On Admissibility of Deterministic and Stochastic Linear Volterra Operators with Applications to Inefficient Financial Markets

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Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Doctor of Philosophy in Mathematics is entirely my own work, that I have exercised reasonable care to ensure the work is original, and does not to the best of my knowledge breach any law of copyright, and has not been taken from the work of others save and to extent that such work has been cited and acknowledged within the text of my work.

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Lastly, I hope you the reader will derive as much benefit from this thesis as I have done in writing it.

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Abstract

This thesis examines the long–run behaviour of both differential and difference, deterministic and stochastic linear Volterra equations. Firstly we consider a stationary autoregressive conditional heteroskedastic (ARCH) process of order infinity. This type of process is used in time series analysis due to its non-constant conditional variance. In describing the extent of the dependence of the current values of the process upon past values we are led to the study of the autocovariance function. Necessary and sufficient conditions are established for the autocovariance function to lie in a particular class of slowly decaying (subexponential) sequences.

We develop sharp conditions for solutions of linear Volterra summation equations to lie in a class of sequences which is characterised by having a subexponential rate of decay coupled with a periodic fluctuation. This theory illustrates and clarifies the effect of the kernel upon the solution of Volterra-type summation equations. In particular this theory is applied to the autocovariance function of ARCH(∞) processes.

A stochastic admissibility theory of stochastic Volterra operators is developed. In particular necessary and sufficient conditions for mean square convergence and sufficient conditions for almost sure convergence are established for stochastic integrals. This theory is then applied to stochastic linear functional equations of Volterra and finite delay type.

Lastly, we introduce a particular stochastic differential equation with an average functional which may be viewed as modelling the demand of traders in an inefficient financial market. The asymptotic behaviour of this process is determined for almost all values of the parameters of the model. A discretisation of this stochastic differential equation is also studied. The asymptotic behaviour of the discretisation is shown to mirror that of the continuous–time equation.

Introduction and Preliminaries

0.1 Motivation and Goals of the Thesis

Financial markets are referred to as weak form efficient when the future movement of asset prices is independent of all historical data of the asset, Fama [50]. However the presence of market bubbles and crashes are indicators of the lack of efficiency of markets, Kirman and Teyssière [74]. The presence of traders who use models of past price information leads to the study of stochastic functional differential equations or stochastic delay differential equations, e.g. Bouchaud and Cont [30]. A useful tool in studying the efficiency (or inefficiency) of a financial market is the autocovariance function because it enables one to study the correlation between asset returns taken over different intervals of time.

Weak form efficiency, together with stationary and independent returns, and an absence of jumps in the price of a risky asset, imply that the asset price is described by a Geometric Brownian motion. This stochastic process can be thought of as the solution of a linear stochastic differential equation. In order to study departures from efficiency, and to preclude the addition of other confounding modelling factors (such as nonlinearities or jumps), in this thesis we will presume that asset prices or returns follow linear or affine stochastic models. In such cases, if returns are stationary, the autocovariance function can be used to determine the degree of dependence across time: indeed it is particularly suited to this task, because the covariance measures the linear association between two random variables. Of course, by making such linearity assumptions, we hope to simplify the mathematical analysis as well. If trading takes place in discrete time, one can argue in a similar manner that the most parsimonious modelling assumption to allow for inefficiency is to model the returns as the solution of an affine or linear stochastic difference equation.

In what follows, we pose a selection of questions concerning the long run behaviour of inefficient financial markets. It will be the goal of this work to address these questions with at least partial positive answers. The most important mathematical tool used to answer these questions turns out to be the admissibility theory of deterministic and stochastic Volterra operators.

Empirical evidence suggests slow decay in the autocovariance function of many real–

life time series (e.g. in the physical sciences, hydrology, climatology and financial time series) Baillie [24]. One class of processes which have been shown to possess slow decay in their autocovariance function are autoregressive processes of order infinity (ARCH(∞)). Such a process is a discrete stochastic process where the latest term in the sequence is a linear functional of all preceding terms in the process. The linear functional attaches a weighting to past realisations of the process. It has been shown that when the weights decay no faster than polynomially then the autocovariance function also decays no faster than polynomially. It is not explored what the effect of weights decaying at a definite rate has on the autocovariance function, or indeed if one can deduce similar results for non–polynomially decaying weights. Indeed while it is desirable to extend the analysis of the autocovariance function of ARCH(∞) in the manner just described consideration must be given to the approach taken toward proving such results as the proofs of e.g. Giraitis, Kokoszka, Leipus, Surgailis, and Zaffaroni [51, 52, 118] are quite complex.

Also, being motivated by econometric time series, one would like the autocovariance function to exhibit a mixture of non–exponential decay and oscillation (and in particular allowing the autocovariance function to undergo regular changes of sign). Such a switch in the term structure of the autocovariance function has been observed in real time series, such as property prices, Cutler, Poterba and Summers [40]. It does not appear one can achieve such behaviour in the autocovariance function of ARCH(∞) processes, not least because their autocovariance functions are always non–negative. However, nothing in the existing theory forbids the autocovariance function of an ARCH(∞) process from experiencing a fluctuation around some positive decaying sequence as the time lag increases. Both these effects are of interest to investigate.

In the analysis of the autocovariance function of a stochastic process one often tests for the presence of long memory. A common definition of long memory or long range dependence, for a stationary process $\{X(n) : n \in \mathbb{Z}\}$ is that

$$\sum_{n=0}^{\infty} |\text{Cov}[X(0), X(n)]| = +\infty.$$

When this condition fails the process is said to possess short memory. It is proved in [51] that ARCH(∞) processes have short memory. It is with the view to classifying a process as either short or long memory that one is interested in the rate of decay of the autocovariance function.

When a process is not stationary but is in some respect "close" to being stationary, then from a heuristic standpoint, it is not clear whether one might prefer to consider the rates of decay in the time lag k of $n \mapsto \text{Cov}[X(n), X(n+k)]$ as $n \to \infty$ (as a function of k), or $k \mapsto \text{Cov}[X(n), X(n+k)]$ as $k \to \infty$ for fixed n, in understanding the asymptotic behaviour of the autocovariance function. For autonomous stochastic functional differential equations, it appears that these questions lead to the same answer. But in general, for non–stationary processes, these limiting objects need not yield non–conflicting results. In light of this ambiguity of the memory properties of non–stationary processes one may speculate that it is possible to observe both long and short memory features in the same process. This is of interest due to empirical disagreement about the presence of long memory in certain time series, c.f. e.g. Cont [35], Mikosch and Stărică [87].

The foregoing discussion concerns moment behaviour of stochastic processes and ascertaining exact rates of decay. We turn now to looking at pathwise features of stochastic processes in inefficient financial markets. In financial markets it is often argued that prices fluctuate about a fundamental value, Poterba and Summers [101]. It is desirable to study models then where one can not only identify the size of the largest fluctuations from the equilibrium but also the value of the equilibrium (which may be at a non–trivial random level). As bubbles and crashes are hallmarks of inefficient markets, one should not only describe conditions under which they will occur but also quantify the rate of growth of such a bubble. One type of bubble dynamics evidenced in financial markets are "switchback rides" whereby the asset price fluctuates with growing amplitude. Also the effect of long range dependence on the bubble growth requires further study; one might expect that such inertia might retard the expansion of a bubble in prices, or forestall the formation of a crash. How such behaviour might be mimicked, and quantified, in the solution of a stochastic (functional) differential equation needs still to be explored further.

It has been observed in this thesis that the asymptotic behaviour of the solutions of a class of affine stochastic equations mirrors the fine asymptotic structure of their underlying resolvent equations providing that an additive noise term is "small" enough. This analysis forms part of Chapter 3, moreover these results are part of a larger class of results which question how large a perturbation in a differential equation may be permitted while still preserving the fine asymptotic structure of the underlying unperturbed equation e.g. Győri and Hartung [54]. A motivation for this general work is that special cases of the analysis studied in this thesis have been used to understand the manner in which financial market bubbles or crashes form.

The above issues require an analysis of the paths of the stochastic process itself (as opposed to the moment behaviour). In determining these stochastic results it is often the case that the solution of the stochastic differential equation is comparable to that of an underlying deterministic resolvent equation. However due to the non–deterministic nature of stochastic equations the same methods of proof will not work. Hence one may ask in what manner is it possible to amend the deterministic theory so that it is applicable to stochastic problems.

Lastly we ask whether answers to the above problems outlined are dependent upon whether one is studying problems in continuous or discrete time. This last question is significant as when one wishes to perform a numerical simulation of a continuous problem one is now interested discrete analysis. Also, one may believe that the price process should be modelled in discrete time. In both cases, it is of interest to ask what happens if the time step is small: for numerical problems, this hopefully leads to more accurate simulations, while in discrete economic modelling, it reflects the fact that trading is happening with increasing frequency. In particular it is to be questioned what restrictions on the discretisation step–size are needed so that salient asymptotic features present in the continuous problem are also present in the discrete analysis.

0.2 Mathematical Framework of the Thesis

The study of the long run behaviour of deterministic differential (and difference) equations is performed throughout this thesis. In the first two chapters these deterministic equations arise from the study of moment behaviour. While in the latter part of the thesis the integrand of stochastic integral is often connected with a deterministic equation. One technique, which is of central importance to this thesis, for understanding the long run behaviour of deterministic Volterra equations is the theory of admissibility by Appleby, Győri and Reynolds [13]. This theory only applies to equations of the form

$$z(n+1) = \sum_{j=0}^{n} k(n-j)z(j), \quad n \geq 0, \quad z(0) = z_0 \tag{0.2.1}$$

where the kernel is, loosely speaking, 'monotonic'. However this theory may be adapted to describe the long run behaviour of equations close to (0.2.1), for example the infinite history equation which arises in Chapter 1,

$$z(n+1) = \sum_{j=-\infty}^{n} k(n-j)z(j), \quad n \geq 0; \quad z(0) = z_0; \quad z(n) = z(-n), \quad n \leq -1$$

or Volterra equations where the kernel has a periodic component. Appleby and Krol [15] study a stochastic process whose memory properties are driven by a kernel sequence which lies in a class of slowly decaying sequences.

While one may be chiefly interested in understanding the pathwise behaviour of a stochastic process the solution of a stochastic differential equation (as previously observed) may often be expressed in terms of the solution of an underlying resolvent equation. To illustrate, consider the affine stochastic functional equation

$$dX(t) = L(X(t))dt + \sigma dB(t), \quad t \geq 0; \quad X(0) = x_0 \in \mathbb{R}, \tag{0.2.2}$$

where L is a linear functional and σ is a non-zero positive constant. The associated deterministic equation arises from setting $\sigma \equiv 0$, giving

$$r'(t) = L(r(t)), \quad t \geq 0; \quad r(0) = 1; \quad r(t) = 0, \quad t < 0. \tag{0.2.3}$$

Providing both (0.2.2) and (0.2.3) have well-defined solutions then X may be expressed in terms of r, i.e.

$$X(t) = r(t)x_0 + \int_0^t r(t-s)\sigma dB(s), \quad t \geq 0. \tag{0.2.4}$$

From (0.2.4) it is clear that one should expect the solution of the deterministic equation to influence the asymptotic behaviour of the solution of the stochastic equation. A result of this nature includes Appleby and Riedle [19] where the location of the roots of a characteristic equation determine the integrability of the solution of a stochastic equation. The leading order behaviour of r may be determined from a variety of methods e.g. while ordinary (and delay) differential equations are generally quite difficult to solve analytically it may be the case that they can be reformulated into a class of equations which have known asymptotic behaviour. Some such asymptotic results used in this thesis include the Birkhoff-Adams Theorem (for discrete second order equations), results of Diekmann et al. [43] and Gripenberg et al. [53] for finite delay and Volterra continuous equations, and the theory of special functions [96]. If for example the leading order behaviour of (0.2.3) is given by exponential polynomials, i.e., e.g.

$$r(t) = t\mathrm{e}^t + O(\mathrm{e}^t), \quad \text{as } t \to \infty,$$

then one should try to retain this exponential polynomial structure within the stochastic integral representation (0.2.4). Moreover one should also utilise the convolution structure of the stochastic integral in (0.2.4). In particular one should, where possible, separate the t and s terms, this separability then allows one to employ martingale theory in the asymptotic analysis.

The above approach of separating the leading order terms of the resolvent from the lower order terms and using the structure of the stochastic integral enables one to deduce the leading order behaviour of the stochastic process. It is then required to show that the remainder terms from the resolvent give rise to the lower order terms in the solution of the stochastic equation. The stochastic analysis of these leading order terms may involve scaling by any growing or decaying factors and the addition (or subtraction) of oscillating terms. Thus, after these adjustments have been made, the remainder terms will typically have very little structure remaining and will appear of the form

$$\int_0^t H(t,s)dB(s),$$

for some function $H(\cdot,\cdot)$. It is thus required to develop a theory which will describe the asymptotic behaviour of such processes.

0.3 Synopsis of the Thesis

The first chapter of this thesis investigates the asymptotic properties of the memory structure of ARCH(∞) equations. ARCH processes are a discrete time stochastic process with non-constant conditional volatility. The autocovariance function of ARCH(∞) equations may be expressed as the solution of a linear Volterra summation equation. The asymptotic analysis of the autocovariance function is then achieved by applying the admissibility theory of linear Volterra operators to this equation and to an associated resolvent equation. It is shown that the autocovariance function decays subexponentially (or geometrically) if and only if the kernel of the resolvent equation has the same decay property. It is also shown that upper subexponential bounds apply to the autocovariance function if and only if similar bounds apply to the kernel. The results of this chapter extend the scrutiny of the autocovariance function conducted in [51]. However the method of proof differs markedly from that of [51, 52, 75, 118] (employing theory of Volterra equations as opposed to studying a closed form solution). It is assumed in our analysis though that the kernel of the conditional volatility of the ARCH(∞) process belongs to a particular class of slowly decaying sequences. In the development of a counter–example to a claim in [118] we required the kernel to have a fluctuation. The adaptation of the admissibility theory of [13] to construct this counter–example forms the basis for the second chapter.

In the second chapter we consider a Volterra convolution summation equation where the kernel decays at a known rate but with a periodic component. By a careful splitting up of the summation we can isolate the periodic components and apply admissibility theory to deal with the decaying component. In general, we show (roughly speaking) that the kernel k decomposing according to $k(n) \sim p(n)\gamma(n)$ as $n \to \infty$ where p is an asymptotically N–periodic function, and γ is in a class of slowly decaying functions, is equivalent to the solution $x(n)$ having asymptotic behaviour given by $x(n) \sim q(n)\gamma(n)$ as $n \to \infty$ where q is an asymptotically N–periodic function. This extends work of [13], in which the kernel does not have a periodic component. Once this problem is understood for the resolvent case the result can be easily generalised to apply to a more general perturbed Volterra convolution summation equation. As noted above this theory is used to provide a counter–example to a result regarding the rate of decay of the autocovariance function of an ARCH(∞) process.

The first chapter concerns itself with the rate of decay of a second moment of a stochastic process, i.e. the solution of a deterministic Volterra equation. The second chapter continues this study of deterministic Volterra equations. However if one wishes to describe the pathwise long run behaviour of a stochastic process (as opposed to the long run behaviour of its moments) then one may reformulate the stochastic differential equation in a manner such that the deterministic Volterra admissibility theory may be applied or alternatively one may develop an authentically stochastic admissibility theory. The latter approach is the motivation for Chapter 3.

The first half of Chapter 3 identifies conditions guaranteeing convergence of linear stochastic Volterra operators. Necessary and sufficient conditions for mean square convergence are established, while almost sure convergence of the linear operator is shown to imply mean square convergence. Sufficient conditions for almost sure convergence of the stochastic linear operator are established. The second half of Chapter 3 applies these almost sure conditions to determine the rate of growth or decay of the solutions of a class of affine stochastic functional differential equations. It is shown that the asymptotic behaviour of Volterra linear functional equations and finite delay linear functional equations are determined from the roots of an associated characteristic equation. This analysis is then in contrast to that of Chapters 1 and 2 where the contribution of the roots of the characteristic equation to the long run behaviour of the solution of the equations under study was dominated by that of the kernel. An example is provided which discusses the sharpness of the conditions guaranteeing the asymptotic results.

The remainder of the thesis is concerned with the pathwise asymptotic analysis and autocovariance asymptotic analysis of two particular stochastic functional equations. Chapters 5 and 6 consider the equation

$$dX(t) = \left(aX(t) + b\frac{1}{1+t}\int_{-1}^{t} X(s)\,ds \right) dt + \sigma\,dB(t), \quad t \geq 0, \tag{0.3.1}$$

where X is given by the continuous function ψ, defined on $[-1,0]$, B is a standard one–dimensional Brownian motion and $\sigma \neq 0$ and a and b are real parameters. While Chapter 7 considers a related stochastic difference equation. Chapter 4 serves as an introduction to these three chapters and discusses the commonalities and differences between them. In particular it is argued that (0.3.1) may be viewed as a simple model of an inefficient financial market in which operate technical analysts and reference traders.

While Chapters 5 and 6 both examine (0.3.1) they differ in their respective approaches to ascertaining sharp asymptotic results for the solution X of (0.3.1). The approach taken in Chapter 5 is to use existing admissibility results, i.e. [13], to determine the long run behaviour of X. The result, for $a > 0$,

$$\lim_{t\to\infty} \frac{X(t)}{e^{at}t^{b/a}} = C, \quad \text{almost surely},$$

is shown, where C is an almost surely finite Gaussian random variable. However it is not so clear as to whether or not C is non-zero (i.e. whether or not the deterministic admissibility theory produces a sharp asymptotic result). One could apply the stochastic admissibility theory which was developed in the first half of Chapter 3, however we choose to perform our analysis instead via a method more specific to the particular features of (0.3.1), this analysis is contained in Chapter 6.

The asymptotic behaviour of the solution X of (0.3.1) is given for all real values of a and b in Chapter 6. When solutions are recurrent, it is shown that the autocovariance function of the solution decays at a polynomial rate, even though the solution is asymptotically equal to another asymptotically stationary process whose autocovariance function decays exponentially. It is shown that when solutions grow, they do so at either a polynomial or exponential rate in time depending on the sign of a parameter of the model, modulo some exceptional parameter sets. On these exceptional sets, solutions are recurrent on the real line with large fluctuations consistent with the Law of the Iterated Logarithm, or exhibit subexponential yet superpolynomial growth. The results of this chapter show that the solution of (0.3.1) has the same asymptotic rate of growth as the solution of an underlying deterministic equation, almost surely.

The last chapter of this thesis considers the growth, large fluctuations and correlation behaviour of an affine stochastic functional difference equation with an average functional which has comparable asymptotic properties to that of (0.3.1). It is shown that when solutions grow, they do so at a polynomial rate in time. Similar to (0.3.1) when solutions of the stochastic difference equation are recurrent, it is shown that the autocorrelation of the solution decays at a non–summable and polynomial rate, even though the solution is asymptotically equal to another asymptotically stationary process whose autocorrelation decays geometrically. The stochastic equation is characterised by two parameters. The limiting behaviour of the solution of the stochastic equation is detailed for all real values of these parameters.

0.4 Mathematical Preliminaries

This section details some notation, definitions and fundamental results which are used throughout this thesis.

0.4.1 Deterministic Preliminaries

The set of integers is denoted by $\mathbb{Z}$, $\mathbb{Z}^+ = \{n \in \mathbb{Z} : n \geq 0\}$ and $\mathbb{R}$ the set of real numbers. We denote by $\mathbb{R}_+$ the half-line $[0, \infty)$. The complex plane is denoted by $\mathbb{C}$ and $\mathbb{C}_0 := \{z \in \mathbb{C} : \Re(z) \geq 0\}$, where $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of any complex number z. If $x \in \mathbb{R}$, then $\lceil x \rceil$, or the *ceiling* of x is the smallest integer greater than or equal to $x \in \mathbb{R}$. $|x|$ denotes the absolute value of $x \in \mathbb{R}$. If d is a positive integer, $\mathbb{R}^d$ is the space of d-dimensional column vectors with real components and $\mathbb{R}^{d\times d}$ is the space of all $d \times d$ real matrices. Similarly, the space of all $d \times d$ matrices with complex–valued entries is denoted by $\mathbb{C}^{d\times d}$.

The Wronskian for any two functions x_1 and x_2, which have domain of definition $[0, \infty)$, is defined as $\mathcal{W}(t) = x_1(t)x_2'(t) - x_1'(t)x_2(t)$, for $t \geq 0$. The Casoratian, C, of two sequences r_1 and r_2, which have domain of definition $\mathbb{Z}^+$, is given by $C(n) = r_1(n)r_2(n+1) - r_1(n+1)r_2(n)$, $n \in \mathbb{Z}^+$.

Let $A \in \mathbb{R}^{d\times d}$ then $\det(A)$ denotes the determinant of the square matrix A. A^T denotes the transpose of any $A \in \mathbb{R}^{d_1\times d_2}$. A matrix $A = (A_{ij})$ in $\mathbb{R}^{d\times d}$ is *non-negative* if $A_{ij} \geq 0$, in which case we write $A \geq 0$. A partial ordering is defined on $\mathbb{R}^{d\times d}$ by letting $A \leq B$ if and only if $B - A \geq 0$. Of course $A \leq B$ and $C \geq 0$ implies that $CA \leq CB$ and $AC \leq BC$. The *absolute value* of $A = (A_{ij})$ in $\mathbb{R}^{d\times d}$ is the matrix given by $(|A|)_{ij} = |A_{ij}|$. The Frobenius norm of a matrix A is denoted $\|A\|_F$.

We use standard Landau notation, (c.f. e.g., [46, Chapter 8.1]), let f and g be two functions defined on $\mathbb{R}$, then we write $f(t) = O(g(t))$, $t \to \infty$ if there exists $T > 0$ and $M > 0$ such that $|f(t)| \leq M|g(t)|$ for all $t > T$. Whereas $f(t) = o(g(t))$, $t \to \infty$ means that $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 0$. Also, $f(t) \sim g(t)$, $t \to \infty$ means that $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1$.

We define a class of real-valued weight functions, which was studied in [13] and is variously used in this thesis.

Definition 0.4.1. Let $r > 0$ be finite. A real-valued sequence $\gamma = \{\gamma(n)\}_{n\geq 0}$ is in $\mathcal{W}(r)$

if $\gamma(n) > 0$ for all $n \geq 0$, and

$$\lim_{n\to\infty} \frac{\gamma(n-1)}{\gamma(n)} = \frac{1}{r}, \quad \sum_{i=0}^{\infty} \gamma(i) r^{-i} < \infty, \tag{0.4.1}$$

$$\lim_{m\to\infty} \left(\limsup_{n\to\infty} \frac{1}{\gamma(n)} \sum_{i=m}^{n-m} \gamma(n-i)\gamma(i) \right) = 0. \tag{0.4.2}$$

Observe that if $r < 1$ and $\gamma \in \mathcal{W}(r)$, then γ decays, whereas if $r > 1$, then γ diverges. If γ is in $\mathcal{W}(1)$, it is called a *subexponential sequence*, one reason being that if γ is in $\mathcal{W}(1)$, then

$$\lim_{n\to\infty} \gamma(n)\kappa^n = \infty \quad \text{for all } \kappa > 1. \tag{0.4.3}$$

Of course if γ is in $\mathcal{W}(r)$ and $\delta(n) = r^{-n}\gamma(n)$, then δ is in $\mathcal{W}(1)$.

Examples of sequences in $\mathcal{W}(r)$ include, but are not limited to, $\gamma(n) = r^n n^{-\alpha}$ for $\alpha > 1$; $\gamma(n) = r^n n^{-\alpha} \exp(-n^{\beta})$ for $\alpha \in \mathbb{R}$, $0 < \beta < 1$; and $\gamma(n) = r^n e^{-n/(\log n)}$. The sequences defined by $\gamma(n) = r^n$ and $\gamma(n) = r^n n^{-\alpha}$, $\alpha \leq 1$ are *not* in $\mathcal{W}(r)$.

We define the Gamma function $\Gamma : \mathbb{C} \to \mathbb{C}$ according to $\Gamma(z) = \int_0^\infty s^{z-1} \mathrm{e}^{-s} \,\mathrm{d}s$ for $\Re(z) > 0$. When $\Re(z) \leq 0$, $\Gamma(z)$ is defined by analytic continuation.

The space of p-summable sequences is denoted as ℓ^p, i.e.

$$\ell^p(\mathbb{Z}^+) = \{u : \mathbb{Z}^+ \to \mathbb{R} : \sum_{j=0}^{\infty} |u(j)|^p < +\infty\}.$$

Sequences $u = \{u(n)\}_{n\geq 0}$ in $\mathbb{R}^d$ or $U = \{U(n)\}_{n\geq 0}$ in $\mathbb{R}^{d\times d}$ are sometimes identified with functions $u : \mathbb{Z}^+ \to \mathbb{R}^d$ and $U : \mathbb{Z}^+ \to \mathbb{R}^{d\times d}$. If $\{U(n)\}_{n\geq 0}$ and $\{V(n)\}_{n\geq 0}$ are sequences in $\mathbb{R}^{d\times d}$, we define the *convolution* of $\{(U * V)(n)\}_{n\geq 0}$ by

$$(U * V)(n) = \sum_{j=0}^{n} U(n-j)V(j), \quad n \geq 0.$$

Moreover using this definition of convolution one may recursively define the *j-fold convolution*, $\{(U^{*j})(n)\}_{j\geq 2, n\geq 0}$, by $(U^{*2})(n) = (U * U)(n)$ and $(U^{*j})(n) = (U^{*(j-1)} * U)(n)$ for $j \geq 3$ and $n \geq 0$.

In this thesis the *Z-transform* of a sequence U in $\mathbb{R}^{d\times d}$ is the function defined by

$$\tilde{U}(\lambda) = \sum_{j=0}^{\infty} U(j)\lambda^j,$$

provided λ is a complex number for which the series converges absolutely. A similar definition pertains for sequences with values in other spaces. We remark that this definition of the Z-transform differs from the more usual definition (see e.g. [46, Chapter 6.1]) in that λ plays the role of λ^{-1} and hence roots and poles of the Z-transform which were outside the unit circle are now inside the unit circle, and vice versa.

For any two functions $U : \mathbb{R}_+ \to \mathbb{R}^{d_1\times d_2}$ and $V : \mathbb{R}_+ \to \mathbb{R}^{d_2\times d_3}$. we define the *convolution* of $\{(U * V)(t)\}_{t\geq 0}$ by

$$(U * V)(t) = \int_0^t U(t-s)V(s)\,ds, \quad t \geq 0.$$

In this thesis the *Laplace transform* of a function U in $\mathbb{R}^{d_1\times d_2}$ is the function defined by

$$\tilde{U}(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda s} U(s)\,ds,$$

provided λ is a complex number for which the integral converges absolutely. A similar definition pertains for the Laplace transform of a measure, [53, Definitions 2.1, 2.2] and for functions with values in other spaces.

Let $BC(\mathbb{R}_+;\mathbb{R}^{d_1\times d_2})$ denote the space of matrices whose elements are bounded continuous functions. Let BC_l be the space of bounded continuous functions with a limit at infinity (although not necessarily the same limit at $-\infty$ as at $+\infty$ if the domain is $\mathbb{R}$). The abbreviation *a.e.* stands for *almost everywhere*, while *a.s.* stands for *almost sure* or *almost surely*. The space of continuous and continuously differentiable functions on $\mathbb{R}_+$ with values in $\mathbb{R}^{d_1\times d_2}$ is denoted by $C(\mathbb{R}_+;\mathbb{R}^{d_1\times d_2})$ and $C^1(\mathbb{R}_+;\mathbb{R}^{d_1\times d_2})$ respectively, while $C^{1,0}(\Delta;\mathbb{R}^{d_1\times d_2})$ represents the space of functions which are continuously differentiable in their first argument and continuous in their second argument, over some two–dimensional space Δ. For any scalar function φ, the space of weighted p^{th}integrable functions is denoted by

$$L^p(\mathbb{R}_+;\mathbb{R}^{d_1\times d_2};\varphi) := \{f : \mathbb{R}_+ \to \mathbb{R}^{d_1\times d_2} : \int_0^\infty \varphi(s)|f(s)_{i,j}|^p\,ds < +\infty, \text{ for all } i,j\}.$$

When $\varphi = 1$, we do not include it in our notation, i.e. $L^p(\mathbb{R}_+;\mathbb{R}^{d_1\times d_2};1) = L^p(\mathbb{R}_+;\mathbb{R}^{d_1\times d_2})$.

0.4.2 Stochastic Preliminaries

Many of the below definitions and theorems may be found in Mao [84], Karatzas and Shreve [72], and Revuz and Yor [104].

Probability Space. Consider the ordered triple (Ω, $\mathcal{F}$, $\mathbb{P}$). Let Ω (referred to as the sample space) be a set of points (or outcomes) ω. A family, $\mathcal{C}$, of subsets (or events) of Ω is referred to as a $\sigma - algebra$ if: $\Omega \in \mathcal{C}$; $A \in \mathcal{C}$ implies $A^C \in \mathcal{C}$; $\{A_i\}_{i\geq 1} \subset \mathcal{C}$ implies $\cup_{i=1}^{\infty} A_i \in \mathcal{C}$. A^C denotes the complement of A in Ω. Let $\mathcal{F}$ denote the family of subsets (or events) of Ω which are a σ–algebra. Elements of $\mathcal{F}$ are called $\mathcal{F}$–measurable sets.

If $\mathcal{C}$ is a family of subsets of Ω then there exists a smallest σ–algebra, $\sigma(\mathcal{C})$, on Ω which contains $\mathcal{C}$. This σ–algebra is called the *$\sigma - algebra$ generated by* $\mathcal{C}$. If $\Omega = \mathbb{R}$ and $\mathcal{C}$ is the family of all open sets in $\mathbb{R}$ then $\mathcal{B} = \sigma(\mathcal{C})$ is called the *Borel $\sigma - algebra$* and the elements of $\mathcal{B}$ are called the *Borel sets*.

A probability measure on the measurable space $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ which obeys the following: $\mathbb{P}[\Omega] = 1$,; for any disjoint sequence $\{A_i\}_{i\geq 1} \subset \mathcal{F}$, $\mathbb{P}[\cup_{i=1}^{\infty} A_i] = \sum_{i=1}^{\infty} \mathbb{P}[A_i]$. If an event has probability one then we say that it is an almost sure (a.s) event. A triple (Ω, $\mathcal{F}$, $\mathbb{P}$) with Ω, $\mathcal{F}$ and $\mathbb{P}$ as described is called a *probability space.* Any measure $\mathbb{P}$ defined on the σ-algebra of Borel sets is called a Borel measure.

A filtration is a family $\{\mathcal{F}(t)\}_{t\geq 0}$ of increasing sub-σ-algebras of $\mathcal{F}$. The filtration at time t represents all of the information available up to time t. The filtered probability space is denoted by $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$.

A filtration is said to satisfy the *usual conditions* if it is right–continuous, i.e. $\mathcal{F}(t) = \cap_{s>t}\mathcal{F}(s)$ for all $t \geq 0$, and $\mathcal{F}(0)$ contains all the $\mathbb{P}$–null events in $\mathcal{F}$. We also define

$$\mathcal{F}(\infty) = \sigma\big(\cup_{t\geq 0}\mathcal{F}(t)\big).$$

Random Variable. A real–valued function $X : \Omega \to \mathbb{R}$ is said to be $\mathcal{F}$–measurable if $\{\omega : X(\omega) \leq a\} \in \mathcal{F}$ for all $a \in \mathbb{R}$. Such a function is called an ($\mathcal{F}$–measurable) random variable.

For random variables U and V defined on the same probability space, and each of which has finite variance, we denote their means (or expectations) by $\mathbb{E}[U]$ and $\mathbb{E}[V]$ and their variances by $\text{Var}[U]$ and $\text{Var}[V]$. Their covariance is denoted by $\text{Cov}(U, V)$.

Stochastic Process. A collection of random variables, $\{X(t)\}_{t\geq 0}$, defined on the same probability space is called a stochastic process. It is $\mathcal{F}(t)$–adapted if $X(t)$ is $\mathcal{F}(t)$–measurable for each t,. It is called *continuous* if for all $\omega \in \Omega$ the function $t \mapsto X(t, \omega)$ is continuous. A stochastic process may also be defined on a discrete time-domain.

Let J be either $\mathbb{Z}$ or $\mathbb{R}$. A stochastic process $X = \{X(k) : k \in J\}$ is referred to as *strictly stationary* if all of its finite-dimensional distribution functions are time-invariant, or more specifically

$$\mathbb{P}[X(t_1 + k) \leq x_1, X(t_2 + k) \leq x_2, ..., X(t_n + k) \leq x_n]$$
$$= \mathbb{P}[X(t_1) \leq x_1, X(t_2) \leq x_2, ..., X(t_n) \leq x_n]$$

for all $k, t_1, t_2, ..., t_n \in J$, $x_1, ..., x_n \in \mathbb{R}$ and for all $n \in \mathbb{Z}^+/\{0\}$. One infers from this definition that the statistical properties of X do not change over time. It is difficult to test for strict stationary from a sample, so for this reason we concern ourselves mainly in this work with weak stationarity. We do this also because weak stationarity is especially effective in describing dependence in affine or linear models.

A stochastic process $X = \{X(k) : k \in J\}$ is said to be *weakly stationary* or *wide sense stationary* if it has constant mean, $\mathbb{E}[X(k)] \in \mathbb{R}$ for all $k \in J$, and there exists a function $\rho : J \to \mathbb{R}$, called the *autocovariance function*, such that,

$$\mathrm{Cov}[X(n), X(k)] = \rho(n - k), \quad \text{for all } n, k \in J. \tag{0.4.4}$$

Throughout this work the qualifiers weak and weakly are dropped, and we refer to such processes as being *stationary* or possessing the property of *stationarity*. The concept of stationarity is that a structure is imposed upon the statistical properties of the process which gives the process a time–invariance property. The *autocorrelation function* of X is defined by $\rho(k)/\mathrm{Var}[X(0)]$ for $k \in J$, where $\mathrm{Var}[X(0)]$ is non–trivial.

It is of special interest in this work to establish the rate at which $\rho(k) \to 0$ as $k \to \infty$ and in particular to investigate whether the process X possesses long memory. A number of definitions of long memory exist in the literature: here we adopt one of the commonest, saying that X, with $J = \mathbb{Z}$, has *long memory* if the autocovariance function is not summable i.e.,

$$\sum_{k \in J} |\rho(k)| = +\infty. \tag{0.4.5}$$

When $J = \mathbb{R}$ then the summation in (0.4.5) is duly replaced with an integral. The underpinning idea of long memory is that realisations far in the past do not fade away quickly and so have a bearing upon the present and future development of the process. The significance of long memory as a measure of the efficiency of a financial market is discussed in e.g. Cont [35].

Continuous Stochastic Preliminaries

Standard Brownian Motion. *Standard Brownian motion* is an almost surely continuous, $\mathcal{F}(t)$-adapted process $B = \{B(t); 0 \leq t \leq \infty\}$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the properties that $B(0) = 0$ a.s. and for $0 \leq s \leq t$, the increment $B(t) - B(s)$ is independent of $\mathcal{F}(s)$ and is normally distributed with mean zero and variance $t - s$. The *natural filtration* generated by $\{B(t)\}_{t \geq 0}$ is defined by

$$\mathcal{F}^B(t) = \sigma\Big(B(s) : 0 \leq s \leq t\Big).$$

We will often take $\{\mathcal{F}^B(t)\}_{t \geq 0}$ as the filtration with respect to which stochastic processes are adapted throughout this thesis.

Theorem 0.4.1 (Hinčin's Law of the Iterated Logarithm (LIL))**.** *For almost every $\omega \in \Omega$, we have*

$$\limsup_{t \to \infty} \frac{B(t)}{\sqrt{2t \log \log t}} = 1, \quad \liminf_{t \to \infty} \frac{B(t)}{\sqrt{2t \log \log t}} = -1.$$

Stochastic Integrals. Let $B(t) = \{B_1(t), ...B_d(t)\}$, where each element of B is a standard Brownian motion. The $n \times d$ dimensional Itô integral is denoted

$$\int_0^t g(s) dB(s),$$

for an $\mathbb{R}^{n \times d}$ dimensional function $g = \{g(t)\}_{0t \geq 0}$ such that

$$\int_0^t \|g(s)\|_F^2 \, ds < \infty.$$

Then g obeys *Itô's isometry*, in particular

$$\mathbb{E}\left[\left\|\int_0^t g(s) dB(s)\right\|_F^2\right] = \int_0^t \|g(s)\|_F^2 \, ds.$$

An n-dimensional Itô-process is an $\mathbb{R}^n$-valued continuous adapted process $X(t) = \{X_1(t), ..., X_n(t)\}^T$ on $t \geq 0$ of the form

$$X(t) = X_0 + \int_0^t f(s) ds + \int_0^t g(s) dB(s),$$

where $f = (f_1, ..., f_n)^T \in L^1(\mathbb{R}_+; \mathbb{R}^n)$ and $g = (g_{i,j})_{n \times d} \in L^2(\mathbb{R}_+; \mathbb{R}^{n \times d})$. We shall say that $X(t)$ has the stochastic differential $dX(t)$ on $t \geq 0$ given by

$$dX(t) = f(t) dt + g(t) dB(t).$$

Martingales. Properties of martingales and their connection to Brownian motion are central to the proof of many of the results in this thesis. We list some important properties here.

Definition 0.4.2. A real valued process $M(t)$, $t \in \mathbb{R}_+$, adapted to $(\mathcal{F}(t))$ is a martingale (with respect to $\mathcal{F}(t)$) if

(i) $\mathbb{E}[|M(t)|] < +\infty$ for every $t \in \mathbb{R}_+$;

(ii) $\mathbb{E}[M(t)|\mathcal{F}(s)] = M(s)$ a.s. for every pair s, t such that $s < t$.

If the process M is a real-valued square integrable martingale then there exists a unique adapted, continuous increasing process $\langle M \rangle = \{\langle M \rangle(t)\}_{t \geq 0}$ such that the process $\{M(t)^2 - \langle M \rangle(t)\}_{t \geq 0}$ is a martingale which vanishes at $t = 0$. The process $\langle M \rangle$ is referred to as the *quadratic variation* of M.

A random variable $\tau : \Omega \to [0, \infty]$ is called an $\mathcal{F}(t)$-*stopping time* if $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}(t)$ for any $t \geq 0$. A right-continuous adapted process $M = \{M(t)\}_{t \geq 0}$ is called a *local martingale* if there exists a non-decreasing sequence $\{\tau_k\}_{k \geq 1}$ of stopping times with $\tau_k \to \infty$ as $k \to \infty$ a.s. such that $\{M(\min(\tau_k, t))\}_{t \geq 0}$ is a martingale.

The following results may be found in [104].

Theorem 0.4.2 (Martingale Convergence Theorem)**.** *For a continuous local martingale M, the sets $\{\langle M \rangle(\infty) < \infty\}$ and $\{\lim_{t \to \infty} M(t)$ exists$\}$ are almost-surely equal. Furthermore, $\limsup_{t \to \infty} M(t) = +\infty$ and $\liminf_{t \to \infty} M(t) = -\infty$ a.s. on the set $\{\langle M \rangle(\infty) = \infty\}$.*

Theorem 0.4.3 (Martingale Time-Change Theorem)**.** *Let M be a continuous local martingale vanishing at zero such that $\lim_{t \to \infty} \langle M \rangle(t) = \infty$. Define, for each $0 \leq s < \infty$,*

$$T(s) = \inf\{t \geq 0; \langle M \rangle(t) > s\}.$$

Then $B(s) = M(T(s))$ is a $(\mathcal{F}(T(s)))$-Brownian motion and $M(t) = B(\langle M \rangle(t))$.

Lemma 0.4.1. *Let M be a continuous local martingale. Then on $\{\langle M \rangle(\infty) = \infty\}$, one has*

$$\limsup_{t \to \infty} \frac{M(t)}{\sqrt{2\langle M \rangle(t) \log\log\langle M \rangle(t)}} = 1, \quad \liminf_{t \to \infty} \frac{M(t)}{\sqrt{2\langle M \rangle(t) \log\log\langle M \rangle(t)}} = -1, \quad a.s.$$

Discrete Stochastic Preliminaries

Many of the below definitions and theorems may be found in Shiryaev [108], Williams [115] and Chow and Thiecher [34]. The definition of a discrete-time martingale is similar to that of a continuous-time martingale, c.f. e.g. [115, Chapter 10], and so is omitted. Of particular importance to the results in this thesis is that the sum of independent zero-mean random variables is a discrete-time martingale. We firstly state a convergence result.

Theorem 0.4.4. *Suppose that $\{X(n)\}_{n\in\mathbb{Z}^+}$ is a sequence of independent random variables such that $\mathbb{E}[X(n)] = 0$, for every n. Then if*

$$\sum_{n\in\mathbb{Z}^+} \mathit{Var}[X(n)] < \infty$$

the series $\sum_{n\in\mathbb{Z}^+} X(n)$ converges with probability one.

The following result is stated as Theorem 2 of [114] or Exercise 3 in [34, pp383, Section 10.2]

Lemma 0.4.2 (Law of the Iterated Logarithm)**.** *Let $\{X_n\}_{n\in\mathbb{Z}^+}$ be a sequence of independent Gaussian random variables where X_n has mean zero and variance σ_n^2. If $s_n^2 = \sum_{i=1}^n \sigma_i^2 \to \infty$ as $n \to \infty$ and $\sigma_n = o(s_n)$ as $n \to \infty$, then*

$$\limsup_{n\to\infty} \frac{\sum_{j=1}^n X_j}{\sqrt{2s_n^2 \log\log s_n^2}} = -\liminf_{n\to\infty} \frac{\sum_{j=1}^n X_j}{\sqrt{2s_n^2 \log\log s_n^2}} = 1, \quad a.s.$$

While the above result is sufficient for the analysis of this article, as Tomkins [114] observes these sufficient conditions may be sharpened. For instance, Hartman [63] requires only $\limsup_{n\to\infty} \sigma_n/s_n < 1$ as opposed to $\sigma_n/s_n \to 0$ as $n \to \infty$ in order to achieve a discrete law of the iterated logarithm result.

Useful Results

Chebyshevs inequality. For $p \in (0,\infty)$, let X be a random variable with $\mathbb{E}[\,|X|^p] < \infty$. If $c > 0$ then

$$\mathbb{P}[\omega : |X(\omega)| \geq c] \leq c^{-p}\,\mathbb{E}[\,|X|^p].$$

Borel-Cantelli Lemma. Let $A_1, A_2, ...$ be a sequence of events in $\mathcal{F}$. Let $\{A_n, i.o.\}$ denote the event that the events A_n are realised infinitely often.

(a) If $\sum_{n=1}^{\infty} \mathbb{P}[A_n] < \infty$ then $\mathbb{P}[A_n, i.o.] = 0$.

(b) If $\sum_{n=1}^{\infty} \mathbb{P}[A_n] = \infty$ and $A_1, A_2, ...$ are independent, then $\mathbb{P}[A_n, i.o.] = 1$.

The Borel-Cantelli Lemma is intermittently used in this thesis to determine the order of fluctuations of a stochastic process from its mean.

Chapter 1

Long run behaviour of the autocovariance function of ARCH(∞) models

1.1 Introduction

The significant influence of past data upon current and future values of a time series is evidenced in many time series from the physical sciences and finance, e.g. tree-ring data series, wheat market prices (cf., e.g., Baillie [24]) and stock market and foreign exchange returns (cf., e.g., Ding and Granger [44]). The influence of past realisations may be defined in terms of the persistence of the autocorrelations of the series, with a stationary series whose autocorrelations decay at a non-summable rate being referred to as a "long memory" process. Furthermore, the presence and application of long memory processes in macroeconomics, asset pricing models and interest rate models is noted in [24] and the references contained therein. Various properties of fractional Brownian motion are illustrated in Mandelbrot and Van Ness [81]: of particular note is that fractional Brownian motion is a self–similar process whose increments are stationary and can exhibit long memory.

Kirman and Teyssière [73, 74] give discrete time series models which are derived from a market which is composed of fundamental and technical analysts, these models are then shown to possess long memory characteristics in the differenced log returns of price processes associated with these models, while other features such as bubbles are demonstrated. Appleby and Krol [15] analyse the long memory properties of a linear stochastic Volterra equation in both continuous and discrete time, with conditions for both subexponential rates of decay and arbitrarily slow decay rates in the autocovariance function being characterised in terms of the decay of the kernel of the Volterra equation. A continuous–time infinite history financial market model is discussed in Anh et al. [2, 3], which is a generalisation of the classic Black-Scholes model, where characterisations for long memory are proved. In each of [2, 3, 15] the equations studied have additive noise, so the size of stochastic shocks are independent of the state of the system.

A widely–employed class of discrete–time stochastic processes in which the shock size depends on the state are the so–called ARCH (autoregressive conditional heteroskedastic) processes. ARCH processes are widely used and studied in financial mathematics to characterise time varying conditional volatility as well as the non–trivial autocovariance functions possessed by autoregressive processes driven by additive noise. In particular, the ARCH formulation captures well the tendency for clustering of volatility Engle [49]. Much of the work on ARCH processes concerns processes with finite memory: if only the last q values of the process determine the dynamics, the process is termed an ARCH(q) process. A property of these finite–memory processes is that their autocovariance functions decay exponentially fast in their time lag. Therefore slow decay or long memory in an ARCH–type process can only be achieved by considering terms from unboundedly far in the past. This naturally leads to the study of ARCH(∞) processes and in this work we study the memory properties of such processes. A standard definition given in e.g., [51], for these processes is:

Definition 1.1.1. A random sequence $X = \{X(k), k \in \mathbb{Z}\}$ is said to satisfy ARCH(∞) equations if there exists a sequence of independent and identically distributed (i.i.d.) non–negative random variables $\xi = \{\xi(k), k \in \mathbb{Z}\}$ such that

$$X(k) = \varsigma(k)\xi(k), \quad \varsigma(k) = a + \sum_{j=1}^{\infty} b(j)X(k-j), \tag{AH}$$

where $a \geq 0$ and $b = \{b(j), j \in \{1, 2, ...\}\}$ satisfies $b(j) \geq 0$, for $j \in \{1, 2, ...\}$.

ARCH(∞) processes were initially introduced by Robinson [106] as an alternative model when testing for serial correlation. This process is a generalisation of the "classical" ARCH(∞) process

$$r(k) = \sigma(k)\epsilon(k), \quad \sigma(k)^2 = \tau + \sum_{j=1}^{\infty} \phi(j)r(k-j)^2,$$

where $\tau, \phi \geq 0$ and ϵ is an i.i.d. random sequence. Moreover (AH) includes models where r and σ are replaced by an arbitrary fractional positive powers of themselves and the 'shocks', ϵ, are taken to be non-negative. The terminology ARCH(∞) is justified, as an ARCH(∞) process is in some sense the limit of an ARCH(q) process as $q \to \infty$. It can be seen, moreover that ARCH(∞) processes are generalisations of the finite order ARCH and GARCH processes: indeed the ARCH(q) process of [49], results when $\phi(j) = 0$ for $j \geq q+1$

and the GARCH(p, q) process of Bollerslev [29] may be rewritten as an ARCH(∞) process with exponentially decaying weights b.

As attested to above, empirical findings indicate the presence of long memory in financial and economic time series, which has resulted in research being focused on the long memory properties of stationary solutions of ARCH-like processes (cf., e.g., Baillie et al. [25]). Of note here are the investigations into necessary and sufficient conditions for the existence of a weakly stationary solution of the ARCH(∞) process, conducted by Giraitis, Kokoszka, Leipus, Surgailis, and Zaffaroni [51, 52, 75, 118]. Moreover, these papers extensively study the autocovariance structure and long memory properties of (AH). Section 1.2 details some of the results of [51, 52, 118] which are applicable to the results of this chapter. Also in Section 1.2 we highlight in particular the importance of an underlying resolvent equation in determining the long term memory characteristics of (AH). Also, a Volterra series representation of the autocovariance function is established.

The main results of this chapter appear in Section 1.3 where conditions on the coefficients a and b and the process ξ in (AH), are given to describe decay rates in a class wider than the class of hyperbolically decaying sequences considered heretofore. Roughly speaking, for the memory, or kernel b, lying in a class of slowing decaying (subexponential) sequences it is shown that the autocovariance function must decay at precisely the rate of b. Furthermore, we prove for the first time converse results which show that such exact non–exponential rates of decay of the autocovariance function result only when b lies in this class. These results strengthen the hypotheses of [118, Theorem 2].

Section 1.4 describes the effect that upper and lower slowly decaying bounds on b have on the autocovariance function. The main result is that a nontrivial subexponential upper bound on the rate of decay of the autocovariance function is equivalent to a nontrivial subexponential upper bound on the decay rate of the kernel b. However, a numerical example demonstrates that a corresponding lower bound on the autocovariance function does not necessarily come from a corresponding lower bound on b, so one cannot readily characterise necessary and sufficient conditions for lower bounds on the memory of (AH). Section 1.4 also gives necessary and sufficient conditions for exponential decay of the autocovariance function. This last result complements the sufficient conditions of [75, Theorem 3.1] while employing a different method of proof.

One of the chief differences in the analysis of this chapter to that of [51, 52, 75] is that rather than analysing an explicit representation of the solution of (AH), we primarily express the autocovariance function and its associated resolvent as the solutions of Volterra equations and then employ admissibility theory of linear Volterra operators to study the asymptotic behaviour. Such admissibility theory has been developed and used by e.g., Appleby, Győri, Horváth, Reynolds [6, 13, 14, 55] to determine rates of convergence to the equilibrium of linear Volterra summation equations. The proofs of results stated in Sections 1.3 and 1.4 are confined to Section 1.5.

In this work, we have concentrated solely on the asymptotic behaviour of stationary solutions of ARCH(∞) equations. It is our belief that many of the asymptotic results presented here are robust to mild departures from stationarity and have some continuous time analogues. A brief analysis of the continuous case is presented in [10]. However, an investigation of non–stationary processes is deferred to a later work. The work of this chapter appears as a joint paper with Appleby [9].

1.2 Discussion of Existing Results on ARCH(∞) Processes

Throughout this chapter we use the notation

$$\lambda_1 = \mathbb{E}[\xi(0)], \quad \lambda_2 = \mathbb{E}[\xi(0)^2], \quad B = \sum_{j=1}^{\infty} b(j), \quad \sigma^2 = \text{Var}[\xi(0)] = \lambda_2 - \lambda_1^2.$$

It is assumed throughout that both the first moment of ξ is finite and non–zero, i.e. $0 < \lambda_1 < \infty$. A zero mean of ξ results in X reducing to the trivial solution, i.e. $X(k) = 0$ a.s. for all $k \in \mathbb{Z}$. Also $\sigma = 0$ is equivalent to the shocks ξ being a.s. constant, and is therefore not of interest. Equally, the case $a = 0$ is not of interest, for it is known in this case that $X(k) = 0$ a.s. for all $k \in \mathbb{Z}$ is the only stationary solution of (AH), see e.g. [51, Theorem 2.1].

Furthermore if $b(j) = 0$ for all $j \geq 1$ then this results in the degenerate case of a constant conditional volatility of X in (AH), thereby defeating the initial motivation for studying ARCH processes. In this case, X degenerates to a constant multiple of the i.i.d. non-negative "shocks". We thus argue it is reasonable to assume that there exists at least one value in the sequence b which is positive. For this reason, we have as a standing hypothesis throughout the chapter that

$$\lambda_1 \in (0,\infty), \quad a > 0, \quad \sigma \in (0,\infty), \quad b \not\equiv 0. \tag{S$_0$}$$

With the added assumption that

$$\lambda_1 B < 1, \tag{S$_1$}$$

it is shown in [51] that $\mathbb{E}[X(k)] = a\lambda_1/(1-\lambda_1 B) < +\infty$ for all $k \in \mathbb{Z}$.

A moving average representation of the solution of (AH) is derived in [51]. We briefly outline the construction of this representation and use it to develop a Volterra equation satisfied by the coefficients of this representation. The results later in this work concur with [118, Theorem 2], namely that these coefficients determine the rate of decay of the autocovariance function.

Let $\psi(L) = 1 - \lambda_1 \sum_{j=1}^{\infty} b(j)L^j$, where L is the *lag* or *backward shift* operator which operates on a process $Y = \{Y(k) : k \in \mathbb{Z}\}$ according to $L\big(Y(k)\big) = Y(k-1)$. Define $\nu(k) := X(k) - \lambda_1 \varsigma(k)$: then from (AH) we have

$$\psi(L)X(k) = a\lambda_1 + \nu(k).$$

A moving average representation for X is then obtained by applying the operator $\psi^{-1}(L)$ across this equation. The existence of such an inverse operator (on the closed unit circle in the complex plane) is given in [51] and the references contained therein. This existence is chiefly guaranteed by the summability of b, a consequence of (S$_1$) which is assumed throughout this work. We now state Lemma 4.1 of [51], which is also [107, Problem 8, Chapter 18].

Lemma 1.2.1. *Suppose $\sum_{j=0}^{\infty} |\psi_j| < \infty$, $\psi(\lambda) := \sum_{j=0}^{\infty} \psi_j \lambda^j$, and $|\psi(\lambda)| > 0$ for $|\lambda| \le 1$. Then there exists a sequence $z = \{z(j) : j \in \mathbb{Z}^+\}$ such that $D(\lambda) := 1/\psi(\lambda) = \sum_{j=0}^{\infty} z(j)\lambda^j$ is well defined for all $|\lambda| \le 1$. Furthermore, $\sum_{j=0}^{\infty} |z(j)| < +\infty$.*

We state the theorem guaranteeing a moving average representation from [51, Theorem 4.1].

Theorem 1.2.1. *If condition* (S$_1$) *holds, then there is a solution X of* (AH) *which admits the representation*

$$X(k) = \mathbb{E}[X(k)] + \sum_{j=0}^{\infty} z(j)\nu(k-j)$$

where $\sum_{j=0}^{\infty}|z(j)| < \infty$ *and the process* ν *satisfies* $\mathbb{E}[\nu(k)|\mathcal{F}(k-1)] = 0$ *for each* k*, where* $(\mathcal{F}(k))_{k\in\mathbb{Z}}$ *is the natural filtration generated by* ξ*.*

Moreover, in [51] it is shown that with the additional assumption

$$\lambda_2^{\frac{1}{2}} \sum_{j=1}^{\infty} b(j) < 1, \tag{1.2.1}$$

then (AH) has a unique *weakly* stationary solution, and hence $\mathbb{E}[\nu(k)^2] < +\infty$.

In both [52] and [118] necessary and sufficient conditions are derived for the existence of a weakly stationary solution of (AH). For completeness we state next a slightly reformulated variant of part of [52, Theorem 3.1], omitting those parts that are not relevant to our investigation.

Theorem 1.2.2. *The following are equivalent*

(a) (S_1) *holds and*

$$\Omega := \frac{\sigma}{\lambda_1}\left(\sum_{j=1}^{\infty} z(j)^2\right)^{1/2} < 1 \tag{S_2}$$

where z *is (well) defined by*

$$\frac{1}{1-\lambda_1\sum_{j=1}^{\infty} b(j)\lambda^j} = \sum_{j=0}^{\infty} z(j)\lambda^j, \quad |\lambda| \le 1;$$

(b) A weakly stationary solution X *of* (AH) *exists.*

Both imply that there exists a unique, ergodic solution of (AH) *which may be written as a convergent orthogonal Volterra series. Moreover,* $\mathrm{Cov}[X(0), X(k)] \ge 0$ *and*

$$\mathrm{Cov}[X(0), X(k)] = \left(\frac{a\sigma}{1-\lambda_1 B}\right)^2 \frac{1}{1-\Omega^2}\,\chi_z(k), \quad \textit{for} \quad k \in \mathbb{Z}, \tag{1.2.2}$$

where

$$\chi_z(k) = \sum_{j=0}^{\infty} z(j)z(j+|k|). \tag{1.2.3}$$

While the explicit representation of X as a convergent orthogonal Volterra series is a key component in the proof of Theorem 1.2.2, in order to keep this chapter concise we do not state this explicit form in the above as it does not form part of our analysis. We further comment that, as observed in [52], the condition (S_2) is weaker than (1.2.1), which is imposed in [51]. Under (S_2), X is weakly stationary and the autocovariance function is a multiple of χ_z and hence is absolutely summable, thus ruling out long memory. Moreover as $b \geq 0$ by hypothesis, this gives, via (1.2.6), that $z \geq 0$ and hence, under the condition (S_2), Theorem 1.2.2 gives $\text{Cov}[X(n), X(n+k)] \geq 0$. This observation concurs with that of [51] for the non-negativity of the autocovariance function under (1.2.1).

Under the conditions of Theorem 1.2.2, the moving average representation of Theorem 1.2.1 and (1.2.2) imply that

$$\mathbb{E}[\nu(0)^2] = \left(\frac{a\sigma}{1-\lambda_1 B}\right)^2 \frac{1}{1-\Omega^2},$$

and also that

$$\text{Var}[X(0)] = \left(\frac{a\sigma}{1-\lambda_1 B}\right)^2 \frac{1}{1-\Omega^2} \sum_{j=0}^{\infty} z(j)^2 = \left(\frac{a\sigma}{1-\lambda_1 B}\right)^2 \frac{1+\lambda_1^2\Omega^2/\sigma^2}{1-\Omega^2}. \tag{1.2.4}$$

The first result of this chapter is the calculation of a Yule-Walker style of representation for the autocovariance of (AH).

Proposition 1.2.1. *Let* (S_1) *and* (S_2) *hold. Then* ρ*, as defined by* (0.4.4)*, obeys*

$$\rho(k) = \begin{cases} \lambda_1 \sum_{j=-\infty}^{k-1} b(k-j)\rho(j), & \text{if } k \in \{1,2,3,...\}, \\ \rho(0), & \text{if } k = 0, \\ \rho(-k), & \text{if } k \in \{-1,-2,-3,...\}, \end{cases} \tag{1.2.5}$$

where $\rho(0)$ *is given by* (1.2.4).

The proof of Proposition 1.2.1, in common with many of the main results of the chapter, is postponed to the end.

Proposition 1.2.1 shows that the autocovariance obeys a Volterra summation equation with infinite delay. Since the chief focus of this chapter is to describe the asymptotic behaviour of ρ, it is interesting to draw a distinction between the potential asymptotic behaviour of ρ and the asymptotic behaviour of the autocovariance function of an equation with a *finite* number of lags. To this end consider an ARCH(q) rather than an ARCH(∞) process. Then the resulting autocorrelation function, as described by e.g., Taylor [113, pp.77,95], corresponds exactly to the autocorrelation function of the AR(q) process

$$W(k) = \sum_{j=1}^{q} \lambda_1 b(j) W(k-j) + e(k), \quad k \in \mathbb{Z},$$

where $e = \{e(k)\}_{k\in\mathbb{Z}}$ is an uncorrelated sequence of random variables with finite constant variance. Hence (1.2.5) reduces to the Yule–Walker equations:

$$\rho(k) = \lambda_1 \sum_{j=1}^{q} b(j)\rho(k-j), \quad k \in \{1, 2, ...\}.$$

Thus, the autocovariance function satisfies a q^{th}–order linear difference equation with constant coefficients. It is well–known that if the ARCH process is to be weakly stationary, all solutions of an auxiliary polynomial equation must lie inside the unit disc in $\mathbb{C}$, and that this condition also forces the autocovariance function to decay geometrically. Hence, for a finite history equation with a stationary solution, the autocovariance function must decay geometrically: polynomial decay is impossible.

Thus, the study of the autocovariance function of AR or ARCH models is bound–up with that of difference equations. It is then natural to ask what the asymptotic features of the solutions of unbounded equations of the form

$$y(k) = \sum_{j=0}^{k-1} u(k-i)y(i), \quad k \geq 1,$$

are for some $u : \mathbb{Z} \to \mathbb{R}$ and initial condition $y(0)$ and whether such an equation could be regarded as an underlying equation for the autocovariance function of some stationary times series. To the former question: it is well known that the dynamics of this equation allow both exponential and slower–than–exponential decay (see e.g., [93] for convergence rates in weighted l^1 spaces, [13] for exact rates in l^∞ spaces, and [47] for the characterisation of exponential decay). As to the latter: while for a stationary time series this is an open question nevertheless for a non-stationary times series such an equation could describe a family of autocovariances indexed by an initial starting time $m \in \mathbb{Z}$ i.e. $k \mapsto \mathrm{Cov}[X(m), X(k)] = y_m(k)$.

The distinction between this work and [51, 52, 75, 118] is that we exploit the fact that z from Lemma 1.2.1 and Theorem 1.2.2 may be written as the solution of a Volterra summation equation.

Lemma 1.2.2. *Suppose, for any $R > 0$, $\lambda_1 \sum_{j=1}^{\infty} b(j)R^j < +\infty$ and also that $\psi(\lambda) = 1 - \lambda_1 \sum_{j=1}^{\infty} b(j)\lambda^j$ for $|\lambda| \leq R$. Then the following are equivalent:*

(i) $D(\lambda) := 1/\psi(\lambda) = \sum_{j=0}^{\infty} z(j)\lambda^j$ is well defined for $|\lambda| \leq R$, $\sum_{j=0}^{\infty} z(j)R^j < \infty$ and

$$z(n) = \lambda_1 \sum_{j=0}^{n-1} b(n-j)z(j), \quad n = 1, 2, ...; \quad z(0) = 1; \tag{1.2.6}$$

(ii) $\lambda_1 \sum_{j=1}^{\infty} b(j)R^j < 1$.

Remark 1.2.1. We remark that in the case $R = 1$ much of the above lemma is covered in Lemma 1.2.1. We note however that in Lemma 1.2.2 the necessity of the condition $\lambda_1 \sum_{j=1}^{\infty} b(j)R^j < 1$ for the summability of z is drawn out.

Remark 1.2.2. It is elementary, using (1.2.6), to show that (1.2.2) is a solution of (1.2.5).

We observe that z may be thought of as a resolvent for (1.2.5) where the summation term is broken into a sum up to time $k-1$ and the remainder of the sum thought of as a perturbation term, i.e.

$$\rho(k) = \lambda_1 \sum_{j=0}^{k-1} b(k-j)\rho(j) + f(k-1), \quad k \geq 1, \tag{1.2.7}$$

where $f(k) = \lambda_1 \sum_{j=1}^{\infty} b(k+j+1)\rho(-j)$ and hence one has the variation of parameters formula

$$\rho(k) = z(k)\rho(0) + \sum_{j=0}^{k-1} z(k-j-1)f(j), \quad k \geq 1, \tag{1.2.8}$$

(see e.g., [46]). We demonstrate the usefulness of this formulation of the autocovariance function in the proof of Theorem 1.4.6. As this chapter primarily uses properties of Volterra equations to derive its results, it is perhaps more intuitive to regard z as the solution of an associated resolvent equation rather than the coefficients of a power series or moving average representation as in [51, 52, 118].

Remark 1.2.3. Using (1.2.6) and (1.2.1), we can show that (S_2) holds. Recalling that (1.2.1) implies (S_1), we can thus independently verify the sufficiency of (1.2.1) for the weak stationarity of the solution of (AH) as shown in [51, Theorem 2.1].

Proof of Remark 1.2.3. Applying the Cauchy–Schwartz inequality to the righthand side of (1.2.6) yields

$$z(n)^2 \leq \lambda_1^2 B \sum_{j=0}^{n-1} b(n-j)z(j)^2, \quad n \geq 1.$$

By summing both sides of this equation, and using the fact that (1.2.1) implies that z^2 is summable, we obtain

$$1 + \sum_{n=1}^{\infty} z(n)^2 \leq 1 + \lambda_1^2 B \sum_{n=1}^{\infty}\sum_{j=0}^{n-1} b(n-j)z(j)^2 = 1 + \lambda_1^2 B^2 \sum_{j=0}^{\infty} z(j)^2.$$

Since $z(0) = 1$, we obtain $\sum_{j=1}^{\infty} z^2(j) \leq 1/(1-\lambda_1^2 B^2) - 1$. Using this bound and (1.2.1) leads to (S_2). □

Remark 1.2.4. We can use the fact that z satisfies (1.2.6) to obtain a condition on b which implies the stationarity of X and which is sometimes weaker than the condition (1.2.1). More precisely, we show that

$$\lambda_2 < \lambda_1^2 + \frac{(1-\lambda_1 B)^2}{\sum_{j=1}^{\infty} b(j)^2} \tag{1.2.9}$$

implies (S_2), and that (1.2.1) implies (1.2.9) if

$$\lambda_1 B < \frac{1-\sum_{j=1}^{\infty} b(j)^2/B^2}{1+\sum_{j=1}^{\infty} b(j)^2/B^2}. \tag{1.2.10}$$

Proof of Remark 1.2.4. We start by noticing that (S_1) implies z is summable, and by summing on both sides of (1.2.6) it can readily be shown that $\sum_{j=0}^{\infty} z(j) = 1/(1-\lambda_1 B)$. Since b and z are non–negative, we may apply the Cauchy–Schwartz inequality to the right–hand side of (1.2.6) to get

$$z(n)^2 \le \lambda_1^2 \sum_{j=0}^{n-1} z(j) \cdot \sum_{j=0}^{n-1} b(n-j)^2 z(j), \quad n \ge 1.$$

Since z^2 is summable, we get

$$\sum_{n=1}^{\infty} z(n)^2 \le \lambda_1^2 \sum_{j=0}^{\infty} z(j) \cdot \sum_{n=1}^{\infty}\sum_{j=0}^{n-1} b(n-j)^2 z(j) = \lambda_1^2 \frac{1}{(1-\lambda_1 B)^2} \sum_{j=1}^{\infty} b(j)^2.$$

Therefore by this estimate and (1.2.9), we have

$$\frac{\sigma^2}{\lambda_1^2} \sum_{j=1}^{\infty} z(j)^2 \le \frac{\lambda_2 - \lambda_1^2}{\lambda_1^2} \cdot \lambda_1^2 \frac{1}{(1-\lambda_1 B)^2} \sum_{j=1}^{\infty} b(j)^2 < 1,$$

which is (S_2). We notice that (1.2.1) can be written as $\lambda_2 B^2 < 1$, so (1.2.1) is stronger than (1.2.9) if

$$1 < \lambda_1^2 B^2 + \frac{(1-\lambda_1 B)^2}{\sum_{j=1}^{\infty} b(j)^2/B^2}.$$

which is equivalent to (1.2.10), because $\lambda_1 B < 1$. □

1.3 Exact Rates of Decay of the Autocovariance Function in the Class $\mathcal{W}(r)$

1.3.1 Subexponential decay in linear Volterra summation equations

In ascertaining rates of decay of Volterra equations we use admissibility theory of Volterra operators, see e.g. [13]. Chapter 2 illustrates this facet of admissibility theory for a discrete time Volterra equation whose solution is an autocovariance function. We mention some pertinent results of this theory. Consider the linear convolution equation

$$x(n+1) = f(n) + \sum_{i=0}^{n} F(n-i)x(i), \quad n \ge 0; \quad x(0) = x_0 \in \mathbb{R}, \tag{1.3.1}$$

where $f : \mathbb{Z}^+ \to \mathbb{R}$ and $F : \mathbb{Z}^+ \to \mathbb{R}$. This problem has a unique solution $x : \mathbb{Z}^+ \to \mathbb{R}$. In the case that $x(n) \to 0$ as $n \to \infty$, our aim is to describe the exact rate of decay of x. Our method is to introduce a suitable sequence $\gamma = \{\gamma(n)\}_{n\geq 0}$ which decays to zero and then to examine the behaviour of

$$\omega(n) = x(n)/\gamma(n), \tag{1.3.2}$$

and show that ω converges to a non-trivial limit. It then follows that $x(n) \to 0$ as $n \to \infty$ at exactly the same rate as $\gamma(n) \to 0$.

We divide the results of this section into a discussion of subexponential rates of decay ($r = 1$) and a discussion of $\mathcal{W}(r)$ rates of decay for $r < 1$. While the proofs of both of these sections are treated together, we choose to present the results separately in order to emphasise the subexponential behaviour in (0.4.1) which falls just short of long memory and which is perhaps of greater interest in the context of time series. The principal difference in the statement of these decay results is that for sequences which are in $\mathcal{W}(1)$ we further require that they are asymptotic to non–increasing sequences, whereas a sequence in the class $\mathcal{W}(r)$, for $r < 1$, is asymptotic to a non-increasing sequence by the first part of (0.4.1). Hence we define a subclass $\mathcal{W}^{\downarrow}(r)$ of $\mathcal{W}(r)$ for $r \in (0, 1]$ by

$$\begin{aligned}\mathcal{W}^{\downarrow}(r) := \{g : \mathbb{Z}^+ \to (0,\infty) : \quad & g \in \mathcal{W}(r) \text{ and there exists } \gamma : \mathbb{Z}^+ \to (0,\infty) \\ & \text{such that } \gamma(n+1) \leq \gamma(n) \text{ for all } n \in \mathbb{Z}^+ \text{ and } g(n) \sim \gamma(n) \text{ as } n \to \infty\}.\end{aligned}$$

We note that $\mathcal{W}^{\downarrow}(r) = \mathcal{W}(r)$ for $r < 1$. This additional monotonicity is in practice quite a mild assumption given that we are interested in determining a rate of *decay* of ρ. We require it to simplify the asymptotic analysis of certain infinite sums.

If γ is a real sequence with $\gamma(n) > 0$ for all $n \geq 0$ and $\{u(n)\}_{n\geq 0}$ is a sequence in $\mathbb{R}^{d_1 \times d_2}$ such that $\lim_{n\to\infty} u(n)/\gamma(n)$ exists, then this limit is denoted by $L_\gamma u$. This notation enables us to state succinctly [13, Theorem 3.2].

Theorem 1.3.1. *Suppose that there is a γ in $\mathcal{W}(r)$ such that $L_\gamma f$ and $L_\gamma F$ both exist, and that*

$$\sum_{i=0}^{\infty} r^{-(i+1)}|F(i)| < 1. \tag{1.3.3}$$

Then the solution x of (1.3.1) *satisfies*

$$L_\gamma x = \left(r - \sum_{i=0}^{\infty} r^{-i}F(i)\right)^{-1} [L_\gamma f + (L_\gamma F)\sum_{j=0}^{\infty} r^{-j}x(j)], \tag{1.3.4}$$

where

$$\sum_{j=0}^{\infty} r^{-j}x(j) = \left(r - \sum_{k=0}^{\infty} r^{-k}F(k)\right)^{-1}[rx_0 + \sum_{l=0}^{\infty} r^{-l}f(l)]. \tag{1.3.5}$$

1.3.2 Necessary and sufficient conditions for subexponential decay.

Our first main results show that subexponential decay in b implies subexponential decay in ρ, and moreover that ρ decays at exactly the same rate as b.

Theorem 1.3.2. *Let* (S_2) *and* $\lambda_1 \sum_{j=1}^{\infty} b(j) < 1$ *hold. If* $b \in \mathcal{W}^{\downarrow}(1)$ *then* $\rho \in \mathcal{W}^{\downarrow}(1)$. *Moreover,*

$$L_b\rho = \frac{\lambda_1}{\left(1-\lambda_1 B\right)} \sum_{j=-\infty}^{\infty} \rho(j) = \frac{\lambda_1 \mathbb{E}[\nu(0)^2]}{\left(1-\lambda_1 B\right)^3}. \tag{1.3.6}$$

The proof of Theorem 1.3.2 is a consequence of Theorems 1.2.2 and 1.3.1. This result is strongly related to [118, Theorem 2], about which we comment presently. The limit on the righthand side of (1.3.6) is zero only when $a\sigma = 0$, which is ruled out under the standing assumptions (S_0) discussed at the beginning of Section 1.2. The limit formulae (1.3.6) highlights the inherent short memory of stationary solutions of ARCH(∞) equations, because the infinite sum can be expressed in terms of a finite quantity.

A simple corollary of this result is that if b obeys $b(k)/k^{-\alpha} \to c > 0$ as $k \to \infty$ for some $\alpha > 1$, and (S_2) and $\lambda_1 \sum_{j=1}^{\infty} b(j) < 1$ also hold, then $b \in \mathcal{W}^{\downarrow}(1)$, and we have

$$\lim_{k\to\infty} \frac{\rho(k)}{k^{-\alpha}} = c' > 0.$$

We notice that this strengthens slightly results in [51] and [52], which give upper and lower polynomial bounds on the rate of decay.

The necessity of subexponential decay in b is captured by the following result, which to the best of the authors' knowledge, is not analogous to known results in the time series literature. It shows, under an additional stability condition to that in Theorem 1.3.2, that if ρ is decaying subexponentially, then b must decay subexponentially, and at the same rate.

Theorem 1.3.3. *Let* (S_2) *and* $\lambda_1 \sum_{j=1}^{\infty} b(j) < 1/2$ *hold. Then* $b \in \mathcal{W}^{\downarrow}(1)$ *if and only if* $\rho \in \mathcal{W}^{\downarrow}(1)$*, and both statements imply* (1.3.6).

In the same spirit, we establish later in the chapter a corresponding pair of results for sequences in $\mathcal{W}(r)$, as well as necessary and sufficient conditions for ρ to be bounded above by a subexponential sequence.

A novel feature of the proof of Theorem 1.3.3 is that we deal with the advanced difference equation (1.2.2), rather than a Volterra equation. The proof of this partial converse is more delicate than that of Theorem 1.3.2 itself. It relies mainly on showing that ρ is asymptotic to z; once this is done, a known result from the theory of Volterra difference equations ensures that z is asymptotic to b.

1.3.3 Connections of Theorem 1.3.2 with extant work

Theorem 1.3.2 (and Lemma 1.5.1) assert that, when b is subexponential, then both ρ (and z) inherit the rate of decay of b. It is remarked in [51, pp.16] and [118, pp.154] that it is the asymptotic behaviour of the coefficients in the moving average representation given in Theorem 1.2.1 that impart the rate of decay of the autocovariance function of (AH). The precise influence of these coefficients is the subject of [118, Theorem 2]. There, it is claimed that if (S_1) holds (which forces b to be summable) and

$$\lim_{k\to\infty} \frac{b(k)}{\zeta^k} = \infty, \qquad \text{for any } 0 < \zeta < 1, \tag{1.3.7}$$

then

$$z(k) \sim C_1 b(k) \quad \text{and} \quad \chi_z(k) \sim C_2 b(k), \quad \text{as } k \to \infty, \tag{1.3.8}$$

where $C_1, C_2 \in (0, \infty)$ and χ_z is as defined in (1.2.3). The first asymptotic estimate appears as part of the proof of [118, Theorem 2], but the statement of the theorem lists only the second estimate as its conclusion.

It should be noted that when $b \in \mathcal{W}(1)$, it obeys the first condition in (0.4.1) (with, by definition, $r = 1$), and therefore obeys (0.4.3) which is equivalent to (1.3.7). Therefore, at a first glance, it would appear that Theorem 1.3.2 proves the same result as in [118, Theorem 2], but requires stronger hypotheses, as $\mathcal{W}(1)$ is merely a subclass of the summable sequences obeying (1.3.7).

Despite this, we now show that there exist sequences b which obey (1.3.7), and which also satisfy the other conditions of [118, Theorem 2], but for which the claimed asymptotic behaviour for z and χ_z in (1.3.8) does not hold. Notably, the sequences we consider are ruled out under the stronger conditions of Theorem 1.3.2 above. In essence, we show that if b does not obey the first condition in (0.4.1) due to the presence of a 2-periodic component in its decay, then this 2–periodic component is present in the rates of decay of

z and of χ_z. Furthermore, this decay is "out of phase", in the sense that neither z nor χ_z are asymptotic to b, and therefore violate (1.3.8).

The example we cite has been explored in detail in Section 2.4 of Chapter 2. We state the result here to make our presentation self–contained.

Example 1.3.1. Let $b(n) = a_1 n^{-2}$ for $n/2 \in \mathbb{N}$ and $b(n) = a_0 n^{-2}$ for $n/2 \notin \mathbb{N}$ where $a_0 = 0.5$ and $a_1 = 0.25$. Also, let $\{\xi(n)\}_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed non–negative random variables with mean $\lambda_1 = 1$. Note that

$$\lim_{n\to\infty} \frac{b(2n+1)}{b(2n)} = 2, \quad \lim_{n\to\infty} \frac{b(2n+2)}{b(2n+1)} = \frac{1}{2},$$

so that b does not obey the first part of (0.4.1) for $r = 1$ (or indeed any value of r), but does obey (1.3.7). Since (S_1) holds, [118, Theorem 2] predicts that there exist $C_1, C_2 \in (0,\infty)$ such that

$$\lim_{n\to\infty} \frac{z(n)}{b(n)} = C_1, \quad \lim_{n\to\infty} \frac{\chi_z(n)}{b(n)} = C_2,$$

while Theorem 1.3.2 does not apply.

Theorem 1.3.4. *Let $b(j) = a_1 j^{-2}$ for $j/2 \in \mathbb{N}$, $b(j) = a_0 j^{-2}$ for $j/2 \notin \mathbb{N}$, where $a_0 := 0.5$ and $a_1 := 0.25$. Set $\lambda_1 = 1$. Then*

$$\lim_{n\to\infty} \frac{z(2n)}{b(2n)} = 18.86796\ldots, \qquad \lim_{n\to\infty} \frac{z(2n+1)}{b(2n+1)} = 9.65210\ldots, \tag{1.3.9}$$

and

$$\lim_{n\to\infty} \frac{\chi_z(2n)}{b(2n)} = 67.9375\ldots, \lim_{n\to\infty} \frac{\chi_z(2n+1)}{b(2n+1)} = 34.1128\ldots \tag{1.3.10}$$

It is apparent from (1.3.9) that the claim of the first statement of (1.3.8) does not hold. While (1.3.10) contradicts the second statement in (1.3.8).

Proof. The proof of Theorem 1.3.4 is gone through in Examples 2.4.1 and 2.4.2 and Remarks 2.4.2 and 2.4.3 of Chapter 2. □

1.3.4 Necessary and sufficient conditions for $\mathcal{W}(r)$ decay.

If it is observed that the autocovariances of the ARCH(∞) equations decay in a manner consistent with the class $\mathcal{W}(r)$ for $r \in (0,1)$, then this can only occur if the memory of the process, b, decays likewise.

Theorem 1.3.5. *Fix $r \in (0,1)$. Let* (S_2) *and* $\lambda_1 \sum_{j=1}^{\infty} b(j) r^{-j} < 1$ *hold. If* $b \in \mathcal{W}(r)$ *then* $\rho \in \mathcal{W}(r)$. *Moreover,*

$$\lim_{n\to\infty} \frac{\rho(n)}{b(n)} = \frac{\mathbb{E}[\nu(0)^2]}{(1-\lambda_1 \sum_{j=0}^{\infty} b(j) r^j)} \cdot \frac{\lambda_1}{(1-\lambda_1 \sum_{j=1}^{\infty} b(j) r^{-j})^2}. \tag{1.3.11}$$

A converse corresponding to Theorem 1.3.3 may also be stated.

Theorem 1.3.6. *Fix $r \in (0,1)$. Let* (S_2) *and* $\lambda_1 \sum_{j=1}^{\infty} b(j) r^{-j} < 1/2$ *hold. Then* $b \in \mathcal{W}(r)$ *if and only if* $\rho \in \mathcal{W}(r)$ *and both imply* (1.3.11).

We remark that the rate of decay exhibited by a function in the weight class of functions $\mathcal{W}(r)$, for $r < 1$, is faster than a purely geometric rate of decay. Let $b \in \mathcal{W}(r)$, for $r < 1$, and suppose that the conditions of Theorem 1.3.5 hold. Consider the open disc $D = \{\lambda \in \mathbb{C} : |\lambda| < 1/r\}$ of radius $1/r$ in the complex plane. Then the Z-transform of b is defined on D and on the boundary of D, $\partial D = \{\lambda \in \mathbb{C} : |\lambda| = 1/r\}$. Thus ψ, of Lemma 1.2.2, is well defined on $\bar{D} = D \cup \partial D$. However, by the conditions of Theorem 1.3.5, ψ has no zeroes in $\bar{D}$. Moreover, because b is in $\mathcal{W}(r)$, and $b(j) \geq 0$, we have $\sum_{j=1}^{\infty} b(j)(1/r+\epsilon)^j = +\infty$ for every $\epsilon > 0$, and therefore neither the Z–transform of b, nor ψ, are defined for real $\lambda > 1/r$. Therefore the characteristic equation $\psi(\lambda) = 0$ excludes the possibility that there are geometrically bounded solutions of z at any rate $(1/|\lambda|)^n$ for $|\lambda| \leq 1/r$. On the other hand, Theorem 1.3.5 ensures that z decays at the rate r^n times a subexponential sequence.

ψ and the Z-transform of b may be well defined in other regions of the complex plane in the complement of $\bar{D}$, and indeed ψ may have zeroes in these other regions. Irrespective of these potential zeroes, it is the $\mathcal{W}(r)$ rate of decay of b which determines the asymptotic behaviour of the resolvent z (i.e., the $\mathcal{W}(r)$ rate of decay dominates the geometrically decaying solutions associated with the zeroes of ψ). This analysis is consistent with Theorem 1.4.6 which describes a geometric decay. However, in light of the above comments, it is apparent that this geometric decay rate need not be given in terms of the roots of the characteristic equation.

1.4 Bounds on the Decay Rate of the Autocovariance Function

In this section we show that if there are decaying bounds imposed upon the kernel of (1.2.5) then this forces the autocovariance function to also be bounded with the same bounding decay rates. While the thrust of Section 1.3 was that specific rates of decay of the kernel imply those same rates of decay arising in the autocovariance function, we present an explicit example where a bound in the rate of decay present in the autocovariance function does not arise from the same rate of decay in the kernel.

Many of the results of this section hinge on the positivity of either b or ρ rather than merely on non–negativity. Following on from the standing assumptions (S_0) at the start of Section 1.2, we may assume that b has at least one positive component. Therefore, we are free to assume that

$$\text{There exists a minimal } 1 \leq j^* < \infty \text{ such that } b(j^*) > 0. \tag{A$_1$}$$

Then assuming (A_1),

$$z(j^*) = \lambda_1 \sum_{l=0}^{j^*-1} b(j^* - l)z(l) \geq \lambda_1 b(j^*) > 0$$

and

$$\rho(j^*) = \mathbb{E}[\nu(0)^2] \sum_{l=0}^{\infty} z(l)z(l + j^*) \geq \mathbb{E}[\nu(0)^2]z(j^*) > 0.$$

By (1.2.5), for $k \geq 0$ we see that

$$\rho(k+1) = \lambda_1 \sum_{l=-\infty}^{k} b(k+1-l)\rho(l) \geq \lambda_1 b(k+1+j^*)\rho(-j^*),$$

so

$$\rho(k+1) \geq \lambda_1 b(k+1+j^*)\rho(j^*). \tag{1.4.1}$$

Similarly, for all $k > j^*$, $z(k) \geq \lambda_1 b(k - j^*)z(j^*)$.

Theorem 1.4.1. *Let $r \in (0,1]$ and suppose that $\lambda_1 \sum_{j=1}^{\infty} b(j)r^{-j} < 1$ and (S_2) hold. Let $\gamma \in \mathcal{W}^{\downarrow}(r)$ be such that $b(n) \leq \gamma(n)$ for all $n \geq 0$. Then*

$$\textit{There exists } C_2 \in (0,\infty) \textit{ such that } \rho(n) \leq C_2\gamma(n), \quad \textit{for all } n \geq 0. \tag{1.4.2}$$

Remark 1.4.1. It is to be observed that Theorem 1.4.1 is concerned in part with bounds in the class of non–increasing functions in $\mathcal{W}(1)$, which is a wider class than the class of summable hyperbolically decaying functions examined in [51, Proposition 3.2] and [52, Corollary 3.2].

We now show that the conditions of Theorem 1.4.1 are sharp if we are to observe an upper bound on ρ in $\mathcal{W}^{\downarrow}(r)$. Then we mention a result concerning lower bounds on the autocovariance function.

Theorem 1.4.2. *Suppose that* (S$_1$) *and* (S$_2$) *hold and suppose that* $\gamma \in \mathcal{W}^{\downarrow}(r)$ *for* $r \in (0,1]$. *Then the following are equivalent*

(a) $\lambda_1 \sum_{j=1}^{\infty} b(j) r^{-j} < 1$ *and there exists* $C_0 \in (0,\infty)$ *such that*

$$b(n) \le C_0 \gamma(n) \quad \text{for all } n \ge 1;$$

(b) There exists $C_2 \in (0,\infty)$ *such that*

$$\rho(n) \le C_2 \gamma(n) \quad \text{for all } n \ge 0.$$

Theorem 1.4.1 asserts that (a) implies (b). In the proof that (b) implies (a) the resulting bound on b is immediate from (1.4.1), while $\lambda_1 \sum_{j=1}^{\infty} b(j) r^{-j} < 1$ must hold, as $z \le C_1\gamma$, and so $\tilde{z}(r^{-1}) < \infty$. Therefore the proof of Theorem 1.4.2 is omitted.

Theorem 1.4.3. *Suppose that* (S$_1$) *and* (S$_2$) *hold and suppose that* $\gamma \in \mathcal{W}^{\downarrow}(r)$ *for* $r \in (0,1]$. *If there exists* $C_0 \in (0,\infty)$ *such that* $b(n) \ge C_0\gamma(n)$ *for all* $n \ge 1$ *then there exists* $C_2 \in (0,\infty)$ *such that* $\rho(n) \ge C_2 \gamma(n)$ *for all* $n \ge 0$.

The proof of Theorem 1.4.3 is similarly omitted as it is immediate from (1.4.1). Combining the last two results gives the main result of this section.

Theorem 1.4.4. *Suppose that* (S$_1$) *and* (S$_2$) *hold and suppose that* $\gamma \in \mathcal{W}^{\downarrow}(r)$ *for* $r \in (0,1]$. *Then the following are equivalent*

(a) $\lambda_1 \sum_{j=1}^{\infty} b(j) r^{-j} < 1$ *and there exists* $C_0^* \in (0,\infty)$ *such that*

$$\limsup_{n\to\infty} \frac{b(n)}{\gamma(n)} = C_0^*;$$

(b) There exists $C_2^* \in (0,\infty)$ *such that*

$$\limsup_{n\to\infty} \frac{\rho(n)}{\gamma(n)} = C_2^*.$$

Remark 1.4.2. Theorem 1.4.4 allows subsequences of b to decay at rates faster than subexponentially, or indeed to be equal to zero. In this respect Theorem 1.4.4 is different from the related result Theorem 1.3.2. Indeed the nature of the decay of b may be quite erratic, yet providing that there is a subexponential decay which is an upper limiting bound for some subsequence of b then this limiting upper bound must be found in the autocovariance function and conversely.

Remark 1.4.3. It is interesting to investigate what Theorem 1.4.4 claims in the case when $r = 1$. Suppose that there is a stationary solution X of (AH). Then Theorem 1.2.2 shows that conditions (S_1) and (S_2) hold. If, from observation of the time series data, a subexponential sequence γ is proposed for which $\limsup_{n\to\infty} \rho(n)/\gamma(n) \in (0,\infty)$, then Theorem 1.4.4 shows that $\limsup_{n\to\infty} b(n)/\gamma(n) \in (0,\infty)$.

Remark 1.4.4. It is interesting to ask whether an analogue of Theorem 1.4.4 can be proven with the limit inferior in place of the limit superior, for even though it is obvious from (1.4.1) that $\liminf_{n\to\infty} b(n)/\gamma(n) > 0$ implies $\liminf_{n\to\infty} \rho(n)/\gamma(n) > 0$, it is not so obvious whether in general the converse holds. In Example 1.4.1 below, we demonstrate via a counterexample that this converse does not hold in general. Therefore, it is also the case that the converse of Theorem 1.4.3 is not generally true.

Example 1.4.1. Define the kernel b so that it exhibits some periodicity:

$$b(n) = \begin{cases} 0, & n/3 \in \mathbb{Z}^+, \\ n^{-2}, & \text{otherwise}. \end{cases}$$

Note that $\sum_{j=1}^{\infty} b(j) = 4\pi^2/27$. Suppose that the sequence of shocks $\xi = \{\xi(n)\}_{n\in\mathbb{Z}}$ is such that $0 < \lambda_1 < 27/(4\pi^2)$, so that ($S_1$) holds. Following the techniques of Chapter 2 and the examples contained therein, we obtain

$$\liminf_{n\to\infty} \frac{z(n)}{n^{-2}} = K \min\{d_0, d_1, d_2\} > 0,$$

where

$$S_i = \lambda_1 \sum_{n=0}^{\infty} b(3n+i+1), \quad i \in \{0,1,2\},$$

and

$$K = \lambda_1/(1 - S_0^3 - 3S_0S_1 - S_1^3)^2,$$

$$d_0 = S_0^4 + 2S_1(1 - S_0^3) + 2S_0(1 - S_1^3) + 3(S_0^2 + S_1^2) + S_1^4,$$

$$d_1 = 1 + 2S_0^3(1 - S_1) + 2S_1 + 2S_1^3 + S_1^4 + 3S_0^2(1 + S_1^2),$$

$$d_2 = 1 + 2S_1^3(1 - S_0) + 2S_0 + 2S_0^3 + S_0^4 + 3S_1^2(1 + S_0^2).$$

Note that the denominator of K is non–zero if $S_0 > 0$, $S_1 > 0$ and $S_0 + S_1 < 1$. Similarly one may show that

$$\liminf_{n\to\infty} \frac{\chi_z(n)}{n^{-2}} = \min\{c_0, c_1, c_2\} > 0,$$

where χ_z is defined by (1.2.3) and

$$c_0 = d_0 \sum_{j=0}^{\infty} z(3j) + d_1 \sum_{j=0}^{\infty} z(3j+1) + d_2 \sum_{j=0}^{\infty} z(3j+2),$$

$$c_1 = d_1 \sum_{j=0}^{\infty} z(3j) + d_2 \sum_{j=0}^{\infty} z(3j+1) + d_0 \sum_{j=0}^{\infty} z(3j+2),$$

$$c_2 = d_2 \sum_{j=0}^{\infty} z(3j) + d_0 \sum_{j=0}^{\infty} z(3j+1) + d_1 \sum_{j=0}^{\infty} z(3j+2).$$

Noticing that $\sum_{j=1}^{\infty} b(j)^2 = 8\pi^4/729$, we see from Remarks 1.2.3 and 1.2.4 that if

$$\lambda_2 \frac{16\pi^4}{729} < 1 + \max\left(0, \lambda_1^2 \frac{16\pi^4}{729} + 2\left(1 - \lambda_1 \frac{4\pi^2}{27}\right)^2 - 1\right),$$

then (S_2) also holds and one has $\liminf_{n\to\infty} \rho(n)/n^{-2} > 0$. Therefore when the autocovariances of a stationary ARCH(∞) process are observed to be bounded from below by a certain rate of decay, then it need *not* follow that this lower bounding rate of decay is present in b.

This example illustrates two further general points made earlier: first, in this example $\limsup_{n\to\infty} b(n)/n^{-2} \in (0,\infty)$, and the above results confirm that

$$\limsup_{n\to\infty} \rho(n)/n^{-2} = \mathbb{E}[\nu(0)^2] \max\{c_0, c_1, c_2\} \in (0,\infty),$$

as claimed in Theorem 1.4.4.

Secondly, we notice from (1.2.10) that whenever $\lambda_1 < 9/(4\pi^2)$, the condition (1.2.9), which implies the stationarity of X, is weaker than condition (1.2.1).

Using the subexponential bounds of Theorems 1.4.2 and 1.4.3, we can weaken the hypothesis that b is subexponential, but still recover results on polynomial and "superpolynomial" decay of ρ. This is achieved at the expense of some lost sharpness in characterising the asymptotic behaviour of ρ.

Theorem 1.4.5. *Let* (S_1) *and* (S_2) *hold and* $\beta \in \big\{(1,\infty) \cup \{\infty\}\big\}$.

$$\text{If} \quad \lim_{n\to\infty} \frac{\log b(n)}{\log n} = -\beta \quad \text{then} \quad \lim_{n\to\infty} \frac{\log \rho(n)}{\log n} = -\beta. \tag{i}$$

$$\limsup_{n\to\infty} \frac{\log b(n)}{\log n} = -\beta \quad \text{if and only if} \quad \limsup_{n\to\infty} \frac{\log \rho(n)}{\log n} = -\beta. \tag{ii}$$

Once again, we notice that the equivalence of the existence of a stationary solution of (AH) and the conditions (S_1) and (S_2) means that the "polynomial–like" decay in the autocovariance function exhibited in Theorem 1.4.5 is possible if and only if similar "polynomial–like" decay is present in b.

Theorem 1.4.5 can be used to determine the asymptotic behaviour for kernels b which are not covered by previous results. We can find examples of kernels b for which

$$\lim_{n\to\infty} \frac{\log b(n)}{\log n} = -\beta, \quad b \notin \mathcal{W}(1)$$

and also b for which

$$\limsup_{n\to\infty} \frac{\log b(n)}{\log n} = -\beta, \quad \lim_{n\to\infty} \frac{\log b(n)}{\log n} \text{ does not exist}, \quad b \notin \mathcal{W}(1).$$

An example of the former is $b(n) = (2+\cos(n\pi))n^{-\beta}$ or $b(n) = n^{-\beta}\log(n+2)(2+\sin(n+2))$ while an example of the latter is $b(n) = n^{-\beta+\sin(n)-1}$ for $n \geq 1$. All these examples are not subexponential sequences as they fail to satisfy the first condition of (0.4.1).

Remark 1.4.5. Example 1.4.1 shows that the first implication in Theorem 1.4.5 cannot be reversed, as $\lim_{n\to\infty} \log\rho(n)/\log n = -2$, but $\lim_{n\to\infty} \log b(n)/\log n$ does not exist.

Remark 1.4.6. Theorem 1.4.4 can be applied when $b(n) = (2+(-1)^n)n^{-1}(\log(n+2))^{-2}$ with e.g., $\gamma(n) = (n+2)^{-1}(\log(n+2))^{-2} \in \mathcal{W}(1)$, by following an adaptation of the proof of [17, Proposition 3.3]. However, Theorem 1.4.5 does *not* apply to this sequence.

Despite the last remark, one may prefer Theorem 1.4.5 over Theorem 1.4.4 if the goal is to fit real–world data to an ARCH(∞) model. In practice, one may not be able to establish a subexponential sequence to which the data is "close". In particular, it may only be possible to identify the exponent of polynomial decay ($-\beta \in (-\infty, -1)$ in Theorem 1.4.5) in b and not any lower order component (for example logarithmic or other more slowly varying factors). Such difficulties might render impossible the detection of the precise form of the subexponential sequence to which the kernel is close, particularly for sequences such as $b(n) = n^{-\beta+\sin(n)-1}$.

In the final result, we show that exponential decay of b is both necessary and sufficient for exponential decay of ρ. Thus we recover a special case of [75, Theorem 3.1], which concerns exponential decay of the autocovariance function, while using a different method of proof.

Theorem 1.4.6. *Let* (S_1) *and* (S_2) *hold. Then the following are equivalent:*

(a) There exist $\alpha_1 \in (0,1)$, $C_1 \in (0,\infty)$ *such that* $b(k) \le C_1\alpha_1^k$ *for all* $k \in \mathbb{Z}^+$*;*

(b) There exist $\alpha_2 \in (0,1)$, $C_2 \in (0,\infty)$ *such that* $\rho(k) \le C_2\alpha_2^k$ *for all* $k \in \mathbb{Z}^+$*.*

1.5 Proofs

Proposition 1.2.1 necessitates that interchange of an infinite summation and an expectation sign. This interchange is made rigorous via standard application of the Monotone–Convergence Theorem (cf. e.g., [115, Theorem 5.3]).

Proof of Proposition 1.2.1. Firstly observe that the identity $\rho(k) = \rho(-k)$, for all $k \in \mathbb{Z}$ holds for the autocovariance function. Now, for $k > 0$ we have

$$\begin{aligned}
\rho(-k) = \text{Cov}[X(n), X(n-k)] &= \text{Cov}[a\xi(n) + \sum_{j=1}^{\infty} b(j)X(n-j)\xi(n), X(n-k)] \\
&= a\,\text{Cov}[\xi(n), X(n-k)] + \sum_{j=1}^{\infty} b(j)\text{Cov}[X(n-j)\xi(n), X(n-k)] \\
&= 0 + \lambda_1 \sum_{j=1}^{\infty} b(j)\text{Cov}[X(n-j), X(n-k)] = \lambda_1 \sum_{j=1}^{\infty} b(j)\rho(k-j).
\end{aligned}$$

The result follows due to the symmetry of the autocovariance function. □

Proof of Lemma 1.2.2. Firstly we note that $\lambda_1 \sum_{j=1}^{\infty} b(j)R^j < +\infty$ ensures that $\psi(\lambda)$ is finite in the region $|\lambda| \le R$.

Suppose now that $\lambda_1 \sum_{j=1}^{\infty} b(j)R^j < 1$. Let $|\lambda| \le R$. Define $\Lambda := \lambda/R$, so that $|\Lambda| \le 1$. Also, define the sequence ψ^* by $\psi_0^* = 1$, $\psi_j^* = -\lambda_1 b(j)R^j$ for $j \ge 1$. Therefore $\sum_{j=0}^{\infty} |\psi_j^*| <$

$+\infty$. Consequently, we may define $\psi^*(\Lambda) = \sum_{j=0}^{\infty} \psi_j^* \Lambda^j$ for $|\Lambda| \le 1$. Furthermore, for $|\Lambda| \le 1$, we may use the non–negativity of b to get

$$|\psi^*(\Lambda)| = |1 - \lambda_1 \sum_{j=1}^{\infty} b(j) R^j \Lambda^j| \ge 1 - \lambda_1 \sum_{j=1}^{\infty} b(j) R^j > 0.$$

Hence we may apply Lemma 1.2.1 to ψ^*, so that there exists a summable sequence $z^* = \{z^*(j) : j \in \mathbb{Z}^+\}$ such that $1/\psi^*(\Lambda) = \sum_{j=0}^{\infty} z^*(j)\Lambda^j$ for $|\Lambda| \le 1$. Therefore, for $|\lambda| \le R$ we have

$$\frac{1}{\psi(\lambda)} = \frac{1}{\psi^*(\Lambda)} = \frac{1}{\sum_{j=0}^{\infty} \psi_j^* \Lambda^j} = \sum_{j=0}^{\infty} z^*(j)\Lambda^j = \sum_{j=0}^{\infty} z^*(j) R^{-j} \lambda^j.$$

Therefore

$$\sum_{j=0}^{\infty} z^*(j) R^{-j} \lambda^j \sum_{k=0}^{\infty} \psi_k^* R^{-k} \lambda^k = 1, \quad |\lambda| \le R.$$

Note that when $R = 1$, we have $z^* = z$ in the notation of Lemma 1.2.1. Rearranging gives

$$\sum_{l=0}^{\infty} \sum_{j=0}^{l} \psi_{l-j}^* z^*(j) R^{-l} \lambda^l = 1.$$

Now comparing powers of λ on both sides of this equality gives

$$\psi_0^* z^*(0) = 1, \quad z^*(n) = -\sum_{j=0}^{n-1} \psi_{n-j}^* z^*(j), \quad n \ge 1. \tag{1.5.1}$$

Rearranging the second equation gives

$$R^{-n} z^*(n) = \lambda_1 \sum_{j=0}^{n-1} b(n-j) R^{-j} z^*(j), \quad n \ge 1.$$

Observe that if $R = 1$, z^* satisfies (1.2.6). Define $w(n) = R^{-n} z^*(n)$ for $n \ge 0$. Then, by the uniqueness of the solution of (1.2.6), it is seen that $w(n) = z(n)$, $n \ge 0$ and so $z^*(n) = R^n z(n)$, $n \ge 0$. Hence $1/\psi(\lambda) = \sum_{j=0}^{\infty} z(j)\lambda^j, \quad |\lambda| \le R$ and $\sum_{j=0}^{\infty} z(j) R^j < +\infty$.

Conversely, suppose that z is defined by (1.2.6) and that $\sum_{j=0}^{\infty} z(j) R^j < +\infty$. Multiplying across (1.2.6) by R^n and summing gives

$$\sum_{n=1}^{\infty} z(n) R^n = \lambda_1 \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} b(n-j) R^{n-j} R^j z(j).$$

Since the summand on the righthand side is non–negative, the order of summation may be exchanged to give

$$\sum_{n=0}^{\infty} z(n) R^n = 1 + \lambda_1 \sum_{j=1}^{\infty} b(j) R^j \sum_{n=0}^{\infty} z(n) R^n.$$

Now, since $\sum_{n=0}^{\infty} z(n)R^n \in [1,\infty)$, it follows that $\lambda_1 \sum_{j=1}^{\infty} b(j)R^j$ is finite, and moreover the identity can be rearranged to give

$$\lambda_1 \sum_{j=1}^{\infty} b(j)R^j = \frac{\sum_{n=0}^{\infty} z(n)R^n - 1}{\sum_{n=0}^{\infty} z(n)R^n} \in [0,1),$$

as required. □

1.5.1 Rates

It is obvious from (1.2.6) that if $\lambda_1 \sum_{j=1}^{\infty} b(j)r^{-j} < 1$ then

$$\sum_{j=0}^{\infty} z(j)r^{-j} = \frac{1}{1-\lambda_1 \sum_{j=1}^{\infty} b(j)r^{-j}} < +\infty$$

and trivially $\sum_{j=0}^{\infty} z(j)r^j < \infty$ and $\lambda_1 \sum_{j=1}^{\infty} b(j)r^j < 1$ for $r \in (0,1]$.

Lemma 1.5.1. *If $b \in \mathcal{W}(r)$ and $\lambda_1 \sum_{j=1}^{\infty} b(j)r^{-j} < 1$, then*

$$\lim_{n\to\infty} \frac{z(n)}{b(n)} = \frac{\lambda_1}{(1-\lambda_1 \sum_{j=1}^{\infty} b(j)r^{-j})^2}.$$

Proof of Lemma 1.5.1. Apply Theorem 1.3.1 to (1.2.6). □

Lemma 1.5.2. *If $b \in \mathcal{W}^{\downarrow}(r)$ for $r \in (0,1]$, $\lambda_1 \sum_{j=1}^{\infty} b(j)r^{-j} < 1$, and χ_z is defined by* (1.2.3)*, then*

$$\lim_{k\to\infty} \frac{\chi_z(k)}{z(k)} = \frac{1}{1-\lambda_1 \sum_{j=1}^{\infty} b(j)r^j}.$$

Proof of Lemma 1.5.2. Firstly, note that $\lambda_1 \sum_{j=1}^{\infty} b(j)r^{-j} < 1$ gives $\sum_{j=0}^{\infty} z(j)r^{-j} < +\infty$. Consider the case $r < 1$. Then for any fixed $M \geq 2$ we have

$$\left|\frac{\chi_z(n)}{z(n)} - \sum_{j=0}^{\infty} z(j)r^j\right| \leq \sum_{j=0}^{M-1} z(j)\left|\frac{z(n+j)}{z(n)} - r^j\right| + \sum_{j=M}^{\infty} z(j)\frac{z(n+j)}{z(n)} + \sum_{j=M}^{\infty} z(j)r^j.$$

Let $\epsilon \in (0,1)$ be such that $r < r(1+\epsilon) < 1 < r^{-1}$. By Lemma 1.5.1 there is an $N(\epsilon) \in \mathbb{Z}^+$ such that $z(n+1)/z(n) < r(1+\epsilon) < 1$ for all $n \geq N(\epsilon)$. Hence for $j \geq 1$, $z(n+j)/z(n) < r^j(1+\epsilon)^j < r^{-j}$ for all $n \geq N(\epsilon)$. Thus for $n \geq N(\epsilon)$,

$$\left|\frac{\chi_z(n)}{z(n)} - \sum_{j=0}^{\infty} z(j)r^j\right| \leq 2\sum_{j=M}^{\infty} z(j)r^{-j} + \sum_{j=0}^{M-1} z(j)\left|\frac{z(n+j)}{z(n)} - r^j\right|.$$

Since $\lim_{n\to\infty} z(n+j)/z(n) = r^j$, we have

$$\limsup_{n\to\infty}\left|\frac{\chi_z(n)}{z(n)} - \sum_{j=0}^{\infty} z(j)r^j\right| \leq 2\sum_{j=M}^{\infty} z(j)r^j.$$

Finally, letting $M \to \infty$ gives the desired result for $r < 1$.

For the case $r = 1$, we split the sums in the same manner as above. From Lemma 1.5.1 we have that $z \in \mathcal{W}(1)$. Then we use the asymptotic monotonicity of b to bound $z(n+j)/z(n)$. We have for $n \geq N_1$, for some N_1 sufficiently large

$$\lim_{n\to\infty} \frac{z(n)}{b(n)} = L \in (0,\infty), \quad \frac{b(n+j)}{b(n)} \leq \frac{b(n+j)}{\gamma(n+j)} \cdot \frac{\gamma(n)}{b(n)} \leq 2 \cdot 2 \text{ for all } j \geq 1.$$

where γ is the non–increasing sequence which is asymptotic to b. Thus for $n \geq N_1$

$$\frac{z(n+j)}{z(n)} = \frac{z(n+j)}{b(n+j)} \cdot \frac{b(n+j)}{b(n)} \cdot \frac{b(n)}{z(n)} \leq 2L \frac{b(n+j)}{b(n)} \frac{1}{L} 2 \leq 2^4.$$

The result follows through as before. □

Proof of Theorems 1.3.2 and 1.3.5. Theorem 1.3.5 and the second limit in Theorem 1.3.2 are an immediate consequence of Lemmas 1.5.1 and 1.5.2 with (S_2) being required to guarantee that $\mathbb{E}[\nu(0)^2]$ is well defined and finite.

Turning to the first limit formula in Theorem 1.3.2, from Lemma 1.5.2 we have that $\rho \in \mathcal{W}(1)$ and hence $\sum_{j=0}^{\infty} \rho(j) < \infty$. From (1.2.7) we have

$$\rho(n+1) = \lambda_1 \sum_{j=0}^{n} b(n-j+1)\rho(j) + f(n), \tag{1.5.2}$$

where $f(n) = \lambda_1 \sum_{j=1}^{\infty} b(n+j+1)\rho(j)$. Letting $F(n) = \lambda_1 b(n+1)$ we can then apply Theorem 1.3.1 to get a representation for $L_b\rho$, providing that $L_\gamma f$ and $L_\gamma F$ both exist, and that $\sum_{j=0}^{\infty} F(j) < 1$. We have the last condition by assumption. To prove that $L_\gamma F$ exists, note that

$$\lim_{n\to\infty} \frac{F(n)}{\gamma(n)} = \lim_{n\to\infty} \frac{\lambda_1 b(n+1)}{\gamma(n)} = \lim_{n\to\infty} \frac{\lambda_1 b(n+1)}{\gamma(n+1)} \frac{\gamma(n+1)}{\gamma(n)} = \lambda_1.$$

As to the existence of $L_\gamma f$, we fix $M \in \mathbb{Z}^+$, and make the estimate

$$\left| \frac{f(n)}{\gamma(n)} - \lambda_1 \sum_{j=1}^{\infty} \rho(j) \right| \leq \lambda_1 \sum_{j=1}^{M} \left| \frac{b(n+j+1)}{\gamma(n)} - 1 \right| \rho(j) + \lambda_1 \sum_{j=M+1}^{\infty} \frac{b(n+1+j)}{\gamma(n)} \rho(j) + \lambda_1 \sum_{j=M+1}^{\infty} \rho(j).$$

For the second term on the right hand side we have

$$\frac{b(n+1+j)}{\gamma(n)} = \frac{b(n+1+j)}{\gamma(n+1+j)} \frac{\gamma(n+1+j)}{\gamma(n)} \leq 2,$$

for all $n \geq N_0$ and some N_0 sufficiently large. Thus for $n \geq N_0$,

$$\left|\frac{f(n)}{\gamma(n)} - \lambda_1 \sum_{j=1}^{\infty} \rho(j)\right| \leq 3\lambda_1 \sum_{j=M+1}^{\infty} \rho(j) + \lambda_1 \sum_{j=1}^{M} \left|\frac{b(n+j+1)}{\gamma(n)} - 1\right| \rho(j).$$

Then

$$\limsup_{n\to\infty} \left|\frac{f(n)}{\gamma(n)} - \lambda_1 \sum_{j=1}^{\infty} \rho(j)\right| \leq 3\lambda_1 \sum_{j=M+1}^{\infty} \rho(j).$$

Letting $M \to \infty$ gives $L_\gamma f = \lambda_1 \sum_{j=1}^{\infty} \rho(j)$.

Thus we may apply Theorem 1.3.1, which gives that $L_b\rho = L_\gamma \rho$ exists. Applying [13, Theorem 4.3] to (1.5.2) gives

$$L_b\rho = \frac{\lambda_1 \sum_{j=0}^{\infty} \rho(j) + \lambda_1 \sum_{j=1}^{\infty} \rho(j)}{1 - \lambda_1 \sum_{j=1}^{\infty} b(j)}.$$

Using the symmetry of the autocovariance function, i.e., $\rho(n) = \rho(-n)$ for all $n \in \mathbb{Z}$, gives (1.3.6) as required. □

We provide a partial converse to Lemma 1.5.1, i.e., that $z \in \mathcal{W}(r)$ implies $b \in \mathcal{W}(r)$. To do so, we state without proof a variant of Theorem 2.3.2 of Chapter 2. The proof of this consists of rewriting (1.2.6) so that the roles of b and z are interchanged, and by then applying Theorem 1.3.1.

Lemma 1.5.3. *Let z be the sequence which satisfies* (1.2.6)*, $z \in \mathcal{W}(r)$ and further suppose that*

$$\lambda_1 \sum_{j=1}^{\infty} b(j) r^{-j} < \frac{1}{2}. \tag{1.5.3}$$

Then

$$\lim_{n\to\infty} \frac{b(n)}{z(n)} = \frac{1}{\lambda_1 \left(\sum_{j=0}^{\infty} z(j) r^{-j}\right)^2}.$$

Remark 1.5.1. If $r \in (0,1]$ and $\lambda_1 \sum_{j=1}^{\infty} b(j) r^{-j} < \frac{1}{2}$, then $\sum_{j=1}^{\infty} z(j) r^{-j} < 1$, and hence $\lambda_1 \sum_{j=1}^{\infty} b(j) r^{j} < \frac{1}{2}$ and $\sum_{j=1}^{\infty} z(j) r^{j} < 1$.

We now state some preparatory lemmata which lead to converses of Theorems 1.3.2 and 1.3.5.

Lemma 1.5.4. *Let z be the solution of* (1.2.6) *and let* (1.5.3) *hold with $r \in (0,1]$. Define the sequences $(U_m)_{m\geq 1}$ and $(L_m)_{m\geq 1}$ by*

$$U_1 = 1, \quad U_{m+1} = 1 - \sum_{j=1}^{m} z(j) r^j L_m, \quad L_m = 1 - \sum_{j=1}^{\infty} z(j) r^j U_m, \quad m \in \mathbb{Z}^+/\{0\}.$$

Then

$$\lim_{m\to\infty} U_m = \lim_{m\to\infty} L_m = 1 - \lambda_1 \sum_{j=1}^{\infty} b(j) r^j.$$

Proof of Lemma 1.5.4. The proof concentrates on verifying that $\lim_{m\to\infty} U_m$ exists. Once this limit is established it is easy to find $\lim_{m\to\infty} L_m$. We have $U_1 = 1$ and

$$U_{m+1} = g(m) + a(m)U_m, \quad m \geq 1,$$

where $g(m) = 1 - \sum_{j=1}^{m} z(j)r^j$ and $a(m) = \sum_{j=1}^{\infty} z(j)r^j \sum_{l=1}^{m} z(l)r^l$. An explicit formula for U is given in e.g. [46, Exercise 2.1.17] and is

$$U_{m+1} = \prod_{j=1}^{m} a(j)U_1 + \sum_{n=1}^{m}\{\prod_{j=n+1}^{m} a(j)\}g(n), \quad m \geq 2, \tag{1.5.4}$$

in which the usual convention $\prod_{j=m+1}^{m} a(j) := 1$ applies. Also we note that $g(m) \to 1 - \sum_{j=1}^{\infty} z(j)r^j$ and $a(m) \to \left(\sum_{j=1}^{\infty} z(j)r^j\right)^2 \in (0,1)$, as $m \to \infty$. Thus the first term on the right–hand side of (1.5.4) tends to zero as $m \to \infty$. We here observe from our standing assumption that b has at least one non-zero component. Thus there exists a minimal j^* such that $b(j^*) > 0$ and hence $a(j) > 0$ for all $j \geq j^*$. For convenience we take $j^* = 1$ (calculations follow similarly for other values of j^*). Turning our attention then to the second term we have

$$A_m := \sum_{n=1}^{m} \frac{\prod_{j=1}^{m} a(j)}{\prod_{j=1}^{n} a(j)} g(n) = \frac{\sum_{n=1}^{m} \frac{1}{\prod_{j=1}^{n} a(j)} g(n)}{\frac{1}{\prod_{j=1}^{m} a(j)}} = \frac{\sum_{n=2}^{m} c(n) + c(1)}{\sum_{n=2}^{m} d(n) + \frac{1}{a(1)}},$$

where

$$d(n) := \frac{1}{\prod_{j=1}^{n} a(j)} - \frac{1}{\prod_{j=1}^{n-1} a(j)}, \quad c(n) := \frac{1}{\prod_{j=1}^{n} a(j)} g(n).$$

Thus $d(n) = \frac{1-a(n)}{\prod_{j=1}^{n} a(j)}$ and hence $c(n) \to \infty$ and $d(n) \to \infty$ as $n \to \infty$. Moreover,

$$\frac{c(n)}{d(n)} = \frac{g(n)}{1-a(n)} = \frac{1 - \sum_{j=1}^{n} z(j)r^j}{1 - \sum_{j=1}^{\infty} z(j)r^j \sum_{l=1}^{n} z(l)r^l}$$

and so

$$\lim_{n\to\infty} \frac{c(n)}{d(n)} = \frac{1 - \sum_{j=1}^{\infty} z(j)r^j}{1 - \left(\sum_{j=1}^{\infty} z(j)r^j\right)^2} = \frac{1}{1 + \sum_{j=1}^{\infty} z(j)r^j}.$$

Applying Toeplitz's Lemma (cf., e.g., [108, 4.3.2 pp.390]) now gives

$$\lim_{m\to\infty} \frac{\sum_{n=2}^{m} c(n)}{\sum_{n=2}^{m} d(n)} = \frac{1}{1 + \sum_{j=1}^{\infty} z(j)r^j}.$$

Therefore

$$\lim_{m\to\infty} U_m = \lim_{m\to\infty} A_m = \lim_{m\to\infty} \frac{\sum_{n=2}^{m} c(n) + c(1)}{\sum_{n=2}^{m} d(n) + \frac{1}{a(1)}} = \frac{1}{1+\sum_{j=1}^{\infty} z(j) r^j}.$$

Finally, z may be written in terms of b using (1.2.6). □

Lemma 1.5.5. *Let* (S_2) *and* (1.5.3) *hold. If* $\rho \in \mathcal{W}^{\downarrow}(r)$*, for* $r \in (0,1]$*, then* z *satisfies*

$$L_m \le \mathbb{E}[\nu(0)^2] \liminf_{n\to\infty} \frac{z(n)}{\rho(n)} \le \mathbb{E}[\nu(0)^2] \limsup_{n\to\infty} \frac{z(n)}{\rho(n)} \le U_{m+1}, \quad m \ge 1, \tag{1.5.5}$$

where U *and* L *are the sequences defined in Lemma 1.5.4.*

Proof of Lemma 1.5.5. The upper and lower bounds on z/ρ are established by an inductive proof. The bounds themselves are constructed recursively. For convenience define $P(n) = \rho(n)/\mathbb{E}[\nu(0)^2]$. We deal with the case when $r \in (0,1)$: the proof for $r = 1$ is largely similar, but employs the asymptotic monotonicity of P to establish estimates for terms of the form $P(n+j)/P(n)$.

From (1.2.2) and using the non-negativity of z and definition of P, we have

$$P(n) = \sum_{j=0}^{\infty} z(j)z(n+j) = z(n) + \sum_{j=1}^{\infty} z(j)z(n+j) \ge z(n). \tag{1.5.6}$$

Thus $z(n)/P(n) \le 1$ and so $\limsup_{n\to\infty} z(n)/P(n) \le 1 = U_1$. As $\lim_{n\to\infty} P(n+1)/P(n) = r$ we have for all $\epsilon > 0$ fixed sufficiently small that there exists an $N_0(\epsilon) \in \mathbb{Z}^+$ such that $P(n+j)/P(n) < r^j(1+\epsilon)^j < 1 < r^{-j}$ for all $n \ge N_0(\epsilon)$. Fix $M \in \mathbb{Z}^+$. Let $n \ge N_0$. Thus by (1.5.6)

$$\begin{aligned}\frac{1}{P(n)} \sum_{j=1}^{\infty} z(j)z(n+j) &\le \frac{1}{P(n)} \sum_{j=1}^{\infty} z(j)P(n+j) \\ &= \sum_{j=1}^{M} z(j)\frac{P(n+j)}{P(n)} + \sum_{j=M+1}^{\infty} z(j)\frac{P(n+j)}{P(n)} \\ &\le \sum_{j=1}^{M} z(j)r^j(1+\epsilon)^j + \sum_{j=M+1}^{\infty} z(j)r^{-j},\end{aligned}$$

which gives

$$1 = \frac{z(n)}{P(n)} + \frac{1}{P(n)} \sum_{j=1}^{\infty} z(j)z(n+j) \le \frac{z(n)}{P(n)} + \sum_{j=1}^{M} z(j)r^j(1+\epsilon)^j + \sum_{j=M+1}^{\infty} z(j)r^{-j}.$$

Thus

$$\frac{z(n)}{P(n)} \ge 1 - \sum_{j=1}^{M} z(j)r^j(1+\epsilon)^j - \sum_{j=M+1}^{\infty} z(j)r^{-j}, \quad n \ge N_0(\epsilon).$$

Hence

$$\liminf_{n\to\infty} \frac{z(n)}{P(n)} \geq 1 - \sum_{j=1}^{M} z(j) r^j (1+\epsilon)^j - \sum_{j=M+1}^{\infty} z(j) r^{-j}.$$

Let $\epsilon \to 0$ from the right, then let $M \to \infty$ to get

$$\liminf_{n\to\infty} \frac{z(n)}{P(n)} \geq 1 - \sum_{j=1}^{\infty} z(j) r^j = L_1 > 0,$$

where the fact that $L_1 > 0$ is a consequence of assumption (1.5.3).

The lower bound L_1 is used then to determine the upper bound U_2: we rewrite (1.5.6) according to

$$z(n) + z(n+1)z(1) = P(n) - \sum_{j=2}^{\infty} z(j) z(n+j) \leq P(n).$$

Since $\liminf_{n\to\infty} z(n)/P(n) \geq L_1$, for all $\epsilon \in (0,1)$ there exists an $N_3(\epsilon) \in \mathbb{Z}^+$ such that for all $n \geq N_3(\epsilon)$

$$\frac{z(n)}{P(n)} \leq 1 - z(1) \frac{P(n+1)}{P(n)} \frac{z(n+1)}{P(n+1)} \leq 1 - z(1) \frac{P(n+1)}{P(n)} L_1 (1-\epsilon).$$

Hence as $P(n+1)/P(n) \to r$ as $n \to \infty$, we get

$$\limsup_{n\to\infty} \frac{z(n)}{P(n)} \leq 1 - z(1) r L_1 (1-\epsilon).$$

Let $\epsilon \to 0$ from the right to get $\limsup_{n\to\infty} z(n)/P(n) \leq 1 - z(1) r L_1 = U_2$. Therefore we have established (1.5.5) for $m = 1$.

Regarding the induction step at level m for $m \geq 2$, assume that (1.5.5) holds, i.e.,

$$\limsup_{n\to\infty} \frac{z(n)}{P(n)} \leq U_m, \quad \liminf_{n\to\infty} \frac{z(n)}{P(n)} \geq L_{m-1}.$$

This implies that, for all $\epsilon > 0$ sufficiently small, there exists $N_1(\epsilon) > 0$ such that $z(n)/P(n) \leq U_m(1+\epsilon)$ for all $n \geq N_1(\epsilon)$.

Fix $M \in \mathbb{Z}^+$, and let $N_0(\epsilon)$ be as defined above. Then for $n \geq \max(N_1(\epsilon), N_0(\epsilon))$, we note that

$$\begin{aligned}
\sum_{j=1}^{\infty} z(j) \frac{z(n+j)}{P(n)} &= \sum_{j=1}^{\infty} z(j) \frac{z(n+j)}{P(n+j)} \frac{P(n+j)}{P(n)} \leq \sum_{j=1}^{\infty} z(j) U_m (1+\epsilon) \frac{P(n+j)}{P(n)} \\
&= \sum_{j=1}^{M} z(j) U_m (1+\epsilon) \frac{P(n+j)}{P(n)} + \sum_{j=M+1}^{\infty} z(j) U_m (1+\epsilon) \frac{P(n+j)}{P(n)} \\
&\leq \sum_{j=1}^{M} z(j) U_m (1+\epsilon) r^j (1+\epsilon)^j + \sum_{j=M+1}^{\infty} z(j) U_m (1+\epsilon) r^{-j}.
\end{aligned}$$

Hence

$$1 = \frac{z(n)}{P(n)} + \frac{1}{P(n)} \sum_{j=1}^{\infty} z(j)z(n+j)$$

$$\leq \frac{z(n)}{P(n)} + \sum_{j=1}^{M} z(j)U_m(1+\epsilon)r^j(1+\epsilon)^j + \sum_{j=M+1}^{\infty} z(j)U_m(1+\epsilon)r^{-j},$$

which rearranges to give

$$\liminf_{n\to\infty} \frac{z(n)}{P(n)} \geq 1 - U_m(1+\epsilon)\left(\sum_{j=1}^{M} z(j)r^j(1+\epsilon)^j + \sum_{j=M+1}^{\infty} z(j)r^{-j}\right),$$

having taken the limit inferior as $n \to \infty$. Letting $\epsilon \to 0$ from the right, and then letting $M \to \infty$, gives

$$\liminf_{n\to\infty} \frac{z(n)}{P(n)} \geq 1 - U_m \sum_{j=1}^{\infty} z(j)r^j = L_m.$$

This yields the lower limit in (1.5.5) at level $m+1$.

It remains to show that the upper limit in (1.5.5) holds at level $m+1$. To prove this, we start by rewriting (1.5.6) in the form

$$z(n) + \sum_{j=1}^{m} z(j)z(n+j) + \sum_{j=m+1}^{\infty} z(j)z(n+j) = P(n),$$

which gives

$$\frac{z(n)}{P(n)} + \frac{1}{P(n)} \sum_{j=1}^{m} z(j)z(n+j) = 1 - \frac{1}{P(n)} \sum_{j=m+1}^{\infty} z(j)z(n+j) \leq 1. \tag{1.5.7}$$

Since $\liminf_{n\to\infty} z(n)/P(n) \geq L_m$, for every $\epsilon \in (0,1)$ there is an $N_2(\epsilon) \in \mathbb{Z}^+$ such that $n \geq N_2(\epsilon)$ implies $z(n)/P(n) > L_m(1-\epsilon)$.

Let $n \geq \max(N_2(\epsilon), N_0(\epsilon))$. Then

$$\frac{1}{P(n)} \sum_{j=1}^{m} z(j)z(n+j) = \sum_{j=1}^{m} z(j)\frac{z(n+j)}{P(n+j)}\frac{P(n+j)}{P(n)} \geq \sum_{j=1}^{m} z(j)\frac{P(n+j)}{P(n)}L_m(1-\epsilon).$$

Inserting this estimate into (1.5.7) and rearranging yields

$$\frac{z(n)}{P(n)} \leq 1 - L_m(1-\epsilon)\sum_{j=1}^{m} z(j)\frac{P(n+j)}{P(n)}, \quad n \geq \max(N_2(\epsilon), N_0(\epsilon)).$$

Therefore, using the positivity of P and z, we get

$$\limsup_{n\to\infty} \frac{z(n)}{P(n)} \leq 1 + \limsup_{n\to\infty}\left(-L_m(1-\epsilon)\sum_{j=1}^{m} z(j)\frac{P(n+j)}{P(n)}\right)$$

$$= 1 - \liminf_{n\to\infty}\left(\sum_{j=1}^{m} z(j)\frac{P(n+j)}{P(n)}\right)L_m(1-\epsilon).$$

Since $P(n+j)/P(n) \to r^j$ as $n \to \infty$, and the sum contains only finitely many terms, we have that

$$\liminf_{n\to\infty}\left(\sum_{j=1}^{m} z(j)\frac{P(n+j)}{P(n)}\right) = \lim_{n\to\infty}\left(\sum_{j=1}^{m} z(j)\frac{P(n+j)}{P(n)}\right) = \sum_{j=1}^{m} z(j)r^j.$$

Hence

$$\limsup_{n\to\infty}\frac{z(n)}{P(n)} \le 1 - \sum_{j=1}^{m} z(j)r^j L_m(1-\epsilon).$$

Letting $\epsilon \to 0^+$ yields

$$\limsup_{n\to\infty}\frac{z(n)}{P(n)} \le 1 - \sum_{j=1}^{m} z(j)r^j L_m = U_{m+1},$$

by the definition of U_{m+1}. Thus we have shown that if the m–th level statement in (1.5.5) holds, then

$$L_m \le \liminf_{n\to\infty}\frac{z(n)}{P(n)} \le \limsup_{n\to\infty}\frac{z(n)}{P(n)} \le U_{m+1},$$

which is the $(m+1)$–th level statement in (1.5.5). This completes the proof of the general induction step, and since we have already shown that (1.5.5) holds for $m=1$, the lemma is true. □

Proof of Theorems 1.3.3 and 1.3.6. The implication that $b \in \mathcal{W}^{\downarrow}(r)$ gives rise to $\rho \in \mathcal{W}^{\downarrow}(r)$, for $r \in (0,1]$ is nothing other than the subject of Theorems 1.3.2 and 1.3.5. The converse result that $\rho \in \mathcal{W}^{\downarrow}(r)$ implies $b \in \mathcal{W}^{\downarrow}(r)$, for $r \in (0,1]$, is an immediate consequence of Remark 1.5.1 and Lemmas 1.5.3, 1.5.4 and 1.5.5 with (S_2) being required to guarantee that $\mathbb{E}[\nu(0)^2]$ is well defined and finite.

It can be seen that the sequence U_m and L_m have the same limit as $m \to \infty$. By virtue of Lemma 1.5.4, we may take the limit as $m \to \infty$ on both sides of (1.5.5), which yields $\lim_{n\to\infty} z(n)/P(n) = \lim_{m\to\infty} L_m = \lim_{m\to\infty} U_{m+1}$, from which the result follows. □

1.5.2 Bounds

The proof of Theorem 1.4.1 uses a result concerning the boundedness of linear Volterra operators in [13, Theorem 5.1]. We state a scalar variant of this theorem. Consider the non–convolution linear Volterra summation equation

$$z(n+1) = \sum_{i=0}^{n} H(n,i)z(i), \quad n \in \mathbb{Z}^+; \tag{1.5.8}$$

where $z(0) = z_0 \in \mathbb{R}$ and $H : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{R}$ with $H(n,i) = 0$ for $i > n$.

Lemma 1.5.6. *Suppose that there are integers M and N with $0 < M < N$ such that*

$$\sup_{n\geq N}\sum_{i=M}^{n}|H(n,i)| < 1, \quad \sup_{n\geq M}\sum_{i=0}^{M}|H(n,i)| < +\infty.$$

Then there is $K > 0$ independent of z_0 such that the solution of equations (1.5.8) *satisfies* $|z(n)| \leq K|z_0|$ *for* $n \geq 0$.

Proof of Theorem 1.4.1. We deal here only with the case $r = 1$. The case $r < 1$ follows the same steps as that of $r = 1$. We firstly show that z/γ is bounded. In order to write (1.2.6) as a convolution equation we define $\beta(n) = \lambda_1 b(n+1)$. Thus $\beta(n) \leq C_0\gamma(n)$ for some $C_0 > 0$ and all n. Then defining $x = z/\gamma$ and using (1.2.6), we have

$$x(n+1) = \sum_{j=0}^{n} H(n,j)x(j), \quad n \geq 0, \quad x(0) = 1/\gamma(0),$$

where

$$H(n,j) := \frac{\beta(n-j)\gamma(j)}{\gamma(n)}\frac{\gamma(n)}{\gamma(n+1)}, \quad n \geq j \geq 0.$$

To show the boundedness of x we apply Lemma 1.5.6. That is, we must show that

$$W_H := \lim_{N\to\infty}\limsup_{n\to\infty}\sum_{j=N}^{n} H(n,j) < 1$$

and $H_M := \sup_{n\geq M}\sum_{j=0}^{M} H(n,j)$ is finite for each $M \in \mathbb{Z}^+$. By the definition of H and (0.4.1) we get

$$\limsup_{n\to\infty}\sum_{j=N}^{n} H(n,j) = \limsup_{n\to\infty}\sum_{j=N}^{n}\frac{\beta(n-j)\gamma(j)}{\gamma(n)}.$$

Let $n \geq 2N$. Then

$$\sum_{j=N}^{n}\frac{\beta(n-j)\gamma(j)}{\gamma(n)} = \sum_{l=0}^{n-N}\beta(l)\frac{\gamma(n-l)}{\gamma(n)} \leq \sum_{l=0}^{N-1}\beta(l)\frac{\gamma(n-l)}{\gamma(n)} + C_0\sum_{l=N}^{n-N}\frac{\gamma(l)\gamma(n-l)}{\gamma(n)}.$$

Thus by (0.4.1)

$$\limsup_{n\to\infty}\sum_{j=N}^{n} H(n,j) \leq \sum_{l=0}^{N-1}\beta(l) + C_0\limsup_{n\to\infty}\sum_{l=N}^{n-N}\frac{\gamma(l)\gamma(n-l)}{\gamma(n)},$$

and by (0.4.2) we get

$$\begin{aligned} W_H &= \lim_{N\to\infty}\limsup_{n\to\infty}\sum_{j=N}^{n} H(n,j) \\ &\leq \sum_{l=0}^{\infty}\beta(l) + C_0\lim_{N\to\infty}\limsup_{n\to\infty}\sum_{l=N}^{n-N}\frac{\gamma(l)\gamma(n-l)}{\gamma(n)} = \sum_{l=0}^{\infty}\beta(l), \end{aligned}$$

so $W_H < 1$ as required. Now to show that for each fixed M, H_M is bounded, we note for $n \geq M$ that

$$\sum_{j=0}^{M} H(n,j) = \sum_{j=0}^{M} \frac{\beta(n-j)}{\gamma(n-j)} \frac{\gamma(j)\gamma(n-j)}{\gamma(n)} \frac{\gamma(n)}{\gamma(n+1)}$$
$$\leq C_0 \sup_{n\geq 0}\left(\frac{\gamma(n)}{\gamma(n+1)}\right) \sum_{j=0}^{M} \frac{\gamma(j)\gamma(n-j)}{\gamma(n)}$$
$$\leq C_0 \sup_{n\geq 0}\left(\frac{\gamma(n)}{\gamma(n+1)}\right) \sup_{n\geq M}\left(\frac{(\gamma*\gamma)(n)}{\gamma(n)}\right)$$

and so $\sup_{n\geq M} H_M(n)$ is finite and therefore x is bounded. As a bound on the resolvent is established, it just remains to deduce the bound on the autocovariance function. Moreover, it is immediate from $x(n) = z(n)/\gamma(n) \leq C_1$ that z is summable. Hence

$$\rho(n) = G\sum_{j=0}^{\infty} z(j)z(n+j) \leq GC_1 \sum_{j=0}^{\infty} z(j)\frac{\gamma(n+j)}{\gamma(n)}\gamma(n) \leq GC_1\gamma(n)\sum_{j=0}^{\infty} z(j),$$

and the desired result holds, where $G = \mathbb{E}[\nu(0)^2]$. □

Proof of Theorem 1.4.4. First let us suppose that $\limsup_{n\to\infty} b(n)/\gamma(n) =: L_3 \in (0,\infty)$. Then from (1.4.1),

$$\limsup_{n\to\infty} \frac{\rho(n)}{\gamma(n)} \geq \lambda_1\rho(j^*)r^{j^*}L_3 > 0,$$

where j^* is the integer introduced in (A_1). Furthermore, for any fixed $\epsilon > 0$ there exists an $N(\epsilon) \in \mathbb{Z}^+$ such that $b(n) < L_3(1+\epsilon)\gamma(n)$ for all $n \geq N(\epsilon)$. Moreover, $b(n) \leq C_\epsilon\gamma(n)$ for all $n \geq 1$, where $C_\epsilon = \max\{L_3(1+\epsilon), \sup_{1\leq j\leq N(\epsilon)} b(j)/\gamma(j)\}$. Therefore, from Theorem 1.4.1 we have that there exists $C_{1,\epsilon} > 0$ such that $\rho(n) \leq C_{1,\epsilon}\gamma(n)$ for all $n \geq 1$. Thus,

$$0 < \lambda_1\rho(j^*)L_3 \leq \limsup_{n\to\infty}\frac{\rho(n)}{\gamma(n)} \leq C_{1,\epsilon} < \infty.$$

Conversely, suppose now that $\limsup_{n\to\infty} \rho(n)/\gamma(n) =: L_2 \in (0,\infty)$. Then from (1.4.1) we have $\limsup_{n\to\infty} b(n)/\gamma(n) \leq L_2/(\lambda_1\rho(j^*)r^{j^*}) < +\infty$.

In order to show that $\limsup_{n\to\infty} b(n)/\gamma(n) > 0$, we suppose the contrary, namely that $\limsup_{n\to\infty} b(n)/\gamma(n) = 0$. Since b and γ are non–negative, $\lim_{n\to\infty} b(n)/\gamma(n) = 0$. Then it is not difficult to see from the proof of Theorem 1.3.2 that $\lim_{n\to\infty}\rho(n)/\gamma(n) = 0$ and hence $\limsup_{n\to\infty}\rho(n)/\gamma(n) = 0$, which contradicts $\limsup_{n\to\infty}\rho(n)/\gamma(n) > 0$. Therefore, as $\limsup_{n\to\infty} b(n)/\gamma(n)$ must exist, we have $\limsup_{n\to\infty} b(n)/\gamma(n) \in (0,\infty)$. □

Proof of Theorem 1.4.5. The proof is largely established by rewriting the limits in terms of their $\epsilon - N$ definition. This delivers upper and lower bounds, γ_-, γ_+ respectively, on b where $\gamma_-(n) = C_-(n+1)^{-\beta(1-\epsilon)}$ and $\gamma_+(n) = C_+(n+1)^{-\beta(1+\epsilon)}$ for $n \geq 0$ and for some constants $C_-, C_+ > 0$. Theorems 1.4.1, 1.4.2 and 1.4.3 are then applied to generate the appropriate bounds on ρ, from which the result follows.

In order to establish (ii), i.e.

$$\limsup_{n\to\infty} \frac{\log \rho(n)}{\log n} = -\beta \text{ implies } \limsup_{n\to\infty} \frac{\log b(n)}{\log n} = -\beta,$$

one uses (1.4.1) and an argument by contradiction, not unlike that employed in the proof of Theorem 1.4.4.

For the case $\beta = \infty$, the bounding function is n^{-K} where $K > 0$ can be chosen arbitrarily large. In all other respects this case follows through as for other values of β. □

Proof of Theorem 1.4.6. Firstly suppose $\rho(k) \leq C_2\alpha_2^k$. By definition, $b \geq 0$ and hence $z \geq 0$ and $\rho \geq 0$. Thus with j^* as defined in (A_1), from (1.4.1) we have

$$b(k+1+j^*) \leq \frac{1}{\lambda_1\rho(j^*)}\rho(k+1) \leq \frac{C_2}{\lambda_1\rho(j^*)}\alpha_2^{k+1} = \frac{C_2}{\lambda_1\rho(j^*)\alpha_2^{j^*}}\alpha_2^{k+1+j^*}.$$

Hence, $b(k) \leq C_3\alpha_2^k$ for all $k \geq j^*+1$ where $C_3 = C_2/(\lambda_1\rho(j^*)\alpha_2^{j^*})$ and so $b(k) \leq C_4\alpha_2^k$ for all $k \geq 1$, where $C_4 = \max(C_3, Q)$ and $Q = \max_{1\leq l\leq j^*} b(l)\alpha_2^{-l} = b(j^*)\alpha_2^{-j^*}$.

Conversely, suppose that $b(k) \leq C_1\alpha_1^k$. As (S_1) holds we have $z(n) \to 0$, as $n \to \infty$. Thus we may use [47, Theorem 4] to conclude that

$$b(k) \leq C_1\alpha_1^k \quad \text{if and only if} \quad z(k) \leq C_4\alpha_4^k, \tag{1.5.9}$$

for some $\alpha_4 \in (0,1)$ and $C_1, C_4 \in (0,\infty)$. Therefore for the sequence f given in (1.2.7), we get

$$f(k) = \lambda_1\sum_{j=1}^{\infty} b(k+j+1)\rho(-j) \leq \lambda_1 C_1 \sum_{j=1}^{\infty}\alpha_1^{k+j+1}\rho(j) < \lambda_1 C_1\alpha_1\alpha_1^k\sum_{j=1}^{\infty}\rho(j).$$

Thus as ρ is summable from Theorem 1.2.2, we have $f(k) \leq \lambda_1 C_1 K\alpha_1^k$, for some $0 < K < \infty$. Using this estimate for f and (1.5.9) in (1.2.8) gives

$$\rho(k) \leq C_5\alpha_4^k + \sum_{j=1}^{k} C_4\alpha_4^{k-j}C_6\alpha_1^j = C_5\alpha_4^k + C_7\alpha_4^k\sum_{j=1}^{k}\left(\frac{\alpha_1}{\alpha_4}\right)^j. \tag{1.5.10}$$

If $\alpha_1 \neq \alpha_4$, with $\alpha_2 =$max(α_1, α_4) we have $\rho(k) \leq C_5\alpha_4^k + C_8|\alpha_4^k - \alpha_1^k| \leq C_5\alpha_4^k + C_8\alpha_4^k + C_8\alpha_1^k \leq C_9\alpha_2^k$. If $\alpha_1 = \alpha_4$, then

$$\rho(k) \leq C_5\alpha_4^k + C_7\alpha_4^k k < C_5\alpha_4^k + C_7C_8(\alpha_4 + \epsilon)^k < C_{10}(\alpha_4 + \epsilon)^k,$$

where $\alpha_2 = \alpha_4 + \epsilon$ and ϵ is chosen sufficiently small so that $\alpha_2 < 1$, and C_8 is given by $C_8 = \sup_{k\geq 1} k/(1 + \epsilon/\alpha_4)^k$. □

Chapter 2

Necessary and Sufficient Conditions for Periodic Decaying Resolvents in Linear Summation Convolution Volterra Equations and Applications to ARCH(∞) Processes

2.1 Introduction

This chapter characterises the exact decay rate of the solution of the discrete linear Volterra equation

$$X(n+1) = f(n+1) + \sum_{j=0}^{n} U(n-j)X(j), \quad n \in \mathbb{Z}^+, \quad X(0) = X_0, \tag{2.1.1}$$

where $f : \mathbb{Z}^+ \to \mathbb{R}^d$, $U : \mathbb{Z}^+ \to \mathbb{R}^{d\times d}$ and $X_0 \in \mathbb{R}^d$. The exact rate of decay of the forcing function, f, is known and the kernel U has known decay and periodic asymptotic behaviour. We define the associated resolvent equation of (2.1.1)

$$Z(n+1) = \sum_{j=0}^{n} U(n-j)Z(j), \quad n \in \mathbb{Z}^+, \quad Z(0) = I, \tag{2.1.2}$$

where $Z : \mathbb{Z}^+ \to \mathbb{R}^{d\times d}$ and I is the identity matrix. By first examining (2.1.2) we can more easily analyse (2.1.1) via a variation of constants representation:

$$X(n) = Z(n)X(0) + \sum_{j=1}^{n} Z(n-j)f(j), \quad n \in \{1, 2, ...\}. \tag{2.1.3}$$

It is shown in [13] that when the kernel of (2.1.1) has a particular rate of slower than exponential decay (e.g., polynomial or regularly varying decay), then the solution of (2.1.2) also has this exact rate of decay. It is from this class of weight function that the rate of decay of U in this present chapter is imposed. It is shown in Song and Baker [110, 111] and Győri and Reynolds [61] that periodicity in the kernel of perturbed summation Volterra equations implies periodicity in the solution of these equations. The stability of solutions of perturbed summation Volterra equations is also shown. Linear Volterra convolution and non-convolution equations are studied in Elaydi and Murakami [48], where conditions on the summability of the resolvent and stability of the solution are used to establish the existence of a unique bounded (in particular periodic and almost periodic) solution. Conditions guaranteeing the existence of asymptotically periodic solutions of linear non-convolution summation Volterra equations are derived in [61] via an application of admissibility theory.

Section 2.2 gives some fundamental definitions as well as various lemmata needed in the proof in Section 2.3. In Section 2.3 the main result establishes that the solution of (2.1.2) also decays at the same rate as the kernel and the periodic component is preserved. This result is achieved by eliminating the effect of the periodicity, by evaluating (2.1.2) at N discrete time points, where N is the value of the period, and lifting the equation to a higher space dimension in which it is asymptotically autonomous. Then by a careful separation of the summation term we can form a system of equations to which we apply the admissibility theory of [13]. Moreover, it can be shown in the case when the kernel is "small" in some $\ell^1(\mathbb{Z}^+)$ sense, that Z has periodic decaying asymptotic behaviour if and only if U does, and indeed both sequences can be majorised by the same weight function and possess the same period. In forthcoming work, it is planned to investigate more general forms of decay in both continuous and discrete equations, where the decay can be separated into a rate and a bounded component with some structure (such as the periodicity studied here). Lastly, in Section 2.4 the results developed in Section 2.3 are applied to demonstrate that if a periodic fluctuation is present in the kernel of an ARCH(∞) processes then this periodic component propagates through to the autocovariance function of the ARCH(∞) process. This example sheds further light on extant research on the memory properties of ARCH(∞) processes (see e.g., [51, 75, 118]).

The work of this chapter appears as a joint paper with Appleby [8].

2.2 Preliminary Results

Let I denote the identity matrix and 0 the zero matrix. $\mathbb{R}^{d\times d}$ can be endowed with many norms, but they are all equivalent. The *spectral radius* of a matrix A is given by $\rho(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$, where $\|\cdot\|$ is any norm on $\mathbb{R}^{d\times d}$; $\rho(A)$ is independent of the norm employed to calculate it. We note that $\rho(A) \leq \rho(|A|)$. Also if $0 \leq A \leq B$, then $\rho(A) \leq \rho(B)$. Also,

$$\rho(A) \leq \|A^k\|^{1/k}, \forall k \in \mathbb{N}. \tag{2.2.1}$$

In this chapter the matrix norm $\|A\|_\infty = \max_{1\le i\le N} \sum_{j=1}^{N} |A_{i,j}|$ is used. Let $C \in \mathbb{R}^{d\times d}$, then we say that C is a circulant matrix if $C_{i,j} = C_{d+i-j+1,1}$ for $i < j$ and $C_{i-j+1,1}$ for $i \ge j$. Such a matrix is a special type of Toeplitz matrix.

In this chapter, we investigate a class of kernels which have the essential rate of decay of a sequence in $\mathcal{W}(r)$, but exhibit a periodic "fluctuation" of period $N \in \mathbb{N}$ around this rate of decay. To encapsulate this idea we give the following definition.

Definition 2.2.1. Let $d, N \in \mathbb{Z}^+/\{0\}$ and $r > 0$ be finite. A sequence $U = \{U(n)\}_{n\ge 0} \in \mathbb{R}^{d\times d}$ is in $\mathcal{WP}(r, N)$ if there exists a function $\phi \in \mathcal{W}(r)$ and a sequence of $d \times d$ matrices $\{A_i\}_{i=0}^{N-1}$ such that $\lim_{n\to\infty} U(Nn + i)/\phi(Nn) = A_i$. We refer to ϕ as a weight function for U.

If we wish to investigate the rate of decay of a function relative to a particular weight function, say γ, then it is desirable to know how $\gamma(Nn)$ relates to $\gamma(n)$.

Lemma 2.2.1. *Let N be a positive integer and $r > 0$. If $\phi \in \mathcal{W}(r)$ then $\Phi \in \mathcal{W}(\tau)$, where $\Phi(n) := \phi(Nn)$ and $\tau := r^N$*

Proof. Note that $\Phi(n) = \phi(Nn) > 0$. We establish (0.4.1) and (0.4.2) for Φ. Since $\Phi(n-1)/\Phi(n) = \phi(Nn-N)/\phi(Nn)$ and ϕ obeys (0.4.1), we get $\lim_{n\to\infty} \Phi(n-1)/\Phi(n) = 1/r^N = 1/\tau$. Also

$$\sum_{i=0}^{\infty} \Phi(i)\tau^{-i} = \sum_{i=0}^{\infty} \phi(Ni) r^{-Ni} \le \sum_{i=0}^{\infty} \phi(i) r^{-i} < \infty.$$

Turning to (0.4.2), by construction we have

$$\sum_{i=m}^{n-m} \frac{\Phi(n-i)\Phi(i)}{\Phi(n)} = \sum_{i=m}^{n-m} \frac{\phi(Nn-Ni)\phi(Ni)}{\phi(Nn)} \le \sum_{i=Nm}^{Nn-Nm} \frac{\phi(Nn-i)\phi(i)}{\phi(Nn)}.$$

Therefore

$$\limsup_{n\to\infty} \sum_{i=m}^{n-m} \frac{\Phi(n-i)\Phi(i)}{\Phi(n)} \le \limsup_{n\to\infty} \sum_{i=Nm}^{Nn-Nm} \frac{\phi(Nn-i)\phi(i)}{\phi(Nn)} \le \limsup_{L\to\infty} \sum_{i=Nm}^{L-Nm} \frac{\phi(L-i)\phi(i)}{\phi(L)}.$$

The last inequality is obtained by letting $L = Nn$ and noting that in the limit the sum to $L - Nm$ will contain more terms than $Nn - Nm$. Finally, as $\phi \in \mathcal{W}(r)$

$$\begin{aligned}\limsup_{m\to\infty} \limsup_{n\to\infty} \sum_{i=m}^{n-m} \frac{\Phi(n-i)\Phi(i)}{\Phi(n)} &\le \limsup_{m\to\infty} \limsup_{L\to\infty} \sum_{i=Nm}^{L-Nm} \frac{\phi(L-i)\phi(i)}{\phi(L)} \\ &\le \limsup_{P\to\infty} \limsup_{L\to\infty} \sum_{i=P}^{L-P} \frac{\phi(L-i)\phi(i)}{\phi(L)} = 0,\end{aligned}$$

with the last inequality holding by reasoning similar to above. □

In determining the results in Section 2.3 we have used [13, Thm.3.2] which we state here for completeness. Note in this result and the rest of the chapter that if γ is a positive real sequence, $f \in \mathbb{R}^{d_1 \times d_2}$, and $\lim_{n\to\infty} f(n)/\gamma(n)$ exists we denote the limit by $L_\gamma f$. The theorem provides an explicit formula for $L_\gamma z$ in terms of the data.

Theorem 2.2.1. *Let $f : \mathbb{Z}^+ \to \mathbb{R}^d$ and $F : \mathbb{Z}^+ \to \mathbb{R}^{d\times d}$ and suppose $\{z(n)\}_{n\geq 0}$ obeys*

$$z(n+1) = f(n) + \sum_{i=0}^{n} F(n-i)z(i), \quad n \geq 0, \quad z(0) = z_0 \in \mathbb{R}^d. \tag{2.2.2}$$

Suppose that there is a γ in $\mathcal{W}(r)$ such that $L_\gamma f$ and $L_\gamma F$ both exist, and that

$$\rho\big(r^{-1}\widetilde{|F|}(r^{-1})\big) = \rho\bigg(\sum_{i=0}^{\infty} r^{-(i+1)}|F(i)|\bigg) < 1. \tag{2.2.3}$$

Then the solution z of (2.2.2) *satisfies*

$$L_\gamma z = (rI - \tilde{F}(r^{-1}))^{-1}[L_\gamma f + (L_\gamma F)\tilde{z}(r^{-1})], \tag{2.2.4}$$

where $\tilde{z}(r^{-1}) = (rI - \tilde{F}(r^{-1}))^{-1}[rz_0 + \tilde{f}(r^{-1})]$.

We provide a preliminary lemma which demonstrates that the inverse of a lower triangular block Toeplitz matrix is also a lower triangular block Toeplitz matrix.

Lemma 2.2.2. *Let $B_{2,1}, B_{3,1}, ..., B_{N,1}$ be $d \times d$ matrices. Let B be a matrix in $\mathbb{R}^{Nd\times Nd}$ with $N, d \in \mathbb{Z}^+$ such that B has the following block structure, for $i, j = \{1, ..., N\}$,*

$$B_{i,j} = \begin{cases} 0_d, & \text{if } i \leq j, \\ B_{i-j+1,1}, & \text{if } i > j, \end{cases}$$

where 0_d represents the $d\times d$ zero matrix. Then $(I-B)^{-1}$ exists and setting $C := (I-B)^{-1}$ we have

$$C_{i,j} = \begin{cases} 0_d, & \text{if } i < j, \\ I_d, & \text{if } i = j, \\ C_{i-1,j-1}, & \text{if } i > j > 1. \end{cases} \tag{2.2.5}$$

and

$$C_{t,1} = \sum_{l=1}^{t-1} B_{l+1,1}C_{t-l,1} = \sum_{l=1}^{t-1} C_{t-l,1}B_{l+1,1} \quad \text{for } t \geq 2. \tag{2.2.6}$$

Proof. Note that $I - B$ has ones on its main diagonal (i.e. $\det(I - B) = 1 \neq 0$) and hence is invertible. The lower triangular structure of C is determined by considering the i, j^{th} element of $(I - B)C$ and using an induction argument. We start by establishing the relation

$$C_{i,j} = \sum_{l=j}^{i-1} B_{i,l} C_{l,j} = \sum_{l=j+1}^{i} C_{i,l} B_{l,j}, \quad \text{for } i > j. \tag{2.2.7}$$

First, we observe that

$$0_d = [C(I - B)]_{i,j} = \sum_{l=j}^{i} C_{i,l}(I - B)_{l,j} = C_{i,j} - \sum_{l=j+1}^{i} C_{i,l} B_{l,j}$$

By similarly considering $[(I-B)C]_{i,j}$, one establishes (2.2.7). We use induction to establish the third equality of (2.2.5), which is equivalent to

$$C_{i,j} = C_{i-j+1,1}, \quad \text{for } i > j. \tag{2.2.8}$$

We first prove $C_{j+1,j} = C_{2,1}$. From (2.2.7)

$$C_{j+1,j} = \sum_{l=j}^{j} B_{j+1,l} C_{l,j} = B_{j+1,j} C_{j,j} = B_{2,1} C_{1,1} = \sum_{l=1}^{2-1} B_{2,l} C_{l,1} = C_{2,1}$$

Now, assume $C_{p,q} = C_{p-q+1,1}$ for all $0 \leq p - q < i - j$ and $p, q \in \{1, ..., N\}$ and i, j are fixed.

$$\begin{aligned} C_{i,j} &= \sum_{l=j}^{i-1} B_{i,l} C_{l,j} = \sum_{l=1}^{i-j} B_{i,l+j-1} C_{l+j-1,j} \\ &= \sum_{l=1}^{i-j+1-1} B_{i-j+1,l} C_{l+j-1,j} = \sum_{l=1}^{i-j+1-1} B_{i-j+1,l} C_{l,1} = C_{i-j+1,1}. \end{aligned}$$

Thus one has $C_{i,j} = C_{i-j+1,1}$ for all $i > j$. With (2.2.7) and (2.2.8) established, we can conclude (2.2.6). □

We supply a Lemma which will be used in the proof of the main result, Theorem 2.3.1.

Lemma 2.2.3. *Let $\{U(n)\}_{n \in \mathbb{Z}^+}$ be a sequence in $\mathbb{R}^{d \times d}$. Suppose*

$$\max_{1 \leq p \leq d} \left(\sum_{q=1}^{d} \sum_{i=0}^{N-1} \sum_{l=0}^{\infty} r^{-N(l+1)} |U(Nl + i)|_{p,q} \right) < 1, \quad r \leq 1. \tag{2.2.9}$$

Define, for some $N \in \{1, 2, \dots\}$, the matrix function $F : \mathbb{Z}^+ \to \mathbb{R}^{N \times N}$ by $F(n) = (I - B)^{-1} J(n)$ for $n \geq 1$, where the $d \times d$ block composition of B and J, for $i, j \in \{1, ..N\}$, *is given by*

$$[I-B]_{i,j}=\begin{cases}0_d, & \text{if } i<j,\\ I_d, & \text{if } i=j,\\ -U(i-j-1), & \text{if } i>j,\end{cases}\quad [J(n)]_{i,j}=\begin{cases}U(Nn+N+i-j-1), & \text{if } i\le j,\\ U(N(n+1)+i-j-1) & \text{if } i>j.\end{cases}\tag{2.2.10}$$

Then

$$\left\|\sum_{i=0}^{\infty} r^{-N(i+1)}|F(i)|\right\|_{\infty}<1.\tag{2.2.11}$$

Although the entries $[J(n)]_{i,j}$ of $J(n)$ have the same form for all i and j, it is convenient in the proof to express them in the slightly differing forms displayed above.

Proof. We use the notation, for $\lambda\in\{0,\cdots N-1\}$, $S_\lambda:=\sum_{l=0}^{\infty}r^{-Nl}|U(Nl+\lambda)|$, $S:=\sum_{l=0}^{N-1}S_l$ and $M:=\sum_{i=0}^{\infty}r^{-N(i+1)}|F(i)|$. Note by (2.2.9) that $\|r^{-N}S\|_\infty<1$. Hence,

$$0\le\left[\sum_{l=0}^{\infty}r^{-N(l+1)}|J(l)|\right]_{i,j}=\begin{cases}r^{-N}S_{N+i-j-1}, & \text{if } i\le j,\\ S_{i-j-1}-|U(i-j-1)|, & \text{if } i>j.\end{cases}$$

Also, for $i>j$ and by noting that $(I-B)$ is a matrix of the form in Lemma 2.2.2, we use (2.2.5)

$$\begin{aligned}M_{i,j}&\le\sum_{k=1}^{N}|(I-B)^{-1}|_{i,k}\left[\sum_{n=0}^{\infty}r^{-N(l+1)}|J(l)|\right]_{k,j}\\&=\sum_{k=1}^{j}|C|_{i,k}\left[\sum_{n=0}^{\infty}r^{-N(l+1)}|J(l)|\right]_{k,j}+\sum_{k=j+1}^{i}|C|_{i,k}\left[\sum_{n=0}^{\infty}r^{-N(l+1)}|J(l)|\right]_{k,j}\\&=\sum_{k=1}^{j}|C|_{i,k}r^{-N}S_{N+k-j-1}+\sum_{k=j+1}^{i}|C|_{i,k}(S_{k-j-1}-|U(k-j-1)|),\end{aligned}$$

where $C:=(I-B)^{-1}$. Similarly for $i\le j$ we have $M_{i,j}\le\sum_{k=1}^{i}|C|_{i,k}r^{-N}S_{N+k-j-1}$. We note that, by definition M is a non-negative matrix, that is, in verifying (2.2.11) we need consider the row sums of M rather than $|M|$. We now compute the sum of each row of M and show that they are all less than one. The sum of the first and second block-rows are special cases. We compute the sum for the first row and also the general case; the sum for the second row is similar to the general case.

For $i=1$,

$$\sum_{j=1}^{N}M_{1,j}\le\sum_{j=1}^{N}|C|_{1,1}r^{-N}S_{N-j}=r^{-N}\sum_{j=1}^{N}S_{N-j}=r^{-N}S.$$

Indeed,

$$\left\|\sum_{j=1}^{N} M_{1,j}\right\|_{\infty} = \max_{1\le p\le d}\sum_{q=1}^{d}[\sum_{j=1}^{N} M_{1,j}]_{p,q} \le \max_{1\le p\le d}\sum_{q=1}^{d} r^{-N}[S]_{p,q} < 1.$$

For $i \ge 3$

$$\begin{aligned}
\sum_{j=1}^{N} M_{i,j} &= \sum_{j=1}^{i-1} M_{i,j} + \sum_{j=i}^{N} M_{i,j} \\
&\le \sum_{j=1}^{i-1}\sum_{k=1}^{j} |C|_{i,k} r^{-N} S_{N+k-j-1} + \sum_{j=1}^{i-1}\sum_{k=j+1}^{i} |C|_{i,k} S_{k-j-1} \\
&\quad - \sum_{j=1}^{i-1}\sum_{k=j+1}^{i} |C|_{i,k}|U(k-j-1)| + \sum_{j=i}^{N}\sum_{k=1}^{i} |C|_{i,k} r^{-N} S_{N+k-j-1} \\
&= \sum_{k=1}^{i-1} |C|_{i,k} \sum_{j=k}^{i-1} r^{-N} S_{N+k-j-1} + \sum_{k=2}^{i} |C|_{i,k} \sum_{j=1}^{k-1} S_{k-j-1} \\
&\quad - \sum_{k=2}^{i} |C|_{i,k} \sum_{j=1}^{k-1} |U(k-j-1)| + \sum_{k=1}^{i} |C|_{i,k} \sum_{j=i}^{N} r^{-N} S_{N+k-j-1}.
\end{aligned}$$

By moving the $k = i$ terms from the second and fourth sum, and combining the first and fourth sum we get

$$\begin{aligned}
\sum_{j=1}^{N} M_{i,j} \le \sum_{k=1}^{i-1} |C|_{i,k} \sum_{j=k}^{N} r^{-N} S_{N+k-j-1} + \sum_{k=2}^{i-1} |C|_{i,k} \sum_{j=1}^{k-1} S_{k-j-1} \\
- \sum_{k=2}^{i} |C|_{i,k} \sum_{j=1}^{k-1} |U(k-j-1)| + \sum_{j=i}^{N} r^{-N} S_{N-j+i-1} + \sum_{j=1}^{i-1} S_{i-j-1} := A_2 - A_3 + A_1.
\end{aligned} \tag{2.2.12}$$

where the first two sums are A_2, the next is A_3 and the last two are A_1. Next, we write A_1 as

$$A_1 = \sum_{l=i-1}^{N-1} r^{-N} S_l + \sum_{l=0}^{i-2} S_l = r^{-N} S + (1 - r^{-N}) \sum_{l=0}^{i-2} S_l. \tag{2.2.13}$$

As for A_2 we rearrange to get

$$\begin{aligned}
A_2 &= \sum_{k=1}^{i-1} |C|_{i,k} \sum_{l=k-1}^{N-1} r^{-N} S_l + \sum_{k=2}^{i-1} |C|_{i,k} \sum_{l=0}^{k-2} S_l \\
&= \sum_{k=2}^{i-1} |C|_{i,k} r^{-N} \sum_{l=0}^{N-1} S_l - \sum_{k=2}^{i-1} |C|_{i,k} r^{-N} \sum_{l=0}^{k-2} S_l + |C|_{i,1} \sum_{l=0}^{N-1} r^{-N} S_l + \sum_{k=2}^{i-1} |C|_{i,k} \sum_{l=0}^{k-2} S_l \\
&= \sum_{k=2}^{i-1} |C|_{i,k} r^{-N} S + (1 - r^{-N}) \sum_{k=2}^{i-1} |C|_{i,k} \sum_{l=0}^{k-2} S_l + |C|_{i,1} r^{-N} S.
\end{aligned} \tag{2.2.14}$$

Regarding A_3, we note that by (2.2.8) and (2.2.6)

$$C_{i,k} = C_{i-k+1,1} = - \sum_{l=0}^{i-k-1} C_{i-k-l,1} U(l)$$

for $i > k$. Therefore

$$
\begin{aligned}
A_3 &= \sum_{l=2}^{i} |C|_{i,l} \sum_{k=1}^{l-1} |U(l-1-k)| = \sum_{k=1}^{i-1} \sum_{l=k+1}^{i} |C|_{i,l} |U(l-k-1)| \\
&= \sum_{k=1}^{i-1} \sum_{l=1}^{i-k} |C|_{i,l+k} |U(l-1)| = \sum_{k=1}^{i-1} \sum_{l=1}^{i-k+1-1} |C|_{i-l-k+1,1} |U(l-1)| \\
&= \sum_{k=1}^{i-1} \sum_{l=0}^{i-k-1} |C|_{i-l-k,1} |U(l)| \geq \sum_{k=1}^{i-1} |C|_{i-k+1,1} = \sum_{k=1}^{i-1} |C|_{i,k}. \qquad (2.2.15)
\end{aligned}
$$

Inserting (2.2.13), (2.2.14) and (2.2.15) into (2.2.12) we can write.

$$
\begin{aligned}
\sum_{j=1}^{N} M_{ij} &\leq r^{-N} S + (1 - r^{-N}) \sum_{l=0}^{i-2} S_l + \sum_{k=2}^{i-1} |C|_{ik} r^{-N} S + (1 - r^{-N}) \sum_{k=2}^{i-1} |C|_{ik} \sum_{l=0}^{k-2} S_l \\
&\quad + |C|_{i1} r^{-N} S - \sum_{k=1}^{i-1} |C|_{ik} \\
&= r^{-N} S + (1 - r^{-N}) \sum_{l=0}^{i-2} S_l + \sum_{k=1}^{i-1} |C|_{i,k} (r^{-N} S - I_d) + (1 - r^{-N}) \sum_{k=2}^{i-1} |C|_{i,k} \sum_{l=0}^{k-2} S_l.
\end{aligned}
$$

We note that by conditions (2.2.9) we have $1 - r^{-N} \leq 0$. Therefore

$$
\sum_{j=1}^{N} M_{i,j} \leq r^{-N} S + \sum_{k=1}^{i-1} |C|_{i,k} (r^{-N} S - I_d).
$$

Letting $[\sum_{j=1}^{N} M_{i,j}]_{p,q}$ denote the p, q^{th} element of the $d \times d$ matrix $\sum_{j=1}^{N} M_{i,j}$, we have

$$
\begin{aligned}
\left\| \sum_{j=1}^{N} M_{i,j} \right\|_{\infty} &= \max_{1 \leq p \leq d} \sum_{q=1}^{d} [\sum_{j=1}^{N} M_{i,j}]_{p,q} \\
&\leq \max_{1 \leq p \leq d} \left(\sum_{q=1}^{d} r^{-N} [S]_{p,q} + \sum_{q=1}^{d} \sum_{\alpha=1}^{d} [\sum_{k=1}^{i-1} |C|_{i,k}]_{p,\alpha} [r^{-N} S - I_d]_{\alpha,q} \right) \\
&= \max_{1 \leq p \leq d} \left(\sum_{q=1}^{d} r^{-N} [S]_{p,q} + \sum_{\alpha=1}^{d} [\sum_{k=1}^{i-1} |C|_{i,k}]_{p,\alpha} (r^{-N} \sum_{q=1}^{d} [S]_{\alpha,q} - 1) \right) \\
&< \max_{1 \leq p \leq d} \big(r^{-N} \sum_{q=1}^{d} [S]_{p,q} \big) < 1.
\end{aligned}
$$

With the last two inequalities holding as $r^{-N} \sum_{q=1}^{d} [S]_{\alpha,q} < 1$ for all $\alpha \in \{1, \cdots, d\}$. Thus $\|M\|_\infty = \max_{1 \leq i \leq N} (\| \sum_{j=1}^{N} M_{i,j} \|_\infty) < 1$ and (2.2.11) is satisfied. □

2.3 Main Results

We next show that the solution Z of equation (2.1.2) is in $\mathcal{WP}(r, N)$ with weight function ϕ, when the kernel U lies in $\mathcal{WP}(r, N)$ with weight function ϕ. Once the behaviour of Z

is known, a variation of constants formula readily enables us to determine the asymptotic behaviour of the solution of (2.1.1). Firstly we give a lemma concerning the summability of Z.

Lemma 2.3.1. *Let Z be the solution of* (2.1.2). *If* (2.3.2) *holds then*

$$\mathbf{S^{(Z)}} := \sum_{i=0}^{N-1}\sum_{n=0}^{\infty} r^{-N(n+1)}|Z(Nn+i)|$$

is finite and the following inequality holds:

$$\mathbf{S^{(Z)}} \le r^{-N}I + \left(\sum_{i=0}^{N-1}\sum_{n=0}^{\infty} r^{-N(n+1)}|U(Nn+i)|\right)\left(\sum_{i=0}^{N-1}\sum_{n=0}^{\infty} r^{-N(n+1)}|Z(Nn+i)|\right).$$

Theorem 2.3.1. *Let $\{Z(n), n \in \mathbb{N}\}$ be the sequence which satisfies* (2.1.2). *Suppose that $U \in \mathcal{WP}(r, N)$ with weight function $\phi \in \mathcal{W}(r)$ such that there exists a sequence of $d \times d$ matrices $\{A_i\}_{i=0}^{N-1}$ and*

$$\lim_{n\to\infty} \frac{1}{\phi(Nn)} U(Nn+i) = A_i, \quad i \in \{0, 1, 2, ..., N-1\}, \tag{2.3.1}$$

$$\max_{1\le p\le d}\left(\sum_{q=1}^{d}\sum_{i=0}^{N-1}\sum_{l=0}^{\infty} r^{-N(l+1)}|U(Nl+i)|_{p,q}\right) < 1, \quad r \le 1, \tag{2.3.2}$$

for some $N \in \mathbb{N}$. Then $Z \in \mathcal{WP}(r, N)$ and there exists a $\{\rho_i\} \in \mathbb{R}^{d\times d}$ such that

$$\lim_{n\to\infty} \frac{1}{\phi(Nn)} Z(Nn+i) =: \rho_i. \tag{2.3.3}$$

Remark 2.3.1. Condition (2.3.1) gives us the rate of decay of the components of $U(Nn+i)$ for each i. Hence it encapsulates both the decay and periodic components of the kernel. Condition (2.3.2) is imposed in order to ensure stability of the problem. While the $||\cdot||_\infty$ is employed here for simplicity and to ease the calculations involved, we speculate that other norms may also be possible while noting the equivalence of norms for scalar functions. The result (2.3.3) is analogous to (2.3.1), that is that the solution of (2.1.2) inherits the same rate of decay as U, and also retains a similar periodic component. We note that while it is possible to calculate an explicit formula for ρ_i, it is in general far more complicated than the constant matrix A_i. That such limits may in general prove rather unilluminating may be seen from the explicit example in Section 2.4.

Remark 2.3.2. Later, we give a partial converse to Theorem 2.3.1 which illustrates the sharpness of (2.3.1), (2.3.2).

Proof of Theorem 2.3.1. We first develop a system of equations from (2.1.2), which can be put into the form of (2.2.2). We then focus on ensuring that all the conditions of Theorem 2.2.1 hold. From (2.1.2) we can write for $i > 0$,

$$\begin{aligned}
Z(Nn+i) &= \sum_{j=0}^{Nn+i-1} U(j)Z(Nn+i-1-j) \\
&= \sum_{k=0}^{n}\sum_{j=0}^{i-1} U(Nk+j)Z(Nn+i-1-Nk-j) \\
&\quad + \sum_{k=0}^{n-1}\sum_{j=i}^{N-1} U(Nk+j)Z(Nn+i-1-Nk-j) \\
&= \sum_{j=0}^{i-1} U(j)Z(Nn+i-j-1) \\
&\quad + \sum_{k=0}^{n-1}\sum_{j=0}^{i-1} U(N(k+1)+j)Z(N(n-k-1)+i-j-1) \\
&\quad + \sum_{j=i}^{N-1}\sum_{k=0}^{n-1} U(Nk+j)Z(N(n-k-1)+N+i-j-1) \\
&= \sum_{j=0}^{i-1} U_j(0)Z_{i-j-1}(n) + \sum_{j=0}^{i-1}\sum_{k=0}^{n-1} \bar{U}_j(k)Z_{i-j-1}(n-1-k) \\
&\quad + \sum_{j=i}^{N-1}\sum_{k=0}^{n-1} U_j(k)Z_{N+i-j-1}(n-1-k)
\end{aligned}$$

where in the last line, we set $Z_i(n) := Z(Nn+i)$; $U_i(n) := U(Nn+i)$; and $\bar{U}_i(n) := U_i(n+1)$. Thus

$$Z_i(n) = \sum_{j=0}^{i-1} U_j(0)Z_{i-j-1}(n) + \sum_{l=0}^{i-1} \left(\bar{U}_{i-1-l} * Z_l\right)(n-1) + \sum_{l=i}^{N-1} \left(U_{N+i-1-l} * Z_l\right)(n-1). \tag{2.3.4}$$

In the case when $i = 0$, a similar result is obtained, but neither the second nor the third term appear in (2.3.4). Thus, for $i \in \{0, 1, ..., N-1\}$ we generate a system of equations

$$\underline{Z}(n) = B \cdot \underline{Z}(n) + (J * \underline{Z})(n-1), \quad n \geq 1, \tag{2.3.5}$$

where $\underline{Z}(n) \in \mathbb{R}^{Nd \times d}$, $B \in \mathbb{R}^{Nd \times Nd}$ and $J(n) \in \mathbb{R}^{Nd \times Nd}$ where for $p, q \in \{1, 2.., N\}$ we define $[\underline{Z}(n)]_p = Z_{p-1}(n)$ and

$$B_{p,q} = \begin{cases} 0, & \text{if } p \leq q, \\ U(p-q-1), & \text{if } p > q. \end{cases} ; \quad J(n)_{p,q} = \begin{cases} U_{N+p-q-1}(n), & \text{if } p \leq q, \\ \bar{U}_{p-q-1}(n), & \text{if } p > q. \end{cases} \tag{2.3.6}$$

Note that $I - B$ is in the form given in (2.2.10) in Lemma 2.2.2, so $(I - B)^{-1}$ exists. Equation (2.3.5) simplifies to

$$\underline{Z}(n) = (F * \underline{Z})(n-1), \quad n \geq 1, \tag{2.3.7}$$

where $F(n) := (I - B)^{-1} J(n)$. In order to satisfy the conditions of Theorem 2.2.1, we need to show that, for some weight function, μ, in $\mathcal{W}(s)$, $L_\mu F$ exists and that

$$\rho\left(\sum_{l=0}^{\infty} s^{-(l+1)} F(l)\right) < 1. \tag{2.3.8}$$

We note that a natural choice of μ is $\{\Phi(n)\}_{n\geq 0} := \{\phi(Nn)\}_{n\geq 0}$ as $L_\Phi F$ is well-defined. We note by Lemma 2.2.1 that Φ is in $\mathcal{W}(r^N)$. Observing $L_\Phi F = (I-B)^{-1}\lim_{n\to\infty} J(n)/\Phi(n)$, where this limit exists because

$$\left[\lim_{n\to\infty}\frac{1}{\Phi(n)}J(n)\right]_{p,q} = \begin{cases} A_{N+p-q-1}, & \text{if } p \leq q, \\ A_{p-q-1} r^N, & \text{if } p > q. \end{cases} \tag{2.3.9}$$

Turning our attention to (2.3.8), we see what is needed is

$$\rho\left(\sum_{l=0}^{\infty} r^{-N(l+1)} |F(i)|\right) < 1. \tag{2.3.10}$$

But, by (2.2.1) we need only check $\|\sum_{i=0}^{\infty} r^{-N(i+1)}|F(i)|\,\|_\infty < 1$. Applying Lemma 2.2.3 we see that (2.3.10) holds. Therefore, $L_\Phi \underline{Z}$ exists and is given by Theorem 2.2.1. Hence, by looking at the components of $\underline{Z}$ we see that $Z(nN + i)/\phi(Nn) \to \rho_i$, as $n \to \infty$. $\square$

Proof of Lemma 2.3.1. Define $Z_i(n) = Z(Nn+i)$, $U_i(n) = U(Nn+i)$ for $i \in \{0, 1 \ldots, N-1\}$. Then by (2.1.2), $Z_0(0) = I$, and for $n \geq 1$, $i \in \{1, \ldots, N-1\}$

$$Z_i(0) = \sum_{p=0}^{i-1} U_{i-p-1}(0) Z_p(0), \quad Z_0(n) = \sum_{l=0}^{n-1}\sum_{p=0}^{N-1} U_{N-p-1}(n-l-1) Z_p(l),$$
$$Z_i(n) = \sum_{l=0}^{n}\sum_{p=0}^{i-1} U_{i-p-1}(n-l) Z_p(l) + \sum_{l=0}^{n-1}\sum_{p=i}^{N-1} U_{N+i-p-1}(n-l-1) Z_p(l).$$

Then taking absolute values across (2.1.2) and summing we have

$$\sum_{i=0}^{N-1}\sum_{n=0}^{T} r^{-N(n+1)}|Z_i(n)| \leq r^{-N}|Z_0(0)| + \sum_{i=1}^{N-1} r^{-N}|Z_i(0)| + \sum_{n=1}^{T} r^{-N(n+1)}|Z_0(n)|$$
$$+ \sum_{i=1}^{N-1}\sum_{n=1}^{T} r^{-N(n+1)}|Z_i(n)|,$$

where T is a large fixed integer. Substituting the above representations for Z into this equation and permuting sums yields

$$\begin{aligned}
&\sum_{i=0}^{N-1}\sum_{n=0}^{T} r^{-N(n+1)}|Z_i(n)| \\
&\le r^{-N}I + \sum_{p=0}^{N-2}\sum_{q=0}^{N-p-2} r^{-N}|U_q(0)||Z_p(0)| + \sum_{p=0}^{N-1}\sum_{l=0}^{T-1}\sum_{n=0}^{T-l-1} r^{-N(n+l+2)}|U_{N-p-1}(n)||Z_p(l)| \\
&\quad + \sum_{p=0}^{N-2}\sum_{q=0}^{N-p-2}\sum_{l=1}^{T}\sum_{n=0}^{T-l} r^{-N(n+l+1)}|U_q(n)||Z_p(l)| \\
&\quad + \sum_{p=0}^{N-2}\sum_{q=0}^{N-p-2}\sum_{n=1}^{T} r^{-N(n+1)}|U_q(n)||Z_p(0)| \\
&\quad + \sum_{p=1}^{N-1}\sum_{q=N-p}^{N-1}\sum_{l=0}^{T-1}\sum_{n=0}^{T-l-1} r^{-N(n+l+2)}|U_q(n)||Z_p(l)|.
\end{aligned}$$

The remainder of the calculation hinges on careful splitting and recombination of these sums, and by replacing $T-c$ by T in various upper limits of summation. Successively, we estimate according to

$$\begin{aligned}
&\sum_{i=0}^{N-1}\sum_{n=0}^{T} r^{-N(n+1)}|Z_i(n)| \\
&\le r^{-N}I + \sum_{p=0}^{N-2}\sum_{q=0}^{N-p-2} r^{-N}|U_q(0)||Z_p(0)| + \sum_{p=0}^{N-1}\sum_{l=0}^{T}\sum_{n=0}^{T} r^{-N(n+l+2)}|U_{N-p-1}(n)||Z_p(l)| \\
&\quad + \sum_{p=0}^{N-2}\sum_{q=0}^{N-p-2}\sum_{l=1}^{T}\sum_{n=0}^{T} r^{-N(n+l+1)}|U_q(n)||Z_p(l)| + \sum_{p=0}^{N-2}\sum_{q=0}^{N-p-2}\sum_{n=1}^{T} r^{-N(n+1)}|U_q(n)||Z_p(0)| \\
&\quad + \sum_{p=1}^{N-1}\sum_{q=N-p}^{N-1}\sum_{l=0}^{T}\sum_{n=0}^{T} r^{-N(n+l+2)}|U_q(n)||Z_p(l)| \\
&= r^{-N}I + \sum_{p=0}^{N-2}\sum_{q=0}^{N-p-2}\sum_{n=0}^{T} r^{-N(n+1)}|U_q(n)||Z_p(0)| \\
&\quad + \sum_{p=0}^{N-2}\sum_{q=0}^{N-p-2}\sum_{l=1}^{T}\sum_{n=0}^{T} r^{-N(n+l+1)}|U_q(n)||Z_p(l)| + \sum_{l=0}^{T}\sum_{n=0}^{T} r^{-N(n+l+2)}|U_{N-1}(n)||Z_0(l)| \\
&\quad + \sum_{p=1}^{N-1}\sum_{q=N-p-1}^{N-1}\sum_{l=0}^{T}\sum_{n=0}^{T} r^{-N(n+l+2)}|U_q(n)||Z_p(l)|.
\end{aligned}$$

Grouping terms from different sums gives

$$
\begin{aligned}
&\sum_{i=0}^{N-1}\sum_{n=0}^{T} r^{-N(n+1)}|Z_i(n)| \\
&\leq r^{-N}I + \sum_{p=0}^{N-2}\sum_{q=0}^{N-p-2}\sum_{l=0}^{T}\sum_{n=0}^{T} r^{-N(n+l+1)}|U_q(n)||Z_p(l)| \\
&\quad + \sum_{l=0}^{T}\sum_{n=0}^{T} r^{-N(n+l+2)}|U_{N-1}(n)||Z_0(l)| + \sum_{p=1}^{N-2}\sum_{q=N-p-1}^{N-1}\sum_{l=0}^{T}\sum_{n=0}^{T} r^{-N(n+l+2)}|U_q(n)||Z_p(l)| \\
&\quad + \sum_{q=0}^{N-1}\sum_{l=0}^{T}\sum_{n=0}^{T} r^{-N(n+l+2)}|U_q(n)||Z_{N-1}(l)| \\
&\leq r^{-N}I + \sum_{p=0}^{N-2}\sum_{q=0}^{N-p-2}\sum_{l=0}^{T}\sum_{n=0}^{T} r^{-N(n+l+2)}|U_q(n)||Z_p(l)| \\
&\quad + \sum_{l=0}^{T}\sum_{n=0}^{T} r^{-N(n+l+2)}|U_{N-1}(n)||Z_0(l)| + \sum_{p=1}^{N-2}\sum_{q=N-p-1}^{N-1}\sum_{l=0}^{T}\sum_{n=0}^{T} r^{-N(n+l+2)}|U_q(n)||Z_p(l)| \\
&\quad + \sum_{q=0}^{N-1}\sum_{l=0}^{T}\sum_{n=0}^{T} r^{-N(n+l+2)}|U_q(n)||Z_{N-1}(l)| \\
&= r^{-N}I + \left(\sum_{q=0}^{N-1}\sum_{n=0}^{T} r^{-N(n+1)}|U_q(n)|\right)\left(\sum_{p=0}^{N-1}\sum_{l=0}^{T} r^{-N(n+1)}|Z_p(l)|\right),
\end{aligned}
$$

where the last inequality holds as $1 \leq r^{-N}$. Therefore by (2.3.2)

$$
\sum_{i=0}^{N-1}\sum_{n=0}^{T} r^{-N(n+1)}|Z_i(n)| \leq r^{-N}I + \left(\sum_{j=0}^{N-1}\sum_{n=0}^{\infty} r^{-N(n+1)}|U_j(n)|\right)\left(\sum_{i=0}^{N-1}\sum_{l=0}^{T} r^{-N(n+1)}|Z_i(l)|\right). \tag{2.3.11}
$$

Due to condition (2.3.2), we have that $\left(I - \sum_{j=0}^{N-1}\sum_{n=0}^{\infty} r^{-N(n+1)}|U_j(n)|\right)^{-1}$ exists and moreover is a non-negative matrix. Hence we have

$$
\sum_{i=0}^{N-1}\sum_{n=0}^{T} r^{-N(n+1)}|Z_i(n)| \leq \left(I - \sum_{j=0}^{N-1}\sum_{n=0}^{\infty} r^{-N(n+1)}|U_j(n)|\right)^{-1} r^{-N}.
$$

Noting that each entry in the left hand side of the above inequality is an increasing function of T and is bounded above by a term which is independent of T, tells us that each entry of the matrix has a finite limit as $T \to \infty$. This proves the result. The inequality in the statement of the lemma follows by letting $T \to \infty$ in (2.3.11). □

The following corollary applies Theorem 2.3.1 to (2.1.1).

Corollary 2.3.1. *Let* $\{X(n) : n \in \mathbb{N}\}$ *be the solution of* (2.1.1)*,* $\{Z(n) : n \in \mathbb{N}\}$ *the solution of* (2.1.2) *and* $\phi \in \mathcal{W}(r)$ *and* (2.3.1)*,* (2.3.2) *hold. Let* $\{\rho_l\}_{l=0}^{N-1}$ *be given by*

Theorem 2.3.1 and $i \in \{0, 1, ..., N-1\}$. *Suppose* $\lim_{n\to\infty} f(Nn+i)/\phi(Nn) = L_i$. *Then* $\lim_{n\to\infty} X(Nn+i)/\phi(Nn)$ *exists and can be calculated.*

Remark 2.3.3. Other results in the direction of Corollary 2.3.1 are certainly possible to state in which the rate of decay of the perturbation is different to that of the kernel or where their periods differ. The proofs follow readily by the variation of constants formula and the facts that (i) the convolution of two sequences which lie in $\mathcal{WP}(r, N)$ also lies in $\mathcal{WP}(r, N)$. (ii) the sum of two sequences in $\mathcal{WP}(r, N)$ is also in $\mathcal{WP}(r, N)$. Therefore, we do not dwell on this issue but leave it instead to the reader's imagination to consider these obvious extensions.

Proof of Corollary 2.3.1. By Theorem 2.3.1 we have $\lim_{n\to\infty} Z(Nn+i)/\phi(Nn) = \rho_i$. Using (2.1.3) and the same argument at the start of the proof of Theorem 2.3.1 we can write

$$X(Nn+i) = Z(Nn+i)X(0) + \sum_{l=0}^{i}(Z_l * F_{i-l})(n) + \sum_{l=i+1}^{N-1}(Z_l * F_{N+i-l})(n-1),$$

where $f(0) := 0$, $Z_a(b) := Z(Nb+a)$ and $F_a(b) := f(Nb+a)$, $a \in \{0, 1, \dots, N-1\}, b \in \mathbb{Z}^+$. Define $\Phi(n) = \phi(Nn)$. Using [13, Thm:4.3] and $\Phi \in \mathcal{W}(r^N)$ we obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{X(Nn+i)}{\phi(Nn)} &= \rho_i X(0) + \sum_{l=0}^{i}\rho_l\sum_{j=0}^{\infty}F_{i-l}(j)r^{-Nj} + \sum_{l=0}^{i}\sum_{j=0}^{\infty}Z_l(j)r^{-Nj}L_{i-l}\\ &\quad + \sum_{l=i+1}^{N-1}\rho_l\sum_{j=0}^{\infty}F_{N+i-l}(j)r^{-N(j+1)} + \sum_{l=i+1}^{N-1}\sum_{j=0}^{\infty}Z_l(j)r^{-N(j+1)}L_{N+i-l}.\end{aligned} \tag{2.3.12}$$

which completes the proof. □

We close this section by noting that $Z \in \mathcal{W}(r, N)$ is in some sense only possible if $U \in \mathcal{W}(r, N)$. This result is a consequence of Theorem 2.3.1 and Corollary 2.3.1.

We note that one may show, via induction, that the solution Z of (2.1.2) can be expressed as $Z(n) = U(n-1) + \sum_{j=2}^{n} U^{(*j)}(n-j)$, for $n \geq 2$, with $Z(1) = U(0)$, $Z(0) = I$. Furthermore this representation allows one to show that Z is also a solution of the equation $W(n+1) = (W * U)(n)$, $n \geq 0$, $W(0) = I$. Hence $(U * Z)(n) = Z(n+1) = W(n+1) = (W * U)(n) = (Z * U)(n)$. By rewriting (2.1.2), we get $U(n+1) = Z(n+2) - \sum_{j=1}^{n+1} U(n+1-j)Z(j)$ for $n \geq 0$. Putting $Y(n) = -Z(n+1)$ we see that

$$U(n+1) = -Y(n+1) + \sum_{l=0}^{n} U(n-l)Y(l), \quad n \geq 0. \tag{2.3.13}$$

We now argue that $U * Y = Y * U$. For $n \geq 0$ we have

$$(U * Y)(n) = -\sum_{j=0}^{n} U(n-j)Z(j+1) = -\sum_{j=0}^{n} U(n-j)(U * Z)(j) = -(U * U * Z)(n).$$

Similarly $(Y*U)(n) = -(U*Z*U)(n)$. But $Z*U = U*Z$, so $(U*Y)(n) = -(U*U*Z)(n) = -(U*(Z*U))(n) = (Y*U)(n)$. Therefore (2.3.13) becomes

$$U(n+1) = -Y(n+1) + \sum_{l=0}^{n} Y(n-l)U(l), \quad n \geq 0. \tag{2.3.14}$$

which is in the form of (2.1.1). We introduce the resolvent R by $R(n+1) = \sum_{j=0}^{n} Y(n-j)R(j)$ for $n \geq 0$, where $R(0) = I$. We now give conditions under which Theorem 2.3.1 can be applied. If we suppose that Z obeys (2.3.3), then for $i = 0, \ldots, N-1$ we have

$$\lim_{n\to\infty} \frac{Y(Nn+i)}{\phi(Nn)} = -\lim_{n\to\infty} \frac{Z(Nn+i+1)}{\phi(Nn)} = \begin{cases} -\rho_{(i+1)}, & i = 0, \ldots, N-2 \\ -r^N \rho_0, & i = N-1. \end{cases} \tag{2.3.15}$$

Moreover, the condition

$$\max_{1\leq p\leq d} \left(\sum_{q=1}^{d} \sum_{i=0}^{N-1} \sum_{l=0}^{\infty} r^{-N(l+1)} |Z(Nl+i+1)|_{p,q} \right) < 1, \quad r \leq 1, \tag{2.3.16}$$

is equivalent to

$$\max_{1\leq p\leq d} \left(\sum_{q=1}^{d} \sum_{i=0}^{N-1} \sum_{l=0}^{\infty} r^{-N(l+1)} |Y(Nl+i)|_{p,q} \right) < 1, \quad r \leq 1,$$

and by applying Theorem 2.3.1 with Y in the role of U and R in the role of Z, there exist $D_i \in \mathbb{R}^{d\times d}$ for $i = 0, \ldots, N-1$ such that $\lim_{n\to\infty} R(Nn+i)/\phi(Nn) =: D_i$. Using this limit in conjunction with (2.3.15), we may now apply Corollary 2.3.1 to (2.3.14) to deduce that there exist $A_i \in \mathbb{R}^{d\times d}$ for $i = 0, \ldots, N-1$ such that $\lim_{n\to\infty} U(Nn+i)/\phi(Nn) =: A_i$. However we would rather replace (2.3.16) with a norm condition on U (see (2.3.17) below) which must be stronger than (2.3.2), as this would then yield a converse with conditions closer to that of Theorem 2.3.1. By virtue of the discussion above, what remains to be proved in the converse below is that (2.3.17) implies (2.3.16).

Theorem 2.3.2. *Let $\{Z(n), n \in \mathbb{N}\}$ be the sequence which satisfies* (2.1.2)*. Suppose that $Z \in \mathcal{WP}(r, N)$ with weight function ϕ in $\mathcal{W}(r)$ so that there is a sequence of $d \times d$ matrices $\{\rho_i\}_{i=0}^{N-1}$ and*

$$\lim_{n\to\infty} \frac{1}{\phi(Nn)} Z(Nn+i) = \rho_i, \quad i \in \{0, 1, 2, ..., N-1\}.$$

Also suppose

$$\max_{1\le p\le d}\left(\sum_{q=1}^{d}\sum_{i=0}^{N-1}\sum_{l=0}^{\infty} r^{-N(l+1)}|U(Nl+i)|_{p,q}\right) < \frac{1}{1+r^{-N}}, \quad r\le 1, \tag{2.3.17}$$

holds for some $N\in\mathbb{N}$. *Then* $U\in\mathcal{WP}(r,N)$ *with weight function* ϕ *i.e., there exists* $\{A_i\}\in\mathbb{R}^{d\times d}$ *such that*

$$\lim_{n\to\infty}\frac{1}{\phi(Nn)}U(Nn+i)=A_i, \quad i\in\{0,1,2,...,N-1\}.$$

Remark 2.3.4. In the special case where there is no periodicity (i.e., $N=1$) the necessary and sufficient nature of Theorems 2.3.1 and 2.3.2 is an improvement on the sufficient nature of the conditions of Theorem 2.2.1.

Proof. We show that (2.3.17) implies (2.3.16). Regrouping the terms in (2.3.16), one deduces

$$\begin{aligned}&\max_{1\le p\le d}\left(\sum_{q=1}^{d}\sum_{i=0}^{N-1}\sum_{l=0}^{\infty} r^{-N(l+1)}|Z(Nl+i+1)|_{p,q}\right)\\ &=\max_{1\le p\le d}\sum_{q=1}^{d}\left(\sum_{j=1}^{N-1}r^{-N}|Z(j)|_{p,q}+\sum_{j=1}^{N-1}\sum_{l=1}^{\infty}r^{-N(l+1)}|Z(Nl+j)|_{p,q}+\sum_{l=1}^{\infty}r^{-Nl}|Z(Nl)|_{p,q}\right).\end{aligned}$$

Hence, using $1\le r^{-N}$,

$$\begin{aligned}&\max_{1\le p\le d}\left(\sum_{q=1}^{d}\sum_{i=0}^{N-1}\sum_{l=0}^{\infty} r^{-N(l+1)}|Z(Nl+i+1)|_{p,q}\right)\\ &\le \max_{1\le p\le d}\sum_{q=1}^{d}\Bigg(\sum_{j=1}^{N-1}r^{-N}|Z(j)|_{p,q}\\ &\qquad+\sum_{j=1}^{N-1}\sum_{l=1}^{\infty}r^{-N(l+1)}|Z(Nl+j)|_{p,q}+\sum_{l=1}^{\infty}r^{-N(l+1)}|Z(Nl)|_{p,q}\Bigg)\\ &=\max_{1\le p\le d}\sum_{q=1}^{d}\left(\sum_{j=0}^{N-1}\sum_{l=0}^{\infty}r^{-N(l+1)}|Z(Nl+j)|_{p,q}-r^{-N}|Z(0)|_{p,q}\right)\\ &=\max_{1\le p\le d}\sum_{q=1}^{d}\left(\sum_{j=0}^{N-1}\sum_{l=0}^{\infty}r^{-N(l+1)}|Z(Nl+j)|_{p,q}\right)-r^{-N},\end{aligned} \tag{2.3.18}$$

with the last equality holding as $Z(0)=I$, whose rows sum to one, which is independent of p. Define the matrices $A=\sum_{i=0}^{N-1}\sum_{n=0}^{\infty}r^{-N(n+1)}|Z(Nn+i)|$ and $B=\sum_{i=0}^{N-1}\sum_{n=0}^{\infty}r^{-N(n+1)}|U(Nn+i)|$. Then Lemma 2.3.1 gives $A\le r^{-N}I+BA$ or equivalently $A\le (I-B)^{-1}r^{-N}$, with the direction of the inequality being preserved due to

$B \geq 0$ and the expression $(I-B)^{-1} = \sum_{l=0}^{\infty} B^l$, which is valid due to (2.3.17). Taking the infinity norm on both sides of this inequality gives

$$\|A\|_\infty \leq \left\| \sum_{l=0}^{\infty} B^l \right\|_\infty r^{-N} \leq r^{-N} \sum_{l=0}^{\infty} \left\| B^l \right\|_\infty \leq r^{-N} \sum_{l=0}^{\infty} \|B\|_\infty^l = r^{-N} \frac{1}{1-\|B\|_\infty}.$$

Combining this with (2.3.18) gives

$$\max_{1\leq p\leq d} \left(\sum_{q=1}^{d} \sum_{i=0}^{N-1} \sum_{l=0}^{\infty} r^{-N(l+1)} |Z(Nl+i+1)|_{p,q} \right) \leq \|A\|_\infty - r^{-N} \leq r^{-N} \frac{1}{1-\|B\|_\infty} - r^{-N}.$$

Thus if $r^{-N}/(1-\|B\|_\infty) - r^{-N} < 1$ we have our result. But this inequality is equivalent to $\|B\|_\infty < 1/(1+r^{-N}) \leq 1/2 < 1$, which is true by hypothesis. □

2.4 Examples

We provide an application of the above theory to analysing the memory characteristics of autoregressive conditional heteroskedastic processes of order infinity. We consider the sufficiently simple case of a scalar Volterra equation where the kernel has a 'two–periodic' $(N = 2)$ component. We believe that this example is instructive in demonstrating the complexity of the calculations for higher d or N, while retaining results which are eminently verifiable.

The following example serves as a proof of Theroem 1.3.4. We follow the notation of Chapter 1, i.e. b is the non-negative sequence in (AH), z is given by (1.2.6) and χ_z by (1.2.3). The idea of the example is that if b obeys (1.3.7) and also contains a periodic component then χ_z will have a similar rate of decay to b but their periodic components will not be in phase and hence $b \not\sim \chi_z$. Our first illustration of the theory deals with the ratio of z/ϕ; the second uses this result to analyse χ_z/ϕ.

Example 2.4.1.

We can take $\lambda_1 := \mathbb{E}[\xi(0)] > 0$ because if $\lambda_1 = 0$ then $\xi(n) = 0$ for all $n \in \mathbb{Z}^+$. Let $\lambda_1 b(2n+i+1)/\phi(2n) \to a_i > 0$ for $i \in \{0,1\}$, for some $\phi \in \mathcal{W}(1)$ and $a_0 \neq a_1$. Let (S_1) hold. Observing that (1.2.6) is of the form of (2.1.2), we apply Theorem 2.3.1 to (1.2.6) giving,

$$d_0 := \lim_{n\to\infty} \frac{z(2n)}{\phi(2n)} = a_0 T_0 + a_1 T_1, \qquad d_1 := \lim_{n\to\infty} \frac{z(2n+1)}{\phi(2n)} = a_1 T_0 + a_0 T_1,$$

where $T_0 = \Lambda(2S_0(1-S_1))$, $T_1 = \Lambda(S_0^2 + (1-S_1)^2)$, $\Lambda = \big((1-S_1)^2 - S_0^2\big)^{-2}$ and $S_i = \lambda_1 \sum_{j=0}^{\infty} b(2j+i+1)$. Also,

$$\lim_{n\to\infty} \frac{z(2n)}{b(2n)} = \lambda_1 d_0/a_1, \quad \lim_{n\to\infty} \frac{z(2n+1)}{b(2n+1)} = \lambda_1 d_1/a_0.$$

Thus, we cannot have $z \sim b$ unless $(a_0 - a_1)(a_0 + a_1)T_0 = 0$, which cannot occur without violating the hypotheses of the example.

Remark 2.4.1. In order to achieve $z \sim \phi$ (or $d_0 = d_1$) one might consider $T_0 = T_1$, this however leads to $S_0 + S_1 = 1$, i.e. a contradiction of (S_1). Hence in general z is not asymptotic to ϕ.

Remark 2.4.2. We provide a numerical illustration where all of the limits in Example 2.4.1 may be computed explicitly. Define $\phi(n) = n^{-2}$ for all $n \geq 1$ and $\phi(0) = 2$. Let $b(j) = a_1 j^{-2}$ for $j/2 \in \mathbb{N}$, $b(j) = a_0 j^{-2}$ for $j/2 \notin \mathbb{N}$, where $a_0 := 0.5$ and $a_1 := 0.25$. Furthermore let $\{\xi(n)\}_{n\in\mathbb{Z}}$ be an i.i.d. non–negative stochastic process with mean equal to unity (i.e. $\lambda_1 = 1$). Thus it is calculated that

$$S_0 = a_0\lambda_1 \sum_{j=0}^{\infty} \frac{1}{(2j+1)^2} = \frac{\pi^2}{16}, \quad S_1 = a_1\lambda_1 \sum_{j=0}^{\infty} \frac{1}{2^2(j+1)^2} = \frac{\pi^2}{96}.$$

Noting that $S_0 + S_1 < 1$, one can evaluate Λ, T_0 and T_1 respectively and hence d_0 and d_1. Indeed $\Lambda = 5.55073...$, $T_0 = 6.14391...$ and $T_1 = 6.58015...$, which gives $d_0 = 4.71699...$ and $d_1 = 4.82605....$ Therefore

$$\lim_{n\to\infty} \frac{z(2n)}{b(2n)} = d_0/a_1 = 4d_0, \quad \lim_{n\to\infty} \frac{z(2n+1)}{b(2n+1)} = d_1/a_0 = 2d_1,$$

and $4d_0 \neq 2d_1$. Hence the claim of the first statement of (1.3.8) does not hold.

Example 2.4.2.

We show that while it is possible for (1.3.7) to hold one need not have that the second part of (1.3.8) holds. We proceed with the same set up as in Example 2.4.1, noting that (1.3.7) is satisfied. Let ϕ be asymptotic to a decreasing sequence. Now observe,

$$\chi_z(2u) = \sum_{j=0}^{\infty} z(2(j+u))z(2j) + \sum_{j=0}^{\infty} z(2(j+u)+1)z(2j+1),$$

$$\chi_z(2u+1) = \sum_{j=0}^{\infty} z(2(j+u)+1)z(2j) + \sum_{j=0}^{\infty} z(2(j+u+1))z(2j+1).$$

Thus for some sufficiently large positive integer M, we have

$$\frac{\chi_z(2u)}{\phi(2u)} = \sum_{j=0}^{M} \frac{z(2(j+u))}{\phi(2u)} z(2j) + \sum_{j=0}^{M} \frac{z(2(j+u)+1)}{\phi(2u)} z(2j+1)$$
$$+ \sum_{j=M+1}^{\infty} \frac{z(2(j+u))}{\phi(2u)} z(2j) + \sum_{j=M+1}^{\infty} \frac{z(2(j+u)+1)}{\phi(2u)} z(2j+1).$$

For the third sum, recalling that $z \in \ell^1(\mathbb{Z}^+)$ as (S_1) holds,

$$\sum_{j=M+1}^{\infty} \frac{z(2(j+u))}{\phi(2u)} z(2j) = \sum_{j=M+1}^{\infty} \frac{z(2(j+u))}{\phi(2(j+u))} \frac{\phi(2(j+u))}{\phi(2u)} z(2j) \leq 4\, d_0 \sum_{j=M+1}^{\infty} z(2j).$$

The fourth sum can be treated similarly. Recalling the non-negativity of z, we have

$$\lim_{M\to\infty} \lim_{u\to\infty} \sum_{j=M+1}^{\infty} \frac{z(2(j+u))}{\phi(2u)} z(2j) = \lim_{M\to\infty} \lim_{u\to\infty} \sum_{j=M+1}^{\infty} \frac{z(2(j+u)+1)}{\phi(2u)} z(2j+1) = 0.$$

For the first sum we see

$$\lim_{M\to\infty} \lim_{u\to\infty} \sum_{j=0}^{M} \frac{z(2(j+u))}{\phi(2u)} z(2j) = \lim_{M\to\infty} d_0 \sum_{j=0}^{M} z(2j) = d_0 \sum_{j=0}^{\infty} z(2j),$$

and a similar calculation applies to the second sum. Thus, after a similar analysis of $\chi_z(2u+1)$ we have

$$\lim_{u\to\infty} \frac{\chi_z(2u)}{\phi(2u)} = d_0 \sum_{j=0}^{\infty} z(2j) + d_1 \sum_{j=0}^{\infty} z(2j+1) = a_0\tau_0 + a_1\tau_1,$$
$$\lim_{u\to\infty} \frac{\chi_z(2u+1)}{\phi(2u)} = d_1 \sum_{j=0}^{\infty} z(2j) + d_0 \sum_{j=0}^{\infty} z(2j+1) = a_0\tau_1 + a_1\tau_0,$$

where

$$\tau_0 = T_0 \sum_{j=0}^{\infty} z(2j) + T_1 \sum_{j=0}^{\infty} z(2j+1), \quad \tau_1 = T_1 \sum_{j=0}^{\infty} z(2j) + T_0 \sum_{j=0}^{\infty} z(2j+1).$$

Thus for $\chi_z \sim b$ we need $\lim_{u\to\infty} \chi_z(2u)/b(2u) = \lim_{u\to\infty} \chi_z(2u+1)/b(2u+1)$, which is equivalent to $\tau_0(a_0 - a_1)(a_0 + a_1)/(a_0 a_1) = 0$, which can only occur if either $a_0 = a_1$ or $\tau_0 = 0$. The first is ruled out by hypothesis. For the second, summing over (1.2.6) for both $z(2n)$ and $z(2n+1)$ gives

$$\sum_{j=0}^{\infty} z(2j) = \frac{(1-S_1)}{(1-S_1)^2 - S_0^2}, \quad \sum_{j=0}^{\infty} z(2j+1) = \frac{S_0}{(1-S_1)^2 - S_0^2},$$

which gives τ_0 the representation

$$\tau_0 = \frac{\Lambda S_0(S_0^2 + 3(1-S_1)^2)}{(1-S_1)^2 - S_0^2}.$$

Thus, τ_0 cannot be equal to zero (as otherwise $a_0 = 0$). Thus, while $b(i)/\zeta^i \to \infty$ as $i \to \infty$ for any $0 < \zeta < 1$ we do not have $\chi_z(u) \sim Cb(u)$, as $u \to \infty$, for any $0 < C < \infty$.

Remark 2.4.3. Following on from Remark 2.4.2 one can compute the various limits and infinite sums in Example 2.4.2, i.e. $\sum_{j=0}^{\infty} z(2j), \sum_{j=0}^{\infty} z(2j+1), \tau_0$ and τ_1 respectively and hence we have

$$\lim_{u\to\infty} \frac{\chi_z(2u)}{b(2u)} = \lambda_1(\frac{a_0}{a_1}\tau_0 + \tau_1) = 67.9375.., \quad \lim_{u\to\infty} \frac{\chi_z(2u+1)}{b(2u+1)} = \lambda_1(\frac{a_1}{a_0}\tau_0 + \tau_1) = 34.1128..$$

Thus as both Λ and τ_0 are positive (approximately 5.55073 and 22.5498 respectively), we have that the above two limits are unequal and hence $\chi_z(u) \not\sim Cb(u)$ as $u \to \infty$ for some $0 < C < \infty$.

Chapter 3

Admissibility of Linear Stochastic Volterra Operators and Exact Asymptotic Behaviour of Affine Stochastic Volterra Equations

3.1 Introduction

Interest in stochastic functional differential equations, including stochastic differential equations with delay, and stochastic Volterra equations, has increased in recent years, in part because of their attraction for modelling real–world systems in which the change in the state of a system is both random and depends on the path of the process in the past. Examples include population biology (Mao [83], Mao and Rassias [85, 86]), neural networks (cf. e.g. Blythe et al. [28]), viscoelastic materials subjected to heat or mechanical stress (Drozdov and Kolmanovskii [45], Caraballo et al. [32], Mizel and Trutzer [89, 90]), or financial mathematics (Anh et al. [2, 3], Appleby et al. [20], Appleby and Daniels [7], Arrojas et al. [22], Hobson and Rogers [68], and Bouchaud and Cont [30]).

Naturally, in all these disciplines, there is a great interest in understanding the long–run behaviour of solutions. In disciplines such as engineering and physics it is often of great importance to know that the system is *stable*, in the sense that the solution of the mathematical model converges in some sense to equilibrium. Consequently, a great deal of mathematical activity has been devoted to the question of stability of point equilibria of stochastic functional differential equations and also to the rate at which solutions converge. The literature is extensive, but a flavour of the work can be found in the monographs of Mao [82, 84], Mohammed [91], and Kolmanovskii and Myskhis [76]. Results are known concerning the asymptotic behaviour of affine stochastic Volterra equations, including rates of convergence (see [19, 18]), but generally upper bounds on the solutions are found, rather than exact rates of decay. In this chapter, we investigate not only the exact rate of convergence of solutions to point equilibria, but also the exact rate of growth of solutions of affine equations, which are of interest in studying the explosive growth or collapse of asset prices in financial market models. This develops results established in [20].

To determine the precise asymptotic results we require, it proves efficient and instructive to ask first a more general question concerning the asymptotic behaviour of stochastic integrals of the form

$$(\mathcal{H}f)(t) := \int_0^t H(t,s)f(s)\,dB(s)$$

where H is a deterministic Volterra kernel and f is a deterministic function on $[0,\infty)$. We require certain continuity and regularity properties on H and f which simplify our analysis and ensure the existence of $\mathcal{H}f$ for every appropriate f. The result we have found of most use is to determine, for fixed sample path, under which conditions $\mathcal{H}$ takes the space of bounded continuous functions on $[0,\infty)$ into the space of bounded continuous functions on $[0,\infty)$ with a limit at infinity.

This may be thought of as an analogue of the theory of admissibility of (deterministic) linear continuous Volterra operators, especially in the important case where the operator takes $BC_l(0,\infty)$ into itself, or when $\mathcal{H}$ takes BC into BC_l. Corduneanu has done significant work on the general theory of admissibility for Volterra integral operators (see [36] and [37]). One motivation for the development of such an admissibility theory in the deterministic case is to give precise asymptotic information regarding the solutions of integral and differential equations. Corduneanu [38] contains a comprehensive survey of progress up to 1991, while further developments in this theory are due to Cushing, Miller and others. More recently, admissibility of continuous linear Volterra operators has been used to determine asymptotic behaviour of a nonlinear integrodifferential equation with infinite memory in Appleby, Győri and Reynolds [14]. Parallel results are also available in discrete time: indeed, recent results on the theory of admissibility of Volterra operators in discrete time, together with applications to Volterra summation equations, include Győri and Reynolds [60] and Song and Baker [112].

Reynolds [105] has established results which characterise certain admissible pairs of spaces, as well as connecting the recent dynamical systems literature with parallel, earlier work in the theory of linear operators.

Once we have developed some general results concerning the asymptotic behaviour of $\mathcal{H}f$, the majority of the chapter is devoted to applying this theory to describe the fine structure of the asymptotic behaviour of affine stochastic functional differential equations of the form

$$dX(t) = L(t, X_t)\,dt + \Sigma(t)\,dB(t)$$

where $L = L(\phi)$ is a linear functional from $C([-\tau, 0])$ to $\mathbb{R}^d$, or $L(t, \phi_t)$ is a linear convolution Volterra functional from $C([0, \infty))$ to $\mathbb{R}^d$. Therefore, we are chiefly interested in the effect of time–dependent stochastic perturbations on the asymptotic behaviour of autonomous (or asymptotically autonomous) linear functional differential equations. It is assumed that the asymptotic behaviour of solutions of the underlying fundamental solution or differential resolvent can be described in terms of the solutions of the characteristic equation. We presume that such solutions lie in the region of existence of the Laplace transform of the measure in the linear functional on the right–hand side.

Results of Mohammed and Scheutzeow [92] show that with respect to white noise perturbations, the Liapunov spectrum of deterministic functional differential equations is preserved, to the extent that the leading positive Liapunov exponent of the deterministic equation becomes the a.s. leading Liapunov exponent of the stochastic equation. However, it is also of interest to ask whether oscillation, or multiplicity of the characteristic equations are preserved when the noise intensity is sufficiently small (or does not grow too rapidly, or decay too slowly, relative to the exponential rate of growth or decay of the resolvent). It is known from [20] in the case of a particular scalar functional differential equation with finite delay, for which the solution of the characteristic equation with largest real part is real and simple, and for which the noise intensity is constant, that the solution of the stochastic equation inherits exactly the rate of growth of the resolvent. It is natural to ask whether a result of this kind can be generalised to deal with finite dimensional equations, of both finite delay and Volterra type, for which there may be many solutions of the characteristic equation which have the same real part, need not be simple, nor even be real solutions.

It is a longstanding theme in the asymptotic theory of differential equations, and especially of linear equations, to ask the question: how large can a forcing or perturbation term be, so that the perturbed differential system preserves the asymptotic behaviour of the underlying unperturbed equation. Investigations of this type were systematically initiated by Hartman and Wintner in the 1950's [64, 65, 66, 67]. More recently, there have been many interesting contributions concerning the asymptotic behaviour of functional differential equations: the literature is quite large, but some important and representative papers include Cruz and Hale [39], Haddock and Sacker [62], Arino and Győri [21], Castillo and Pinto [33], Győri and Pituk [58], Pituk [99, 100], and Győri and Hartung [54] among many others. Already, some results for stochastic Volterra equations with state–independent perturbations suggest that results of this type may also be available in the random case Appleby [5].

It is one of the goals of this paper to demonstrate that very sharp conditions can be identified on the intensity of the perturbations under which the asymptotic behaviour of the deterministic equations is preserved. Moreover, we show that the results apply to a wide class of affine stochastic functional differential equation, and examples and underlying admissibility results show that there is the potential for our work to apply to a wider class yet.

Our results for the solution X of functional differential equations have the form

$$\lim_{t\to\infty}\left\{\frac{X(t,\omega)}{\gamma(t)} - S(t,\omega)\right\} = 0, \quad \text{a.s. and in mean square} \tag{3.1.1}$$

where $\gamma : (0,\infty) \to (0,\infty)$ is a deterministic real exponential polynomial, and S is a random sinusoidal vector, whose "frequencies" are deterministic but whose "amplitudes" or "multipliers" are multidimensional normal random variables which are path–dependent (in the case where the zeros of the characteristic equation with largest real part are real, S is a constant random vector). These "multipliers" turn out to be identifiable linear functionals of the Brownian motion, the noise intensity Σ, and of the initial function or condition, because we have an explicit formula for these multipliers in terms of the solutions of the characteristic equation with largest real part. Similar multipliers emerge in papers of Appleby, Devin and Reynolds on stochastic Volterra equations whose solutions have Gaussian limits [11, 12]. Moreover, the joint distribution of these random limits is known exactly, because the mean and covariance matrix of the Gaussian limit can be computed explicitly in terms of the components of the random vector. This has already proved of interest in [20] where the form of the multiplier can be used to describe the mechanism by which financial market bubbles can start. Our results here are also superior to those in Appleby and Daniels [7] (i.e. Chapter 5) in which a limit formula for asset returns of the form (3.1.1) is found for a nonautonomous stochastic functional differential equation. The method of asymptotic analysis, which applies the *deterministic* admissibility theory pathwise, shows that the distribution of S is Gaussian, but does not enable a formula for the variance to be determined. These examples from finance demonstrate the utility of an authentically stochastic admissibility theory in finding the exact form of the limiting multiplier.

3.1.1 Preliminaries

If d is a positive integer, $\mathbb{R}^d$ is the space of d-dimensional column vectors with real components and $\mathbb{R}^{d_1 \times d_2}$ is the space of all $d_1 \times d_2$ real matrices. The identity matrix on $\mathbb{R}^{d\times d}$ is denoted by I_d, while $0_{d_1,d_2}$ represents the matrix of zeros in $\mathbb{R}^{d_1\times d_2}$.

For any vector $x \in \mathbb{R}^d$ the norm $\|\cdot\|$ denotes the Euclidean norm, $\|x\|^2 = \sum_{j=1}^d x_j^2$. While for a matrix norm we use the Frobenius norm, for any $A = (a_{i,k}) \in \mathbb{R}^{n\times d}$

$$\|A\|_F^2 = \sum_{i=1}^n \sum_{k=1}^d |a_{i,k}|^2.$$

As both $\mathbb{R}^d$ and $\mathbb{R}^{d\times d}$ are finite dimensional Banach spaces all norms are equivalent in the sense that for any other norm, $\|\cdot\|$, one can find universal constants $d_1(n,d) \le d_2(n,d)$ such that

$$d_1 \|A\|_F \le \|A\| \le d_2 \|A\|_F .$$

Thus there is no loss of generality in using the Euclidean and Frobenius norms, which for ease of calculation, are used throughout the proofs of this chapter. Moreover we remark that the Frobenius norm is a *consistent matrix norm*, i.e. for any $A \in \mathbb{R}^{n_1\times n_2}, B \in \mathbb{R}^{n_2\times n_3}$

$$\|AB\|_F \le \|A\|_F \|B\|_F .$$

For any matrix $C \in \mathbb{R}^{n\times d}$ we say $C \ge 0$ if $(C)_{i,j} \ge 0$ for all i, j. Also, we say for any matrices $A, B \in \mathbb{R}^{n\times d}$ that $A \le B$ if $B - A \ge 0$. We will use the fact that $\|A\| \le \|B\|$ whenever $0 \le A \le B$.

Definition 3.1.1. A positive function φ defined on $\mathbb{R}$ is called submultiplicative, if $\varphi(0) = 1$, and

$$\varphi(s + t) \le \varphi(s)\varphi(t),$$

for all $s, t \in \mathbb{R}$.

We also define the limits

$$\alpha_\varphi := -\lim_{t\to-\infty}\frac{\ln(\varphi(t))}{t}, \quad \omega_\varphi := -\lim_{t\to\infty}\frac{\ln(\varphi(t))}{t}.$$

Which always exist when φ is a submultiplicative function, c.f. [53, Lemma 4.1].

We define the following modes of convergence:

Definition 3.1.2. The $\mathbb{R}^n$-valued stochastic process $\{X(t)\}_{t\geq 0}$ converges in mean-square to X_∞ if

$$\lim_{t\to\infty}\mathbb{E}[\|X(t)-X_\infty\|^2]=0.$$

Definition 3.1.3. If there exists a $\mathbb{P}$–null set Ω_0 such that for every $\omega\notin\Omega_0$ the following holds

$$\lim_{t\to\infty}X(t,\omega)=X_\infty(\omega),$$

then we say X converges almost surely (a.s.) to X_∞.

3.2 Stochastic Limit Relation

3.2.1 Mean Square Convergence

Let $B(t)=\{B_1(t),B_2(t),...,B_d(t)\}$ be a vector of mutually independent standard Brownian motions. For the definition of a stochastic integral in higher dimensions and the result corresponding to Itô's isometry we refer the reader to [84, Definition 1.5.20 and Theorem 1.5.21].

We consider the following hypotheses: let $\Delta\subset\mathbb{R}^2$ be defined by

$$\Delta=\{(t,s):0\leq s\leq t<+\infty\}$$

suppose that

$$H:\Delta\to\mathbb{R}^{n\times n} \text{ is continuous.} \tag{3.2.1}$$

We first characterise, for $f\in C([0,\infty);\mathbb{R}^{n\times d})$ with bounded norm, the convergence of the stochastic process $X_f=\{X_f(t):t\geq 0\}$ defined by

$$X_f(t)=\int_0^t H(t,s)f(s)\,dB(s), \quad t\geq 0$$

to a limit as $t\to\infty$ in *mean–square*.

Before discussing this convergence, we note that (3.2.1) is sufficient to guarantee that $X_f(t)$ is a well–defined random variable for each fixed t. Therefore the family of random variables $\{X_f(t) : t \geq 0\}$ is well–defined, and X_f is indeed a process. Condition (3.2.1) also guarantees that $\mathbb{E}[X_f(t)^2] < +\infty$ for each $t \geq 0$. Since $f \mapsto X_f$ is linear, and the family $(X_f(t))_{t\geq 0}$ is Gaussian for each fixed f, the limit should also be Gaussian and linear in f, as well as being an $\mathcal{F}^B(\infty)$–measurable random variable. Therefore, a reasonably general form of the limit should be

$$X_f^* := \int_0^\infty H_\infty(s) f(s)\, dB(s),$$

where we would expect H_∞ to be a function independent of f. For each fixed t the random variable $X_f(t)$ is $\mathcal{F}^B(t)$-adapted. In our first main result, we show that $X_f(t) \to X_f^*$ in mean square as $t \to \infty$ for each f.

Theorem 3.2.1. *Suppose that H obeys* (3.2.1). *Then the statements*

(A) There exists $H_\infty \in C([0,\infty);\mathbb{R}^{n\times n})$ such that $\int_0^\infty \|H_\infty(s)\|^2\, ds < +\infty$ and

$$\lim_{t\to\infty} \int_0^t \|H(t,s) - H_\infty(s)\|^2\, ds = 0. \tag{3.2.2}$$

(B) There exists $H_\infty \in C([0,\infty);\mathbb{R}^{n\times n})$ such that for each $f \in BC(\mathbb{R}_+;\mathbb{R}^{n\times d})$,

$$\lim_{t\to\infty} \mathbb{E}\left[\left\| \int_0^t H(t,s) f(s)\, dB(s) - \int_0^\infty H_\infty(s) f(s)\, dB(s) \right\|^2\right] = 0 \tag{3.2.3}$$

are equivalent.

In the deterministic admissibility theory, the assumptions for convergence are given in a different form from (3.2.2), c.f. e.g. Theorem A.1 from [14]. Our next result shows that the natural analogues of those assumptions are equivalent to (3.2.2).

Proposition 3.2.1. *Suppose that H obeys* (3.2.1). *Then the following are equivalent:*

(A) H obeys (3.2.2);

(B) There exists $H_\infty \in C([0,\infty);\mathbb{R}^{n\times n})$ such that

$$\lim_{T\to\infty} \limsup_{t\to\infty} \int_T^t \|H(t,s)\|^2\, ds = 0, \tag{3.2.4}$$

$$\lim_{t\to\infty} \int_0^T \|H(t,s) - H_\infty(s)\|^2\, ds = 0, \quad \textit{for every } T > 0. \tag{3.2.5}$$

3.2.2 Necessary Condition for Almost Sure Convergence

We now consider the almost sure convergence of $X_f(t)$ as $t \to \infty$ to a limit. Our next main result shows that if we have convergence in an a.s. sense, we must also have convergence in a mean square sense.

Theorem 3.2.2. *Suppose that H obeys (3.2.1) and there exists $H_\infty \in C([0,\infty);\mathbb{R}^{n\times n})$ such that for each $f \in BC([0,\infty);\mathbb{R}^{n\times d})$,*

$$\lim_{t\to\infty} \int_0^t H(t,s)f(s)\,dB(s) = \int_0^\infty H_\infty(s)f(s)\,dB(s), \quad a.s. \tag{3.2.6}$$

Then (3.2.2) and (3.2.3) hold.

Theorem 3.2.1 is concerned with moment behaviour of $X_f(t) = \int_0^t H(t,s)f(s)\,dB(s)$, indeed the continuity of these moments is guaranteed by the assumption (3.2.1). In Theorem 3.2.2 the condition (3.2.6) may implicitly impose continuity of the sample paths of X_f. The issue of continuous sample paths of X_f is addressed in Lemma 2.D. of [27]. Specifically, let H obey (3.2.1). Suppose that H obeys a Hölder continuity condition of the following form: there exists a function $K(s)$ and a constant $\alpha > 0$ such that

$$\int_0^T |K(s)|^2 ds < +\infty$$

and

$$|H(t_2,s) - H(t_1,s)| \le K(s)\,(t_2 - t_1)^\alpha, \quad \text{for } 0 \le s \le t_1 \le t_2 \le T. \tag{3.2.7}$$

Since H is continuous, it follows that there exist constants $\epsilon > 0$, $D > 0$ such that

$$\sup_{t\in[0,T]} \int_0^T |H(t,s)|^{2+\epsilon} ds \le D.$$

Lemma 2.D. of [27] now guarantees that a continuous version of

$$\int_0^t H(t,s)f(s)dB(s)$$

exists on $[0,T]$.

Remark 3.2.1. Therefore from Theorem 3.2.2 we have shown that (3.2.2) is a necessary condition for a.s. convergence. It is of course natural to then ask whether (3.2.2) is sufficient. We show by a simple example that in general additional conditions are needed in order for (3.2.6) to hold. It is further noted that the assumed continuity and structure of H is Examples 3.2.1 and 3.2.2 immediately gives the continuity of the sample paths of $\int_0^t H(t,s)f(s)dB(s)$, and that the sufficient condition (3.2.7) is not needed.

Example 3.2.1. Suppose that $H^\sharp : [0,\infty) \to \mathbb{R}$ and $H_\infty : [0,\infty) \to \mathbb{R}$ are continuous functions, and define

$$H(t,s) = H_\infty(s) + H^\sharp(t), \quad (t,s) \in \Delta.$$

Then H is continuous. Suppose also that $H_\infty \in L^2([0,\infty);\mathbb{R})$. We have that

$$\int_0^t (H(t,s) - H_\infty(s))^2 \, ds = tH^\sharp(t)^2.$$

By Theorem 3.2.1, it follows that

$$\lim_{t\to\infty} \sqrt{t} H^\sharp(t) = 0 \tag{3.2.8}$$

is a necessary and sufficient condition for (3.2.3). It is also a necessary condition for (3.2.6).

Since H_∞ is in $L^2([0,\infty);\mathbb{R})$, for each $f \in BC([0,\infty);\mathbb{R})$ we have that the integral $\int_0^t H_\infty(s) f(s)\, dB(s)$ tends almost surely as $t \to \infty$ to $\int_0^\infty H_\infty(s) f(s)\, dB(s)$. Therefore, we have that

$$\lim_{t\to\infty} \int_t^\infty H_\infty(s) f(s)\, dB(s) = 0, \quad \text{a.s.}$$

and thus

$$\begin{aligned}\int_0^t H(t,s) f(s)\, dB(s) &- \int_0^\infty H_\infty(s) f(s)\, dB(s) \\ &= H^\sharp(t) \int_0^t f(s)\, dB(s) - \int_t^\infty H_\infty(s) f(s)\, dB(s). \end{aligned} \tag{3.2.9}$$

Suppose that $H^\sharp$ obeys

$$\lim_{t\to\infty} \sqrt{t \log\log t}\, H^\sharp(t) = 0, \tag{3.2.10}$$

so that, in particular, $H^\sharp(t) \to 0$ as $t \to \infty$. If $f \in L^2([0,\infty);\mathbb{R})$, then $\int_0^t f(s)\, dB(s)$ tends to a finite limit a.s., and therefore both terms on the righthand side of (3.2.9) tend to zero as $t \to \infty$ a.s., and (3.2.6) holds.

On the other hand, if $f \notin L^2([0,\infty);\mathbb{R})$, then the martingale time change theorem and the Law of the Iterated Logarithm, c.f. e.g [104, Exercise 5.1.15], give that

$$\limsup_{t\to\infty} \frac{\left|\int_0^t f(s)\, dB(s)\right|}{\sqrt{2\int_0^t f^2(s)\, ds \log\log \int_0^t f^2(s)\, ds}} = 1, \quad \text{a.s.}$$

Since f is bounded, we have that

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t f^2(s)\, ds \le \limsup_{t\to\infty} f^2(t) =: f_*^2,$$

so we have

$$\limsup_{t\to\infty} \frac{\left|\int_0^t f(s)\,dB(s)\right|}{\sqrt{2t\log\log t}} \le f_*, \quad \text{a.s.}$$

Therefore, using this estimate and (3.2.10), we have $H^\sharp(t)\int_0^t f(s)\,dB(s) \to 0$ as $t\to\infty$ a.s., and we have that (3.2.6) holds.

Obviously the conditions (3.2.10) and (3.2.8) do not coincide; in fact, (3.2.10) implies (3.2.8). This provides an example of the veracity of Theorem 3.2.2 which can be verified independently of the general proof of that result.

We note also that it is very difficult to relax (3.2.10) and still have the integral $\int_0^t H(t,s)f(s)\,dB(s)$ tending to a limit a.s. as $t\to\infty$. Indeed, there exist functions $H^\sharp$ which do not obey (3.2.10), and so must satisfy

$$\limsup_{t\to\infty} \sqrt{t\log\log t}|H^\sharp(t)| > 0,$$

for which

$$\mathbb{P}\left[\lim_{t\to\infty}\int_0^t H(t,s)\,dB(s) \quad \text{exists}\right] = 0, \tag{3.2.11}$$

while at the same time we still have (3.2.3).

A choice of $H^\sharp$ which satisfies these conditions can readily be made. Consider a continuous function $H^\sharp$ which obeys $H^\sharp(n) = 1/\sqrt{n\log\log(n+2)}$ for all integers $n\ge 1$ but for which $\sqrt{t}H^\sharp(t)\to 0$ as $t\to\infty$ and $\limsup_{t\to\infty}\sqrt{t\log\log t}|H^\sharp(t)| < +\infty$.

By virtue of the fact that $H^\sharp$ obeys (3.2.8), we have that (3.2.3) holds. By the Law of the Iterated Logarithm, we have that

$$\limsup_{t\to\infty} |H^\sharp(t)B(t)| < +\infty, \quad \text{a.s.}$$

However,

$$\limsup_{t\to\infty} |H^\sharp(t)B(t)| \ge \limsup_{n\to\infty} |H^\sharp(n)||B(n)| = \sqrt{2}\limsup_{n\to\infty} \frac{|B(n)|}{\sqrt{2n\log\log(n+2)}} = \sqrt{2},$$

a.s., by the discrete version of the Law of the iterated logarithm, cf. e.g. [34, Theorem 10.2.1]. If $f(t)=1$ for all $t\ge 0$ in (3.2.9), we have

$$\int_0^t H(t,s)\,dB(s) - \int_0^\infty H_\infty(s)\,dB(s) = H^\sharp(t)B(t) - \int_t^\infty H_\infty(s)\,dB(s).$$

The second term on the righthand side has zero limit as $t\to\infty$ a.s., but by the above argument, the first term obeys

$$0 < \limsup_{t\to\infty} |H^\sharp(t)B(t)| < +\infty, \quad \text{a.s.}$$

and therefore (3.2.11) holds, as claimed.

The next example shows that sometimes the conditions which give mean square convergence and a.s. convergence are the same.

Example 3.2.2. Suppose that $H^\sharp : [0,\infty) \to \mathbb{R}$ and $H_\infty : [0,\infty) \to \mathbb{R}$ are continuous functions, and define

$$H(t,s) = H_\infty(s)H^\sharp(t), \quad (t,s) \in \Delta.$$

Then H is continuous. Suppose also that $H_\infty \in L^2([0,\infty);\mathbb{R})$. We have that

$$\int_0^t (H(t,s) - H_\infty(s))^2 \, ds = (H^\sharp(t) - 1)^2 \int_0^t H_\infty^2(s) \, ds.$$

Therefore, by Theorem 3.2.1, we have (3.2.3) if and only if

$$\lim_{t\to\infty} H^\sharp(t) = 1.$$

We know by Theorem 3.2.2 that this condition is also necessary for a.s. convergence.

To show that it is sufficient, suppose $f \in BC([0,\infty);\mathbb{R})$. Then, as $H_\infty \in L^2([0,\infty);\mathbb{R})$, we have that

$$\lim_{t\to\infty} \int_0^t H_\infty(s) f(s) \, dB(s) \quad \text{exists and is finite a.s., and}$$
$$\lim_{t\to\infty} \int_t^\infty H_\infty(s) f(s) \, dB(s) = 0, \quad \text{a.s.}$$

Therefore, we have the identity

$$\int_0^t H(t,s) f(s) \, dB(s) - \int_0^\infty H_\infty(s) f(s) \, dB(s)$$
$$= (H^\sharp(t) - 1) \int_0^t H_\infty(s) f(s) \, dB(s) - \int_t^\infty H_\infty(s) f(s) \, dB(s).$$

Since $H^\sharp(t) \to 1$ as $t \to \infty$, the limit as $t \to \infty$ of the righthand side is zero, and so we have (3.2.6). Therefore, the condition $H^\sharp(t) \to 1$ as $t \to \infty$ is necessary and sufficient both for (3.2.6) and for (3.2.3).

3.2.3 Sufficient Conditions for Almost Sure Convergence

We now investigate sufficient conditions for a.s. convergence for functions H which need not necessarily be of the form

$$H(t,s) = \sum_{j=1}^n H_j(s) H_j^\sharp(t), \quad (t,s) \in \Delta,$$

and which are covered by explicit and direct calculations in Examples 3.2.1 and 3.2.2. Firstly the scalar case is looked at, after which the multi–dimensional case follows.

Theorem 3.2.3. *Suppose that* $H \in C(\Delta;\mathbb{R})$*, and also that* $H \in C^{1,0}(\Delta;\mathbb{R})$*. Let* $H_\infty \in C([0,\infty);\mathbb{R}) \cap L^2([0,\infty);\mathbb{R})$*. Suppose also that*

$$\lim_{t\to\infty} \int_0^t (H(t,s) - H_\infty(s))^2\, ds \cdot \log t = 0, \tag{3.2.12}$$

and

There exists $q \geq 0$ *and* $c_q > 0$ *such that*

$$\int_0^t H_1(t,s)^2\, ds \leq c_q(1+t)^{2q}, \quad H(t,t)^2 \leq c_q(1+t)^{2q}. \tag{3.2.13}$$

Then

$$\lim_{t\to\infty} \int_0^t H(t,s)f(s)dB(s) = \int_0^\infty H_\infty(s)f(s)dB(s), \quad a.s.$$

for each $f \in BC([0,\infty);\mathbb{R})$*.*

Remark 3.2.2. We notice that (3.2.12) implies a given rate of decay to zero of $\int_0^t (H(t,s) - H_\infty(s))^2\, ds$ as $t \to \infty$. This strengthens the hypothesis (3.2.2) which is known, by Theorem 3.2.2, to be necessary.

Remark 3.2.3. The continuity of the sample paths of $\int_0^t H(t,s)f(s)dB(s)$ in Theorem 3.2.3 is assured by the derivative condition (3.2.13). Fix $T > 0$ and let $0 \leq s \leq t_1 \leq t_2 \leq T$. Then, as $H \in C^{1,0}(\Delta;\mathbb{R})$ by the Mean Value Theorem, we get

$$|H(t_2,s) - H(t_1,s)| = |H_1(t^*,s)|\,|t_2 - t_1|,$$

for some $t^* = t^*(s) \in [t_1,t_2]$. Next, define $K(s) := \sup_{t_1\leq t\leq t_2} |H_1(t,s)|$. This is well–defined and finite by the continuity of H_1. Therefore

$$|H(t_2,s) - H(t_1,s)| \leq K(s)\,|t_2 - t_1|, \text{ for all } 0 \leq s \leq t_1 \leq t_2 \leq T,$$

which is (3.2.7) with $\alpha = 1$. Note moreover that the continuity of $s \mapsto K(s)$ ensures that

$$\int_0^T |K(s)|^2\, ds < +\infty,$$

and therefore all the conditions of Berger and Mizel's continuity lemma are satisfied.

Remark 3.2.4. While the pointwise bound on $H(t,t)$ in (3.2.13) may appear quite mild, one may prefer an integral condition to this pointwise bound as this would allow for $H(t,t)$ to potentially have "thin spikes" of larger than polynomial order. Scrutiny of the proof of Theorem 3.2.3 reveals that the condition $H(t,t)^2 \leq c_q(1+t)^{2q}$ can be replaced by

$$\lim_{k\to\infty} \int_{k^\theta}^{(k+1)^\theta} H(s,s)^2\,ds \cdot \log k = 0, \qquad \text{for any } 0<\theta<1/(1+2q), \tag{3.2.14}$$

where the limit is taken through the integers. Condition (3.2.14) shall be used in the proof of Proposition 3.3.1 in preference to $H(t,t)^2 \leq c_q(1+t)^{2q}$. Nevertheless for simplicity we retain the condition on $H(t,t)$ in the statement of Theorem 3.2.3.

We give the multi–dimensional version of Theorem 3.2.3.

Theorem 3.2.4. *Suppose that H obeys* (3.2.1) *and also that $H \in C^{1,0}(\Delta;\mathbb{R}^{n\times n})$. Suppose also that there exists $H_\infty \in C([0,\infty);\mathbb{R}^{n\times n})$ such that $\int_0^\infty \|H_\infty(s)\|^2\,ds < +\infty$ and*

$$\lim_{t\to\infty} \int_0^t \|H(t,s)-H_\infty(s)\|^2\,ds \cdot \log t = 0, \tag{3.2.15}$$

and

There exists $q \geq 0$ and $c_q > 0$ such that

$$\int_0^t \|H_1(t,s)\|^2\,ds \leq c_q(1+t)^{2q}, \quad \|H(t,t)\|^2 \leq c_q(1+t)^{2q}. \tag{3.2.16}$$

Then H obeys (3.2.6)*.*

Proof. This proof relies upon the established sufficient conditions of Theorem 3.2.3 for almost sure convergence. As Theorem 3.2.3 applies to scalar valued functions we firstly see the implications of the norm conditions of Theorem 3.2.4 upon the elements of their respective matrices.

As $f \in BC([0,\infty);\mathbb{R}^{n\times d})$ each element of f is continuous. Also $f \in BC([0,\infty);\mathbb{R}^{n\times d})$ is equivalent to $\sup_{s\geq 0}\|f(s)\|_F^2 < +\infty$, then $f_{k,j} \in BC([0,\infty);\mathbb{R})$ for all $1 \leq k \leq n$, $1 \leq j \leq d$.

The condition $H(t,s) \in C^{1,0}([0,\infty);\mathbb{R})$ holds true element-wise. Also, $\|H_\infty\|_F \in L^2([0,\infty);\mathbb{R})$ clearly implies $H_{\infty_{i,k}} \in L^2([0,\infty);\mathbb{R})$ for all $1 \leq i \leq n$, $1 \leq k \leq n$.

Now (3.2.15) is equivalent to

$$\lim_{t\to\infty} \sum_{i=1}^n \sum_{k=1}^n \int_0^t \Big(H(t,s)_{i,k} - H_\infty(s)_{i,k}\Big)^2\,ds \cdot \log t = 0$$

which clearly implies

$$\lim_{t\to\infty}\int_0^t \left(H(t,s)_{i,k} - H_\infty(s)_{i,k}\right)^2 ds \cdot \log t = 0, \quad \text{for all } 1 \le i \le n,\ 1 \le k \le n. \tag{3.2.17}$$

Now $\int_0^t \|H_1(t,s)\|_F^2\, ds \le c_q(1+t)^{2q}$, is equivalent to

$$\sum_{i=1}^n \sum_{k=1}^n \int_0^t \left[\frac{\partial}{\partial t} H(t,s)\right]_{i,k}^2 ds \le c_q(1+t)^{2q}$$

which clearly implies

$$\int_0^t \left[\frac{\partial}{\partial t} H(t,s)\right]_{i,k}^2 ds \le c_q(1+t)^{2q}, \quad \text{for all } 1 \le i \le n,\ 1 \le k \le n. \tag{3.2.18}$$

Similarly, $\|H(t,t)\|_F^2\, ds \le c_q(1+t)^{2q}$, is equivalent to

$$\sum_{i=1}^n \sum_{k=1}^n H(t,t)_{i,k}^2 ds \le c_q(1+t)^{2q}$$

which clearly implies

$$H(t,t)_{i,k}^2 ds \le c_q(1+t)^{2q}, \quad \text{for all } 1 \le i \le n,\ 1 \le k \le n. \tag{3.2.19}$$

Hence, considering (3.2.17), (3.2.18),(3.2.19) and proceeding discussion regarding continuity of H and boundedness of f one may apply Theorem 3.2.3 to get

$$\lim_{t\to\infty}\int_0^t H(t,s)_{i,k} f(s)_{k,j} dB_j(s) = \int_0^\infty H_{\infty_{i,k}}(s) f(s)_{k,j} dB_j(s), \quad a.s.$$

for all $1 \le i \le n$, $1 \le k \le n$, $1 \le j \le d$. Thus,

$$\lim_{t\to\infty}\sum_{k=1}^n \sum_{j=1}^d \int_0^t H(t,s)_{i,k} f(s)_{k,j} dB_j(s) = \sum_{k=1}^n \sum_{j=1}^d \int_0^\infty H_{\infty_{i,k}}(s) f(s)_{k,j} dB_j(s), \quad a.s.$$

for all $1 \le i \le n$. Equivalently one may write

$$\lim_{t\to\infty}\left(\int_0^t H(t,s) f(s) dB(s)\right)_i = \left(\int_0^\infty H_{\infty(s)} f(s) dB(s)\right)_i, \quad \text{a.s.}$$

for all $1 \le i \le n$ which is (3.2.6). □

Remark 3.2.5. Analogous to Remark 3.2.4, the condition $\|H(t,t)\|^2 \le c_q(1+t)^{2q}$ in Theorem 3.2.4 may be replaced with

$$\lim_{k\to\infty}\int_{k^\theta}^{(k+1)^\theta} \|H(s,s)\|^2\, ds \cdot \log k = 0, \quad \text{for } 0 < \theta < 1/(1+2q), \tag{3.2.20}$$

with the limit taken through the integers.

3.3 Applications to affine equations

3.3.1 Asymptotic behaviour of a stochastic convolution integral

This section applies the theory of stochastic admissibility, developed in the Section 3.2, to SFDEs. We consider *Volterra linear SFDEs* and *linear SFDEs with finite delay.* These equations both have the form

$$dX(t) = \big(f(t) + L(X_t)\big)dt + \Sigma(t)\,dB(t), \quad t \geq 0,$$

where L is a linear functional, $\Sigma \in C(\mathbb{R}_+;\mathbb{R}^{d\times d'})$, $f \in C(\mathbb{R}_+;\mathbb{R}^d)$, B is a standard d'-dimensional Brownian vector and the solution X lies in $\mathbb{R}^d$. For any $y : \mathbb{R} \to \mathbb{R}^{d\times n}$ we define the *segment* $y_t : \mathbb{R} \to \mathbb{R}^{d\times n} : s \mapsto y(t+s)$ for any $n, d \in \mathbb{Z}^+$. An appropriate initial condition is also imposed. The associated deterministic equation is

$$x'(t) = L(x_t), \quad t \geq 0,$$

with the same initial value as the stochastic equation. Also defining the differential resolvent, r,

$$r'(t) = L(r_t), \quad t \geq 0, \qquad r(0) = I_d, \tag{3.3.1}$$

allows one to write the variation of parameters formula, for $t \geq 0$,

$$X(t) = x(t) + \int_0^t r(t-s)f(s)\,ds + \int_0^t r(t-s)\Sigma(s)\,dB(s).$$

The asymptotic behaviour of x and r is primarily known from the theory of deterministic linear differential equations and so one may now apply the admissibility theory of Section 3.2 to determine the asymptotic behaviour of the stochastic convolution integral, $\int_0^t r(t-s)\Sigma(s)\,dB(s)$, and hence of X, providing that the diffusion, Σ, does not grow too rapidly.

Proposition 3.3.1. *Let $\alpha \in \mathbb{R}$, N be some finite positive integer, $\{\beta_j\}_{j=1}^N$ be a sequence of some real constants and $(P_j)_{j=1}^N$ and $(Q_j)_{j=1}^N$ be sequences of $d \times d$ matrix–polynomials of degree n , for some positive integer n, and in particular*

$$P_j(t) = t^n P_j^* + O(t^{n-1}), \quad Q_j(t) = t^n Q_j^* + O(t^{n-1}).$$

where at least one of $P_j^*, Q_j^* \neq 0$ *for all* $j \in \{1, \dots, N\}$. *Suppose* R *is a.e. absolutely continuous and is defined such that it obeys, for some* $\epsilon > 0$, *the asymptotic estimates*

$$R(t) = \begin{cases} O(\mathrm{e}^{(\alpha-\epsilon)t}), & \text{if } n = 0 \\ O(\mathrm{e}^{\alpha t} t^{n-1}), & \text{if } n \geq 1 \end{cases}, \quad \text{as } t \to \infty, \tag{3.3.2}$$

$$R'(t) = \begin{cases} O(\mathrm{e}^{(\alpha-\epsilon)t}), & \text{if } n = 0 \\ O(\mathrm{e}^{\alpha t} t^{n}), & \text{if } n \geq 1 \end{cases}, \quad \text{as } t \to \infty. \tag{3.3.3}$$

and suppose that r *is given by*

$$r(t) = \sum_{j=1}^{N} \mathrm{e}^{\alpha t}\{P_j(t)\cos(\beta_j t) + Q_j(t)\sin(\beta_j t)\} + R(t), \quad t \geq 0. \tag{3.3.4}$$

Let $\Sigma \in C([0,\infty); \mathbb{R}^{d \times d'})$ *be continuous with*

$$\int_0^\infty e^{-2\alpha t}\|\Sigma(t)\|^2 \, dt < +\infty. \tag{3.3.5}$$

Let Y *be the process defined by*

$$Y(t) = \int_0^t r(t-s)\Sigma(s)\, dB(s), \quad t \geq 0, \quad Y(0) = 0. \tag{3.3.6}$$

Then

$$\lim_{t\to\infty}\left(\frac{Y(t)}{t^n \mathrm{e}^{\alpha t}} - \sum_{j=1}^{N}\{L_{1,j}\sin(\beta_j t) + L_{2,j}\cos(\beta_j t)\}\right) = 0, \quad a.s. \tag{3.3.7}$$

where

$$L_{1,j} := \int_0^\infty \mathrm{e}^{-\alpha s}\{P_j^*\sin(\beta_j s) + Q_j^*\cos(\beta_j s)\}\Sigma(s)\, dB(s), \tag{3.3.8a}$$

$$L_{2,j} := \int_0^\infty \mathrm{e}^{-\alpha s}\{P_j^*\cos(\beta_j s) - Q_j^*\sin(\beta_j s)\}\Sigma(s)\, dB(s). \tag{3.3.8b}$$

The square integrability, $L^2(0,\infty)$, of the noise term, i.e. (3.3.5), is a usual condition to have when dealing with stochastic terms. When ascertaining asymptotic behaviour of deterministic forcing functions it is more typical to require an absolute integrability condition, $L^1(0,\infty)$. This is indeed what is required in Corollary 3.3.1, i.e. (3.3.9). Proposition 3.3.1 is shown to be robust with respect to deterministic perturbations.

Corollary 3.3.1. *Let* $\alpha \in \mathbb{R}$, $N \in \mathbb{Z}^+/\{0\}$. *Let* $\{\beta_j\}_{j=1}^N, \{P_j\}_{j=1}^N, \{Q_j\}_{j=1}^N, r, R, \Sigma$ *and* Y *be as defined in Proposition 3.3.1, with* (3.3.5) *holding. Let* $f \in C([0,\infty), \mathbb{R}^d)$ *with*

$$\int_0^\infty \mathrm{e}^{-\alpha t}|f(t)|dt < +\infty. \tag{3.3.9}$$

Let V be the process defined by

$$V(t) = \int_0^t r(t-s)f(s)ds + Y(t), \quad t \geq 0, \quad V(0) = 0. \tag{3.3.10}$$

Then

$$\lim_{t\to\infty} \left(\frac{V(t)}{t^n e^{\alpha t}} - \sum_{j=1}^{N} \{M_{1,j}\sin(\beta_j t) + M_{2,j}\cos(\beta_j t)\} \right) = 0, \quad a.s.$$

where

$$M_{1,j} = L_{1,j} + \int_0^\infty e^{-\alpha s}\{P_j^* \sin(\beta_j s) + Q_j^* \cos(\beta_j s)\} f(s)\, ds,$$
$$M_{2,j} = L_{2,j} + \int_0^\infty e^{-\alpha s}\{P_j^* \cos(\beta_j s) - Q_j^* \sin(\beta_j s)\} f(s)\, ds$$

and where $L_{1,j}$ and $L_{2,j}$ are given by Proposition 3.3.1.

3.3.2 Preliminaries

Let $M(J, \mathbb{R}^{d\times d'})$ be the space of finite Borel measures on J with values in $\mathbb{R}^{d\times d'}$, where J shall be either $\mathbb{R}_+$ or $[-\tau, 0]$. The total variation of a measure ν in $M(J, \mathbb{R}^{d\times d'})$ on a Borel set $B \subseteq J$ is defined by

$$|\nu|(B) := \sup \sum_{i=1}^{N} |\nu(E_i)|,$$

where $(E_i)_{i=1}^N$ is a partition of B and the supremum is taken over all partitions. The total variation defines a positive scalar measure $|\nu|$ in $M(J, \mathbb{R})$. If one specifies temporarily the norm $|\cdot|$ as the l^1-norm on the space of real-valued sequences and identifies $\mathbb{R}^{d\times d'}$ by $\mathbb{R}^{dd'}$ one can easily establish for the measure $\nu = (\nu_{i,j})_{i,j=1}^{d,d'}$ the inequality

$$|\nu|(B) \leq C \sum_{i=1}^{d} \sum_{j=1}^{d'} |\nu_{i,j}|(B) \qquad \text{for every Borel set } B \subseteq \mathbb{R}_+ \tag{3.3.11}$$

with $C = 1$. Then, by the equivalence of every norm on finite-dimensional spaces, the inequality (3.3.11) holds true for the arbitrary norms $|\cdot|$ and some constant $C > 0$. Moreover, as in the scalar case we have the fundamental estimate

$$\left| \int_J \nu(ds)\, f(s) \right| \leq \int_J |\nu|(ds)\ |f(s)|$$

for every function $f : J \to \mathbb{R}^{d'\times d''}$ which is $|\nu|$-integrable.

3.3.3 Volterra linear functional equations

A Shea-Wainger theorem is developed in [19] which relates the location of the roots of a characteristic equation to the solution of a Volterra linear SFDE lying in a weighted L^p-space. We reproduce the set-up of those equations here.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a filtration $(\mathcal{F}(t))_{t\geq 0}$, and let $(B(t))_{t\geq 0}$ be a standard d'-dimensional Brownian motion on this probability space. Consider the stochastic integro-differential equation with stochastic perturbations of the form

$$\begin{aligned} dX(t) &= \left(f(t) + \int_{[0,t]} \mu(ds)\, X(t-s) \right) dt + \Sigma(t)\, dB(t) \quad \text{for } t \geq 0, \\ X(0) &= X_0, \end{aligned} \tag{3.3.12}$$

where μ is a measure in $M(\mathbb{R}_+, \mathbb{R}^{d\times d})$, $\Sigma \in C(\mathbb{R}_+; \mathbb{R}^{d\times d'})$, $f \in C(\mathbb{R}_+; \mathbb{R}^d)$. The initial condition X_0 is an $\mathbb{R}^d$-valued, $\mathcal{F}(0)$-measurable random variable with $\mathbb{E}\,|X_0|^2 < \infty$. The existence and uniqueness of a continuous solution X of (3.3.12) with $X(0) = X_0$ P-a.s. is covered in Berger and Mizel [27], for instance. Independently, the existence and uniqueness of solutions of stochastic functional equations was established in Itô and Nisio [69] and Mohammed [91].

The so-called *fundamental solution or resolvent of* (3.3.12) is the matrix-valued function $r : \mathbb{R}_+ \to \mathbb{R}^{d\times d}$, which is the unique solution of

$$r'(t) = \int_{[0,t]} \mu(ds)\, r(t-s) \quad \text{for } t \geq 0, \quad r(0) = \mathrm{I}_d\,. \tag{3.3.13}$$

Adapting Reiß, Riedle and van Gaans [103, Lemma 6.1] for deterministic perturbations gives that the solution X obeys the variation of constants formula for $t \geq 0$:

$$X(t) = r(t)X_0 + \int_0^t r(t-s)f(s)\, ds + \int_0^t r(t-s)\Sigma(s)\, dB(s) \quad P\text{-a.s.} \tag{3.3.14}$$

To see this, define w to be the unique solution of

$$w'(t) = f(t) + \int_{[0,t]} \mu(ds) w(t-s), \quad t \geq 0, \quad w(0) = 0.$$

Then $w(t) = \int_0^t r(t-s)f(s)ds$. Defining $Z(t) := X(t) - w(t)$ for $t \geq 0$ gives

$$dZ(t) = \int_{[0,t]} \mu(ds) Z(t-s) + \Sigma(t)dB(t), \quad t \geq 0, \quad Z(0) = X_0.$$

Hence [103, Lemma 6.1] gives

$$Z(t) = r(t)X_0 + \int_0^t r(t-s)\Sigma(s)dB(s), \quad P\text{-a.s.} \quad t \geq 0,$$

which rearranges to yield (3.3.14).

Define

$$\alpha^* = \inf\{a \in \mathbb{R} : \int_{[0,\infty)} \mathrm{e}^{-as}|\mu|(ds) \text{ is well–defined and finite}\}. \tag{3.3.15}$$

Then the function $h_\mu : \mathbb{C} \to \mathbb{C}$ defined by

$$h_\mu(\lambda) = \det\left(\lambda I_d - \int_{[0,\infty)} \mathrm{e}^{-\lambda s}\mu(ds)\right)$$

is well–defined for $\Re(\lambda) > \alpha^*$.

Define also the set

$$\Lambda = \{\lambda \in \mathbb{C} : h_\mu(\lambda) = 0\}.$$

The function h_μ is analytic, and so the elements of Λ are isolated. Define

$$\alpha := \sup\{\Re(\lambda) : h_\mu(\lambda) = 0\}. \tag{3.3.16}$$

It is always the case that such an α is finite, we assume however that $\alpha^* < \alpha$. Because the solution r obeys an exponentially growing or decaying upper bound, this is equivalent to assuming that there exists $\lambda \in \mathbb{C}$ with $\Re(\lambda) > \alpha^*$ for which $h_\mu(\lambda) = 0$.

With the assumption $\alpha^* < \alpha$, there exists $\delta \in (0, \alpha - \alpha^*)$. By the Riemann–Lebesgue lemma, cf. e.g. [53, Theorem. 2.2.7 (i)], for such a $\delta > 0$ there exists $M = M(\delta) > 0$ such that $h_\mu(\lambda) \neq 0$ for all $\lambda \in \mathbb{C}$ such that $\alpha^* < \alpha - \delta \leq \Re(\lambda) \leq \alpha + \delta$ and $|\Im(\lambda)| \geq M(\delta)$. If $K = \{\lambda \in \mathbb{C} : 0 < |\Re(\lambda) - \alpha| < \delta, |\Im(\lambda)| \leq M(\delta)\}$, the fact that h_μ is analytic ensures that there are at most finitely many zeros of h_μ in K. Therefore, there exists a minimal $\varepsilon \in (0, \delta]$ such that $h_\mu(\lambda) \neq 0$ for all $\alpha - \varepsilon \leq \Re(\lambda) < \alpha$, and therefore there exists $\delta' = \alpha - \varepsilon$ such that $h_\mu(z) \neq 0$ for all $\Re(z) = \delta'$. Define $\varphi(t) = e^{-\delta' t}$ for $t \in \mathbb{R}$. Then φ is a submultiplicative weight function on $\mathbb{R}$ for which $\omega_\varphi = \alpha_\varphi = \delta' = \alpha - \varepsilon$. Define $\Lambda_\varepsilon = \{\lambda \in \Lambda : \Re(\lambda) > \alpha - \varepsilon\}$. Clearly Λ_ε is a set with only finitely many elements, as is $\Lambda' = \{\lambda \in \Lambda : \Re(\lambda) = \alpha\}$. Then by Theorem 7.2.1 in [53], there exists an a.e. absolutely continuous function q such that $q, q' \in L^1(\mathbb{R}_+; \varphi; \mathbb{R}^{d\times d})$ and

$$r(t) = \sum_{\lambda_j \in \Lambda_\varepsilon, \Im(\lambda_j) \geq 0} e^{\alpha_j t}\{P_j(t)\cos(\beta_j t) + Q_j(t)\sin(\beta_j t)\} + q(t), \quad t \geq 0. \tag{3.3.17}$$

where $\Re(\lambda_j) = \alpha_j$ and $\Im(\lambda_j) = \beta_j$, and where P_j and Q_j are matrix–valued polynomials of degree n_j, with $n_j + 1$ being the order of the pole $\lambda_j = \alpha_j + i\beta_j$ of $[h_\mu]^{-1}$. We remark that n_j (the ascent of λ_j) is less than or equal to the multiplicity of the zero λ_j of h_μ.

Let n denote the highest degree of all polynomials associated with roots in Λ' and let $\lambda_1, ..., \lambda_N$ be the finitely many roots in Λ' which have associated polynomials of this degree and have $\Im(\lambda_j) = \beta_j \geq 0$. We associate with each such $\lambda_j = \alpha + i\beta_j$ the matrix polynomials P_j and Q_j of degree n in (3.3.17). Therefore we may write

$$P_j(t) = t^n P_j^* + O(t^{n-1}), \quad Q_j(t) = t^n Q_j^* + O(t^{n-1}). \tag{3.3.18}$$

where at least one of P_j^* and Q_j^* are not equal to the zero matrix, for each $j \in \{1, ..., N\}$. The precise values of P_j^* and Q_j^* can be determined from the Laurent series of the inverse of the characteristic function, h_μ, expanded about λ_j, i.e.

$$\left[\lambda I_d - \int_{[0,\infty)} \mathrm{e}^{-\lambda s}\mu(ds)\right]^{-1} = \sum_{m=0}^{n} \frac{m!\, K_{j,m}}{(\lambda - \lambda_j)^{m+1}} + \hat{q}_j(\lambda), \tag{3.3.19}$$

where the remainder term $\hat{q}_j(\lambda)$ is analytic at λ_j. If λ_j is real then $P_j^* = K_{j,n}$, otherwise $P_j^* := 2\,\Re(K_{j,n})$ and $Q_j^* := -2\,\Im(K_{j,n})$. We note that (3.3.19) defines the value of n.

Define

$$R(t) = r(t) - \sum_{j=1}^{N} \mathrm{e}^{\alpha t}\{P_j(t)\cos(\beta_j t) + Q_j(t)\sin(\beta_j t)\}, \quad t \geq 0. \tag{3.3.20}$$

Then R is a.e. absolutely continuous. We determine asymptotic properties of R and R'.

Lemma 3.3.1. *Let R be defined by* (3.3.20)*. Suppose that α^* and α, defined by* (3.3.15) *and* (3.3.16) *respectively, obey $\alpha^* < \alpha$. Then there exists $\varepsilon \in (0, \alpha - \alpha^*)$ such that*

(i) If $n = 0$, then $R(t) = O(e^{(\alpha-\varepsilon)t})$ as $t \to \infty$.

(ii) If $n = 0$, then $R'(t) = O(e^{(\alpha-\varepsilon)t})$ as $t \to \infty$.

(iii) If $n \geq 1$, then $R(t) = O(t^{n-1}e^{\alpha t})$ as $t \to \infty$.

(iv) If $n \geq 1$, then $R'(t) = O(t^n \mathrm{e}^{\alpha t})$, as $t \to \infty$.

3.3.4 Finite delay linear functional equations.

The exact rate of growth of the running maxima of solutions of affine SFDEs with finite memory is discussed in [16] We reproduce the set-up of those equations here.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a filtration $(\mathcal{F}(t))_{t\geq 0}$, and let $(B(t))_{t\geq 0}$ be a standard d'-dimensional Brownian motion on this probability space. Consider the stochastic integro-differential equation of the form

$$\begin{aligned} dX(t) &= \left(f(t) + \int_{[-\tau,0]} \nu(ds)\, X(t+s) \right) dt + \Sigma(t)\, dB(t) \quad \text{for } t \geq 0, \\ X(t) &= \phi(t), \quad t \in [-\tau, 0], \end{aligned} \tag{3.3.21}$$

where ν is a measure in $M([-\tau,0], \mathbb{R}^{d\times d})$, $\Sigma \in C(\mathbb{R}_+; \mathbb{R}^{d\times d'})$, $f \in C(\mathbb{R}_+; \mathbb{R}^d)$. For every $\phi \in C([-\tau,0], \mathbb{R}^d)$ there exists a unique, adapted strong solution $(X(t,\phi) : t \geq -\tau)$ with finite second moments of (3.3.21) (cf., e.g., Mao [84]). The dependence of the solution on the initial condition ϕ is neglected in our notation in what follows; that is, we will write $X(t) = X(t,\phi)$ for the solution of (3.3.21).

Turning our attention to the deterministic equation in $\mathbb{R}^d$ underlying (3.3.21). For fixed constant $\tau \geq 0$:

$$x'(t) = \int_{[-\tau,0]} \nu(ds)\, x(t+s) \quad \text{for } t \geq 0, \quad x(t) = \phi(t) \quad t \in [-\tau, 0]. \tag{3.3.22}$$

For every $\phi \in C([-\tau,0], \mathbb{R}^d)$ there is a unique $\mathbb{R}^d$-valued function $x = x(\cdot,\phi)$ which satisfies (3.3.22).

The so-called *fundamental solution or resolvent of* (3.3.21) is the matrix-valued function $r : \mathbb{R}_+ \to \mathbb{R}^{d\times d}$, which is the unique solution of

$$r'(t) = \int_{[\max\{-\tau,-t\},0]} \nu(ds)\, r(t+s) \quad \text{for } t \geq 0, \quad r(0) = \mathrm{I}_d\,. \tag{3.3.23}$$

For convenience one could set $r(t) = 0_{d,d}$ for $t \in [-\tau, 0)$.

The solution $x(\cdot,\phi)$ of (3.3.22) for an arbitrary initial segment ϕ exists, is unique, and can be represented as

$$x(t,\phi) = r(t)\phi(0) + \int_{-\tau}^{0} \int_{[-\tau,u]} \nu(ds) r(t+s-u)\phi(u) du, \qquad \text{for } t \geq 0;$$

cf. Diekmann et al. [43, Chapter I].

By Reiß, Riedle and van Gaans [103, Lemma 6.1] the solution $(X(t) : t \geq -\tau)$ obeys a variation of constants formula:

$$X(t) = \begin{cases} x(t) + \int_0^t r(t-s) f(s)\, ds + \int_0^t r(t-s)\Sigma(s)\, dB(s), & t \geq 0, \\ \phi(t), & t \in [-\tau, 0]. \end{cases} \tag{3.3.24}$$

The process X defined by (3.3.24) obeys (3.3.21) pathwise on an almost sure event.

Define the function $g_\nu : \mathbb{C} \to \mathbb{C}$ by

$$g_\nu(\lambda) = \det\left(\lambda I_d - \int_{[-\tau,0]} \mathrm{e}^{\lambda s}\nu(ds)\right)$$

and also the set of its zeros

$$\Lambda = \{\lambda \in \mathbb{C} : g_\nu(\lambda) = 0\}.$$

The function g_ν is analytic, and so the elements of Λ are isolated. Define

$$\alpha := \sup\{\Re(\lambda) : g_\nu(\lambda) = 0\}. \tag{3.3.25}$$

Once again α is finite. Furthermore the cardinality of $\Lambda' = \{\Re(\lambda) = \alpha : \lambda \in \Lambda\}$ is finite. Then, following a similar argument as in Subsection 3.3.3, there exists $\varepsilon_0 > 0$ such that $g_\nu(\lambda) \neq 0$ for $\alpha - \varepsilon_0 \leq \Re(\lambda) < \alpha$ and hence $g_\nu(\lambda) \neq 0$ on the line $\Re(\lambda) = \varepsilon$ for every $\varepsilon \in (0, \varepsilon_0)$. Thus we have

$$r(t)e^{-\alpha t} = \sum_{\lambda_j \in \Lambda', \Im(\lambda_j) \geq 0} \left(\tilde{P}_j(t)\cos(\beta_j)t) + \tilde{Q}_j(t)\sin(\beta_j)t)\right) + o(e^{-\varepsilon t}),\ t \to \infty, \tag{3.3.26}$$

where $\Re(\lambda_j) = \alpha$ and $\Im(\lambda_j) = \beta_j$, and where $\tilde{P}_j$ and $\tilde{Q}_j$ are matrix–valued polynomials of degree n_j, with $n_j + 1$ being the order of the pole $\lambda_j = \alpha + i\beta_j$ of $[g_\nu]^{-1}$. This is a restatement of Diekmann et al [43, Theorem 5.4].

Let n denote the highest degree of all polynomials associated with roots in Λ' and let $\lambda_1, ..., \lambda_N$ be the finitely many roots in Λ' which have associated polynomials of this degree and have $\Im(\lambda_j) = \beta_j \geq 0$. We associate with each characteristic root $\lambda_j = \alpha + i\beta_j$ the matrix polynomials P_j and Q_j in (3.3.26) above, each of which has degree n. Therefore we may write

$$P_j(t) = t^n P_j^* + O(t^{n-1}), \quad Q_j(t) = t^n Q_j^* + O(t^{n-1}). \tag{3.3.27}$$

where at least one of P_j^* and Q_j^* are not equal to the zero matrix, for each $j \in \{1, ..., N\}$. The precise values of P_j^* and Q_j^* can be determined from the Laurent series of the inverse of the characteristic function, g_ν, expanded about λ_j, c.f. [43, pp.31] i.e.

$$\left[\lambda I_d - \int_{[-\tau,0]} \mathrm{e}^{\lambda s}\nu(ds)\right]^{-1} = \sum_{m=0}^{n} \frac{m!\, K_{j,m}}{(\lambda - \lambda_j)^{m+1}} + \hat{q}_j(\lambda), \tag{3.3.28}$$

where the remainder term $\hat{q}_j(\lambda)$ is analytic at λ_j. If λ_j is real then $P_j^* = K_{j,n}$, otherwise $P_j^* := 2\,\Re(K_{j,n})$ and $Q_j^* := -2\,\Im(K_{j,n})$. We note that (3.3.28) defines the value of n.

Define

$$R(t) = r(t) - \sum_{j=1}^{N} e^{\alpha t}\{P_j(t)\cos(\beta_j t) + Q_j(t)\sin(\beta_j t)\}, \quad t \geq 0. \tag{3.3.29}$$

Then R is a.e. absolutely continuous. We determine asymptotic properties of R and R'.

Lemma 3.3.2. *Let R be defined by* (3.3.29)*. Suppose that α is as defined by* (3.3.25)*. Then there exists $\varepsilon > 0$ such that*

(i) If $n = 0$, then $R(t) = O(e^{(\alpha-\varepsilon)t})$ as $t \to \infty$.

(ii) If $n = 0$, then $R'(t) = O(e^{(\alpha-\varepsilon)t})$ as $t \to \infty$.

(iii) If $n \geq 1$, then $R(t) = O(t^{n-1}e^{\alpha t})$ as $t \to \infty$.

(iv) If $n \geq 1$, then $R'(t) = O(t^n e^{\alpha t})$, as $t \to \infty$.

Remark 3.3.1. We observe that the differential resolvent of (3.3.23) may be regarded as the solution of a Volterra equation. Define $\nu_+(E) = \nu(-E)$ where $-E = \{x : -x \in E\}$ for all sets E which are subsets of the Borel sets formed from the interval $[0, \tau]$ and $\nu_+(E) = 0$ for all sets E which are subsets of the Borel sets formed from the interval (τ, ∞). Then

$$r'(t) = \int_{[0,\tau]} \nu_+(ds)r(t-s) \text{ for } t \geq 0, \quad r(0) = I_d.$$

For $t > \tau$,

$$\begin{aligned} r'(t) &= \int_{[0,t]} \nu_+(ds)r(t-s) - \int_{(\tau,t]} \nu_+(ds)r(t-s) \\ &= \int_{[0,t]} \nu_+(ds)r(t-s) \end{aligned}$$

as $\nu_+ = 0$ in the second term on the right–hand side. On the other hand, for $0 \leq t \leq \tau$, it is true that $\max\{-\tau, -t\} = -t$ and hence

$$r'(t) = \int_{[0,t]} \nu_+(ds)r(t-s) \quad \text{for } t \geq 0, \quad r(0) = I_d.$$

We will use this fact in the proof of Lemma 3.3.2.

3.3.5 Main Results

We now state the main results for the Volterra equation and affine SFDE with finite memory

Theorem 3.3.1. *Let α^* and α, as defined by* (3.3.15) *and* (3.3.16) *respectively, obey $\alpha^* < \alpha$. Let n be given by* (3.3.19) *(i.e. $n+1$ denotes the highest order of all roots in $\Lambda'' = \Lambda \cap \{\Re(\lambda) = \alpha, \Im(\lambda) \geq 0\}$) and let $(\lambda_j)_{j=1}^N$ be the finitely many roots in Λ'' with this order. Define $\beta_j = \Im(\lambda_j)$, $j = 1, \dots, N$. Suppose that P_j^*, Q_j^* for $j = 1, \dots, N$ are given by* (3.3.18). *Let $f \in C([0,\infty); \mathbb{R}^d)$ be such that*

$$\int_0^\infty \mathrm{e}^{-\alpha t} |f(t)|\, dt < +\infty \tag{3.3.30}$$

and let $\Sigma \in C([0,\infty); \mathbb{R}^{d \times d'})$ be such that

$$\int_0^\infty e^{-2\alpha t} \|\Sigma(t)\|^2 \, dt < +\infty. \tag{3.3.31}$$

Let X be the unique solution of (3.3.12). *Then*

$$\lim_{t\to\infty} \left(\frac{X(t)}{t^n \mathrm{e}^{\alpha t}} - \sum_{j=1}^N \{ (Q_j^* X_0 + M_{1,j}) \sin(\beta_j t) + (P_j^* X_0 + M_{2,j}) \cos(\beta_j t) \} \right) = 0, \quad a.s. \tag{3.3.32}$$

where $M_{1,j}$ and $M_{2,j}$ are given by Corollary 3.3.1.

Observe that from the conclusions of Lemma 3.3.1, R and R' of (3.3.20) obey equations (3.3.2) and (3.3.3). Also a rearrangement of r given by (3.3.20) yields the form of (3.3.4). Thus, the proof of Theorem 3.3.1 is an immediate consequence of Lemma 3.3.1, Corollary 3.3.1 and Remark 3.3.5 and so is omitted.

Remark 3.3.2. The condition $\alpha^* < \alpha$ is imposed as in order to apply Theorem 7.2.1 of [53], it is needed that the Laplace transform of μ in h_μ is well–defined over an open region of the complex plane which contains the critical line $\Re(\lambda) = \alpha$. Theorem 7.2.1 of [53] then allows one to conclude the asymptotic behaviour of the deterministic resolvent, (3.3.17). This condition is also required in determining the asymptotic behaviour of the remainder term R of (3.3.20).

In the case that $\alpha^* = \alpha$ (i.e. the line on which lie the zeros of h with largest real part co–incides with the boundary of the region of existence of the Laplace transform of $|\mu|$), then the deterministic theory differs to that as describes by Theorem 7.2.1 of [53]. The asymptotic behaviour in this case is examined in great depth in Jordan et al. [70], Kriszten and Terjéki [77] and Miller [88]. In particular, in order to apply successfully our stochastic admissibility results, we need good asymptotic information about both the resolvent and its derivative. For the cases covered here, existing deterministic results for the resolvent suffice, but new work has been required, and is supplied, for the derivative. Thus, in this case the stochastic theory as described by Theorem 3.3.1 would not necessarily hold.

Some articles which examine the case when the line containing the *leading* characteristic exponents of the characteristic equation co–incides with the boundary of the domain of the transform of the measure are e.g. [53, Chapter 7.3], [77] for deterministic theory and [11], [12], for stochastic theory.

The corresponding result for the affine SFDE is as follows.

Theorem 3.3.2. *Let α be as defined by* (3.3.25). *Let n be given by* (3.3.19) *(i.e. $n+1$ denotes the highest order of all roots in $\Lambda'' = \Lambda \cap \{\Re(\lambda) = \alpha, \Im(\lambda) \geq 0\}$) and let $(\lambda_j)_{j=1}^N$ be the finitely many roots in Λ'' with this order. Define $\beta_j = \Im(\lambda_j)$, $j = 1, \ldots, N$. Suppose that P_j^*, Q_j^* for $j = 1, \ldots, N$ are given by* (3.3.27). *Let $f \in C([0,\infty);\mathbb{R}^d)$ be such that*

$$\int_0^\infty \mathrm{e}^{-\alpha t}|f(t)|\,dt < +\infty$$

and let $\Sigma \in C([0,\infty);\mathbb{R}^{d\times d'})$ be such that

$$\int_0^\infty e^{-2\alpha t}\|\Sigma(t)\|^2\,dt < +\infty. \tag{3.3.33}$$

Let X be the unique solution of (3.3.21). *Then*

$$\lim_{t\to\infty}\left(\frac{X(t)}{t^n \mathrm{e}^{\alpha t}} - \sum_{j=1}^N \{J_{1,j}\sin(\beta_j t) + J_{2,j}\cos(\beta_j t)\}\right) = 0, \quad a.s. \tag{3.3.34}$$

where

$$\begin{aligned}
J_{1,j} &= Q_j^*\phi(0) + G_{1,j} + M_{1,j}, \quad J_{2,j} = P_j^*\phi(0) + G_{2,j} + M_{2,j},\\
G_{1,j} &= \int_{-\tau}^0 \int_{[-\tau,u]} \mathrm{e}^{\alpha u}\nu(ds)\{Q_j^*\cos(\beta_j u) - P_j^*\sin(\beta_j u)\}\phi(s-u)du,\\
G_{2,j} &= \int_{-\tau}^0 \int_{[-\tau,u]} \mathrm{e}^{\alpha u}\nu(ds)\{P_j^*\cos(\beta_j u) + Q_j^*\sin(\beta_j u)\}\phi(s-u)du,
\end{aligned}$$

and where $M_{1,j}$ and $M_{2,j}$ are given by Corollary 3.3.1.

Observe that from the conclusions of Lemma 3.3.2, R and R' of (3.3.29) obey equations (3.3.2) and (3.3.3). Also a rearrangement of r given by (3.3.29) yields the form of (3.3.4). Thus, the proof of Theorem 3.3.2 is an immediate consequence of Lemma 3.3.2, Corollary 3.3.1 and Remark 3.3.5 and so is omitted.

Remark 3.3.3. Theorem 3.3.2 differs from Theorem 3.3.1 with respect to the region of existence of the characteristic equation g_ν, i.e. $\int_{[-\tau,0]} e^{as}|\nu|(ds)$ exists for all $a \in (-\infty, \infty)$ and thus the condition $\alpha^* < \alpha$, present in Theorem 3.3.1, has no analogue in Theorem 3.3.2.

Remark 3.3.4. While Theorems 3.3.1 and 3.3.2 give a rate of growth or decay in an almost sure sense, it is observed, via Theorem 3.2.2, that this convergence also holds in mean square. That is, for the solution of the Volterra equation (3.3.12), with the assumptions of Theorem 3.3.1,

$$\lim_{t\to\infty} \mathbb{E}\left[\left\|\frac{X(t)}{t^n e^{\alpha t}} - \sum_{j=1}^{N}\{(Q_j^* X_0 + M_{1,j})\sin(\beta_j t) + (P_j^* X_0 + M_{2,j})\cos(\beta_j t)\}\right\|^2\right] = 0.$$

Also, for the solution of the finite delay equation (3.3.21), under the assumptions of Theorem 3.3.2,

$$\lim_{t\to\infty} \mathbb{E}\left[\left\|\frac{X(t)}{t^n e^{\alpha t}} - \sum_{j=1}^{N}\{J_{1,j}\sin(\beta_j t) + J_{2,j}\cos(\beta_j t)\}\right\|^2\right] = 0.$$

Remark 3.3.5. The asymptotic behaviour of the deterministic functional differential equations (3.3.13) or (3.3.23), each of which obey

$$\lim_{t\to\infty}\left(\frac{r(t)}{t^n e^{\alpha t}} - \sum_{j=1}^{N}\{Q_j^*\sin(\beta_j t) + P_j^*\cos(\beta_j t)\}\right) = 0, \tag{3.3.35}$$

where P_j^* and Q_j^* are determined by (3.3.18) (in the case of the Volterra equation) and (3.3.27) (for the equation with finite delay) is analogous to the asymptotic behaviour of X as given by (3.3.32) and (3.3.34) respectively.

It can therefore be seen, despite the presence of the stochastic integral, that X inherits the asymptotic behaviour of r, provided that the intensity of the noise perturbation does not grow too rapidly.

Regarding the multipliers of the trigonometric terms we remark that $M_{1,j}$ and $M_{2,j}$ are Gaussian distributed random variables and hence their values and, in particular, sign will depend upon the sample path. Moreover these random variables depend on the coefficients of the trigonometric terms in (3.3.35) i.e. P_j^* and Q_j^*.

Remark 3.3.6. The conditions (3.3.5) and (3.3.9) on the growth of Σ and f are, in some sense, unimprovable if the asymptotic behaviour of X is to be recovered.

Consider, for example, the scalar ordinary affine stochastic equation

$$dX(t) = \big(\alpha X(t) + f(t)\big)\,dt + \Sigma(t)\,dB(t), \quad t \geq 0, \quad X(0) = X_0 \in \mathbb{R},$$

where $\alpha \in \mathbb{R}$, $\Sigma \in C([0,\infty);\mathbb{R})$ and f is a non–negative function, i.e. $f \in C([0,\infty);[0,\infty))$. Then we have the following equivalent conditions:

(i) (3.3.5) and (3.3.9) hold.

(ii) There exists an a.s. finite random variable L such that

$$\mathbb{P}\left[\lim_{t\to\infty} \mathrm{e}^{-\alpha t} X(t) = L \in (-\infty,\infty)\right] > 0. \tag{3.3.36}$$

(iii) There exists an a.s. finite random variable L such that

$$\lim_{t\to\infty} \mathrm{e}^{-\alpha t} X(t) = L, \quad \text{a.s.} \tag{3.3.37}$$

The proof of Remark 3.3.6 is deferred to Section 3.6.

Remark 3.3.7. The asymptotic behaviour of the solution of (3.3.21) in the case when $\alpha < 0$ and the diffusion coefficient is time independent, i.e. $\Sigma(t) = \Sigma \in \mathbb{R}^{d\times d'}$ for all $t \geq 0$, is considered in [16]. It is argued that asset prices in financial markets fluctuate and therefore it is of interest to describe the order of the oscillations about the mean in particular the rate of growth of the running maximum of this asset price. In this case the resolvent function decays exponentially to zero resulting in the process X behaving asymptotically like a Gaussian process. Specifically, it is shown that

$$\limsup_{t\to\infty} \frac{|X(t)|_\infty}{\sqrt{2\log t}} = \max_{i=1,\dots,d} \sqrt{\sum_{k=1}^{m} \left(r(s)\Sigma\right)^2_{i,k} ds}, \quad a.s.$$

However for constant coefficient of diffusion, condition (3.3.33) is violated and hence Theorem 3.3.2 does not apply.

Remark 3.3.8. The asymptotic behaviour of the solution of the scalar equation (3.3.21), with $d = 1$, is considered in [20] with $\alpha \geq 0$, the zero of g which has this real part is a simple real zero and all other zeros of g have real parts less than α. Thus [20, Theorem 3.1 (b)], which considers the case of $\alpha > 0$, is a special case of Theorem 3.3.2. Moreover, as in practice it is quite difficult to determine the zeroes of g a subclass of measures is looked at which give the desired properties on the zeroes of g. Also, the economic interpretations of these impositions are discussed. To summarise the results: it is shown that if $\alpha = 0$ then the market behaves similar to a Black-Scholes model, in particular X undergoes fluctuations according to the law of the iterated logarithm.

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = -\liminf_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = C_1,$$

where C_1 is a positive constant. On the other hand, the case $\alpha > 0$ gives

$$\lim_{t\to\infty} \mathrm{e}^{-\alpha t} X(t) = C_2,$$

where C_2 is a random variable. This regime is interpreted as the market undergoing a bubble or crash, depending upon the sign of C_2, with both events being possible.

However the case $\alpha = 0$ studied in [20] also has a constant diffusion coefficient, thus (3.3.33) is not satisfied and so Theorem 3.3.2 does not apply.

3.3.6 Examples

We give some illustrative examples of Theorems 3.3.1 and 3.3.2 and Proposition 3.3.1. The first three examples consider the situation where the resolvent is of the especially simple form

$$\mu(ds) = A\delta_0(ds),$$

where A is a $d \times d$ matrix with real entries. In this case, the resolvent is nothing other than the principal matrix solution

$$r'(t) = Ar(t), \quad r(0) = I_d$$

and the stochastic equation is just the affine stochastic differential equation

$$dX(t) = AX(t)\,dt + \Sigma(t)\,dB(t), \quad t \geq 0; \quad X(0) = \xi.$$

Since there are no more than d eigenvalues, the resolvent r and its derivative can be expressed as finite sums, and so there is no need for a detailed analysis of remainder terms.

Our first example looks at the case when the leading eigenvalue (or zero of the characteristic equation) has algebraic multiplicity equal to the geometric multiplicity.

Example 3.3.1. Suppose that $A = \gamma I$ where I is the 2×2 identity matrix. Then $Y(t) = \mathrm{e}^{-\gamma t} X(t)$ obeys $dY(t) = \mathrm{e}^{-\gamma t}\Sigma(t)\,dB(t)$, so

$$Y(t) = \xi + \int_0^t \mathrm{e}^{-\gamma s}\Sigma(s)\,dB(s), \quad t \geq 0.$$

In this case, applying our results to Y, we have $\alpha = 0$. If $s \mapsto \mathrm{e}^{-\gamma s}\Sigma(s) \in L^2(0,\infty)$, by the martingale convergence theorem we have

$$\lim_{t\to\infty} \frac{X(t)}{\mathrm{e}^{\gamma t}} = \lim_{t\to\infty} Y(t) = \xi + \int_0^\infty \mathrm{e}^{-\gamma s}\Sigma(s)\,dB(s), \quad \text{a.s.}$$

Let $\lambda_j = 0$. Since $A - \gamma I = 0$, we see that, with $n = 0$, $K_{j,0} = I$ and $\hat{q}_j(\lambda) = 0$, we have

$$(\lambda I - (A - \gamma I_2))^{-1} = \lambda^{-1} I = \sum_{m=0}^{n} \frac{m!\, K_{j,m}}{\lambda^{m+1}} + \hat{q}_j(\lambda).$$

Thus, we may set $P_j^* = I$, and therefore the limit for Y has the form predicted by Theorem 3.3.2 with $\alpha = 0$.

We now demonstrate the resulting asymptotic behaviour of the solution of the stochastic equation when the leading eigenvalue has geometric multiplicity less than the algebraic multiplicity.

Example 3.3.2. Suppose that

$$A = \begin{pmatrix} \gamma & 1 \\ 0 & \gamma \end{pmatrix}.$$

Consider $Y(t) = \mathrm{e}^{-\gamma t} X(t)$. Then

$$dY(t) = (A - \gamma I) Y(t)\, dt + \mathrm{e}^{-\gamma t} \Sigma(t)\, dB(t).$$

Then, applying our theory to Y, we find that $\alpha = 0$, because $\lambda = 0$ is an eigenvalue of multiplicity 2. In this case r is given by

$$r(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Since $\det(r(t)) = 1$ for all $t \geq 0$, $r(t)$ is invertible, and we may write $r(t-s) = r(t) r^{-1}(s) = r(t) r(-s)$ for all $0 \leq s \leq t$. Therefore

$$Y(t) = r(t)\xi + \int_0^t r(t-s) \mathrm{e}^{-\gamma s} \Sigma(s)\, dB(s) = r(t)\xi + r(t) \int_0^t r(-s) \mathrm{e}^{-\gamma s} \Sigma(s)\, dB(s).$$

Notice that $r(t) = I_d + t(A - \gamma I)$ and $(A - \gamma I) r(-s) = A - \gamma I$. Then

$$\begin{aligned}
&\frac{r(t)}{t} \int_0^t r(-s) \mathrm{e}^{-\gamma s} \Sigma(s)\, dB(s) \\
&= \frac{1}{t} \int_0^t r(-s) \Sigma(s)\, dB(s) + \int_0^t (A - \gamma I) r(-s) \mathrm{e}^{-\gamma s} \Sigma(s)\, dB(s) \\
&= \frac{1}{t} \int_0^t r(-s) \mathrm{e}^{-\gamma s} \Sigma(s)\, dB(s) + (A - \gamma I) \int_0^t \mathrm{e}^{-\gamma s} \Sigma(s)\, dB(s).
\end{aligned}$$

Using Lemma 3.6.1, the first term has zero limit as $s \mapsto \mathrm{e}^{-\gamma s} \Sigma(s)$ is in $L^2(0, \infty)$, and $r(-s)/s \to -(A - \gamma I)$ as $s \to \infty$. The second term converges by the martingale convergence theorem. Thus

$$\lim_{t \to \infty} \frac{X(t)}{t \mathrm{e}^{\gamma t}} = (A - \gamma I)\xi + (A - \gamma I) \int_0^\infty \mathrm{e}^{-\gamma s} \Sigma(s)\, dB(s).$$

This is exactly the form of the limit predicted in Theorem 3.3.2, because for $\lambda_j = 0$ with $n = 1$, we have

$$P_j^* = K_{j,1} = \lim_{\lambda \to 0} \lambda^2 \left(\lambda I_d - (A - \gamma I)\right)^{-1} = A - \gamma I.$$

This next example demonstrates the case when the leading eigenvalues are complex solutions of the characteristic equation.

Example 3.3.3. Suppose that

$$A = \begin{pmatrix} \gamma & -1 \\ 1 & \gamma \end{pmatrix}.$$

Suppose that $Y(t) = \mathrm{e}^{-\gamma t} X(t)$. If $J = A - \gamma I$, then

$$dY(t) = JY(t)\, dt + \mathrm{e}^{-\gamma t}\Sigma(t)\, dB(t).$$

For the equation solved by Y, we have $\alpha = 0$, because $\lambda = \pm i$ are eigenvalues of multiplicity 1. In this case r is given by

$$r(t) = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}.$$

Since $\det(r(t)) = 1$ for all $t \geq 0$, $r(t)$ is invertible, and we may write $r(t-s) = r(t)r^{-1}(s) = r(t)r(-s)$ for all $0 \leq s \leq t$. Therefore

$$X(t) = r(t)\xi + \int_0^t r(t-s)\mathrm{e}^{-\gamma s}\Sigma(s)\, dB(s) = r(t)\xi + r(t)\int_0^t r(-s)\mathrm{e}^{-\gamma s}\Sigma(s)\, dB(s).$$

Since $r(-s)$ is bounded, and $s \mapsto e^{-\gamma s}\Sigma(s) \in L^2(0,\infty)$, it follows that

$$\lim_{t\to\infty} \int_0^t r(-s)\mathrm{e}^{-\gamma s}\Sigma(s)\, dB(s) = \int_0^\infty r(-s)\mathrm{e}^{-\gamma s}\Sigma(s)\, dB(s), \quad \text{a.s.}$$

Therefore

$$\lim_{t\to\infty} \left\{ Y(t) - r(t)\left(\xi + \int_0^\infty r(-s)\Sigma(s)\, dB(s)\right)\right\} = 0, \quad \text{a.s.}$$

We now see that $r(t) = \cos(t)I + \sin(t)J$, and so the following limit holds almost surely,

$$\lim_{t\to\infty} \left\{ \frac{X(t)}{\mathrm{e}^{\gamma t}} - (\cos(t)I + \sin(t)J)\left(\xi + \int_0^\infty (\cos(s)I - \sin(s)J)\mathrm{e}^{-\gamma s}\Sigma(s)\, dB(s)\right)\right\} = 0.$$

Since $J^2 = -I$, this yields

$$\lim_{t\to\infty} \left\{ \frac{X(t)}{\mathrm{e}^{\gamma t}} - \cos(t)\left(\xi + \int_0^\infty \cos(s)\mathrm{e}^{-\gamma s}\Sigma(s)\, dB(s) - J\int_0^\infty \sin(s)\mathrm{e}^{-\gamma s}\Sigma(s)\, dB(s)\right)\right.$$
$$\left. - \sin(t)\left(J\xi + J\int_0^\infty \cos(s)\mathrm{e}^{-\gamma s}\Sigma(s)\, dB(s) + \int_0^\infty \sin(s)\mathrm{e}^{-\gamma s}\Sigma(s)\, dB(s)\right)\right\} = 0, \quad \text{a.s.}$$

To show that this asymptotic expansion agrees exactly with formula (3.3.34) derived in Theorem 3.3.2 we notice for $\lambda_j = (-1)^{j-1} i$ for $j = 1, 2$ where each of which has multiplicity $n + 1 = 1$, that

$$K_{j,0} = \lim_{\lambda \to \lambda_j} (\lambda - \lambda_j)(\lambda I - J)^{-1} = \lim_{\lambda \to \lambda_j} (\lambda - \lambda_j) \frac{1}{1 + \lambda^2} \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}.$$

Since $(\lambda - \lambda_j)(\lambda - \overline{\lambda}_j) = 1 + \lambda^2$, we have

$$K_{j,0} = \frac{1}{\lambda_j - \overline{\lambda}_j} \begin{pmatrix} \lambda_j & -1 \\ 1 & \lambda_j \end{pmatrix} = \frac{1}{2\lambda_j} \begin{pmatrix} \lambda_j & -1 \\ 1 & \lambda_j \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & \lambda_j \\ -\lambda_j & 1 \end{pmatrix}$$

Hence $2K_{1,0} = I - iJ$ and $2K_{2,0} = I + iJ$. Therefore $P_1^* = I$ and $Q_1^* = J$.

We provide an example of a convolution Volterra integro–differential equation where the zeros of the characteristic equation do not lie in the domain of the transform of the measure, i.e. $\alpha^* > \alpha$. Nevertheless an explicit formula for the resolvent may obtained and hence one may deduce the asymptotic behaviour of the solution of the stochastic equation.

Example 3.3.4. Let X be the unique solution of

$$dX(t) = \int_{[0,t]} \mu(ds) X(t-s) dt + \Sigma(t) dB(t), \quad t \geq 0$$

where $X(0) = X_0 \in \mathbb{R}^d$ and $\mu(ds) = -6\,\delta_0(ds) I_d - 4\,\mathrm{e}^{-s}\, ds I_d$. Hence $\alpha^* = -1$ and h is given by

$$h(\lambda) = \det\left(\lambda I_d - \int_{[0,\infty)} \mu(ds) \mathrm{e}^{-\lambda s} I_d\right) = \frac{(\lambda+2)^d (\lambda+5)^d}{(\lambda+1)^d}.$$

Thus $\alpha^* = -1 > -2 = \alpha$ and so we cannot apply Theorem 3.3.1 to this problem.

Nevertheless, the differential resolvent, (3.3.13), may rewritten as the solution of a second order equation and solved to give

$$r(t) = -\frac{1}{3}\mathrm{e}^{-2t} I_d + \frac{4}{3}\mathrm{e}^{-5t} I_d.$$

Therefore $n = 0$ and $P_1^* = -1/3$ and one can now apply Proposition 3.3.1 to determine the asymptotic behaviour of X, i.e.

$$\lim_{t \to \infty} \frac{X(t)}{\mathrm{e}^{-2t}} = -\frac{1}{3} X_0 - \frac{1}{3} \int_0^\infty \mathrm{e}^{2s} \Sigma(s) dB(s).$$

Thus, in instances where Theorem 3.3.1 does not apply, providing that the asymptotic behaviour of r may be estimated to agree with (3.3.4), then via Proposition 3.3.1 the asymptotic behaviour of the solution of the stochastic equation can still be recovered.

We finish with an example where the underlying deterministic functional differential equation is not equivalent to a linear ordinary differential equation, but for which it is possible, owing to the special structure of the equation, to determine exactly the leading order asymptotic behaviour.

Example 3.3.5. Suppose that X obeys

$$dX(t) = a(X(t) - X(t - 1/3))\,dt + \Sigma(s)\,dB(t), \quad t \geq 0,$$

where $\Sigma \in C(\mathbb{R}_+; \mathbb{R}^{1\times d'})$, $X(t) = \phi(t)$ for $t \in [-1/3, 0]$, where $\phi \in C([-1/3, 0], \mathbb{R})$. Let $a = 3/(1 - 1/\mathrm{e}) > 0$. This is equivalent to choosing $\tau = 1/3$ and the finite measure $\nu(ds) = a\delta_0(ds) - a\delta_{-1/3}(ds)$. Then it can be shown that $\nu([-t, 0]) \geq 0$ for all $t \in [0, 1/3]$ with $\nu([-1/3, 0]) = 0$. Also

$$\int_{[-1/3,0]} s\,\nu(ds) = \frac{1}{1 - \mathrm{e}^{-1}} > 1.$$

Consequently, all the conditions of part (i), Theorem 3.3 in [20] hold, and therefore there is a unique positive real solution $\lambda_1 > 0$ of $g_\nu(\lambda_1) = 0$ where $g_\nu(\lambda) = \lambda - a + a\mathrm{e}^{-\lambda/3}$, and moreover $\alpha = \lambda_1$. Since $a = 3/(1-1/\mathrm{e})$, it is easily verified that $\alpha = \lambda_1 = 3$. Furthermore, as $g'_\nu(\lambda_1) = 1 - a\mathrm{e}^{-1}/3 \neq 0$, it can be shown that $n = 0$ in Theorem 3.3.2, and moreover by l'Hôpital's rule that

$$P_1^* = \lim_{\lambda\to\lambda_1} \frac{\lambda - \lambda_1}{g_\nu(\lambda)} = \frac{1}{g'_\nu(3)} = \frac{1 - \mathrm{e}^{-1}}{1 - 2\mathrm{e}^{-1}}.$$

Therefore, assuming (3.3.33) holds, then all the conditions of Theorem 3.3.2 apply, we have that

$$\lim_{t\to\infty} \frac{X(t)}{\mathrm{e}^{3t}} = P_1^*\phi(0) + P_1^* \int_{-\tau}^{0} \int_{[-\tau,u]} \mathrm{e}^{3u}\nu(ds)\phi(s-u)du + P_1^* \int_0^\infty \mathrm{e}^{-3s}\Sigma(s)dB(s).$$

3.4 Proofs of Admissibility Results

3.4.1 Proof of Theorem 3.2.1

It is not difficult to see using Itô's isometry that

$$\begin{aligned}\mathbb{E}&\left[\left\|\int_0^t H(t,s)f(s)dB(s) - \int_0^\infty H_\infty(s)f(s)dB(s)\right\|^2\right] \\ &= \int_0^t \left\|\Big(H(t,s) - H_\infty(s)\Big)f(s)\right\|_F^2 ds + \int_t^\infty \|H_\infty(s)f(s)\|_F^2\,ds, \qquad (3.4.1)\end{aligned}$$

where the independence of the elements of the Brownian vector and of stochastic integrals over non-overlapping intervals has been used.

Firstly we show that (A) implies (B). Let f be such that $\sup_{s\geq 0}\|f(s)\|_F^2 < +\infty$. Recalling the submultiplicative property of the Frobenius norm,

$$\begin{aligned}
&\mathbb{E}\left[\left\|\int_0^t H(t,s)f(s)dB(s) - \int_0^\infty H_\infty(s)f(s)dB(s)\right\|^2\right] \\
&\quad \leq 2\int_0^t \|H(t,s)-H_\infty(s)\|_F^2\,\|f(s)\|_F^2\,ds + 2\int_t^\infty \|H_\infty(s)\|_F^2\,\|f(s)\|_F^2\,ds \\
&\quad \leq 2\left(\int_0^t \|H(t,s)-H_\infty(s)\|_F^2\,ds + \int_t^\infty \|H_\infty(s)\|_F^2\,ds\right)\sup_{s\geq 0}\|f(s)\|_F^2 .
\end{aligned}$$

By hypothesis both terms on the right–hand side of the above inequality tend to zero as $t\to\infty$ and so

$$\lim_{t\to\infty}\mathbb{E}\left[\left\|\int_0^t H(t,s)f(s)\,dB(s) - \int_0^\infty H_\infty(s)f(s)\,dB(s)\right\|^2\right] = 0.$$

Conversely suppose that (B) holds. Then it is implicit that the stochastic integral

$$\int_0^\infty H_\infty(s)f(s)dB(s)$$

exists a.s. for each $f\in C([0,\infty);\mathbb{R}^{n\times d})$ with the property $\sup_{s\geq 0}\|f(s)\|_F^2 < +\infty$. We view this stochastic integral as the pathwise limit of the finite–dimensional martingale $M=\{M(t):0\leq t<\infty;\mathcal{F}^B(t)\}$ defined by

$$M(t)=\int_0^t H_\infty(s)f(s)dB(s).$$

The i–th component of M, denoted by M_i, is a scalar martingale, and given by

$$M_i(t)=\sum_{j=1}^d\int_0^t [H_\infty(s)f(s)]_{i,j}dB_j(s).$$

We have that $M_i(t)$ tends to a finite limit as $t\to\infty$ a.s. Therefore it follows that $\lim_{t\to\infty}\langle M_i\rangle(t)$ tends to a finite limit as $t\to\infty$. This is equivalent to

$$\int_0^\infty \sum_{j=1}^d [H_\infty(s)f(s)]_{i,j}^2\,ds < +\infty, \quad \text{a.s.}$$

Since this holds for each $i=1,\ldots,n$, we have that

$$\int_0^\infty \|H_\infty(s)f(s)\|_F^2\,ds < +\infty.$$

Rearranging (3.4.1),

$$\int_0^t \left\| \Big(H(t,s) - H_\infty(s) \Big) f(s) \right\|_F^2 ds$$
$$= \mathbb{E}\left[\left\| \int_0^t H(t,s) f(s) dB(s) - \int_0^\infty H_\infty(s) f(s) dB(s) \right\|^2 \right] - \int_t^\infty \|H_\infty(s) f(s)\|_F^2 \, ds,$$

By our assumption B and the argument made above we have that

$$\lim_{t\to\infty} \int_0^t \left\| \Big(H(t,s) - H_\infty(s) \Big) f(s) \right\|_F^2 ds = 0 \tag{3.4.2}$$

for each $f \in C([0,\infty); \mathbb{R}^{n\times d})$ with the property $\sup_{s\geq 0} \|f(s)\|_F^2 < +\infty$. Define

$$(f^i)_{l,m} = \begin{cases} 1, & l = i \text{ and } m = 1, \\ 0, & \text{otherwise}, \end{cases}$$

for all $1 \leq i \leq n$ i.e. an n by d matrix with 1 in the i^{th} position on the first column and zeroes in all other locations. Then (3.4.2) holds true when $f = f^i$ for any $1 \leq i \leq n$. But

$$\lim_{t\to\infty} \int_0^t \left\| \Big(H(t,s) - H_\infty(s) \Big) f^i(s) \right\|_F^2 ds = 0$$

being true is equivalent to $\sum_{l=1}^n \lim_{t\to\infty} \int_0^t \Big((H(t,s))_{l,i} - (H_\infty(s))_{l,i} \Big)^2 ds = 0$ and so $\lim_{t\to\infty} \int_0^t \Big((H(t,s))_{l,i} - (H_\infty(s))_{l,i} \Big)^2 ds = 0$ for all $1 \leq l \leq n$. Now as this is true for any $1 \leq i \leq n$ we have

$$\lim_{t\to\infty} \int_0^t \|H(t,s) - H_\infty(s)\|_F^2 \, ds = 0.$$

Knowing that $\int_0^\infty \|H_\infty(s) f(s)\|_F^2 \, ds < \infty$, for all f with the assumed properties, one can make a similar argument as just outlined to show that $\int_0^\infty \|H_\infty(s)\|_F^2 \, ds < \infty$.

3.4.2 Proof of Proposition 3.2.1

We prove that (A) implies (B) first. To prove (3.2.4), note for any $t \geq T$ we have the estimate

$$\begin{aligned} \int_T^t \|H(t,s)\|_F^2 \, ds &= \int_T^t \|H(t,s) - H_\infty(s) + H_\infty(s)\|_F^2 \, ds \\ &\leq 2\int_T^t \|H(t,s) - H_\infty(s)\|_F^2 \, ds + 2\int_T^t \|H_\infty(s)\|_F^2 \, ds \\ &\leq 2\int_0^t \|H(t,s) - H_\infty(s)\|_F^2 \, ds + 2\int_T^t \|H_\infty(s)\|_F^2 \, ds. \end{aligned}$$

Since $\|H_\infty\| \in L^2([0,\infty);\mathbb{R})$ and (3.2.2) holds, we have

$$\limsup_{t\to\infty} \int_T^t \|H(t,s)\|_F^2 \, ds \le 2\int_T^\infty \|H_\infty(s)\|_F^2 \, ds.$$

Since the lefthand side is monotone in T, we may take the limit as $T \to \infty$ on both sides, using the fact that $\|H_\infty\| \in L^2([0,\infty);\mathbb{R})$ to obtain the desired conclusion.

To show (3.2.5), let $T > 0$ be arbitrary. Then, for any $t \ge T$ we have

$$\int_0^T \|H(t,s) - H_\infty(s)\|_F^2 \, ds \le \int_0^t \|H(t,s) - H_\infty(s)\|_F^2 \, ds,$$

whence the result taking limits as $t \to \infty$ and applying by (3.2.2). Therefore (A) implies (B).

To prove that (B) implies (A), we first must show that $\|H_\infty\| \in L^2([0,\infty);\mathbb{R})$. We start by observing that (3.2.4) is nothing other than $\lim_{T\to\infty} L(T) = 0$ where

$$L(T) := \limsup_{t\to\infty} \int_T^t \|H(t,s)\|_F^2 \, ds.$$

Since L is non–increasing, for every $\epsilon > 0$ there exists $T_0(\epsilon) > 0$ such that $L(T) < \epsilon$ for all $T \ge T_0(\epsilon)$. Now, let $T \ge T_0$. Suppose also that $t \ge T$. Then

$$\begin{aligned}
\int_{T_0}^T \|H_\infty(s)\|_F^2 \, ds &= \int_{T_0}^T \|H_\infty(s) - H(t,s) + H(t,s)\|_F^2 \, ds \\
&\le 2\int_{T_0}^T \|H_\infty(s) - H(t,s)\|_F^2 \, ds + 2\int_{T_0}^T \|H(t,s)\|_F^2 \, ds \\
&\le 2\int_{T_0}^T \|H_\infty(s) - H(t,s)\|_F^2 \, ds + 2\int_{T_0}^t \|H(t,s)\|_F^2 \, ds.
\end{aligned}$$

Now

$$\begin{aligned}
&\int_{T_0}^T \|H_\infty(s) - H(t,s)\|_F^2 \, ds \\
&\qquad = \int_0^T \|H_\infty(s) - H(t,s)\|_F^2 \, ds - \int_0^{T_0} \|H_\infty(s) - H(t,s)\|_F^2 \, ds,
\end{aligned}$$

so by (3.2.5) we have

$$\lim_{t\to\infty} \int_{T_0}^T \|H_\infty(s) - H(t,s)\|_F^2 \, ds = 0.$$

Hence

$$\int_{T_0}^T \|H_\infty(s)\|_F^2 \, ds \le 2L(T_0),$$

and since the righthand side is independent of T, it follows that one has $\|H_\infty\|_F \in L^2([0,\infty);\mathbb{R})$, which is one part of (3.2.2).

To prove the other part, let $t \geq T > 0$. Then we have the estimate

$$\begin{aligned}
&\int_0^t \|H(t,s) - H_\infty(s)\|_F^2 \, ds \\
&= \int_0^T \|H(t,s) - H_\infty(s)\|_F^2 \, ds + \int_T^t \|H(t,s) - H_\infty(s)\|_F^2 \, ds \\
&\leq \int_0^T \|H(t,s) - H_\infty(s)\|_F^2 \, ds + 2\int_T^t \|H(t,s)\|_F^2 \, ds + 2\int_T^t \|H_\infty(s)\|_F^2 \, ds.
\end{aligned}$$

Since $\|H_\infty\|_F \in L^2([0,\infty);\mathbb{R})$ and H obeys (3.2.5), we have

$$\begin{aligned}
&\limsup_{t\to\infty} \int_0^t \|H(t,s) - H_\infty(s)\|_F^2 \, ds \\
&\qquad \leq 2\limsup_{t\to\infty} \int_T^t \|H(t,s)\|_F^2 \, ds + 2\int_T^\infty \|H_\infty(s)\|_F^2 \, ds.
\end{aligned}$$

Now let $T \to \infty$ on both sides of the inequality; since $\|H_\infty\| \in L^2([0,\infty);\mathbb{R})$, this yields

$$\limsup_{t\to\infty} \int_0^t \|H(t,s) - H_\infty(s)\|_F^2 \, ds \leq 2\lim_{T\to\infty} \limsup_{t\to\infty} \int_T^t \|H(t,s)\|_F^2 \, ds = 0,$$

where the limit on the righthand side is a consequence of (3.2.4). This proves the other part of (3.2.2).

3.4.3 Proof of Theorem 3.2.2

As before, we remark that from the form of (3.2.6) it is implied that $\int_0^\infty \|H_\infty(s)\|_F^2 \, ds < \infty$. Thus we need only show (3.2.2). Condition (3.2.6) is equivalent to

$$\lim_{t\to\infty} \int_0^t \Big(H(t,s) - H_\infty(s) \Big) f(s) \, dB(s) = 0_n, \quad \text{a.s.}$$

where 0_n denotes the $n \times 1$ vector of zeroes. But $\left(\int_0^t \big(H(t,s) - H_\infty(s) \big) f(s) \, dB(s) \right)_i$ is a Gaussian random variable which converges to zero a.s. for each $1 \leq i \leq n$. Since it converges a.s., it does so to a Gaussian random variable which has zero mean and zero variance, and by the argument of pp304-305 in Shiryaev [108], we have that

$$\lim_{t\to\infty} \mathbb{E}\left[\left(\int_0^t \big(H(t,s) - H_\infty(s) \big) f(s) \, dB(s) \right)_i^2 \right] = 0$$

for all $1 \leq i \leq n$ and so

$$\lim_{t\to\infty} \sum_{i=1}^n \mathbb{E}\left[\left(\int_0^t \big(H(t,s) - H_\infty(s) \big) f(s) \, dB(s) \right)_i^2 \right] = 0.$$

Or equivalently

$$\lim_{t\to\infty} \sum_{i=1}^n \mathbb{E}\left[\left(\sum_{k=1}^d \int_0^t \Big(\big(H(t,s) - H_\infty(s) \big) f(s) \Big)_{i,k} \, dB_k(s) \right)^2 \right] = 0.$$

But

$$\mathbb{E}\left[\left(\sum_{k=1}^{d}\int_0^t\Big(\big(H(t,s)-H_\infty(s)\big)f(s)\Big)_{i,k}\,dB_k(s)\right)^2\right]$$
$$=\mathbb{E}\left[\sum_{k=1}^{d}\left(\int_0^t\Big(\big(H(t,s)-H_\infty(s)\big)f(s)\Big)_{i,k}\,dB_k(s)\right)^2\right].$$

Thus

$$\lim_{t\to\infty}\sum_{i=1}^{n}\sum_{k=1}^{d}\mathbb{E}\left[\left(\int_0^t\Big(\big(H(t,s)-H_\infty(s)\big)f(s)\Big)_{i,k}\,dB_k(s)\right)^2\right]=0$$

and so

$$\lim_{t\to\infty}\sum_{i=1}^{n}\sum_{k=1}^{d}\int_0^t\Big(\big(H(t,s)-H_\infty(s)\big)f(s)\Big)_{i,k}^2\,ds=0.$$

Which is equivalent to

$$\lim_{t\to\infty}\int_0^t\left\|\Big(H(t,s)-H_\infty(s)\Big)f(s)\right\|_F^2\,ds=0$$

for each $f\in C([0,\infty);\mathbb{R}^{n\times d})$ with the property $\sup_{s\geq 0}\|f(s)\|_F^2<+\infty$ and so as argued at the end of the proof of Theorem 3.2.1 we have

$$\lim_{t\to\infty}\int_0^t\|H(t,s)-H_\infty(s)\|_F^2\,ds=0,$$

as required.

3.5 Proof of Theorem 3.2.3

Define $\tilde{H}=H-H_\infty$. Notice that $H_\infty\in L^2([0,\infty);\mathbb{R})$ implies that

$$\lim_{t\to\infty}\int_t^\infty H_\infty(s)f(s)\,dB(s)=0,\quad\text{a.s.}$$

so that proving

$$\lim_{t\to\infty}\int_0^t\tilde{H}(t,s)f(s)\,dB(s)=0,\quad\text{a.s.}\tag{3.5.1}$$

is equivalent to establishing (3.2.6).

Since $H\in C^{1,0}$, we have $\tilde{H}_1=H_1$. Therefore, we have

$$\tilde{X}_f(t):=\int_0^t\tilde{H}(t,s)f(s)\,dB(s)=\int_0^t\left(\tilde{H}(s,s)f(s)+\int_s^t\tilde{H}_1(u,s)f(s)\,du\right)dB(s).$$

By a stochastic Fubini theorem, [102, Theorem 4.6.64, pp.210–211], we have

$$\tilde{X}_f(t)=\int_0^t\tilde{H}(s,s)f(s)\,dB(s)+\int_0^t\left(\int_0^u H_1(u,s)f(s)\,dB(s)\right)du.$$

Now, let $(t_n)_{n\geq 0}$ be an increasing sequence with $t_0 = 0$ and $t_n \to \infty$ as $n \to \infty$. In fact, choose

$$t_n = n^\theta, \quad \text{for some } \theta \in (0, 1/(1+q) \wedge 1/(1+2q)) \subset (0,1), \tag{3.5.2}$$

where q is the number in (3.2.13).

Therefore for $t \in [t_n, t_{n+1})$, we have

$$\tilde{X}_f(t) = \tilde{X}_f(t_n) + \int_{t_n}^t H(s,s)f(s)\,dB(s) - \int_{t_n}^t H_\infty(s)f(s)\,dB(s) + \int_{t_n}^t \left(\int_0^u H_1(u,s)f(s)\,dB(s)\right)du.$$

Hence

$$\sup_{t_n\leq t\leq t_{n+1}} |\tilde{X}_f(t)| \leq |\tilde{X}_f(t_n)| + \sup_{t_n\leq t\leq t_{n+1}} \left|\int_{t_n}^t H_\infty(s)f(s)\,dB(s)\right| + \sup_{t_n\leq t\leq t_{n+1}} \left|\int_{t_n}^t H(s,s)f(s)\,dB(s)\right| + \int_{t_n}^{t_{n+1}} \left|\int_0^u H_1(u,s)f(s)\,dB(s)\right| du. \tag{3.5.3}$$

We now show that each of the four terms on the righthand side of (3.5.3) tends to zero as $n \to \infty$ a.s.

STEP 1: First term on the righthand side of (3.5.3)**.** First we prove that

$$\lim_{n\to\infty} \tilde{X}_f(t_n) = 0, \quad \text{a.s.} \tag{3.5.4}$$

Notice that $\tilde{X}_f(t_n)$ is normally distributed with mean zero and variance v_n^2 where

$$v_n^2 := \int_0^{t_n} \tilde{H}^2(t_n,s)f^2(s)\,ds \leq \int_0^{t_n} \tilde{H}^2(t_n,s)\,ds \cdot \sup_{s\geq 0} f^2(s).$$

Using (3.2.12) and the fact that $t_n \to \infty$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \int_0^{t_n} \tilde{H}(t_n,s)^2\,ds \cdot \log t_n = 0,$$

Therefore

$$\limsup_{n\to\infty} v_n^2 \log t_n \leq \limsup_{n\to\infty} \int_0^{t_n} \tilde{H}(t_n,s)^2\,ds \cdot \sup_{s\geq 0} f^2(s) \cdot \log t_n = 0. \tag{3.5.5}$$

Since $X_n := \tilde{X}_f(t_n)/v_n$ is a standardised normal random variable, we have that

$$\limsup_{n\to\infty} \frac{|\tilde{X}_f(t_n)|}{\sqrt{2}v_n(\log n)^{1/2}} = \limsup_{n\to\infty} \frac{|X_n|}{\sqrt{2\log n}} \leq 1, \quad \text{a.s.},$$

the last inequality being a routine consequence of the Borel–Cantelli lemma. Thus

$$\limsup_{n\to\infty}|\tilde{X}_f(t_n)| = \limsup_{n\to\infty}\frac{|\tilde{X}_f(t_n)|}{\sqrt{2}v_n(\log n)^{1/2}}\cdot\sqrt{2}v_n(\log n)^{1/2}$$
$$\leq \sqrt{2}\limsup_{n\to\infty} v_n(\log t_n)^{1/2}\sqrt{\frac{\log n}{\log t_n}} = 0,$$

due to (3.5.2) and (3.5.5), proving (3.5.4).

STEP 2: Second term on the righthand side of (3.5.3) Next we show that

$$\lim_{n\to\infty}\sup_{t_n\leq t\leq t_{n+1}}\left|\int_{t_n}^t H_\infty(s)f(s)\,dB(s)\right| = 0, \quad \text{a.s.} \tag{3.5.6}$$

To do this, notice for every $\epsilon > 0$ by Chebyshev's inequality and the Birkholder–Davis–Gundy inequality, c.f. e.g. [84, Theorem 1.3.8, Theorem 1.7.3] that

$$\begin{aligned}
&\mathbb{P}\left[\sup_{t_n\leq t\leq t_{n+1}}\left|\int_{t_n}^t H_\infty(s)f(s)\,dB(s)\right| > \epsilon\right]\\
&\leq \frac{1}{\epsilon^2}\mathbb{E}\left[\sup_{t_n\leq t\leq t_{n+1}}\left|\int_{t_n}^t H_\infty(s)f(s)\,dB(s)\right|^2\right]\\
&\leq \frac{4}{\epsilon^2}\mathbb{E}\left[\left|\int_{t_n}^{t_{n+1}} H_\infty(s)f(s)\,dB(s)\right|^2\right]\\
&= \frac{4}{\epsilon^2}\int_{t_n}^{t_{n+1}} H_\infty^2(s)f^2(s)\,ds.
\end{aligned}$$

Since $H_\infty \in L^2([0,\infty);\mathbb{R})$, and $f \in BC([0,\infty);\mathbb{R})$, we have

$$\sum_{n=0}^{\infty}\mathbb{P}\left[\sup_{t_n\leq t\leq t_{n+1}}\left|\int_{t_n}^t H_\infty(s)f(s)\,dB(s)\right| > \epsilon\right] \leq \frac{4}{\epsilon^2}\int_0^\infty H_\infty^2(s)f^2(s)\,ds.$$

By the Borel–Cantelli Lemma, we have that (3.5.6) holds.

STEP 3: Third term on the righthand side of (3.5.3)**.**

$$\lim_{n\to\infty} U_n = 0, \quad \text{a.s.}$$

where

$$U_n = \sup_{t_n\leq t\leq t_{n+1}}\left|\int_{t_n}^t H(s,s)f(s)\,dB(s)\right|.$$

Note that $(U_n)_{n\geq 0}$ is a sequence of independent random variables.

Notice that on the interval $[t_n, t_{n+1}]$, by the martingale time change theorem, there exists a Brownian motion $\tilde{B}$ such that

$$\begin{aligned} U_n &= \sup_{t_n \le t \le t_{n+1}} \left| \tilde{B}_n \left(\int_{t_n}^t H^2(s,s) f^2(s)\, ds \right) \right| \\ &= \sup_{0 \le \tau \le \int_{t_n}^{t_{n+1}} H^2(s,s) f^2(s)\, ds} |\tilde{B}_n(\tau)| \\ &\le \sup_{0 \le \tau \le \int_{t_n}^{t_{n+1}} H^2(s,s)\, ds \cdot \sup_{v \ge 0} f^2(v)} |\tilde{B}_n(\tau)|. \end{aligned}$$

Therefore, with $w_n := \int_{t_n}^{t_{n+1}} H^2(s,s)\, ds \cdot \sup_{v \ge 0} f^2(v)$, we have for some Brownian motion W that

$$\mathbb{P}[U_n > \epsilon] \le \mathbb{P}[\sup_{0 \le \tau \le w_n} |W(\tau)| > \epsilon].$$

Using the symmetry of the distribution function leads to the estimate

$$\mathbb{P}[U_n > \epsilon] \le 2\mathbb{P}[|W(w_n)| > \epsilon] \le 4\mathbb{P}[W(w_n) > \epsilon] = 4\mathbb{P}[Z > \epsilon/\sqrt{w_n}],$$

where Z is a standard normal random variable, and we interpret the right hand side as zero if $w_n = 0$. Hence if Φ is the distribution function of a standard normal random variable and

$$\sum_{n=0}^{\infty} \left\{ 1 - \Phi\left(\frac{\varepsilon}{\sqrt{\int_{t_n}^{t_{n+1}} H^2(s,s)\, ds}} \right) \right\} < +\infty, \quad \text{for all } \varepsilon > 0,$$

we have that $\lim_{n\to\infty} U_n = 0$, a.s. The sum is finite provided

$$\lim_{n\to\infty} \int_{t_n}^{t_{n+1}} H^2(s,s)\, ds \cdot \log n = 0.$$

Since $H(t,t)^2 \le c_q(1 + t^{2q})$, we have that

$$\int_{t_n}^{t_{n+1}} H^2(s,s)\, ds \cdot \log n \le c_q \int_{n^\theta}^{(n+1)^\theta} \{1 + s^{2q}\}\, ds \cdot \log n,$$

so the right hand side is of the order $n^{-1+\theta} n^{2q\theta} \log n = n^{-1+(2q+1)\theta} \log n \to 0$ as $n \to \infty$, because $\theta < 1/(1+2q)$.

STEP 4: Fourth term on the righthand side of (3.5.3)**.** Finally, we show that

$$\lim_{n\to\infty} Z_n = 0, \quad \text{a.s.} \tag{3.5.7}$$

where

$$Z_n := \int_{t_n}^{t_{n+1}} \left| \int_0^u H_1(u,s) f(s)\, dB(s) \right| du. \tag{3.5.8}$$

By (3.2.13) there exists $c_q > 0$ such that

$$\int_0^t H_1^2(t,s)\,ds \le c_q(1+t)^{2q}, \quad t \ge 0.$$

By (3.5.2), $\theta < 1/(1+q) \le 1$, so we can choose $p \in \mathbb{N}$ so large that $2p[1-(1+q)\theta] > 1$. Clearly for such a $p \in \mathbb{N}$ we have, via Jensen's inequality

$$Z_n^{2p} \le (t_{n+1}-t_n)^{2p-1} \int_{t_n}^{t_{n+1}} \left(\int_0^u H_1(u,s)f(s)\,dB(s)\right)^{2p} du,$$

so there exists $C_p > 0$ such that

$$\begin{aligned}
\mathbb{E}[Z_n^{2p}] &\le C_p(t_{n+1}-t_n)^{2p-1} \int_{t_n}^{t_{n+1}} \left(\int_0^u H_1^2(u,s)f^2(s)\,ds\right)^p du \\
&\le C_p \sup_{s\ge 0} f^{2p}(s)(t_{n+1}-t_n)^{2p-1} \int_{t_n}^{t_{n+1}} \left(\int_0^u H_1^2(u,s)\,ds\right)^p du.
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{E}[Z_n^{2p}] &\le C_p \sup_{s\ge 0} f^{2p}(s)(t_{n+1}-t_n)^{2p-1} \int_{t_n}^{t_{n+1}} \left(c_q(1+u)^{2q}\right)^p du \\
&\le C_p c_q^p \sup_{s\ge 0} f^{2p}(s)\cdot(t_{n+1}-t_n)^{2p-1} \int_{t_n}^{t_{n+1}} (1+u)^{2qp}\,du \\
&\le C_p c_q^p \sup_{s\ge 0} f^{2p}(s)\cdot(t_{n+1}-t_n)^{2p}(1+t_{n+1})^{2qp}.
\end{aligned}$$

Since $t_n = n^\theta$, the right hand side is of the order $[n^{\theta-1}]^{2p}n^{2pq\theta} = n^{-2p[1-(1+q)\theta]}$ as $n \to \infty$. By Chebyshev's inequality, for any $\epsilon > 0$ we have

$$\mathbb{P}[|Z_n| > \epsilon] \le \frac{1}{\epsilon^{2p}}\mathbb{E}[Z_n^{2p}] \le C_{\epsilon,p}n^{-2p[1-(1+q)\theta]},$$

and because $2p[1-(1+q)\theta] > 1$, the righthand side is summable. Therefore, by the Borel–Cantelli lemma, we have (3.5.7).

3.6 Proof from Section 3.3

3.6.1 Proof of Proposition 3.3.1

We start with the proof of a preliminary lemma.

Lemma 3.6.1. *Suppose $f \in L^2([0,\infty),\mathbb{R}^{d\times r})$. If $k > 0$, then*

$$\lim_{t\to\infty} \frac{1}{(1+t)^k}\int_0^t s^k f(s)\,dB(s) = 0, \quad a.s.$$

Proof. Define

$$K(t) = \frac{1}{(1+t)^k} \int_0^t s^k f(s)\, dB(s), \quad t \geq 0.$$

Then $dK(t) = -k(1+t)^{-1}K(t)\,dt + (1+t)^{-k}t^k f(t)\,dB(t)$. Hence for $i = 1, \ldots, d$ with $K_i(t) := \langle K(t), \mathbf{e}_i \rangle$, we have

$$dK_i(t) = -k(1+t)^{-1}K_i(t)\,dt + \sum_{j=1}^{r} \frac{t^k}{(1+t)^k} f_{ij}(t)\,dB_j(t).$$

Therefore

$$d\|K(t)\|^2 = \left(-2k(1+t)^{-1}\|K(t)\|^2 + \frac{t^{2k}}{(1+t)^{2k}}\|f(t)\|_F^2\right) dt \\ + \sum_{i=1}^{d} 2K_i(t) \sum_{j=1}^{r} \frac{t^k}{(1+t)^k} f_{ij}(t)\,dB_j(t).$$

Now define the non–decreasing processes A_1 and A_2 by

$$A_1(t) = \int_0^t \frac{s^{2k}}{(1+s)^{2k}}\|f(s)\|_F^2\,ds, \quad A_2(t) = \int_0^t 2k(1+s)^{-1}\|K(s)\|^2\,ds.$$

and the martingale M by

$$M(t) = \sum_{j=1}^{r} \int_0^t \sum_{i=1}^{d} 2K_i(s) \frac{s^k}{(1+s)^k} f_{ij}(s)\,dB_j(s).$$

Then we have

$$\|K(t)\|^2 = A_1(t) - A_2(t) + M(t), \quad t \geq 0.$$

Since f is in $L^2(0,\infty)$, we notice that $A_1(t)$ tends to a finite limit as $t \to \infty$. Therefore, we have that $\|K(t)\|^2 \to \kappa$ as $t \to \infty$ a.s where $\kappa \in [0,\infty)$ a.s. (It is known that $\lim_{t\to\infty}\|K(t)\|^2$ exists and is finite due to [78, Theorem 7, pp.139]). Then by l'Hôpital's rule we have

$$\lim_{t\to\infty} \frac{A_2(t)}{\log t} = 2k\kappa.$$

Notice now that M has quadratic variation

$$\langle M \rangle(t) = \int_0^t \sum_{j=1}^{r} \left(\sum_{i=1}^{d} 2K_i(s) \frac{s^k}{(1+s)^k} f_{ij}(s) \right)^2 ds.$$

Therefore by the Cauchy–Schwartz inequality

$$\langle M \rangle(t) \leq \int_0^t \sum_{j=1}^{r} 4 \sum_{l=1}^{d} K_l^2(s) \sum_{i=1}^{d} \frac{s^{2k}}{(1+s)^{2k}} f_{ij}^2(s)\,ds \leq 4\int_0^t \|K(s)\|^2 \|f(s)\|_F^2\,ds.$$

Since f is in $L^2(0,\infty)$, we see that $\lim_{t\to\infty}\langle M\rangle(t)$ is finite and hence that M tends to a finite limit a.s. Let $A = \{\omega : \kappa(\omega) > 0\}$ and suppose that $\mathbb{P}[A] > 0$. Then on A we have $\lim_{t\to\infty} \|K(t,\omega)\|^2 = -\infty$, which is a contradiction. Hence $\mathbb{P}[A] = 0$, or $\kappa = 0$ a.s. Therefore $K(t) \to 0$ as $t \to \infty$, a.s., as required. □

3.6.2 Proof of Proposition 3.3.1 for n ≥ 1

In the following M denotes a positive constant whose value may change from line to line. Using (3.3.4) and (3.3.6) we may write

$$Y(t) = \int_0^t S(t-s)\Sigma(s)\,dB(s) + \int_0^t R(t-s)\Sigma(s)\,dB(s), \quad t \geq 0,$$

where

$$S(t) = \sum_{j=1}^{N} \mathrm{e}^{\alpha t}\{P_j(t)\cos(\beta_j t) + Q_j(t)\sin(\beta_j t)\}.$$

Thus,

$$\frac{Y(t)}{t^n \mathrm{e}^{\alpha t}} = \int_0^t \frac{S(t-s)}{t^n \mathrm{e}^{\alpha t}}\Sigma(s)\,dB(s) + \int_0^t \frac{R(t-s)}{t^n \mathrm{e}^{\alpha t}}\Sigma(s)\,dB(s). \tag{3.6.1}$$

We show using Theorem 3.2.4 that the second stochastic integral term on the right–hand side above converges to zero almost surely. So in the notation of Section 3.2 we define

$$H(t,s) := \frac{R(t-s)}{t^n \mathrm{e}^{\alpha t}}\Sigma(s).$$

Now as $R(t) = O(t^{n-1}\mathrm{e}^{\alpha t})$ as $t \to \infty$ from (3.3.2) it is natural to choose $H_\infty(s) = 0_{n,d}$. Thus we need only verify conditions (3.2.15) and (3.2.16). Now, from (3.3.2) we have

$$\int_0^t \|H(t,s)\|_F^2\,ds \\ \leq \left(\frac{1+t}{t}\right)^{2n} \frac{1}{(1+t)^{2n}\mathrm{e}^{2\alpha t}} \int_0^t M(1+t-s)^{2n-2}\mathrm{e}^{2\alpha(t-s)}\|\Sigma(s)\|_F^2\,ds,$$

for some $M > 0$. Hence for $t \geq 1$ we have

$$\begin{aligned}\int_0^t \|H(t,s)\|_F^2\,ds &\leq 2^{2n} M \frac{1}{(1+t)^{2n}} \int_0^t (1+t-s)^{2n-2}\mathrm{e}^{-2\alpha s}\|\Sigma(s)\|_F^2\,ds \\ &\leq 2^{2n} M \frac{1}{(1+t)^2} \int_0^t \mathrm{e}^{-2\alpha s}\|\Sigma(s)\|_F^2\,ds \\ &\leq 2^{2n} M \frac{1}{(1+t)^2} \int_0^\infty \mathrm{e}^{-2\alpha s}\|\Sigma(s)\|_F^2\,ds,\end{aligned}$$

where we use the fact that $\int_0^\infty e^{-2\alpha s}\|\Sigma(s)\|_F^2\,ds$ is finite. Therefore

$$\lim_{t\to\infty} \int_0^t \|H(t,s)\|_F^2\,ds \cdot \log t = 0.$$

Next, we consider

$$\int_{k^\theta}^{(1+k)^\theta} \|H(s,s)\|_F^2\, ds \le \int_{k^\theta}^{(1+k)^\theta} K s^{-2n} \mathrm{e}^{-2\alpha s} \|\Sigma(s)\|_F^2 \, ds$$
$$\le K k^{-2n\theta} \int_{k^\theta}^{(1+k)^\theta} \mathrm{e}^{-2\alpha s} \|\Sigma(s)\|_F^2 \, ds,$$

for some $K > 0$. Since $n \ge 1$, $\theta > 0$ and $\int_0^\infty \mathrm{e}^{-2\alpha s}\|\Sigma(s)\|_F^2\, ds$ is finite, we have that

$$\lim_{k\to\infty} \int_{k^\theta}^{(1+k)^\theta} \|H(s,s)\|_F^2\, ds \cdot \log k = 0.$$

Turning then to the derivative condition of (3.2.13) we see

$$H_1(t,s) = t^{-n}\mathrm{e}^{-\alpha t} R'(t-s)\Sigma(s) - \alpha\, t^{-n}\mathrm{e}^{-\alpha t} R(t-s)\Sigma(s)$$
$$- n\, t^{-n-1}\mathrm{e}^{-\alpha t} R(t-s)\Sigma(s). \quad (3.6.2)$$

Therefore we have

$$\|H_1(t,s)\|_F \le t^{-n}\mathrm{e}^{-\alpha t}\Big(\|R'(t-s)\|_F + |\alpha|\, \|R(t-s)\|_F$$
$$+ nt^{-1}\|R(t-s)\|_F \Big) \|\Sigma(s)\|_F,$$

and so as $\|R(t)\|_F \le M(1+t)^{n-1}\mathrm{e}^{\alpha t}$, $\|R'(t)\|_F \le M(1+t)^n \mathrm{e}^{\alpha t}$ we have for $t \ge 1$

$$\|H_1(t,s)\|_F \le M t^{-n}\mathrm{e}^{-\alpha s}\Big((1+t-s)^n + |\alpha|(1+t-s)^{n-1}$$
$$+ nt^{-1}(1+t-s)^{n-1}\Big) \|\Sigma(s)\|_F$$
$$\le M t^{-n}(1+t-s)^n \left(1 + (|\alpha|+n)(1+t-s)^{-1}\right) \mathrm{e}^{-\alpha s}\|\Sigma(s)\|_F$$
$$\le M\,(1+|\alpha|+n)\cdot t^{-n}(1+t-s)^n \mathrm{e}^{-\alpha s}\|\Sigma(s)\|_F.$$

Thus for $t \ge 1$ we have

$$\int_0^t \|H_1(t,s)\|_F^2\, ds \le M_1^2 t^{-2n} \int_0^t (1+t-s)^{2n}\mathrm{e}^{-2\alpha s}\|\Sigma(s)\|_F^2\, ds$$
$$\le M_1^2 \left(\frac{1+t}{t}\right)^{2n} \int_0^t \mathrm{e}^{-2\alpha s}\|\Sigma(s)\|_F^2\, ds$$
$$\le M_1^2 2^{2n} \int_0^\infty \mathrm{e}^{-2\alpha s}\|\Sigma(s)\|_F^2\, ds.$$

Hence $\int_0^t \|H_1(t,s)\|_F^2\, ds$ may easily be bounded above by a polynomially growing function. So we have shown that

$$\lim_{t\to\infty} \int_0^t \frac{R(t-s)}{t^n \mathrm{e}^{\alpha t}}\Sigma(s)\, dB(s) = 0, \quad \text{a.s.} \quad (3.6.3)$$

Next write

$$P_j(t) = t^n P_j^* + P_{j,n-1}(t) \text{ and } Q_j(t) = t^n Q_j^* + Q_{j,n-1}(t),$$

where $P_{j,n-1}$ and $Q_{j,n-1}$ are matrix polynomials of order $n-1$. Then S can be expressed according to

$$S(t) = \sum_{j=1}^{N} \mathrm{e}^{\alpha t} t^n \{P_j^* \cos(\beta_j t) + Q_j^* \sin(\beta_j t)\} + \sum_{j=1}^{N} \mathrm{e}^{\alpha t} \{P_{j,n-1}(t)\cos(\beta_j t) + Q_{j,n-1}(t)\sin(\beta_j t)\}.$$

Thus,

$$\begin{aligned} &\int_0^t \frac{S(t-s)}{t^n \mathrm{e}^{\alpha t}} \Sigma(s)\, dB(s) \qquad (3.6.4) \\ &= \int_0^t \sum_{j=1}^{N} \mathrm{e}^{-\alpha s} \frac{(t-s)^n}{t^n} \{P_j^* \cos(\beta_j(t-s)) + Q_j^* \sin(\beta_j(t-s))\}\Sigma(s)\, dB(s) \\ &+ \int_0^t \sum_{j=1}^{N} \mathrm{e}^{-\alpha s} \frac{P_{j,n-1}(t-s)}{t^n} \cos(\beta_j(t-s))\Sigma(s)\, dB(s) \\ &+ \int_0^t \sum_{j=1}^{N} \mathrm{e}^{-\alpha s} \frac{Q_{j,n-1}(t-s)}{t^n} \sin(\beta_j(t-s))\Sigma(s)\, dB(s). \end{aligned}$$

We now argue that the second and third stochastic integrals on the right–hand side in (3.6.4) tend to zero as $t \to \infty$. We focus on the second integral. Note that it suffices to show for any degree $n-1$ polynomial P that

$$\int_0^t \frac{P(t-s)}{(1+t)^n} \cos(\beta(t-s)) \mathrm{e}^{-\alpha s} \Sigma(s)\, dB(s) \to 0, \quad \text{as } t \to \infty, \quad \text{a.s.}$$

By recalling the trigonometric identity, for any $a_1, a_2 \in \mathbb{R}$,

$$\begin{aligned} \cos(a_1 - a_2) &= \cos(a_1)\cos(a_2) + \sin(a_1)\sin(a_2), \qquad (3.6.5) \\ \sin(a_1 - a_2) &= \sin(a_1)\cos(a_2) - \cos(a_1)\sin(a_2), \end{aligned}$$

we see that it suffices to show that the process

$$a(t) = \int_0^t \frac{P(t-s)}{(1+t)^n} f(s)\, dB(s),$$

obeys $a(t) \to 0$ as $t \to \infty$ where f is in $L^2(\mathbb{R}_+; \mathbb{R}^{d\times d'})$ and P is a matrix–valued polynomial of degree $n-1$. Define $H(t,s) = P(t-s)(1+t)^{-n} f(s)$. Define $H_\infty(s) = 0$. Since P is a polynomial, there exists M such that $|P(t)| \le M(1+t)^{n-1}$ and $|P'(t)| \le M(1+t)^{n-1}$ for all $t \ge 0$.

Using Theorem 3.2.4 and the same procedure as used to establish (3.6.3), we get

$$\lim_{t\to\infty}\int_0^t \sum_{j=1}^N \mathrm{e}^{-\alpha s}\frac{P_{j,n-1}(t-s)}{t^n}\cos(\beta_j(t-s))\Sigma(s)\,dB(s) = 0, \quad \text{a.s.}$$

One can argue similarly that

$$\lim_{t\to\infty}\int_0^t \sum_{j=1}^N \mathrm{e}^{-\alpha s}\frac{Q_{j,n-1}(t-s)}{t^n}\cos(\beta_j(t-s))\Sigma(s)\,dB(s) = 0, \quad \text{a.s.}$$

We now turn our attention to the first integral term on the right–hand side of (3.6.4). Consider the integral

$$A_j(t) = \int_0^t \mathrm{e}^{-\alpha s}\frac{(t-s)^n}{t^n}P_j^*\cos(\beta_j(t-s))\Sigma(s)\,dB(s), \tag{3.6.6}$$

and define

$$A_{j,0}(t) = P_j^*\cos(\beta_j t)\int_0^t \cos(\beta_j s)\mathrm{e}^{-\alpha s}\Sigma(s)\,dB(s) \\ + P_j^*\sin(\beta_j t)\int_0^t \sin(\beta_j s)\mathrm{e}^{-\alpha s}\Sigma(s)\,dB(s).$$

Since $s\mapsto \mathrm{e}^{-\alpha s}\Sigma(s)$ is in $L^2(\mathbb{R}_+;\mathbb{R}^{d\times d'})$, if we define

$$A_{j,0}^*(t) = P_j^*\cos(\beta_j t)\int_0^\infty \cos(\beta_j s)\mathrm{e}^{-\alpha s}\Sigma(s)\,dB(s) \\ + P_j^*\sin(\beta_j t)\int_0^\infty \sin(\beta_j s)\mathrm{e}^{-\alpha s}\Sigma(s)\,dB(s). \tag{3.6.7}$$

we have that $A_{j,0}(t) - A_{j,0}^*(t) \to 0$ as $t\to\infty$ a.s. By Newton's binomial expansion theorem $(t-s)^n = \sum_{m=0}^n \binom{n}{m} t^m(-s)^{n-m}$ and using (3.6.5), we get

$$\begin{aligned} A_j(t) &= \sum_{m=0}^n P_j^*(-1)^{n-m}\binom{n}{m}\frac{1}{t^{n-m}}\int_0^t s^{n-m}\cos(\beta_j(t-s))\mathrm{e}^{-\alpha s}\Sigma(s)\,dB(s) \\ &= \sum_{m=0}^{n-1} P_j^*(-1)^{n-m}\binom{n}{m}A_{j,n-m}(t) + A_{j,0}(t), \end{aligned}$$

where we have defined for $k = 1,\dots,n$

$$A_{j,k}(t) = \frac{1}{t^k}\int_0^t s^k\left(\cos(\beta_j t)\cos(\beta_j s) + \sin(\beta_j t)\sin(\beta_j s)\right)\mathrm{e}^{-\alpha s}\Sigma(s)\,dB(s).$$

This can be expressed as

$$A_{j,k}(t) = \cos(\beta_j t)\frac{1}{t^k}\int_0^t s^k\cos(\beta_j s)\mathrm{e}^{-\alpha s}\Sigma(s)\,dB(s) \\ + \sin(\beta_j t)\frac{1}{t^k}\int_0^t s^k\sin(\beta_j s)\mathrm{e}^{-\alpha s}\Sigma(s)\,dB(s).$$

Now by applying Lemma 3.6.1 to each of the terms on the righthand side, we get

$$\lim_{t\to\infty} A_{j,k}(t) = 0, \quad \text{a.s.}$$

Therefore we see that

$$A_j(t) - A^*_{j,0}(t) \to 0, \quad \text{as } t \to \infty \text{ a.s.} \tag{3.6.8}$$

Define

$$C_j(t) = \int_0^t \mathrm{e}^{-\alpha s} \frac{(t-s)^n}{t^n} Q^*_j \sin(\beta_j(t-s))\Sigma(s)\, dB(s) \tag{3.6.9}$$

and

$$C_{j,0}(t) = Q^*_j \int_0^t \sin(\beta_j(t-s))\mathrm{e}^{-\alpha s}\Sigma(s)\, dB(s).$$

Then

$$C_{j,0}(t) = Q^*_j \sin(\beta_j t) \int_0^t \cos(\beta_j s)\mathrm{e}^{-\alpha s}\Sigma(s)\, dB(s) \\ - Q^*_j \cos(\beta_j t) \int_0^t \sin(\beta_j s)\mathrm{e}^{-\alpha s}\Sigma(s)\, dB(s),$$

and define

$$C^*_{j,0}(t) = Q^*_j \sin(\beta_j t) \int_0^\infty \cos(\beta_j s)\mathrm{e}^{-\alpha s}\Sigma(s)\, dB(s) \\ - Q^*_j \cos(\beta_j t) \int_0^\infty \sin(\beta_j s)\mathrm{e}^{-\alpha s}\Sigma(s)\, dB(s). \tag{3.6.10}$$

Then $C_{j,0}(t) - C^*_{j,0}(t) \to 0$ as $t \to \infty$ a.s., and by proceeding as before we obtain

$$C_j(t) - C^*_{j,0}(t) \to 0, \quad \text{as } t \to \infty \text{ a.s.} \tag{3.6.11}$$

Therefore, returning to (3.6.4) and using (3.6.6), (3.6.9) we have

$$\int_0^t \frac{S(t-s)}{t^n \mathrm{e}^{\alpha t}} \Sigma(s)\, dB(s) - \sum_{j=1}^N \{A^*_{j,0}(t) + C^*_{j,0}(t)\} \tag{3.6.12}$$
$$= \sum_{j=1}^N \left\{A_j(t) - A^*_{j,0}(t)\right\} + \sum_{j=1}^N \left\{C_j(t) - C^*_{j,0}(t)\right\}$$
$$+ \int_0^t \sum_{j=1}^N \mathrm{e}^{-\alpha s} \frac{P_{j,n-1}(t-s)}{t^n} \cos(\beta_j(t-s))\Sigma(s)\, dB(s)$$
$$+ \int_0^t \sum_{j=1}^N \mathrm{e}^{-\alpha s} \frac{Q_{j,n-1}(t-s)}{t^n} \sin(\beta_j(t-s))\Sigma(s)\, dB(s),$$

so by (3.6.8) and (3.6.11) we have

$$\lim_{t\to\infty}\left(\int_0^t \frac{S(t-s)}{t^n e^{\alpha t}}\Sigma(s)\,dB(s)-\sum_{j=1}^{N}\{A^*_{j,0}(t)+C^*_{j,0}(t)\}\right)=0,\quad \text{a.s.} \tag{3.6.13}$$

Using (3.6.1), (3.6.3), (3.6.13) together with the definitions (3.6.7) and (3.6.10), we have

$$\lim_{t\to\infty}\left(\frac{Y(t)}{t^n e^{\alpha t}}-\sum_{j=1}^{N}\{\sin(\beta_j t)L_{1,j}+\cos(\beta_j t)L_{2,j}\}\right)=0,\quad \text{a.s.} \tag{3.6.14}$$

where $L_{1,j}$ and $L_{2,j}$ are given by (3.3.8a) and (3.3.8b), which is (3.3.7).

3.6.3 Proof of Proposition 3.3.1 for $\mathbf{n = 0}$

The proof of Proposition 3.3.1, in the case $n = 0$, uses Lemma 3 from Appleby [4]. We state this lemma for completeness.

Lemma 3.6.2. *Suppose $x : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, integrable function, and $\eta > 0$ is any fixed constant. Then, the sequence $\{a_n\}_{n=0}^{\infty}$ given by $a_0 = 0$ and*

$$a_{n+1}=\inf\left\{t\in[a_n+\eta/2,a_n+3\eta/4]: x(t)=\min_{a_n+\eta/2\le\tau\le a_n+3\eta/4}x(\tau)\right\},\quad n\in\mathbb{Z}^+,$$

satisfies

$$\frac{\eta}{4}<a_{n+1}-a_n<\eta \quad \textit{for all } n\in\mathbb{Z}^+,\quad \lim_{n\to\infty}a_n=\infty,$$

together with

$$\sum_{n=0}^{\infty}x(a_n)<\infty.$$

The following lemma, to be used in the proof of Proposition 3.3.1 ($n = 0$), is a mild adaptation of Lemma 5.2 from [20].

Lemma 3.6.3. *Let $k : \mathbb{R}_+ \to \mathbb{R}$ be such that $k, k' \in L^2([0,\infty);\mathbb{R})$. Define for $f \in L^2([0,\infty);\mathbb{R})$ the Gaussian process $\{K(t) : t \ge 0\}$ by*

$$K(t)=\int_0^t k(t-s)f(s)\,dB(s).$$

Then $\lim_{t\to\infty} K(t) = 0$, a.s.

Proof of Lemma 3.6.3. Re–express K, using the stochastic Fubini's Theorem, e.g. [102, Theorem 4.6.64, pp.210–211], according to

$$\begin{aligned} K(t) &= \int_0^t \left(k(0) + \int_0^{t-s} k'(u)\,du \right) f(s)\,dB(s) \\ &= \int_0^t k(0) f(s)\,dB(s) + \int_0^t \int_s^t k'(v-s)\,dv\, f(s)\,dB(s) \\ &= k(0) \int_0^t f(s)\,dB(s) + \int_0^t \int_0^v k'(v-s) f(s)\,dB(s)\,dv. \end{aligned}$$

Then for any increasing sequence $\{a_n\}_{n=0}^{\infty}$ we have, for $t \in [a_n, a_{n+1})$,

$$K(t) = K(a_n) + k(0) \int_{a_n}^t f(s)\,dB(s) + \int_{a_n}^t \int_0^v k'(v-s) f(s)\,dB(s)\,dv.$$

Squaring, taking suprema and finally an expectation across this inequality gives

$$\begin{aligned} \mathbb{E}\left[\sup_{a_n \le t \le a_{n+1}} |K(t)|^2 \right] &\le 3\,\mathbb{E}\left[K(a_n)^2\right] + 3\,k(0)^2\,\mathbb{E}\left[\sup_{a_n \le t \le a_{n+1}} \left| \int_{a_n}^t f(s)\,dB(s) \right|^2 \right] \\ &\quad + 3\,\mathbb{E}\left[\sup_{a_n \le t \le a_{n+1}} \left| \int_{a_n}^t \int_0^v k'(v-s) f(s)\,dB(s)\,dv \right|^2 \right]. \end{aligned} \tag{3.6.15}$$

We consider each term on the right–hand side separately. Now for the second term, applying Doob's inequality, c.f. e.g. [84, Theorem 1.38] yields

$$\mathbb{E}\left[\sup_{a_n \le t \le a_{n+1}} \left| \int_{a_n}^t f(s)\,dB(s) \right|^2 \right] \le 4 \int_{a_n}^{a_{n+1}} f(s)^2\,ds$$

and thus

$$\sum_{n=0}^{\infty} \mathbb{E}\left[\sup_{a_n \le t \le a_{n+1}} \left| \int_{a_n}^t f(s)\,dB(s) \right|^2 \right] < +\infty. \tag{3.6.16}$$

For the third term, applying the Cauchy–Schwarz inequality gives

$$\begin{aligned} &\mathbb{E}\left[\sup_{a_n \le t \le a_{n+1}} \left| \int_{a_n}^t \int_0^v k'(v-s) f(s)\,dB(s)\,dv \right|^2 \right] \\ &\qquad \le \mathbb{E}\left[\sup_{a_n \le t \le a_{n+1}} (t-a_n) \int_{a_n}^t \left| \int_0^v k'(v-s) f(s)\,dB(s) \right|^2 dv \right] \\ &\qquad = (a_{n+1}-a_n) \int_{a_n}^{a_{n+1}} \mathbb{E}\left[\left| \int_0^v k'(v-s) f(s)\,dB(s) \right|^2 \right] dv \\ &\qquad = (a_{n+1}-a_n) \int_{a_n}^{a_{n+1}} \int_0^v k'(v-s)^2 f(s)^2\,ds\,dv. \end{aligned}$$

Now suppose that $0 < a_{n+1} - a_n < \eta$ for some $\eta > 0$, then

$$\begin{aligned} &\sum_{n=1}^{\infty} \mathbb{E}\left[\sup_{a_n \le t \le a_{n+1}} \left| \int_{a_n}^t \int_0^v k'(v-s) f(s)\,dB(s)\,dv \right|^2 \right] \\ &\qquad \le \eta \sum_{n=1}^{\infty} \int_{a_n}^{a_{n+1}} \int_0^v k'(v-s)^2 f(s)^2\,ds\,dv < +\infty. \end{aligned} \tag{3.6.17}$$

Now the first term, $t \mapsto x(t) = \mathbb{E}[K(t)^2]$, is continuous and non–negative, also

$$\int_0^\infty x(t)\,dt = \int_0^\infty k(t)^2\,dt \int_0^\infty f(s)^2\,ds < +\infty.$$

Therefore by Lemma 3.6.2, for all $\eta > 0$ there exists a sequence $\{a_n\}_{n=0}^\infty$ such that

$$\sum_{n=0}^\infty x(a_n) = \sum_{n=0}^\infty \mathbb{E}[K(a_n)^2] < +\infty. \tag{3.6.18}$$

So, using (3.6.16), (3.6.17) and (3.6.18) in (3.6.15) yields

$$\sum_{n=0}^\infty \mathbb{E}\left[\sup_{a_n \le t \le a_{n+1}} |K(t)|^2\right] < +\infty.$$

By the Monotone Convergence Theorem, c.f. e.g. [115, Theorem 5.3],

$$\mathbb{E}\left[\sum_{n=0}^\infty \sup_{a_n \le t \le a_{n+1}} |K(t)|^2\right] < +\infty.$$

and hence

$$\sum_{n=0}^\infty \sup_{a_n \le t \le a_{n+1}} |K(t)|^2 < +\infty, \quad \text{a.s.}$$

Thus,

$$\lim_{n\to\infty} \sup_{a_n \le t \le a_{n+1}} |K(t)|^2 = 0, \quad a.s.$$

and therefore $\lim_{t\to\infty} K(t) = 0$, a.s. □

3.6.4 Proof of Proposition 3.3.1 for n = 0

Using (3.3.4) and (3.3.6) we may write

$$Y(t) = \int_0^t S(t-s)\Sigma(s)\,dB(s) + \int_0^t R(t-s)\Sigma(s)\,dB(s), \quad t \ge 0,$$

where

$$S(t) = \sum_{j=1}^N \mathrm{e}^{\alpha t}\{P_j^* \cos(\beta_j t) + Q_j^* \sin(\beta_j t)\}.$$

Thus,

$$\mathrm{e}^{-\alpha t} Y(t) = \int_0^t \mathrm{e}^{-\alpha t} S(t-s)\Sigma(s)\,dB(s) + \int_0^t \mathrm{e}^{-\alpha t} R(t-s)\Sigma(s)\,dB(s). \tag{3.6.19}$$

Defining $k(t) = \mathrm{e}^{-\alpha t} R(t)$, then from (3.3.2) and (3.3.3), $k(t) = O(\mathrm{e}^{-\varepsilon t})$ and

$$|k'(t)| \le |\alpha||k(t)| + \mathrm{e}^{-\alpha t}|R'(t)| = O(\mathrm{e}^{-\varepsilon t})$$

Thus

$$\int_0^t \mathrm{e}^{-\alpha t} R(t-s)\Sigma(s)\,dB(s) = \int_0^t k(t-s)\mathrm{e}^{-\alpha s}\Sigma(s)\,dB(s)$$

and so Lemma 3.6.3 applied element–wise gives

$$\lim_{t\to\infty}\int_0^t e^{-\alpha t}R(t-s)\Sigma(s)\,dB(s)=0 \quad \text{a.s.} \tag{3.6.20}$$

Moreover,

$$\begin{aligned}\lim_{t\to\infty}\Bigg(\int_0^t e^{-\alpha t}S(t-s)\Sigma(s)\,dB(s) \\ -\cos(\beta_j t)\int_0^\infty e^{-\alpha s}\{P_j^*\cos(\beta_j s)-Q_j^*\sin(\beta_j s)\}\Sigma(s)\,dB(s) \\ -\sin(\beta_j t)\int_0^\infty e^{-\alpha s}\{P_j^*\sin(\beta_j s)+Q_j^*\cos(\beta_j s)\}\Sigma(s)\,dB(s)\Bigg)=0.\end{aligned} \tag{3.6.21}$$

Using (3.6.20) and (3.6.21) in (3.6.19), gives the required result.

3.6.5 Proof of Corollary 3.3.1

Lemma 3.6.4. *For any* $\phi\in L^1([0,\infty);\mathbb{R}^d)$,

$$\lim_{t\to\infty}\frac{1}{t^j}\int_0^t s^j\phi(s)\,ds=0, \quad j=1,...,n.$$

Proof. For any $\theta\in(0,1)$,

$$\begin{aligned}\left|\frac{1}{t^j}\int_0^t s^j\phi(s)ds\right| &\le \frac{1}{t^j}\int_0^{\theta t} s^j|\phi(s)|ds+\frac{1}{t^j}\int_{\theta t}^t s^j|\phi(s)|ds \\ &\le \theta^j\int_0^\infty|\phi(s)|ds+\int_{\theta t}^\infty|\phi(s)|ds\end{aligned}$$

Thus,

$$\limsup_{t\to\infty}\left|\frac{1}{t^j}\int_0^t s^j\phi(s)ds\right|\le\theta^j\int_0^\infty|\phi(s)|ds.$$

Letting $\theta\to 0$ gives the result. □

3.6.6 Proof of Corollary 3.3.1

Firstly consider the case $n\ge 1$. The asymptotic behaviour of Y is known from Proposition 3.3.1. Thus we concentrate solely upon the term $\int_0^t r(t-s)f(s)\,ds$ in (3.3.10) in determining the asymptotic behaviour of V. Defining

$$S(t)=\sum_{j=1}^N e^{\alpha t}\{P_j(t)\cos(\beta_j t)+Q_j(t)\sin(\beta_j t)\}, \quad t\ge 0.$$

Then we have

$$\int_0^t\frac{r(t-s)}{t^n e^{\alpha t}}f(s)\,ds=\int_0^t\frac{S(t-s)}{t^n e^{\alpha t}}f(s)\,ds+\int_0^t\frac{R(t-s)}{t^n e^{\alpha t}}f(s)\,ds$$

Then,

$$\left|\int_0^t \frac{R(t-s)}{t^n e^{\alpha t}} f(s)\,ds\right| \le \frac{1}{(1+t)^n} M \int_0^t (1+t-s)^{n-1} e^{-\alpha s}|f(s)|ds$$
$$\le \frac{1}{1+t} M \int_0^t e^{-\alpha s}|f(s)|ds$$

Taking the limit superior, as $t \to \infty$, over this inequality yields,

$$\lim_{t\to\infty} \int_0^t \frac{R(t-s)}{t^n e^{\alpha t}} f(s)\,ds = 0.$$

In analysising the term $S(t-s)$ one may decompose the trigonometric terms via (3.6.5), whilst the polynomial terms, P_j and Q_j may be dealt with using Newton's binomial expansion, i.e.

$$(t-s)^n = \sum_{m=0}^{n} \binom{n}{m} t^{n-m}(-s)^m.$$

This, together with Lemma 3.6.4, yields

$$\lim_{t\to\infty}\left(\int_0^t \frac{r(t-s)}{t^n e^{\alpha t}} f(s)\,ds - \sum_{j=1}^{N}\{\sin(\beta_j t)D_{1,j} + \cos(\beta_j t)D_{2,j}\}\right) = 0.$$

with

$$D_{1,j} = \int_0^\infty e^{-\alpha s}\{P_j^* \sin(\beta_j s) + Q_j^* \cos(\beta_j s)\} f(s)\,ds,$$
$$D_{2,j} = \int_0^\infty e^{-\alpha s}\{P_j^* \cos(\beta_j s) - Q_j^* \sin(\beta_j s)\} f(s)\,ds.$$

Combining this with Proposition 3.3.1 yields the result for V.

For the case $n = 0$, the proof follows as for the case $n \ge 1$. However in the analysis of the remainder term, R, it is required to understand the asymptotic behaviour of the integral

$$\int_0^t e^{-\varepsilon(t-s)} e^{-\alpha s} f(s)\,ds.$$

This integral is the convolution of a term in $L^1(0,\infty)$ with a term which tends to zero. Hence this integral itself tends to zero, [53, Theorem 2.2.2 (i)].

3.6.7 Proof of Lemma 3.3.1

We start with the proof of a preliminary lemma.

Lemma 3.6.5. *Let $K_{j,0}$ be defined by* (3.3.19) *with $n = 0$. Then*

$$\left(\lambda_j I_d - \int_{[0,\infty)} e^{-\lambda_j s}\mu(ds)\right) K_{j,0} = 0_{d,d},$$

where $\lambda_j \in \Lambda'$ are zeroes of $h_\mu(\lambda)$.

A corresponding result can be shown for the zeroes of the characteristic equation, g_ν, of the finite delay equation using (3.3.28) and is omitted.

Proof of Lemma 3.6.5. Multiply (3.3.19) on the left by $(\lambda-\lambda_j)\left(\lambda I_d - \int_{[0,\infty)} \mathrm{e}^{-\lambda s}\mu(ds)\right)$ to get

$$(\lambda-\lambda_j)I_d = \left(\lambda I_d - \int_{[0,\infty)} \mathrm{e}^{-\lambda s}\mu(ds)\right) K_{j,0} + (\lambda-\lambda_j)\left(\lambda I_d - \int_{[0,\infty)} \mathrm{e}^{-\lambda s}\mu(ds)\right)\hat{q}_j(\lambda).$$

Now let $\lambda \to \lambda_j$, recalling that $\hat{q}_j(\lambda)$ is analytic at λ_j, to get the result. □

3.6.8 Proof of Lemma 3.3.1

Define $\tilde{q}(t) = \mathrm{e}^{-\alpha t} q(t)$ for $t \geq 0$. Then $\tilde{q}$ is differentiable a.e. and

$$\int_0^\infty \mathrm{e}^{\varepsilon t}|\tilde{q}(t)|\,dt < +\infty,$$

where ε is defined as in Subsection 3.3.3. Also $|\tilde{q}'(t)| \leq \mathrm{e}^{-\alpha t}|q'(t)| + |\alpha|\mathrm{e}^{-\alpha t}|q(t)|$ for $t \geq 0$. Since $q, q' \in L^1(\mathbb{R}^+;\varphi;\mathbb{R}^{d\times d})$, we have

$$\int_0^\infty \mathrm{e}^{\varepsilon t}|\tilde{q}'(t)|\,dt \leq \int_0^\infty \mathrm{e}^{\varepsilon t}\mathrm{e}^{-\alpha t}|q'(t)|\,dt + \int_0^\infty |\alpha|\,\mathrm{e}^{\varepsilon t}\mathrm{e}^{-\alpha t}|q(t)|\,dt < +\infty.$$

Finally, we have that

$$\tilde{q}(t)\mathrm{e}^{\varepsilon t} = \tilde{q}(0) + \int_0^t \tilde{q}'(s)\mathrm{e}^{\varepsilon s}\,ds + \varepsilon\int_0^t \tilde{q}(s)\mathrm{e}^{\varepsilon s}\,ds,$$

so $|\tilde{q}(t)| \leq C\mathrm{e}^{-\varepsilon t}$ for all $t \geq 0$.

Let $\Lambda_n' = \{\lambda_1, ..., \lambda_N\}$. Then from (3.3.17) and (3.3.20), we get

$$\begin{aligned}
&\mathrm{e}^{-\alpha t}R(t)\\
&= \sum_{\lambda_j\in\Lambda_\varepsilon\setminus\Lambda_n',\Im(\lambda_j)\geq 0} \mathrm{e}^{-(\alpha-\Re(\lambda_j))t}\{P_j(t)\cos(\Im(\lambda_j)t) + Q_j(t)\sin(\Im(\lambda_j)t)\} + \tilde{q}(t)\\
&= \sum_{\lambda_j\in\Lambda'\setminus\Lambda_n',\Im(\lambda_j)\geq 0} \mathrm{e}^{-(\alpha-\Re(\lambda_j))t}\{P_j(t)\cos(\Im(\lambda_j)t) + Q_j(t)\sin(\Im(\lambda_j)t)\}\\
&\quad + \sum_{\lambda_j\in\Lambda_\varepsilon\setminus\Lambda',\Im(\lambda_j)\geq 0} \mathrm{e}^{-(\alpha-\Re(\lambda_j))t}\{P_j(t)\cos(\Im(\lambda_j)t) + Q_j(t)\sin(\Im(\lambda_j)t)\} + \tilde{q}(t).
\end{aligned}$$

If $n=0$, then $R(t) = O(\mathrm{e}^{(\alpha-\varepsilon)t})$ as $t\to\infty$. If $n \geq 1$, and $\Lambda_n' = \Lambda' \cap \{\Im(\lambda)\geq 0\}$, then $R(t) = O(\mathrm{e}^{(\alpha-\varepsilon)t})$. If $n\geq 1$, and $\Lambda_n' \subset \Lambda'\cap\{\Im(\lambda)\geq 0\}$, then $R(t) = O(t^{n-1}\mathrm{e}^{\alpha t})$ as $t\to\infty$. Therefore if $n \geq 1$, we always have $R(t) = O(t^{n-1}\mathrm{e}^{\alpha t})$ as $t\to\infty$.

We now prove the estimate on the derivative. We deal here with the case $n \geq 1$. From (3.3.13) we know that r is differentiable and hence from (3.3.20) so too is R. Defining

$$S(t) := \sum_{j=1}^{N} \mathrm{e}^{\alpha t}\{P_j(t)\cos(\beta_j t) + Q_j(t)\sin(\beta_j t)\}$$

and using (3.3.13) and (3.3.20) we have

$$R'(t) = r'(t) - S'(t) = \int_{[0,t]} \mu(ds)\, r(t-s) - S'(t)$$

It is clear from (3.3.17) that $r(t) = O(t^n \mathrm{e}^{\alpha t})$ and from the definition of S that $S'(t) = O(t^n \mathrm{e}^{\alpha t})$. Therefore, it follows that $\|r(t)\| \leq M(1+t)^n \mathrm{e}^{\alpha t}$ and $\|S'(t)\| \leq M(1+t)^n \mathrm{e}^{\alpha t}$ for $t \geq 0$ and some $M > 0$. Hence as $|\mu| \in M(\mathbb{R}_+;\mathbb{R})$ and $\int_{[0,\infty)} \mathrm{e}^{-\alpha s}|\mu|(ds) < +\infty$, we have

$$\begin{aligned}
\left\|R'(t)\right\| &\leq \int_{[0,t]} |\mu|(ds)\, \|r(t-s)\| + \left\|S'(t)\right\| \\
&\leq \int_{[0,t]} |\mu|(ds)\, M(1+t-s)^n \mathrm{e}^{\alpha(t-s)} + M(1+t)^n \mathrm{e}^{\alpha t} \\
&\leq \int_{[0,t]} |\mu|(ds)\, M(1+t)^n \mathrm{e}^{\alpha(t-s)} + M(1+t)^n \mathrm{e}^{\alpha t} \\
&\leq M(1+t)^n \mathrm{e}^{\alpha t} \int_{[0,\infty)} \mathrm{e}^{-\alpha s}|\mu|(ds) + M(1+t)^n \mathrm{e}^{\alpha t},
\end{aligned}$$

and therefore $R'(t) = O(t^n \mathrm{e}^{\alpha t})$ for $n \geq 1$.

For the case $n = 0$, we define

$$S(t) := \sum_{j=1}^{N} \mathrm{e}^{\alpha t}\{P_j^* \cos(\beta_j t) + Q_j^* \sin(\beta_j t)\},$$

then the real function S can be rewritten concisely using complex constants as

$$S(t) = \sum_{\lambda_j \in \Lambda'} \mathrm{e}^{\lambda_j t} K_{j,0}.$$

As $R(t) = r(t) - S(t)$ we have

$$\begin{aligned}
R'(t) = r'(t) - S'(t) &= \int_{[0,t]} \mu(ds) r(t-s) - \lambda_j \sum_{\lambda_j \in \Lambda'} \mathrm{e}^{\lambda_j t} K_{j,0} \\
&= \int_{[0,t]} \mu(ds) R(t-s) + \int_{[0,t]} \mu(ds) \sum_{\lambda_j \in \Lambda'} \mathrm{e}^{\lambda_j (t-s)} K_{j,0} - \sum_{\lambda_j \in \Lambda'} \lambda_j\, \mathrm{e}^{\lambda_j t} K_{j,0} \\
&= \int_{[0,t]} \mu(ds) R(t-s) - \sum_{\lambda_j \in \Lambda'} \mathrm{e}^{\lambda_j t} \left(\lambda_j\, I_d - \int_{[0,t]} \mathrm{e}^{-\lambda_j s} \mu(ds) \right) K_{j,0} \\
&= \int_{[0,t]} \mu(ds) R(t-s) - \sum_{\lambda_j \in \Lambda'} \mathrm{e}^{\lambda_j t} \left(\lambda_j\, I_d - \int_{[0,\infty)} \mathrm{e}^{-\lambda_j s} \mu(ds) \right) K_{j,0} \\
&\quad - \sum_{\lambda_j \in \Lambda'} \mathrm{e}^{\lambda_j t} \int_{(t,\infty)} \mathrm{e}^{-\lambda_j s} \mu(ds) K_{j,0}.
\end{aligned}$$

By Lemma 3.6.5 the second term on the right–hand side is equal to zero, and so

$$|R'(t)| \le \left|\int_{[0,t]} \mu(ds)R(t-s)\right| + \left|\sum_{\lambda_j\in\Lambda'} \mathrm{e}^{\lambda_j t}\int_{(t,\infty)} \mathrm{e}^{-\lambda_j s}\mu(ds)K_{j,0}\right|. \tag{3.6.22}$$

Now,

$$\begin{aligned}\left|\int_{[0,t]} \mu(ds)R(t-s)\right| &\le \int_{[0,t]} |\mu|(ds)M\mathrm{e}^{(\alpha-\varepsilon)(t-s)} \\ &= \mathrm{e}^{(\alpha-\varepsilon)t}\int_{[0,t]} \mathrm{e}^{-(\alpha-\varepsilon)s}|\mu|(ds)M \\ &\le \mathrm{e}^{(\alpha-\varepsilon)t}\int_{[0,\infty)} \mathrm{e}^{-(\alpha-\varepsilon)s}|\mu|(ds)M.\end{aligned}$$

Thus, $\int_{[0,t]}\mu(ds)R(t-s) = O(\mathrm{e}^{(\alpha-\varepsilon)t})$. Recalling that $\lambda_j = \alpha + i\beta_j$ and so $|\mathrm{e}^{\lambda_j t}| = \mathrm{e}^{\alpha t}$. Thus,

$$\begin{aligned}\left|\sum_{\lambda_j\in\Lambda'} \mathrm{e}^{\lambda_j t}\int_{(t,\infty)} \mathrm{e}^{-\lambda_j s}\mu(ds)K_{j,0}\right| &\le \mathrm{e}^{\alpha t}\sum_{\lambda_j\in\Lambda'}\int_{(t,\infty)} \mathrm{e}^{-\alpha s}|\mu|(ds)M \\ &= \mathrm{e}^{\alpha t}\sum_{\lambda_j\in\Lambda'}\int_{(t,\infty)} \mathrm{e}^{-\varepsilon s}\mathrm{e}^{-(\alpha-\varepsilon)s}|\mu|(ds)M \\ &\le \mathrm{e}^{(\alpha-\varepsilon)t}\sum_{\lambda_j\in\Lambda'}\int_{(t,\infty)} \mathrm{e}^{-(\alpha-\varepsilon)s}|\mu|(ds)M \\ &\le \mathrm{e}^{(\alpha-\varepsilon)t}M_1,\end{aligned}$$

where it is noted that Λ' contains finitely many elements. Therefore, (3.6.22) gives

$$R'(t) = O(\mathrm{e}^{(\alpha-\varepsilon)t}), \quad t\to\infty.$$

3.6.9 Proof of Lemma 3.3.2

We now use (3.3.26) to determine properties of R of (3.3.29). From (3.3.29)

$$r(t) = \sum_{j=1}^{N} \mathrm{e}^{\alpha t}\{P_j(t)\cos(\beta_j t) + Q_j(t)\sin(\beta_j t)\} + R(t), \quad t\ge 0.$$

In the case when $\{\lambda_1, ..., \lambda_N\} = \Lambda' \cap \{\Im(\lambda)\ge 0\}$, we have that $R(t) = e^{(\alpha-\varepsilon)t}$ for all $\varepsilon\in(0,\varepsilon_0)$. If $n\ge 1$, and $\{\lambda_1, ..., \lambda_N\}\subset \Lambda'\cap\{\Im(\lambda)\ge 0\}$, then $R(t) = O(t^{n-1}e^{\alpha t})$ as $t\to\infty$. Therefore if $n\ge 1$, we always have $R(t) = O(t^{n-1}e^{\alpha t})$ as $t\to\infty$. If $n = 0$, then $R(t) = O(e^{(\alpha-\varepsilon)t})$ as $t\to\infty$.

We deal here with the case $n\ge 1$. From (3.3.23) we know that r is differentiable and hence from (3.3.29) so too is R. Defining

$$S(t) := \sum_{j=1}^{N} \mathrm{e}^{\alpha t}\{P_j(t)\cos(\beta_j t) + Q_j(t)\sin(\beta_j t)\}$$

and using (3.3.23) and (3.3.29) we have

$$R'(t) = r'(t) - S'(t) = \int_{[-\tau,0]} \nu(ds)\, r(t+s) - S'(t), \qquad \text{for all } t \geq \tau.$$

It is clear from (3.3.26) that $r(t) = O(t^n \mathrm{e}^{\alpha t})$ and from the definition of S that $S'(t) = O(t^n \mathrm{e}^{\alpha t})$. Thus, there exists $t_0 \geq 0$ and positive constant matrices M_1, M_2 such that for $t \geq t_0 + \tau$,

$$\begin{aligned} |R'(t)| &\leq \int_{[-\tau,0]} |\nu|(ds)\, |r(t+s)| + t^n \mathrm{e}^{\alpha t} M_2 \\ &\leq \int_{[-\tau,0]} |\nu|(ds)\, (s+t)^n \mathrm{e}^{\alpha(t+s)} M_1 + t^n \mathrm{e}^{\alpha t} M_2 \\ &\leq t^n \mathrm{e}^{\alpha t} \int_{[-\tau,0]} \mathrm{e}^{\alpha s}\, |\nu|(ds) M_1 + t^n \mathrm{e}^{\alpha t} M_2. \end{aligned}$$

Thus, $R'(t) = O(t^n \mathrm{e}^{\alpha t})$.

The case $n = 0$ follows from Remark 3.3.1 and a similar proof to that of Lemma 3.3.1.

Proof of Remark 3.3.6. In this case $r(t) = \mathrm{e}^{\alpha t}$ and X obeys, for $t \geq 0$,

$$e^{-\alpha t} X(t) = X_0 + \int_0^t \mathrm{e}^{-\alpha s} f(s)\, ds + \int_0^t e^{-\alpha s} \Sigma(s)\, dB(s). \tag{3.6.23}$$

Define the Gaussian martingale M by $M(t) = \int_0^t e^{-\alpha s}\Sigma(s)\, dB(s)$ and the deterministic function d by $d(t) = X_0 + \int_0^t \mathrm{e}^{-\alpha s} f(s)\, ds$. Then from (3.3.36), we have on this event of positive probability that

$$\lim_{t\to\infty} \{M(t) + d(t)\} = L \in (-\infty, \infty).$$

Suppose that $\lim_{t\to\infty} \langle M\rangle(t) = +\infty$. Consequently $\limsup_{t\to\infty} M(t) = +\infty$ and $\liminf_{t\to\infty} M(t) = -\infty$. Also, $\limsup_{t\to\infty} d(t) = +\infty$, otherwise, if $d(t) \leq D$ for all $t \geq 0$, we have

$$L = \liminf_{t\to\infty} \{d(t) + M(t)\} \leq D + \liminf_{t\to\infty} M(t) = -\infty,$$

which is a contradiction. (Similarly one can show that $\liminf_{t\to\infty} d(t) = -\infty$).

Then there exists a deterministic sequence $\{t_n\}_{n\in\mathbb{Z}^+}$, with $t_0 = 0$ and $t_n \to \infty$ as $n \to \infty$, such that $d(t_{n+1}) > d(t_n)$ and $d(t_n) \to \infty$ as $n \to \infty$. Then $M(t_n) \to -\infty$ as $n \to \infty$.

Now,

$$\tilde{M}(n) := M(t_n) = \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \mathrm{e}^{-\alpha s} \Sigma(s)\, dB(s) = \sum_{j=1}^n G_j,$$

where each $G_j = \int_{t_{j-1}}^{t_j} \mathrm{e}^{-\alpha s}\Sigma(s)\,dB(s)$ is a Gaussian distributed random variable with mean zero and variance $\int_{t_{j-1}}^{t_j} \mathrm{e}^{-2\alpha t}\Sigma(t)^2\,ds$, each G_j is measurable with respect to the filtration $\mathcal{G}_n = \mathcal{F}^B(t_n)$, $n \geq 1$, and $\{G_j\}_{j\in\mathbb{Z}^+}$ are independent and $\langle \tilde{M}\rangle(n) = \langle M\rangle(t_n) = \int_0^{t_n} \mathrm{e}^{-2\alpha t}\Sigma(t)^2\,ds \to \infty$ as $n \to \infty$.

Therefore by arguments akin to that used in Shiryeav [108, Section 4.1]

$$\mathbb{P}\left[\limsup_{n\to\infty} \frac{\tilde{M}(n)}{\sqrt{\langle\tilde{M}\rangle(n)}} = +\infty\right] = 1, \quad \mathbb{P}\left[\liminf_{n\to\infty} \frac{\tilde{M}(n)}{\sqrt{\langle\tilde{M}\rangle(n)}} = -\infty\right] = 1, \tag{3.6.24}$$

which implies that $\mathbb{P}\left[\limsup_{n\to\infty} \tilde{M}(n) = +\infty\right] = 1$ and so that

$$\mathbb{P}\left[\limsup_{n\to\infty} M(t_n) = +\infty\right] = 1.$$

But our assumption gave that $\lim_{n\to\infty} M(t_n) = -\infty$, with positive probability. Thus a contradiction. Hence $\langle M\rangle(t) \to L' \in (-\infty,\infty)$ as $t \to \infty$, i.e.

$$\int_0^\infty \mathrm{e}^{-2\alpha t}\Sigma(t)^2\,dt < +\infty.$$

Therefore $M(t) \to M(\infty) \in (-\infty,\infty)$ as $t \to \infty$ a.s. and so $\lim_{t\to\infty} d(t) = \lim_{t\to\infty}\{d(t) + M(t) - M(t)\} = L - M(\infty) \in (-\infty,\infty)$. Hence

$$\lim_{t\to\infty} \int_0^t \mathrm{e}^{-\alpha s} f(s)\,ds = \lim_{t\to\infty}\{d(t) - X_0\} \in (-\infty,\infty).$$

All that remains to be shown is the validity of (3.6.24), i.e. we need to show that

$$A' = \left\{\limsup_{n\to\infty} \frac{\tilde{M}(n)}{\sqrt{\langle\tilde{M}\rangle(n)}} = +\infty\right\}, \quad A'' = \left\{\liminf_{n\to\infty} \frac{\tilde{M}(n)}{\sqrt{\langle\tilde{M}\rangle(n)}} = -\infty\right\}$$

are almost sure events. Let

$$A'_c = \left\{\limsup_{n\to\infty} \frac{\tilde{M}(n)}{\sqrt{\langle\tilde{M}\rangle(n)}} > c\right\}, \quad A''_c = \left\{\liminf_{n\to\infty} \frac{\tilde{M}(n)}{\sqrt{\langle\tilde{M}\rangle(n)}} < -c\right\}.$$

Then $A'_c \to A'$ and $A''_c \to A''$ as $c \to \infty$ and A', A'', A'_c, A''_c are tail events. We show that $\mathbb{P}[A'_c] = \mathbb{P}[A''_c] = 1$ for all $c > 0$.

Using Section 4.1.5 Problem 5, pp.383 of [108] gives

$$\mathbb{P}[A'_c] = \mathbb{P}\left[\limsup_{n\to\infty} \frac{\tilde{M}(n)}{\sqrt{\langle\tilde{M}\rangle(n)}} > c\right] \geq \limsup_{n\to\infty} \mathbb{P}\left[\frac{\tilde{M}(n)}{\sqrt{\langle\tilde{M}\rangle(n)}} > c\right] = 1 - \Phi(c) > 0$$

and

$$\mathbb{P}[A''_c] = \mathbb{P}\left[\liminf_{n\to\infty} \frac{\tilde{M}(n)}{\sqrt{\langle\tilde{M}\rangle(n)}} < -c\right] = \mathbb{P}\left[\limsup_{n\to\infty} \frac{-\tilde{M}(n)}{\sqrt{\langle\tilde{M}\rangle(n)}} > c\right] \geq 1 - \Phi(c) > 0.$$

So, $\mathbb{P}[A'_c] > 0$ and $\mathbb{P}[A''_c] > 0$, then since the G_j's are independent an application of Kolomogrov's Zero-One Law, c.f. e.g. [108, Theorem 4.1.1], implies $\mathbb{P}[A'_c] = \mathbb{P}[A''_c] = 1$. Therefore $\mathbb{P}[A'] = \lim_{c\to\infty} \mathbb{P}[A'_c] = 1$ and $\mathbb{P}[A''] = \lim_{c\to\infty} \mathbb{P}[A''_c] = 1$. □

Chapter 4

Introduction: Long Memory and Financial Market Bubble Dynamics in Affine Stochastic Differential Equations with Average Functionals

4.1 Introduction and overview

This present chapter serves as in introduction to Chapters 5, 6 and 7. In particular, the common problem to be studied in both Chapters 5 and 6 is introduced as well as motivation for the study of this problem. The chief results of each of the chapters are also discussed. In Chapters 5 and 6, we consider the asymptotic behaviour of an affine scalar stochastic functional differential equation where the average of the process over its entire history appears on the right–hand side. Accordingly, we study

$$dX(t) = \left(aX(t) + b\frac{1}{1+t}\int_{-1}^{t} X(s)\,ds\right)\,dt + \sigma\,dB(t), \quad t \geq 0, \tag{4.1.1}$$

where X is given by the continuous function ψ, defined on $[-1,0]$, B is a standard one–dimensional Brownian motion and $\sigma \neq 0$. Here a and b are real parameters. There is a unique strong solution of (4.1.1) which is a Gaussian process. The goal of Chapters 5 and 6 is to describe for all pairs of the parameters a and b the asymptotic behaviour of the paths, as well as information about the autocovariance function of X in the case that the solution is recurrent on $\mathbb{R}$.

4.1.1 Organisation of results and methods of proof

Chapter 5 considers the case $a > 0$. Under this condition the solution X is shown to grow at a well–defined exponential rate, with a polynomial correction. Specifically, the rate of growth is given by

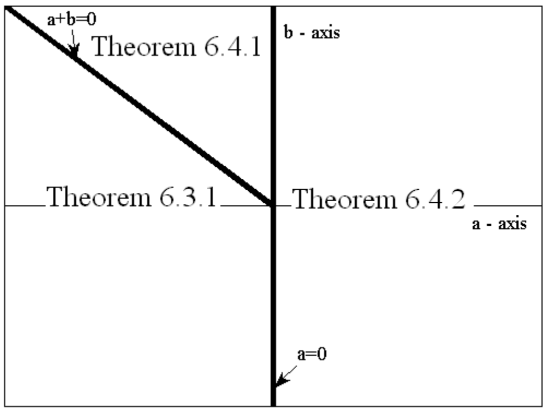
$$\lim_{t\to\infty} \frac{X(t)}{e^{at}t^{b/a}} = C, \quad \text{a.s.} \tag{4.1.2}$$

where C is an almost surely finite and Gaussian distributed random variable. The results in Chapter 5 rely on the theory of admissibility of linear deterministic Volterra operators. The work of Chapter 5 appears as a joint paper with Appleby [7].

While Chapter 6 establishes some new results concerning the case when $a > 0$, for the most part it is concerned with the case when $a \leq 0$, where the solution need not have a well-defined growth rate but rather may fluctuate. This behaviour is not wholly unexpected; in the case when $a < 0$ and $b = 0$, for example, the solution of (4.1.1) is an asymptotically stationary Ornstein–Uhlenbeck process, while when $a = 0$ and $b = 0$, it is a scaled standard Brownian motion.

A complete asymptotic dynamical picture of the solution X is determined for all real values of a and b in Chapter 6. Our analysis shows that there are only three principal regions in the '$a-b$' parameter space, within which the process X undergoes different path-wise asymptotic behaviour. For clarity we provide a bifurcation diagram of the parameter space:

Figure 4.1: Bifurcation diagram of '$a - b$' parameter space



- In Theorem 6.3.1, corresponding to $a < 0$ and $a + b \leq 0$, the solution X is asymptotically equal to an Ornstein–Uhlenbeck process and has oscillations of magnitude described by

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2\log t}} = \frac{\sigma}{\sqrt{2|a|}}, \quad \liminf_{t\to\infty} \frac{X(t)}{\sqrt{2\log t}} = -\frac{\sigma}{\sqrt{2|a|}}, \quad \text{a.s.} \tag{4.1.3}$$

- In Theorem 6.4.1, corresponding to $a < 0$ and $a + b > 0$, the solution X tends to plus or minus infinity at a polynomial rate

$$\lim_{t\to\infty} \frac{X(t)}{t^{-(1+\frac{b}{a})}} = C, \quad \text{a.s.} \tag{4.1.4}$$

where C is an almost surely finite proper random variable.

- In Theorem 6.4.2, corresponding to $a > 0$, the solution X is shown to obey (4.1.2)

- In Theorem 6.4.3, corresponding to $a = 0$ and $b > 0$, the solution grows at a rate which is faster than the polynomial growth of (4.1.4) yet slower than the exponential growth given by (4.1.2).

- In Theorem 6.3.6, corresponding to $a = 0$ and $b < 0$, the solution X is recurrent on $\mathbb{R}$ and its largest fluctuations are described by a result reminiscent of the Law of the Iterated Logarithm.

In analysing the solution of the stochastic equation it is helpful first to ask how the underlying deterministic equation behaves asymptotically. This deterministic equation is attained from (4.1.1) by letting $\sigma = 0$. The solution of this underlying equation (which corresponds to the mean of X) may be expressed in terms of confluent hypergeometric, modified Bessel and Bessel functions. Properties of these special functions are well–documented, c.f. e.g. [1, 95, 96]. An associated differential resolvent may also be decomposed in terms of these special functions. In Theorems 5.2.1, 6.4.1, 6.4.2 and 6.4.3 the asymptotic behaviour of the solution X may then be shown to mirror that of the deterministic equations, i.e. the asymptotic rates of growth or decay of the solutions of the deterministic equations are preserved under the addition of a stochastic perturbation. However, as Theorem 6.3.1 demonstrates the stochastic perturbation can for particular values of the parameters produce asymptotic behaviour which is distinct from that of the solution of the associated deterministic equation. The analysis is achieved via this decomposition of the resolvent and a variation of parameters formula.

Many of the asymptotic results concern pathwise behaviour. However, many of the growth results also hold true in mean or in mean square. Furthermore, in the main case where there are fluctuations (i.e., when $a < 0$ and $a + b \leq 0$), we show that the autocovariance function of the process X decays at a polynomial rate in time, i.e. for any fixed $t > 0$,

$$\lim_{\Delta\to\infty} \frac{\gamma_t(\Delta)}{\Delta^{-1-\frac{b}{a}}} = c_t \in (0,\infty),$$

where $\gamma_t(\cdot) = \mathrm{Cov}[X(t), X(t+\Delta)]$. Thus X may be viewed as possessing *long memory*, in the sense that for any fixed t,

$$\int_0^\infty \gamma_t(\Delta)\, d\Delta = +\infty, \quad a < 0, \quad b > 0, \quad a + b < 0.$$

This result is all the more striking as Theorem 6.3.1 proves that X is asymptotically equal to a process whose autocovariance function decays exponentially quickly, i.e. a "short memory" process. Moreover, it can be shown that X is transiently non–stationary, and has limiting autocovariance function equal to that of the stationary Ornstein–Uhlenbeck process to which it converges pathwise. We comment more on this result in the next section.

4.1.2 Motivation for the work

One of the motivations of this work is to develop a parameterised stochastic functional differential equation whose asymptotic behaviour is completely characterised, as such an equation can act as a test equation for simulation methods for SFDEs. Another mathematical motivation is to demonstrate that the general approach of admissibility theory developed in Chapter 5 can generate the same results as the special function theory outlined in Chapter 6 (at least in some cases), thus supporting the conjecture that it can prove a sharp tool in studying the asymptotic behaviour of linear, quasilinear or affine stochastic functional differential equations.

However, one of the main interests in examining this equation is to gain insight into some features of price dynamics in inefficient financial markets. First, we argue that (4.1.1) may be considered as a simple model of such a market. Suppose that there is a class of technical analysts who compare the current returns of a risky asset with the average of historical returns. This leads to an instantaneous excess demand of

$$\alpha \left(X_1(t) - \frac{1}{1+t} \int_{-1}^{t} X_1(s)\ \mathrm{d}s \right)$$

per unit time at time t. A class of feedback traders compare the returns to a reference level $\bar{X}$, leading to an instantaneous excess demand of

$$\beta(X_1(t) - \bar{X})$$

per unit time at time t. Unplanned demand by the traders arises from "news", where the news in each period is independent of that in previous periods. The contribution of this news to overall excess demand is $\sigma(B(t_2) - B(t_1))$ over the time interval $[t_1, t_2]$, where B is a standard one–dimensional Brownian motion. If we presume that returns respond linearly to the excess demand of the market, then X_1 obeys the stochastic functional differential equation

$$\mathrm{d}X_1(t) = \left(\alpha\left(X_1(t) - \frac{1}{1+t}\int_{-1}^{t} X_1(s)\ \mathrm{d}s\right) + \beta(X_1(t) - \bar{X})\right)\ \mathrm{d}t + \sigma\ \mathrm{d}B(t),$$

for $t \geq 0$, where $X_1(t) = \psi_1(t)$ for $t \in [-1, 0]$. The price of the risky asset at time $t \geq 0$ is denoted by $S(t)$ and defined by

$$\mathrm{d}S(t) = \mu S(t)\ \mathrm{d}t + S(t)\ \mathrm{d}X_1(t), \quad t \geq 0$$

with $S(0) = s_0$. Now define $X(t) = X_1(t) - \bar{X}$ for $t \geq 0$ and $\psi(t) = \psi_1(t) - \bar{X}$ for $t \in [-1, 0]$. Then X obeys (4.1.1) with $a = \alpha + \beta$ and $b = -\alpha$.

Motivation and literature for such models, as well as alternative inefficient market models may be found in [20], in which a market with finite memory is considered. In common with [20], in this work X_1 can grow to plus or minus infinity, with both events being possible. In terms of the mathematics, this happens if and only if

- $a > 0$;
- $a < 0$ and $a + b > 0$;
- $a = 0$ and $b > 0$.

From an economic perspective, the first case corresponds to the situation where the feedback traders chase trends ($\alpha > 0$) and either dominate the fundamental investors, who have mean–reverting expectations about price movements ($\alpha + \beta > 0$, $\beta < 0$) or both classes of agents have trend chasing type expectations ($\alpha > 0$, $\beta > 0$). The other two cases, while interesting mathematically, are less likely within the scope of the model: the second case requires $\beta > 0$, which implies that fundamental investors are bullish about higher than average returns, but $\alpha + \beta < 0$, which indicates these investors dominate the technical traders, who now have mean reverting expectations about returns. Nonetheless, this case serves to demonstrate that if at least one of the investor classes believes that high and rising returns are a signal of higher returns in the future, and that that class of agent dominates, then bubbles are likely outcomes. The third case occurs if the two classes of traders have equal strength, ($\beta + \alpha = 0$), with the technical traders having mean reverting expectations, and the fundamental investors being bullish about higher than average returns ($\alpha < 0$, $\beta > 0$).

In all these cases, the limiting random variable is path dependent, so it follows that the initial behaviour of the market determines whether there is a bubble or a crash. This picture is consistent with the mechanism proposed for the formation of bubbles with those formed in models of mimetic contagion, first introduced by Orléan [98].

From a modelling and time series perspective, the behaviour in the "non–bubble" case when $a < 0$ and $a + b \leq 0$, or $a = 0$ and $b < 0$ is also of interest. The former corresponds to the situation where $\alpha + \beta < 0$, $\beta \leq 0$, in which the fundamental investors have mean reverting expectations, and either dominate the technical investors (if they have trend chasing expectations) or the technical traders also have mean reverting expectations themselves. In this case as we observed the size of the largest fluctuations of the process is given by $\sigma/\sqrt{2|\alpha+\beta|}$. Thus as the process is actually mean reverting in this scenario it is in the interests of the trend chasing traders to ensure that $\alpha + \beta$ is as close to zero as possible so that the process undergoes as large fluctuations as possible. This phenomenon is observed in financial markets, i.e. when there is a large proportion of uninformed investors in a market then the volatility of the market tends to be higher than in their absence c.f. e.g. De Long et al. [41]. If however the uninformed investors where to force $\alpha + \beta > 0$ then this, as already observed, will result in the formation of an uncontrollable bubble.

The case when $a = 0$ and $b < 0$ is consistent with solutions obeying the law of the iterated logarithm, and so may be roughly associated with Gaussian processes that are non–stationary, but possess stationary increments. However, in the former case, not only (as we have already pointed out) is X is asymptotically indistinguishable from an asymptotically stationary process, it can be shown that X itself is asymptotically stationary (or *transiently non–stationary*), i.e.

$$\lim_{t\to\infty} \mathrm{Cov}(X(t), X(t+\Delta)) = \gamma(\Delta),$$

for some function $\gamma : \mathbb{R} \to \mathbb{R}$. Moreover this limiting autocovariance, as a function of Δ, decays exponentially and so is indicative of a short memory process. At the same time, we have already seen that when t is fixed and $\Delta \to \infty$, then $\Delta \to \mathrm{Cov}(X(t), X(t+\Delta))$ tends to zero at a polynomial rate, and is indeed non–integrable when $b > 0$. In a sense therefore, the process exhibits "long–memory" and "short–memory" characteristics. Of course, it is not unheard of that reversing the order of these limits leads to different answers, and while this is an interesting mathematical example of this phenomenon, it is otherwise not noteworthy. However, given that there is considerable debate among empiricists in finance concerning the presence or absence of long memory in certain financial time series, it is interesting to note that Chapter 6 presents an asymptotically stationary process in a (highly simplified, indeed unrealistic) market model, which also possesses somewhat ambiguous memory properties. For autonomous equations it is typically the case that one can permute these limits. We thus speculate that it is the non-autonomous nature of (4.1.1) which gives rises to this ambiguity.

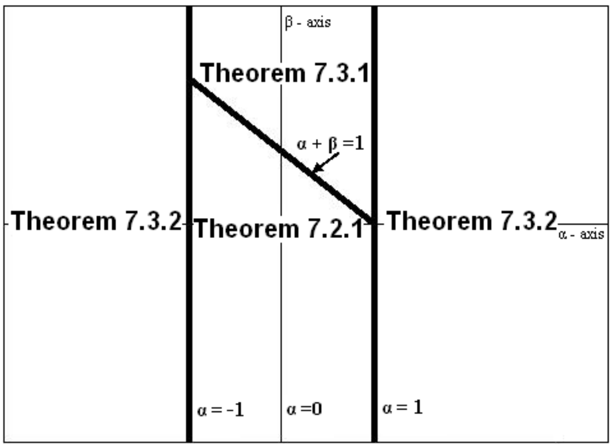
From the perspective of numerical simulation it is desirable that one can perform a discretisation of (4.1.1) which yields a discrete equation which preserves the asymptotic features (fluctuations, polynomial and exponential growth) of (4.1.1). It is argued in Chapter 7 that

$$X(n+1) = \alpha X(n) + \frac{\beta}{n+1}\sum_{j=0}^{n} X(j) + \sigma\xi(n+1), \quad n \in \{0,1,2,...\} \tag{4.1.5a}$$

$$X(0) = x_0 \in \mathbb{R}, \tag{4.1.5b}$$

serves as such a discretisation. A complete description of the pathwise asymptotic behaviour of (4.1.5) is given. For clarity we provide a bifurcation diagram of the parameter space:

Figure 4.2: Bifurcation diagram of '$\alpha - \beta$' parameter space



- In Theorem 7.2.1, corresponding to $\alpha \in (-1,1)$ and $\alpha + \beta \leq 1$, the solution X of (4.1.5) is asymptotically equal to the solution of an autoregressive process of order one and has oscillations of magnitude described by

$$\limsup_{n\to\infty} \frac{X(n)}{\sqrt{2\log n}} = -\limsup_{n\to\infty} \frac{X(n)}{\sqrt{2\log n}} = \frac{\sigma}{\sqrt{1-\alpha^2}}, \quad \text{a.s.} \tag{4.1.6}$$

- In Theorem 7.3.1, corresponding to $\alpha \in (-1,1)$ and $\alpha+\beta > 1$, the solution X of (4.1.5) tends to plus or minus infinity at a polynomial rate
$$\lim_{n\to\infty} \frac{X(n)}{n^{-1-\frac{\beta}{\alpha-1}}} = C_1, \quad \text{a.s.} \tag{4.1.7}$$
where C_1 is an almost surely finite proper random variable.

- In Theorem 7.3.2, corresponding to $|\alpha| > 1$, the solution X of (4.1.5) tends to plus or minus infinity at a geometric rate
$$\lim_{n\to\infty} \frac{X(n)}{\alpha^n n^{\frac{\beta}{\alpha-1}}} = C_2, \quad \text{a.s.} \tag{4.1.8}$$
where C_2 is an almost surely finite proper random variable.

- In Theorem 7.4.2, corresponding to $\alpha = 1$ and $\beta > 0$, the solution grows at a rate which is faster than the polynomial growth of (4.1.7) yet slower than the exponential growth given by (4.1.8).

- In Theorem 7.4.2, corresponding to $\alpha = 1$ and $\beta < 0$, the solution is recurrent on $\mathbb{Z}^+$.

- In Theorem 7.4.1, corresponding to $\alpha = -1$, the solution of (4.1.5) has asymptotic dynamical behaviour which, depending upon the value of β may be polynomial growth akin to (4.1.7), geometric growth akin to (4.1.8) or recurrent.

Areas of asymptotic behaviour of (4.1.5) are identified which are qualitatively and quantitatively analogous to areas of asymptotic behaviour of (4.1.1), e.g. the recurrence of (4.1.6) is akin to the recurrence of (4.1.3). Also it is shown that X is asymptotically equal to the solution of an autoregressive process of order one, with similar comments upon the ambiguity of the memory properties. For (4.1.1) there is a regime shift in the asymptotic properties depending whether a and $a+b$ are positive or negative, whereas for (4.1.5) the corresponding regime shift depends upon whether $|\alpha|$ and $\alpha+\beta$ are greater or less than unity.

It is observed however that there are regions and types of pathwise asymptotic behaviour of (4.1.5) which do not have a counterpart in continuous time (specifically when $\alpha \leq -1$). As for the continuous equation while Chapter 7 is primarily concerned with establishing almost sure asymptotic results it is noted that these asymptotic results also hold in mean square.

Exponential Growth in the Solution of an Affine Stochastic Differential Equation with an Average Functional and Financial Market Bubbles

5.1 Introduction

In this chapter, we determine the exact almost sure rate of growth for solutions of the affine stochastic functional differential equation (SFDE)

$$\mathrm{d}X(t) = \left(aX(t) + \frac{b}{1+t}\int_{-1}^{t} X(s)\ \mathrm{d}s\right)\ \mathrm{d}t + \sigma\ \mathrm{d}B(t), \quad t \geq 0, \tag{5.1.1}$$

where B is a one–dimensional standard Brownian motion, $X(t) = \psi(t)$ for $t \in [-1, 0]$, ψ is a continuous function, and a and σ are positive, and b is a non–zero real parameter. The equation is termed affine by virtue of the linearity of the functional in the drift and the fact that the diffusion is independent of the state. This forces solutions of the equation to be Gaussian processes, a fact which is exploited in our analysis. We exclude the case $a \leq 0$ from our analysis here, as solutions in this regime do not have a definite deterministic asymptotic rate of growth.

In our main result, it is shown that the solution obeys

$$\lim_{t\to\infty} \frac{X(t)}{e^{at}t^{b/a}} = C, \quad \text{almost surely,} \tag{5.1.2}$$

where C, known in terms of a, b, ψ and σ, is a Gaussian random variable with mean c which is known in terms of the data (i.e., in terms of a, b and ψ). Generally, c is non–zero, so on almost every sample path, $X(t)$ is asymptotic to $e^{at}t^{b/a}$ as $t \to \infty$. This result is established by employing the admissibility theory for linear Volterra operators developed by Corduneanu (cf, e.g., [14, 37]), applied pathwise to the solution of a random C^1 dynamical system related to (5.1.1). Such admissibility theory has recently been used in a series of papers by Appleby, Győri, Horváth and Reynolds [6, 14, 13, 55, 56, 57, 59, 60] to determine convergence or rates of convergence to the equilibrium of linear Volterra integral or summation equations. A novel feature of this work is that we use this admissibility theory to determine rates of *growth*, rather than *decay*, of solutions, and that the equations considered are *stochastic*, rather than *deterministic*.

It is interesting to question whether using the rather general admissibility theory of Volterra operators enables us to determine a sharp rate of growth of the solution of (5.1.1), or whether (5.1.2) over–estimates the rate of growth (which would be the case if $C = 0$ a.s. in (5.1.2)). One reason for studying (5.1.1) is that we can independently use results on the asymptotic behaviour of confluent hypergeometric functions to determine the exact growth rate of solutions of the deterministic equation underlying (5.1.1), namely $x'(t) = ax(t) + b/(1+t)\int_{-1}^{t} x(s)\ ds$ for $t > 0$ with initial conditions $x(t) = \psi(t)$ for $t \in [-1, 0]$. These results show that $x(t)$ is asymptotic to a constant times $e^{at}t^{b/a}$ as $t \to \infty$, with the constant generally being non–trivial. We have $x(t) = \mathbb{E}[X(t)]$ and the asymptotic behaviour of the mean is then inherited by the solution of (5.1.1). This not only demonstrates the sharpness of the admissibility theory, but also that the limit in (5.1.2) is non–trivial. This latter remark is of interest, because the admissibility approach does not readily reveal the nature of the limiting constant.

The chapter is organised as follows: the equations to be analysed are introduced in Section 5.1.1, together with notation. In Section 5.1.2 the representation of solutions of the underlying deterministic equation are given in terms of confluent hypergeometric functions. The main results are given in Section 5.2. Results from the admissibility theory of Volterra operators are given in Section 5.3. The proofs are deferred to Section 5.4.

5.1.1 Preliminaries

We consider the affine scalar SFDE with an average functional

$$dX(t) = \left(aX(t) + b\frac{1}{1+t}\int_{-1}^{t} X(s)\ ds\right)\ dt + \sigma\ dB(t), \quad t \geq 0; \tag{5.1.3a}$$

$$X(t) = \psi(t), \quad t \in [-1, 0]. \tag{5.1.3b}$$

Here $\sigma > 0$, $a > 0$, $b \in \mathbb{R}$ and $\psi \in C([-1, 0], \mathbb{R})$. Then by Berger and Mizel [27] or Mao [84] there is a unique continuous adapted process which obeys (6.1.1), hereinafter referred to as the *solution* of (6.1.1). There is also a unique continuous solution of

$$x'(t) = ax(t) + b\frac{1}{1+t}\int_{-1}^{t} x(s)\ ds, \quad t \geq 0, \quad x(t) = \psi(t), \quad t \in [-1, 0]. \tag{5.1.4}$$

We define the differential resolvent r associated with (5.1.4) by

$$\frac{\partial r}{\partial t}(t,s) = a\, r(t,s) + b\frac{1}{1+t}\int_s^t r(u,s)\ \mathrm{d}u, \quad t > s; \tag{5.1.5a}$$

$$r(t,s) = 0, \quad t < s; \quad r(s,s) = 1. \tag{5.1.5b}$$

Then, as is shown in Lemma 6.1.1, the solution of (5.1.3) is given by

$$X(t) = x(t) + \sigma \int_0^t r(t,s)\ \mathrm{d}B(s), \quad t \geq 0. \tag{5.1.6}$$

Therefore X is a Gaussian process with $\mathbb{E}[X(t)] = x(t)$ for $t \geq 0$.

5.1.2 Explicit formulae for solution of (5.1.4)

The solution of (5.1.4) can be rewritten as the solution of the second–order linear differential equation

$$x''(t) + \left(\frac{1}{1+t} - a\right)x'(t) - \frac{a+b}{1+t}x(t) = 0, \quad t \geq 0; \tag{5.1.7}$$
$$x(0) = \psi(0), \quad x'(0) = a\psi(0) + b\int_{-1}^0 \psi(s)\ \mathrm{d}s.$$

Therefore (for $b/a \notin \{-1,-2,...\}$) the solution of (5.1.4) can be expressed in terms of confluent hypergeometric functions, according to:

$$x(t) = c_1 U\left(1 + \frac{b}{a}, 1, a(1+t)\right) + c_2 M\left(1 + \frac{b}{a}, 1, a(1+t)\right), \quad t \geq 0, \tag{5.1.8}$$

where U and M are two linearly independent solutions of *Kummer's differential equation*, which is given by $zw''(z)+(\beta-z)w'(z)-\alpha w(z) = 0$, where α and β are real and z complex v. See [97, Chapter 13.2.1] and following sections. We use various properties of confluent hypergeometric functions (i.e. U and M) to analyse the mean, x, of X. This knowledge of the mean aids us in our analysis of the stochastic process X. Section 6.2 of Chapter 6 gives a variety of properties and identities satisfied by both U and M. In order to avoid repetition we do not state these properties here but rather reference those in Section 6.2 of Chapter 6 as needed.

We do note here however that when $b/a \in \{-1,-2...\}$, the two functions on the right–hand side of (5.1.8) are no longer linearly independent, and x may be represented by (6.2.38).

5.2 Main Results

When $a > 0$, $b = 0$, (6.1.1) collapses to the Ornstein–Uhlenbeck equation $dX(t) = aX(t)\,dt + \sigma\,dB(t)$, for $t \geq 0$, $X(0) = c$, and it can be shown by the martingale convergence theorem (cf., e.g., [104, Proposition IV.1.26] that

$$\lim_{t\to\infty} \frac{X(t)}{e^{at}} = c + \int_0^\infty \sigma e^{-as}\ \mathrm{d}B(s), \quad \text{a.s.}$$

Note that the limiting random variable on the right–hand side is normally distributed with mean c and variance $\sigma^2/(2a)$. We focus now on the case when $b \neq 0$.

Theorem 5.2.1. *Let $a > 0$, $\sigma \neq 0$ and $\psi \in C([-1, 0], \mathbb{R})$. Suppose that X is the solution of* (6.1.1) *and x is the unique continuous solution to* (5.1.4)*. Then*

(a) X obeys

$$\lim_{t\to\infty} \frac{X(t)}{e^{at}t^{b/a}} = C\left(a, b, \psi(0), \int_{-1}^0 \psi(s)\ \mathrm{d}s, \sigma\right), \quad \textit{a.s.,} \tag{5.2.1}$$

where C is an a.s. finite normal random variable with mean c.

(b) The solution x of (5.1.4) *obeys*

$$\lim_{t\to\infty} \frac{x(t)}{e^{at}t^{b/a}} = c\left(a, b, \psi(0), \int_{-1}^0 \psi(s)\ \mathrm{d}s\right). \tag{5.2.2}$$

(c) c in (5.2.2) *is given by*

$$c\left(a, b, \psi(0), \int_{-1}^0 \psi(s)\ \mathrm{d}s\right) = a^{\frac{b}{a}}\left\{\psi(0)U\left(\frac{b}{a}, 0, a\right) + b\int_{-1}^0 \psi(s)\ \mathrm{d}s\,U\left(1 + \frac{b}{a}, 1, a\right)\right\}. \tag{5.2.3}$$

Parts (a) and (b) of Theorem 5.2.1 are proven using a result, stated in Section 5.3, from the admissibility theory of linear Volterra operators. The proof of Theorem 5.2.1 is postponed to Section 5.4.

By (5.1.6) and Theorem 5.2.1, c is linear in $\psi(0)$, $\int_{-1}^0 \psi(s)\ \mathrm{d}s$ and C in (5.2.1) depends on the parameters according to

$$C = c\left(a, b, \psi(0), \int_{-1}^0 \psi(s)\ \mathrm{d}s\right) + \sigma G(a, b)$$

where G is a zero mean normal random variable. This leads us to ask whether there are values of a and b for which C is almost surely non–zero. Since proper Gaussian random variables possess a density, this will clearly be true if $c \neq 0$. It transpires that this can be ensured for almost all initial functions ψ in the case that $b>0$. These remarks are made precise in the following results.

Proposition 5.2.1. *Let $a>0$. If c is as given by part (b) of Theorem 5.2.1 is non–zero, then $C(a,b,\psi(0),\int_{-1}^{0}\psi(s)\,\mathrm{d}s,\sigma)\neq 0$ a.s.*

While Proposition 5.2.1 shows that $C\neq 0$ a.s. by analysising the mean of C, the same result is obtained in Theorem 6.4.2 by showing that the variance of C is positive (without the need for restrictions upon b).

Proposition 5.2.2. *Let $a>0$. If $b>0$ and ψ obeys $\psi(0)>0$ and $\int_{-1}^{0}\psi(s)\,\mathrm{d}s>0$ then $c(a,b,\psi(0),\int_{-1}^{0}\psi(s)\,\mathrm{d}s)>0$ and therefore $C(a,b,\psi(0),\int_{-1}^{0}\psi(s)\,\mathrm{d}s,\sigma)\neq 0$ a.s.*

The case when $b<0$ is more delicate. By (5.2.3), it can be seen that $c\neq 0$ for almost all initial functions provided that at least one of $U(b/a,0,a)$ and $U(1+b/a,1,a)$ is non–zero. However, the case that $U(b/a,0,a)=U(1+b/a,1,a)=0$ is not generic. This is because each of the functions $x\mapsto U(b/a,0,x)$ and $x\mapsto U(1+b/a,1,x)$ possess exactly $\lceil -1-b/a\rceil$ positive zeros. Thus, for most values of b/a, it is unlikely that a is a zero of either function. Of course, for particular values of a,b and initial conditions ψ one can calculate c using (5.2.3), and check whether it is non–zero.

In the case when $b/a\in\{-1,-2,...\}$, $U(b/a,0,a)$ and $U(1+b/a,1,a)$ are polynomials, in a of order $-b/a$ and $-1-b/a$ respectively. Hence one can calculate their zeros more readily than the non-polynomial case. For instance when $b/a=-2$, $U(-2,0,a)=a(a-2)$, $U(-1,1,a)=a-1$, so $ac=\psi(0)(a-2)-2(a-1)\int_{-1}^{0}\psi(s)\,\mathrm{d}s$.

5.3 Admissibility Results

The results on the asymptotic behaviour of (6.1.1) in this chapter rely on applying an existing admissibility result for linear Volterra operators. It is stated here to make this work self–contained. A variant of the result is cited in [14].

Theorem 5.3.1. *Suppose that $H:\Delta\to\mathbb{R}$ is continuous on $\Delta=\{(t,s)\in\mathbb{R}^2:0\le s\le t<\infty\}$, and that there is a $H_\infty\in L^1(0,\infty)$ such that*

$$\lim_{t\to\infty}\int_0^T |H(t,s)-H_\infty(s)|\,\mathrm{d}s=0\quad \textit{for all } T>0. \tag{5.3.1}$$

Assume also that

$$W := \lim_{T\to\infty} \limsup_{t\to\infty} \int_T^t |H(t,s)| \, \mathrm{d}s < 1, \tag{5.3.2}$$

and that there is $V \in \mathbb{R}$ *such that* $\lim_{T\to\infty} \limsup_{t\to\infty} \left| \int_T^t H(t,s) \, \mathrm{d}s - V \right| = 0$. *Let* ξ *be in* $BC_l(\mathbb{R}^+;\mathbb{R})$, *and let* $\eta : [0,\infty) \to \mathbb{R}$ *be the continuous solution of*

$$\eta(t) = \xi(t) + \int_0^t H(t,s)\eta(s) \, \mathrm{d}s, \quad t \geq 0. \tag{5.3.3}$$

Then $\lim_{t\to\infty} \eta(t) =: \eta(\infty)$ *exists and*

$$\eta(\infty) = (1-V)^{-1}\left[\xi(\infty) + \int_0^\infty H_\infty(s)\eta(s) \, \mathrm{d}s\right]. \tag{5.3.4}$$

Remark 5.3.1. Because $|V| \leq W$, $W < 1$ implies $V < 1$. Also if $W = 0$, $V = 0$.

5.4 Proofs of Main Results

We give an outline of the strategy of the proof of Theorem 5.2.1. We cannot apply directly admissibility theory for Volterra equations to equation (6.1.1). Moreover we would like to exploit second order features in equation (6.1.1) which aid asymptotic analysis of the underlying deterministic equation. However, we cannot do so, owing to presence of the non–differentiable Brownian motion. To avoid this, we decompose X into a stochastic term (which turns out to be asymptotically dominated by X) and a random process which itself is not twice differentiable but whose asymptotic behaviour is governed by a second order ordinary differential equation. This second process, denoted by Z below, is appropriately scaled (to give a process W) in order to capture its asymptotic behaviour. W obeys a Volterra integral equation to which the admissibility theory can be applied.

The function x is twice differentiable, and Lemma 5.4.1 allows us to rewrite X so that the extra smoothness of x can be exploited.

Lemma 5.4.1. *Let x obey (5.1.4) and r obey (5.1.5). Define $K = \int_{-1}^0 \psi(s)\,\mathrm{d}s$,*

$$D_1(t) := \int_0^t \left(-\frac{b}{a}r(t,s) + b\int_s^t r(u,s)\,\mathrm{d}u\right)\left(a\psi(0) + \frac{b}{1+s}(s\psi(0)+K)\right)\,\mathrm{d}s$$
$$+ a(1+t)\psi(0) + b(t\psi(0)+K), \quad t \ge 0,$$

$$G_1(t,s) := \int_s^t \left(-\frac{b}{a}r(t,m) + b\int_m^t r(u,m)\,\mathrm{d}u\right)\left(a + \frac{b(m-s)}{1+m}\right)\,\mathrm{d}m$$
$$+ a(1+t) + b(t-s), \quad 0 \le s \le t,$$

$$D_2(t) := bK(1+t)^{-\frac{b}{a}} + a\psi(0)(1+t)$$
$$-\frac{b^2}{a^2}(1+t)^{-\frac{b}{a}}\int_0^t\int_s^t (1+u)^{\frac{b}{a}-1}r(u,s)\,\mathrm{d}u\left(a\psi(0) + \frac{b}{1+s}(s\psi(0)+K)\right)\,\mathrm{d}s,$$

and

$$G_2(t,s) := a(1+t) - \frac{b^2}{a^2}(1+t)^{-\frac{b}{a}}\int_s^t\int_m^t (1+u)^{\frac{b}{a}-1}r(u,m)\,\mathrm{d}u\left(a + \frac{b(m-s)}{1+m}\right)\,\mathrm{d}m.$$

Then

$$D_1(t) = D_2(t), \quad t \ge 0, \tag{5.4.1}$$

$$G_1(t,s) = G_2(t,s), \quad 0 \le s \le t. \tag{5.4.2}$$

Proof. Let $y(t) = \psi(0)$ for $t \ge 0$, $y(t) = \psi(t)$ for $-1 \le t < 0$, and $z(t) := x(t) - y(t)$. Thus

$$z'(t) = az(t) + \frac{b}{1+t}\int_0^t z(s)\,\mathrm{d}s + f(t), \quad t \ge 0, \tag{5.4.3}$$

where

$$f(t) = a\psi(0) + \frac{b}{1+t}\int_{-1}^0 \psi(s)\,\mathrm{d}s + b\psi(0)\frac{t}{1+t}.$$

Our method of rewriting (5.4.3) so that Theorem 5.3.1 can be applied is inspired by [26, Example 3.8.7]. Converting (5.4.3) to the second order equation $z''(t)+(1/(1+t)-a)z'(t) = (a+b)z(t)/(1+t) + (a+b)\psi(0)/(1+t)$ and substituting

$$w(t) = z(t)/(\mathrm{e}^{at}(1+t)^{\frac{b}{a}}) \tag{5.4.4}$$

gives

$$w''(t) + \left(\frac{2b/a+1}{1+t} + a\right)w'(t) = -\frac{b^2}{a^2}\frac{w(t)}{(1+t)^2} + (a+b)\psi(0)\mathrm{e}^{-at}(1+t)^{-\frac{b}{a}-1}.$$

Multiplying both sides by $\mathrm{e}^{at}(1+t)^{\frac{2b}{a}+1}$, and using an integrating factor, we get

$$\frac{d}{dt}\left(\mathrm{e}^{at}(1+t)^{\frac{2b}{a}+1}w'(t)\right) = -\frac{b^2}{a^2}\mathrm{e}^{at}(1+t)^{\frac{2b}{a}-1}w(t) + (a+b)\psi(0)(1+t)^{\frac{b}{a}}.$$

Integrating on both sides and recalling the resolvent representation

$$z(t) = \int_0^t r(t,s)f(s)\,\mathrm{d}s = \int_0^t r(t,s)\left(a\psi(0) + \frac{b}{1+s}(s\psi(0)+K)\right)\,\mathrm{d}s \tag{5.4.5}$$

we get

$$\begin{aligned} w'(t) &= \mathrm{e}^{-at}(1+t)^{-\frac{2b}{a}-1}bK + \mathrm{e}^{-at}(1+t)^{-\frac{b}{a}}a\psi(0) \\ &- \frac{b^2}{a^2}\frac{\mathrm{e}^{-at}}{(1+t)^{\frac{2b}{a}+1}}\int_0^t (1+s)^{\frac{b}{a}-1}\int_0^s r(s,u)\left(a\psi(0) + \frac{b(u\,\psi(0)+K)}{1+u}\right)\,\mathrm{d}u\ \mathrm{d}s. \end{aligned} \tag{5.4.6}$$

We now obtain an alternative representation for w', without using a second order equation. Differentiating (5.4.4) and using (5.4.5) gives

$$\begin{aligned} w'(t) &= a\psi(0)\mathrm{e}^{-at}(1+t)^{-\frac{b}{a}} + bK\mathrm{e}^{-at}(1+t)^{-\frac{b}{a}-1} + b\psi(0)t\mathrm{e}^{-at}(1+t)^{-\frac{b}{a}-1} \\ &- \frac{b}{a}\mathrm{e}^{-at}(1+t)^{-\frac{b}{a}-1}\int_0^t r(t,s)\left(a\psi(0) + \frac{b}{1+s}(K+s\psi(0))\right)\,\mathrm{d}s \\ &+ b\mathrm{e}^{-at}(1+t)^{-\frac{b}{a}-1}\int_0^t\int_0^s r(s,u)\left(a\psi(0) + \frac{b}{1+u}(K+u\psi(0))\right)\,\mathrm{d}u\ \mathrm{d}s. \end{aligned} \tag{5.4.7}$$

Thus comparing (5.4.6) and (5.4.7) we deduce (5.4.1).

In establishing (5.4.2), we consider r, the solution of (5.1.5) in place of x. Let $y(t,s) = 1$ for $t \geq s$ and $y(t,s) = 0$ for $t < s$ and $z(t,s) := r(t,s) - y(t,s)$, which leads to

$$\frac{\partial z}{\partial t}(t,s) = az(t,s) + \frac{b}{1+t}\int_s^t z(u,s)\,\mathrm{d}u + f(t,s), \tag{5.4.8}$$

where $f(t,s) = a + b(t-s)/(1+t)$. Therefore $z(t,s) = \int_s^t r(t,u)f(u,s)\,\mathrm{d}u$. Following the steps used to prove (5.4.1) (i.e., considering $w(t,s) = z(t,s)/(\mathrm{e}^{at}(1+t)^{b/a})$; obtaining a representation for $\partial w/\partial t$ via a second order equation; and examining the scaled version of (5.4.8))) one derives (5.4.2). □

Proof of parts (a) and (b) of Theorem 5.2.1. Define

$$Y(t) = \begin{cases} \psi(0) + \sigma B(t), & t \geq 0, \\ \psi(t), & t \in [-1,0). \end{cases} \tag{5.4.9}$$

and $Z(t) := X(t) - Y(t)$. Thus

$$Z'(t) = aZ(t) + \frac{b}{1+t}\int_0^t Z(s)\,\mathrm{d}s + f(t), \tag{5.4.10}$$

where $K := \int_{-1}^0 \psi(s)\ \mathrm{d}s$ and f is given by

$$f(t) = a\psi(0) + \frac{b}{1+t}(t\psi(0)+K) + \sigma\int_0^t \left(a + \frac{b(t-s)}{1+t}\right)\,\mathrm{d}B(s), \quad t \geq 0.$$

Define $W(t) := Z(t)/(e^{at}(1+t)^{\frac{b}{a}})$. By (5.1.5), and the fact that $Z(0) = 0$, Z is given by $Z(t) = \int_0^t r(t,s)f(s)\,ds$ for $t \geq 0$. Using this representation for Z and (5.4.10) gives

$$\begin{aligned}W'(t) &= -\frac{\frac{b}{a}e^{-at}}{(1+t)^{\frac{b}{a}+1}}\int_0^t r(t,s)f(s)\,ds \\ &\quad + b\frac{e^{-at}}{(1+t)^{\frac{b}{a}+1}}\int_0^t\int_0^s r(s,u)f(u)\,du\,ds + e^{-at}(1+t)^{-\frac{b}{a}}f(t),\end{aligned}$$

or

$$W'(t) = e^{-at}(1+t)^{-\frac{b}{a}-1}\left(D_1(t) + \sigma\int_0^t G_1(t,s)\,dB(s)\right) \tag{5.4.11}$$

where D_1 and G_1 are defined in Lemma 5.4.1. By Lemma 5.4.1 and (5.4.11), we have

$$\begin{aligned}W'(t) &= e^{-at}(1+t)^{-\frac{b}{a}-1}\left(D_2(t) + \sigma\int_0^t G_2(t,s)\,dB(s)\right) \qquad (5.4.12)\\ &= \frac{a\psi(0)+bK}{e^{at}(1+t)^{1+\frac{2b}{a}}} + e^{-at}(1+t)^{-1-\frac{2b}{a}}\left(\int_0^t (1+s)^{\frac{b}{a}}(a+b)(\psi(0)+\sigma B(s))\,ds\right.\\ &\quad \left. + \int_0^t a(1+s)^{\frac{b}{a}+1}\sigma\,dB(s)\right) - \frac{b^2}{a^2}\frac{e^{-at}}{(1+t)^{1+\frac{2b}{a}}}\int_0^t (1+s)^{\frac{2b}{a}-1}e^{as}W(s)\,ds.\end{aligned}$$

Integrating across (5.4.12) gives us an equation of the form of (5.3.3), namely

$$W(t) = h(t) + \int_0^t H(t,s)W(s)\,ds, \quad t \geq 0, \tag{5.4.13}$$

where

$$\begin{aligned}h(t) &= \int_0^t (a_2(s)+a_3(s)+a_4(s))\,ds\\ H(t,s) &= -\frac{b^2}{a^2}(1+s)^{\frac{2b}{a}-1}e^{as}\int_s^t e^{-au}(1+u)^{-1-\frac{2b}{a}}\,du,\end{aligned}$$

and $a_2(t) - a_4(t)$ are defined for $t \geq 0$ by

$$\begin{aligned}a_2(t) &= bKe^{-at}(1+t)^{-1-\frac{2b}{a}} + a\psi(0)e^{-at}(1+t)^{-\frac{b}{a}},\\ a_3(t) &= \sigma(a+b)e^{-at}(1+t)^{-1-\frac{2b}{a}}\int_0^t (1+s)^{\frac{b}{a}}B(s)\,ds,\\ a_4(t) &= \sigma a e^{-at}(1+t)^{-1-\frac{2b}{a}}\int_0^t (1+s)^{\frac{b}{a}+1}\,dB(s).\end{aligned}$$

Having put (5.4.13) in the form of (5.3.3), we next verify the conditions of Theorem 5.3.1.

Since $a > 0$, it is obvious that $a_2 \in L^1(0,\infty)$. We bound a_3 according to

$$|a_3(t)| \leq |\sigma(a+b)|e^{-at}(1+t)^{-1-\frac{2b}{a}}\int_0^t (1+s)^{\frac{b}{a}+1}\frac{|B(s)|}{1+s}\,ds.$$

By the law of large numbers for Brownian motion, [72, Problem 2.9.3] , the integral term above is polynomially bounded, so $a_3 \in L^1(0,\infty)$, a.s. For a_4, stochastic integration by parts yields

$$a_4(t) = \sigma a \mathrm{e}^{-at}(1+t)^{-\frac{b}{a}} B(t) - a\sigma \left(\frac{b}{a}+1\right) \mathrm{e}^{-at}(1+t)^{-1-\frac{2b}{a}} \int_0^t (1+s)^{\frac{b}{a}} B(s)\, \mathrm{d}s,$$

and hence a_4 is also in $L^1(0,\infty)$, a.s. Thus $\lim_{t\to\infty} h(t)$ exists.

We note that H is continuous and because $a > 0$, H_∞ given by

$$H_\infty(s) := -\frac{b^2}{a^2}(1+s)^{\frac{2b}{a}-1} \mathrm{e}^{as} \int_s^\infty \mathrm{e}^{-au}(1+u)^{-1-\frac{2b}{a}}\, \mathrm{d}u$$

is well–defined. By L'Hôpital's Rule, we get $\lim_{s\to\infty} |H_\infty(s)|/(1+s)^{-2} = b^2/a^3$, so $H_\infty \in L^1(0,\infty)$. To check (5.3.2), let $G(u) = \mathrm{e}^{-au}(1+u)^{-\frac{2b}{a}-1} \int_0^u \mathrm{e}^{as}(1+s)^{\frac{2b}{a}-1}\, \mathrm{d}s$. Then using L'Hôpital's Rule we get $\lim_{u\to\infty} G(u)/(1+u)^{-2} = 1/a$ and thus $G \in L^1(0,\infty)$. Also, $\int_T^t |H(t,s)|\ \mathrm{d}s \le b^2/a^2 \int_T^t G(u)\ \ \mathrm{d}u$, which gives $\lim_{T\to\infty} \limsup_{t\to\infty} \int_T^t |H(t,s)|\ \ \mathrm{d}s = 0 < 1$ as required. To check (5.3.1), for any $T > 0$, by the definition of H_∞ and $u \mapsto \mathrm{e}^{-au}(1+u)^{-\frac{2b}{a}-1} \in L^1(0,\infty)$, we have

$$\limsup_{t\to\infty} \int_0^T |H(t,s) - H_\infty(s)|\, \mathrm{d}s \le \limsup_{t\to\infty} \frac{b^2}{a^2} \int_t^\infty \mathrm{e}^{-au}(1+u)^{-\frac{2b}{a}-1}\ \mathrm{d}u \int_0^T \mathrm{e}^{as}(1+s)^{\frac{2b}{a}-1}\ \mathrm{d}s = 0.$$

Hence Theorem 5.3.1 applies, so $\lim_{t\to\infty} W(t) =: C$ is finite and so we have that $\lim_{t\to\infty} Z(t)/(e^{at}(1+t)^{b/a}) = C$. (5.4.9) and the law of large numbers for Brownian motion gives $\lim_{t\to\infty} Y(t)/(e^{at}(1+t)^{b/a}) = 0$ a.s., so (5.2.1) holds as required.

The proof outlined above suffices to prove (5.2.2), with the following changes: from (5.1.4) one can write down equation (5.4.10) with $Z(t)$ replaced by $x(t)$ and $Y(t) \equiv 0$. By Theorem 5.3.1, it follows that $w(t) := x(t)/(e^{at}(1+t)^{b/a})$ has a finite limit as $t \to \infty$.

It remains to show that the limit in part (a) is Gaussian. Since $X(t)$ is Gaussian for each t, $\xi_n := \mathrm{e}^{-an}(1+n)^{-\frac{b}{a}} X(n)$ is a Gaussian random variable for each $n \in \{1,2,...\}$. Let each ξ_n have mean m_n and variance ς_n^2. By Theorem 5.2.1, we have $\lim_{n\to\infty} \xi_n = C$ a.s., so (ξ_n) also converges to C in probability. By [108, Chap. 2.13.5, pp.304-305], it follows that $m := \lim_{n\to\infty} m_n$ and $\varsigma^2 := \varsigma_n^2$ exist, and that C is normally distributed with mean m and variance ς^2. Since $\lim_{n\to\infty} \mathbb{E}[\xi_n] = \lim_{n\to\infty} \mathrm{e}^{-an}(1+n)^{-\frac{b}{a}} \mathbb{E}[X(n)] = \lim_{n\to\infty} \mathrm{e}^{-an}(1+n)^{-\frac{b}{a}} x(n)$, it follows from part (b) of Theorem 5.2.1 that $c = m$, so C is normally distributed with mean c. □

Proof of part (c) of Theorem 5.2.1. First, let $b/a \notin \{-1,-2,...\}$. From the notation of Theorem 5.2.1, using (5.1.8) and (6.2.3), we obtain $c = \lim_{t\to\infty} x(t)/(\mathrm{e}^{at}(1+t)^{\frac{b}{a}}) = c_2 \mathrm{e}^a a^{b/a}/\Gamma(1+b/a)$, where

$$c_2 = \frac{-(1+\frac{b}{a})\psi(0)U(2+\frac{b}{a},2,a) - a\psi(0)U(1+\frac{b}{a},1,a) - bK\,U(1+\frac{b}{a},1,a)}{\mathcal{W}(a,b,0)},$$

$K = \int_{-1}^{0} \psi(s)\,\mathrm{d}s$, and $\mathcal{W}(a,b,t)$ is the Wronskian of the solutions $U(1+\frac{b}{a},1,a(1+t))$ and $M(1+\frac{b}{a},1,a(1+t))$. From (6.2.7), we have $\mathcal{W}(a,b,t) = \mathrm{e}^{a(1+t)}/((1+t)\Gamma(1+b/a))$. Using the above and (6.2.35), c must obey (5.2.3). In the case when $b/a \in \{-1,-2,...\}$, we have a solution given by (6.2.10),

$$U\left(1+\frac{b}{a},1,a(1+t)\right) = (-1)^{-1-\frac{b}{a}} \sum_{j=0}^{-1-\frac{b}{a}} \frac{((-1-\frac{b}{a})!)^2(-1)^j}{(-1-\frac{b}{a}-j)!(j!)^2} a^j(1+t)^j. \tag{5.4.14}$$

$M(1+\frac{b}{a},1,a(1+t))$ and $U(1+\frac{b}{a},1,a(1+t))$ are linearly dependent. Using Abel's Theorem [31, Ch.3.3.2], the Wronskian associated with (6.2.1a) is $\mathcal{W}(a,b,t) = \mathcal{W}(a,b,0)\mathrm{e}^{at}(1+t)^{-1}$. This allows us to derive a second solution, linearly independent of (5.4.14). Hence our general solution is given by (6.2.38). Thus,

$$c = \lim_{t\to\infty} \frac{x(t)}{\mathrm{e}^{at}(1+t)^{\frac{b}{a}}} = c_2 a^{\frac{b}{a}} = a^{\frac{b}{a}} \left(\psi(0)U(\frac{b}{a},0,a) + b\int_{-1}^{0} \psi(s)\,\mathrm{d}s U(1+\frac{b}{a},1,a)\right),$$

which is the same formula for c as (5.2.3), proving the result. □

Proof of Proposition 5.2.1. By hypothesis, c is non-zero. If the variance of C is zero, $C = c$ a.s. If the variance of C is non-zero, as C is normal it has a probability density function on $\mathbb{R}$, and therefore $\mathbb{P}[C=0]=0$. Thus $\mathbb{P}[C\neq 0]=1$, as required. □

Proof of Proposition 5.2.2. By Theorem 5.2.1, c is finite. By hypothesis $a,b>0$, and $\psi(0), \int_{-1}^{0}\psi(s)\,\mathrm{d}s > 0$, so by (6.2.28), $U(\frac{b}{a},0,a) > 0$, $U(1+\frac{b}{a},1,a) > 0$. By (5.2.3), $c>0$. □

Long Memory and Financial Market Bubble Dynamics in Affine Stochastic Differential Equations with Average Functionals

6.1 Introduction

This chapter continues the study of (4.1.1) which was commenced in Chapter 5. While Chapter 5 uses a rather general theory of admissibility of Volterra operators to attain its results this chapter instead uses techniques which are more tailored to specifically analyse (4.1.1). In spite of the loss of generality of the approach, the methods of this chapter extend the asymptotic analysis of (4.1.1) to all real values of the parameters of the equation. Moreover in contrast to Chapter 5 the methods of this chapter produce unambiguously sharp asymptotic rates of growth, decay, etc.

6.1.1 Organisation of the chapter and mathematical preliminaries

This chapter is organised as follows. In this section (Section 6.1.1), we formally introduce the equation under scrutiny and define some notation. Section 6.2 gives a detailed description of the decomposition of the solution of the deterministic equation into special functions, and in particular details the differing functions which are used depending on the values of a and b. In order to make our presentation self–contained, various properties of these functions which are needed in the analysis of the asymptotic behaviour, are listed. Section 6.3 deals with recurrent dynamics of X, with Subsection 6.3.1 giving results on the almost sure pathwise asymptotic behaviour of the process, while Subsection 6.3.2 discusses the memory properties when X has these recurrent dynamics. Section 6.4 gives results concerning transient dynamical behaviour of the process. Proofs of the results are deferred to Section 6.5.1 and sections thereafter.

Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ satisfying the usual conditions and let $B = \{B(t) : t \geq 0\}$ be a one–dimensional Brownian motion adapted to $\{\mathcal{F}(t)\}_{t\geq 0}$ on this space. The probability measure induces an expectation $\mathbb{E}$ in the usual manner, in the sense that if Y is an $\mathcal{F}$–measurable random variable such that $\int_\Omega |Y(\omega)|\ d\mathbb{P}\{\omega\} < +\infty$, then $\mathbb{E}[Y] = \int_\Omega Y(\omega)\ d\mathbb{P}\{\omega\}$.

Exactly as in Section 5.1.1 of Chapter 5 we consider the affine scalar stochastic functional differential equation with an average functional

$$dX(t) = \left(aX(t) + b\frac{1}{1+t}\int_{-1}^{t} X(s)\,ds\right)\,dt + \sigma\,dB(t), \quad t \geq 0; \tag{6.1.1a}$$

$$X(t) = \psi(t), \quad t \in [-1, 0], \tag{6.1.1b}$$

Here $\sigma > 0$, a, $b \in \mathbb{R}$ and $\psi \in C([-1,0],\mathbb{R})$. Then by Berger and Mizel [27] or Mao [84, Theorem 2.3.1] there is a unique continuous adapted process which obeys (6.1.1), hereinafter referred to as the *solution* of (6.1.1) and denoted X. There is also a unique continuous solution of

$$x'(t) = ax(t) + b\frac{1}{1+t}\int_{-1}^{t} x(s)\,ds, \quad t \geq 0, \tag{6.1.2a}$$

$$x(t) = \psi(t), \quad t \in [-1, 0]. \tag{6.1.2b}$$

The differential resolvent r associated with (6.1.2) is defined according to

$$\frac{\partial r}{\partial t}(t,s) = a\,r(t,s) + b\frac{1}{1+t}\int_{s}^{t} r(u,s)\,du, \quad t > s; \tag{6.1.3a}$$

$$r(t,s) = 0, \quad t < s; \quad r(s,s) = 1. \tag{6.1.3b}$$

Then with x being the solution of (6.1.2), the solution of (6.1.1) has a variation of parameters representation.

Lemma 6.1.1. *Suppose that* $\psi \in C([-1,0];\mathbb{R})$. *Let* X *be the unique solution of* (6.1.1), x *the unique solution of* (6.1.2) *and* r *the unique solution of* (6.1.3). *Then* X *is a Gaussian process and obeys*

$$X(t) = x(t) + \sigma\int_{0}^{t} r(t,s)\,dB(s), \quad t \geq 0. \tag{6.1.4}$$

A proof of the validity of this representation is provided in Section 6.5.

Using the representation (6.1.4) for X, we deduce formulae for the mean and autocovariance of X. By considering for $t \geq 0$ fixed and $\tau \geq 0$ the process

$$M(\tau) = \int_{0}^{\tau} r(t,s)\,dB(s), \quad \tau \geq 0,$$

we can see that M is a martingale and moreover a Gaussian process, so therefore $X(t) = x(t) + M(t)$ is Gaussian distributed. Since $\mathbb{E}[M(\tau)^2] < +\infty$ for all $\tau \geq 0$, we have that $\mathbb{E}[M(\tau)] = 0$ for all $\tau \geq 0$, and hence $\mathbb{E}[M(t)] = 0$. Hence

$$\mathbb{E}[X(t)] = x(t), \quad t \geq 0. \tag{6.1.5}$$

Since $\mathbb{E}[X(t)^2]$ is finite for all $t \geq 0$, it follows that $\mathrm{Cov}(X(t), X(t+\Delta))$ is well–defined for all $t \geq 0$ and $\Delta \geq 0$. We also see that

$$\begin{aligned}\mathrm{Cov}(X(t), X(t+\Delta)) &= \sigma^2 \mathbb{E}[M(t)M(t+\Delta)] \\ &= \sigma^2 \mathbb{E}[\int_0^{t+\Delta} r(t,s)\chi_{[0,t]}(s)\, dB(s) \int_0^{t+\Delta} r(t+\Delta, s)\, dB(s)].\end{aligned}$$

Considering t and Δ as fixed, we may apply Itô's isometry to obtain the variance of

$$V_1 := \int_0^{t+\Delta} r(t+\Delta, s)\, dB(s), \quad V_2 := \int_0^{t+\Delta} r(t,s)\chi_{[0,t]}(s)\, dB(s) \quad \text{and}$$
$$\int_0^{t+\Delta} \{r(t,s)\chi_{[0,t]} + r(t+\Delta, s)\}\, dB(s) = V_1 + V_2,$$

and using the fact that $2\mathrm{Cov}(V_1, V_2) = \mathrm{Var}[V_1 + V_2] - \mathrm{Var}[V_1] - \mathrm{Var}[V_2]$, we obtain

$$\mathrm{Cov}(X(t), X(t+\Delta)) = \sigma^2 \int_0^t r(t,s) r(t+\Delta, s)\, ds, \quad t \geq 0, \quad \Delta \geq 0. \tag{6.1.6}$$

We have already seen that mean and resolvent obey functional differential equations involving an average functional. This also holds true for the autocovariance function, and the result is recorded below.

Proposition 6.1.1. *Suppose that $\psi \in C([-1,0];\mathbb{R})$. Let X be the unique solution of* (6.1.1) *and r the unique solution of* (6.1.3)*. Fix $t \geq 0$ and define*

$$\gamma_t(\Delta) := \sigma^2 \int_0^t r(t,s) r(t+\Delta, s)\, ds, \quad \Delta \geq -t. \tag{6.1.7}$$

If $\Delta \geq 0$, then $\gamma_t(\Delta) = \mathrm{Cov}(X(t), X(t+\Delta))$ and,

$$\gamma_t'(\Delta) = a\gamma_t(\Delta) + \frac{b}{1+t+\Delta} \int_{-t}^{\Delta} \gamma_t(w)\, dw, \quad \Delta \geq 0, \tag{6.1.8}$$
$$\gamma_t'(\Delta) = a\gamma_t(\Delta) + \frac{b}{1+t+\Delta} \int_{-t}^{\Delta} \gamma_t(w)\, dw + \sigma^2 r(t, t+\Delta), \quad -t \leq \Delta < 0. \tag{6.1.9}$$

This result is proven in Section 6.5.1. The differential equation (6.1.8) may be thought of as a Yule–Walker–type representation of the autocovariance function.

In this work, we could equally have studied the equation

$$dX(t) = \left(aX(t) + b\frac{1}{t}\int_0^t X(s)\,ds \right)\,dt + \sigma\,dB(t), \quad t \geq 0; \quad X(0) = \xi.$$

However, this equation is more delicate to analyse, on account of the potential singularity in the average functional at $t = 0$. We obviate such complications by considering an equation with an initial history on a non–trivial compact interval. Taking this to be $[-1, 0]$ leads to (6.1.1).

From a modelling perspective, recalling the financial interpretation of (6.1.1) from Chapter 4, it may be natural for traders to initially observe the market before they commence trading (as opposed to trading without using any past data, i.e. an initial value problem). This period of observation could then be normalised to be of length one.

6.2 Formulae and Asymptotic Behaviour of Solutions of (6.1.2) and (6.1.3)

The solution of (6.1.2) can be rewritten as the solution of an initial value problems for a second–order differential equation. The equation is

$$x''(t) + \left(\frac{1}{1+t} - a\right) x'(t) - \frac{a+b}{1+t} x(t) = 0, \quad t \geq 0; \tag{6.2.1a}$$

$$x(0) = \psi(0), \quad x'(0) = a\psi(0) + b\int_{-1}^{0} \psi(s)\ \mathrm{d}s. \tag{6.2.1b}$$

There are three cases to consider: $a < 0$, $a > 0$ and $a = 0$. We discuss each case and their subcases, conditioned by b, in turn. In the case when $b = 0$, the stochastic differential equation (6.1.1) reduces to an Ornstein–Uhlenbeck SDE, and so the behaviour of x, r, and indeed X, are well–understood. Therefore, we exclude the case $b = 0$ from our analysis. In the exposition below the asymptotic behaviour of the solution of (6.2.1) is deduced from the known asymptotics of certain functions. It is here observed however that a general theory concerning the asymptotic behaviour of linear second order equations with analytic coefficients may be found in e.g. [95, Ch. 7.1 and 7.2].

6.2.1 $a < 0$

When $a < 0$, the solution of (6.2.1) can be expressed in terms of two linearly independent confluent hypergeometric functions, according to:

$$x(t) = c_1 r_1(t) + c_2 r_2(t) \quad \text{for } a < 0 \text{ and } b/a \notin \{1, 2, ...\} \tag{6.2.2}$$

where

$$r_1(t) = \mathrm{e}^{at}U(-b/a, 1, -a(1+t)), \quad r_2(t) = \mathrm{e}^{at}M(-b/a, 1, -a(1+t)).$$

Here $U(\alpha, \beta, \cdot)$ and $M(\alpha, \beta, \cdot)$ are two linearly independent solutions of *Kummer's differential equation* which is given by

$$zw''(z) + (\beta - z)w'(z) - \alpha w(z) = 0,$$

where α and β are real and z a complex number. M is sometimes referred to as Kummer's function (of the first kind) or a *confluent hypergeometric function*, while U is sometimes called the Tricomi confluent hypergeometric function. See [96, Chapter 13.2.1] and following sections.

To see that r_1 and r_2 are solutions of (6.2.1a), observe that as $z \mapsto U(\alpha, \beta, z)$ is a solution of Kummer's equation then $t \mapsto U(-b/a, 1, -a(1+t))$ satisfies

$$\begin{aligned} &-a(1+t)U''(-b/a, 1, -a(1+t)) + (1 + a(1+t))U'(-b/a, 1, -a(1+t)) \\ &\quad + \frac{b}{a}U(-b/a, 1, -a(1+t)) = 0. \end{aligned}$$

Therefore

$$\begin{aligned} &r_1''(t) + \left(\frac{1}{1+t} - a\right) r'(t) - \frac{a+b}{1+t} r_1(t) \\ &= a^2\mathrm{e}^{at}U''(-\frac{b}{a}, 1, -a(1+t)) - 2a^2\mathrm{e}^{at}U'(-\frac{b}{a}, 1, -a(1+t)) \\ &\quad + a^2\mathrm{e}^{at}U(-\frac{b}{a}, 1, -a(1+t)) - \frac{a+b}{1+t}U(-\frac{b}{a}, 1, -a(1+t)) \\ &\quad + \left(\frac{1}{1+t} - a\right)\left(-a\mathrm{e}^{at}U'(-\frac{b}{a}, 1, -a(1+t)) + a\mathrm{e}^{at}U(-\frac{b}{a}, 1, -a(1+t))\right) \\ &= \frac{-a\,\mathrm{e}^{at}}{1+t}\Big[-a(1+t)U''(-\frac{b}{a}, 1, -a(1+t)) + (1 + a(1+t))U'(-\frac{b}{a}, 1, -a(1+t)) \\ &\quad + \frac{b}{a}U(-\frac{b}{a}, 1, -a(1+t))\Big] = 0, \end{aligned}$$

as required. A similar calculation shows that r_2 is a solution of (6.2.1a).

As we are chiefly interested in the long–run behaviour of X it is necessary to have information on the asymptotic behaviour of both U and M. This is given by [1, 13.1.4 & 13.1.8], or

$$M(\alpha, \beta, t) = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\mathrm{e}^t t^{\alpha-\beta}[1 + O(t^{-1})], \quad \text{as } t \to \infty, \tag{6.2.3a}$$

$$U(\alpha, \beta, t) = t^{-\alpha}[1 + O(t^{-1})], \quad \text{as } t \to \infty. \tag{6.2.3b}$$

This immediately gives asymptotic information about r_1 and r_2:

$$r_1(t) \sim e^{at}|a|^{b/a}t^{b/a}, \text{ as } t \to \infty, \tag{6.2.4}$$

$$r_2(t) \sim \frac{1}{\Gamma(-b/a)}e^{-a}|a|^{-b/a-1}t^{-b/a-1}, \text{ as } t \to \infty \tag{6.2.5}$$

To determine the asymptotic behaviour of x, we need values for c_1 and c_2 in (6.2.2) in terms of the initial conditions of (6.2.1a). As usual, by using (6.2.1b), these values are obtained by solving

$$c_1 r_1(0) + c_2 r_2(0) = \psi(0), \quad c_1 r_1'(0) + c_2 r_2'(0) = a\psi(0) + b\int_{-1}^{0} \psi(s)\ \mathrm{d}s. \tag{6.2.6}$$

Clearly, these values can be expressed in terms of the Wronskian of r_1 and r_2, evaluated at $t = 0$, as well as the derivatives of r_1 and r_2. Since r_1 and r_2 depend on M and U, it is of value to have a general formula for the Wronskian and the derivatives of U and M. A formula for the Wronskian, W, of M and U is given by [96, 13.2.34]:

$$W\{M(\alpha,\beta,z), U(\alpha,\beta,z)\} = -\Gamma(\beta)z^{-\beta}\mathrm{e}^{z}/\Gamma(\alpha). \tag{6.2.7}$$

Expressions for the derivatives of U and M are given by [96, 13.3.15 & 13.3.22]:

$$M'(\alpha,\beta,z) = \frac{\alpha}{\beta}M(\alpha+1,\beta+1,z), \quad U'(\alpha,\beta,z) = -\alpha U(\alpha+1,\beta+1,z). \tag{6.2.8}$$

Using these results, we obtain the following formulae for c_1 and c_2:

$$c_1 = \Gamma(-\frac{b}{a})\mathrm{e}^{a}b\left(\psi(0)M(1-\frac{b}{a},2,-a) - \int_{-1}^{0}\psi(s)\ \mathrm{d}s\, M(-\frac{b}{a},1,-a)\right),$$
$$c_2 = \Gamma(-\frac{b}{a})\mathrm{e}^{a}b\left(\psi(0)U(1-\frac{b}{a},2,-a) + \int_{-1}^{0}\psi(s)\ \mathrm{d}s\, U(-\frac{b}{a},1,-a)\right). \tag{6.2.9}$$

We now consider the case when $b/a \in \{1,2,\dots\}$. As alluded to earlier, in this case $t \mapsto M(-b/a,1,-a(1+t))$ and $t \mapsto U(-b/a,1,-a(1+t))$ are linearly dependent, and therefore the representation (6.2.2) for x is not valid. It is however known that $t \mapsto U(-b/a,1,-a(1+t))$ is a polynomial in $|a|(1+t)$ of degree b/a. We even have an explicit formula for this polynomial. Indeed, for $n \in \{0,1,2,...\}$, we have from [96, 13.2.7] that

$$U(-n,1,z) = (-1)^n \sum_{j=0}^{n} \frac{(n!)^2}{(n-j)!(j!)^2}(-z)^j. \tag{6.2.10}$$

Note that $z \mapsto U(-n,1,z)$ is analytic, and so its (at most n) zeros are isolated. Therefore, the zeros of the real–valued polynomial $t \mapsto U(-b/a,1,-a(1+t))$ are also isolated.

Suppose now we take $r_1(t) = e^{at}U(-b/a, 1, -a(1+t))$ for $t \geq 0$. We know from standard theory (cf., e.g. [31]) that there exists a second solution, $\tilde{r}_2$, of (6.2.1a) which is linearly independent of r_1. Next, by Abel's Theorem (cf., e.g. [31, Ch.3.3.2]), the Wronskian of r_1 and $\tilde{r}_2$, which is associated with (6.2.1a) obeys

$$\mathcal{W}(a,b,t) = \mathcal{W}(a,b,0)\mathrm{e}^{at}(1+t)^{-1}, \quad t \geq 0,$$

where $\mathcal{W}(a,b,0) = r_1(0)\tilde{r}_2'(0) - r_1'(0)\tilde{r}_2'(0) \neq 0$.

This expression is equivalent to

$$r_1(t)\tilde{r}_2'(t) - r_1'(t)\tilde{r}_2(t) = \mathcal{W}(a,b,0)\mathrm{e}^{at}(1+t)^{-1}, \quad t \geq 0.$$

We now wish to find a representation for $\tilde{r}_2$ which allows us to deduce its asymptotic properties.

Notice that because r_1 has finitely many zeros, it must have a maximal real zero. Let $t_1 = 1 + \max(0, \sup\{t \in \mathbb{R} : r_1(t) = 0\})$, where we define $\sup\{t \in \mathbb{R} : r_1(t) = 0\} = -\infty$ if $r_1(t) \neq 0$ for all $t \geq 0$. Then for $t \geq t_1$ we have

$$\tilde{r}_2'(t) - \frac{r_1'(t)}{r_1(t)}\tilde{r}_2(t) = \mathcal{W}(a,b,0)\frac{\mathrm{e}^{at}(1+t)^{-1}}{r_1(t)}, \quad t \geq t_1. \tag{6.2.11}$$

Since $r_1(t) \neq 0$ for all $t \geq t_1$, we have that $t \mapsto r_1'(t)/r_1(t)$ and $t \mapsto e^{at}(1+t)^{-1}/r_1(t)$ are continuous on $[t_1, \infty)$, and therefore we may solve (6.2.11) for $\tilde{r}_2$ to obtain the following representation for $\tilde{r}_2$ on $[t_1, \infty)$:

$$\tilde{r}_2(t) = r_1(t)\frac{\tilde{r}_2(t_1)}{r_1(t_1)} + \mathcal{W}(a,b,0)r_1(t)\int_{t_1}^{t} \frac{\mathrm{e}^{as}(1+s)^{-1}}{r_1^2(s)}\,\mathrm{d}s, \quad t \geq t_1. \tag{6.2.12}$$

Since t_1 exceeds the maximal zero of r_1, the integral on the right hand side of (6.2.12) is well–defined for $t \geq t_1$. Moreover, using l'Hôpital's rule together with (6.2.3b) or (6.2.10), one may show that

$$\lim_{t\to\infty} t^{1+\frac{b}{a}}\tilde{r}_2(t) = \mathcal{W}(a,b,0)|a|^{-1-\frac{b}{a}}, \quad a < 0, \quad -\frac{b}{a} \in \{1,2,\ldots\}. \tag{6.2.13}$$

Note that this recovers the asymptotic behaviour of r_2 above in (6.2.5) the case $a < 0$ and $b/a \notin \{1, 2, \ldots\}$.

It is also useful to determine some asymptotic information about $\tilde{r}_2'$. Notice that $r_1(t) \sim e^{at}t^{b/a}|a|^{b/a}$ as $t \to \infty$. Also we have

$$\frac{r_1'(t)}{r_1(t)} - a = \frac{r_1'(t) - ar_1(t)}{r_1(t)} = \frac{-aU'(-b/a, 1, -a(1+t))}{U(-b/a, 1, -a(1+t))}, \quad t \geq t_1,$$

so using the fact that $t \mapsto U(-b/a, 1, -a(1+t))$ is a polynomial of degree $b/a \in \mathbb{N}$, we have that $\lim_{t\to\infty} r_1'(t)/r_1(t) = a$. By (6.2.13), it follows that there is $t_1' > 0$ such that $\tilde{r}_2(t) \neq 0$ for all $t \geq t_1'$. Let $t_1'' = \max(t_1', t_1)$. Then we may rewrite (6.2.11) for $t \geq t_1''$ to get

$$\frac{\tilde{r}_2'(t)}{\tilde{r}_2(t)} = \frac{r_1'(t)}{r_1(t)} + \mathcal{W}(a,b,0)\frac{\mathrm{e}^{at}(1+t)^{-1}}{r_1(t)\tilde{r}_2(t)}.$$

Using the fact that $r_1(t) \sim e^{at}t^{b/a}|a|^{b/a}$ as $t \to \infty$ together with (6.2.13) shows that the second term has limit $|a| = -a$, and therefore

$$\lim_{t\to\infty} \frac{\tilde{r}_2'(t)}{\tilde{r}_2(t)} = 0. \tag{6.2.14}$$

Finally, we see that the solution of (6.2.1) is given by

$$x(t) = \tilde{c}_1 r_1(t) + \tilde{c}_2 \tilde{r}_2(t), \quad t \geq 0, \qquad \text{for } a < 0 \text{ and } b/a \in \{1, 2, ...\} \tag{6.2.15}$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are found using (6.2.1b). Note that $\tilde{c}_2$ is known entirely in terms of r_1 and its dependence on $\tilde{r}_2$ is solely through the value of the Wronskian, because

$$\tilde{c}_2 = \frac{1}{\mathcal{W}(a,b,0)}\left(b\psi(0)\,U(1-\frac{b}{a}, 2, |a|) + b\int_{-1}^{0}\psi(s)\,ds\,U(-\frac{b}{a}, 1, |a|)\right).$$

Note also that for $b = 0$, (6.2.15) reduces to $x(t) = \psi(0)\mathrm{e}^{at}$.

We now turn our attention to the representation of the resolvent r defined by (6.1.3). In a manner similar to the treatment of the solution x of (6.1.2), it can be shown for every fixed $s \geq 0$, the solution $t \mapsto r(t,s) =: r_s(t)$ of the resolvent equation (6.1.3) is also the solution of the second order differential equation

$$r_s''(t) + \left(\frac{1}{1+t} - a\right) r_s'(t) - \frac{a+b}{1+t} r_s(t) = 0, \quad t \geq s, \tag{6.2.16}$$

with initial conditions $r_s(s) = 1$ and $r_s'(s) = a$. It is to be noted that (6.2.16) is the same differential equation as (6.2.1a) apart from the fact that the argument of the solution is restricted to the interval $[s, \infty)$, a subinterval of the interval of existence of the equation (6.2.1a). Therefore, $r(t,s) = r_s(t)$ can be represented as a linear combination of the linearly independent solutions of (6.2.1a) according to

$$r(t,s) = \begin{cases} d_1(s)r_1(t) + d_2(s)r_2(t), & t \geq s \geq 0, \quad a < 0, \quad b/a \notin \{1, 2...\}, \\ \tilde{d}_1(s)r_1(t) + \tilde{d}_2(s)\tilde{r}_2(t), & t \geq s \geq 0, \quad a < 0, \quad b/a \in \{1, 2...\}. \end{cases} \tag{6.2.17}$$

The multipliers d_1, d_2 etc are s–dependent, because initial data for the problem (6.2.16) is specified at s. Considering first the non–degenerate case when $b/a \notin \{1, 2...\}$, it can be seen that expressions for the coefficients d_1 and d_2 are obtained from the initial conditions (6.1.3b) and (6.2.7), i.e.

$$d_1(s)r_1(s) + d_2(s)r_2(s) = 1, \quad d_1(s)r_1'(s) + d_2(s)r_2'(s) = a. \tag{6.2.18}$$

From these equations, and using (6.2.7) and (6.2.8), we obtain the formulae

$$d_1(s) = \Gamma(-\frac{b}{a})(1+s)\mathrm{e}^a b\, M(1-\frac{b}{a}, 2, -a(1+s)), \tag{6.2.19}$$
$$d_2(s) = \Gamma(-\frac{b}{a})(1+s)\mathrm{e}^a b\, U(1-\frac{b}{a}, 2, -a(1+s)). \tag{6.2.20}$$

Using the fact that $\Gamma(1-b/a) = -b/a\Gamma(-b/a)$ and employing (6.2.3), we get

$$d_1(s) \sim b\frac{\Gamma(-\frac{b}{a})}{\Gamma(1-\frac{b}{a})}|a|^{-1-\frac{b}{a}}e^{-as}s^{-\frac{b}{a}} = |a|^{-\frac{b}{a}}e^{-as}s^{-\frac{b}{a}}, \text{ as } s \to \infty, \tag{6.2.21}$$
$$d_2(s) \sim \Gamma(-\frac{b}{a})\mathrm{e}^a b|a|^{\frac{b}{a}-1}s^{\frac{b}{a}}, \text{ as } s \to \infty. \tag{6.2.22}$$

In the degenerate case when $b/a \in \{1, 2, ...\}$, we have

$$\tilde{d}_1(s) = \frac{\tilde{r}_2'(s) - a\tilde{r}_2(s)}{\mathcal{W}(a,b,0)\mathrm{e}^{as}(1+s)^{-1}} = -\frac{a}{\mathcal{W}(a,b,0)}\tilde{r}_2(s)(1+s)\mathrm{e}^{-as}\left(1 + \frac{1}{-a}\frac{\tilde{r}_2'(s)}{\tilde{r}_2(s)}\right),$$
$$\tilde{d}_2(s) = -\frac{r_1'(s) - ar_1(s)}{\mathcal{W}(a,b,0)\mathrm{e}^{as}(1+s)^{-1}} = \frac{1}{\mathcal{W}(a,b,0)}b(1+s)U(1-\frac{b}{a}, 2, -a(1+s)).$$

We notice by (6.2.13) and (6.2.14) that

$$\tilde{d}_1(s) \sim |a|^{-\frac{b}{a}}s^{-b/a}\mathrm{e}^{-as}, \quad \text{as } s \to \infty, \tag{6.2.23}$$

which mirrors the asymptotic behaviour for d_1 in (6.2.21) in the non–degenerate case. As to the asymptotic behaviour of $\tilde{d}_2$, we may use (6.2.3b) to obtain

$$\tilde{d}_2(s) \sim \frac{1}{\mathcal{W}(a,b,0)}b|a|^{\frac{b}{a}-1}s^{\frac{b}{a}} \quad \text{as } s \to \infty, \tag{6.2.24}$$

and so $\tilde{d}_2$ has the same asymptotic behaviour as d_2 given in (6.2.22) in the non–degenerate case.

Using the fact that $\mathrm{Cov}(X(t), X(t+\Delta))$ obeys (6.1.6) for $t \geq 0$ and $\Delta \geq 0$, and $r(t,s)$ is given by (6.2.17), we have

$$\mathrm{Cov}(X(t), X(t+\Delta)) = \begin{cases} c_{1,t}r_1(t+\Delta) + c_{2,t}r_2(t+\Delta), & a < 0, \quad b/a \notin \{1, 2...\}, \\ \tilde{c}_{1,t}r_1(t+\Delta) + \tilde{c}_{2,t}\tilde{r}_2(t+\Delta), & a < 0, \quad b/a \in \{1, 2...\}, \end{cases} \tag{6.2.25}$$

for $t \geq 0$ and $\Delta \geq 0$, where

$$c_{1,t} = \sigma^2 \int_0^t r(t,s) d_1(s)\, ds, \quad c_{2,t} = \sigma^2 \int_0^t r(t,s) d_2(s)\, ds, \tag{6.2.26}$$

and

$$\tilde{c}_{1,t} = \sigma^2 \int_0^t r(t,s) \tilde{d}_1(s)\, ds, \quad \tilde{c}_{2,t} = \sigma^2 \int_0^t r(t,s) \tilde{d}_2(s)\, ds. \tag{6.2.27}$$

In order that certain limiting constants in our analysis are non–zero, we find it useful to employ the following integral representation of U:

$$U(\alpha, \beta, t) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \mathrm{e}^{-tu} u^{\alpha-1} (1+u)^{\beta-\alpha-1} du, \quad \alpha > 0. \tag{6.2.28}$$

It appears as [96, 13.4.4].

6.2.2 $a > 0$

When $a > 0$, the solution of (6.2.1a) can be expressed in terms of confluent hypergeometric functions, according to:

$$x(t) = c_3 r_3(t) + c_4 r_4(t) \quad \text{ for } a > 0 \text{ and } b/a \notin \{-1, -2, ...\} \tag{6.2.29}$$

where

$$r_3(t) = U(1 + \frac{b}{a}, 1, a(1+t)), \quad r_4(t) = M(1 + \frac{b}{a}, 1, a(1+t)). \tag{6.2.30}$$

Using (6.2.3b), we get

$$r_3(t) \sim a^{-1-\frac{b}{a}} t^{-1-\frac{b}{a}}, \quad \text{as } t \to \infty,\ a > 0, \tag{6.2.31}$$

and using (6.2.3a), we obtain

$$r_4(t) \sim \frac{1}{\Gamma(1+\frac{b}{a})} e^a a^{\frac{b}{a}} e^{at} t^{\frac{b}{a}}, \quad \text{ as } t \to \infty,\ a > 0,\ \frac{b}{a} \notin \{-1, -2, \ldots\} \tag{6.2.32}$$

The initial conditions (6.2.1b) can be used to determine c_3 and c_4; the relevant formulae are:

$$\begin{aligned} c_3 &= \Gamma(1+\frac{b}{a}) \mathrm{e}^{-a} \left(b\psi(0) M(1+\frac{b}{a}, 2, a) - b \int_{-1}^0 \psi(s)\, \mathrm{d}s\, M(1+\frac{b}{a}, 1, a) \right), \\ c_4 &= \Gamma(1+\frac{b}{a}) \mathrm{e}^{-a} \left(a\psi(0) U(1+\frac{b}{a}, 2, a) + b \int_{-1}^0 \psi(s)\, \mathrm{d}s\, U(1+\frac{b}{a}, 1, a) \right). \end{aligned} \tag{6.2.33}$$

One may verify, as before, that r_3 and r_4 solve (6.2.1a). In the determination of these formulae for c_3 and c_4, we have used the fact that one may deduce from Kummer's differential equation the identities [96, 13.3.13 & 13.3.14], which are

$$(\alpha+1)zM(\alpha+2,\beta+2,z)+(\beta+1)(\beta-z)M(\alpha+1,\beta+1,z) \\ -\beta(\beta+1)M(\alpha,\beta,z)=0 \tag{6.2.34}$$

$$(\alpha+1)zU(\alpha+2,\beta+2,z)+(z-\beta)U(\alpha+1,\beta+1,z)-U(\alpha,\beta,z)=0. \tag{6.2.35}$$

Moreover, letting $\beta\to 0$ in (6.2.34) and (6.2.35) gives

$$(\alpha+1)zM(\alpha+2,2,z)-zM(\alpha+1,1,z)-\alpha zM(\alpha+1,2,z)=0, \tag{6.2.36}$$

$$(\alpha+1)zU(\alpha+2,2,z)+zU(\alpha+1,1,z)-zU(\alpha+1,2,z)=0. \tag{6.2.37}$$

as [96, 13.2.5] in conjunction with [96, 5.2.1] gives $\lim_{\beta\to 0}\beta M(\alpha,\beta,z)=\alpha zM(\alpha+1,2,z)$ and [96, 13.2.11] gives $U(\alpha,0,z)=zU(\alpha+1,2,z)$.

Again, for certain values of a and b (i.e., if $-b/a\in\{1,2,3\ldots\}$), the two functions on the right–hand side of (6.2.29) are no longer linearly independent. Nevertheless the second–order equation (6.2.1a) has two linearly independent solutions r_3 (still given by (6.2.30)) and $\tilde{r}_4$, and so the solution of (6.1.2) obeys

$$x(t)=\tilde{c}_3r_3(t)+\tilde{c}_4\tilde{r}_4(t),\quad t\geq 0,\qquad \text{for } a>0 \text{ and } b/a\in\{-1,-2,...\}. \tag{6.2.38}$$

By (6.2.31), $r_3(t)>0$ for all t sufficiently large. Therefore we may define $t_2=1+\max(0,\sup\{t\in\mathbb{R}:r_3(t)=0\})$, where $\sup\{t\in\mathbb{R}:r_3(t)=0\}:=-\infty$ if $r_3(t)\neq 0$ for all $t\geq 0$. By considering the Wronskian of r_3 and $\tilde{r}_4$ for $t\geq t_2$ we have

$$\tilde{r}_4'(t)-\frac{r_3'(t)}{r_3(t)}\tilde{r}_4(t)=\mathcal{W}(a,b,0)\frac{\mathrm{e}^{at}(1+t)^{-1}}{r_3(t)},\quad t\geq t_2, \tag{6.2.39}$$

where $\mathcal{W}(a,b,0)\neq 0$ is the Wronskian of r_3 and $\tilde{r}_4$ at $t=0$. (6.2.39) yields the representation

$$\tilde{r}_4(t)=r_3(t)\frac{\tilde{r}_4(t_2)}{r_3(t_2)}+\mathcal{W}(a,b,0)r_3(t)\int_{t_2}^{t}\frac{\mathrm{e}^{as}(1+s)^{-1}}{r_3^2(s)}\,\mathrm{d}s,\quad t\geq t_2$$

for $\tilde{r}_4$. By means of l'Hôpital's rule and (6.2.31) we can deduce from this representation for $\tilde{r}_4$ that

$$\lim_{t\to\infty}\mathrm{e}^{-at}t^{-\frac{b}{a}}\tilde{r}_4(t)=\mathcal{W}(a,b,0)a^{\frac{b}{a}}. \tag{6.2.40}$$

This is consistent with the asymptotic behaviour we established for r_4 in (6.2.32).

It is also useful to determine some asymptotic information about $\tilde{r}_4'$. Notice that $t \mapsto U(1+\frac{b}{a}, 1, a(1+t))$ is a polynomial of degree $-1-b/a \in \mathbb{N}$, and so $\lim_{t\to\infty} r_3'(t)/r_3(t) = 0$. By (6.2.40), it follows that $\tilde{r}_4(t) \neq 0$ for all $t \geq t_3$. Letting $t_4 = \max(t_2, t_3)$, we rewrite (6.2.39) for $t \geq t_4$ to get

$$\frac{\tilde{r}_4'(t)}{\tilde{r}_4(t)} = \frac{r_3'(t)}{r_3(t)} + \mathcal{W}(a,b,0)\frac{\mathrm{e}^{at}(1+t)^{-1}}{r_3(t)\tilde{r}_4(t)}.$$

Using the fact that $r_3(t) \sim a^{-1-\frac{b}{a}} t^{-1-b/a}$ as $t \to \infty$ together with (6.2.40) shows that the second term has limit a, and therefore

$$\lim_{t\to\infty} \frac{\tilde{r}_4'(t)}{\tilde{r}_4(t)} = a. \tag{6.2.41}$$

Since r_3 and $\tilde{r}_4$ are linearly independent, we can use the representation (6.2.38) for x to find $\tilde{c}_3$ and $\tilde{c}_4$ such that the initial conditions of (6.2.1b) (or (6.1.2)) are satisfied. In particular, $\tilde{c}_4$ can be expressed according to

$$\tilde{c}_4 = \frac{1}{\mathcal{W}(a,b,0)}\left(a\psi(0)\,U(1+\frac{b}{a},2,a) + b\int_{-1}^{0}\psi(s)ds\,U(1+\frac{b}{a},1,a)\right).$$

An argument, which is identical in all respects to that used to deduce the representation (6.2.17) of the solution r of the resolvent equation (6.1.3) in the case when $a < 0$, can be used to justify the formulae

$$r(t,s) = \begin{cases} d_3(s)r_3(t) + d_4(s)r_4(t), & a > 0, \quad b/a \notin \{-1,-2...\}, \\ \tilde{d}_3(s)r_3(t) + \tilde{d}_4(s)\tilde{r}_4(t), & a > 0, \quad b/a \in \{-1,-2...\}. \end{cases} \tag{6.2.42}$$

Conditions for d_3 and d_4, and for $\tilde{d}_3$ and $\tilde{d}_4$, are obtained from the initial conditions (6.1.3b) and (6.2.7), just as was done to obtain the equations (6.2.18) for d_1 and d_2 in the case when $a < 0$. Solving the corresponding equations to (6.2.18), we obtain

$$\begin{aligned} d_3(s) &= \Gamma(1+\frac{b}{a})\mathrm{e}^{-a(1+s)}(1+s)bM(1+\frac{b}{a},2,a(1+s)), \\ d_4(s) &= \Gamma(1+\frac{b}{a})\mathrm{e}^{-a(1+s)}(1+s)aU(1+\frac{b}{a},2,a(1+s)). \end{aligned} \tag{6.2.43}$$

Proceeding in the same manner in the degenerate case when $b/a \in \{-1,-2,...\}$ yields the expressions

$$\begin{aligned} \tilde{d}_3(s) &= \frac{\tilde{r}_4'(s) - a\tilde{r}_4(s)}{\mathcal{W}(a,b,0)\mathrm{e}^{as}(1+s)^{-1}} = -\frac{a}{\mathcal{W}(a,b,0)}\tilde{r}_4(s)(1+s)\mathrm{e}^{-as}\left(1+\frac{1}{-a}\frac{\tilde{r}_4'(s)}{\tilde{r}_4(s)}\right), \\ \tilde{d}_4(s) &= -\frac{r_3'(s) - ar_3(s)}{\mathcal{W}(a,b,0)\mathrm{e}^{as}(1+s)^{-1}} = \frac{1}{\mathcal{W}(a,b,0)}\mathrm{e}^{-as}(1+s)aU(1+\frac{b}{a},2,a(1+s)). \end{aligned}$$

We now turn our attention to the asymptotic behaviour of d_3, d_4 etc. Using (6.2.3), we can show that

$$d_3(s) \sim b a^{b/a-1} s^{b/a}, \qquad \text{as } s \to \infty, \tag{6.2.44}$$

$$d_4(s) \sim \Gamma(1+b/a) e^{-a} a^{-b/a} s^{-b/a} \mathrm{e}^{-as}, \qquad \text{as } s \to \infty. \tag{6.2.45}$$

In the degenerate case when $b/a \in \{-1, -2, ...\}$, we may use (6.2.40) and (6.2.41) to establish that

$$\tilde{d}_3(s) = o(s^{\frac{b}{a}+1}), \quad \text{as } s \to \infty. \tag{6.2.46}$$

(6.2.46) is consistent with, but weaker than, the asymptotic estimate obtained for d_3 in (6.2.44) in the non–degenerate case. As to the asymptotic behaviour of $\tilde{d}_4$, we may use (6.2.3b) to give

$$\tilde{d}_4(s) \sim \frac{1}{\mathcal{W}(a,b,0)} a^{-\frac{b}{a}} s^{-\frac{b}{a}} \mathrm{e}^{-as} \quad \text{as } s \to \infty, \tag{6.2.47}$$

which is consistent with the asymptotic behaviour in (6.2.45) in the non–degenerate case.

6.2.3 $a = 0$

When $a = 0$ and $b > 0$, it transpires that the solution of (6.2.1a) can be expressed in terms of *modified Bessel functions*. To be more precise, we have

$$x(t) = c_5 r_5(t) + c_6 r_6(t), \quad \text{for } t \geq 0, \text{ when } a = 0 \text{ and } b > 0 \tag{6.2.48}$$

where

$$r_5(t) = I_0(2\sqrt{b(t+1)}), \quad r_6(t) = K_0(2\sqrt{b(t+1)}) \tag{6.2.49}$$

and I_ν and K_ν are two linearly independent solutions of *modified Bessel's equation*

$$z^2 w''(z) + z w'(z) - (z^2 + \nu^2) w(z) = 0,$$

with ν a real parameter. See e.g. [96, Chapter 10.25.1] for details. I_ν and K_ν are referred to as *modified Bessel functions of the first kind* and *second kind* respectively. One may verify that r_5 and r_6 are linearly independent solutions of (6.2.1a) by a direct calculation.

The constants c_5 and c_6 in (6.2.48) can be found using the initial conditions (6.2.1b) or (6.1.2b). Doing this yields the formulae

$$c_5 = 2\left(\psi(0)\sqrt{b}K_1(2\sqrt{b}) + b\int_{-1}^{0}\psi(s)ds K_0(2\sqrt{b})\right), \tag{6.2.50}$$

$$c_6 = 2\left(\psi(0)\sqrt{b}I_1(2\sqrt{b}) - b\int_{-1}^{0}\psi(s)ds I_0(2\sqrt{b})\right).$$

In finding these expressions for c_5 and c_6, we have exploited the fact that the Wronskian of I_ν and K_ν obeys the identity

$$W\{K_\nu(z), I_\nu(z)\} = 1/z \tag{6.2.51}$$

(which appears as [96, 10.28.2], for example) and the derivatives of I_0 and K_0 obey

$$I_0'(z) = I_1(z), \quad K_0'(z) = -K_1(z). \tag{6.2.52}$$

(cf., e.g. [96, 10.29.3]). We will also employ in the sequel the asymptotic behaviour of I_ν and K_ν. The relevant results are

$$I_\nu(t) = \frac{\mathrm{e}^t}{\sqrt{2\pi t}}\{1 + O(t^{-1})\}, \quad K_\nu(t) = \sqrt{\frac{\pi}{2t}}\mathrm{e}^{-t}\{1 + O(t^{-1})\}, \quad \text{as } t \to \infty, \tag{6.2.53}$$

which appear as [1, 9.7.1 & 9.7.2], for example.

As in the cases when $a < 0$ or $a > 0$, the solution to the resolvent equation (6.1.3) can be represented as the sum of products of functions in t and s. Indeed, $r(t, s)$ can be written in the form

$$r(t, s) = d_5(s)r_5(t) + d_6(s)r_6(t), \quad t \geq s \geq 0, \text{ for } a = 0 \text{ and } b > 0. \tag{6.2.54}$$

As in e.g., (6.2.18), d_5 and d_6 may be found by solving a pair of linear simultaneous equations formulated from (6.1.3b). This leads to the formulae

$$d_5(s) = 2\sqrt{b(s+1)}K_1(2\sqrt{b(s+1)}), \quad d_6(s) = 2\sqrt{b(s+1)}I_1(2\sqrt{b(s+1)}), \tag{6.2.55}$$

by making use of the identities (6.2.51) and (6.2.52).

In the case when $a = 0$ and $b < 0$, it turns out that the solution of (6.2.1a) can be expressed in terms of *Bessel functions.* Indeed, we have

$$x(t) = c_7 r_7(t) + c_8 r_8(t) \quad \text{for } t \geq 0, \text{ when } a = 0 \text{ and } b < 0 \tag{6.2.56}$$

where

$$r_7(t) = J_0(2\sqrt{-b(t+1)}), \quad r_8(t) = Y_0(2\sqrt{-b(t+1)}) \tag{6.2.57}$$

and J_ν and Y_ν are two linearly independent solutions of *Bessel's Equation*

$$z^2 w''(z) + z w'(z) + (z^2 - \nu^2)w(z) = 0,$$

where ν is a real parameter (cf., e.g. [96, Chapter 10.2.1] for details). J_ν and Y_ν are referred to as the *Bessel functions of the first kind* and *second kind* respectively. We remark that the Bessel functions are oscillatory, convergent to zero and real–valued for positive arguments. Moreover as the argument $t \to +\infty$, $Y_\nu(t)$ and $J_\nu(t)$ share the same amplitude, and are out of phase by $\frac{1}{2}\pi$, [95, pp.242, Ch.7.5.1]. We make this precise in (6.2.60) below. One may verify by direct calculation that r_7 and r_8 are linearly independent solutions of (6.2.1a).

From (6.2.56) and (6.2.1b), we can find expressions for the constants c_7 and c_8. In fact, one obtains

$$c_7 = \pi\left(\psi(0)\sqrt{|b|}Y_1(2\sqrt{|b|}) - b\int_{-1}^{0}\psi(s)ds\, Y_0(2\sqrt{|b|})\right), \tag{6.2.58}$$

$$c_8 = \pi\left(\psi(0)\sqrt{|b|}J_1(2\sqrt{|b|}) + b\int_{-1}^{0}\psi(s)ds\, J_0(2\sqrt{|b|})\right). \tag{6.2.59}$$

In deducing these formulae, we have used the fact that the Wronskian of J_ν and Y_ν obeys

$$W\{J_\nu(z), Y_\nu(z)\} = 2/(\pi z)$$

(cf., e.g., [96, 10.5.2]) and also that the derivatives of J_ν and Y_ν obey

$$J_0'(z) = -J_1(z), \quad Y_0'(z) = Y_1(z)$$

cf., e.g. [96, 10.6.3]. In asymptotic analysis of the solution of the stochastic equation, we will need information about the asymptotic behaviour of $J_\nu(t)$ and $Y_\nu(t)$ as $t \to \infty$. The required asymptotic information is furnished by [1, 9.2.1, 9.2.2, 9.2.5, 9.2.6], which we record now for convenience:

$$J_\nu(t) = \sqrt{2/(\pi t)}\{\cos(t - \frac{1}{2}\nu\pi - \frac{1}{4}\pi) + O(t^{-1})\}, \quad \text{as } t \to \infty, \tag{6.2.60a}$$

$$Y_\nu(t) = \sqrt{2/(\pi t)}\{\sin(t - \frac{1}{2}\nu\pi - \frac{1}{4}\pi) + O(t^{-1})\}, \quad \text{as } t \to \infty. \tag{6.2.60b}$$

Once again the solution to the resolvent equation (6.1.3) can be written as a sum of products of functions depending on t and s. Indeed, $r(t,s)$ can be written in the form

$$r(t,s) = d_7(s)r_7(t) + d_8(s)r_8(t), \quad t \geq s \geq 0, \quad a = 0, \quad b < 0, \tag{6.2.61}$$

and expressions for d_7 and d_8 may be obtained from this representation and (6.1.3b). This yields

$$d_7(s) = \pi\sqrt{|b|(1+s)}Y_1(2\sqrt{|b|(s+1)}), \quad d_8(s) = \pi\sqrt{|b|(s+1)}\, J_1(2\sqrt{|b|(s+1)}), \tag{6.2.62}$$

upon use of the identities for the Wronskian of J_0 and Y_0 and formulae for the derivatives of J_0 and Y_0.

6.3 Recurrent Asymptotic Behaviour

6.3.1 Pathwise asymptotic stationary behaviour

The asymptotic behaviour of (6.1.1) in the case when $a < 0$ and $a + b < 0$ is very similar to the Ornstein–Uhlenbeck process U given by

$$dU(t) = aU(t)\,dt + \sigma\,dB(t), \quad t \geq 0; \quad U(0) = 0. \tag{6.3.1}$$

There is a unique continuous adapted process which obeys (6.3.1) and it is given by

$$U(t) = e^{at}\int_0^t \sigma e^{-as}\,dB(s), \quad t \geq 0. \tag{6.3.2}$$

Theorem 6.3.1. *Let $a < 0$ and $a + b \leq 0$. Suppose that $\psi \in C([-1, 0]; \mathbb{R})$. Let X be the unique continuous adapted process which obeys* (6.1.1) *and let U be the unique continuous adapted process which obeys* (6.3.1)*. Then:*

(i) X obeys

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2\log t}} = \frac{\sigma}{\sqrt{2|a|}}, \quad \liminf_{t\to\infty} \frac{X(t)}{\sqrt{2\log t}} = -\frac{\sigma}{\sqrt{2|a|}}, \quad a.s. \tag{6.3.3}$$

(ii) In the case that $a + b < 0$, we have

$$\lim_{t\to\infty}\{X(t) - U(t)\} = 0, \quad a.s. \tag{6.3.4}$$

and that

$$\lim_{t\to\infty} \frac{1}{1+t}\int_{-1}^t X(s)\,ds = 0, \quad a.s. \tag{6.3.5}$$

(iii) In the case that $a + b = 0$, we have

$$\lim_{t\to\infty}\{X(t) - U(t)\} = L, \quad a.s. \tag{6.3.6}$$

where L is a proper Gaussian random variable with mean and variance given by

$$\begin{aligned}
\mathbb{E}[L] &= b^2\Gamma(-\frac{b}{a})\left(\int_{-1}^0 \psi(u)du\right)\int_0^\infty U(1-\frac{b}{a}, 2, -a(1+s))\,ds \\
&\quad + b^2\Gamma(-\frac{b}{a})\psi(0)\int_0^\infty \mathrm{e}^{au}\int_u^\infty U(1-\frac{b}{a}, 2, -a(1+s))\,ds\,du, \\
Var[L] &= \sigma^2\int_0^\infty \mathrm{e}^{-2au}\left(\int_u^\infty \mathrm{e}^{aw}\int_w^\infty b^2\Gamma(-\frac{b}{a})\,U(1-\frac{b}{a}, 2, -a(1+s))\,ds\,dw\right)^2 du.
\end{aligned}$$

and that

$$\lim_{t\to\infty} \frac{1}{1+t}\int_{-1}^t X(s)\,ds = L, \quad a.s. \tag{6.3.7}$$

The result (6.3.5) shows that, when $a < 0$ and $a+b < 0$, the average value of the process X tends to zero, i.e. the fluctuations of X, which are of order $\sqrt{\log t}$, occur symmetrically about zero. The result (6.3.7) however shows that, when $a < 0$ and $a + b = 0$, the fluctuations of X occur about the level L (which is random and so will appear different for each sample path).

It is of interest to ask if we can provide an upper bound on the a.s. rate of convergence of $X - U$ to zero when $a + b < 0$. Of course the case when $a + b = 0$ is excluded, because in that case $X - U$ tends to a non–trivial limit. We show that in all cases, the bound on the closeness decays polynomially.

Theorem 6.3.2. *Let $a < 0$ and $a + b < 0$. Suppose that $\psi \in C([-1, 0]; \mathbb{R})$. Let X be the unique continuous adapted process which obeys* (6.1.1) *and let U be the unique continuous adapted process which obeys* (6.3.1)*. Then:*

(i) If $a + b < 0$ and $2b + a > 0$, then

$$\limsup_{t\to\infty} \frac{|X(t) - U(t)|}{t^{-1-\frac{b}{a}}} \in [0, \infty), \quad a.s.$$

(ii) If $2b + a < 0$, then

$$\limsup_{t\to\infty} \frac{|X(t) - U(t)|}{t^{-1/2}\sqrt{\log\log t}} \in [0, \infty), \quad a.s.$$

(iii) If $2b + a = 0$, then

$$\limsup_{t\to\infty} \frac{|X(t) - U(t)|}{t^{-1/2}\log t\sqrt{\log\log t}} \in [0, \infty), \quad a.s.$$

While we conjecture that these estimates are sharp, i.e. the limits superior in Theorem 6.3.2 are positive, such an analysis would involve, amongst other things, a sharper analysis of the leading order terms in the expansions in (6.2.3), as well as lower estimates of certain integrals in the proof. Such analysis goes beyond the scope of the present work.

6.3.2 Asymptotic behaviour of the autocovariance function

Theorem 6.3.1 shows that X is a Gaussian process which is asymptotically close to the asymptotically stationary Gaussian process U (for $b = 0$, X is itself an Ornstein-Uhlenbeck process). Since U is given by (6.3.2), its autocovariance function may be shown to obey

$$\text{Cov}(U(t), U(t + \Delta)) = \sigma^2 e^{a\Delta} e^{2at} \int_0^t e^{-2as}\, ds = e^{a\Delta}\sigma^2 \frac{1}{2|a|}\left(1 - e^{2at}\right).$$

Therefore, for each fixed $t > 0$ we have $\Delta \mapsto \text{Cov}(U(t), U(t+\Delta))$ decays *exponentially* to zero as $\Delta \to \infty$. It is therefore reasonable to expect that the autocovariance function of X defined by (6.1.6) to behave according to $\lim_{\Delta\to\infty} \text{Cov}(X(t), X(t+\Delta)) = 0$ for every $t \geq 0$. However, as is shown below, although $X(t) - U(t) \to 0$ as $t \to \infty$ a.s., for each fixed $t > 0$, the autocovariance $\Delta \mapsto \text{Cov}(X(t), X(t+\Delta))$ decays *polynomially* to zero as $\Delta \to \infty$.

We have already seen in (6.2.25) that it is possible to represent the autocovariance function in terms of r_1, r_2, d_1, d_2 etc. Using the information about the asymptotic behaviour of these functions, we can readily describe how rapidly the autocovariance function decays in the time lag Δ.

Theorem 6.3.3. *Suppose that $a < 0$ and $a + b \leq 0$. Suppose that $\psi \in C([-1, 0]; \mathbb{R})$. Let X be the unique continuous adapted process which obeys* (6.1.1). *Let $t \geq 0$ be fixed. Then*

$$\lim_{\Delta\to\infty} \frac{\text{Cov}(X(t), X(t+\Delta))}{\Delta^{-(1+\frac{b}{a})}} = c_t(a, b), \tag{6.3.8}$$

where $c_t = c_t(a, b)$ is given by

$$c_t(a, b) = \sigma^2 b|a|^{-1-b/a} \int_0^t r(t, s)(1+s)U(1 - b/a, 2, -a(1+s))\, ds. \tag{6.3.9}$$

Hence the process X defined by (6.1.1) is a long memory process (i.e., for each fixed t, $\int_0^\infty \text{Cov}(X(t), X(t+\Delta)\, d\Delta = +\infty)$ when $a < 0$, $b > 0$ and $a + b < 0$.

In the case when $a + b = 0$, the covariance does not tend to zero as $\Delta \to \infty$; instead

$$\lim_{\Delta\to\infty} \text{Cov}(X(t), X(t+\Delta)) = c_t(a, b). \tag{6.3.10}$$

In the special case $a < 0$ and $b = 0$, equation (6.1.1) reduces to an Ornstein-Uhlenbeck equation and hence its autocovariance function is decays exponentially. This is consistent with the result of Theorem 6.3.3, because the value of c_t is zero in (6.3.9). This leads us to question under what conditions will the limit obtained in Theorem 6.3.3 be nonzero.

Proposition 6.3.1. *Let $b > 0$. Then Cov$(X(t), X(t+\Delta)) > 0$ for all $\Delta > 0$.*

Proposition 6.3.2. *If $a < 0$, $b > 0$ and $a + b < 0$, then the limiting constant in* (6.3.9) *obeys $c_t(a, b) > 0$.*

The case when $b < 0$ is more delicate to analyse. However, it can be shown that if t is sufficiently large, then $c_t(a, b)$ is negative. We can also show that $c_t(a, b) \to 0$ as $t \to \infty$

in the case when $b > 0$ and that $c_t(a,b) \to -\infty$ as $t \to \infty$ in the case that $b < 0$. We also see that $\lim_{t\to\infty} c_t(a,b)$ is nontrivial in the case when $a+b=0$, and its limit will be of interest later in this section. Accordingly, the asymptotic behaviour of c_t is recorded in the next result.

Proposition 6.3.3. *Suppose that $a < 0$ and $a+b \le 0$ and let $c_t(a,b)$ be defined by* (6.3.9).

(a) If $b < 0$ and $a+b < 0$, then

$$\lim_{t\to\infty} \frac{c_t(a,b)}{t^{b/a}} = \sigma^2 b|a|^{-3} \frac{|b|+|a|}{2|b|+|a|} < 0, \tag{6.3.11}$$

and so $c_t \to -\infty$ as $t \to \infty$.

(b) If $b > 0$ and $a+b < 0$, then $c_t \to 0$ as $t \to \infty$. Furthermore

(i) If $2b+a > 0$, then

$$\lim_{t\to\infty} \frac{c_t(a,b)}{t^{-b/a-1}} = \frac{\sigma^2 b^2}{|a|^{2+2b/a}} \int_0^\infty (1+s)^2 U^2\left(1-\frac{b}{a}, 2, -a(1+s)\right) ds > 0; \tag{6.3.12}$$

(ii) If $2b+a = 0$, then

$$\lim_{t\to\infty} \frac{c_t(a,b)}{t^{-1/2}\log t} = \sigma^2 \frac{1}{4}|a|^{-2} > 0; \tag{6.3.13}$$

(iii) If $2b+a < 0$, then c_t obeys (6.3.11) *with the limit on the righthand side being positive.*

(c) If $a+b=0$, then

$$\lim_{t\to\infty} c_t(a,b) = \sigma^2 \frac{b^2}{|a|^{2+2b/a}} \int_0^\infty (1+s)^2 U(1-b/a, 2, |a|(1+s))^2\, ds. \tag{6.3.14}$$

In Theorem 6.3.3 we held the starting time, t, fixed and observed the behaviour of the auto–covariance function as the time lag, Δ tended to infinity. However it is perhaps more typical, when testing for long memory (c.f. e.g. [10]), to fix the time lag and let the starting time tend to infinity. It is then observed that this limiting auto–covariance function depends only on the time lag Δ (so that the process is transiently non–stationary) and the limiting autocovariance function is integrable over Δ, so that X does not have long memory.

Theorem 6.3.4. *Suppose that $a < 0$ and $a+b \le 0$. Suppose that $\psi \in C([-1,0];\mathbb{R})$. Let X be the unique continuous adapted process which obeys* (6.1.1). *Then, for all $\Delta \ge 0$,*

(a) *If $a+b<0$, then*

$$\lim_{t\to\infty} Cov(X(t), X(t+\Delta)) = \frac{\sigma^2}{2|a|} e^{a\Delta}. \tag{6.3.15}$$

(b) *If $a+b=0$, then*

$$\lim_{t\to\infty} Cov(X(t), X(t+\Delta)) \\ = \frac{\sigma^2}{2|a|} e^{a\Delta} + \sigma^2 \frac{b^2}{|a|^{2+2\frac{b}{a}}} \int_0^\infty (1+s)^2 U(1-\frac{b}{a}, 2, |a|(1+s))^2\, ds. \tag{6.3.16}$$

It is interesting to remark that the differing rates of decay of the autocovariance function recorded for the solution of (6.1.1) when $a<0$ and $a+b<0$ in the limits (6.3.15) and (6.3.8) are *not* generally seen in autonomous affine differential equations. We show below for asymptotically stationary scalar affine SFDEs which are either finite delay or of Volterra type, that one is in a position to characterise short or long memory by means of a single limiting autocovariance function. Therefore, in the case of autonomous affine equations, it does not matter whether one takes $\Delta \to \infty$ or $t \to \infty$: as both limits lead to the same function, both give the same classification of the process as being short or long memory.

To make this claim more precise, and to find notation to connect the behaviour of the autocovariance function of the solution of (6.1.1) with autocovariance functions of solutions of such autonomous affine SFDEs, and to also contrast these behaviours, we start by examining, for example, the solution X of an affine SFDE with finite delay. Such a process X would be the solution of

$$dX(t) = L(X_t)\, dt + \sigma\, dB(t), \quad t \ge 0; \quad X(t) = \psi(t), \quad t \in [-\tau, 0], \tag{6.3.17}$$

where $L : C([-\tau,0];\mathbb{R}) \to \mathbb{R}$ is a linear functional and $\psi \in C([-\tau,0];\mathbb{R})$. Suppose that r is the differential resolvent given by

$$r'(t) = L(r_t), \quad t>0; \quad r(0)=1; \quad r(t)=0 \text{ for } t \in [-\tau, 0).$$

We now summarise the situation in the following claim.

Remark 6.3.1. If X is the solution of (6.3.17), and the differential resolvent r associated with the drift of (6.3.17) obeys $r(t) \to 0$ as $t \to \infty$ and $r(t)$ is of one sign for all t sufficiently large, then there are functions γ and c such that

$$\lim_{t\to\infty} \frac{\mathrm{Cov}(X(t), X(t+\Delta))}{\gamma(\Delta)} = 1, \tag{6.3.18a}$$

$$\lim_{\Delta\to\infty} \frac{\mathrm{Cov}(X(t), X(t+\Delta))}{\gamma(\Delta)} = c_t, \tag{6.3.18b}$$

$$\lim_{t\to\infty} c_t = 1. \tag{6.3.18c}$$

A similar result pertains to Volterra equations with slowly decaying autocovariance function. For instance, if X is the solution of

$$dX(t) = \left(-aX(t) + \int_0^t k(t-s)X(s)\,ds\right)\,dt + \sigma\,dB(t), \quad t \geq 0; \quad X(0) = \xi, \tag{6.3.19}$$

and we suppose that k is a continuous, positive and integrable function. Let the differential resolvent r be the solution of

$$r'(t) = -ar(t) + \int_0^t k(t-s)r(s)\,ds, \quad t \geq 0; \quad r(0) = 1.$$

Remark 6.3.2. Suppose that k is a positive, continuous and integrable function which is subexponential and asymptotic to a decreasing function, and moreover obeys $a > \int_0^\infty k(s)\,ds$. Then the autocovariance function of the solution X of (6.3.19) obeys (6.3.18).

We are now in a position to compare and contrast the situation with (6.3.18), which pertains for solutions of affine autonomous equations. For the average equation the autocovariance function obeys

$$\lim_{t\to\infty} \frac{\mathrm{Cov}(X(t), X(t+\Delta))}{\gamma_1(\Delta)} = 1, \tag{6.3.20a}$$

$$\lim_{\Delta\to\infty} \frac{\mathrm{Cov}(X(t), X(t+\Delta))}{\gamma_2(\Delta)} = c_t, \tag{6.3.20b}$$

$$\lim_{t\to\infty} c_t = \begin{cases} 0, & b > 0, \\ -\infty, & b < 0 \end{cases} \tag{6.3.20c}$$

where $\gamma_1(\Delta) = \sigma^2/2|a| \cdot e^{a\Delta}$ and $\gamma_2(\Delta) = \Delta^{-(1+b/a)}$. Therefore, the situation in (6.3.20) differs from the case in (6.3.18), because there are two different rates of decay in Δ in (6.3.20a) and (6.3.20b) and the function c_t in (6.3.20c) does not tend to a non–trivial finite limit as $t \to \infty$.

Theorem 6.3.4 part (a) is consistent with Theorem 6.3.1 part (ii), because in the case when $a + b < 0$, the latter result shows that X is pathwise asymptotic to a process whose limiting autocovariance function is given in part (a). The result of part (b) is also consistent with Theorem 6.3.1, because when $a+b=0$, we know from part (iii) of Theorem 6.3.1 that the solution is asymptotically equal to U plus a non–trivial limiting random variable, whose presence is suggested by the form of the limiting autocovariance function in part (b).

It is tempting to remark that when $b>0$, Proposition 6.3.3 part (b) may be thought of as partly reconciling the differing asymptotic behaviour of $\text{Cov}(X(t), X(t+\Delta))$ recorded in Theorem 6.3.3 and 6.3.4 according as to whether $\Delta \to \infty$ or $t \to \infty$. This is because $c_t(a,b) \to 0$ as $t \to \infty$, so that the "long memory" recorded in (6.3.8) becomes ever weaker as the start time t becomes greater, and therefore becomes closer to the "short memory" or exponential decay in Δ in the limiting autocovariance function determined in part (a) of Theorem 6.3.4.

This heuristic explanation of the reconciliation of the asymptotic behaviour of the autocovariance must however be taken with caution. In particular, in the case when $b<0$, it is harder to forward with equal confidence the same explanation as to the differing asymptotic behaviour recorded in Theorem 6.3.3 and 6.3.4. In this case, Proposition 6.3.3 part (b) shows that $c_t(a,b) \to -\infty$ as $t \to \infty$, suggesting that the polynomial decay in the autocovariance function given in (6.3.8) tends to become *stronger* as the start time is chosen to be very large. On the other hand, the fact that $|c_t|$ has power law growth which is less rapid as $t \to \infty$ (at a rate $t^{b/a}$ according to (6.3.11)) compared to the power law decay of $\text{Cov}(X(t), X(t+\Delta))$ as $\Delta \to \infty$ (which is at the rate $\Delta^{-(1+b/a)}$) may point to a weakening overall correlation.

One situation in which it does not seem to matter in which order limits are taken is when $a+b=0$. Taking the limit as $\Delta \to \infty$ in (6.3.16) leads to

$$\lim_{\Delta\to\infty}\lim_{t\to\infty} \text{Cov}(X(t), X(t+\Delta)) = \sigma^2 \frac{b^2}{|a|^{2+2\frac{b}{a}}} \int_0^\infty (1+s)^2 U(1-\frac{b}{a}, 2, |a|(1+s))^2 \, ds.$$

On the other hand, by (6.3.10) and (6.3.14) we have that

$$\lim_{t\to\infty}\lim_{\Delta\to\infty} \text{Cov}(X(t), X(t+\Delta)) = \sigma^2 \frac{b^2}{|a|^{2+2\frac{b}{a}}} \int_0^\infty (1+s)^2 U(1-\frac{b}{a}, 2, |a|(1+s))^2 \, ds,$$

so the limits are equal.

6.3.3 Non-stationary asymptotic behaviour

In the case when $a < 0$ and $b < 0$, we have already seen that the solution of (6.1.1) is asymptotically stationary, and when $a > 0$ and $b < 0$ (see Chapter 5), the solution exhibits a.s. exponential growth. Therefore, we expect to see intermediate asymptotic behaviour on the boundary of these two parameter regions, where $a = 0$ and $b < 0$. In broad terms, we can establish that the solution behaves in some ways like a standard Brownian motion, in the sense that the solution is a Gaussian process which has asymptotically vanishing mean, variance which grows linearly in time, and experiences a.s. large fluctuations which satisfy the Law of the iterated logarithm.

Theorem 6.3.5. *Suppose that $\psi \in C([-1,0];\mathbb{R})$. Let X be the unique continuous adapted process which obeys* (6.1.1)*. If $a = 0$ and $b < 0$, then $\mathbb{E}[X(t)] \to 0$ as $t \to \infty$ and*

$$\lim_{t\to\infty} \frac{Var[X(t)]}{t} = \frac{1}{3}\sigma^2.$$

We now state the result which deals with the magnitude of the large fluctuations of X.

Theorem 6.3.6. *Suppose that $\psi \in C([-1,0];\mathbb{R})$. Let X be the unique continuous adapted process which obeys* (6.1.1)*. If $a = 0$ and $b < 0$, then*

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = \frac{1}{\sqrt{3}}\sigma, \quad \liminf_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = -\frac{1}{\sqrt{3}}\sigma, \quad a.s.$$

Remark 6.3.3. Both Theorems 6.3.5 and 6.3.6 show that, asymptotically, X behaviours somewhat akin to standard Brownian motion. In particular it is observed that the limiting constant in Theorem 6.3.5 is the square of that in Theorem 6.3.6. We are then drawn to conjecture that the increments of X, under the hypothesises of Theorems 6.3.5 and 6.3.6, are asymptotically stationary.

6.4 Transient Asymptotic Behaviour

From (6.1.4) we see that as X depends upon x, we then expect the asymptotic behaviour of X to also depend upon x, especially in the case when $|x(t)| \to \infty$ as $t \to \infty$. This arises in two main situations: when $a < 0$ and $a + b > 0$, and when $a > 0$. We deal with the first of these cases first, and establish that $|X(t)| \to \infty$ as $t \to \infty$ like a power of t. In fact, X can tend to $+\infty$ or to $-\infty$, each with positive probability. Moreover, the choice of which limit is attained depends on the path of the Brownian motion driving X, with the increments of B earlier in the path generally proving to be more influential in deciding which limit is attained. The key to the proof of this result, and to the others in this Section, hinge on the representation of the solution X of (6.1.1) in terms of the resolvent r and mean x, as well as the asymptotic analysis of these functions given in Section 6.2.

Theorem 6.4.1. *Suppose that $a < 0$, $a + b > 0$. Suppose also that $\psi \in C([-1, 0]; \mathbb{R})$. Let X be the unique continuous adapted process which obeys* (6.1.1). *Then*

(a) There exists an $\mathcal{F}^B(\infty)$-measurable normal random variable C such that

$$\lim_{t\to\infty} \frac{X(t)}{t^{-(1+\frac{b}{a})}} = C, \quad a.s. \tag{6.4.1}$$

(b) C is given by

$$C = |a|^{-1-\frac{b}{a}} b \left\{ \psi(0)\, U\left(1 - \frac{b}{a}, 2, |a|\right) + \int_{-1}^{0} \psi(s) ds\, U\left(-\frac{b}{a}, 1, |a|\right) \right\} + \sigma \int_0^\infty \frac{b}{|a|^{1+\frac{b}{a}}} (1+s) U(1 - \frac{b}{a}, 2, |a|(1+s))\, dB(s).$$

(c) The mean and variance of C are given by

$$\mathbb{E}[C] = |a|^{-1-\frac{b}{a}} b \left\{ \psi(0)\, U\left(1 - \frac{b}{a}, 2, |a|\right) + \int_{-1}^{0} \psi(s) ds\, U\left(-\frac{b}{a}, 1, |a|\right) \right\}, \tag{6.4.2}$$

$$Var[C] = \sigma^2 \frac{b^2}{|a|^{2+2\frac{b}{a}}} \int_0^\infty (1+s)^2 U(1 - \frac{b}{a}, 2, |a|(1+s))^2 ds > 0. \tag{6.4.3}$$

(d) The mean and variance of X obey

$$\lim_{t\to\infty} \frac{\mathbb{E}[X(t)]}{t^{-1-\frac{b}{a}}} = \mathbb{E}[C], \quad \lim_{t\to\infty} \frac{Var[X(t)]}{t^{-2-2\frac{b}{a}}} = Var[C].$$

Once the formula (6.4.3) is established, it is clear that C is a proper Gaussian random variable, because $s \mapsto U(1 - \frac{b}{a}, 2, |a|(1+s))^2$ is asymptotic to a positive function and so is itself eventually positive. Thus we have $C \neq 0$ a.s.

In the case when $a > 0$, we show that X grows to plus or minus infinity at an exponential rate, with a power law correction growth factor. Once again, there is a positive probability of each of the events $\{\lim_{t\to\infty} X(t) = +\infty\}$ and $\{\lim_{t\to\infty} X(t) = -\infty\}$ occurring.

Theorem 6.4.2. *Suppose that $a > 0$. Suppose also that $\psi \in C([-1,0];\mathbb{R})$. Let X be the unique continuous adapted process which obeys* (6.1.1).

(a) There exists an $\mathcal{F}^B(\infty)$-measurable normal random variable C such that

$$\lim_{t\to\infty} \frac{X(t)}{e^{at}t^{b/a}} = C, \quad a.s. \tag{6.4.4}$$

(b) C is given by

$$C = a^{\frac{b}{a}} \left\{ a\psi(0)U\left(1+\frac{b}{a},2,a\right) + b\int_{-1}^{0} \psi(s)\,\mathrm{d}s\, U\left(1+\frac{b}{a},1,a\right)\right\} + \sigma a^{1+\frac{b}{a}} \int_0^\infty e^{-as}(1+s)U(1+\frac{b}{a},2,a(1+s))\,dB(s).$$

(c) The mean and variance of C are given by

$$\mathbb{E}[C] = a^{\frac{b}{a}} \left\{ a\psi(0)U\left(1+\frac{b}{a},2,a\right) + b\int_{-1}^{0} \psi(s)\,\mathrm{d}s\, U\left(1+\frac{b}{a},1,a\right)\right\}. \tag{6.4.5}$$

and

$$Var[C] = \sigma^2 a^{2+2\frac{b}{a}} \int_0^\infty \mathrm{e}^{-2as}(1+s)^2 U(1+\frac{b}{a},2,a(1+s))^2\,ds > 0.$$

(d) The mean and variance of X obey

$$\lim_{t\to\infty}\frac{\mathbb{E}[X(t)]}{e^{at}t^{b/a}} = \mathbb{E}[C], \quad \lim_{t\to\infty}\frac{Var[X(t)]}{e^{2at}t^{2b/a}} = Var[C].$$

It can be seen from part (b) of Theorem 6.4.2 that the limiting random variable in (6.4.4) is a linear functional of (the increments of) the Brownian motion B. The formula for $\mathbb{E}[C]$, given in part (c) of Theroem 6.4.2, is discussed in Theorem 5.2.1 of Chapter 5 where it is shown that in certain regions of the parameter space $\mathbb{E}[C]$ is non–zero and hence the continuous random variable C is non-zero almost surely . While part (a) is also dealt with in Theorem 5.2.1 of Chapter 5 we present an alternative method of proof in this chapter, with the chief difference being that a simpler formula for C is attained in this chapter from the variation of parameters representation (rather using an admissibility approach as in Chapter 5).

In the ab-parameter space the line $a = 0$ and $b > 0$ is bordered by a region wherein X undergoes polynomial growth (covered by Theorem 6.4.1) and a region of exponential growth (which is described by Theorem 6.4.2). As neither the representation (6.2.17) nor (6.2.42) of the resolvent r are valid on this line, it therefore seems somewhat apt that X

should have a rate of faster then polynomial yet slower than exponential growth on this line. A precise asymptotic result is recorded in the next theorem.

Theorem 6.4.3. *Suppose that* $a = 0$ *and* $b > 0$. *Suppose also that* $\psi \in C([-1,0];\mathbb{R})$. *Let* X *be the unique continuous adapted process which obeys* (6.1.1). *Then*

(a) There exists an $\mathcal{F}^B(\infty)$*-measurable normal random variable* C *such that*

$$\lim_{t\to\infty} \frac{X(t)}{t^{-1/4}e^{2\sqrt{bt}}} = C, \quad a.s.$$

(b) C *is given by*

$$C = \frac{1}{b^{1/4}\sqrt{\pi}}\left(\psi(0)\sqrt{b}K_1(2\sqrt{b}) + b\int_{-1}^{0}\psi(s)dsK_0(2\sqrt{b})\right) + \frac{\sigma b^{1/4}}{\sqrt{\pi}}\int_0^\infty \sqrt{s+1}K_1(2\sqrt{b(s+1)})\,dB(s),$$

where K_0 *and* K_1 *are modified Bessel functions of the second kind.*

(c) The mean and variance of C *are given by*

$$\mathbb{E}[C] = \frac{1}{\sqrt{\pi}}\left(\psi(0)b^{1/4}K_1(2\sqrt{b}) + b^{3/4}\int_{-1}^{0}\psi(s)dsK_0(2\sqrt{b})\right),$$

$$Var[C] = \frac{\sigma^2 b^{1/2}}{\pi}\int_0^\infty (s+1)K_1^2(2\sqrt{b(s+1)})\,ds > 0.$$

(d) The mean and variance of X *obey*

$$\lim_{t\to\infty}\frac{\mathbb{E}[X(t)]}{t^{-1/4}e^{2\sqrt{bt}}} = \mathbb{E}[C], \quad \lim_{t\to\infty}\frac{Var[X(t)]}{t^{-1/2}e^{4\sqrt{bt}}} = Var[C].$$

We see from part (c) that C has positive variance, so we have that $C \neq 0$ a.s. Therefore the limit in part (a) is nontrivial a.s.

Remark 6.4.1. If one scales (6.1.4) by r_2 then we have

$$X(t)/r_2(t) = x(t)/r_2(t) + \sigma\int_0^t H(t,s)dB(s)$$

where $H(t,s) = r(t,s)/r_2(t)$. Under the hypothesis of Theorem 6.4.1, it is immediate from Theorem 3.2.2 that as the stochastic integral $\int_0^t H(t,s)dB(s)$ converges to C almost surely then the convergence must take place in mean square also. Similarly each of the results of Theorems 6.4.2 and 6.4.3 for almost sure convergence hold true for mean square convergence also.

6.5 Proofs from Section 6.1.1 and 6.3.2

6.5.1 Proof of Lemma 6.1.1

Existence and uniqueness of the solution of (6.1.1) is known from general theory of SFDEs, c.f. e.g. [27, 84]. Thus we need only demonstrate that the representation (6.1.4) satisfies the SFDE (6.1.1).

Firstly observe that the resolvent equation, (6.1.3), may be re–expressed as

$$r(t,s) = 1 + a\int_s^t r(u,s)\,du + \int_s^t \frac{b}{1+u}\int_s^u r(w,s)\,dw\,du, \quad t \geq s.$$

Defining $Z = X - x$, we have that Z obeys

$$Z(t) = a\int_0^t Z(s)ds + \int_0^t \frac{b}{1+s}\int_0^s Z(u)\,du\,ds + \sigma B(t), \quad t \geq 0, \tag{6.5.1a}$$

$$Z(t) = 0, \quad t \in [-1,0]. \tag{6.5.1b}$$

From the definition of Z it is apparent that demonstrating the validity of (6.1.4) is equivalent to showing that Z obeys

$$Z(t) = \sigma\int_0^t r(t,s)dB(s), \quad t \geq 0. \tag{6.5.2}$$

Let $Z^*(t) = \sigma\int_0^t r(t,s)dB(s)$, $t \geq 0$ and so $Z^*(0) = 0$ as required. Now using the stochastic Fubini theorem

$$\begin{aligned}
&a\int_0^t Z^*(s)ds + \int_0^t \frac{b}{1+s}\int_0^s Z^*(u)\,du\,ds + \sigma B(t)\\
&= a\sigma\int_0^t\int_0^s r(s,w)\,dB(w)\,ds\\
&\qquad + \int_0^t \frac{b}{1+s}\int_0^s \sigma\int_{w=0}^u r(u,w)dB(w)\,du\,ds + \sigma B(t)\\
&= \sigma\int_0^t \left(a\int_w^t r(s,w)\,ds + \int_w^t \frac{b}{1+s}\int_w^s r(u,w)\,du\,ds\right) dB(w) + \sigma B(t)\\
&= \sigma\int_0^t (r(t,w)-1)\,dB(w) + \sigma B(t) = \sigma\int_0^t r(t,w)\,dB(w) = Z^*(t).
\end{aligned}$$

As Z is the unique solution of (6.5.1) we have $Z = Z^*$ and hence X has the representation (6.1.4).

6.5.2 Proof of Proposition 6.1.1

Let $t \geq 0$ and $\Delta \geq 0$. Differentiating (6.1.7) with respect to Δ, using (6.1.3a), and by exchanging the order of integration and decomposing the integral, we get

$$\begin{aligned}\gamma_t'(\Delta) &= \sigma^2 \int_0^t r(t,s) \frac{\partial}{\partial \Delta} r(t+\Delta, s)\, ds \\ &= \sigma^2 a \int_0^t r(t,s) r(t+\Delta, s)\, ds + \sigma^2 \frac{b}{1+t+\Delta} \int_0^t \int_s^{t+\Delta} r(t,s) r(u,s)\, du\, ds \\ &= a\gamma_t(\Delta) + \frac{b\sigma^2}{1+t+\Delta} \int_0^t \int_0^u r(t,s) r(u,s) ds\, du \\ &\quad + \frac{b\sigma^2}{1+t+\Delta} \int_t^{t+\Delta} \int_0^t r(t,s) r(u,s) ds\, du.\end{aligned}$$

Now, because $r(w,s) = 0$ for $0 \leq w < s$, we see that $\int_0^u r(t,s) r(u,s)\, ds = \int_0^t r(t,s) r(u,s)\, ds$ for $u \in [0,t]$. Hence the two integrals on the right hand side can be combined. By making the substitution $w = u - t$, and then splitting the integral, we get

$$\begin{aligned}\gamma_t'(\Delta) &= a\gamma_t(\Delta) + \frac{b\sigma^2}{1+t+\Delta} \int_0^{t+\Delta} \int_0^u r(t,s) r(u,s) ds\, du \\ &= a\gamma_t(\Delta) + \frac{b\sigma^2}{1+t+\Delta} \int_{-t}^{\Delta} \int_0^t r(t,s) r(t+w,s) ds\, dw \\ &\quad + \frac{b\sigma^2}{1+t+\Delta} \int_{-t}^{\Delta} \int_t^{w+t} r(t,s) r(t+w,s) ds\, dw \\ &= a\gamma_t(\Delta) + \frac{b\sigma^2}{1+t+\Delta} \int_{-t}^{\Delta} \gamma_t(w)\, dw \\ &\quad + \frac{b\sigma^2}{1+t+\Delta} \int_{-t}^{\Delta} \int_t^{w+t} r(t,s) r(t+w,s) ds\, dw,\end{aligned}$$

where we have used the definition of $\gamma_t(w)$ at the last step. It now suffices to show that the last integral is zero. We first decompose it according to

$$\begin{aligned}&\int_{-t}^{\Delta} \int_t^{w+t} r(t,s) r(t+w,s) ds\, dw \\ &\quad = \int_{-t}^{0} \int_t^{w+t} r(t,s) r(t+w,s) ds\, dw + \int_0^{\Delta} \int_t^{w+t} r(t,s) r(t+w,s) ds\, dw \\ &\quad = \int_{-t}^{0} \int_t^{w+t} r(t,s) r(t+w,s) ds\, dw,\end{aligned}$$

where the last integral is zero as when $w > 0$, $r(t,s) = 0$ for $s \in (t, t+w]$. Since $t \geq 0$ and $w \in [-t, 0]$, we have that $s \in (t+w, t]$ in the remaining integral and therefore $r(t+w, s) = 0$. Thus,

$$\int_{-t}^{\Delta} \int_t^{w+t} r(t,s) r(t+w,s) ds\, dw = 0,$$

which proves (6.1.8).

For $t \geq 0$ and $-t \leq \Delta \leq 0$, we prove (6.1.9) in a similar manner to (6.1.8). However, since $\Delta \in [-t, 0]$, we can show that γ_t can be written in the form

$$\gamma_t(\Delta) = \sigma^2 \int_0^{t+\Delta} r(t,s) r(t+\Delta, s)\, ds, \quad \Delta \in [-t, 0].$$

The function on the righthand side is differentiable with respect to Δ on $(-t, 0)$, because $\Delta \mapsto r(t+\Delta, s)$ is differentiable on $(-t, 0)$. Now, differentiating with respect to Δ, we get

$$\gamma_t'(\Delta) = \sigma^2 \int_0^{t+\Delta} r(t,s) \frac{\partial}{\partial \Delta} r(t+\Delta, s)\, ds + \sigma^2 r(t, t+\Delta) r(t+\Delta, t+\Delta), \quad \Delta \in (-t, 0),$$

and proceeding in a manner similar to the proof of (6.1.8) above, we establish (6.1.9).

6.5.3 Proof of Theorem 6.3.3

In the case when $b/a \notin \{1, 2, \ldots\}$, from (6.2.25), we have

$$\frac{\text{Cov}(X(t), X(t+\Delta))}{\Delta^{-(1+b/a)}} = c_{1,t} \frac{r_1(t+\Delta)}{\Delta^{-(1+b/a)}} + c_{2,t} \frac{r_2(t+\Delta)}{(t+\Delta)^{-(1+b/a)}} \cdot \left(\frac{t+\Delta}{\Delta}\right)^{-(1+b/a)}.$$

By (6.2.4) and (6.2.5) we have that

$$\lim_{\Delta \to \infty} \frac{\text{Cov}(X(t), X(t+\Delta))}{\Delta^{-(1+b/a)}} = c_{2,t} \frac{1}{\Gamma(-b/a)} e^{-a} |a|^{-1-b/a}.$$

Since $c_{2,t}$ is given by (6.2.26) and d_2 by (6.2.20), we obtain

$$\lim_{\Delta \to \infty} \frac{\text{Cov}(X(t), X(t+\Delta))}{\Delta^{-(1+\frac{b}{a})}} = c_t(a, b)$$

where c_t is given by (6.3.9). The proof in the case when $b/a \in \{1, 2, \ldots\}$ proceeds in the same manner, making use of (6.2.4) and (6.2.13) to obtain

$$\lim_{\Delta \to \infty} \frac{\text{Cov}(X(t), X(t+\Delta))}{\Delta^{-(1+b/a)}} = \tilde{c}_{2,t} \mathcal{W}(a, b, 0) |a|^{-1-b/a}.$$

From this and the formula for $\tilde{c}_{2,t}$ in (6.2.27) we obtain the desired representation.

6.5.4 Proof of Proposition 6.3.1

Since $\text{Cov}(X(t), X(t+\Delta)$ obeys (6.1.6) for $t \geq 0$ and $\Delta \geq 0$, we see that it suffices to show that $r(t, s) > 0$ for all $t \geq s > 0$.

To this end, fix $s > 0$ and write $r_s(t) = r(t, s)$ for $t \geq s$. Then (6.1.3a) and (6.1.3b) are equivalent to

$$r_s'(t) = a r_s(t) + b \frac{1}{1+t} \int_s^t r_s(u)\, \mathrm{d}u, \quad t \geq s; \quad r_s(s) = 1.$$

Note that $r_s \in \mathrm{C}^1(s, \infty)$. Hence there exists some $\epsilon > 0$ such that $r(t,s) > 0$ for $t \in (s, s+\epsilon)$. Suppose there exists a minimal $t_0 > s$ such that $r_s(t_0) = 0$, but $r_s(t) > 0$ for $s \le t \le t_0$. Then $r_s'(t) \le 0$ and

$$0 \ge r_s'(t_0) = a r_s(t_0) + b\frac{1}{1+t}\int_s^{t_0} r_s(u)\,\mathrm{d}u = b\frac{1}{1+t}\int_s^{t_0} r_s(u)\,\mathrm{d}u > 0,$$

a contradiction. Hence $r(t,s) = r_s(t) > 0$ for all $t \ge s$, and so $\mathrm{Cov}(X(t), X(t+\Delta)) > 0$ for all $t > 0$ and $\Delta \ge 0$.

6.5.5 Proof of Proposition 6.3.2

Since $a < 0$, by Theorem 6.3.3 we have that $c_t(a,b)$ obeys (6.3.9). In the proof of Proposition 6.3.1 we showed that $r(t,s) > 0$ for all $t \ge s > 0$. Therefore, to show that $c_t(a,b) > 0$ for all $t > 0$, by examining the integral in (6.3.9), it suffices to show that $U(1 - b/a, 2, |a|(1+t)) > 0$ for $t \ge 0$. Since $a < 0$ and $b > 0$, we have $1 - b/a > 0$, so by the integral representation (6.2.28), we have

$$U(1-\frac{b}{a}, 2, -a(1+t)) = \frac{1}{\Gamma(1-\frac{b}{a})}\int_0^\infty \mathrm{e}^{a(1+t)s} s^{-\frac{b}{a}}(1+s)^{\frac{b}{a}}\,\mathrm{d}s, \qquad \text{for } t \ge 0.$$

Thus $U(1 - b/a, 2, -a(1+t)) > 0$ for all $t \ge 0$ and $a < 0 < b$, and the claim is proven.

6.5.6 Proof of Proposition 6.3.3

Suppose that $b/a \notin \{1, 2, \ldots\}$. We estimate the asymptotic behaviour of c_t in (6.3.9) by substituting $r(t,s) = r_1(t)d_1(s) + r_2(t)d_2(s)$ and estimating the asymptotic behaviour of each resulting integral in

$$\begin{aligned} c_t/(\sigma^2 b|a|^{-1-b/a}) = r_1(t)&\int_0^t d_1(s)(1+s)U(1-b/a, 2, -a(1+s))\,ds \\ &+ r_2(t)\int_0^t d_2(s)(1+s)U(1-b/a, 2, -a(1+s))\,ds. \end{aligned} \tag{6.5.3}$$

We start with the first integral in (6.5.3). By (6.2.3b) we have that

$$(1+s)U(1-b/a, 2, -a(1+s)) \sim |a|^{b/a-1} s^{b/a} \text{ as } s \to \infty. \tag{6.5.4}$$

Therefore by (6.2.21) we have that

$$d_1(s)(1+s)U(1-b/a, 2, -a(1+s)) \sim \frac{1}{|a|}e^{-as} \text{ as } s \to \infty.$$

Using the fact that $a < 0$, by (6.2.4) we get

$$r_1(t)\int_0^t d_1(s)(1+s)U(1-b/a,2,-a(1+s))\,ds \sim |a|^{b/a-2}t^{b/a}, \qquad \text{as } t\to\infty. \qquad (6.5.5)$$

In the case when $b/a \in \{1,2,\dots\}$, c_t is given by

$$c_t/(\sigma^2 b|a|^{-1-b/a}) = r_1(t)\int_0^t \tilde{d}_1(s)(1+s)U(1-b/a,2,-a(1+s))\,ds + \tilde{r}_2(t)\int_0^t \tilde{d}_2(s)(1+s)U(1-b/a,2,-a(1+s))\,ds. \qquad (6.5.6)$$

Again, we estimate the asymptotic behaviour of the first integral. By (6.5.4) and (6.2.23) we have that

$$\tilde{d}_1(s)(1+s)U(1-b/a,2,-a(1+s)) \sim |a|^{-1}\mathrm{e}^{-as}, \text{ as } s\to\infty.$$

Using the fact that $a < 0$ and that r_1 obeys (6.2.4), we get

$$r_1(t)\int_0^t \tilde{d}_1(s)(1+s)U(1-b/a,2,-a(1+s))\,ds \sim |a|^{b/a-2}t^{b/a}, \qquad \text{as } t\to\infty. \qquad (6.5.7)$$

We next prepare estimates of the integrand in the second integral in (6.5.3) and (6.5.6). When $b/a \notin \{1,2,\dots\}$, we use (6.2.22) and (6.5.4) to obtain

$$d_2(s)(1+s)U(1-b/a,2,-a(1+s)) \sim \Gamma(-\frac{b}{a})\mathrm{e}^a b|a|^{2b/a-2}\, s^{2b/a} \text{ as } s\to\infty. \qquad (6.5.8)$$

When $b/a \in \{1,2,\dots\}$, we use (6.2.24) and (6.5.4) to obtain

$$\tilde{d}_2(s)(1+s)U(1-b/a,2,-a(1+s)) \sim \frac{1}{\mathcal{W}(a,b,0)} b|a|^{2b/a-2}s^{2b/a} \text{ as } s\to\infty. \qquad (6.5.9)$$

We now prove part (a). If $b < 0$ and $b/a \notin \{1,2,\dots\}$, we have that $2b/a > 0$, so using (6.5.8)

$$\int_0^t d_2(s)(1+s)U(1-b/a,2,-a(1+s))\,ds \sim \Gamma(-\frac{b}{a})\mathrm{e}^a b|a|^{2b/a-2}\, t^{2b/a+1}\frac{1}{2b/a+1},$$

as $t\to\infty$. Therefore by (6.2.5), as $t\to\infty$, we have that

$$r_2(t)\int_0^t d_2(s)(1+s)U(1-b/a,2,-a(1+s))\,ds \sim b|a|^{b/a-3}\,\frac{1}{2b/a+1}t^{b/a}. \qquad (6.5.10)$$

In the case that $b < 0$ and $b/a \in \{1,2,\dots\}$ using (6.5.9) gives

$$\int_0^t \tilde{d}_2(s)(1+s)U(1-b/a,2,-a(1+s))\,ds \sim \frac{1}{\mathcal{W}(a,b,0)} b|a|^{2b/a-2}t^{2b/a+1}\frac{1}{2b/a+1},$$

as $t \to \infty$. Therefore by (6.2.13) we have that

$$\tilde{r}_2(t)\int_0^t \tilde{d}_2(s)(1+s)U(1-b/a,2,-a(1+s))\,ds \sim b|a|^{b/a-3}t^{b/a}\frac{1}{2b/a+1}, \qquad \text{as } t\to\infty. \tag{6.5.11}$$

Examining (6.5.10) and (6.5.11), we see that the second integrals on the righthand sides of (6.5.3) and (6.5.6) have the same asymptotic behaviour. Similarly, by (6.5.5) and (6.5.7), we see that the first integrals on the righthand sides of (6.5.3) and (6.5.6) have the same asymptotic behaviour. Hence, if $b<0$, we have that

$$\frac{c_t}{\sigma^2 b|a|^{-1-b/a}} \sim |a|^{b/a-2}\left(b|a|^{-1}\frac{1}{2b/a+1}+1\right)t^{b/a}, \qquad \text{as } t\to\infty,$$

which implies (6.3.11).

We now prove part (b). In this case $b>0$. Therefore, $b/a \notin \{1,2\ldots\}$, so we estimate the asymptotic behaviour of each integral on the right hand side of (6.5.3). In particular, the estimate (6.5.5) holds for the first integral. To analyse the asymptotic behaviour of the second term, we must consider three subcases: $2b/a<-1$, $2b/a=-1$ and $2b/a>-1$.

Case 1: $2b/a<-1$**.** If $2b/a<-1$, by (6.5.8) we have

$$\lim_{t\to\infty}\int_0^t d_2(s)(1+s)U(1-b/a,2,-a(1+s))\,ds \\ = \Gamma(-\frac{b}{a})e^a b\int_0^\infty (1+s)^2U(1-b/a,2,-a(1+s))^2\,ds,$$

where we have used (6.2.20) to obtain the formula for the limit. Hence by (6.2.5) we have

$$r_2(t)\int_0^t d_2(s)(1+s)U(1-b/a,2,-a(1+s))\,ds \\ \sim b|a|^{-b/a-1}\int_0^\infty (1+s)^2U(1-b/a,2,-a(1+s))^2\,ds\cdot t^{-b/a-1} \text{ as } t\to\infty. \tag{6.5.12}$$

Since $2b/a<-1$, we have that $b/a<-1-b/a<0$, so using the last estimate, (6.5.12) and (6.5.5) we have (6.3.12). Notice also that $c_t\to 0$ as $t\to\infty$.

Case 2: $2b/a=-1$**.** If $2b/a=-1$, by (6.5.8) and (6.2.5) we have

$$r_2(t)\int_0^t d_2(s)(1+s)U(1-b/a,2,-a(1+s))\,ds \sim b|a|^{b/a-3}t^{-1/2}\log t \\ = \frac{1}{2}|a|^{-5/2}t^{-1/2}\log t, \quad \text{as } t\to\infty.$$

Using this estimate, (6.5.3) and (6.5.5), together with the fact that $b/a=-1/2$, we have (6.3.13). Notice also that $c_t\to 0$ as $t\to\infty$.

Case 3: $2b/a > -1$**.** If $2b/a > -1$, then by (6.5.8) and (6.2.5) we have

$$r_2(t)\int_0^t d_2(s)(1+s)U(1-b/a,2,-a(1+s))\,ds \sim b|a|^{b/a-3}\,t^{b/a}\frac{1}{2b/a+1} \quad \text{as } t\to\infty.$$

Using this estimate, (6.5.3) and (6.5.5), we have (6.3.11). Since $b>0$ and $a<0$, we have $c_t \to 0$ as $t\to\infty$.

Finally we prove part (c), or (6.3.14), in the case that $a+b=0$. We consider the asymptotic behaviour of the first term on the right hand side of (6.5.3). We can still apply (6.5.5) so that

$$r_1(t)\int_0^t d_1(s)(1+s)U(1-b/a,2,|a|(1+s))\,ds \sim |a|^{b/a-2}t^{b/a} = |a|^{b/a-2}t^{-1}, \qquad \text{as } t\to\infty.$$

Therefore

$$\lim_{t\to\infty} r_1(t)\int_0^t d_1(s)(1+s)U(1-b/a,2,|a|(1+s))\,ds = 0. \tag{6.5.13}$$

Since $a+b=0$ and r_2 obeys (6.2.5), we have $r_2(t)\to \frac{1}{\Gamma(-b/a)}e^{-a}|a|^{-1-b/a}$ as $t\to\infty$. Since d_2 is given by (6.2.20), we have that

$$\begin{aligned}&\int_0^t d_2(s)(1+s)U(1-b/a,2,|a|(1+s))\,ds\\ &\qquad = \mathrm{e}^a b\Gamma(-\frac{b}{a})\int_0^t (1+s)^2U^2(1-b/a,2,|a|(1+s))\,ds.\end{aligned}$$

By (6.2.3b), we have that $(1+s)^2U^2(1-b/a,2,|a|(1+s)) \sim (|a|s)^{2b/a} = (|a|s)^{-2}$ as $s\to\infty$. Therefore it follows that the integral tends to a finite limit and therefore

$$\begin{aligned}&\lim_{t\to\infty} r_2(t)\int_0^t d_2(s)(1+s)U(1-b/a,2,-a(1+s))\,ds\\ &\qquad = |a|^{-1-b/a}b\int_0^\infty (1+s)^2U^2(1-b/a,2,|a|(1+s))\,ds.\end{aligned}$$

Combining this limit with (6.5.13) and taking the limit as $t\to\infty$ in (6.5.3), we obtain (6.3.14).

6.5.7 Proof of Theorem 6.3.4

Let $t\geq 0$ and $\Delta \geq 0$. Suppose first that $b/a \notin \{1,2,\ldots\}$. Using (6.1.6) and (6.2.17) one obtains

$$\begin{aligned}&\mathrm{Cov}(X(t),X(t+\Delta))\\ &\quad = \sigma^2 r_1(t)r_1(t+\Delta)\int_0^t d_1^2(s)\,\mathrm{d}s + \sigma^2 r_1(t)r_2(t+\Delta)\int_0^t d_1(s)d_2(s)\,\mathrm{d}s\\ &\quad + \sigma^2 r_2(t)r_2(t+\Delta)\int_0^t d_2^2(s)\,\mathrm{d}s + \sigma^2 r_1(t+\Delta)r_2(t)\int_0^t d_1(s)d_2(s)\,\mathrm{d}s.\end{aligned} \tag{6.5.14}$$

Our plan is now to determine the exact asymptotic behaviour of each of the four terms in (6.5.14) as $t \to \infty$ (for fixed $\Delta \geq 0$). Since $a < 0$ from (6.2.21) we have

$$d_1^2(t) \sim |a|^{-2b/a} e^{-2at} t^{-2b/a}, \qquad \text{as } t \to \infty.$$

Therefore, one can use the last limit and l'Hôpital's rule to show that

$$\int_0^t d_1^2(s)\, ds \sim \frac{1}{2|a|} \cdot |a|^{-2b/a} e^{-2at} t^{-2b/a}, \qquad \text{as } t \to \infty.$$

By (6.2.4), and the above limit, we have

$$\begin{aligned}
&\lim_{t\to\infty} r_1(t) r_1(t+\Delta) \int_0^t d_1(s)^2\, ds \\
&= e^{a\Delta} \lim_{t\to\infty} \Big\{ \frac{r_1(t)}{e^{at}|a|^{b/a} t^{b/a}} \frac{r_1(t+\Delta)}{e^{a(t+\Delta)}|a|^{b/a}(t+\Delta)^{b/a}} e^{2at}|a|^{2b/a} t^{b/a}(t+\Delta)^{b/a} \\
&\quad \times \frac{1}{2|a|} \cdot |a|^{-2b/a} e^{-2at} t^{-2b/a} \frac{\int_0^t d_1(s)^2\, ds}{\frac{1}{2|a|} \cdot |a|^{-2b/a} e^{-2at} t^{-2b/a}} \Big\} \\
&= \frac{1}{2|a|} e^{a\Delta} \lim_{t\to\infty} \Big\{ (t+\Delta)^{b/a} \cdot t^{-b/a} \Big\} = \frac{1}{2|a|} e^{a\Delta}.
\end{aligned} \tag{6.5.15}$$

For the second and fourth terms in (6.5.14), we use (6.2.21) and (6.2.22) to get

$$\int_0^t d_1(s) d_2(s)\, \mathrm{d}s \sim |a|^{-2} \mathrm{e}^a b\, \Gamma(-\frac{b}{a}) \mathrm{e}^{-at}, \quad \text{as } t \to \infty.$$

Thus, using (6.2.4) and (6.2.5), we get

$$\begin{aligned}
&\lim_{t\to\infty} r_1(t) r_2(t+\Delta) \int_0^t d_1(s) d_2(s)\, \mathrm{d}s \\
&= \lim_{t\to\infty} \Big\{ \frac{r_1(t)}{e^{at}|a|^{b/a} t^{b/a}} \frac{r_2(t+\Delta)}{\frac{1}{\Gamma(-b/a)} e^{-a}|a|^{-b/a-1}(t+\Delta)^{-b/a-1}} \\
&\quad \times e^{at}|a|^{b/a} t^{b/a} \frac{1}{\Gamma(-b/a)} e^{-a}|a|^{-b/a-1}(t+\Delta)^{-b/a-1} \\
&\quad \times |a|^{-2} \mathrm{e}^a b\, \Gamma(-\frac{b}{a}) \mathrm{e}^{-at} \frac{\int_0^t d_1(s) d_2(s)\, \mathrm{d}s}{|a|^{-2} \mathrm{e}^a b\, \Gamma(-\frac{b}{a}) \mathrm{e}^{-at}} \Big\} \\
&= b|a|^{-3} \lim_{t\to\infty} t^{b/a}(t+\Delta)^{-b/a-1} = 0.
\end{aligned} \tag{6.5.16}$$

Similarly, we can show that the fourth term on the righthand side of (6.5.14) obeys

$$\lim_{t\to\infty} r_1(t+\Delta) r_2(t) \int_0^t d_1(s) d_2(s)\, \mathrm{d}s = 0. \tag{6.5.17}$$

Finally, we consider the third term on the righthand side of (6.5.14). Using (6.2.22) we have

$$d_2^2(s) \sim \Gamma(-\frac{b}{a}) \mathrm{e}^{2a} b^2 |a|^{2\frac{b}{a}-2} s^{2b/a}, \text{ as } s \to \infty.$$

If $2b/a < -1$, we have that $d_2^2 \in L^1(0,\infty)$. In the case that $a+b<0$, we have that $r_2(t) \to 0$ as $t \to \infty$, so

$$\lim_{t\to\infty} r_2(t)r_2(t+\Delta)\int_0^t d_2^2(s)\,\mathrm{d}s = 0. \tag{6.5.18}$$

In the case $2b/a < -1$ and $a+b=0$, we have from (6.2.5) that $r_2(t) \to \frac{1}{\Gamma(-b/a)}|a|^{-b/a-1}e^{-a}$ as $t \to \infty$. Then from (6.2.20) we have

$$\begin{aligned}\lim_{t\to\infty} r_2(t)r_2(t+\Delta)\int_0^t d_2^2(s)\,\mathrm{d}s &= \frac{1}{\Gamma(-b/a)^2}|a|^{-2b/a-2}e^{-2a}\int_0^\infty d_2^2(s)\,\mathrm{d}s \\ &= b^2|a|^{-2b/a-2}\int_0^\infty (1+s)^2U^2(1-\frac{b}{a},2,|a|(1+s))\,\mathrm{d}s. \end{aligned}\tag{6.5.19}$$

If $2b/a=-1$, we have that

$$\int_0^t d_2^2(s)\,ds \sim \Gamma(-\frac{b}{a})\mathrm{e}^{2a}b^2|a|^{2\frac{b}{a}-2}\log t, \quad \text{as } t\to\infty.$$

Since $b/a=-1/2$, we have that $r_2(t) \sim kt^{-3/2}$ as $t\to\infty$ for some $k \neq 0$, and therefore (6.5.18) holds. If $2b/a > -1$, then

$$\int_0^t d_2^2(s)\,ds \sim \Gamma(-\frac{b}{a})\mathrm{e}^{2a}b^2|a|^{2\frac{b}{a}-2}\,t^{2b/a+1}\frac{1}{2b+a}, \qquad \text{as } t\to\infty.$$

Using (6.2.5) we have

$$r_2(t)r_2(t+\Delta)\int_0^t d_2^2(s)\,ds \sim \frac{1}{\Gamma(-b/a)}|a|^{-4}b^2\frac{1}{2b+a}t^{-1},$$

as $t\to\infty$. Hence (6.5.18) holds.

Next, in the case when $b/a \notin \{1,2,\dots\}$ and $a+b<0$, by taking the limit as $t\to\infty$ on both sides of (6.5.14), using the limits (6.5.15), (6.5.16) and (6.5.17) on the first, second and fourth terms, and (6.5.18) on the third term on the righthand side of (6.5.14), we obtain (6.3.15).

On the other hand, when $a+b=0$, by taking the limit as $t\to\infty$ on both sides of (6.5.14), using the limits (6.5.15), (6.5.16) and (6.5.17) on the first, second and fourth terms, and (6.5.19) on the third term on the righthand side of (6.5.14), we obtain (6.3.16).

For the case when $b/a \in \{1,2,...\}$, then one decomposes $\mathrm{Cov}(X(t),X(t+\Delta))$ as in (6.5.14) above but where $\tilde{r}_2$, $\tilde{d}_1$ and $\tilde{d}_2$ play the role of r_2, d_1 and d_2. Moreover as can be seen from (6.2.13), (6.2.23) and (6.2.24), $\tilde{r}_2, \tilde{d}_1$ and $\tilde{d}_2$ have the same asymptotic behaviour as r_2, d_1 and d_2 (to within a multiplicative constant) and so one can deduce the limits (6.3.15) and (6.3.16) as before.

6.5.8 Proof of Remark 6.3.2

Since $a > \int_0^\infty k(s)\,ds$, we have that r is in $L^1(0,\infty)$, and moreover that $\int_0^\infty r(s)\,ds = 1/(a - \int_0^\infty k(s)\,ds)$. Therefore, we have that

$$\lim_{t\to\infty} \text{Cov}(X(t), X(t+\Delta)) = \sigma^2 \int_0^\infty r(s)r(s+\Delta)\,ds =: \gamma(\Delta).$$

Next, suppose that k is a subexponential function. Then

$$\lim_{t\to\infty} \frac{r(t)}{k(t)} = \frac{1}{(a - \int_0^\infty k(s)\,ds)^2}.$$

We determine the asymptotic behaviour of $\gamma(\Delta)$ as $\Delta \to \infty$ under the additional assumption that k is asymptotic to a decreasing function. We then have

$$\begin{aligned}
&\frac{\gamma(\Delta)}{k(\Delta)} - \sigma^2 \int_0^\infty r(s)\,ds \cdot \frac{1}{(a - \int_0^\infty k(s)\,ds)^2} \\
&\quad = \sigma^2 \int_0^\infty r(s) \left(\frac{r(s+\Delta)}{k(s+\Delta)} - \frac{1}{(a - \int_0^\infty k(s)\,ds)^2} \right) \cdot \frac{k(s+\Delta)}{k(\Delta)}\,ds \\
&\qquad + \sigma^2 \int_0^\infty r(s) \left(\frac{k(s+\Delta)}{k(\Delta)} - 1 \right) ds \cdot \frac{1}{(a - \int_0^\infty k(s)\,ds)^2}.
\end{aligned}$$

The first term has zero limit as $\Delta \to \infty$. The second term can be shown to have a zero limit as $\Delta \to \infty$ by splitting the integral over the intervals $[0, T)$ and $[T, \infty)$ for $T > 0$ so large that $\int_T^\infty |r(s)|\,ds < \epsilon(a - \int_0^\infty k(s)\,ds)^2$, where $\epsilon > 0$ is taken arbitrarily small. Then, letting $\Delta \to \infty$, we see that the first of these two integrals tends to zero, while for the second using the monotonicity of k, the limit superior of the absolute value is less than $2\sigma^2\epsilon$. Letting $\epsilon \to 0$ confirms that

$$\lim_{\Delta\to\infty} \frac{\gamma(\Delta)}{k(\Delta)} = \sigma^2 \frac{1}{(a - \int_0^\infty k(s)\,ds)^3}.$$

Now we fix t and compute the autocovariance function. We have

$$\begin{aligned}
\lim_{\Delta\to\infty} \frac{\text{Cov}(X(t), X(t+\Delta))}{\gamma(\Delta)} &= \lim_{\Delta\to\infty} \frac{k(\Delta)}{\gamma(\Delta)} \cdot \sigma^2 \int_0^t r(s) \frac{r(s+\Delta)}{k(s+\Delta)} \cdot \frac{k(s+\Delta)}{k(\Delta)}\,ds \\
&= \frac{(a - \int_0^\infty k(s)\,ds)^3}{\sigma^2} \sigma^2 \frac{1}{(a - \int_0^\infty k(s)\,ds)^2} \int_0^t r(s)\,ds.
\end{aligned}$$

Therefore, we have

$$\lim_{\Delta\to\infty} \frac{\text{Cov}(X(t), X(t+\Delta))}{\gamma(\Delta)} = \frac{\int_0^t r(s)\,ds}{\int_0^\infty r(s)\,ds} =: c_t,$$

so clearly $c_t \to 1$ as $t \to \infty$. Therefore the autocovariance function obeys (6.3.18).

6.5.9 Proof of Remark 6.3.1

If $r(t) \to 0$ as $t \to \infty$, it is known that $r \in L^1(0,\infty)$ and that r decays to zero exponentially. As a consequence

$$\lim_{t\to\infty} \mathrm{Cov}(X(t), X(t+\Delta)) = \sigma^2 \int_0^\infty r(s) r(s+\Delta)\, ds =: \gamma(\Delta).$$

Let us further suppose, for example, that r is asymptotic to a function of one sign. Then there exists $n \in \mathbb{Z}^+$ and $\alpha > 0$ such that $r(t)/(t^{n-1}e^{-\alpha t}) \to C \neq 0$ as $t \to \infty$. We now determine the asymptotic behaviour of $\gamma(\Delta)$ as $\Delta \to \infty$. We start by writing

$$\begin{aligned}
&\frac{\gamma(\Delta)}{\Delta^{n-1}e^{-\alpha\Delta}} - \sigma^2 C \int_0^\infty e^{-\alpha s} r(s)\, ds \\
&\quad = \sigma^2 \int_0^\infty e^{-\alpha s} r(s) \left(\frac{r(s+\Delta)}{(s+\Delta)^{n-1} e^{-\alpha(\Delta+s)}} - C \right) \cdot \frac{(s+\Delta)^{n-1}}{\Delta^{n-1}}\, ds \\
&\qquad + \left\{ \sigma^2 C \int_0^\infty e^{-\alpha s} r(s) \cdot \frac{(s+\Delta)^{n-1}}{\Delta^{n-1}}\, ds - \sigma^2 C \int_0^\infty e^{-\alpha s} r(s)\, ds \right\}.
\end{aligned}$$

It can then be shown that the limits as $\Delta \to \infty$ of the two terms on the righthand side is zero, so that

$$\lim_{\Delta\to\infty} \frac{\gamma(\Delta)}{\Delta^{n-1}e^{-\alpha\Delta}} = \sigma^2 C \int_0^\infty e^{-\alpha s} r(s)\, ds =: c^*.$$

Considering now the limit when $\Delta \to \infty$ for t fixed, we have

$$\begin{aligned}
&\frac{\mathrm{Cov}(X(t), X(t+\Delta))}{\gamma(\Delta)} \\
&= \frac{\mathrm{Cov}(X(t), X(t+\Delta))}{\Delta^{n-1}e^{-\alpha\Delta}} \cdot \frac{\Delta^{n-1}e^{-\alpha\Delta}}{\gamma(\Delta)} \\
&= \sigma^2 \int_0^t r(s) e^{-\alpha s} \frac{r(s+\Delta)}{(s+\Delta)^{n-1} e^{-\alpha(\Delta+s)}} \cdot \frac{(s+\Delta)^{n-1}}{\Delta^{n-1}}\, ds \cdot \frac{\Delta^{n-1}e^{-\alpha\Delta}}{\gamma(\Delta)}.
\end{aligned}$$

Therefore we have

$$\lim_{\Delta\to\infty} \frac{\mathrm{Cov}(X(t), X(t+\Delta))}{\gamma(\Delta)} = \frac{1}{c^*} C\sigma^2 \int_0^t r(s) e^{-\alpha s}\, ds =: c_t.$$

We see that $c_t \to 1$ as $t \to \infty$. Therefore (6.3.18) holds.

6.6 Proof of Results in Section 6.4

In this section, we give the proofs of the growth rates of X stated in Section 6.4.

6.6.1 Proof of Theorem 6.4.2

For $b/a \notin \{-1,-2,...\}$, from (6.2.29), (6.2.42) and (6.1.4), we can write X according to

$$X(t) = r_3(t)c_3 + r_4(t)c_4 + \sigma r_3(t)\int_0^t d_3(s)\,dB(s) + \sigma r_4(t)\int_0^t d_4(s)\,dB(s). \qquad (6.6.1)$$

We have already deduced the asymptotic behaviour of r_3, r_4, d_3 and d_4 in (6.2.31), (6.2.32), (6.2.44) and (6.2.45). We recapitulate their limiting behaviour now:

$$r_3(t) \sim a^{-1-\frac{b}{a}} t^{-1-\frac{b}{a}}, \quad r_4(t) \sim \frac{1}{\Gamma(1+\frac{b}{a})} \mathrm{e}^{a(1+t)} a^{\frac{b}{a}} t^{\frac{b}{a}}, \qquad \text{as } t \to \infty,$$
$$d_3(s) \sim b\, a^{-1+\frac{b}{a}} s^{\frac{b}{a}}, \quad d_4(s) \sim \Gamma(1+\frac{b}{a}) a^{-\frac{b}{a}} \mathrm{e}^{-a(1+s)} s^{-\frac{b}{a}}, \qquad \text{as } s \to \infty.$$

Dividing across (6.6.1) by $r_4(t)$ yields

$$\frac{X(t)}{r_4(t)} = \frac{r_3(t)}{r_4(t)} c_3 + c_4 + \sigma \frac{r_3(t)}{r_4(t)} \int_0^t d_3(s)\,dB(s) + \sigma \int_0^t d_4(s)\,dB(s). \qquad (6.6.2)$$

The asymptotic behaviour of the first and last terms is readily estimated. Since $a > 0$, $r_3(t)/r_4(t) \to 0$ as $t \to \infty$. $a > 0$ also implies $d_4 \in L^2(0,\infty)$. Therefore by the Martingale Convergence Theorem for continuous martingales (cf., e.g., [104, Thm. V.1.8]) we have that

$$\lim_{t\to\infty} \sigma \int_0^t d_4(s)\,\mathrm{d}B(s) = \sigma \int_0^\infty d_4(s)\,\mathrm{d}B(s), \quad \text{a.s.}$$

If

$$\lim_{t\to\infty} \frac{r_3(t)}{r_4(t)} \int_0^t d_3(s)\,\mathrm{d}B(s) = 0, \quad \text{a.s.} \qquad (6.6.3)$$

then we obtain

$$\lim_{t\to\infty} \frac{X(t)}{r_4(t)} = c_4 + \sigma \int_0^\infty d_4(s)\,\mathrm{d}B(s) =: C_4, \quad \text{a.s.}$$

By (6.2.32) we therefore have

$$\lim_{t\to\infty} \frac{X(t)}{\mathrm{e}^{at} t^{\frac{b}{a}}} = \Gamma(1+\frac{b}{a}) e^a a^{\frac{b}{a}} c_4 + \Gamma(1+\frac{b}{a}) e^a a^{\frac{b}{a}} \sigma \int_0^\infty d_4(s)\,\mathrm{d}B(s) = C, \quad \text{a.s.}$$

which implies (6.4.4) and also part (b), due to the definitions of c_4 and d_4 in (6.2.33) and (6.2.43) and of C in part (b).

Moreover, it follows from [108, Ch. 2.13.5, p.304-305] that

$$\mathbb{E}[C] = \lim_{t\to\infty} \frac{\mathbb{E}[X(t)]}{\mathrm{e}^{at} t^{\frac{b}{a}}} = \lim_{t\to\infty} \frac{x(t)}{\mathrm{e}^{at} t^{\frac{b}{a}}} = c_4 \Gamma(1+\frac{b}{a}) e^a a^{\frac{b}{a}},$$

and that

$$\mathrm{Var}[C] = \lim_{t\to\infty} \frac{\mathrm{Var}[X(t)]}{\mathrm{e}^{2at}t^{2\frac{b}{a}}} = \sigma^2\Gamma(1+\frac{b}{a})^2 e^{2a}a^{2\frac{b}{a}} \int_0^\infty d_4^2(s)\,\mathrm{d}s > 0.$$

These results and (6.2.33) and (6.2.43) establish the validity of parts (c) and (d).

All that remains to show is that (6.6.3) is indeed true. If $\frac{2b}{a} < -1$, then $d_3 \in L^2(0,\infty)$, and the stochastic integral tends to a finite limit by the Martingale Convergence Theorem. Since $r_3(t)/r_4(t) \to 0$ as $t \to \infty$, we obtain

$$\lim_{t\to\infty} \frac{r_3(t)}{r_4(t)} \int_0^t d_3(s)\,\mathrm{d}B(s) = 0, \quad \text{a.s.}$$

If $\frac{2b}{a} > -1$, then $d_3 \notin L^2(0,\infty)$. Indeed, the quadratic variation of $\int_0^t d_3(s)dB(s)$ is given by

$$v(t) := \int_0^t d_3^2(s)\,\mathrm{d}s \sim b^2 a^{\frac{2b}{a}} (\frac{2b}{a}+1)^{-1} t^{\frac{2b}{a}+1}, \quad \text{as } t\to\infty,$$

and hence $\log\log v(t) \sim \log\log t$ as $t \to \infty$. Therefore the stochastic integral $\int_0^t d_3(s)dB(s)$ obeys the Law of the Iterated Logarithm for continuous martingales (cf., e.g., [104, Exercise V.1.15]), so

$$\limsup_{t\to\infty} \frac{\int_0^t d_3(s)dB(s)}{\sqrt{2v(t)\log\log v(t)}} = -\liminf_{t\to\infty} \frac{\int_0^t d_3(s)dB(s)}{\sqrt{2v(t)\log\log v(t)}} = 1, \quad \text{a.s.}$$

These asymptotic estimates for the stochastic integral and v, together with (6.2.31) and (6.2.32) yield

$$\lim_{t\to\infty} \frac{r_3(t)}{r_4(t)} \int_0^t d_3(s)dB(s) = 0, \quad \text{a.s.}$$

as required. The above argument holds similarly for the case when $\frac{2b}{a} = -1$.

The case $b/a \in \{-1,-2,-3,...\}$ can be dealt with similarly. While we only have the crude estimate (6.2.46) for the asymptotic behaviour of $\tilde{d}_3$, it is nevertheless the case that the quadratic variation of $\int_0^t \tilde{d}_3(s)dB(s)$ can grow no faster than a power of t as $t \to \infty$ (or indeed may converge as $t \to \infty$). Thus we obtain

$$\lim_{t\to\infty} \frac{r_3(t)}{\tilde{r}_4(t)} \int_0^t \tilde{d}_3(s)\,\mathrm{d}B(s) = 0, \quad \text{a.s.}$$

as before.

6.6.2 Proof of Theorem 6.4.1

Since $a < 0$ and $a + b > 0$, we have $b/a \notin \{1, 2, \ldots\}$. Therefore, from (6.2.2), (6.2.17) and (6.1.4) one has,

$$X(t) = r_1(t)c_1 + r_2(t)c_2 + \sigma r_1(t) \int_0^t d_1(s)dB(s) + \sigma r_2(t) \int_0^t d_2(s)dB(s). \tag{6.6.4}$$

We have already deduced the asymptotic behaviour of r_1, r_2, d_1 and d_2 in (6.2.4), (6.2.5), (6.2.21) and (6.2.22). We recapitulate their limiting behaviour now:

$$r_1(t) \sim \mathrm{e}^{at}|a|^{\frac{b}{a}} t^{\frac{b}{a}}, \quad r_2(t) \sim \frac{\mathrm{e}^{-a}}{\Gamma(-\frac{b}{a})}|a|^{-1-\frac{b}{a}} t^{-1-\frac{b}{a}}, \qquad \text{as } t \to \infty,$$
$$d_1(s) \sim |a|^{-\frac{b}{a}} \mathrm{e}^{-as} s^{-\frac{b}{a}}, \quad d_2(s) \sim \Gamma(-\frac{b}{a}) b\, \mathrm{e}^{a} |a|^{-1+\frac{b}{a}} s^{\frac{b}{a}}, \qquad \text{as } s \to \infty.$$

Dividing across (6.6.4) by $r_2(t)$ yields

$$\frac{X(t)}{r_2(t)} = \frac{r_1(t)}{r_2(t)} c_1 + c_2 + \sigma \frac{r_1(t)}{r_2(t)} \int_0^t d_1(s)\, dB(s) + \sigma \int_0^t d_2(s)\, dB(s). \tag{6.6.5}$$

The asymptotic behaviour of the first and last terms is readily estimated. Since $a < 0$, we have from (6.2.4) and (6.2.5) that $r_1(t)/r_2(t) \to 0$ as $t \to \infty$. Also, since $a < 0$ and $a + b > 0$, we have $2b/a < -2$. Hence $d_2 \in L^2(0, \infty)$ and therefore by the martingale convergence theorem for continuous martingales (cf., e.g., [104, Thm. V.1.8]) we have

$$\lim_{t\to\infty} \int_0^t d_2(s) dB(s) = \int_0^\infty d_2(s) dB(s), \quad \text{a.s.} \tag{6.6.6}$$

We now examine the asymptotic behaviour of the third term on the righthand side of (6.6.5). Firstly observe that $\int_0^t d_1(s) dB(s)$ is normally distributed with mean zero and variance given by

$$v_1(t) = \int_0^t d_1^2(s) ds.$$

By l'Hôpital's rule we have

$$v_1(t) \sim \frac{1}{2}|a|^{-1-\frac{2b}{a}} \mathrm{e}^{-2at}(1+t)^{-\frac{2b}{a}}, \quad \log\log v_1(t) \sim \log t, \text{ as } t \to \infty,$$

and so we have by the Law of the Iterated Logarithm for continuous martingales (cf., e.g., [104, Exercise V.1.15]) that

$$\limsup_{t\to\infty} \frac{\int_0^t d_1(s) dB(s)}{\sqrt{2 v_1(t) \log\log v_1(t)}} = -\liminf_{t\to\infty} \frac{\int_0^t d_1(s) dB(s)}{\sqrt{2 v_1(t) \log\log v_1(t)}} = 1, \quad \text{a.s.}$$

Thus we have

$$\limsup_{t\to\infty} \sigma \frac{r_1(t) \int_0^t d_1(s)\, \mathrm{d}B(s)}{\sqrt{\log t}} = -\liminf_{t\to\infty} \sigma \frac{r_1(t) \int_0^t d_1(s)\, \mathrm{d}B(s)}{\sqrt{\log t}} = \frac{\sigma}{\sqrt{|a|}}. \tag{6.6.7}$$

Using (6.6.7), the fact that $\log t / r_2(t) \to 0$ as $t \to \infty$, together with (6.6.6), we arrive at

$$\lim_{t\to\infty} \frac{X(t)}{r_2(t)} = c_2 + \sigma \int_0^\infty d_2(s)\, \mathrm{d}B(s), \quad \text{a.s.}$$

By (6.2.5) we therefore obtain

$$\lim_{t\to\infty}\frac{X(t)}{t^{-1-\frac{b}{a}}}$$
$$=\frac{e^{-a}}{\Gamma(-\frac{b}{a})}|a|^{-1-\frac{b}{a}}c_2+\sigma\frac{e^{-a}}{\Gamma(-\frac{b}{a})}|a|^{-1-\frac{b}{a}}\int_0^\infty d_2(s)\,dB(s)=C,\quad \text{a.s.} \quad (6.6.8)$$

which implies part (a) and also part (b), due to the definitions of c_2 and d_2 in (6.2.9) and (6.2.20) and of C in part (b).

Moreover, it follows from [108, Ch. 2.13.5, p.304-305] that

$$\mathbb{E}[C]=\lim_{t\to\infty}\frac{\mathbb{E}\left[X(t)\right]}{t^{-1-\frac{b}{a}}}=\lim_{t\to\infty}\frac{x(t)}{t^{-1-\frac{b}{a}}}=c_2\frac{e^{-a}}{\Gamma(-\frac{b}{a})}|a|^{-1-\frac{b}{a}},$$

and that

$$\mathrm{Var}[C]=\lim_{t\to\infty}\frac{\mathrm{Var}[X(t)]}{t^{-2-2\frac{b}{a}}}=\sigma^2\frac{e^{-2a}}{\Gamma^2(-\frac{b}{a})}|a|^{-2-2\frac{b}{a}}\int_0^\infty d_2^2(s)\,ds>0$$

These results and (6.2.9) and (6.2.20) establish the validity of parts (c) and (d).

6.6.3 Proof of Theorem 6.4.3

From (6.2.48), (6.2.54) and (6.1.4), we can write X according to

$$X(t)=r_5(t)c_5+r_6(t)c_6+\sigma r_5(t)\int_0^t d_5(s)\,dB(s)+\sigma r_6(t)\int_0^t d_6(s)\,dB(s). \quad (6.6.9)$$

We can deduce the asymptotic behaviour of r_5, r_6, d_5 and d_6 using (6.2.53), (6.2.49) and (6.2.55). Hence

$$r_5(t)\sim\frac{1}{2b^{1/4}\sqrt{\pi}}e^{2\sqrt{bt}}t^{-1/4},\ \text{as } t\to\infty,\quad r_6(t)\sim\frac{\sqrt{\pi}}{2b^{1/4}}e^{-2\sqrt{bt}}t^{-1/4},\ \text{as } t\to\infty,$$
$$d_5(s)\sim\sqrt{\pi}b^{1/4}s^{1/4}e^{-2\sqrt{bs}},\ \text{as } s\to\infty,\quad d_6(s)\sim\frac{1}{\sqrt{\pi}}b^{1/4}s^{1/4}e^{2\sqrt{bs}},\ \text{as } s\to\infty.$$

Dividing across (6.6.9) by $r_5(t)$ yields

$$\frac{X(t)}{r_5(t)}=c_5+\frac{r_6(t)}{r_5(t)}c_6+\sigma\int_0^t d_5(s)\,dB(s)+\sigma\frac{r_6(t)}{r_5(t)}\int_0^t d_6(s)\,dB(s). \quad (6.6.10)$$

The asymptotic behaviour of the second and third terms is readily estimated. First as $b>0$, $r_6(t)/r_5(t)\to 0$ as $t\to\infty$. Also, $d_5\in L^2(0,\infty)$ and therefore by the martingale convergence theorem for continuous martingales (cf., e.g., [104, Thm. V.1.8]) we have

$$\lim_{t\to\infty}\int_0^t d_5(s)dB(s)=\int_0^\infty d_5(s)dB(s),\quad \text{a.s.} \quad (6.6.11)$$

We now examine the asymptotic behaviour of the fourth term on the righthand side of (6.6.10). Firstly observe that $\int_0^t d_6(s)dB(s)$ is normally distributed with mean zero and variance given by

$$v_3(t) := \int_0^t d_6(s)^2\,ds.$$

By using l'Hôpital's rule, the asymptotic behaviour of $v_3(t)$ as $t \to \infty$ can be found:

$$\lim_{t\to\infty} \frac{v_3(t)}{te^{4\sqrt{bt}}} = \frac{1}{2\pi}, \quad \lim_{t\to\infty} \frac{\log\log v_3(t)}{\log t} = \frac{1}{2}.$$

Thus by the Law of the Iterated Logarithm for continuous martingales (cf., e.g., [104, Exercise V.1.15]) we have that

$$\limsup_{t\to\infty} \frac{\int_0^t d_6(s)dB(s)}{\sqrt{2v_3(t)\log\log v_3(t)}} = -\liminf_{t\to\infty} \frac{\int_0^t d_6(s)dB(s)}{\sqrt{2v_3(t)\log\log v_3(t)}} = 1, \quad \text{a.s.}$$

Thus we have

$$\int_0^t d_6(s)dB(s) = O\left(t^{1/2}e^{2\sqrt{bt}}\sqrt{\log t}\right), \quad \text{as } t \to \infty.$$

Therefore

$$\frac{r_6(t)}{r_5(t)}\int_0^t d_6(s)dB(s) = O\left(t^{1/2}e^{-2\sqrt{bt}}\sqrt{\log t}\right), \quad \text{as } t \to \infty,$$

and so

$$\lim_{t\to\infty} \frac{r_6(t)}{r_5(t)}\int_0^t d_6(s)dB(s) = 0 \quad \text{a.s.} \tag{6.6.12}$$

Taking the limit as $t \to \infty$ in (6.6.10) and using (6.6.12) together with (6.6.11), we arrive at

$$\lim_{t\to\infty} \frac{X(t)}{r_5(t)} = c_5 + \sigma\int_0^\infty d_5(s)\,\mathrm{d}B(s), \quad \text{a.s.}$$

Using the asymptotic behaviour of r_5 we therefore obtain

$$\begin{aligned}\lim_{t\to\infty} \frac{X(t)}{e^{2\sqrt{bt}}t^{-1/4}} &= \lim_{t\to\infty} \frac{X(t)}{r_5(t)} \cdot \frac{r_5(t)}{e^{2\sqrt{bt}}t^{-1/4}} \\ &= \frac{1}{2b^{1/4}\sqrt{\pi}}\left(c_5 + \sigma\int_0^\infty d_5(s)\,dB(s)\right) = C, \quad \text{a.s.}\end{aligned} \tag{6.6.13}$$

which implies part (a) and also part (b), due to the definitions of c_5 and d_5 in (6.2.50) and (6.2.55) and of C in part (b).

Moreover, it follows from [108, Ch. 2.13.5, p.304-305] that

$$\mathbb{E}[C] = \lim_{t\to\infty} \frac{\mathbb{E}[X(t)]}{e^{2\sqrt{bt}}t^{-1/4}} = \lim_{t\to\infty} \frac{x(t)}{e^{2\sqrt{bt}}t^{-1/4}} = \frac{1}{2b^{1/4}\sqrt{\pi}}c_5,$$

and that

$$\mathrm{Var}[C] = \lim_{t\to\infty} \frac{\mathrm{Var}[X(t)]}{e^{4\sqrt{bt}}t^{-1/2}} = \frac{1}{4b^{1/2}\pi}\sigma^2\int_0^\infty d_5^2(s)\,ds > 0.$$

These results and (6.2.50) and (6.2.55) establish the validity of parts (c) and (d).

6.7 Proof of Theorem 6.3.1 and Theorem 6.3.2

We note that similar asymptotic analysis as that above would give us, for $a+b \le 0$,

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2\log t}} = \frac{\sigma}{\sqrt{2|a|}}.$$

We choose however to prove this result via Theorem 6.3.1, as it provides an interesting result regarding the asymptotic behaviour of the process.

6.7.1 A preliminary lemma

Lemma 6.7.1. *Let $a<0$ and $a+b=0$. Define H by*

$$H(t,u) = \int_u^t d_2(s)\frac{b}{1+s}\mathrm{e}^{-au}\int_u^s \sigma \mathrm{e}^{aw}\,dw\,ds, \quad 0 \le u \le t, \tag{6.7.1}$$

where d_2 is as given by (6.2.20)*. Define H_∞*

$$H_\infty(u) = \frac{\sigma}{|a|}\int_u^\infty \frac{d_2(s)}{1+s}ds - \frac{\sigma}{|a|}\mathrm{e}^{-au}\int_u^\infty \mathrm{e}^{as}\frac{d_2(s)}{1+s}ds, \quad u \ge 0. \tag{6.7.2}$$

Then

$$\lim_{t\to\infty}\int_0^t H(t,u)dB(u) = \int_0^\infty H_\infty(u)\,dB(u), \quad a.s.$$

Proof. The proof of this almost sure convergence result is an application of Theorem 3.2.3 H simplifies to

$$H(t,u) = \frac{\sigma}{|a|}\int_u^t \frac{d_2(s)}{1+s}ds - \frac{\sigma}{|a|}\mathrm{e}^{-au}\int_u^t \mathrm{e}^{as}\frac{d_2(s)}{1+s}ds.$$

H_∞ given by (6.7.2) is well–defined by virtue of (6.2.22). To estimate the rate of decay of H_∞ to zero, we use (6.2.22) to get

$$\int_u^\infty \frac{d_2(s)}{1+s}ds \sim |a|^{-1}\mathrm{e}^a u^{-1}, \qquad \text{as } u\to\infty, \tag{6.7.3a}$$

$$\mathrm{e}^{-au}\int_u^\infty \mathrm{e}^{as}\frac{d_2(s)}{1+s}ds \sim |a|^{-2}\mathrm{e}^a u^{-2}, \qquad \text{as } u\to\infty. \tag{6.7.3b}$$

Thus $H_\infty(u) \sim \sigma|a|^{-2}\mathrm{e}^a u^{-1}$ as $u\to\infty$ and so $H_\infty \in L^2(0,\infty)$.

We now wish to show that

$$\lim_{t\to\infty}\int_0^t (H(t,u)-H_\infty(u))^2\,du\cdot\log t = 0. \tag{6.7.4}$$

Define

$$f(t) := \frac{\sigma}{|a|}\int_t^\infty \frac{d_2(s)}{1+s}ds, \quad g(t) := \frac{\sigma}{|a|}\int_t^\infty \mathrm{e}^{as}\frac{d_2(s)}{1+s}ds.$$

Then the Cauchy-Schwarz inequality gives

$$\begin{aligned}\int_0^t (H(t,u) - H_\infty(u))^2\,du &= \int_0^t \left(-f(t) + \mathrm{e}^{-au}g(t)\right)^2 du \\ &\le \int_0^t 2f(t)^2 du + \int_0^t 2\mathrm{e}^{-2au}g(t)^2 du \\ &= 2tf(t)^2 + \frac{1}{2|a|}2g(t)^2\left(\mathrm{e}^{-2at}-1\right).\end{aligned}$$

The asymptotic relations (6.7.3) determine completely the asymptotic behaviour of f and g, and this, together with the last inequality, gives (6.7.4).

We show now that there exist $q \ge 0$ and $c_q > 0$ such that

$$\int_0^t \left[\frac{\partial}{\partial t}H(t,u)\right]^2 du \le c_q(1+t)^{2q}, \quad t \ge 0. \tag{6.7.5}$$

To do this we estimate according to

$$\int_0^t \left[\frac{\partial}{\partial t}H(t,u)\right]^2 du = \frac{\sigma^2}{|a|^2}\frac{d_2^2(t)}{(1+t)^2}\int_0^t \left(1-\mathrm{e}^{-au}\mathrm{e}^{at}\right)^2 du,$$

and using (6.2.22), we see that H obeys (6.7.5) for any $q \ge 0$ and $c_q > 0$. Also as $H(t,t) = 0$ for all $t \ge 0$ then all of the conditions of Theorem 3.2.3 are satisfied and so we conclude $\lim_{t\to\infty}\int_0^t H(t,u)dB(u) = \int_0^\infty H_\infty(u)dB(u)$ a.s. as required. □

6.7.2 Proof of Theorem 6.3.1

We start by defining a process $Y = \{Y(t) : t \ge -1\}$, which is related to U defined by (6.3.1). It will be used in proving Theorems 6.3.1 and 6.4.1. Y is defined by $Y(t) = \psi(t)$ for $t \in [-1,0]$ and it obeys

$$dY(t) = aY(t)\,dt + \sigma\,dB(t), \quad t \ge 0. \tag{6.7.6}$$

Note that (6.3.3) is an immediate consequence of (6.3.4) or (6.3.6) and the fact that

$$\limsup_{t\to\infty}\frac{U(t)}{\sqrt{2\log t}} = \frac{\sigma}{\sqrt{2|a|}}, \quad \liminf_{t\to\infty}\frac{U(t)}{\sqrt{2\log t}} = -\frac{\sigma}{\sqrt{2|a|}}, \quad \text{a.s.} \tag{6.7.7}$$

Therefore it remains to prove (6.3.4) and (6.3.6). Firstly extend U to $[-1,0)$ by $U(t) = 0$ for $t \in [-1,0)$. Then for Y defined by (6.7.6), for $t \ge 0$ we have $Y(t) - U(t) = \psi(0)e^{at}$. Therefore $U(t) - Y(t) \to 0$ as $t \to \infty$, a.s. Hence it remains to prove that $X(t) - Y(t) \to 0$ as $t \to \infty$ a.s. in order to establish (6.3.4) and (6.3.6).

Define $Z(t) = X(t) - Y(t)$ for $t \ge -1$. Then $Z(t) = 0$ for $t \in [-1,0]$ and

$$Z'(t) = aZ(t) + b\frac{1}{1+t}\int_0^t Z(s)\,ds + f(t), \quad t > 0, \tag{6.7.8}$$

where

$$f(t) := b\frac{1}{t+1}\int_{-1}^{0}\psi(s)\,ds + b\frac{1}{t+1}\int_0^t Y(s)\,ds, \quad t\geq 0. \tag{6.7.9}$$

Next we show that $f(t)\to 0$ as $t\to\infty$, a.s. This clearly follows if $\int_0^t Y(s)\,ds/t \to 0$ as $t\to\infty$ a.s. To prove this, note that

$$Y(t) = \psi(0) + a\int_0^t Y(s)\,ds + \sigma B(t), \quad t\geq 0. \tag{6.7.10}$$

Since U obeys (6.7.7), $Y(t)-U(t)\to 0$ as $t\to\infty$, Y obeys

$$\limsup_{t\to\infty}\frac{|Y(t)|}{\sqrt{2\log t}} = \frac{\sigma}{\sqrt{2|a|}} \quad \text{a.s.}$$

Therefore by this limit and the strong law of large numbers for standard Brownian motion [72, 2.9.3], we get from (6.7.10) that $\int_0^t Y(s)\,ds/t\to 0$ as $t\to\infty$ a.s., and therefore that $f(t)\to 0$ as $t\to\infty$ a.s. Indeed, by using the Law of the iterated logarithm for standard Brownian motion [72], we have

$$\limsup_{t\to\infty}\frac{f(t)}{t^{-1/2}\sqrt{2\log\log t}} = -\liminf_{t\to\infty}\frac{f(t)}{t^{-1/2}\sqrt{2\log\log t}} = \frac{|b|\sigma}{|a|}, \quad \text{a.s.} \tag{6.7.11}$$

Recalling that the resolvent r obeys (6.1.3), by applying the conventional variation of constants formula to (6.7.8), and using (6.2.17) in the case that $b/a\notin\{1,2,\ldots\}$, we get

$$Z(t) = \int_0^t r(t,s)f(s)\,\mathrm{d}s = r_1(t)\int_0^t d_1(s)f(s)ds + r_2(t)\int_0^t d_2(s)f(s)ds \tag{6.7.12}$$

and hence

$$|Z(t)| \leq |r_1(t)|\int_0^t |d_1(s)||f(s)|ds + |r_2(t)|\int_0^t |d_2(s)||f(s)|\,ds. \tag{6.7.13}$$

The first integral on the righthand side of (6.7.13) converges to zero using (6.2.4), (6.2.21) and (6.7.11), on application of l'Hôpital's rule.

It transpires that the limiting behaviour as $t\to\infty$ of the second integral on the righthand side of (6.7.13) differs according to whether $a+b<0$ or $a+b=0$. We consider first the case when $a+b<0$. Using (6.2.22) and (6.7.11) in the case that $2b+a>0$, there exists an a.s. finite positive random variable M such that

$$\limsup_{t\to\infty}\int_0^t |d_2(s)||f(s)|ds \leq \limsup_{t\to\infty} M\int_0^\infty (1+s)^{\frac{b}{a}-1/2}\sqrt{\log\log(e+s)}\,ds < \infty.$$

Hence

$$\lim_{t\to\infty}\int_0^t d_2(s)f(s)\,ds = \int_0^\infty d_2(s)f(s)ds \in (-\infty,\infty) \quad \text{a.s.} \tag{6.7.14}$$

Since r_2 obeys (6.2.5), we have

$$\lim_{t\to\infty} |r_2(t)| \int_0^t |d_2(s)||f(s)|\,ds = 0, \quad \text{a.s.} \tag{6.7.15}$$

In the case when $2b+a \le 0$, notice from (6.7.11) that for any $\epsilon < 1/2$ that $f(t)/t^{-1/2+\epsilon} \to 0$ as $t \to \infty$ on the a.s. event Ω_1, say. Therefore, by the continuity of f and this relation, there is an a.s. finite and positive random variable K_ϵ such that $|f(t,\omega)| \le K_\epsilon(\omega)(1+t)^{-1/2+\epsilon}$ for all $t \ge 0$. Therefore, by virtue of the continuity of r_2, d_2 and (6.2.5) and (6.2.22), there exists an a.s. finite and positive random variable M_ϵ such that, for all $t \ge 0$, we have

$$\begin{aligned} |r_2(t)| \int_0^t |d_2(s)||f(s,\omega)|ds &\le M_\epsilon(\omega)(1+t)^{-1-\frac{b}{a}} \int_0^t (1+s)^{\frac{b}{a}-1/2+\epsilon}\,ds \\ &\le M_\epsilon(\omega)(1+t)^{-1-\frac{b}{a}}(1+t)^{\frac{b}{a}+1/2+\epsilon}\frac{1}{b/a+1/2+\epsilon}, \end{aligned}$$

for each $\omega \in \Omega_1$, with the last inequality holding because $b/a - 1/2 + \epsilon > -1$. Since Ω_1 is an a.s. event. Thus we again have (6.7.15) and so, using this limit and (6.7.13), we see that $Z(t) \to 0$ as $t \to \infty$ a.s. in the case that $b/a \notin \{1,2,...\}$. We can demonstrate that $Z(t) \to 0$ as $t \to \infty$ a.s. in a similar manner when $b/a \in \{1,2,...\}$ by using the asymptotic behaviour of r_1, $\tilde{r}_2$, $\tilde{d}_1$ and $\tilde{d}_2$. Hence the proof of parts (i) and (ii) are complete.

For the proof of part (iii), we consider the case $a+b=0$. Recall that Y can be written in the form

$$Y(t) = \psi(0)\mathrm{e}^{at} + \sigma e^{at}\int_0^t \mathrm{e}^{-as}dB(s), \quad t \ge 0.$$

In this case, we wish to show that Z tends to a non–trivial limit. Arguing as above, we have that the first integral on the right hand side of (6.7.12) tends to zero as $t \to \infty$ a.s. As to the second term on the right hand side of (6.7.12), by using a stochastic Fubini theorem, it is seen that

$$\begin{aligned} \int_0^t d_2(s)f(s)\,ds = &\int_0^t \frac{b}{1+s}d_2(s)\,ds \int_{-1}^0 \psi(u)\,du \\ &+ \int_0^t d_2(s)\frac{b}{1+s}\int_0^s \psi(0)\mathrm{e}^{au}du\,ds + \int_0^t H(t,u)\,dB(u), \end{aligned}$$

where H is given by (6.7.1). The two Riemann integrals on the right–hand side of the above equation converge to finite limits as $t \to \infty$. Moreover as (6.7.14) holds therefore the stochastic integral on the right–hand side above converges almost surely. Recalling from (6.2.5) that $\lim_{t\to\infty} r_2(t) = e^{-a}|a|^{-b/a-1}$ in the case when $a+b=0$, and by applying Lemma 6.7.1, we have that

$$\lim_{t\to\infty} Z(t) = \frac{e^{-a}}{|a|^{b/a+1}} \int_0^\infty \frac{b}{1+s} d_2(s)\, ds \int_{-1}^0 \psi(u)\, du$$
$$+ \frac{e^{-a}}{|a|^{b/a+1}} \int_0^\infty d_2(s) \frac{b}{1+s} \int_0^s \psi(0) e^{au} du\, ds + \frac{e^{-a}}{|a|^{b/a+1}} \int_0^\infty H_\infty(u)\, dB(u), \quad \text{a.s.,}$$

where H_∞ is given by (6.7.2). We call the limit on the righthand side L. Therefore $X(t) - U(t) \to L$ as $t \to \infty$ a.s. Clearly L is an $\mathcal{F}^B(\infty)$–measurable normal random variable. In order to see that L is nontrivial, we may use Itô's isometry to show that its mean and variance are given by the formulae in the statement of part (ii) of the theorem.

The proof of (6.3.5) and (6.3.7) is deferred to Lemma 6.7.2.

Lemma 6.7.2. *Let $a < 0$. If $a+b < 0$ then* (6.3.5) *holds, whereas if $a+b = 0$ then* (6.3.7) *holds.*

Proof of Lemma 6.7.2. Firstly observe the following result

Lemma 6.7.3. *Let the function $f : (0,\infty) \to \mathbb{R}$ be such that f is continuous and obeys $\lim_{t\to\infty} f(t) = L_1 \in (-\infty, \infty)$. Then*

$$\lim_{t\infty} \frac{1}{t} \int_0^t f(s)\, ds = L_1.$$

Define $f(t) := X(t) - U(t)$. Then as already shown we have

$$\lim_{t\to\infty} f(t) = L_1 = \begin{cases} 0, & \text{a.s., if } a+b < 0, \\ L, & \text{a.s., if } a+b = 0, \end{cases}$$

where L is as given by (iii) of Theorem 6.3.1. Therefore we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(s) ds = L_1, \quad \text{a.s.}$$

Rewriting (6.3.1) gives

$$U(t) = a \int_0^t U(s) ds + \sigma B(t), \quad t \geq 0.$$

Thus as $U(t) = O(\sqrt{\log t})$,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t U(s) ds = 0, \quad \text{a.s.}$$

Therefore,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t X(s) ds = \lim_{t\to\infty} \frac{1}{t} \int_0^t f(s) ds + \lim_{t\to\infty} \frac{1}{t} \int_0^t U(s) ds = L_1, \quad \text{a.s.}$$

□

6.7.3 Proof of Theorem 6.3.2

Let Y and Z be as defined in the proof of Theorem 6.3.1. To attain a bound on the rate of $X - U$ tending to zero, the integral terms in (6.7.13) need to be analysed more carefully. From (6.7.11), and by using the continuity of f, it follows for every ω in an almost sure event Ω_1 that there exists an a.s. finite and positive random variable $K = K(\omega) > 0$ such that such that

$$|f(t,\omega)| \leq K(\omega)(1+t)^{-1/2}\sqrt{\log\log(t+e)}, \quad t \geq 0.$$

For the first integral in (6.7.13), we start by using l'Hôpital's rule to show that

$$\lim_{t\to\infty} \frac{\int_0^t \mathrm{e}^{-as}(1+s)^{-\frac{b}{a}-1/2}\sqrt{\log\log(e+s)}\,ds}{\mathrm{e}^{-at}(1+t)^{-\frac{b}{a}-1/2}\sqrt{\log\log(e+t)}} \in (0,\infty).$$

Therefore, there is $K_3 > 0$ such that

$$\int_0^t \mathrm{e}^{-as}(1+s)^{-\frac{b}{a}-1/2}\sqrt{\log\log(e+s)}\,ds \leq K_3\mathrm{e}^{-at}(1+t)^{-\frac{b}{a}-1/2}\sqrt{\log\log(e+t)},$$

for all $t \geq 0$. Now, by using (6.2.4) and (6.2.21) and the continuity of r_1 and d_1, we have that there exist $K_1 > 0$ and $K_2 > 0$ such that

$$|r_1(t)| \leq K_1\mathrm{e}^{at}(1+t)^{\frac{b}{a}}, \quad t \geq 0; \quad |d_1(s)| \leq K_2\mathrm{e}^{-as}(1+s)^{-\frac{b}{a}}, \quad s \geq 0.$$

Therefore for all $\omega \in \Omega_1$ and $t \geq 0$ we have

$$|r_1(t)|\int_0^t |d_1(s)||f(s,\omega)|ds \leq K_4(\omega)(1+t)^{-1/2}\sqrt{\log\log(t+e)},$$

where $K_4(\omega) = K_1K_2K(\omega)K_3$. Hence

$$\limsup_{t\to\infty} \frac{|r_1(t)|\int_0^t |d_1(s)||f(s)|ds}{(1+t)^{-1/2}\sqrt{\log\log(1+t)}} \in [0,\infty), \quad \text{a.s.} \tag{6.7.16}$$

For the second integral in (6.7.13), we showed in the proof of Theorem 6.3.1 that $\limsup_{t\to\infty}\int_0^t |d_2(s)f(s)|\,ds < +\infty$ a.s. in the case when $2b + a > 0$. Hence

$$\limsup_{t\to\infty} \frac{|r_2(t)|\int_0^t |d_2(s)||f(s)|ds}{(1+t)^{-1-\frac{b}{a}}} \in [0,\infty). \tag{6.7.17}$$

Moreover in this parameter regime $-1/2 < -1 - b/a < 0$, and so comparing the decay rates in (6.7.16) and (6.7.17) gives (i).

When $2b + a < 0$, we may use l'Hôpital's rule to get

$$\lim_{t\to\infty} \frac{\int_0^t (1+s)^{\frac{b}{a}-1/2}\sqrt{\log\log(e+s)}\,ds}{(1+t)^{\frac{b}{a}+1/2}\sqrt{\log\log(e+t)}} \in (0,\infty).$$

Hence there exists $K_7 > 0$ such that

$$\int_0^t (1+s)^{\frac{b}{a}-1/2}\sqrt{\log\log(e+s)}\,ds \le K_7(1+t)^{\frac{b}{a}+1/2}\sqrt{\log\log(e+t)}, \quad t \ge 0.$$

Since r_2 and d_2 obey (6.2.5) and (6.2.22), we have that there exist $K_5 > 0$ and $K_6 > 0$ such that

$$|r_2(t)| \le K_5(1+t)^{-1-\frac{b}{a}}, \quad t \ge 0; \quad |d_2(s)| \le K_6(1+s)^{\frac{b}{a}}, \quad s \ge 0.$$

Therefore for all $\omega \in \Omega_1$ and $t \ge 0$ we have

$$|r_2(t)| \int_0^t |d_2(s)||f(s,\omega)|\,ds \le K_5 K_6 K(\omega) K_7 (1+t)^{-1/2}\sqrt{\log\log(e+t)},$$

and so

$$\limsup_{t\to\infty} \frac{|r_2(t)| \int_0^t |d_2(s)||f(s)|ds}{(1+t)^{-1/2}\sqrt{\log\log(1+t)}} \in [0,\infty). \tag{6.7.18}$$

Applying (6.7.17) and (6.7.18) in (6.7.13) proves (ii).

In the case $2b + a = 0$, we have the estimate

$$\lim_{t\to\infty} \frac{\int_0^t (1+s)^{-1}\sqrt{\log\log(\mathrm{e}+s)}\,ds}{\log t\sqrt{\log\log t}} = 1.$$

Now following the same procedure as for the proof of (ii) gives the result.

6.8 Proof of Theorem 6.3.6 and 6.3.5

We begin this section with the statement and proof of some preparatory lemmata.

Lemma 6.8.1. *Let $b < 0$. Then the following limits hold:*

$$\lim_{t\to\infty} \frac{\pi\sqrt{|b|}\int_0^t (1+s)^{1/2}\sin^2\left(2\sqrt{|b|(s+1)} - \frac{3}{4}\pi\right)ds}{\frac{\pi}{3}|b|^{1/2}(1+t)^{3/2}} = 1, \tag{6.8.1}$$

and

$$\lim_{t\to\infty} \frac{\pi\sqrt{|b|}\int_0^t (1+s)^{1/2}\cos^2\left(2\sqrt{|b|(s+1)} - \frac{3}{4}\pi\right)ds}{\frac{\pi}{3}|b|^{1/2}(1+t)^{3/2}} = 1. \tag{6.8.2}$$

While this lemma amounts to little more than integration by parts, it serves as an asymptotic estimate of the rate of growth of the quadratic variation of stochastic integrals to be considered later.

Proof of Lemma 6.8.1. Consider first the limit (6.8.1). Making the substitution $w = 2\sqrt{|b|(s+1)} - \frac{3}{4}\pi$ in the integral, we get

$$\pi\sqrt{|b|}\int_0^t (1+s)^{1/2}\sin^2\left(2\sqrt{|b|(s+1)} - \frac{3}{4}\pi\right)ds$$
$$= \frac{\pi}{4|b|}\int_{2\sqrt{|b|}-3\pi/4}^{2\sqrt{|b|(1+t)}-3\pi/4}\left(w+\frac{3\pi}{4}\right)^2\sin^2(w)\,dw.$$

Since $\int_0^x (w+3\pi/4)^2\sin^2(w)\,dw$ can be computed explicitly for $x \geq 0$, and this leads to

$$\lim_{x\to\infty}\frac{1}{x^3}\int_0^x (w+\frac{3\pi}{4})^2\sin^2(w)\,dw = \frac{1}{6},$$

(6.8.1) holds. Similar calculations confirm the limit (6.8.2) □

We next introduce functions which correspond to the leading order asymptotic behaviour of r_7, r_8, d_7 and d_8. Define the functions, for $t \geq 0$

$$g_1(t) = \frac{1}{\sqrt{\pi}}|b|^{-1/4}(1+t)^{-1/4}\cos(2\sqrt{|b|(1+t)} - \pi/4), \tag{6.8.3a}$$
$$g_2(t) = \sqrt{\pi}|b|^{1/4}(1+t)^{1/4}\sin\left(2\sqrt{|b|(1+t)} - \frac{3}{4}\pi\right), \tag{6.8.3b}$$
$$g_3(t) = \frac{1}{\sqrt{\pi}}|b|^{-1/4}(1+t)^{-1/4}\sin(2\sqrt{|b|(1+t)} - \pi/4), \tag{6.8.3c}$$
$$g_4(t) = \sqrt{\pi}|b|^{1/4}(1+t)^{1/4}\cos\left(2\sqrt{|b|(1+t)} - \frac{3}{4}\pi\right). \tag{6.8.3d}$$

We aim to show that these leading order terms describe a continuous time process which obeys the Law of the Iterated Logarithm along many carefully designed sequences. These sequences will later be used to extrapolate the asymptotic behaviour of the continuous time process to the positive real line.

Lemma 6.8.2. *Fix $\eta \in [0, \pi/2)$. Define the sequence $\{t_n : n \in \mathbb{Z}^+\}$ such that*

$$t_0 = 0, \quad t_n = |b|^{-1}(n\pi + \pi/8 + \lceil\sqrt{|b|}/\pi - 1/8\rceil\pi + \eta/2)^2 - 1, \quad n \geq 1.$$

If g_1, g_2, g_3 and g_4 are defined by (6.8.3)*, then*

$$\limsup_{n\to\infty}\frac{g_1(t_n)\int_0^{t_n} g_2(s)dB(s) + g_3(t_n)\int_0^{t_n} g_4(s)dB(s)}{\sqrt{2t_n\log\log t_n}} = \frac{1}{\sqrt{3}}, \quad a.s., \tag{6.8.4}$$
$$\liminf_{n\to\infty}\frac{g_1(t_n)\int_0^{t_n} g_2(s)dB(s) + g_3(t_n)\int_0^{t_n} g_4(s)dB(s)}{\sqrt{2t_n\log\log t_n}} = -\frac{1}{\sqrt{3}}, \quad a.s. \tag{6.8.5}$$

Proof of Lemma 6.8.2. We start by noticing that $t_n \geq 0$ for all $n \geq 0$ and therefore $(t_n)_{n\geq 1}$ is a increasing sequence. Note also that

$$2\sqrt{|b|(t_n+1)} = 2n\pi + \frac{\pi}{4} + \eta + 2\pi L_b, \tag{6.8.6}$$

where

$$L_b := \lceil \sqrt{|b|}/\pi - 1/8 \rceil \geq \sqrt{|b|}/\pi - 1/8 \geq -1/8. \tag{6.8.7}$$

Therefore, as $L_b \in \mathbb{Z}$, we see that we must have L_b a non–negative integer. For all $n \in \mathbb{Z}$ let $\beta = \beta_\eta$ be the number such that $\cos(2n\pi + \eta) = \beta \in (0,1]$ and it is to be noted that β does not depend upon n. Then (6.8.6) implies

$$\cos(2\sqrt{|b|(1+t_n)} - \pi/4) = \beta, \quad \text{and hence} \quad \sin(2\sqrt{|b|(1+t_n)} - \pi/4) = \sqrt{1-\beta^2}. \tag{6.8.8}$$

Our plan now is to establish that

$$\int_0^{t_n} [g_2(s)g_1(t_n) + g_4(s)g_3(t_n)]dB(s)$$

gives rise to a discrete–time Gaussian martingale, to which Lemma 0.4.2 can be applied. To do this, we write

$$\begin{aligned} &\frac{\int_0^{t_n}[g_2(s)g_1(t_n) + g_4(s)g_3(t_n)]dB(s)}{\sqrt{2t_n \log\log t_n}} \\ &= (1+t_n)^{-1/4}\Bigg(\frac{\int_0^{t_n}(s+1)^{1/4}\sin\left(2\sqrt{|b|(s+1)} - \frac{3}{4}\pi\right)\beta dB(s)}{\sqrt{2t_n \log\log t_n}} \\ &\quad + \frac{\int_0^{t_n}(s+1)^{1/4}\cos\left(2\sqrt{|b|(s+1)} - \frac{3}{4}\pi\right)\sqrt{1-\beta^2}dB(s)}{\sqrt{2t_n \log\log t_n}}\Bigg) \\ &= (1+t_n)^{-1/4}\frac{\int_0^{t_n}(s+1)^{1/4}\sin\left(2\sqrt{|b|(s+1)} - \frac{3}{4}\pi\ + \eta\right)dB(s)}{\sqrt{2t_n \log\log t_n}}, \end{aligned} \tag{6.8.9}$$

where we have used (6.8.8) at the last step. As the last stochastic integral on the right hand side does not depend upon n in the integrand, we can decompose the integral and apply Lemma 0.4.2 to it. We therefore define for $n \geq 1$

$$S_n := \sum_{j=1}^n Y_j, \text{ where } Y_j = \int_{t_{j-1}}^{t_j} (s+1)^{1/4}\sin\left(2\sqrt{|b|(s+1)} - \frac{3}{4}\pi\ + \eta\right) dB(s).$$

Then Y_j is a Gaussian distributed random variable with mean zero and variance

$$\sigma_j^2 := \int_{t_{j-1}}^{t_j} (s+1)^{1/2}\sin^2\left(2\sqrt{|b|(s+1)} - \frac{3}{4}\pi\ + \eta\right) ds$$

and S_n is a Gaussian distributed random variable with mean zero and variance

$$s_n^2 = \sum_{j=0}^n \sigma_j^2 = \int_0^{t_n} (s+1)^{1/2}\sin^2\left(2\sqrt{|b|(s+1)} - \frac{3}{4}\pi\ + \eta\right) ds.$$

We wish to ascertain the rate of growth of both σ_j^2 and s_n^2. Define

$$M_\eta(t) = \int_0^t (1+s)^{1/4}\sin(2\sqrt{|b|(1+s)} - 3\pi/4 + \eta)dB(s), \quad t \geq 0.$$

Then M_η is a continuous martingale and its quadratic variation is given by

$$\langle M_\eta\rangle(t) = \int_0^t (1+s)^{1/2}\sin^2(2\sqrt{|b|(1+s)} - 3\pi/4+\eta)\,ds, \quad t\ge 0.$$

Therefore we have that

$$\langle M_\eta\rangle(t) = \frac{1}{4|b|^{3/2}}\int_{2\sqrt{|b|}-\frac{3\pi}{4}+\eta}^{2\sqrt{|b|(1+t)}-\frac{3\pi}{4}+\eta}\left(w+\frac{3\pi}{4}-\eta\right)^2\sin^2(w)\,dw, \quad t\ge 0.$$

An explicit calculation following exactly the model of Lemma 6.8.1 shows that

$$\langle M_\eta\rangle(t) \sim \frac{1}{3}t^{3/2}, \quad \text{as } t\to\infty.$$

We remark that the asymptotic behaviour of the quadratic variation is independent of η. Thus, since $t_n \sim n^2\pi^2/|b|$ as $n\to\infty$, we have that

$$s_n^2 = \langle M_\eta\rangle(t_n) \sim \frac{1}{3}t_n^{3/2} \sim \frac{n^3\pi^3}{3|b|^{3/2}} \quad \text{as } n\to\infty.$$

For $n\ge 1$, by (6.8.6) we have

$$\begin{aligned}
\sigma_n^2 &= \langle M_\eta\rangle(t_n) - \langle M_\eta\rangle(t_{n-1}) \\
&= \frac{1}{4|b|^{3/2}}\int_{2\sqrt{|b|(1+t_{n-1})}-\frac{3\pi}{4}+\eta}^{2\sqrt{|b|(1+t_n)}-\frac{3\pi}{4}+\eta}\left(w+\frac{3\pi}{4}-\eta\right)^2\sin^2(w)\,dw \\
&\le \frac{1}{4|b|^{3/2}}\int_{2(n-1)\pi-\pi/2+2\eta+2\pi L_b}^{2n\pi-\pi/2+2\eta+2\pi L_b}\left(w+\frac{3\pi}{4}-\eta\right)^2 dw \\
&= \frac{1}{12|b|^{3/2}}\left((2n\pi+\pi/4+\eta+2\pi L_b)^3 - (2(n-1)\pi+\pi/4+\eta+2\pi L_b)^3\right).
\end{aligned}$$

Therefore we have that $\sigma_n^2 = O(n^2) = O(t_n)$ as $n\to\infty$. Hence $\lim_{n\to\infty}\sigma_n/s_n = 0$. Thus all the conditions of Lemma 0.4.2 are satisfied and so the discrete Law of the Iterated Logarithm may be applied to S_n (or equivalently, to $M_\eta(t_n)$). Therefore by (6.8.9), and by using the fact that

$$\lim_{n\to\infty}\frac{t_n^{-1/4}}{\sqrt{2t_n\log\log t_n}}\sqrt{2\langle M_\eta\rangle(t_n)\log\log\langle M_\eta\rangle(t_n)} = \frac{1}{\sqrt{3}},$$

gives the limit superior in (6.8.4). The limit inferior in (6.8.5) may be obtained via a symmetry argument. □

Remark 6.8.1. Although Lemma 6.8.2 fixes η in the interval $[0,\pi/2)$, it is apparent from the proof of this lemma that one is free to choose η in any of the non–overlapping intervals $[\pi/2,\pi)$, $[\pi,3\pi/2)$ or $[3\pi/2,2\pi)$. The only amendments in the proof that would result from choosing η in these other intervals would be changes in the signs of the cosine and sine terms in (6.8.8).

Lemma 6.8.3. *Fix $k \in \mathbb{Z}^+$. Define the sequence $\{t_n^{(k)} : n \in \mathbb{Z}^+\}$ by $t_0^{(k)} = 0$ and*

$$t_j^{(k)} = \frac{1}{|b|}\left(N_j\pi + \left\lceil \frac{\sqrt{|b|}}{\pi} - \frac{1}{8} \right\rceil \pi + \frac{\eta^{(j,k)}}{2} + \frac{\pi}{8}\right)^2 - 1, \quad j \geq 1 \tag{6.8.10}$$

where

$$N_j = \left\lceil \frac{j}{2^{2+k}} \right\rceil - 1, \quad i_j = j - 2^{2+k}N_j - 1, \quad \eta^{(j,k)} = \frac{i_j}{2^k}\frac{\pi}{2},$$

so that $i_j \in \{0, 1, \dots, 2^2 2^k - 1\}$. Then

$$\limsup_{n\to\infty} \frac{g_1(t_n^{(k)}) \int_0^{t_n^{(k)}} g_2(s)dB(s) + g_3(t_n^{(k)}) \int_0^{t_n^{(k)}} g_4(s)dB(s)}{\sqrt{2t_n^{(k)} \log\log t_n^{(k)}}} = \frac{1}{\sqrt{3}}, \quad a.s., \tag{6.8.11a}$$

$$\liminf_{n\to\infty} \frac{g_1(t_n^{(k)}) \int_0^{t_n^{(k)}} g_2(s)dB(s) + g_3(t_n^{(k)}) \int_0^{t_n^{(k)}} g_4(s)dB(s)}{\sqrt{2t_n^{(k)} \log\log t_n^{(k)}}} = -\frac{1}{\sqrt{3}}, \quad a.s. \tag{6.8.11b}$$

where g_1, g_2, g_3 and g_4 are as defined in (6.8.3)*. Also,*

$$N_j \sim \frac{j}{2^2 2^k}, \quad t_j^{(k)} \sim \frac{1}{|b|}N_j^2\pi^2 \sim \frac{1}{|b|}\frac{1}{2^4 2^{2k}}j^2\pi^2, \quad \text{as } j \to \infty, \tag{6.8.12}$$

$$\Delta t_j^{(k)} := t_{j+1}^{(k)} - t_j^{(k)} \sim \frac{1}{|b|}N_j\frac{1}{2^k}\frac{\pi^2}{2} \sim \frac{1}{|b|}\frac{j}{2^{2k}}\frac{\pi^2}{2^3} \quad \text{as } j \to \infty. \tag{6.8.13}$$

Proof of Lemma 6.8.3. Define $\beta_{j,k}^{(i)} := \cos(\eta_i^{(j,k)})$, where

$$\eta_i^{(j,k)} := (i-1)\frac{\pi}{2} + \frac{j}{2^k}\frac{\pi}{2}, \quad i \in \{1,2,3,4\}, \quad j \in \{0, 1, ...2^k - 1\}.$$

Now define the following 4×2^k sequences. For each $j \in \{0, 1, ...2^k - 1\}$, we define for $n \geq 0$

$$\tau_n^{(j,k)} = |b|^{-1}(n\pi + \pi/8 + \eta_1^{(j,k)}/2)^2 - 1,$$
$$T_n^{(j,k)} = |b|^{-1}(n\pi + \pi/8 + \eta_2^{(j,k)}/2)^2 - 1,$$
$$\theta_n^{(j,k)} = |b|^{-1}(n\pi + \pi/8 + \eta_3^{(j,k)}/2)^2 - 1,$$
$$\Theta_n^{(j,k)} = |b|^{-1}(n\pi + \pi/8 + \eta_4^{(j,k)}/2)^2 - 1.$$

Notice that each of these sequences is increasing. Then the sequence $\{\tau_n^{(j,k)}\}_{n\geq 0}$ may be expressed in terms of $\beta_{j,k}^{(1)}$ (which is independent of n) according to

$$\beta_{j,k}^{(1)} = \cos(\eta_i^{(j,k)}) = \cos\left(2\sqrt{|b|(\tau_n^{(j,k)} + 1)} - \pi/4\right).$$

Similarly $T_n^{(j,k)}, \theta_n^{(j,k)}, \Theta_n^{(j,k)}$ may be expressed in terms of $\beta_{j,k}^{(2)}, \beta_{j,k}^{(3)}, \beta_{j,k}^{(4)}$ respectively.

Define

$$\bar{Y}(t) := \frac{g_1(t)\int_0^t g_2(s)dB(s) + g_3(t)\int_0^t g_4(s)dB(s)}{\sqrt{2t\log\log t}}, \quad t \geq e^e.$$

Then, from Lemma 6.8.2, for each $j \in \{0, 1, ..., 2^k - 1\}$,

$$\limsup_{n\to\infty} \bar{Y}(\tau_n^{(j,k)}) = -\liminf_{n\to\infty} \bar{Y}(\tau_n^{(j,k)}) = \frac{1}{\sqrt{3}},$$

on an event of probability one, $\Omega_1^{(j,k)}$. Using Lemma 6.8.2 in conjunction with Remark 6.8.1 gives

$$\begin{aligned}
\limsup_{n\to\infty} \bar{Y}(T_n^{(j,k)}) &= -\liminf_{n\to\infty} \bar{Y}(T_n^{(j,k)}) = \frac{1}{\sqrt{3}}, \\
\limsup_{n\to\infty} \bar{Y}(\theta_n^{(j,k)}) &= -\liminf_{n\to\infty} \bar{Y}(\theta_n^{(j,k)}) = \frac{1}{\sqrt{3}}, \\
\limsup_{n\to\infty} \bar{Y}(\Theta_n^{(j,k)}) &= -\liminf_{n\to\infty} \bar{Y}(\Theta_n^{(j,k)}) = \frac{1}{\sqrt{3}},
\end{aligned}$$

on almost sure events, $\Omega_2^{(j,k)}$,$\Omega_3^{(j,k)}$ and $\Omega_4^{(j,k)}$ respectively. Now,

$$\begin{aligned}
&\tau_n^{(0,k)} < \tau_n^{(1,k)} < ... < \tau_n^{(2^k-1,k)} < T_n^{(0,k)} < ... < T_n^{(2^k-1,k)} < \theta_n^{(0,k)} < ... < \theta_n^{(2^k-2,k)} \\
&< \theta_n^{(2^k-1,k)} < \Theta_n^{(0,k)} < ... < \Theta_n^{(2^k-1,k)}
\end{aligned}$$

and $\Theta_n^{(2^k-1,k)} < \tau_{n+1}^{(0,k)}$. Observe that the sequence $\{t_n^{(k)}\}_{n\geq 0}$, defined in the statement of this Lemma, obeys, for $j \geq 1$

$$t_j^{(k)} = \begin{cases} \tau^{(i_j,k)}_{N_j+\lceil\sqrt{|b|}/\pi-1/8\rceil}, & i_j \in \{0, ..., 2^k - 1\}, \\ T^{(i_j-2^k,k)}_{N_j+\lceil\sqrt{|b|}/\pi-1/8\rceil}, & i_j \in \{2^k, ..., 2.2^k - 1\}, \\ \theta^{(i_j-2.2^k,k)}_{N_j+\lceil\sqrt{|b|}/\pi-1/8\rceil}, & i_j \in \{2.2^k, ..., 3.2^k - 1\}, \\ \Theta^{(i_j-3.2^k,k)}_{N_j+\lceil\sqrt{|b|}/\pi-1/8\rceil}, & i_j \in \{3.2^k, ..., 4.2^k - 1\}, \end{cases}$$

Hence, defining $\Omega_5^{(k)} = \bigcap_{i=1}^4 \bigcap_{j=0}^{2^k-1} \Omega_i^{(j,k)}$ and noting that $\Omega_5^{(k)}$ is an almost sure event, we have that

$$\limsup_{n\to\infty} \bar{Y}(t_n^{(k)}) = -\liminf_{n\to\infty} \bar{Y}(t_n^{(k)}) = \frac{1}{\sqrt{3}},$$

on the event $\Omega_5^{(k)}$, which is (6.8.11).

We turn next to determining the asymptotic behaviour of the sequences N_j, $t_j^{(k)}$, $\Delta t_j^{(k)}$ as $j \to \infty$. We start with N_j. By definition, we have $j/(2^2.2^k) - 1 \leq N_j < j/(2^2.2^k)$, and thus, $1/(2^2.2^k) - 1/j \leq N_j/j < 1/(2^2.2^k)$. Now letting j tend to infinity and we have $\lim_{j\to\infty} N_j/j = 1/2^{2+k}$. Moreover as $\eta^{(j,k)}$ is bounded we have $\lim_{j\to\infty} \eta^{(j,k)}/j = 0$. Then from the definition of the sequence $\{t_n^{(k)}\}_{n\geq 0}$ it follows that

$$t_j^{(k)} \sim \frac{1}{|b|} N_j^2 \pi^2 \sim \frac{1}{|b|} \frac{1}{2^4 2^{2k}} j^2 \pi^2, \quad \text{as } j \to \infty.$$

In determining the asymptotic behaviour of $\Delta t_j^{(k)}$ we first consider the asymptotic behaviour of $\Delta\eta^{(j+1,k)} := \eta^{(j+1,k)} - \eta^{(j,k)}$ for large j. From the definition of $\eta^{(j,k)}$ it is trivially true that $\Delta\eta^{(j,k)} = \pi/2^{1+k}$ whenever $N_{j+1} = N_j$. Moreover the only values of j for which $N_{j+1} \neq N_j$ are values of the type $j = m.2^{2+k}$ for $m \in \{1, 2, ...\}$. So, if $j \neq m.2^{2+k}$ and $j \geq 1$, we get

$$\begin{aligned}\Delta t_j^{(k)} &= \frac{1}{|b|}\left(N_j\pi + L_b\pi + \frac{\pi}{8} + \frac{\eta^{(j,k)}}{2} + \frac{\Delta\eta^{(j,k)}}{2}\right)^2 - 1 \\ &\quad - \frac{1}{|b|}\left(N_j\pi + L_b\pi + \frac{\pi}{8} + \frac{\eta^{(j,k)}}{2}\right)^2 + 1 \\ &= \frac{2}{|b|}\left(N_j\pi + L_b\pi + \frac{\pi}{8} + \frac{\eta^{(j,k)}}{2}\right)\frac{\Delta\eta^{(j,k)}}{2} + \frac{1}{|b|}\frac{(\Delta\eta^{(j,k)})^2}{4}.\end{aligned}$$

Thus,

$$\Delta t_j^{(k)} \sim \frac{2}{|b|}N_j\pi\frac{\Delta\eta^{(j,k)}}{2} = \frac{N_j\pi^2}{|b|2.2^k} \sim \frac{j\pi^2}{|b|2^3.2^{2k}}, \quad \text{as } j \to \infty. \tag{6.8.14}$$

If $j = m.2^{2+k}$ for $m \in \{1, 2, ...\}$, we have $N_{j+1} = N_j + 1 = m$ (as we are interested in the asymptotic behaviour of $\eta^{(j,k)}$ for large j we may exclude $m = 0$ from our analysis). In this case,

$$\eta^{(j+1,k)} = \frac{(j + 1 - 2^{2+k}N_{j+1} - 1)}{2^k}\frac{\pi}{2} = \frac{(m.2^{2+k} + 1 - m.2^{2+k} - 1)}{2^k}\frac{\pi}{2} = 0$$

while

$$\eta^{(j,k)} = \frac{(j - 2^{2+k}N_j - 1)}{2^k}\frac{\pi}{2} = \frac{(m.2^{2+k} - (m-1)2^{2+k} - 1)}{2^k}\frac{\pi}{2} = \frac{(2^{2+k} - 1)}{2^k}\frac{\pi}{2}.$$

This gives

$$\begin{aligned}\Delta t_j^{(k)} &= \frac{1}{|b|}\left(N_{j+1}\pi + L_b\pi + \frac{\pi}{8} + \frac{\eta^{(j+1,k)}}{2}\right)^2 - 1 \\ &\quad - \frac{1}{|b|}\left(N_j\pi + L_b\pi + \frac{\pi}{8} + \frac{\eta^{(j,k)}}{2}\right)^2 + 1 \\ &= \frac{1}{|b|}\left(N_j\pi + L_b\pi + \frac{\pi}{8} + \pi\right)^2 \\ &\quad - \frac{1}{|b|}\left(N_j\pi + L_b\pi + \frac{\pi}{8} + \frac{(2^{2+k} - 1)}{2.2^k}\frac{\pi}{2}\right)^2 \\ &= \frac{2}{|b|}\left(N_j\pi + L_b\pi + \frac{\pi}{8}\right)\frac{\pi}{2^2.2^k} + \frac{\pi^2}{|b|}\frac{2^{3+k} + 1}{2^{4+2k}}.\end{aligned}$$

Thus, as $j = m.2^{2+k}$,

$$\Delta t_{m.2^{2+k}}^{(k)} \sim \frac{2}{|b|}m\frac{\pi^2}{2^2.2^k}, \quad \text{as } m \to \infty. \tag{6.8.15}$$

Therefore (6.8.15) together with (6.8.14) yields (6.8.13). □

Lemma 6.8.4. *Let g_1,g_2,g_3 and g_4 be as defined in* (6.8.3). *Let*

$$Y(t) := g_1(t)\int_0^t g_2(s)dB(s) + g_3(t)\int_0^t g_4(s)dB(s), \quad t \geq 0.$$

Then,

$$\limsup_{t\to\infty} \frac{Y(t)}{\sqrt{2t\log\log t}} = \frac{1}{\sqrt{3}}, \quad \liminf_{t\to\infty} \frac{Y(t)}{\sqrt{2t\log\log t}} = -\frac{1}{\sqrt{3}}, \quad a.s.$$

Proof of Lemma 6.8.4. A lower bound on the limit superior may easily be obtained from Lemma 6.8.3. We have

$$\limsup_{t\to\infty} \frac{Y(t)}{\sqrt{2t\log\log t}} \geq \limsup_{n\to\infty} \frac{Y(t_n)}{\sqrt{2t_n\log\log t_n}} = \frac{1}{\sqrt{3}}, \quad \text{a.s.,} \tag{6.8.16}$$

where the sequence $\{t_n\}_{n\in\mathbb{Z}^+}$ is as defined by (6.8.10) (for ease of notation we omit the k-dependence). We now turn our attention to obtaining an upper bound.

Define $\tilde{Y}(t) := \sqrt{\pi}|b|^{1/4}(1+t)^{1/4}Y(t)$ for $t \geq 0$. Then from Lemma 6.8.3 we have

$$\limsup_{n\to\infty} \frac{|\tilde{Y}(t_n)|}{\sqrt{2}t_n^{3/4}\sqrt{\log\log t_n}} = \limsup_{n\to\infty} \frac{\sqrt{\pi}|b|^{1/4}|Y(t_n)|}{\sqrt{2t_n\log\log t_n}} = \frac{\sqrt{\pi}|b|^{1/4}}{\sqrt{3}}, \quad \text{a.s.,} \tag{6.8.17}$$

where the limit superior is taken through the sequence $\{t_n\}_{n\in\mathbb{Z}^+}$ defined in (6.8.10) (again for ease of notation we omit the k-dependence). Now, for $t_n \leq t \leq t_{n+1}$,

$$\begin{aligned}\frac{\tilde{Y}(t)}{\sqrt{2}t^{3/4}\sqrt{\log\log t}} &= \frac{\tilde{Y}(t)-\tilde{Y}(t_n)}{\sqrt{2}t_n^{3/4}\sqrt{\log\log t_n}}\frac{\sqrt{2}t_n^{3/4}\sqrt{\log\log t_n}}{\sqrt{2}t^{3/4}\sqrt{\log\log t}} \\ &\quad + \frac{\tilde{Y}(t_n)}{\sqrt{2}t_n^{3/4}\sqrt{\log\log t_n}}\frac{\sqrt{2}t_n^{3/4}\sqrt{\log\log t_n}}{\sqrt{2}t^{3/4}\sqrt{\log\log t}},\end{aligned}$$

and so

$$\frac{\tilde{Y}(t)}{\sqrt{2}t^{\frac{3}{4}}\sqrt{\log\log t}} \leq \frac{\sup_{t_n\leq t\leq t_{n+1}}|\tilde{Y}(t)-\tilde{Y}(t_n)|}{\sqrt{2}t_n^{\frac{3}{4}}\sqrt{\log\log t_n}} + \frac{|\tilde{Y}(t_n)|}{\sqrt{2}t_n^{\frac{3}{4}}\sqrt{\log\log t_n}}, \; t\in[t_n,t_{n+1}]. \tag{6.8.18}$$

We firstly examine the asymptotic behaviour of $\sup_{t_n\leq t\leq t_{n+1}}|\tilde{Y}(t)-\tilde{Y}(t_n)|$. Define

$$\tilde{Y}_1(t) = \sqrt{\pi}|b|^{1/4}(1+t)^{1/4}g_1(t)\int_0^t g_2(s)dB(s), \quad t \geq 0,$$
$$\tilde{Y}_2(t) = \sqrt{\pi}|b|^{1/4}(1+t)^{1/4}g_3(t)\int_0^t g_4(s)dB(s), \quad t \geq 0.$$

Then $\tilde{Y}(t) = \tilde{Y}_1(t) + \tilde{Y}_2(t)$ and for $t \in [t_n, t_{n+1}]$ we have

$$
\begin{aligned}
&|\tilde{Y}_1(t) - \tilde{Y}_1(t_n)| \\
&\le |\cos(2\sqrt{|b|(1+t)} - \pi/4)| \left| \int_0^t g_2(s)dB(s) - \int_0^{t_n} g_2(s)dB(s) \right| \\
&\quad + |\cos(2\sqrt{|b|(1+t)} - \pi/4) - \cos(2\sqrt{|b|(1+t_n)} - \pi/4)| \left| \int_0^{t_n} g_2(s)dB(s) \right| \\
&\le \left| \int_{t_n}^t g_2(s)dB(s) \right| + |2\sqrt{|b|(1+t)} - 2\sqrt{|b|(1+t_n)}| \left| \int_0^{t_n} g_2(s)dB(s) \right| \\
&= \left| \int_{t_n}^t g_2(s)dB(s) \right| + 2\sqrt{|b|} \left(\frac{1+t-(1+t_n)}{\sqrt{1+t} + \sqrt{1+t_n}} \right) \left| \int_0^{t_n} g_2(s)dB(s) \right|,
\end{aligned}
$$

where the Lipschitz continuity of $\cos(2\sqrt{|b|(1+\cdot)} - \pi/4)$ on $\mathbb{R}$ has been used. A similar inequality can be developed for $|\tilde{Y}_2(t) - \tilde{Y}_2(t_n)|$ for $t \in [t_n, t_{n+1}]$. Using the fact that $|\tilde{Y}(t) - \tilde{Y}(t_n)| \le |\tilde{Y}_1(t) - \tilde{Y}_1(t_n)| + |\tilde{Y}_2(t) - \tilde{Y}_2(t_n)|$, we obtain

$$
\begin{aligned}
\sup_{t_n \le t \le t_{n+1}} |\tilde{Y}(t) - \tilde{Y}(t_n)| &\le \sup_{t_n \le t \le t_{n+1}} \left| \int_{t_n}^t g_2(s)dB(s) \right| + \sup_{t_n \le t \le t_{n+1}} \left| \int_{t_n}^t g_4(s)dB(s) \right| \\
&\quad + 2\sqrt{|b|} \left(\frac{t_{n+1} - t_n}{2\sqrt{1+t_n}} \right) \left\{ \left| \int_0^{t_n} g_2(s)dB(s) \right| + \left| \int_0^{t_n} g_4(s)dB(s) \right| \right\}, \quad (6.8.19)
\end{aligned}
$$

where we have used the fact that $1/(\sqrt{1+t} + \sqrt{1+t_n}) \le 1/(2\sqrt{1+t_n})$ for $t \ge t_n$. We now estimate the order of the largest fluctuations of each term on the right hand side of (6.8.19). We show that, for $i \in \{2, 4\}$

$$
\limsup_{n \to \infty} \frac{\sup_{t_n \le t \le t_{n+1}} \left| \int_{t_n}^t g_i(s)dB(s) \right|}{t_n^{3/4} \sqrt{\log\log t_n}} = 0, \quad \text{a.s.} \quad (6.8.20)
$$

Now, let $\epsilon_n > 0$. By the martingale time change theorem, for every n, there exists a standard Brownian motion $\tilde{B}_{i,n}$ such that

$$
\begin{aligned}
\mathbb{P}\left[\sup_{t_n \le t \le t_{n+1}} \left| \int_{t_n}^t g_i(s)\, dB(s) \right| \ge \epsilon_n \right] &= \mathbb{P}\left[\sup_{t_n \le t \le t_{n+1}} \left| \tilde{B}_{i,n}\left(\int_{t_n}^t g_i(s)^2\, ds \right) \right| \ge \epsilon_n \right] \\
&= \mathbb{P}\left[\sup_{0 \le u \le \int_{t_n}^{t_{n+1}} g_i(s)^2 ds} \left| \tilde{B}_{i,n}(u) \right| \ge \epsilon_n \right]
\end{aligned}
$$

Hence there is a Brownian motion $B^*_{i,n}$ such that

$$
\begin{aligned}
&\mathbb{P}\left[\sup_{t_n \le t \le t_{n+1}} \left| \int_{t_n}^t g_i(s)\, dB(s) \right| \ge \epsilon_n \right] \\
&\le 2\,\mathbb{P}\left[\sup_{0 \le u \le \int_{t_n}^{t_{n+1}} g_i(s)^2 ds} B^*_{i,n}(u) \ge \epsilon_n \right] = 2\,\mathbb{P}\left[\left| B^*_{i,n}\left(\int_{t_n}^{t_{n+1}} g_i(s)^2 ds \right) \right| \ge \epsilon_n \right] \\
&= 4\,\mathbb{P}\left[B^*_{i,n}\left(\int_{t_n}^{t_{n+1}} g_i(s)^2 ds \right) \ge \epsilon_n \right] = 4\left\{ 1 - \Phi\left(\frac{\epsilon_n}{\sqrt{\int_{t_n}^{t_{n+1}} g_i(s)^2 ds}} \right) \right\}, \quad (6.8.21)
\end{aligned}
$$

where we have used the fact that $\max_{0\le s\le t} W(s)$ has the same distribution as $|W(t)|$ when W is a standard Brownian motion, the symmetry of the distribution of a standard Brownian motion, and Φ denotes the distribution function of a standard normal random variable. Now,

$$\begin{aligned} g_2(t)^2 &= \pi|b|^{1/2}(1+t)^{1/2}\sin^2\left(2\sqrt{|b|(1+t)}-\frac{3}{4}\pi\right) \le \pi|b|^{1/2}(1+t)^{1/2}, \\ g_4(t)^2 &= \pi|b|^{1/2}(1+t)^{1/2}\cos^2\left(2\sqrt{|b|(1+t)}-\frac{3}{4}\pi\right) \le \pi|b|^{1/2}(1+t)^{1/2}. \end{aligned}$$

Thus, by (6.8.12) we have

$$\int_{t_n}^{t_{n+1}} g_i(s)^2 ds \le \pi|b|^{1/2}(1+t_{n+1})^{1/2}(t_{n+1}-t_n) \sim \pi|b|^{1/2}t_n^{1/2}\Delta t_n, \quad \text{as } n\to\infty, \tag{6.8.22}$$

and therefore by (6.8.12) and (6.8.13)

$$\limsup_{n\to\infty} \frac{\sqrt{\int_{t_n}^{t_{n+1}} g_i(s)^2 ds}}{N_n} \le \limsup_{n\to\infty} \frac{\pi^{1/2}|b|^{1/4}t_n^{1/4}(\Delta t_n)^{1/2}}{N_n} = \frac{1}{|b|^{1/2}}\pi^2\frac{1}{2^{1/2+k/2}}.$$

So letting $\epsilon_n = t_n^{5/8}\sqrt{\log\log t_n}$, and using the last relation and (6.8.12) gives

$$\begin{aligned} &\liminf_{n\to\infty} \frac{\epsilon_n}{\sqrt{\int_{t_n}^{t_{n+1}} g_i(s)^2 ds}\cdot n^{1/4}\sqrt{\log\log n}} \\ &= \liminf_{n\to\infty} \frac{t_n^{5/8}\sqrt{\log\log t_n}}{N_n n^{1/4}\sqrt{\log\log n}} \frac{N_n}{\sqrt{\int_{t_n}^{t_{n+1}} g_i(s)^2 ds}} \\ &\ge \liminf_{n\to\infty} \frac{(\frac{1}{|b|}\frac{1}{2^4 2^{2k}}n^2\pi^2)^{5/8}}{\frac{n}{2^2 2^k}n^{1/4}} \frac{1}{\frac{1}{|b|^{1/2}}\pi^2\frac{1}{2^{1/2+k/2}}} =: C_k' > 0. \end{aligned}$$

Therefore there exists a positive constant C_k such that

$$\frac{\epsilon_n}{\sqrt{\int_{t_n}^{t_{n+1}} g_i(s)^2 ds}} \ge C_k(1+n)^{1/4}\sqrt{\log\log(n+e^e)}, \quad n\ge 1.$$

By (6.8.21), this implies

$$\begin{aligned} &\mathbb{P}\left[\sup_{t_n\le t\le t_{n+1}}\left|\int_{t_n}^t g_i(s)dB(s)\right| \ge \epsilon_n\right] \\ &\qquad\qquad \le 4\left\{1-\Phi\left(C_k(1+n)^{1/4}\sqrt{\log\log(n+e^e)}\right)\right\}, \quad n\ge 1. \end{aligned}$$

Now from [72, Problem 2.9.22],

$$1-\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty \mathrm{e}^{-u^2/2}du \le \frac{1}{\sqrt{2\pi}}\frac{1}{x}\mathrm{e}^{-x^2/2}, \quad x>0.$$

Thus, for $n \geq 1$

$$\mathbb{P}\left[\sup_{t_n \leq t \leq t_{n+1}} \left|\int_{t_n}^t g_i(s)dB(s)\right| \geq \epsilon_n\right]$$
$$\leq \frac{4}{\sqrt{2\pi}} \frac{1}{C_k(1+n)^{1/4}\sqrt{\log\log(n+e^e)}} \mathrm{e}^{-\frac{1}{2}C_k^2(1+n)^{1/2}\log\log(n+e^e)}.$$

Therefore

$$\sum_{n=0}^{\infty} \mathbb{P}\left[\sup_{t_n \leq t \leq t_{n+1}} \left|\int_{t_n}^t g_i(s)dB(s)\right| \geq \epsilon_n\right] < +\infty.$$

The Borel-Cantelli Lemma then gives that

$$\limsup_{n\to\infty} \frac{\sup_{t_n \leq t \leq t_{n+1}} \left|\int_{t_n}^t g_i(s)dB(s)\right|}{t_n^{5/8}\sqrt{\log\log t_n}} \leq 1, \quad \text{a.s.}$$

Therefore (6.8.20) holds. We now show for $i \in \{2, 4\}$ that

$$\limsup_{n\to\infty} \frac{\left|\int_0^{t_n} g_i(s)dB(s)\right|}{\sqrt{2}t_n^{3/4}\log\log t_n} = \frac{\sqrt{\pi}|b|^{1/4}}{\sqrt{3}}, \quad \text{a.s.} \tag{6.8.23}$$

Define

$$X_n^{(i)} := \int_{t_{n-1}}^{t_n} g_i(s)dB(s), \quad n \geq 1.$$

Then

$$S_n^{(i)} := \int_0^{t_n} g_i(s)dB(s) = \sum_{j=1}^n X_j^{(i)}.$$

Now from (6.8.13), (6.8.12) and (6.8.22) we get

$$\sigma_n^2 := \mathrm{Var}[X_n^{(i)}] = \int_{t_{n-1}}^{t_n} g_i(s)^2 ds = O(t_n), \quad \text{as } n \to \infty,$$

while, from Lemma 6.8.1

$$s_n^2 := \mathrm{Var}[S_n^{(i)}] = \int_0^{t_n} g_i(s)^2 ds \sim \frac{\pi}{3}|b|^{1/2}t_n^{3/2}, \quad \text{as } n \to \infty.$$

and so $\sigma_n/s_n \to 0$ as $n \to \infty$. Hence we may apply Lemma 0.4.2 to $S_n^{(i)}$ to obtain

$$\limsup_{n\to\infty} \frac{|\int_0^{t_n} g_i(s)dB(s)|}{\sqrt{2\int_0^{t_n} g_i(s)^2 ds \, \log\log \int_0^{t_n} g_i(s)^2 ds}} = 1, \quad \text{a.s.}$$

which is equivalent to (6.8.23).

Lastly observe from (6.8.13) and (6.8.12) that

$$\lim_{n\to\infty} 2\sqrt{|b|}\left(\frac{t_{n+1}-t_n}{2\sqrt{1+t_n}}\right) = \frac{\pi}{2.2^k}. \tag{6.8.24}$$

Scaling (6.8.19), taking limit superiors across the resulting inequality, and employing (6.8.20), (6.8.23) and (6.8.24) gives

$$\limsup_{n\to\infty} \frac{\sup_{t_n \le t \le t_{n+1}} |\tilde{Y}(t) - \tilde{Y}(t_n)|}{\sqrt{2} t_n^{3/4} \sqrt{\log\log t_n}} \le \frac{\pi}{2^k} \frac{\sqrt{\pi}|b|^{1/4}}{\sqrt{3}}, \quad \text{a.s.} \tag{6.8.25}$$

Next, define

$$K_n := \frac{\sup_{t_n \le t \le t_{n+1}} |\tilde{Y}(t) - \tilde{Y}(t_n)|}{\sqrt{2} t_n^{3/4} \sqrt{\log\log t_n}} + \frac{|\tilde{Y}(t_n)|}{\sqrt{2} t_n^{3/4} \sqrt{\log\log t_n}}.$$

Since for every $t > 0$ there exists $N(t)$ such that $t_{N(t)} \le t < t_{N(t)+1}$, it follows from (6.8.18) that

$$\frac{\tilde{Y}(t)}{\sqrt{2} t^{3/4} \sqrt{\log\log t}} \le K_{N(t)}.$$

Now, by (6.8.25) and (6.8.17) we have that

$$\limsup_{n\to\infty} K_n \le \frac{\sqrt{\pi}|b|^{1/4}}{\sqrt{3}} \left(\frac{\pi}{2^k} + 1 \right)$$

and since $N(t) \to +\infty$ as $t \to \infty$, we have

$$\limsup_{t\to\infty} \frac{\tilde{Y}(t)}{\sqrt{2} t^{3/4} \sqrt{\log\log t}} \le \frac{\sqrt{\pi}|b|^{1/4}}{\sqrt{3}} \left(\frac{\pi}{2^k} + 1 \right)$$

holding on an almost sure set Ω_k. This result also holds on the almost sure set $\Omega^* = \bigcap_{k\in\mathbb{Z}^+} \Omega_k$ and hence

$$\limsup_{t\to\infty} \frac{\tilde{Y}(t)}{\sqrt{2} t^{3/4} \sqrt{\log\log t}} \le \frac{\sqrt{\pi}|b|^{1/4}}{\sqrt{3}}, \quad \text{a.s.}$$

Since $\tilde{Y}(t) = \sqrt{\pi}|b|^{1/4}(1+t)^{1/4} Y(t)$, we have that

$$\limsup_{t\to\infty} \frac{Y(t)}{\sqrt{2t\log\log t}} \le \frac{1}{\sqrt{3}}, \quad \text{a.s.}$$

Combining this upper bound on the limit superior with (6.8.16) gives the required limit superior.

The limit inferior result may be obtained by considering the process $Z(t) = -Y(t)$. Then

$$Z(t) = g_1(t) \int_0^t g_2(s) dW(s) + g_3(t) \int_0^t g_4(s) dW(s), \quad t \ge 0,$$

where $W(t) := -B(t)$ is a standard Brownian motion. One then may apply the foregoing argument to deduce that

$$-\liminf_{t\to\infty} \frac{Y(t)}{\sqrt{2t\log\log t}} = \limsup_{t\to\infty} \frac{-Y(t)}{\sqrt{2t\log\log t}} = \limsup_{t\to\infty} \frac{Z(t)}{\sqrt{2t\log\log t}} = \frac{1}{\sqrt{3}}, \quad \text{a.s.}$$

as required. □

The proof of Theorem 6.3.6 can now given. It is chiefly concerned with identifying the leading order terms which contribute to the overall asymptotic behaviour of X. The asymptotic behaviour of these leading order terms are then known from Lemma 6.8.4.

Proof of Theorem 6.3.6. By (6.1.4), (6.2.56), and (6.2.61) the solution X of (6.1.1) has the representation

$$X(t) = r_7(t)c_7 + r_8(t)c_8 + \sigma r_7(t)\int_0^t d_7(s)dB(s) + \sigma r_8(t)\int_0^t d_8(s)dB(s). \tag{6.8.26}$$

By (6.2.57) and (6.2.60), r_7 and r_8 have asymptotic behaviour given by

$$r_7(t) = \frac{1}{\sqrt{\pi}}|b|^{-1/4}(1+t)^{-1/4}\{\cos(2\sqrt{|b|(1+t)} - \pi/4) + O(t^{-1/2})\}, \quad \text{as } t \to \infty,$$
$$r_8(t) = \frac{1}{\sqrt{\pi}}|b|^{-1/4}(1+t)^{-1/4}\{\sin(2\sqrt{|b|(1+t)} - \pi/4) + O(t^{-1/2})\}, \quad \text{as } t \to \infty.$$

Also by (6.2.55) and (6.2.60), d_7 and d_8 have asymptotic behaviour given by

$$d_7(s) = \sqrt{\pi}|b|^{1/4}(s+1)^{1/4}\left(\sin\left(2\sqrt{|b|(s+1)} - \frac{3}{4}\pi\right) + O(s^{-1/2})\right), \quad \text{as } s \to \infty,$$
$$d_8(s) = \sqrt{\pi}|b|^{1/4}(s+1)^{1/4}\left(\cos\left(2\sqrt{|b|(s+1)} - \frac{3}{4}\pi\right) + O(s^{-1/2})\right), \quad \text{as } s \to \infty.$$

Define the functions R_7, R_8, D_7 and D_8 so that, for $s \geq 0$ and $t \geq 0$ we have

$$r_7(t) = g_1(t) + R_7(t), \quad r_8(t) = g_3(t) + R_8(t), \tag{6.8.27a}$$

$$d_7(s) = g_2(s) + D_7(s), \quad d_8(s) = g_4(s) + D_8(s), \tag{6.8.27b}$$

where g_1,g_2,g_3 and g_4 are as defined in (6.8.3). Notice that R_7, R_8, D_7 and D_8 are continuous functions. Since

$$R_7(t) = O(t^{-3/4}), \quad R_8(t) = O(t^{-3/4}) \quad \text{as } t \to \infty,$$
$$D_7(s) = O(s^{-1/4}), \quad D_8(s) = O(s^{-1/4}) \quad \text{as } s \to \infty,$$

it follows that there exists $M > 0$ such that

$$|R_7(t)| \leq M(1+t)^{-3/4}, \quad |R_8(t)| \leq M(1+t)^{-3/4} \quad t \geq 0,$$
$$|D_7(s)| \leq M(1+s)^{-1/4}, \quad |D_8(s)| \leq M(1+s)^{-1/4}, \quad s \geq 0.$$

Next, we decompose X according to

$$\frac{X(t)}{\sqrt{2t\log\log t}} = \frac{r_7(t)c_7 + r_8(t)c_8}{\sqrt{2t\log\log t}} + \sigma\frac{g_1(t)\int_0^t g_2(s)dB(s)}{\sqrt{2t\log\log t}} + \sigma\frac{g_3(t)\int_0^t g_4(s)dB(s)}{\sqrt{2t\log\log t}}$$
$$+\sigma\frac{R_7(t)\int_0^t g_2(s)dB(s)}{\sqrt{2t\log\log t}} + \sigma\frac{R_8(t)\int_0^t g_4(s)dB(s)}{\sqrt{2t\log\log t}}$$
$$+\sigma\frac{r_7(t)\int_0^t D_7(s)dB(s)}{\sqrt{2t\log\log t}} + \sigma\frac{r_8(t)\int_0^t D_8(s)dB(s)}{\sqrt{2t\log\log t}}. \quad (6.8.28)$$

Since $r_7(t) \to 0$ and $r_8(t) \to 0$ as $t \to \infty$, the first term on the righthand–side of (6.8.28) tends to zero as $t \to \infty$. The asymptotic behaviour of the second and third terms is described by Lemma 6.8.4. We now proceed to demonstrate that the remaining terms have do not contribute to size of the largest oscillations of X.

We start by considering the last two terms on the right hand side of (6.8.28). If $\int_0^\infty D_7(s)^2 ds < \infty$ then because $r_7(t) \to 0$ as $t \to \infty$, we have

$$\lim_{t\to\infty} \frac{r_7(t)\int_0^t D_7(s)dB(s)}{\sqrt{2t\log\log t}} = 0, \quad \text{a.s.} \quad (6.8.29)$$

On the other hand, if $\lim_{t\to\infty}\int_0^t D_7(s)^2 ds = +\infty$, by using the estimate on D_7, for all $t \geq 0$ we have

$$\int_0^t D_7(s)^2 ds \leq M^2 \int_0^t (1+s)^{-1/2} ds \leq 2M^2(1+t)^{1/2}.$$

Therefore

$$\limsup_{t\to\infty} \frac{2\int_0^t D_7(s)^2 ds \log\log \int_0^t D_7(s)^2 ds}{t^{1/2}\log\log t} \leq 4M^2.$$

Hence by the Law of the Iterated Logarithm for continuous martingales, we have

$$\limsup_{t\to\infty} \frac{\left|r_7(t)\int_0^t D_7(s)dB(s)\right|}{\sqrt{2t\log\log t}}$$
$$= \limsup_{t\to\infty} \frac{|r_7(t)|\left|\int_0^t D_7(s)dB(s)\right|}{\sqrt{2\int_0^t D_7(s)^2 ds\log\log\int_0^t D_7(s)^2 ds}} \frac{\sqrt{2\int_0^t D_7(s)^2 ds\log\log\int_0^t D_7(s)^2 ds}}{\sqrt{2t\log\log t}}$$
$$= \limsup_{t\to\infty} \frac{|r_7(t)|\sqrt{2\int_0^t D_7(s)^2 ds\log\log\int_0^t D_7(s)^2 ds}}{\sqrt{2t\log\log t}}.$$

Now,

$$\limsup_{t\to\infty} \frac{\left|r_7(t)\int_0^t D_7(s)dB(s)\right|}{\sqrt{2t\log\log t}}$$
$$\leq M \limsup_{t\to\infty} \frac{t^{-1/4}\sqrt{2\int_0^t D_7(s)^2 ds\log\log\int_0^t D_7(s)^2 ds}}{\sqrt{t^{1/2}\log\log t}} \frac{t^{1/4}\sqrt{\log\log t}}{\sqrt{2t\log\log t}}$$
$$\leq 2M^2 \limsup_{t\to\infty} \frac{\sqrt{\log\log t}}{\sqrt{2t\log\log t}} = 0.$$

Hence (6.8.29) holds. One may similarly show that

$$\lim_{t\to\infty} \frac{r_8(t)\int_0^t D_8(s)dB(s)}{\sqrt{2t\log\log t}} = 0, \quad \text{a.s.} \tag{6.8.30}$$

To estimate the asymptotic behaviour of the fourth and fifth terms on the right hand side of (6.8.28), we note from Lemma 6.8.1, we have that

$$\lim_{t\to\infty} \frac{\int_0^t g_2^2(s)\,ds}{t^{3/2}} = \frac{1}{3}\pi|b|^{1/2}, \quad \lim_{t\to\infty} \frac{\int_0^t g_4^2(s)\,ds}{t^{3/2}} = \frac{1}{3}\pi|b|^{1/2}.$$

Therefore by the Law of the Iterated Logarithm for continuous martingales we have

$$\limsup_{t\to\infty} \frac{\left|\int_0^t g_2(s)dB(s)\right|}{\sqrt{2}t^{3/4}\sqrt{\log\log t}} = \sqrt{\frac{\pi}{3}}|b|^{1/4}, \quad \text{a.s.}$$

Therefore, using the estimate on R_7 we have

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{\left|R_7(t)\int_0^t g_2(s)dB(s)\right|}{\sqrt{2t\log\log t}} \\
&\le M \limsup_{t\to\infty} \frac{t^{-3/4}\left|\int_0^t g_2(s)dB(s)\right|}{\sqrt{2}t^{3/4}\sqrt{\log\log t}} \frac{\sqrt{2}t^{3/4}\sqrt{\log\log t}}{\sqrt{2t\log\log t}} \\
&= M\sqrt{\frac{\pi}{3}}|b|^{1/4} \limsup_{t\to\infty} \frac{\sqrt{2\log\log t}}{\sqrt{2t\log\log t}} = 0.
\end{aligned}$$

Thus,

$$\lim_{t\to\infty} \frac{R_7(t)\int_0^t g_2(s)dB(s)}{\sqrt{2t\log\log t}} = 0, \quad \text{a.s.} \tag{6.8.31}$$

Similarly it may be shown that

$$\lim_{t\to\infty} \frac{R_8(t)\int_0^t g_4(s)dB(s)}{\sqrt{2t\log\log t}} = 0, \quad \text{a.s.} \tag{6.8.32}$$

Then due to (6.8.29), (6.8.30), (6.8.31), (6.8.32), and Lemma 6.8.4, by taking the limit superior across (6.8.28) we get

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = \frac{\sigma}{\sqrt{3}}, \quad \text{a.s.}$$

Taking the limit inferior and applying these preparatory estimates along with Lemma 6.8.4 secures the corresponding limit inferior result. □

Proof of Theorem 6.3.5. By (6.1.6) and (6.2.61) we have that

$$\begin{aligned}
\frac{1}{\sigma^2}\text{Var}[X(t)] &= \int_0^t r(t,s)^2\,ds \\
&= r_7(t)^2\int_0^t d_7(s)^2\,ds + 2r_7(t)r_8(t)\int_0^t d_7(s)d_8(s)\,ds + r_8(t)^2\int_0^t d_8(s)^2\,ds.
\end{aligned} \tag{6.8.33}$$

We deduce the asymptotic behaviour of the terms on the righthand side of (6.8.33). By the definition of d_7 we have the identity

$$\frac{1}{t^{3/2}}\int_0^t d_7(s)^2\,ds - \frac{1}{t^{3/2}}\int_0^t g_2^2(s)\,ds = 2\frac{1}{t^{3/2}}\int_0^t g_2(s)D_7(s)\,ds + \frac{1}{t^{3/2}}\int_0^t D_7(s)^2.$$

By the definition of g_2 and D_7, we have that $g_2(t) = O(t^{1/4})$ and $D_7(t) = O(t^{-1/4})$ as $t \to \infty$, so the limit as $t \to \infty$ of the two terms on the right hand side is zero. Since the second term on the left hand side has limit $\pi|b|^{1/2}/3$ as $t \to \infty$, we have

$$\lim_{t\to\infty}\frac{1}{t^{3/2}}\int_0^t d_7(s)^2\,ds = \frac{\pi}{3}|b|^{1/2}. \tag{6.8.34}$$

Similarly, we may establish

$$\lim_{t\to\infty}\frac{1}{t^{3/2}}\int_0^t d_8(s)^2\,ds = \frac{\pi}{3}|b|^{1/2}. \tag{6.8.35}$$

We determine the asymptotic behaviour of the integral in the second term on the right hand side of (6.8.33). First, we express d_7 and d_8 in terms of g_2, g_4, D_7 and D_8 to get

$$\int_0^t d_7(s)d_8(s)\,ds$$
$$= \int_0^t g_2(s)g_4(s)\,ds + \int_0^t \{g_2(s)D_8(s) + g_4(s)D_7(s) + D_7(s)D_8(s)\}\,ds.$$

Since $g_2(t) = O(t^{1/4})$, $g_4(t) = O(t^{1/4})$, $D_7(t) = O(t^{-1/4})$ and $D_8(t) = O(t^{-1/4})$ as $t \to \infty$, the second integral on the right hand side is of order t as $t \to \infty$. Finally,

$$\int_0^t g_2(s)g_4(s)\,ds = \frac{1}{2}\pi\int_0^t |b|^{1/2}(1+s)^{1/2}\sin\left(4\sqrt{|b|(1+s)} - \frac{3}{2}\pi\right)\,ds.$$

Making a substitution in the integral leads to

$$\int_0^t |b|^{1/2}(1+s)^{1/2}\sin\left(4\sqrt{|b|(1+s)} - \frac{3}{2}\pi\right)\,ds$$
$$= \frac{1}{32|b|}\int_{4\sqrt{|b|}-\frac{3}{2}\pi}^{4\sqrt{|b|(1+t)}-\frac{3}{2}\pi}(u+3\pi/2)^2\sin(u)\,du.$$

Since the last integral can be evaluated exactly, we see that

$$\int_0^t |b|^{1/2}(1+s)^{1/2}\sin\left(4\sqrt{|b|(1+s)} - \frac{3}{2}\pi\right)\,ds = O(t), \qquad \text{as } t \to \infty,$$

so it follows that

$$\int_0^t d_7(s)d_8(s)\,ds = O(t), \quad \text{as } t \to \infty. \tag{6.8.36}$$

We prepare one final estimate; it is on $r_7^2(t) + r_8^2(t)$ as $t \to \infty$. First we observe that because $g_1(t) = O(t^{-1/4})$, $g_3(t) = O(t^{-1/4})$, $R_7(t) = O(t^{-3/4})$ and $R_8(t) = O(t^{-3/4})$ as $t \to \infty$, it follows that

$$2g_1(t)R_7(t) + 2g_3(t)R_8(t) + R_7^2(t) + R_8^2(t) = O(t^{-1}), \qquad \text{as } t \to \infty.$$

Therefore

$$\begin{aligned} r_7^2(t) + r_8^2(t) &= g_1^2(t) + g_3^2(t) + 2g_1(t)R_7(t) + 2g_3(t)R_8(t) + R_7^2(t) + R_8^2(t) \\ &= \frac{1}{\pi}|b|^{-1/2}(1+t)^{-1/2} + O(t^{-1}), \end{aligned}$$

or

$$\lim_{t\to\infty} \frac{r_7^2(t) + r_8^2(t)}{t^{-1/2}} = \frac{1}{\pi}|b|^{-1/2}. \tag{6.8.37}$$

Now, we return to estimate the asymptotic behaviour of $\text{Var}[X(t)]$ in (6.8.33) using the estimates established above. We start by rewriting the identity (6.8.33) according to

$$\begin{aligned} \frac{1}{\sigma^2 t}\text{Var}[X(t)] = \frac{r_7(t)^2}{t^{-1/2}} &\left(\frac{\int_0^t d_7(s)^2\,ds}{t^{3/2}} - \frac{\pi|b|^{1/2}}{3} \right) \\ &+ 2\frac{r_7(t)}{t^{-1/4}}\frac{r_8(t)}{t^{-1/4}}\frac{\int_0^t d_7(s)d_8(s)\,ds}{t} \cdot \frac{1}{t^{1/2}} \\ &+ \frac{r_8(t)^2}{t^{-1/2}}\left(\frac{\int_0^t d_8(s)^2\,ds}{t^{3/2}} - \frac{\pi|b|^{1/2}}{3} \right) + \frac{r_7(t)^2 + r_8^2(t)}{t^{-1/2}} \cdot \frac{\pi|b|^{1/2}}{3}. \end{aligned}$$

Since $r_7(t) = O(t^{-1/4})$ and $r_8(t) = 0(t^{-1/4})$, by (6.8.34) and (6.8.35), the first and third terms on the right hand side have each limit zero as $t \to \infty$. Using these estimates on r_7 and r_8, along with (6.8.36), confirms that the second term has zero limit as $t \to \infty$. The fourth term has limit $1/3$ as $t \to \infty$, by (6.8.37), and therefore we have

$$\lim_{t\to\infty} \frac{\text{Var}[X(t)]}{t} = \frac{1}{3}\sigma^2,$$

as claimed. □

Long Memory and Asymptotic Behaviour in an Affine Stochastic Difference Equation with an Average Functional

7.1 Introduction

In this chapter, we consider the asymptotic behaviour of an affine scalar stochastic functional difference equation where the average of the process over its entire history appears on the right hand side. Accordingly, we study

$$X(n+1) = \alpha X(n) + \frac{\beta}{n+1}\sum_{j=0}^{n} X(j) + \sigma\xi(n+1), \quad n \in \{0, 1, 2, ...\} \tag{7.1.1a}$$

$$X(0) = x_0 \in \mathbb{R}, \tag{7.1.1b}$$

where X is given by the known value x_0 at time $n = 0$, and $\xi = \{\xi(n)\}_{n=1}^{\infty}$ is a sequence of independent and identically distributed random variables with zero mean and unit variance. There is a unique strong solution of (7.1.3) which is a Gaussian process in the case that the ξ's are normally distributed.

The motivation for studying (7.1.1) is that it may be viewed as a discretisation or numerical method of the continuous time equation which was analysed in Chapters 5 and 6. It is argued in Chapter 4 that this continuous equation may be viewed as an inefficient market model. As we are largely interested in the long–run behaviour of the process, we ask then whether the asymptotic properties of the continuous process are preserved under the discretisation. While the analysis in Chapter 5 was chiefly conducted using admissibility theory, c.f. e.g. [13], the methods of this chapter decompose the solution of (7.1.1) into martingales, and then uses the asymptotic theory of discrete time martingales [16, 34, 115]. Use of martingale techniques to analyse the asymptotic behaviour in the continuous case appears in Chapter 6.

An important step in analysing the asymptotic behaviour of (7.1.1a) is to understand the asymptotic behaviour of the linear deterministic equation underlying (7.1.1a), and in particular the fundamental solution or resolvent of this deterministic equation. It can be readily shown that the resolvent obeys a second–order linear difference equation with analytic coefficients. Therefore, we can deduce the asymptotic behaviour of the resolvent entirely using the Birkhoff–Adams Theorem, [46, Theorem 8.36]. Using a variation of parameters representation, the solution of the stochastic equation, X, can be expressed in terms of the resolvent and the noise sequence. Using results for the convergence of sums of random variables and a law of the iterated logarithm–like result we are able to characterise the asymptotic behaviour of the stochastic equation for almost all values of its parameters (α, β). While the asymptotic results in this chapter are shown to hold almost surely this mode of convergence is shown to imply convergence in mean square.

Of particular note is the regime when $|\alpha| < 1$, $\alpha + \beta < 1$. In this parameter region the stochastic process undergoes large fluctuations and moreover is asymptotically equal to a short memory process, in spite of the fact that the process has long memory characteristics. We also demonstrate that solutions can grow polynomially or exponentially fast in other parameter regimes. On the boundaries of these parameter regimes the solutions exhibit a variety of behaviours, including non–stationary fluctuations, growth which is neither exponential nor polynomial, and these results are also recorded.

The chapter is organised as follows: the equations to be analysed are introduced in Section 7.1.1, together with notation. Section 7.2 details pathwise recurrent dynamics of X and also memory (or autocovariance function) properties of X. Section 7.3 looks at parameter regions where the process undergoes growth, while Section 7.4 completes the asymptotic analysis of X by looking at parameter regions not considered in Sections 7.2 and 7.3. It shown in Section 7.5 that the almost sure asymptotic results of Sections 7.2, 7.3 and 7.4 also hold in mean square. Section 7.6 discusses how (7.1.1a) may be viewed as a discretisation of the continuous equation looked at in Chapters 5 and 6 and compares the asymptotic results which arise from these equations. The proofs are deferred to Section 7.7 and subsequent sections.

7.1.1 Preliminaries

Asymptotic expansions or asymptotic power series are defined in the usual way (cf. e.g., [26]). The power series $\sum_{n=0}^{\infty} a_n t^{-n}$ is said to be asymptotic to the function $y(t)$ as $t \to \infty$ and we write $y(t) \sim \sum_{n=0}^{\infty} a_n t^{-n}$ as $t \to \infty$ if

$$y(t) - \sum_{n=0}^{N} a_n t^{-n} = o(t^{-N}), \qquad \text{as } t \to \infty, \text{ for every } N.$$

We note that it will be clear from our workings in which sense the symbol "$\sim$" which denotes asymptotic equivalence, is being used.

We now turn to introducing precisely the average functional process. Let $\sigma > 0$, $\alpha, \beta \in \mathbb{R}$ and suppose the stochastic process $\xi = \{\xi(n) : n \in \mathbb{Z}^+/\{0\}\}$ is a sequence of independent Gaussian random variables such that

$$\mathbb{E}[\xi(n)] = 0, \quad \mathrm{Var}[\xi(n)] = 1, \quad n \geq 1, \quad \mathrm{Cov}(\xi(n), \xi(m)) = 0 \text{ for all } n \neq m. \tag{7.1.2}$$

This is a standing assumption throughout the chapter, and is not always given as a hypothesis in the statement of our main results. We consider the affine stochastic functional difference equation with an average functional given by

$$X(n+1) = \alpha X(n) + \frac{\beta}{n+1} \sum_{j=0}^{n} X(j) + \sigma \xi(n+1), \quad n \in \{0, 1, 2, ...\} \tag{7.1.3a}$$

$$X(0) = x_0 \in \mathbb{R}, \tag{7.1.3b}$$

where x_0 is a deterministic constant. There exists a unique solution of (7.1.3), which may be found via iteration. There also exists a unique solution of the associated deterministic equation

$$x(n+1) = \alpha x(n) + \frac{\beta}{n+1} \sum_{j=0}^{n} x(j), \quad n \in \mathbb{Z}^+, \tag{7.1.4a}$$

$$x(0) = x_0 \in \mathbb{R}. \tag{7.1.4b}$$

In order to obtain a variation of parameters representaion of the solution of (7.1.3), we define the difference–resolvent r associated with (7.1.4) according to

$$r(n+1, m) = \alpha\, r(n, m) + \frac{\beta}{n+1} \sum_{j=m}^{n} r(j, m), \quad 0 \leq m \leq n, \tag{7.1.5a}$$

$$r(n, m) = 0, \quad n < m, \qquad r(n, n) = 1. \tag{7.1.5b}$$

For each fixed m, a sequence $r(\cdot, m) = \{r(n, m) : n \in \mathbb{Z}^+\}$ which obeys (7.1.5) exists and is uniquely defined. Then with x being the solution of (7.1.4), the solution of (7.1.3) obeys a variation of parameters representation.

Lemma 7.1.1. *Let X be the unique solution of* (7.1.3)*, x be the unique solution of* (7.1.4) *and r be the unique solution of* (7.1.5)*. Then X obeys*

$$X(n) = x(n) + \sigma \sum_{m=1}^{n} r(n, m) \xi(m), \quad n \in \{1, 2, \dots\} \tag{7.1.6}$$

Note that x_0 is deterministic and hence uncorrelated with $\{\xi(n)\}_{n\geq 1}$. Thus we have

$$\mathbf{E}[X(n)] = x(n), \quad n \geq 0.$$

Moreover, as ξ is a white noise process, from (7.1.6) we have

$$\mathrm{Cov}(X(n), X(n+k)) = \sigma^2 \sum_{m=1}^{n} r(n,m)r(n+k,m), \quad n \geq 1, \quad k \geq 0. \tag{7.1.7}$$

We have already seen that the mean and resolvent obey difference equations involving an average functional. This also holds true for the autocovariance function, and the result is recorded below. It may be thought of as a type of Yule–Walker equation.

Proposition 7.1.1. *Define*

$$\gamma_n(k) := \sigma^2 \sum_{m=1}^{n} r(n,m)r(n+k,m), \quad n \geq 1,\, k \geq -n.$$

If $n \geq 1$ *and* $k \geq 0$ *then* $\gamma_n(k) = \mathrm{Cov}(X(n), X(n+k))$ *and,*

$$\gamma_n(k+1) = \alpha\gamma_n(k) + \frac{\beta}{k+n+1}\sum_{j=-n}^{k}\gamma_n(j), \quad \text{for } n \geq 1, \quad k \geq 0 \tag{7.1.8}$$

$$\gamma_n(0) = \alpha\gamma_n(-1) + \frac{\beta}{n}\sum_{j=-n}^{-1}\gamma_n(j) + \sigma^2, \quad \text{for } n \geq 1 \tag{7.1.9}$$

$$\gamma_0(k) = 0, \quad \text{for all } k \geq 0 \tag{7.1.10}$$

$$\gamma_n(k+1) = \alpha\gamma_n(k) + \frac{\beta}{n}\sum_{j=-n-k-1}^{-k-2}\gamma_n(j), \quad \text{for } n \geq 1, \quad -n \leq k \leq -2.$$

Remark 7.1.1. Due to x_0 being deterministic we get (7.1.10). Indeed we note then that $\gamma_n(-n) = 0$ and hence for $n + k \geq 1$, (7.1.8) becomes

$$\gamma_n(k+1) = \alpha\gamma_n(k) + \frac{\beta}{k+n+1}\sum_{j=1-n}^{k}\gamma_n(j), \quad \text{for } n \geq 1, \quad k \geq 0 \tag{7.1.11}$$

The extra term at the end of (7.1.9) illustrates the Yule-Walker nature of the autocovariance function.

7.1.2 Asymptotic behaviour of solution of (7.1.4).

The solution of (7.1.4) can be rewritten as the solution of the second order linear difference equation

$$x(n+2) - \left(1 + \alpha + \frac{\beta - 1}{n+2}\right)x(n+1) + \alpha\left(1 - \frac{1}{n+2}\right)x(n) = 0, \quad n \geq 0, \tag{7.1.12a}$$

$$x(0) = x_0, \quad x(1) = (\alpha + \beta)x_0, \tag{7.1.12b}$$

and as we will shortly show, the resolvent r defined by (7.1.5) solves the same second order difference equation (7.1.12a) on a subset of the positive integers. Since it is reasonable to suppose that the asymptotic behaviour of X depends on the asymptotic behaviour of both r and x, a study of this behaviour is now presented.

To do this, we appeal to a general result concerning the asymptotic behaviour of second order linear difference equations with time varying coefficients, which can be applied to the solution of (7.1.12a). It is generally referred to as the Birkhoff–Adams Theorem [46, 117]. This theorem is used in the study of the asymptotics of a discrete Schrödinger equation (which is a fourth order linear difference equation) in [23] and in [109] to characterise the spectral structure of a particular linear difference operator. Other methods exist for determining the asymptotics of linear difference equations (or recurrence relations) c.f. e.g. Chapter 8 of [46], while a tutorial on asymptotics of linear difference equations is given in Wimp and Zeilberger [116]. A closed form solution of a second order linear recurrence equation is given in Mallik [79] while [80] develops an explicit solution of unbounded (or Volterra) and higher order linear recurrence equations.

We cite as much of the Birkhoff–Adams Theorem as is used in this work.

Theorem 7.1.1 (Birkhoff–Adams)**.** *Consider the linear difference equation*

$$y(n+2)+p_1(n)y(n+1)+p_2(n)y(n)=0, \tag{7.1.13}$$

where p_1 and p_2 have the asymptotic expansions

$$p_1(n)\sim\sum_{j=0}^{\infty}\frac{a_j}{n^j}\quad p_2(n)\sim\sum_{j=0}^{\infty}\frac{b_j}{n^j},\qquad \text{as } n\to\infty \tag{7.1.14}$$

and $b_0\neq 0$, and let λ_1,λ_2 be the roots of the characteristic equation $\lambda^2+a_0\lambda+b_0=0$.

(i) If $\lambda_1\neq\lambda_2$, then equation (7.1.13) *has two linearly independent solutions, y_1, y_2, whose asymptotic behaviour is described by*

$$y_i(n)=\lambda_i^n n^{\eta_i}\left(1+\frac{c_i(1)}{n}+O\left(\frac{1}{n^2}\right)\right),\quad i=1,2,\quad n\to\infty$$

where $\eta_i=\frac{a_1\lambda_i+b_1}{a_0\lambda_i+2b_0}$, $i=1,2$ and

$$c_i(1)=\frac{-2\lambda_i^2\eta_i(\eta_i-1)-\lambda_i(a_2+\lambda_i a_1+\eta_i(\eta_i-1)a_0/2)-b_2}{2\lambda_i^2(\eta_i-1)+\lambda_i(a_1+(\lambda_i-1)a_0)+b_1}.$$

(ii) If $\lambda_1 = \lambda_2 = \lambda$, but $a_1\lambda + b_1 \neq 0$, then equation (7.1.13) *has two linearly independent solutions, y_1, y_2, whose asymptotic behaviour is described by*

$$y_i(n) = \lambda^n e^{\varpi_i \sqrt{n}} n^\eta \left(1 + \frac{c_i}{\sqrt{n}} + O\left(\frac{1}{n}\right)\right), \quad i = 1, 2, \quad n \to \infty, \tag{7.1.15}$$

where

$$\eta = \frac{1}{4} + \frac{b_1}{2b_0}, \quad \varpi_1 = 2\sqrt{\frac{a_0 a_1 - 2b_1}{2b_0}}, \quad \varpi_2 = -\varpi_1,$$

and c_1 and c_2 are calculable constants known in terms of a_0, a_1, b_0, b_1, b_2.

We now demonstrate that this result can be applied to (7.1.12a). We first notice that

$$\frac{1}{n+2} = \sum_{j=1}^{\infty} \frac{(-2)^{j-1}}{n^j}, \quad \text{for all } n > 2.$$

Therefore, if we identify

$$p_1(n) = -(1+\alpha) + \frac{1-\beta}{n+2}, \quad p_2(n) = \alpha\left(1 - \frac{1}{n+2}\right),$$

we see that

$$p_1(n) = -(1+\alpha) + \sum_{j=1}^{\infty} \frac{(-2)^{j-1}(1-\beta)}{n^j}, \quad p_2(n) = \alpha + \sum_{j=1}^{\infty} \frac{-\alpha(-2)^{j-1}}{n^j}, \quad n > 2.$$

Therefore p_1 and p_2 obey the asymptotic relations (7.1.14). We identify $a_0 = -(1+\alpha)$, $b_0 = \alpha$, $a_1 = 1 - \beta$ and $b_1 = -\alpha$. We have $b_0 \neq 0$ provided $\alpha \neq 0$.

For the equation (7.1.12a), the associated characteristic equation is $\lambda^2 - (1+\alpha)\lambda + \alpha = 0$. Thus, $\lambda_1 = \alpha$, $\lambda_2 = 1$. Thus if $\alpha \notin \{0, 1\}$, we may apply part (i) of the theorem to the solution of (7.1.12a). If $\alpha = 1$, then $\lambda_1 = \lambda_2 = 1$, and $a_1\lambda_1 + b_1 = -\beta$. Therefore, part (ii) can be applied in the case when $\alpha = 1$ and $\beta \neq 0$. Hence the Birkhoff–Adams theorem can be applied unless $\alpha = 0$, or $\alpha = 1$ and $\beta = 0$ (in the latter case, X collapses to a random walk, while the former case can be treated by a direct approach which is independent of the Birkhoff–Adams theory). In all other cases (7.1.12a) has two linearly independent solutions r_1 and r_2 and therefore the solution of (7.1.12) may be written $x(n) = \tilde{c}_1 r_1(n) + \tilde{c}_2 r_2(n)$, where $\tilde{c}_1, \tilde{c}_2$ are determined from the initial conditions and the initial values of the fundamental solutions r_1 and r_2.

In the cases when there are two linearly independent solutions of (7.1.13), the Casoratian obeys a first order difference equation. It is $C(n+1) = p_2(n)C(n)$ (see e.g., Theorem 2.13 of [46]). This leads to the following result.

Lemma 7.1.2. *The Casoratian of* (7.1.12a) *is given by*

$$C(n) = C(0) \prod_{i=0}^{n-1} \alpha \left(1 - \frac{1}{i+2}\right) = \frac{\alpha^n}{n+1} C(0).$$

7.1.3 $\alpha \notin \{0, 1\}$

Applying the Birkhoff–Adams Theorem to (7.1.12a) gives for $\alpha \notin \{0, 1\}$,

$$r_1(n) = n^{-1-\frac{\beta}{\alpha-1}} \left(1 + \frac{c_1(1)}{n} + O\left(\frac{1}{n^2}\right)\right), \quad n \to \infty, \tag{7.1.16a}$$

$$r_2(n) = \alpha^n n^{\frac{\beta}{\alpha-1}} \left(1 + \frac{c_2(1)}{n} + O\left(\frac{1}{n^2}\right)\right), \quad n \to \infty. \tag{7.1.16b}$$

Furthermore it is observed that for every fixed $m \geq 0$ the solution $n \mapsto r(n, m) := r^{(m)}(n)$ is a solution of the second order difference equation

$$r^{(m)}(n+2) - \left(\alpha + 1 + \frac{\beta - 1}{n+2}\right) r^{(m)}(n+1) + \alpha \left(1 - \frac{1}{n+2}\right) r^{(m)}(n) = 0, \quad n \geq m, \tag{7.1.17}$$

with the initial conditions $r^{(m)}(m) = 1$, $r^{(m)}(m+1) = \alpha + \beta/(m+1)$. This second–order difference equation is the same as (7.1.12a) with the only distinction being that the domain of the solution is now $\{m, m+1, ...\}$ rather than $\mathbb{Z}^+$. Therefore the solution of (7.1.17) may be represented as a linear combination of the fundamental solutions of (7.1.12a), according to

$$r(n, m) = d_1(m) r_1(n) + d_2(m) r_2(n), \quad 0 \leq m \leq n, \tag{7.1.18}$$

where the co-efficients d_1 an d_2 are m-dependent as the initial conditions commence at the value m. Indeed $x(n) = r(n, 0)x_0$. To find expressions for d_1 and d_2 one solves the equations

$$d_1(m) r_1(m) + d_2(m) r_2(m) = 1, \; d_1(m) r_1(m+1) + d_2(m) r_2(m+1) = \alpha + \frac{\beta}{m+1}.$$

This defines $d_1(m)$ and $d_2(m)$ uniquely, due to the linear independence of r_1 and r_2. Therefore

$$d_1(m) = \frac{1}{C(0)} \alpha^{-m} (m+1) \left\{ r_2(m+1) - (\alpha + \frac{\beta}{m+1}) r_2(m) \right\}, \tag{7.1.19a}$$

$$d_2(m) = \frac{1}{C(0)} \alpha^{-m} (m+1) \left\{ -r_1(m+1) + (\alpha + \frac{\beta}{m+1}) r_1(m) \right\} \tag{7.1.19b}$$

and we also have $\tilde{c}_1 = x_0 d_1(0)$ and $\tilde{c}_2 = x_0 d_2(0)$. From the known asymptotic behaviour of r_1 and r_2 it can be deduced that

$$d_1(m) = \frac{\beta}{(\alpha-1)^2} m^{\frac{\beta}{\alpha-1}} [1 + O(m^{-1})], \quad \text{as } m \to \infty, \tag{7.1.20a}$$

$$d_2(m) = \alpha^{-m} m^{-\frac{\beta}{\alpha-1}} [1 + O(m^{-1})], \text{ as } m \to \infty. \tag{7.1.20b}$$

Indeed, it can be shown that $C(0) = \alpha - 1$. These results are established in Lemma 7.7.1.

Remark 7.1.2. Since $\mathrm{Cov}(X(n), X(n+k))$ is given by (7.1.7), and r obeys (7.1.18), we have

$$\mathrm{Cov}(X(n), X(n+k)) = c_{1,n} r_1(n+k) + c_{2,n} r_2(n+k), \tag{7.1.21}$$

where

$$c_{1,n} = \sigma^2 \sum_{m=1}^{n} r(n,m) d_1(m), \quad c_{2,n} = \sigma^2 \sum_{m=1}^{n} r(n,m) d_2(m). \tag{7.1.22}$$

7.1.4 $\alpha = 1$

For the case $\alpha = 1$, $\beta > 0$, if we denote the linearly independent solutions by r_3 and r_4, then Theorem 7.1.1 gives

$$r_3(n) = \mathrm{e}^{2\sqrt{\beta n}} n^{-1/4} \left(1 + \frac{c_1}{\sqrt{n}} + O\left(\frac{1}{n}\right)\right), \quad \text{as } n \to \infty, \tag{7.1.23}$$

$$r_4(n) = \mathrm{e}^{-2\sqrt{\beta n}} n^{-1/4} \left(1 + \frac{c_2}{\sqrt{n}} + O\left(\frac{1}{n}\right)\right), \quad \text{as } n \to \infty. \tag{7.1.24}$$

Therefore r can be written in the form

$$r(n,m) = d_3(m) r_3(n) + d_4(m) r_4(n), \quad 0 \le m \le n. \tag{7.1.25}$$

where d_3 and d_4 are sequences which obey

$$d_3(m) = \frac{1}{C(m)} \left\{ r_4(m+1) - \left(1 + \frac{\beta}{m+1}\right) r_4(m) \right\},$$
$$d_4(m) = \frac{1}{C(m)} \left\{ -r_3(m+1) + \left(1 + \frac{\beta}{m+1}\right) r_3(m) \right\},$$

where the Casoratian is $C(m) = r_4(m+1) r_3(m) - r_3(m+1) r_4(m) \neq 0$. The asymptotic behaviour of d_3 and d_4 are required. We prove in Lemma 7.12.1 that

$$d_3(m) = \frac{1}{2} m^{\frac{1}{4}} \mathrm{e}^{-2\sqrt{\beta m}} \left(1 + O(m^{-1/2})\right), \quad \text{as } m \to \infty, \tag{7.1.26a}$$

$$d_4(m) = \frac{1}{2} m^{\frac{1}{4}} \mathrm{e}^{2\sqrt{\beta m}} \left(1 + O(m^{-1/2})\right), \quad \text{as } m \to \infty. \tag{7.1.26b}$$

It can be seen from a fuller statement of the Birkhoff-Adams Theorem [117] that one may generate recursive formulae for the constant multipliers of lower order terms in the expansions of (i) and (ii) in Theorem 7.1.1. In particular in (ii) of Theorem 7.1.1 when λ, ϖ_1, ϖ_2, η, $\{a_j\}_{j\in\mathbb{Z}^+}$ and $\{b_j\}_{j\in\mathbb{Z}^+}$ are all real–valued then so too are all of the multipliers of the lower order terms in the expansion (7.1.15).

In the case when $\beta < 0$, then $\varpi_1 = 2\sqrt{|\beta|}i = \overline{\varpi_2}$. Following the example outlined on page 70–72 of [117] we have that the multipliers of the lower order terms in the expansion (7.1.15) of one of the linearly independent solutions are complex conjugates of the multipliers of the lower order terms in the expansion (7.1.15) of the other linearly independent solution. Thus it is seen that $r_3(n)$ and $r_4(n)$ are complex conjugates of one another. It is also observed that linear combinations of asymptotic series are also asymptotic series. In this instance one may wish to consider the two linearly independent real valued solutions r_5 and r_6 whose asymptotic behaviour is given by

$$r_5(n) = \cos(2\sqrt{|\beta|n})\, n^{-1/4}\left(1 + \frac{c_1'}{\sqrt{n}} + O(n^{-1})\right), \quad \text{as } n \to \infty \tag{7.1.27a}$$

$$r_6(n) = \sin(2\sqrt{|\beta|n})\, n^{-1/4}\left(1 + \frac{c_2'}{\sqrt{n}} + O(n^{-1})\right), \quad \text{as } n \to \infty. \tag{7.1.27b}$$

Therefore r can be written in the form

$$r(n,m) = d_5(m)r_5(n) + d_6(m)r_6(n), \quad 0 \le m \le n,$$

and with the Casoratian given by $C(m) = r_6(m+1)r_5(m) - r_6(m)r_5(m+1) \neq 0$, we have

$$d_5(m) = \frac{1}{C(m)}\left\{r_6(m+1) - \left(1 + \frac{\beta}{m+1}\right) r_6(m)\right\},$$
$$d_6(m) = \frac{1}{C(m)}\left\{-r_5(m+1) + \left(1 + \frac{\beta}{m+1}\right) r_5(m)\right\}.$$

Then, it is shown in Lemma 7.13.1 that the sequences d_5 and d_6 obey

$$d_5(m) = m^{1/4}\cos(2\sqrt{|\beta|m})\left(1 + O(m^{-1/2})\right), \quad \text{as } m \to \infty \tag{7.1.28}$$

$$d_6(m) = m^{1/4}\sin(2\sqrt{|\beta|m})\left(1 + O(m^{-1/2})\right), \quad \text{as } m \to \infty. \tag{7.1.29}$$

The case when $\alpha = 1$ and $\beta = 0$ is easily dealt with: we have $x(n) = x_0$ for all $n \ge 0$ and $r(n,m) = 1$ for all $n \ge m \ge 0$.

7.1.5 $\alpha = 0$

The case $\alpha = 0$ is dealt with in Lemma 7.7.2. Excluding points where r may be zero for specific values of β, we have that $r(n, m) = D_1(m)R_1(n)$, where

$$R_1(n) = \frac{\Gamma(\beta + n)}{n!} \sim n^{\beta-1}, \quad \text{as } n \to \infty, \tag{7.1.30}$$

$$D_1(m) = \beta \frac{m!}{\Gamma(\beta + m + 1)} \sim \beta m^{-\beta}, \quad \text{as } m \to \infty. \tag{7.1.31}$$

Notice that this agrees with the asymptotic behaviour of r_1 and d_1 in (7.1.16) and (7.1.20) respectively.

7.1.6 Order arithmetic

In ascertaining many of the asymptotic estimates (7.1.20), (7.1.26) and (7.1.28) it is necessary to be able to add and multiply terms of known size.

Lemma 7.1.3. *Let f and g be real–valued seuqences such that*

$$f(n) = O(n^{\alpha_1}) \quad g(n) = O(n^{\alpha_2}), \quad n \to \infty$$

and let $\alpha_1 > \alpha_2$. Then

$$f(n) + g(n) = O(n^{\alpha_1})$$

and

$$f(n)g(n) = O(n^{\alpha_1+\alpha_2}), \quad n \to \infty.$$

The proof of this lemma is immediate from the definition of the Landau notation.

7.2 Recurrent Asymptotic Behaviour

In the case $\beta = 0$, (7.1.3) reduces to an autoregressive process of order one, or AR(1) process, which is a process with well–understood asymptotic behaviour. Thus throughout this chapter, we generally take $\beta \neq 0$.

7.2.1 Pathwise asymptotic stationary behaviour

The discrete analogue of an Ornstein-Uhlenbeck process is an AR(1) process. We define U to be the solution of the autoregressive equation

$$U(n + 1) = \alpha U(n) + \sigma\xi(n + 1), \quad \text{for } n \in \{0, 1, 2, ...\}, \quad U(0) = 0, \tag{7.2.1}$$

where $\{\xi(n) : n \in \mathbb{Z}^+/\{0\}\}$ is the same process as that in equation (7.1.3a). An explicit representation for the solution of (7.2.1) is

$$U(n) = \begin{cases} \alpha^n \sum_{j=1}^n \sigma\alpha^{-j}\xi(j), & n \geq 1, \quad \alpha \neq 0, \\ \sigma\xi(n), & n \geq 1, \quad \alpha = 0. \end{cases} \tag{7.2.2}$$

Theorem 7.2.1. *Let $\alpha \in (-1, 1)$ and $\alpha + \beta \leq 1$. Let X be the unique solution of (7.1.3) and let U be the unique solution of (7.2.1). Suppose also that $(\xi(n))_{n\geq 1}$ obeys (7.1.2) and is a Gaussian process.*

(i) X obeys

$$\limsup_{n\to\infty} \frac{X(n)}{\sqrt{2\log n}} = \frac{\sigma}{\sqrt{1-\alpha^2}}, \quad \liminf_{n\to\infty} \frac{X(n)}{\sqrt{2\log n}} = -\frac{\sigma}{\sqrt{1-\alpha^2}}, \quad a.s. \tag{7.2.3}$$

(ii) In the case $\alpha + \beta < 1$, we have

$$\lim_{n\to\infty} \{X(n) - U(n)\} = 0, \quad a.s. \tag{7.2.4}$$

and that

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} X(j) = 0, \quad a.s. \tag{7.2.5}$$

(iii) In the case $\alpha + \beta = 1$, we have

$$\lim_{n\to\infty} \{X(n) - U(n)\} = L, \quad a.s., \tag{7.2.6}$$

where L is the proper Gaussian random variable given by

$$L = \sum_{m=1}^{\infty} \frac{\beta}{m} d_1(m) \sum_{j=0}^{m-1} \alpha^j x_0 + \sigma \sum_{l=1}^{\infty} \left(\sum_{m=l+1}^{\infty} \frac{\beta}{m} d_1(m) \sum_{j=l}^{m-1} \alpha^j \right) \alpha^{-l}\xi(l),$$

for $\alpha \neq 0$ and

$$L = x_0 + \sigma \sum_{m=1}^{\infty} \frac{1}{m+1}\xi(m),$$

for $\alpha = 0$.

(iv) If L is as defined in part (iii), for $\alpha \neq 0$ we have

$$\mathbb{E}[L] = \sum_{m=1}^{\infty} \frac{\beta}{m} d_1(m) \sum_{j=0}^{m-1} \alpha^j x_0,$$

$$Var[L] = \sigma^2 \sum_{l=1}^{\infty} \left(\sum_{m=l+1}^{\infty} \frac{\beta}{m} d_1(m) \sum_{j=l}^{m-1} \alpha^{j-l} \right)^2,$$

for $\alpha = 0$ we have

$$\mathbb{E}[L] = x_0, \quad Var[L] = \sigma^2 \sum_{m=1}^{\infty} \frac{1}{(m+1)^2}.$$

and that

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} X(j) = L, \quad a.s. \tag{7.2.7}$$

We note that the case $\alpha = 0$ is proven in a slightly different manner to the case when $\alpha \neq 0$. This is because when $\alpha = 0$ the second order ordinary difference equation reduces to a first order equation, and one should not expect to apply the Birkhoff-Adams asymptotic theory of second order equations. Nonetheless, the results are of the same form.

Having established (7.2.4), i.e. that $X(n) - U(n)$ tends to zero as $n \to \infty$, it is interesting to ask at what rate this convergence occurs. We provide an upper bound on this rate of decay.

Theorem 7.2.2. *Let $\alpha \in (-1,1)/\{0\}$. Let X be the unique solution of* (7.1.3) *and let U be the unique solution of* (7.2.1)*. Suppose also that $(\xi(n))_{n\geq 1}$ obeys* (7.1.2) *and is a Gaussian process. Then*

(i) If $\alpha + \beta < 1$ and $\alpha + 2\beta > 1$, then

$$\limsup_{n\to\infty} \frac{|X(n) - U(n)|}{n^{-1-\frac{\beta}{\alpha-1}}} \in [0,\infty), \quad a.s..$$

(ii) If $\alpha + 2\beta < 1$, then

$$\limsup_{n\to\infty} \frac{|X(n) - U(n)|}{n^{-1/2}\sqrt{\log\log n}} \in [0,\infty), \quad a.s..$$

(iii) If $\alpha + 2\beta = 1$, then

$$\limsup_{n\to\infty} \frac{|X(n) - U(n)|}{n^{-1/2}\log n\sqrt{\log\log n}} \in [0,\infty), \quad a.s..$$

We conjecture that the decay rates in Theorem 7.2.2 are sharp, i.e. that the limiting values are non–zero, but a proof of this conjecture lies beyond the scope of this thesis. We do remark however that when $\alpha = 0$ the variation of parameters formula for X is simpler than when $\alpha \neq 0$ and one can show that the rates of decay are indeed sharp in this case (although the $\alpha + 2\beta = 1$ seems to be somewhat different).

7.2.2 Asymptotic behaviour of the autocovariance function

The autocovariance function of U is given, for $k \geq 0$, by

$$\text{Cov}(U(n), U(n+k)) = \frac{\sigma^2}{1-\alpha^2}\,\alpha^k(1-\alpha^{2(n+1)}), \quad n \geq 1.$$

Therefore, for each fixed $n > 0$ we have $k \mapsto \text{Cov}(U(n), U(n+k))$ decays *exponentially* to zero as $k \to \infty$. Thus, as X and U are asymptotically equal to one another it might be expected that the autocovariance function of X decays toward zero and does so 'quickly'.

As the forthcoming Theorems 7.2.3 and 7.2.5 will demonstrate, when analysing the memory properties of X, the order in which one takes limits in n and k to infinity is of crucial importance. We start by demonstrating that X possesses a polynomial decay in its autocovariance function, which, in the case when $\beta > 0$ is consistent with a long memory process.

Theorem 7.2.3. *Let $\alpha \in (-1, 1)/\{0\}$. Let $n \geq 1$. Then,*

$$\lim_{k\to\infty} \frac{\text{Cov}(X(n), X(n+k))}{k^{-1-\frac{\beta}{\alpha-1}}} = c_{1,n},$$

where $c_{1,n}$ is given by (7.1.22).

This result follows immediately from Remark 7.1.2. In the case when $\alpha = 0$ and $\beta \notin \{-1, -2, \ldots\}$ the same result holds with $c_{1,n} = \sigma^2 R_1(n)\sum_{m=1}^{n} D_1(m)^2$. Hence the process X defined by (7.1.3) is a *long memory* process when $0 < |\alpha| < 1$, $\beta > 0$ and $\alpha + \beta < 1$. This is of course providing that the limit obtained in Theorem 7.2.3 is indeed non-zero. Indeed, it is difficult to determine the value of $c_{1,n}$, for any n, due to the fact that Theorem 7.1.1 gives only asymptotic information on r_1 and r_2 and in particular does not specify initial values for these sequences. Proposition 7.2.1 somewhat addresses this question.

Proposition 7.2.1. *Let $\alpha \in (-1, 1)/\{0\}$ and $\alpha + \beta \leq 1$. Let $c_{1,n}$ be given by* (7.1.22).

(a) If $\beta < 0$ and $\alpha + \beta < 1$, then

$$\lim_{n\to\infty} \frac{c_{1,n}}{n^{\frac{\beta}{\alpha-1}}} = \sigma^2 \frac{\beta}{(1-\alpha)^3}\left(\frac{1-\alpha-\beta}{1-\alpha-2\beta}\right) < 0. \quad (7.2.8)$$

(b) (i) If $\alpha + \beta < 1$ and $\alpha + 2\beta > 1$, then

$$\lim_{n\to\infty} \frac{c_{1,n}}{n^{-1-\frac{\beta}{\alpha-1}}} = \sigma^2 \sum_{m=1}^{\infty} d_1(m)^2 \in (0, \infty).$$

(ii) If $\alpha + 2\beta = 1$, then

$$\lim_{n\to\infty} \frac{c_{1,n}}{n^{-1/2}\log n} = \frac{1}{4}\sigma^2 \frac{1}{(\alpha-1)^2} > 0.$$

(iii) If $\beta > 0$ and $\alpha + 2\beta < 1$, then $c_{1,n}$ obeys (7.2.8) *with the limit on the righthand side being positive.*

(c) If $\alpha + \beta = 1$, then

$$\lim_{n\to\infty} c_{1,n} = \sigma^2 \sum_{m=1}^{\infty} d_1(m)^2 \in (0,\infty).$$

In the case when $\alpha = 0$ then $c_{1,n}$ is given by $c_{1,n} = \sigma^2 R_1(n) \sum_{m=1}^{n} D_1(m)^2$ for $\beta \notin \{-1,-2,\ldots\}$, while $c_{1,n}$ is given by $c_{1,n} = \sigma^2 R_1(n) \sum_{m=-\beta}^{n} D_1(m)^2$ for $\beta \in \{-1,-2,\ldots\}$ and n large enough. In this case it can be shown that $c_{1,n}$ has the same rates of growth or decay, as $n \to \infty$. It can be seen in the formulae below that these constants in the special case $\alpha = 0$ agree with the results obtained by substituting $\alpha = 0$ into the formulae in Proposition 7.2.1. Specifically, we have

$$\begin{aligned}
c_{1,n} &\sim \sigma^2 \frac{\beta(1-\beta)}{1-2\beta} n^{-\beta}, \quad \text{as } n\to\infty, \text{ if } 2\beta < 1 \\
c_{1,n} &\sim \sigma^2 \frac{1}{4} n^{-1/2}\log n, \quad \text{as } n\to\infty, \text{ if } 2\beta = 1 \\
c_{1,n} &\sim \sigma^2 n^{\beta-1} \sum_{m=1}^{\infty} D_1(m)^2, \quad \text{as } n\to\infty, \text{ if } 2\beta > 1.
\end{aligned}$$

In all the parameter regimes considered in Proposition 7.2.1 in which $\beta > 0$ it is seen that $c_{1,n} \to 0$ as $n \to \infty$. Moreover, $c_{1,n}$ is seen to be asymptotic to a *positive* function. Thus for any fixed value of n large enough the value of $c_{1,n}$ is positive and hence Theorem 7.2.3 describes the correct rate of decay of the autocovariance function. In the cases when $\beta < 0$ we still see that $c_{1,n}$ is non–trivial for all n sufficiently large, so therefore, once again, Theorem 7.2.3 appears to identify the correct rate of decay of the autocovariance function.

We give a result concerning the positivity of the autocovariance function.

Theorem 7.2.4. *Let $\alpha \geq 0$ and $\beta > 0$. Then* $\text{Cov}(X(n), X(n+k)) > 0$ *for $n \geq 1$, $k \geq 0$.*

However, if one lets the starting-time n tend to infinity first and then considers the autocovariance function as a function solely of the time-lag k then the process is observed to have short-memory of the same form as that of U.

Theorem 7.2.5. *Let $\alpha \in (-1,1)/\{0\}$.*

(i) If $\alpha + \beta < 1$, then

$$\lim_{n\to\infty} Cov(X(n), X(n+k)) = \frac{\sigma^2}{1-\alpha^2}\alpha^k, \quad k \geq 0.$$

(ii) If $\alpha + \beta = 1$, then

$$\lim_{n\to\infty} Cov(X(n), X(n+k)) = \sigma^2 \sum_{m=1}^{\infty} d_1(m)^2 + \frac{\sigma^2}{1-\alpha^2}\alpha^k, \quad k \geq 0.$$

Once again in the case when $\alpha = 0$ we have essentially identical results. Part (i) holds for $\beta < 1$ as $\lim_{n\to\infty} \text{Cov}(X(n), X(n+k)) = 0$ for $k \geq 1$ and $\lim_{n\to\infty} \text{Cov}(X(n), X(n)) = \sigma^2$. When $\beta = 1$ we have that

$$\lim_{n\to\infty} \text{Cov}(X(n), X(n+k)) = \sigma^2 \sum_{m=1}^{\infty} D_1(m)^2, \quad k \geq 0.$$

We notice that

$$\lim_{n\to\infty} \text{Cov}(U(n), U(n+k)) = \frac{\sigma^2}{1-\alpha^2}\,\alpha^k,$$

so not only is X pathwise asymptotic to U, but both X and U have the same limiting autocovariance function, despite the long memory characteristics that X exhibits.

As observed for the autocovariance function of the continuous analogue of (7.1.3), in contrast to many non–autonomous equations one cannot permute the order in which the limits $n \to \infty$ and $k \to \infty$ are taken. It is noted however that the order in which the limits are taken does not yield conflicting results when $\alpha + \beta = 1$ (and also trivially when $\beta = 0$).

7.3 Transient Asymptotic Behaviour

The stochastic process X undergoes polynomial asymptotic growth where the exact rate of growth is inherited from the deterministic equation. Whether the process grows to plus infinity or decays to minus infinity depends upon the sample path.

Theorem 7.3.1. *Let $\alpha \in (-1,1)$ and $\alpha + \beta > 1$. Let X be the unique solution of the stochastic difference equation* (7.1.3)*. Suppose also that $(\xi(n))_{n\geq 1}$ obeys* (7.1.2) *and is a Gaussian process.*

(a) *There is a Gaussian* $\mathcal{F}^\xi(\infty)$ *measurable non–trivial normal random variable such that*

$$\lim_{n\to\infty} \frac{X(n)}{n^{-1-\frac{\beta}{\alpha-1}}} = C, \quad a.s.$$

(b) *If* $\alpha \neq 0$ *then*

$$C = x_0 d_1(0) + \sigma \sum_{j=1}^{\infty} d_1(j)\xi(j),$$

while $\alpha = 0$ *implies*

$$C = x_0 \frac{\beta}{\Gamma(\beta+1)} + \sigma \sum_{m=1}^{\infty} \frac{m!}{\Gamma(\beta+m+1)} \xi(m).$$

(c) *The mean and variance of* X *obey*

$$\lim_{n\to\infty} \frac{\mathbb{E}[X(n)]}{n^{-1-\frac{\beta}{\alpha-1}}} = \mathbb{E}[C], \quad \lim_{n\to\infty} \frac{Var[X(n)]}{n^{-2-2\frac{\beta}{\alpha-1}}} = Var[C] > 0.$$

When the solution of the deterministic equation undergoes exponential growth then so too does the solution of the stochastic equation. This exponential growth is tempered by a polynomial factor.

Theorem 7.3.2. *Let* $|\alpha| > 1$*. Let* X *be the unique solution of the stochastic difference equation* (7.1.3)*. Suppose also that* $(\xi(n))_{n\geq 1}$ *obeys* (7.1.2) *and is a Gaussian process. Then*

(a) *There is a Gaussian* $\mathcal{F}^\xi(\infty)$ *measurable non–trivial normal random variable such that*

$$\lim_{n\to\infty} \frac{X(n)}{\alpha^n n^{\frac{\beta}{\alpha-1}}} = C, \quad a.s.$$

(b) C *is given by*

$$C = x_0 d_2(0) + \sigma \sum_{j=1}^{\infty} d_2(j)\xi(j).$$

(c) *The mean and variance of* X *obey*

$$\lim_{n\to\infty} \frac{\mathbb{E}[X(n)]}{\alpha^n n^{\frac{\beta}{\alpha-1}}} = x_0 d_2(0), \quad \lim_{n\to\infty} \frac{Var[X(n)]}{\alpha^{2n} n^{2\frac{\beta}{\alpha-1}}} = \sigma^2 \sum_{j=1}^{\infty} d_2(j)^2 > 0.$$

We further remark that when $\alpha < -1$, α^n alternates between being positive and negative as n increases. Therefore, one could state the above result as

$$\lim_{m\to\infty} \frac{X(2m)}{|\alpha|^{2m}(2m)^{\frac{\beta}{\alpha-1}}} = -\lim_{m\to\infty} \frac{X(2m+1)}{|\alpha|^{2m+1}(2m+1)^{\frac{\beta}{\alpha-1}}} = x_0 d_2(0) + \sigma \sum_{j=1}^{\infty} d_2(j)\xi(j).$$

Remark 7.3.1. The positivity of the variance of the almost sure limits in Theorems 7.3.1 and 7.3.2 gives that the limiting random variables are non-zero almost surely. This observation allows us to conclude that the rates of growth of X in both Theorems 7.3.1 and 7.3.2 are exact.

7.4 Boundary Cases

In this section, we detail the dynamics of the stochastic equation for the cases $\alpha = \{-1, 1\}$. These values of α form boundaries between the open regions $|\alpha| < 1$ and $|\alpha| > 1$ in the (α,β) parameter space. We know that solutions grow exponentially in the region $|\alpha| > 1$. For $0 < \alpha < 1$ and $\beta > 0$ we have that solutions grow polynomially, provided that α is sufficiently close to unity; also, we see that the solution fluctuates for $0 < \alpha < 1$ and $\beta < 0$. Therefore, there is a change in the asymptotic behaviour when α crosses the boundary $\alpha = 1$. Similarly, we can see that there is a change in the asymptotic behaviour when α crosses the boundary $\alpha = -1$. Therefore, it is a natural question to ask whether the asymptotic behaviour of the solution on these boundaries may be analogous to that of one of the adjacent open regions or whether it may, in some fashion, have dynamics which are intermediate to those in the adjacent regions, or dynamics which are entirely unrelated to the behaviour in the bordering regions.

We first consider the case $\alpha = -1$. Here the asymptotic behaviour of the deterministic equation is known, (7.1.16), via the Birkhoff–Adams Theorem. However the asymptotic behaviour of r_1 fails to be geometric thus precluding the method of proof of Theorems 7.2.1, 7.3.1 and 7.3.2.

Theorem 7.4.1. *Let $\alpha = -1$. Let X be the unique solution of the stochastic difference equation* (7.1.3)*.*

(i) If $\beta > 3$, then

$$\lim_{n\to\infty} \frac{X(n)}{n^{-1+\beta/2}} = x_0 d_1(0) + \sigma \sum_{j=1}^{\infty} d_1(j)\xi(j), \quad a.s.$$

(ii) If $-1 < \beta \leq 3$, then

$$\limsup_{n\to\infty} \frac{X(n)}{\sqrt{2n\log\log n}} = \sigma\frac{1}{\sqrt{\beta+1}}, \quad \liminf_{n\to\infty} \frac{X(n)}{\sqrt{2n\log\log n}} = -\sigma\frac{1}{\sqrt{\beta+1}}, \quad a.s.$$

(iii) If $\beta = -1$, then

$$\limsup_{n\to\infty} \frac{X(n)}{\sqrt{2n\log n \log\log\log n}} = \sigma, \quad \liminf_{n\to\infty} \frac{X(n)}{\sqrt{2n\log n \log\log\log n}} = -\sigma, \quad a.s.$$

(iv) If $\beta < -1$, then

$$\lim_{n\to\infty} \frac{(-1)^n X(n)}{n^{-\beta/2}} = x_0 d_2(0) + \sigma \sum_{j=1}^{\infty} d_2(j)\xi(j), \quad a.s.$$

It is observed here that when $\alpha = -1$, X inherits the asymptotic behaviour of all three open regions which border it.

The case $\alpha = 1$ is a special case of the Birkhoff–Adams Theorem which gives faster than polynomial growth when $\beta > 0$ and damped fluctuations when $\beta < 0$. This asymptotic behaviour propagates through to the stochastic equation.

Theorem 7.4.2. *Let $\alpha = 1$. Let X be the unique solution of the stochastic difference equation* (7.1.3)*.*

(i) If $\beta > 0$, then

$$\lim_{n\to\infty} \frac{X(n)}{e^{2\sqrt{\beta n}} n^{-1/4}} = x_0 d_1(0) + \sum_{j=1}^{\infty} d_1(j)\xi(j), \quad a.s.$$

(ii) If $\beta < 0$, then

$$-\sigma\frac{2}{\sqrt{3}} \le \liminf_{n\to\infty} \frac{X(n)}{\sqrt{2n\log\log n}} \le \limsup_{n\to\infty} \frac{X(n)}{\sqrt{2n\log\log n}} \le \sigma\frac{2}{\sqrt{3}}, \quad a.s.$$

(iii) If $\beta = 0$, then

$$\limsup_{n\to\infty} \frac{X(n)}{\sqrt{2n\log\log n}} = -\liminf_{n\to\infty} \frac{X(n)}{\sqrt{2n\log\log n}} = \sigma, \quad a.s.$$

We note that for $\beta > 0$ the two bordering open regions in the parameter space exhibit polynomial and exponential growth, so the asymptotic behaviour on the boundary is intermediate to that seen on either side of the boundary. When $\beta < 0$, there is a transition from stationary fluctuations ($\alpha < 1$) to exponential growth ($\alpha > 1$); the boundary behaviour seems more consistent with non–stationary fluctuations.

Remark 7.4.1. The limiting result of Theorem 7.4.2 (ii) is reminiscent of the Law of the Iterated Logarithm. Moreover we conjecture that the more precise statement

$$-\sigma\frac{1}{\sqrt{3}} = \liminf_{n\to\infty} \frac{X(n)}{\sqrt{2n\log\log n}} \le \limsup_{n\to\infty} \frac{X(n)}{\sqrt{2n\log\log n}} = \sigma\frac{1}{\sqrt{3}}, \quad \text{a.s.}$$

holds. This conjecture is reliant upon two observations. Firstly, it is argued in Section 7.6 that (7.1.3) may serve as a discretisation of a continuous time equation. With (7.1.3) regarded in this way it is seen that Theorems 7.2.1, 7.3.1, 7.3.2 and 7.4.2 (i) identify corresponding regions of qualitatively analogous asymptotic behaviour to that of the continuous problem and moreover the rates of growth and decay of X in continuous time are analogous to those in discrete time, i.e. fluctuations mirror fluctuations, polynomial growth mirrors polynomial growth (with the exponents matching), and exponential growth mirrors exponential growth. Thus it appears that the discretisation is robust in describing asymptotic behaviour between continuous and discrete processes for almost the entire half-plane ($\alpha > 0$). Thus, as the continuous region corresponding to $\alpha = 1$, $\beta < 0$ (see Theorem 6.3.6) undergoes exact Law of the Iterated Logarithm–like fluctuations, one expects this to also appear in Theorem 7.4.2 (ii).

Secondly, we identify two parts to the proof of Theorem 7.4.2 (ii) where estimations have been made which are not optimal. If these estimates could be improved then one should be able to markedly improve the result of Theorem 7.4.2 (ii). We expand on this point after the proof of Theorem 7.4.2 (ii). Moreover if these difficulties are overcome one may prove a discrete result analogous to Theorem 6.3.5.

7.5 Almost sure convergence implies convergence in mean square

While all the Theorems outlined so far in this chapter give almost sure pathwise results for the solution of (7.1.3) it is interesting to ask whether other modes of convergence can also be obtained. We commence with a result which characterises necessary and sufficient conditions for mean square convergence

Theorem 7.5.1. *Let the function H obey*

$$H : \{(n, j) : 0 \le j \le n,\ n, j \in \mathbb{Z}^+\} \to \mathbb{R}.$$

Let $\{\xi(n)\}_{n\in\mathbb{Z}^+}$ be an i.i.d. sequence of standard Gaussian random variables. Then the statements

(A) There exists $H_\infty : \mathbb{Z}^+ \to \mathbb{R}$ such that $H_\infty \in \ell^2(\mathbb{Z}^+)$ and

$$\lim_{n\to\infty} \sum_{j=0}^{n} (H(n, j) - H_\infty(j))^2 = 0. \tag{7.5.1}$$

(B) There exists $H_\infty : \mathbb{Z}^+ \to \mathbb{R}$ such that

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\sum_{j=0}^{n} H(n,j)\xi(j) - \sum_{j=0}^{\infty} H_\infty(j)\xi(j)\right)^2\right] = 0 \tag{7.5.2}$$

are equivalent.

Theorem 7.5.2. *Suppose that H obeys* (7.5.1) *and there exists $H_\infty : \mathbb{Z}^+ \to \mathbb{R}$ such that*

$$\lim_{n\to\infty} \sum_{j=0}^{n} H(n,j)\xi(j) = \sum_{j=0}^{\infty} H_\infty(j)\xi(j), \quad a.s. \tag{7.5.3}$$

Then (7.5.1) *and* (7.5.2) *hold, where $\{\xi(n)\}_{n\in\mathbb{Z}^+}$ is an i.i.d. sequence of standard Gaussian random variables.*

Remark 7.5.1. Theorem 7.5.2 may be applied to many of the pathwise results of this chapter. For example part (a) of Theorem 7.3.1 may be restated as

$$\lim_{n\to\infty} \sum_{j=1}^{n} \frac{\sigma r(n,j)}{n^{-1-\frac{\beta}{\alpha-1}}}\xi(j) = \sum_{j=1}^{\infty} \sigma d_1(j)\xi(j), \quad \text{a.s.}$$

Using the nomenclature of Theorem 7.5.2 we have

$$H(n,j) = \sigma n^{1+\frac{\beta}{\alpha-1}} r(n,j), \quad H_\infty(j) = \sigma d_1(j).$$

Therefore Theorem 7.5.2 gives that

$$\sigma n^{1+\frac{\beta}{\alpha-1}} \sum_{j=1}^{n} r(n,j)\xi(j) \text{ converges in mean square to } \sum_{j=1}^{\infty} \sigma d_1(j)\xi(j) \text{ as } n \to \infty.$$

One may argue similarly for Theorem 7.3.2 etc.

7.6 Discretisation of Continuous Average Functional Equation

In Chapter 6 the equation

$$dX(t) = \left(aX(t) + \frac{b}{1+t}\int_{-1}^{t} X(s)ds\right)dt + \varsigma dB(t), \quad t \geq 0, \tag{7.6.1a}$$

$$X(t) = \phi(t), \quad t \in [-1, 0]. \tag{7.6.1b}$$

was studied for some initial function ϕ and constants $a, b \in \mathbb{R}$ and $\varsigma > 0$. For the purposes of numerical simulation it behoves one to ask whether a discrete equation may be deduced which is a discretised version of (7.6.1) and preserves the qualitative and quantitative asymptotic features of the continuous equation.

We suppose for fixed $h > 0$ that $X_h(n)$ is an approximation of $X(nh)$ for $n \in \{0, 1, 2, \ldots\}$. Then using standard discretisation techniques, an implicit discretisation of (7.6.1) yields

$$X_h(n+1) - X_h(n) = ahX_h(n+1) + \frac{bh}{hn+1}\sum_{j=0}^{n} X_h(j)h + \varsigma\sqrt{h}\xi(n+1).$$

Observing that

$$\frac{bh}{hn+1} = \frac{bh}{n+1}\frac{1+n}{1+nh} = \frac{b}{n+1}\frac{1+n}{(\frac{1}{h}+n)}$$

and that

$$\lim_{n\to\infty} \frac{1+n}{(\frac{1}{h}+n)} = 1,$$

we argue, for large n, that

$$X_h(n+1) - X_h(n) = ahX_h(n+1) + \frac{bh}{n+1}\sum_{j=0}^{n} X_h(j) + \varsigma\sqrt{h}\xi(n+1).$$

is an acceptable discretisation. Then, letting $\sigma := \varsigma\sqrt{h}/(1-ah)$, $\beta = bh/(1-ah)$ and $\alpha = 1/(1-ah)$ yields (7.1.3).

An implicit discretisation is appropriate in the case that $a < 0$. We demonstrate this by comparing pathwise results only. In this situation, for every $h > 0$, we have $\alpha \in (0, 1)$. Since $\alpha + \beta = (1 + bh)/(1 - ah)$, we also see that

(i) $a + b > 0$ implies $\alpha + \beta > 1$;

(ii) $a + b < 0$ implies $\alpha + \beta < 1$;

(iii) $a + b = 0$ implies $\alpha + \beta = 1$.

In case (i), the discrete solution obeys

$$\lim_{n\to\infty} \frac{X_h(n)}{n^{-1-\beta/(\alpha-1)}} = L_h$$

where L_h is a non–trivial normal random variable. Note that $-1 - \beta/(\alpha - 1) = -1 - b/a$, so we have

$$\lim_{n\to\infty} \frac{X_h(n)}{(nh)^{-1-b/a}} = L'_h.$$

Recalling that the corresponding continuous time result is

$$\lim_{t\to\infty} \frac{X(t)}{t^{-1-b/a}} = L,$$

we can see that the rate of growth of the continuous solution has been recovered perfectly by the discrete scheme. We conjecture that the limiting random variable L_h' tends to a nontrivial limit as $h \to 0$ (though perhaps not to L itself because we have made an approximation of the discretisation).

In the case (ii), we have that the discrete solution obeys

$$-\liminf_{n\to\infty} \frac{X_h(n)}{\sqrt{2\log n}} = \limsup_{n\to\infty} \frac{X_h(n)}{\sqrt{2\log n}} = \frac{\sigma}{\sqrt{1-\alpha^2}}.$$

Since

$$\frac{\sigma}{\sqrt{1-\alpha^2}} = \frac{\varsigma}{\sqrt{2|a|+a^2h}},$$

This implies

$$\limsup_{n\to\infty} \frac{X_h(n)}{\sqrt{2\log nh}} = \frac{\varsigma}{\sqrt{2|a|+a^2h}},$$

while the continuous process obeys

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2\log t}} = \frac{\varsigma}{\sqrt{2|a|}}.$$

Thus, for any $h > 0$, the solutions of the discretised equation fluctuate on the real line with large deviations growing logarithmically, which is precisely the behaviour exhibited by the corresponding continuous time equation. Therefore, the qualitative form of the dynamics is correctly predicted irrespective of the step size h. Moreover, it can be seen that the limiting constant $\varsigma/\sqrt{2|a|+a^2h}$ for the discretisation converges to the limiting constant $\varsigma/\sqrt{2|a|}$ as $h \to 0$, so that the asymptotic rate of growth of the large fluctuations are more precisely recovered as computational effort increases.

The implicit discretisation also recovers the dynamics in the case that $a = 0$. If this is so, then $\alpha = 1$, and we have $\beta = bh > 0$ whenever $b > 0$ and $\beta < 0$ whenever $b < 0$. In the case when $\beta > 0$ we have that the discrete equation behaves according to

$$\lim_{n\to\infty} \frac{X_h(n)}{e^{2\sqrt{\beta n}} n^{-1/4}} = L_h.$$

Noting that $\beta n = b \cdot nh$, we have

$$\lim_{n\to\infty} \frac{X_h(n)}{e^{2\sqrt{b\cdot nh}} (nh)^{-1/4}} = L_h'.$$

Therefore, because

$$\lim_{t\to\infty} \frac{X(t)}{e^{2\sqrt{bt}} t^{-1/4}} = L,$$

it can be seen that the rate of growth of the discrete and continuous equations agree. Finally, in the case that $\beta < 0$, we have that the discrete equation obeys

$$\limsup_{n\to\infty} \frac{X_h(n)}{\sqrt{2n\log\log n}} \le \frac{2\sigma}{\sqrt{3}}$$

or

$$\limsup_{n\to\infty} \frac{X_h(n)}{\sqrt{2nh\log\log nh}} \le \frac{2\varsigma}{\sqrt{3}}$$

This corresponds to the continuous limit

$$\limsup_{t\to\infty} \frac{X(t)}{\sqrt{2t\log\log t}} = \frac{\varsigma}{\sqrt{3}}$$

so it can be seen that the discrete equation correctly determines an upper bound on the rate of growth of the largest fluctuations, and that the limit on the righthand side is independent of b and h, and linear in ς. If we could prove the conjecture

$$\limsup_{n\to\infty} \frac{X_h(n)}{\sqrt{2n\log\log n}} = \frac{\sigma}{\sqrt{3}}$$

then there would be an exact agreement in the pathwise large fluctuation behaviour of the discretised and continuous equations.

An explicit discretisation is effective in the case when $a > 0$, giving the correct qualitative asymptotic behaviour for all step sizes $h > 0$. Such a discretisation yields

$$X_h(n+1) - X_h(n) = ahX_h(n) + \frac{bh}{hn+1}\sum_{j=0}^{n} X_h(j)h + \varsigma\sqrt{h}\xi(n+1)$$

and once again, considering n large, we modify this to obtain

$$X_h(n+1) - X_h(n) = ahX_h(n) + \frac{bh}{n+1}\sum_{j=0}^{n} X_h(j) + \varsigma\sqrt{h}\xi(n+1).$$

Letting $\sigma = \varsigma\sqrt{h}$, $\beta = bh$ and $\alpha = 1 + ah$ yields (7.1.3). In the case that $a > 0$, we have that $\alpha > 1$. Therefore the solution of the discrete equation obeys

$$\lim_{n\to\infty} \frac{X_h(n)}{\alpha^n n^{\beta/(\alpha-1)}} = L_h.$$

Now, as $\beta/(\alpha - 1) = b/a$, if we define $a_h = \log(1 + ah)/h$, then

$$\lim_{n\to\infty} \frac{X_h(n)}{e^{a_h nh}(hn)^{b/a}} = L'_h.$$

Notice that $\lim_{h\to 0} a_h = a$, so as the step size tends to zero, we recover the rate of growth of the continuous equation, given by

$$\lim_{t\to\infty} \frac{X(t)}{e^{at}t^{b/a}} = L.$$

Also, for any $h > 0$, the discretised equation has solutions which grow exponentially with a polynomial growth correction, which is precisely the behaviour exhibited by the corresponding continuous time equation. Therefore, the qualitative form of the dynamics is correctly predicted irrespective of the step size h.

It is worth remarking that an explicit discretisation may be applied to the continuous equation in the case that $a < 0$, but that restrictions on the step size are now required. Since $\alpha = 1 + ah$ in this case, we request that $h < 2/|a|$ in order to give $|\alpha| < 1$. If $\alpha < -1$, the solution oscillates unboundedly with exponentially growing amplitude, rather than fluctuating logarithmically or growing to infinity polynomially. We have that $\alpha + \beta = 1 + (a + b)h$, so $\alpha + \beta < 1$ whenever $a + b < 0$, $\alpha + \beta > 1$ whenever $a + b > 0$ and $\alpha + \beta = 1$ whenever $a + b = 0$. In the first and last cases, we obtain logarithmic fluctuations, consistent with the continuous time dynamics, while in the second case, as

$$-1 - \frac{\beta}{\alpha - 1} = -1 - \frac{b}{a}$$

we have that

$$\lim_{n\to\infty} \frac{X_h(n)}{(nh)^{-1-b/a}} = L'_h$$

as in the implicit case. When $a = 0$, we automatically have $\alpha = 1$ in the explicit case, and then $\beta = bh$ is positive or negative according as to whether b is positive or negative. Therefore, the explicit scheme correctly recovers the dynamics in this case without a step size restriction.

Similarly, if an implicit discretisation is applied to the continuous equation in the case when $a > 0$, then we need once more a restriction on the step size. Notice first that $\alpha > 1$ provided $h < 1/a$. This restriction is necessary, because in the case that $h > 2/a$, then $\alpha \in (-1, 0)$ and the discretised equation exhibits fluctuations or polynomial growth rather than the exponential growth present in the continuous case. On the other hand, if $1/a < h < 2/a$, although the amplitude of the solution of the discretised equation grows exponentially, it alternates in sign at each time step, which is inconsistent with the continuous equation. In the case that $h < 1/a$, the asymptotic behaviour in the continuous case is recovered in a manner similar to that of the discrete problem. Since $\beta/(\alpha - 1) = b/a$, if we define $a_h = -\log(1 - ah)/h$, then

$$\lim_{n\to\infty} \frac{X_h(n)}{e^{a_h \cdot nh}(nh)^{b/a}} = L'_h$$

so the solution exhibits exponential growth with the appropriate modifying polynomial factor. Furthermore, as the step size $h \to 0$, we have that $a_h \to a$, so the growth exponent is recovered exactly as the computational effort involved in the simulation increases.

We note that we do not give (7.1.3) an initial history, as in the continuous case, for the reason that there is no potential for a singularity in (7.1.3) at $n = 0$.

7.7 Asymptotic Behaviour of Deterministic Sequences

Before giving proofs of the main stochastic results, we first estimate the asymptotic behaviour of the sequences d_1 and d_2 and sums of sequences which depend on them.

Lemma 7.7.1. *Let $\alpha \in \mathbb{R}/\{0,1\}$, then d_1 and d_2 as given by* (7.1.19)*, obey* (7.1.20)

Proof. Firstly we remark that the asymtptoic behaviours provided by Theorem 7.1.1 specifies $C(0)$, even though the values of $r_1(0)$ and $r_2(0)$ are not known. Since r_1 and r_2 obey (7.1.16), we have

$$C(n) = r_1(n)r_2(n+1) - r_1(n+1)r_2(n) = \alpha^n n^{-1}\left((\alpha-1) + O(n^{-1})\right), \quad \text{as } n \to \infty.$$

From Lemma 7.1.2 we have $C(n) = C(0)\alpha^n/(n+1)$. Therefore

$$C(0) = \lim_{n\to\infty}(n+1)\alpha^{-n}C(n) = \alpha - 1.$$

The asymptotic behaviour of d_1 and d_2 is now determined. First notice that, as $m \to \infty$,

$$(m+1)^{\beta/(\alpha-1)} = m^{\beta/(\alpha-1)}\left(1 + \frac{\beta}{\alpha-1}m^{-1} + O(m^{-2})\right),$$
$$(m+1)^{-1} = m^{-1}\left(1 + O(m^{-1})\right),$$

Therefore inserting (7.1.16) and these estimates into (7.1.19) gives

$$\begin{aligned}
(\alpha-1)\frac{d_1(m)}{m+1} &= \alpha m^{\beta/(\alpha-1)}\left(1 + \frac{\beta}{\alpha-1}m^{-1} + O(m^{-2})\right)\left(1 + \frac{c_2}{m+1} + O(\frac{1}{m^2})\right)\\
&\quad - \alpha m^{\frac{\beta}{\alpha-1}}\left(1 + \frac{c_2}{m} + O(\frac{1}{m^2})\right)\\
&\quad - \beta m^{-1}(1 + O(m^{-1}))m^{\frac{\beta}{\alpha-1}}\left(1 + \frac{c_2}{m} + O(\frac{1}{m^2})\right)\\
&= \alpha m^{\beta/(\alpha-1)}\left(1 + \frac{\beta}{\alpha-1}m^{-1} + \frac{c_2}{m} + O(m^{-2})\right)\\
&\quad - \alpha m^{\frac{\beta}{\alpha-1}}\left(1 + \frac{c_2}{m} + O(\frac{1}{m^2})\right) - \beta m^{\frac{\beta}{\alpha-1}-1}\left(1 + O(\frac{1}{m})\right)\\
&= m^{\beta/(\alpha-1)-1}\left(\frac{\alpha\beta}{\alpha-1} - \beta + O(m^{-1})\right).
\end{aligned}$$

Thus

$$d_1(m) = \frac{\beta}{(\alpha-1)^2} m^{\beta/(\alpha-1)} \left(1 + O(m^{-1})\right), \qquad \text{as } m \to \infty.$$

The asymptotic behaviour for d_2 can be determined in a similar manner. However, the analysis is simpler because there is no cancellation of the leading order terms in (7.1.19).

□

Lemma 7.7.2. *Let $\alpha = 0$. If $\beta \in \{0, -1, -2, ...\}$ and $1 \le m+1 \le -\beta \le n-1$ then*

$$r(n, m) = 0. \tag{7.7.1}$$

Otherwise,

$$r(n, m) = \frac{\Gamma(\beta+n)}{n!} \frac{\beta m!}{\Gamma(\beta+m+1)}, \quad n \ge m+1 \ge 1 \quad \textit{and} \quad r(m, m) = 1,\ m \ge 0. \tag{7.7.2}$$

Moreover,

$$\lim_{n\to\infty} \frac{\Gamma(n+\beta)}{n!} \frac{1}{n^{\beta-1}} = 1. \tag{7.7.3}$$

Proof. As before re–expressing (7.1.5) as a second order difference equation gives

$$r(n+2, m) = \frac{n+1+\beta}{n+2} r(n+1, m), \quad n \ge m \ge 0.$$

This gives

$$r(n, m) = \frac{\beta}{m+1} \prod_{k=m+2}^{n} \frac{\beta+k-1}{k}, \quad n \ge m+2 \ge 2. \tag{7.7.4}$$

Thus if $\beta \in \{0, -1, -2, ...\}$ then (7.7.1) holds.

On the other hand, for $\beta \notin \{0, -1, -2, ...\}$, the formula (7.7.4) may be rewritten as

$$r(n, m) = \beta \frac{\Gamma(\beta+n)}{n!} \frac{m!}{\Gamma(\beta+m+1)}, \quad n \ge m+1 \ge 1.$$

The asymptotic result (7.7.3) follows from Stirling's formula

$$\Gamma(t) \sim \sqrt{2\pi}\, \mathrm{e}^{-t}\, t^{t-1/2}, \quad \text{as } t \to \infty,$$

(see e.g., [95, Ch. 3.8.3]). Applying this, we get

$$\begin{aligned}
\lim_{n\to\infty} \frac{\Gamma(n+\beta)}{n!} \frac{1}{n^{\beta-1}} &= \lim_{n\to\infty} \frac{\sqrt{2\pi}\, \mathrm{e}^{-n-\beta} (n+\beta)^{n+\beta-1/2}}{\sqrt{2\pi}\, \mathrm{e}^{-n-1} (n+1)^{n+1-1/2} n^{\beta-1}} \\
&= \lim_{n\to\infty} \mathrm{e}^{-\beta+1} \left(1 + \frac{\beta-1}{n+1}\right)^{n} \\
&= \lim_{n\to\infty} \mathrm{e}^{-\beta+1} \left(1 + \frac{\beta-1}{n+1}\right)^{n+1} \left(1 + \frac{\beta-1}{n+1}\right)^{-1} = 1,
\end{aligned}$$

as required. □

In order to attain rates of growth or decay of various summations we use the Stolz–Cesàro Theorem which maybe viewed as a discrete analogue of L'Hôpital's Rule. We state the Stolz–Cesàro Theorem for completeness. A proof is given in [94, pp. 85] or may be inferred from Toeplitz's Lemma [108, Lemma 4.3.1].

Theorem 7.7.1 (Stolz–Cesàro Theorem)**.** *Let a_n be a sequence of real numbers and b_n a strictly increasing and divergent sequence for n sufficiently large. Then*

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = l \in [-\infty,\infty]$$

implies

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l.$$

The theorem is also valid if a_n is strictly decreasing and convergent to zero, and b_n is positive, strictly decreasing for n sufficiently large and convergent to zero.

Lemma 7.7.3. *Suppose that d_1 is given by* (7.1.19) *then*

(i) *If $\alpha < 1$, $\alpha \neq 0$ and $\alpha + 2\beta > 1$ or $\alpha > 1$ and $\alpha + 2\beta < 1$. Then $d_1 \in \ell^2(\mathbb{Z}^+)$ and hence $\lim_{n\to\infty} \sigma \sum_{m=1}^n d_1(m)\xi(m)$ exists a.s. and is an a.s. finite random variable.*

(ii) *If $\alpha < 1$, $\alpha \neq 0$ and $\alpha + 2\beta < 1$ or $\alpha > 1$ and $\alpha + 2\beta > 1$. Then*

$$\sum_{j=1}^n d_1^2(j) \sim \frac{\beta^2}{(\alpha-1)^3(\alpha+2\beta-1)} n^{\frac{2\beta}{\alpha-1}+1}, \quad \text{as } n\to\infty.$$

(iii) *If $\alpha \neq \{0,1\}$ and $\alpha + 2\beta = 1$. Then*

$$\sum_{j=1}^n d_1^2(j) \sim \frac{\beta^2}{(\alpha-1)^4} \log n, \quad \text{as } n\to\infty.$$

Proof. We prove each part of the lemma in turn:

(i) Consequence of (7.1.20) and the discrete martingale convergence theorem [108, Theorem 4.2.1].

(ii) Consequence of (7.1.20) and Theorem 7.7.1, with $b_n = n^{\frac{2\beta}{\alpha-1}+1}$. From the Mean Value Theorem

$$b_{n+1} - b_n = \left(\frac{2\beta}{\alpha-1}+1\right) \kappa_n^{\frac{2\beta}{\alpha-1}}$$

for some $\kappa_n \in (n, n+1)$. Thus $\kappa_n \sim n$ as $n\to\infty$ and so

$$b_{n+1} - b_n \sim \left(\frac{2\beta}{\alpha-1}+1\right) n^{\frac{2\beta}{\alpha-1}}, \quad \text{as } n\to\infty.$$

(iii) Follows again from use of the Mean Value Theorem with $b_n = \log n$.

□

Lemma 7.7.4. *Suppose that d_2 is given by* (7.1.19)*.*

(i) *If $|\alpha| > 1$, then $d_2 \in \ell^2(\mathbb{Z}^+)$ and hence $\lim_{n\to\infty} \sigma \sum_{m=1}^{n} d_2(m)\xi(m)$ converges to a finite limit with probability one.*

(ii) *If $\alpha \in (-1,1)/\{0\}$, then*

$$\sum_{j=1}^{n} d_2^2(j) \sim \frac{1}{1-\alpha^2}\alpha^{-2n} n^{-\frac{2\beta}{\alpha-1}}, \quad \text{as } n \to \infty.$$

(iii) *If $\alpha = -1$ and $\beta < -1$, then $d_2 \in \ell^2(\mathbb{Z}^+)$ and $\lim_{n\to\infty} \sigma \sum_{j=1}^{n} d_2(j)\xi(j)$ exists a.s. and is an a.s. finite random variable.*

(iv) *If $\alpha = -1$ and $\beta > -1$, then*

$$\sum_{j=1}^{n} d_2^2(j) \sim \frac{1}{1+\beta} n^{1+\beta}, \quad \text{as } n \to \infty.$$

(v) *If $\alpha = -1$ and $\beta = -1$, then*

$$\sum_{j=1}^{n} d_2^2(j) \sim \log n, \quad \text{as } n \to \infty.$$

Proof. Proof follows from the known asymptotic behaviour of d_2 given by (7.1.20) and by use of Theorem 7.7.1. □

7.8 Proofs

7.8.1 Proof of Lemma 7.1.1

We demonstrate that (7.1.6) obeys (7.1.3). Define $Z(0) = x_0$ and

$$Z(n) = r(n,0)x_0 + \sigma \sum_{m=1}^{n} r(n,m)\xi(m), \quad n \geq 1.$$

We show that $Z = X$. Firstly, note that $Z(1) = (\alpha+\beta)x_0 + \sigma\xi(1)$ and also the recurrence relation (7.1.3) gives

$$X(1) = (\alpha + \beta)x_0 + \sigma\xi(1).$$

Now rewrite (7.1.3a) as

$$X(n+1) = \alpha X(n) + \frac{\beta}{n+1} x_0 + \frac{\beta}{n+1} \sum_{j=1}^{n} X(j) + \sigma \xi(n+1), \quad n \geq 1. \tag{7.8.1}$$

Then for $n \geq 1$,

$$\begin{aligned}
&\alpha Z(n) + \frac{\beta}{n+1} x_0 + \frac{\beta}{n+1} \sum_{j=1}^{n} Z(j) + \sigma \xi(n+1) \\
&= \alpha \left[r(n,0)x_0 + \sigma \sum_{m=1}^{n} r(n,m)\xi(m) \right] + \frac{\beta}{n+1} x_0 \\
&\quad + \frac{\beta}{n+1} \sum_{j=1}^{n} \left[r(j,0)x_0 + \sigma \sum_{m=1}^{j} r(j,m)\xi(m) \right] + \sigma \xi(n+1) \\
&= r(n+1,0)x_0 + \sigma \sum_{m=1}^{n} \left[\alpha\, r(n,m) + \frac{\beta}{n+1} \sum_{j=m}^{n} r(j,m) \right] \xi(m) + \sigma \xi(n+1) \\
&= r(n+1,0)x_0 + \sigma \sum_{m=1}^{n+1} r(n+1,m)\xi(m) = Z(n+1).
\end{aligned}$$

By the uniqueness of the solution of (7.1.3) we must have $Z(n) = X(n)$, for $n \geq 0$. If one further observes that $r(\cdot, 0)x_0$ satisfies the initial value problem (7.1.4), then as (7.1.4) has a unique solution we must have $x(\cdot) = r(\cdot, 0)x_0$.

7.8.2 Proof of Proposition 7.1.1

Recall

$$X(n) = \alpha X(n-1) + \frac{\beta}{n} \sum_{j=0}^{n-1} X(j) + \sigma \xi(n) \quad \text{for } n \geq 1.$$

Regarding (7.1.8) and (7.1.9), we have, for $n + k + 1 \geq 1$ (or equivalently $n + k \geq 0$),

$$\begin{aligned}
\gamma_n(k+1) &= \text{Cov}(X(n), X(n+k+1)) \\
&= \text{Cov}\left(X(n), \alpha X(n+k) + \frac{\beta}{n+k+1} \sum_{j=0}^{n+k} X(j) + \sigma \xi(n+k+1) \right) \\
&= \alpha \text{Cov}(X(n), X(n+k)) + \frac{\beta}{k+n+1} \sum_{j=0}^{n+k} \text{Cov}(X(n), X(j+n-n)) \\
&\quad + \sigma \text{Cov}(X(n), \xi(n+k+1)) \\
&= \alpha \gamma_n(k) + \frac{\beta}{k+n+1} \sum_{j=0}^{n+k} \gamma_n(j-n) + \sigma^2 \delta_{k+1}.
\end{aligned}$$

Hence

$$\gamma_n(k+1) = \alpha \gamma_n(k) + \frac{\beta}{k+n+1} \sum_{l=-n}^{k} \gamma_n(l) + \sigma^2 \delta_{k+1}$$

where

$$\delta_{k+1} := \begin{cases} 1, & k+1=0, \\ 0, & \text{otherwise.} \end{cases}$$

Regarding (7.1.10), we recall that $X(0) = x_0$, which is assumed to be deterministic.

7.8.3 Proof of Theorem 7.2.5

From (7.1.21) we have

$$\begin{aligned} \text{Cov}(X(n), X(n+k)) &= \sigma^2 r_1(n+k) r_1(n) \sum_{m=1}^{n} d_1(m)^2 \\ &+ \sigma^2 r_1(n+k) r_2(n) \sum_{m=1}^{n} d_1(m) d_2(m) + \sigma^2 r_1(n) r_2(n+k) \sum_{m=1}^{n} d_1(m) d_2(m) \\ &+ \sigma^2 r_2(n+k) r_2(n) \sum_{m=1}^{n} d_2(m)^2. \end{aligned}$$

For the case $\alpha + \beta < 1$ it is apparent from Lemma 7.7.3 and Lemma 7.7.4 that the first three terms in the sum on the right–hand side above tend to zero as $n \to \infty$. This gives

$$\lim_{n\to\infty} \text{Cov}(X(n), X(n+k)) = \sigma^2 \lim_{n\to\infty} r_2(n+k) r_2(n) \sum_{m=1}^{n} d_2(m)^2 = \frac{\sigma^2}{1-\alpha^2}\alpha^k.$$

For the case $\alpha + \beta = 1$ we find that $\lim_{n\to\infty} r_1(n+k) r_1(n) = 1$. Hence,

$$\lim_{n\to\infty} \text{Cov}(X(n), X(n+k)) = \sigma^2 \sum_{m=1}^{\infty} d_1(m)^2 + \frac{\sigma^2}{1-\alpha^2}\alpha^k,$$

as required.

7.8.4 Proof of Proposition 7.2.1

From (7.1.22) we have the formula for $c_{1,n}$,

$$c_{1,n} = \sigma^2 r_1(n) \sum_{m=1}^{n} d_1(m)^2 + \sigma^2 r_2(n) \sum_{m=1}^{n} d_1(m) d_2(m). \tag{7.8.2}$$

The asymptotic behaviour of d_1 and d_2 is known form (7.1.26a). Using Theorem 7.7.1 one can deduce

$$\sum_{m=1}^{n} d_1(m) d_2(m) \sim \frac{\beta}{(\alpha-1)^2} \frac{\alpha^{-n}}{1-\alpha}, \qquad \text{as } n \to \infty.$$

Thus,

$$\lim_{n\to\infty} \frac{r_2(n) \sum_{m=1}^{n} d_1(m) d_2(m)}{n^{\frac{\beta}{\alpha-1}}} = \frac{\beta}{(1-\alpha)^3}.$$

For the first term on the right hand side of (7.8.2) the asymptotics of the summation is detailed in Lemma 7.7.3. From the asymptotics of r_1 given in (7.1.16) this gives the result.

7.8.5 Proof of Theorem 7.2.4

Observe $r(m, m) = 1$ and $r(m+1, m) = \alpha + \beta/(m+1) > 0$. Assume $r(n, m) > 0$ for all $m \leq n \leq k$. Then

$$r(k+1, m) = \alpha r(k, m) + \frac{\beta}{n+1} \sum_{j=m}^{k} r(j, m) > 0.$$

Hence $r(n, m) > 0$ for all $n \geq m$. Equation (7.1.7) for the auto–covariance function implies $\text{Cov}(X(n), X(n+k)) > 0$ for $n > 0$, $k \geq 1$.

7.9 Proof of Theorems 7.3.1 and 7.3.2

7.9.1 Proof of Theorem 7.3.2

Observing, from (7.1.16), that for sufficiently large n, $r_2(n) \neq 0$, equation (7.1.18) gives

$$\frac{X(n)}{r_2(n)} = d_1(0)\frac{r_1(n)}{r_2(n)}x_0 + d_2(0)x_0 + \sigma\frac{r_1(n)}{r_2(n)}\sum_{j=1}^{n} d_1(j)\xi(j) + \sigma\sum_{j=1}^{n} d_2(j)\xi(j).$$

The first term on the right hand side converges to zero due to (7.1.16), while the last term converges with probability one as n tends to infinity, due to Lemma 7.7.4 (i).

Now, suppose $\alpha < -1$ and $\alpha + 2\beta > 1$ or $\alpha > 1$ and $\alpha + 2\beta < 1$. Regarding the third term: $\sum_{j=1}^{n} d_1(j)\xi(j)$ converges with probability one as n tends to infinity, due to Lemma 7.7.3 (i). Hence the third term converges to zero as $n \to \infty$.

Now consider the case $\alpha < -1$ and $\alpha + 2\beta < 1$ or $\alpha > 1$ and $\alpha + 2\beta > 1$. Again we have

$$\lim_{n\to\infty} \frac{X(n)}{r_2(n)} = d_2(0)x_0 + \sigma\sum_{j=1}^{\infty} d_2(j)\xi(j), \quad \text{a.s.} \tag{7.9.1}$$

provided we can show

$$\lim_{n\to\infty} \frac{r_1(n)}{r_2(n)}\sum_{j=1}^{n} d_1(j)\xi(j) = 0, \quad \text{a.s.}$$

This is evidenced by an upper bound on the size of the fluctuations of a sequence of normal random variables. Define $A_1(n) = \sum_{j=1}^{n} d_1(j)\xi(j)$ and

$$v_{1,n}^2 := \sum_{j=1}^{n} d_1^2(j) \sim \frac{\beta^2}{(\alpha-1)^3(\alpha+2\beta-1)} n^{\frac{2\beta}{\alpha-1}+1}, \quad \text{as } n \to \infty. \tag{7.9.2}$$

Since $A_1(n)/v_{1,n}$ is a standardised normal random variable, we have that

$$\limsup_{n\to\infty} \frac{|A_1(n)|}{v_{1,n}\sqrt{2\log n}} \leq 1, \quad \text{a.s.},$$

as a routine consequence of the Borel–Cantelli lemma. Thus

$$\begin{aligned}\limsup_{n\to\infty}\left|\frac{r_1(n)}{r_2(n)}A_1(n)\right| &= \limsup_{n\to\infty}\left|\frac{r_1(n)v_{1,n}\sqrt{2\log n}}{r_2(n)}\right|\left|\frac{A_1(n)}{v_{1,n}\sqrt{2\log n}}\right| \\ &\le \limsup_{n\to\infty}\left|\frac{r_1(n)v_{1,n}\sqrt{2\log n}}{r_2(n)}\right| = 0\end{aligned}$$

with the last equality holding due to the geometric growth of r_2, (7.1.16) and (7.9.2).

In the case $|\alpha| > 1$ and $\alpha + 2\beta = 1$, it is seen that

$$v_{1,n}^2 \sim \frac{\beta^2}{(\alpha-1)^4}\log n, \quad \text{as } n\to\infty.$$

The result then holds by the same argument as above.

Therefore it has been shown that (7.9.1) holds and so from an argument of Shiryaev [108, Chap. 2.13.5, pp.304-305], we have that the limiting random variable is Gaussian and that part (c) is true.

7.9.2 Proof of Theorem 7.3.1

We consider first the case where $\alpha \neq 0$. Equation (7.1.18) gives

$$\frac{X(n)}{r_1(n)} = d_1(0)x_0 + \frac{r_2(n)}{r_1(n)}d_2(0)x_0 + \sigma\sum_{j=1}^{n} d_1(j)\xi(j) + \sigma\frac{r_2(n)}{r_1(n)}\sum_{j=1}^{n} d_2(j)\xi(j).$$

The second term tends to zero as n tends to infinity and the third term converges to a finite limit with probability one due to (7.1.16) and Lemma 7.7.3. The result follows provided we can show

$$\lim_{n\to\infty}\frac{r_2(n)}{r_1(n)}\sum_{j=1}^{n} d_2(j)\xi(j) = 0, \quad \text{a.s.}$$

Define $A_1(n) := \sum_{j=1}^{n} d_2(j)\xi(j)$ and

$$v_{2,n}^2 := \sum_{j=1}^{n} d_2^2(j) \sim \frac{1}{1-\alpha^2}\,\alpha^{-2n}n^{-\frac{2\beta}{\alpha-1}}, \quad \text{as } n\to\infty. \tag{7.9.3}$$

Since $A_2(n)/v_{2,n}$ is a standardised normal random variable, we have that

$$\limsup_{n\to\infty}\frac{|A_2(n)|}{v_{2,n}\sqrt{2\log n}} \le 1, \quad \text{a.s.},$$

as a routine consequence of the Borel-Cantelli lemma. Thus

$$\begin{aligned}\limsup_{n\to\infty}\left|\frac{r_2(n)}{r_1(n)}A_2(n)\right| &= \limsup_{n\to\infty}\left|\frac{r_2(n)v_{2,n}\sqrt{2\log n}}{r_1(n)}\right|\left|\frac{A_2(n)}{v_{2,n}\sqrt{2\log n}}\right| \\ &\le \limsup_{n\to\infty}\left|\frac{r_2(n)v_{2,n}\sqrt{2\log n}}{r_1(n)}\right| = 0\end{aligned}$$

with the last equality holding due to the polynomial growth of r_1, (7.1.16).

We have now shown that $X(n)/n^{-1-\frac{\beta}{\alpha-1}}$ tends to a finite limit almost surely as n tends to infinity. Thus from [108, Chap.2.13.5, pp.304-305] we have that this limit is Gaussian and that part (c) holds. The positivity of the variance is due to d_1 being asymptotic to a positive function, (7.1.20).

Suppose now that $\alpha = 0$ and $\beta > 1$. Then by Lemma 7.7.2 we have $r(n,m) = R_1(n)D_1(m)$, $1 \leq m+1 \leq n$ where

$$R_1(n) = \frac{\Gamma(\beta+n)}{n!} \sim n^{\beta-1}, \quad \text{as } n \to \infty, \tag{7.9.4a}$$

$$D_1(m) = \beta\frac{m!}{\Gamma(\beta+m+1)} \sim \beta m^{-\beta}, \quad \text{as } m \to \infty. \tag{7.9.4b}$$

Therefore X has the representation, for $n \geq 1$

$$X(n) = \frac{\beta x_0}{\Gamma(\beta+1)}R_1(n) + \sigma R_1(n)\sum_{m=1}^{n-1} D_1(m)\xi(m) + \sigma\xi(n).$$

As D_1 behaves asymptotically polynomially then we have $D_1 \in \ell^2(\mathbb{Z}^+)$ when $\beta > 1/2$. We therefore have

$$\lim_{n\to\infty} n^{1-\beta}R_1(n)\sum_{m=1}^{n-1} D_1(m)\xi(m) = \sum_{m=1}^{\infty} D_1(m)\xi(m), \quad \text{a.s.} \quad \beta \geq 1.$$

Therefore for $\beta > 1$, we have

$$\lim_{n\to\infty}\frac{X(n)}{n^{\beta-1}} = \frac{\beta}{\Gamma(\beta+1)}x_0 + \sigma\sum_{m=1}^{\infty} D_1(m)\xi(m), \quad \text{a.s.,}$$

as required.

7.10 Proof of Theorems 7.2.1 and 7.2.2

In this section, we show that the solutions of the average functional equation are coupled to those of an equation whose solution is a Markov process.

7.10.1 Preparatory results

We firstly begin with a lemma concerning the convergence of Gaussian summations.

Lemma 7.10.1. *Suppose the function H obeys*

$$H : \{(n,j) : 0 \leq j \leq n,\ n,j \in \mathbb{Z}^+\} \to \mathbb{R}.$$

If there exists a function $H_\infty : \mathbb{Z}^+ \to \mathbb{R}$ such that $H_\infty \in \ell^2(\mathbb{Z}^+)$ and

$$\lim_{n\to\infty} \sum_{j=1}^{n} \left(H(n,j) - H_\infty(j)\right)^2 \cdot \log n = 0. \tag{7.10.1}$$

Then

$$\lim_{n\to\infty} \sum_{j=1}^{n} H(n,j)\xi(j) = \sum_{j=1}^{\infty} H_\infty(j)\xi(j), \quad a.s. \tag{7.10.2}$$

where $\{\xi(n)\}_{n\in\mathbb{Z}^+}$ is an i.i.d. sequence of standard Gaussian random variables.

Proof. As $H_\infty \in \ell^2(\mathbb{Z}^+)$ then

$$\lim_{n\to\infty} \sum_{j=1}^{n} H_\infty(j)\xi(j) = \sum_{j=1}^{\infty} H_\infty(j)\xi(j), \quad \text{a.s.}$$

Thus showing (7.10.2) holds is equivalent to showing

$$\lim_{n\to\infty} \sum_{j=1}^{n} \{H(n,j) - H_\infty(j)\}\xi(j) = 0, \quad \text{a.s.}$$

Define $U(n) := \sum_{j=1}^{n}\{H(n,j) - H_\infty(j)\}\xi(j)$, $n \geq 1$ then $U(n)$ is a Gaussian distributed random variable with mean zero and variance equal to

$$v_n^2 := \sum_{j=1}^{n} \left(H(n,j) - H_\infty(j)\right)^2.$$

Moreover $U(n)/v_n$ is Gaussian distributed with mean zero and variance one.

It is a routine consequence of the Borel-Cantelli Lemma that for a sequence of standard Gaussian random variables, $\{Z(n) : n \in \mathbb{Z}^+\}$, c.f. e.g. [16, Lemma 8] that

$$\limsup_{n\to\infty} \frac{|Z(n)|}{\sqrt{2\log n}} \leq 1, \quad \text{a.s.}$$

Therefore,

$$\limsup_{n\to\infty} |U(n)| = \limsup_{n\to\infty} \frac{|U(n)|}{\sqrt{2v_n^2 \log n}} \cdot \sqrt{2}\sqrt{v_n^2 \log n} \leq 0,$$

with the last inequality being a consequence of the condition (7.10.1). □

Lemma 7.10.2. *Let $0 < |\alpha| < 1$ and $\alpha + \beta = 1$. Let $\{\xi(n)\}_{n\in\mathbb{Z}^+}$ be an i.i.d. sequence of standard Gaussian random variables. Define $H(\cdot,\cdot)$ by*

$$H(n,l) = \sum_{m=l+1}^{n} d_1(m)\frac{\beta}{m}\sum_{j=l}^{m-1} \alpha^j \alpha^{-l}, \quad 1 \leq l \leq n-1, \quad n \geq 2$$

$$H(n,n) = 0, \quad n \geq 0.$$

where d_1 *is given by* (7.1.19). *Then*

$$\lim_{n\to\infty}\sum_{l=1}^{n} H(n,l)\xi(l) = \sum_{l=1}^{\infty}\left(\sum_{m=l+1}^{\infty}\frac{\beta}{m}d_1(m)\sum_{j=l}^{m-1}\alpha^j\right)\alpha^{-l}\xi(l), \quad a.s. \tag{7.10.3}$$

Proof. H simplifies to

$$H(n,l) = \frac{1}{1-\alpha}\sum_{m=l+1}^{n} d_1(m)\frac{\beta}{m} - \alpha^{-l}\frac{1}{1-\alpha}\sum_{m=l+1}^{n} d_1(m)\frac{\beta}{m}\alpha^m.$$

Define

$$H_\infty(l) = \frac{1}{1-\alpha}\sum_{m=l+1}^{\infty} d_1(m)\frac{\beta}{m} - \alpha^{-l}\frac{1}{1-\alpha}\sum_{m=l+1}^{\infty} d_1(m)\frac{\beta}{m}\alpha^m$$

Due to the known asymptotic behaviour of d_1 we have that H_∞ is well–defined. Consider

$$h(l) = \sum_{m=l+1}^{\infty}\frac{\beta}{m}d_1(m) \sim l^{-1}, \quad \text{as } l\to\infty,$$
$$g(l) = \sum_{m=l+1}^{\infty}\frac{\beta}{m}d_1(m)\alpha^m \sim \frac{1}{1-\alpha}l^{-2}\alpha^{l+1}, \quad \text{as } l\to\infty.$$

Then using the Cauchy-Schwarz inequality gives

$$H_\infty(l)^2 \le 2\frac{1}{1-\alpha}h(l)^2 + 2\frac{1}{(1-\alpha)^2}\alpha^{-2l}g(l)^2.$$

So clearly we have $H_\infty \in \ell^2(Z^+)$. Now,

$$\begin{aligned}(H(n,l)-H_\infty(l))^2 &= \frac{1}{(1-\alpha)^2}\left(-h(n)+\alpha^{-l}g(n)\right)^2\\ &\le \frac{2}{(1-\alpha)^2}\left(h(n)^2+\alpha^{-2l}g(n)^2\right).\end{aligned}$$

Therefore,

$$\sum_{l=1}^{n}(H(n,l)-H_\infty(l))^2 \le \frac{2}{(1-\alpha)^2}\left(n\,h(n)^2 + \frac{\alpha^{-2}}{1-\alpha^{-2}}g(n)^2(1-\alpha^{-2n})\right).$$

From the known asymptotic behaviour of h and g we have

$$\lim_{n\to\infty}\sum_{l=1}^{n}(H(n,l)-H_\infty(l))^2\cdot\log n = 0.$$

Hence all the conditions of Lemma 7.10.1 hold and so (7.10.3) is true. □

7.10.2 Proof of Theorem 7.2.1

We start by considering the case when $\alpha \neq 0$. We firstly show a result concerning the large deviations of an AR(1) process. This result is facilitated by Lemmas 2&3 of [16].

Define the stochastic process $Y(n+1) = \alpha Y(n) + \sigma\xi(n+1)$ for $n \geq 0$ and $Y(0) = x_0$, noting that Y is a Gaussian process. Then $Y(n) = \alpha^n x_0 + \alpha^n \sigma \sum_{j=1}^{n} \alpha^{-j}\xi(j)$ and (7.2.2) gives $Y(n) - U(n) = \alpha^n x_0$. Therefore $U(n) - Y(n) \to 0$ as $n \to \infty$, a.s.

Observe, without loss of generality, for $n > m$,

$$\mathrm{Cov}(Y(n), Y(m)) = \frac{\sigma^2}{(1-\alpha^2)}\alpha^{n-m}(1-\alpha^{2m}),$$

while the variance is given by

$$\sigma_{n,Y}^2 := \mathrm{Cov}(Y(n), Y(n)) = \frac{\sigma^2}{(1-\alpha^2)}(1-\alpha^{2n}).$$

Indeed $Y(n) \sim N(0, \sigma_{n,Y}^2)$. $\tilde{Y}(n) := Y(n)/\sigma_{n,Y}$ is a sequence of standard normal random variables. Thus we have,

$$|\mathrm{Cov}(\tilde{Y}(n), \tilde{Y}(m))| = |\alpha|^{n-m}\sqrt{\frac{1-\alpha^{2m}}{1-\alpha^{2n}}}.$$

Observe that for $n > m$

$$\sqrt{\frac{1-\alpha^{2m}}{1-\alpha^{2n}}} < 1.$$

We define $\theta := |\alpha|$ hence $\theta \in (0,1)$. Then

$$|\mathrm{Cov}(\tilde{Y}(n), \tilde{Y}(m))| \leq \theta^{n-m}, \qquad \text{for } n \geq m+1 \geq 1.$$

For the case $n = m$, we have $|\mathrm{Cov}(\tilde{Y}(n), \tilde{Y}(m))| = 1$. Thus for any $n, m \in \mathbb{Z}^+$

$$|\mathrm{Cov}(\tilde{Y}(n), \tilde{Y}(m))| \leq \theta^{|n-m|}, \qquad \text{for } n, m \in \mathbb{Z}^+.$$

Thus it is shown in Lemma 3 of [16], (this result also appears as Theorem 3 of [42] and in [71])

$$\lim_{n\to\infty} \frac{\max_{1\leq j\leq n} \tilde{Y}(j)}{\sqrt{2\log n}} = 1, \quad \text{a.s.}$$

Hence $\limsup_{n\to\infty} \tilde{Y}(n) = +\infty \geq 0$. By Lemma 2 of [16],

$$\limsup_{n\to\infty} \frac{\tilde{Y}(n)}{\sqrt{2\log n}} = \limsup_{n\to\infty} \frac{\max_{1\leq j\leq n} \tilde{Y}(j)}{\sqrt{2\log n}} = \lim_{n\to\infty} \frac{\max_{1\leq j\leq n} \tilde{Y}(j)}{\sqrt{2\log n}} = 1, \quad \text{a.s.}$$

or

$$\limsup_{n\to\infty} \frac{Y(n)}{\sqrt{2\log n}} = \frac{\sigma}{\sqrt{1-\alpha^2}}, \quad \text{a.s.}$$

The lim inf result of (7.2.3) is achieved via a symmetry argument. Hence it remains to prove that $X(n) - Y(n) \to 0$ as $n \to \infty$ a.s. in order to establish (7.2.4).

Define $Z(n) = X(n) - Y(n)$ for $n \geq 0$. Then $Z(0) = 0$ and

$$Z(n+1) = \alpha Z(n) + \frac{\beta}{n+1}\sum_{j=0}^{n} Z(j) + f(n+1), \quad n \geq 0 \tag{7.10.4}$$

where

$$f(n+1) = \frac{\beta}{n+1}\sum_{j=0}^{n} Y(j). \tag{7.10.5}$$

We examine now the rate at which f tends to zero as n tends to infinity, to do so requires an understanding of $\sum_{j=0}^{n} Y(j)$. Summing across the defining equation of the AR(1) process Y, gives

$$\sum_{j=0}^{n} Y(j) + Y(n+1) - Y(0) = \alpha \sum_{j=0}^{n} Y(j) + \sigma \sum_{j=0}^{n} \xi(j+1). \tag{7.10.6}$$

We have just shown

$$\limsup_{n\to\infty} \frac{|Y(n)|}{\sqrt{2\log n}} = \frac{\sigma}{\sqrt{1-\alpha^2}}, \quad \text{a.s.}$$

Therefore by this limit and Kolmogorov's strong law of large numbers [34, 5.2 Cor.2], we get from (7.10.6) that $\sum_{j=0}^{n} Y(j)/(n+1) \to 0$ as $n \to \infty$ a.s., and therefore that $f(n+1) \to 0$ as $n \to \infty$ a.s. Indeed from the Law of the Iterated Logarithm for independent Gaussian random variables we have

$$\limsup_{n\to\infty} \frac{|f(n+1)|}{n^{-1/2}\sqrt{2\log\log n}} = \frac{\sigma\beta}{1-\alpha}, \quad \text{a.s.} \tag{7.10.7}$$

From (7.10.4) we have, for $n \geq 1$,

$$Z(n) = \sum_{m=1}^{n} r(n,m) f(m) = r_1(n) \sum_{m=1}^{n} d_1(m) f(m) + r_2(n) \sum_{m=1}^{n} d_2(m) f(m) \tag{7.10.8}$$

and

$$|Z(n)| \leq |r_1(n)| \sum_{m=1}^{n} |d_1(m)||f(m)| + |r_2(n)| \sum_{m=1}^{n} |d_2(m)||f(m)|. \tag{7.10.9}$$

The second term on the righthand side converges to zero via a standard application of the Stolz–Cesàro Theorem (applicable as $|\alpha| < 1$) and (7.10.7). The limiting behaviour of the first term on the righthand side however differs depending on whether or not $\alpha + \beta$ is less than or equal to zero.

Firstly consider the case $\alpha + \beta < 1$. If $\frac{\beta}{\alpha-1} - \frac{1}{2} < -1$ (equivalently $\alpha + 2\beta > 1$), then there exists an almost surely finite random variable M such that

$$\limsup_{n\to\infty} \sum_{m=1}^{n} |d_1(m)||f(m)| \leq M \limsup_{n\to\infty} \sum_{m=1}^{n} m^{\frac{\beta}{\alpha-1} - 1/2} \sqrt{2\log\log m} < \infty.$$

Therefore,

$$\lim_{n\to\infty}\sum_{m=1}^{n} d_1(m)f(m) = \sum_{m=1}^{\infty} d_1(m)f(m) \in (-\infty,\infty), \quad \text{a.s.}$$

Indeed in this regime $r_1(n) \to 0$ as $n \to \infty$ and hence $|Z(n)| \to 0$ as $n \to \infty$.

If $\alpha+2\beta \leq 1$ then from (7.10.7) it can be seen that for any $0 < \epsilon < 1/2$, $f(n)/n^{-1/2+\epsilon} \to 0$ as $n \to \infty$ on some almost sure event Ω_2. Thus, there exists an almost surely finite positive random variable K_ϵ such that $|f(n,\omega)| \leq K_\epsilon(\omega)n^{-1/2+\epsilon}$ for all $n \geq 1$ and all $\omega \in \Omega_2$ and for some $0 < \epsilon < -1/2 - \beta/(\alpha-1)$ small enough. Also there exists an almost surely finite positive random variable M_ϵ such that

$$\begin{aligned}|r_1(n)|\sum_{m=1}^{n}|d_1(m)||f(m)| &\leq M_\epsilon(\omega)n^{-1-\frac{\beta}{\alpha-1}}\sum_{m=1}^{n} m^{\frac{\beta}{\alpha-1}-1/2+\epsilon}\\ &\leq M_\epsilon(\omega)M_1\frac{1}{\frac{\beta}{\alpha-1}+1/2+\epsilon}n^{-1/2+\epsilon},\end{aligned}$$

for each $\omega \in \Omega_2$, with the last inequality being a consequence of Theorem 7.7.1 for some positive constant M_1 as $\frac{\beta}{\alpha-1} - 1/2 < -1$. Thus we have

$$\lim_{n\to\infty} r_1(n)\sum_{m=1}^{n} d_1(m)f(m) = 0, \quad \text{a.s.}$$

Applying this in (7.10.8) gives that $\lim_{n\to\infty} Z(n) = 0$ a.s.

The remaining case $|\alpha| < 1$ and $\alpha+\beta = 1$. Here we have $r_1(n) \to 1$ as $n \to \infty$ while $d_1(m) \sim \frac{1}{\beta}m^{-1}$ as $m \to \infty$. As above $\sum_{j=1}^{\infty} d_1(j)f(j)$ is an almost surely finite random variable. Thus $\lim_{n\to\infty} Z(n) = \sum_{m=1}^{\infty} d_1(m)f(m)$ a.s. Indeed as Y may be expressed explicitly in terms of its initial value and a series of the noise terms we have for $n \geq 2$,

$$\sum_{m=1}^{n} d_1(m)f(m) = \sum_{m=1}^{n}\frac{\beta}{m}d_1(m)\sum_{j=0}^{m-1}\alpha^j x_0 + \sigma\sum_{l=1}^{n-1}\left(\sum_{m=l+1}^{n}\frac{\beta}{m}d_1(m)\sum_{j=l}^{m-1}\alpha^j\right)\alpha^{-l}\xi(l).$$

From Lemma 7.10.2 we have a formula for $\lim_{n\to\infty}\sum_{m=1}^{n} d_1(m)f(m)$ from which one may deduce the mean and variance as required.

It remains to consider the case when $\alpha = 0$. We start by presuming that $\beta \notin \{0,-1,-2,...\}$. Then from Lemma 7.7.2 we have $r(n,m) = R_1(n)D_1(m)$, $1 \leq m+1 \leq n$ where

$$\begin{aligned}R_1(n) &= \frac{\Gamma(\beta+n)}{n!} \sim n^{\beta-1}, \quad \text{as } n \to \infty,\\ D_1(m) &= \beta\frac{m!}{\Gamma(\beta+m+1)} \sim \beta m^{-\beta}, \quad \text{as } m \to \infty.\end{aligned}$$

Therefore X has the representation, for $n \geq 1$

$$X(n) = \frac{\beta x_0}{\Gamma(\beta+1)} R_1(n) + \sigma R_1(n) \sum_{m=1}^{n-1} D_1(m)\xi(m) + \sigma \xi(n).$$

As D_1 behaves asymptotically polynomially then we have $D_1 \in \ell^2(\mathbb{Z}^+)$ when $\beta > 1/2$, while for values of $\beta < 1/2$ application of Lemma 0.4.2 yields

$$\limsup_{n\to\infty} \frac{\sigma R_1(n) \sum_{m=1}^{n-1} D_1(m)\xi(m)}{n^{-1/2}\sqrt{\log\log n}} = \frac{\sigma\beta\sqrt{2}}{\sqrt{1-2\beta}}, \quad \text{a.s.}$$

A similar result holds for $\beta = 1/2$. Altogether this gives the result

$$\lim_{n\to\infty} R_1(n) \sum_{m=1}^{n-1} D_1(m)\xi(m) = 0, \quad \text{a.s.} \quad \beta < 1,$$
$$\lim_{n\to\infty} n^{1-\beta} R_1(n) \sum_{m=1}^{n-1} D_1(m)\xi(m) = \sum_{m=1}^{\infty} D_1(m)\xi(m), \quad \text{a.s.} \quad \beta = 1.$$

Recalling that a sequence of independent standard Gaussian random variables has a known asymptotic limit given by

$$\limsup_{n\to\infty} \sigma \frac{\xi(n)}{\sqrt{2\log n}} = -\liminf_{n\to\infty} \sigma \frac{\xi(n)}{\sqrt{2\log n}} = \sigma, \quad \text{a.s.} \tag{7.10.10}$$

See e.g., Problem IV.4.3.1 of [108]. Therefore for $\beta < 1$, we have

$$\lim_{n\to\infty} \{X(n) - \sigma\xi(n)\} = 0, \quad \text{a.s.,} \tag{7.10.11}$$

and for $\beta = 1$

$$\lim_{n\to\infty} \{X(n) - \sigma\xi(n)\} = \frac{\beta}{\Gamma(\beta+1)} x_0 + \sigma \sum_{m=1}^{\infty} D_1(m)\xi(m), \quad \text{a.s.,}$$

Thus, for $\beta \leq 1$,

$$\limsup_{n\to\infty} \frac{X(n)}{\sqrt{2\log n}} = -\liminf_{n\to\infty} \frac{X(n)}{\sqrt{2\log n}} = \sigma, \quad \text{a.s.} \tag{7.10.12}$$

Now consider the case $\beta = 0$. Then $X(n) = \sigma\xi(n)$, $n \geq 1$ and so the asymptotic behaviour of X is described by (7.10.10).

Lastly consider the case $\beta \in \{-1, -2, ...\}$. From (7.7.1), we have for $n \geq 2 - \beta$

$$X(n) = \sigma R_1(n) \sum_{m=-\beta}^{n-1} D_1(m)\xi(m) + \sigma\xi(n),$$

where R_1 and D_1 are given by (7.9.4). Again due to the polynomial asymptotic behaviour of R_1 and D_1 one can generate the results (7.10.11) and (7.10.12).

The proof of (7.2.5) and (7.2.7) is deferred to Lemma 7.10.3.

Lemma 7.10.3. *Let $|\alpha| < 1$. If $\alpha + \beta < 1$ then* (7.2.5) *holds, if $\alpha + \beta = 1$ then* (7.2.7) *holds.*

Proof of Lemma 7.10.3. Define the sequence $b_n = n$ for $n \geq 0$. Now it has already been shown that

$$\lim_{n\to\infty} \{X(n) - U(n)\} = L_2 = \begin{cases} 0, & \text{a.s., if } \alpha + \beta < 1, \\ L, & \text{a.s., if } \alpha + \beta = 1, \end{cases}$$

Define the sequence $a_n := \sum_{j=0}^{n} \{X(j) - U(j)\}$ for $n \geq 0$. Then a direct application of Theorem 7.7.1 gives

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n} \{X(j) - U(j)\} = L_2, \quad \text{a.s.}$$

Now from (7.10.6) and $U(n) = O(\sqrt{\log n})$ we have

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} U(j) = 0, \quad \text{a.s.}$$

Thus,

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} X(j) = \lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} \{X(j) - U(j)\} + \lim_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} U(j) = L_2, \quad \text{a.s.}$$

□

7.10.3 Proof of Theorem 7.2.2

Let Y and Z be as defined in the proof of Theorem 7.2.1. To attain a bound on the rate of $X - U$ tending to zero, the summation terms in (7.10.9) need to be analysed more carefully. From (7.10.7) we have

$$\limsup_{n\to\infty} \frac{f(n)}{n^{-1/2}\sqrt{2\log\log n}} = \frac{\sigma\beta}{1-\alpha} = -\liminf_{n\to\infty} \frac{f(n)}{n^{-1/2}\sqrt{2\log\log n}} = \frac{\sigma\beta}{1-\alpha}.$$

Observe that

$$\limsup_{n\to\infty} \frac{|\sum_{m=1}^{n} d_2(m) f(m)|}{\alpha^{-n} n^{-\frac{\beta}{\alpha-1}-1/2}\sqrt{\log\log n}} \leq \limsup_{n\to\infty} \frac{C\sum_{m=1}^{n} |d_2(m)| m^{-1/2}\sqrt{2\log\log m}}{\alpha^{-n} n^{-\frac{\beta}{\alpha-1}-1/2}\sqrt{\log\log n}} = C\frac{1}{1-\alpha},$$

for some positive random variable $C = C(\omega) > 0$. Consider first the case $\alpha + 2\beta > 1$, $\alpha + \beta < 1$. Then

$$\lim_{n\to\infty} \sum_{m=1}^{n} d_1(m) f(m) = \sum_{m=1}^{\infty} d_1(m) f(m)$$

and

$$\lim_{n\to\infty}\frac{|r_2(n)|\sum_{m=1}^{n}|d_2(m)||f(m)|}{r_1(n)}=0.$$

Therefore,

$$\limsup_{n\to\infty}\frac{|Z(n)|}{n^{-1-\frac{\beta}{\alpha-1}}}\le\sum_{m=1}^{\infty}|d_1(m)||f(m)|<+\infty,$$

whereas when $\alpha+2\beta<1$, we have

$$\limsup_{n\to\infty}\frac{|Z(n)|}{n^{-1/2}\sqrt{\log\log n}}<\infty.$$

While for $\alpha+2\beta=1$,

$$|r_1(n)\sum_{m=1}^{n}d_1(m)f(m)|\le C_2 n^{-1/2}\sum_{m=1}^{n}m^{-1}\sqrt{\log\log(m+\mathrm{e})},$$

for some positive random variable $C_2=C_2(\omega)$. As $g(m)=m^{-1}\sqrt{\log\log(m+\mathrm{e})}$ is an eventually decreasing sequence we may estimate the summation term via an integral, i.e. there exists an integer $N\in\mathbb{Z}^+$ such that

$$\sum_{m=N+1}^{n}m^{-1}\sqrt{\log\log(m+\mathrm{e})}\le\int_N^n s^{-1}\sqrt{\log\log(s+\mathrm{e})}\,ds$$

Now as

$$\int_N^n s^{-1}\sqrt{\log\log(s+\mathrm{e})}\,ds\sim\log n\sqrt{\log\log n},\quad n\to\infty,$$

we have

$$\limsup_{n\to\infty}\frac{|r_1(n)\sum_{m=1}^{n}d_1(m)f(m)|}{n^{-1/2}\log n\sqrt{\log\log n}}\in[0,\infty).$$

which concludes the proof.

7.11 Proof of Theorem 7.4.1

In advance of presenting the proof of Theorem 7.4.1, we must first state and prove some preliminary asymptotic results.

7.11.1 Preparatory results

In the proof of theorem 7.4.1 and 7.4.2, we require a Lemma giving Law of the Iterated Logarithm–type behaviour for a sum of weighted independent normal random variables. The variance of these random variables grows unboundedly. Such a result is Lemma 0.4.2

We also will require some elementary lemmata on the asymptotic behaviour of sequences. The proofs are provided here for completeness.

Lemma 7.11.1. *Let* $f : \mathbb{Z}^+ \to \mathbb{R}$, $g : \mathbb{Z}^+ \to (0, \infty)$, $h : \mathbb{Z}^+ \to (0, \infty)$ *and suppose*

$$\limsup_{n\to\infty} \frac{f(n)}{g(n)} = -\liminf_{n\to\infty} \frac{f(n)}{g(n)} = 1, \qquad g = o(h),$$

Then $\lim_{n\to\infty} f(n)/h(n) = 0$.

Proof. The first statement in the lemma is equivalent to the following: for all $\epsilon > 0$ there exists an $N_1 \in \mathbb{Z}^+$ such that,

$$1-\epsilon < \sup_{j\geq n}\left(\frac{f(j)}{g(j)}\right) < 1+\epsilon, \quad -1-\epsilon < \inf_{j\geq n}\left(\frac{f(j)}{g(j)}\right) < -1+\epsilon, \quad -\epsilon < \frac{g(n)}{h(n)} < \epsilon, \quad n \geq N_1.$$

Thus,

$$-(1+\epsilon) < \inf_{j\geq n}\left(\frac{f(j)}{g(j)}\right) \leq \frac{f(n)}{g(n)} \leq \sup_{j\geq n}\left(\frac{f(j)}{g(j)}\right) < 1+\epsilon, \quad n \geq N_1.$$

Hence by defining $\varepsilon := (1+\epsilon)\epsilon$,

$$-\varepsilon = -(1+\epsilon)\epsilon < -(1+\epsilon)\frac{g(n)}{h(n)} < \frac{f(n)}{h(n)} < (1+\epsilon)\frac{g(n)}{h(n)} < (1+\epsilon)\epsilon = \varepsilon, \quad n \geq N_1,$$

which gives $\lim_{n\to\infty} f(n)/h(n) = 0$, as required. ☐

Lemma 7.11.2. $g : \mathbb{Z}^+ \to (0, \infty)$, $h : \mathbb{Z}^+ \to (0, \infty)$ *and*

$$\lim_{n\to\infty} g(n) = g \in (0,\infty), \quad \limsup_{n\to\infty} h(n) = -\liminf_{n\to\infty} h(n) = h \in (0,\infty), \tag{7.11.1}$$

then

$$\limsup_{n\to\infty} g(n)h(n) = -\liminf_{n\to\infty} g(n)h(n) = gh.$$

Proof. From (7.11.1), we have that for all $\epsilon > 0$ there exists an $N(\epsilon) \in \mathbb{Z}^+$ such that for all $n \geq N$ we have

$$g-\epsilon < g(n) < g+\epsilon, \quad h-\epsilon < \sup_{j\geq n} h(j) < h+\epsilon, \quad -h-\epsilon < \inf_{j\geq n} h(j) < -h+\epsilon.$$

Observe that

$$\sup_{j\geq n}\{g(j)h(j)\} \leq \sup_{j\geq n}\{g(j)\}\sup_{j\geq n}\{h(j)\} < (g+\epsilon)(h+\epsilon) = gh + \epsilon(g+h) + \epsilon^2.$$

Also note that $(g-\epsilon)h(n) < g(n)h(n)$ for all $n \geq N$.

$$\begin{aligned}\sup_{j\geq n}\{g(j)h(j)\} \geq \sup_{j\geq n}\{(g-\epsilon)h(j)\} = (g-\epsilon)\sup_{j\geq n}\{h(j)\} &> (g-\epsilon)(h-\epsilon) \\ &= gh - \epsilon(g+h) + \epsilon^2 > gh - \epsilon(g+h) - \epsilon^2.\end{aligned}$$

Defining $\theta := \epsilon(g+h) + \epsilon^2$ gives

$$-\theta < \limsup_{n\to\infty}\{g(n)h(n)\} - gh < \theta.$$

A limit inferior argument follows similarly, which concludes the proof. ☐

7.11.2 Proof of Theorem 7.4.1

From (7.1.16) and the initial conditions of (7.1.5b) we have

$$r_1(n) \sim n^{-1+\beta/2}, \quad r_2(n) \sim (-1)^n n^{-\beta/2}, \quad \text{as } n \to \infty,$$
$$d_1(m) \sim \frac{\beta}{4} m^{-\beta/2}, \quad d_2(m) \sim (-1)^{-m} m^{\beta/2}, \quad \text{as } m \to \infty.$$

Thus using the Martingale Convergence Theorem, [108, 4.2.1], and Lemma 0.4.2 one has: If $\beta > 1$ then $d_1 \in \ell^2(\mathbb{Z}+)$ and $\lim_{n\to\infty} \sum_{j=1}^n d_1(j)\xi(j) = \sum_{j=1}^\infty d_1(j)\xi(j)$ a.s. If $\beta < 1$ then

$$\sum_{j=1}^n d_1^2(j) \sim \frac{\beta^2}{4^2} \frac{n^{1-\beta}}{(1-\beta)} \to \infty, \text{ as } n \to \infty$$

and so $d_1(n) = o\left(\sqrt{\sum_{j=1}^n d_1^2(j)}\right)$ as $n \to \infty$. Thus,

$$\limsup_{n\to\infty} \frac{\sum_{j=1}^n d_1(j)\xi(j)}{\sqrt{2\frac{\beta^2}{16}\frac{n^{1-\beta}}{(1-\beta)}\log\log n}} = -\liminf_{n\to\infty} \frac{\sum_{j=1}^n d_1(j)\xi(j)}{\sqrt{2\frac{\beta^2}{16}\frac{n^{1-\beta}}{(1-\beta)}\log\log n}} = 1, \quad \text{a.s.}$$

In the case $\beta = 1$, we have

$$\sum_{j=1}^n d_1^2(j) \sim \frac{\beta^2}{16} \log n, \quad \text{as } n \to \infty$$

and so

$$\limsup_{n\to\infty} \frac{\sum_{j=1}^n d_2(j)\xi(j)}{\sqrt{2\log n \log\log\log n}} = -\liminf_{n\to\infty} \frac{\sum_{j=1}^n d_2(j)\xi(j)}{\sqrt{2\log n \log\log\log n}} = \frac{\beta}{4}, \quad \text{a.s.}$$

Similarly when $\beta < -1$, we have $d_2 \in \ell^2(\mathbb{Z}^+)$ and so

$$\lim_{n\to\infty} \sum_{j=1}^n d_2(j)\xi(j) = \sum_{j=1}^\infty d_2(j)\xi(j) \in (-\infty, \infty), \quad \text{a.s.}$$

Also, if $\beta > -1$ then

$$\sum_{j=1}^n d_2^2(j) \sim \frac{n^{1+\beta}}{1+\beta} \to \infty, \text{ as } n \to \infty$$

and so $d_2(n) = o\left(\sqrt{\sum_{j=1}^n d_2^2(j)}\right)$ as $n \to \infty$. Thus,

$$\limsup_{n\to\infty} \frac{\sum_{j=1}^n d_2(j)\xi(j)}{\sqrt{2\frac{n^{1+\beta}}{(1+\beta)}\log\log n}} = -\liminf_{n\to\infty} \frac{\sum_{j=1}^n d_2(j)\xi(j)}{\sqrt{2\frac{n^{1+\beta}}{(1+\beta)}\log\log n}} = 1, \quad \text{a.s.} \tag{7.11.2}$$

In the case when $\beta = -1$, we have that

$$\sum_{j=1}^n d_2^2(j) \sim \log n, \quad \text{as } n \to \infty$$

and so

$$\limsup_{n\to\infty} \frac{\sum_{j=1}^n d_2(j)\xi(j)}{\sqrt{2\log n \log\log\log n}} = -\liminf_{n\to\infty} \frac{\sum_{j=1}^n d_2(j)\xi(j)}{\sqrt{2\log n \log\log\log n}} = \frac{\beta}{4}, \quad \text{a.s.} \tag{7.11.3}$$

(i): We next use Lemma 7.11.1 and 7.11.2. First recall the representation (7.1.6) for X which leads to

$$\frac{X(n)}{r_1(n)} = x_0 d_1(0) + x_0 d_2(0)\frac{r_2(n)}{r_1(n)} + \sigma\sum_{j=1}^n d_1(j)\xi(j) + \sigma\frac{r_2(n)}{r_1(n)}\sum_{j=1}^n d_2(j)\xi(j).$$

For $\beta > 3$, the second and third terms on the right–hand side converge to zero and to $\sigma\sum_{j=1}^\infty d_1(j)\xi(j)$ respectively, as $n\to\infty$. We show the fourth term converges to zero. Define

$$g(n) = \sqrt{2\frac{n^{1+\beta}}{(1+\beta)}\log\log n}, \quad h(n) = \frac{n^{-1+\beta/2}}{n^{-\beta/2}}.$$

Then we have $g, h > 0$ and $g = o(h)$. For the last term we write

$$\sigma\frac{r_2(n)}{r_1(n)}\sum_{j=1}^n d_2(j)\xi(j) = \sigma\frac{r_2(n)}{r_1(n)}h(n)\frac{\sum_{j=1}^n d_2(j)\xi(j)}{h(n)}.$$

Observing that $\{r_2(n)h(n)/r_1(n)\}$ is a bounded sequence on $\mathbb{Z}^+$, we may use Lemma 7.11.1 to obtain

$$\lim_{n\to\infty}\sigma\frac{r_2(n)}{r_1(n)}h(n)\frac{\sum_{j=1}^n d_2(j)\xi(j)}{h(n)} = 0,$$

as required.

(ii) For $1 < \beta \le 3$, define $w(n) = \sqrt{\frac{2}{\beta+1}}\sqrt{n\log\log n}$. The representation of X yields

$$\frac{X(n)}{w(n)} = d_1(0)\frac{r_1(n)}{w(n)}x_0 + d_2(0)\frac{r_2(n)}{w(n)}x_0 + \sigma\frac{r_1(n)}{w(n)}\sum_{j=1}^n d_1(j)\xi(j) \tag{7.11.4}$$
$$+\sigma r_2(n)(-1)^n n^{\beta/2}\frac{(-1)^n\sum_{j=1}^n d_2(j)\xi(j)}{n^{\beta/2}w(n)}.$$

Observe that $\lim_{n\to\infty} r_1(n)/w(n) = 0$. We thus have

$$\limsup_{n\to\infty}\frac{X(2n)}{\sqrt{\frac{2}{\beta+1}}\sqrt{2n\log\log n}} = \sigma\limsup_{n\to\infty}\frac{V(2n)}{(2n)^{\beta/2}\sqrt{\frac{2}{\beta+1}}\sqrt{2n\log\log n}}, \tag{7.11.5}$$

where $V(n) = (-1)^n\sum_{j=1}^n d_2(j)\xi(j)$, and a similar equation holds for the liminf, and for limit superiors and limit inferiors taken through the odd integers. We now seek to remove the alternating sign of the numerator in (7.11.4). Then

$$V(2n) = \sum_{j=1}^{2n} d_2(j)\xi(j) = \sum_{l=1}^n \delta_2(j)\xi_2(j) =: V_2(n)$$

where $\delta_2(j) = \sqrt{d_2(2j)^2 + d_2(2j-1)^2}$ and

$$\xi_2(j) = \frac{d_2(2j-1)}{\delta_2(j)}\xi(2j-1) + \frac{d_2(2j)}{\delta_2(j)}\xi(2j).$$

Lemma 0.4.2 can be applied to V_2. To see this, note that $\delta_2(n)^2 = d_2(2n)^2 + d_2(2n-1)^2 \sim 2(2n)^\beta$ as $n \to \infty$, and

$$\sum_{j=1}^{n} \delta_2(j)^2 = \sum_{j=1}^{2n} d_2(j)^2 \sim \sum_{j=1}^{2n} j^\beta \sim \frac{1}{1+\beta}(2n)^{\beta+1}, \qquad \text{as } n \to \infty.$$

Therefore

$$\limsup_{n\to\infty} \frac{V(2n)}{\sqrt{2\frac{1}{1+\beta}(2n)^{\beta+1}\log\log n}} = -\liminf_{n\to\infty} \frac{V(2n)}{\sqrt{2\frac{1}{1+\beta}(2n)^{\beta+1}\log\log n}} = 1.$$

Similarly we have

$$\limsup_{n\to\infty} \frac{V(2n+1)}{\sqrt{2\frac{1}{1+\beta}(2n)^{\beta+1}\log\log n}} = -\liminf_{n\to\infty} \frac{V(2n+1)}{\sqrt{2\frac{1}{1+\beta}(2n)^{\beta+1}\log\log n}} = 1.$$

Using these limits together with (7.11.5) gives

$$\limsup_{n\to\infty} \frac{X(n)}{\sqrt{\frac{2}{\beta+1}}\sqrt{n\log\log n}} = -\liminf_{n\to\infty} \frac{X(n)}{\sqrt{\frac{2}{\beta+1}}\sqrt{n\log\log n}} = \sigma, \quad \text{a.s.}$$

as required.

For $\beta \in (-1, 1)$, we do not have $\sum_{j=1}^{n} d_1(j)\xi(j)$ converging to a limit, as $n \to \infty$. We proceed as above, with w defined as before, so that one considers the asymptotic behaviour of X/w. Observe that

$$\sqrt{2\frac{\beta^2}{16}\frac{n^{1-\beta}}{(1-\beta)}\log\log n} = o\left(\frac{w(n)}{r_1(n)}\right), \quad \text{as } n \to \infty.$$

Thus, we have

$$\lim_{n\to\infty} \sigma\frac{r_1(n)}{w(n)}\sum_{j=1}^{n} d_1(j)\xi(j) = 0. \tag{7.11.6}$$

Following from (7.11.4) we have our result (with the same argument for the elimination of the alternating sign). For $\beta = 1$ the result follows similarly as

$$\sqrt{2\log n \log\log\log n} = o\left(\frac{w(n)}{r_1(n)}\right), \quad (n \to \infty).$$

and so (7.11.6) also holds.

(iv) For $\beta < -1$, define $u(n) = (-1)^n n^{-\beta/2}$. Firstly note that $r_1(n)/u(n)$ tends to zero as $n \to \infty$. Secondly note that

$$\sqrt{2\frac{\beta^2}{16}\frac{n^{1-\beta}}{(1-\beta)}\log\log n} = o\left(\frac{u(n)}{r_1(n)}\right), \qquad \text{as } n \to \infty.$$

Hence

$$\lim_{n\to\infty} \frac{r_1(n)}{u(n)} \sum_{j=1}^{n} d_1(j)\xi(j) = 0.$$

Thus

$$\lim_{n\to\infty} \frac{(-1)^n X(n)}{n^{-\beta/2}} = x_0 d_2(0) + \sigma \sum_{j=1}^{\infty} d_2(j)\xi(j), \quad \text{a.s.}$$

as required.

(iii) For $\beta = -1$, the largest fluctuations of $r_2(n)\sum_{j=1}^n d_2(j)\xi(j)$ dominate the growth of all other terms and so (7.11.3), together with the known asymptotic behaviour of r_2, gives

$$\limsup_{n\to\infty} \frac{X(n)}{(-1)^n\sqrt{2n\log n\log\log\log n}} = \limsup_{n\to\infty} \frac{r_2(n)\sum_{j=1}^n d_2(j)\xi(j)}{(-1)^n\sqrt{2n\log n\log\log\log n}} = \sigma.$$

Arguing in a similar fashion to that applied to the case $1 < \beta \le 3$, one can acquire the required result. The limit inferior result follows analogously.

7.12 Proof of Theorem 7.4.2 parts (i) and (iii)

This short section covers the rate of growth of solutions in the case $\alpha = 1$ and $\beta > 0$.

7.12.1 A preliminary lemma

We start with a preliminary estimate on the asymptotic behaviour rate of growth of d_3 and d_4.

Lemma 7.12.1. *If $\alpha = 1$ and $\beta > 0$ then d_3 and d_4 defined in* (7.1.25) *obey* (7.1.26).

Proof. Using the initial conditions (7.1.5b) one can solve (7.1.18) for d_3 and d_4, this yields

$$d_3(m) = \frac{1}{C(m)}\left(r_4(m+1) - \big(\alpha + \beta/(m+1)\big)r_4(m)\right),$$
$$d_4(m) = \frac{1}{C(m)}\left(-r_3(m+1) + \big(\alpha + \beta/(m+1)\big)r_3(m)\right),$$
$$C(m) = r_3(m)r_4(m+1) - r_3(m+1)r_4(m).$$

Although Lemma 7.1.2 gives $C(m) = C(0)(m+1)^{-1} \sim C(0)m^{-1}$ as $m \to \infty$, in order to obtain the value of $C(0)$ is it necessary to use the explicit asymptotic estimates of r_3 and r_4. Inserting these estimates into the formula for the Casoratian yields

$$C(m) = \frac{m^{-1/4}}{(m+1)^{\frac{1}{4}}}\left((\mathrm{e}^{K_m} - \mathrm{e}^{-K_m})(1 + \frac{c_1 + c_2}{\sqrt{m}}) + \mathrm{e}^{K_m} O\left(\frac{1}{m}\right) - \mathrm{e}^{-K_m} O\left(\frac{1}{m}\right)\right),$$

where $K_m = 2\sqrt{\beta}(\sqrt{m} - \sqrt{m+1})$. Observe

$$\begin{aligned} K_m &= -\sqrt{\beta} m^{-1/2}\left(1 + O(m^{-1})\right), \quad \text{as } m \to \infty, \\ \mathrm{e}^x &= 1 + x + O(x^2), \text{ as } x \to 0 \\ \mathrm{e}^x - \mathrm{e}^{-x} &= 2\sinh(x) = 2x + O(x^3), \text{ as } x \to 0. \end{aligned} \tag{7.12.1}$$

Thus $\mathrm{e}^{K_m} O(m^{-1}) - \mathrm{e}^{-K_m} O(m^{-1}) = O(m^{-1})$ as $m \to \infty$, and hence we obtain

$$\begin{aligned} &\frac{C(m)}{m^{-1/4}(m+1)^{-1/4}} \\ &= 2\sinh(K_m)(1 + \frac{c_1 + c_2}{\sqrt{m}}) + O(m^{-1}) \\ &= 2K_m(1 + O(m^{-1}))\left(1 + \frac{c_1 + c_2}{\sqrt{m}}\right) + O(m^{-1}) \\ &= -2\sqrt{\beta} m^{-1/2}\left(1 + O(m^{-1})\right)\left(1 + O(m^{-1})\right)\left(1 + O(m^{-1/2})\right) + O(m^{-1}) \\ &= -2\sqrt{\beta} m^{-1/2} + O(m^{-1}). \end{aligned}$$

Therefore

$$C(m) = -2\sqrt{\beta} m^{-1} + O(m^{-3/2}), \qquad \text{as } m \to \infty. \tag{7.12.2}$$

Hence $C(0) = \lim_{m \to \infty} m\, C(m) = -2\sqrt{\beta}$. Next we define

$$\begin{aligned} D_3(m) &:= r_4(m+1) - \left(1 + \frac{\beta}{m+1}\right) r_4(m) \\ &= \mathrm{e}^{-2\sqrt{\beta(m+1)}}(m+1)^{-1/4}\left(1 + c_2 m^{-1/2} + O(m^{-1})\right) \\ &\quad - \mathrm{e}^{-2\sqrt{\beta m}} m^{-1/4}\left(1 + c_2 m^{-1/2} + O(m^{-1})\right) \\ &\quad\quad - \frac{\beta}{m+1}\mathrm{e}^{-2\sqrt{\beta m}} m^{-1/4}\left(1 + c_2 m^{-1/2} + O(m^{-1})\right). \end{aligned}$$

Observing that $(m+1)^r = m^r + r\,m^{r-1} + O(m^{r-2})$ as $m \to \infty$ gives

$$\begin{aligned}
D_3(m) &= \mathrm{e}^{-2\sqrt{\beta(m+1)}}[m^{-1/4} + c_2 m^{-3/4} + O(m^{-5/4})] \\
&\quad - \mathrm{e}^{-2\sqrt{\beta m}}[m^{-1/4} + c_2 m^{-3/4} + O(m^{-5/4})] \\
&\quad\quad - \beta \mathrm{e}^{-2\sqrt{\beta m}}[m^{-5/4} + c_2 m^{-7/4} + O(m^{-9/4})] \\
&= (m^{-1/4} + c_2 m^{-3/4})(\mathrm{e}^{-2\sqrt{\beta(m+1)}} - \mathrm{e}^{-2\sqrt{\beta m}}) + \mathrm{e}^{-2\sqrt{\beta(m+1)}} O(m^{-5/4}) \\
&\quad + \mathrm{e}^{-2\sqrt{\beta m}} O(m^{-5/4}) \\
&= (m^{-1/4} + c_2 m^{-3/4})(\mathrm{e}^{-2\sqrt{\beta(m+1)}} - \mathrm{e}^{-2\sqrt{\beta m}}) + O(\mathrm{e}^{-2\sqrt{\beta m}} m^{-5/4}).
\end{aligned}$$

By the mean value theorem, we have that

$$\mathrm{e}^{-2\sqrt{\beta(m+1)}} - \mathrm{e}^{-2\sqrt{\beta m}} \sim -\sqrt{\beta} m^{-1/2} \mathrm{e}^{-2\sqrt{\beta m}}, \quad \text{as } m \to \infty.$$

This then gives

$$D_3(m) = -\sqrt{\beta} m^{-3/4} \mathrm{e}^{-2\sqrt{\beta m}} + O(\mathrm{e}^{-2\sqrt{\beta m}} m^{-5/4}). \tag{7.12.3}$$

From (7.12.2) and (7.12.3) we thus have

$$d_3(m) = \frac{D_3(m)}{C(m)} = \frac{1}{2} m^{1/4} \mathrm{e}^{-2\sqrt{\beta m}}(1 + O(m^{-1/2}), \quad \text{as } m \to \infty.$$

A similar analysis yields the asymptotic behaviour of d_4. □

7.12.2 Proof of Theorem 7.4.2 parts (i) and (iii)

We start with the proof of part (i).

(i) We begin with the case $\beta > 0$. We have (7.1.23) from the Birkhoff–Adams Theorem. From Lemma 7.12.1 we have that (7.1.26) holds. We thus remark that $d_3 \in \ell^2(\mathbb{Z}^+)$ and hence $\sum_{j=1}^{\infty} d_3(j)\xi(j)$ exists. Regarding d_4, use of Theorem 7.7.1 gives

$$\sum_{j=1}^{n} d_4^2(j) \sim \frac{1}{8\sqrt{\beta}} n\, \mathrm{e}^{4\sqrt{\beta n}}, \quad \text{as } n \to \infty,$$

and hence $d_4(n) = o\left(\sqrt{\sum_{j=1}^{n} d_4^2(j)}\right)$ as $n \to \infty$. From Lemma 0.4.2 we have

$$\sum_{j=1}^{n} d_4(j)\xi(j) = O\left(\sqrt{2\sum_{j=1}^{n} d_4^2(j) \log\log\left(\sum_{j=1}^{n} d_4^2(j)\right)}\right) = O\left(n^{1/2}\mathrm{e}^{2\sqrt{\beta n}}\sqrt{\log n}\right),$$

as $n \to \infty$, so we see that

$$\frac{r_4(n)}{r_3(n)} \sum_{j=1}^{n} d_4(j)\xi(j) = O(n^{1/2}\mathrm{e}^{-2\sqrt{\beta n}}\sqrt{\log n}), \quad \text{as } n \to \infty.$$

Using the representation

$$\frac{X(n)}{r_3(n)} = d_3(0)x_0 + d_4(0)\frac{r_4(n)}{r_3(n)}x_0 + \sigma\sum_{j=1}^{n} d_3(j)\xi(j) + \sigma\frac{r_4(n)}{r_3(n)}\sum_{j=1}^{n} d_4(j)\xi(j)$$

and applying the estimates deduced above, we arrive at

$$\lim_{n\to\infty}\frac{X(n)}{r_3(n)} = d_3(0)x_0 + \sigma\sum_{j=1}^{\infty} d_3(j)\xi(j).$$

(iii) Let $\beta = 0$. Then X has the representation $X(n) = x_0 + \sigma\sum_{j=1}^{n}\xi(j)$ for $n \geq 1$, so X clearly obeys the Law of the Iterated Logarithm.

7.13 Proof of Theorem 7.4.2 (ii)

Lemma 7.13.1. *Let $\alpha = 1$ and $\beta < 0$. Then d_5 and d_6 have the asymptotic expansions* (7.1.28).

Proof. We firstly determine the value of $C(0)$ in Lemma 7.1.2. Recall the Taylor series representations of sine and cosine, i.e. $\sin x = x + O(x^3)$ and $\cos x = 1 + O(x^2)$ as $x \to 0$. Since $C(m) = r_5(m)r_6(m+1) - r_5(m+1)r_6(m)$ we have

$$\begin{aligned}
&\frac{C(m)}{m^{-1/4}(m+1)^{-1/4}} \\
&= \Bigg(\cos(2\sqrt{|\beta|m})\sin(2\sqrt{|\beta|(m+1)})\left(1 + \frac{c_1' + c_2'}{\sqrt{m}} + O(m^{-1})\right) \\
&\qquad - \sin(2\sqrt{|\beta|m})\cos(2\sqrt{|\beta|(m+1)})\left(1 + \frac{c_1' + c_2'}{\sqrt{m}} + O(m^{-1})\right)\Bigg) \\
&= \sin(2\sqrt{|\beta|(m+1)} - 2\sqrt{|\beta|m})\left(1 + \frac{c_1' + c_2'}{\sqrt{m}}\right) + O(m^{-1}) \\
&= \left(\sqrt{|\beta|}m^{-1/2} + O(m^{-3/2})\right)\left(1 + \frac{c_1' + c_2'}{\sqrt{m}}\right) + O(m^{-1}) \\
&= m^{-1/2}\sqrt{|\beta|} + O(m^{-1}),
\end{aligned}$$

where we have used the fact that $2\sqrt{|\beta|(m+1)} - 2\sqrt{|\beta|m} = \sqrt{|\beta|}m^{-1/2} + O(m^{-3/2})$ as $m \to \infty$. Hence $C(m) = m^{-1}\sqrt{|\beta|} + O(m^{-3/2})$ as $m \to \infty$. From Lemma 7.1.2 we have $C(0) = (m+1)C(m)$, so by letting $m \to \infty$ across this equation gives $C(0) = \sqrt{|\beta|}$. Define

$$D_6(m) := r_6(m+1) - \left(1 + \frac{\beta}{m+1}\right)r_6(m).$$

Then as $(m+1)^{-1/4} = m^{-1/4}(1+O(m^{-1}))$, we have

$$\begin{aligned}
D_6(m) &= (m+1)^{-1/4}\sin(2\sqrt{|\beta|(m+1)})\left(1+\frac{c_2'}{\sqrt{m}}+O(m^{-1})\right) \\
&\quad -\left(1+\frac{\beta}{m+1}\right)m^{-1/4}\sin(2\sqrt{|\beta|m})\left(1+\frac{c_2'}{\sqrt{m}}+O(m^{-1})\right) \\
&= m^{-1/4}(1+O(m^{-1}))\sin(2\sqrt{|\beta|(m+1)})\left(1+\frac{c_2'}{\sqrt{m}}+O(m^{-1})\right) \\
&\quad - m^{-1/4}\sin(2\sqrt{|\beta|m})\left(1+\frac{c_2'}{\sqrt{m}}+O(m^{-1})\right)+O(m^{-5/4}) \\
&= m^{-1/4}\sin(2\sqrt{|\beta|(m+1)})\left(1+\frac{c_2'}{\sqrt{m}}\right) \\
&\quad - m^{-1/4}\sin(2\sqrt{|\beta|m})\left(1+\frac{c_2'}{\sqrt{m}}\right)+O(m^{-5/4}) \\
&= m^{-1/4}\left(\sin(2\sqrt{|\beta|(m+1)})-\sin(2\sqrt{|\beta|m})\right)\left(1+\frac{c_2'}{\sqrt{m}}\right)+O(m^{-5/4}).
\end{aligned}$$

Since $\sqrt{m+1}-\sqrt{m} = 1/2m^{-1/2}(1+O(m^{-1})) \to 0$ as $m\to\infty$, we have

$$\begin{aligned}
&\sin(2\sqrt{|\beta|(m+1)})-\sin(2\sqrt{|\beta|m}) \\
&= \cos(2\sqrt{|\beta|m})\left(2\sqrt{|\beta|(m+1)}-2\sqrt{|\beta|m}\right)+O(m^{-1}) \\
&= \cos(2\sqrt{|\beta|m})\sqrt{|\beta|}m^{-1/2}(1+O(m^{-1}))+O(m^{-1}).
\end{aligned}$$

Hence

$$\begin{aligned}
D_6(m) &= m^{-3/4}\left(\cos(2\sqrt{|\beta|m})\sqrt{|\beta|}(1+O(m^{-1}))+O(m^{-1/2})\right)\left(1+\frac{c_2'}{\sqrt{m}}\right) \\
&\quad + O(m^{-5/4}) \\
&= \sqrt{|\beta|}m^{-3/4}\cos(2\sqrt{|\beta|m})\left(1+O(m^{-1})\right)\left(1+\frac{c_2'}{\sqrt{m}}\right)+O(m^{-5/4}) \\
&= \sqrt{|\beta|}m^{-3/4}\cos(2\sqrt{|\beta|m})\left(1+O(m^{-1/2})\right)+O(m^{-5/4}) \\
&= \sqrt{|\beta|}m^{-3/4}\cos(2\sqrt{|\beta|m})+O(m^{-5/4}).
\end{aligned}$$

Thus we have

$$d_5(m) = \frac{D_6(m)}{C(m)} = m^{\frac{1}{4}}\cos(2\sqrt{|\beta|m})+O(m^{-\frac{1}{4}}) = m^{\frac{1}{4}}\cos(2\sqrt{|\beta|m})(1+O(m^{-\frac{1}{2}})),$$

as required. One can similarly obtain the asymptotic behaviour of d_6. □

Lemma 7.13.2. *Define the real–valued sequences f_5,f_6,g_5 and g_6 for $n \geq 0$ by*

$$f_5(n) = n^{-1/4}\cos(2\sqrt{|\beta|n}),\quad f_6(n) = n^{-1/4}\sin(2\sqrt{|\beta|n}),$$
$$g_5(n) = n^{1/4}\cos(2\sqrt{|\beta|n}),\quad g_6(n) = n^{1/4}\sin(2\sqrt{|\beta|n}).$$

Then

$$\limsup_{n\to\infty} \frac{f_5(n)\sum_{j=0}^{n} g_5(j)\xi(j) + f_6(n)\sum_{j=0}^{n} g_6(j)\xi(j)}{\sqrt{2n\log\log n}} \le \frac{2}{\sqrt{3}}, \quad a.s. \tag{7.13.1}$$

and

$$\liminf_{n\to\infty} \frac{f_5(n)\sum_{j=0}^{n} g_5(j)\xi(j) + f_6(n)\sum_{j=0}^{n} g_6(j)\xi(j)}{\sqrt{2n\log\log n}} \ge -\frac{2}{\sqrt{3}}, \quad a.s. \tag{7.13.2}$$

Proof. Define for $i = \{5, 6\}$ and $n \ge 1$,

$$M_i(n) = \sum_{j=1}^{n} g_i(j)\xi(j).$$

In order to show that $M_i(n)$ obeys the Law of the Iterated Logarithm we now establish bounds upon the rate of growth of the quadratic variation of M_5 and M_6. Firstly consider M_6.

$$\langle M_6\rangle(n) = \sum_{j=1}^{n} j^{1/2}\sin^2(2\sqrt{|\beta|j}) \le \sum_{j=1}^{n} j^{1/2}.$$

Now from Theorem 7.7.1 it is seen that

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} j^{1/2}}{n^{3/2}} = \frac{2}{3}.$$

Thus,

$$\limsup_{n\to\infty} \frac{\langle M_6\rangle(n)}{n^{3/2}} \le \frac{2}{3}. \tag{7.13.3}$$

To establish a lower bound, define the sequence

$$t_j = \left[\left(\frac{2j\pi + \pi/2}{2\sqrt{|\beta|}}\right)^2\right], \quad j \ge j_1. \tag{7.13.4}$$

and the function $N = N(n)$, for $n \ge n_1$

$$N = \left[\frac{2\sqrt{|\beta|n} - \pi/2}{2\pi}\right], \tag{7.13.5}$$

where

$$j_1 := \max(1, \left[\frac{|\beta|}{\pi^2} - \frac{1}{4}\right] + 1), \quad n_1 = \max(j_1, \left[\frac{\pi^2}{16|\beta|}\right] + 1).$$

Here j_1 is chosen so that $t_{j+1} > t_j$ and n_1 is chosen so that $N > j_1$.

Then $t_j \sim j^2\pi^2/|\beta|$ as $j \to \infty$ and $N(n) \sim \sqrt{|\beta|}\sqrt{n}/\pi$ as $n \to \infty$ and $t_N \le n$. Now,

$$\langle M_6\rangle(n) \ge \sum_{j=j_1}^{N} t_j^{1/2}\sin^2(2\sqrt{|\beta|t_j}). \tag{7.13.6}$$

The sequence $\{t_j\}$ is defined in such a way that the trigonometric term on the righthand side of the above inequality is close to one. Now define the sequence

$$t_j^* = \left(\frac{2j\pi + \pi/2}{2\sqrt{|\beta|}}\right)^2, \quad j \geq 1$$

so that $\sin(2\sqrt{|\beta| t_j^*}) = 1$ and $\cos(2\sqrt{|\beta| t_j^*}) = 0$. Also, $t_j \sim \frac{j^2\pi^2}{|\beta|}$ as $j \to \infty$. Now,

$$\begin{aligned}
\sin\left(2\sqrt{|\beta| t_j}\right) &= \sin\left(2\sqrt{|\beta|}\left[\sqrt{t_j} - \sqrt{t_j^*}\right] + 2\sqrt{|\beta|}\sqrt{t_j^*}\right) \\
&= \cos\left(2\sqrt{|\beta|}\left[\sqrt{t_j} - \sqrt{t_j^*}\right]\right)\sin\left(2\sqrt{|\beta|}\sqrt{t_j^*}\right) \\
&\quad - \sin\left(2\sqrt{|\beta|}\left[\sqrt{t_j} - \sqrt{t_j^*}\right]\right)\cos\left(2\sqrt{|\beta|}\sqrt{t_j^*}\right) \\
&= \cos\left(2\sqrt{|\beta|}\left[\sqrt{t_j} - \sqrt{t_j^*}\right]\right).
\end{aligned}$$

Observe that $-1 < t_j - t_j^* \leq 0$ and so

$$\left|\sqrt{t_j} - \sqrt{t_j^*}\right| = \frac{|t_j - t_j^*|}{\sqrt{t_j} + \sqrt{t_j^*}} < \frac{1}{\sqrt{t_j} + \sqrt{t_j^*}}.$$

From the known asymptotic behaviour of both t_j and t_j^* we have

$$\limsup_{j\to\infty} \frac{\left|\sqrt{t_j} - \sqrt{t_j^*}\right|}{\sqrt{|\beta|}/(2j\pi)} \leq 1$$

and in particular $\lim_{j\to\infty}\left\{\sqrt{t_j} - \sqrt{t_j^*}\right\} = 0$. Therefore

$$\lim_{j\to\infty} \sin\left(2\sqrt{|\beta| t_j}\right) = \lim_{j\to\infty} \cos\left(2\sqrt{|\beta|}\left[\sqrt{t_j} - \sqrt{t_j^*}\right]\right) = 1.$$

Then for any fixed $\epsilon \in (0,1)$ there exists an $T(\epsilon) \in \mathbb{Z}^+$ such that $T(\epsilon) > j_1$ and for all $j \geq T(\epsilon)$

$$\sin\left(2\sqrt{|\beta| t_j}\right) > 1 - \epsilon.$$

Therefore, from (7.13.6) we get

$$\langle M_6\rangle(n) \geq (1-\epsilon)^2 \sum_{j=T+1}^{N} t_j^{1/2} + \sum_{j=j_1}^{T} t_j^{1/2} \sin^2\left(2\sqrt{|\beta| t_j}\right).$$

It can be seen from Theorem 7.7.1 that

$$\sum_{j=T+1}^{N} t_j^{1/2} \sim \frac{\pi}{\sqrt{|\beta|}}\frac{N^2}{2}, \quad \text{as } N \to \infty,$$

and hence

$$\sum_{j=T+1}^{N(n)} t_j^{1/2} \sim \frac{\sqrt{|\beta|}}{2\pi} n, \quad \text{as } n \to \infty.$$

Therefore

$$\liminf_{n\to\infty} \frac{\langle M_6\rangle(n)}{n} \geq (1-\epsilon)^2 \frac{\sqrt{|\beta|}}{2\pi}$$

and letting ϵ tend to zero gives

$$\liminf_{n\to\infty} \frac{\langle M_6\rangle(n)}{n} \geq \frac{\sqrt{|\beta|}}{2\pi}. \tag{7.13.7}$$

Thus $\langle M_6\rangle(n) \to \infty$ as $n \to \infty$. Defining $\sigma_n^2 := n^{1/2}\sin^2(2\sqrt{|\beta|\sqrt{n}})$ then from (7.13.7) we have

$$\limsup_{n\to\infty} \frac{\sigma_n^2}{\langle M_6\rangle(n)} \leq \limsup_{n\to\infty} \frac{n^{1/2}\sin^2(2\sqrt{|\beta|\sqrt{n}})}{\frac{\sqrt{|\beta|}}{2\pi}n} = 0,$$

Hence

$$\lim_{n\to\infty} \frac{\sigma_n^2}{\langle M_6\rangle(n)} = 0$$

and so Lemma 0.4.2 may be applied to M_6 to conclude

$$\limsup_{n\to\infty} \frac{M_6(n)}{\sqrt{2\langle M_6\rangle(n)\log\log\langle M_6\rangle(n)}} = 1, \quad \text{a.s.} \tag{7.13.8a}$$

$$\liminf_{n\to\infty} \frac{M_6(n)}{\sqrt{2\langle M_6\rangle(n)\log\log\langle M_6\rangle(n)}} = -1, \quad \text{a.s.} \tag{7.13.8b}$$

If in place of the sequence (7.13.4) and the function (7.13.5) one considers

$$t_j^{(5)} = \left[\left(\frac{2j\pi}{2\sqrt{|\beta|}}\right)^2\right], \quad j \geq j_2$$

and

$$N^{(5)}(n) = \left[\frac{2\sqrt{|\beta|}\sqrt{n}}{2\pi}\right], \quad n \geq n_2.$$

respectively, where j_2 and n_2 are chosen so that $t_{j+1}^{(5)} > t_j^{(5)}$ and $N^{(5)}(n) > j_2$. Then it can be demonstrated that the limits

$$\limsup_{n\to\infty} \frac{M_5(n)}{\sqrt{2\langle M_5\rangle(n)\log\log\langle M_5\rangle(n)}} = 1, \quad \text{a.s.} \tag{7.13.9a}$$

$$\liminf_{n\to\infty} \frac{M_5(n)}{\sqrt{2\langle M_5\rangle(n)\log\log\langle M_5\rangle(n)}} = -1, \quad \text{a.s.} \tag{7.13.9b}$$

hold, with the proof being analogous to that of (7.13.8). Moreover, at an intermediate stage, it can be shown that

$$\limsup_{n\to\infty} \frac{\langle M_5\rangle(n)}{n^{3/2}} \leq \frac{2}{3}, \quad \liminf_{n\to\infty} \frac{\langle M_5\rangle(n)}{n} \geq \frac{\sqrt{|\beta|}}{2\pi}. \tag{7.13.10}$$

So, (7.13.3) together with (7.13.8) gives

$$\limsup_{n\to\infty}\frac{|M_6(n)|}{\sqrt{2}n^{3/4}\sqrt{\log\log n}}$$
$$\leq \limsup_{n\to\infty}\frac{\sqrt{2\langle M_6\rangle(n)\log\log\langle M_6\rangle(n)}}{\sqrt{2}n^{3/4}\sqrt{\log\log n}}\cdot\frac{|M_6(n)|}{\sqrt{2\langle M_6\rangle(n)\log\log\langle M_6\rangle(n)}}\leq\frac{\sqrt{2}}{\sqrt{3}}.$$

Also, (7.13.10) together with (7.13.9) gives

$$\limsup_{n\to\infty}\frac{|M_5(n)|}{\sqrt{2}n^{3/4}\sqrt{\log\log n}}\leq\frac{\sqrt{2}}{\sqrt{3}}.$$

Thus for any fixed $\epsilon>0$ there exists $T_1\in\mathbb{Z}^+$ such that for all $n\geq T_1$

$$\left(\frac{f_5(n)\sum_{j=1}^n g_5(j)\xi(j)+f_6(n)\sum_{j=1}^n g_6(j)\xi(j)}{\sqrt{2n\log\log n}}\right)^2$$
$$\leq 2\cos^2(2\sqrt{|\beta|n})\left(\frac{n^{-1/4}M_5(n)}{\sqrt{2n\log\log n}}\right)^2+2\sin^2(2\sqrt{|\beta|n})\left(\frac{n^{-1/4}M_6(n)}{\sqrt{2n\log\log n}}\right)^2$$
$$\leq 2\cos^2(2\sqrt{|\beta|n})\frac{2}{3}(1+\epsilon)^2+2\sin^2(2\sqrt{|\beta|n})\frac{2}{3}(1+\epsilon)^2=\frac{4}{3}(1+\epsilon)^2. \quad (7.13.11)$$

Taking square roots across this inequality, and then taking the limit superior, we arrive at

$$\limsup_{n\to\infty}\frac{\left|f_5(n)\sum_{j=1}^n g_5(j)\xi(j)+f_6(n)\sum_{j=1}^n g_6(j)\xi(j)\right|}{\sqrt{2n\log\log n}}\leq\frac{2(1+\epsilon)}{\sqrt{3}},\quad \text{a.s.}$$

Finally, letting ϵ tend to zero allows one to obtain the desired results (7.13.1) and (7.13.2).

□

Proof of Theorem 7.4.2 (ii). X has the representation, for $n\geq 1$

$$X(n)=r_5(n)d_5(0)x_0+r_6(n)d_6(0)x_0+\sigma r_5(n)\sum_{j=1}^n d_5(j)\xi(j)+\sigma r_6(n)\sum_{j=1}^n d_6(j)\xi(j),$$

where r_5,r_6,d_5 and d_6 are given by (7.1.27) and (7.1.28). Thus, for $1\leq m\leq n$, we define the remainder terms R_5,R_6,D_5 and D_6 by

$$r_5(n)=f_5(n)+R_5(n),\quad d_5(m)=g_5(m)+D_5(m)$$
$$r_6(n)=f_6(n)+R_6(n),\quad d_6(m)=g_6(m)+D_6(m)$$

where f_5,f_6,g_5 and g_6 are given by Lemma 7.13.2. Therefore we have $R_5(n)=O(n^{-3/4})$, $R_6(n)=O(n^{-3/4})$, $D_5(m)=O(m^{-1/4})$ and $D_6(m)=O(m^{-1/4})$. We now decompose X

as

$$\frac{X(n)}{\sqrt{2n\log\log n}} = \frac{r(n,0)x_0}{\sqrt{2n\log\log n}} + \frac{\sigma f_5(n)\sum_{j=1}^n g_5(j)\xi(j)}{\sqrt{2n\log\log n}} + \frac{\sigma f_6(n)\sum_{j=1}^n g_6(j)\xi(j)}{\sqrt{2n\log\log n}}$$
$$+ \frac{\sigma f_5(n)\sum_{j=1}^n D_5(j)\xi(j)}{\sqrt{2n\log\log n}} + \frac{\sigma f_6(n)\sum_{j=1}^n D_6(j)\xi(j)}{\sqrt{2n\log\log n}}$$
$$+ \frac{\sigma R_5(n)\sum_{j=1}^n d_5(j)\xi(j)}{\sqrt{2n\log\log n}} + \frac{\sigma R_6(n)\sum_{j=1}^n d_6(j)\xi(j)}{\sqrt{2n\log\log n}}. \tag{7.13.12}$$

We now wish to ascertain an upper bound in the largest fluctuations of X. The first term on the lefthand side of the above tends to zero as $n \to \infty$ and so does not contribute to the limit superior. The limits superior and inferior of the second and third terms are described in Lemma 7.13.2. We now show that all other terms have a zero limit as $n \to \infty$.

Considering the fourth term we have

$$\sum_{j=1}^n D_5(j)^2 \leq M \sum_{j=1}^n j^{-1/2} \sim 2Mn^{1/2}, \quad \text{as } n \to \infty,$$

for some positive constant M. Then it is a consequence of the Borel–Cantelli Lemma that

$$\limsup_{n\to\infty} \frac{|\sum_{j=1}^n D_5(j)\xi(j)|}{\sqrt{2\sum_{j=1}^n D_5(j)^2 \log n}} \leq 1, \quad \text{a.s.}$$

Hence

$$\limsup_{n\to\infty} \frac{|\sum_{j=1}^n D_5(j)\xi(j)|}{\sqrt{4M}n^{1/4}\sqrt{\log n}} \leq 1.$$

Thus

$$\limsup_{n\to\infty} \left| \frac{f_5(n)\sum_{j=1}^n D_5(j)\xi(j)}{\sqrt{2n\log\log n}} \right|$$
$$= \limsup_{n\to\infty} \frac{n^{-1/4}|\cos(2\sqrt{|\beta n|})|}{\sqrt{2n\log\log n}} \sqrt{4M}n^{1/4}\sqrt{\log n} \frac{\left|\sum_{j=1}^n D_5(j)\xi(j)\right|}{\sqrt{4M}n^{1/4}\sqrt{\log n}}$$
$$\leq \limsup_{n\to\infty} \frac{n^{-1/4}|\cos(2\sqrt{|\beta n|})|}{\sqrt{2n\log\log n}} \sqrt{4M}n^{1/4}\sqrt{\log n} = 0$$

or

$$\lim_{n\to\infty} \frac{f_5(n)\sum_{j=1}^n D_5(j)\xi(j)}{\sqrt{2n\log\log n}} = 0, \quad \text{a.s.} \tag{7.13.13}$$

One similarly argues that

$$\lim_{n\to\infty} \frac{f_6(n)\sum_{j=1}^n D_6(j)\xi(j)}{\sqrt{2n\log\log n}} = 0, \quad \text{a.s.} \tag{7.13.14}$$

Now for the sixth term on the right hand side of (7.13.12), observe that from (7.1.28) and Theorem 7.7.1 we have

$$\sum_{j=1}^n d_5(j)^2 = O(n^{3/2})$$

and so

$$\limsup_{n\to\infty} \frac{|\sum_{j=1}^n d_5(j)\xi(j)|}{\sqrt{2M}n^{3/4}\sqrt{\log n}} \leq \limsup_{n\to\infty} \frac{|\sum_{j=1}^n d_5(j)\xi(j)|}{\sqrt{2\sum_{j=1}^n d_5(j)^2 \log n}} \leq 1,$$

for some positive constant M. Thus,

$$\begin{aligned}
&\limsup_{n\to\infty} \left| \frac{R_5(n)\sum_{j=1}^n d_5(j)\xi(j)}{\sqrt{2n\log\log n}} \right| \\
&= \limsup_{n\to\infty} \frac{K_1 n^{-3/4}}{\sqrt{2n\log\log n}} \sqrt{2M}n^{3/4}\sqrt{\log n} \frac{\left|\sum_{j=1}^n d_5(j)\xi(j)\right|}{\sqrt{2M}n^{3/4}\sqrt{\log n}} \\
&\leq \limsup_{n\to\infty} \frac{K_1 n^{-3/4}}{\sqrt{2n\log\log n}} \sqrt{2M}n^{3/4}\sqrt{\log n} = 0,
\end{aligned}$$

for some positive constant K_1. That is we have

$$\lim_{n\to\infty} \frac{R_5(n)\sum_{j=1}^n d_5(j)\xi(j)}{\sqrt{2n\log\log n}} = 0, \quad \text{a.s.} \tag{7.13.15}$$

One can similary show that

$$\lim_{n\to\infty} \frac{R_6(n)\sum_{j=1}^n d_6(j)\xi(j)}{\sqrt{2n\log\log n}} = 0, \quad \text{a.s.} \tag{7.13.16}$$

Now applying Lemma 7.13.2 and (7.13.13), (7.13.14), (7.13.15) and (7.13.16) in (7.13.12) gives the desired upper bound upon the limit superior.

The lower bound on the limit inferior may be established using the same argument but applied to $-X$ rather than X. □

Remark 7.13.1. There are two points in the proof of Theorem 7.4.2 (ii) where estimations on rates of growth have been made. The first is in the estimation of a rate of growth on the quadratic variation of the martingales M_5 and M_6, see e.g. (7.13.10). It may be possible to improve the estimate, which is of a deterministic function, of the lower bound on the rate of growth by sampling the function along a sequence where the terms in the sequence are closer together. The second estimate appears when considering the limit superior of two terms, which are of the same relative size, added together, see (7.13.11). If it possible to amalgamate these two terms together first and find the limit superior of this new term it should serve to improve the estimate.

As to the second estimate we observe that if one could obtain a lower bound on the growth of the quadratic variation which is within a multiplicative constant of the upper bounding growth rate then this would enable one to deduce a lower bound upon the limit superior (and an upper bound on the limit inferior) of the largest fluctuations of X. To illustrate suppose that

$$C_1 \le \liminf_{n\to\infty} \frac{\langle M_5\rangle(n)}{n^{3/2}} \le \limsup_{n\to\infty} \frac{\langle M_5\rangle(n)}{n^{3/2}} \le C_2,$$

Then there exists an integer sequence $\{t_n : n \in \mathbb{Z}^+\}$ such that for any $0 < \epsilon < 1$,

$$\sqrt{1-\epsilon} < \sin(2\sqrt{\beta t_n}) < 1, \quad \cos(2\sqrt{\beta t_n}) < \epsilon.$$

Thus,

$$|f_5(t_n)M_5(t_n) + f_6(t_n)M_6(t_n)| \ge \sqrt{1-\epsilon}\left|t_n^{-1/4}M_6(t_n)\right| - \sqrt{\epsilon}\left|t_n^{-1/4}M_6(t_n)\right| \tag{7.13.17}$$

In the special case that β is a rational multiple of π^2 then the choice of the sequence t_n is obvious and the estimation (7.13.17) is unnecessary. Moreover the rates on the size of the largest fluctuations of X would be the same (although the limiting constants would be different).

Furthermore if one could establish an exact rate of growth of $\langle M_5\rangle(\cdot)$ and $\langle M_6\rangle(\cdot)$ then not only could a lower bound on the largest fluctuations of X be established but also the upper bound (7.13.11) could be improved.

7.14 Proofs of Theorems 7.5.1 and 7.5.2

Proof of Theorem 7.5.1. Define the martingale $M = \{M(n) : n \in \mathbb{Z}^+\}$ by

$$M(n) = \sum_{j=0}^{n} H_\infty(j)\xi(j).$$

We now establish an identity connecting parts (A) and (B).

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{j=0}^{n} H(n,j)\xi(j) - \sum_{j=0}^{\infty} H_\infty(j)\xi(j)\right)^2\right] \\
&= \mathbb{E}\left[\left(\sum_{j=0}^{n}(H(n,j) - H_\infty(j))\xi(j) - \sum_{j=n+1}^{\infty} H_\infty(j)\xi(j)\right)^2\right] \\
&= \mathbb{E}\left[\left(\sum_{j=0}^{n}(H(n,j) - H_\infty(j))\xi(j)\right)^2\right] + \mathbb{E}\left[\left(\sum_{j=n+1}^{\infty} H_\infty(j)\xi(j)\right)^2\right] \\
&= \sum_{j=0}^{n}(H(n,j) - H_\infty(j))^2 + \sum_{j=n+1}^{\infty} H_\infty(j)^2.
\end{aligned} \tag{7.14.1}$$

where the independence of ξ has been used.

Suppose (A) holds. Then as $H_\infty \in \ell^2(\mathbb{Z}^+)$ we have that

$$\lim_{n\to\infty} \sum_{j=n+1}^{\infty} H_\infty(j)^2 = 0, \tag{7.14.2}$$

both terms on the righthand side of (7.14.1) tend to zero as $n \to \infty$ and therefore so too must the term on the lefthand side, which is nothing other than (B).

Conversely suppose that (B) holds. Observe that it is implicit in the statement of (B) that $M(\infty)$ is a well defined and finite random variable. Regarding $M(\infty)$ as the pathwise limit of $M(\cdot)$, i.e.

$$\lim_{n\to\infty} M(n) = M(\infty) \in (-\infty, \infty), \quad a.s.,$$

we have therefore that a sequence of Gaussian random variables, $M(\cdot)$, converges almost surely to a finite limit, $M(\infty)$. Thus we may conclude from an argument of Shiryaev [108, Chap.2.13.5, pp.304-305] that $M(\infty)$ is Gaussian and moreover

$$\lim_{n\to\infty} \sum_{j=0}^{n} H_\infty(j)^2 = \lim_{n\to\infty} \text{Var}[M(n)] = \text{Var}[M(\infty)] = \sum_{j=0}^{\infty} H_\infty(j)^2 < +\infty.$$

Thus we again have (7.14.2). Thus rearranging (7.14.1) and taking limits gives (A). □

Proof of Theorem 7.5.2. As before remarked it is implict in the statement of Theorem 7.5.2 that $\sum_{j=0}^{\infty} H_\infty(j)\xi(j)$ is a well–defined finite random variable and therefore that $H_\infty \in \ell^2(\mathbb{Z}^+)$. Thus

$$\lim_{n\to\infty} \sum_{j=n+1}^{\infty} H_\infty(j)\xi(j) = 0, \quad a.s.$$

Then the given statement (7.5.3) is equivalent to

$$\lim_{n\to\infty} \sum_{j=0}^{n} \left(H(n,j) - H_\infty(j)\right)\xi(j) = 0, \quad a.s.$$

Thus we have a sequence of Gaussian random variables which converges almost surely to a finite limit. Again applying [108, Chap.2.13.5, pp.304-305] gives that

$$\lim_{n\to\infty} \text{Var}\left[\sum_{j=0}^{n} \left(H(n,j) - H_\infty(j)\right)\xi(j)\right] = 0,$$

which is equivalent to (7.5.1). The equivalence to (7.5.2) is given by Theorem 7.5.1. □

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