Worst-Case-Optimal Dynamic Reinsurance for Large Claims

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Abstract

We control the surplus process of a non-life insurance company by dynamic proportional reinsurance. The objective is to maximize expected (utility of the) surplus under the worst-case claim development. In the large claim case with a worst-case upper limit on claim numbers and claim sizes, we find the optimal reinsurance strategy in a differential game setting where the insurance company plays against mother nature. We analyze the resulting strategy and illustrate its characteristics numerically. A crucial feature of our result is that the optimal strategy is robust to claim number and size modeling and robust to the choice of utility function. This robustness makes a strong case for our approach. Numerical examples illustrate the characteristics of the new approach. We analyze the optimal strategy, e.g. in terms of the more conventional, in the insurance context, objective of minimizing the probability of ruin. Finally, we calculate the intrinsic risk-free return of the model and we show that the principle of Markowitz – don't put all your eggs in one basket – does not hold in this setting.

Keywords: dynamic proportional reinsurance, reserve process, worst-case scenario approach, Cramér-Lundberg model, differential game, robust optimization

Mathematics Subject Classification: 91B30, 93E20

1 Introduction

The optimal control of a non-life business has been an object for research for around two decades. The industrial case is a non-life insurance company with access to a reinsurance market and/or an investment market and seeking for optimal decision making, statically or dynamically, in order to optimize the expectation of some function of the surplus. This paper takes the same starting point but with some specific features that takes us away from the main roads.

Most importantly, we consider the decision making of the insurance company as one opponent's part in a differential game against mother nature or whomever causes the claims on the books. At first glance this may seem odd, given the usual perception that mother nature plays randomly and not with the objective to hurt insurance companies. However, this is also just the mathematical-economic way of interpreting the setup of the problem. A consequence of the formulation, and this is its real motivation, is that the insurance company controls optimally for the worst-case or, more generally, for some 'bad case' specified by the insurance company. Worst-case optimization is finding its way to theory and practice these years where risk-based solvency rules are implemented, typically, e.g. in Solvency II, in terms of stress test (i.e. bad case) measures. There is no convention about how the worst-case optimization – or solvency capital optimization – problems are properly formulated, but we present in this paper one approach. Before we give an overview over some more standard optimization problem formulations, we explain here a couple of nice features of our formulation:

- The problem formulation and solution is independent of the claim number and size distributions. The only piece of information that one needs is some worst-case claim number and size levels. Whether that level is a true upper bound or e.g. a quantile in some distribution or something else influences, of course, the exact interpretation of the solution, but it does not influence the mathematics that lead to it. **We speak of robustness against claim number and size modeling.**
- In the problem formulation, the insurance company maximizes utility of the surplus at some given deterministic time horizon. However, a delicate consequence of our setup is that the result does not depend on the utility function as long as it is increasing. Note, that it does not even have to be concave, even allowing for utility function of prospect theory type. One of

the drawbacks of practical utility optimization is the challenge of estimating one's utility function, so this feature is a great relief in our approach. **We speak of robustness against choice of utility function.**

The features listed above make a strong case for our approach compared with the alternative more standard approaches that we very briefly mention here.

A standard objective is the probability of ruin which is often sought to be minimized (against mother nature playing randomly). This objective has a long history of research and as part of a control problem it has been studied, among many others, by Hipp and Vogt [11], Schmidli [20, 21, 22], and Eisenberg and Schmidli [5].

Another standard objective is allowing the surplus process to pay out dividends and then maximizing the expected discounted sum of dividend payments until ruin. Whereas, the ruin probability minimization is in the spirit of the prudent actuary's pattern of thinking, maximizing dividends until (typically certain!) ruin is more in the spirit of the greedy economist's pattern of thinking. Important contributions are found in Gerber [9], Azcue and Muler [2], and Albrecher and Thonhauser [1].

Finally, a standard objective in economics, but less standard in the context of controlling a non-life insurance company, is expected utility maximization of the surplus. Although utility maximization is a classical tool in insurance mathematics for pricing and optimal contract design, it is rarely used as a criterion for dynamic decision making. Exceptions include, for instance, Liu and Ma [17] and Liang and Guo [16]. Other papers using this approach justify it by relying on the conjecture that maximizing expected exponential utility is the same as or similar to minimizing the probability of ruin, see e.g. Ferguson [6], Browne [4], and Fernández et al. [7].

Our approach has, of course, different similarities with each of the standard ideas, e.g. by viewing the expected utility (for any reasonable utility function) as something we want to maximize. However, the differential game setup leading to the worst-case optimization interpretation makes the situation quite different. Worst-case optimization has a longer history in the context of engineering and medicine (drug development) but it was introduced in the context of portfolio optimization by Korn and Wilmott [15]. This approach has been extended in various ways so far, including with respect to the utility function by Korn and Menkens [13], with respect to optimizing the investment portfolio of an insurance company by Korn [12], conceptually, studied as a stochastic differential game, by Korn and Steffensen [14], and methodologically by Seifried [23]. The worst-case approach has strong connotations to the increasingly important notion of robust optimization (see e.g. the survey of Bertsimas et al. [3] and the references therein). The standard robust optimization approach is to calculate the worst-case of some perturbation analysis. Whereas the approach taken in this paper is to consider the worst-case of all possible scenarios. Indeed, our approach is per se the most robust optimization approach possible.

In this paper the main objective is to solve the worst-case problem with respect to proportional reinsurance and to illustrate numerically and analyze its solution. In particular, we study the dynamics of the optimally controlled surplus process and we calculate, in order to put our approach in perspective, the ruin probability following from our optimal strategy. Moreover, we calculate the minimum return for the insurance company given that it has sufficient initial reserves. We name this minimum return the intrinsic risk-free return of the model. Finally, we compare different business strategies with each other. In doing so, we show that the principle of Markowitz (don't put all your eggs in one basket) does not hold in this setting.

The outline of the paper is as follows: Section 2 introduces the model considered while Section 3 formulates and solves the optimization problem. Numerical examples and sensitivities of the optimal worst-case reinsurance strategy are given in Section 4. In Section 5, the evolution of the net reserve process is described and in Section 6 the probability of ruin is calculated. Section 7 derives the guaranteed intrinsic risk–free rate of return if the optimal worst-case reinsurance strategy is used. In Section 8 different business strategies are compared with each other. Finally, Section 9 concludes the paper and gives an outlook.

2 The Model

We consider a simple model for non-life reinsurance consisting of a constant continuously paid premium π and claims which occur at random times. Additionally, it is assumed that all insurance policies are terminated at time T and that no claim is possible thereafter. Further, instead of making a particular assumption about the exact form of the distribution of the claims arrival process $N^c(t)$, we assume that the number of claims arriving in $[0,T]$ are bounded, i.e. we assume

$$N^c\,(T) \leq N \tag{1}$$

with N being the maximum possible number of claims in $[0,T]$. Apart from these assumptions, the exact dynamics of the arrival process N^c plays no role, as long as the process satisfies Assumption (1) and has paths which are RCLL (i.e. right-continuous with left limits). Moreover, we suppose that all claims have a non-negative size which is bounded above by β. More specifically, let b_n (with $n = 1, \ldots, N$) denote the actual observed claim size of the $N - n + 1$–st claim arriving at the random time τ^n (which is determined by the claims arrival process N^c). Furthermore, let us assume that the insurance company can reinsure the fraction $p(t) \in [0,1]$ at time t of its business which is also known as dynamic reinsurance. We require $p(t)$ to be predictable with regard to the filtration generated by N^c. To reinsure the fraction $p(t)$ of its business, the insurance company has to pay the rate $\pi\, p(t)\,[1+\varepsilon]$, where $\varepsilon \geq 0$ is the load or premium. The case of $\varepsilon = 0$ is known as cheap reinsurance compared to $\varepsilon > 0$ which is called non-cheap reinsurance. Thus, the dynamics of the reserve process $R(t)$, the income/outflow from reinsurance $I^p(t)$, and the net reserve process $R^p(t)$ are specifically given by

$$\begin{aligned} dR(t) &= \pi\, dt - \beta\, dN^c \\ dI^p(t) &= -\pi\, p(t)\, [1+\varepsilon]\, dt + \beta\, p(t)\, dN^c \\ dR^p(t) &= dR(t) + dI^p(t) \\ &= \pi\, [1 - p(t)\, [1+\varepsilon]]\, dt - \beta\, [1 - p(t)]\, dN^c\ . \end{aligned} \tag{2}$$

In order to simplify the notation, the reserve processes defined here are only the worst-case reserve processes, since we assume that any incoming claim will have the worst-case claim size β. It is straightforward to extend this to the general reserve processes. All the following results hold for both, the worst-case reserve processes and the general reserve processes. If the results are different, this is made clear (see e.g. Corollary 5.2 and Proposition 5.4 below.)

Observe that Assumption (1) can be included into our setting above by assuming

$$dR(t) = \pi\, dt$$

after having observed N claims.

Suppose that τ is a time where a claim comes in. Clearly, the following holds at time τ if a claim of size $b \in [0, \beta]$ occurs:

$$R^p(\tau) = R^p(\tau-) - b\, [1 - p\,(\tau)]\ . \tag{3}$$

Here we have spoken of both N and β as strict worst-case bounds on claim numbers and sizes. If one has a probabilistic view on claim numbers and sizes, one can, of course, replace the strict bounds by some quantiles in the distributions. This was already suggested in the q-quantile approach by Menkens [19] for the claim size. The flexibility in the interpretation of the bounds leads to a similar flexibility in the interpretation of the results.

3 Worst Case Scenario Optimization

Our aim is to maximize the worst-case expected utility of final reserves

$$\sup_{p\in\mathcal{A}} \inf_{N^c\in\mathcal{B}} \mathbb{E}\left[U\left(R^p\left(T\right)\right)\right]\ , \tag{4}$$

where $\mathcal{A}$ and $\mathcal{B}$ are the sets of admissible controls for p and N^c, respectively. More specific, $\mathcal{A}$ is the set of all predictable processes with respect to the $\sigma-$algebra generated by the jump process which determines how many claims are still possible. The set $\mathcal{B}$ denotes the set of all such possible jump processes. We further assume that the utility function $U(x)$ is strictly increasing and defined on $\mathbb{R}$.

Remark 3.1
Allowing the opponent of the game to choose among all jump processes may seem to be in strict opposition to the standard robust optimization approach which is sometimes also called a worst-case approach, namely to let the opponent choose a jump intensity in a band of allowed intensities. But our worst-case approach can actually also be formalized in that setting. Letting λ_i denote the intensity for a claim occurring given that i claims have already occurred (note that we are not assuming a Poisson claims arrival process), our worst-case claims are similar to the worst-case intensity where the λ_i is chosen in $[0,\infty]$ for $i \leq N$ and $\lambda_i = 0$ for $i > N$. After this observation, our approach can be viewed in the light of the general trend towards capital control where decisions are made in order to e.g. minimize solvency capital requirements where such requirements may be formalized by stressed scenarios.

With the objective given by (4), the value function $V^n(t,x)$ of our problem is given by

$$V^n(t,x) = \sup_{p\in\mathcal{A}} \inf_{N^c\in\mathcal{B}} \mathbb{E}^{t,x,n}\left[U\left(R^p\left(T\right)\right)\right] ,$$

where $\mathbb{E}^{t,x,n}$ is the conditional expectation given that $X(t) = x$ and given that there are at most n claims possible left. We are now in the position to formulate the main theorem of this paper.

Theorem 3.2 (Verification Theorem)
Let the function $U(x)$ be strictly increasing and be defined on $\mathbb{R}$. Given that $n \in \{1,...,N\}$ possible claims can still occur, the optimal worst-case reinsurance strategy $p^n(t)$ is defined through the following property of the value function

$$\begin{aligned} V^n(t,x) &= V^{n-1}\left(t, x-\beta\left(1-p^n(t)\right)\right) &(5)\\ &= U\left(x+\pi\left(T-t\right)-\beta\sum_{i=1}^{n}\left(1-p^i(t)\right)\right). &(6)\end{aligned}$$

It is given as the unique solution of the (system of) ordinary differential equation

$$p_t^n\left(t\right) = \frac{\pi}{\beta}\left[1+\varepsilon\right]\left[p^n\left(t\right)-p^{n-1}\left(t\right)\right] \qquad (7)$$

with boundary conditions

$$\begin{aligned} p^n(T) &= 1 \qquad \text{for } n=1,2,\ldots,N &(8)\\ p^0(t) &\equiv 0 \qquad \text{for all } t\in[0,T]. &(9)\end{aligned}$$

In particular, we have that the value functions are monotonically increasing. Further, with the notation of $\alpha = \frac{\pi}{\beta}\left[1+\varepsilon\right]$, the optimal worst-case reinsurance strategy p^n has the explicit form of

$$p^n(t) = \exp\left(-\alpha\left[T-t\right]\right)\sum_{k=0}^{n-1}\frac{1}{k!}\left(\alpha\left[T-t\right]\right)^k, \quad n=1,2,\ldots,N \qquad (10)$$

$$p^0(t) = 0. \tag{11}$$

Note that, due to our assumptions, there is no further claim possible in the case of $n = 0$. We then have $p^0(t) \equiv 0$ as there is no need to buy reinsurance. Hence, the insurance company can and will enjoy the still incoming premia without worrying about any potential claims. This also yields $V^0(t,x) = U(x + \pi(T-t))$, the value function if no more claim can occur. This can be included in Equation (6) by interpreting it for $n = 0$ in an obvious way.

Proof of Theorem 3.2: The case of $n = 0$ follows directly from the remark preceding the proof. We prove the general case by induction over n, the number of claims still possible before T.

Case $n = 1$:

Let $p^1(t)$ be the reinsurance strategy such that the insurer is indifferent between the largest possible claim coming in now and no claim coming in at all. This requirement yields the relation

$$\begin{aligned} & U\left(x + \pi(T-t) - \beta\left(1 - p^1(t)\right)\right) \\ &= U\left(x + \pi(T-t) - \pi(1+\epsilon)\int_t^T p^1(s)\,ds\right). \end{aligned} \tag{12}$$

Observe that it is necessary to integrate from t to T, because nature might want to play a trick on the insurance company at any time and let the insurance company leave in uncertainty up to the last second.

Using that U is strictly monotone (that is the arguments of $U(\cdot)$ are equal in the above equation), and differentiating the arguments with respect to t and denoting the derivative of p^1 with respect to t by p^1_t, leads to

$$\begin{aligned} p^1(T) &= 1 \qquad \text{and} \\ p^1_t(t) &= \pi\frac{1+\epsilon}{\beta}p^1(t) = \alpha\left(p^1(t) - p^0(t)\right). \end{aligned} \tag{13}$$

Consequently, we have

$$p^1(t) = \exp(-\alpha(T-t)). \tag{14}$$

Note in particular that by relation (12) the insurer is indifferent between the worst claim happening **at any time** or never.

Worst-case optimality of $p^1(t)$ can be seen by explicit comparison with an arbitrary admissible reinsurance strategy $p(t)$. Due to our measurability assumptions $p(t)$ has to be a deterministic function between two claims and satisfies $p(t) \in [0,1]$. We can further use a left-continuous version of it without changing the value of the reserves process.

For $p(t)$ to be a better worst-case strategy than $p^1(t)$, we must have

$$p(0) > p^1(0) \tag{15}$$

as otherwise an immediate claim of size β yields a worst-case performance which is not better than that of $p^1(.)$ (note that this situation also yields the worst-case bound for $p^1(.)$).

On the other hand if there is no claim in $t=0$, there is a $0<\bar{t}<T$ with

$$\int_0^{\bar{t}} p(s)\,ds \geq \int_0^{\bar{t}} p^1(s)\,ds \quad \textbf{and} \quad p(\bar{t}) \leq p^1(\bar{t}). \tag{16}$$

To see this note that $\bar{t}=0$ would contradict relation (15), $\bar{t}=T$ would contradict the worst-case optimality of $p(.)$ as it would then not outperform $p^1(.)$ in the no-claim scenario. However, a claim of size β at time $\bar{t}$ shows also that $p(.)$ cannot outperform $p^1(.)$ in the worst-case sense as $p^1(.)$ is indifferent between all those scenarios.

We have thus proved every assertion as $V^1(t,x)$ by construction has the asserted form and is then monotonically increasing in x.

Case $n-1 \Rightarrow n$:

We now denote by $p^n(t)$ the indifference strategy between the highest claim occurring now and no claim occurring until T given that still at most n claims can come in. As by induction we already know the optimal reinsurance strategy and value function with still $(n-1)$ possible claims, that is $p^{n-1}(t)$ and $V^{n-1}(t,x)$, respectively, the indifference requirement yields:

$$\begin{aligned} V^n(t,x) &= V^{n-1}(t, x-\beta(1-p^n(t))) \qquad \Longleftrightarrow \\ U\left(x+\pi(T-t)-\pi(1+\epsilon)\int_t^T p^n(s)\,ds\right) & \\ &= U\left((x-\beta(1-p^n(t)))+\pi(T-t)-\beta\sum_{i=1}^{n-1}\left(1-p^i(t)\right)\right) \\ &= U\left(x+\pi(T-t)-\beta\sum_{i=1}^{n}\left(1-p^i(t)\right)\right) \end{aligned} \tag{17}$$

As in the case $n=1$, using that U is strictly monotone (which implies the equality of the arguments of $U(\cdot)$), differentiating the arguments with respect to t, and using induction for $n-1$, we arrive at the differential equation for $p^n(t)$:

$$\begin{aligned} p_t^n(t) &= \alpha\left(p^n(t)-p^{n-1}(t)\right) \\ &= \alpha p^n(t)-\exp(-\alpha(T-t))\sum_{k=0}^{n-2}\frac{1}{k!}(\alpha(T-t))^k, \\ p^n(T) &= 1. \end{aligned}$$

Solving this equation by the variation of constants formula yields

$$p^n(t) = \exp(-\alpha(T-t))\left(1+\int_t^T\sum_{k=0}^{n-2}\frac{1}{k!}(\alpha(T-s))^k ds\right)$$

$$
\begin{aligned}
&= \exp\left(-\alpha\left(T-t\right)\right)\left(1+\sum_{k=0}^{n-2}\frac{1}{(k+1)!}\left(\alpha\left(T-t\right)\right)^{k+1}\right)\\
&= \exp\left(-\alpha\left(T-t\right)\right)\sum_{k=0}^{n-1}\frac{1}{k!}\left(\alpha\left(T-t\right)\right)^{k}.
\end{aligned}
$$

Clearly, this implies that $p^n(t) > p^{n-1}(t)$ for all $t \in [0,T]$, since α is positive. By the already proved monotonicity of $V^{n-1}(t,x)$ in x, worst-case optimality of $p^n(t)$, if still n claims are possible, is proved totally similar to the case of $n = 1$. Further, the form of $V^n(t,x)$ and its monotonicity follow by induction. □

Remark 3.3

a) The strategy p^n can be interpreted as the probability of experiencing less than N claims in an artificial model where the claim numbers are Poisson distributed with intensity α, i.e.

$$
p^n(t) = P\left(N^P(T) < N \,\middle|\, N^P(t) = N-n\right),
$$

where $\left\{N^P(t)\right\}_{t\geq 0}$ is a Poisson process with intensity α. This is a convenient way to think about the result in the form of a rule-of-thumb: **Calculate an intensity α as the worst-case premium to claim ratio. Then, reinsure the proportion of your portfolio corresponding to the probability of experiencing less than a given number N of claims.**

In particular, note that this Poisson process N^P has nothing to do with and is therefore independent from the claims arrival process N^c.

b) Note that, due to the form of the differential equation (7) for $p^n(\cdot)$, it is clear that the worst-case scenario optimal strategy is independent of the utility function used. This is due to the fact that the randomness in our model only enters the claim occurrence and claim size, but does not affect the dynamics of the risk process apart from jump times. However, the jump events (i.e. the claim size and time) only have an effect on the value function, not the reinsurance strategy. Hence, the value function depends on U, but the worst-case scenario optimal strategy is independent of U.

Moreover, note that U does not need to be a utility function for Theorem 3.2 to hold. U just needs to be strictly increasing.

c) The requirement to consider a utility function that is defined on the whole real line can be relaxed if we assume a kind of **prudent insurance condition**, namely the assumption of

$$
N \leq \frac{\pi T}{\beta}.
$$

This is of course a very strong condition as in any state of the world the premia cover all possible losses. However, if an insurer with a utility function such as the log-utility or the power utility wants to consider a worst-case approach, he actually has to ask for such a high premium π as otherwise he cannot ensure to have a finite expected utility.

An alternative view to this requirement is given by looking at the initial reserve level $R(0)$. So far, we assumed tacitly that $R(0) = 0$. For $R(0) = y$ with $y > 0$, however, the **prudent insurance condition** can be rewritten as

$$N \leq \frac{\pi T + y}{\beta}.$$

Hence, the premia cover no longer all possible losses, which is more realistic. This implies that the insurance company faces the risk of losing money if too many claims come in. That is, there exist an n_0 with $n_0 < N$ such that

$$n > \frac{\pi T}{\beta} \quad \text{for all } n > n_0 \text{ and } n \leq N.$$

However, a careful study of the proof of Theorem 3.2 reveals that the optimal worst-case reinsurance strategy p^n is indeed independent from the initial reserve level $R(0)$ – just as it is independent from the current reserve level.

d) Note that we again have an indifference requirement:

$$V^n(t,x) = V^{n-1}(t, x - \beta\,(1 - p^n(t))). \tag{18}$$

This condition is no longer multiplicative as in the case of worst-case portfolio optimization of stocks (see e.g. Korn and Wilmott [15], Korn and Menkens [13], or Korn and Steffensen [14]). Indeed it is additive and it can be rewritten in a multiplicative form where the impact of the claim depends on the reserve level.

$$V^{n-1}(t, x - \beta\,(1 - p^n(t))) = V^{n-1}\left(t, x\left(1 - \frac{\beta}{x}\,(1 - p^n(t))\right)\right). \tag{19}$$

Additionally, as Equation (18) has to hold for all $t \in [0,T]$, it directly implies that $p^n(T) = 1$ for all $n \in \{1, 2, \ldots, N\}$.

4 Numerical examples and sensitivity analysis

Figure 1 depicts the worst-case optimal reinsurance strategy for $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0$, whereas Figure 2 pictures the worst-case optimal reinsurance strategy for $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0.5$. The figures are read as follows. The insurance company uses the worst-case reinsurance strategy p^4 until the first claim comes in. At the time the first claim comes in (e.g. $\tau^4 = 0.5$), the insurance company switches from the worst-case optimal reinsurance strategy p^4 to the worst-case optimal reinsurance strategy p^3. The insurance follows p^3 until another claim is made (e.g. at time $\tau^3 = 2$), where it switches to the worst-case optimal reinsurance strategy p^2, etc.

Clearly, one can see that the increasing costs of reinsurance in Figure 2 have to be compensated for by the insurer via taking more claim risk (compare Figure 1 with Figure 2).

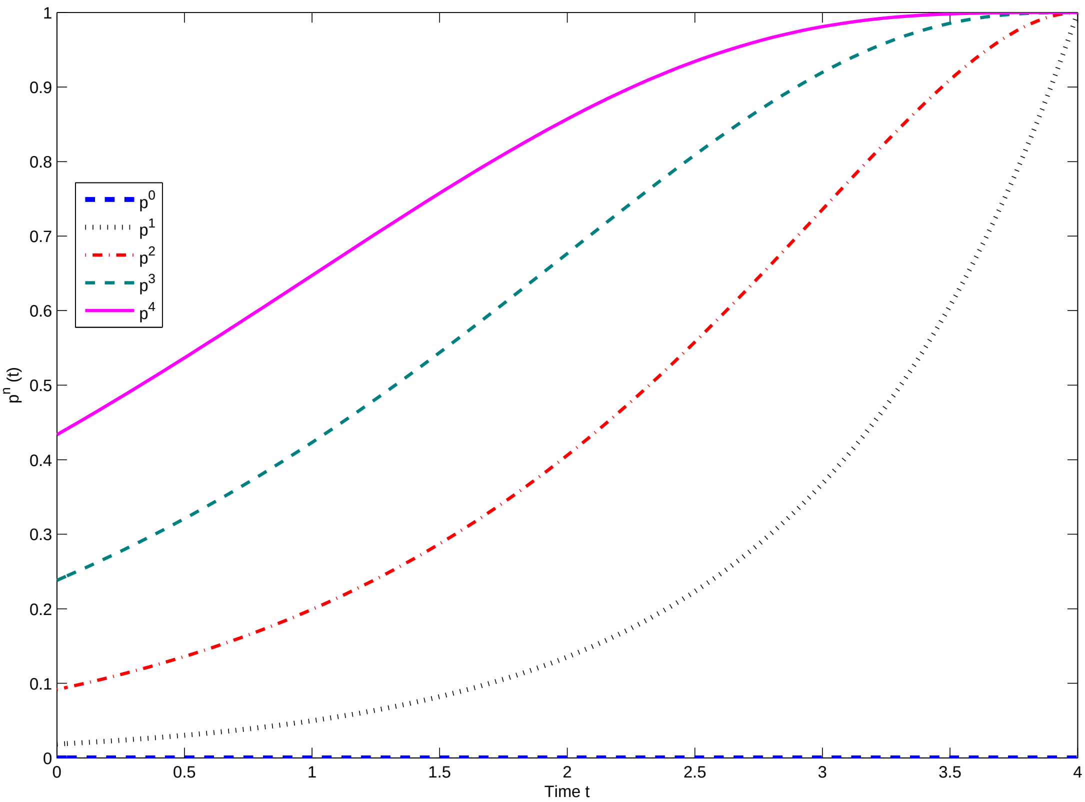

Figure 1: Worst-Case Optimal Reinsurance (p^n) for $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0$

This graphic shows the worst-case optimal reinsurance strategy (p^n) for $n = 0, 1, \ldots, N$ and with $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0$.

Here, we discuss the sensitivity of the strategy $p^n(t)$ with respect to various parameters.

(i) As already seen above, if ε increases (meaning that reinsurance becomes dearer), the reinsurance level declines (which is in line with practice).

$$\frac{\partial p^n(t)}{\partial \varepsilon} = -\frac{\pi}{\beta}\,[T-t]\,\frac{(\alpha\,[T-t])^{n-1}}{(n-1)!}\,\exp\left(-\alpha\,[T-t]\right)$$

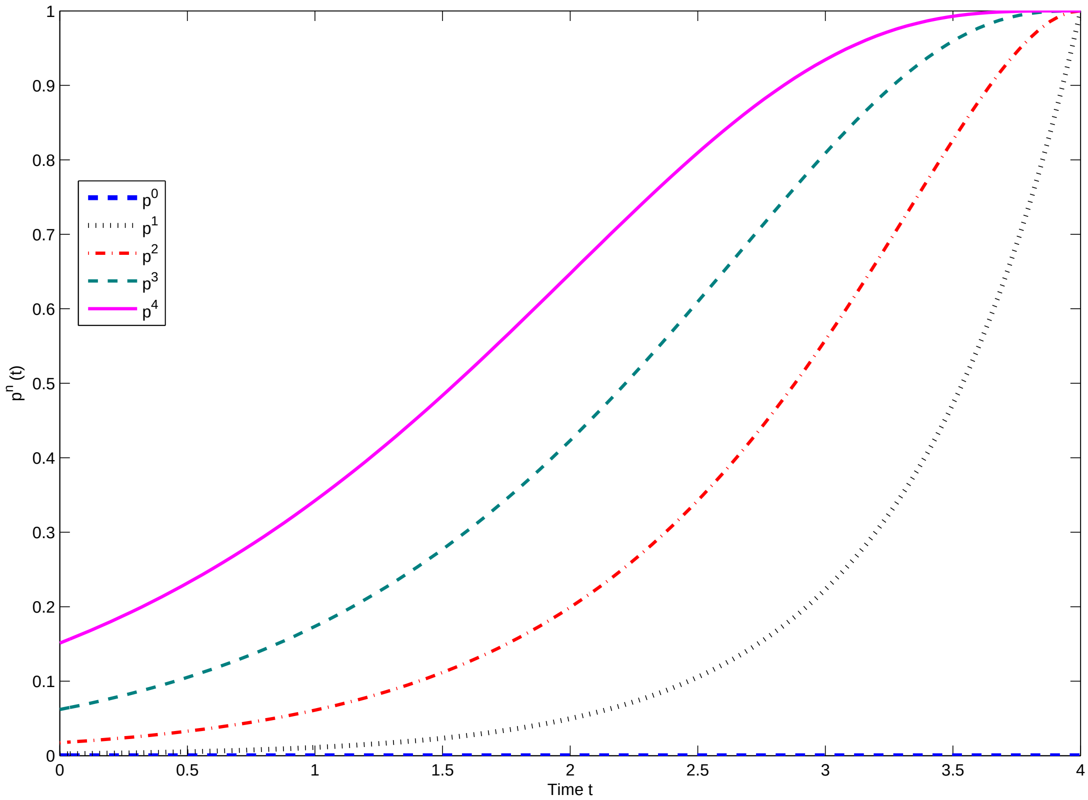

Figure 2: Worst-Case Optimal Reinsurance (p^n) for $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0.5$

This graphic shows the worst-case optimal reinsurance strategy (p^n) for $n = 0, 1, \ldots, N$ and with $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0.5$.

$$= \frac{\pi}{\beta} [T - t] \left[p^{n-1}(t) - p^n(t)\right] \leq 0 .$$

(ii) Similarly, if π increases, the reinsurance level declines as well.

$$\begin{aligned} \frac{\partial p^n(t)}{\partial \pi} &= -\frac{1}{\beta} [1 + \varepsilon] [T - t] \frac{(\alpha [T - t])^{n-1}}{(n-1)!} \exp(-\alpha [T - t]) \\ &= \frac{1}{\beta} [1 + \varepsilon] [T - t] \left[p^{n-1}(t) - p^n(t)\right] \leq 0 . \end{aligned}$$

(iii) If β increases, the reinsurance level increases which is intuitively clear as larger claim sizes should be considered riskier than smaller ones.

$$\begin{aligned} \frac{\partial p^n(t)}{\partial \beta} &= \frac{1}{\beta} \exp(-\alpha [T - t]) \frac{(\alpha [T - t])^n}{(n-1)!} \\ &= \frac{n}{\beta} \left[p^{n+1}(t) - p^n(t)\right] \geq 0 . \end{aligned}$$

(iv) Further, p^n decreases in $T-t$:

$$\begin{aligned}\frac{\partial p^n(t)}{\partial(T-t)} &= -\,\alpha\frac{(\alpha\,[T-t])^{n-1}}{(n-1)!}\exp\left(-\alpha\,[T-t]\right)\\ &= \alpha\left[p^{n-1}(t)-p^n(t)\right]\leq 0\,. \qquad (20)\end{aligned}$$

Observe that all inequalities above hold for $t\in[0,T]$ and are strict if $t<T$.

5 Evolution of the Net Reserve Process

If at most n (with $n\geq 1$) claims are still possible and the optimal worst-case reinsurance strategy p^n is used, then the worst-case net reserve process is given by (compare with Equation (2))

$$dR^{p^n}(t)=\pi\left[1-p^n(t)\left[1+\varepsilon\right]\right]dt-\beta\left[1-p^n(t)\right]dN^P\ .$$

For the remainder of this article, we use the following definition.

Definition 5.1
Let $0\leq s\leq t\leq T$. Denote by $R\left(s,t;p,n\right)$ the net reserve process between time s and t which consists of the incoming premia and the outgoing reinsurance premia given that the insurance company uses the reinsurance strategy $p(u)$ with $u\in[s,t]$, given that at most n claims can come in, and given that no claim occurs in $[s,t]$. If the optimal worst-case reinsurance strategy p^n is considered, the notation is simplified to $R\left(s,t;p^n\right):=R\left(s,t;p^n,n\right)$.

With this definition, one has that

$$R\left(s,t;p^n\right)=\int_s^t\pi\left[1-p^n(u)\left[1+\varepsilon\right]\right]du=\pi\left[t-s\right]-\pi\left[1+\varepsilon\right]\int_s^t p^n(u)\,du\,,$$

Using Equation (10) in combination with $\alpha=\frac{\pi}{\beta}\left[1+\varepsilon\right]$, the substitution $x=T-u$, and formula 3.351.1. in Gradshteyn and Ryzhik [10], p. 340, the latter integral computes to

$$\begin{aligned}\int_s^t p^n(u)\,du &= \int_s^t e^{-\alpha[T-u]}\sum_{k=0}^{n-1}\frac{1}{k!}\left(\alpha\left[T-u\right]\right)^k du\\ &= \sum_{k=0}^{n-1}\sum_{l=0}^{k}\alpha^{l-1}\,\frac{e^{-\alpha[T-t]}\left[T-t\right]^l-e^{-\alpha[T-s]}\left[T-s\right]^l}{l!}\\ &= \frac{1}{\alpha}\sum_{k=1}^{n}\left[p^k\left(t\right)-p^k\left(s\right)\right]\ .\end{aligned}$$

Altogether, the net reserve process between time s and t is given as

$$R(s,t;p^n) = \pi[t-s] - \beta \sum_{k=1}^{n} \left[p^k(t) - p^k(s)\right] , \tag{21}$$

provided the reinsurance strategy p^n is used and no claims occur in $[s,t]$. There are two special cases which will be considered more closely in the following. The first one is given by setting $s=0$, that is

$$R(0,t;p^n) = \pi t - \beta \sum_{k=1}^{n} \left[p^k(t) - p^k(0)\right] , \tag{22}$$

which will simply be called the *forward net reserve process without claims* and second,

$$R(t,T;p^n) = \pi(T-t) - \beta \sum_{k=1}^{n} \left[1 - p^k(t)\right] , \tag{23}$$

which will simply be called the *backward net reserve process without claims*. Before continuing let us establish an important property of $R(t,T;p^n)$ which can be verified either by direct computation or by using Theorem 3.2 with $U(x)=x$.

Corollary 5.2 (Backward Reserve Process)
For any $t \in [0,T]$ and $n \in \{1,\ldots,N\}$ the following holds

$$R(t,T;p^n) \le R\left(t,T;p^{n-1}\right) - b_n(1-p^n(t)) , \tag{24}$$

where $b_n \le \beta$ denotes the actually observed claim size for the $N-n+1$–st claim. Equality holds in (24) if and only if $b_n = \beta$.

Examples for the evolution of $R(0,t,p^n)$ are given in Figures 3 and 4 while examples for $R(t,T,p^n)$ are given in Figures 5 and 6. Note that $R(t,T,p^n)$ can become negative if $\varepsilon > 0$. This is, because $\pi T = \beta N$; thus $\pi(T-t) - \beta N < 0$ for all $t > 0$. Here it has been assumed that all claims made are of the worst-case size β. However, it is straightforward to verify that $\pi t + R(t,T,p^n) \ge 0$ for all $t \in [0,T]$.

Remark 5.3
Comparing Equation (23) with Equation (6), it is straightforward to verify that

$$V^n(t,x) = U(x + R(t,T;p^n)) .$$

In particular, for $U = id$ and $x = 0$, one has

$$V^n(t,0) = R(t,T;p^n) .$$

Therefore, Figures 5 and 6 depict also $V^n(t,0)$ if $U = id$ and can therefore be used to get an idea of $V^n(t,x)$.

Notice that it is straightforward – using Equation (5) and (6) – to verify that

$$V^n(t,x) \leq V^m(t,x) \qquad \text{for all } t \in [0,T],\ x \geq 0, \text{ and } 0 \leq m < n \leq N.$$

Equality holds only for $t = T$ as this is the only case where $p^n(t) = 1$, which is a necessary condition for equality to hold. An example of the value function is given for the case $U = id$ in Figure 7, compare also with Figures 5 and 6.

Let us calculate the reserve process up to time t^* assuming that $N - n$ claims occurred up to time t^*.

Proposition 5.4 (Forward Reserve Process)
Denote by τ^i the arrival time of the $N-i+1$–st claim and its size by b_i, then $0 \leq \tau^N < \ldots < \tau^i < \ldots < \tau^{n+1} \leq t^* \leq T$ and $b_i \leq \beta$ with $i = n+1, \ldots, N$. Assuming that the insurance follows the worst-case reinsurance strategy – that is up to the first claim p^N, between the first and second claim p^{N-1}, etc., the reserve process at time t^* computes to

$$\begin{aligned} & R\left(0,\tau^N;p^N\right) - b_N\left[1-p^N\left(\tau^N\right)\right] + \sum_{i=n+1}^{N-1}\left\{R\left(\tau^{i+1},\tau^i;p^i\right) - b_i\left[1-p^i\left(\tau^i\right)\right]\right\} + \\ & \quad + R\left(\tau^{n+1},t^*;p^n\right) \\ = \ & R\left(0,t^*;p^N\right) + \sum_{k=n+1}^{N}(\beta-b_k)\left[1-p^k\left(\tau^k\right)\right] - \beta\sum_{i=n+1}^{N}\left[1-p^i\left(t^*\right)\right] . \end{aligned} \tag{25}$$

*In particular, the lower bound or the **worst-case bound** for the reserve process at time T is given by*

$$R\left(0,T;p^N\right) \;=\; \pi T - \beta\sum_{k=1}^{N}\left[1-p^k(0)\right] , \tag{26}$$

assuming that the insurance follows the worst-case reinsurance strategy (p^n), $n = 0,1,\ldots,N$. This bound is reached if either no claim is made or if all claims that are made are of the worst-case size β.

The last part in the above proposition is straightforward to verify by Equation (25) for $t^* = T$ in combination with the fact that $p^k(T) = 1$ for all $k \geq 1$. Intuitively, the worst case is given if the claims are made when the reinsurance level is lowest. Since p^k is strictly increasing in t (see Equation (20)), the reinsurance level is lowest at time $t = 0$.

Moreover, observe that this comment means that the insurer is indifferent towards a claim of the worst-case size coming in or not – and if a claim of the worst-case size comes in, the insurer is also indifferent towards at which time the claim is made. This is exactly what the worst-case scenario approach claims to do and which can be verified in this case directly. However, if the claim size is strictly less than the worst-case size, the insurance prefers a claim is made over no claim is made and it is advantageous if the claim is made sooner than later (this is Equation (25)) in combination with the fact that p^n is strictly increasing in t (see Equation (20)).

Unlike Corollary 5.2 Equation (25) does not follow directly from Theorem 3.2 but needs to be verified by direct calculation.

Proof of Proposition 5.4: Equation (25) is verified by straightforward but tedious computations, which show that

$$\begin{aligned} & R\left(0,\tau^N;p^N\right) - b_N\left[1-p^N\left(\tau^N\right)\right] + \sum_{i=n+1}^{N-1}\left\{R\left(\tau^{i+1},\tau^i;p^i\right) - b_i\left[1-p^i\left(\tau^i\right)\right]\right\} + \\ & \quad + R\left(\tau^{n+1},t^*;p^n\right) \\ = & \quad \pi t^* - \beta\sum_{k=1}^{N}\left[1-p^k(0)\right] + \beta\sum_{k=1}^{n}\left[1-p^k(t^*)\right] + \sum_{k=n+1}^{N}(\beta-b_k)\left[1-p^k\left(\tau^k\right)\right] \end{aligned} \tag{27}$$

which gives the assertion. □

Examples for $R\left(0,t^*;p^N\right) - \beta\sum_{k=n+1}^{N}\left[1-p^k(t^*)\right]$ as given in Equation (25) (assuming that $b_i=\beta$ for all i) are given in Figures 8 and 9. The figures show that the insurance company arrives always at the same final net reserve $R\left(0,T;p^N\right)$, if all claims that are actually made are of the worst-case claim size.

Note, however, that these figures also reveal the possibility that the net reserve process might be negative for some time before getting positive in the end. Thus, if not bankruptcy, the insurance is facing at least a liquidity problem. The next section analyses this in more detail.

6 Computing the Probability of Ruin

So far, the initial reserve level was not considered because the worst-case reinsurance strategy is independent of the reserve level at any time. While it is clear that the insurance would not go bankrupt in the long run if $\beta N \leq \pi T$ (or if inequality (28) holds given that the insurance uses the worst-case reinsurance strategy), it is possible that the insurance has a liquidity problem (that is the reserve level is temporarily negative) for some scenarios (see e.g. Figures 8 and 9).

First, note that the minimum requirement of a reinsurance strategy is that it is not possible for the insurance to go bankrupt by using the reinsuance strategy $p(t)$ and no claim comes in. Mathematically speaking, this weak solvency condition

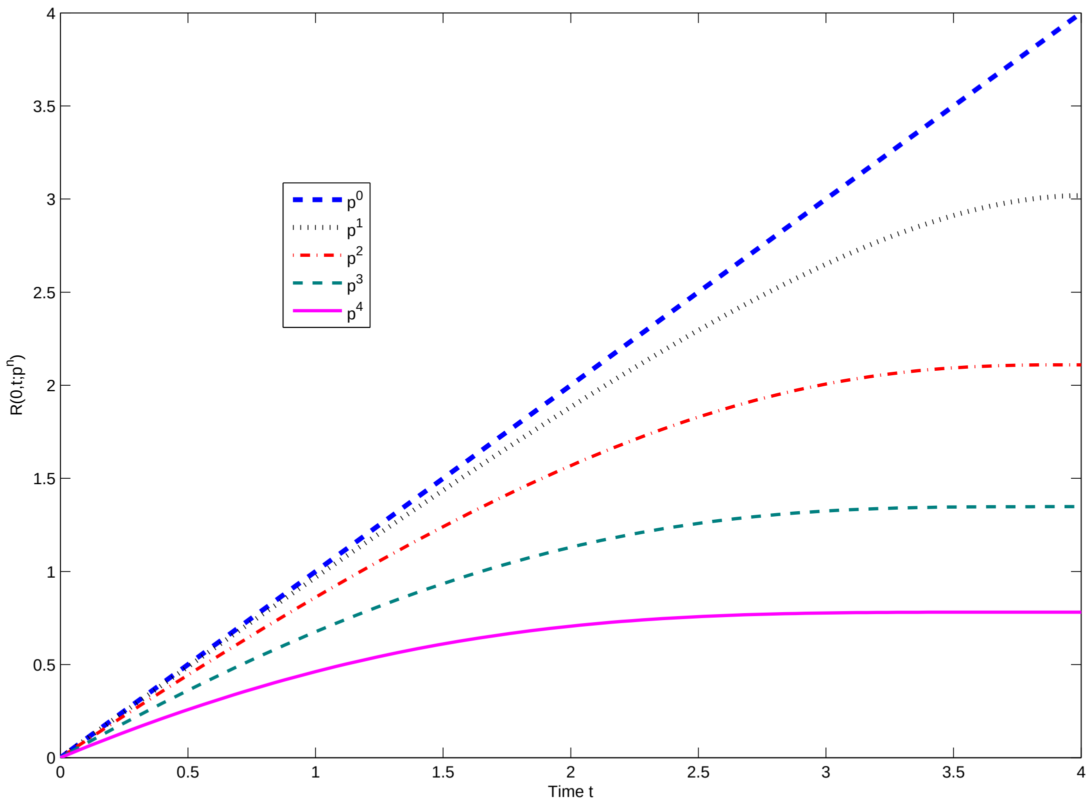

Figure 3: $R(0,t,p^n)$ with $\pi = \beta = 1$, $T = N = 4$, and $\varepsilon = 0$
This graphic shows the forward net reserve process without claims for $n = 0, 1, \ldots, N$ and with $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0$.

is not a necessary condition, because it might be possible that the insurance company can minimize its ruin probability by assuming that from some fixed time onwards there would have claims being made. However, in practice, it might be difficult (not to say impossible) for an insurance company to say that it went bankrupt because there were too few claims made. The weak solvency condition gives the following average upper bound for any reinsurance strategy.

Lemma 6.1 (Average Upper Bound for the Reinsurance Strategy)
*Given that the initial reserve is $R(0) = y \geq 0$ and that the **weak solvency condition** $R(0,t;p,N,y) := y + R(0,t;p,N) \geq 0$ holds for all $t \in [0,T]$, then the reinsurance strategy p satisfies*

$$\frac{1}{t}\int_0^t p(s)\,ds \;\leq\; \frac{1}{1+\varepsilon} + \frac{y}{\pi\,(1+\varepsilon)\,t} \qquad \text{for all } t \in [0,T]\,,$$

which is a strict bound for $\varepsilon > 0$ and $y = 0$. Moreover, the worst-case reinsurance

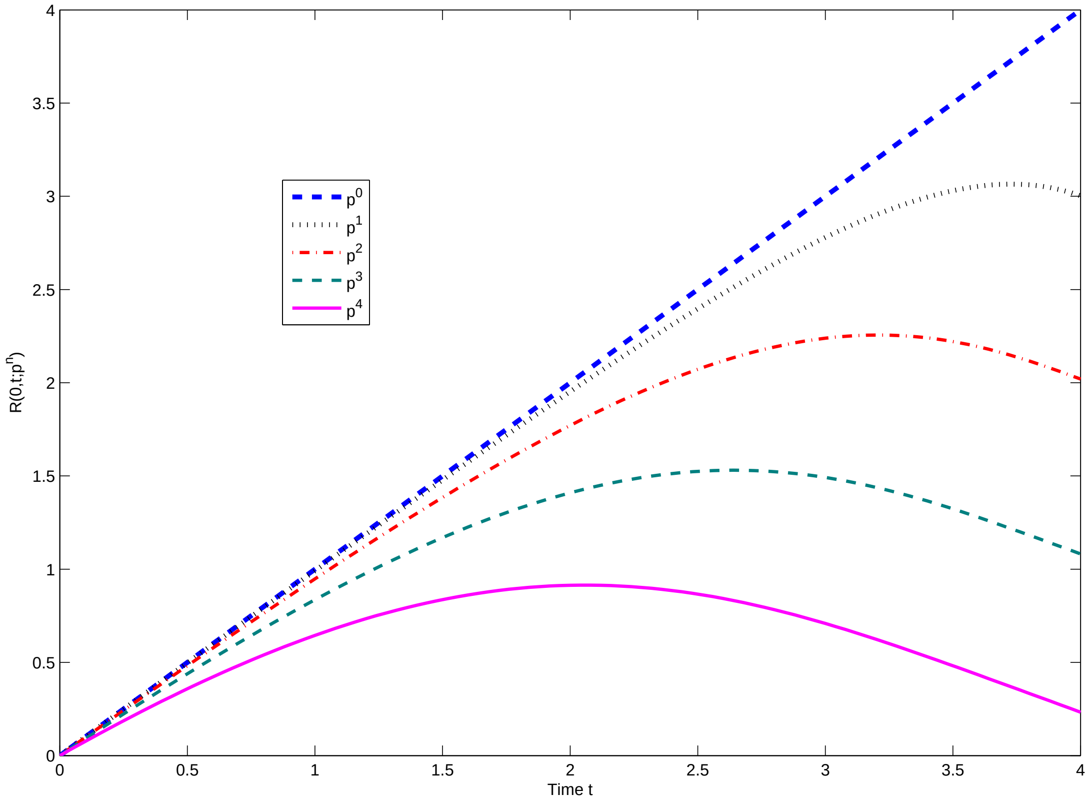

Figure 4: $R(0,t,p^n)$ with $\pi = \beta = 1$, $T = N = 4$, and $\varepsilon = 0.5$
This graphic shows the forward net reserve process without claims for $n = 0, 1, \ldots, N$ and with $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0.5$.

strategy satisfies this weak solvency condition if and only if

$$\beta N \leq y + \pi T + \beta \sum_{k=1}^{N} \left[1 - p^k(0)\right] . \tag{28}$$

Proof : The first part follows directly from Equation (2), assuming that no claim occurs. The weak solvency condition for the worst-case reinsurance strategy is given by $R\left(0, t; p^N, y\right) \geq 0$ for all $t \in [0, T]$. To see this assertion, note that $R\left(0, 0; p^N, y\right) = y \geq 0$ and Equation (26) gives the value for $R\left(0, T; p^N\right)$ (to be adjusted by the initial reserve y). Furthermore, differentiating representation (22) of the forward net reserve process without claims twice, we see that $R\left(0, t; p^N, y\right)$ is concave in t, since

$$\frac{d^2}{dt^2} R\left(0, t; p^N, y\right) = -\alpha^2 \beta \left[p^N(t) - p^{N-1}(t)\right] \leq 0 ,$$

where it has been used that $p^N(t) \geq p^{N-1}(t)$ for all $t \in [0, T]$. Hence the minimum is attained either in $t = 0$ or $t = T$, which concludes the proof. □

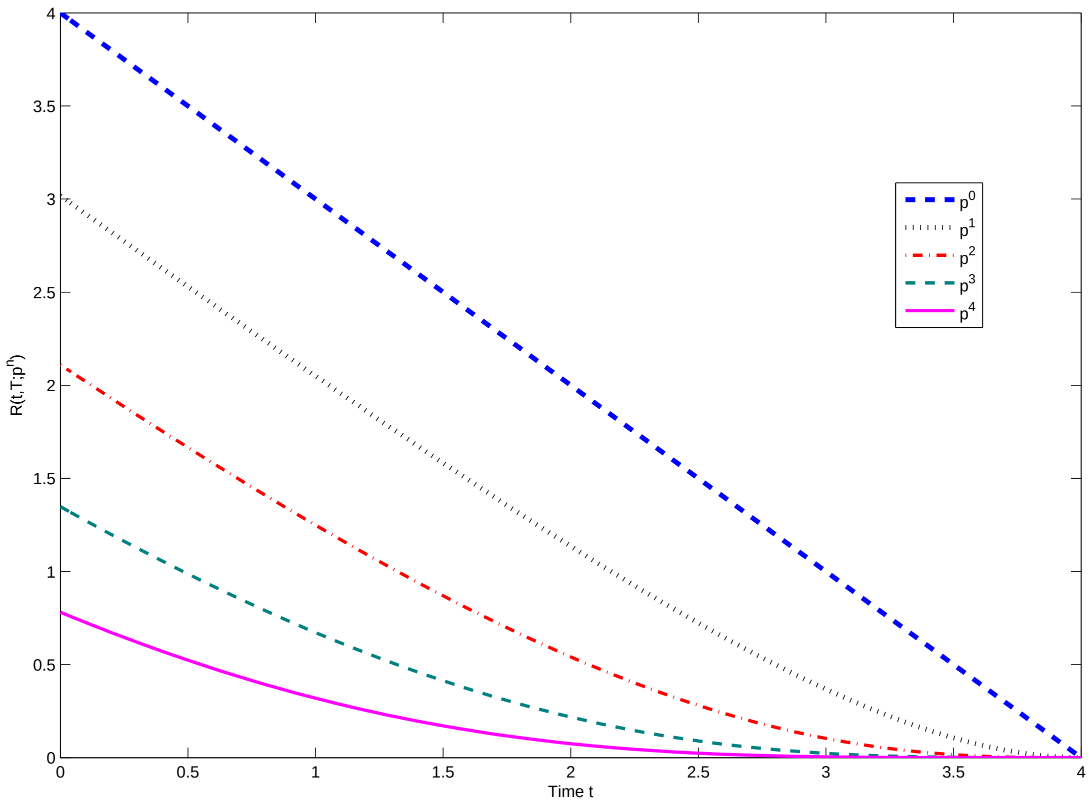

Figure 5: $R(t, T, p^n)$ with $\pi = \beta = 1$, $T = N = 4$, and $\varepsilon = 0$
This graphic shows the backward net reserve process without claims $R(t, T, p^n)$ for $n = 0, 1, \ldots, N$ and with $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0$.

In order to identify the scenarios where the reserve level is negative, the zeros of Equation (25), adjusted by the initial capital y, are calculated for $n = 0, 1, \ldots, N-1$. The zeros are denoted by $t_0^n(y)$ or simply t_0^n as they clearly depend on the initial capital y. This means that $N - n$ claims have been made so far (at times $0 \leq \tau^N \leq \tau^{N-1} \leq \ldots \leq \tau^{n+1} \leq t_0^n \leq T$) and n claims might still be made. Using Equation (27) with $t^* = t_0^n$, this can be written as

$$\pi t_0^n + \beta \sum_{k=1}^{n} \left[1 - p^k\left(t_0^n\right)\right] \quad = \quad \beta \sum_{k=1}^{N} \left[1 - p^k\left(0\right)\right] - \sum_{k=n+1}^{N} \left(\beta - b_k\right) \left[1 - p^k\left(\tau^k\right)\right] - y\,,$$

which can be solved explicitly just for $n = 0$:

$$t_0^0 = \frac{\beta}{\pi} \sum_{i=1}^{N} \left[1 - p^i(0)\right] - \frac{y}{\pi} - \frac{1}{\pi} \sum_{k=1}^{N} \left(\beta - b_k\right) \left[1 - p^k\left(\tau^k\right)\right] . \tag{29}$$

Given the initial reserve y and the reinsurance strategy p, denote by $\psi_p(y)$ the probability of ruin. Sometimes, the probability of survival is considered instead

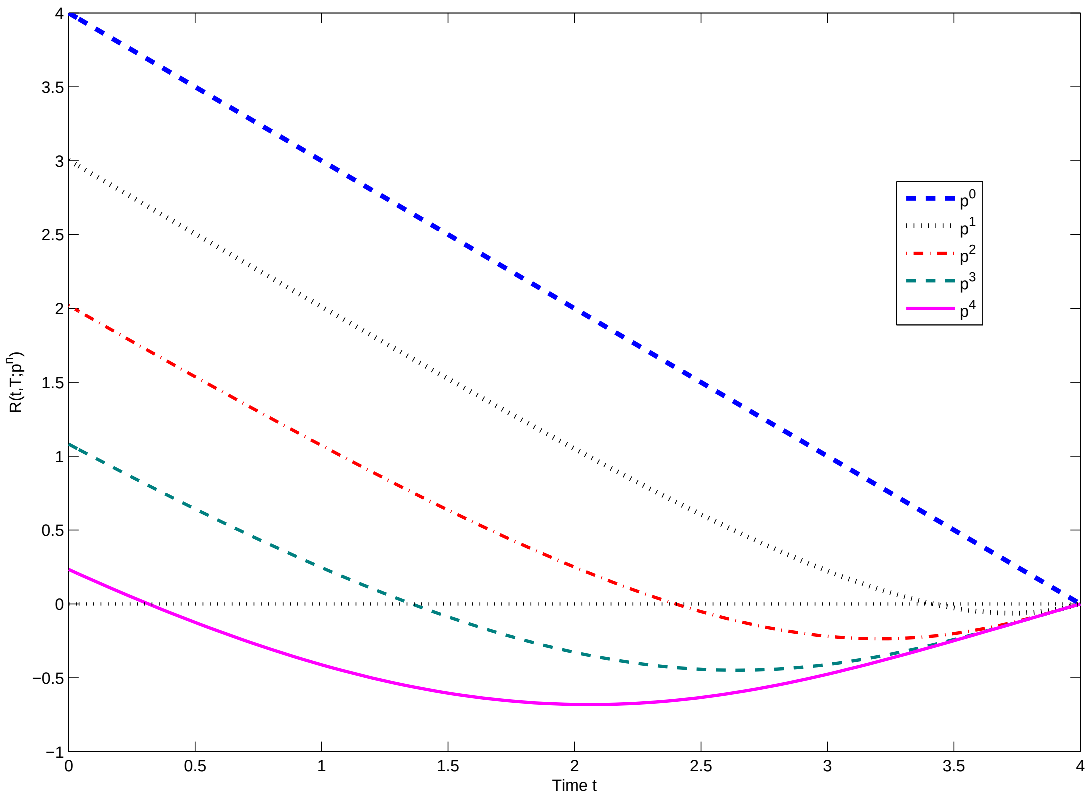

Figure 6: $R(t,T,p^n)$ with $\pi = \beta = 1$, $T = N = 4$, and $\varepsilon = 0.5$
This graphic shows the backward net reserve process without claims $R(t,T,p^n)$ for $n = 0,1,\ldots,N$ and with $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0.5$.

(e.g. see Schmidli [20, 21]), which is given by $\delta_p(y) = 1 - \psi_p(y)$. Ruin occurs for instance, if one claim is made between time 0 and t_0^{N-1}, or two claims are made between time 0 and t_0^{N-2}, and so on. In general, ruin occurs if $N-n$ claims are made between time 0 and t_0^n for $n = 0,1,\ldots,N-1$. Hence,

$$\psi_{(p^n)}(y) = \frac{\mathbb{P}\left(R\left(0,t;p^N,y\right)<0\right) + \sum_{n=0}^{N-1} \mathbb{P}\left(N^c\left(t_0^n(y)\right) = N-n\right)}{\mathbb{P}\left(N^c\left(T\right) \leq N\right)}. \tag{30}$$

The first term on the right side is due to the possibility that ruin may occur even with no claims being made. The denominator is due to the assumption that at most N claims can be made. Note that $\mathbb{P}\left(R\left(0,t;p^N,y\right)<0\right) = 0$ if $t_0^N < 0$ or if Lemma 6.1 holds.

Example 6.2
Assuming that Condition (28) holds (that is Lemma 1 holds) and that N^c is Poisson distributed with parameter λ, the **worst-case bound of the probability**

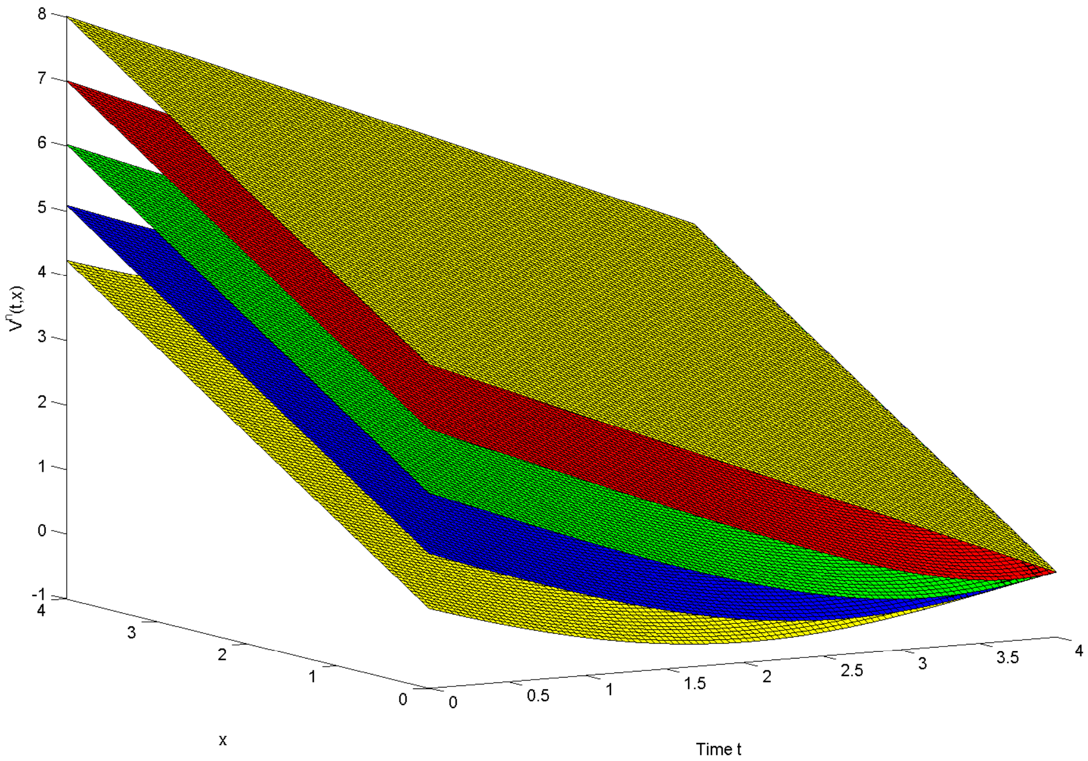

Figure 7: $V^n(t,x)$ with $\pi = \beta = 1$, $T = N = 4$, and $\varepsilon = 0.5$
This graphic shows the value function $V^n(t,x)$ for $n = 0, 1, \dots, N$ with $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, $\varepsilon = 0.5$, and $U = id$. Observe that $V^0(t,x)$ is the highest surface and $V^4(t,x)$ is the lowest surface.

of ruin (that means setting $b_i = \beta$ for all i) calculates to

$$\psi_{(p^n)}(y) = \frac{\sum_{n=0}^{N-1} \frac{\left(\lambda t_0^n(y)\right)^{N-n}}{(N-n)!}}{\sum_{k=0}^{N} \frac{(\lambda T)^k}{(k)!}} .$$

While Figures 10 and 11 give some examples for t_0^n for various initial reserves y, Figures 12 and 13 give some examples for the **worst-case bound of the probability of survival**.

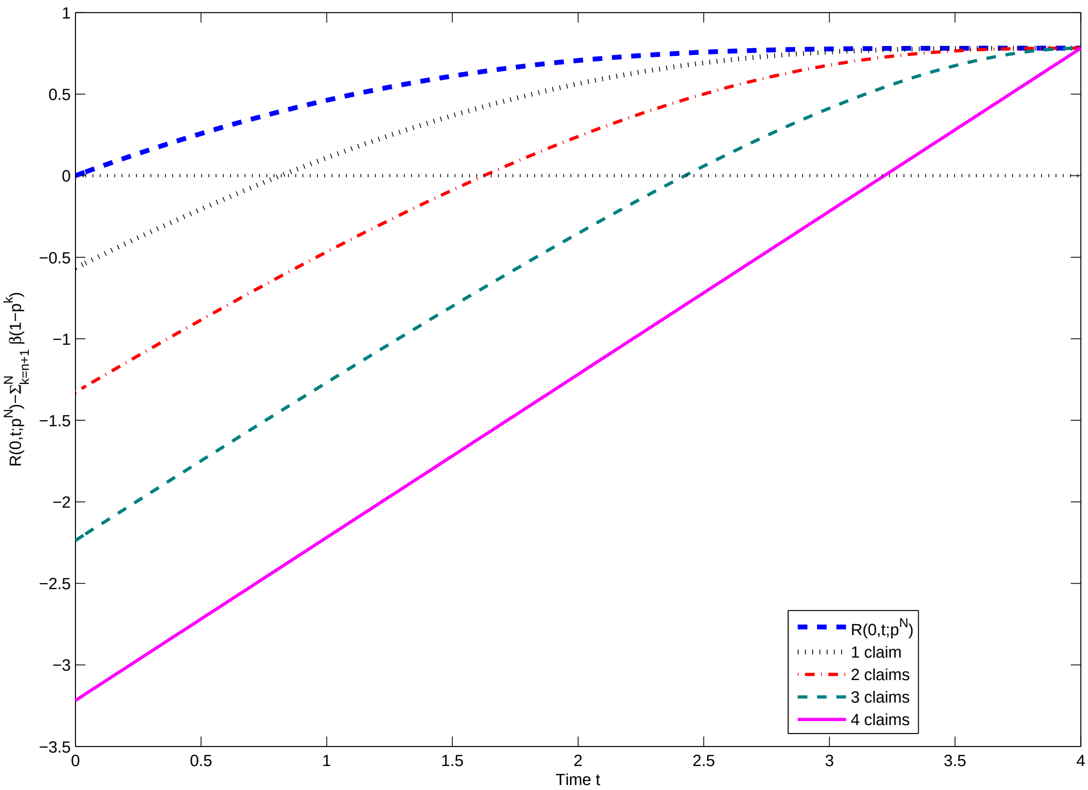

Figure 8: Possible Scenarios for the Net Reserve Process with $\varepsilon = 0$

This graphic shows the possible evolution of the net reserve process $R(0,t;p^N) - \beta \sum_{k=n+1}^{N} \left[1 - p^k\left(t_0^n\right)\right]$ using the worst-case reinsurance strategy (p^n) with $n = 0, 1, \ldots, N$ for $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0$.

7 The Intrinsic Risk–free Rate of Return

Equation (29) also implies that the insurance company cannot go bankrupt if the initial reserve satisfies

$$y \geq \beta \sum_{k=1}^{N} \left[1 - p^k(0)\right] =: y^* .$$

Note, that $y^* \in [0, \beta N]$. Hence, the **worst-case bound for the return** or the **intrinsic risk–free rate of return** is given by

$$\frac{R\left(0, T; p^N\right)}{y^*} = \frac{\pi T}{\beta \sum_{k=1}^{n} \left[1 - p^k(0)\right]} - 1 = \frac{\pi T}{y^*} - 1 .$$

This is, because $R\left(0, T; p^N\right)$ is lower bound of the profit the insurance company makes if it follows the optimal worst-case reinsurance strategy. To get the actual

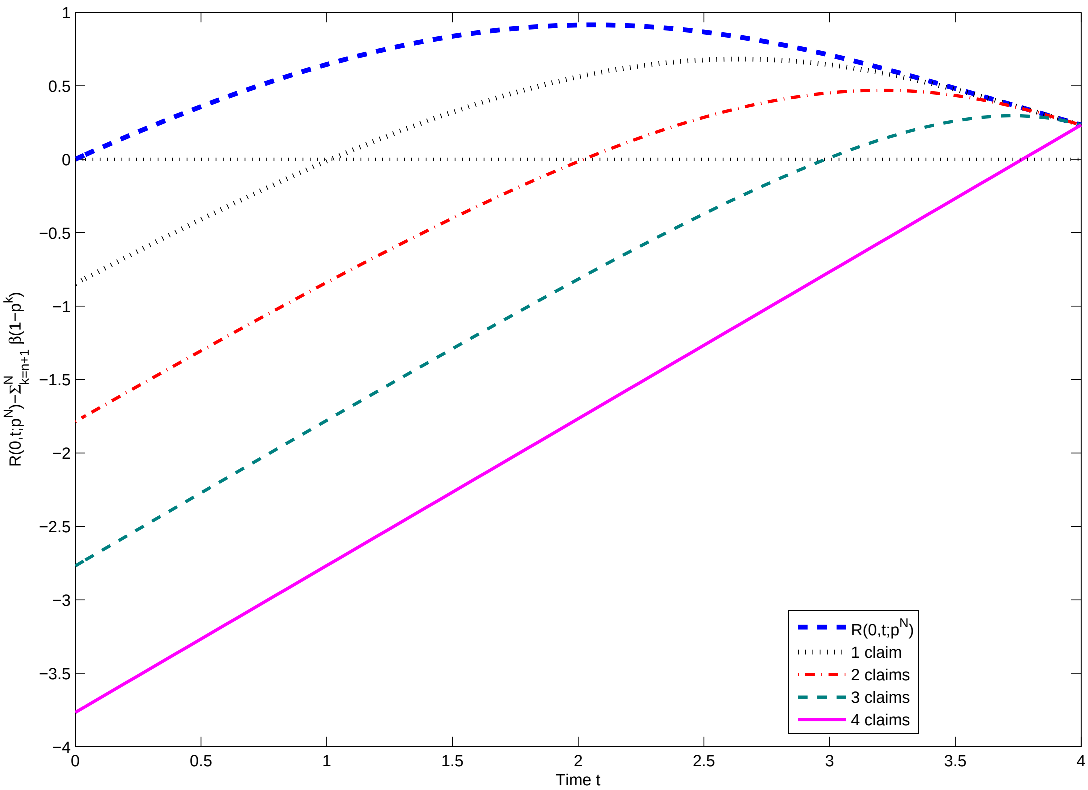

Figure 9: Possible Scenarios for the Net Reserve Process with $\varepsilon = 0.5$
This graphic shows the possible evolution of the net reserve process $R(0,t;p^N) - \beta \sum_{k=n+1}^{N} \left[1 - p^k\left(t_0^n\right)\right]$ using the worst-case reinsurance strategy (p^n) with $n = 0, 1, \ldots, N$ for $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0.5$.

profit, $R\left(0, T; p^N\right)$ has to be adjusted by $\sum_{k=n+1}^{N} (\beta - b_k)\left[1 - p^k\left(\tau^k\right)\right]$. Moreover, this profit has a deterministic worst-case bound if the initial reserve of the insurance company is given by y^*. This is, because it is no longer possible that ruin occurs in this case – hence the worst-case bound of the net reserve process is deterministic. Table 1 gives some examples.

8 Comparing Different Business Strategies

Observe that it is possible to follow different business strategies. One basic strategy is to concentrate on large businesses, that is the claim size β is large and the number of possible claims N is small. On the other hand, it is possible to concentrate on small businesses, that is the claim size β is small and the number of possible claims N is large. If the theory of mean-variance portfolios (Markowitz) applies here (don't put all your eggs in one basket), the latter strategy is less risky

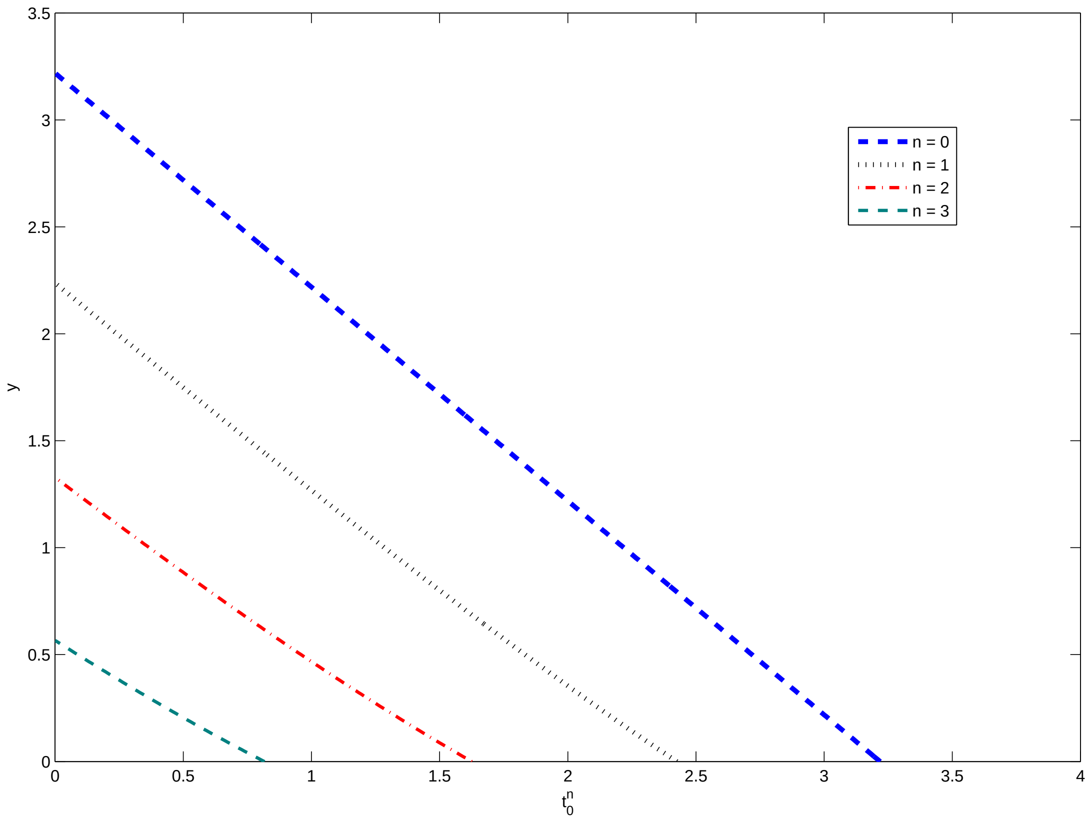

Figure 10: Zero Lines for Various Initial Reserves with $\varepsilon = 0$
This graphic shows the zero lines t_0^n for various initial reserves y given that the worst-case reinsurance strategy (p^n) with $n = 0, 1, \ldots, N$ is used for $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0$.

and should therefore require less reinsurance and should have a higher worst-case bound. However, this is **not** the case for the optimal worst-case reinsurance strategy.

To see this, let be K, $L \in \mathbb{N}$ with K, $L \geq 1$. Now, compare the case of K contracts (or possible claims) with potential worst-case claim size $\beta = b$ with the case of $K + L$ contracts with potential worst-case claim size $\beta = \frac{Kb}{K+L}$. Setting $\pi T = bK = \frac{Kb}{K+L}(K + L)$, it is clear that both business strategies have the same turnover volume. Under these conditions, it is interesting to check if Markowitz's principle (that is if it is true within this approach, then it is always better to spread your risk) holds. If Markowitz's principle is **not** true, the following inequality should hold

$$\begin{aligned} & R\left(0, T; p^K\right) && \geq && R\left(0, T; p^{K+L}\right) \\ \Longleftrightarrow \quad & \frac{1}{K}\sum_{l=1}^{K} p^l\left(0; b\right) && \geq && \frac{1}{K+L}\sum_{l=1}^{K+L} p^l\left(0; \frac{Kb}{K+L}\right) \end{aligned}$$

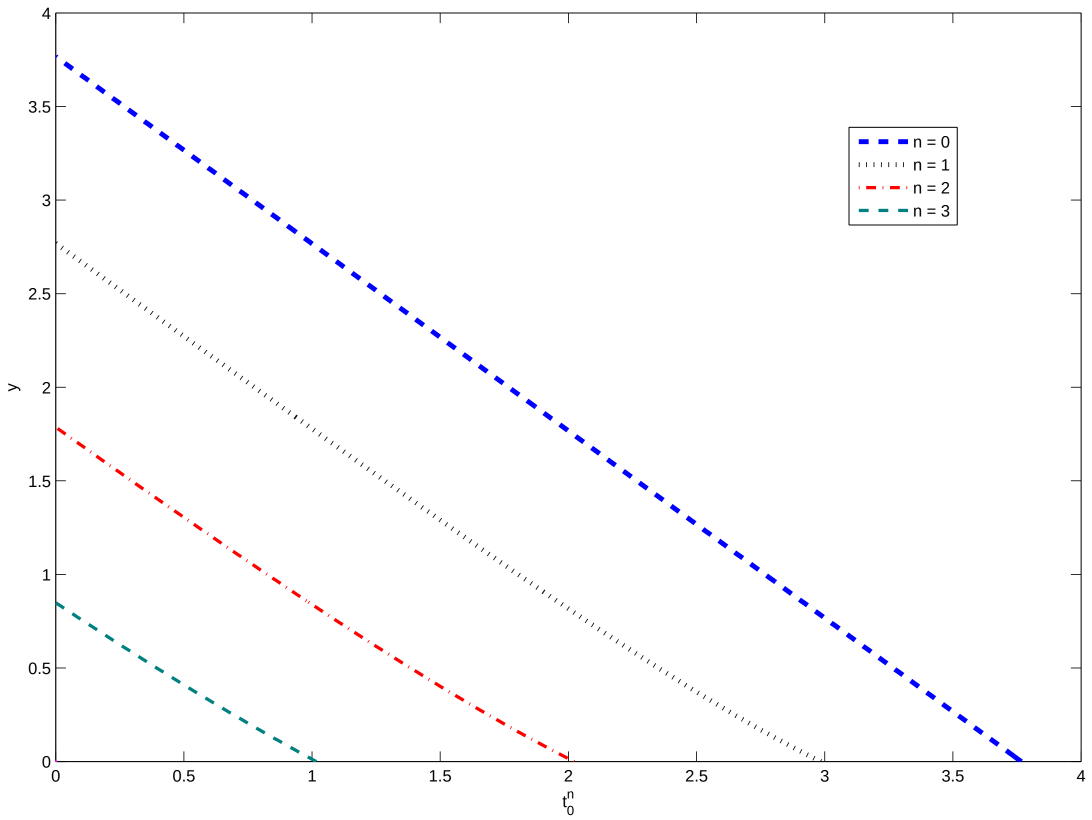

Figure 11: Zero Lines for Various Initial Reserves with $\varepsilon = 0.5$
This graphic shows the zero lines t_0^n for various initial reserves y given that the worst-case reinsurance strategy (p^n) with $n = 0, 1, \ldots, N$ is used for $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0.5$.

$$\begin{aligned}
\iff \quad e^{-\alpha T} \sum_{l=1}^{K} \sum_{k=0}^{l-1} \frac{[\alpha T]^k}{k!} &\geq e^{-\frac{K+L}{K}\alpha T} \frac{K}{K+L} \sum_{l=1}^{K+L} \sum_{k=0}^{l-1} \frac{\left[\frac{K+L}{K}\alpha T\right]^k}{k!} \\
\iff \quad \sum_{l=1}^{K} \sum_{k=0}^{l-1} \frac{[\alpha T]^k}{k!} &\geq e^{-\frac{L}{K}\alpha T} \frac{K}{K+L} \sum_{l=1}^{K+L} \sum_{k=0}^{l-1} \frac{\left[\frac{K+L}{K}\alpha T\right]^k}{k!} \\
\iff \quad \sum_{l=0}^{K-1} (K-l) \frac{[\alpha T]^l}{l!} &\geq e^{-\frac{L}{K}\alpha T} \frac{K}{K+L} \sum_{l=0}^{K+L-1} (K+L-l) \frac{\left[\frac{K+L}{K}\alpha T\right]^l}{l!} .
\end{aligned}$$

Setting

$$f_{K,L}(x) := \sum_{l=0}^{K-1} (K-l) \frac{x^l}{l!} - e^{-\frac{L}{K}x} \frac{K}{K+L} \sum_{l=0}^{K+L-1} (K+L-l) \frac{\left[\frac{K+L}{K}x\right]^l}{l!} ,$$

it is sufficient to investigate under which conditions $f_{K,L}(x) \geq 0$, since

$$f_{K,L}(\alpha T) = R\left(0, T; p^K\right) - R\left(0, T; p^{K+L}\right) .$$

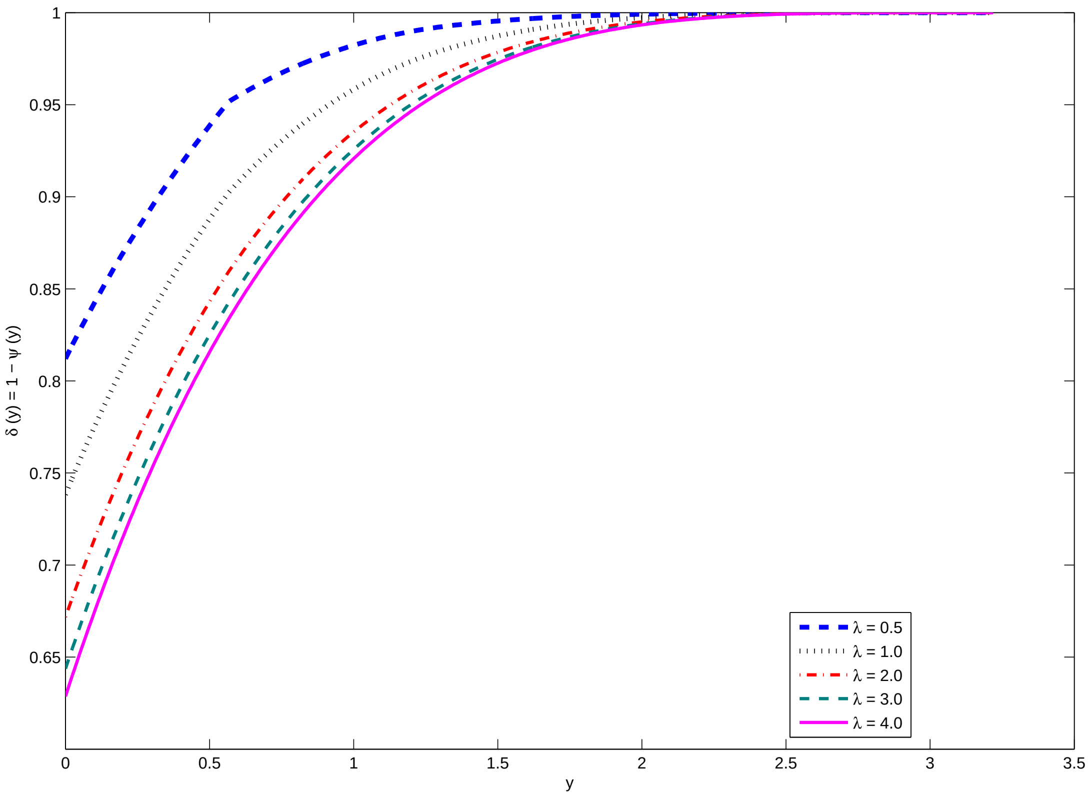

Figure 12: Probability of Survival with $\varepsilon = 0$
This graphic shows the worst-case bound for the probability of survival for various initial reserves y and various λ if the worst-case reinsurance strategy (p^n) with $n = 0, 1, \ldots, N$ is used for $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0$.

First, it is straightforward to verify, that $f_{K,L}(0) = 0$ and

$$\lim_{x\to\infty} f_{K,L}(x) = \left\{ \begin{array}{ll} \infty & \text{for } K > 1 \\ 1 & \text{for } K = 1 \end{array} \right\} .$$

This verifies that the business strategy of having K contracts gives eventually (that is for αT large enough) a higher worst-case bound than the strategy of having $K + L$ contracts with the same turnover volume.

This proves the first part of the following proposition.

Proposition 8.1

(i) The business strategy of having K contracts gives a higher worst-case bound than the strategy of having $K + L$ contracts with the same turnover volume, if αT is sufficiently large.

(ii) In the special case $K = 1$, the business strategy of having only $K = 1$ contract gives a higher worst-case bound than having $1 + L$ contracts (with $L \geq 1$) given that both business strategies generate the same turnover volume.

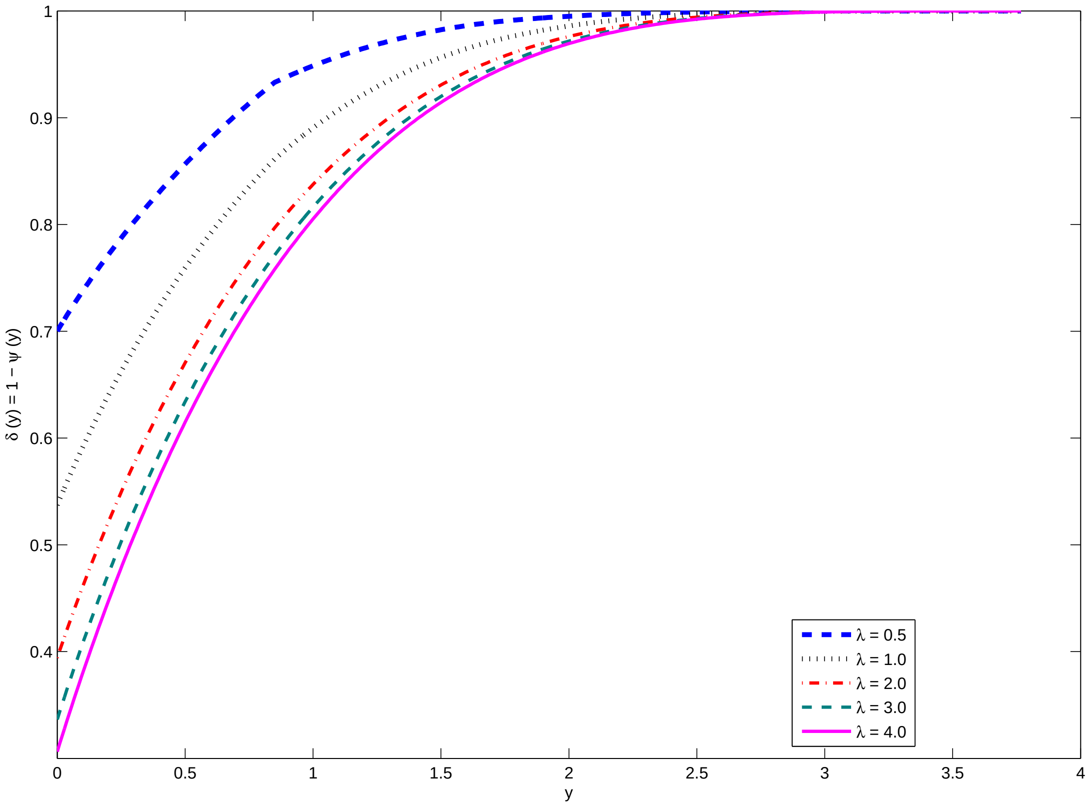

Figure 13: Probability of Survival with $\varepsilon = 0.5$
This graphic shows the worst-case bound for the probability of survival for various initial reserves y and various λ if the worst-case reinsurance strategy (p^n) with $n = 0, 1, \ldots, N$ is used for $\pi = 1$, $\beta = 1$, $T = 4$, $N = 4$, and $\varepsilon = 0.5$.

Proof : It remains to prove the second part of the assertion. For $K = 1$ the first derivative of $f_{1,L}$ computes to

$$\begin{aligned} \frac{d}{dx} f_{1,L}(x) &= \frac{L}{1+L} e^{-Lx} \frac{[(1+L)x]^L}{L!} + e^{-Lx} \sum_{l=0}^{L-1} \frac{[(1+L)x]^l}{l!} \left\{ \frac{1}{1+L}\, l \right\} \\ &= e^{-Lx} \frac{1}{1+L} \sum_{l=1}^{L} \frac{[(1+L)x]^l}{(l-1)!} > 0 \qquad \text{for all } x > 0\,. \end{aligned}$$

This proves that $f_{1,L}(x)$ is always strictly positive for $x > 0$, meaning that the business strategy of having only 1 contract gives always (that is for every αT) a higher worst-case bound than the strategy of having $1 + L$ contracts with the same volume, since $f_{K,L}(0) = 0$. $\square$

This means that from a worst-case scenario maximization viewpoint the business strategy of having only 1 contract is always superior to having $1 + L$ contracts. Numerical implementation of the function $f_{K,L}(x)$ indicate that this is true for any $K \in \mathbb{N}$ (see Figure 14).

Table 1: Profitability of the Worst Case Reinsurance strategy with $\pi T = \beta N$

ε	0	0.1	0.5	0	0.1	0.5
β	1	1	1	2	2	2
N	4	4	4	2	2	2
$R(0,T;p^N)$	0.7815	0.6232	0.2330	1.0827	0.9307	0.4979
y^*	3.2185	3.3768	3.7670	2.9173	3.0693	3.5021
risk–free return	0.2428	0.1845	0.0619	0.3711	0.3032	0.1422

This table is calculated using $\pi = 1$ and $T = 4$.

Table 2: Profitability of the Worst Case Reinsurance strategy with $\pi T = \beta N$

ε	0	0.1	0.5	0	0.1	0.5	0.1	0.5
β	1	1	1	2	2	2	10	10
N	10	10	10	5	5	5	1	1
$R(0,T;p^N)$	1.2511	0.8533	0.1368	1.7547	1.3565	0.4334	3.3287	2.2313
y^*	8.7489	9.1467	9.8632	8.2453	8.6435	9.5666	6.6713	7.7687
risk–free return	0.1430	0.0933	0.0139	0.2128	0.1569	0.0453	0.4990	0.2872

This table is calculated using $\pi = 1$ and $T = 10$.

Finally, notice that this result is confirmed by comparing the various optimal worst-case reinsurance strategies with each other. More specifically, the optimal reinsurance strategies for the small business strategy are presented in solid lines and denoted by p^n, the ones for the large(r) business strategy are drawn in dashed lines and denoted by $p2^n$ (see Figure 15), and the ones for the very large business strategy are depicted in dash–dotted lines and denoted by $p10^n$ (see Figure 16).

Note that $p2^n$ can be compared only with p^{2n} and $p10^n$ can be compared only with $p2^{5n}$ and p^{10n}. Observe that the reinsurance level is higher (implying that this business is more risky) for the large business strategy on the long run (see Figure 16). This is in line with Markowitz. However, on the short horizon this is no longer true. For instance, in Figure 15, p^4 is always larger than $p2^2$ and p^2 is larger than $p2^1$ for $t \geq 1.5$ or so. Moreover, Tables 1 and 2 confirm that it is more profitable for an insurance to concentrate on fewer contracts with larger claim size potential per claim instead of having more contracts with lower claim size potential per claim. Additionally, the initial capital needed to avoid bankruptcy (denoted as y^* above) reduces as well if an insurance company concentrates on fewer contracts but with larger claim size potential. Altogether, this has the impact that the worst-case bound for the return increases (as the profit does and the initial capital needed decreases). **Thus, it is no longer true that it is always better to spread your risk.**

A similar result is obtained in a different setting. Diversification is disadvantageous if only the losses of a portfolio are considered which are supposed to have heavy tails with a tail index smaller than one. However, if the tail index is larger than one, diversification is again advantageous. Moreover, if the value change of the portfolio (that is gains **and** losses) is considered, nothing can be said of the diversification effect (see e.g. Mainik and Rüschendorf [18] and references therein). Comparing those results with the results in this paper, neither a regime shift (with respect to the diversification effect) happens in the model described in this paper nor is it necessary to have a claims process with heavy tails to obtain a negative diversification effect in our model. Furthermore, the expected (utility of the) net reserve (which contains gain and losses) is maximized. That is, no restriction to the losses is necessary to obtain negative diversification effects.

Additionally, notice that some practitioners do not think that diversification is necessarily a good thing. Most famous is a bon mot which is supposed to go back to Andrew Carnegie but made famous by the Mark Twain character David Wilson: **Put all your eggs in the one basket, and — WATCH THAT BASKET** (see Fox [8], p. 56).

9 Conclusion and Outlook

We have introduced the worst-case scenario approach to reinsurance decision making. The results are derived and analyzed theoretically but also illustrated in a series of numerical examples showing the optimal strategy, the optimally controlled surplus process, ruin probabilities and other features. Importantly, we have demonstrated its attractive properties, specifically

- explicitly computable, worst-case optimal reinsurance strategies,

- robustness against choice of utility function,
- robustness against modeling of claim sizes and claim numbers, and
- giving fresh insights on the aspects of diversification.

It is interesting, in future studies to include further aspects of the non-life insurance company's decision making, including

- investment risk modeling and control,
- small claims modeling and control, e.g. by a Gaussian process for the small claims surplus, thereby adding noise to the system and the results, and
- alternative ways of formalizing the worst-case, e.g. comparing with worst-case bounds on the (claim–number independent) intensity.

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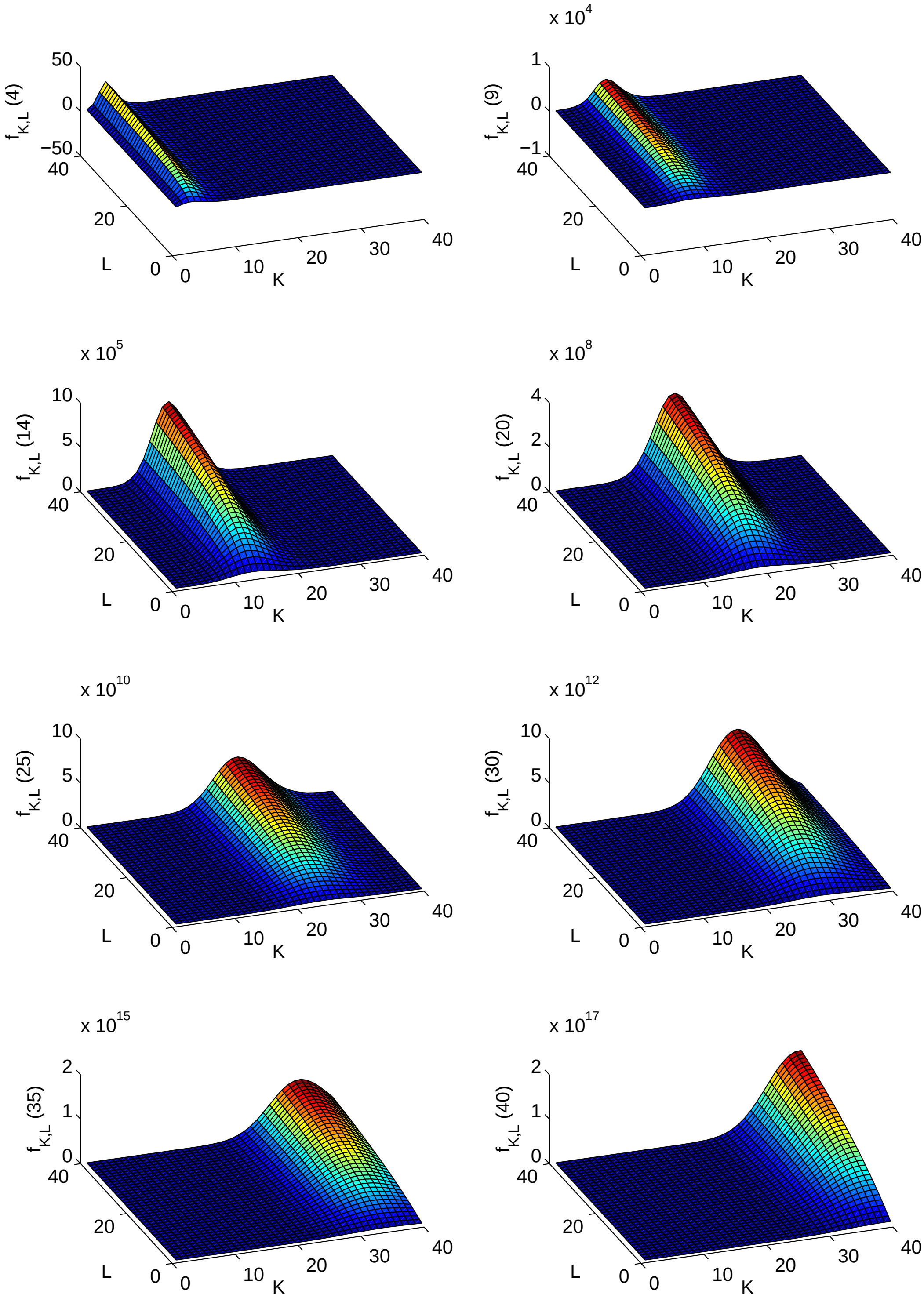

Figure 14: The function $f_{K,L}(x)$ for various x

This graphic shows the function $f_{K,L}(x)$ for $x = 4, 9, 14, 20, 25, 30, 35$, and 40.

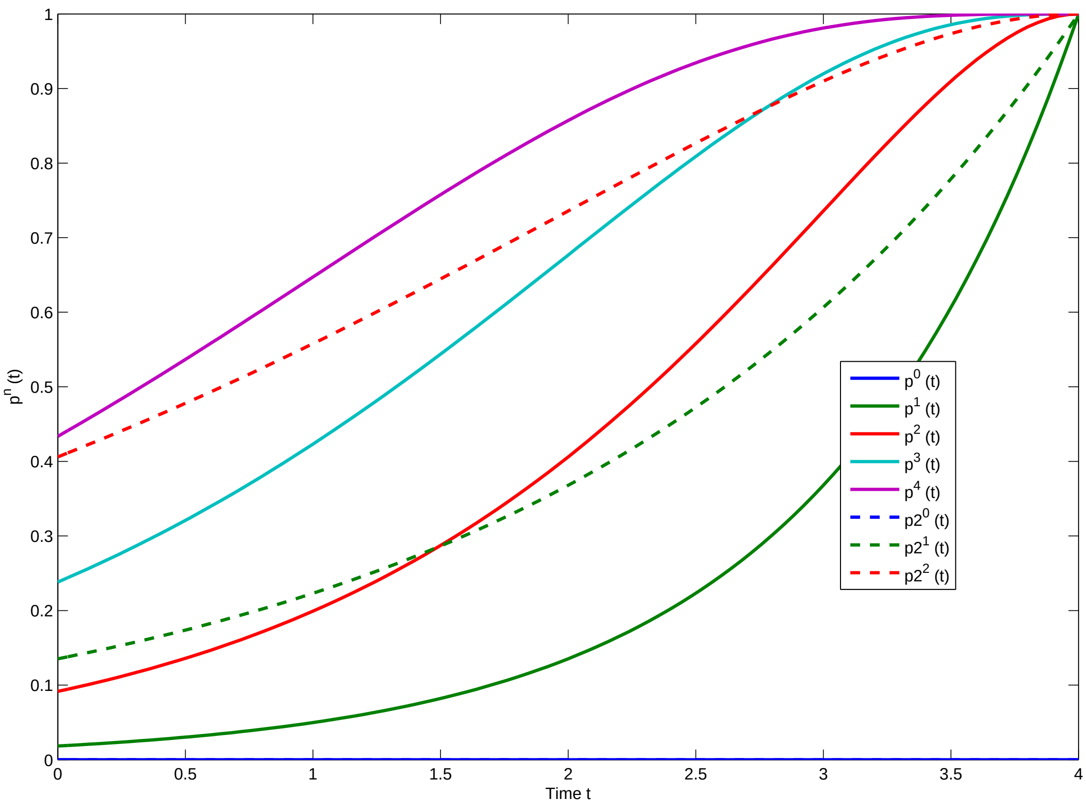

Figure 15: Comparing Different Business Strategies with $\varepsilon = 0$

This graphic shows the worst-case optimal reinsurance strategy for $\pi = 1$, $T = 4$, and $\varepsilon = 0$ for $N = 4$, $\beta = 1$ (solid lines) and $\beta = 2$, $N = 2$ (dashed lines).

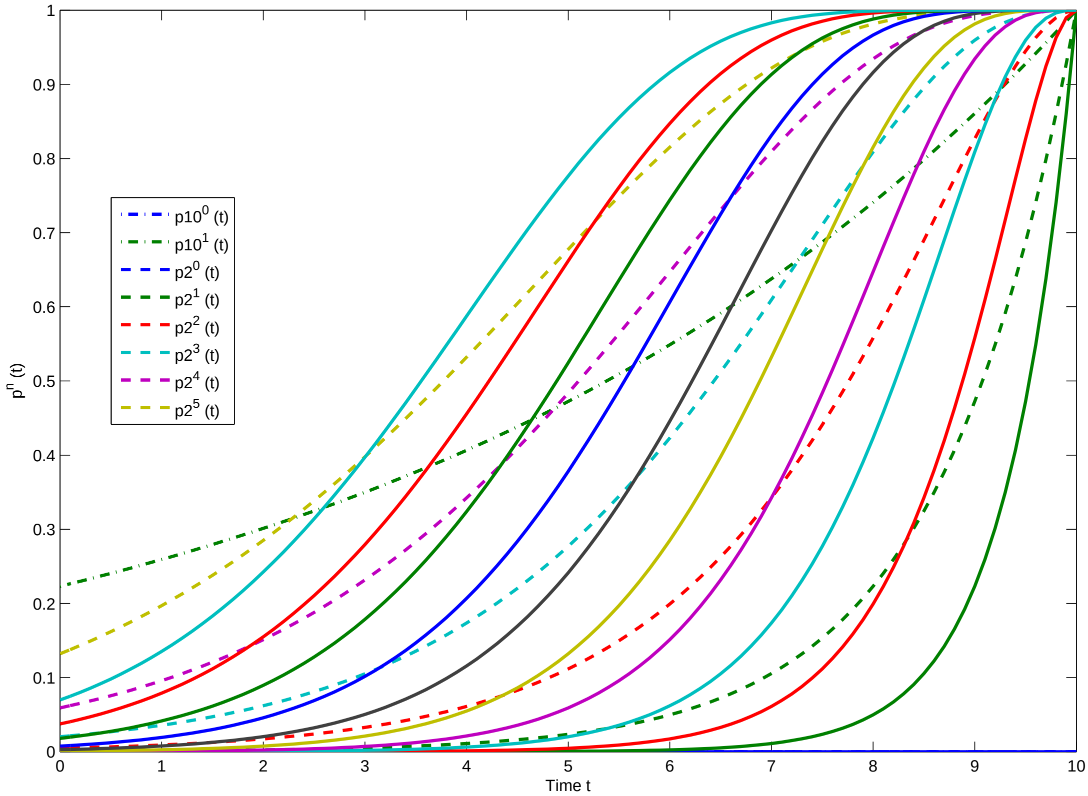

Figure 16: Comparing Different Business Strategies with $\varepsilon = 0.5$
This graphic shows the worst-case optimal reinsurance strategy for $\pi = 1$, $T = 10$, and $\varepsilon = 0.5$ for $\beta = 1$, $N = 10$ (solid lines); $\beta = 2$, $N = 5$ (dashed lines); and $\beta = 10$, $N = 1$ (dashed–dotted line).