -\frac{2\exp(-s)}{\mu_0^2},$$

and so we have $\tau \to -\infty$ as $s \to -\infty$.

Proof. The proof of part (i) follows directly from the existence of (5.1.5). The proof of part (ii) follows from the integration of $-\int_s^{s_0} \frac{\mathrm{d}\bar{s}}{x(x-y)}$ using (5 1 5). □

From the previous lemma it is clear that τ is a valid time coordinate. So we can recast (5.1.4) as follows

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = yx(x-y), \tag{5.1.9a}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = zx(x-y), \tag{5.1.9b}$$

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = yx(x-y) - 2x(y-z)^2 + \frac{(x-y)^2(x-2y+z)(x^2+1)}{(x^2-1)} - \frac{2x^3(x-y)(y-z)}{(x^2-1)}, \tag{5.1.9c}$$

where

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = x(x-y) \tag{5.1.9d}$$

with the condition

$$\lim_{\tau \to -\infty} \vec{x}(\tau) = \vec{0}$$

If we linearise (5.1.9a)–(5.1.9c) about $\vec{x} = \vec{0}$ we get three zero eigenvalues and so centre manifold analysis is not applicable. With a view to obtaining a hyperbolic equilibrium point corresponding to $\lim_{\tau \to -\infty} \vec{x}(\tau) = \vec{0}$ we define new dependent variables by

$$A(\tau) = x|\tau|^{\frac{1}{2}}, \quad B(\tau) = y|\tau|^{\frac{1}{2}}, \quad C(\tau) = z|\tau|^{\frac{1}{2}}, \tag{5.1 10}$$

and introduce another time variable, T, defined by

$$w(T) = e^T = |\tau|^{-1}.$$

This implies that

$$\frac{\mathrm{d}w(T)}{\mathrm{d}T} = w(T).$$

We then obtain a 4-dim non-linear autonomous dynamical system

$$\begin{aligned}
\frac{\mathrm{d}A}{\mathrm{d}T} &= -\frac{A}{2} - AB(A-B), && (5.1.11a)\\
\frac{\mathrm{d}B}{\mathrm{d}T} &= -\frac{B}{2} - AC(A-B), && (5.1.11b)\\
\frac{\mathrm{d}C}{\mathrm{d}T} &= -\frac{C}{2} - AB(A-B) - 2A(B-C)^2 - \frac{2A^3(A-B)(B-C)w}{(wA^2-1)} \\
&+ \frac{(A-B)^2(A-2B+C)(wA^2+1)}{(wA^2-1)}, && (5.1.11c)\\
\frac{\mathrm{d}w}{\mathrm{d}T} &= w. && (5.1.11d)
\end{aligned}$$

We consider the equilibrium point $\vec{A_0} : \lim_{T\to-\infty}(A, B, C, w) \to (\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}, 0)$[1]. We linearise (5.1 11) about $\vec{A_0}$ to get

$$\frac{\mathrm{d}(\vec{A} - \vec{A_0})}{\mathrm{d}\tau} = J \cdot (\vec{A} - \vec{A_0}), \tag{5.1.12}$$

[1] There are three equilibrium points that (5 1.11) could approach as $T \to -\infty$.

1. $(A, B, C, w) \to (\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}, 0) \equiv \vec{A_0}$
2. $(A, B, C, w) \to (-\sqrt{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}, 0) \equiv \vec{A_1}$
3. $(A, B, C, w) \to (0, 0, 0, 0) \equiv \vec{A_2}$.

When we analyse the equilibrium point $\vec{A_2}$ we find the 1-dim unstable manifold corresponds to

$$\vec{A} \equiv \vec{0}, \quad w(T) = e^T, \quad T \geq -\infty.$$

This represents the trivial solution $\mu \equiv 0$. When we analyse the equilibrium point $\vec{A_1}$ we find a non-trivial 3-dim unstable manifold, but this corresponds to solutions with $\mu < 0$ on a neighbourhood of the axis, which we have ruled out by definition. We have only one remaining equilibrium point, $\vec{A_0}$, which we will analyse in detail and show that it corresponds to the axis.

where, J, the Jacobian of the system, is

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{3}{4} & -1 & 1 & 0 \\ \frac{1}{4} & -\frac{1}{2} & 1 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which has the following eigenvalues and eigenvectors

$$\begin{aligned} \lambda_1 =&1, & \vec{v_1} =&(-8\sqrt{2}, -3\sqrt{2}, 0, 1), \\ \lambda_2 =&1, & \vec{v_2} =&(4, 2, 1, 0), \\ \lambda_3 =&\frac{1}{\sqrt{2}}, & \vec{v_3} =&(0, 2+\sqrt{2}, 1, 0), \\ \lambda_4 =&\frac{-1}{\sqrt{2}}, & \vec{v_4} =&(0, 2-\sqrt{2}, 1, 0). \end{aligned}$$

Therefore, $\vec{A_0}$ is a hyperbolic equilibrium point. The solution to the non-linear system (5.1.11) is given by the flow ϕ_τ. By the Hartman-Grobman Theorem [31] ϕ_τ is locally topologically equivalent to the flow of the linearised system (5.1.12). By the Stable Manifold Theorem [31] there exists a 3-dim unstable manifold, U, tangent to the 3-dim unstable subspace, E^U, of the linear system (5 1 12) at $\vec{A_0}$, spanned by $\vec{v_1}$, $\vec{v_2}$ and $\vec{v_3}$, such that, for all $\tau \leq 0$, $\phi_\tau \subset S$ and for all $\vec{A_i} \in S$

$$\lim_{\tau \to -\infty} \phi_\tau(\vec{A_i}) = \vec{A_0}.$$

Trajectories in this unstable manifold have the asymptotic behavior·

$$\vec{A} = \vec{A_0} + c_3\vec{v_3}e^{\frac{T}{\sqrt{2}}} + (c_1\vec{v_1} + c_2\vec{v_2})e^T + \mathcal{O}(e^{(1+\frac{1}{\sqrt{2}})T}), \quad \tau \to -\infty, \qquad (5.1.13)$$

which is a three-parameter family of solutions. However, we defined

$$w(T) = e^T,$$

which implies that $c_1 = 1$, and so this is a two-parameter family of solutions.

Using (5.1.10) we can write (5.1.5) in terms of A, B, C, w.

$$\begin{aligned} A &= \sqrt{2} + \mathcal{O}(|\tau|^{-1}), \quad \tau \to -\infty, \\ B &= \frac{1}{\sqrt{2}} + \mathcal{O}(|\tau|^{-1}), \quad \tau \to -\infty, \\ C &= \frac{1}{2\sqrt{2}} + \mathcal{O}(|\tau|^{-1}), \quad \tau \to -\infty. \end{aligned}$$

Therefore, (5.1.5) corresponds to $\vec{A} \to \vec{A_0}$ as $\tau \to -\infty$.

Proposition 5.1.5. *Existence of the regular solution* (5.1.1) *of* (4.4.11c) *implies existence of the solution with* α*-limit* $\vec{A_0}$ *of* (5.1.11).

Proof. The proposition is proved by application of Lemma 5.1.3 and Lemma 5.1.4. □

Next, we want to show that among the solutions with α-limit $\vec{A_0}$ of (5.1.11) there is a particular solution which is equivalent to the regular solution (5.1.1). Solutions to (5.1.11) with α-limit $\vec{A_0}$ have the following asymptotic form in terms of τ

$$\begin{aligned} A(\tau) &= \sqrt{2} + (4c_2 + 8\sqrt{2})|\tau|^{-1} + \mathcal{O}(|\tau|^{-(1+\frac{1}{\sqrt{2}})}), \quad \tau \to -\infty, \\ B(\tau) &= \frac{1}{\sqrt{2}} + c_3(2+\sqrt{2})|\tau|^{\frac{-1}{\sqrt{2}}} + (2c_2 - 3\sqrt{2})|\tau|^{-1} + \mathcal{O}(|\tau|^{-(1+\frac{1}{\sqrt{2}})}), \quad \tau \to -\infty, \\ C(\tau) &= \frac{1}{2\sqrt{2}} + c_3|\tau|^{\frac{-1}{\sqrt{2}}} + c_2|\tau|^{-1} + \mathcal{O}(|\tau|^{-(1+\frac{1}{\sqrt{2}})}), \quad \tau \to -\infty. \end{aligned}$$

We can then use the definitions (5.1 10) to write the corresponding x, y, z:

$$x(\tau) = |\tau|^{-\frac{1}{2}}\left(\sqrt{2} + (4c_2 + 8\sqrt{2})|\tau|^{-1} + \mathcal{O}(|\tau|^{-(1+\frac{1}{\sqrt{2}})})\right), \quad \tau \to -\infty, \tag{5.1 14a}$$

$$y(\tau) = |\tau|^{-\frac{1}{2}}\left(\frac{1}{\sqrt{2}} + c_3(2+\sqrt{2})|\tau|^{\frac{-1}{\sqrt{2}}} + (2c_2 - 3\sqrt{2})|\tau|^{-1} + \mathcal{O}(|\tau|^{-(1+\frac{1}{\sqrt{2}})})\right), \quad \tau \to -\infty, \tag{5.1.14b}$$

$$z(\tau) = |\tau|^{-\frac{1}{2}}\left(\frac{1}{2\sqrt{2}} + c_3|\tau|^{\frac{-1}{\sqrt{2}}} + c_2|\tau|^{-1} + \mathcal{O}(|\tau|^{-(1+\frac{1}{\sqrt{2}})})\right), \quad \tau \to -\infty \tag{5.1.14c}$$

Lemma 5.1.6. *There exists an independent variable* s, *which is a monotone increasing function of* τ *and has the asymptotic behaviour* $s \to -\infty$ *as* $\tau \to -\infty$ *Let* x, y, z *correspond to* (5 1.14) *Then,* *(i). there exists a* $\tau_0 > -\infty$ *such that* $x > 0$, $x - y > 0$, *for all* $\tau \in (-\infty, \tau_0]$

(ii). the function

$$s = -\int_{\tau}^{\tau_0} (x(x-y))\, d\bar{\tau}$$

is defined on $\tau \in (-\infty, \tau_0]$ *and satisfies*

$$s(\tau_0) = 0, \quad s \sim -\ln|\tau|,$$

and so we have that $s \to -\infty$ *as* $\tau \to -\infty$.

Proof The proof of part (i) follows directly from the existence of (5.1.14) The proof of part (ii) follows from the integration of $-\int_{\tau}^{\tau_0} (x(x-y))\mathrm{d}\bar{\tau}$ using (5.1.14). □

We can now write our solutions in terms of s as follows

$$\begin{aligned}
x(s) &= \frac{\sqrt{2}}{c_*^{\frac{1}{2}}} \exp\left(\frac{s}{2}\right) + \frac{c_3 \exp\left(-(\frac{1}{2}+\frac{1}{\sqrt{2}})s\right)}{\sqrt{2} c_*^{(\frac{1}{2}+\frac{1}{\sqrt{2}})}} + \frac{4c_2 - 8\sqrt{2}}{c_*^{\frac{3}{2}}} \exp\left(\frac{3s}{2}\right) + \cdots, \quad (5.1.15)\\
y(s) &= \frac{\exp\left(\frac{s}{2}\right)}{\sqrt{2} c_*^{\frac{1}{2}}} + \frac{c_3 \exp\left(-(\frac{1}{2}+\frac{1}{\sqrt{2}})s\right)(3-4\sqrt{2})}{\sqrt{2} c_*^{(\frac{1}{2}+\frac{1}{\sqrt{2}})}} + \frac{2c_2 - 3\sqrt{2}}{c_*^{\frac{3}{2}}} \exp\left(\frac{3s}{2}\right) + \cdots,\\
z(s) &= \frac{\exp\left(\frac{s}{2}\right)}{2\sqrt{2} c_*^{\frac{1}{2}}} + \frac{c_3 \exp\left(-(\frac{1}{2}+\frac{1}{\sqrt{2}})s\right)(4\sqrt{2}-1)}{4\sqrt{2} c_*^{(\frac{1}{2}+\frac{1}{\sqrt{2}})}} + \frac{c_2}{c_*^{\frac{3}{2}}} \exp\left(\frac{3s}{2}\right) + \cdots,
\end{aligned}$$

where c_* is a constant of integration.

Lemma 5.1.7. *Among the two-parameter family of solutions with* α*-limit* $\vec{A}_0$ *of* (5.1.11) *there exists a particular regular solution whose density is an even, smooth function of the proper radius* r_ϕ.

Proof. In order to have an analytic solution at the regular axis we require that the density $\rho(\xi)$ must be an even smooth function of $r\sigma(\xi)$. This is because $r\sigma(\xi) = r_\phi$ is the proper radius and for a Lorentzian spacetime we can identify this with the proper radius in flat Cartesian coordinates, i.e., $(x^2+y^2+z^2)^{\frac{1}{2}}$. If the density ρ is not an even smooth function of r_ϕ then there will be a cusp in the density function or one of its derivatives. But when we write out expressions for these quantities using our two-parameter solution we see that

$$\rho(\xi) = \frac{-2|\xi|\mu'}{\mu} = 1 + 2c_3\left(\frac{|\xi|}{c_*}\right)^{\frac{1}{\sqrt{2}}} + \mathcal{O}(\xi), \quad \xi \to 0,$$

$$r\sigma(\xi) = \sigma_0|\xi|^{\frac{1}{2}}(1 + c_3|\xi|^{\frac{1}{\sqrt{2}}} + \mathcal{O}(|\xi|)), \quad \xi \to 0,$$

where σ_0 can be found using the second regularity condition. Thus, we must have $c_3 = 0$. We now have a one-parameter family of solutions

A regular solution to (4.4.11c) will have the form (5.1 1) but we can specify that the coefficient of the $|\xi|^{\frac{3}{2}}$ term have the following form

$$x(\xi) = \mu_0|\xi|^{\frac{1}{2}} + \frac{\mu_0^3}{2}|\xi|^{\frac{3}{2}} + \mathcal{O}(|\xi|^{\frac{5}{2}}), \quad \xi \to 0.$$

This information is obtained from the ODE (4 4 11c) Comparing this with (5.1.15) we get a value of $c_2 = \frac{16\sqrt{2}-1}{8}$. □

Finally we are in a position to prove Proposition 5.1.1,

Proof of Proposition 5.1.1 We have shown that the existence of a solution of (5.1.1) is equivalent to the existence of a particular solution of the two-parameter family of solutions with α-limit $\vec{A}_0$ of (5.1.11), using the Proposition 5.1.5 and Lemma 5.1.3, Lemma 5.1.6 and Lemma 5 1 7. But this particular solution is guaranteed to exist by our analysis of (5.1.11) □

5.1.1 Scaling of solution

The solution has the form

$$\mu(\xi) = |\xi|^{-1}\left(\mu_0|\xi|^{\frac{1}{2}} + \frac{\mu_0^3}{2}|\xi|^{\frac{3}{2}} - \frac{3\mu_0^5}{56}|\xi|^{\frac{5}{2}} + \cdots\right). \tag{5.1.16}$$

But we have the option of rescaling the comoving coordinate, r, by $r \rightarrow \alpha\bar{r}$, without changing our line element

$$\begin{aligned}
\mathrm{d}s^2 &= -\mathrm{d}t^2 + \mu^2(\xi)\mathrm{d}r^2 + r^2\left(\nu^2(\xi)^2\mathrm{d}z^2 + \sigma^2(\xi)^2\mathrm{d}\phi^2\right) \\
&= -\mathrm{d}t^2 + \alpha^2\mu^2(\xi)\mathrm{d}\bar{r}^2 + \alpha^2\bar{r}^2\left(\nu^2(\xi)\mathrm{d}z^2 + \sigma^2(\xi)\mathrm{d}\phi^2\right) \\
&= -\mathrm{d}t^2 + \bar{\mu}^2(\bar{\xi})\mathrm{d}\bar{r}^2 + \bar{r}^2\left(\bar{\nu}^2(\bar{\xi})\mathrm{d}z^2 + \bar{\sigma}^2(\bar{\xi})\mathrm{d}\phi^2\right),
\end{aligned}$$

where $\xi = \alpha\bar{\xi} = \alpha\left(\frac{\bar{r}}{t}\right)$ and

$$\bar{\mu}(\bar{\xi}) = \alpha\mu(\xi), \quad \bar{\nu}(\bar{\xi}) = \alpha\nu(\xi), \quad \bar{\sigma}(\bar{\xi}) = \alpha\sigma(\xi).$$

From (5.1.16) we can write

$$\begin{aligned}
\bar{\mu}(\bar{\xi}) &= |\bar{\xi}|^{-1}\left(\bar{\mu_0}|\bar{\xi}|^{\frac{1}{2}} + \frac{\bar{\mu_0}^3}{2}|\bar{\xi}|^{\frac{3}{2}} - \frac{3\bar{\mu_0}^5}{56}|\bar{\xi}|^{\frac{5}{2}} + \cdots\right) \\
&= \alpha|\xi|^{-1}\left(\bar{\mu_0}\alpha^{\frac{-1}{2}}|\xi|^{\frac{1}{2}} + \frac{\bar{\mu_0}^3}{2}\alpha^{\frac{-3}{2}}|\xi|^{\frac{3}{2}} - \frac{3\bar{\mu_0}^5}{56}\alpha^{\frac{-5}{2}}|\xi|^{\frac{5}{2}} + \cdots\right) \\
&= \alpha|\xi|^{-1}\left(\mu_0|\xi|^{\frac{1}{2}} + \frac{\mu_0^3}{2}|\xi|^{\frac{3}{2}} - \frac{3\mu_0^5}{56}|\xi|^{\frac{5}{2}} + \cdots\right) \\
&= \alpha\mu(\xi) \\
\Leftrightarrow \bar{\mu_0} &= \alpha^{\frac{1}{2}}\mu_0.
\end{aligned}$$

Therefore, the parameter μ_0 may be rescaled to an arbitrary value without affecting the dynamics, and therefore there is a unique solution that is regular at $r = 0$.

We will not choose a specific value for μ_0 until §5.2.3 when an appropriate choice of μ_0 (or choice of scaling for r) will be very convenient. So we write out our unique solution

$$\mu(\xi) = \mu_0|\xi|^{-\frac{1}{2}} + \frac{\mu_0^3}{2}|\xi|^{\frac{1}{2}} - \frac{3\mu_0^5}{56}|\xi|^{\frac{3}{2}} + \cdots. \tag{5.1.17a}$$

We also have

$$x = \mu_0|\xi|^{\frac{1}{2}} + \frac{\mu_0^3}{2}|\xi|^{\frac{3}{2}} + \mathcal{O}\left(|\xi|^{\frac{5}{2}}\right), \quad \xi \to 0, \tag{5.1.17b}$$

$$y = \frac{\mu_0}{2}|\xi|^{\frac{1}{2}} + \frac{3\mu_0^3}{4}|\xi|^{\frac{3}{2}} + \mathcal{O}\left(|\xi|^{\frac{5}{2}}\right), \quad \xi \to 0, \tag{5 1.17c}$$

$$z = \frac{\mu_0}{4}|\xi|^{\frac{1}{2}} + \frac{9\mu_0^3}{8}|\xi|^{\frac{3}{2}} + \mathcal{O}\left(|\xi|^{\frac{5}{2}}\right), \quad \xi \to 0 \tag{5.1.17d}$$

5.2 The solution in the region $0 < |\xi| \leq 1$

Next we consider how this solution (5.1.17) will evolve It is convenient to remove the singularities in (5.1.4) at $x = 1$, $x = 0$ and $x = y$ by introducing an auxiliary tıme coordinate τ (which is different to the τ defined by (5.1.9d)) defined by

$$\frac{ds}{d\tau} = x(x-y)(1-x^2). \tag{5.2.1a}$$

We can integrate this using (5.1.17) to show that $\tau(-\infty) = -\infty$. Then (5.1.4) becomes

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = yx(x-y)(1-x^2), \tag{5.2.1b}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} = zx(x-y)(1-x^2), \tag{5.2.1c}$$

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}\tau} &= yx(x-y)(1-x^2) - 2x(y-z)^2(1-x^2) + (x-y)^2(x-2y+z)(x^2+1) \\ &- 2x^3(x-y)(y-z).\end{aligned} \tag{5.2.1d}$$

We have the new initial value problem, with $F_{(2)} \in C^1(\mathbb{R}^3)$, defined by (5.2.1),

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau} = F_{(2)}(\vec{x}), \quad \vec{x}(-\infty) = \vec{0}.$$

In order to ascertain how our unique solution evolves we wıll derive some properties of the system

Proposition 5.2.1. *There exists $\tau_0 > -\infty$ such that*

$$x > 0 \quad \textit{for} \quad \tau \in (-\infty, \tau_0].$$

Proof. We have the initial condition $x = 0$ ($\vec{x} = 0$) at $\xi = 0$ We can use this and (5.1.17a) to show that there exists $\xi_0 \neq 0$ such that

$$x > 0 \quad \text{for} \quad |\xi| \in (0, |\xi_0|].$$

We have defined

$$\frac{ds}{d\tau} = x(x-y)(1-x^2) \geq 0.$$

Therefore, $s = s(\tau)$ (and therefore $|\xi|$) is an increasing function of τ. By integrating this using the local solution (5 1.17), we can show that as $s \to -\infty$ ($|\xi| \to 0$), $\tau \to -\infty$. Thus, by continuity there exists a $\tau(|\xi_0|) = \tau_0$ such that $\tau_0 > -\infty$ and

$$x > 0 \quad \text{for} \quad \tau \in (-\infty, \tau_0].$$

□

Proposition 5.2.2. *The following identities hold for $-\infty < \tau < \infty$:*

$$x(\tau) > 0, \tag{5.2.2a}$$
$$x(\tau) < 1, \tag{5.2.2b}$$
$$x(\tau) - y(\tau) > 0, \tag{5.2.2c}$$
$$y(\tau) - z(\tau) > 0, \tag{5.2.2d}$$
$$x(\tau) - 2y(\tau) + z(\tau) > 0. \tag{5.2.2e}$$

Proof. By Proposition 5.2.1 we know that

$$x > 0 \quad \text{for} \quad \tau \in (-\infty, \tau_0].$$

If x changes sign at some $\tau = \tau_1 < \infty$, then $x(\tau_1) = 0$ and by (5.2.1) we have

$$\left.\frac{dx}{d\tau}\right|_{\tau=\tau_1} = 0.$$

But if this is true then $x = 0$ for all τ. Therefore x can only reach zero as $\tau \to \pm\infty$ and

$$x(\tau) > 0 \;\text{ for }\; \tau \in (-\infty, \infty).$$

Similarly if $x = 1$ at some $\tau = \tau_1 < \infty$, then by (5 2.1) we have that

$$\left.\frac{dx}{d\tau}\right|_{\tau=\tau_1} = 0.$$

But if this is true then $x = 1$ for all τ which contradicts Proposition 5.2.1 This proves (5.2.2b). Moreover $x = 1$ can only occur as $\tau \to \infty$

To prove (5 2.2c) we use (5.1.17a) to show that as $|\xi| \to 0$ we have $x - y > 0$. If $x - y = 0$ at some $\tau = \tau_2 < \infty$ then by (5.2.1) we have

$$\left.\frac{d(x-y)}{d\tau}\right|_{\tau=\tau_2} = 0.$$

But if this is true then $x - y = 0$ for all τ. This contradiction implies $x - y = 0$ can only occur as $\tau \to \infty$.

To prove (5.2.2d) we use (5.1.17a) to show that as $|\xi| \to 0$ we have $y - z > 0$. If $y - z = 0$ at some $\tau = \tau_3 < \infty$ then by (5.2.1) we have

$$\left.\frac{d(y-z)}{d\tau}\right|_{\tau=\tau_3} = \frac{(x-y)^2(x^2+1)}{x(1-x^2)} > 0.$$

Thus $y - z > 0$ for all τ

To prove (5 2.2e) we use (5.1.17a) to show that as $|\xi| \to 0$ we have $x - 2y + z > 0$. If $x - 2y + z = 0$ at some $\tau = \tau_4 < \infty$ then by (5.2.1) we have

$$\left.\frac{d(x-2y+z)}{d\tau}\right|_{\tau=\tau_4} = \frac{2x^2(x-y)}{(1-x^2)} > 0.$$

Thus $x - 2y + z > 0$ or $(x - y) > (y - z)$ for all τ. □

Proposition 5.2.3.

$$y(\tau) > 0, \quad \text{for} \quad -\infty < \tau < \infty.$$

Proof. We can use (5.1 17c) to show

$$\lim_{\tau \to -\infty} y = 0$$

and there exists a τ_*, $-\infty < \tau_* < \infty$, such that $y > 0$ for $\tau \in (-\infty, \tau_*]$. Next let $u = \frac{y}{x}$ and by (5.1.17b) and (5.1.17c) we have $\lim_{\tau \to -\infty} u = \frac{1}{2}$. Next we consider

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}\tau} = u &= \frac{xy - y\dot{x}}{x^2} \\ &= \frac{(x-y)(1-x^2)}{x}\left(xz - y^2\right) = \frac{(x-y)(1-x^2)v}{x}, \end{aligned} \tag{5.2.3}$$

where overdot now refers to differentiation with respect to τ and where we have let $v = xz - y^2$. By using (5.1.17b)-(5.1.17d) we can show that there exists a τ_0, $-\infty < \tau_0 < \infty$ such that

$$v > 0 \quad \text{for} \quad \tau \in (-\infty, \tau_0].$$

Therefore, u increases from $u = \frac{1}{2}$ on $\tau \in (-\infty, \tau_0]$ by (5.2.3) and (5.2.2a)–(5.2.2c).

We have shown that $x > y$ on $-\infty < \tau < \infty$

$$\Rightarrow u = 0 \quad \Leftrightarrow \quad (y = 0 \quad \text{and} \quad x = x_0 \quad \text{where} \quad 0 < x_0 < 1).$$

So if there exists τ_1 such that

$$y(\tau_1) = 0 \quad \Rightarrow \quad y < \infty \quad \text{for} \quad -\infty < \tau < \tau_1,$$

i.e., $u(\tau)$ evolves smoothly from $u = \frac{1}{2}$ to $u = 0$. We also require that $u(\tau_1)$ (and consequently $y(\tau_1)$) is the first zero of $u(\tau)$ on $-\infty < \tau < \infty$. This implies that we must have a τ_2, $-\infty < \tau_2 < \tau_1$ such that

$$\dot{u}(\tau_2) = 0, \quad \Rightarrow v(\tau_2) = 0,$$

using (5.2.2a)–(5.2.2c) and where $\dot{u}(\tau_2) = 0$ is the first turning point of $u(\tau)$ on $-\infty < \tau < \infty$

$$
\begin{aligned}
\Rightarrow \dot{u} &> 0 \quad \text{for} \quad \tau \in (-\infty, \tau_2), \\
\Rightarrow u &> \frac{1}{2} \quad \text{for} \quad \tau \in (-\infty, \tau_2], \\
\Rightarrow \frac{y}{x} &> \frac{1}{2} \Rightarrow x - 2y < 0 \quad \text{for} \quad \tau \in (-\infty, \tau_2], \\
\Rightarrow y &< x < 2y < 3y \quad \text{for} \quad \tau \in (-\infty, \tau_2].
\end{aligned} \tag{5.2.4}
$$

But $v(\tau_2) = 0$ is necessarily the first zero of $v(\tau)$ Therefore $\dot{v}(\tau_2) \leq 0$. However,

$$
\begin{aligned}
\dot{v} &= x\dot{z} + zx - 2y\dot{y}, \\
\Rightarrow \dot{v}|_{v=0} &= -(x-y)^2[(x-y)(x-2y) + x^3(x-3y)] > 0.
\end{aligned}
$$

Hence, using (5 2 4), there cannot be a first zero of $v(\tau)$ and

$$
\begin{aligned}
&\Rightarrow \dot{u} > 0, \quad \text{for} \quad -\infty < \tau < \infty, \\
&\Rightarrow u > \frac{1}{2}, \quad \text{for} \quad -\infty < \tau < \infty, \\
&\Rightarrow y > 0, \quad \text{for} \quad -\infty < \tau < \infty.
\end{aligned}
$$

□

5.2.1 Existence and uniqueness of the solution

Local existence

We now apply Theorem 2.4.1 to the trajectory that was shown to emerge from the axis (5.1.17). We can construct the initial data problem

$$
\begin{aligned}
\frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau} &= \vec{F}_{(1)}(\vec{x}), \quad \text{from (5.1.9)} \\
\vec{x}(\tau_0) &= \vec{x_0},
\end{aligned}
$$

where $\vec{F}_{(1)} \in C^1(\mathbb{R}^n)$ and the initial data $\vec{x_0} \in \mathbb{R}^n$ is obtained from (5.1.17). We can apply the local existence and uniqueness theorem to show that there exists an

$a > 0$ such that the initial value problem has a unique solution $\vec{x}(\tau)$ on the interval $(\tau_0 - a, \tau_0 + a)$.

Global existence

Using the properties derived in the previous section, we can apply Theorem 2.4.4 to get a global existence and uniqueness result. A solution to the dynamical system (5.2.1) has the following properties for all τ.

$$x \in (0,1), \quad y \in (0,1), \quad z \in (-1,1),$$

which are derived from Propositions 5.2.2 and 5.2 3. Therefore, we can define a compact manifold

$$K = \{y \in \mathbb{R}^3 | y \in [-1,1]^3\},$$

and state our initial value problem

$$\begin{aligned} \frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau} &= \vec{F}_{(2)}(\vec{x}), \quad \text{from (5.2.1)} \\ \vec{x}(\tau_0) &= \vec{x_0} \end{aligned}$$

where $\vec{F}_{(2)} \in C^1(K)$ and $\vec{x_0} \in K$, which satisfies Theorem 2.4.4 and therefore (5.2.1) has a unique solution $x(\tau)$ defined for all $\tau \in \mathbb{R}$

5.2.2 Equilibrium sets

Our unique solution must approach one of the following equilibrium sets as $\tau \to \infty$:

$$\begin{aligned} E_0 &: (x,y,z) \to (0,0,0) \\ E_1 &: (x,y,z) \to (0,0,\hat{z}) \\ E_2 &: (x,y,z) \to (0,\hat{y},2\hat{y}) \\ E_3 &: (x,y,z) \to (1,1,\hat{z}) \\ E_4 &: (x,y,z) \to (\hat{x},\hat{x},\hat{x}) \\ E_5 &: (x,y,z) \to (1,\hat{y},f(\hat{y})) \end{aligned}$$

where the hat, $\hat{x}$, denotes that x is a parameter and $f(y) = \frac{2y^2-4y+1}{y-2}$

Proposition 5.2.4. *The trajectory corresponding to* (5.1.17) *cannot approach* E_0, E_1, E_2, E_3 *or* E_4.

Proof. Firstly we note that at E_1 we have

$$\begin{aligned} y - z = -\hat{z} \geq 0 \quad &\Rightarrow \ \hat{z} \leq 0, \ \text{by (5.2.2d)}, \\ x - 2y + z = \hat{z} \geq 0 \quad &\Rightarrow \ \hat{z} \geq 0, \ \text{by (5.2.2e)}, \\ &\Rightarrow \ \hat{z} = 0. \end{aligned}$$

So E_1 reduces to E_0. To eliminate E_0 and E_2 we note that x is increasing initially. Therefore, if x is to approach zero it must begin decreasing at some $\tau_* < \infty$. However,

$$\begin{aligned} \frac{dx}{d\tau} &= yx(x-y)(1-x^2), \\ \text{so if} \quad \left.\frac{dx}{d\tau}\right|_{\tau=\tau_*} &< 0, \\ \text{then} \quad y|_{\tau=\tau_*} &< 0, \ \text{by (5.2.2a), (5.2.2b) and (5.2.2c).} \end{aligned}$$

This contradicts Proposition 5.2.3. To eliminate E_3 we note

$$\begin{aligned} y - z &= 1 - \hat{z} \geq 0 \quad \text{so} \quad \hat{z} \leq 1, \ \text{by (5.2.2d)}, \\ x - 2y + z &= -1 + \hat{z} \geq 0 \quad \text{so} \quad \hat{z} \geq 1, \ \text{by (5.2.2e)}, \\ \Rightarrow \hat{z} &= 1. \end{aligned}$$

Thus E_3 reduces to E_4 with $\hat{x} = 1$. To eliminate E_4 we can show using (5.1.13) that

$$\lim_{\tau \to -\infty} (x - y) = 0^+,$$

and that there exists τ_0, $-\infty < \tau_0 < \infty$ such that

$$(x - y) > 0 \quad \text{for} \quad \tau \in (-\infty, \tau_0]$$

$$\text{and} \quad (x-y)|_{\tau=\tau_0} = \epsilon > 0$$

However,

$$
\begin{aligned}
\frac{d(x-y)}{d\tau} &= x(1-x^2)(x-y)(y-z) > 0 \quad \text{for} \quad -\infty < \tau < \infty, \\
&\Rightarrow \frac{d(x-y)}{d\tau} > 0 \quad \text{on} \quad [\tau_0\,,\infty)\,, \\
&\Rightarrow (x-y) \geq \epsilon \quad \text{on} \quad [\tau_0\,,\infty)\,, \\
&\Rightarrow \lim_{\tau\to+\infty}(x-y) \geq \epsilon > 0.
\end{aligned}
$$

But if our solution approached E_4 we would have

$$\lim_{\tau\to+\infty}(x-y) = \hat{x} - \hat{x} = 0,$$

which we have showed cannot occur □

Thus the only equilibrium set which can be reached is E_5. We conclude that our solution must approach the locus $E_5 \cdot \vec{x_0} = (1, \hat{y}, f(\hat{y}))$ where $\hat{y} \in (\frac{1}{2}, 1)$, for simplicity we shall rename our parameter as follows, $\hat{y} = k$. We linearise about this locus to get

$$\frac{\mathrm{d}\vec{\tilde{x}}}{\mathrm{d}\tau} = J \cdot \vec{\tilde{x}} + u(\vec{\tilde{x}}), \tag{5.2.5}$$

where

$$\vec{\tilde{x}} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} x-1 \\ y-k \\ z-f(k) \end{pmatrix}$$

We obtain the Jacobian

$$J = \begin{pmatrix} 2k(k-1) & 0 & 0 \\ \frac{2(k-1)(2k^2-4k+1)}{(k-2)} & 0 & 0 \\ \frac{2k(k-1)(3-2k)^2}{(k-2)^2} & \frac{2(2k^3-10k^2-15k-7)}{(k-2)} & 2(2-k)(k-1) \end{pmatrix}$$

Which has the following eigenvalues and eigenvectors

$$
\begin{aligned}
\lambda_1 &= 2(2-k)(k-1), & \vec{v_1} &= (0,0,1),\\
\lambda_2 &= 2k(k-1), & \vec{v_2} &= (v_2^1, v_2^2, 1),\\
\lambda_3 &= 0, & \vec{v_3} &= (0, v_3^2, 1),
\end{aligned}
$$

where

$$
v_2^1 = \frac{2k(k-2)^2}{(8k^3-28k^2+29k-7)}, \quad v_2^2 = \frac{2(k-2)(2k^2-4k+1)}{(8k^3-28k^2+29k-7)}, \quad v_3^2 = \frac{(k-2)^2}{(2k^2-8k+7)},
$$

with $\lambda_1 \in \left(-\frac{3}{2}, 0\right)$ and $\lambda_2 \in \left(-\frac{1}{2}, 0\right)$ as $k \in \left(\frac{1}{2}, 1\right)$ and $\lambda_2 < \lambda_1$ for all k, and where the denominators $(8k^3-28k^2+29k-7) \neq 0$ and $(2k^2-8k+7) \neq 0$ in the range $k \in \left(\frac{1}{2}, 1\right)$. We have named the non trivial vector components for convenience. The existence of a zero eigenvalue implies that the equilibrium set, E_5, is non-hyperbolic and thus we cannot apply the Hartman-Grobman and Stable Manifold Theorems as before But the eigenvector corresponding to the zero eigenvalue is tangent to E_5 which indicates that this is a normally hyperbolic equilibrium set. As described in §2.4.2, Theorem 2.4.8 shows that a solution approaching E_5 will lie in the stable manifold, S, of E_5. In Appendix B we describe how we can recast (5.2.5) in coordinates appropriate for the application of Theorem 2.4.9. We arrive at the system

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = q_1(a), \tag{5.2.6a}$$

$$\frac{\mathrm{d}b}{\mathrm{d}\tau} = \lambda_1 b, \tag{5.2.6b}$$

$$\frac{\mathrm{d}c}{\mathrm{d}\tau} = \lambda_2 c. \tag{5.2.6c}$$

where

$$
\vec{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \left(\frac{-f_3(k)}{f_1(k)f_2(k)}\right)\tilde{x} + \frac{\tilde{y}}{f_2(k)} \\ \left(\frac{f_3(k)-f_2(k)}{f_1(k)f_2(k)}\right)\tilde{x} - \frac{\tilde{y}}{f_2(k)} + \tilde{z} \\ \frac{\tilde{x}}{f_1(k)} \end{pmatrix}
$$

Theorem 2.4.9 proves the topological equivalence of solutions to (5.2.5) (trajectories in S) and solutions to the linear system (5.2.6) (trajectories in E^S). Trajectories in the 2-dim stable manifold E^S have the asymptotic behaviour.

$$a = 0, \tag{5.2.7a}$$
$$b = c_1 e^{\lambda_1 \tau} + O(e^{2\lambda_1 \tau}), \tag{5.2.7b}$$
$$c = c_2 e^{\lambda_2 \tau} + O(e^{2\lambda_2 \tau}), \tag{5.2.7c}$$

This is a two-parameter family of solutions, where c_1 and c_2 are the two parameters, we can write this in terms of the original dependant variables as follows

$$x = 1 + c_2 v_2^1 e^{\lambda_2 \tau} + O(e^{2\lambda_2 \tau}), \tag{5.2 8a}$$
$$y = k + c_2 v_2^2 e^{\lambda_2 \tau} + O(e^{2\lambda_2 \tau}), \tag{5.2.8b}$$
$$z = f(k) + c_2 v_2^3 e^{\lambda_2 \tau} + c_1 e^{\lambda_1 \tau} + O(e^{2\lambda_2 \tau}), \tag{5.2.8c}$$

The equilibrium point E_5 is clearly approached as $\tau \to \infty$. We can define

$$\lim_{\tau \to \infty} |\xi(\tau)| = \xi_\infty$$

as the value of $|\xi|$ when the solution reaches the equilibrium set.

Proposition 5.2.5. *The solution (x, y, z) reaches the equilibrium set E_5 at $|\xi| = \xi_\infty < \infty$.*

Proof. We prove by demonstrating that

$$(x, y, z) \to (1, k, f(k)) \quad \text{at} \quad \xi_\infty = \infty$$

generates a contradiction. In the limit $|\xi| \to \infty$ we can take the following asymptotic expressions from our dynamical systems solution (5.2.8):

$$\mu(\xi) \sim \frac{1}{|\xi|} \quad \text{as} \quad |\xi| \to \infty, \tag{5 2.9}$$
$$\mu'(\xi) \sim \frac{k-1}{|\xi|^2} \quad \text{as} \quad |\xi| \to \infty, \tag{5.2.10}$$
$$\mu''(\xi) \sim \frac{2 - 3k + f(k)}{|\xi|^3} \quad \text{as} \quad |\xi| \to \infty, \tag{5.2.11}$$

We can integrate these expressions to obtain the following

$$\int \mu'(\xi) d\xi \sim \mu_0 + \frac{1-k}{\xi} \quad \text{as} \quad |\xi| \to \infty.$$

If we equate this expression with (5.2.9) we obtain the result $\mu_0 = 0$ and $k = 0$ But we have shown that $k \in (\frac{1}{2}, 1)$ by Proposition 5.2.3. The result follow immediately. □

We know that this equilibrium set is approached as $\tau \to \infty$. However, to find the value of $|\xi| = \xi_\infty$ (or $s = s_\infty$) at which the equilibrium set occurs we can integrate

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = x(x-y)(1-x^2),$$

using our locally valid solution to get

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = a_1 e^{\lambda_2 \tau} + a_2 e^{2\lambda_2 \tau} + O(e^{3\lambda_2 \tau}),$$

where $a_1 = -2c_2 v_2^1 (1-k)$ and $a_2 = (c_2 v_2^1)^2$, which we integrate to get

$$s + C \quad = \quad a_1 \lambda_2 e^{\lambda_2 \tau} + a_2 2\lambda_2 e^{2\lambda_2 \tau} + O(e^{3\lambda_2 \tau}),$$

where C is a constant of integration which we can choose to be zero as we have the freedom to rescale ξ. This implies that in the limit $\tau \to \infty$ we get $s \to 0$ and $|\xi| \to 1$. We can invert our expression to obtain

$$e^{\lambda_2 \tau} = b_1 s + b_2 s^2 + \cdots .$$

We finally arrive at

$$\Rightarrow \mu(\xi) \quad = \quad \frac{1}{|\xi|} + c_2 v_2^1 b_1 \frac{\ln(|\xi|)}{|\xi|} + \cdots . \qquad (5.2\ 12a)$$

and we can substitute back into (5.2.1) to get a consistent solution,

$$
\begin{aligned}
x &= 1 + ks + \mathcal{O}(s^2), && (5.2.12b)\\
y &= k + f(k)s + \mathcal{O}(s^2), && (5.2\ 12c)\\
z &= f(k) + \frac{(8k^3 - 28k^2 + 29k - 7)}{2(2k^2 - 8k + 7)}s + \mathcal{O}(s^2). && (5.2.12d)
\end{aligned}
$$

We have shown in the region to the past of E_5 (at $|\xi| = 1$) that x is a monotone increasing function and

$$\lim_{\tau \to \infty} x = 1 \Rightarrow \lim_{|\xi| \to 1} \mu|\xi| = 1,$$

But this defines $\mathcal{N}$ as $|\xi| = 1$ or $\xi = -1$ is therefore the first negative solution of the equation $\xi = -\frac{1}{\mu}$. So we can identify E_5 with $\mathcal{N}$ occuring at $\xi = -1$.

5.2.3 Numerical evaluation of the parameter k

We simulate the evolution of equation (4.4.11c) using a numerical method. The default solver in Mathematica suffices for our purposes We can use our solution (5.1.17a) to construct some realistic initial data:

$$
\begin{aligned}
\mu(\xi_0; \mu_0) &= \mu_0|\xi_0|^{-\frac{1}{2}} + \frac{\mu_0^3}{2}|\xi_0|^{\frac{1}{2}} + \cdots,\\
\mu'(\xi_0; \mu_0) &= \frac{-\mu_0}{2}|\xi_0|^{-\frac{3}{2}} + \frac{\mu_0^3}{4}|\xi_0|^{\frac{-1}{2}} + \cdots,\\
\mu''(\xi_0; \mu_0) &= \frac{3\mu_0}{4}|\xi_0|^{-\frac{5}{2}} + \frac{-\mu_0^3}{8}|\xi_0|^{\frac{-3}{2}} + \cdots,
\end{aligned}
$$

where ξ_0 is some small initial value for ξ, which we shall take as $\xi_0 = 0\ 000001$ throughout this subsection. As we have noted in §5.1.1 fixing the value of μ_0 is equivalent to fixing a scale for r. This can be seen in Figure 5.1 where we have plotted some numerical solutions for different values of μ_0.

For convenience we wish to identify the value of μ_0 for which $x(|\xi| = 1) = 1$. The reason for this is that with this scaling choice the equilibrium set occurs at $s = 0$. One way to numerically estimate the required value of μ_0 is to view the problem (4.4.11c) together with the condition $x(|\xi| = 1) = 1$ as a boundary value problem and employ a shooting method. This is an iterative procedure which generates a sequence of values

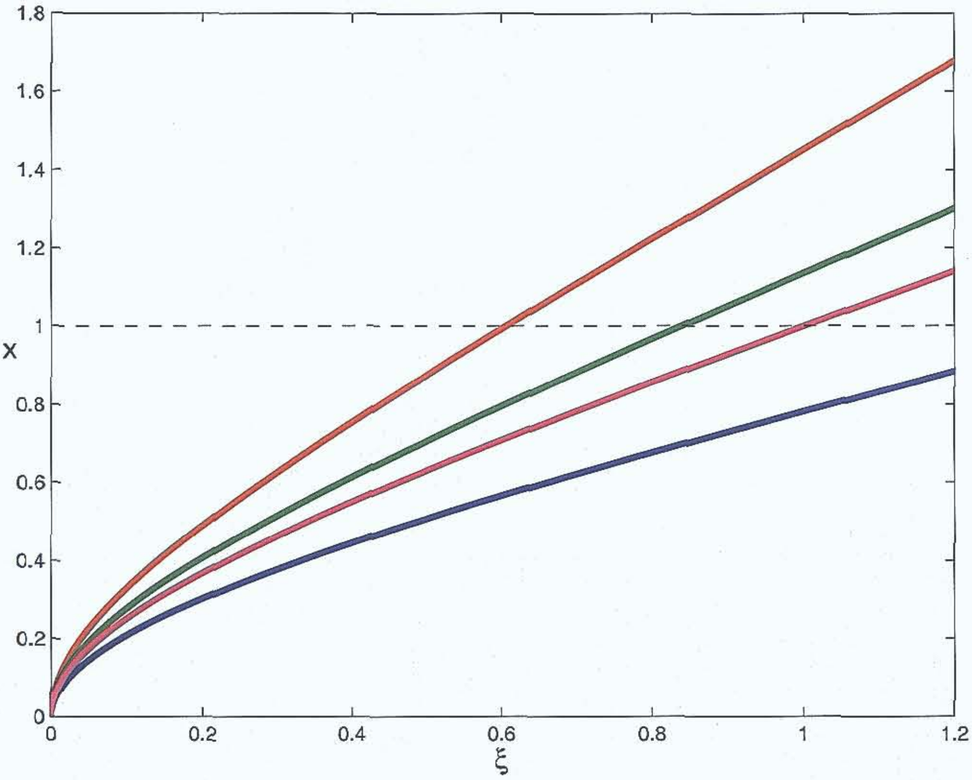

Figure 5.1: Numerical solutions of (4.4.11c) for various values of μ_0.

of μ_0, $\{\mu_0^n\}$. These are chosen in such a way so that the corresponding values of $w \equiv x(|\xi| = 1)$, $\{w^n\}$ converge to 1. This amounts to finding the root of the equation $f(\mu_0) = w(\mu_0) - 1$. We use the Secant Method to determine successive values of μ_0^n with $\mu_0^0 = 1.0$ and $\mu_0^1 = 0.9$ as our initial guesses. The results are shown in Table 5.1. The stopping criterion used was: find N such that $|\mu_0^N - \mu_0^{N+1}| < 10^{-5}$.

Now we are in a position to find an estimate for the value of k. Recall that

$$
\begin{aligned}
y &= \frac{\mathrm{d}x}{\mathrm{d}s} = |\xi| \frac{\mathrm{d}(\mu|\xi|)}{\mathrm{d}|\xi|} = \mu'|\xi|^2 + \mu|\xi|, \\
k &= y(|\xi| = 1).
\end{aligned}
$$

n	μ_0^n	w^n
0	1.0	1.452484
1	0.9	1.235785
2	0.791193	1.023398
3	0.779206	1.001426
4	0.778428	1.000009
5	0.778423	1.000000

Table 5.1: Values of μ_0^n and w^n generated by the Secant Method

So using the final value of μ_0^n in Table 5.1 we get that $k = 0.70838$

5.2.4 Examination of the physical properties

First we calculate the Ricci scalar R:

$$R = \rho = \frac{2(x-y)}{r^2\zeta^2 x} = \frac{2}{r^2\zeta^2}\left((1-k) + \mathcal{O}(s)\right)$$

which diverges as we approach p_o We look at the expansion scalars, which we can rewrite in terms of x, y and z for convenience

$$\begin{aligned}
\theta_+ &= \frac{\beta_+}{r|\zeta|}\left(\left(-1 - \frac{y-z}{x-y}\right) + \frac{1}{x}\left(1 - \frac{y-z}{x-y}\right)\right), \\
\theta_- &= \frac{\beta_-}{r|\zeta|}\left(\left(-1 - \frac{y-z}{x-y}\right) - \frac{1}{x}\left(1 - \frac{y-z}{x-y}\right)\right), \\
\theta_-\theta_+ &= \frac{\beta_-\beta_+}{r^2\zeta^2}\left(\left(-1 - \frac{y-z}{x-y}\right)^2 - \frac{1}{x^2}\left(1 - \frac{y-z}{x-y}\right)^2\right).
\end{aligned}$$

Using the solution (5.1.17) valid near the axis $r = 0$ we obtain untrapped cylinders (as required)

$$\begin{aligned}
\lim_{\xi,r\to 0} \theta_+ &= \lim_{\xi,r\to 0} \frac{\beta_+}{|t|}\left(\frac{1}{2}|\xi|^{-\frac{1}{2}} - \frac{3}{2} + \mathcal{O}(|\xi|^{\frac{1}{2}})\right) > 0, \\
\lim_{\xi,r\to 0} \theta_- &= \lim_{\xi,r\to 0} \frac{\beta_-}{|t|}\left(-\frac{1}{2}|\xi|^{-\frac{1}{2}} - \frac{3}{2} + \mathcal{O}(|\xi|^{\frac{1}{2}})\right) < 0, \\
\lim_{\xi,r\to 0} \theta_+\theta_- &= \frac{\beta_+\beta_-}{|t|^2}\left(-\frac{1}{4}|\xi|^{-1} + \frac{9}{4} + \mathcal{O}(|\xi|^{1})\right) < 0.
\end{aligned}$$

Using the solution (5.2.12) valid near the past null cone we obtain trapped cylinders

$$\begin{aligned}
\lim_{r\to -t} \theta_+ &= \lim_{r\to -t} \frac{\beta_+}{r|\zeta|}(-1.225 + 0.774) \approx \frac{-0.541\beta_+}{r|\zeta|} < 0, \\
\lim_{r\to -t} \theta_- &= \lim_{r\to -t} \frac{\beta_-}{r|\zeta|}(-1.225 - 0\ 774) \approx \frac{-2\beta_-}{r|\zeta|} < 0, \\
\lim_{r\to -t} \theta_+\theta_- &\approx \lim_{r\to -t} \frac{\beta_+\beta_-}{r^2|\zeta|^2}(1.082) > 0.
\end{aligned}$$

Chapter 6

The future of the $\mathcal{N}$

6.1 To prove the existence of a solution emanating from $\mathcal{N}$

We now consider how our unique solution (5.2.12) emanates from the equilibrium point $(x, y, z) = (1, k, f(k))$ or $\mathcal{N}$ [1]. To do this we introduce the functions $\alpha(s)$, $\beta(s)$ and $\gamma(s)$ as follows

$$\begin{aligned} x &= 1 + s\alpha(s), \\ y &= k + s\beta(s), \\ z &= f(k) + s\gamma(s). \end{aligned}$$

Rewriting (5.1.4) in terms of $\alpha(s)$, $\beta(s)$ and $\gamma(s)$, we get

$$\begin{aligned} \frac{d\alpha}{ds} &= \frac{(k - \alpha(s) + s\beta(s))}{s}, \\ \frac{d\beta}{ds} &= \frac{(f(k) - \beta(s) + s\gamma(s))}{s}, \\ \frac{d\gamma}{ds} &= \frac{(F(\alpha, \beta, \gamma, s) - \gamma(s))}{s}, \end{aligned}$$

[1] We have not considered the spacetime matching conditions across $\mathcal{N}$ here but we note that the existence of the solution described below is a minimal condition for a metric that extends continuously across $\mathcal{N}$

where

$$
\begin{aligned}
F(\alpha,\beta,\gamma,s) &= k + s\beta - \frac{2(k+f(k)+s\beta-s\gamma)^2}{(1-k+s(\alpha-\beta))} \\
&- \frac{2(1+s\alpha)^2(k-f(k)+s\beta-s\gamma)}{(1+s\alpha)^2-1} \\
&+ \frac{(1+(1+s\alpha)^2)(1-k+s(\alpha-\beta))(1+\frac{1}{-2+k}+s(\alpha-2\beta+\gamma))}{-1-s\alpha+(1+s\alpha)^3}.
\end{aligned}
$$

These equations are singular at $s=0$ so we define a new stretched time coordinate T to regularise the system

$$
\frac{ds}{dT} = s\alpha(s), \quad \lim_{s\to 0^+} T(s) = -\infty. \tag{6.1.1a}
$$

We finally obtain the 4-dim dynamical system

$$
\begin{aligned}
\frac{d\alpha}{dT} &= (k-\alpha(s)+s\beta(s))\alpha(s), && (6.1.1b) \\
\frac{d\beta}{dT} &= (f(k)-\beta(s)+s\gamma(s))\alpha(s), && (6.1.1c) \\
\frac{d\gamma}{dT} &= (F-\gamma(s))\alpha(s), && (6.1.1d) \\
\frac{ds}{dT} &= s\alpha(s). && (6.1.1e)
\end{aligned}
$$

This dynamical system has two equilibria at $s=0$:

$$
\begin{aligned}
E_1 &: (\alpha,\beta,\gamma,s) \to (k, f(k), g(k), 0) = \vec{\alpha_1}, \\
E_2 &: (\alpha,\beta,\gamma,s) \to (0, \hat{\beta}, h(\hat{\beta},k), 0) = \vec{\alpha_2},
\end{aligned}
$$

where $k \cong 0.708$ as we found above, $\hat{\beta}$ is a new parameter and

$$
\begin{aligned}
f(k) &= \frac{2k^2-4k+1}{k-2}, \\
g(k) &= \frac{(-7+29k-28k^2+8k^3)}{2(-2+k)^2}, \\
h(\hat{\beta},k) &= \frac{(7\beta-8k\hat{\beta}+2k^2\hat{\beta})}{(-2+k)^2}.
\end{aligned}
$$

6.1.1 Stability analysis of equilibria

To analyse the stability of equilibrium set E_2 we linearise (6.1.1) about E_2 to get 2 non-zero eigenvalues and 2 zero eigenvalues

$$\begin{aligned}
\lambda_1 =& k, & \vec{v_1} =&(v_1^1, v_1^2, 1, 0), \\
\lambda_2 =& 2-k, & \vec{v_2} =&(0,0,1,0), \\
\lambda_3 =& 0 & \vec{v_3} =&(0, v_3^2, 0, 1), \\
\lambda_4 =& 0 & \vec{v_4} =&(0, v_4^2, 1, 0),
\end{aligned}$$

where

$$\begin{aligned}
v_1^1 &= \frac{(-2+k)^2}{(2k^3 - 7\hat{\beta} - 2k^2(3+\hat{\beta}) + k(5+8\hat{\beta}))}, \\
v_1^2 &= \frac{(k-2)^2(k-\hat{\beta})}{(2k^3 - 7\hat{\beta} - 2k^2(3+\hat{\beta}) + k(5+8\hat{\beta}))}, \\
v_3^2 &= \frac{\hat{\beta}^2}{(14 - 23k + 12k^2 - 2k^3)}, \quad v_4^2 = \frac{(k-2)^2}{(7-8k+2k^2)},
\end{aligned}$$

where the denominators in v_3^2 and v_4^2 are non-zero, and if the denominator in v_1^1 and v_1^2 is zero (i.e. if $\hat{\beta} \sim -0.920$) this just makes one of the non-diagonal components in the Jacobian equal to zero (see Appendix C) and then $v_1^1 = \frac{-2k(k-1)(k-2)}{\hat{\beta}(7-8k+2k^2)}$, $v_1^2 = \frac{2(k-1)(k-2)}{(7-8k+2k^2)}$ and v_3^2 and v_4^2 remain the same.

Proposition 6.1.1. *The equilibrium set E_2 does not correspond to $\mathcal{N}$.*

Proof. E_2 is an unstable equilibrium set, with a 2-dim unstable manifold, U, tangent to the 2-dim unstable subspace, E^U spanned by $\vec{v_1}$ and $\vec{v_2}$ and a 2-dim centre manifold, C, tangent to the 2-dim subspace, E^C spanned by $\vec{v_3}$ and $\vec{v_4}$. In Appendix C we will show by using an analytical approximation to the centre manifold that $\alpha(s)$ is independent of the coordinates of the centre manifold up to third order, so we can write

$$\alpha = c_1 \exp(kT) + \mathcal{O}(\exp(2kT)).$$

We know $c_1 > 0$. Then $\Rightarrow \alpha > 0$ for $T < T_*$, some $T_* \in \mathrm{R}$, and

$$\frac{ds}{dT} = s\alpha(s) = s\alpha(T) \quad \text{for} \quad s > 0,$$

then

$$\frac{ds}{dT} > 0 \quad \text{for} \quad T \in (-\infty, T_*),$$
$$\Rightarrow s(T) < s_0 \quad \text{for} \quad T < T_0, \quad s_0 = s(T_0),$$

where $T_0 \leq T_*$ is any fixed value. Consequently

$$\frac{\mathrm{d}s}{\mathrm{d}T} < s_0 \alpha(T),$$

and so

$$\int_s^{s_0} \mathrm{d}s' < s_0 \int_T^{T_0} \alpha(T')\mathrm{d}T'. \qquad (6.1.2)$$

We can choose T_0 so that

$$\alpha(T) = c_1 \exp(kT) + \mathcal{O}(\exp(2kT)), \quad T \to -\infty,$$

satisfies

$$\alpha(T) < 2c_1 \exp(kT) \quad \forall \quad T \leq T_0$$

Then

$$\int_T^{T_0} \alpha(T')\mathrm{d}T' < \frac{2c_1}{k}(e^{kT_0} - e^{kT}),$$

and from (6.1.2)

$$\begin{aligned} s_0 - s &< \frac{2c_1}{k}(e^{kT_0} - e^{kT})s_0, \\ \Rightarrow s &> s_0\left(1 - \frac{2c_1}{k}e^{kT_0}\right) + \frac{2c_1}{k}e^{kT}s_0, \end{aligned}$$

and taking the limit as $T \to -\infty$ we get

$$\lim_{T \to -\infty} s(T) \geq s_0 \left(1 - \frac{2c_1}{k}e^{kT_0}\right).$$

Now choose T_0 sufficiently large and negative so that

$$1 - \frac{2c_1}{k} e^{kT_0} > 0,$$
$$\Rightarrow \lim_{T \to -\infty} s(T) > 0,$$

giving the required contradiction (see (6.1.1a)). $\square$

Therefore our solution must approach E_1 (as $T \to -\infty$). To analyse this equilibrium point we first linearise $F(\alpha, \beta, \gamma, s)$ about $s = 0$ as it has a singularity at that point. Now we have our dynamical system in a convenient form, and when we linearise this system about E_1 we get 4 non-zero eigenvalues (so this is a hyperbolic equilibrium point).

$$\begin{aligned}
\lambda_1 &= 2 - 2k, & \vec{v_1} &= (0, 0, 1, 0),\\
\lambda_2 &= -k, & \vec{v_2} &= (v_2^1, 0, 1, 0),\\
\lambda_3 &= -k & \vec{v_3} &= (v_3^1, 1, 0, 0),\\
\lambda_4 &= k & \vec{v_4} &= (v_4^1, v_4^2, v_4^3, 1),
\end{aligned}$$

where

$$\begin{aligned}
v_2^1 &= \frac{(-2+k)^3}{7 - 11k + 4k^2}, \quad v_3^1 = \frac{2(-2+k)(7 - 8k + 2k^2)}{(7 - 11k + 4k^2)}, \quad v_4^1 = \frac{(-1 + 4k - 2k^2)}{2(k-2)},\\
v_4^2 &= -\frac{(7 - 29k + 28k^2 - 8k^3)}{4(k-2)^2}, \quad v_4^3 = -\frac{(-22 + 106k - 155k^2 + 93k^3 - 20k^4)}{4(k-2)^3}.
\end{aligned}$$

By the Stable Manifold Theorem there exists a 2-dim unstable manifold, U tangent to a 2-dim unstable subspace, E^U spanned by $\vec{v_1}$ and $\vec{v_4}$. Trajectories in this unstable manifold have the asymptotic behavior:

$$\vec{\alpha} = \vec{\alpha_1} + c_1 \vec{v_4} \exp(kT) + c_2 \vec{v_1} \exp(2 - 2k)T + O(\exp 2kT), \qquad (6.1\ 3)$$

which is a two-parameter family of solutions, with parameters c_1 and c_2. We then obtain

$$\alpha = k + c_1 v_4^1 \exp(kT) + \mathcal{O}(\exp(2kT)), \quad (6.1.4a)$$

$$\beta = f(k) + c_1 v_4^2 \exp(kT) + \mathcal{O}(\exp(2kT)), \quad (6.1\ 4b)$$

$$\gamma = g(k) + c_1 v_4^3 \exp(kT) + c_2 \exp(2-2k)T + \mathcal{O}(\exp(2kT)), \quad (6.1.4c)$$

$$s = c_1 \exp(kT) + \mathcal{O}(\exp(2kT)). \quad (6.1.4d)$$

We can invert (6.1.4d) to obtain

$$\exp(kT) = \frac{s}{c_1} + \mathcal{O}(s^2),$$

and we finally arrive at

$$x = 1 + s\alpha = 1 + ks + v_4^1 s^2 + \mathcal{O}(s^3), \quad (6.1.5a)$$

$$y = k + s\beta = k + f(k)s + v_4^2 s^2 + \mathcal{O}(s^3), \quad (6.1.5b)$$

$$z = f(k) + s\gamma = f(k) + g(k)s + v_4^3 s^2 + \frac{c_2 s^{\left(\frac{2-k}{k}\right)}}{c_1^{\frac{2(1-k)}{k}}} + \mathcal{O}(s^{\left(\frac{4-3k}{k}\right)}), \quad (6.1.5c)$$

where $\frac{2-k}{k} \simeq 1.825$. In order to ensure that our system is consistent substitute (6 1.5) into the following equations of (5.1.4)

$$\frac{\mathrm{d}x}{\mathrm{d}s} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}s} = z$$

and we get the necessary agreement

$$v_4^1 = 2f(k), \quad v_4^2 = 2g(k).$$

Proposition 6.1.2. *There exists a unique analytic solution emanating from $\mathcal{N}$.*

Proof. To ensure that the solution is analytic we require that $c_2 = 0$ (this eliminates terms with non integer powers in (6.1.5c)). The remaining parameter c_1 has cancelled out of (6.1.5), this is a consequence of the fact that we introduced the independent variable T by

$$\frac{\mathrm{d}s}{\mathrm{d}T} = s\alpha,$$

so it was defined up to a constant of integration. We can easily show that this constant of integration is directly related to c_1 $\square$

We can then rewrite our solution as

$$\mu(\xi) = \frac{1}{|\xi|} + k\frac{\ln|\xi|}{|\xi|} + \frac{f(k)(\ln|\xi|)^2}{2|\xi|} + \mathcal{O}\left(\frac{(\ln|\xi|)^3}{|\xi|}\right) \tag{6.1.6a}$$

$$x = 1 + s\alpha = 1 + ks + \mathcal{O}(s^2), \tag{6.1.6b}$$

$$y = k + s\alpha = k + f(k)s + \mathcal{O}(s^2), \tag{6.1.6c}$$

$$z = f(k) + s\alpha = f(k) + g(k)s + \mathcal{O}(s^2). \tag{6.1.6d}$$

6.2 Analysis of the behaviour of the solution emanating from $\mathcal{N}$

Recall that $\xi = \frac{r}{t}$ and $s = \ln(|\xi|)$, and we are now considering the interval $s \in (0, \infty)$. We consider our original dynamical system

$$\begin{aligned}
\frac{dx}{ds} &= y, \\
\frac{dy}{ds} &= z, \\
\frac{dz}{ds} &= y - \frac{2(y-z)^2}{x-y} - \frac{2x^2(y-z)}{(x^2-1)} \\
&+ \frac{(x^2+1)(x-y)(x-2y+z)}{(x^3-x)} = f(x,y,z).
\end{aligned}$$

We must remove the singularities at $x = 1$ and $x = y$ by introducing an auxiliary time coordinate τ defined to be the solution of the initial value problem

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = (x-y)(x^2-1), \quad s(-\infty) = 0. \tag{6 2 1a}$$

We can verify that this independent variable has the property $\tau(-\infty) = -\infty$ by simply integrating (6.2.1a) using the solution (6.1.6). We obtain the following dynamical system

$$\frac{dx}{d\tau} = y(x-y)(x^2-1), \tag{6.2.1b}$$
$$\frac{dy}{d\tau} = z(x-y)(x^2-1), \tag{6.2.1c}$$
$$\frac{dz}{d\tau} = y(x-y)(x^2-1) - 2(y-z)^2(x^2-1) - \frac{(x-y)^2(x-2y+z)(x^2+1)}{x} + 2x^2(x-y)(y-z) \tag{6.2.1d}$$

Lemma 6.2.1. *The system* (6.2.1) *has the following properties:*

$$(x-1) > 0 \quad \textit{for} \quad \tau \in (-\infty, \infty), \tag{6.2.2}$$
$$(x-y) > 0 \quad \textit{for} \quad \tau \in (-\infty, \infty) \tag{6.2.3}$$

Proof. The result follows from the fact that these two quantities are invariant submanifolds of (6.2 1). Using our regular solution at $\mathcal{N}$, $(s = 0)$ we find that there exists an $s_0 > 0$ such that,

$$(x-1) = ks + \mathcal{O}(s^2) > 0, \quad \text{for} \quad s \in (0, s_0)$$

if $x-1$ changes sign at some $\tau = \tau_1 < \infty$, then $x(\tau_1) = 1$ then by (6.2.1) we have that

$$\left.\frac{dx}{d\tau}\right|_{\tau=\tau_0} = 0$$

But if this is true then $x = 1$ for all τ. Therefore $(x-1)$ can only reach zero as $\tau \to \pm\infty$ and

$$x(\tau) - 1 > 0 \ \text{ for } \ \tau \in (-\infty, \infty).$$

Similarly for $(x-y)$. □

Lemma 6.2.2. *The behaviour of the unique solution* (6.1.6a) *can be divided into two cases:*

Case 1. $(y-z) > 0$ *for* $\tau \in (-\infty, \tau_0)$ *and* $(y-z) < 0$ *for* $\tau \in (\tau_0, \infty)$ *where* $\tau_0 < \infty$, *or*

Case 2. $(y-z) > 0$ *for* $\tau \in (-\infty, \infty)$

Proof. The proof is simply based on the fact that $(y - z)$ is initially positive and can only change signs once.

$$\begin{aligned}\lim_{s\to0^+}(y-z) &= k - f(k) = \frac{(k-1)^2}{(2-k)} > 0, \\ \left.\frac{d(y-z)}{ds}\right|_{y=z} &= -\frac{(x^2+1)\,(x-y)^2}{(x^3-x)} < 0 \quad \forall \quad \tau.\end{aligned}$$

□

Lemma 6.2.3. *The unique solution* (6.1.6a) *also has the property that* $(x - 2y + z)$ *and* $(x - z)$ *can only change sign once.*

Proof. The proof is simply based on the fact that $x - 2y + z$ and $x - z$ are initially positive and cannot increase at a root.

$$\begin{aligned}\lim_{s\to0^+}(x-z) &= 1 - f(k) > 0, \\ \lim_{s\to0^+}(x-2y+z) &= 1 - 2k + f(k) > 0, \\ \left.\frac{d(x-z)}{d\tau}\right|_{x=z} &= -2(x-y)^2\left(1 + \frac{(x^2+1)}{x}(x-y)\right) < 0 \quad \forall \quad \tau, \\ \left.\frac{d(x-2y+z)}{d\tau}\right|_{x-y=y-z} &= -2x^2(x-y)^2 < 0 \quad \forall \quad \tau,\end{aligned}$$

□

6.2.1 Analysis of Equilibrium sets

The equilibrium sets of (6 2.1) can be reduced to the following two sets using the Lemmas (6.2.2) and (6.2.3).

$$\begin{aligned}L_1 &: (x,y,z) \to (\hat{x},\hat{x},\hat{x}), \quad \tau\to\infty \quad \text{where} \quad \hat{x} > 1, \\ L_2 &: (x,y,z) \to (1,\hat{y},f(\hat{y})), \quad \tau\to\infty \quad \text{where} \quad \hat{y} < 0.\end{aligned}$$

If we approach L_2 then

$$\lim_{\tau\to\infty}(y-z) = (\hat{y} - f(\hat{y})) \geq \frac{1}{2},$$

but in **Case 1** $(y - z) \leq 0$ as $\tau \to \infty$. Therefore L_2 can only be approached in **Case 2**. Furthermore, in **Case 2** we have that

$$\begin{aligned} (y-z) &> 0, \\ \Rightarrow \frac{\mathrm{d}(x-y)}{\mathrm{d}\tau} &> 0, \\ \Rightarrow (x-y) &> 1-k \approx 0.292, \end{aligned}$$

for all τ but if we approach L_1 then

$$\lim_{\tau\to\infty}(x-y) = \hat{x} - \hat{x} = 0.$$

Therefore L_1 can only be approached in **Case 1**

Proposition 6.2.4. *If our solution approaches L_1 then $x(\tau)$ increases to $x_0 \in (1, \infty)$.*

Proof. If our solution approaches L_1 then

$$\lim_{\tau\to\infty} y = x_0 \geq 1 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\tau} > 0 \quad \text{as } \tau \to \infty.$$

□

Next we will describe each case, but we will provide numerical evidence in §6.2.3 to rule out L_2.

6.2.2 Case 2: Solution approaches L_2

If we linearise (6.2.1) about L_2 we can show that this equilibrium set is non-hyperbolic with one positive, one negative and one zero eigenvalue.

$$\begin{aligned} \lambda_1 =&0, & \vec{v_1} =&\left(0, \frac{(\hat{y}-2)^2}{7-8\hat{y}+2\hat{y}^2}, 1\right), \\ \lambda_2 =&2(\hat{y}-2)(\hat{y}-1), & \vec{v_2} =&(0,0,1), \\ \lambda_3 =&2\hat{y}(1-\hat{y}) & \vec{v_3} =&\left(\frac{2\hat{y}(\hat{y}-2)^2}{(-7+29\hat{y}-28\hat{y}^2+8\hat{y}^3)}, \frac{2(\hat{y}-2)(1-4\hat{y}+2\hat{y}^2)}{(-7+29\hat{y}-28\hat{y}^2+8\hat{y}^3)}, 1\right). \end{aligned}$$

This equilibrium set is a normally hyperbolic saddle and according to Theorem 2.4.8 solutions approaching L_2 will lie in the 1-dim stable manifold of L_2. We know from the previous subsection that if $(y-z)$ becomes negative then L_2 cannot be approached. In the following we conjecture that this is a property of the unique analytic solution and provide some numerical evidence in the subsequent subsection.

Conjecture 6.2.5. *The unique analytic solution from $\mathcal{N}$ has the property that $(y-z)$ becomes negative. This solution can therefore only approach the equilibrium set L_1.*

6.2.3 Numerical simulation of solution emanating from $\mathcal{N}$

In this subsection we examine numerically the evolution of the unique solution emanating from $\mathcal{N}$. The unique solution (6.1.5) was used to create reasonable initial data for the dynamical system (5.1.4) at $s = s_0$ where s_0 is some small initial value of s, which we shall take as $s_0 = 0.000001$. We are interested in the function $(y-z)$, specifically whether is crosses the axis If it does let s_1 be the value of s for which $y - z = 0$.

To generate numerical solutions we employ a variable order Adams-Bashforth-Moulton method This is a multistep solver appropriate for non-stiff systems of ODEs, such as (5 1 4) MATLAB's **ode113** routine is an implementation of this method [34]. At each step, the method estimates the local error, e, in each component of the solution. This error must be less than or equal to the acceptable error, specified by the user, which is a function of the specified relative tolerance, RelTol, and the specified absolute tolerance, AbsTol·

$$\|e\| \leq \max\{\text{RelTol}\|\text{size of solution components}\|, \text{AbsTol}\}.$$

Roughly speaking, this means that you want RelTol correct digits in all solution components except those smaller than the threshold AbsTol. For our purposes we choose a value for AbsTol of essentially zero and generate estimates to the solution components for successively smaller values of RelTol. Let N be the total number of mesh points used and let s_1^N be the approximation of s_1. This is calculated using linear interpolation of the numerical solution The results are presented in Table 6.1. We see that the numerical approximation to $(y-z)$ does indeed cross the axis. The sequences of values s_1^N for the range of RelTol considered appear to converge to a value of roughly 0.77. In Figure 6.2 3 we plot representative numerical approximations of $(y-z)$ and $(x-y)$.

The results demonstrate that at least numerically the statement in Conjecture 6.2.5 holds.

RelTol	N	s_1^N
10^{-3}	14	0.75338
10^{-4}	17	0.76143
10^{-5}	20	0 76604
10^{-6}	24	0.77018
10^{-7}	30	0.77306
10^{-8}	36	0.77559
10^{-9}	49	0.77769
10^{-10}	63	0.77869
10^{-11}	80	0.77952
10^{-12}	104	0.77871
10^{-13}	128	0.77952

Table 6.1 *Values of* s_1^N *generated by the* Adams-Bashforth-Moulton Method

6.3 Case 1: Solution approaches L_1

6.3.1 Stability and Liapunov functions

If we linearise (6.2.1) about L_1 we can show that this equilibrium set is non-hyperbolic with three zero eigenvalues. In order to be able to make a statement about the stability properties we construct a Liapunov function $V(x,y,z) \in C^1(E)$ where E is a subset of the solution space S, ($S \subset \mathbb{R}^3$), containing L_1 and where

$$S = \{x \in (1,\infty), y \in (-\infty,\infty), z \in (-\infty,\infty)\}.$$

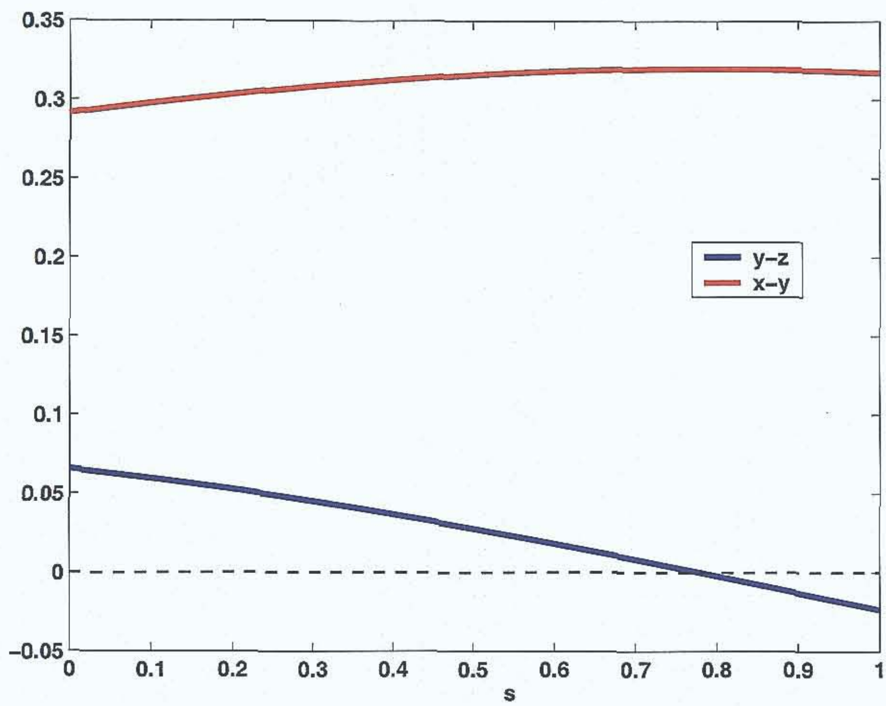

Figure 6.1: Numerical approximations of $(y - z)$ and $(x - y)$.

We are considering a solution with the following asymptotic behaviour,

$$L_1 : (x, y, z) \to (x_0, x_0, x_0), \quad \text{as} \quad \tau \to \infty \quad \text{where} \quad x_0 > 1,$$

we will define new coordinates as follows

$$\vec{\bar{x}} = \begin{pmatrix} \bar{x} \\ \bar{u} \\ \bar{w} \end{pmatrix} = \begin{pmatrix} x - x_0 \\ x - y \\ y - z \end{pmatrix}$$

where $\lim_{\tau \to \infty} \vec{\bar{x}} = \vec{0}$ thus represents L_1. We can recast our dynamical system (6.2.1) as

$$\frac{\mathrm{d}\vec{\bar{x}}}{\mathrm{d}\tau} = \bar{f}(\vec{\bar{x}}).$$

Theorem 6.3.1. *There exists positive numbers α_1, α_3 and α_4 such that the function*

$$V(\bar{x}, \bar{u}, \bar{w}) = (\bar{x} - \bar{u})^2 + \bar{w}\bar{x}$$

satisfies

1. $V(\vec{0}) = 0$,
2. $V(\vec{\bar{x}}) > 0, \quad \forall \quad \vec{\bar{x}} \in E \backslash \{\vec{0}\}$,
3. $DV(\vec{\bar{x}})\bar{f}(\vec{\bar{x}}) = \dot{V}(\vec{\bar{x}}) < 0, \quad \forall \quad \vec{\bar{x}} \in E \backslash \{\vec{0}\}$,

for

$$E = \left\{ \bar{x} \in \left[-\frac{\alpha_1}{\alpha_4}, 0 \right], \bar{u} \in \left[0, \frac{\alpha_1}{\alpha_3} \right], \bar{w} \in (-\infty, 0] \right\}$$

and so $V(\vec{\bar{x}})$ is a Liapunov function and $\vec{0}$ is asymptotically stable on E.

Proof. We know from Lemmas 6.2.1 and 6.2.2 that for all $\tau \in (\tau_0, \infty)$ we have

$$\bar{x} < 0, \quad \bar{u} > 0, \quad \bar{w} < 0,$$

which ensures that $V(\vec{0}) = 0$ and $V(\vec{\bar{x}}) > 0, \quad \forall \quad \vec{\bar{x}} \in E \backslash \{\vec{0}\}$. To show that $\dot{V}(\vec{\bar{x}}) < 0$ on the required interval we write out $\dot{V}(\vec{\bar{x}})$, it has 26 terms which we can simplify by writing the second, third, fourth, fifth and sixth order terms on separate lines. The second order terms are all negative and we will essentially use these dominant terms to cancel out any higher order terms which are positive by defining a sufficiently small region about the origin.

$$\begin{aligned}
\dot{V}(\vec{\bar{x}}) \;=\; & \bar{w}\bar{u}(x_0^4 - x_0^2) + \bar{u}\bar{x}(2(x_0^4 - x_0^2)) + \bar{u}^2(-2(x_0^4 - x_0^2)) \quad \} \, \mathrm{Or}(\vec{\bar{x}}^2), \\
& \left.\begin{aligned} &+\bar{w}\bar{u}\bar{x}(3x_0^3 + x_0) + \bar{w}\bar{u}^2(x_0^3 - x_0) + \bar{w}^2\bar{x}(2(x_0^3 - x_0)) \\ &+\bar{u}^3(2(x_0^3 - x_0)) + \bar{u}^2\bar{x}(-10x_0^3 + 6x_0) + \bar{x}^2\bar{u}(8x_0^3 - 4x_0) \end{aligned}\right\} \mathrm{Or}(\vec{\bar{x}}^3) \\
& \left.\begin{aligned} &+\bar{u}^2\bar{x}^2(-18x_0^2 + 4) + \bar{x}^3\bar{u}(12x_0^2 - 2) + \bar{u}^3\bar{x}(5x_0^2 - 3) \\ &+\bar{w}\bar{u}\bar{x}^2(3x_0^2 + 2) + \bar{w}^2\bar{x}^2(6x_0^2 - 2) + \bar{x}\bar{w}\bar{u}^2(4x_0^2) \end{aligned}\right\} \mathrm{Or}(\vec{\bar{x}}^4) \\
& \left.\begin{aligned} &+4\bar{x}^2\bar{u}^3x_0 + 5\bar{x}^2\bar{w}\bar{u}^2x_0 - 14\bar{x}^3\bar{u}^2x_0 \\ &+\bar{x}^3\bar{w}\bar{u}x_0 + 6\bar{x}^3\bar{w}^2x_0 + 8\bar{x}^4\bar{u}x_0 \end{aligned}\right\} \mathrm{Or}(\vec{\bar{x}}^5) \\
& +\bar{u}^3\bar{x}^3 + 2\bar{u}^2\bar{w}\bar{x}^3 - 4\bar{x}^4\bar{u}^2 + 2\bar{u}\bar{x}^5 + 2\bar{w}^2\bar{x}^4, \quad \} \, \mathrm{Or}(\vec{\bar{x}}^6)
\end{aligned}$$

where $\mathrm{Or}(\vec{x}^n)$ denotes all the homogeneous nth order terms. We will begin by describing explicitly how the second order terms will dominate the third order terms. Let

$$\alpha_1 = (x_0^4 - x_0^2), \quad \alpha_2 = (3x_0^3 + x_0), \quad \alpha_3 = (10x_0^3 - 6x_0), \quad \alpha_4 = (8x_0^3 - 4x_0),$$

as $x_0 > 1$ it is clear that $\alpha_i > 0$ for $i = 1 - 4$. Now we can write the second and third order terms as follows;

$$\begin{aligned} \dot{V}(\vec{x}) &= \bar{w}\bar{u}(\alpha_1 + \alpha_2\bar{x}) + \bar{u}\bar{x}(\alpha_1 - \alpha_3\bar{u}) + \bar{u}\bar{x}(\alpha_1 + \alpha_4\bar{x}) + \\ & \quad \bar{u}^2\left(-2\alpha_1 + \frac{2\alpha_1}{x_0}\bar{u}\right) + \bar{w}\bar{u}^2\left(\frac{\alpha_1}{x_0}\right) + \bar{w}^2\bar{x}\left(\frac{2\alpha_1}{x_0}\right) + \mathcal{O}(\vec{x}^4). \quad (6.3.1) \end{aligned}$$

By inspection of each term we can state that this expression is negative to fourth order if

$$(\alpha_1 + \alpha_2\bar{x}) > 0, \quad (\alpha_1 - \alpha_3\bar{u}) > 0 \quad (\alpha_1 + \alpha_4\bar{x}) > 0, \quad \left(-2\alpha_1 + \frac{2\alpha_1}{x_0}\bar{u}\right) < 0,$$

which leads to some bounds on $\bar{u}$ and $\bar{x}$, respectively,

$$-\frac{\alpha_1}{\alpha_2} < \bar{x} < 0, \quad 0 < \bar{u} < \frac{\alpha_1}{\alpha_3}, \quad -\frac{\alpha_1}{\alpha_4} < \bar{x} < 0, \quad 0 < \bar{u} < x_0.$$

For convenience we let

$$\begin{aligned} \alpha_5 &= 18x_0^2 - 4, \quad \alpha_6 = 12x_0^2 - 2, \quad \alpha_7 = 5x_0^2 - 3, \\ \alpha_8 &= x_0^2, \quad \alpha_9 = 3x_0^2 + 2, \quad \alpha_{10} = 6x_0^2 - 2, \end{aligned}$$

where $\alpha_i > 0$, for $i = 5 - 10$. The last two third order terms in (6.3.1) are negative and we can add these to the only two positive fourth order terms to rewrite all the fourth order terms as,

$$\bar{w}\bar{u}^2\left(\frac{\alpha_1}{x_0} + 4\alpha_8\bar{x}\right) + \bar{w}^2\bar{x}\left(\frac{2\alpha_1}{x_0} + \alpha_{10}\bar{x}\right) - \alpha_5\bar{u}^2\bar{x}^2 + \alpha_6\bar{u}\bar{x}^3 + \alpha_7\bar{u}^3\bar{x} + \alpha_9\bar{w}\bar{u}\bar{x}^2. \quad (6.3.2)$$

If $x > -\frac{\alpha_1}{4x_0\alpha_8}$ and $x > -\frac{2\alpha_1}{x_0\alpha_{10}}$ then (6.3 2) will be negative. Next we add the four negative fourth order terms (the last four terms in (6.3.2)) to the only four positive fifth order terms to rewrite all the fifth order terms as

$$-\bar{x}^2\bar{u}^2(\alpha_5 + 14x_0\bar{x}) + \bar{x}^3\bar{u}(\alpha_6 + 8x_0\bar{x}) + \bar{x}\bar{u}^3(3x_0^2 - 3)$$
$$+\bar{x}\bar{u}^3(2x_0^2 + 4x_0\bar{x}) + \bar{w}\bar{x}^2\bar{u}(\alpha_9 + x_0\bar{x}) + 5x_0\bar{w}\bar{x}^2\bar{u}^2 + 6x_0\bar{w}^2\bar{x}^3. \quad (6.3.3)$$

If $\bar{x} > -\frac{\alpha_5}{14x_0}$, $\bar{x} > -\frac{\alpha_6}{8x_0}$, $\bar{x} > -\frac{x_0}{2}$ and $\bar{x} > -\frac{\alpha_9}{x_0}$ then (6.3.3) is negative. Finally we add the two negative fifth order terms (the last two terms in (6.3.3))to the only two positive sixth order terms to rewrite all the sixth order terms as

$$\bar{x}^2\bar{u}^2\bar{w}(5x_0 + 2\bar{x}) + \bar{x}^3\bar{w}^2(6x_0 + 2\bar{x}) + \bar{x}^3\bar{u}^3 - 4\bar{x}^4\bar{u}^2 + 2\bar{x}^5\bar{u} \quad (6.3.4)$$

which is negative if $\bar{x} > -\frac{5x_0}{2}$ and $\bar{x} > -3x_0$. The region in which all the bounds are satisfied is

$$E = \left\{\bar{x} \in \left[-\frac{\alpha_1}{\alpha_4}, 0\right], \bar{u} \in \left[0, \frac{\alpha_1}{\alpha_3}\right], \bar{w} \in (-\infty, 0]\right\}.$$

Therefore we can rewrite $V(\vec{\bar{x}})$ as follows

$$\begin{aligned}
\dot{V}(\vec{\bar{x}}) &= \bar{w}\bar{u}(\alpha_1 + \alpha_2\bar{x}) + \bar{u}\bar{x}(\alpha_1 - \alpha_3\bar{u}) + \bar{u}\bar{x}(\alpha_1 + \alpha_4\bar{x}) + \\
&\quad \bar{u}^2\left(-2\alpha_1 + \frac{2\alpha_1}{x_0}\bar{u}\right) + \bar{w}\bar{u}^2\left(\frac{\alpha_1}{x_0} + 4\alpha_8\bar{x}\right) + \\
&\quad +\bar{w}^2\bar{x}\left(\frac{2\alpha_1}{x_0} + \alpha_{10}\bar{x}\right) - \bar{x}^2\bar{u}^2(\alpha_5 + 14x_0\bar{x}) + \bar{x}^3\bar{u}(\alpha_6 + 8x_0\bar{x}) + \\
&\quad +\bar{x}\bar{u}^3(3x_0^2 - 3) + \bar{x}\bar{u}^3(2x_0^2 + 4x_0\bar{x}) + \bar{w}\bar{x}^2\bar{u}(\alpha_9 + x_0\bar{x}) + \\
&\quad +\bar{x}^2\bar{u}^2\bar{w}(5x_0 + 2\bar{x}) + \bar{x}^3\bar{w}^2(6x_0 + 2\bar{x}) + \bar{x}^3\bar{u}^3 - 4\bar{x}^4\bar{u}^2 + 2\bar{x}^5\bar{u}
\end{aligned}$$

So this expression is negative by inspection and $\dot{V}(\vec{\bar{x}}) < 0 \ \forall \ \ \vec{\bar{x}} \in E\backslash\{\vec{0}\}$. □

6.3.2 Further analysis of L_1 - Compactification of the state space

As we would like to calculate some physical properties of the final state of (6.2.1) (e.g. density and expansion scalars) we need quantitative information about this equilibrium point. To do this we redefine our solution space as a compact cube $[0,1]^3$ in the following way, see [19]. Define

$$\begin{aligned} a &= x-1>0, \\ b &= x-y>0, \\ c &= z-y>0, \end{aligned}$$

using (6.2.2), (6.2.3) and the definition of Case 1, valid for $\tau \in (\tau_0, \infty)$. Define

$$A = \frac{a}{a+1} \qquad a = \frac{A}{1-A}, \tag{6.3.5}$$

$$B = \frac{b}{b+1} \quad \Leftrightarrow \quad b = \frac{B}{1-B}, \tag{6.3.6}$$

$$C = \frac{c}{c+1} \qquad c = \frac{C}{1-C}, \tag{6 3.7}$$

so that $(A, B, C) \in (0,1)^3$ (valid for $\tau \in (\tau_0, \infty)$). We can introduce a new independent variable, λ, via

$$\frac{\mathrm{d}s}{\mathrm{d}\lambda} = AB(2-A)(1-B)^2(1-C), \quad \lambda(s_1) = \lambda_0 < \infty, \tag{6.3.8}$$

where $s = s_1$ when $y - z = 0$ and $y - z < 0$ for $s > s_1$. Therefore the solution is confined to the state space $(A, B, C) \in (0,1)^3$ and $\frac{\mathrm{d}s}{\mathrm{d}\lambda} \geq 0$ for all $\lambda > \lambda_0$. Once we obtain the solution for A, B, C in terms of λ we will prove in §6.3.5 that λ does have the correct asymptotic behaviour i.e

$$\lambda(\infty) = \infty$$

We assume it does and state that $(A, B, C) \in (0,1)^3$ is valid for $\lambda \in (\lambda_0, \infty)$. We obtain the redefined dynamical system:

$$
\begin{aligned}
\frac{\mathrm{d}A}{\mathrm{d}\lambda} &= AB(1-A)(1-B)(1-C)(2-A)(1+(-2+A)B), && \text{(6.3.9a)}\\
\frac{\mathrm{d}B}{\mathrm{d}\lambda} &= -ABC(2-A)(1-B)^4, && \text{(6.3.9b)}\\
\frac{\mathrm{d}C}{\mathrm{d}\lambda} &= CB(1-B)(1-AB)(A^2-2A+2)(1-C)^2 && \text{(6 3.9c)}\\
&+ B^3(A^2-2A+2)(1-A)(1-C)^3 - 2AC^2(2-A)(1-B)^3(1-C).
\end{aligned}
$$

The right hand side of (6 3.9) is a polynomial in A, B, C so it is natural to smoothly extend the system to the side faces of the cube so we obtain the state space $\Omega = [0,1]^3$. The mapping

$$(x, y, z) \xrightarrow{\alpha} (A, B, C)$$

can be shown to be one to one and onto, and from (6.3.5),(6 3 6),(6.3.7) we know the following

$$
\begin{aligned}
A &\to 1 \text{ as } x \to \infty, & A \to 0 \text{ as } x \to 1,\\
B &\to 1 \text{ as } x - y \to \infty, & B \to 0 \text{ as } x - y \to 0,\\
C &\to 1 \text{ as } z - y \to \infty, & C \to 0 \text{ as } z - y \to 0,
\end{aligned}
$$

As we have applied a singular transformation (6.3.8) there exists the possibility that we have introduced spurious equilibrium sets, so although we know that $(x, y, z) \to (x_0, x_0, x_0)$ as $\tau \to \infty$ and this corresponds to $(A, B, C) \to (A_0, 0, 0)$ it is important to establish that none of the other equilibrium sets are approached. The system (6.3.9) has the following equilibrium sets:

$$
\begin{aligned}
&L_1 . (A_0, 0, 0), && L_2 : (A_0, 1, 1),\\
&L_3 . (1, B_0, 0), && L_4 \cdot (A_0, 0, 1),\\
&L_5 . (0, 0, C_0), && L_6 \cdot (0, B_0, 1),\\
&L_7 . (1, 1, C_0),
\end{aligned}
$$

where $(A_0, B_0, C_0) \in \Omega$. For convenience we will label each face of our state space,

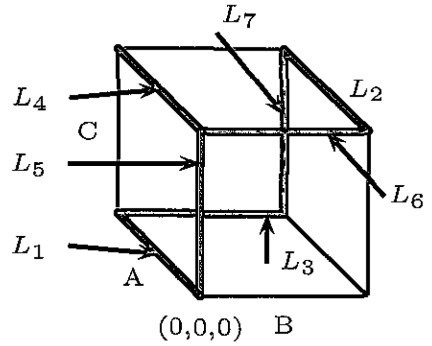

Figure 6.2. Diagram of compactified state space Ω.

note the edges are not included.

$$\begin{aligned}
F_1 &= \{A = 0, 0 < B < 1, 0 < C < 1\}, \\
F_2 &= \{A = 1, 0 < B < 1, 0 < C < 1\}, \\
F_3 &= \{B = 0, 0 < A < 1, 0 < C < 1\}, \\
F_4 &= \{B = 1, 0 < A < 1, 0 < C < 1\}, \\
F_5 &= \{C = 0, 0 < A < 1, 0 < B < 1\}, \\
F_6 &= \{C = 1, 0 < A < 1, 0 < B < 1\}.
\end{aligned}$$

6.3.3 Properties of the compactified system

1. From (6 3.9b) we see that
$$\frac{dB}{d\lambda} < 0 \quad \text{in} \quad \tilde{\Omega}.$$
Then we can apply the Monotonicity principle [41] to show that there are no equilibrium points in the interior of the state space, which we will refer to as $\tilde{\Omega} = (0,1)^3$ We note that the following sets are invariant, $A = 0$, $A = 1$, $B = 0$, $B = 1$ and $C = 1$. The set $C = 0$ is reflecting, i.e.,
$$\left.\frac{dC}{d\lambda}\right|_{C=0} > 0.$$

Thus, any trajectory that enters Ω cannot leave Ω and must terminate at a boundary equilibrium point as $\lambda \to \infty$.

2. B is a monotonically decreasing function, $\frac{dB}{d\lambda} < 0 \Rightarrow B \nrightarrow 1$, this rules out L_2 and L_7 and implies that $B_0 \in [0, 1)$ for L_3 and L_6.

3. Let $f = 1 - 2B + AB$. If $A \to 0 \Rightarrow \frac{dA}{d\lambda} < 0 \Leftrightarrow f < 0 \Leftrightarrow B > \frac{1}{2-A}$, this rules out L_5 and $L_6|_{B<\frac{1}{2}}$ and implies that $A_0 \in (0, 1]$ in L_1 and L_4.

4. To rule out $L_3 : (1, B_0, 0)$ and $L_4 : (A_0, 0, 1)$, we first prove that no interior solution $((A, B, C) \in \tilde{\Omega} = (0, 1)^3)$ can terminate on the faces

$$F_5 : \{C = 0, 0 < A < 1, 0 < B < 1\} \quad \text{or} \quad F_6 : \{C = 1, 0 < A < 1, 0 < B < 1\}.$$

To do this we write

$$(1 - C)^{-3} \frac{dC}{d\lambda} = \beta_1 \vartheta^2 + \beta_2 \vartheta + \beta_3,$$

where $\vartheta = \frac{C}{1-C}$

$$\begin{aligned}
\beta_1 &= -2A(2 - A)(1 - B)^3 < 0, \\
\beta_2 &= B(2 - 2A + A^2)(1 - B)(1 - AB) > 0, \\
\beta_3 &= B^3(2 - 2A + A^2)(1 - A) > 0,
\end{aligned}$$

for $(A, B, C) \in \tilde{\Omega} = (0, 1)^3$. Then $\frac{dC}{d\lambda} = 0$ iff $\vartheta = \vartheta_1, \vartheta_2$ where

$$\begin{aligned}
\vartheta_1 &= \frac{-\beta_2 - \sqrt{\beta_2^2 - 4\beta_1\beta_3}}{2\beta_1} > 0, \\
\vartheta_2 &= \frac{-\beta_2 + \sqrt{\beta_2^2 - 4\beta_1\beta_3}}{2\beta_1} < 0.
\end{aligned}$$

But $C_i = \frac{\vartheta_i}{1+\vartheta_i}$ so $C_1 = \frac{\vartheta_1}{1+\vartheta_1} \in (0, 1)$ and $C_2 = \frac{\vartheta_2}{1+\vartheta_2} \notin (0, 1)$ so $C = C_2$ is not of relevance. So we have that $\frac{dC}{d\lambda} = 0$ in $\tilde{\Omega} \Leftrightarrow C = C_1 = C_1(A, B)$. But

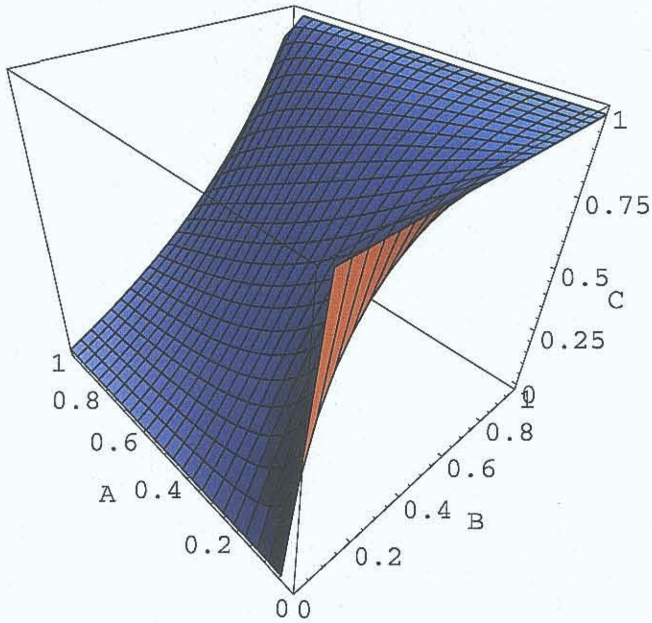

Figure 6.3: The surface $C = C_1$.

$$
\begin{aligned}
\left.\frac{\mathrm{d}C}{\mathrm{d}\lambda}\right|_{C=0} &> 0 \\
\Rightarrow \frac{\mathrm{d}C}{\mathrm{d}\lambda} &> 0 \Leftrightarrow C < C_1, \\
\Rightarrow \frac{\mathrm{d}C}{\mathrm{d}\lambda} &< 0 \Leftrightarrow C > C_1.
\end{aligned}
$$

Therefore a solution from $\tilde{\Omega}$ cannot terminate on F_5 or F_6.

5. The surface $C = C_1$ divides $\tilde{\Omega}$ into two distinct regions:

$$
\begin{aligned}
R^- \quad &\text{where} \quad \frac{\mathrm{d}C}{\mathrm{d}\lambda} < 0, \\
R^+ \quad &\text{where} \quad \frac{\mathrm{d}C}{\mathrm{d}\lambda} > 0,
\end{aligned}
$$

where we can show that:

R^- is the region bounded by the faces F_3, F_6, the surface $C = C_1$ and the part of the face F_2 for which $g(B, C) = B - 2C + BC < 0$.
R^+ is the region bounded by the faces F_5, F_4, F_1, the surface $C = C_1$ and the part of the face F_2 for which $g(B, C) = B - 2C + BC > 0$.

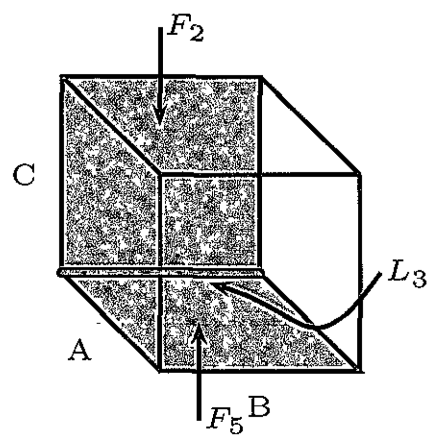

Figure 6.4. Equilibrium set L_3.

6. We can prove that $L_3 : (1, B_0, 0)$, $B_0 \in (0, 1)$ cannot be reached by an interior solution. We are not considering the endpoints of L_3, $B_0 = 0$ as this is contained in the line L_1 which is allowed, or $B_0 = 1$ as this cannot be reached by monotonicity.
Assume an interior solution has an ω-limit point on L_3 at say, p, $p = (1, p_B, 0)$. Consider the set of points, $\mathcal{B}$, lying in the ball of radius $\delta > 0$, centred at p and the cube Ω, i e ,

$$\mathcal{B} = B(p, \delta) \cap \Omega.$$

Define

$$\mathcal{B}_1 = \mathcal{B} \cap \tilde{\Omega}, \quad \mathcal{B}_2 = \mathcal{B} \cap F_2, \quad \mathcal{B}_3 = \mathcal{B} \cap F_5, \quad \mathcal{B}_4 = \mathcal{B} \cap L_3,$$

so that

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4.$$

Then,

$$\begin{aligned}
\frac{\mathrm{d}C}{\mathrm{d}\lambda} &> 0 \text{ on } \mathcal{B}_1 \text{ as } \mathcal{B}_1 \in R^+, \\
\frac{\mathrm{d}C}{\mathrm{d}\lambda} &= B^3(2 - 4A + 3A^2 - A^3) > 0 \text{ on } \mathcal{B}_2, \\
\frac{\mathrm{d}C}{\mathrm{d}\lambda} &= C(1-C)(1-B)^2(B - 2C + BC) \text{ on } \mathcal{B}_3,
\end{aligned}$$

But if we consider a point on the circumference of the semicircle $\mathcal{B}_3$ it has coordinates $(B, C) = (p_B + \delta\cos\theta, \delta\sin\theta)$ where $\theta \in (0, \pi)$, then we can write

$$(B - 2C + BC) = p_B + \delta\left(\cos\theta - 2\sin\theta + p_B\sin\theta + \delta\cos\theta\sin\theta\right),$$

and as we can choose $\delta > 0$ sufficiently small so that

$$\begin{aligned}
(B - 2C + BC) &> 0 \text{ on } \mathcal{B}_3, \\
\Rightarrow \frac{\mathrm{d}C}{\mathrm{d}\lambda} &> 0 \text{ on } \mathcal{B}_3, \\
\Rightarrow C &\nrightarrow 0 \quad \text{through} \quad \mathcal{B},
\end{aligned}$$

and so no interior solution can reach p.

7. We can prove that $L_4 : (A_0, 0, 1)$, $A_0 \in (0, 1]$ cannot be reached by an interior solution.

Case 1. $A_0 \in (0, 1)$. Assume an interior solution has an ω-limit point on L_4 at say, p, $p = (p_A, 0, 1)$. Consider the set of points, $\mathcal{B}$, lying in the ball of radius $\delta > 0$, centred at p and the cube Ω, i.e.,

$$\mathcal{B} = B(p, \delta) \cap \Omega.$$

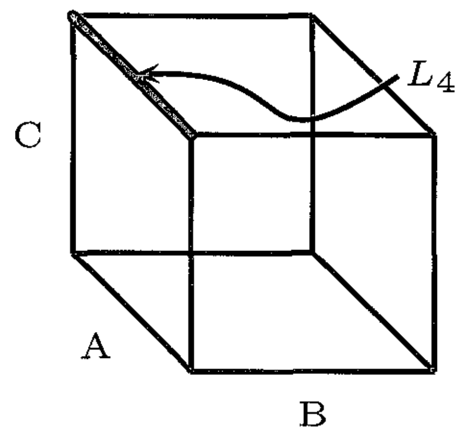

Figure 6.5: Equilibrium set L_4.

Define

$$\mathcal{B}_1 = \mathcal{B} \cap \tilde{\Omega}, \quad \mathcal{B}_2 = \mathcal{B} \cap F_3, \quad \mathcal{B}_3 = \mathcal{B} \cap F_6, \quad \mathcal{B}_4 = \mathcal{B} \cap L_4,$$

so that

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4.$$

Then,

$$\begin{aligned}
\frac{\mathrm{d}C}{\mathrm{d}\lambda} &< 0 \quad \text{on} \quad \mathcal{B}_1 \quad \text{as} \quad \mathcal{B}_1 \in R^-, \\
\frac{\mathrm{d}C}{\mathrm{d}\lambda} &= -2AC^2(2-A)(1-C) < 0 \quad \text{on} \quad \mathcal{B}_2, \\
\frac{\mathrm{d}C}{\mathrm{d}\lambda} &= 0 \quad \text{on} \quad \mathcal{B}_3, \\
\Rightarrow C &\nrightarrow 1 \quad \text{through} \quad \mathcal{B}.
\end{aligned}$$

Therefore no interior solution can reach p.

Case 2. $A_0 = 1$. Consider the endpoint of $L_4 \cdot (A, B, C) = (1, 0, 1)$. Assume an interior solution has an ω-limit point on $p = (1, 0, 1)$. Consider the set of points, $\mathcal{B}$, lying in the ball of radius $\delta > 0$, centred at p and the cube

Ω, i.e.,

$$\mathcal{B} = B(p, \delta) \cap \Omega.$$

Define

$$\mathcal{B}_1 = \mathcal{B} \cap \tilde{\Omega}, \quad \mathcal{B}_2 = \mathcal{B} \cap F_3, \quad \mathcal{B}_3 = \mathcal{B} \cap F_6, \quad \mathcal{B}_4 = \mathcal{B} \cap F_2, \quad \mathcal{B}_5 = \mathcal{B} \cap p$$

so that

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4 \cup \mathcal{B}_5.$$

Then,

$$\begin{aligned}
\frac{\mathrm{d}C}{\mathrm{d}\lambda} &< 0 \quad \text{on} \quad \mathcal{B}_1 \quad \text{as} \quad \mathcal{B}_1 \in R^-, \\
\frac{\mathrm{d}C}{\mathrm{d}\lambda} &= -2AC^2(2-A)(1-C) < 0 \quad \text{on} \quad \mathcal{B}_2, \\
\frac{\mathrm{d}C}{\mathrm{d}\lambda} &= 0 \quad \text{on} \quad \mathcal{B}_3, \\
\frac{\mathrm{d}C}{\mathrm{d}\lambda} &= C(1-C)(1-B)^2(B-2C+BC) < 0 \quad \text{on} \quad \mathcal{B}_4 \\
& \quad \text{as} \quad (B-2C+BC) < 0 \quad \text{close to} \quad p, \\
\Rightarrow C &\nrightarrow 1 \quad \text{through} \quad \mathcal{B}.
\end{aligned}$$

Therefore no interior solution can reach p.

Proposition 6.3.2. *The system* (6.3.9) *can only have a solution approaching* L_1 *(with* $A_0 \in (0,1]$*) or* L_6 *(with* $B_0 \geq \frac{1}{2}$*) as* $\lambda \to \infty$.

Proof. We use the properties we derived above: Property 1 proves that there are no equilibrium points in the interior of the state space, Property 2 rules out L_2 and L_7, Property 3 rules out L_5 and L_6 (for $B_0 < \frac{1}{2}$) and proves that $A_0 \in (0,1]$ in L_1, Property 6 rules out L_3 and finally Property 7 rules out L_4. Thus, we can only have a solution approaching L_1 (with $A_0 \in (0,1]$) or L_6 (with $B_0 \geq \frac{1}{2}$). □

The equilibrium set $L_6|_{B_0 \geq \frac{1}{2}}$ in the original x, y, z variables corresponds to

$$(x, y, z) \to (1, \hat{y}, \infty) \quad \text{where} \quad -\infty < \hat{y} < 0.$$

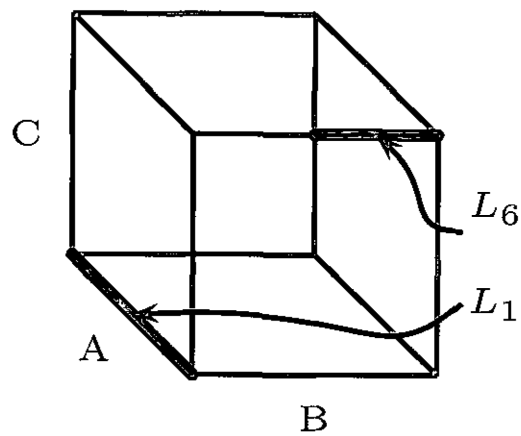

Figure 6.6: Equilibrium sets L_1 and L_6

From the numerics it is clear that this point cannot be reached. The initial values for (A, B, C) at $\lambda = \lambda(\tau_0) = \lambda_0$ are given by the numerical values for (x, y, z) at the point $\lambda = \lambda_0$ and they are approximately

$$(A, B, C) \sim (0.33, 0.23, 0)$$

However, B is monotonically decreasing for $\lambda > \lambda_0$ so we cannot reach the equilibrium point with $B_0 \in [\frac{1}{2}, 1)$. This rules out L_6 on the basis of numerical evidence

6.3.4 Analysis of L_1 using polar-blow up method

We have deduced that the only equilibrium point that our solution can approach is L_1, which corresponds to

$$(x, y, z) \to (\hat{x}, \hat{x}, \hat{x}), \quad \hat{x} \in (1, \infty).$$

When we linearise (6 2.1) about this equilibrium line we find that it is non-hyperbolic (with three zero eigenvalues), so centre manifold analysis is not applicable. In our new compactified coordinates we also have that L_1 is a non-hyperbolic equilibrium set with three zero eigenvalues. We use a polar blowing up technique. Define cylindrical coordinates centred at $L_1 : (A_0, 0, 0)$, $A_0 \in (0, 1]$

$$r = \sqrt{B^2 + C^2}, \quad \theta = \arctan\left(\frac{C}{B}\right), \quad z = A,$$

where $\theta \in (0, \frac{\pi}{2})$, $r \in (0, r_{max})$ and $z \in (0, 1]$. We can choose $r_{max} = 0.4$ for convenience. We can let

$$\frac{d\lambda}{d\tilde{\lambda}} = \frac{1}{r},$$

to obtain an equivalent system in $\tilde{\lambda}$

$$\frac{dr}{d\tilde{\lambda}} = r \sin\theta \left(f_{r,0}(\theta, z) + r f_{r,1}(\theta, z) + r^2 f_{r,2}(\theta, z) + r^3 f_{r,3}(\theta, z) + r^4 f_{r,4}(\theta, z) \right), \tag{6.3.10a}$$

$$\frac{d\theta}{d\tilde{\lambda}} = \cos\theta \left(f_{\theta,0}(\theta, z) + r f_{\theta,1}(\theta, z) + r^2 f_{\theta,2}(\theta, z) + r^3 f_{\theta,3}(\theta, z) + r^4 f_{\theta,4}(\theta, z) \right), \tag{6.3.10b}$$

$$\frac{dz}{d\tilde{\lambda}} = z \cos\theta \left(f_{z,0}(\theta, z) + r f_{z,1}(\theta, z) + r^2 f_{z,2}(\theta, z) + r^3 f_{z,3}(\theta, z) \right). \tag{6.3.10c}$$

Where $f_{i,j}(\theta, z)$, $i = (r, \theta, z), j = 1, 2, \cdots$, are polynomial in z, $\cos\theta$ and $\sin\theta$, so our system (6.3.10) can be smoothly extended to include $r = 0$, $\theta = 0$ and $\theta = \frac{\pi}{2}$. The state space, S, for this dynamical system is now the cuboid $\Lambda : [0, r_{max}) \times [0, \frac{\pi}{2}] \times [0, 1]$ By construction, the subset $r = 0$ can be regarded as a blow up of the fixed line L_1. Also $r = 0$, $\theta = \frac{\pi}{2}$, $z = 0$ and $z = 1$ are invariant subsets.

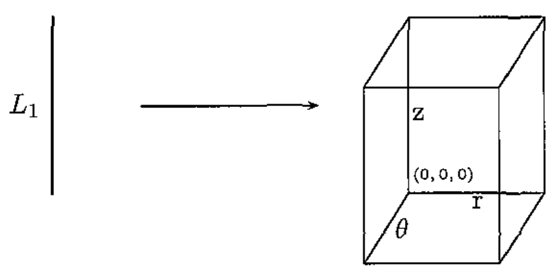

Figure 6 7: Polar blow-up coordinate change.

	(r,θ,z)	Eigenvalues	Eigenvectors
L_1	$(0,\frac{\pi}{2},z_0)$	$z_0(2-z_0)$	$(0,-1,1-z_0)$
	$z_0 \in (0,1]$	$-2z_0(2-z_0)$	$(1,0,0)$
		0	$(0,0,1)$
P_1	$(0,\frac{\pi}{4},1)$	$\frac{-1}{\sqrt{2}}$	$(1,0,0)$
		$\frac{-1}{\sqrt{2}}$	$(0,1,0)$
		$\frac{-1}{\sqrt{2}}$	$(0,0,1)$
P_2	$(0,0,1)$	1	$(0,1,0)$
		-1	$(0,0,1)$
		0	$(1,0,0)$
P_3	$(0,0,0)$	2	$(0,0,1)$
		2	$(0,1,0)$
		0	$(-1,1,0)$

Table 6.2: Eigenvalues and Eigenvectors of L_1, P_1, P_2 and P_3

Stability analysis of polar blow up equilibrium sets

An equilibrium point analysis of $r = 0$ reveals the following four equilibrium sets and the eigenvalues and eigenvectors corresponding to their linearisations, see Table 6.2. L_1 for $z_0 \in (0,1]$ is a non-hyperbolic equilibrium set, it has a 1-dim stable subspace, a 1-dim unstable subspace and a 1-dim centre manifold along the z direction (the centre manifold *is* L_1). We will show in Lemma 6.3 3 that no interior solution can terminate on this line. P_2 is a non-hyperbolic equilibrium point, it has a positive eigenvalues along the θ direction, a negative eigenvalue tangent to the z axis and a 1-dim centre manifold is tangent to the r axis. We will show in Lemma 6.3.4 that no interior solution can terminate on P_2 P_3 is also a non-hyperbolic equilibrium point, it has two positive eigenvalues tangent to the θ and z directions, and a centre manifold tangent to the r direction. We will show in Lemma 6.3.5 that no interior solution can terminate on P_3. In order to examine these equilibrium sets we consider our state space Λ,

$$\Lambda = [0, r_{max}] \times \left[0, \frac{\pi}{2}\right] \times [0,1],$$

we can define the interior of this cuboid as

$$\tilde{\Lambda} = (0, r_{max}) \times \left(0, \frac{\pi}{2}\right) \times (0, 1).$$

Next we define the faces and edges as follows,

$$\begin{aligned}
F_1 =& \{r = 0, 0 < \theta < \frac{\pi}{2}, 0 < z < 1\}, & L_1 =& \left(0, \frac{\pi}{2}, z_0\right), \\
F_2 =& \{r = r_m, 0 < \theta < \frac{\pi}{2}, 0 < z < 1\}, & L_2 =& (0, 0, z_0), \\
F_3 =& \{\theta = 0, 0 < r < r_{max}, 0 < z < 1\}, & L_3 =& (0, \theta_0, 0), \\
F_4 =& \{\theta = \frac{\pi}{2}, 0 < r < r_{max}, 0 < z < 1\}, & L_4 =& (0, \theta_0, 1), \\
F_5 =& \{z = 0, 0 < \theta < \frac{\pi}{2}, 0 < r < r_{max}\}, & L_5 =& (r_0, 0, 0), \\
F_6 =& \{z = 1, 0 < \theta < \frac{\pi}{2}, 0 < r < r_{max}\}, & L_6 =& \left(r_0, \frac{\pi}{2}, 0\right), \\
L_7 =& (r_0, 0, 1), & L_8 =& \left(r_0, \frac{\pi}{2}, 1\right),
\end{aligned}$$

For completion we write out (6.3 10b) and (6.3.10c) in detail:

$$\begin{aligned}
\frac{d\theta}{d\tilde{\lambda}} &= r(2-z)\sin\theta(-z\cos\theta^2(r\cos\theta-1)^4 - \frac{1}{z-2}((r\sin\theta-1)(-2(z-2)\sin\theta^2 + \\
&\quad \cos\theta\sin\theta(-2+2z-z^2+r\sin\theta(2-14z+7z^2)) \\
&\quad -r\cos^2\theta\sin\theta(-2+2z^2-z^3+r\sin\theta(2-12z+5z^2+z^3)) + \\
&\quad r\cos^3\theta(-2+4z-3z^2+z^3-r\sin\theta(-4+10z-8z^2+3z^3) + \\
&\quad +r^2\sin^2\theta(-2+2z-3z^2+2z^3))))), \\
\frac{dz}{d\tilde{\lambda}} &= z(z-2)(z-1)\cos\theta(r\cos\theta-1)(r\sin\theta-1)(1+r(-2+z)\cos\theta).
\end{aligned}$$

Lemma 6.3.3. *An interior solution cannot have an ω-limit point on L_1*

Proof. **Case 1.** $z_0 = 1$ Assume an interior solution has an ω-limit point at the endpoint, p, of L_1 where $p = (0, \frac{\pi}{2}, 1)$. Consider the set of points, $\mathcal{B}$, lying in the ball of radius $\delta > 0$, centred at p and the cuboid Λ, i.e ,

$$\mathcal{B} = B(p, \delta) \cap \Lambda.$$

Define

$$\begin{aligned}\mathcal{B}_1 &= \mathcal{B}\cap\tilde{\Lambda}, \quad \mathcal{B}_2=\mathcal{B}\cap F_1, \quad \mathcal{B}_3=\mathcal{B}\cap F_4, \quad \mathcal{B}_4=\mathcal{B}\cap F_6,\\ \mathcal{B}_5 &= \mathcal{B}\cap L_1, \quad \mathcal{B}_6=\mathcal{B}\cap L_4, \quad \mathcal{B}_7=\mathcal{B}\cap L_8,\end{aligned}$$

so that

$$\mathcal{B}=\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_3\cup\mathcal{B}_4\cup\mathcal{B}_5\cup\mathcal{B}_6\cup\mathcal{B}_7.$$

We will consider $\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}}$ for each $\mathcal{B}_i$, where $i=1-7$. First we have

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}}=0 \quad \text{on} \quad \mathcal{B}_3,\mathcal{B}_5,\mathcal{B}_7,$$

so θ cannot reach $\frac{\pi}{2}$ along these surfaces, unless $\theta=\frac{\pi}{2}$ for some $\tilde{\lambda}_1<\infty$, which is not possible as $\theta=\frac{\pi}{2}$ is an invariant manifold. Next we look at

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}}=\cos\theta\sin\theta(\cos\theta-\sin\theta) \quad \text{on} \quad \mathcal{B}_6,$$

but for $\theta\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)$, which holds for $\delta>0$ sufficiently small, we have that,

$$\begin{aligned}\cos\theta\sin\theta(\cos\theta-\sin\theta)<0,\\ \Rightarrow\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}}<0 \quad \text{on} \quad \mathcal{B}_6,\\ \theta\nrightarrow\frac{\pi}{2} \quad \text{through} \quad \mathcal{B}_6.\end{aligned}$$

Next we look at

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}}=\frac{\cos\theta}{4}\left((r\cos\theta-1)^2(\cos2\theta+\sin2\theta-1)(2-4r\sin\theta+r^2\sin2\theta)\right) \quad \text{on} \quad \mathcal{B}_4,$$

we can show that

$$\frac{\cos\theta}{4}\left((r\cos\theta-1)^2(\cos2\theta+\sin2\theta-1)(2-4r\sin\theta+r^2\sin2\theta)\right)<0,$$

for $\theta\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)$ and $r\in(0,0.4)$, which holds for $\delta>0$ sufficiently small, we have the *following*

$$\begin{aligned}
\frac{\cos\theta}{4}(r\cos\theta-1)^2(\cos 2\theta+\sin 2\theta-1) &< 0,\\
2-4r\sin\theta+r^2\sin 2\theta > 2-4r\sin\theta &> 0,\\
\Rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} &< 0,\\
\Rightarrow \theta \nrightarrow \frac{\pi}{2} \quad \text{through} \quad \mathcal{B}_4
\end{aligned}$$

Next we consider

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} = \cos\theta\sin\theta\left((z^2-2z+2)\cos\theta+z(z-2)\sin\theta\right) \quad \text{on} \quad \mathcal{B}_2,$$

we can show that

$$\cos\theta\sin\theta\left((z^2-2z+2)\cos\theta+z(z-2)\sin\theta\right) < 0,$$

by considering a point on the circular boundary of $\mathcal{B}_2$ (which we define as the boundary of $\mathcal{B}_2$ less L_1 and L_4 i.e. $\partial\mathcal{B}_2\backslash\{L_1\cup L_4\}$) which can be written in polar coordinates, (δ,ψ), where

$$\frac{\pi}{2}-\theta=\delta\cos\psi, \quad 1-z=\delta\sin\psi \quad \text{for} \quad \psi\in\left(0,\frac{\pi}{2}\right)$$

as follows

$$\begin{aligned}
(z^2-2z+2)\cos\theta+z(z-2)\sin\theta &= -1+\delta\cos\psi+\mathcal{O}(\delta^2),\\
\cos\theta\sin\theta &= \delta\cos\psi+\mathcal{O}(\delta^3),\\
\Rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} &= -\delta\cos\psi+\delta^2\cos^2\psi+\mathcal{O}(\delta^3)
\end{aligned}$$

and we can choose $\delta_1>0$ sufficiently small so that for all $\delta\in(0,\delta_1)$,

$$\begin{aligned}
\Rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} &< 0,\\
\Rightarrow \theta &\nrightarrow \frac{\pi}{2} \quad \text{through} \quad \mathcal{B}_2,
\end{aligned}$$

where we have used the identities

$$\begin{aligned}\cos(A+B) &= \cos A\cos B - \sin A\sin B,\\ \sin(A+B) &= \sin A\cos B + \cos A\sin B,\end{aligned}$$

and the approximations

$$\cos(A) \approx 1 - \frac{A^2}{2}, \quad \sin(A) \approx A - \frac{A^3}{6}, \tag{6.3.11}$$

which are valid for A close to zero. Finally we use a Taylor expansion for θ about $\frac{\pi}{2}$ to get

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} = f_1(r,z)\left(\frac{\pi}{2}-\theta\right) + f_2(r,z)\left(\frac{\pi}{2}-\theta\right)^2 + \mathcal{O}\left(\left(\frac{\pi}{2}-\theta\right)^3\right) \quad \text{on} \quad \mathcal{B}_1,$$

where

$$\begin{aligned}f_1(r,z) &= -z(z-2)(2r-1),\\ f_2(r,z) &= (2-2z+z^2-4r(z-1)^2+r^2(2-14z+7z^2))\end{aligned}$$

But if we consider a point on the spherical surface of $\mathcal{B}_1$ (which we define as $\partial\mathcal{B}_1\backslash\{F_1 \cup F_4 \cup F_6\}$), which can be written in spherical polar coordinates, (δ,ψ,ϕ), where

$$r = \delta\cos\psi\sin\phi, \quad \left(\frac{\pi}{2}-\theta\right) = \delta\sin\psi\sin\phi, \quad 1-z = \delta\cos\phi,$$

and $(\phi,\psi) \in (0,\frac{\pi}{2})$, then we can write

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} = \delta\sin\psi\sin\phi\left(-1+\delta\left(2\cos\psi+\sin\psi\right)\sin\phi + \mathcal{O}(\delta^2)\right),$$

then there exists $\delta_2 > 0$ such that for all $\delta \in (0,\delta_2)$

$$\begin{aligned}\left(-1+\delta\left(2\cos\psi+\sin\psi\right)\sin\phi+\mathcal{O}(\delta^2)\right) &< 0,\\ \Rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} &< 0 \quad \text{on} \quad \mathcal{B}_1,\\ \Rightarrow \theta &\nrightarrow \frac{\pi}{2} \quad \text{through} \quad \mathcal{B}_1\end{aligned}$$

Therefore no interior solution can reach p.

Case 2: $z_0 \in (0,1)$ Assume an interior solution has an ω-limit point on L_1 at say, $p = (0, \frac{\pi}{2}, z_L)$, where $z_L \in (0,1)$. Consider the set of points, $\mathcal{B}$, lying in the ball of radius $\delta > 0$, centred at p and the cuboid Λ, i.e.,

$$\mathcal{B} = B(p, \delta) \cap \Lambda.$$

Define

$$\mathcal{B}_1 = \mathcal{B} \cap \tilde{\Lambda}, \quad \mathcal{B}_2 = \mathcal{B} \cap F_1, \quad \mathcal{B}_3 = \mathcal{B} \cap F_4, \quad \mathcal{B}_4 = \mathcal{B} \cap L_1$$

so that

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4.$$

We will consider $\frac{d\theta}{d\tilde{\lambda}}$ for each $\mathcal{B}_i$, where $i = 1 - 4$:

$$\frac{d\theta}{d\tilde{\lambda}} = 0 \quad \text{on} \quad \mathcal{B}_3, \mathcal{B}_4,$$

so θ cannot reach $\frac{\pi}{2}$ along these surfaces, unless $\theta = \frac{\pi}{2}$ for some $\tilde{\lambda}_1 < \infty$ which is not possible as this is an invariant manifold. Next we consider

$$\frac{d\theta}{d\tilde{\lambda}} = \cos\theta \sin\theta \left((z^2 - 2z + 2)\cos\theta + z(z-2)\sin\theta\right) \quad \text{on} \quad \mathcal{B}_2,$$

we can show that

$$\cos\theta \sin\theta \left((z^2 - 2z + 2)\cos\theta + z(z-2)\sin\theta\right) < 0,$$

by considering a point on the circular boundary of $\mathcal{B}_2$, which can be written in polar coordinates, (δ, ψ), where

$$\frac{\pi}{2} - \theta = \delta \sin\psi, \quad z_L - z = \delta \cos\psi \quad \text{for} \quad \psi \in (0, \pi)$$

as follows

$$\begin{aligned}(z^2-2z+2)\cos\theta + z(z-2)\sin\theta &= z_L(z_L-2)\\ &+ \delta(2(z_L-1)\cos\psi + 2\sin\psi) + \mathcal{O}(\delta^2),\\ \cos\theta\sin\theta &= \delta\sin\psi + \mathcal{O}(\delta^3),\end{aligned}$$

$$\Rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} = \delta z_L(z_L-2)\sin\psi + \delta^2\sin\psi(2(z_L-1)\cos\psi + 2\sin\psi) + \mathcal{O}(\delta^3),$$

and we can choose $\delta_1 > 0$ sufficiently small so that for all $\delta \in (0, \delta_1)$,

$$\begin{aligned}\Rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} &< 0,\\ \Rightarrow \theta &\nrightarrow \frac{\pi}{2} \quad \text{through} \quad \mathcal{B}_2.\end{aligned}$$

Finally we use a Taylor expansion for θ about $\frac{\pi}{2}$ to get

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} = f_1(r,z)\left(\frac{\pi}{2}-\theta\right) + f_2(r,z)\left(\frac{\pi}{2}-\theta\right)^2 + \mathcal{O}\left(\left(\frac{\pi}{2}-\theta\right)^3\right) \quad \text{on} \quad \mathcal{B}_1,$$

where

$$\begin{aligned}f_1(r,z) &= -z(z-2)(2r-1),\\ f_2(r,z) &= \left(2-2z+z^2-4r(z-1)^2+r^2(2-14z+7z^2)\right)\end{aligned}$$

But if we consider a point on the spherical surface of $\mathcal{B}_1$, which can be written in spherical polar coordinates, (δ, ψ, ϕ), where

$$r = \delta\cos\psi\sin\phi, \quad \left(\frac{\pi}{2}-\theta\right) = \delta\sin\psi\sin\phi, \quad z_L - z = \delta\cos\phi$$

and $\phi \in (0, \pi)$, $\psi \in (0, \frac{\pi}{2})$, then we can write

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} &= \delta\sin\psi\sin\phi\left(-2z_L + z_L^2 + \delta\left(2z_L(z_L-1)\cos\phi\right.\right.\\ &\quad \left.\left. + \left(2z_L(z_L-1)\cos\psi + (z_L^2-2z_L+2)\sin\psi\right)\sin\phi\right) + \mathcal{O}(\delta^2)\right),\\ &= \delta\sin\psi\sin\phi\left(z_L(z_L-2) + \delta g_1(\psi,\phi,z_L) + \mathcal{O}(\delta^2)\right),\end{aligned}$$

then there exists $\delta_1 > 0$ such that for all $\delta \in (0, \delta_1)$,

$$\begin{aligned} \left(z_L(z_L - 2) + \delta g_1(\psi, \phi, z_L) + \mathcal{O}(\delta^2)\right) &< 0, \\ \Rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} &< 0 \quad \text{on} \quad \mathcal{B}_1, \\ \Rightarrow \theta &\nrightarrow \frac{\pi}{2} \quad \text{through} \quad \mathcal{B}_1 \end{aligned}$$

Therefore no interior solution can reach p □

Lemma 6.3.4. *An interior solution cannot have an ω-limit point on P_2.*

Proof. Assume an interior solution has an ω-limit point at P_2 where $P_2 = (0, 0, 1)$. Consider the set of points, $\mathcal{B}$, lying in the ball of radius $\delta > 0$, centred at P_2 and the cuboid Λ, i e.,

$$\mathcal{B} = B(P_2, \delta) \cap \Lambda$$

Define

$$\begin{aligned} \mathcal{B}_1 &= \mathcal{B} \cap \tilde{\Lambda}, \quad \mathcal{B}_2 = \mathcal{B} \cap F_1, \quad \mathcal{B}_3 = \mathcal{B} \cap F_3, \quad \mathcal{B}_4 = \mathcal{B} \cap F_6, \\ \mathcal{B}_5 &= \mathcal{B} \cap L_2 \quad \mathcal{B}_6 = \mathcal{B} \cap L_4, \quad \mathcal{B}_7 = \mathcal{B} \cap L_7, \end{aligned}$$

so that

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4 \cup \mathcal{B}_5 \cup \mathcal{B}_6 \cup \mathcal{B}_7.$$

We will consider $\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}}$ for each $\mathcal{B}_i$, where $i = 1 - 7$: Firstly

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} = 0 \quad \text{on} \quad \mathcal{B}_5, \mathcal{B}_7,$$

so θ cannot reach 0 along these surfaces, unless $\theta = 0$ and $z = 1$, or $\theta = 0$ and $r = 0$ for some $\tilde{\lambda}_1 < \infty$, which is not possible as $z = 1$ and $r = 0$ are invariant manifolds. Next

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} = \frac{\cos\theta}{4}\left((r\cos\theta - 1)^2(\cos 2\theta + \sin 2\theta - 1)(2 - 4r\sin\theta + r^2\sin 2\theta)\right) \quad \text{on} \quad \mathcal{B}_4,$$

we can choose $\delta > 0$ sufficiently small so that $\theta \in \left(0, \frac{\pi}{4}\right)$ and $r \in (0, 0\,4)$, then we have that

$$\begin{aligned}
\frac{\cos\theta}{4}(r\cos\theta-1)^2(\cos 2\theta+\sin 2\theta-1) &> 0, \\
2-4r\sin\theta+r^2\sin 2\theta &> 2-4r\sin\theta>0, \\
\Rightarrow\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} &> 0.
\end{aligned}$$

Next we can show that,

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}}=\cos\theta\sin\theta(2\cos\theta+z(z-2)(\cos\theta+\sin\theta))>0 \quad \text{on} \quad \mathcal{B}_2,$$

by considering a point on the circular boundary of $\mathcal{B}_2$, which can be written in polar coordinates, (δ,ψ), where

$$\theta=\delta\cos\psi, \quad 1-z=\delta\sin\psi \quad \text{for} \quad \psi\in\left(0,\frac{\pi}{2}\right)$$

as follows

$$\begin{aligned}
(z^2-2z+2)\cos\theta+z(z-2)\sin\theta &= 1-\delta\cos\psi+\mathcal{O}(\delta^2), \\
\cos\theta\sin\theta &= \delta\cos\psi+\mathcal{O}(\delta^3), \\
\Rightarrow\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} &= \delta\cos\psi-\delta^2\cos^2\psi+\mathcal{O}(\delta^3)
\end{aligned}$$

and we can choose $\delta_1>0$ sufficiently small so that for all $\delta\in(0,\delta_1)$,

$$\begin{aligned}
\Rightarrow\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} &> 0, \\
\Rightarrow\theta &\nrightarrow 0 \quad \text{through} \quad \mathcal{B}_2,
\end{aligned}$$

where we have used the approximations (6 3.11) which are valid for A close to zero. Next we consider

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}}=r(1-z)(2-2z+z^2) \quad \text{on} \quad \mathcal{B}_3,$$

but for $z \in (0,1)$ and $r > 0$ we have that

$$r(1-z)(2-2z+z^2) = r(1-z)((1-z)^2+1) > 0,$$
$$\Rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} > 0 \quad \text{on} \quad \mathcal{B}_3, \quad \Rightarrow \theta \nrightarrow 0 \quad \text{through} \quad \mathcal{B}_3$$

Next we consider
$$\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} = \cos\theta\sin\theta(\cos\theta - \sin\theta) \quad \text{on} \quad \mathcal{B}_6,$$
but we can choose $\delta > 0$ sufficiently small so that $\theta \in (0, \frac{\pi}{4})$ and therefore

$$\cos\theta\sin\theta(\cos\theta - \sin\theta) > 0,$$
$$\Rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} > 0 \quad \text{on} \quad \mathcal{B}_6, \quad \Rightarrow \theta \nrightarrow 0 \quad \text{through} \quad \mathcal{B}_6.$$

Finally we use a Taylor expansion for θ about 0 to get

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} = f_0(r,z) + f_1(r,z)\theta + f_2(r,z)\theta^2 + \mathcal{O}(\theta^3) \quad \text{on} \quad \mathcal{B}_1,$$

where

$$\begin{aligned} f_0(r,z) &= r(2-4z+3z^2-z^3), \\ f_1(r,z) &= (2-2z+z^2)(1-r(1+z)+r^2(4z-3)), \\ f_2(r,z) &= -1+(1-z)^2+\mathcal{O}(r), \end{aligned}$$

if we consider a point on the spherical surface of $\mathcal{B}_1$, which can be written in spherical polar coordinates, (δ,ψ,ϕ), where

$$r = \delta\cos\psi\sin\phi, \quad \theta = \delta\sin\psi\sin\phi, \quad 1-z = \delta\cos\phi,$$

and $(\phi,\psi) \in (0, \frac{\pi}{2})$, then we can write

$$\begin{aligned} \frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} &= \delta\sin\psi\sin\phi + \delta^2\sin\phi\left(\cos\psi\cos\phi - \sin^2\psi\sin\phi - 2\cos\psi\sin\psi\sin\phi\right) + \mathcal{O}(\delta^3), \\ &= \delta\sin\psi\sin\phi + \delta^2 g(\psi,\phi) + \mathcal{O}(\delta^3), \end{aligned}$$

then there exists $\delta_2 > 0$ such that for all $\delta \in (0, \delta_2)$

$$\begin{aligned} \delta \sin\psi \sin\phi + \delta^2 g(\psi, \phi) + \mathcal{O}(\delta^3) &> 0, \\ \Rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} &> 0 \quad \text{on} \quad \mathcal{B}_1, \\ \Rightarrow \theta &\nrightarrow 0 \quad \text{through} \quad \mathcal{B}_1. \end{aligned}$$

Therefore no interior solution can reach P_2.

□

Lemma 6.3.5. *An interior solution cannot have an ω-limit point on P_3*

Proof. Assume an interior solution has an ω-limit point at P_3 where $P_3 = (0,0,0)$. Consider the set of points, $\mathcal{B}$, lying in the ball of radius $\delta > 0$, centred at P_3 and the cuboid Λ, i.e.,

$$\mathcal{B} = B(P_3, \delta) \cap \Lambda.$$

Define

$$\begin{aligned} \mathcal{B}_1 &= \mathcal{B} \cap \tilde{\Lambda}, \quad \mathcal{B}_2 = \mathcal{B} \cap F_1, \quad \mathcal{B}_3 = \mathcal{B} \cap F_3, \quad \mathcal{B}_4 = \mathcal{B} \cap F_5, \\ \mathcal{B}_5 &= \mathcal{B} \cap L_2 \quad \mathcal{B}_6 = \mathcal{B} \cap L_3, \quad \mathcal{B}_7 = \mathcal{B} \cap L_5, \end{aligned}$$

so that

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4 \cup \mathcal{B}_5 \cup \mathcal{B}_6 \cup \mathcal{B}_7.$$

We will consider $\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}}$ for each $\mathcal{B}_i$, where $i = 1 - 7$: Firstly

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} = 0 \quad \text{on} \quad \mathcal{B}_5,$$

so θ cannot reach 0 along these surfaces, unless $\theta = 0$ and $r = 0$ for some $\tilde{\lambda}_1 < \infty$, which is not possible as $z=1$ is an invariant manifold. Next we look at

$$\begin{aligned}
\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} &= r(1-z)(2-2z+z^2) = r(1-z)((1-z)^2+1) > 0 \quad \text{on} \quad \mathcal{B}_3, \\
\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} &= 2\cos^2\theta\sin\theta > 0 \quad \text{on} \quad \mathcal{B}_6, \\
\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} &= 2r > 0 \quad \text{on} \quad \mathcal{B}_7
\end{aligned}$$

Next we consider

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} = \cos\theta\sin\theta(2\cos\theta + z(z-2)(\cos\theta+\sin\theta)) > 0 \quad \text{on} \quad \mathcal{B}_2$$

by considering a point on the circular boundary of $\mathcal{B}_2$, which can be written in polar coordinates, (δ, ψ), where

$$\theta = \delta\cos\psi, \quad z = \delta\sin\psi \quad \text{for} \quad \psi \in \left(0, \frac{\pi}{2}\right)$$

as follows

$$\begin{aligned}
(z^2-2z+2)\cos\theta + z(z-2)\sin\theta &= 2 - 2\delta\sin\psi + \mathcal{O}(\delta^2), \\
\cos\theta\sin\theta &= \delta\cos\psi + \mathcal{O}(\delta^3), \\
\Rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} &= 2\delta\cos\psi - 2\delta^2\cos^2\psi + \mathcal{O}(\delta^3),
\end{aligned}$$

and we can choose $\delta_1 > 0$ sufficiently small so that for all $\delta \in (0, \delta_1)$,

$$\Rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} < 0, \quad \Rightarrow \theta \nrightarrow \frac{\pi}{2} \quad \text{through} \quad \mathcal{B}_2,$$

where we have used the approximations (6.3.11) which are valid for A close to zero Next we consider

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} = -\frac{\cos^2\theta}{2}(r\sin\theta - 1)^2\left(-2r\cos 2\theta + (r^2-4)\sin\theta + r(-2+2\sin 2\theta + r\sin 3\theta)\right)$$

on $\mathcal{B}_4$, by looking at its components,

$$\begin{aligned}
-\frac{\cos^2\theta}{2}(r\sin\theta-1)^2 &< 0, \\
-2r\cos 2\theta &< 0, \quad \text{for} \quad \theta\in\left(0,\frac{\pi}{4}\right) \\
(r^2-4)\sin\theta &< 0, \quad \text{for} \quad r\in(0,2) \\
-1+2\sin 2\theta &< 0, \quad \text{for} \quad \theta\in\left(0,\frac{\pi}{16}\right) \\
-1+r\sin 3\theta &< 0, \quad \text{for} \quad \theta\in\left(0,\frac{\pi}{3}\right) \quad r\in(0,0.4) \\
\Rightarrow r(-2+2\sin 2\theta+r\sin 3\theta) &= r((-1+2\sin 2\theta)+(-1+r\sin 3\theta))<0,
\end{aligned}$$

and as we can choose $\delta>0$ sufficiently small so that $\theta\in(0,\frac{\pi}{16})$ and $r\in(0,0.4)$ we have,

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}}>0, \quad \Rightarrow \theta \not\to 0 \quad \text{through} \quad \mathcal{B}_4.$$

Finally we use a Taylor expansion for θ about 0 to get

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} = f_0(r,z)+f_1(r,z)\theta+f_2(r,z)\theta^2+\mathcal{O}(\theta^3) \quad \text{on} \quad \mathcal{B}_1,$$

where

$$\begin{aligned}
f_0(r,z) &= r(2-4z+3z^2-z^3), \\
f_1(r,z) &= (2-2z+z^2)(1-r(1+z)+r^2(4z-3)), \\
f_2(r,z) &= -2z+z^2+\mathcal{O}(r),
\end{aligned}$$

if we consider a point on the spherical surface of $\mathcal{B}_1$, which can be written in spherical polar coordinates, (δ,ψ,ϕ), where

$$r=\delta\cos\psi\sin\phi, \quad \theta=\delta\sin\psi\sin\phi, \quad z=\delta\cos\phi,$$

and $(\phi, \psi) \in (0, \frac{\pi}{2})$, then we can write

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} &= 2\delta(\cos\psi + \sin\psi)\sin\phi \\ &+ \delta^2 \sin\phi\,(-4\cos\psi\cos\phi - 2\sin\psi\sin\phi(\cos\psi + \sin\psi)) + \mathcal{O}(\delta^3), \\ &= 2\delta(\cos\psi + \sin\psi)\sin\phi + \delta^2 g(\psi, \phi) + \mathcal{O}(\delta^3),\end{aligned}$$

then there exists $\delta_2 > 0$ such that for all $\delta \in (0, \delta_2)$

$$\delta\sin\psi\sin\phi + \delta^2 g(\psi, \phi) + \mathcal{O}(\delta^3) > 0,$$
$$\Rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}\tilde{\lambda}} > 0 \quad \text{on} \quad \mathcal{B}_1, \quad \Rightarrow \theta \nrightarrow 0 \quad \text{through} \quad \mathcal{B}_1.$$

Therefore no interior solution can reach P_3. □

The only equilibrium point remaining is P_1 which is a hyperbolic equilibrium point. When we linearise (6.3.10) about P_1 we obtain the following eigenvalues and eigenvectors:

$$\begin{aligned}\lambda_1 &= \frac{-1}{\sqrt{2}}, & \vec{v_1} &= (1, 0, 0), \\ \lambda_2 &= \frac{-1}{\sqrt{2}}, & \vec{v_2} &= (0, 1, 0), \\ \lambda_3 &= \frac{-1}{\sqrt{2}}, & \vec{v_3} &= (0, 0, 1).\end{aligned}$$

By the Stable Manifold Theorem [31] there exists a 3-dim stable manifold S tangent to the 3-dim stable subspace E^S of the linear system at P_1, spanned by $\vec{v_1}$, $\vec{v_2}$ and $\vec{v_3}$, such that for all $\tilde{\lambda} \geq 0$, $\phi_{\tilde{\lambda}} \subset S$ and for all $\vec{r_i} \in S$

$$\lim_{\tau \to \infty} \phi_\tau(\vec{r_i}) = \vec{r_0}$$

where $\vec{r_0} = P_1$ and ϕ_τ is the flow of the linear system. Trajectories in this stable manifold have the asymptotic behavior:

$$
\begin{aligned}
r &= 0 + c_1 \exp\left(\frac{-\tilde{\lambda}}{\sqrt{2}}\right) + c_1^* \exp\left(\frac{-2\tilde{\lambda}}{\sqrt{2}}\right) + \mathcal{O}\left(\exp\left(\frac{-3\tilde{\lambda}}{\sqrt{2}}\right)\right), && (6.3.12a)\\
\theta &= \frac{\pi}{4} + c_2 \exp\left(\frac{-\tilde{\lambda}}{\sqrt{2}}\right) + \mathcal{O}\left(\exp\left(\frac{-2\tilde{\lambda}}{\sqrt{2}}\right)\right), && (6.3.12b)\\
z &= 1 + c_3 \exp\left(\frac{-\tilde{\lambda}}{\sqrt{2}}\right) + \mathcal{O}\left(\exp\left(\frac{-2\tilde{\lambda}}{\sqrt{2}}\right)\right), && (6.3.12c)
\end{aligned}
$$

which is a three-parameter family of solutions valid for $\tilde{\lambda} \to \infty$, where the three parameters are c_1, c_2 and c_3, and c_1^* is not a new parameter but we will show later that c_1^* is actually related to c_1 and c_2.

6.3.5 Rewriting the solution in terms of original variables

In order to write (6 3 12) in terms of the original dependant and independent variables we first integrate (6 3.8) to prove that $\tilde{\lambda}(s)$, (equivalent to λ), has the correct asymptotic behaviour, i.e. $\tilde{\lambda}(\infty) = \infty$,

$$
\begin{aligned}
\frac{\mathrm{d}s}{\mathrm{d}\lambda} &= AB(2-A)(1-B)^2(1-C),\\
\Rightarrow \frac{\mathrm{d}s}{\mathrm{d}\tilde{\lambda}} &= \frac{\mathrm{d}s}{\mathrm{d}\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}\tilde{\lambda}} = \frac{\mathrm{d}s}{\mathrm{d}\lambda}\frac{1}{r}.
\end{aligned}
$$

We can use (6.3.12) to get an expression for $\frac{\mathrm{d}s}{\mathrm{d}\tilde{\lambda}}$ which is locally valid.

$$
\begin{aligned}
\frac{\mathrm{d}s}{\mathrm{d}\tilde{\lambda}} &= \left(\frac{1}{r}\right) AB(2-A)(1-B)^2(1-C)\\
&= \left(\frac{1}{r}\right) zr\cos\theta\,(2-z)\left(1 - 2r\cos\theta - r^2\cos\theta^2\right)(1 - r\sin\theta)\\
&= z\cos\theta\,(2-z)\left[1 - r(\sin\theta + 2\cos\theta) + \mathcal{O}(r^2)\right].
\end{aligned}
$$

Let $v = e^{\frac{-\tilde{\lambda}}{\sqrt{2}}}$. Then we have that

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}\tilde{\lambda}} &= \left(1 + c_3 v + \mathcal{O}(v^2)\right)\left(1 - c_3 v + \mathcal{O}(v^2)\right)\cos\left(\frac{\pi}{4} + c_2 v + \mathcal{O}(v^2)\right) \\ &\times \left[1 - (c_1 v + \mathcal{O}(v^2))\left(\sin\left(\frac{\pi}{4} + c_2 v + \mathcal{O}(v^2)\right) + 2\cos\left(\frac{\pi}{4} + c_2 v + \mathcal{O}(v^2)\right)\right)\right]\end{aligned}$$

We use the identities

$$\begin{aligned}\cos(A+B) &= \cos A \cos B - \sin A \sin B, \\ \sin(A+B) &= \sin A \cos B + \cos A \sin B,\end{aligned}$$

and the approximations (6.3.11) which are valid for A close to zero, to obtain

$$\frac{\mathrm{d}s}{\mathrm{d}\tilde{\lambda}} = \frac{1}{\sqrt{2}}\left(1 + \left(-c_2 - \frac{3c_1}{\sqrt{2}}\right)v + \mathcal{O}(v^2)\right), \qquad (6.3.13)$$

we let $c_4 = (-c_2 - \frac{3c_1}{\sqrt{2}})$ and we can deduce that

$$\frac{\mathrm{d}s}{\mathrm{d}\tilde{\lambda}} \sim \frac{1}{\sqrt{2}} \quad \text{as} \quad \tilde{\lambda} \to \infty,$$

it follows that $s \to \infty$ as $\tilde{\lambda} \to \infty$ which is the desired asymptotic behaviour Next we can rewrite our solution for $(r, \theta, z, \tilde{\lambda})$ in terms of our original variables (x, y, z, s) and ultimately return to our metric function $\mu(\zeta)$. When we integrate (6.3 13) we obtain

$$s = c_5 + \frac{\tilde{\lambda}}{\sqrt{2}} - \frac{c_4 e^{\frac{-\tilde{\lambda}}{\sqrt{2}}}}{2} + \mathcal{O}(e^{\frac{-2\tilde{\lambda}}{\sqrt{2}}}),$$

where c_5 is a constant of integration. We can think of this as a one-parameter family of solutions of (6.3.13) and by making an allowable transformation of $\tilde{\lambda}$ we can choose $c_5 = 0$. We recall that from our definition of $s = \ln|\xi|$ we have,

$$e^s = |\xi| = \frac{r}{|t|} = \frac{1}{|\zeta|},$$

where this radial coordinate r is not to be confused with the r used in the polar blow-up analysis of the previous section Thus, we can write

$$\begin{aligned} |\xi| = e^s &= \exp\left(\frac{\tilde{\lambda}}{\sqrt{2}} - \frac{c_4}{2}e^{\frac{-\tilde{\lambda}}{\sqrt{2}}} + \mathcal{O}(e^{\frac{-2\tilde{\lambda}}{\sqrt{2}}})\right) \\ &= \exp\left(\frac{\tilde{\lambda}}{\sqrt{2}}\right)\exp\left(-\frac{c_4}{2}e^{\frac{-\tilde{\lambda}}{\sqrt{2}}} + \mathcal{O}(e^{\frac{-2\tilde{\lambda}}{\sqrt{2}}})\right) \\ &= \frac{1}{v}\left(1 - \frac{c_4 v}{2} + \mathcal{O}(v^2)\right), \end{aligned}$$

We can invert this to obtain

$$\begin{aligned} v &= \frac{1}{|\xi|}\left(1 - \frac{c_4}{2|\xi|} + \mathcal{O}(|\xi|^{-2})\right), \\ &= |\zeta|\left(1 - \frac{c_4|\zeta|}{2} + \mathcal{O}(|\zeta|^2)\right), \end{aligned}$$

Next we rewrite A in terms of x,

$$\begin{aligned} A &= 1 - \frac{1}{x} = 1 + c_3 v + \mathcal{O}(v^2) \text{ by (6.3.12c)} \\ x &= \frac{-1}{c_3|\zeta|}\left(1 + \mathcal{O}\left(|\zeta|\right)\right), \end{aligned} \qquad (6\ 3.14)$$

where we assume $c_3 < 0$; when $c_3 = 0$ we get $\lim_{\zeta \to 0^+} \rho = 0$ so we shall not focus on that case. Next we use that fact that $\mu(\zeta) = x|\zeta| = -x\zeta$ to write

$$\mu(\zeta) = \frac{1}{c_3}\left(1 + \mathcal{O}\left(\zeta\right)\right).$$

Similarly we rewrite

$$B = r\cos\theta, \quad C = r\sin\theta,$$

in terms of v and then using

$$x - y = \frac{B}{1-B}, \quad z - y = \frac{C}{1-C}$$

we arrive at

$$x - y = y_1|\zeta| + y_2\zeta^2 + \mathcal{O}(\zeta^3), \qquad (6.3.15)$$

$$z - y = z_1|\zeta| + z_2\zeta^2 + \mathcal{O}(\zeta^3). \qquad (6.3.16)$$

where

$$y_1 = z_1 = \frac{c_1}{\sqrt{2}}, \quad y_2 = \frac{c_1^*}{\sqrt{2}} + \frac{5c_1^2}{4} - \frac{c_1c_2}{2\sqrt{2}}, \quad z_2 = \frac{c_1^*}{\sqrt{2}} + \frac{5c_1^2}{4} + \frac{3c_1c_2}{2\sqrt{2}}.$$

6.4 Existence and uniqueness of solution

We define the hypersurface Σ_0 as $\{t, r|t = 0, r > 0\}$.

Theorem 6.4.1. *There is a unique solution, analytic in ζ at $\mathcal{N}$, which persists to Σ_0. In terms of the similarity variable $\zeta = \frac{t}{r}$ this is the interval $\zeta \in (-1, 0)$.*

Proof. We begin by considering local existence and then we extend this result using the compactified system.

Local existence

We can construct the initial data problem

$$\frac{d\vec{x}}{d\tau} = F_{(3)}(\vec{x}), \quad \text{from (6.2.1)}$$
$$\vec{x}(\tau_0) = \vec{x_0}$$

where $f \in C^1(\mathbb{R}^n)$ and the unique initial data $\vec{x_0} \in \mathbb{R}^n$ is obtained from (6.1 6). We can apply the local existence and uniqueness theorem to show that there exists an $a > 0$ such that the initial value problem has a unique solution $\vec{x}(\tau)$ on the interval $(\tau_0 - a, \tau_0 + a)$. We denote the maximal interval on which this unique solution is defined as (α, β).

Global existence

We proved in Lemma 6.2.1 that the dynamical system (6 2.1) has the properties for all τ:

$$x > 1, \quad x > y$$

We showed numerically in §6.2.3 that there exists $\tau_1 < \infty$ such that

$$\begin{aligned} x(\tau_1) &= 1 + \bar{\delta_1}, \\ y(\tau_1) &= k + \bar{\delta_2}, \\ z(\tau_1) &= k + \bar{\delta_2}, \end{aligned}$$

where $\bar{\delta_1} > 0$ and $\bar{\delta_2} > 0$ are both small. This indicated that the important quantity $(y - z)$ changes sign. We may also take $\tau_2 = \tau_1 + \epsilon$, where $\epsilon > 0$ is small, such that

$$\begin{aligned} x(\tau_2) &= 1 + \delta_1, \\ y(\tau_2) &= k + \delta_2, \\ z(\tau_2) &= k + \delta_3, \end{aligned}$$

where $\delta_1 > \bar{\delta_1}$ and $\delta_3 > \delta_2 > \bar{\delta_2}$ are all small. We consider the maximal interval of existence, if $\beta = \infty$ then we have a unique solution for all $\tau \in \mathbb{R}$. If $\beta < \infty$ then

$$\lim_{\tau \to \beta} \|\vec{x}\| = \infty,$$

if that limit exists, but we know that our solution is bounded for all $\tau \in (\tau_0, \tau_2)$ so $\beta > \tau_2$ and we have uniqueness up until τ_2. Next we consider the dynamical system (6.2.1), on the interval $[\tau_2, \infty]$, we can not rule out the possibility that the unique solution $\vec{x}(\tau)$ diverges in finite time, say τ^*. We can prove the following:

Lemma 6.4.2. *If any component of the solution vector $\vec{x}(\tau)$ diverges in finite time, $\tau_2 < \tau^* < \infty$, then all components do.*

Proof. We note that $x - y$ is monotonically decreasing for $\tau_2 < \tau^* < \infty$ and so

$$\lim_{\tau \to \tau^*} (x - y) \nrightarrow \infty \tag{6.4.1}$$

We consider each component diverging separately:

case 1 If $\lim_{\tau \to \tau^*} x = \infty$ and $\lim_{\tau \to \tau^*} y = y^* < \infty$, then $\lim_{\tau \to \tau^*} x - y = \infty$ which contradicts (6.4.1). So if x diverges in finite time, y must diverge and this leads to case 2.

case 2(a) If $\lim_{\tau \to \tau^*} y = \infty$, we use the fact that $x > y, z > y$ for all $\tau > \tau_2$ to deduce that $\lim_{\tau \to \tau^*} x = \infty$ and $\lim_{\tau \to \tau^*} z = \infty$.

case 2(b) If $\lim_{\tau \to \tau^*} y = -\infty$, then $\lim_{\tau \to \tau^*} x - y = \infty$, whether x diverges or not, and this contradicts (6.4 1)

case 3(a) If $\lim_{\tau \to \tau^*} z = \infty$ and $x(\tau^*)$ and $y(\tau^*)$ are finite then we write $\lim_{\tau \to \tau^*} \frac{dz}{d\tau} = -2z^2(x^2 - 1) + \mathcal{O}(z) = -\infty$, this contradiction implies that this case cannot occur.

case 3(b) If $\lim_{\tau \to \tau^*} z = -\infty$ and $x(\tau^*)$ and $y(\tau^*)$ are finite then the property $z > y$ implies that $\lim_{\tau \to \tau^*} y = -\infty$ and this leads to case 2(b).

□

If the solution does diverge in finite time, then we will not be able to get qualitative information about the solution - as it will not be in the form of an equilibrium point However, if we compactify the system we will not have any divergences and we will be guaranteed uniqueness.

We can construct the initial data problem

$$\begin{aligned} \frac{d\vec{A}}{d\lambda} &= F_{(4)}(\vec{A}), \quad \text{from (6.3.9)} \\ \vec{A}(\lambda_0) &= \vec{A}_0. \end{aligned}$$

where $f \in C^1(\Omega) = [0,1]^3$ and the unique initial data $\vec{A}_0 \in \Omega$ is obtained from §6.2.3. We can apply the global existence theorem and therefore (6.3.9) has a unique solution $\vec{A}(\lambda)$ defined for all $\lambda \in \mathbb{R}$. We also have the quantitative information about our unique solution which we need for the next section. We can use our quantitative information to show the asymptotic relationship between the different independent variables used in this section: We showed that as $\lambda \to \infty$, $t \to 0$, and we can integrate

$$\begin{aligned} \frac{\mathrm{d}\tau}{\mathrm{d}\lambda} &= (1-B)^3(1-C)(1-A)^2 = \frac{1}{(x-y+1)^3(z-y+1)(x^2)}, \\ \tau + C &= \frac{-c_3^2}{\sqrt{2}c_1}\exp\left(\frac{-\lambda}{\sqrt{2}}\right) + \mathcal{O}\left(\exp\frac{-2\lambda}{\sqrt{2}}\right), \end{aligned}$$

This implies that as $\lambda \to \infty$ we have $\tau \to C < \infty$, which corresponds to the system (6.2.1) diverging in finite τ. □

6.5 Examination of the physical properties of this solution

From Properties (6.2.2) and (6.2 3) we can conclude that

$$c_3 < 0, \quad c_1 > 0 \quad c_1 c_3 < 0, \tag{6.5.1}$$

where the constants c_1 and c_3 appear in the expression for x, $x-y$ and $z-y$ in (6.3.14)-(6.3.16). We can write out an expression for the density ρ as follows,

$$\rho r^2 = \frac{2(x-y)}{x\zeta^2} = -\sqrt{2}c_1c_3 + \mathcal{O}(\zeta)$$

and so by (6.5.1) we deduce that the density as $t \to 0$, $r \neq 0$ is finite and positive

$$\lim_{\zeta \to 0+} \rho r^2 = -\sqrt{2}c_1c_3 > 0.$$

For convenience we rename our coefficients in (6.3.14) as follows:

$$\begin{aligned} x &= \frac{x_0}{|\zeta|} - x_1 + x_2|\zeta| - x_3|\zeta|^2 + x_4|\zeta|^3 + \cdots, \\ \Rightarrow \mu(\zeta) &= x_0 - x_1|\zeta| + x_2|\zeta|^2 - x_3|\zeta|^3 + x_4|\zeta|^4 + \cdots, \\ &= x_0 + x_1\zeta + x_2\zeta^2 + x_3\zeta^3 + x_4\zeta^4 + \cdots, \end{aligned} \tag{6.5 2}$$

where $x_0 = \frac{-1}{c_3} > 0$. From our definition of x, y and z, we can write

$$\begin{aligned} x - y &= -\frac{\mathrm{d}\mu}{\mathrm{d}\zeta}, \\ z - y &= -\frac{\mathrm{d}^2\mu}{\mathrm{d}\zeta^2}\zeta, \end{aligned}$$

and so if we differentiate (6 5 2) then we can get

$$\begin{aligned} -\frac{\mathrm{d}\mu}{\mathrm{d}\zeta} &= -x_1 - 2x_2\zeta + \mathcal{O}(\zeta^2), \\ -\frac{\mathrm{d}^2\mu}{\mathrm{d}\zeta^2}\zeta &= -2x_2\zeta + \mathcal{O}(\zeta^2). \end{aligned}$$

If we compare with (6.3.15) and (6.3.16) and equate the relevant coefficients we see that for consistency we must require

$$x_1 = 0, \quad -2x_2 = y_1 = z_1 \quad -3x_3 = y_2, \quad -6x_3 = z_2.$$

the last two equations imply that $z_2 = 2y_2$ which we can use to get an expression for c_1^*,

$$\begin{aligned} z_2 = 2y_2 \quad \Rightarrow \quad & \frac{c_1^*}{\sqrt{2}} + \frac{5c_1^2}{4} - \frac{c_1 c_2}{2\sqrt{2}} = \frac{c_1^*}{\sqrt{2}} + \frac{5c_1^2}{4} + \frac{3c_1 c_2}{2\sqrt{2}}, \\ \Rightarrow \quad & \frac{c_1^*}{\sqrt{2}} = -\frac{5c_1^2}{4} - \frac{5c_1 c_2}{2\sqrt{2}}, \end{aligned}$$

which we can use to simplify y_2 and z_2,

$$y_2 = -\frac{3c_1 c_2}{\sqrt{2}}, \quad z_2 = -\frac{c_1 c_2}{\sqrt{2}}.$$

It is convenient to write out, using (6 3.15) and (6.3.16),

$$\frac{y - z}{x - y} = -1 + \left(\frac{y_2 - z_2}{z_1}\right)\zeta + \mathcal{O}(\zeta^2) = -1 - (4c_2\zeta) + \mathcal{O}(\zeta^2)$$

and we see how the final parameter c_2 comes into play. Finally we can look at the expansion scalars, which we can rewrite in terms of x, y and z for convenience.

$$\begin{aligned}
\theta_+ &= \frac{\beta_-}{r|\zeta|}\left(\left(-1-\frac{y-z}{x-y}\right)+\frac{1}{x}\left(1-\frac{y-z}{x-y}\right)\right), \\
\theta_- &= \frac{\beta_+}{r|\zeta|}\left(\left(-1-\frac{y-z}{x-y}\right)-\frac{1}{x}\left(1-\frac{y-z}{x-y}\right)\right), \\
\theta_-\theta_+ &= \frac{\beta_-\beta_+}{r^2\zeta^2}\left(\left(1+\frac{y-z}{x-y}\right)^2-\frac{1}{x^2}\left(1-\frac{y-z}{x-y}\right)^2\right),
\end{aligned}$$

and we finally arrive at

$$\begin{aligned}
\lim_{\zeta,t\to 0^+}\theta_+ &= \frac{\beta_+}{r}\left(-4c_2-2c_3+\mathcal{O}(\zeta)\right), \\
\lim_{\zeta,t\to 0^+}\theta_- &= \frac{\beta_-}{r}\left(-4c_2+2c_3+\mathcal{O}(\zeta)\right), \\
\lim_{\zeta,t\to 0^+}\theta_-\theta_+ &= \frac{\beta_-\beta_+}{r^2}\left((-4c_2)^2-(2c_3)^2+\mathcal{O}(\zeta)\right),
\end{aligned}$$

recall $c_3 < 0$, but as we do not know the sign of c_2, we do not know the sign of $\lim_{\zeta,t\to 0^+}\theta_+$ either We can conclude that there exists values of c_2 (i e. $c_2 < -\frac{c_3}{2}$) for which we do not have a trapped cylinder as $t \to 0^+$. However, our numerical simulation suggests that $c_2 > 0$, and that θ_+ decreases from its negative value at $\mathcal{N}$ to a final value of

$$\theta_+ \propto \left(\left(-1-\frac{y-z}{x-y}\right)+\frac{1}{x}\left(1-\frac{y-z}{x-y}\right)\right) \approx -3.1.$$

We can examine our solution (6.5 2) from another perspective by returning to our original third order ODE - gleaned from our field equations. If (6.5 2) is to satisfy the ODE as $\zeta \to 0^+$ we insert (6.5.2) into the ODE to get,

$$f_1\zeta^3+f_2\zeta^4+f_3\zeta^5+f_4\zeta^6+\mathcal{O}(\zeta^7)=0,$$

where $f_1 = f_1(x_0, x_2, x_3, x_4)$ etc., and this is satisfied if,

$$f_1 = 0, \quad f_2 = 0, \quad f_3 = 0, \quad f_4 = 0, \quad \text{etc.}$$

But as these equations have the form

$$
\begin{aligned}
f_1(x_0, x_2, x_3, x_4) &= 0, \\
f_2(x_0, x_2, x_3, x_4, x_5) &= 0, \\
f_3(x_0, x_2, x_3, x_4, x_5, x_6) &= 0, \\
f_4(x_0, x_2, x_3, x_4, x_5, x_6, x_7) &= 0 \quad \text{etc.,}
\end{aligned}
$$

we can see that (6.5.2) represents a three-parameter family of solutions, where x_0, x_2 and x_3 are the parameters and,

$$
x_0 = \frac{-1}{c_3}, \quad x_2 = -\frac{c_1}{2\sqrt{2}}, \quad x_3 = \frac{c_1 c_2}{\sqrt{2}},
$$

show how they relate to the parameters c_1, c_2 and c_3 which we derived from the dynamical systems analysis

Chapter 7

The future of $\sum_0$

In this chapter we describe the solution emanating from $\sum_0$ into the future [1]. Firstly we will prove the existence of a one-parameter family of solutions emanating from $\sum_0$, using the Hartman-Grobman theorem. Then we derive properties of this dynamical system and describe the asymptotic behaviour.

7.1 Proof of the existence of a solution emanating from $\sum_0$

We introduce new dependent and independent variables as follows

$$a = \frac{1}{x}, \quad b = x - y, \quad c = \frac{y - z}{-x + y}, \quad \bar{s} = \ln(\zeta) = \ln\left(\frac{t}{r}\right),$$

[1]We have not considered the spacetime matching conditions across $\sum_0$ here but we note that the existence of the solution described below is a minimal condition for a metric that extends continuously across $\mathcal{N}$.

and we recast our original dynamical system (5 1.4) in terms of a, b, c and $\bar{s}$.

$$\frac{da}{d\bar{s}} = a - a^2 b, \tag{7 1.1a}$$

$$\frac{db}{d\bar{s}} = bc, \tag{7.1.1b}$$

$$\frac{dc}{d\bar{s}} = c^2 - \left(\frac{1+a^2}{1-a^2}\right)(ab(1+c)+c) \tag{7.1.1c}$$

This dynamical system has the following equilibria as $\bar{s} \to -\infty$

1. $(a, b, c) \to (0, 0, 0)$
2. $(a, b, c) \to (0, B_0, 0), \quad B_0 \in (0, \infty)$
3. $(a, b, c) \to (0, 0, 1)$.

For continuity with the result of the previous chapter we require that our solution has the following behaviour

$$\lim_{\bar{s}\to-\infty} a = 0, \quad \lim_{\bar{s}\to-\infty} b = 0, \quad \lim_{\bar{s}\to-\infty} c = 1.$$

So our system must be emerging from equilibrium point (3). We linearise (7.1.1) about this point to obtain the following Jacobian

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, equilibrium point (3) is a hyperbolic equilibrium point, and by the Hartman-Grobman theorem the linearised system has a 3-dim unstable space E^U, spanned by the eigenvectors of J. Trajectories in E^U have the following asymptotic behaviour, as $\bar{s} \to -\infty$:

$$a = C_1 \exp(\bar{s}) + \mathcal{O}(\exp(2\bar{s})), \tag{7.1 2a}$$

$$b = C_2 \exp(\bar{s}) + \mathcal{O}(\exp(2\bar{s})), \tag{7.1.2b}$$

$$c = 1 + C_3 \exp(\bar{s}) + \mathcal{O}(\exp(2\bar{s})) \tag{7.1.2c}$$

From the analysis of the previous chapter, the values of C_1 and C_2 can be determined from the limiting values of the metric function μ and the comoving density $\rho(\zeta)$ as $\bar{s} \to -\infty$ (or $t \to 0^-$). From this we know that $C_1 > 0$ and $C_2 > 0$ So C_3 is a parameter. Our solution undergoes a bifurcation at $t = 0$, and (7.1.2) is a one-parameter family of solutions emerging into the future.

7.2 Analysis of the behaviour of the solution emanating from $\sum_0$

We are interested now in the evolution of the solution (7.1.2). To this end we define a new independent variable τ by

$$\frac{\mathrm{d}\bar{s}}{\mathrm{d}\tau} = (1 - a^2). \tag{7.2.1a}$$

Then our dynamical system becomes

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = (1 - a^2)(a - a^2 b), \tag{7.2.1b}$$

$$\frac{\mathrm{d}b}{\mathrm{d}\tau} = (1 - a^2)bc, \tag{7.2.1c}$$

$$\frac{\mathrm{d}c}{\mathrm{d}\tau} = (1 - a^2)c^2 - \left(1 + a^2\right)\left(ab(1 + c) + c\right), \tag{7.2.1d}$$

which we denote by

$$\frac{\mathrm{d}\vec{a}}{\mathrm{d}\tau} = \vec{F}_{(5)}(\vec{a}, C_3),$$

where we can see that $\vec{F}_{(5)}$ depends on the vector $\vec{a}$ and also the parameter C_3. Next we derive certain properties of our solution.

7.2.1 Properties of the solution emanating from $\sum_0$

Property 1. We know that $\lim_{\bar{s}\to-\infty} a = 0^-$ and from our solution (7.1.2) we know that $a > 0$ as $\bar{s} \to -\infty$. But if

$$\begin{aligned} a(\tau_1) &= 0 \quad \text{for some} \quad \tau_1 < \infty, \\ \Rightarrow \left.\frac{\mathrm{d}a}{\mathrm{d}\tau}\right|_{\tau=\tau_1} &= 0, \\ \Rightarrow a(\tau) &= 0 \quad \text{for all} \quad \tau. \end{aligned}$$

Similarly if

$$\begin{aligned} a(\tau_2) &= 1 \quad \text{for some} \quad \tau_2 < \infty, \\ \Rightarrow \left.\frac{\mathrm{d}a}{\mathrm{d}\tau}\right|_{\tau=\tau_2} &= 0, \\ \Rightarrow a(\tau) &= 1 \quad \text{for all} \quad \tau. \end{aligned}$$

Therefore $0 < a < 1$ for all $\tau < \infty$.

Property 2. We know that $\lim_{\bar{s}\to-\infty} b = 0^-$ and from our solution (7.1.2) we know that $b > 0$ as $\bar{s} \to -\infty$. But if

$$\begin{aligned} b(\tau_1) &= 0 \quad \text{for some} \quad \tau_1 < \infty, \\ \Rightarrow \left.\frac{db}{\mathrm{d}\tau}\right|_{\tau=\tau_1} &= 0, \\ \Rightarrow b(\tau) &\equiv 0 \quad \text{for all} \quad \tau. \end{aligned}$$

Therefore $0 < b < \infty$ for all $\tau < \infty$.

Property 3. The function $c(\tau)$ can only approach zero from above i.e.,

$$\left.\frac{\mathrm{d}c}{\mathrm{d}\tau}\right|_{c=0} = -ab(1+a^2) < 0.$$

Equilibria of (7.2.1)

The dynamical system (7 2 1) has the following equilibria as $\tau \to \infty$

$$\begin{aligned} E_1 &\cdot (a,b,c) \to (0,0,1) = \vec{a_1}, \\ E_2 &: (a,b,c) \to (0,B_1,0) = \vec{a_2}, \\ E_3 &. (a,b,c) \to \left(1, B_0, -\frac{B_0}{1+B_0}\right) = \vec{a_3}, \end{aligned}$$

where B_0 and B_1 are parameters which will be described below.

Equilibrium point E_1: $(a,b,c) \to (0,0,1) = \vec{a_1}$

Consider our solution (7.1.2) approaching $\vec{a_1}$. Then

$$\begin{aligned} \lim_{\tau\to\infty} b(\tau) = 0 \ &\Rightarrow \ \frac{\mathrm{d}b}{\mathrm{d}\tau} < 0 \ \text{ as } \ \tau \to \infty, \\ &\Rightarrow \ c < 0 \text{ as } \ \tau \to \infty. \end{aligned}$$

But to reach this equilibrium point we must have $\lim_{\tau\to\infty} c = 1$, which contradicts the above.

Equilibrium set E_2: $(a,b,c) \to (0,B_1,0) = \vec{a_2}$

Consider our solution (7.1.2) approaching $\vec{a_2}$. Then

$$\begin{aligned} \lim_{\tau\to\infty} a(\tau) = 0 \ &\Rightarrow \ \frac{\mathrm{d}a}{\mathrm{d}\tau} < 0 \quad \text{as} \quad \tau \to \infty, \\ &\Rightarrow \ (1-ab) < 0 \quad \text{as} \quad \tau \to \infty, \\ &\Rightarrow \ b > \frac{1}{a} \quad \text{as} \ \tau \to \infty, \\ \lim_{\tau\to\infty} \frac{1}{a(\tau)} = \infty \ &\Rightarrow \ \lim_{\tau\to\infty} b(\tau) \geq \lim_{\tau\to\infty} \frac{1}{a(\tau)} = \infty, \\ &\Rightarrow \ \lim_{\tau\to\infty} b(\tau) = B_1 = \infty. \end{aligned}$$

In order to prove that this equilibrium point cannot be reached we rewrite our system in terms of new coordinates

$$a = \frac{1}{x}, \quad h = \frac{x-y}{x} = ab, \quad c = -\frac{y-z}{x-y},$$

where we are using the same independent variable τ

$$\frac{\mathrm{d}\bar{s}}{\mathrm{d}\tau} = (1-a^2) \tag{7.2.2a}$$

Then our dynamical system becomes

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = a(1-h)(1-a^2), \tag{7.2.2b}$$

$$\frac{\mathrm{d}h}{\mathrm{d}\tau} = (hc+h-h^2)(1-a^2), \tag{7 2.2c}$$

$$\frac{\mathrm{d}c}{\mathrm{d}\tau} = (1-a^2)c^2 - \left(1+a^2\right)\left(h(1+c)+c\right). \tag{7.2.2d}$$

From our previous analysis we know that

$$\lim_{\bar{s}\to-\infty}(a,h,c) = (0,0,1),$$

and from (7 1.2) we have

$$\begin{aligned} a &= C_1\exp(\bar{s}) + \mathcal{O}(\exp(2\bar{s}), \\ h = ab &= C_1C_2\exp(2\bar{s}) + \mathcal{O}(\exp(3\bar{s}), \\ c &= 1 + C_3\exp(\bar{s}) + \mathcal{O}(\exp(2\bar{s})). \end{aligned}$$

As $\bar{s} \to -\infty$ we can say that a and h are initially increasing from $\tau_0(\bar{s}) = \tau(-\infty)$ Thus we can derive the following properties for (7 2.2)·

Property 1.

$$\begin{aligned} \text{If} \quad a(\tau_1) &= 0 \quad \text{for some} \quad \tau_1 < \infty, \\ \text{then} \quad \frac{\mathrm{d}a(\tau_1)}{\mathrm{d}\tau} &= 0, \\ \Rightarrow a(\tau_1) &\equiv 0 \quad \text{for all} \quad \tau. \end{aligned}$$

But we know from (7 1.2) that a is initially increasing This implies that $0 \leq a \leq 1$ for all τ.

Property 2. Similarly we get $0 \leq h < \infty$ for all τ.

Next we consider the equilibria of (7.2.2) as $\tau \to \infty$

$$\begin{aligned}
E_1^* &: (a,h,c) \to (0,0,0), \\
E_2^* &: (a,h,c) \to (0,0,1), \\
E_3^* &: (a,h,c) \to \left(0, \frac{2}{3}, -\frac{1}{3}\right), \\
E_4^* &: (a,h,c) \to \left(1, H_0, -\frac{H_0}{1+H_0}\right),
\end{aligned}$$

where $H_0 \in (0, \infty)$. To reach the equilibria E_1^*, E_2^* and E_3^* we must have

$$\begin{aligned}
\lim_{\tau\to\infty} a(\tau) = 0 \quad &\Rightarrow \quad \frac{\mathrm{d}a}{\mathrm{d}\tau} < 0 \quad \text{as} \quad \tau \to \infty, \\
&\Rightarrow \quad (1-h) < 0 \quad \text{as} \quad \tau \to \infty, \\
&\Rightarrow \quad h > 1 \quad \text{as} \quad \tau \to \infty, \\
&\Rightarrow \quad \lim_{\tau\to\infty} h(\tau) \geq 1.
\end{aligned}$$

But equilibrium points E_1^*, E_2^* and E_3^* have respectively

$$\lim_{\tau\to\infty} h(\tau) = 0, \quad \lim_{\tau\to\infty} h(\tau) = 0, \quad \lim_{\tau\to\infty} h(\tau) = \frac{2}{3}$$

This contradiction shows that these equilibria cannot be reached unless $a(\tau_1) = 0$ for some $\tau_1 < \infty$ but this violates Property 1 ($a(\tau)$ can only approach 0 as $\tau \to \infty$). Only equilibrium set E_4^* can be approached and this corresponds exactly to equilibrium set E_3 *of* (7.2.1).

Equilibrium set E_3: $\vec{a_3}$. $(a,b,c) \to (1, B_0, -\frac{B_0}{1+B_0})$

To reach this set we must have

$$\begin{aligned}
\lim_{\tau\to\infty} a(\tau) = 1 \;&\Rightarrow\; \frac{\mathrm{d}a}{\mathrm{d}\tau} > 0 \quad \text{as} \quad \tau \to \infty, \\
&\Rightarrow\; (1-ab) > 0 \quad \text{as} \quad \tau \to \infty, \\
&\Rightarrow\; b < \frac{1}{a} \quad \text{as} \quad \tau \to \infty, \\
\text{Then} \quad B_0 = \lim_{\tau\to\infty} b(\tau) \;&\leq\; \lim_{\tau\to\infty} \frac{1}{a(\tau)} = 1, \\
&\Rightarrow\; B_0 \in [0,1],
\end{aligned}$$

where we have used Property 2 to obtain the lower limit. There are three separate cases to consider: $B_0 = 0$, $B_0 = 1$ and $B_0 \in (0,1)$. We will eliminate the first and second cases with the following two propositions.

Proposition 7.2.1. *If $B_0 = 0$ then $\vec{a_3} = (1,0,0)$ cannot be reached as $\tau \to \infty$.*

Proof. To reach $\vec{a_3} = (1,0,0)$ we must have

$$\begin{aligned}
\lim_{\tau\to\infty} c \;&=\; 0 \quad \Rightarrow c > 0 \quad \text{as} \quad \tau \to \infty \quad \text{(by Property 3)}. \\
\text{But if} \quad \lim_{\tau\to\infty} b \;&=\; 0 \quad \Rightarrow \frac{\mathrm{d}b}{\mathrm{d}\tau} < 0 \quad \text{as} \quad \tau \to \infty, \\
\text{then} \quad c \;&<\; 0 \quad \text{as} \quad \tau \to \infty, \quad \text{(by Property 2)}.
\end{aligned}$$

This contradiction rules out the case $B_0 = 0$ □

Proposition 7.2.2. *If $B_0 = 1$ then $\vec{a_3} = (1,1,-\frac{1}{2})$ cannot be reached as $\tau \to \infty$.*

Proof If $B_0 = 1$ then $\vec{a_3}$ is a non-hyperbolic equilibrium point with a 2-dim centre manifold. To show that this special case of $\vec{a_3}$ cannot be reached we use a centre manifold reduction (see Appendix D) □

So we now consider $\vec{a_3} : (a,b,c) \to (1, B_0, -\frac{B_0}{1+B_0})$ with $B_0 \in (0,1)$. When we linearise (7.2.1) about this set the Jacobian is

$$J = \begin{pmatrix} 2(B_0-1) & 0 & 0 \\ \frac{2B_0^2}{1+B_0} & 0 & 0 \\ \frac{2B_0(1+2B_0)}{(1+B_0)^2} & \frac{-2}{1+B_0} & -2(B_0+1) \end{pmatrix}.$$

The matrix J has the following eigenvalues and eigenvectors

$$\begin{aligned} \lambda_1 &= -2(B_0+1), & \vec{v_1} &= (0,0,1), \\ \lambda_2 &= 2(B_0-1), & \vec{v_2} &= (f_1(B_0), f_2(B_0), 1), \\ \lambda_3 &= 0, & \vec{v_3} &= (0, f_3(B_0), 1), \end{aligned}$$

where

$$f_1(B_0) = -\frac{2(B_0+1)^2(B_0-1)}{2B_0^2-1}, \quad f_2(B_0) = -\frac{2B_0^2(1+B_0)}{2B_0^2-1}, \quad f_3(B_0) = -(1+B_0)^2.$$

This set of equilibrium points is normally hyperbolic. We have a 2-dim stable subspace, E^S and a 1-dim centre manifold. When we perform a centre manifold reduction we obtain an expression for an analytic approximation to the 1-dim centre manifold but it has no dynamics along it, in accordance with Theorem 2.4.8. The stability of the whole system is the same as that for the 2-dim stable subspace, E^S, spanned by $\vec{v_1}$ and $\vec{v_2}$.

7.2.2 Existence of a two-parameter family of solutions at $\mathcal{F}$

Solutions approaching the equilibrium set $\vec{a_3}$ exist in E^S and trajectories in E^S have the following asymptotic behaviour, as $\tau \to \infty$:

$$a = 1 + C_4 f_1(B_0)\exp(\lambda_2\tau) + \cdots, \qquad (7.2.3a)$$

$$b = B_0 + C_4 f_2(B_0)\exp(\lambda_2\tau) + \cdots, \qquad (7.2.3b)$$

$$c = -\left(\frac{B_0}{1+B_0}\right) + C_4\exp(\lambda_2\tau) + C_5\exp(\lambda_1\tau) + \cdots \qquad (7.2.3c)$$

For each B_0 this is a two-parameter (C_4 and C_5) family of solutions. We define $\zeta = \zeta_e$ as the value of the similarity variable when the equilibrium point is reached, i.e.,

$$\lim_{\tau \to \infty} \zeta = \zeta_e.$$

Proposition 7.2.3. *$\zeta = \zeta_e$ occurs at the future null cone, $\mathcal{F}$*

Proof We have seen that the future null cone, $\mathcal{F}$, for the metric

$$ds^2 = -dt^2 + \mu(\zeta)^2 dr^2 + r^2 \left(\nu(\zeta)^2 dz^2 + A(\zeta)^2 d\phi^2\right)$$

corresponds to the first positive root, $\zeta = \zeta_{\mathcal{F}}$, of the equation

$$\frac{\mu(\zeta)}{|\zeta|} - 1 \equiv x - 1 = 0 \Rightarrow 1 - a = 0. \tag{7.2.4}$$

By Property 1 we know that $a = 1$ occurs for the first time as $\tau \to \infty$. We can use our solution (7.2.3) to show that

$$\lim_{\tau \to \infty} (1 - a) \;\; = \;\; 0.$$

But this represents the future null cone. Therefore $\zeta = \zeta_{\mathcal{F}} = \zeta_e$. □

Proposition 7.2.4. *$\zeta = \zeta_e$ is finite.*

Proof. We prove by demonstrating that

$$(a, b, c) \to \left(1, B_0, -\left(\frac{B_0}{1 + B_0}\right)\right) \quad \text{at} \quad \zeta_e = \infty$$

generates a contradiction. In the limit $\zeta \to \infty$ we can take the following asymptotic expressions from our dynamical systems solution (7.2.3).

$$\mu(\zeta) \sim \zeta \quad \text{as} \quad \zeta \to \infty, \tag{7.2.5}$$

$$\dot{\mu}(\zeta) \sim B_0 \quad \text{as} \quad \zeta \to \infty, \tag{7.2.6}$$

$$\ddot{\mu}(\zeta) \sim -\frac{B_0^2}{(1 + B_0)\zeta} \quad \text{as} \quad \zeta \to \infty, \tag{7.2.7}$$

We can integrate these expressions to obtain the following

$$\int \mu(\zeta)d\zeta \sim B_0\zeta + c_1 \quad \text{as} \quad \zeta \to \infty.$$

If we equate this expression with (7.2.5) we obtain the result $c_1 = 0$ and $B_0 = 1$

$$\int \ddot{\mu}(\zeta)d\zeta \sim -\frac{B_0^2}{(1+B_0)}\ln(\zeta) + c_2 \quad \text{as} \quad \zeta \to \infty.$$

If we equate this expression with (7 2 6) we obtain the result $c_2 = B_0$ and $B_0 = 0$, where c_1 and c_2 are constants of integration. The result follows immediately. □

We use (7.2.3) to integrate (7.2.1a). This gives us the relation $\tau = \tau(\bar{s})$,

$$\begin{aligned}
\frac{\mathrm{d}\bar{s}}{\mathrm{d}\tau} &= (1-a^2) = -2C_*\exp(\lambda_2\tau) - C_*^2\exp(2\lambda_2\tau) + \cdots, \\
\int_{\bar{s}_0}^{\bar{s}} d\bar{s}' &= \int_{\tau_0}^{\tau} d\tau'(-2C_*\exp(\lambda_2\tau) - C_*^2\exp(2\lambda_2\tau) + \cdots), \\
\bar{s} - C &= -\frac{2C_*}{\lambda_2}\exp(\lambda_2\tau) - \frac{C_*^2}{2\lambda_2}\exp(2\lambda_2\tau) + \cdots,
\end{aligned}$$

where $C_* = C_4 f_1(B_0)$ and C is a constant of integration, we can write $\tilde{s} = \bar{s} - C = \ln\left(\frac{\zeta}{e^C}\right)$ where

$$\lim_{\tau\to\infty}\tilde{s} = 0, \quad \Rightarrow \lim_{\tau\to\infty}\left(\frac{\zeta}{e^C}\right) = \left(\frac{\zeta_e}{e^C}\right) = 1.$$

We can invert this to get

$$C_*\exp(\lambda_2\tau) = -\frac{\lambda_2\tilde{s}}{2}\left(1 + \frac{\lambda_2\tilde{s}}{8} + \mathcal{O}(\tilde{s}^2)\right), \tag{7 2.8}$$

giving us the local form of the solution

$$a = 1 - (B_0 - 1)\tilde{s} + \mathcal{O}(\tilde{s}^2), \tag{7.2 9a}$$

$$b = B_0 - \frac{B_0^2}{(B_0+1)}\tilde{s} + \mathcal{O}(\tilde{s}^2), \tag{7.2.9b}$$

$$c = -\left(\frac{B_0}{1+B_0}\right) - \left(\frac{2B_0^2-1}{2(1+B_0)^2}\right)\tilde{s} + \frac{C_5}{C_4^m}\left(\frac{2B_0^2-1}{2(1+B_0)^2}\right)^m \tilde{s}^m + \mathcal{O}(\tilde{s}^2), \tag{7.2.9c}$$

where $m = \frac{\lambda_1}{\lambda_2}$. This is a two-parameter (C_4 and C_5) family of solutions

7.2.3 Existence of a one-parameter solution

Local existence

Next we apply Theorem 2.4.3 to prove that the one-parameter family of solutions (7.1.2) that was shown to emerge from Σ_0 exists for all τ and remains a one-parameter family of solutions for all τ. Let E be an open subset of $\mathbb{R}^3 \times \mathbb{R}$ containing the point $(a_0, C_{3(0)})$ where $a_0 \in \mathbb{R}^3$ and $C_{3(0)} \in \mathbb{R}$ and where $\vec{F}_{(5)} \in C^1(E)$. Then it follow that there exists $\gamma > 0$ and $\delta > 0$ such that for all $\vec{y} \in N_\delta(a_0)$ and $C_3 \in N_\delta(C_{3(0)})$ the initial value problem

$$\begin{aligned} \frac{d\vec{a}}{d\tau} &= \vec{F}_{(5)}(\vec{a}, C_3), \quad \text{from (7.2.1)} \\ \vec{a}(\tau_0) &= \vec{y}, \end{aligned}$$

where C_3 is our parameter, $C_{3(0)}$ is a specific initial value of the parameter and the initial data $\vec{a_0} \in \mathbb{R}^n$ is obtained from (7.1.2), has a unique solution $u(\tau, y, C_3)$ with $u \in C^1(G)$ where $G = [-\gamma, \gamma] \times N_\delta(a_0) \times N_\delta(C_{3(0)})$. We do not know what $C_{3(0)}$ is but we are merely proving that the parametrisation is preserved.

Global existence

A solution to the dynamical system (7.2.1) is bounded for all τ, so by Theorem 2.4.2 our maximal interval of existence will be (τ_0, ∞), which we obtain with repeated applications of Theorem 2 4.3. So there exists a one-parameter family of solutions emanating from Σ_0 for all τ.

7.2.4 Numerical demonstration of the stability of equilibrium set E_3

We have shown analytically that our solution can only approach the stable equilibrium set E_3, namely $(a,b,c) \to \left(1, B_0, \frac{-B_0}{(1+B_0)}\right)$, where $B_0 \in (0,1)$ is a parameter In this section we will numerically demonstrate that for a range of initial data we do see the analytically predicted asymptotic behaviour. Using our one-parameter solution (7.1.2) for $\zeta \gtrapprox 0$, $(\zeta = \frac{t}{\tau})$ we can write

$$\begin{aligned}
\mu &= \mu_0 + \mu_1\zeta^2 + \mu_2\zeta^3 + \mathcal{O}(\zeta^4), \\
a &= \frac{\zeta}{\mu_0} - \frac{\mu_1\zeta^3}{\mu_0^2} - \frac{\mu_2\zeta^4}{\mu_0^2} + \mathcal{O}(\zeta^5), \\
b &= 2\mu_1\zeta + 3\mu_2\zeta^2 + \mathcal{O}(\zeta^3), \\
c &= 1 + \frac{3\mu_2\zeta}{2\mu_1} - \frac{(9{\mu_2}^2)\zeta^2}{4\mu_1^2} + \mathcal{O}(\zeta^3)
\end{aligned}$$

We perform a numerical simulation for a range of initial data $(\zeta_0, \mu_0, \mu_1, \mu_2)$, where

$$\mu_0 = \frac{1}{C_1} > 0, \quad \mu_1 = \frac{C_2}{2} > 0, \quad \mu_2 = \frac{C_2C_3}{3}.$$

For this purpose we employed Mathematica's default solver. In Figure 7.1 we can see the system approaching equilibrium set E_3 for the initial data $\mu_0 = 0.5$, $\mu_1 = 0.1$, $\mu_2 = 0.1$. In Figure 7.2 we can see then system approaching equilibrium set E_3 for the initial data $\mu_0 = 1$, $\mu_1 = 2$, $\mu_2 = -1$.

We have pointed out in the previous chapter that the values of C_1 and C_2 are specified by our unique solution, but we do not actually know these as we are unable to track the solution numerically to $\zeta = 0$ The constant C_3 however is a parameter.

In Table 7.1 we have performed the simulation using a selection of different values for μ_0, μ_1 and μ_2. In the fifth and sixth columns we show the numerical values of the variables b_{num} and c_{num} evaluated at the value of s at which $a_{num} = 1 \pm (10^{-5})$. In the last two columns we are measuring how close the simulation is coming to the

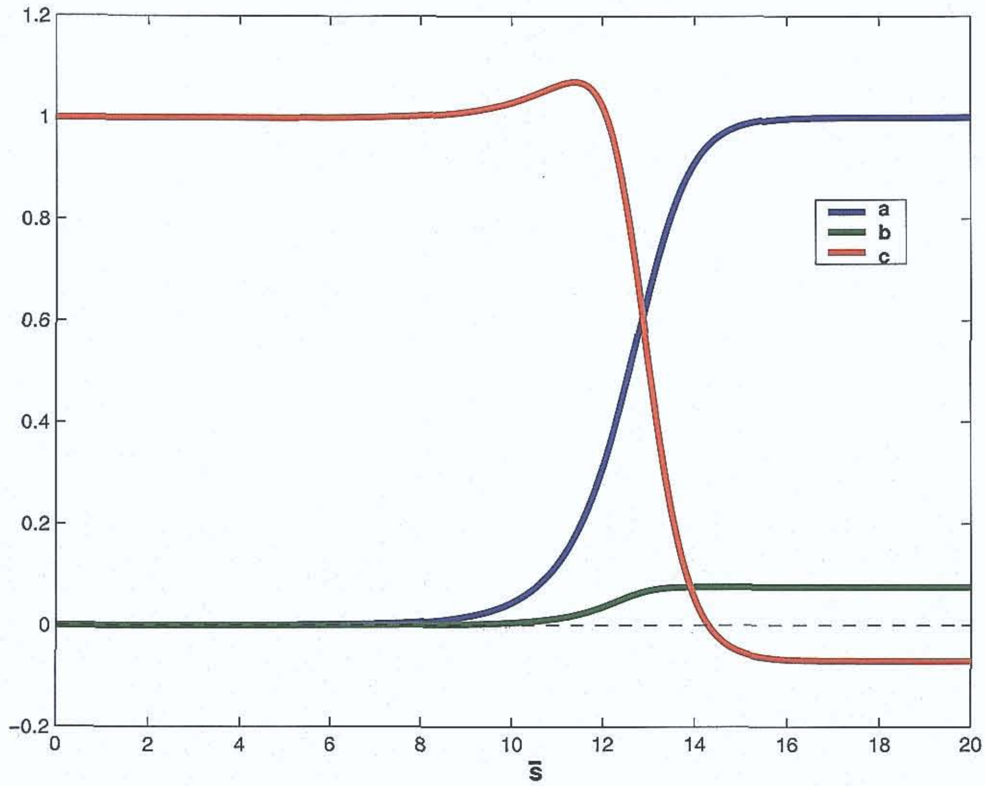

Figure 7.1: Numerical approximation of solution approaching E_3 for the initial data $\mu_0 = 0.5$, $\mu_1 = 0.1$,$\mu_2 = 0.1$.

analytically predicted point

$$(a, b, c) \to \left(1, B_0, \frac{-B_0}{(1 + B_0)}\right).$$

To do this we calculate the quantity $\frac{-b_{num}}{1+b_{num}}$ and compare it with c_{num}. So we introduce

$$\delta = c_{num} - \left(\frac{-b_{num}}{1 + b_{num}}\right),$$

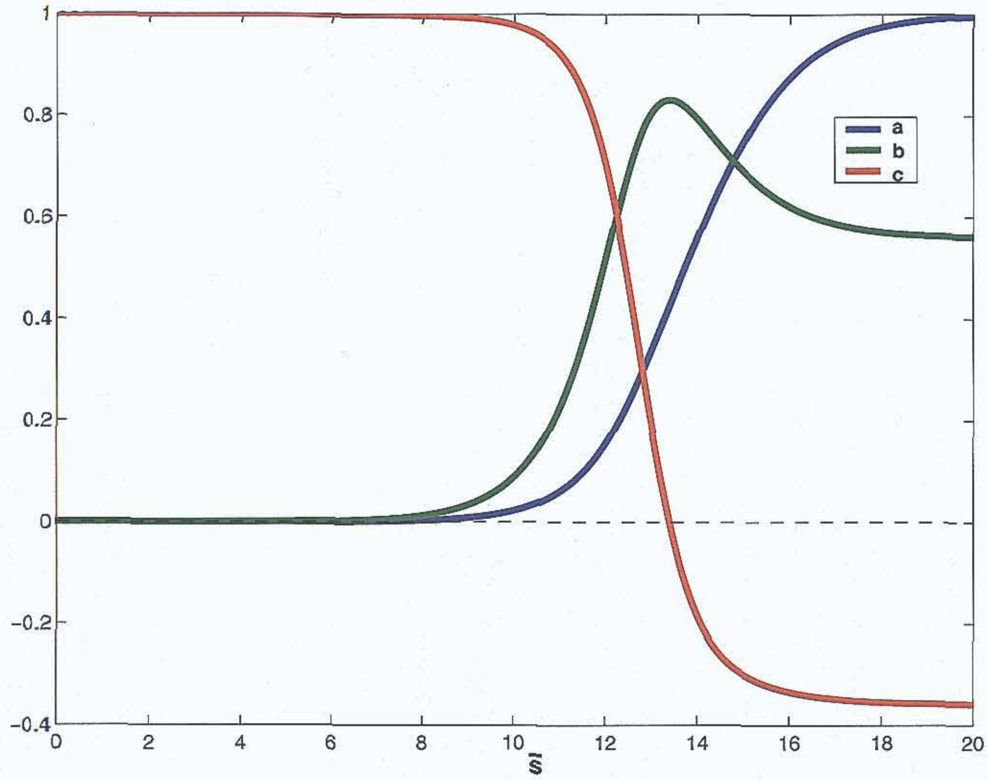

Figure 7.2: Numerical approximation of solution approaching E_3 for the initial data $\mu_0 = 1$, $\mu_1 = 2$, $\mu_2 = -1$.

The smaller the value of δ the closer the numerical simulation has come to the analytically predicted point. In the first five rows of Table 7.1 we see excellent agreement with numerical and analytical results. In the last five rows of the Table (for larger values of μ_0 and μ_1) the agreement is still apparent but the numerical and analytical results are not as close.

y_0	μ_0	μ_1	μ_2	b_{num}	c_{num}	$\frac{-b_{num}}{1+b_{num}}$	δ
0.000001	0.5	0.1	0 1	0.0748508	-0.069638	-0 069638	-8.2179×10^{-8}
0 000001	0.5	0.1	-0.1	0.035328	-0.034124	-0.034123	-7.8888×10^{-7}
0.000001	0 5	0 2	0.1	0.11138	-0.100219	-0.10021	-1.7904×10^{-7}
0.000001	0 5	0 2	-0.1	0.078752	-0.073003	-0.0730035	$-2\ 2279 \times 10^{-7}$
0 000001	0.5	0.2	-0.2	0.068672	-0 064259	-0.064259	-2.5560×10^{-7}
0.000001	1	1	1	0.62717	-0.38562	-0.38544	-0.000183556
0.000001	1	1	-1	0.37598	-0 27322	-0.27325	0.000023724
0.000001	1	2	-1	0.55796	-0.35924	-0.35813	-0.0011123
0.000001	1	2	1	0 61980	-0.38286	-0.38264	-0 00022887
0.000001	3	2.5	5	0 63618	-0.38985	-0.38882	-0 0010271

Table 7.1. Numerical demonstration of solution approaching equilibrium point E_3.

7.3 Examination of the physical properties

First we calculate the Ricci scalar R:

$$R = \rho = \frac{2ab}{r^2\zeta^2} = \frac{2}{r^2\zeta^2}\left(B_0 + \mathcal{O}(\bar{s})\right)$$

which diverges as we approach p_o. Next we look at the expansion scalars, which we can rewrite in terms of a, b and c for convenience

$$\begin{aligned}
\theta_+ &= \frac{\beta_+}{r|\zeta|}\left((-1+c) + a(1+c)\right), \\
\theta_- &= \frac{\beta_-}{r|\zeta|}\left((-1+c) - a(1+c)\right), \\
\theta_-\theta_+ &= \frac{\beta_-\beta_+}{r^2\zeta^2}\left((-1+c)^2 - a^2(1+c)^2\right).
\end{aligned}$$

Using the solution (7.2.9) valid near $\mathcal{F}$ we obtain trapped cylinders

$$\begin{aligned}
\lim_{\zeta\to 1}\theta_+ &= \lim_{\zeta\to 1}\frac{\beta_+}{r|\zeta|}\left(\frac{-2B_0}{1+B_0} + \mathcal{O}(\log|\zeta|)\right) = \frac{\beta_+}{r|\zeta|}\left(\frac{-2B_0}{1+B_0}\right) < 0, \\
\lim_{\zeta\to 1}\theta_- &= \lim_{\zeta\to 1}\frac{\beta_-}{r|\zeta|}\left(-2 + \mathcal{O}(\log|\zeta|)\right) = \frac{-2\beta_-}{r|\zeta|} < 0, \\
\lim_{\zeta\to 1}\theta_+\theta_- &= \lim_{\zeta\to 1}\frac{\beta_+\beta_-}{r^2|\zeta|^2}\left[\left(\frac{4B_0}{1+B_0}\right) + \mathcal{O}(\log|\xi|) = \frac{\beta_+\beta_-}{r^2|\zeta|^2}\left(\frac{4B_0}{1+B_0}\right) > 0\right] > 0.
\end{aligned}$$

Therefore, we have trapped cylinders at $\mathcal{F}$. As we have seen the presence of a trapped surface suggests the presence of a singularity. A singularity is a point in spacetime at which the gravitational field diverges. A naked singularity is one from which null geodesics can escape and so is observable from the outside.

Theorem 7.3.1. *There is a naked singularity at p_o.*

Proof. We proved in §4.7 that the only radial null geodesics through p_o are given by solutions to

$$(1 \pm \mu\xi) = 0,$$

and that the first positive value of $\xi = \xi_{\mathcal{F}}$ which satisfies

$$\xi = \frac{1}{\mu}, \tag{7.3.1}$$

represents the outgoing radial null geodesics through p_o, (the Cauchy horizon or future null cone). We have proved the existence of a one-parameter family of solutions (7.2.9), which satisfy this requirement. We proved in Proposition 7.2 4 that this occurs at a finite value of the similarity variable ξ Therefore we do have a naked singularity at p_o. □

Chapter 8

Conclusions

In summary, we present what we believe to be the main results and findings of this thesis.

- We proved that it is not possible to match a cylindrically symmetric perfect fluid to a vacuum. This demonstrates the impossibility of generalising the Oppenheimer-Snyder model of gravitational collapse to the cylindrically symmetric case.

- We presented a full solution to the field equations for a self-similar cylindrically symmetric vacuum spacetime, and described the nature of this solution

- Next we reduced the field equations for a self-similar cylindrically symmetric dust filled spacetime to the solution of a 3-dim autonomous dynamical system.

- We proved the existence of a unique solution emanating from the axis to Σ_0, where we assumed that the axis was regular, the density was an even, smooth function of the proper radius and in the region to the future of $\mathcal{N}$ we used a numerical simulation to provide evidence to rule out the possibility that the solution tend to an unstable equilibrium set. This unique solution is trapped at $\mathcal{N}$ and Σ_0.

- A one-parameter family of solutions was shown to reach $\mathcal{F}$, thus proving that the singularity at p_o is naked

Further work will include an analysis of the solution in the region to the future of $\zeta = \zeta_{\mathcal{F}}$. Then spacetime matching conditions will be considered. We need to check matching conditions across the null hypersurfaces $\mathcal{N}$ and $\mathcal{F}$ and the spacelike hypersurface Σ_0 in order to ascertain the differentiability of the spacetime across these three hypersurfaces and prove the extendibility of the metric beyond $\mathcal{F}$.

Further work will also involve examining the asymptotic behaviour of the outgoing radial null geodesics, and also the behaviour of the ingoing radial null geodesics, with a view to describing the global structure of the naked singularity in this spacetime. We also intend to study self-similar cylindrically symmetric spacetimes with more general energy momentum tensors.

Part IV

Appendices

Appendix A

Field equations

The G_{23} field equation component for unpolarised vacuum spacetime $\alpha(T,R) \neq R$ is

$$\begin{aligned}
\frac{\partial^2\omega}{\partial R^2} - \frac{\partial^2\omega}{\partial T^2} &= \frac{3\omega e^{4\psi}}{2\alpha^2}\left(\left(\frac{\partial\omega}{\partial R}\right)^2 - \left(\frac{\partial\omega}{\partial T}\right)^2\right) + \frac{1}{\alpha}\frac{\partial\alpha}{\partial R}\left(\frac{\partial\omega}{\partial R} - 4\omega\frac{\partial\psi}{\partial R}\right) - \frac{1}{\alpha}\frac{\partial\alpha}{\partial T}\frac{\partial\omega}{\partial T} \\
&+ 2\omega\left(\frac{\partial\psi}{\partial R}^2 - \frac{\partial\psi}{\partial T}^2 + \frac{\partial^2\gamma}{\partial R^2} - \frac{\partial^2\gamma}{\partial T^2} - 2\frac{\partial^2\psi}{\partial R^2} + 2\frac{\partial^2\psi}{\partial T^2}\right) \\
&+ \frac{4\omega}{\alpha}\left(\frac{\partial\alpha}{\partial T}\frac{\partial\psi}{\partial T}\right) - 4\left(\frac{\partial\omega}{\partial R}\frac{\partial\psi}{\partial R} - \frac{\partial\omega}{\partial T}\frac{\partial\psi}{\partial T}\right).
\end{aligned} \tag{A.1}$$

The field equations for polarised vacuum spacetime $\alpha \neq R$ are

$$\begin{aligned}
\frac{\partial^2\alpha}{\partial R^2} - \frac{\partial^2\alpha}{\partial T^2} &= 0, \\
\frac{\partial\alpha}{\partial R}\frac{\partial\gamma}{\partial R} + \frac{\partial\alpha}{\partial T}\frac{\partial\gamma}{\partial T} - \alpha\left(\left(\frac{\partial\psi}{\partial R}\right)^2 + \left(\frac{\partial\psi}{\partial T}\right)^2\right) = \frac{\partial^2\alpha}{\partial T^2} &= \frac{\partial^2\alpha}{\partial R^2}, \\
\left(\frac{\partial\psi}{\partial R}\right)^2 - \left(\frac{\partial\psi}{\partial T}\right)^2 + \frac{\partial^2\gamma}{\partial R^2} - \frac{\partial^2\gamma}{\partial T^2} &= 0, \\
\frac{\partial\gamma}{\partial R}\frac{\partial\alpha}{\partial T} + \frac{\partial\alpha}{\partial R}\frac{\partial\gamma}{\partial T} - 2\alpha\frac{\partial\psi}{\partial R}\frac{\partial\psi}{\partial T} - \frac{\partial^2\alpha}{\partial R\partial T} &= 0.
\end{aligned}$$

The field equations for polarised vacuum spacetime $\alpha = R$ are

$$
\begin{aligned}
\frac{\partial^2\psi}{\partial T^2} - \frac{1}{R}\frac{\partial\psi}{\partial R} - \frac{\partial^2\psi}{\partial R^2} &= 0, \\
\frac{\partial\gamma}{\partial T} - 2R\frac{\partial\psi}{\partial R}\frac{\partial\psi}{\partial T} &= 0, \\
\frac{\partial\gamma}{\partial R} - R\left(\left(\frac{\partial\psi}{\partial T}\right)^2 + \left(\frac{\partial\psi}{\partial R}\right)^2\right) &= 0, \\
\frac{\partial^2\gamma}{\partial R^2} - \frac{\partial^2\gamma}{\partial T^2} - \left(\frac{\partial\psi}{\partial T}\right)^2 + \left(\frac{\partial\psi}{\partial R}\right)^2 &= 0.
\end{aligned}
\tag{A.2}
$$

Appendix B

Centre manifold reduction for E_5 of (5.2.1)

To obtain some quantitative information about the equilibrium set E_5 we apply Theorem 2.4.9. We can linearise our dynamical system (5.2.1) about the point E_5 to obtain

$$\begin{aligned} \frac{\mathrm{d}(\vec{x}-\vec{x_0})}{\mathrm{d}\tau} &= J\cdot(\vec{x}-\vec{x_0}) + u(\vec{x}-\vec{x_0}), \\ \Rightarrow \frac{\mathrm{d}(\vec{\tilde{x}})}{\mathrm{d}\tau} = \dot{\vec{\tilde{x}}} &= J.\vec{\tilde{x}} + u(\vec{\tilde{x}}), \end{aligned}$$

where J is the Jacobian

$$J = \begin{pmatrix} 2k(k-1) & 0 & 0 \\ \frac{2(k-1)(2k^2-4k+1)}{(k-2)} & 0 & 0 \\ \frac{2k(k-1)(3-2k)^2}{(k-2)^2} & \frac{2(2k^3-10k^2-15k-7)}{(k-2)} & 2(2-k)(k-1) \end{pmatrix}$$

and

$$\vec{\tilde{x}} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} x-1 \\ y-k \\ z-f(k) \end{pmatrix}.$$

We build the matrix $\mathbf{P}$ necessary to put $\mathbf{J}$ into Jordan Normal form

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & f_1(k) \\ f_2(k) & 0 & f_3(k) \\ 1 & 1 & 1 \end{pmatrix}, \quad \mathbf{P}^{-1}\mathbf{J}\mathbf{P} = \mathbf{M} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix},$$

where

$$\begin{aligned} f_1(k) &= \frac{2k(2-k)^2}{(-7+29k-28k^2+8k^3)}, \quad f_2(k) = \frac{(2-k)^2}{(7-8k+2k^2)}, \\ f_3(k) &= \frac{f_1(k)(1-4k+2k^2)}{k(2-k)}. \end{aligned}$$

Next we reorganise our dynamical system in terms of our new coordinates

$$\vec{a} = \mathbf{P}^{-1}\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \left(\frac{-f_3(k)}{f_1(k)f_2(k)}\right)\tilde{x} + \frac{\tilde{y}}{f_2(k)} \\ \left(\frac{f_3(k)-f_2(k)}{f_1(k)f_2(k)}\right)\tilde{x} - \frac{\tilde{y}}{f_2(k)} + \tilde{z} \\ \frac{\tilde{x}}{f_1(k)} \end{pmatrix},$$

$$\begin{aligned} \mathbf{P}^{-1}\frac{\mathrm{d}\vec{\tilde{x}}}{\mathrm{d}\tau} &= \mathbf{P}^{-1}\mathbf{J}\vec{\tilde{x}} + \mathbf{P}^{-1}u(\vec{\tilde{x}}) = \mathbf{P}^{-1}\mathbf{J}\mathbf{P}\vec{a} + \mathbf{P}^{-1}u(\vec{\tilde{x}}), \\ \frac{\mathrm{d}\vec{a}}{\mathrm{d}\tau} &= \mathbf{M}\vec{a} + \mathbf{P}^{-1}u(\mathbf{P}\vec{a}) = \mathbf{M}\vec{a} + q(\vec{a}), \end{aligned}$$

and this gives us

$$\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}\tau} &= q_1(\vec{a}) = j_1(\vec{a}), \\ \frac{\mathrm{d}b}{\mathrm{d}\tau} &= \lambda_1 b + q_2(\vec{a}) = j_2(\vec{a}), \\ \frac{\mathrm{d}c}{\mathrm{d}\tau} &= \lambda_2 c + q_3(\vec{a}) = j_3(\vec{a}). \end{aligned}$$

We thus have a one dimensional centre manifold tangent to a and a two dimensional stable manifold tangent to the $b-c$ plane. To see the behaviour of solutions in the centre manifold we do a centre manifold reduction [10]. We assume that an analytic approximation to the centre manifold exists and has the form

$$\begin{aligned} b &= h_1(a) = B_1 a^2 + B_2 a^3 + \cdots , &\text{(B.1)}\\ c &= h_2(a) = C_1 a^2 + C_2 a^3 + \cdots , &\text{(B.2)}\end{aligned}$$

where the coefficients B_i and C_i, $i = 1, 2, 3...$ are to be determined. Then the dynamics of a solution in this 1-dim centre manifold will be given by the two equations

$$\begin{aligned} \frac{\mathrm{d}b}{\mathrm{d}\tau} = \frac{\partial h_1}{\partial a}\frac{\mathrm{d}a}{\mathrm{d}\tau} &\Leftrightarrow j_2(a, h_1(a), h_2(a)) = \frac{\partial h_1}{\partial a} j_1(a, h_1(a), h_2(a)),\\ \frac{\mathrm{d}c}{\mathrm{d}\tau} = \frac{\partial h_2}{\partial a}\frac{\mathrm{d}a}{\mathrm{d}\tau} &\Leftrightarrow j_3(a, h_1(a), h_2(a)) = \frac{\partial h_2}{\partial a} j_1(a, h_1(a), h_2(a)).\end{aligned}$$

These equations are satisfied by

$$B_1 = \frac{k-2}{(2k^2-8k+7)^2}, \quad B_2 = \frac{(k-2)^2(2k-3)}{(k-1)(2k^2-8k+7)^3}, \quad C_i = 0.$$

However, when we try to find the dynamics along the centre manifold by writing

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = j_1(a, h_1(a), h_2(a)),$$

we just get

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = 0$$

So there are no dynamics along the centre manifold, which we expected as this is a normally hyperbolic equilibrium point. Hence the asymptotic behaviour of the solution near E_5 is given by

$$\begin{aligned} a &= 0,\\ b &= c_1 e^{\lambda_1 \tau} + O(e^{2\lambda_1 \tau}),\\ c &= c_2 e^{\lambda_2 \tau} + O(e^{2\lambda_2 \tau}).\end{aligned}$$

Appendix C

Centre manifold reduction for E_2 of (6.1.1)

To analyse equilibrium set E_2 we linearise $F(\alpha, \beta, \gamma, s)$ about $s = 0$ Now we have our dynamical system in a convenient form, and when we linearise this system about E_2 as follows

$$\begin{aligned} \frac{\mathrm{d}(\vec{\alpha} - \vec{\alpha_2})}{\mathrm{d}T} &= J.(\vec{\alpha} - \vec{\alpha_2}) + f(\vec{\alpha} - \vec{\alpha_2}), \\ \Rightarrow \frac{\mathrm{d}(\vec{\tilde{\alpha}})}{\mathrm{d}T} &= J.\vec{\tilde{\alpha}} + f(\vec{\tilde{\alpha}}), \end{aligned}$$

where J is the Jacobian

$$J = \begin{pmatrix} k & 0 & 0 & 0 \\ k - \beta & 0 & 0 & 0 \\ f_1(k, \hat{\beta}) & f_2(k, \hat{\beta}) & 2 - k & f_3(k, \hat{\beta}) \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned} \text{where} \quad f_1(k,\hat{\beta}) &= \frac{(4k^3 - 7\hat{\beta} - 2k^2(6+\hat{\beta}) + k(9+8\hat{\beta})}{(k-2)^2}, \\ f_2(k,\hat{\beta}) &= \frac{(7-8k+2k^2)}{(k-2)}, \\ f_3(k,\hat{\beta}) &= \frac{\hat{\beta}^2}{(k-2)^2}. \end{aligned}$$

As J yields 2 non-zero eigenvalues, E_2 is a non hyperbolic equilibrium point. Denote

$$\vec{\tilde{\alpha}} = \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \\ \tilde{\gamma} \\ s \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta - \hat{\beta} \\ \gamma - h(k,\hat{\beta}) \\ s \end{pmatrix}.$$

We build the matrix $\mathbf{P}$ necessary to put $\mathbf{J}$ into Jordan Normal form,

$$\mathbf{P} = \begin{pmatrix} f_4(k,\hat{\beta}) & 0 & 0 & 0 \\ f_4(k,\hat{\beta})(k-\hat{\beta}) & f_5(k,\hat{\beta}) & \frac{k-2}{f_2(k,\hat{\beta})} & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\mathbf{P}^{-1}\mathbf{J}\mathbf{P} = \mathbf{M} = \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2-k \end{pmatrix},$$

$$\begin{aligned} \text{where} \quad f_4(k,\hat{\beta}) &= \frac{(2-k)^2}{(2k^3 - 7\hat{\beta} - 2k^2(3+\hat{\beta}) + k(5+8\hat{\beta}))}, \\ \text{and} \quad f_5(k,\hat{\beta}) &= \frac{\hat{\beta}^2}{(14-23k+12k^2-2k^3)}. \end{aligned}$$

Next we reorganise our dynamical system in terms of our new coordinates

$$\vec{A} = \mathbf{P}^{-1}\vec{\alpha} = \begin{pmatrix} A \\ B \\ C \\ S \end{pmatrix} = \begin{pmatrix} \frac{\tilde{\alpha}}{f_4} \\ \tilde{s} \\ \frac{-f_2(k-\hat{\beta})\tilde{\alpha}}{k-2} + \frac{f_2\tilde{\beta}}{k-2} + \frac{f_3\tilde{s}}{k-2} \\ \tilde{f_6}\tilde{\alpha} + \frac{-f_2\tilde{\beta}}{k-2} + \tilde{\gamma} + \frac{-f_3\tilde{s}}{k-2} \end{pmatrix},$$

$$\begin{aligned} \mathbf{P}^{-1}\frac{\mathrm{d}\vec{\hat{\alpha}}}{\mathrm{d}T} &= \mathbf{P}^{-1}\mathbf{J}\vec{\hat{\alpha}} + \mathbf{P}^{-1}f(\vec{\hat{\alpha}}) = \mathbf{P}^{-1}\mathbf{J}\mathbf{P}\vec{A} + \mathbf{P}^{-1}f(\vec{\hat{\alpha}}), \\ \frac{\mathrm{d}\vec{A}}{\mathrm{d}T} &= \mathbf{M}\vec{A} + \mathbf{P}^{-1}f(\mathbf{P}\vec{A}) = \mathbf{M}\vec{A} + q(\vec{A}), \end{aligned}$$

and this gives us

$$\begin{aligned} \frac{dA}{dT} &= kA + q_1(A,B,C,S) = j_1(A,B,C,S), \\ \frac{dB}{dT} &= q_2(A,B,C,S) = j_2(A,B,C,S), \\ \frac{dC}{dT} &= q_3(A,B,C,S) = j_3(A,B,C,S), \\ \frac{dS}{dT} &= (2-k)S + q_4(A,B,C,S) = j_4(A,B,C,S) \end{aligned}$$

We thus have a 2 dimensional centre manifold tangent to the $B-C$ plane and a two dimensional stable subspace tangent to the $A-S$ plane. To see the behaviour of solutions in the centre manifold we do a centre manifold reduction. We assume that an analytic approximation to the centre manifold exists and has the form

$$\begin{aligned} A &= h_1(B,C) = A_1B^2 + A_2C^2 + A_3BC + \cdots, \quad (C.1) \\ S &= h_2(B,C) = S_1B^2 + S_2C^2 + S_3BC + S_4B^3 + S_5C^3 + S_6B^2C + S_7BC^2 + \cdots, \end{aligned}$$

where the coefficients A_i and S_i, $i = 1, 2, 3..$ are to be determined by the two equations

$$\frac{\mathrm{d}A}{\mathrm{d}T} = j_1(h_1(B,C), B, C, h_2(B,C)) = \frac{\partial h_1}{\partial B}\frac{\mathrm{d}B}{\mathrm{d}T} + \frac{\partial h_1}{\partial C}\frac{\mathrm{d}C}{\mathrm{d}T}, \quad (C.2)$$

$$\frac{\mathrm{d}S}{\mathrm{d}T} = j_4(h_1(B,C), B, C, h_2(B,C)) = \frac{\partial h_2}{\partial B}\frac{\mathrm{d}B}{\mathrm{d}T} + \frac{\partial h_2}{\partial C}\frac{\mathrm{d}C}{\mathrm{d}T}, \quad (C.3)$$

where we use

$$\frac{\mathrm{d}B}{\mathrm{d}T} = j_2(h_1(B,C), B, C, h_2(B,C)), \qquad \frac{\mathrm{d}C}{\mathrm{d}T} = j_3(h_1(B,C), B, C, h_2(B,C)).$$

These equations are satisfied by

$$\begin{aligned} S_4 &= \frac{(17-16k+4k^2)\hat{\beta}^4}{((k-2)^5(7-8k+2k^2)^2)}, \quad S_6 = \frac{3(9-8k+2k^2)\hat{\beta}^2}{(14-23k+12k^2-2k^3)^2}, \\ S_7 &= \frac{2(k-2)(5-4k+k^2)}{(7-8k+2k^2)^2}, \quad A_i = 0. \end{aligned}$$

However, when we try to find the dynamics in the centre manifold by writing the 2-dim system

$$\begin{aligned} \frac{\mathrm{d}B}{\mathrm{d}T} &= j_2(h_1(B,C), B, C, h_2(B,C)), \\ \frac{\mathrm{d}C}{\mathrm{d}T} &= j_3(h_1(B,C), B, C, h_2(B,C)), \end{aligned}$$

we just get

$$\frac{\mathrm{d}B}{\mathrm{d}T} = A(f_4(k,\hat{\beta})B), \quad \frac{\mathrm{d}C}{\mathrm{d}T} = A(f_7(k,\hat{\beta})B - f_4(k,\hat{\beta})C + \cdots),$$

$$\text{where} \quad f_7(k,\hat{\beta}) = \frac{2(\hat{\beta})^2}{(2-k)((2k^3-7\hat{\beta}-2k^2(3+\hat{\beta}))+k(5+8\hat{\beta})).}$$

Therefore we have shown that A $(=\frac{\alpha}{f_4})$ has no B or C dependance when we 'project' A into the centre manifold, in the analytic approximation, up to third order, so we can write

$$\alpha = c_1 \exp(kT) + \mathcal{O}(\exp(2kT)).$$

Appendix D

Centre manifold reduction for E_2 of (7.2.1)

We introduce new coordinates

$$\hat{a} = a - 1, \quad \hat{b} = b - 1, \quad \hat{c} = c - \left(-\frac{1}{2}\right), \quad \vec{\hat{a}} = (a, b, c)^T,$$

so that $\vec{a_3}$ is situated at the origin in these coordinates We linearise about $\vec{a_3}$ to get

$$\frac{\mathrm{d}\vec{\hat{a}}}{\mathrm{d}\tau} = \mathbf{J}\vec{\hat{a}} + \mathcal{O}(\vec{\hat{a}}^2), \quad \mathbf{J} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -\frac{3}{2} & -1 & -4 \end{pmatrix}$$

We build the matrix $\mathbf{P}$ necessary to put $\mathbf{J}$ into Jordan Normal form

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & -4 \\ 0 & -4 & 5 \\ 1 & 1 & 0 \end{pmatrix}, \quad \mathbf{P}^{-1} = \frac{1}{16}\begin{pmatrix} 5 & 4 & 16 \\ -5 & -4 & 0 \\ -4 & 0 & 0 \end{pmatrix},$$

$$\text{such that} \quad \mathbf{M} = \mathbf{P}^{-1}\mathbf{J}\mathbf{P} = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Next we transform our dynamical system

$$\begin{aligned}\mathbf{P}^{-1}\frac{\mathrm{d}\hat{a}}{\mathrm{d}\tau} &= \mathbf{P}^{-1}\mathbf{J}\vec{\hat{a}} + \mathbf{P}^{-1}g(\vec{\hat{a}}) = \mathbf{P}^{-1}\mathbf{J}\mathbf{P}\vec{x} + \mathbf{P}^{-1}g(\vec{\hat{a}}),\\ \frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau} &= \mathbf{M}\vec{x} + \mathbf{P}^{-1}g(\mathbf{P}\vec{x}),\end{aligned}$$

where $\vec{x} = P^{-1}\vec{\hat{a}}$. Setting $\vec{x} = (x_1, x_2, x_3)^T$ we get

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}\tau} &= -4x_1 + g_1(\vec{x}) = j_1(\vec{x}),\\ \frac{\mathrm{d}x_2}{\mathrm{d}\tau} &= x_3 + g_2(\vec{x}) = j_2(\vec{x}),\\ \frac{\mathrm{d}x_3}{\mathrm{d}\tau} &= g_3(\vec{x}) = j_3(\vec{x}).\end{aligned}$$

We thus have a one dimensional stable manifold tangent to x_1 and a two dimensional centre manifold tangent to the $x_2 - x_3$ plane. To see the behaviour of solutions in the centre manifold we do a centre manifold reduction. We assume that an analytic approximation to the centre manifold exists and has the form

$$x_1 = A_1x_2^2 + A_2x_3^2 + A_3x_2x_3 + \mathcal{O}(\vec{x}^3) = h(x_2, x_3). \qquad \text{(D.1)}$$

Then the dynamics of a solution in this 2-dim centre manifold will be given by

$$\frac{\mathrm{d}x_1}{\mathrm{d}\tau} = \frac{\partial h}{\partial x_2}\frac{\mathrm{d}x_2}{\mathrm{d}\tau} + \frac{\partial h}{\partial x_3}\frac{\mathrm{d}x_3}{\mathrm{d}\tau}$$

This implies

$$j_1(h(x_2,x_3),x_2,x_3) = \frac{\partial h}{\partial x_2}j_2(h(x_2,x_3),x_2,x_3) + \frac{\partial \vec{x}}{\partial x_3}j_3(h(x_2,x_3),x_2,x_3). \qquad \text{(D.2)}$$

This equation is satisfied by

$$A_1 = 1, \quad A_2 = -\frac{3}{4}, \quad A_3 = -\frac{7}{4}.$$

We can find the form of (D.1) to any desired order by reverting to (D.2) and solving for the lowest order terms, once you have (D.1) for subsequent orders. We have found the analytic approximation to the centre manifold to third order

$$x_1 = x_2^2 - \frac{3}{4}x_3^2 - \frac{7}{4}x_2x_3 + x_2^3 + \frac{27}{32}x_3^3 - \frac{169}{16}x_2x_3^2 - \frac{61}{4}x_2^2x_3 + \cdots$$

We can now use this approximation to calculate the dynamics in the centre manifold:

$$\begin{aligned}
\frac{dx_2}{d\tau} &= x_3 - 8x_2x_3 - \frac{61}{4}x_3^2 + 7x_2^2x_3 + \left(\frac{735}{4}\right)x_2x_3^2 - \left(\frac{6048}{64}\right)x_3^3 + \cdots, \\
\frac{dx_3}{d\tau} &= -4x_2x_3 - 11x_3^2 + 80x_2x_3^2 - 52x_3^3 + \cdots
\end{aligned}$$

As we are only interested in local behaviour we truncate at $\mathcal{O}(\vec{x}^3)$ and look at the phase space dynamics of our centre manifold

$$\frac{dx_3}{dx_2} = \frac{-4x_2x_3 - 11x_3^2}{x_3 - 8x_2x_3 - \frac{61}{4}x_3^2 + 7x_2^2x_3} \approx \frac{-4x_2 - 11x_3}{1 - 8x_2 - \frac{61}{4}x_3} = \frac{L_1}{L_2}.$$

Consider next the possible values that x_2 and x_3 can take.
We know that $a \in (0,1)$ $\forall\tau$ from Property 1, and we know that

$$x_3 = -\frac{\hat{a}}{4} = -\frac{(a-1)}{4} \Rightarrow x_3 \in \left(0, \frac{1}{4}\right) \quad \forall\tau$$

Next we consider x_2. We know that $b \in (0,\infty)$ $\forall\tau$ from Property 2 and we know that

$$\begin{aligned}
x_2 &= -\frac{5}{16}\hat{a} - \frac{4}{16}\hat{b} = -\frac{5}{16}(a-1) - \frac{4}{16}(b-1), \\
\Rightarrow 16x_2 &= -5(a-1) - 4(b-1) = -5a - 4b + 9.
\end{aligned}$$

But we know that as we approach the equilibrium point $\vec{a_3}$, $a \uparrow 1$. Then there exists $\tau_1 < \infty$ such that for all $\tau \in (\tau_1, \infty)$ we have

$$\frac{da}{d\tau} > 0, \quad \Rightarrow (1 - ab) > 0 \quad \Rightarrow b < \frac{1}{a}.$$

Thus if we are approaching the equilibrium point $\vec{a_3}$, we have, for all $\tau \in (\tau_1, \infty)$

$$16x_2 = -5a - 4b + 9 > -5a - \frac{4}{a} + 9 = -\frac{(a-1)(5a-4)}{a} \equiv \tilde{f}(a).$$

Thus $\tilde{f}(a) > 0$ for $a \in \left(\frac{4}{5}, 1\right)$. There exists $\tau_2 \in (\tau_1, \infty)$ such that for all $\tau \in (\tau_2, \infty)$ we have

$$\tilde{f}(a) \geq 0 \quad \Rightarrow x_2 > 16\tilde{f}(a) \geq 0.$$

So we can restrict ourselves to the first quadrant $Q1 = \{x_2, x_3 | x_2 \geq 0, x_3 \geq 0\}$.
It is useful to divide $Q1$ into two regions. R_1 is the region above the line L_2 and R_2 is the region below the line L_2, where $L_2 = \{x_2, x_3 | 1 - 8x_2 - \frac{61}{4}x_3 = 0\}$. Thus $Q1 = \{R_1 \cup R_2\}$.

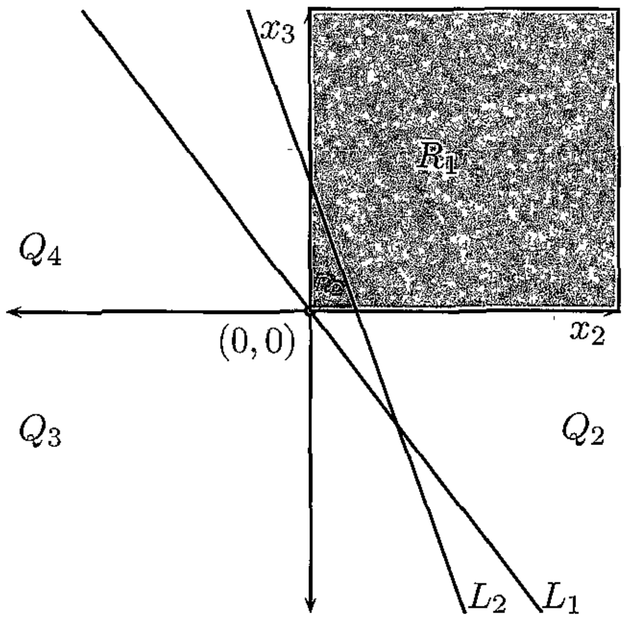

Figure D.1: Phase space for equilibrium set E_3.

Approach to origin in centre manifold

If our solution is at a point $p \in R_1$ for some $\tau_3 \in (\tau_2, \infty)$ then

$$\frac{dx_3}{dx_2} > 0, \quad \forall p \in R_1.$$

This implies that this solution cannot reach the origin.
If our solution is at a point $p \in R_2$ for some $\tau_3 \in (\tau_2, \infty)$ then

$$\frac{\mathrm{d}x_3}{\mathrm{d}x_2} < 0, \quad \forall p \in R_2.$$

This implies that we can only reach the point $(x_2, x_3) = (B_e, 0)$ where $B_e \in (0, \frac{1}{8}]$. So we cannot reach the origin from this point either.
As $B_e \in (0, \frac{1}{8}] \Leftrightarrow B_0 \in \left[\frac{1}{2}, 1\right)$ these centre manifold dynamics could lead to the endpoint

$$(a, b, c) \to \left(1, B_0, -\frac{B_0}{B_0 + 1}\right) \text{ with } B_0 \in \left[\frac{1}{2}, 1\right).$$

Bibliography

[1] T. A. Apostolatos and K. Thorne. Rotation halts cylindrical, relativistic gravitational collapse. *Phys. Rev.*, D46:2435, 1992.

[2] A. Ashtekar, J. Bicak, and B. G. Schmidt. Asymptotic structure of symmetry-reduced general relativity. *Phys. Rev.*, D55(2):669–686, 1997.

[3] B Aulbach *Continuous and Discrete Dynamics near Manifolds of Equilibria.* Lecture Notes in Mathematics No. 1058. Springer, 1984.

[4] A. Barnes. A comment on a paper: "On the definition of cylindrical symmetry". *Class Quantum Grav.*, 17(13):2605, 2000.

[5] C Barrabés, W. Israel, and P. S. Letelier. Analytic models of nonspherical collapse, cosmic censorship and the hoop conjecture *Phys. Lett. A*, 160(1):41, 1991.

[6] B K. Berger, P. T. Chrusciel, and V. Moncrief. On "asymptotically flat" space-times with G^2-invariant Cauchy surfaces. *Ann. Phys. (N.Y.)*, 237(2):322–354, 1995.

[7] J. Carot, L Mas, and A M. Sintes. Space-times admitting a three-parameter similarity group. *J. Math. Phys.*, 35(7):3560–3570, 1994.

[8] J. Carot, J. M. M. Senovilla, and R. Vera. On the definition of cylindrical symmetry. *Class. Quantum Grav*, 16(9):3025–3034, 1999.

[9] B. J. Carr and A. A. Coley. The similarity hypothesis in general relativity. *Gen. Relativity Gravitation*, 37(12):2165–2188, 2005.

[10] J. Carr. *Applications of Centre Manifold Theory.* Springer-Verlag, 1981.

[11] T. Chiba. Cylindrical dust collapse in general relativity—toward higher-dimensional collapse *Progr. Theoret. Phys.*, 95(2):321–338, 1996.

[12] F. Echeverria. Gravitational collapse of an infinite cylindrical dust shell *Phys Rev.*, D47:2271, 1993.

[13] A. Einstein and N. Rosen. On gravitational waves. *Journal Franklin Institute*, 223·43–54, 1937.

[14] A. Einstein and E. G. Straus. The influence of the expansion of space on the gravitation fields surrounding the individual stars. *Rev. Mod. Phys.*, 17.120, 1945.

[15] F. Fayos, J. M. M. Senovilla, and R. Torres General matching of two spherically symmetric spacetimes. *Phys. Rev.*, D54(8):4862, 1996.

[16] C. Gundlach. Critical phenomena in gravitational collapse. *Living Rev. Relativity 2*, (4), 1999 URL (cited on 1/7/2007): http://www.livingreviews.org/lrr-1999-4.

[17] S. W. Hawking and G. F. R. Ellis. *The large scale structure of space-time.* Cambridge University Press, London, 1973. Cambridge Monographs on Mathematical Physics, No. 1.

[18] S A. Hayward Gravitational waves, black holes and cosmic strings in cylindrical symmetry. *Class Quantum Gravity*, 17(8)·1749–1764, 2000.

[19] J. M. Heinzle, A. D. Rendall, and C. Uggla. Theory of Newtonian self-gravitating stationary spherically symmetric systems. *Math. Proc. Cambridge Philos. Soc.*, 140(1):177–192, 2006.

[20] U. Kirchgraber and K J Palmer. *Geometry in the Neighbourhood of invariant manifolds of maps and flows and linearization.* Longman, 1990

[21] D. Kramer, H. Stephani, M. A. H. MacCallum, C. Hoenselaers, and E. Herlt. *Exact Solutions of Einstein's Field Equations.* Cambridge University Press, 2003.

[22] M. Mars. Axially symmetric Einstein-Straus models *Phys Rev. D(3)*, 57(6):3389-3400, 1998

[23] M. Mars. On the uniqueness of the Einstein-Straus model *Class. Quantum Grav.*, 18(17):3645–3663, 2001.

[24] M. Mars and J. M. M. Senovilla Geometry of general hypersurfaces in spacetime junction conditions. *Class Quantum Grav.*, 10:1865–1897, 1993.

[25] K-I. Nakao and Y Morisawa. High speed dynamics of collapsing cylindrical dust fluid. *Class. Quantum Grav*, 21(8):2101, 2004.

[26] B. C. Nolan. Naked singularities in cylindrical collapse of counter-rotating dust shells. *Phys. Rev.*, D65(10):104006, 2002.

[27] B. C. Nolan and L. V. Nolan On isotropic cylindrically symmetric stellar models. *Class. Quantum Grav.*, 21(15):3693, 2004

[28] B C. Nolan and R. Vera. Axially symmetric equilibrium regions of Friedmann-Lemaître-Robertson-Walker universes. *Class. Quantum Gravity*, 22(19):4031–4050, 2005.

[29] J. R. Oppenheimer and H. Snyder. On continued gravitational contraction. *Phys. Rev.*, 56:455, 1939.

[30] R Penrose Gravitational collapse: the role of general relativity. *Classical Quantum Gravity*, 1:252–276, 1969

[31] L. Perko. *Differential Equations and Dynamical Systems.* Springer-Verlag, 1991.

[32] T Piran. Cylindrical general relativistic collapse. *Phys. Rev. Lett.*, 41(16):1085–1088, Oct 1978.

[33] J. M. M. Senovilla and R Vera. *Phys. Rev. Lett.*, 78:2284, 1996.

[34] L. F. Shampine and M W Reichelt. The MATLAB ODE suite. *SIAM Journal on Scientific Computing*, 18:1–22, 1997.

[35] S L. Shapiro and S. A Teukolsky Formation of naked singularities: the violation of cosmic censorship. *Phys. Rev. Lett.*, 66(8):994–997, 1991.

[36] M. Sharif and S. Aziz. Kinematic self-similar cylindrically symmetric solutions *Int. J. Modern Phys.*, D14(9):1527–1543, 2005

[37] K. Thorne. Energy of infinitely long, cylindrically symmetric systems in general relativity. *Phys Rev. (2)*, 138:B251–B266, 1965.

[38] K. S. Thorne. in Magic without Magic: John Archibald Wheeler, edited by J. Klauder (Freeman, San Francisco, 1972), Box 32 3.

[39] K. S. Thorne. *Geometrodynamics of cylindrical systems*. PhD thesis, Princeton University, 1965. University Microfilms Inc., Ann Arbor, MI.

[40] R Vera Symmetry-preserving matchings. *Class. Quantum Grav.*, 19:5249–5264, 2002.

[41] J. Wainwright and G. F R. Ellis *Dynamical Systems in Cosmology*. Cambridge University Press, 1997

[42] A. Wang. Critical collapse of a cylindrically symmetric scalar field in four-dimensional Einstein's theory of gravity. *Phys. Rev.*, D68(6):064006, 2003.