On the Oscillatory Behaviour of Stochastic Delay Equations.

Cónall Kelly

B.Sc.

A thesis submitted to

Dublin City University

for the degree of

Doctor of Philosophy

Thesis Supervisors:

Dr. John Appleby

Prof. Emmanuelle Buffet

School of Mathematical Sciences

Dublin City University

July 2005

Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of *Doctor of Philosophy*, is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

Signed: Conall Kelly

ID Number: 97308994

Date: 4 July 2005

Acknowledgements

My supervisor, Dr. John Appleby, has not wavered in his support over the past four years. His guidance, insight and limitless enthusiasm have been much appreciated.

I am indebted to all those who took an interest in this material. I found myself constantly challenged with great questions and interesting discussion, and it all made a difference. For their helpful comments and suggestions, I am particularly grateful to the following people: my co-supervisor, Prof. Emmanuel Buffet of DCU, Dr. Evelyn Buckwar and Dr. Markus Riedle, both of Humboldt University, and Prof. István Györi of the University of Veszprem.

Many thanks also to my family and friends, in DCU and elsewhere. They kept me sane.

Contents

Abstract

This is an investigation of the causes of oscillatory behaviour in solutions of stochastic delay differential equations. Delay equations are used to study phenomena in which some part of the history of the system determines its evolution. Real–world interactions are often characterised by inefficiency and such equations are therefore widely used in applications. Real–world processes are also subject to interference in the form of random external perturbations or feedback noise. This interference can have a dramatic effect on the qualitative behaviour of these processes and so should be included in the mathematical analysis.

Specifically, we consider the roles played by delayed feedback and noise perturbation in the onset of oscillation around an equilibrium solution. To this end, we consider a nonlinear equation with fixed delay, and a linear equation with asymptotically vanishing delay. Where necessary, results guaranteeing the global existence and uniqueness of solutions are presented. To facilitate the analysis of the linear equation, we present two difference schemes that are designed to mimic the oscillatory behaviour of its solutions. The first, a discretisation on a uniform mesh, is unsuccessful. We determine the reasons for this failure, and design a successful scheme based on this analysis.

These choices allow the empirical manipulation of the relative involvement of the delay in the behaviour of solutions. In this way, and by comparison with the known qualitative behaviour of the corresponding deterministic delay differential equation, a picture of the mechanisms underlying oscillatory behaviour can be developed.

List of Figures

Chapter 1

Introduction

1.1 Discussion

Feedback mechanisms guide the evolution of many real–world phenomena. Such mechanisms rarely process information flawlessly.

For instance, a neural network is a system of nodes, each producing output that is a function of the outputs of other nodes in the network. In the physical implementation of a neural network, communication delays due to capacitance arise, and imperfect data transmission tends to disrupt the input signals of each node.

The population dynamics of a single species are also determined by a feedback mechanism. Changes in biomass will affect the reproductive potential, and therefore the growth rate, of the species within the confines of its habitat. Ordinary differential equations are a natural mathematical tool for describing these dynamics, assuming that feedback conditions are ideal. In reality, gestation, maturation, and incubation periods result in feedback delays, and disease, weather, and other environmental factors disturb the feedback mechanism.

Clearly then, processes that model real–world phenomena are often characterised by interactions that are inefficient in the sense that some part of their history determines their evolution. They can also be subject to interference in the form of random external perturbations or feedback noise. This interference can have a dramatic qualitative effect on these processes and should therefore be included in any analysis of their behaviour. Returning to the example of population dynamics, it has been shown in Mao, Marion & Renshaw [31] that a stochastic perturbation of a population model can radically change its qualitative behaviour. Physically impossible characteristics, such as a finite–time explosion in the unperturbed model, can be suppressed.

The primary focus of this work is the effect that noise and delay can have on the qualitative behaviour of feedback processes. More specifically, we seek to describe the roles played by feedback delay and noise perturbation in the oscillatory behaviour of solutions of scalar stochastic delay differential equations. Substantial evidence can be found in the literature to suggest that delayed feedback and noise perturbation are the main players in this behaviour, and that their roles are complementary.

The importance of delayed feedback in oscillatory behaviour can be seen by considering the first order linear differential equation

$$\begin{aligned} x'(t) &= bx(t), \quad t > 0, \\ x(0) &\in \mathbb{R}^+, \end{aligned}$$

with solution $x(t) = x(0)e^{bt}$, a strictly positive function. Contrast this with Gopalsamy's analysis [11] of the solutions of the linear delay differential equation

$$x'(t) = bx(t-\tau), \quad t > 0, \tag{1.1.1a}$$

$$x(t) = \psi(t), \quad t \in [-\tau, 0]. \tag{1.1.1b}$$

When $b < 0$, the combination of a long delay and a weak feedback intensity, in the sense that $-be\tau > 1$, are sufficient to guarantee that all nontrivial solutions of (1.1.1) are oscillatory. However, if the delay is short and the feedback intensity strong, in the sense that $-be\tau < 1$, then nonoscillatory solutions exist. It is the introduction of a sufficiently long delay that underlies the onset of oscillation in the solutions of (1.1.1).

Noise perturbations in themselves are not the cause of oscillatory behaviour in equations that admit an equilibrium solution. For example, where B is a standard Brownian motion, a geometric Brownian motion satisfying

$$dX(t) = bX(t)dt + \sigma X(t)dB(t), \quad t > 0, \tag{1.1.2a}$$

$$X(0) \in [0, \infty), \tag{1.1.2b}$$

can be written in closed form as the strictly positive process $X(t) = X(0)e^{(b-\frac{\sigma^2}{2})t+\sigma B(t)}$. Note that (1.1.2) admits the equilibrium solution $X(t) \equiv 0$ when $X(0) = 0$. By contrast, Appleby & Buckwar [1] give an analysis of the oscillatory behaviour of the linear stochastic differential equation with fixed delay

$$\begin{aligned} dX(t) &= bX(t-\tau)dt + \sigma X(t)dB(t), \quad t > 0, \\ X(t) &= \psi(t), \quad t \in [-\tau, 0]. \end{aligned}$$

This equation also admits the equilibrium solution $X(t) \equiv 0$, when $\psi(t) \equiv 0$. For $b < 0$ all solutions are a.s. oscillatory regardless of the magnitude of the delay τ, or the strength of the feedback b. Under the influence of a stochastic perturbation, any nonzero delay is enough to guarantee oscillation. It appears that oscillation is in some way facilitated by the presence of noise. It is the nature of this relationship that we seek to describe.

Continuous-time stochastic processes driven by standard Brownian motion naturally fluctuate as they evolve. For this reason it is important to distinguish between stochastic fluctuation and true oscillation, which occurs around an equilibrium solution and so cannot be attributed solely to the presence of noise. A precise definition of this kind of oscillatory behaviour will be given in Section 1.4.

In order to investigate the roles played by feedback delay and noise perturbation in the oscillatory behaviour of stochastic differential equations, we consider a nonlinear equation with fixed delay in Chapter 3, and a linear equation with asymptotically vanishing delay in Chapters 5, 6, and 7. These choices allows us to empirically manipulate the relative involvement of the delay in the behaviour of solutions. In this way, and by comparison with the known qualitative behaviour of the corresponding deterministic delay differential equation, due to Gopalsamy [11] and Ladde, Lakshmikantham & Zhang [9], we develop a picture of the mechanisms underlying oscillatory behaviour. It is important to emphasise that these choices are illustrative. We do not present a comprehensive treatment of oscillation in stochastic delay equations. Equations, both linear and nonlinear, with unbounded delay, multiple delays, distributed delays and so on are equally deserving of attention.

We begin in Chapter 2 by developing the global existence and uniqueness theory for solutions of nonlinear stochastic delay differential equations to include all continuous processes analysed in this thesis. We consider equations that can be guaranteed to have unique continuous local solutions up to a possible explosion time, and investigate the circumstances under which the explosion time cannot be finite.

Once the existence theory is in place, we generalise the analysis of the oscillatory behaviour of linear stochastic equations with fixed delay, due to Appleby & Buckwar [1], to equations with nonlinear coefficients in Chapter 3.

Once again, it emerges that the strength of the restoring force is the primary engine for oscillation. This strength is represented as a specific property of the nonlinear drift coefficient. It turns out that, even if the restoring force is strong, the unperturbed delay equation requires that the delay be of a minimum length for oscillation to occur. This length is determined by the strength of the equilibrium–restoring force. However, the inclusion of a stochastic perturbation allows oscillation to occur in the presence of the same restoring force, but with a feedback delay of any length.

In Chapter 3 we also determine sufficient conditions on a weak equilibrium–restoring force that allow for nonoscillatory behaviour to take place with positive probability.

The asymptotic behaviour of increments of standard Brownian motion will be seen to play an important role in the remaining analysis. Technical lemmata describing this behaviour are gathered together in Chapter 4, there to be referenced from the chapters that follow.

An attempt to characterise the oscillatory behaviour of a linear stochastic differential equation with vanishing delay is met with partial success in Chapter 5. We find clear indications that the solutions of this equation display singular behaviour in a two–fold way, in that the limiting stochastic equation without delay displays nonoscillatory behaviour, as does the corresponding deterministic equation with vanishing delay. However, by allowing the delay to vanish slowly enough, the presence of a noise perturbation is sufficient to induce oscillation in the solution. Nonetheless, the picture that develops is incomplete in several important ways.

Throughout our research we have used simulation to inform our intuition as to the roles of feedback delay and noise perturbation in oscillatory behaviour. A welcome development of this practice was the discovery that a more detailed picture of the qualitative behaviour of the linear vanishing delay equation could be determined by the construction of a discrete process defined by an Euler–type difference scheme. However, the randomness inherent in these processes gave us reason to be cautious in our choice of discretisation.

A nice illustration of the care that must be taken when using Euler methods to discretise even the simplest of ordinary differential equations can be found in the introduction to the paper by Mohamad and Gopalsamy [26]. There, examples are given of ordinary differential equations, including the logistic equation and the simple linear equation

$$y'(t) = -y(t), \quad t > 0,$$

that have an Euler discretisation displaying spurious behaviour that arises from the discretisation process. This uncharacteristic behaviour is misleading, and its occurrence must be carefully avoided if our intent is to develop a clearer picture of the behaviour of the original differential equation. Indeed, in Chapter 6, we show that a successful method for discretising a deterministic differential equation with vanishing delay, found in Karoui & Vaillancourt [18], induces spurious oscillatory behaviour in a similar stochastic difference equation.

However, by identifying and preserving the essential characteristics of the stochastic process over its lifetime, this problem can be fixed. In Chapter 7, we develop an alternative difference scheme yielding a complete picture of the oscillatory behaviour of the process, and a full description of the roles of the equation parameters in this behaviour. Thus, the picture of oscillatory behaviour that is begun in Chapter 5 can be completed.

Where possible, we have illustrated our analysis through the simulation of examples. This was done in Java, using the IEEE–754 floating–point standard to represent real numbers with 64–bit precision. To make qualitative comparisons easier, all plots appearing in Chapters 2 and 3 were generated from the same set of Gaussian numbers with a standard Euler–Maruyama scheme, running on a uniform mesh of size 10^{-4}, and with constant, unit–valued initial data. All plots appearing in Chapter 6 were generated with a direct implementation of the discrete process described in that chapter, again with constant, unit–valued initial data. Other relevant information is provided on a case–by–case basis.

The remainder of this chapter gives a summary of ideas and results in stochastic analysis and dynamical systems theory that will be used throughout this thesis.

1.2 Fundamentals of stochastic processes.

In order to provide a background for the analysis in the chapters ahead, we begin by considering some of the fundamental ideas underlying the theory of stochastic processes. All of the material presented here can be found in such texts as Lamberton & Lapeyere [23], Mao [25], or Øksendal [27]. The primary focus in the chapters to follow is on the qualitative behaviour of scalar stochastic processes, and the background given will reflect that. Generalisation to finite dimensions is possible, specifically for the global existence theory in Chapter 2, and an example is given in Appendix A. For this reason, the definition of a standard Brownian motion in finite dimensions is presented, and it should be noted that finite dimensional versions of Lemmata 1.3.2 and 1.3.3 exist, and can be found in any of the texts referenced above.

σ–algebras and measurable spaces. If Ω is a given set, then a *σ–algebra* $\mathcal{F}$ on Ω is a family $\mathcal{F}$ of subsets of Ω with the following properties:

1. $\Omega \in \mathcal{F}$
2. If $A \in \mathcal{F}$ then $\overline{A} \in \mathcal{F}$, where $\overline{A} = \Omega \backslash A$
3. If $A_1, A_2 \cdots \in \mathcal{F}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Given any family $\mathcal{U}$ of subsets of Ω there is a smallest σ–algebra $\mathcal{H}_{\mathcal{U}}$ containing $\mathcal{U}$, namely

$$\mathcal{H}_{\mathcal{U}} = \bigcap \{\mathcal{H};\ \mathcal{H} \text{ a } \sigma\text{–algebra of } \Omega,\ \mathcal{U} \subset \mathcal{H}\}.$$

This is called the *σ–algebra generated by $\mathcal{U}$*. For example, if $\mathcal{U}$ is the collection of all open subsets of a topological space Ω, say $\Omega = \mathbb{R}^d$, then the *Borel σ-algebra $\mathcal{B}$* is the σ–algebra generated by $\mathcal{U}$. The pair $(\Omega, \mathcal{F})$ is called a *measurable space.*

Probability spaces. A *probability measure* $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a function $\mathbb{B} : \mathcal{F} \mapsto [0, 1]$ such that

1. $\mathbb{P}[\emptyset] = 0$, and $\mathbb{P}[\Omega] = 1$

2. If $A_1, A_2 \cdots \in \mathcal{F}$ and $\{A_i\}_{i=1}^{\infty}$ is disjoint, then

$$\mathbb{P}\left[\bigcup_{i=1}^{\infty} A_i\right] = \sum_{i=1}^{\infty} \mathbb{P}[A_i].$$

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space.* It is *complete* if $\mathcal{F}$ contains all $\mathbb{P}$–zero valued subsets of Ω.

Events, measurability and random variables. The subsets A of Ω which belong to $\mathcal{F}$ are called $\mathcal{F}$–measurable sets, or *events.* The number $\mathbb{P}[A]$ represents the probability of A occurring. In particular, if $\mathbb{P}[A] = 1$, then we say that A occurs *almost surely (a.s.).*

Let $\{A_n\}_{n\geq 0}$ be a sequence of events. If infinitely many of the events A_n occur, then we say that the event 'A_n *infinitely often (i.o.)*' has occurred, where

$$\text{'}A_n \text{ i.o.'} = \bigcap_{n=0}^{\infty} \bigcup_{j=n}^{\infty} A_j.$$

If all the events A_n occur from a certain rank on, then we say that the event 'A_n *eventually (e.v.)*' has occurred, where

$$\text{'}A_n \text{ e.v.'} = \bigcup_{n=0}^{\infty} \bigcap_{j=n}^{\infty} A_j.$$

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we say that a function $X : \Omega \mapsto \mathbb{R}^d$ is $\mathcal{F}$*–measurable* if

$$X^{-1}(U) := \{\omega \in \Omega; X(\omega) \in U\} \in \mathcal{F}$$

for all open sets $U \in \mathbb{R}^d$. X is called a *random variable.*

Stochastic processes. A *stochastic process* $X = \{X(t) : t \in \mathcal{T}\}$ is a collection of random variables parameterised over the time set $\mathcal{T}$. Every random variable is defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assumes a value in $\mathbb{R}^d$.

For every fixed $\omega \in \Omega$, The process $X(\cdot, \omega)$ is a function

$$\mathcal{T} \ni t \mapsto X(t, \omega) \in \mathbb{R}^d,$$

called a *sample path,* or simply *path,* of the process. For every fixed $t \in \mathcal{T}$, $X(t, \cdot)$ is a random variable

$$\Omega \ni \omega \mapsto X(t, \omega) \in \mathbb{R}^d.$$

Thus the stochastic process may be regarded as a function of two variables (t, ω) from $\mathcal{T} \times \Omega$ to $\mathbb{R}^d$.

A stochastic process is *continuous* if for almost all $\omega \in \Omega$ the function $X(t, \omega)$ is continuous on $t \geq 0$. Left and right continuity can be defined similarly. It is an *increasing process* if, for almost all $\omega \in \Omega$, $X(t, \omega)$ is nonnegative, nondecreasing, and right-continuous on $t \geq 0$. It is *integrable* if for every $t \geq 0$, $X(t)$ is an integrable random variable. It is a *process of finite variation* if $X(t) = \overline{X}(t) - \widetilde{X}(t)$, with $\overline{X}$ and $\widetilde{X}$ both increasing processes. It is *square integrable* if $\mathbb{E}|X(t)|^2 < \infty$ for every $t \geq 0$.

Filtered stochastic processes. We can associate a family of increasing sub–σ–algebras of $\mathcal{F}$ with the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. This family represents the increasing set of information available to an observer regarding the evolution of the process.

A *filtration* is a family $\{\mathcal{F}(t)\}_{t \geq 0}$ of increasing sub–σ–algebras of $\mathcal{F}$. Therefore, for all $0 \leq t < s < \infty$, $\mathcal{F}(t) \subset \mathcal{F}(s) \subset \mathcal{F}$. The filtration is *right–continuous* if $\mathcal{F}(t) = \bigcap_{s>t} \mathcal{F}(s)$ for all $t \geq 0$.

A stochastic process $\{X(t)\}_{t \geq 0}$ is said to be *adapted* to a filtration $\{\mathcal{F}(t)\}_{t \geq 0}$ if, for every t, $X(t)$ is $\mathcal{F}(t)$–measurable. Let $\mathcal{P}$ denote the smallest σ–algebra on $\mathbb{R}^+ \times \Omega$ with respect to which every left continuous process is a measurable function of (t, ω). Then a stochastic process is *predictable* if it is $\mathcal{P}$–measurable.

Brownian motion. A *Brownian motion* is a real valued stochastic process $\{B(t)\}_{t \geq 0}$ satisfying the following:

1. Almost all sample paths of B are continuous: $s \mapsto B(s)$ is a.s. continuous.

2. Nonoverlapping increments are independent: If $s \leq t$, then $B(t) - B(s)$ is independent of $\mathcal{F}(s)$.

3. Increments are stationary: If $s \leq t$, then $B(t) - B(s)$ and $B(t-s) - B(0)$ have the same distribution.

4. Increments are Gaussian: $B(t) - B(0)$ is Normally distributed with mean μt and variance $\sigma^2 t$, where μ and σ are constant real numbers.

In fact the first three properties are sufficient to guarantee the fourth [10]. However, for convenience, we include it here as part of the definition.

This definition can be extended to finite dimensions: A d–dimensional process $\{\mathbf{B}(t) = (B_1(t), \ldots, B_d(t))\}_{t\geq 0}$ is a *d–dimensional Brownian motion* if every component $\{B_i(t)\}$ is a Brownian motion, and $\{B_1(t)\}, \ldots, \{B_d(t)\}$ are independent.

If the Normally distributed increments of a Brownian motion B satisfy $\mu = 0$ and $\sigma = 1$ in Property 4 above, then B is a *standard Brownian motion.*

Properties of Brownian motion. A Brownian motion $\{B(t)\}$ has the following properties which we will find useful in our analysis:

1. $\{-B(t)\}$ is a Brownian motion with respect to the same filtration $\{\mathcal{F}(t)\}$.

2. For all $t > 0$, the random variables $|B(t)|$ and $\max_{0\leq s\leq t} B(s)$ are identically distributed.

3. The strong law of large numbers states that $\lim_{t\to\infty} B(t)/t = 0$ a.s.

4. For almost every $\omega \in \Omega$, the Brownian path $t \mapsto B(t,\omega)$ is nowhere differentiable.

We also have the following lemma which can be found, for example, in Karatzas & Shreve [17], describing the asymptotic behaviour of the deviations of Brownian motion.

Lemma 1.2.1 (Law of the Iterated Logarithm). *For almost every $\omega \in \Omega$, we have*

$$(i)\ \limsup_{t\to 0^+} \frac{B(t)(\omega)}{\sqrt{2t\log\log(\frac{1}{t})}} = 1, \quad (ii)\ \liminf_{t\to 0^+} \frac{B(t)(\omega)}{\sqrt{2t\log\log(\frac{1}{t})}} = -1,$$

$$(iii)\ \limsup_{t\to\infty} \frac{B(t)(\omega)}{\sqrt{2t\log\log t}} = 1, \quad (iv)\ \liminf_{t\to\infty} \frac{B(t)(\omega)}{\sqrt{2t\log\log t}} = -1.$$

Stopping times and martingales. A random variable $\Upsilon : \Omega \mapsto [0,\infty]$ is called an *$\mathcal{F}(t)$–stopping time* if $\{\omega : \Upsilon(\omega) \leq t\} \in \mathcal{F}(t)$ for any $t \geq 0$. A real–valued $\{\mathcal{F}(t)\}$–adapted integrable process $\{M(t)\}_{t\geq 0}$ is called a *martingale* with respect to $\{\mathcal{F}(t)\}$ if

$$\mathbb{E}[M(t)|\mathcal{F}(s)] = M(s), \quad \text{a.s. for all } 0 \leq s < t < \infty.$$

The *quadratic variation* of a martingale is the unique continuous integrable adapted increasing process $\{\langle M\rangle(t)\}_{t\geq 0}$, where $M = \{M(t)\}_{t\geq 0}$ is a real–valued square integrable continuous martingale, such that $\{M(t)^2 - \langle M\rangle(t)\}$ is a continuous martingale vanishing at $t = 0$.

The following result is a consequence of the strong law of large numbers, given as Property 3 of Brownian motion.

Lemma 1.2.2 (Law of large numbers for martingales.). *Let $M = \{M(t)\}_{t\geq 0}$ be a real-valued continuous martingale vanishing at $t = 0$. If $\lim_{t\to\infty}\langle M\rangle(t) = \infty$ a.s., then*

$$\lim_{t\to 0} \frac{M(t)}{\langle M\rangle(t)} = 0, \quad a.s.$$

Also, if $\limsup_{t\to\infty}\langle M\rangle(t)/t < \infty$ a.s., then

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0, \quad a.s.$$

The next well known result shows how to express a martingale as a standard Brownian motion on a new time scale.

Lemma 1.2.3 (Martingale time–change theorem.). *Let $M = \{M(t)\}_{t\geq 0}$ be a real–valued continuous martingale such that $M(0) = 0$ and $\lim_{t\to\infty}\langle M\rangle(t) = \infty$ a.s. For each $t \geq 0$, define the stopping time*

$$\vartheta(t) = \inf\{s : \langle M\rangle(s) > t\}.$$

Then $\{M(\vartheta(t))\}_{t\geq 0}$ is a standard Brownian motion with respect to the new filtration $\{\mathcal{F}(\vartheta(t))\}_{t\geq 0}$. In other words, there exists a standard Brownian motion $\widetilde{B}$ such that

$$M(t) = \widetilde{B}(\langle M\rangle(t)), \text{ for all } t \geq 0 \text{ a.s.}$$

1.3 Stochastic calculus.

In order to describe a stochastic process as the solution of a stochastic delay differential equation, it is necessary to develop a valid calculus within which we can work. It will be necessary to integrate with respect to a standard Brownian motion. The lack of regularity in the paths of Brownian motion excludes the possibility of developing a differential calculus. There is more than one way to define a stochastic integral. For example, a discussion of the definition of the Stratonovich integral can be found in Øksendal [27]. Nonetheless we will exclusively use the Itô integral in our analysis.

1.3.1 Definition of the stochastic integral.

It is first necessary to develop a definition of the stochastic integral

$$I(t)(f) = \int_0^t f(s)dB(s), \tag{1.3.1}$$

for simple stochastic processes. A stochastic process $\{Y(t); 0 \leq t \leq T\}$ is a *simple* process if, for some $0 = t_0 < t_1 < \cdots < t_p = T$,

$$Y(t) = Y(t_j), \quad t_j \leq t \leq t_{j+1}.$$

The *Itô integral of a simple stochastic process* adapted to the filtration $\{\mathcal{F}(t)\}_{t\geq 0}$ is defined to be

$$I_T(Y) = \int_0^T Y(s)dB(s) = \sum_{j=0}^{p-1} Y(t_j)(B(t_{j+1}) - B(t_j)).$$

Similarly, for $t < T$, we put

$$I_t(Y) = \int_0^t Y(s)dB(s) = \sum_{j=0}^{l-1} Y(t_j)(B(t_{j+1}) - B(t_j)) + Y_{t_l}(B(t) - B(t_l)),$$

where l is the last, t–dependent, index of the last jump before t.

It is now possible to define the Itô integral for more general processes, by expressing it as the limit of a sequence of integrals of simple processes. Thus, the *Itô integral of a $\mathcal{B} \times \mathcal{F}$–measurable, $\{\mathcal{F}(t)\}$–adapted, square-integrable process X* is defined by

$$\int_0^t X(s)dB(s) = \lim_{n\to\infty} \int_0^t Y_n(s)dB(s), \tag{1.3.2}$$

where $\{Y_n\}$ is a sequence of simple processes such that

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^t (X(s) - Y_n(s))^2 ds\right] = 0. \tag{1.3.3}$$

Øksendal [27] shows that a sequence $\{Y_n\}$ satisfying (1.3.3) exists for every X, and that the limit on the right hand side of (1.3.2) exists and is independent of the actual choice of $\{Y_n\}$, as long as (1.3.3) holds.

Finally, we show the conditions under which the stochastic integral defined in (1.3.1) is a martingale. The following is the martingale representation theorem, and can be found in, for example, Mao [25].

Lemma 1.3.1 (Martingale representation theorem.). *Let $\{f(t)\}_{t\geq 0}$ be a real-valued, mean square-integrable adapted process. Then the process*

$$J(t) = \int_0^t f(s)dB(s), \quad t \geq 0,$$

is a square-integrable martingale, null-valued at zero, with quadratic variation

$$\langle J\rangle(t) = \int_0^t f(s)^2 ds, \quad t \geq 0.$$

1.3.2 Itô processes.

Let B be a one-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. A *one-dimensional Itô process*, also known as a *semimartingale*, or *stochastic integral*, is a stochastic process $X = \{X(t)\}_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ of the form

$$X(t) = X(0) + \int_0^t u(s)ds + \int_0^t v(s)dB(s), \tag{1.3.4}$$

where v is $\mathcal{B} \times \mathcal{F}$-measurable, $\{\mathcal{F}(t)\}$-adapted, and square-integrable, so that

$$\mathbb{P}\left[\int_0^t v(s)^2 ds < \infty \text{ for all } t \geq 0\right] = 1.$$

We also assume that u is adapted to an increasing family of σ-algebras $\{\mathcal{H}(t)\}_{t\geq 0}$ such that, for all $t \geq 0$, B is a martingale with respect to $\mathcal{H}_t$, and that

$$\mathbb{P}\left[\int_0^t |u(s)|ds < \infty \text{ for all } t \geq 0\right] = 1.$$

(1.3.4) is sometimes written in the shorter *differential form*

$$dX(t) = u(t)dt + v(t)dB(t),$$

and in fact this notational convention will be used almost exclusively from here on.

Semimartingale form of solutions of stochastic functional differential equations. We can now explicitly represent the processes that are the focus of this work as Itô processes. Given any standard Brownian motion B, and a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\geq 0}, \mathbb{P})$, where $\mathcal{F}(t) = \sigma(B(s) : 0 \leq s \leq t)$ is the natural filtration of B, we can define X as the the adapted stochastic process that satisfies the semimartingale decomposition

$$X(t) = X(0) + \int_0^t f(X_s)ds + \int_0^t h(X(s))dB(s), \tag{1.3.5a}$$
$$X(t) = \psi(t), \quad t \in [-\overline{\tau}, 0], \tag{1.3.5b}$$

almost everywhere. According to the standard differential notation, (1.3.5) is written as

$$\begin{aligned} dX(t) &= f(X_t)dt + h(X(t))dB(t), \\ X(t) &= \psi(t), \quad t \in [-\overline{\tau}, 0]. \end{aligned}$$

The initial data function $\psi : [-\overline{\tau}, 0] \mapsto \mathbb{R}^+$ is an $\mathcal{F}(0)$–measurable random variable such that $\mathbb{E}||\psi||^2 < \infty$. Since the probability space and the Brownian motion are predefined, X is a *strong* solution of (1.3.5).

The functional $f : C([-\overline{\tau}, s] \times \Omega; \mathbb{R}) \mapsto \mathbb{R}$ is known as the *drift coefficient*, or simply *drift*, and represents a deterministic feedback component processing information from subsets of the sample path, $X_s = \{X(s+\theta) : -\overline{\tau} - s \leq \theta \leq 0\}$. In fact it will be seen in Chapter 2 that there will always be a gap separating historical feedback from instantaneous feedback: hence the 'delay' of the title.

The function $h : \mathbb{R} \mapsto \mathbb{R}$ is known as the *diffusion coefficient*, or simply *diffusion*, and represents the state–dependent stochastic perturbation processing instantaneous information, denoted $X(s)$ in (1.3.5) for a particular time s. The integral $\int_0^t h(X(s))dB(s)$ taken as a whole is often called the *noise*.

1.3.3 Properties of Itô processes.

We state the change–of–variable formula for stochastic integrals in the standard differential notation.

Lemma 1.3.2 (Itô's Rule). *Let $X(t)$ be a continuous scalar Itô process given by*

$$dX(t) = u(t)dt + v(t)dB(t),$$

where $\int_0^\infty u(s)ds < \infty$, and $\int_0^\infty v(s)^2 ds < \infty$. Let $V \in C^2(\mathbb{R};\mathbb{R})$. Then $\widetilde{V} = \{V(X(t)\}_{t\geq 0}$ is again an Itô process, and

$$dV(X(t)) = V'(X(t))dX(t) + \frac{1}{2}V''(X(t))u(t)^2 dt.$$

We also present the formula for stochastic integration by parts.

Lemma 1.3.3 (Stochastic integration by parts formula). *Let $\{X(t)\}_{t\geq 0}$ be a continuous scalar Itô process satisfying*

$$dX(t) = u(t)dt + v(t)dB(t).$$

Let $\{Y(t)\}_{t\geq 0}$ be a continuous, real-valued adapted process of finite variation. Then

$$d[X(t)Y(t)] = Y(t)dX(t) + X(t)dY(t).$$

Lemmata 1.3.2 and 1.3.3 are proved as Theorems 6.2 and 6.5, respectively, in [25].

1.4 Oscillation of stochastic processes.

The phenomenon of oscillation must be distinguished from the path fluctuation that prominently distinguishes most stochastic processes. Oscillation occurs around an equilibrium solution (which is $X(t) \equiv 0$ in our analysis) and as such cannot be attributed solely to the effect of a multiplicative, and therefore equilibrium–preserving, noise perturbation. This distinction is illustrated in Figure 1.4.1.

Definition 1.4.1. We say that a nontrivial continuous function $y : [t_0, \infty) \to \mathbb{R}$ is *oscillatory* if the set

$$Z_y = \{t \geq t_0 \;:\; y(t) = 0\}$$

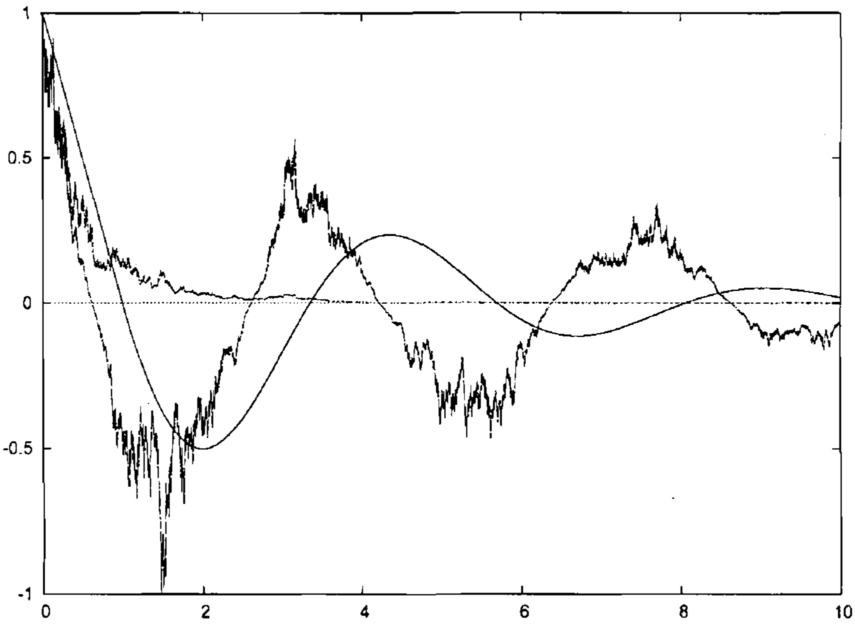


Figure 1.4.1: Red: An oscillatory solution of the delay differential equation $x'(t) = -x(t-1)$. Green: An oscillatory path of the stochastically perturbed equation $dX(t) = -X(t-1)dt + X(t)dB(t)$. Blue: When the delay is zero, solutions of the equation $dX(t) = -X(t)dt + X(t)dB(t)$ are a.s. nonoscillatory, although the paths fluctuate.

---

satisfies $\sup Z_y = \infty$.

A function which is not oscillatory is called *nonoscillatory.*

In [2], a continuous stochastic process was determined to be a.s. oscillatory if these notions were extended in the following intuitive manner:

Definition 1.4.2. A stochastic process $\{X(t,\omega)\}_{t\geq t_0}$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and with continuous sample paths, is said to be *a.s. oscillatory* if there exists $\Omega^* \subseteq \Omega$ with $\mathbb{P}[\Omega^*] = 1$ such that for all $\omega \in \Omega^*$, the path $X(\cdot, \omega)$ is oscillatory.

A stochastic process is *a.s. nonoscillatory* if there exists $\Omega^* \subseteq \Omega$ with $\mathbb{P}[\Omega^*] = 1$ such that for all $\omega \in \Omega^*$, the path $X(\cdot, \omega)$ is nonoscillatory.

1.5 Classifying oscillatory behaviour in deterministic delay equations.

Our analysis relies heavily on a transformation that results in a continuous stochastic process with oscillatory behaviour that is identical to that of the solutions of (1.3.5), but the paths of which satisfy

$$y'(t,\omega) = -p(t,\omega)y(t-\tau(t),\omega), \tag{1.5.1a}$$

$$y(t) = \psi(t), \quad t \in [-\overline{\tau}, 0]. \tag{1.5.1b}$$

As a result, we can apply results from deterministic theory on a pathwise basis to develop an understanding of the qualitative behaviour of the solutions of (1.3.5). The following result concerning oscillatory solutions of (1.5.1), is a special case of Theorem 2 in Staikos & Stavroulakis [30], and originally appeared in Ladas et al [8].

Theorem 1.5.1. *Suppose p is a continuous, nonnegative function defined on $[t_0, \infty)$ which satisfies*

$$\limsup_{t\to\infty} \int_{t-\tau(t)}^{t} p(s)\,ds > 1 \tag{1.5.2}$$

where τ is a continuous function with the properties that $t \mapsto t-\tau(t)$ is nondecreasing on $[t_0,\infty)$, $\tau(t) > 0$ on $[t_0,\infty)$, and $t-\tau(t) \to \infty$ as $t \to \infty$. Then all solutions of (1.5.1) are oscillatory.

Proof. Without loss of generality, let y be a nonoscillatory solution of (1.5.1) such that $y(t) > 0$ for all $t \geq t_0$, and therefore that $y(t-\tau(t)) > 0$, for all $t \geq t_1 \geq t_0$. Integrating (1.5.1a) from $t-\tau(t)$ to t, we have that

$$y(t) - y(t-\tau(t)) + \int_{t-\tau(t)}^{t} p(s)y(t-\tau(s))ds = 0.$$

Since y is decreasing and positive, we have

$$y(t) + y(t-\tau(t))\left[\int_{t-\tau(t)}^{t} p(s)ds - 1\right] \leq 0. \tag{1.5.3}$$

By (1.5.2),

$$\int_{t-\tau(t)}^{t} p(s)ds > 1,$$

for t sufficiently large, thereby contradicting (1.5.3) under the assumption that y is a nonoscillatory solution. $\square$

Similarly, a method of proving the existence of nonoscillatory solutions is furnished by the following result, which may be found in Ladde, Lakshmikantham & Zhang [9]. The result originally appeared in Ladde [22].

Theorem 1.5.2. *Suppose that p is a continuous, nonnegative function defined on $[0, \infty)$ which satisfies*

$$\limsup_{t\to\infty} \int_{t-\tau(t)}^{t} p(s)\,ds < \frac{1}{e},$$

where τ is a continuous function with the properties that $\tau(t) > 0$, and $t - \tau(t) \to \infty$ as $t \to \infty$. Then (1.5.1) has a positive solution.

The proof of Theorem 1.5.2 is somewhat longer than that of Theorem 1.5.1, and we do not present it in full here.

Sketch of Proof. It is sufficient to find a solution of (1.5.1) that has the form

$$y(t) = e^{\int_{t_0}^{t} \lambda(s)ds}, \quad t \geq t_0,$$

for some $t_0 > 0$. Substituting into (1.5.1a) gives

$$\lambda(t) = -p(t)e^{-\int_{t-\tau(t)}^{t} \lambda(s)ds}. \tag{1.5.4}$$

The right hand side of (1.5.4) is used to construct a nondecreasing and continuous operator T on a particular space of continuous functions. The remainder of the proof involves showing that T possesses a fixed point, and hence that λ exists. $\square$

Chapter 2

Global Existence and Uniqueness

Before any consideration is given to the qualitative behaviour of the solutions of stochastic delay differential equations, we must know the circumstances under which unique solutions exist.

There is a well-developed theory of the existence and uniqueness of strong solutions of stochastic functional differential equations of Itô type, due to Itô & Nisio [15] and Berger & Mizel [5], among others. The existence of such unique global solutions is often proved under the assumption that the functional coefficients of the equation are locally Lipschitz continuous, and satisfy global linear growth bounds. These requirements are indicated in Mao [25] and Kolmanovskii & Myskis [20].

In this chapter, we drop the requirement that the drift coefficient satisfy a global linear bound. Mao [24] has given examples of stochastic delay differential equations which have unique global solutions, despite the absence of a linear growth bound, by assuming the presence of strongly nonlinear negative feedback in an instantaneous drift term. Such conditions are often called one–sided linear bounds. In these cases however, the corresponding deterministic equation has a global solution. It is also interesting, therefore, to investigate the circumstances under which solutions of a stochastic delay differential equations exist globally if solutions of the unperturbed equation explode.

Protter [28] has shown that unique strong solutions can exist up to an explosion time under the assumption that the functionals are locally Lipschitz continuous. This result gives us a starting point, and can be used as the basis of an investigation into the minimal set of regularity and growth hypotheses on the coefficients that will ensure that a stochastic functional differential equation has a unique and almost surely nonexploding solution.

We begin with a description of the delay structure that characterises our equations, and define a mesh that will play a vital role in our analysis, not only in this chapter, but throughout the thesis. Following this, our primary goal is to develop a global existence and uniqueness theory for a class of functional differential equations with drift coefficients that are completely characterised by delayed feedback. In order to motivate our method of proof, and to allow a direct comparison to be made with the stochastic equation, we will describe the properties of the deterministic equation. Our secondary goal is to consider equations with instantaneous feedback in the drift coefficient. Since the solutions of such equations can explode in finite time, we seek to develop a theory describing the roles of noise and feedback delays in guaranteeing the absence of such explosions, and therefore global existence of the solutions. We compare the properties of equations constructed from varying combinations of instantaneous feedback terms, delayed feedback terms, and noise perturbation, some of which are already described in the literature. The analysis in this chapter can be found in Appleby & Kelly [4].

The delay structure that we consider is quite general, and is characterised by a distinct time lag between the most recent information in the feedback and the present. We begin by defining the delay structure precisely in finite dimensions. Several special cases of this general structure in one dimension will be used to construct the equations under investigation throughout the the remainder of this thesis.

2.1 Structure of the delay.

Let $r \geq 0$. For each $t \geq 0$ denote by $C([-r,t];\mathbb{R}^d)$ the family of continuous functions from $[-r,t]$ to $\mathbb{R}^d$ with the norm $\|\varphi\| = \sup_{-r\leq s\leq t, i=1,\dots,d} |\varphi_i(s)|$. For $\varphi \in C([-r,t];\mathbb{R}^d)$, the *history of φ, up to time t*, will be defined by

$$\varphi_t = \{\varphi(s) : -r \leq s \leq t\}.$$

Our investigations deal with functional differential equations in which the influence of the past is distinct from that of the present. To make this idea precise, define the functional $g_{(\tau,\overline{\tau})}$ to be a mapping from $C([-\overline{\tau},t];\mathbb{R}^d)$ to $\mathbb{R}^d$ where $\overline{\tau}$ is a real number which obeys

$$\overline{\tau} \in [0,\infty), \tag{2.1.1}$$

and τ is a function satisfying

$$\tau : [0,\infty) \to (0,\infty) : t \mapsto \tau(t) \text{ is continuous,} \tag{2.1.2}$$

such that

for all $t \geq 0$ and $\varphi \in C([-\overline{\tau}, t]; \mathbb{R}^d)$

$$(t, \varphi) \mapsto \mathbf{g}_{(\tau,\overline{\tau})}(t, \varphi_t) \quad \text{depends only on} \quad \{\varphi(s) : -\overline{\tau} \leq s \leq t - \tau(t)\}. \quad (2.1.3)$$

There is always a distinction between the dependence of solutions on the past and dependence on the present, because (2.1.2) implies that the function τ obeys

$$\tau(t) > 0, \quad \text{for all } t \geq 0. \quad (2.1.4)$$

We impose (2.1.1) so as to exclude equations with infinite delay. However, this does not rule out the study of equations with unbounded delay, in which $t - \tau(t) \to \infty$ as $t \to \infty$.

Finally, we address the continuity requirements on the functional $\mathbf{g}_{(\tau,\overline{\tau})}$. We suppose that

for each continuous function $\varphi \in C([-\overline{\tau}, \infty); \mathbb{R}^d)$

$$t \mapsto \mathbf{g}_{(\tau,\overline{\tau})}(t, \varphi_t) \quad \text{is a continuous function from } [0, \infty) \text{ to } \mathbb{R}^d. \quad (2.1.5)$$

We say that a functional $\mathbf{g}_{(\tau,\overline{\tau})}$ is *continuous with delay structure* $(\tau, \overline{\tau})$ if there is a function τ and a number $\overline{\tau}$ which obey (2.1.1), (2.1.2) and (2.1.4), so that $\mathbf{g}_{(\tau,\overline{\tau})}$ obeys (2.1.3) and (2.1.5).

Examples of the delay functional $\mathbf{g}_{(\tau,\overline{\tau})}$. There are many specific delay types that are characterised by the functional $g_{(\tau,\overline{\tau})}$. For example, we may consider very general equations with discrete delay. Suppose $\tilde{g} \in C(\mathbb{R}^d; \mathbb{R})$, and $\tau_1, \tau_2, \ldots \tau_n$ are n functions each of which obeys (2.1.2) and (2.1.4). Then, with $\tau(t) = \min_{j=1\ldots n} \tau_j(t)$, $-\overline{\tau} = \inf_{t\geq 0}\{t - \max_{j=1\ldots n} \tau_j(t)\}$, and $\varphi \in C([-\overline{\tau}, t]; \mathbb{R})$, the functional

$$g_{(\tau,\overline{\tau})}(t, \varphi_t) = \tilde{g}(\varphi(t - \tau_1(t)), \varphi(t - \tau_2(t)), \ldots, \varphi(t - \tau_n(t)))$$

is continuous with delay structure $(\tau, \overline{\tau})$. It is this case, with $n = 1$, that will characterise the processes in this thesis.

Note that equations with distributed delay also have this general structure. As an example, suppose that τ_1, τ_2 are continuous nonnegative functions on $\mathbb{R}^+$ obeying $\tau_1(t) \leq$

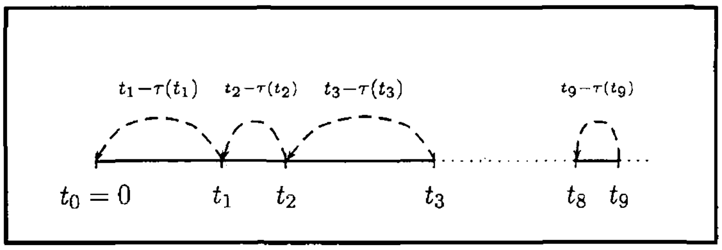


Figure 2.2.1: Construction of the sequence $\{t_n\}_{n=0}$.

$\tau_2(t)$ for $t \geq 0$, $\tau_1(t) > 0$ for all $t \geq 0$. Then τ_1 and τ_2 obey (2.1.2) and (2.1.4). Set $\tau = \tau_1$, and $-\overline{\tau} = \inf_{t\geq 0} t - \tau_2(t)$. Let $\tilde{g} \in C([-\overline{\tau},\infty) \times [-\overline{\tau},\infty) \times \mathbb{R};\mathbb{R})$. Then with $\varphi \in C([-\overline{\tau}, t];\mathbb{R})$, the functional

$$g_{(\tau,\overline{\tau})}(t,\varphi_t) = \int_{t-\tau_2(t)}^{t-\tau_1(t)} \tilde{g}(s,t,\varphi(s))\,ds.$$

is continuous with delay structure $(\tau, \overline{\tau})$.

2.2 Discussion – The method of steps.

The primary technique used in this chapter is the so–called method of steps. The application of this technique requires the partitioning of $\mathbb{R}^+$ into intervals so that at every time point on a given interval the drift coefficient receives feedback from the previous interval. We can thus use the existence of a continuous local solution on any interval to guarantee the existence of a continuous local solution on the next interval. In a sense, this technique is analogous to proof by induction.

We must construct an appropriate mesh with which to partition $\mathbb{R}^+$. Define the sequence $\{t_n\}_{n\geq 0}$ by

$$t_0 = 0, \quad t_{n+1} = \inf\{t \geq 0 \,:\, t - \tau(t) = t_n\},\ n \geq 0. \tag{2.2.1}$$

A schematic of this sequence for a nonspecific delay function τ is given in Figure 2.2.1.

Either there are finitely many members of this sequence, or there are infinitely many. Suppose first that there are infinitely many. Then we see that the sequence $\{t_n\}_{n\geq 0}$ diverges. A similar result appears as Lemma 1.1 in Shreve [29].

Lemma 2.2.1. *Suppose that the function τ obeys (2.1.2), and (2.1.4). Then the sequence $\{t_n\}_{n\geq 0}$ defined by (2.2.1) is strictly increasing and obeys $\lim_{n\to\infty} t_n = \infty$.*

Proof. By (2.2.1), $t_{n+1} = t_n + \tau(t_{n+1})$, so by (2.1.4), $t_{n+1} > t_n$. Suppose that $t_n \to L^-$ where $0 \leq L < \infty$ as $n \to \infty$. Then, by (2.1.2) and (2.2.1)

$$0 = \lim_{n\to\infty} t_{n+1} - \tau(t_{n+1}) - t_n = L - \tau(L) - L = -\tau(L).$$

But by (2.1.4), $\tau(t) > 0$ for all $t \geq 0$, so no such finite L can exist. Hence $t_n \to \infty$. □

On the other hand, if there are finitely many members of the sequence, there exists $N \in \mathbb{N}$ and $t_N \geq 0$ such that $t - \tau(t) \leq t_N$ for all $t \geq 0$.

2.3 Global existence theory.

In order to properly illustrate the method of steps, and motivate its use, we consider here a deterministic equation with delayed feedback. This result will also serve as a useful comparison to the stochastic equation.

2.3.1 The deterministic equation.

Suppose that $\mathbf{g}_{(\tau,\overline{\tau})}$ is continuous with delay structure τ, and consider the deterministic functional differential equation

$$\mathbf{x}'(t) = \mathbf{g}_{(\tau,\overline{\tau})}(t, \mathbf{x}_t), \quad t \geq 0, \tag{2.3.1a}$$

$$\mathbf{x}(t) = \boldsymbol{\psi}(t), \quad t \in [-\overline{\tau}, 0] \tag{2.3.1b}$$

where $\boldsymbol{\psi} \in C([-\overline{\tau}, 0]; \mathbb{R}^d)$. We avail of the continuity and positivity properties of the the delay function τ expressed in (2.1.2) and (2.1.4), in order to solve (2.3.1). We will apply the method of steps over a mesh defined by the sequence (2.2.1).

If $\{t_n\}_{n=0}$ has infinitely many members: Define $I_n = [t_n, t_{n+1}]$, so $\cup_{n\in\mathbb{N}} I_n = \mathbb{R}^+$. Suppose a unique solution of (2.3.1) exists on $[-\overline{\tau}, t_n]$. Then, for each $t \in [t_n, t_{n+1}]$,

$$\mathbf{x}(t) = \mathbf{x}(t_n) + \int_{t_n}^{t} \mathbf{g}_{(\tau,\overline{\tau})}(s, \mathbf{x}_s)\, ds. \tag{2.3.2}$$

We now wish to show that the quantity on the righthand side of (2.3.2) is finite, ensuring that $\mathbf{x}(t)$ can be uniquely defined. This guarantees the uniqueness and existence of a solution on the subinterval $[t_n, t_{n+1}]$. For each $t \in [t_n, t_{n+1}]$, $\mathbf{g}_{(\tau,\overline{\tau})}(t, \mathbf{x}_t)$ does not depend on $\{\mathbf{x}(s) : t_n \leq s \leq t_{n+1}\}$, by (2.1.3) and Lemma 2.2.1. Furthermore, by (2.1.5) $t \mapsto \mathbf{g}_{(\tau,\overline{\tau})}(t, \mathbf{x}_t)$ is a continuous function. So (2.3.2) uniquely defines a C^1 solution of (2.3.1) on $[t_n, t_{n+1}]$, and consequently on $[-\overline{\tau}, t_{n+1}]$. Therefore, by induction, a unique continuous solution exists on $[-\overline{\tau}, \infty)$.

If $\{t_n\}_{n=0}$ has finitely many members: The same argument as above guarantees that (2.3.1) has a unique continuous solution on $[-\overline{\tau}, t_N]$. For $t > t_N$, notice that $t - \tau(t) \leq t_N$, so referring to (2.3.2), the delay property (2.1.3), the delay continuity property (2.1.5), and Lemma 2.2.1, we see that a unique continuous solution is again guaranteed on $[-\overline{\tau}, \infty)$.

Hereinafter we assume, without loss of generality, that the sequence $\{t_n\}$ contains infinitely many members.

2.3.2 The stochastic equation.

We ask whether a scalar stochastic perturbation of the scalar deterministic functional differential equation (2.3.1a), (2.3.1b) still has a *unique* global solution. To provide a source of randomness for this perturbation, let $B = \{B(t); \mathcal{F}^B(t); 0 \leq t < \infty\}$ be a standard one-dimensional Brownian motion defined on the probability triple $(\Omega, \mathcal{F}, \mathbb{P})$. Here $\{\mathcal{F}^B(t)\}_{t\geq 0}$ is the natural filtration of B in the sense that $\mathcal{F}^B(t) = \sigma(\{B(s) : 0 \leq s \leq t\})$.

We study the scalar stochastic functional differential equation

$$\begin{aligned} dX(t) &= g_{(\tau,\overline{\tau})}(t, X_t)\, dt + h(X(t))\, dB(t), && \text{(2.3.3a)} \\ X(t) &= \psi(t), \quad t \in [-\overline{\tau}, 0]. && \text{(2.3.3b)} \end{aligned}$$

We request that $h \in C(\mathbb{R};\mathbb{R})$ be a locally Lipschitz continuous function, $\psi \in C([-\overline{\tau}, 0];\mathbb{R})$ and that $g_{(\tau,\overline{\tau})}$ is continuous with delay structure $(\tau, \overline{\tau})$, obeying (2.1.1), (2.1.2), and (2.1.4). By Theorem 5.3 in Protter [28], the existence of a unique strong solution is guaranteed up to a, possibly random, *explosion time* τ_e^{ψ}. Note the dependence on the initial function ψ. At time $0 \leq t < \tau_e^{\psi}$, this solution can be denoted $X(t, [-\overline{\tau}, 0], \psi)$, but we will generally suppress the dependence on the initial function by writing the solution at time t as $X(t)$. The explosion time is defined to be

$$\tau_e^{\psi} = \inf\{t > 0 : \lim_{s\to t^-} |X(s, [-\overline{\tau}, 0], \psi)| = \infty\}. \tag{2.3.4}$$

Theorem 2.3.1. *Suppose that $g_{(\tau,\overline{\tau})}$ is continuous with delay structure $(\tau, \overline{\tau})$ obeying (2.1.1), (2.1.2), and (2.1.4). Let h be a locally Lipschitz continuous function. If $\overline{\tau}$ is defined by (2.1.1), and $\psi \in C([-\overline{\tau}, 0]; \mathbb{R})$, then there is a unique continuous adapted process X which is a strong solution of (2.3.3) on $\mathbb{R}^+$.*

The statement of the Theorem is equivalent to saying that τ_e^{ψ} defined by (3.5.14) obeys $\tau_e^{\psi} = \infty$, almost surely.

Given the hypotheses (2.1.2) and (2.1.4), the sequence $\{t_n\}$ defined in (3.5.7) is increasing, and, by the assumption at the end of Section 2.3.1, infinite. So, by Lemma 2.2.1, it obeys $\lim_{n\to\infty} t_n = \infty$. Just as for the deterministic equation (2.3.1), we prove the existence of a unique strong solution of (2.3.3) on each of the subintervals $I_n = [t_n, t_{n+1}]$ successively.

A generalisation of this theorem in finite dimensions can be found in Appendix A.

Proof of Theorem 2.3.1. Define

$$\tau_k^{\psi} = \inf\{t \in [0, \tau_e^{\psi}) : X(t, [-\overline{\tau}, 0], \psi) \notin (-k, k)\} \tag{2.3.5}$$

and, as $\{\tau_k\}$ is increasing, we may define $\tau_\infty^{\psi} = \lim_{k\to\infty} \tau_k^{\psi}$. Clearly $\tau_\infty^{\psi} \le \tau_e^{\psi}$, a.s. Therefore, if $\tau_\infty^{\psi} = \infty$ a.s. for each ψ, then $\tau_e^{\psi} = \infty$. We prove $\tau_\infty^{\psi} = \infty$ by contradiction, assuming that $\tau_\infty^{\psi} = \infty$, a.s. is false. First, we show that $\tau_\infty^{\psi} > t_1$, a.s. Consider the negation of this statement, namely that there exists some ψ such that $\mathbb{P}[\tau_\infty^{\psi} > t_1] < 1$. Hence there is $k_0 \in \mathbb{N}$ sufficiently large, and $\varepsilon = \varepsilon(\psi) \in (0, 1)$ such that $\mathbb{P}[\tau_k^{\psi} \le t_1] \ge \varepsilon$, $k \ge k_0$.

Consider a function $V \in C^2(\mathbb{R};\mathbb{R})$ which obeys

$$V(x) > 0, \quad \text{for all } x \in \mathbb{R}, \tag{V1}$$

$$V(x) = V(-x), \quad x > 0, \tag{V2}$$

$$V' \text{ is bounded on } \mathbb{R}, \tag{V3}$$

$$\text{There is } x^* > 0 \text{ such that } V''(x) < 0 \text{ for all } |x| \geq x^*, \tag{V4}$$

$$\lim_{|x|\to\infty} V(x) = \infty. \tag{V5}$$

We note that functions obeying these properties exist. Examples are provided at the end of the proof.

Define $F(x,\mu) = V'(x)\mu + \frac{1}{2}V''(x)h(x)^2$ and $G(x) = V'(x)h(x)$, where $x, \mu \in \mathbb{R}$. Then Itô's rule gives

$$V(X(\tau_k^\psi \wedge t_1)) = V(\psi(0)) + \int_0^{\tau_k^\psi \wedge t_1} G(X(s))\,dB(s) + \int_0^{\tau_k^\psi \wedge t_1} F(X(s), g_{(\tau,\overline{\tau})}(s, X_s))\,ds. \tag{2.3.6}$$

It can be shown by (V3) and (V4) that $\sup_{x\in\mathbb{R}} |F(x,\mu)| \leq \overline{F}(\mu) := C_1|\mu| + C_2$, where C_1, C_2 are μ-independent constants. Using this in (2.3.6) gives

$$V(X(\tau_k^\psi \wedge t_1)) \leq V(\psi(0)) + \int_0^{\tau_k^\psi \wedge t_1} G(X(s))\,dB(s) + \int_0^{t_1} \overline{F}(g_{(\tau,\overline{\tau})}(s, X_s))\,ds.$$

This is true as $\tau_k^\psi \wedge t_1 \leq t_1$. The $\mathcal{F}(0)$-measurability of $F_1(t) = \overline{F}(g_{(\tau,\overline{\tau})}(t, X_t))$ for $t \in [0, t_1]$ implies that

$$\mathbb{E}[V(X(\tau_k^\psi \wedge t_1))] \leq V(\psi(0)) + \int_0^{t_1} F_1(s)\,ds < \infty.$$

Hence for $k \geq k_0$

$$\begin{aligned}\infty > V(\psi(0)) + \int_0^{t_1} F_1(s)\,ds &\geq \mathbb{E}[V(X(\tau_k^\psi \wedge t_1))] \geq \mathbb{E}[1_{\{\tau_k^\psi \leq t_1\}} V(X(\tau_k^\psi))] \\ &\geq \mathbb{P}[\tau_k^\psi \leq t_1] V(k) \geq \varepsilon V(k) \to \infty \text{ as } k \to \infty,\end{aligned}$$

by (V2) and (V5), yielding a contradiction. We therefore conclude that (2.3.3) has a unique adapted continuous solution on $[0,t_1]$.

On $[t_1,t_2]$ we need to show that

$$\mathbb{E}[1_{\{\tau_\infty^\psi > t_2\}} | \mathcal{F}(t_1)] = 1, \quad \text{a.s.} \tag{2.3.7}$$

for each $\psi \in C([-\overline{\tau},0];\mathbb{R})$, implying that $\tau_\infty^\psi > t_2$, a.s. If we can do this, it can be established, by induction and an identical argument, that there is a unique continuous adapted process which obeys (2.3.3) on $\mathbb{R}^+$. We suppose, contradicting (2.3.7), that there exists ψ and $A \subset \Omega$, with $\mathbb{P}[A] > 0$ such that

$$\mathbb{E}[1_{\{\tau_\infty^\psi > t_2\}} | \mathcal{F}(t_1)](\omega) < 1 \quad \text{for all } \omega \in A.$$

So there exists $\varepsilon \in (0,1)$ and $A^\varepsilon \subseteq A$ with $\mathbb{P}[A^\varepsilon] > 0$ such that $\mathbb{E}[1_{\{\tau_\infty^\psi \le t_2\}} | \mathcal{F}(t_1)] > \varepsilon$ on A^ε. Now define the sequence of random variables

$$P_k = \mathbb{E}[1_{\{\tau_k^\psi \le t_2\}} | \mathcal{F}(t_1)] \tag{2.3.8}$$

and the limiting random variable of (2.3.8) in k, namely $P = \mathbb{E}[1_{\{\tau_\infty^\psi \le t_2\}} | \mathcal{F}(t_1)]$. So $P_k \ge P$ and P_k is decreasing in k. So for all k sufficiently large, $P_k \ge P > \varepsilon$ on A^ε. Consequently, we can say that there is $\psi \in C([-\overline{\tau},0];\mathbb{R})$ such that there is an $\varepsilon \in (0,1)$ and a set of positive probability A^ε such that $\mathbb{E}[1_{\{\tau_k^\psi \le T_2\}} | \mathcal{F}(t_1)] > \varepsilon$ on A^ε for $k \in \mathbb{N}$. Now

$$\begin{aligned} V(X(\tau_k^\psi \wedge t_2)) = V(X(t_1)) + \int_{t_1}^{\tau_k^\psi \wedge t_2} F(X(s), g_{(\tau,\overline{\tau})}(s, X_s))\, ds \\ + \int_{t_1}^{\tau_k^\psi \wedge t_2} G(X(s))\, dB(s). \end{aligned}$$

If we consider in turn the three possibilities $\tau_k^\psi \le t_1$, $t_1 \le \tau_k^\psi \le t_2$, and $\tau_k^\psi \ge t_2$, we get

$$\begin{aligned} &\int_{t_1}^{\tau_k^\psi \wedge t_2} F(X(s), g_{(\tau,\overline{\tau})}(s, X_s))\, ds \\ &\qquad \le \max_{t \in [0,t_1]} \int_{t_1}^{t} F(X(s), g_{(\tau,\overline{\tau})}(s, X_s))\, ds \bigvee \int_{t_1}^{t_2} F_1(s)\, ds. \end{aligned} \tag{2.3.9}$$

The right hand side of (2.3.9) is $\mathcal{F}(t_1)$-measurable. Because X does not explode on $[-\overline{\tau}, t_1]$, the conditional expectation relative to $\mathcal{F}(t_1)$ is bounded by an a.s. finite random variable. Therefore, if we can bound

$$\mathbb{E}\left[\int_{t_1}^{\tau_k^\psi \wedge t_2} G(X(s))\,dB(s)\middle|\mathcal{F}(t_1)\right]$$

we are done.

Consider the process $G(t) = G(X(t)) = V'(X(t))h(X(t))$. Define a martingale M on $[0, (\tau_k^\psi \wedge t_2) \vee t_1]$ by

$$M(t) = \int_0^t G(s)\,dB(s). \tag{2.3.10}$$

By Lemma 2.3.3

$$\mathbb{E}\left[\int_{t_1}^{\tau_k^\psi \wedge t_2} G(s)\,dB(s)\middle|\mathcal{F}(t_1)\right] = \int_{t_1}^{\tau_k^\psi \wedge t_1} G(s)\,dB(s) = 1_{\{\tau_k^\psi < t_1\}}\int_{t_1}^{\tau_k^\psi} G(s)\,dB(s)$$

since $\int_0^{t_1} G(s)\,dB(s)$ is $\mathcal{F}(t_1)$-measurable. Thus

$$\mathbb{E}\left[\int_{t_1}^{\tau_k^\psi \wedge t_2} G(s)\,dB(s)\middle|\mathcal{F}(t_1)\right] \le \max_{0\le t\le t_1}\int_{t_1}^{t} G(s)\,dB(s).$$

So

$$\begin{aligned}\mathbb{E}\left[V(X(\tau_k^\psi \wedge t_2))\middle|\mathcal{F}(t_1)\right] &\le V(X(t_1)) + \max_{0\le t\le t_1}\int_{t_1}^{t} G(s)\,dB(s)\\ &+ \int_{t_1}^{t_2} |\overline{F}(g_{(\tau,\overline{\tau})}(s, X_s))|\,ds \bigvee \max_{0\le t\le t_1}\int_{t_1}^{t} F(X(s), g_{(\tau,\overline{\tau})}(s, X_s))\,ds.\end{aligned} \tag{2.3.11}$$

The right hand side of (2.3.11) is $\mathcal{F}(t_1)$-measurable and independent of k. By (V1), we have

$$\begin{aligned}\mathbb{E}\left[V(X(\tau_k^\psi \wedge t_2))\middle|\mathcal{F}(t_1)\right] &\ge \mathbb{E}\left[1_{\{\tau_k^\psi \le t_2\}}V(X(\tau_k^\psi \wedge t_2))|\mathcal{F}(t_1)\right]\\ &= \mathbb{E}\left[1_{\{\tau_k^\psi \le t_2\}}V(X(\tau_k^\psi))|\mathcal{F}(t_1)\right] \ge V(k)\mathbb{E}[1_{\{\tau_k^\psi \le t_2\}}|\mathcal{F}(t_1)] = V(k)\varepsilon \quad \text{on } A^\varepsilon.\end{aligned}$$

Since $\lim_{|k|\to\infty} V(k)\varepsilon = \infty$, by (V5) and (V2), the induced contradiction implies that $\tau_\infty^\psi > t_2$ a.s. for each ψ. We can proceed in this way, as in Section (2.3.1), to show that $\tau_\infty^\psi = \infty$ a.s. for each ψ. $\square$

The same method of proof ensures the existence of a global solution of a more general stochastic functional differential equation.

Theorem 2.3.2. *Let $h \in C(\mathbb{R};\mathbb{R})$ be locally Lipschitz continuous, and $g_{(\tau,\overline{\tau})}$ and $h_{(\tau',\overline{\tau}')}$ be continuous functionals with delay structures $(\tau,\overline{\tau})$ and $(\tau',\overline{\tau}')$ respectively, obeying (2.1.1), (2.1.2), (2.1.4). Define $\tilde{\tau} = \overline{\tau} \vee \overline{\tau}'$, and $\psi \in C([-\tilde{\tau},0];\mathbb{R})$. Then the stochastic functional differential equation*

$$dX(t) = g_{(\tau,\overline{\tau})}(t, X_t)dt + \left[h(X(t)) + h_{(\tau',\overline{\tau}')}(t, X_t)\right] dB(t) \qquad (2.3.12a)$$

$$X(t) = \psi(t), \quad t \in [-\tilde{\tau}, 0]. \qquad (2.3.12b)$$

has a unique adapted continuous strong solution on $\mathbb{R}^+$.

Observe that the function V defined for $x \geq 0$ by

$$V(x) = \begin{cases} \log(x), & x \geq e \\ \frac{1}{4} + e^{-2}x^2 - \frac{1}{4}e^{-4}x^4, & 0 \leq x \leq e, \end{cases} \qquad (2.3.13)$$

and which obeys $V(-x) = V(x)$, for $x \leq 0$, satisfies all the conditions (V1)-(V5) in Theorem 2.3.1.

Finally we prove a technical result required in the proof of Theorem 2.3.1, which will be recognised as part of the Doob martingale stopping theorem [17].

Lemma 2.3.3. *Let $\tau > 0$, and ρ a bounded stopping time for the process M, which is well defined on $[0, \rho \vee \tau]$. If moreover, M is a martingale adapted to $\{\mathcal{F}(t)\}_{t\geq 0}$, we have*

$$\mathbb{E}[M(\rho)|\mathcal{F}(\tau)] = M(\rho \wedge \tau).$$

Proof. Since $M(\tau) + M(\rho) = M(\rho \wedge \tau) + M(\rho \vee \tau)$, we have

$$\begin{aligned} \mathbb{E}[M(\rho)|\mathcal{F}(\tau)] &= \mathbb{E}[M(\rho \wedge \tau) + M(\rho \vee \tau) - M(\tau)|\mathcal{F}(\tau)] \\ &= \mathbb{E}[M(\rho \vee \tau)|\mathcal{F}(\tau)] + \mathbb{E}[M(\rho \wedge \tau)|\mathcal{F}(\tau)] - \mathbb{E}[M(\tau)|\mathcal{F}(\tau)] \\ &= M(\tau) + M(\rho \wedge \tau) - M(\tau) = M(\rho \wedge \tau), \end{aligned}$$

since $M(\tau)$, $M(\rho \wedge \tau)$ are $\mathcal{F}(\tau)$-measurable, and M is a martingale. □

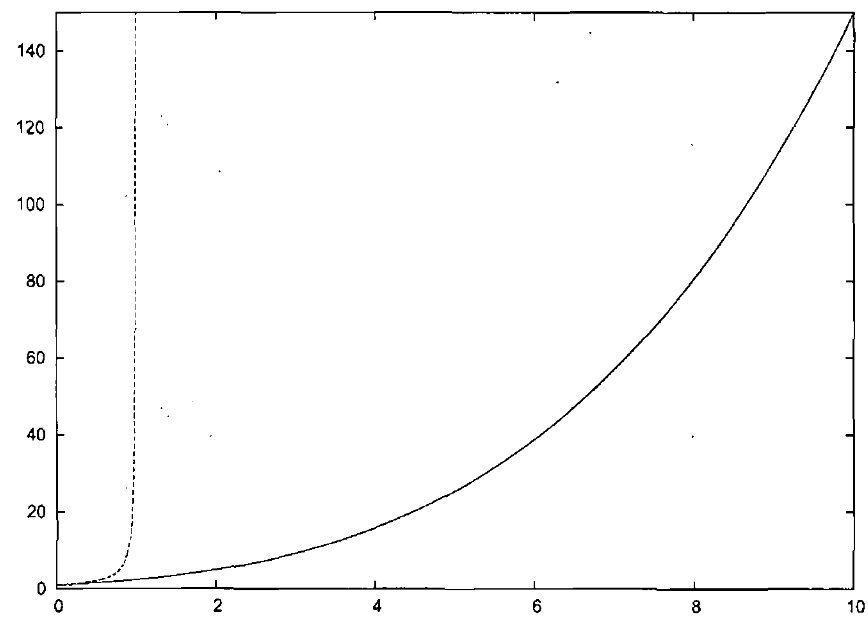


Figure 2.4.1: Red: The solution of (2.4.2), with $f(x) = x^{3/4}$, does not explode. Green: When $f(x) = x^2$, the solution explodes.

2.4 Prevention of explosion by noise.

Consider the scalar equation

$$dX(t) = \left(f(X(t)) + g_{(\tau,\overline{\tau})}(t, X_t)\right) dt + h(X(t))\, dB(t), \tag{2.4.1a}$$

$$X(t) = \psi(t), \quad t \in [-\overline{\tau}, 0]. \tag{2.4.1b}$$

We suppose here that f and h are locally Lipschitz continuous functions on $\mathbb{R}$. We assume that the functional $g_{(\tau,\overline{\tau})}$ is nonnegative, so that $g_{(\tau,\overline{\tau})}(t, \varphi_t) \geq 0$ for all $t \geq 0$ and $\varphi \in C([-\overline{\tau}, t]; \mathbb{R})$, and that it is locally Lipschitz continuous in the second argument.

Our goal is to understand the manner in which the interactions between noise and delay affect the explosive behaviour of the solutions of (2.4.1). Initially, we will consider the behaviour of a deterministic equation with instantaneous feedback. The effect of adding a delayed feedback term and, subsequently, a stochastic perturbation to this equation will be investigated. Figures 2.4.1 to 2.4.4 illustrate the discussion with examples.

The deterministic equation with instantaneous feedback. We remove the delayed feedback and noise perturbation from (2.4.1). Let f be a locally Lipschitz continuous function such that there is some $x^* \in \mathbb{R}$ such that $f(x) > 0$ for $x > x^*$. Consider the initial value problem

$$x'(t) = f(x(t)), \quad t \geq 0, \quad x(0) = x_0, \tag{2.4.2}$$

where $x_0 \geq x^*$. The following proposition gives a well–known sufficient condition for the solutions of (2.4.2) to explode in finite time.

Proposition 2.4.1. *If*

$$\int_{x_0}^{\infty} \frac{1}{f(x)}\, dx < \infty, \tag{2.4.3}$$

then there exists $0 < T < \infty$ *such that*

$$\lim_{t \to T^-} x(t) = \infty, \tag{2.4.4}$$

where x *is the unique continuous solution of (2.4.2).*

Proof. By (2.4.2), and since $x_0 > 0$, $f(x(t)) > f(x_0) > 0$ for all $t > 0$. Hence, by (2.4.2), $x'(t)/f(x(t)) = 1$. Integrating both sides with respect to t, and applying the change of variables $u = x(t)$ gives

$$\int_0^t \frac{x'(s)}{f(x(s))} ds = \int_{x_0}^{x(t)} \frac{1}{f(u)} du = t. \tag{2.4.5}$$

Assume that x exists on $[0, \infty)$. There are two possibilities.

Case 1: Suppose $\lim_{t\to\infty} x(t) = L < \infty$. Then, by (2.4.5),

$$\lim_{t\to\infty} \int_{x_0}^{x(t)} \frac{1}{f(u)} du = \int_{x_0}^{L} \frac{1}{f(u)} du = \infty.$$

But this leads to a contradiction, as the integral of a continuous, positive function over a compact interval cannot be infinite.

Case 2: Suppose $\lim_{t\to\infty} x(t) = \infty$. By (2.4.5),

$$\lim_{t\to\infty} \int_{x_0}^{x(t)} \frac{1}{f(u)} du = \int_{x_0}^{\infty} \frac{1}{f(u)} du = \infty.$$

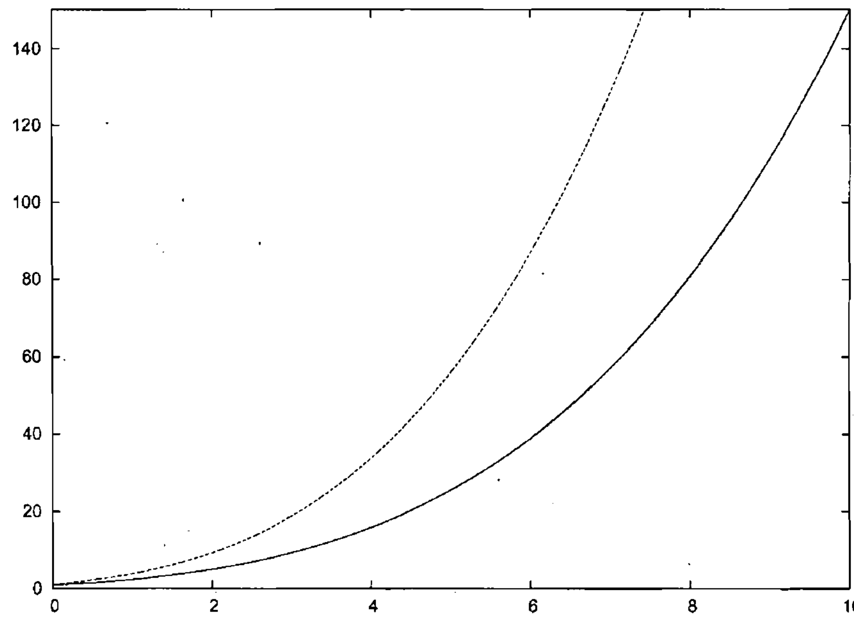


Figure 2.4.2: Red: The solution of (2.4.2) with $f(x) = x^{3/4}$. Green: When a delay term $g_{(1,1)}(x(t)) = (x(t-1))^{1/2}$ is introduced, an explosion is not induced.

But this directly contradicts (2.4.3). Therefore, (2.4.4) holds. □

Including delayed feedback in the deterministic equation. It is natural to ask how the addition of the delay component $g_{(\tau,\overline{\tau})}$ to (2.4.2) affects the explosion of solutions of the perturbed equation, when $f(x) > 0$ for all $x > 0$, and satisfies (2.4.3).

Consider the functional differential equation

$$x'(t) = f(x(t)) + g_{(\tau,\overline{\tau})}(t, x_t), \quad t \geq 0, \quad x(t) = \psi(t), \quad t \in [-\overline{\tau}, 0], \tag{2.4.6}$$

where $\psi \in C([-\overline{\tau}, 0]; \mathbb{R})$ and $g_{(\tau,\overline{\tau})} : [0,\infty) \times C([-\overline{\tau}, t]; \mathbb{R}) \to [0,\infty)$. The local Lipschitz continuity on h guarantees that there is a unique solution to this equation, up to an explosion time. A comparison argument establishes that the solution of (2.4.6) dominates that of (2.4.2) pointwise if the initial value of (2.4.6) is greater than or equal to the initial value of (2.4.2). Therefore, since the solution of (2.4.2) explodes in finite time, the

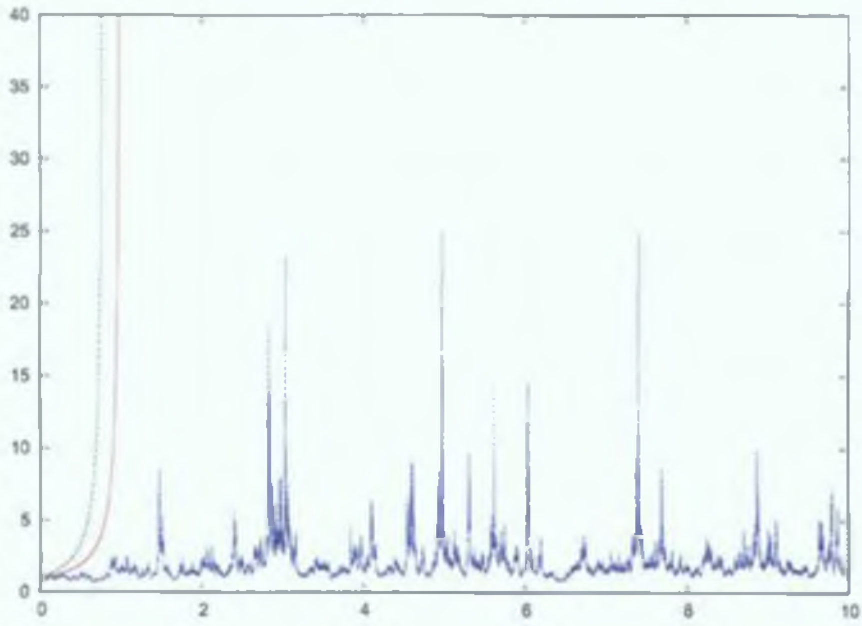


Figure 2.4.3: Red: The solution of (2.4.2) with $f(x) = x^2$. Green: When a delay term $g_{(1,1)}(x(t)) = (x(t-1))^{1/2}$ is introduced, the explosion is not qualitatively affected. Blue: The explosion can be suppressed by the introduction of a noise term $h(x) = x^2$.

---

solution of (2.4.6) explodes in finite time. Moreover, (2.4.3), the condition which ensures the explosion of (2.4.6), does not involve the delay functional $g_{(\tau,\overline{\tau})}$. It does not appear that the explosion of (2.4.6) is influenced by the delay term.

Including a stochastic perturbation. If we perturb (2.4.6) stochastically to form (2.4.1) then the following theorem applies.

Theorem 2.4.2. *Suppose that $g_{(\tau,\overline{\tau})}$ is continuous with delay structure $(\tau,\overline{\tau})$ obeying (2.1.1), (2.1.2), and (2.1.4). If $\psi \in C([-\overline{\tau},0];\mathbb{R})$, f and h are locally Lipschitz continuous, and*

$$\sup_{x\in\mathbb{R}} \frac{xf(x) - \frac{1}{2}h(x)^2}{1+|x|^2} < \infty, \tag{2.4.7}$$

then there is a unique, continuous, adapted process which is a strong solution of (2.4.1) *on* $[0, \infty)$.

This result can be proved in a manner similar to that of Theorem 2.3.1, using the function V defined in (2.3.13).

The condition (2.4.7) ensures that if the intensity of the diffusion term h is sufficiently large relative to f, the explosion exhibited by the solution of (2.4.6) can always be suppressed. Moreover, since the condition (2.4.7) does not involve the delay functional $g_{(\tau,\overline{\tau})}$, it appears that the delay term does not play a role in causing or preventing explosions of (2.4.1).

Removing the delayed feedback from the stochastic equation. By comparison, consider the equation

$$dX(t) = f(X(t))dt + h(X(t))dB(t), \quad t \in [t_0, T] \tag{2.4.8}$$

where f and h obey a local Lipschitz condition. Mao [25] guarantees the existence of a unique solution to (2.4.8) on $[t_0, \infty)$ by requiring that, for every $T > t_0$, there exists a positive constant K_T such that for all $x \in \mathbb{R}$

$$xf(x) + \frac{1}{2}|h(x)|^2 \leq K_T(1 + |x|^2). \tag{2.4.9}$$

This result is interesting because of the similar forms of (2.4.7) and (2.4.9). However, in (2.4.7) it can be seen that the counteracting effect of the diffusion coefficient prevents explosion in the solutions of (2.4.1), whereas (2.4.9) guarantees existence through the action of a drift coefficient that is strongly directed toward equilibrium. Existence in this case is guaranteed despite, rather than because of, the noise perturbation.

In fact, if the relative sizes of f and h in (2.4.8) are as given in Feller's test for explosion and nonexplosion, then we can again suppress an explosion in the corresponding deterministic equation (2.4.2).

Proposition 2.4.3 (Feller's test). *There is a unique solution of (2.4.8) on* $[0, \infty)$ *almost*

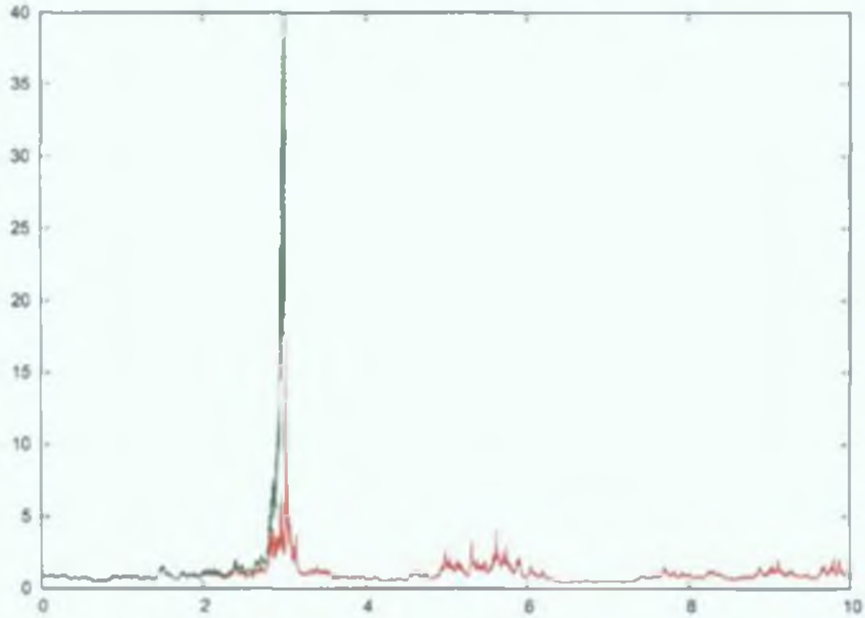


Figure 2.4.4: Green: The solution of (2.4.8) with $f(x) = x^2$, $h(x) = x^{3/2}$. The noise perturbation is insufficient to suppress the explosion. Red: When $h(x) = x^2$, the explosion is suppressed.

---

surely if and only if both of the following conditions hold for all x_0:

$$\int_{-\infty}^{x_0} \exp\Big(-\int_{x_0}^{x} \frac{2f(s)}{h(s)^2}ds\Big)\Big(\int_{x}^{x_0} \frac{\exp(\int_{x_0}^{y} \frac{2f(s)}{h(s)^2}ds)}{h(y)^2}dy\Big)dx = \infty, \tag{2.4.10a}$$

$$\int_{x_0}^{\infty} \exp\Big(-\int_{x_0}^{x} \frac{2f(s)}{h(s)^2}ds\Big)\Big(\int_{x_0}^{x} \frac{\exp(\int_{x_0}^{y} \frac{2f(s)}{h(s)^2}ds)}{h(y)^2}dy\Big)dx = \infty. \tag{2.4.10b}$$

The statement of Feller's test as given in Lemma 2.4.3 can be found in Klebaner [19]. A proof can be found in Karatzas & Shreve [17].

When the delay functional is absent, the condition (2.4.7) is sufficient to guarantee the Feller's test conditions for the solutions of (2.4.8).

Proposition 2.4.4. *Let f and h be continuous, real-valued functions. If (2.4.7) holds, then the Feller's test conditions (2.4.10a) and (2.4.10b) are satisfied.*

Proof. In the context of the discussion, only (2.4.10b) is relevant, and so we restrict ourselves to showing that (2.4.10b) is satisfied. The integral on the left hand side of (2.4.10b) can be rewritten as

$$I(x_0) = \int_{x_0}^{\infty} \int_{x_0}^{x} e^{-2\int_y^x \frac{f(u)}{h(u)^2}du} \frac{1}{h(y)^2} dy\, dx.$$

By (2.4.7), there exists $c < \infty$ large enough that

$$-\frac{2f(x)}{h(x)^2} \geq \frac{-2c(1+x^2)}{xh(x)^2} - \frac{1}{x}.$$

Since $u \mapsto \frac{1+u^2}{u^2}$ is a strictly decreasing function on $(0,\infty)$, we can define a constant $\tilde{c}(x_0) = \sup_{u \geq x_0} c\frac{1+u^2}{u^2} \in (0,\infty)$ so that

$$\begin{aligned} e^{-2\int_y^x \frac{f(u)}{h(u)^2}} &\geq \frac{y}{x} e^{-2c\int_y^x \frac{1+u^2}{u^2}\cdot\frac{u}{h(u)^2}du} \\ &\geq \frac{y}{x} e^{-2\tilde{c}(x_0)\int_y^x \frac{u}{h(u)^2}du}. \end{aligned}$$

Therefore

$$I(x_0) \geq \int_{x_0}^{\infty} \frac{1}{x}\left[e^{-2\tilde{c}(x_0)\int_{x_0}^x \frac{u}{h(u)^2}du} \int_{x_0}^{x} e^{2\tilde{c}(x_0)\int_{x_0}^y \frac{u}{h(u)^2}du} \frac{y}{h(y)^2} dy\right] dx. \tag{2.4.11}$$

Define $L(x) = \int_{x_0}^x \frac{u}{h(u)^2}du$. We can rewrite (2.4.11) as

$$I(x_0) \geq \int_{x_0}^{\infty} \frac{1}{x}\left[e^{-2\tilde{c}(x_0)L(x)} \int_{x_0}^{x} e^{2\tilde{c}(x_0)L(y)} L'(y)dy\right] dx. \tag{2.4.12}$$

Applying a change of variables,

$$\begin{aligned} \int_{x_0}^{x} e^{2\tilde{c}(x_0)L(y)} L'(y)dy &= \int_{L(x_0)}^{L(x)} e^{2\tilde{c}(x_0)u} du \\ &= \frac{1}{2\tilde{c}(x_0)}[e^{2\tilde{c}(x_0)L(x)} - e^{2\tilde{c}(x_0)L(x_0)}]. \end{aligned}$$

Therefore, regardless of whether $\lim_{x\to\infty} L(x)$ is finite or infinite,

$$\lim_{x\to\infty} \frac{\int_{x_0}^x e^{2\tilde{c}(x_0)L(y)} L'(y)dy}{e^{2\tilde{c}(x_0)L(x)}} < \infty. \tag{2.4.13}$$

Now define

$$H(x) = e^{-2\bar{c}(x_0)L(x)} \int_{x_0}^{x} e^{2\bar{c}(x_0)L(y)} L'(y)dy, \tag{2.4.14}$$

so that $H(x) \to H^* > 0$ as $x \to \infty$. By (2.4.12) and (2.4.14)

$$I(x_0) \geq \int_{x_0}^{\infty} \frac{1}{x} H(x)dx.$$

Hence, by (2.4.13), $I(x_0) = \infty$ for all x_0, and (2.4.10a) holds. □

Final comment. The analysis of (2.4.1), (2.4.2), (2.4.6), and (2.4.8) suggests that a sufficiently large state-dependent noise perturbation can suppress an explosion in the solution of a scalar deterministic equation. Moreover, if the delay term is strictly in the past, in the sense that it is present as a functional $g_{(\tau,\overline{\tau})}$ with continuous delay structure $(\tau, \overline{\tau})$, then the presence of an explosion does not depend on the structure of the delay term, but purely on the instantaneous terms.

Chapter 3

Oscillatory Behaviour – The Nonlinear Stochastic Differential Equation with Fixed Delay

In order to begin to sketch the influence of feedback delay and noise perturbation in the oscillatory behaviour of solutions of stochastic delay differential equations, we fix the delay and the noise, and adjust the nonlinearity of the drift coefficient. We find that this illustrates the role played by the noise perturbation. To this end, we analyse the oscillatory properties of the stochastic delay differential equation

$$dX(t) = -g(X(t-\tau))dt + \sigma h(X(t))dB(t), \tag{3.0.1a}$$

$$X(t) = \psi(t), \quad t \in [-\tau, 0] \tag{3.0.1b}$$

where $\sigma \neq 0$, $\tau > 0$ are real constants. This analysis can also be found in Appleby & Kelly [3]. The initial data ψ is a continuous function on $[-\tau, 0]$. Suppose that $g \in C(\mathbb{R};\mathbb{R})$, and both g and h are locally Lipschitz continuous on $\mathbb{R}$. It is shown in Chapter 2 that (3.0.1) has a unique strong continuous solution on $[0, \infty)$, almost surely. By way of comparison, we describe the oscillatory behaviour of the corresponding deterministic equation.

3.1 The deterministic equation.

Gopalsamy [11] and Ladde, Lakshmikantham & Zhang [9], consider the oscillatory behaviour of the nonlinear delay differential equation

$$x'(t) = -g(x(t-\tau)). \tag{3.1.1}$$

The existence of an equilibrium solution $x(t) \equiv 0$ is ensured by requiring that $g(0) = 0$. The continuous forcing function g must act towards the equilibrium in order to generate an environment conducive to oscillatory behaviour, and therefore it is required that $xg(x) > 0$ for $x \neq 0$.

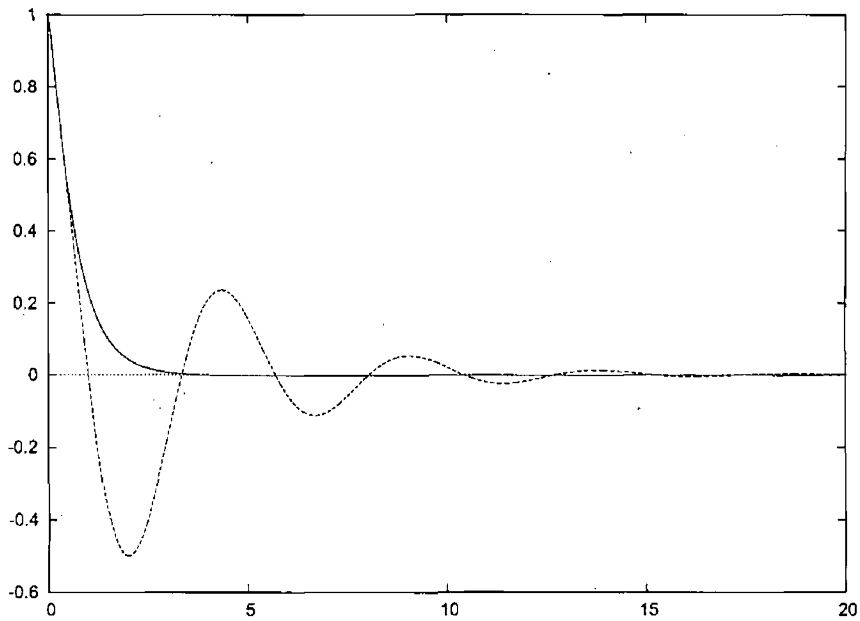


Figure 3.1.1: Red: The solution of (3.1.1), where $g(x) = x$ is **linear**, does not oscillate when there is **short delay**, $\tau = 0.3$. Green: When there is **long delay**, $\tau = 1$, oscillation occurs.

From Theorem 1.5.2 in Gopalsamy [11], a linearisable g at equilibrium is the primary guarantor of oscillatory solutions of (3.1.1). That is to say, if there is $\infty \geq L > 0$ such that

$$\lim_{x \to 0} \frac{g(x)}{x} = L \tag{3.1.2}$$

then oscillatory solutions exist. Oscillation can be guaranteed for every solution by ensuring that the delay term τ is large enough. More specifically, if $\tau L > \frac{1}{e}$ and (3.1.2) is true, then every solution of (3.1.1) oscillates.

From Theorem 3.2.8 in Ladde, Lakshmikantham & Zhang [9], it can be seen that if g does not obey (3.1.2), but the restoring force towards the zero equilibrium is weaker, in the sense that

$$\lim_{x \to 0} \frac{|g(x)|}{|x|^{\gamma}} = \overline{L} \tag{3.1.3}$$

for some $\gamma > 1$ and $\overline{L} > 0$, solutions of (3.1.1) do not have to oscillate. In fact, if ψ is any positive, continuous function on $[-\tau, 0]$, there exists $\alpha^* > 0$ such that the solution of (3.1.1), with $x(t) = \alpha\psi(t)$ for $t \in [-\tau, 0]$ and $0 < \alpha < \alpha^*$, is nonoscillatory.

Nonlinear equations with forcing coefficients are also considered by, among others, Ladas et al [8], Shreve [29], and Staikos & Stavroulakis [30]. For example, Shreve gives an analysis of the equation

$$x'(t) = -a(t)g(x(t - \tau(t))), \tag{3.1.4}$$

where τ is a continuous positive function satisfying $\lim_{t\to\infty} t - \tau(t) = \infty$ and a is a locally integrable function that is nonnegative almost everywhere. He finds that if g is sublinear in the sense that $\lim_{x\to\infty} x/g(x) = M < \infty$, and $\limsup_{t\to\infty} \int_{t-\tau(t)}^{t} a(s)ds > M$, then every solution of (3.1.4) is oscillatory. However if g is superlinear in the sense that $\lim_{x\to 0} x/g(x) = \infty$, then the first zero can be delayed for an arbitrarily long time by choosing a sufficiently small constant initial data function. In fact, if $\int_b^\infty a(s)ds < \infty$, then the existence of a nonoscillatory solution can be guaranteed. It is the strength of the forcing function a, rather than the length of the delay, that drives the oscillatory behaviour of solutions of (3.1.4).

3.2 Properties of the coefficients of the stochastic equation.

Our goal is to determine the behaviour of the solutions of (3.1.1) following the inclusion of a noise perturbation, generating a stochastic process solving (3.0.1). The question of how to structure this perturbation is an important one. In order that we may apply the definition of oscillation for stochastic processes given in Definition 1.4.2, it is crucial that the equilibrium solution be preserved, and this will partially determine the conditions we place on the diffusion coefficient.

We impose the following hypotheses on the continuous function h. Let $h(0) = 0$, and suppose there exists $0 < \underline{h} \leq 1 \leq \overline{h}$ such that

$$\underline{h}|x|^2 \leq xh(x) \leq \overline{h}|x|^2, \tag{3.2.1}$$

and

$$\lim_{x\to 0} \frac{h(x)}{x} = 1. \tag{3.2.2}$$

In addition, the continuous function g has the properties

$$g(0) = 0, \quad xg(x) > 0, \quad x \neq 0. \tag{3.2.3}$$

Notice now that if $\psi(t) = 0$ for all $t \in [-\tau, 0]$ that the unique solution of (3.0.1) is $X(t) = 0$ for all $t \geq 0$, a.s. It is the oscillation, or absence of oscillation, about this equilibrium solution that we intend to study. The existence of this equilibrium solution also rules out any possibility of oscillation if $\tau(t) \equiv 0$.

3.3 The decomposition of solutions.

Our proofs of oscillation and nonoscillation rely upon the representation of the solution of (3.0.1) as the product of a nowhere differentiable but positive process, with asymptotic behaviour that is readily understood, and a process with continuously differentiable sample paths, obeying a scalar random delay differential equation. To this end, we introduce the continuous function $\tilde{h}$

$$\tilde{h}(x) = \begin{cases} \frac{h(x)}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}, \tag{3.3.1}$$

so that $\underline{h} \leq \tilde{h}(x) \leq \overline{h}$, $x \in \mathbb{R}$. We may then define the process $\{\varphi(t)\}_{t \geq -\tau}$ by $\varphi(t) = 1,\ t \in [-\tau, 0]$ and for $t \geq 0$ by

$$\varphi(t) = e^{\sigma \int_0^t \tilde{h}(X(s))dB(s) - \frac{1}{2}\sigma^2 \int_0^t \tilde{h}(X(s))^2 ds}. \tag{3.3.2}$$

The process is uniquely defined on $[0, \infty)$, as X is a well-defined process. We call the almost sure set on which φ exists $\Omega_1^* \subseteq \Omega_0^*$, with $\mathbb{P}[\Omega_1^*] = 1$. Observe further that φ satisfies

$$d\varphi(t) = \sigma \tilde{h}(X(t))\varphi(t)dB(t). \tag{3.3.3}$$

Since φ is positive, we may define

$$Z(t) = X(t)\varphi(t)^{-1}, \quad t \geq -\tau. \tag{3.3.4}$$

Then $Z(t) = \psi(t)$ for $t \in [-\tau, 0]$, and using stochastic integration by parts, (3.0.1) and (3.3.3) imply that

$$Z(t) = \psi(0) + \int_0^t -g(X(s-\tau))\varphi(s)^{-1}ds, \quad t \geq 0.$$

The continuity of the integrand implies that Z is continuously differentiable, and satisfies

$$Z'(t) = -\varphi(t)^{-1}g(X(t-\tau)), \ t > 0. \tag{3.3.5}$$

The following lemma places upper and lower bounds on the rate of decay of the process φ.

Lemma 3.3.1. *Let φ be defined by (3.3.2), where $\tilde{h}$ is given by (3.3.1). Then*

$$\limsup_{t\to\infty} \frac{1}{t}\log\varphi(t) \leq -\frac{1}{2}\sigma^2\underline{h}^2, \quad a.s. \tag{3.3.6}$$

$$\liminf_{t\to\infty} \frac{1}{t}\log\varphi(t) \geq -\frac{1}{2}\sigma^2\overline{h}^2, \quad a.s. \tag{3.3.7}$$

Proof. Define

$$M(t) = \int_0^t \tilde{h}(X(s))dB(s), \quad t \geq 0, \tag{3.3.8}$$

and the associated quadratic variation process

$$\langle M\rangle(t) = \int_0^t \tilde{h}(X(s))^2\, ds. \tag{3.3.9}$$

Then φ may be rewritten as $\varphi(t) = e^{\sigma M(t) - \frac{1}{2}\sigma^2\langle M\rangle(t)}$. By (3.2.1) and (3.3.1),

$$\underline{h}^2 t \leq \langle M\rangle(t) \leq \overline{h}^2 t. \tag{3.3.10}$$

To prove (3.3.6) and (3.3.7), note that, by (3.3.10), $\lim_{t\to\infty}\langle M\rangle(t) = \infty$, a.s., so by Lemma 1.2.2, the law of large numbers for martingales, $M(t)/\langle M\rangle(t) \to 0$ as $t \to \infty$, a.s., and therefore, as

$$\left|\frac{M(t)}{t}\right| = \frac{\langle M\rangle(t)}{t}\cdot\left|\frac{M(t)}{\langle M\rangle(t)}\right| \leq \overline{h}^2\left|\frac{M(t)}{\langle M\rangle(t)}\right|,$$

we get $M(t)/t \to 0$ as $t \to \infty$, a.s. Since $\underline{h}^2 t \leq \langle M\rangle(t) \leq \overline{h}^2 t$, the estimates (3.3.6) and (3.3.7) follow. □

3.4 Almost sure oscillation of solutions.

Consider the stochastic delay differential equation (3.0.1) where, in addition to the earlier hypotheses on g and h, we request that there exists $L \in (0, \infty]$ such that

$$\liminf_{x \to 0} \frac{g(x)}{x} = L. \tag{3.4.1}$$

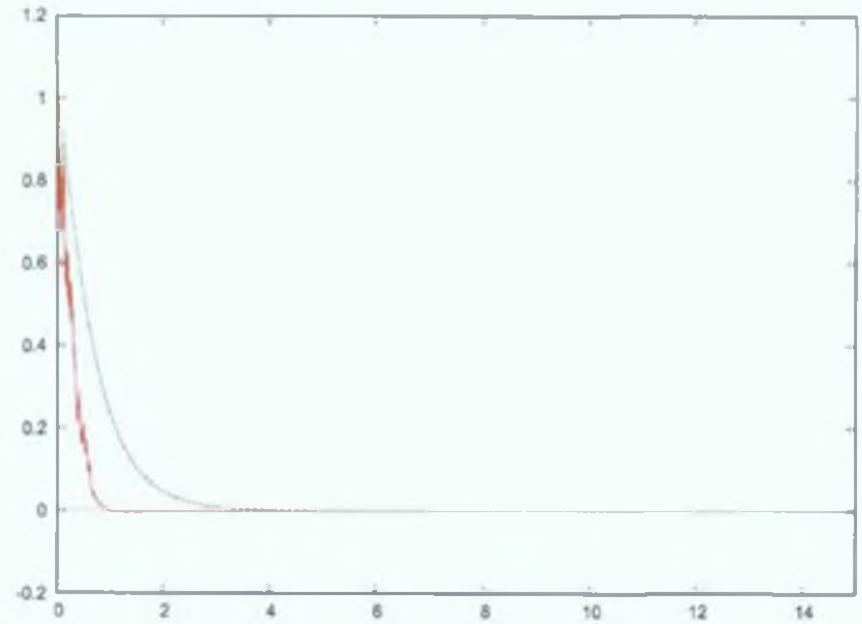
A drift coefficient satisfying (3.4.1) is said to be *sublinear* at zero. This definition is similar to that used in Shreve [29].

3.4.1 Main result.

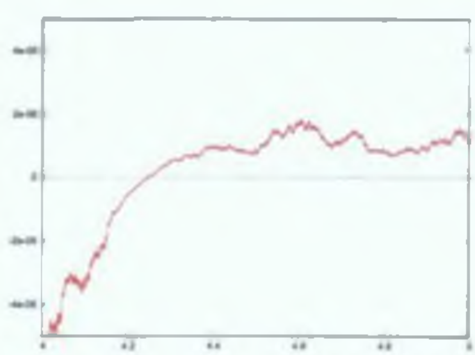
Theorem 3.4.1. *Suppose that the continuous function g obeys (3.2.3) and (3.4.1), and the locally Lipschitz continuous function h obeys (3.2.1) and (3.2.2). If $\psi \in C([-\tau, 0], \mathbb{R})$, then all solutions of (3.0.1) are a.s. oscillatory.*

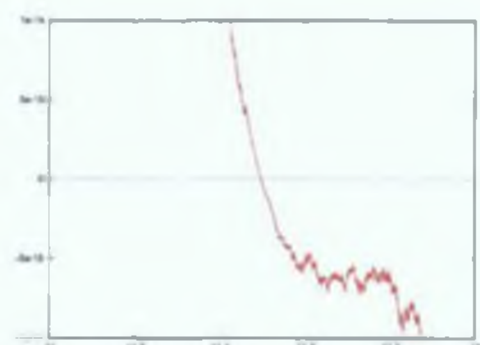
Proof. Note that if $\psi(t) \equiv 0$ on $[-\tau, 0)$, then $X(t) \equiv 0$, for all $t \geq 0$. So, in this case, the solution is oscillatory. Therefore we assume that $\psi(t) \not\equiv 0$ on $[-\tau, 0)$. By Theorem 2.3.1, the solution exists on a set $\Omega^* \subseteq \Omega$, where $\mathbb{P}[\Omega^*] = 1$. So $\Omega^* = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \Omega_2 = \emptyset$ such that the solution is a.s. oscillatory on Ω_1 and a.s. nonoscillatory on Ω_2. Suppose $\mathbb{P}[\Omega_2] > 0$, contradicting the statement of the theorem. Take $\omega \in \Omega^*$. Let φ be the process defined in (3.3.2) which obeys (3.3.3). Let Z be the process defined in (3.3.4) which satisfies the random delay differential equation (3.3.5). Now suppose $\omega \in \Omega_2$. Then there exists $\tau^*(\omega) < \infty$ such that, for all $t > \tau^*(\omega)$, $X(t, \omega) \neq 0$. Therefore, either $X(t, \omega) > 0$ for all $t > \tau^*(\omega)$, or $X(t, \omega) < 0$ for all $t > \tau^*(\omega)$. Suppose, without loss of generality, that $X(t, \omega) > 0$ for all $t > \tau^*(\omega)$. Then, for all $t > \tau^*(\omega) + \tau$, (3.2.3), (3.3.4) and (3.3.5) imply that

$$Z'(t, \omega) < 0, \quad Z(t, \omega) > 0.$$



(a)





(b) (c)

Figure 3.4.1: Green: The solution of (3.1.1), where $g(x) = x$ is **linear** and $\tau = 0.3$, does not oscillate. Red: The inclusion of a noise term $h(x) = x$ does not *appear* to cause oscillation in plot **(a)**. However, a closer inspection of the simulated path highlights sign–changes on $[0, 15]$ at $t = 1.3048$, 4.2498, 6.466, 10.3591, and 11.4957. Close up views of two of these are presented in **(b)** and **(c)**. Extending the path further yields sign changes on [15,30] at $t = 16.2373$, 18.6332, 19.0507, 22.2824, 23.0114, 24.4985, and 26.5472.

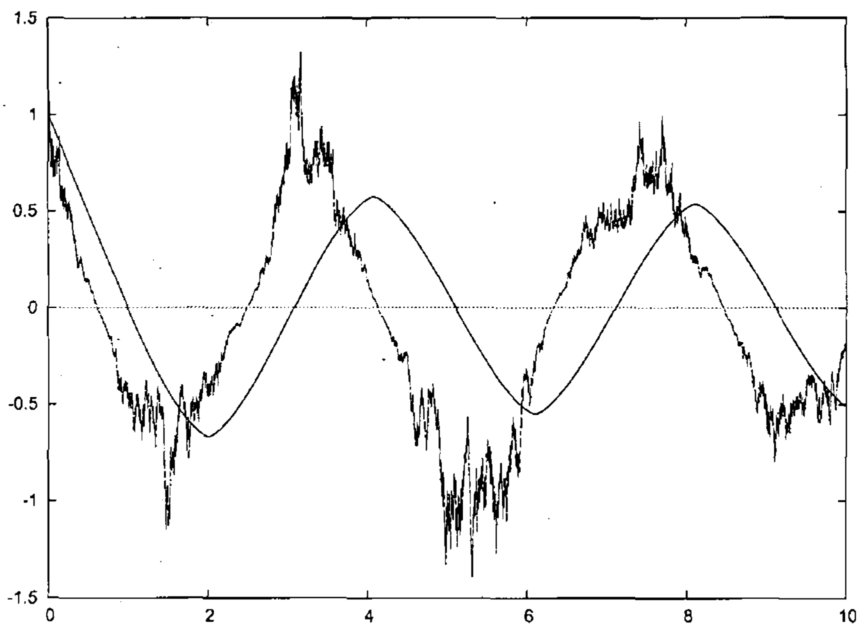


Figure 3.4.2: Red: The solution of (3.1.1), where $g(x) = \operatorname{sgn} x\sqrt{x}$ is **sublinear**, oscillates when there is **long delay**, $\tau = 1$. Green: The introduction of a noise term $h(x) = x$ does not qualitatively affect the oscillatory behaviour.

Hence $0 < X(t) < \varphi(t)Z(\tau^* + \tau)$ for all $t > \tau^* + \tau$. Therefore, for all $t > \tau^* + \tau$

$$|X(t)| \leq \left|\frac{X(\tau^* + \tau)}{\varphi(\tau^* + \tau)}\right| \varphi(t).$$

Since $\tau^* < \infty$ a.s. on Ω_2 and $t \mapsto X(t)$, $t \mapsto \varphi(t)$ are continuous, and therefore bounded, on $[0, \tau^* + \tau]$, the quantity

$$C(\omega) := \left|\frac{X(\tau^* + \tau, \omega)}{\varphi(\tau^* + \tau, \omega)}\right|,$$

is positive and finite for all $\omega \in \Omega_2$, and

$$|X(t,\omega)| \leq C(\omega)\varphi(t,\omega), \quad t > \tau^*(\omega) + \tau.$$

By (3.3.6), $\varphi(t) \to 0$ as $t \to \infty$. Thus $X(t) \to 0$ as $t \to \infty$, on Ω_2. For $t \geq \tau^* + 2\tau$, the

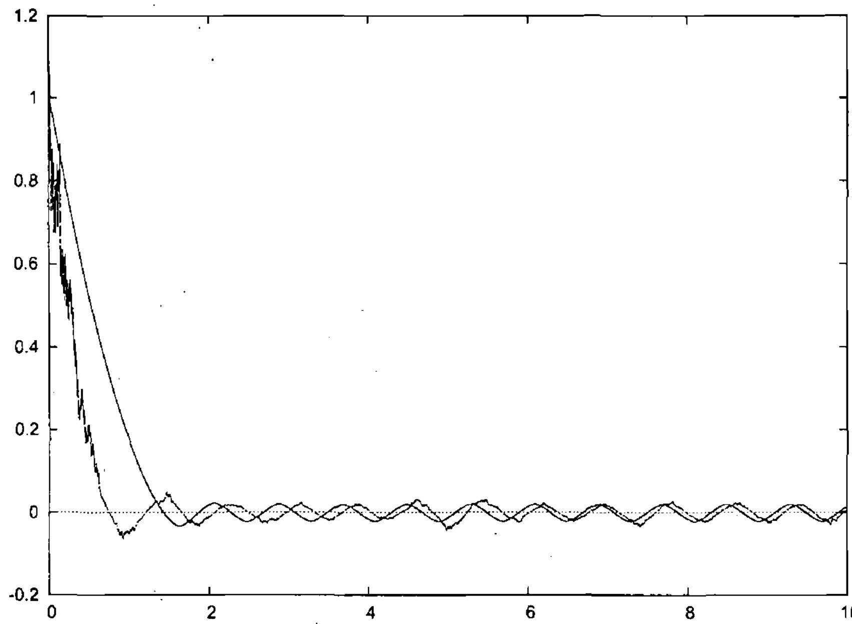


Figure 3.4.3: Red: The solution of (3.1.1), where $g(x) = \operatorname{sgn} x\sqrt{x}$ is **sublinear**, oscillates when there is **short delay**, $\tau = 0.3$. Green: The introduction of a noise term $h(x) = x$ does not qualitatively affect the oscillatory behaviour.

---

function $\tilde{g}$, given by

$$\tilde{g}(t) = \frac{g(X(t-\tau))}{X(t-\tau)},$$

is well defined. Then $\tilde{g}(t) > 0$ for $t \geq \tau^* + 2\tau$ and, as $X(t) \to 0$ as $t \to \infty$, (3.4.1) implies

$$\liminf_{t\to\infty} \tilde{g}(t,\omega) = L > 0, \quad \omega \in \Omega_2. \tag{3.4.2}$$

Letting $P(t) = \varphi(t)^{-1}\varphi(t-\tau)\tilde{g}(t)$, $t \geq \tau^* + 2\tau$, we see that $P(t,\omega) > 0$ for all $\omega \in \Omega_2$, $t \geq \tau^*(\omega) + 2\tau$, and (3.3.5) can be rewritten as

$$Z'(t) = -P(t)Z(t-\tau), \quad t > \tau^* + 2\tau. \tag{3.4.3}$$

Therefore, if we can show

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} P(s,\omega)\, ds = \infty, \tag{3.4.4}$$

for almost all $\omega \in \Omega_2$, then $t \mapsto Z(t,\omega)$ is oscillatory for a.a. $\omega \in \Omega_2$, by applying Lemma 1.5.1 for each $\omega \in \Omega_2$. But as the zeros of $X(t,\omega)$ and $Z(t,\omega)$ coincide, this implies that $t \mapsto X(t,\omega)$ is oscillatory for a.a. $\omega \in \Omega_2$. This contradicts the construction of Ω_2, and so the result follows from (3.4.4). By (3.4.2), we see that (3.4.4) is true if

$$\limsup_{t\to\infty}\left(\int_{t-\tau}^{t} \varphi(s)^{-1}\varphi(s-\tau)\,ds\right)(\omega) = \infty, \quad \text{a.a. } \omega \in \Omega_2. \tag{3.4.5}$$

We now turn to proving this claim. For $t > \tau$, we have

$$\int_{t-\tau}^{t} \varphi(s)^{-1}\varphi(s-\tau)\,ds = \int_{t-\tau}^{t} e^{\frac{1}{2}\sigma^2(\langle M\rangle(s)-\langle M\rangle(s-\tau))}e^{\sigma(M(s-\tau)-M(s))}\,ds.$$

But $\langle M\rangle(s) - \langle M\rangle(s-\tau) \geq \underline{h}^2\tau$, so

$$\int_{t-\tau}^{t} \varphi(s)^{-1}\varphi(s-\tau)\,ds \geq e^{\frac{1}{2}\sigma^2\underline{h}^2\tau}\int_{t-\tau}^{t} e^{-\sigma(\widetilde{B}(\langle M\rangle(s))-\widetilde{B}(\langle M\rangle(s-\tau))}\,ds.$$

Note that $t \mapsto \langle M\rangle(t)$ is a strictly increasing and C^1 function, with $\underline{h}^2 \leq \langle M\rangle'(t) \leq \overline{h}^2$. Therefore, $\langle M\rangle(s) > \overline{h}^2\tau$ for $s > \overline{h}^2\tau/\underline{h}^2$, and moreover,

$$\widetilde{B}(\langle M\rangle(s-\tau)) \leq \max_{u\in[\underline{h}^2\tau,\overline{h}^2\tau]} \widetilde{B}(\langle M\rangle(s)-u), \quad s > \overline{h}^2\tau/\underline{h}^2. \tag{3.4.6}$$

Next we suppose, without loss of generality, that $\sigma < 0$. It then follows from (3.4.6) for $t > \tau + \overline{h}^2\tau/\underline{h}^2$ that

$$\begin{aligned}
&\overline{h}^2 \int_{t-\tau}^{t} e^{-\sigma(\widetilde{B}(\langle M\rangle(s))-\widetilde{B}(\langle M\rangle(s-\tau)))}\,ds \\
&\geq \int_{t-\tau}^{t} e^{-\sigma(\widetilde{B}(\langle M\rangle(s))-\widetilde{B}(\langle M\rangle(s-\tau)))}\langle M\rangle'(s)\,ds \\
&\geq \int_{t-\tau}^{t} e^{-\sigma(\widetilde{B}(\langle M\rangle(s))-\max_{u\in[\underline{h}^2\tau,\overline{h}^2\tau]}\widetilde{B}(\langle M\rangle(s)-u)}\langle M\rangle'(s)\,ds \\
&\geq \int_{\langle M\rangle(t-\tau)}^{\langle M\rangle(t)} e^{-\sigma(\widetilde{B}(v)-\max_{u\in[\underline{h}^2\tau,\overline{h}^2\tau]}\widetilde{B}(v-u))}\,dv.
\end{aligned}$$

Since $\langle M\rangle(t-\tau) \le \langle M\rangle(t) - \underline{h}^2\tau$ for $t > \tau + \overline{h}^2\tau/\underline{h}^2$, we have

$$\begin{aligned}&\int_{t-\tau}^{t} \varphi(s)^{-1}\varphi(s-\tau)\,ds \\ &\geq e^{\frac{1}{2}\sigma^2\underline{h}^2\tau}\int_{t-\tau}^{t} e^{-\sigma(\widetilde{B}(\langle M\rangle(s)-\widetilde{B}(\langle M\rangle(s-\tau))}\,ds \\ &\geq e^{\frac{1}{2}\sigma^2\underline{h}^2\tau}\frac{1}{\overline{h}^2}\int_{\langle M\rangle(t)-\underline{h}^2\tau}^{\langle M\rangle(t)} e^{-\sigma(\widetilde{B}(v)-\max_{u\in[\underline{h}^2\tau,\overline{h}^2\tau]}\widetilde{B}(v-u))}\,dv.\end{aligned}$$

Thus, as $\langle M\rangle(t) \to \infty$ as $t \to \infty$, a.s., and since it is clearly true that for $\sigma < 0$ and any standard Brownian motion W

$$\limsup_{t\to\infty}\int_{t-\underline{h}^2\tau}^{t} e^{-\sigma(W(s)-\max_{u\in[\underline{h}^2\tau,\overline{h}^2\tau]}W(s-u))}\,ds = \infty, \quad \text{a.s.}, \tag{3.4.7}$$

we have established (3.4.5), and completed the proof. A statement similar to (3.4.7) is proved in Lemma 1 of Appleby and Buckwar [1]. □

3.4.2 Some remarks on Theorem 3.4.1.

The zero set of $X(\omega)$. It is possible to comment upon the structure of the zero set $Z_X(\omega) = \{t \geq 0 : X(t,\omega) = 0\}$. From (3.4.3), and the fact that $P(t,\omega) > 0$ for all $t > \tau^* + 2\tau$, it can readily be seen that the zeros of $Z(\omega)$, and hence of $X(\omega)$, must be isolated. In fact, this property of the zero set is also clearly visible in Figures 3.4.1, 3.4.2, and 3.4.3.

Complementary analysis in the literature. Theorem 1 in Gushchin and Küchler [12] guarantees the oscillation of solutions of (3.0.1) in the special case where $h(x) \equiv x$. The following condition is imposed, requiring a weaker regularity, but stronger monotonicity, condition on g:

If g is nondecreasing on $\mathbb{R}$, and there exist real numbers δ and b with $\delta > 0$, $b > 0$, such that

$$\frac{g(x)}{x} \geq b \tag{3.4.8}$$

for all $x \neq 0$ satisfying $|x| < \delta$.

For example, $g(x) = \text{sgn}(x)$ satisfies (3.4.8), and Gushchin and Küchler guarantee the existence of a unique strong solution of (3.0.1) for this choice of g, and $h(x) \equiv x$.

In Chapter 2, the existence of unique solutions of (3.0.1) has been established with a continuity requirement on g, excluding functions with discontinuities, like $g(x) = \text{sgn}(x)$, from consideration. But, given the precedent set by the deterministic theory, it may be possible to prove the existence and uniqueness of a strong solution of (3.0.1) with a weaker regularity requirement on g.

3.5 Nonoscillation of solutions.

We now study nonoscillation of solutions of (3.0.1). Naturally, this requires that g act weakly towards equilibrium in the vicinity of equilibrium. Suppose that there exists $\gamma > 1$ and $0 < L \leq \overline{L}$ such that

$$\lim_{x \to 0} \frac{|g(x)|}{|x|^\gamma} = L, \tag{3.5.1}$$

and

$$|g(x)| \leq \overline{L}|x|^\gamma, \quad x \in \mathbb{R}. \tag{3.5.2}$$

As g obeys condition (3.5.1), it is *superlinear* at zero. The significance of this property of g is emphasised in Shreve [29], where examples are given of deterministic equations with nonlinear coefficient satisfying (3.5.1) and (3.5.2) which do not oscillate. In fact the conditions imposed on f here are somewhat stronger than those imposed in [29].

If g satisfies a global linear bound of the form $|g(x)| \leq K(1+|x|)$ for all $x \in \mathbb{R}$ and some $K > 0$ then this along with (3.5.1) implies (3.5.2). Such a global linear bound, together with the local Lipschitz continuity of g, and with similar global linear bounds and local Lipschitz continuity on h, guarantees that (3.0.1) has a unique strong solution [25], without any need for the analysis in Chapter 2.

However, a global linear bound on g is not required for (3.5.2) to hold, and therefore the main result in this section, Theorem 3.5.2, applies to processes for which the existence theory in Chapter 2 is required.

3.5.1 Preliminary analysis.

In advance of proving a result on the nonoscillation of solutions of (3.0.1), we require a technical result, the proof of which will require some further auxiliary processes. If M is the process defined in (3.3.8), we see that, by (3.3.10), $\lim_{t\to\infty}\langle M\rangle(t) \to \infty$ a.s. Therefore, by Lemma 1.2.3, the martingale time change theorem, there exists a standard Brownian motion $\widetilde{B}$ such that

$$M(t) = \widetilde{B}(\langle M\rangle(t)) \text{ for all } t \geq 0 \ a.s. \tag{3.5.3}$$

We also introduce the process $\widehat{B}$ given by

$$\widehat{B}(t) = \min_{t\leq w\leq t+\tau\overline{h}^2} \widetilde{B}(w). \tag{3.5.4}$$

Lemma 3.5.1. *Let $\widetilde{B}$ and $\widehat{B}$ be the processes defined in (3.5.3) and (3.5.4). If*

$$I = \int_0^\infty \varphi(s-\tau)^\gamma \varphi(s)^{-1}\, ds, \tag{3.5.5}$$

and

$$\overline{I} = \frac{1}{\underline{h}^2}\left\{\int_0^{\tau\overline{h}^2} e^{\frac{\sigma^2}{2}u-\sigma\widetilde{B}(u)}\, du + e^{\frac{1}{2}\sigma^2\overline{h}^2\tau}\int_0^\infty e^{-\frac{1}{2}\sigma^2(\gamma-1)u+\sigma\gamma\widetilde{B}(u)-\sigma\hat{B}(u)}\, du\right\}, \tag{3.5.6}$$

then

$$I \leq \overline{I} < \infty. \tag{3.5.7}$$

Proof. We assume, without loss of generality, that $\sigma > 0$. First, we prove that

$$\int_0^\tau \varphi(s-\tau)^\gamma \varphi(s)^{-1}\, ds \leq \frac{1}{\underline{h}^2}\int_0^{\tau\overline{h}^2} e^{\frac{\sigma^2}{2}u-\sigma\widetilde{B}(u)}\, du, \quad \text{a.s.} \tag{3.5.8}$$

where $\widetilde{B}$ is defined via (3.3.8) and (3.5.3). Let M be the process defined by (3.3.8), with quadratic variation given by (3.3.9). As $t \mapsto \langle M\rangle(t)$ is strictly increasing, and continuously differentiable, we may define $S(u) = \langle M\rangle^{-1}(u)$. Note also that $\langle M\rangle'(t) \geq \underline{h}^2$, a.s. Hence

$$\begin{aligned}\int_0^\tau \varphi(s-\tau)^\gamma\varphi(s)^{-1} ds &= \int_0^\tau \varphi(s)^{-1} ds = \int_0^\tau e^{\frac{\sigma^2}{2}\langle M\rangle(s)-\sigma M(s)} ds \\ &= \int_0^{\langle M\rangle(\tau)} e^{\frac{\sigma^2}{2}u-\sigma M(\langle M\rangle^{-1}(u))}\frac{1}{\langle M\rangle'(S(u))} du.\end{aligned}$$

Now by (3.5.3), (3.5.8) is immediate. Next, we prove that

$$\int_0^\infty \varphi(s)^\gamma \varphi(s+\tau)^{-1} ds \leq \frac{1}{\underline{h}^2} e^{\frac{1}{2}\sigma^2 \overline{h}^2 \tau} \int_0^\infty e^{\{-\frac{1}{2}\sigma^2(\gamma-1)u+\sigma\gamma\widetilde{B}(u)-\sigma\widehat{B}(u)\}} du \tag{3.5.9}$$

where $\widehat{B}$ satisfies (3.5.4) above. Define $\tau(u) = \langle M\rangle(S(u)+\tau) - \langle M\rangle(S(u))$. Then $\tau(u) \leq \tau\overline{h}^2$, a.s. Let $\widehat{B}$ be defined by (3.5.4). Observe that

$$\lim_{u\to\infty} \frac{\widehat{B}(u)}{u} = 0, \quad \text{a.s.} \tag{3.5.10}$$

and

$$\widehat{B}(u) \leq \widetilde{B}(u+\tau(u)), \quad u \geq 0, \quad \text{a.s.}$$

Thus (3.5.10) and $\gamma > 1$ imply that

$$\int_0^\infty e^{-\frac{1}{2}\sigma^2(\gamma-1)u+\sigma\gamma\widetilde{B}(u)-\sigma\widehat{B}(u)}\, du < \infty, \quad \text{a.s.}$$

Now, as $\sigma > 0$

$$\begin{aligned}
&\int_0^\infty e^{-\frac{1}{2}\sigma^2(\gamma-1)u+\sigma\gamma\widetilde{B}(u)-\sigma\widehat{B}(u)}\, du \\
\geq\ & \int_0^\infty e^{-\frac{1}{2}\sigma^2(\gamma-1)u+\sigma\gamma\widetilde{B}(u)-\sigma\widetilde{B}(u+\tau(u))}\, du \\
=\ & \int_0^\infty e^{-\frac{1}{2}\sigma^2(\gamma-1)\langle M\rangle(s)+\sigma\gamma\widetilde{B}(\langle M\rangle(s))-\sigma\widetilde{B}(\langle M\rangle(s+\tau))} \langle M\rangle'(s)\, ds \\
=\ & \int_0^\infty \varphi(s)^\gamma \varphi(s+\tau)^{-1} e^{\frac{1}{2}\sigma^2(\langle M\rangle(s)-\langle M\rangle(s+\tau))} \langle M\rangle'(s)\, ds \\
\geq\ & \underline{h}^2 e^{-\frac{1}{2}\sigma^2\overline{h}^2\tau} \int_0^\infty \varphi(s)^\gamma \varphi(s+\tau)^{-1}\, ds
\end{aligned}$$

which is (3.5.9). Combining (3.5.8) and (3.5.9) yields (3.5.7). □

3.5.2 Main result.

We now prove the main result in this section. To show that solutions of (3.0.1) do not oscillate with positive probability when g obeys (3.5.1) and (3.5.2), we show, for certain positive initial data, that solutions can remain positive with nonzero probability. This result highlights the distinction between the memory driven processes that are the focus of this thesis, and the Markovian processes that obey classical zero–one laws. Suppose $\psi(t) > 0$ for all $t \in [-\tau, 0]$ and define the stopping time

$$\tau_\psi = \inf\{t > 0 : X(t, \psi) = 0\}, \tag{3.5.11}$$

where we set $\tau_\psi(\omega) = +\infty$ if $X(t, \omega) > 0$ for all $t \geq 0$. Suppose the solution of (3.0.1) is defined on Ω_0^*, with $\mathbb{P}[\Omega_0^*] = 1$, and define, as before $\Omega_1^* \subseteq \Omega_0^*$ the almost sure set on which φ exists, is strictly positive, and obeys conditions (3.3.6) and (3.3.7).

Theorem 3.5.2. *Let $\{X(t)\}_{t\geq 0}$ be the unique continuous strong solution of (3.0.1). Suppose g satisfies (3.2.3), (3.5.1) and (3.5.2) and h satisfies (3.2.1) and (3.2.2). Suppose that $\psi(t) > 0$ for all $t \in [-\tau, 0]$ and τ_ψ is defined by (3.5.11). Then*

1. *There exists α^*, possibly infinite, such that for all $\alpha < \alpha^*$*

$$\mathbb{P}[\tau_{\alpha\psi} < \infty] < 1.$$

2. *Moreover*

$$\lim_{\alpha \to 0} \mathbb{P}[\tau_{\alpha\psi} < \infty] = 0. \tag{3.5.12}$$

Proof. Define $\|\psi\| := \sup_{t\in[-\tau,0]} \psi(t) > 0$, $\Omega_\psi = \{\omega \in \Omega_1^* : \tau_\psi(\omega) < \infty\}$ and also

$$D_\psi = \left\{\omega \in \Omega_1^* : \overline{I}(\omega) < \frac{\psi(0)}{\|\psi\|^\gamma \overline{L}}\right\} \tag{3.5.13}$$

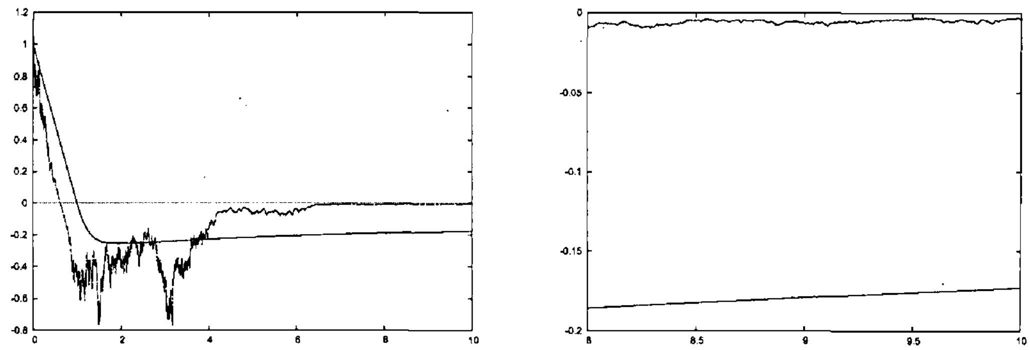
where $\overline{I}$ is given by (3.5.6), and $\overline{L}$ is given by (3.5.2). The main step in the analysis is to prove the following: if I is defined by (3.5.5), then

$$I(\omega) \geq \frac{1}{\overline{L}} \frac{\psi(0)}{\|\psi\|^\gamma}, \quad \omega \in \Omega_\psi. \tag{3.5.14}$$

If (3.5.14) is true, by (3.5.13), Lemma 3.5.1, and (3.5.7), we have $\Omega_\psi \subseteq \overline{D_\psi}$.

Since $\overline{I} < \infty$, a.s., there exists some $C > 0$ such that

$$A := \{\omega \in \Omega_1^* : \overline{I}(\omega) > C\}$$



(a) (b)

Figure 3.5.1: Red: The solution of (3.1.1), where $g(x) = x^3$ is **superlinear**, and $\tau = 1$, is nonoscillatory. Green: The introduction of a noise term $h(x) = x$ does not qualitatively affect the behaviour. **(b)** is a close up view of **(a)**, confirming that the simulated path of $dX(t) = -X(t-1)^3dt + X(t)dB(t)$ remains negative.

---

satisfies $\mathbb{P}[A] < 1$. Now, define α^* by $\frac{\alpha^*\psi(0)}{\overline{L}\alpha^{*\gamma}\|\psi\|^\gamma} = C$. Hence, for $\alpha < \alpha^*$, $\omega \in \overline{D_{\alpha\psi}}$ implies

$$\overline{I}(\omega) \geq \frac{1}{\overline{L}}\frac{\alpha\psi(0)}{\alpha^\gamma\|\psi\|^\gamma} > \frac{\alpha^*\psi(0)}{\overline{L}\alpha^{*\gamma}\|\psi\|^\gamma} = C$$

so $\omega \in A$, or $\overline{D_{\alpha\psi}} \subseteq A$. Therefore, for $\alpha < \alpha^*$,

$$\mathbb{P}[\Omega_{\alpha\psi}] \leq \mathbb{P}[\overline{D_{\alpha\psi}}] \leq \mathbb{P}[A] < 1,$$

as required for Part 1.

To prove Part 2, note from (3.5.6), that $\overline{I} < \infty$, a.s. Hence, by (3.5.13), as $\gamma > 1$,

$$\lim_{\alpha\to 0}\mathbb{P}[\overline{D_{\alpha\psi}}] = \lim_{\alpha\to 0}\mathbb{P}\left[\overline{I} \geq \frac{\alpha\psi(0)}{\alpha^\gamma\|\psi\|^\gamma\overline{L}}\right] = 0.$$

Since $\mathbb{P}[\Omega_{\alpha\psi}] \leq \mathbb{P}[\overline{D_{\alpha\psi}}]$, (3.5.12) now follows.

It remains to justify (3.5.14). Now, suppose that $\omega \in \Omega_\psi$ and for $t \leq \tau_\psi + \tau$ define

$$L(t) = \begin{cases} \frac{g(X(t-\tau))}{X(t-\tau)^\gamma}, & X(t-\tau) \neq 0 \\ L, & X(t-\tau) = 0. \end{cases}$$

By (3.5.1) and (3.5.2), $t \mapsto L(t)$ is continuous, strictly positive and bounded, with $0 < L(t) \le \overline{L}$. Also define

$$P(t) = L(t)\varphi(t-\tau)^{\gamma}\varphi(t)^{-1}, \quad 0 \le t \le \tau_{\psi} + \tau.$$

For $0 \le t < \tau_{\psi} + \tau$, we have $X(t-\tau) > 0$, so

$$Z'(t) = -P(t)Z(t-\tau)^{\gamma}, \; 0 < t < \tau_{\psi} + \tau. \tag{3.5.15}$$

For $t = \tau_{\psi} + \tau$, $X(t-\tau) = 0$. Then $Z(t-\tau) = 0$, so $Z'(t) = 0$. But $-P(t)Z(t-\tau)^{\gamma} = 0$, so $Z'(t) = -P(t)Z(t-\tau)^{\gamma}$ once more. Hence $Z'(t) < 0$ for $t \le \tau_{\psi} + \tau$. Thus $Z(t) \le \psi(0)$ for all $t \in [0, \tau_{\psi} + \tau]$. Another way of writing this is to say that $Z(t-\tau) \le \psi(0)$ for all $t \in [\tau, 2\tau + \tau_{\psi}]$. Also, $Z(t-\tau) = \psi(t-\tau) \le \|\psi\|$ for all $t \in [0, \tau]$. So $Z(t-\tau) \le \|\psi\|$ for all $t \in [0, \tau_{\psi} + 2\tau]$, which implies that $Z(t-\tau) \le \|\psi\|$ for all $t \in [0, \tau_{\psi}]$. Using this, and (3.5.15), we get

$$\begin{aligned}\psi(0) = -Z(\tau_{\psi}) + Z(0) &= \int_0^{\tau_{\psi}} P(s)Z(s-\tau)^{\gamma}\,ds \le \int_0^{\tau_{\psi}} P(s)\|\psi\|^{\gamma}\,ds \\ &\le \overline{L}\|\psi\|^{\gamma}\int_0^{\tau_{\psi}} \varphi(s-\tau)^{\gamma}\varphi(s)^{-1}\,ds \le \overline{L}\|\psi\|^{\gamma}\int_0^{\infty} \varphi(s-\tau)^{\gamma}\varphi(s)^{-1}\,ds.\end{aligned}$$

Therefore

$$\frac{\psi(0)}{\overline{L}\|\psi\|^{\gamma}} \le \int_0^{\infty} \varphi(s-\tau)^{\gamma}\varphi(s)^{-1}ds = I(\omega),$$

as required. □

3.5.3 Further Remarks.

The use of $\overline{I}$ in the proof of Theorem 3.5.2. On a first viewing, it is perhaps not immediately apparent why the random variable $\overline{I}$ is introduced, as one might expect to be able to prove the results of Theorem 3.5.2 with

$$D'_{\psi} = \left\{\omega \in \Omega_1^* : I(\omega) < \frac{\psi(0)}{\|\psi\|^{\gamma}\overline{L}}\right\}, \quad \text{and} \quad A' = \{\omega \in \Omega_1^* : I(\omega) > C\}.$$

It is not automatic that $\lim_{\alpha\to 0}\mathbb{P}[D'_{\alpha\psi}] = 0$, as the random variable I depends on α, because it depends on X through the initial data $\alpha\psi$. However, the random variable $\overline{I}$ has the same distribution as a random variable independent of the initial data, and therefore independent of the scaling factor α. Indeed,

$$\mathbb{P}[D_{\alpha\psi}] = \mathbb{P}\left[\overline{I}' \geq \frac{\psi(0)}{\alpha^{\gamma-1}\|\psi\|^{\gamma}\overline{L}}\right]$$

where the random variable $\overline{I}'$ is given by

$$\overline{I}' = \frac{1}{\underline{h}^2}\left\{\int_0^{\tau\overline{h}^2} e^{\frac{\sigma^2}{2}u-\sigma B'(u)}\,du \right.$$
$$\left. + e^{\frac{1}{2}\sigma^2\overline{h}^2\tau}\int_0^\infty e^{-\frac{1}{2}\sigma^2(\gamma-1)u+\sigma\gamma B'(u)-\sigma\min_{u\le w\le u+\tau}B'(w)}\,du\right\}$$

and B' is any standard Brownian motion. As $\overline{I}'$ is independent of the initial data,

$$\lim_{\alpha\to 0}\mathbb{P}\left[\overline{I}' \geq \frac{\psi(0)}{\alpha^{\gamma-1}\|\psi\|^{\gamma}\overline{L}}\right] = 0,$$

so $\mathbb{P}[\overline{D_{\alpha\psi}}] \to 0$ as $\alpha \to 0$.

If $\overline{I}$ is supported on $(0,\infty)$ then there is a positive probability of nonoscillation for any positive and continuous initial function ψ. $\overline{I}$ is an integral function of Brownian motion, and the distributions of some similar functionals are known. For example, Dufresne has shown in [7] that if B^* is a standard Brownian motion, $a > 0$ and $\sigma \neq 0$, then the random variable

$$L = \int_0^\infty e^{-as+\sigma B^*(s)}ds$$

is a.s. finite, continuous and supported on $(0,\infty)$. In fact, Dufresne has determined the probability density of L.

While this is not direct evidence for $\overline{I}$ to be supported on $[0,\infty)$, the similar functional forms of L and $\overline{I}$ mean that the possibility cannot be automatically ruled out.

For any positive initial function ψ, a knowledge of the distribution of $\overline{I}$ allows us to construct explicitly the scaling factor α^* which guarantees the existence of nonoscillatory solutions with positive probability.

Final comment. The most obvious consequence of adding a noise perturbation to the solutions of (3.1.1) is that a.s. oscillation can be guaranteed without any minimum requirement on the length of τ. Clearly then, noise facilitates the onset of oscillation in circumstances where it does not otherwise occur. Because of the existence of an equilibrium solution, a nonzero delay is nonetheless still required, indicating that an equilibrium–preserving noise perturbation is insufficient on its own to induce oscillation. This is analogous to the analysis of of the linear stochastic differential equation with fixed delay by Appleby & Buckwar in [1]. We will now study in further detail the interdependent roles played by noise and delay in the oscillatory behaviour of stochastic delay equations, by allowing the delay to fade asymptotically.

Chapter 4

Asymptotic Behaviour – Brownian Increments

In Chapter 3 we varied the emphasis placed on the delayed feedback by the equation, rather than altering the delay itself. The remainder of this thesis will consider the effect of a variable delay on the qualitative behaviour of a linear stochastic delay differential equation. We will study the solutions of

$$dX(t) = (aX(t) + bX(t-\tau(t)))\,dt + \sigma X(t)\,dB(t) \tag{4.0.1a}$$

$$X(t) = \psi(t), \quad -\overline{\tau} \leq t \leq 0, \tag{4.0.1b}$$

where $b < 0$, and τ is a continuous function. By varying τ appropriately, we can build up a picture of the complementary roles of delay and noise in oscillatory behaviour. Specifically, we fade out the delay, allowing it to vanish asymptotically, while keeping the intensity of the noise fixed.

The manner in which the variation of τ affects the behaviour of solutions of (4.0.1) is therefore of crucial importance. Just as in Chapter 3, our methods rely on applying a transformation to (4.0.1) yielding an auxiliary process with identical oscillatory behaviour, and differentiable sample paths. For this linear equation, the qualitative behaviour of this auxiliary process is explicitly governed by the evolution of Brownian increments that are scaled by the length of τ. The purpose of this chapter is to develop a theory describing the asymptotic behaviour of these increments and their large deviations, for various decay rates of τ, over continuous and discrete time sets. We will draw upon this theory over the next three chapters.

4.1 The role of Brownian increments in oscillation.

Geometric Brownian motion plays the role of the strictly positive process in the decomposition from here on. Define $\{\varphi(t)\}_{t\geq -\overline{\tau}}$ to be the solution of

$$
\begin{aligned}
d\varphi(t) &= a\varphi(t)dt + \sigma\varphi(t)dB(t), && \text{(4.1.1a)}\\
\varphi(t) &= 1, \quad t \in [-\overline{\tau}, 0]. && \text{(4.1.1b)}
\end{aligned}
$$

Define $y(t) = X(t)/\varphi(t)$ for $t \geq -\overline{\tau}$, where X solves (4.0.1). Hence, by Lemma 1.3.3, stochastic integration by parts, y satisfies

$$
y(t) = y(0) + \int_0^t b\, y(s-\tau(s))\ \varphi(s-\tau(s))\ \varphi(s)^{-1}\, ds, \quad t \geq 0,
$$

which can be written as

$$
\begin{aligned}
y'(t) &= b\varphi(t)^{-1}\varphi(t-\tau(t))y(t-\tau(t)), \ t > 0 && \text{(4.1.2a)}\\
y(t) &= \psi(t), \ t \in [-\overline{\tau}, 0]. && \text{(4.1.2b)}
\end{aligned}
$$

Clearly, $y \in C^1((0,\infty);\mathbb{R})$, and moreover, we have

$$
y'(t) = -p(t)y(t-\tau(t)), \ t > 0. \tag{4.1.3}
$$

Since $t \mapsto t - \tau(t)$ is nondecreasing, there exists $t^* = \inf\{t > 0 : t - \tau(t) = 0\}$, so that $t - \tau(t) \geq 0$ for all $t > t^*$. Then, letting $\lambda = a - \frac{1}{2}\sigma^2$, the path dependent function p satisfies

$$
p(t)(\omega) = \begin{cases} -be^{-\lambda\tau(t)}e^{-\sigma(B(t)(\omega)-B(t-\tau(t))(\omega))}, & t > t^* \\ -be^{-\lambda t-\sigma B(t)(\omega)}, & t \leq t^*. \end{cases} \tag{4.1.4}
$$

Therefore, the solution of (4.0.1) can be written as the product of the geometric Brownian motion φ and the solution of a random delay differential equation which admits a continuously differentiable solution.

The significance of this transformation lies in the relationship that y bears to X. Since $y(t) = X(t)/\varphi(t)$, and φ is an a.s. strictly positive process, the zeros of the process y correspond almost surely to the zeros of the process X. Therefore it is sufficient to analyse the oscillatory behaviour of y in order to determine the oscillatory behaviour of X. The advantage of this approach is that there is a set of deterministic results that apply directly to the paths of the solutions of (4.1.3), given by Theorems 1.5.1 and 1.5.2.

Applying deterministic theory. We will require knowledge of the asymptotic behaviour of the process p. If p takes on values that are large enough, often enough, it will induce oscillation in the process y, and therefore X. Notice in the definition of p, given in (4.1.4), the presence of the Brownian increment $B(t) - B(t - \tau(t))$. The asymptotic behaviour of the large deviations of these increments, for delay functions τ exhibiting various behaviours, will determine the behaviour of p. This chapter is devoted to analysis describing the behaviour of these increments.

Mills' estimate. An estimate of the rate of decay of the tail of the distribution of a standard Gaussian random variable will be useful in our analysis of Brownian increments. If

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{1}{2}y^2}\, dy, \tag{4.1.5}$$

then

$$\frac{1}{\sqrt{2\pi}} \frac{x}{1+x^2} e^{-\frac{1}{2}x^2} \le \Psi(x) \le \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-\frac{1}{2}x^2}, \quad x \ge 0. \tag{4.1.6}$$

The result may be found, for example, in Chapter 2.9.22 of Karatzas & Shreve [17].

4.2 Continuous time processes.

We prove some results on the asymptotic behaviour of the large deviations of the Brownian increment $B(t) - B(t - \tau(t))$ when the delay function τ obeys

$$\lim_{t\to\infty} \tau(t) = 0,\ \tau(t) > 0,\ t \mapsto t - \tau(t) \text{ is nondecreasing.} \tag{4.2.1}$$

These results have been published previously by Appleby & Kelly in [2].

Lemma 4.2.1. *If τ is a continuous, eventually decreasing function satisfying (4.2.1) and*

$$\lim_{t\to\infty} \frac{\log(1/\tau(t))}{\sqrt{\tau(t)\log t}} = 0,$$

then

$$\limsup_{t\to\infty} \int_{t-\tau(t)}^t e^{-\sigma(B(s)-B(s-\tau(s)))}\, ds = \infty, \quad a.s.$$

Proof. Since τ is eventually decreasing and $\tau(t) \to 0$ as $t \to \infty$, there exists $N \in \mathbb{N}$ such that for all $t > N$, $\tau(t) < \frac{1}{2}$, and τ is decreasing on (N, ∞). Let $a_n = n$ for all $n > N+1$ and consider the sequence of random variables $Z_n = Y(a_n)$, where Y is defined by (5.2.4). We may use the independence of nonoverlapping increments of Brownian motion to observe that $\{Z_n\}_{n>N}$ is a sequence of independent random variables, as each Z_n is a functional of increments of Brownian motion on a subinterval of $[n-1, n)$. It now suffices to show that $\limsup_{n\to\infty} Z_n = \infty$, a.s.

Suppose now that τ_1 is a continuous function such that $0 < \tau_1(t) < \tau(t)$. Let $n > N+1$. The monotonicity of τ implies that

$$\max_{s\in[n-\tau_1(n),n]} \tau(s) = \tau(n-\tau_1(n)), \qquad \min_{s\in[n-\tau_1(n),n]} \tau(s) = \tau(n) > \tau_1(n).$$

Therefore for $n - \tau_1(n) \le s \le n$,

$$B(s) - B(s-\tau(s)) \ge \min_{u\in[n-\tau_1(n),n]} B(u) - \max_{u\in[n-\tau_1(n)-\tau(n-\tau_1(n)),n-\tau(n)]} B(u).$$

Observe that the intervals $[n-\tau_1(n), n]$ and $[n-\tau_1(n)-\tau(n-\tau_1(n)), n-\tau(n)]$ do not overlap as $n-\tau_1(n) > n-\tau(n)$.

Without loss of generality, we consider $\sigma < 0$ in the sequel. Next, let $\beta > 0$, and define $\alpha_n(\beta)$ so that $\tau_1(n)e^{-\sigma\alpha_n(\beta)} = \beta$. Define the event

$$\begin{aligned} &A_n(\beta) \\ &\quad = \left\{\omega : \min_{u\in[n-\tau_1(n),n]} B(u,\omega) - \max_{u\in[n-\tau_1(n)-\tau(n-\tau_1(n)),n-\tau(n)]} B(u,\omega) > \alpha_n(\beta)\right\}. \end{aligned}$$

Then for $\omega \in A_n(\beta)$, we have

$$\begin{aligned} Z_n(\omega) &= \int_{n-\tau(n)}^{n} e^{-\sigma(B(s,\omega)-B(s-\tau(s),\omega))}\,ds \\ &\ge \int_{n-\tau_1(n)}^{n} e^{-\sigma(B(s,\omega)-B(s-\tau(s),\omega))}\,ds \\ &\ge \int_{n-\tau_1(n)}^{n} e^{-\sigma\alpha_n(\beta)}\,ds = \tau_1(n)e^{-\sigma\alpha_n(\beta)} = \beta. \end{aligned}$$

Therefore $\mathbb{P}[Z_n \geq \beta] \geq \mathbb{P}[A_n(\beta)]$, and so, by the second Borel–Cantelli Lemma (given, along with the first Borel–Cantelli Lemma, as Theorem 2.7 in Gīhman & Skorohod [10]), proving

$$\sum_{n=N+1}^{\infty} \mathbb{P}[A_n(\beta)] = \infty, \quad \text{for all } \beta > 0 \tag{4.2.2}$$

is enough to show $\limsup_{n\to\infty} Z_n = \infty$, a.s.

Next, we see that the random variable

$$U_n = \min_{u\in[n-\tau_1(n),n]} B(u) - \max_{u\in[n-\tau_1(n)-\tau(n-\tau_1(n)),n-\tau(n)]} B(u)$$

may be rewritten as

$$\min_{u\in[n-\tau_1(n),n]} (B(u) - B(n-\tau_1(n))) + (B(n-\tau_1(n)) - B(n-\tau(n))) + \min_{u\in[n-\tau_1(n)-\tau(n-\tau_1(n)),n-\tau(n)]} (B(n-\tau(n)) - B(u)).$$

The independence of the increments of B means that U_n has the same distribution as

$$\min_{s\in[0,\tau_1(n)]} W^{(1)}(s) + W^{(2)}(\tau(n)-\tau_1(n)) + \min_{s\in[0,\tau_1(n)+\tau(n-\tau_1(n))-\tau(n)]} W^{(3)}(s)$$

where $W^{(1)}$, $W^{(2)}$, $W^{(3)}$ are independent Brownian motions. Recalling that the distribution of $\max_{s\in[0,t]} W(s)$ is the same as $|W(t)|$, when W is a standard Brownian motion, and that $\min_{s\in[0,t]} W(s)$ has the same distribution as $-\max_{s\in[0,t]} W(s)$, we see that U_n has the same distribution as

$$-\sqrt{\tau_1(n)}|Z_1| + \sqrt{\tau(n)-\tau_1(n)}Z_2 - \sqrt{\tau(n-\tau_1(n)) - \tau(n) + \tau_1(n)}|Z_3|$$

where Z_1, Z_2, Z_3 are independent standard Gaussian random variables. Thus, by defining $\tau_1(t) = \frac{1}{2}\tau(t)$, and setting

$$p_n = \sqrt{\frac{\tau(n-\tau_1(n)) - \tau(n) + \tau_1(n)}{\tau_1(n)}},$$

we see that

$$\begin{aligned}\mathbb{P}[A_n(\beta)] &= \mathbb{P}[-\sqrt{\tau_1(n)}|Z_1| + \sqrt{\tau(n)-\tau_1(n)}Z_2 \\ &\qquad -\sqrt{\tau(n-\tau_1(n))-\tau(n)+\tau_1(n)}|Z_3| > \alpha_n(\beta)] \\ &= \mathbb{P}[-|Z_1| + Z_2 - p_n|Z_3| > \frac{\alpha_n(\beta)}{\sqrt{\tau_1(n)}}].\end{aligned}$$

The independence of Z_1, Z_2, Z_3 now implies

$$\begin{aligned}\mathbb{P}[A_n(\beta)] &\geq \mathbb{P}[-|Z_1| \geq -1, Z_2 \geq \frac{\alpha_n(\beta)}{\sqrt{\tau_1(n)}} + 1 + p_n, -p_n|Z_3| \geq -p_n] \\ &\geq \mathbb{P}[|Z_1| \leq 1]\mathbb{P}[Z_2 \geq \frac{\alpha_n(\beta)}{\sqrt{\tau_1(n)}} + 1 + p_n]\mathbb{P}[|Z_3| \leq 1].\end{aligned}$$

Hence there exists $c > 0$ such that

$$\mathbb{P}[A_n(\beta)] > c\Psi\left(\frac{\alpha_n(\beta)}{\sqrt{\tau_1(n)}} + 1 + p_n\right),$$

where Ψ is as given earlier. Since $p_n\sqrt{\tau_1(n)} \to 0$ as $n \to \infty$, it now follows that (4.2.2) is true if we can show, for some $N_1 > N+1$, that,

$$\sum_{n=N_1}^{\infty} \Psi(\gamma_n) = \infty, \tag{4.2.3}$$

where $\gamma_n = \frac{\alpha_n(\beta)+1}{\sqrt{\tau_1(n)}}$. Now, condition (5.2.3) implies that $\lim_{t\to\infty} \tau(t)\log t = \infty$, so for every $\beta > 0$, $\sigma < 0$, we get

$$\limsup_{t\to\infty} \frac{-\frac{1}{\sigma}\log(\beta/\tau_1(t)) + 1}{\sqrt{\tau_1(t)}\sqrt{2\log t}} = 0.$$

Hence $\lim_{n\to\infty} \frac{\gamma_n}{\sqrt{2\log n}} = 0$, and so $\gamma_n < \sqrt{2\log n}$ for all $n > N_2$. Hence, by (4.1.6), for all $n > N_3 \geq N_2$, we get

$$\Psi(\gamma_n) \geq \frac{1}{\sqrt{2\pi}}\frac{\gamma_n}{1+\gamma_n^2}e^{-\frac{1}{2}\gamma_n^2} > \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2}\frac{1}{\sqrt{2\log n}} \cdot \frac{1}{n}.$$

This estimate now yields (4.2.3), and hence the result. □

Lemma 4.2.2. *If τ is a continuous function satisfying (4.2.1) and*

$$\lim_{t\to\infty} \tau(t)\log t = 0,$$

then

$$\lim_{t\to\infty} B(t) - B(t-\tau(t)) = 0, \quad a.s.$$

Proof. Note that $W(t) = tB(\frac{1}{t})$ is standard Brownian motion, so we may rewrite $B(t) - B(t-\tau(t))$ as

$$B(t) - B(t-\tau(t)) = \tau(t)W\left(\frac{1}{t}\right) + (t-\tau(t))\left(W\left(\frac{1}{t}\right) - W\left(\frac{1}{t-\tau(t)}\right)\right).$$

Because $\tau(t) \to 0$ as $t \to \infty$, $\tau(t)W(\frac{1}{t}) \to 0$ as $t \to \infty$. Hence proving that

$$\lim_{t\to\infty} (t-\tau(t))\left(W\left(\frac{1}{t}\right) - W\left(\frac{1}{t-\tau(t)}\right)\right) = 0, \ a.s. \tag{4.2.4}$$

will suffice. $\tau(t) \to 0$ implies both that $\tau(t)\log(\tau(t)) \to 0$ as $t \to \infty$ and that $\frac{t-\tau(t)}{t} \to 1$ as $t \to \infty$. For $t > T_0$, we have $1 < t - \tau(t) < t$. Therefore

$$\tau(t)\log(t) < \tau(t)\log[t(t-\tau(t))] < 2\tau(t)\log(t),$$

so $\tau(t)\log[t(t-\tau(t))] \to 0$ as $t \to \infty$, since $\tau(t)\log(t) \to 0$. Thus

$$\begin{aligned}(t-\tau(t))&\sqrt{\frac{\tau(t)}{t(t-\tau(t))}\log\left(\frac{t(t-\tau(t))}{\tau(t)}\right)} \\ &= \sqrt{\frac{t-\tau(t)}{t}}\sqrt{\tau(t)\log[t(t-\tau(t))] - \tau(t)\log\tau(t)} \to 0 \text{ as } t \to \infty. \end{aligned} \tag{4.2.5}$$

For all $t > T$ and $0 < \frac{1}{t} < \frac{1}{t-\tau(t)} < 1$, let $\delta(u) = \frac{\tau(\frac{1}{u})}{\frac{1}{u}(\frac{1}{u}-\tau(\frac{1}{u}))}$, for $0 < u < \frac{1}{T}$. Note that $\frac{1}{t-\tau(t)} - \frac{1}{t} = \delta(\frac{1}{t})$, and $\delta(u) \to 0$ as $u \to 0$.

For $t > T$

$$\left|W\left(\frac{1}{t}\right) - W\left(\frac{1}{t-\tau(t)}\right)\right| \le \max_{0\le s<r\le 1,\, r-s\le\delta(\frac{1}{t})} |W(s) - W(r)|.$$

Therefore, for $t > T$

$$\begin{aligned}
&(t-\tau(t))\left|W\left(\frac{1}{t}\right)-W\left(\frac{1}{t-\tau(t)}\right)\right| \\
\leq\ & (t-\tau(t))\sqrt{2\delta\left(\frac{1}{t}\right)\log\left(\frac{1}{\delta(\frac{1}{t})}\right)}\frac{\max_{0\leq s<r\leq 1,\, r-s\leq\delta(\frac{1}{t})}|W(s)-W(r)|}{\sqrt{2\delta(\frac{1}{t})\log(\frac{1}{\delta(t)})}} \qquad (4.2.6)\\
=\ & (t-\tau(t))\sqrt{2\frac{\tau(t)}{t(t-\tau(t))}\log\left(\frac{t(t-\tau(t))}{\tau(t)}\right)} \\
& \cdot\frac{\max_{0\leq s<r\leq 1,\, r-s\leq\delta(\frac{1}{t})}|W(s)-W(r)|}{\sqrt{2\delta(\frac{1}{t})\log(\frac{1}{\delta(t)})}}
\end{aligned}$$

Because $\delta(\frac{1}{t}) \to 0^+$ as $t \to \infty$, Lévy's result on the maximum modulus of continuity of standard Brownian motion (see, for example, Theorem 2.9.25 in Karatzas & Shreve, 1991) implies that

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{\max_{0\leq s<r\leq 1,\, r-s\leq\delta(\frac{1}{t})}|W(s)-W(r)|}{\sqrt{2\delta(\frac{1}{t})\log(\frac{1}{\delta(t)})}} \\
&\qquad = \limsup_{\delta\to 0^+}\frac{\max_{0\leq s<r\leq 1,\, r-s\leq\delta}|W(r)-W(s)|}{\sqrt{2\delta\log(\frac{1}{\delta})}} = 1, \quad a.s. \qquad (4.2.7)
\end{aligned}$$

We see from (4.2.7), (4.2.6) and (4.2.5) that

$$\begin{aligned}
&\limsup_{t\to\infty}(t-\tau(t))\left|W\left(\frac{1}{t}\right)-W\left(\frac{1}{t-\tau(t)}\right)\right| \\
\leq\ & \limsup_{t\to\infty}(t-\tau(t))\sqrt{2\frac{\tau(t)}{t(t-\tau(t))}\log\left(\frac{t(t-\tau(t))}{\tau(t)}\right)} \\
& \cdot\limsup_{t\to\infty}\frac{\max_{0\leq s<r\leq 1,\, r-s\leq\delta(\frac{1}{t})}|W(s)-W(r)|}{\sqrt{2\delta(\frac{1}{t})\log(\frac{1}{\delta(t)})}} \\
=\ & 0, \quad a.s.,
\end{aligned}$$

which proves (4.2.4), and the result. □

Lemma 4.2.3. *If $b > 0$, τ is continuous and obeys $\tau(t) \to 0$ as $t \to \infty$ then*

$$\limsup_{t\to\infty}\frac{B(t)-B(t-\tau(t))}{\sqrt{2\tau(t)\log t}} \geq 1, \quad a.s.$$

Proof. Since $\tau(t) \to 0$ as $t \to \infty$, we have $\tau(t_n) < 1$ for all $n > N_1$, and therefore there exists $N_2 > N_1$ such that $t_n \leq n$ for all $n > N_2$.

By the construction of $\{t_n\}$, $t_n - t_{n-1} = \tau(t_n)$. Thus the sequence of random variables

$$Y_n = \frac{B(t_n) - B(t_{n-1})}{\sqrt{\tau(t_n)}}$$

is a sequence of independent standard Normal random variables. Therefore

$$\limsup_{n\to\infty} \frac{Y_n}{\sqrt{2\log n}} = 1, \text{ a.s.}$$

Hence

$$\begin{aligned}
\limsup_{n\to\infty} \frac{B(t_n) - B(t_{n-1})}{\sqrt{2\tau(t_n)\log t_n}} &\geq \limsup_{n\to\infty} \frac{B(t_n) - B(t_{n-1})}{\sqrt{2\tau(t_n)\log n}} \\
&= \limsup_{n\to\infty} \frac{Y_n}{\sqrt{2\log n}} = 1, \text{ a.s.}
\end{aligned}$$

□

If $\tau(t)$ vanishes quickly enough, we can also derive an upper bound.

Lemma 4.2.4. *If τ is continuous, $\tau(t) \to 0$ and $\tau(t)\log t \to \infty$ as $t \to \infty$, then*

$$\limsup_{t\to\infty} \frac{|B(t) - B(t-\tau(t))|}{\sqrt{2\tau(t)\log t}} \leq \sqrt{2}, \text{ a.s.} \tag{4.2.8}$$

Proof. Just as in the proof of Lemma 4.2.2, $W(t) = tB(\frac{1}{t})$ is standard Brownian motion, so we may rewrite (4.2.8) as

$$\begin{aligned}
\frac{B(t) - B(t-\tau(t))}{\sqrt{2\tau(t)\log t}} &= \sqrt{\frac{\log\log t}{\log t}}\sqrt{\frac{\tau(t)}{t}}\frac{B(t)}{\sqrt{2t\log\log t}} \\
&\quad + \frac{t-\tau(t)}{\sqrt{2\tau(t)\log t}}\left(W\left(\frac{1}{t}\right) - W\left(\frac{1}{t-\tau(t)}\right)\right).
\end{aligned} \tag{4.2.9}$$

By Lemma 1.2.1 – the law of the iterated logarithm – and because $\tau(t) \to 0$, we only need consider

$$\frac{t-\tau(t)}{\sqrt{2\tau(t)\log t}}\left(W\left(\frac{1}{t}\right) - W\left(\frac{1}{t-\tau(t)}\right)\right),$$

since the rest of the right hand side of (4.2.9) vanishes as $t \to \infty$.

$\tau(t) \to 0$ implies that $t - \tau(t) \to \infty$ and that there exists $T > 0$ such that, for all $t > T$,

$$0 < \frac{1}{t} < \frac{1}{t - \tau(t)} < 1$$

so

$$0 < \frac{\tau(t)}{t(t - \tau(t))} < 1.$$

We can define function $\delta : [0, T^{-1}) \to \mathbb{R}$, by $\delta(0) = 0$ and

$$\delta(u) = \frac{u^{-1}}{u^{-1}(u^{-1} - \tau(u^{-1}))}, \quad 0 < u < \frac{1}{T}. \tag{4.2.10}$$

Note that $\delta(\frac{1}{t}) \to 0$ as $t \to \infty$. Again, we appeal to Lévy's result on the maximum modulus of continuity of standard Brownian motion,

$$\lim_{t\to\infty} \frac{\max_{0\le s<r\le 1,\, r-s\le \delta(\frac{1}{t})} |W(s) - W(r)|}{\sqrt{2\delta(\frac{1}{t}) \log\left(\frac{1}{\delta(\frac{1}{t})}\right)}} = \lim_{\delta\to 0^+} \frac{\max_{0\le s<r\le 1,\, r-s\le \delta} |W(s) - W(r)|}{\sqrt{2\delta \log(\frac{1}{\delta})}} = 1, \quad a.s. \tag{4.2.11}$$

It is also true that

$$\begin{aligned} &\lim_{t\to\infty} \frac{t - \tau(t)}{\sqrt{2\tau(t)\log t}} \sqrt{2\frac{\tau(t)}{t(t-\tau(t))} \log\left(\frac{t(t-\tau(t))}{\tau(t)}\right)} \\ = &\lim_{t\to\infty} \sqrt{\frac{t-\tau(t)}{t}} \sqrt{1 + \frac{\log(t-\tau(t))}{\log t} - \frac{\tau(t)\log\tau(t)}{\tau(t)\log t}} \\ = &\sqrt{2}. \end{aligned} \tag{4.2.12}$$

Now, for $t > T$,

$$\begin{aligned} &\frac{t - \tau(t)}{\sqrt{2\tau(t)\log t}} \left| W\left(\frac{1}{t}\right) - W\left(\frac{1}{t-\tau(t)}\right) \right| \\ \le\ &\frac{t - \tau(t)}{\sqrt{2\tau(t)\log t}} \sqrt{2\frac{\tau(t)}{t(t-\tau(t))} \log\left(\frac{t(t-\tau(t))}{\tau(t)}\right)} \\ &\cdot \frac{\max_{0\le s\le r\le 1,\, r-s\le \delta(\frac{1}{t})} |W(s) - W(r)|}{\sqrt{2\delta(\frac{1}{t}) \log(\frac{1}{\delta(\frac{1}{t})})}}. \end{aligned}$$

So, letting $t \to \infty$ we see, by (4.2.11) and (4.2.12), that

$$\limsup_{t\to\infty} \frac{|B(t) - B(t-\tau(t))|}{\sqrt{2\tau(t)\log t}} \leq \sqrt{2}, \quad \text{a.s.}$$

□

Note that if we require that $t - \tau(t)$ be strictly increasing, then, when $\tau(t)\log t \to \infty$,

$$\limsup_{t\to\infty} \frac{|B(t) - B(t-\tau(t))|}{\sqrt{2\tau(t)\log t}} = 1, \quad \text{a.s.}$$

4.3 Discrete time processes.

In Chapters 6 and 7, we will find that a useful description of qualitative behaviour can be developed by considering discrete processes that mimic the properties of the solutions of (4.0.1a). For this reason, we require results which give bounds on the asymptotic behaviour of the Brownian increment $B(t) - B(t - \tau(t))$ as discretised on the delay dependent time set defined in (2.2.1). The delay function τ is continuous and obeys

$$\lim_{t\to\infty} \tau(t) = 0,\ \tau(t) > 0,\ t \mapsto \tau(t) \text{ is strictly decreasing on } [0,\infty). \tag{4.3.1}$$

The following result is a consequence of Lemma 4.2.3.

Lemma 4.3.1. *If τ is continuous and obeys $\tau(t) \to 0$ as $t \to \infty$ then*

$$\limsup_{n\to\infty} \frac{B(t_n) - B(t_{n-1})}{\sqrt{2\tau(t_n)\log t_n}} \geq 1, \quad a.s.$$

If we further require that $\tau(t)\log t \to \infty$ when $t \to \infty$, and that τ is strictly decreasing, then the following result holds.

Lemma 4.3.2. *If τ is continuous, $\tau(t) \to 0$ as $t \to \infty$, τ is strictly decreasing, and $\tau(t)\log t \to \infty$ as $t \to \infty$, then*

$$\limsup_{n\to\infty} \frac{|B(t_n) - B(t_{n-1})|}{\sqrt{2\tau(t_n)\log t_n}} \leq 1, \quad a.s.$$

Proof. It is sufficient to show that

$$\limsup_{t\to\infty} \frac{B(t)-B(t-\tau(t))}{\sqrt{2\tau(t)\log t}} \le 1, \quad a.s.,$$

as this bound will also apply over the discrete time set defined in (2.2.1). First, define $\tau_1 \doteq \max\{\tau_1 - \tau(\tau_1) = 0\}$. $W(t) = tB(\frac{1}{t})$ is standard Brownian motion. For $t > \tau_1$ we may rewrite $B(t) - B(t-\tau(t))$ as

$$B(t) - B(t-\tau(t)) = \tau(t)W\left(\frac{1}{t}\right) + (t-\tau(t))\left(W\left(\frac{1}{t}\right) - W\left(\frac{1}{t-\tau(t)}\right)\right)$$

and so

$$\begin{aligned}\frac{B(t)-B(t-\tau(t))}{\sqrt{2\tau(t)\log t}} &= \sqrt{\frac{\log\log t}{\log t}}\sqrt{\frac{\tau(t)}{t}}\frac{B(t)}{\sqrt{2t\log\log t}} \\ &\quad + \frac{t-\tau(t)}{\sqrt{2\tau(t)\log t}}\left(W\left(\frac{1}{t}\right) - W\left(\frac{1}{t-\tau(t)}\right)\right).\end{aligned}$$

The first term has zero limit, by $\tau(t) \to 0$ and Lemma 1.2.1, the law of the iterated logarithm. Therefore it suffices to prove that

$$\limsup_{t\to\infty} \frac{t-\tau(t)}{\sqrt{2\tau(t)\log t}}\left(W\left(\frac{1}{t}\right) - W\left(\frac{1}{t-\tau(t)}\right)\right) \le 1, \quad a.s.$$

Now, because $\tau(t) \to 0$ and $\tau(t)\log t \to \infty$ as $t \to \infty$, for every $\varepsilon > 0$ there is a $T(\varepsilon) > 0$ such that for $t > T(\varepsilon)$ we have

$$\tau(t) < \varepsilon, \quad \tau(t)\log t > \frac{1}{\varepsilon}.$$

So

$$\varepsilon > \tau(t) > \frac{1}{\varepsilon \log t}$$

and

$$\frac{\log\varepsilon}{\log t} > \frac{\log\tau(t)}{\log t} > \frac{-(\log\varepsilon + \log\log t)}{\log t}.$$

Hence

$$\lim_{t\to\infty} \frac{\log\tau(t)}{\log t} = 0. \tag{4.3.2}$$

Let $g(t) = t - \tau(t)$, so $t \mapsto g(t)$ is increasing and $g(t) \to \infty$ as $t \to \infty$, $g(t) > 0$ for all $t > \tau_1$. Define $\delta : [0, \infty) \to \mathbb{R}$ by $\delta(0) = 0$ and

$$\delta(u) = u - \frac{1}{g^{-1}(u^{-1})}, \quad u > 0.$$

Note that $\delta(u) > 0$ for $u > 0$ and $\delta(u) \to 0$ as $u \to 0^+$. Define $u(t) = \frac{1}{g(t)}$ for $t > \tau_1$. Then $t \mapsto u(t)$ is a positive and decreasing function on (τ_1, ∞) with $\lim_{t\to\infty} u(t) = 0$. For $t > \tau_1$, we note that

$$0 < \delta(u(t)) = \frac{1}{t - \tau(t)} - \frac{1}{t} = \frac{\tau(t)}{t(t - \tau(t))}.$$

Hence

$$\lim_{u \to 0^+} \frac{\delta(u)}{u} = \lim_{t\to\infty} \frac{\delta(u(t))}{u(t)} = \lim_{t\to\infty} \frac{\tau(t)}{t} = 0.$$

Next we have the identity

$$\frac{t - \tau(t)}{2\tau(t) \log t} \sqrt{2 \frac{\tau(t)}{t(t - \tau(t))} \log\left(\frac{t(t - \tau(t))}{\tau(t)} \frac{1}{t - \tau(t)} \right)} \\ = \sqrt{\frac{t - \tau(t)}{t}} \sqrt{1 - \frac{\log \tau(t)}{\log t}}. \tag{4.3.3}$$

By (4.3.2), this has limit one when $t \to \infty$. Since $\frac{\delta(u)}{u} \to 0$ as $u \to 0^+$. Lévy's result on the maximum modulus of continuity of standard Brownian motion yields

$$\limsup_{t\to\infty} \frac{\max_{0 \le s < r \le u(t),\, r - s \le \delta(u(t))} |W(s) - W(r)|}{\sqrt{2\delta(u(t)) \log(u(t)/\delta(u(t)))}} \\ = \limsup_{u \to 0^+} \frac{\max_{0 \le s < r \le u,\, r - s \le \delta(u)} |W(s) - W(r)|}{\sqrt{2\delta(u) \log(u/\delta(u))}} = 1, \quad a.s. \tag{4.3.4}$$

Taking (4.3.3) and (4.3.4) together gives the result. □

We can combine these lemmata into a Corollary.

Corollary 4.3.3. *If τ is continuous, $\tau(t) \to 0$ as $t \to \infty$, τ is strictly decreasing, and $\tau(t) \log t \to \infty$ as $t \to \infty$, then*

$$\limsup_{n\to\infty} \frac{B(t_n) - B(t_{n-1})}{\sqrt{2\tau(t_n) \log t_n}} = 1, \quad a.s.$$

We can also prove the following result.

Lemma 4.3.4. *Suppose that τ is continuous, $\tau(t) \to 0$ as $t \to \infty$, τ is strictly decreasing,*

$$\lim_{t\to\infty} \frac{\log \tau(t)}{\log\log t} = -1, \tag{4.3.5}$$

and

$$\lim_{t\to\infty} \frac{\tau(t)\log t}{(\log\log t)^2} = 0. \tag{4.3.6}$$

Then

$$\limsup_{t\to\infty} \frac{|B(t) - B(t-\tau(t))|}{\log\log t} = 0, \quad a.s.$$

Proof. $W(t) = tB(\frac{1}{t})$ is standard Brownian motion. Since $\lim_{t\to\infty} \tau(t) W(\frac{1}{t-\tau(t)}) = 0$, and

$$B(t) - B(t-\tau(t)) = t\left(W\left(\frac{1}{t}\right) - W\left(\frac{1}{t-\tau(t)}\right)\right) + \tau(t)W\left(\frac{1}{t-\tau(t)}\right),$$

it is sufficient to show that

$$\lim_{t\to\infty} \frac{t(W(\frac{1}{t}) - W(\frac{1}{t-\tau(t)}))}{\log\log t} = 0, \quad a.s.$$

Define the new time variable $T = \frac{1}{t-\tau(t)}$, and the transformed delay function δ so that $\delta(\frac{1}{t-\tau(t)}) = \frac{\tau(t)}{t(t-\tau(t))}$. Thus $T - \delta(T) = \frac{1}{t}$ and $\lim_{T\to 0^+} \delta(T) = 0$. It is also true that

$$\lim_{T\to 0^+} \frac{\delta(T)}{T} = 0, \quad \lim_{T\to 0^+} \frac{T-\delta(T)}{T} = 1, \quad \lim_{T\to 0^+} \frac{\log\log\frac{1}{T}}{\log\log\frac{1}{T-\delta(T)}} = 1. \tag{4.3.7}$$

Therefore we now require that

$$\lim_{T\to 0^+} \frac{\frac{1}{T}(W(T) - W(T-\delta(T)))}{\log\log(\frac{1}{T})} = 0, \quad a.s. \tag{4.3.8}$$

Next we have that

$$\frac{\delta(T)\log(\frac{1}{T})}{(T\log\log(\frac{1}{T}))^2} = \frac{t-\tau(t)}{t}\,\frac{\log(t-\tau(t))}{\log t}\left(\frac{\log\log t}{\log\log(t-\tau(t))}\right)^2 \frac{\tau(t)\log t}{(\log\log t)^2}.$$

which, by (4.3.1), (4.3.6), and (4.3.7), has limit zero when $t \to \infty$. We also have that

$$\frac{\log\delta(T)}{\log T} = \frac{1 + \frac{\log\tau(t)}{\log t} + \frac{\log(t-\tau(t))}{\log t}}{\frac{\log(t-\tau(t))}{\log t}}.$$

Since, by (4.3.1), and (4.3.5),

$$\lim_{t\to\infty} \frac{\log \tau(t)}{\log t} = \lim_{t\to\infty} \frac{\log \tau(t)}{\log\log t} \frac{\log\log t}{\log t} = 0,$$

we can therefore say that

$$\lim_{T\to 0^+} \frac{\log \delta(T)}{\log T} = 2. \tag{4.3.9}$$

Since

$$\frac{|W(T-\delta(T)) - W(T)|}{T\log\log(\frac{1}{T})} \leq \max_{0<r\leq s<T,\ 0<r-s<\delta(T)} \frac{|W(r)-W(s)|}{T\log\log(\frac{1}{T})},$$

and if we define new variables $r' = r/T$ and $s' = s/T$ so that $r'T - s'T \leq \delta(T)$, we can say that

$$\frac{|W(T-\delta(T)) - W(T)|}{T\log\log(\frac{1}{T})} \leq \frac{\sqrt{T}}{T\log\log(\frac{1}{T})} \max_{0<r'<s'\leq 1,\ 0<r'-s'\leq\delta(T)/T} |\widetilde{W}(r') - \widetilde{W}(s')|$$

where $\widetilde{W}(r') = \frac{1}{\sqrt{T}} W(r)$ is a standard Brownian motion. Again, Lévy's result on the maximum modulus of continuity of standard Brownian motion yields

$$\lim_{\delta\to 0} \frac{\max_{0<r'<s'<1,\ 0<r'-s'<\delta} |\widetilde{W}(r') - \widetilde{W}(s')|}{\sqrt{2\delta\log\frac{1}{\delta}}} = 1, \quad a.s. \tag{4.3.10}$$

So, by (4.3.10),

$$\begin{aligned}
&\lim_{T\to 0^+} \frac{|W(T-\delta(T)) - W(T)|}{T\log\log(\frac{1}{T})} \\
&\leq \lim_{T\to 0^+} \frac{\sqrt{T}}{T\log\log(\frac{1}{T})} \sqrt{2\frac{\delta(T)}{T}\log\left(\frac{1}{\frac{\delta(T)}{T}}\right)} \\
&\qquad \cdot \frac{\max_{0<r'<s'<1,\ r'-s'<\delta(T)/T} |\widetilde{W}(r') - \widetilde{W}(s')|}{\sqrt{2\frac{\delta(T)}{T}}} \\
&= \lim_{T\to 0^+} \sqrt{2\frac{\delta(T)\log(\frac{1}{T})}{(T\log\log(\frac{1}{T}))^2} \frac{\log(\frac{T}{\delta(T)})}{\log(\frac{1}{T})}} = 0, \quad a.s.,
\end{aligned}$$

as required. □

Chapter 5

Oscillatory Behaviour – The Linear Stochastic Differential Equation with Vanishing Delay

We are now ready to turn our attention to the oscillatory behaviour of the scalar linear stochastic delay differential equation

$$dX(t) = (aX(t) + bX(t-\tau(t)))\,dt + \sigma X(t)\,dB(t) \tag{5.0.1a}$$

$$X(t) = \psi(t), \quad -\overline{\tau} \le t \le 0, \tag{5.0.1b}$$

where $b < 0$, and τ is a continuous function which vanishes as $t \to \infty$. As indicated in Chapter 4, we specifically require that the delay function satisfy

$$\lim_{t\to\infty} \tau(t) = 0,\ \tau(t) > 0,\ t \mapsto t - \tau(t) \text{ is nondecreasing.} \tag{5.0.2}$$

Roughly speaking, when the delay function τ vanishes sufficiently slowly, then all solutions of (5.0.1) are a.s. oscillatory. A quickly vanishing τ allows for the existence of a nonoscillatory solution. This behaviour contrasts dramatically with that of the corresponding deterministic equation, which always admits a nonoscillatory solution, and with that of the stochastic equation with zero delay, which is a.s. nonoscillatory. The analysis in this chapter has been published in Appleby & Kelly [2].

5.1 The deterministic equation.

In [2], we give an analysis of the asymptotic behaviour of the solutions of

$$x'(t) = ax(t) + bx(t-\tau(t)), \quad t \ge 0, \tag{5.1.1}$$

where $b < 0$ and $\tau(t) \to 0$ as $t \to \infty$, under the additional technical restriction that $\int_0^\infty \tau(s)ds < \infty$. We reproduce the statement of the relevant theorem here.

Theorem 5.1.1. *Suppose that $\tau(t) \to 0$ as $t \to \infty$ is continuous, and $\int_0^\infty \tau(s)ds < \infty$. Then for every solution of (5.1.1), there exists $c \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} \frac{x(t)}{g(t)} = c \tag{5.1.2}$$

where g is a positive function (independent of the initial function) which satisfies

$$\lim_{t\to\infty} \frac{1}{t} \log g(t) = a + b. \tag{5.1.3}$$

Moreover, for every finite $c \in \mathbb{R}$ there is a solution which satisfies (5.1.1), where g satisfies (5.1.3).

The proof makes use of results due to Castillo & Pinto [6], and is a special case of the proof of Theorem 4 in [2], which describes the asymptotic behaviour of the solutions of (5.0.1), when $\int_0^\infty \tau(s)ds < \infty$, and (5.0.2) holds.

5.2 The stochastic equation.

Theorem 5.2.1, below, gives us a general idea of the nature of the switch from oscillatory to nonoscillatory behaviour. However, as it stands, the analysis is incomplete in several ways that will be discussed in Section 5.3. Chapters 6 and 7 describe our attempt to develop a complete picture of the qualitative behaviour of the solutions of (4.0.1) using difference equations.

Theorem 5.2.1. *Let $b < 0$. Suppose τ is a continuous, positive function, which is eventually decreasing. Suppose further that $t \mapsto t - \tau(t)$ is increasing, and that there is $\alpha \in (0, \infty]$ such that*

$$-\alpha = \lim_{t\to\infty} \frac{\log \tau(t)}{\log\log t}. \tag{5.2.1}$$

Let $T > 0$, and suppose that $X = \{X(t); T \le t < \infty; \mathcal{F}^B(t)\}$ is the strong solution of the *equation*

$$dX(t) = (aX(t) + bX(t - \tau(t)))\,dt + \sigma X(t)\,dB(t), \quad t > T \tag{5.2.2a}$$

$$X(t) = 1, \quad t \in [T - \tau(T)), T]. \tag{5.2.2b}$$

Then we have the following case distinction:

1. *If $\alpha < 1$, then for every $T \geq 0$, the path $X(\omega)$ is oscillatory for all ω in a set which has probability one.*

2. *If $\alpha > 1$, then for each $\varepsilon > 0$, there is a $T = T(\varepsilon) > 0$ such that the path $X(\omega)$ is positive for all ω in a set which has probability at least $1 - \varepsilon$.*

A better result for nonoscillation can be obtained if it is stipulated that $\int_0^\infty \tau(s)ds < \infty$. In Appleby & Kelly [2], it is shown that, for every outcome ω in an almost sure set $\Omega^* \subseteq \Omega$, there exists an initial data function $\psi(\omega)$ such that $X(\omega)$, the realisation of the process satisfying (5.2.2a) with $T = 0$ and initial data $\psi(\omega)$, is nonoscillatory. However, we present the result with fewer technical restrictions here.

The two cases of Theorem 5.2.1 are a summary of the results obtained in Theorems 5.2.3 and 5.2.2. We consider each case in turn.

5.2.1 Case 1: Oscillatory behaviour.

Theorem 5.2.2. *Let $b < 0$. Suppose τ is eventually decreasing, satisfies (4.2.1), and (5.2.1) with $\alpha < 1$. Then all nontrivial solutions of (4.0.1) are a.s. oscillatory.*

Note that if 5.2.1 holds with $\alpha < 1$, then

$$\lim_{t\to\infty} \frac{\log(1/\tau(t))}{\sqrt{\tau(t)\log t}} = 0. \tag{5.2.3}$$

We proceed by showing that (5.2.3) is sufficient for X to be a.s. oscillatory.

A sufficient condition that implies (5.2.3) is $\lim_{t\to\infty} \tau(t)(\log t)^\gamma = \infty$, for any $\gamma \in (0, 1)$. This illustrates the idea that solutions of (5.0.1) oscillate if we require that τ converge to zero slowly. Notice moreover that (5.2.3) implies $\lim_{t\to\infty} \tau(t)\log t = \infty$. This indicates that the critical rate of decay of the delay function is around $(\log t)^{-1}$. If the delay function decays more slowly, in some sense, then all solutions are oscillatory. If it decays more rapidly, then by choosing an initial interval with a sufficiently large minimal element, an arbitrarily high proportion of the paths of the process can be shown to be positive.

Proof of Theorem 5.2.2. With φ defined by (4.1.1) and $y(t) = X(t)/\varphi(t)$, y obeys (4.1.3) for $t \geq 0$, and p in (4.1.3) is given by (4.1.4). If we can show that the process

$$Y(t) = \int_{t-\tau(t)}^{t} e^{-\sigma(B(s)-B(s-\tau(s)))}\,ds \tag{5.2.4}$$

satisfies $\limsup_{t\to\infty} Y(t) = \infty$, a.s., then $\limsup_{t\to\infty} \int_{t-\tau(t)}^{t} p(s)\,ds = \infty$. This holds as $\lim_{t\to\infty} \tau(t) = 0$ and

$$\int_{t-\tau(t)}^{t} p(s)\,ds = -b\int_{t-\tau(t)}^{t} e^{-(a-\sigma^2)\tau(s)}e^{-\sigma(B(s)-B(s-\tau(s)))}\,ds.$$

By Theorem 1.5.1, and the equivalence of the oscillation of $y(\omega)$ and $X(\omega)$, all solutions of (4.0.1) are a.s. oscillatory. The result is therefore secured by Lemma 4.2.1. □

5.2.2 Case 2: Nonoscillatory behaviour.

If (5.2.1) holds when $\alpha > 1$, then

$$\lim_{t\to\infty} \tau(t)\log t = 0. \tag{5.2.5}$$

We cannot prove that there is a solution that is nonoscillatory on almost all sample paths. The best that can be achieved with this approach is to show that an arbitrarily high proportion of paths are nonoscillatory if the initial value problem starts at a sufficiently large and deterministic time.

Theorem 5.2.3. *Let $b < 0$. Suppose that τ is continuous, positive, $t \mapsto t - \tau(t)$ is increasing, and (5.2.5) holds. Then, for every $\varepsilon > 0$, there is a $t_0(\varepsilon) > 0$, and a set*

$\Omega_\varepsilon \in \mathcal{F}^B(\infty)$ *with* $\mathbb{P}[\Omega_\varepsilon] \geq 1 - \varepsilon$ *such that for each* $\omega \in \Omega_\varepsilon$, $X(\omega)$ *is a positive function, where* $X = \{X(t); \mathcal{F}^B(t); t_0(\varepsilon) \leq t < \infty\}$ *is the strong solution of*

$$\begin{aligned} dX(t) &= (aX(t) + bX(t-\tau(t)))\,dt + \sigma X(t)\,dB(t), \quad t > t_0(\varepsilon) \\ X(t) &= 1, \quad t \in [t_0(\varepsilon) - \tau(t_0(\varepsilon)), t_0(\varepsilon)]. \end{aligned}$$

Proof. Let $\tau_0 > 0$ be given by $\tau_0 - \tau(\tau_0) = 0$, $\tau_1 > \tau_0$ by $\tau_1 - \tau(\tau_1) = \tau_0$, and $\tau_2 > \tau_1$ by $\tau_2 - \tau(\tau_2) = \tau_1$. Then for $t > \tau_0$, $t - \tau(t) > 0$. Hence, for $t > \tau_0$, we can define the $\mathcal{F}^B(\infty)$-measurable random variable

$$C(t) = \sup_{s \geq t} |B(s) - B(s - \tau(s))|.$$

This is well-defined on a set of probability one as $\lim_{t\to\infty} B(t) - B(t-\tau(t)) = 0$ a.s. Thus $\lim_{t\to\infty} C(t) = 0$, a.s. Therefore, for every $\varepsilon \in (0,1)$, there is $T_0^*(\varepsilon) > 0$ such that $t > T_0^*(\varepsilon)$ implies $\mathbb{P}[|C(t) > 1|] \leq \frac{\varepsilon}{2}$ for $t > T_0^*(\varepsilon)$. Since $\tau(t) \to 0$ as $t \to \infty$, for every $\varepsilon \in (0,1)$, there is $T_1^*(\varepsilon) > 0$ such that $t > T_1^*(\varepsilon) > 0$ implies

$$\Psi\left(\frac{1}{\sqrt{\tau(t)}}\right) < \frac{\varepsilon}{4},$$

where Ψ is as defined in (4.1.5). Since $\tau(t) \to 0$ as $t \to \infty$, and $t - \tau(t) \to \infty$ as $t \to \infty$, there exists $T_2 > 0$ such that for $t > T_2$ we have

$$|b|\tau(t-\tau(t)) < \frac{1}{3e}, \quad \tau(t) < 1, \quad \text{and} \quad |b|e^{|\lambda|+|\sigma|}\tau(t) < \frac{1}{3e}.$$

Define $t^*(\varepsilon) = \tau_2 + (T_0^*(\varepsilon) \vee T_1^*(\varepsilon) \vee T_2)$. Since $t^*(\varepsilon) > \tau_2$ and $t \mapsto t - \tau(t)$ is increasing, there exists a unique $t_0(\varepsilon) > \tau_1$ such that

$$t^*(\varepsilon) - \tau(t^*(\varepsilon)) = t_0(\varepsilon).$$

Note morever that $t_0(\varepsilon) - \tau(t_0(\varepsilon)) > \tau_0$ and $|b|\tau(t_0(\varepsilon)) = |b|\tau(t^*(\varepsilon) - \tau(t^*(\varepsilon))) < \frac{1}{3e}$, because $t^*(\varepsilon) > T_2$. Define the sets

$$\Omega_\varepsilon^{(1)} = \left\{\omega : \sup_{s\geq t^*(\varepsilon)} |B(s,\omega) - B(s-\tau(s),\omega)| \leq 1\right\},$$

$$\Omega_\varepsilon^{(2)} = \left\{\omega : \sup_{0\leq s-t_0(\varepsilon)\leq \tau(t^*(\varepsilon))} B(s,\omega) - B(t_0(\varepsilon),\omega) \leq 1\right\}.$$

Let $\Omega_\varepsilon = \Omega_\varepsilon^{(1)} \cap \Omega_\varepsilon^{(2)}$. We already have $\mathbb{P}[\Omega_\varepsilon^{(1)}] \geq 1 - \frac{\varepsilon}{2}$. Next, if W is another standard Brownian motion, then

$$\begin{aligned}
1 - \mathbb{P}[\Omega_\varepsilon^{(2)}] &= \mathbb{P}\left[\sup_{0\leq s-t_0(\varepsilon)\leq\tau(t^*(\varepsilon))} B(s) - B(t_0(\varepsilon)) > 1\right] \\
&= \mathbb{P}\left[\sup_{0\leq s'\leq\tau(t^*(\varepsilon))} W(s') > 1\right] \\
&= \mathbb{P}[|W(\tau(t^*(\varepsilon)))| > 1] = 2\mathbb{P}[W(\tau(t^*(\varepsilon))) > 1] \\
&= 2\Psi\left(\frac{1}{\sqrt{\tau(t^*(\varepsilon))}}\right) < \frac{\varepsilon}{2},
\end{aligned}$$

where Ψ is given by (4.1.5). These equalities hold because $\max_{0\leq s\leq t} W(s)$ has the same distribution as $|W(t)|$, and $t^*(\varepsilon) > T_1^*(\varepsilon)$. Thus $\mathbb{P}[\Omega_\varepsilon^{(2)}] > 1 - \frac{\varepsilon}{2}$, so $\mathbb{P}[\Omega_\varepsilon] > 1 - \varepsilon$.

Define $\{\varphi(t)\}_{t\geq 0}$ by $\varphi(t) = 1$ for $0 \leq t \leq t_0(\varepsilon)$, and

$$\varphi(t) = 1 + \int_{t_0(\varepsilon)}^t a\varphi(s)\,ds + \int_{t_0(\varepsilon)}^t \sigma\varphi(s)\,dB(s), \quad t \geq t_0(\varepsilon).$$

Then for $t > t_0(\varepsilon) - \tau(t_0(\varepsilon))$, as $t > \tau_0$, we have $t - \tau(t) > \tau_0 - \tau(\tau_0) = 0$. Hence we can define

$$p(t) = |b|\varphi(t)^{-1}\varphi(t-\tau(t)), \quad t \geq t_0(\varepsilon) - \tau(t_0(\varepsilon)).$$

Without loss of generality, let $\sigma < 0$. Then, with $\lambda = a - \frac{1}{2}\sigma^2$, we have

$$p(t) = \begin{cases} |b|e^{-\lambda\tau(t)}e^{-\sigma(B(t)-B(t-\tau(t)))}, & t \geq t^*(\varepsilon) \\ |b|e^{-\lambda(t-t_0(\varepsilon))}e^{-\sigma(B(t)-B(t_0(\varepsilon)))}, & t_0(\varepsilon) \leq t \leq t^*(\varepsilon) \\ |b|, & t_0(\varepsilon) - \tau(t_0(\varepsilon)) \leq t \leq t_0(\varepsilon). \end{cases}$$

Now, let $\omega \in \Omega_\varepsilon$ and $t \geq t_0(\varepsilon)$. We consider three cases:

Case 1: $t - \tau(t) \geq t^*(\varepsilon)$.

For $\omega \in \Omega_\varepsilon$, $|B(s,\omega) - B(s-\tau(s),\omega)| \leq 1$, for $s \geq t - \tau(t)$ since $t - \tau(t) \geq t^*(\varepsilon) > T_0^*(\varepsilon)$. Also $\tau(s) < 1$, as $s \geq t - \tau(t)$ and $t - \tau(t) \geq t^*(\varepsilon) > T_2$. Hence, $\omega \in \Omega_\varepsilon$, $t - \tau(t) \geq t^*(\varepsilon)$ gives

$$\begin{aligned}\left(\int_{t-\tau(t)}^{t} p(s)\,ds\right)(\omega) &\leq \int_{t-\tau(t)}^{t} |b| e^{|\lambda|\tau(s)} e^{|\sigma||B(s,\omega)-B(s-\tau(s),\omega)|}\,ds \\ &\leq \int_{t-\tau(t)}^{t} |b| e^{|\lambda|} e^{|\sigma|}\,ds = \tau(t)|b|e^{|\lambda|+|\sigma|} < \frac{1}{3e},\end{aligned}$$

as $t \geq t^*(\varepsilon) + \tau(t) \geq t^*(\varepsilon) > T_2$. Therefore

$$\omega \in \Omega_\varepsilon,\ t \geq t_0(\varepsilon),\ t - \tau(t) \geq t^*(\varepsilon) \text{ implies } \left(\int_{t-\tau(t)}^{t} p(s)\,ds\right)(\omega) < \frac{1}{e}. \tag{5.2.6}$$

Case 2: $t - \tau(t) \leq t^*(\varepsilon)$, $t < t^*(\varepsilon)$.

First, as $t < t^*(\varepsilon)$,

$$\begin{aligned}\sup_{t_0(\varepsilon)\leq s\leq t} B(s,\omega) - B(t_0(s),\omega) &\leq \sup_{0\leq s-t_0(\varepsilon)\leq t^*(\varepsilon)-t_0(\varepsilon)} B(s,\omega) - B(t_0(\varepsilon),\omega) \\ &= \sup_{0\leq s-t_0(\varepsilon)\leq \tau(t^*(\varepsilon))} B(s,\omega) - B(t_0(\varepsilon),\omega) \leq 1.\end{aligned}$$

Therefore, as $t - \tau(t) \geq t_0(\varepsilon) - \tau(t_0(\varepsilon))$, $t < t^*(\varepsilon)$, $t^*(\varepsilon) - \tau(t^*(\varepsilon)) = t_0(\varepsilon)$, we get

$$\begin{aligned}\left(\int_{t-\tau(t)}^{t} p(s)\,ds\right)(\omega) &= |b|(t_0(\varepsilon) - (t-\tau(t))) \\ &\quad + \int_{t_0(\varepsilon)}^{t} |b| e^{-\lambda(s-t_0(\varepsilon))} e^{-\sigma(B(s,\omega)-B(t_0(\varepsilon),\omega))}\,ds \\ &\leq |b|\tau(t_0(\varepsilon)) + |b|e^{|\lambda|(t-t_0(\varepsilon))} e^{-\sigma}(t - t_0(\varepsilon)) \\ &\leq \frac{1}{3e} + |b|e^{|\lambda|\tau(t^*(\varepsilon))+|\sigma|}\tau(t^*(\varepsilon)) < \frac{2}{3e}.\end{aligned}$$

Thus,

$$\omega \in \Omega_\varepsilon,\ t_0(\varepsilon) \leq t < t^*(\varepsilon),\ t - \tau(t) \leq t^*(\varepsilon) \text{ implies } \left(\int_{t-\tau(t)}^{t} p(s)\,ds\right)(\omega) < \frac{1}{e}. \tag{5.2.7}$$

Case 3: $t - \tau(t) \leq t^*(\varepsilon)$, $t > t^*(\varepsilon)$.

Note that $\sup_{t_0(\varepsilon)\le s\le t^*(\varepsilon)} B(s,\omega) - B(t_0(\varepsilon),\omega) \le 1$, and $B(s,\omega) - B(s-\tau(s),\omega) \le 1$ for $s \ge t^*(\varepsilon)$. Furthermore $t \ge t_0(\varepsilon)$ implies $t-\tau(t) \ge t_0(\varepsilon)-\tau(t_0(\varepsilon))$, so $|b|(t_0(\varepsilon)-(t-\tau(t))) \le |b|\tau(t_0(\varepsilon)) < (3e)^{-1}$. Therefore

$$\begin{aligned}
&\left(\int_{t-\tau(t)}^{t} p(s)\,ds\right)(\omega) \\
&= |b|(t_0(\varepsilon) - (t-\tau(t))) + \int_{t_0(\varepsilon)}^{t^*(\varepsilon)} |b|e^{-\lambda(s-t_0(\varepsilon))}e^{-\sigma(B(s,\omega)-B(t_0(\varepsilon),\omega))}\,ds \\
&\quad + \int_{t^*(\varepsilon)}^{t} |b|e^{-\lambda\tau(s)}e^{-\sigma(B(s,\omega)-B(s-\tau(s),\omega))}\,ds \\
&< \frac{1}{3e} + |b|e^{|\lambda|\tau(t^*(\varepsilon))+|\sigma|}(t^*(\varepsilon)-t_0(\varepsilon)) + |b|e^{|\lambda|+|\sigma|}(t-t^*(\varepsilon)) \\
&\le \frac{1}{3e} + |b|e^{|\lambda|+|\sigma|}\tau(t^*(\varepsilon)) + |b|e^{|\lambda|+|\sigma|}\tau(t) \le \frac{1}{e}.
\end{aligned}$$

Thus,

$$\omega \in \Omega_\varepsilon,\ t > t^*(\varepsilon),\ t-\tau(t) \le t^*(\varepsilon) \text{ implies } \left(\int_{t-\tau(t)}^{t} p(s)\,ds\right)(\omega) < \frac{1}{e}. \tag{5.2.8}$$

Combining (5.2.6), (5.2.7) and (5.2.8), for $\omega \in \Omega_\varepsilon$, and $t \ge t_0(\varepsilon)$

$$\left(\int_{t-\tau(t)}^{t} p(s)\,ds\right)(\omega) < \frac{1}{e}. \tag{5.2.9}$$

Next, consider the following delay differential equation, which has a unique continuous solution for almost all $\omega \in \Omega$:

$$y'(t,\omega) = -p(t,\omega)y(t-\tau(t),\omega), \quad t > t_0(\varepsilon) \tag{5.2.10a}$$

$$y(t) = 1, \quad t \in [t_0(\varepsilon)-\tau(t_0(\varepsilon)), t_0(\varepsilon)]. \tag{5.2.10b}$$

Then, for $\omega \in \Omega_\varepsilon$, by Theorem 1.5.2, $y(t,\omega) > 0$ for all $t \ge t_0(\varepsilon)$ and $\omega \in \Omega_\varepsilon$. Consider the process X defined by $X(t) = \varphi(t)y(t)$, $t \ge t_0(\varepsilon)-\tau(t_0(\varepsilon))$. Then $X(t,\omega) > 0$ for $t \ge t_0(\varepsilon)$ and $\omega \in \Omega_\varepsilon$. Moreover, as $X(t) = 1$ for $t \in [t_0(\varepsilon)-\tau(t_0(\varepsilon)), t_0(\varepsilon)]$, and for $t \ge t_0(\varepsilon)$, integration by parts gives

$$X(t) = 1 + \int_{t_0(\varepsilon)}^{t} aX(s) + bX(s-\tau(s))\,ds + \int_{t_0(\varepsilon)}^{t} \sigma X(s)\,dB(s),$$

thereby proving the result. □

5.3 A critique of Theorem 5.2.1.

The statement of Theorem 5.2.1 provides an incomplete classification of the qualitative behaviour of the solutions of (5.0.1). First, when $\alpha > 1$, where α is as defined in (5.2.1) the positivity result given in Case 2 of the theorem applies only to a process that is similar to that satisfying (5.0.1), but with strong technical restrictions on the initial data. Additionally, the positivity result cannot apply with probability one to any such process. Second, we have no information regarding the qualitative behaviour of solutions of (5.0.1) when $\alpha = 1$. In other words, we have not found with any precision the rate of decay of the delay function where the behaviour of solutions of (5.0.1) switches from oscillatory to nonoscillatory.

In order to develop our picture of the behaviour of this process, we require a different approach. The continuity of the sample paths of solutions of (5.0.1) imposes some limitations on the effectiveness of our analysis. However it is not necessary that all of the information in the evolving filtration be available to us. Our pathwise definition of oscillation does not involve a continuous use of information from the path – we place no emphasis on amplitude or period and we only require that its value change sign ad infinitum. Therefore we seek to show that the fundamental oscillatory behaviour of the solutions of (5.0.1) can survive an appropriate discretisation. Following such a discretisation, oscillatory behaviour will prove to be more vulnerable to analysis.

There is evidence suggesting that this is a reasonable line of attack. The disparity in the behaviours of the solutions of (5.0.1) for fast and slow decay rates of τ is foreshadowed by differences in the asymptotic behaviour of the *feedback ratio*

$$\rho(t) = \frac{X(t - \tau(t))}{X(t)}, \quad t \geq 0, \tag{5.3.1}$$

for fast and slow decay rates of τ. As a consequence, ρ will play a key role in the construction of a nonuniform mesh in Chapter 7, suggesting a distribution of mesh points that will be sufficient to capture the oscillatory behaviour of the solutions of (5.0.1).

5.4 The feedback ratio.

It must be emphasised that since the initial data function ψ, defined in (5.0.1b), is continuous and strictly positive, $\rho(t)$ is only well defined for all $t \in [0, \infty)$ if X is strictly positive. To guarantee this, we set $b > 0$ for the duration of this analysis. Initially we consider the asymptotic behaviour of ρ when τ obeys (5.2.5).

5.4.1 Asymptotic behaviour of X via ρ.

We will see that ρ is in some sense well behaved when τ decays quickly. In fact it is possible to calculate a Lyapunov exponent for X using a knowledge of the asymptotic behaviour of ρ.

Theorem 5.4.1. *Suppose $\tau(t)\log(t) \to 0$ as $t \to \infty$, $b > 0$ and $\psi(t) > 0$. Then there is a process ρ^* such that*

$$\lim_{t\to\infty} \rho^*(t) = 0, \quad a.s.$$

and

$$X(t) = X(0)e^{(a+b-\frac{1}{2}\sigma^2)t+\sigma B(t)+\int_0^t \rho^*(s)ds}.$$

Therefore

$$\lim_{t\to\infty} \frac{1}{t}\log|X(t)| = a + b - \frac{1}{2}\sigma^2, \ \ a.s.$$

Proof. Since $\tau(t) \to 0$ as $t \to \infty$, there is a $t_0 \geq 0$ such that $t - \tau(t) > 0$ for all $t > t_0$. Let ρ be given by (5.3.1), a well defined and positive process if $X(t) > 0$ for all $t \geq -\overline{\tau}$.

The definition of ρ means that (5.0.1a) can be restated as

$$dX(t) = (a + b\rho(t))X(t)dt + \sigma X(t)dB(t),$$

so

$$X(t) = X(0)e^{(a+\frac{1}{2}\sigma^2)t+\sigma B(t)+b\int_0^t \rho(s)ds}, \quad t \geq 0.$$

Thus, for $t \geq t_0$, we get, with $\lambda = a - \frac{1}{2}\sigma^2$,

$$\rho(t) = e^{-\lambda\tau(t)}e^{-\sigma(B(t)-B(t-\tau(t)))}e^{-b\int_{t-\tau(t)}^{t}\rho(s)ds}.$$

If $\tau(t)\log(t) \to 0$ as $t \to \infty$, we have $\rho(t) = \mu(t)e^{-b\int_{t-\tau(t)}^{t}\rho(s)ds}$, where $\mu(t) \to 1$ as $t \to \infty$, by Lemma 4.2.2. Therefore, as $\rho(t) > 0$ for all $t \geq 0$, and $b > 0$, $\rho(t) \leq \mu(t)$. This means that $\limsup_{t\to\infty}\rho(t) \leq 1$, and for every $\varepsilon > 0$ and $\omega \in \Omega^*$ there is $T^*(\varepsilon,\omega) > 0$ such that $\mu(t,\omega) > 1-\varepsilon$ and $\rho(t,\omega) < 1+\varepsilon$ for all $t - \tau(t) > T^*$. Then,

$$\rho(t) > (1-\varepsilon)e^{-b\int_{t-\tau(t)}^{t}\rho(s)ds},$$

and

$$0 < b\int_{t-\tau(t)}^{t}\rho(s)ds < b(1+\varepsilon)\tau(t).$$

Hence $\rho(t) > (1-\varepsilon)e^{-b(1+\varepsilon)\tau(t)}$. Therefore, for each $\omega \in \Omega^*$,

$$\liminf_{t\to\infty}\rho(t,\omega) \geq 1-\varepsilon.$$

Letting $\varepsilon \to 0^+$ yields

$$\liminf_{t\to\infty}\rho(t,\omega) = 1$$

for all $\omega \in \Omega^*$. Thus $\rho(t) \to 1$ as $t \to \infty$, a.s. So, when $b > 0$, the result follows with $\rho^*(t) = \rho(t) - 1$. □

5.4.2 Asymptotic behaviour of ρ when $\tau(t)\log t \to \infty$.

In Theorem 5.4.1, we saw that the asymptotic behaviour of ρ is mild when $\tau(t)\log t \to 0$ as $t \to \infty$. However, when the rate of decay of the delay is slower, the asymptotic behaviour of ρ is more complicated. We will show here that almost all paths of ρ, when $\tau(t)\log t \to \infty$ as $t \to \infty$, are recurrent on $(0,\infty)$, in the sense that

$$\liminf_{t\to\infty}\rho(t) = 0, \quad \limsup_{t\to\infty}\rho(t) = \infty, \quad \text{a.s.} \tag{5.4.1}$$

Although it is possible to define ρ only when X does not oscillate, this recurrence arises at a similar rate of decay of τ as the onset of a.s. oscillation of X. It appears as though the rapid fluctuation in ρ, arising when the rate of decay of the delay is slower than $(\log t)^{-1}$, may be symptomatic of the oscillation of all solutions of (5.0.1).

Theorem 5.4.2. *If $b > 0$, τ is continuous and obeys $\tau(t) \to 0$ and $\tau(t)\log t \to \infty$, then ρ, defined by (5.3.1), obeys (5.4.1).*

Proof. We start by proving $\limsup_{t\to\infty} \rho(t) = \infty$, a.s. Note first that, as has been shown in Lemmata 4.2.3 and 4.2.4, when $\tau(t)\log t \to \infty$,

$$1 \leq \limsup_{t\to\infty} \frac{|B(t) - B(t-\tau(t))|}{\sqrt{2\tau(t)\log t}} \leq \sqrt{2}, \quad a.s. \tag{5.4.2}$$

We assume that the set $\{\omega : \limsup_{t\to\infty} \rho(t,\omega) < \infty\}$ has positive probability. This implies that there exists $m \in \mathbb{N}$ such that

$$A_m = \left\{\omega : \limsup_{t\to\infty} \rho(t,\omega) < m\right\}$$

has positive probability. For each $\omega \in A_m$ there is a $T(\omega, m) > 0$ such that for all $t > T(\omega, m)$, $\rho(t) \leq 2m$. Since $t - \tau(t) \to \infty$, for all $t > T(\omega, m)$

$$\begin{aligned} 2m &\geq \rho(t,\omega) \\ &= e^{-\lambda\tau(t) - \sigma(B(t,\omega) - B(t-\tau(t),\omega))} e^{-b\int_{t-\tau(t)}^{t} \rho(s)\,ds} \\ &\geq e^{-\lambda\tau(t) - \sigma(B(t,\omega) - B(t-\tau(t),\omega))} e^{-2bm\tau(t)}. \end{aligned}$$

So

$$\limsup_{t\to\infty} e^{-(\lambda+2bm)\tau(t) - \sigma(B(t) - B(t-\tau(t)))} < \infty \text{ on } A_m.$$

Without loss of generality, we now let $\sigma < 0$. Then the exponent

$$\begin{aligned} &-(\lambda + 2bm)\tau(t) - \sigma(B(t) - B(t-\tau(t))) \\ = &\sqrt{\tau(t)\log t}\left\{-(\lambda + 2bm)\sqrt{\frac{\tau(t)}{\log t}} - \sigma\frac{B(t) - B(t-\tau(t))}{\sqrt{\tau(t)\log t}}\right\}, \end{aligned}$$

has infinite limsup as $t \to \infty$, because $\tau(t)\log t \to \infty$, and (5.4.2) holds. Therefore

$$\limsup_{t\to\infty} e^{-(\lambda+2bm)\tau(t)-\sigma(B(t)-B(t-\tau(t)))} = \infty, \quad a.s.,$$

leading us to a contradiction, and proving the result.

We now turn to the proof of $\liminf_{t\to\infty}\rho(t) = 0$, a.s. Note that, as stated in (5.4.2),

$$\sqrt{2} \geq \limsup_{t\to} \frac{|B(t) - B(t-\tau(t))|}{\sqrt{2\tau(t)\log t}} \geq \liminf_{t\to\infty} \frac{|B(t) - B(t-\tau(t))|}{\sqrt{2\tau(t)\log t}}.$$

We can assume that $\rho(t) \geq 0$ for all t. This time, we assume that the set

$$\left\{\omega : \liminf_{t\to\infty}\rho(t) > 0\right\}$$

has positive probability. That is, there exists $m \in \mathbb{N}$ such that

$$A'_m = \left\{\omega : \liminf_{t\to\infty}\rho(t,\omega) > m\right\}$$

has positive probability. A similar approach to that used when $\limsup_{t\to\infty}\rho(t) = \infty$ a.s., yields the required contradiction. $\square$

Chapter 6

A Uniform Discretisation of the Auxiliary Process Yields Spurious Oscillatory Behaviour

The most natural way to construct a discrete process with identical properties to those of the solutions of (5.0.1), would appear to be to discretise the equation directly, using a stochastic Euler–Maruyama scheme on a uniform mesh. We have not done this. We only need to reproduce a single property of the solutions of (5.0.1) – oscillatory behaviour – and the techniques employed in Chapter 5 are easily adapted to a discrete setting. As a consequence we have chosen to discretise the auxiliary process (4.1.3).

Nonetheless, our first attempt turns out to be a failure, albeit an interesting failure. It is tempting to assume that the most straightforward route to take would be to use a piecewise Euler–Maruyama scheme on a uniform mesh, replacing the delay term with an instantaneous term once the delay has become sufficiently small. Karoui and Vaillancourt [18] take this approach for general deterministic nonlinear vanishing delay equations, and we apply a similar methodology in this chapter. This technique, although valid in the deterministic case, cannot reasonably be expected to work in the stochastic case. Since it appears that the presence of a noise perturbation allows the delay to affect the qualitative behaviour of the process regardless of how small it has become, the asymptotic effect of the delay must be present after discretisation. By removing the delay after a finite interval we are effectively ignoring the very phenomenon we are trying to reproduce – the interaction between the delay and the noise.

In this chapter we present this failed attempt to construct an Euler process on a uniform mesh. It will be seen that a discretisation of this kind induces spurious behaviour that directly contradicts the statement of Theorem 5.2.1, and that is dependent on the mesh size. A successful discretisation is developed in Chapter 7.

6.1 An Euler difference scheme.

A discrete process is characterised by the sequence $\{Y_n\}$ representing the solution of a difference equation, with appropriate initial data, and the set of discrete points, called the mesh, on which $\{Y_n\}$ evolves. Our method involves the use of an Euler scheme to discretise the paths of the auxiliary process (4.1.3) with initial data (4.1.2b). We study the solutions of a class of difference equations of the form

$$Y_{n+1} = Y_n - \Delta_n P_n(\omega) Y_{n-r_M(n)}, \;\; n \geq 0 \tag{6.1.1}$$

evolving on some mesh M, with positive initial data on a finite discrete subset of $[-\overline{\tau}, 0]$ that includes the endpoints. The form of the delay function r_M depends on the structure of the mesh. Each term of the sequence of random variables $\{P_n(\omega)\}_{n=0}^{\infty}$ is defined to be (4.1.4) on the corresponding path, sampled at the n^{th} mesh point. We generally suppress the ω-dependence and write $\{P_n\}_n$. The lack of differentiability in almost every path of the process p, defined in (4.1.4), ensures that the convergence results in [14] do not apply.

Finally, as indicated in Chapter 4, the definition of the delay function must be restricted slightly. Let τ be a continuous function obeying

$$\lim_{t\to\infty} \tau(t) = 0, \; \tau(t) > 0, \; t \mapsto \tau(t) \text{ is strictly decreasing on } [0, \infty). \tag{6.1.2}$$

6.2 Definitions of oscillation.

We cannot use the same definition of oscillation for discrete and continuous processes. When we develop the definition of oscillation of a discrete process, we cannot automatically base it on the zeros of the process. A discrete process can jump across an equilibrium without ever taking on its value. Because of this, we prefer to use sign changes rather than zeros to form a valid definition of the a.s. oscillation of a discrete process defined by (6.1.1). This approach is taken for deterministic difference equations in, for example, Koplatadze [21].

Definition 6.2.1. Let $\{Y_n\}_{n\geq 0}$ be a real valued stochastic process such that

$$\mathbb{P}[Y_n \neq 0 \text{ for all } n] = 1.$$

We say that $\{Y_n\}_{n\geq 0}$ is *a.s. oscillatory* if

$$\mathbb{P}\left[\frac{Y_{n+1}}{Y_n} < 0 \text{ i.o.}\right] = 1.$$

The process is *a.s. nonoscillatory* if $\mathbb{P}[\Upsilon < \infty] = 1$, where

$$\Upsilon(\omega) = \inf\left\{v \in \mathbb{N} : \frac{Y_{n+1}}{Y_n}(\omega) > 0 \quad \text{for all } n \geq v\right\}. \tag{6.2.1}$$

6.3 Constructing a discrete stochastic process.

Initial construction of the discrete process. Let M_Δ be a uniform mesh of mesh size Δ. The difference equation (6.1.1) becomes

$$Y_{n+1} = Y_n - \Delta P_n(\omega) Y_{n-r(n)} \tag{6.3.1}$$

where

$$r(n) = \sup\{k > 0 : k\Delta \leq \tau(n\Delta)\}. \tag{6.3.2}$$

By (6.1.2), we can define a constant $N_0 < \infty$ to be

$$N_0 = \inf\{n \in \mathbb{N} : n - r(n) > 0\}.$$

Thus, the initial data for (6.3.1) is an ordered set

$$\psi = (\psi_0, \psi_1, \ldots, \psi_{N_0-1}, \widetilde{Y}_0), \tag{6.3.3}$$

where $\psi_i = \widetilde{Y}_{i-r(i)}$ for all $i < N_0$, and $\psi_{N_0-1} = \widetilde{Y}_0$ if $(N_0 - 1)\Delta - r((N_0 - 1)\Delta) = 0$, a condition that can be guaranteed for any τ satisfying (6.1.2) by choosing Δ appropriately. However, in general it will not hold. Nonetheless, (6.3.3) is well defined for any given τ, regardless of the size of Δ. Note that is enough to associate an initial data value with each mesh point up to and including the mesh point at $N_0\Delta$, without specifying the location of the initial data values on $\mathbb{R}$. However, as an aid to visualisation, Figure 6.3.1 shows the placement of initial data for a nonspecific vanishing delay function τ defined so that $N_0 = 3$ and $\widetilde{N} = 6$, in order to motivate the structure of (6.3.3) when $(N_0-1)\Delta - r((N_0-1)\Delta) \neq 0$.

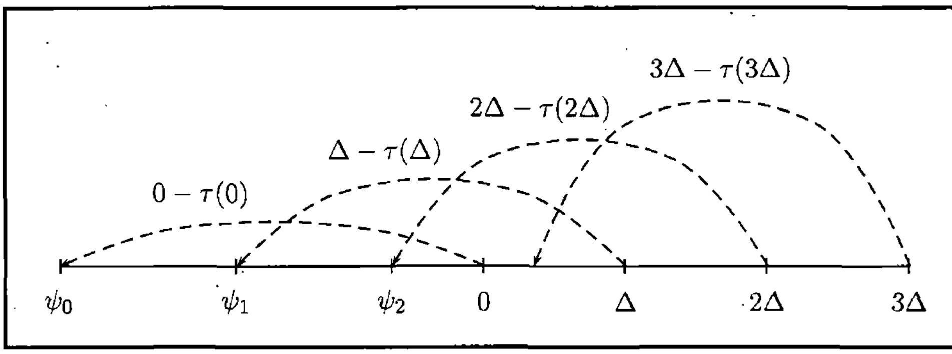


Figure 6.3.1: Visualising the possible placement of initial data points, when $N_0 = 3$, and $2\Delta - \tau(2\Delta) \neq 0$.

For technical reasons, we require that

$$\psi_0 \neq \frac{\widetilde{Y}_0}{\Delta|b|}. \tag{6.3.4}$$

Once again, (6.3.4) can be satisfied by choosing Δ appropriately. We also require that

$$\psi_j \in \mathbb{R}^+, \text{ for all } j < N_0. \tag{6.3.5}$$

The random variable P_n satisfies

$$P_n = \begin{cases} |b|e^{-\lambda\tau(n\Delta)}e^{|\sigma|(B(n\Delta)-B(n\Delta-\tau(n\Delta)))}, & N_0 \leq n, \\ |b|e^{-\lambda n\Delta}e^{|\sigma|B(n\Delta)}, & 0 < n < N_0, \\ |b|, & n = 0. \end{cases}$$

By (6.1.2), we can define a constant $N_1 > N_0$ to be

$$N_1 = \inf\{n > N_0 : n - r(n) - 1 - r(n - r(n) - 1) > 0\}.$$

By (6.3.2) and (6.1.2), there exists $N^* < \infty$ large enough that for all $n > N^*$, $\tau(n\Delta) < \Delta$, and therefore $r(n) = 0$. So, for all $n > N^*$, each term of $\{Y_n\}$ satisfies

$$Y_{n+1} = Y_n(1 - \Delta P_n).$$

The problems with the mesh M_Δ begin at N^*, when the length of the delay drops below the mesh size. From this point onwards $r(n) \equiv 0$, and we have once again ignored the interaction between the delay and the noise. In fact we see that the process which evolves on this mesh is capable of producing spurious behaviour, both oscillatory and nonoscillatory. This behaviour is a result of the discretisation rather than any interaction between the delay and the noise.

A discrete–time filtration. Rather than prove a result describing the behaviour of solutions of (6.3.1) directly, we will replace each P_n with a random variable $\widetilde{P}_n$ of identical distribution. In order to do this, we must define a new filtration.

Consider a sequence of independent standard normal random variables $\{\xi_k\}_{k=0}^{\infty}$. The filtration generated by this sequence is $\{\mathcal{G}_k^{\xi}\}_{k=0}^{\infty}$, where

$$\mathcal{G}_k^{\xi} = \sigma(\{\xi_i\}_{i=0}^{j}; 0 \le j \le k).$$

We associate with each ξ_k a number δ_k^2, where the sequence $\{\delta_k^2\}_{k=0}^{\infty}$ is defined as follows:

1. Define a sequence $\{a_k\}_{k=0}^{\infty}$, where $a_k = k\Delta$ for every k.
2. Define a sequence $\{b_k\}_{k=0}^{\infty}$, where $b_k = (k+N_0)\Delta - \tau((k+N_0)\Delta)$ for every k.
3. From $\{a_k\}_{k=0}^{\infty}$ and $\{b\}_{k=0}^{\infty}$ we can construct a new sequence $\{c_k\}_{k=0}^{\infty}$ as follows: For every $n < \infty$, let $c_{2n} = a_n$, and
   - if there exists $0 \le j < \infty$ such that $b_j \in (a_n, a_{n+1})$, then $c_{2n+1} = b_j$.
   - Otherwise $c_{2n+1} = \frac{a_n + a_{n+1}}{2}$.
4. Now, for every $0 \le k < \infty$, let

$$\delta_k^2 = c_{k+1} - c_k.$$

A visualisation of this construction is given for an arbitrary vanishing delay function τ in the set of schematics given in Figures 6.3.2, 6.3.3, and 6.3.4.

Finally, we define a sequence of independent, $\mathcal{G}_k^{\xi}$–measurable random variables $\{\zeta_k\}_{k=0}^{\infty}$, so that, for each k,

$$\zeta_k = e^{|\sigma|\delta_k \xi_k}.$$

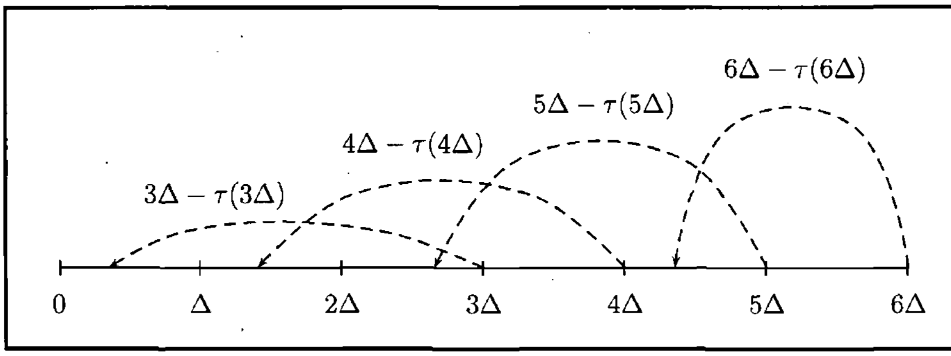


Figure 6.3.2: A uniform mesh of size Δ overlaid with the feedback positions of the continuous delay function τ at each mesh point. $N_0 = 3$, $\widetilde{N} = 6$.

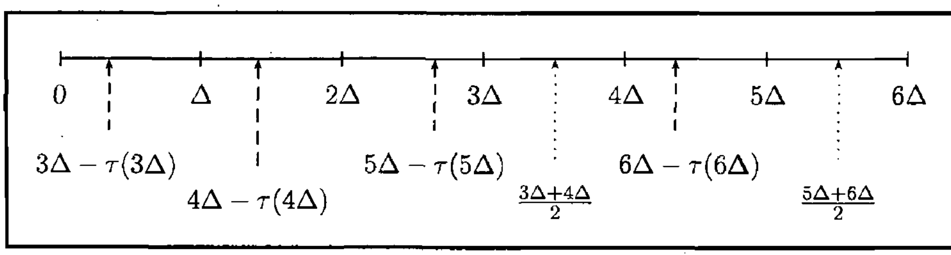


Figure 6.3.3: Addition of artificial separators where there are no feedback positions on the intervals $(3\Delta, 4\Delta)$ and $(5\Delta, 6\Delta)$.

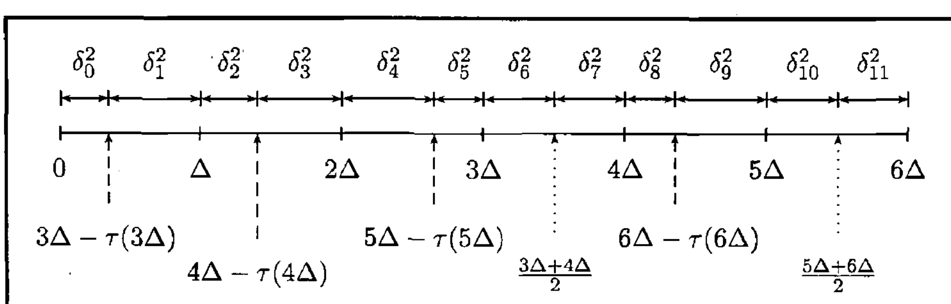


Figure 6.3.4: Labelling the lengths of the newly defined intervals.

Final construction of the difference equation. We introduce the functions $h, i, j : \mathbb{N}_0 \to \mathbb{N}_0$, where

$$\begin{aligned} h(n) &= 2n-1, \\ i(n) &= \begin{cases} 2(n-r(n)), & \tau(n\Delta)/\Delta \in \mathbb{N}_0, \\ 2(n-r(n))-1, & \text{otherwise}, \end{cases} \\ j(n) &= \begin{cases} 2(n-r(n)-1-r(n-r(n)-1)), & \tau((n-r(n)-1)\Delta)/\Delta \in \mathbb{N}_0, \\ 2(n-r(n)-1-r(n-r(n)-1))-1, & \text{otherwise}. \end{cases} \end{aligned}$$

For every $n < N^*$, there is a $\mathcal{G}^{\xi}_{2n}$–measurable random variable $\widetilde{P}_n$ with an identical distribution to that of P_n, defined to be

$$\widetilde{P}_n = \begin{cases} |b|e^{-\lambda\tau(n\Delta)}\zeta_{h(n)}, & n \geq N^*, \\ |b|e^{-\lambda\tau(n\Delta)}\zeta_{i(n)}\cdots\zeta_{h(n)}, & N_0 \leq n < N^*, \\ |b|e^{-\lambda n\Delta}\zeta_0\cdots\zeta_{h(n)}, & 0 < n < N_0, \\ |b|, & n = 0, \end{cases}$$

We consider the oscillatory behaviour of the sequence of random variables $\{\widetilde{Y}_n\}_{n\geq 0}$ obeying

$$\widetilde{Y}_{n+1} = \widetilde{Y}_n - \Delta\widetilde{P}_n\widetilde{Y}_{n-r(n)}, \tag{6.3.6}$$

with initial data (6.3.3) satisfying (6.3.4) and (6.3.5). By (6.3.6) and the definition of $\widetilde{P}_n$, each $\widetilde{Y}_n$ is $\mathcal{G}^{\xi}_{2(n-1)}$–measurable.

6.4 Main Result.

The main result in this section shows that, in many cases, the oscillatory behaviour of $\widetilde{Y}$ depends on the mesh size Δ. Figures 6.4.1 and 6.4.2 illustrate the statement of Theorem 6.4.1 with examples.

Theorem 6.4.1. *Let M_Δ be a uniform mesh. Then the behaviour of the discrete process defined by the solution of the difference equation (6.3.6) with initial data (6.3.3) obeying (6.3.4) and (6.3.5) can be classified as follows:*

1. *Let* $\lim_{t\to\infty} \tau(t)\log t = 0$. *If*

   (a) $\Delta > \frac{1}{|b|}$, *then* $\widetilde{Y}$ *is a.s. oscillatory.*

   (b) $\Delta < \frac{1}{|b|}$, *then* $\widetilde{Y}$ *is a.s. nonoscillatory.*

2. *Let* $\lim_{t\to\infty} \tau(t)\log t = c > 0$. *If*

   (a) $\Delta > \frac{1}{|b|e^{|\sigma|\sqrt{2c}}}$, *then* $\widetilde{Y}$ *is a.s. oscillatory.*

   (b) $\Delta < \frac{1}{|b|e^{|\sigma|\sqrt{2c}}}$, *then* $\widetilde{Y}$ *is a.s. nonoscillatory.*

3. *Let* $\lim_{t\to\infty} \tau(t)\log t = \infty$. *Then* $\widetilde{Y}$ *is a.s. oscillatory.*

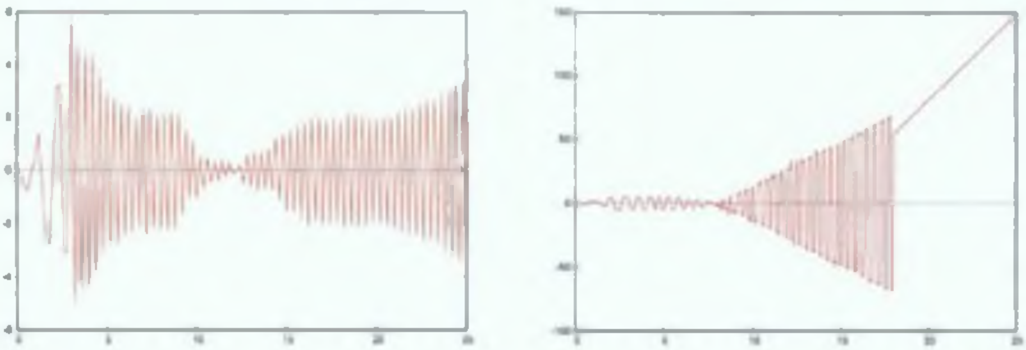
Note the dependence on the mesh size in Cases 1 and 2. It is only when the rate of decay of τ is very slow, in Case 3, that the oscillatory behaviour of this difference equation corresponds to that of the solutions of (5.0.1).

Proof. By Lemma 6.5.7, the quotient $\frac{\widetilde{Y}_{n+1}}{\widetilde{Y}_n}$ exists for all $n \geq 0$. By the definitions of the delay functions τ and r, there exists $N^* < \infty$ large enough that, for all $n \geq N^*$, $r(n) = 0$. Therefore, for all $n \geq N^*$, (6.3.1) can be rewritten as

$$\frac{\widetilde{Y}_{n+1}}{\widetilde{Y}_n} = 1 - \Delta \widetilde{P}_n.$$

Therefore, in order to prove that $\widetilde{Y}$ is a.s. oscillatory, it is enough to show that $\mathbb{P}[\widetilde{P}_n > \frac{1}{\Delta} \text{ i.o.}] = 1$. Similarly, in order to prove that $\widetilde{Y}$ is a.s. nonoscillatory, it is enough to show that $\mathbb{P}[\widetilde{P}_n < \frac{1}{\Delta} \text{ ev.}] = 1$. Define $\vartheta_n = \xi_{h(n)}$ for all $n \geq N^*$, and consider that, since $\tau(t) \to 0$,

$$\begin{aligned}
\limsup_{n\to\infty} \widetilde{P}_n &= |b| \limsup_{n\to\infty} e^{-\lambda\tau(n\Delta)}\zeta_{h(n)}, \\
&= |b| e^{|\sigma| \limsup_{n\to\infty} \delta_{h(n)}\vartheta_n}, \\
&= |b| e^{|\sigma| \limsup_{n\to\infty} \sqrt{\tau(n\Delta)}\vartheta_n}, \\
&= |b| e^{|\sigma| \limsup_{n\to\infty} \sqrt{2\tau(n\Delta)\log n}\frac{\vartheta_n}{\sqrt{2\log n}}},
\end{aligned}$$



(a) (b)

Figure 6.4.1: **Spurious behaviour. Case 1:** Simulations of $\text{sgn}(\tilde{Y})\log|\tilde{Y}|$, where $\tilde{Y}$ is the solution of (6.3.6), with $a = 1$, $b = -10$, $\sigma = 1$, and $\tau(t) = 1/(t+2)$. In **(a)**, a **large mesh size** of $\Delta = 0.2 > \frac{1}{|b|}$ yields spurious oscillatory behaviour. In **(b)**, a **small mesh size** of $\Delta = 0.05 < \frac{1}{|b|}$ results in nonoscillation. Observe that in **(a)** and **(b)**, qualitative behavioural changes are visible at $t = 3$ and $t = 18$ respectively. These are the times at which the delay length drops below the mesh size.

Since $\{\vartheta_n\}_{n\geq N^*}$ is a sequence of independent, standard Normal random variables, we know that $\limsup_{n\to\infty}\frac{\vartheta_n}{\sqrt{2\log n}} = 1$, a.s., and therefore we need only consider the asymptotic behaviour of $\sqrt{2\tau(n\Delta)\log n}$ under the conditions laid out in the statement of the theorem.

Case 1: Where $\lim_{n\to\infty}\tau(n\Delta)\log n\Delta = 0$. Since

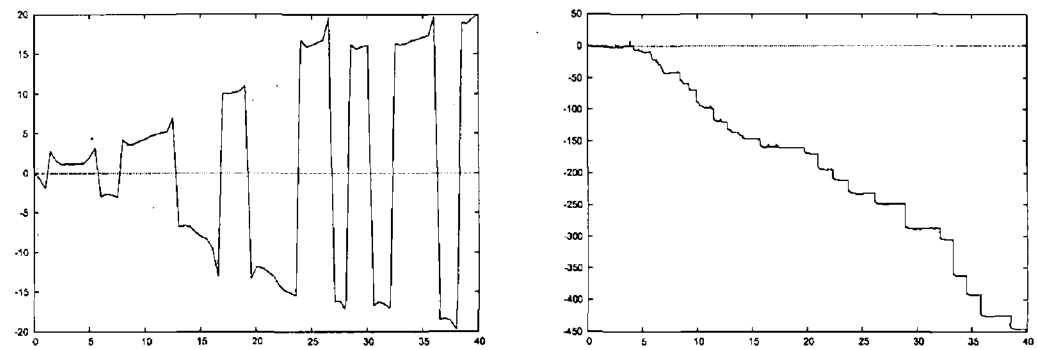
$$\tau(n\Delta)\log n\Delta = \tau(n\Delta)\log n + \tau(n\Delta)\log\Delta,$$

it follows that $\limsup_{n\to\infty}\tau(n\Delta)\log n = 0$. Thus

$$\limsup_{n\to\infty}\tilde{P}_n = |b|, \quad a.s.$$

Therefore, if $\Delta > \frac{1}{|b|}$, $\tilde{Y}$ is a.s. oscillatory, and if $\Delta < \frac{1}{|b|}$, $\tilde{Y}$ is a.s. nonoscilatory.

Case 2: We have $\lim_{n\to\infty}\tau(n\Delta)\log n\Delta = c > 0$. As in Case 1, this implies that



(a) (b)

Figure 6.4.2: **Spurious behaviour. Case 2:** Simulations of $\text{sgn}(\widetilde{Y})\log|\widetilde{Y}|$, where $\widetilde{Y}$ is the solution of (6.3.6), with $a = 1$, $b = -1$, $\sigma = 1$, and $\tau(t) = 1/\log\sqrt{t+3}$, so that $c = 2$. In **(a)**, a **large mesh size** of $\Delta = 0.5 > \frac{1}{|b|e^{\sigma}\sqrt{2c}}$ yields spurious oscillatory behaviour. In **(b)**, a **small mesh size** of $\Delta = 0.005 < \frac{1}{|b|e^{\sigma}\sqrt{2c}}$ results in nonoscillation.

$\lim_{n\to\infty} \tau(n\Delta)\log n = c$. Thus

$$\limsup_{n\to\infty} \widetilde{P}_n = |b|e^{|\sigma|\sqrt{2c}}, \quad a.s..$$

Therefore, if $\Delta > \frac{1}{|b|e^{|\sigma|\sqrt{2c}}}$, $\widetilde{Y}$ is a.s. oscillatory, and if $\Delta < \frac{1}{|b|e^{|\sigma|\sqrt{2c}}}$, $\widetilde{Y}$ is a.s. nonoscillatory.

Case 3: Let $\lim_{n\to\infty} \tau(n\Delta)\log n\Delta = \infty$. Letting $c \to \infty$ in Case 2 implies that $\widetilde{Y}$ is a.s. oscillatory. $\square$

6.5 Technical Lemmata.

6.5.1 Motivating discussion.

The reason for the structure of the filtration $\{\mathcal{G}_k^{\xi}\}_{k=0}^{\infty}$ may not be immediately clear. Neither may it be clear why the sequence $\{\widetilde{Y}\}$ must be considered, rather than $\{Y\}$. These seemingly arbitrary constructions appear unnecessary at first glance, simply because we

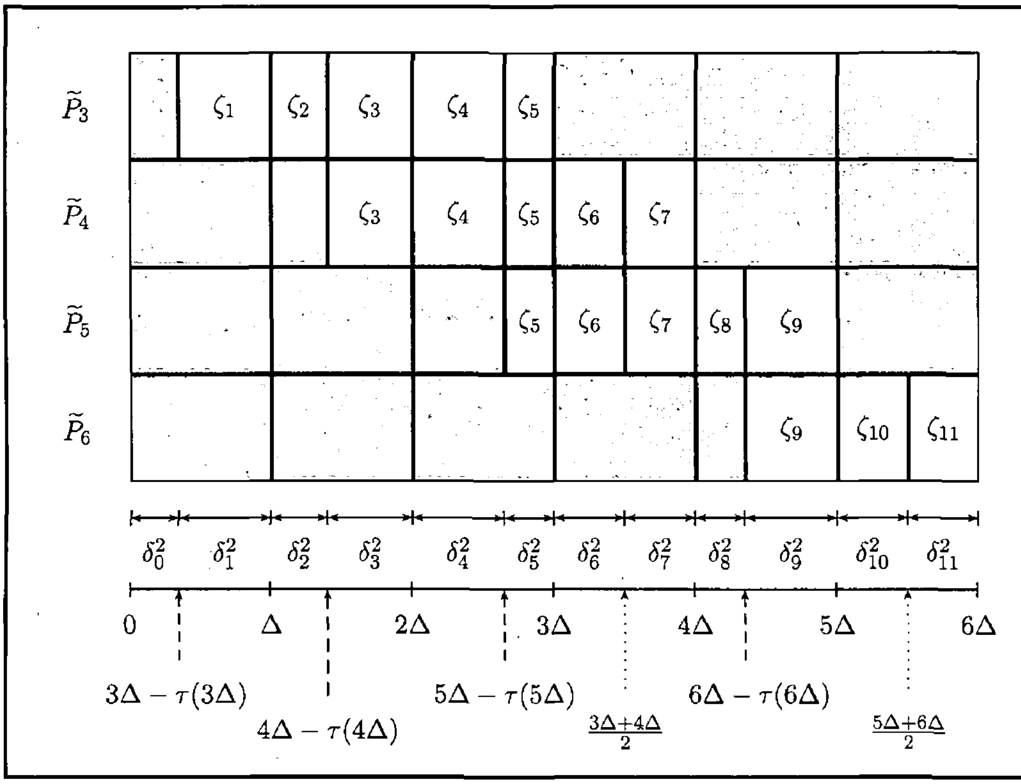


Figure 6.5.1: A representation of the decomposition of the nonindependent distributions of $\widetilde{P}_3$, $\widetilde{P}_4$, $\widetilde{P}_5$, and $\widetilde{P}_6$, into functions of independent Lognormal random variables.

---

have not yet demonstrated an explicit use for them. However, one important question remains, and the rest of this chapter is devoted to answering it. Can Definition 6.2.1 be applied to the sequence $\{\widetilde{Y}_n\}_{n\geq 0}$ that obeys (6.3.6)?

The main barrier to analysis here is the lack of independence between each Y_n, P_n, and $Y_{n-\tau(n)}$on the right hand side of (6.3.1). P_n depends on an increment of Brownian motion longer than Δ. Therefore, each successive P_n is not independent of its predecessors. Neither is P_n independent of Y_n. These dependencies must be explicitly handled. Thus we consider the distribution of the Brownian increment following its subdivision into a sequence of independent standard Normal random variables.

It can be seen in Figures 6.3.2 to 6.3.4 that although the mesh itself imposes a natural partition on the Brownian increment, we must go further, splitting each subdivision of the Brownian increment of length Δ into two smaller subdivisions. This is necessary because, in general, $n\Delta - \tau(n\Delta)$ is not a multiple of Δ, and thus the mesh M_Δ is not sufficient to define a sequence of *independent* Normal random variables that will allow us to analyse the dependencies of the components of (6.3.1). Consider also that (6.1.2) places no upper limit on the rate at which τ can converge to zero, and therefore the delay function may 'jump' across mesh intervals. A need to define $\{\mathcal{G}_k^\xi\}_{k=0}^\infty$ precisely, in spite of these considerations, determines the structure of the filtration. The sequence $\{\widetilde{Y}_n\}_{n\geq 0}$ is then simply an expression of the solutions of (6.3.1) evolving on the new filtration.

We can explicitly show the dependencies between $\widetilde{Y}_n$, $\widetilde{P}_n$, and $\widetilde{Y}_{n-\tau(n)}$ with the example illustrated in Figure 6.5.1. This figure gives a schematic representation of the dependence of each random variable $\widetilde{P}_k$, for $N_0 \geq k \leq \widetilde{N}$, on Lognormal random variables, for some arbitrary delay function τ. For instance, let $n = 5$. The term $\widetilde{Y}_5$ can be explicitly written in terms of the sequence of Lognormal random variables $\{\zeta_i\}_{i\geq 0}$ by iterating (6.3.6) back to the zero–th term. We know from (6.3.6) and Figure 6.5.1 that

$$\widetilde{Y}_5 = \widetilde{Y}_4 - \Delta \widetilde{P}_4 \widetilde{Y}_2.$$

Clearly $\widetilde{Y}_4$ depends on $\{\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\}$, $\widetilde{P}_4$ depends on $\{\zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_7\}$, and $\widetilde{Y}_2$ depends on $\{\zeta_0, \zeta_1\}$. Although $\widetilde{Y}_4$, $\widetilde{P}_4$, and $\widetilde{Y}_2$ are not fully independent, we can see that $\widetilde{Y}_4$ depends on $\{\zeta_2\}$, whereas $\widetilde{P}_4$ and $\widetilde{Y}_2$ do not, and $\widetilde{P}_4$ depends on $\{\zeta_6, \zeta_7\}$, whereas $\widetilde{Y}_4$ and $\widetilde{Y}_2$ do not. It is possible to use this partial independence to prove that $\widetilde{Y}_5 \neq 0$ a.s. In fact, if we can prove that it is a characteristic of all terms of the sequence defined by (6.3.6), then we can prove that all terms are nonzero almost surely. This provides the motivation for the analysis that remains in this chapter.

6.5.2 Analysis.

The following lemma is an application of Jacobi's Transformation Formula [16] to the density functions of random variables.

Lemma 6.5.1. *Let $X = (X_0, \ldots, X_N)$ have joint density f. Let $g : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ be continuously differentiable and injective, with nonvanishing Jacobian. Then $Y = g(X)$*

has density

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y))|\det(J_{g^{-1}(y)})|, & \textit{if } y \textit{ is in the range of } g, \\ 0, & \textit{otherwise.} \end{cases}$$

Lemma 6.5.2. *Suppose that the independent random variables $\eta_0, \eta_1, \dots, \eta_N$ have joint density $f_{(\eta_0,\dots,\eta_N)}$, and*

$$X = h(\eta_0, \eta_1, \dots, \eta_N) := h_1(\eta_0, \dots, \eta_{N-1}) + h_2(\eta_0, \dots, \eta_{N-1})\eta_N.$$

Define

$$S_0 := \{(y_0, y_2, \dots, y_N) : h_2(y_0, \dots, y_{N-1}) = 0,\ y_N \in \mathbb{R}^+\}. \tag{6.5.1}$$

If S_0 has measure 0 in $\mathbb{R}^{N+1}$, then

1. *X has density f_X defined by*

$$f_X(x) = \int_{y_0}\int_{y_1}\cdots\int_{y_{N-1}} f_{(\eta_0,\eta_1,\dots,\eta_N)}\left(y_0, y_1, \dots, y_{N-1}, \frac{x - h_1(y_0, y_1, \dots y_{N-1})}{h_2(y_0, y_1, \dots, y_{N-1})}\right) \times \frac{1}{|h_2(y_0, y_1, \dots, y_{N-1})|} dy_{N-1}dy_{N-2}\cdots dy_0.$$

2. *$\mathbb{P}[X = 0] = 0$.*

Proof. Part 1 relies on Lemma 6.5.1. Define the continuous function $g : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ to be

$$g(y_0, y_1, \dots, y_N) = (y_0, y_1, \dots, y_{N-1}, h(y_0, \dots, y_N)).$$

Thus, if $(y_0, y_1, \dots, y_N) \notin S_0$, then g is injective in its last argument, and the inverse is given by

$$g^{-1}(y_0, y_1, \dots, y_N) = \left(y_0, y_1, \dots, y_{N-1}, \frac{y_N - h_1(y_0, \dots, y_{N-1})}{h_2(y_0, \dots, y_{N-1})}\right).$$

The Jacobian of g^{-1} is given by

$$J_{g^{-1}}(y_0, y_1, \ldots, y_N) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{h_2(y_0,\ldots,y_{N-1})} \end{pmatrix}$$

Thus, $\det(J_{g^{-1}}(y_0, \ldots, y_N)) = \frac{1}{h_2(y_0,\ldots,y_{N-1})}$, and so, by Lemma 6.5.1,

$$f_{(\eta_0,\ldots,\eta_N)}(y_0, \ldots, y_N) = \begin{cases} \dfrac{f_{(\eta_0,\ldots,\eta_N)}(y_0,\ldots,\frac{y_N - h_1(y_0,\ldots,y_{N-1})}{h_2(y_0,\ldots,y_{N-1})})}{h_2(y_0,\ldots,y_{N-1})}, & h_2(y_0, \ldots, y_{N-1}) \neq 0, \\ 0, & h_2(y_0, \ldots, y_{N-1}) = 0. \end{cases}$$

Thus

$$f_X(x) = \int_{y_0} \int_{y_1} \cdots \int_{y_{N-1}} f_{(\eta_0,\ldots,\eta_N)}\left(y_0, \ldots, y_{N-1}, \frac{x - h_1(y_0, y_1, \ldots y_{N-1})}{h_2(y_0, y_1, \ldots, y_{N-1})}\right) \times \frac{1}{|h_2(y_0, y_1, \ldots, y_{N-1})|} dy_{N-1} dy_{N-2} \cdots dy_0.$$

Part 2 is an immediate consequence of the existence of a continuous density function as established in Part 1, since

$$\mathbb{P}[X = 0] = \mathbb{P}\left[\lim_{a \to 0^+} \int_{-a}^{a} f_X(s) ds = 0\right] = 0.$$

□

Note that, by (6.1.2) and (6.3.2), there exists a constant $N_0 < N_1 \leq N^*$ defined to be

$$N_1 = \inf\{n > N_0 : n - r(n) - 1 - r(n - r(n) - 1) > 0\}. \tag{6.5.2}$$

In order to account for varying degrees of dependence on the initial data, it turns out that it is actually necessary to prove that $\mathbb{P}[\widetilde{Y}_n = 0] = 0$ separately over $0 < n \leq N_0$, $N_0 < n \leq N_1$, $N_1 < n < N^*$, and $n \geq N^*$.

Preliminary sequences of functions. Given any sequence of positive real numbers $\{x_k\}_{k=0}^{h(N^*)}$ we can define the following interdependent sequences of functions. Note that where required in parts 1–4, the sequence $\{\tilde{y}_k^{\psi}\}$ is defined over (6.5.12)–(6.5.17) in part 5.

1. For each $2 \le k \le N_0$, define the function $q_k : \mathbb{R}^{h(k-1)+1} \to \mathbb{R}$ so that

$$q_k(x_0, \dots, x_{h(k-1)}) = -\Delta|b|e^{-\lambda k\Delta} x_0 \cdots x_{h(k-1)}\psi_k. \tag{6.5.3}$$

If $N_0 - r(N_0) = 1$, then further define the function $q_{N_0+1} : \mathbb{R}^{h(N_0)+1} \to \mathbb{R}$ so that

$$q_{N_0}(x_0, \dots, x_{h(N_0-1)}) = -\Delta|b|e^{-\lambda N_0\Delta} x_0 \cdots x_{h(N_0-1)}\tilde{y}_1^{\psi}. \tag{6.5.4}$$

By (6.5.3) and (6.5.4) for each $2 \le k \le N_0 + 1$, the surface

$$S_1^k := \{(x_0, \dots, x_{h(k)}) : q_k(x_0, \dots, x_{h(k-1)}) = 0,\ x_{h(k)-1}x_{h(k)} \in \mathbb{R}^+\} \tag{6.5.5}$$

has measure 0 in $\mathbb{R}^{h(k)+1}$.

2. For each $N_0 + 1 \le k < N^*$, define the function $f_k : \mathbb{R}^{h(k-1)-2} \to \mathbb{R}$ so that

$$f_k(x_0, \dots, x_{i(k)-4}, x_{i(k)}, \dots, x_{h(k-1)})$$
$$= -\Delta\tilde{y}_{k-r(k)-1}^{\psi}(x_0, \dots, x_{i(k)-4})|b|e^{-\lambda k\Delta} x_{i(k)} \cdots x_{h(k)}. \tag{6.5.6}$$

3. For each $N_0 + 1 \le k \le N_1$, define the function $g_k : \mathbb{R}^{h(k-1)} \to \mathbb{R}$ so that

$$g_k(x_0, \dots, x_{i(k)-2}, x_{i(k)}, \dots, x_{h(k-1)})$$
$$= -\Delta(-\Delta)|b|e^{-\lambda k\Delta} x_0 \cdots x_{i(k)-2}|b|e^{-\lambda\tau(k\Delta)} x_{i(k)} \cdots x_{h(k)}\psi_{k-r(k)-1}. \tag{6.5.7}$$

If $N_0 - r(N_0) = 1$ and $N_1 - r(N_1) = N_0 + 1$ then further define the function $g_{N_1} : \mathbb{R}^{h(N_1-1)} \to \mathbb{R}$ so that

$$g_{N_1}(x_0, \dots, x_{i(N_1)-2}, x_{i(N_1)}, \dots, x_{h(N_1-1)})$$
$$= -\Delta(-\Delta)|b|e^{-\lambda N_1\Delta} x_0 \cdots x_{i(N_1)-2}|b|e^{-\lambda\tau(N_1\Delta)} x_{i(k)} \cdots x_{h(N_1)}\tilde{y}_1^{\psi}. \tag{6.5.8}$$

By (6.5.6), (6.5.7), and (6.5.8) for each $N_0 + 1 \le k \le N_1 + 1$, the surface

$$S_2^k := \{(x_0, \dots, x_{h(k)}) : f_k(x_0, \dots, x_{h(k-1)})$$
$$+ g_k(x_0, \dots, x_{h(k-1)}) = 0,\ x_{h(k)-1}x_{h(k)} \in \mathbb{R}^+\} \tag{6.5.9}$$

has measure 0 in $\mathbb{R}^{h(k)+1}$.

4. For each $N_1 + 1 \leq k < N^*$, define the function $w_k : \mathbb{R}^{h(k-1)-1} \to \mathbb{R}$ so that

$$w_k(x_0, \ldots, x_{j(k)-2}, x_{j(k)}, \ldots, x_{i(k)-2}, x_{i(k)}, \ldots, x_{h(k-1)})$$
$$= -\Delta(-\Delta)|b|e^{-\lambda k\Delta}x_0 \cdots x_{i(k)-2}|b|e^{-\lambda\tau(k\Delta)}x_{i(k)} \cdots x_{h(k)}$$
$$\times \tilde{y}^{\psi}_{k-r(k)-1-r(k-r(k)-1)}(x_0, \ldots, x_{j(k)-2}). \quad (6.5.10)$$

By (6.5.6) and (6.5.10) for each $N_1 + 1 \leq k < N^*$, the surface

$$S_3^k := \{(x_0, \ldots, x_{h(k)}) : f_k(x_0, \ldots, x_{h(k-1)})$$
$$+ k_k(x_0, \ldots, x_{h(k-1)}) = 0,\ x_{h(k)-1}x_{h(k)} \in \mathbb{R}^+\} \quad (6.5.11)$$

has measure 0 in $\mathbb{R}^{h(k)+1}$.

5. Now define the sequence of functions $\{\tilde{y}^{\psi}_k\}_{k=2}^{N^*-1}$ recursively so that

$$\tilde{y}^{\psi}_1 = \widetilde{Y}_0 - \Delta|b|\psi_0, \quad (6.5.12)$$

and

(a) For $2 \leq k < N_0$,

$$\tilde{y}^{\psi}_{k+1}(x_0, \ldots, x_{h(k)-2}, x_{h(k)-1}x_{h(k)})$$
$$= \tilde{y}^{\psi}_k(x_0, \ldots, x_{h(k-1)-2}, x_{h(k-1)-1}x_{h(k-1)})$$
$$+ q_k(x_0, \ldots, x_{h(k-1)})x_{h(k)-1}x_{h(k)}. \quad (6.5.13)$$

(b) If $N_0 - r(N_0) = 1$ then,

$$\tilde{y}^{\psi}_{N_0+1}(x_0, \ldots, x_{h(N_0)-2}, x_{h(N_0)-1}x_{h(N_0)})$$
$$= \tilde{y}_{N_0}(x_0, \ldots, x_{h(N_0-1)-2}, x_{h(N_0-1)-1}x_{h(N_0-1)})$$
$$+ q_{N_0}(x_0, \ldots, x_{h(N_0-1)})x_{h(N_0)-1}x_{h(N_0)} \quad (6.5.14)$$

and for $N_0 + 2 \leq k < N_1$,

$$\tilde{y}^{\psi}_{k+1}(x_0, \dots, x_{h(k)-2}, x_{h(k)-1}x_{h(k)})$$
$$= \tilde{y}^{\psi}_{k}(x_0, \dots, x_{h(k-1)-2}, x_{h(k-1)-1}x_{h(k-1)})$$
$$+ [f_k(x_0, \dots, x_{i(k)-4}, x_{i(k)}, \dots, x_{h(k-1)})$$
$$+ g_k(x_0, \dots, x_{i(k)-2}, x_{i(k)}, \dots, x_{h(k-1)})]x_{h(k)-1}x_{h(k)}. \quad (6.5.15)$$

Otherwise $\tilde{y}^{\psi}_{k}$ satisfies (6.5.15) for $N_0 + 1 \leq k < N_1$.

(c) If $N_0 - r(N_0) = 1$ and $N_1 - r(N_1) = N_0 + 1$, then,

$$\tilde{y}^{\psi}_{N_1+1}(x_0, \dots, x_{h(N_1)-2}, x_{h(N_1)-1}x_{h(N_1)})$$
$$= \tilde{y}^{\psi}_{N_1}(x_0, \dots, x_{h(N_1-1)-2}, x_{h(N_1-1)-1}x_{h(N_1-1)})$$
$$+ [f_{N_1}(x_0, \dots, x_{i(N_1)-4}, x_{i(N_1),\dots,x_{h(N_1)}})$$
$$+ g_{N_1}(x_0, \dots, x_{i(N_1)-2}, x_{i(N_1)}, \dots, x_{h(N_1)})]$$
$$\times x_{h(N_1)-1}x_{h(N_1)}, \quad (6.5.16)$$

and for $N_1 + 2 \leq k < N^* - 1$,

$$\tilde{y}^{\psi}_{k+1}(x_0, \dots, x_{h(k)-2}, x_{h(k)-1}x_{h(k)})$$
$$= \tilde{y}^{\psi}_{k}(x_0, \dots, x_{h(k-1)-2}, x_{h(k-1)-1}x_{h(k-1)})$$
$$+ [f_k(x_0, \dots, x_{i(k)-4}, x_{i(k)}, \dots, x_{h(k-1)})$$
$$+ w_k(x_0, \dots, x_{j(k)-2}, x_{j(k)}, \dots, x_{i(k)-2}, x_{i(k)}, \dots, x_{h(k-1)})]$$
$$\times x_{h(k)-1}x_{h(k)}. \quad (6.5.17)$$

Otherwise $\tilde{y}^{\psi}_{k}$ satisfies (6.5.17) for $N_1 + 1 \leq k < N^* - 1$.

Lemma 6.5.3. *If $\{\widetilde{Y}_n\}_{n \geq 0}$ is the sequence of random variables defined by (6.3.6), and with an initial data set (6.3.3) obeying (6.3.4) and (6.3.5), then for all $0 < n \leq N_0$,* *1.* $\widetilde{Y}_n = \tilde{y}_n^{\psi}(\zeta_0, \ldots, \zeta_{h(n-1)-2}, \zeta_{h(n-1)-1}\zeta_{h(n-1)})$,

2. $\mathbb{P}[\widetilde{Y}_n = 0] = 0$.

Proof. For every $0 < n \le N_0$,

$$\widetilde{P}_n = |b|e^{-\lambda n\Delta}\zeta_0 \cdots \zeta_{h(n)}.$$

Note first that $\widetilde{Y}_1 = \widetilde{Y}_0 - \Delta|b|\widetilde{Y}_{0-r(0)} = \widetilde{Y}_0 - \Delta|b|\psi_0$. By (6.3.4) and (6.3.5), $\widetilde{Y}_1 \neq 0$.

We proceed by induction. Assume that, for $2 \le k < N_0$,

$$\widetilde{Y}_k = \tilde{y}_k^{\psi}(\zeta_0, \ldots, \zeta_{h(k-1)-2}, \zeta_{h(k-1)-1}\zeta_{h(k-1)}),$$

Since $k < N_0$, $\widetilde{Y}_{k-r(k)} = \psi_k$. Thus

$$\begin{aligned} \widetilde{Y}_{k+1} &= \widetilde{Y}_k - \Delta\widetilde{P}_k\widetilde{Y}_{k-r(k)} \\ &= \tilde{y}_k^{\psi}(\zeta_0, \ldots, \zeta_{h(k-1)-2}, \zeta_{h(k-1)-1}\zeta_{h(k-1)}) - \Delta|b|e^{-\lambda k\Delta}\zeta_0 \cdots \zeta_{h(k)}\psi_k. \end{aligned}$$

Since $k < N_0$, $\widetilde{Y}_{k-r(k)} \in \psi$. Therefore, by (6.5.3) and (6.5.13),

$$\begin{aligned} \widetilde{Y}_{k+1} &= \tilde{y}_k^{\psi}(\zeta_0, \ldots, \zeta_{h(k-1)-2}, \zeta_{h(k-1)-1}\zeta_{h(k-1)}) \\ &\qquad + q_k(\zeta_0, \ldots, \zeta_{h(k-1)})\zeta_{h(k)-1}\zeta_{h(k)}, \\ &= \tilde{y}_{k+1}^{\psi}(\zeta_0, \ldots, \zeta_{h(k)-2}, \zeta_{h(k)-1}\zeta_{h(k)}). \end{aligned}$$

We now consider the base case when $k = 2$. By (6.5.12),

$$\begin{aligned} \widetilde{Y}_2 &= \widetilde{Y}_1 - \Delta\widetilde{P}_1\widetilde{Y}_{1-r(1)}, \\ &= \tilde{y}_1^{\psi} - \Delta|b|e^{n\Delta}\zeta_0\zeta_1\psi_1, \\ &= \tilde{y}_2^{\psi}(\zeta_0, \zeta_1). \end{aligned}$$

Therefore, by induction, for all $0 < n \le N_0$,

$$\widetilde{Y}_n = \tilde{y}_n^{\psi}(\zeta_0, \ldots, \zeta_{h(n-1)-2}, \zeta_{h(n-1)-1}\zeta_{h(n-1)}). \tag{6.5.18}$$

Since the surface S_1^n defined in (6.5.5) has measure 0 in $\mathbb{R}^{h(n)+1}$, Lemma 6.5.2 gives that $\mathbb{P}[\widetilde{Y}_n = 0] = 0$ for $0 < n \le N_0$, where q_{n-1} plays the role of h_2 in Lemma 6.5.2. $\square$

Lemma 6.5.4. *If $\{\widetilde{Y}_n\}_{n\ge 0}$ is the sequence of random variables defined by (6.3.6), and with an initial data set (6.3.3) obeying (6.3.4) and (6.3.5), then for all $N_0 < n < N_1$,*

1. $\widetilde{Y}_n = \tilde{y}_n^{\psi}(\zeta_0, \dots, \zeta_{h(n-1)-2}, \zeta_{h(n-1)-1}\zeta_{h(n-1)})$,
2. $\mathbb{P}[\widetilde{Y}_n = 0] = 0$.

Proof. For every $N_0 < n \le N_1$,

$$\widetilde{P}_n = |b|e^{-\lambda\tau(n\Delta)}\zeta_{i(n)} \cdots \zeta_{h(n)}.$$

Additionally, $n - r(n) - 1 < N_0$, and therefore $\widetilde{Y}_{n-r(n)-1-r(n-r(n)-1)} \in \psi$. Iterating once, we can say that

$$\begin{aligned}
\widetilde{Y}_{n+1} &= Y_n - \Delta\widetilde{P}_n\widetilde{Y}_{n-r(n)} \\
&= \widetilde{Y}_n - \Delta\widetilde{P}_n\widetilde{Y}_{n-r(n)-1} - \Delta(-\Delta)\widetilde{P}_n\widetilde{P}_{n-r(n)-1}\widetilde{Y}_{n-r(n)-1-r(n-r(n)-1)}.
\end{aligned}$$

By (6.5.18) in Lemma 6.5.3, for all $N_0 < n \le N_1$,

$$\begin{aligned}
\widetilde{Y}_{n-r(n)-1} &= \tilde{y}^{\psi}_{n-r(n)-1}(\zeta_0, \dots, \zeta_{h(n-r(n)-1-1)-1}\zeta_{h(n-r(n)-1-1)}), \\
&= \tilde{y}^{\psi}_{n-r(n)-1}(\zeta_0, \dots, \zeta_{i(n)-6}, \zeta_{i(n)-5}\zeta_{i(n)-4}),
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{P}_{n-r(n)-1} &= |b|e^{-\lambda n\Delta}\zeta_0 \cdots \zeta_{h(n-r(n)-1)}, \\
&= |b|e^{-\lambda n\Delta}\zeta_0 \cdots \zeta_{i(n)-2}.
\end{aligned}$$

For $N_0 + 1 < n \le N_1$, we proceed by induction. Assume that

$$\widetilde{Y}_k = \tilde{y}_k^{\psi}(\zeta_0, \dots, \zeta_{h(k-1)-2}, \zeta_{h(k-1)-1}\zeta_{h(k-1)}). \tag{6.5.19}$$

Then, by (6.5.6), (6.5.7), and (6.5.15),

$$
\begin{aligned}
&\widetilde{Y}_{k+1} \\
&= \tilde{y}_k^{\psi}(\zeta_0, \ldots, \zeta_{h(k-1)-2}, \zeta_{h(k-1)-1}\zeta_{h(k-1)}) \\
&\qquad +[-\Delta \tilde{y}^{\psi}_{k-r(k)-1}(\zeta_0, \ldots, \zeta_{i(k)-6}, \zeta_{i(k)-5}\zeta_{i(k)-4}) \\
&\qquad\qquad \times |b| e^{-\lambda k\Delta}\zeta_{i(k)}\cdots\zeta_{h(k)}] + [-\Delta(-\Delta)|b|e^{-\lambda k\Delta}\zeta_0\cdots\zeta_{i(k)-2} \\
&\qquad\qquad\qquad \times |b|e^{-\lambda\tau(k\Delta)}\zeta_{i(k)}\cdots\zeta_{h(k)}\widetilde{Y}_{k-r(k)-1-r(k-r(k)-1)}], \\
&= \tilde{y}_k^{\psi}(\zeta_0, \ldots, \zeta_{h(k-1)-2}, \zeta_{h(k-1)-1}\zeta_{h(k-1)}) \\
&\qquad +[f_k(\zeta_0, \ldots, \zeta_{i(k)-4}, \zeta_{i(k)}, \ldots, \zeta_{h(k-1)}) \\
&\qquad\qquad\qquad +g_k(\zeta_0, \ldots, \zeta_{i(k)-2}, \zeta_{i(k)}, \ldots, \zeta_{h(k-1)})]\zeta_{h(k)-1}\zeta_{h(k)}, \\
&= \tilde{y}^{\psi}_{k+1}(\zeta_0, \ldots, \zeta_{h(k)-2}, \zeta_{h(k)-1}\zeta_{h(k)}).
\end{aligned}
$$

We first determine the form of $\widetilde{Y}_{N_0+1}$. We can then show that the induction proposition (6.5.19) holds for the base case $\widetilde{Y}_{N_0+2}$. By (6.3.6),

$$\widetilde{Y}_{N_0+1} = \widetilde{Y}_{N_0} - \Delta \widetilde{P}_{N_0}\widetilde{Y}_{N_0-r(N_0)}$$

There are two possible forms for $\widetilde{Y}_{N_0+1}$.

Form 1: Let $N_0 - r(N_0) - 1 = 0$. Since $\widetilde{Y}_{N_0-r(N_0)} = \widetilde{Y}_1$, and, by Lemma 6.5.3, $\widetilde{Y}_{N_0} = \tilde{y}^{\psi}_{N_0}(\zeta_0, \ldots, \zeta_{h(N_0-1)})$, we have, by (6.5.4) and (6.5.14),

$$
\begin{aligned}
\widetilde{Y}_{N_0+1} &= \tilde{y}^{\psi}_{N_0}(\zeta_0, \ldots, \zeta_{h(N_0-1)}) - \Delta|b|e^{-\lambda\tau(N_0\Delta)}\zeta_{i(N_0)}\cdots\zeta_{h(N_0)}\tilde{y}^{\psi}_1, \\
&= \tilde{y}^{\psi}_{N_0}(\zeta_0, \ldots, \zeta_{h(N_0-1)-2}, \zeta_{h(N_0-1)-1}\zeta_{h(N_0-1)}) \\
&\qquad +q_{N_0}(\zeta_0, \ldots, \zeta_{h(N_0-1)})\zeta_{h(N_0)-1}\zeta_{h(N_0)}, \\
&= \tilde{y}^{\psi}_{N_0+1}(\zeta_0, \ldots, \zeta_{h(N_0)-2}, \zeta_{h(N_0)-1}\zeta_{h(N_0)}).
\end{aligned}
$$

Form 2: Let $N_0 - r(N_0) - 1 > 0$. From Lemma 6.5.3, we know that

$$\begin{aligned}
\widetilde{Y}_{N_0-r(N_0)} &= \tilde{y}^{\psi}_{N_0-r(N_0)} \\
&\qquad (\zeta_0, \dots, \zeta_{h(N_0-r(N_0)-1)-2}, \zeta_{h(N_0-r(N_0)-1)-1}\zeta_{h(N_0-r(N_0)-1)}), \\
&= \tilde{y}^{\psi}_{N_0-r(N_0)}(\zeta_0, \dots, \zeta_{i(N_0)-4}, \zeta_{i(N_0)-3}\zeta_{i(N_0)-2}).
\end{aligned}$$

So, by (6.5.6), (6.5.8), and (6.5.14),

$$\begin{aligned}
&\widetilde{Y}_{N_0+1} \\
&= \tilde{y}^{\psi}_{N_0}(\zeta_0, \dots, \zeta_{h(N_0-1)-2}, \zeta_{h(N_0-1)-1}\zeta_{h(N_0-1)}) \\
&\qquad +[-\Delta\tilde{y}^{\psi}_{N_0-r(N_0)-1}(\zeta_0, \dots, \zeta_{i(N_0)-6}, \zeta_{i(N_0)-5}\zeta_{i(N_0)-4}) \\
&\qquad\qquad \times|b|e^{-\lambda N_0\Delta}\zeta_{i(N_0)}\cdots\zeta_{h(N_0)}] \\
&\qquad\qquad\qquad +[-\Delta(-\Delta)|b|e^{-\lambda N_0\Delta}\zeta_0\cdots\zeta_{i(N_0)-2} \\
&\qquad\qquad\qquad\quad \times|b|e^{-\lambda\tau(N_0\Delta)}\zeta_{i(N_0)}\cdots\zeta_{h(N_0)} \\
&\qquad\qquad\qquad\qquad \times\widetilde{Y}_{N_0-r(N_0)-1-r(N_0-r(N_0)-1)}], \\
&= \tilde{y}^{\psi}_{N_0}(\zeta_0, \dots, \zeta_{h(N_0-1)-2}, \zeta_{h(N_0-1)-1}\zeta_{h(N_0-1)}) \\
&\qquad +[f_{N_0}(\zeta_0, \dots, \zeta_{i(N_0)-4}, \zeta_{i(N_0)}, \dots, \zeta_{h(N_0-1)}) \\
&\qquad\qquad +g_{N_0}(\zeta_0, \dots, \zeta_{i(N_0)-2}, \zeta_{i(N_0)}, \dots, \zeta_{h(N_0-1)})] \\
&\qquad\qquad\qquad \times\zeta_{h(N_0)-1}\zeta_{h(N_0)}, \\
&= \tilde{y}^{\psi}_{N_0+1}(\zeta_0, \dots, \zeta_{h(N_0)-2}, \zeta_{h(N_0)-1}\zeta_{h(N_0)}).
\end{aligned}$$

Now we can say, by (6.5.6), (6.5.7), and (6.5.15), that

$$\begin{aligned}
&\widetilde{Y}_{N_0+2} \\
&= \tilde{y}^{\psi}_{N_0+1}(\zeta_0,\dots,\zeta_{h(N_0)-2},\zeta_{h(N_0)-1}\zeta_{h(N_0)}) \\
&\quad +[-\Delta\tilde{y}^{\psi}_{N_0+1-r(N_0+1)-1}(\zeta_0,\dots,\zeta_{i(N_0+1)-6},\zeta_{i(N_0+1)-5}\zeta_{i(N_0+1)-4}) \\
&\qquad \times|b|e^{-\lambda(N_0+1)\Delta}\zeta_{i(N_0+1)}\cdots\zeta_{h(N_0+1)}] \\
&\qquad\quad +[-\Delta(-\Delta)|b|e^{-\lambda(N_0+1)\Delta}\zeta_0\cdots\zeta_{i(N_0+1)-2} \\
&\qquad\qquad \times|b|e^{-\lambda\tau((N_0+1)\Delta)}\zeta_{i(N_0+1)}\cdots\zeta_{h(N_0+1)} \\
&\qquad\qquad\quad \times\widetilde{Y}_{N_0+1-r(N_0+1)-1-r(N_0+1-r(N_0+1)-1)}], \\
&= \tilde{y}^{\psi}_{N_0+1}(\zeta_0,\dots,\zeta_{h(N_0)-2},\zeta_{h(N_0)-1}\zeta_{h(N_0)}) \\
&\qquad +[f_{N_0+1}(\zeta_0,\dots,\zeta_{i(N_0+1)-4},\zeta_{i(N_0+1)},\dots,\zeta_{h(N_0)}) \\
&\qquad\quad +g_{N_0+1}(\zeta_0,\dots,\zeta_{i(N_0+1)-2},\zeta_{i(N_0+1)},\dots,\zeta_{h(N_0)})] \\
&\qquad\qquad\qquad \times\zeta_{h(N_0+1)-1}\zeta_{h(N_0+1)}, \\
&= \tilde{y}^{\psi}_{N_0+2}(\zeta_0,\dots,\zeta_{h(N_0+1)-2},\zeta_{h(N_0+1)-1}\zeta_{h(N_0+1)}).
\end{aligned}$$

Therefore, for all $N_0 < n \le N_1$,

$$\widetilde{Y}_n = \tilde{y}^{\psi}_n(\zeta_0,\dots,\zeta_{h(n-1)-2},\zeta_{h(n-1)-1}\zeta_{h(n-1)}).$$

Since the surfaces S_1^n and S_2^n defined in (6.5.5) and (6.5.9) have measure 0 in $\mathbb{R}^{h(n)+1}$, Lemma 6.5.2 gives that $\mathbb{P}[\widetilde{Y}_n = 0] = 0$ for $N_0 < n \le N_1$. □

Lemma 6.5.5. *If $\{\widetilde{Y}_n\}_{n\ge0}$ is the sequence of random variables defined by (6.3.6), and with an initial data set (6.3.3) obeying (6.3.4) and (6.3.5), then for all $N_1 \le n < N^*$,*

1. $\widetilde{Y}_n = \tilde{y}^{\psi}_n(\zeta_0,\dots,\zeta_{h(n-1)-2},\zeta_{h(n-1)-1}\zeta_{h(n-1)})$,
2. $\mathbb{P}[\widetilde{Y}_n = 0] = 0$.

Proof. Iterating once, we can say that

$$\begin{aligned}
\widetilde{Y}_{n+1} &= Y_n - \Delta \widetilde{P}_n \widetilde{Y}_{n-r(n)} \\
&= \widetilde{Y}_n - \Delta \widetilde{P}_n \widetilde{Y}_{n-r(n)-1} - \Delta(-\Delta)\widetilde{P}_n \widetilde{P}_{n-r(n)-1} \widetilde{Y}_{n-r(n)-1-r(n-r(n)-1)},
\end{aligned}$$

For all $N_1 < n < N^*$,

$$\begin{aligned}
\widetilde{Y}_{n-r(n)-1} &= \tilde{y}^{\psi}_{n-r(n)-1}(\zeta_0, \dots, \zeta_{h(n-r(n)-1-1)}), \\
&= \tilde{y}^{\psi}_{n-r(n)-1}(\zeta_0, \dots, \zeta_{i(n)-4}), \\
\widetilde{Y}_{n-r(n)-1-r(n-r(n)-1)} &= \tilde{y}^{\psi}_{n-r(n)-1-r(n-r(n)-1)} \\
&\qquad (\zeta_0, \dots, \zeta_{h(n-r(n)-1-r(n-r(n)-1)-1)-2}, \\
&\qquad\quad \zeta_{h(n-r(n)-1-r(n-r(n)-1)-1)-1} \\
&\qquad\qquad \times \zeta_{h(n-r(n)-1-r(n-r(n)-1)-1)}), \\
&= \tilde{y}^{\psi}_{n-r(n)-1-r(n-r(n)-1)} \\
&\qquad (\zeta_0, \dots, \zeta_{j(n)-4}, \zeta_{j(n)-3}\zeta_{j(n)-2}), \\
\widetilde{P}_n &= |b| e^{-\lambda\tau(n\Delta)} \zeta_{i(n-r(n)-1)} \cdots \zeta_{h(n-r(n)-1)}, \\
&= |b| e^{-\lambda\tau(n\Delta)} \zeta_{j(n)} \cdots \zeta_{i(n)-2}.
\end{aligned}$$

If $N_0 - r(N_0) = 1$ and $N_1 - r(N_1) = N_0 + 1$ then, by (6.5.6) and (6.5.8), we can calculate that

$$\begin{aligned}
\widetilde{Y}_{N_1+1} &= \tilde{y}^{\psi}_{N_1}(\zeta_0, \dots, \zeta_{h(N_1-1)-2}, \zeta_{h(N_1-1)-1}\zeta_{h(N_1-1)}) \\
&\quad + [f_{N_1}(\zeta_0, \dots, \zeta_{i(N_1)-4}, \zeta_{i(N_1)}, \dots, \zeta_{h(N_1)}) \\
&\qquad + g_{N_1}(\zeta_0, \dots, \zeta_{i(N_1)-2}, \zeta_{i(N_1)}, \dots, \zeta_{h(N_1)})] \\
&\qquad\qquad \times \zeta_{h(N_1)-1}\zeta_{h(N_1)}, \\
&= \tilde{y}^{\psi}_{N_1+1}(\zeta_0, \dots, \zeta_{h(N_1)-2}, \zeta_{h(N_1)-1}\zeta_{h(N_1)}).
\end{aligned}$$

Just as in Lemmata 6.5.3 and 6.5.4, we can show, by induction, (6.5.6), (6.5.10), (6.5.17), and Lemma 6.5.4, that for all $N_1 < n < N^*$, and for $n = N_1 + 1$ if either $N_0 - r(N_0) \neq 1$ or $N_1 - r(N_1) \neq N_0 + 1$, that

$$\begin{aligned}
\widetilde{Y}_n &= \tilde{y}^{\psi}_{n-1}(\zeta_0, \dots, \zeta_{h(n-2)-2}, \zeta_{h(n-2)-1}\zeta_{h(n-2)}) \\
&\quad + [f_{n-1}(\zeta_0, \dots, \zeta_{i(n-1)-4}, \zeta_{i(n-1),\dots,\zeta_{h(n-1)}}) \\
&\qquad + w_{n-1}(\zeta_0, \dots, \zeta_{j(n-1)-2}, \zeta_{j(n-1)}, \dots, \zeta_{i(n-1)-2}, \zeta_{i(n-1)}, \dots, \zeta_{h(n-1)})] \\
&\qquad\qquad \times \zeta_{h(n-1)-1}\zeta_{h(n-1)}, \\
&= \tilde{y}^{\psi}_{n}(\zeta_0, \dots, \zeta_{h(n-1)-2}, \zeta_{h(n-1)-1}\zeta_{h(n-1)}).
\end{aligned}$$

So, for $N_1 < n < N^*$,

$$\widetilde{Y}_n = \tilde{y}^{\psi}_{n}(\zeta_0, \dots, \zeta_{h(n-1)-2}, \zeta_{h(n-1)-1}\zeta_{h(n-1)}).$$

Since the surfaces S_2^n and S_3^n defined in (6.5.9) and (6.5.11) have measure 0 in $\mathbb{R}^{h(k)+1}$, Lemma 6.5.2 gives that $\mathbb{P}[\widetilde{Y}_n = 0] = 0$ for $N_1 < n < N^*$. □

Lemma 6.5.6. *If $\{\widetilde{Y}_n\}_{n\geq 0}$ is the sequence of random variables defined by (6.3.6), and with an initial data set (6.3.3) obeying (6.3.4) and (6.3.5), then for all $n > N^*$,*

$$\mathbb{P}[\widetilde{Y}_n = 0] = 0.$$

Proof. When $n \geq N^*$, $\tau(n\Delta) < \Delta$, and therefore $r_{M_\Delta}(n) = 0$. Additionally, by Lemma 6.5.5, $\widetilde{Y}_{N^*-1} \neq 0$ a.s. So (6.3.1) can be rewritten as

$$\begin{aligned}
\widetilde{Y}_{n+1} &= \widetilde{Y}_n(1 - \Delta\widetilde{P}_n), \quad n \geq N^*, \\
\widetilde{Y}_{N^*-1} &\neq 0,
\end{aligned}$$

where $\widetilde{P}_n = |b|e^{-\lambda\tau(n\Delta)}\zeta_{h(n)}$.

Since $\widetilde{P}_n$ is independently distributed for all $n \geq N^*$, each $\widetilde{P}_n$ is $\mathcal{G}^{\xi}_{h(n)}$–measurable and $\mathcal{G}^{\xi}_{h(n-1)}$–independent. $\widetilde{Y}_{n+1}$ is $\mathcal{G}^{\xi}_{h(n)}$–measurable for all n and therefore $\widetilde{Y}_n$ is $\mathcal{G}^{\xi}_{h(n-1)}$–measurable. Thus $\widetilde{P}_n$ is independent of $\widetilde{Y}_n$ for all $n \geq N^*$.

Define the complementary events $A_n = \{\omega : Y_n(\omega) = 0\}$, and $\overline{A_n} = \{\omega : Y_n(\omega) \neq 0\}$, for $n \geq N^*$. Note that, for each $n > N^*$, and by the definition of $\{\widetilde{P}_n\}$,

$$\mathbb{P}\left[A_n \middle| \overline{A_{n-1}}\right] = \mathbb{P}\left[\widetilde{P}_n = \frac{1}{\Delta}\right] = 0.$$

Since $Y_{N^*-1} \neq 0$, we can infer, by induction, that $\mathbb{P}[A_n] = 0$ for all $n \geq N^*$. □

Lemma 6.5.7. *If $\{\widetilde{Y}_n\}_{n\geq 0}$ is the sequence of random variables defined by (6.3.6), and with an initial data set (6.3.3) obeying (6.3.4) and (6.3.5), then*

$$\mathbb{P}[\widetilde{Y}_n \neq 0 \textit{ for all } n] = 1.$$

Proof. By Lemmata 6.5.3, 6.5.4, 6.5.5, and 6.5.6, $\mathbb{P}[\widetilde{Y}_n = 0] = 0$ for all $n > 0$. We extend the definitions of A_n and $\overline{A_n}$ from Lemma 6.5.6 to define the complementary events $B_n = \{\omega : \widetilde{Y}_n(\omega) = 0\}$, and $\overline{B_n} = \{\omega : \widetilde{Y}_n(\omega) \neq 0\}$, for $n \geq 0$. Now,

$$\begin{aligned}
&\mathbb{P}[\widetilde{Y}_n \neq 0 \text{ for all } n] \\
&= \mathbb{P}\left[\bigcap_n \overline{B_n}\right] = \mathbb{P}\left[\overline{\bigcup_n B_n}\right] = 1 - \mathbb{P}\left[\bigcup_n B_n\right] \\
&\geq 1 - \sum_n \mathbb{P}[B_n] = 1.
\end{aligned}$$

□

Chapter 7

A Nonuniform Discretisation of the Auxiliary Process

The solution to the problem of avoiding spurious oscillatory behaviour is to allow the structures responsible for the true oscillatory behaviour to remain in place. In Chapters 3 and 5, we compared the qualitative behaviour of stochastic delay differential equations to that of deterministic delay equations, and to stochastic equations without delay. All the evidence indicates that the delay structure and the noise perturbation must be kept intact if the properties of the solutions are to be preserved. In fact, the failure to allow the feedback delay to persist indefinitely was the cause of the spurious qualitative behaviour observed in Theorem 6.4.1. In this chapter, we avoid this problem by building the effect of the delay into the structure of the mesh. Since we are concerned solely with the replication of oscillatory behaviour, we will continue to use the auxiliary process satisfying (4.1.3) as the basis for an Euler–type discretisation.

It must be emphasised that although there is no mechanism for a global tightening of the nonuniform mesh presented in this chapter, the mesh size becomes arbitrarily small after a finite number of time steps. Since oscillation is a tail phenomenon (in the sense that it is not an $\mathcal{F}(t)$–measurable event for any finite value of t) a strong argument can be made that the oscillatory behaviour of this nonuniform discretisation is characteristic of the continuous process.

Alternatively, this chapter can be viewed as an attempt to develop a discrete–time model that is behaviourally consistent with the solutions of (5.0.1), by identifying and preserving the essential characteristics of the solution through the discretisation.

7.1 Constructing a discrete process displaying nonspurious behaviour.

We consider the oscillatory behaviour of a two–step Euler difference equation evolving on a nonuniform mesh.

7.1.1 A nonuniform mesh.

In order to avoid a situation where spurious behaviour arises, it seems that the mesh must adapt itself to the decay of τ. Recall the definition of the feedback ratio ρ in (5.3.1). The asymptotic behaviour of ρ appears to mirror the onset of oscillatory behaviour in the solutions of (5.0.1). This can be taken as a good indication of the frequency at which we must sample information from the path in order to preserve qualitative behaviour. Because of this, we take ρ as a guide to the distribution of mesh points.

We define the mesh M_τ to be a sequence of points $\{t_n\}_{n\geq 0}$ defined by

$$t_0 = 0, \quad t_{n+1} = \inf\{t > 0 : t - \tau(t) = t_n\}.$$

$t \mapsto t - \tau(t)$ is a strictly increasing function, $\{t_n\}$ is strictly increasing in n and it was shown in Section 2.2 of Chapter 1.1 that $\lim_{n\to\infty} t_n = \infty$. Therefore the sequence $\{t_n\}$ partitions the time set into into a union of disjoint intervals: $\mathbb{R}^+ = \bigcup_{j=0}^{\infty}[t_j, t_{j+1})$. The length of the n^{th} interval $[t_{n-1}, t_n)$ is denoted

$$\Delta_n = \tau(t_n) = t_n - t_{n-1}.$$

A schematic of M_τ for a nonspecific vanishing delay function τ is given in Figure 7.1.1

From Definition 6.2.1, and the construction of this mesh, it should now be clear that the ratio of consecutive terms in the final difference equation will be the discrete–time equivalent of the feedback ratio ρ, and will form the basis of our analysis of the oscillatory behaviour of the final discrete process.

The construction of this mesh for a general vanishing delay function τ, as part of a numerical implementation of this discretisation, will require the approximation of successive mesh points by fixed point iteration. Given only the knowledge of a single mesh point,

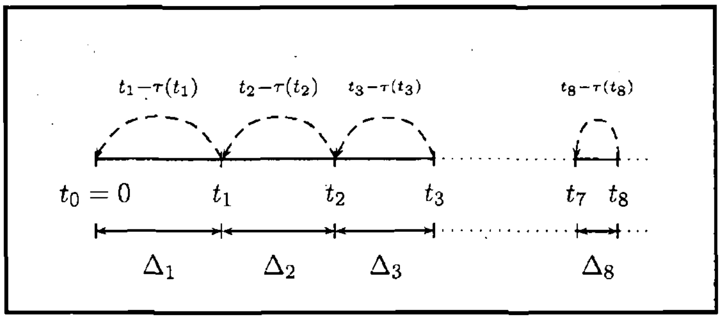


Figure 7.1.1: Construction of the nonuniform mesh M_τ.

and the form of τ, the problem of choosing an interval on which the next mesh point must be unique is nontrivial, and we have not attempted to present a solution here.

7.1.2 The difference equation evolving on M_τ.

On the mesh $M_\tau = \{t_0, t_1, ...\}$, $r_{M_\tau}(n) \equiv 1$. The length of each step of the difference equation is equal to the value of the delay function τ at the end of the step, and so the delay has been precisely encoded in the mesh rather than approximately encoded in the difference equation. We can now refine (6.3.1). In fact the equation that we will study can now be fully defined as

$$Y_{n+1} = Y_n - \Delta_{n+1}P_n(\omega)Y_{n-1}, \quad n \geq 0, \tag{7.1.1a}$$

$$(Y_{-1}, Y_0) \in \mathbb{R}^+ \times \mathbb{R}^+, \tag{7.1.1b}$$

with $P_n = p(t_n)$.

7.1.3 Useful properties of the difference equation (7.1.1).

In order to validate the use of Definition 6.2.1, it is necessary to show that the process obeying (7.1.1) is never zero valued. This will prove an easier task here than in Chapter 6.

Lemma 7.1.1. *If $\{Y_n\}_{n\geq 0}$ is the sequence of random variables defined by (6.1.1), then*

$$\mathbb{P}[Y_n \neq 0 \text{ for all } n] = 1.$$

Proof. On the left hand side of (7.1.1), Y_{n+1} is an $\mathcal{F}(t_n)$–measurable random variable. Therefore, Y_n and Y_{n-1} are $\mathcal{F}(t_{n-1})$–measurable and $\mathcal{F}(t_{n-2})$–measurable random variables respectively. Clearly

$$P_n = |b|e^{-\lambda\tau(t_n)}e^{|\sigma|B(t_n)-B(t_{n-1})},$$

being $\mathcal{F}(t_n)$–measurable and $\mathcal{F}(t_{n-1})$–independent, is independent of all other random variables in the right hand side of (7.1.1).

Define the sequence of events $\{C_n\}_{n\geq 0}$ by

$$C_n = \{\omega : Y_{n-1}(\omega) = 0,\, Y_n(\omega) = 0\}.$$

Since $(Y_{-1}, Y_0) \in \mathbb{R}^+ \times \mathbb{R}^+$, $\mathbb{P}[C_0] = 0$. Assume now that $\mathbb{P}[C_k] = 0$ for $k \geq 0$. If we can show, under this assumption, that $\mathbb{P}[Y_{k+1} = 0|\overline{C}_k] = 0$, then we have that $\mathbb{P}[C_{k+1}] = 0$ for $k \geq 0$, and we can infer, by induction, that

$$\mathbb{P}[A_n] = 0 \text{ for all } n \geq 0,$$

where $\{A_n\}_{n\geq 0}$ is a sequence of events, each defined as $A_n = \{\omega : Y_n(\omega) = 0\}$. The remainder of the proof then follows that of Lemma 6.5.7.

It remains to show, under the induction hypothesis, that $\mathbb{P}[Y_{k+1} = 0|\overline{C}_k] = 0$ for $k \geq 0$. By (6.1.1),

$$\mathbb{P}[Y_{k+1} = 0|\overline{C}_k] = \mathbb{P}[Y_k + \Delta_{k+1}P_kY_{k-1} = 0|\overline{C}_k].$$

Note that the event $\overline{C}_k$ admits the partitioning $\overline{C}_k = \overline{C}'_k \cup \overline{C}''_k \cup \overline{C}'''_k$, where

$$\begin{aligned}
\overline{C}'_k &= \{\omega : Y_{k-1}(\omega) = 0,\, Y_k(\omega) \neq 0\},\\
\overline{C}''_k &= \{\omega : Y_{k-1}(\omega) \neq 0,\, Y_k(\omega) = 0\},\\
\overline{C}'''_k &= \{\omega : Y_{k-1}(\omega) \neq 0,\, Y_k(\omega) \neq 0\}.
\end{aligned}$$

So, since P_n is a continuous random variable supported on $\mathbb{R}^+$,

$$\mathbb{P}[Y_k + \Delta_{k+1}P_kY_{k-1} = 0|\overline{C}_k''] = 0,$$

and

$$\begin{aligned}\mathbb{P}[Y_k + \Delta_{k+1}P_kY_{k-1} = 0|\overline{C}_k'''] &= \mathbb{P}\left[P_k = -\frac{Y_k}{\Delta_{k+1}Y_{k-1}}|\overline{C}_k' \cup \overline{C}_k'''\right] \\ &= 0.\end{aligned}$$

□

The $\mathcal{F}(t_{n-1})$–measurability of Y_n in the proof of Lemma 7.1.1 deserves explicit mention because it highlights the fact that the random process satisfying (7.1.1) is a *predictable* process. If this were not the case then P_n would not be independent of Y_n, and the proof would be invalid. In fact, the independence of P_n and Y_n removes the need to construct a new discrete–time filtration. It is enough to sample the natural filtration of Brownian motion $\{\mathcal{F}(t)\}_{t\geq 0}$ at each mesh point. The predictability of Y also suggests that an implicit discretisation of the auxiliary process satisfying (4.1.3) may prove to be a fruitful research topic in the future.

The following lemma shows that if any path of a solution of (7.1.1) oscillates, its value cannot cross equilibrium twice over any two consecutive time steps, except possibly for paths contained in a subset of the sample space with probability zero.

Lemma 7.1.2. *Consider the solution of (7.1.1), evolving on M_τ. For all $n \geq 0$,*

$$\mathbb{P}\left[\frac{Y_{n+1}}{Y_n} < 0 \,\middle|\, \frac{Y_n}{Y_{n-1}} < 0\right] = 0.$$

Proof. For all n,

$$\begin{aligned}&\mathbb{P}\left[\frac{Y_{n+1}}{Y_n} < 0 \,\middle|\, \frac{Y_n}{Y_{n-1}} < 0\right] \\ &= \mathbb{P}\left[(1-\Delta_{n+1}P_n) + \Delta_{n+1}P_n\left(1 - \frac{Y_{n-1}}{Y_n}\right) < 0 \,\middle|\, \frac{Y_n}{Y_{n-1}} < 0\right] \\ &= \mathbb{P}\left[-\frac{Y_{n-1}}{Y_n} + \frac{1}{\Delta_{n+1}P_n} < 0 \,\middle|\, \frac{Y_n}{Y_{n-1}} < 0\right] \\ &= 0,\end{aligned}$$

by Lemma 7.1.1, and the a.s. positivity of $\Delta_{n+1}P_n$ for all n. □

7.2 Oscillation and nonoscillation.

We consider the effect of varying decay rates of τ on the oscillatory asymptotic behaviour of solutions of (7.1.1), evolving on the mesh M_τ.

The following lemma is a special case of Corollary 4.1(a) in Györi [13], and will be useful in proving nonoscillation results.

Lemma 7.2.1. *Let $N \geq 0$ be a nonnegative integer. For all $n \geq N$ assume that*

$$0 < p_n \leq \frac{1}{4},$$

Then almost all solutions of the difference equation

$$\begin{aligned} y_{n+1} &= y_n - p_n y_{n-1}, \quad n \geq N, \\ (y_N, y_{N-1}) &\in \mathbb{R}^+ \times \mathbb{R}^+, \end{aligned}$$

are positive.

We know from Chapter 5 that the main factor influencing the qualitative behaviour of solutions of (7.1.1) is the decay rate of τ. Assume that τ is such that

$$\lim_{t\to\infty} \frac{\log \tau(t)}{\log\log t} = -\alpha, \quad \alpha \in (0, \infty] \tag{7.2.1}$$

and, when $\alpha = 1$,

$$\lim_{t\to\infty} \frac{\tau(t)\log t}{(\log\log t)^2} = \beta, \quad \beta \in [0, \infty]. \tag{7.2.2}$$

The values of α and β allow us to classify the behaviour of Y.

Theorem 7.2.2. *Let α and β be defined as in (7.2.1) and (7.2.2). Let $b < 0$. If τ is a continuous function satisfying (4.3.1), then the following classification holds. If*

1. *$\alpha = 1$, and*

(a) $\beta > \frac{1}{2\sigma^2}$, *then all solutions of (7.1.1) evolving on* M_τ, *are a.s. oscillatory.*

(b) $\beta < \frac{1}{2\sigma^2}$ *then all solutions of (7.1.1) evolving on* M_τ, *are a.s. nonoscillatory.*

2. $\alpha < 1$, *then all solutions of (7.1.1) evolving on* M_τ, *are a.s. oscillatory.*

3. $\alpha > 1$, *then all solutions of (7.1.1) evolving on* M_τ, *are a.s. nonoscillatory.*

For purposes of comparing our knowledge of the behaviour of the discrete and continuous processes, we reproduce the corresponding oscillation result from Chapter 5 here.

Theorem 7.2.3. *Let* $b < 0$. *Suppose* τ *is a continuous function satisfying (4.2.1), and that (7.2.1) holds. Let* $T > 0$, *and suppose that* $X = \{X(t); T \le t < \infty; \mathcal{F}^B(t)\}$ *is the strong solution of the equation*

$$\begin{aligned} dX(t) &= (aX(t) + bX(t-\tau(t)))dt + \sigma X(t)dB(t), \ \ t > T \\ X(t) &= 1, \ \ t \in [T-\tau(T), T]. \end{aligned}$$

Then the following classification holds. If

1. $\alpha < 1$, *then for every* $T \ge 0$, *the path* $X(\omega)$ *is oscillatory for all* ω *in a set which has probability one.*

2. $\alpha > 1$, *then for each* $\varepsilon > 0$, *there is a* $T = T(\varepsilon) > 0$ *such that the path* $X(\omega)$ *is positive for all* ω *in a set which has probability at least* $1 - \varepsilon$.

Notwithstanding the different monotonicity conditions on τ, there is a clear correspondence between the results in Theorems 7.2.3 and 7.2.2. However, in Theorem 7.2.3, positivity in the solutions of the differential equation with arbitrarily high probability is achieved by translating the necessarily positive and constant initial data forwards in time. The lower bound on this probability depends on the magnitude of the translation. By contrast, in Theorem 7.2.2, we consider a.s. nonoscillation of the solutions of the difference equation with untranslated positive initial data, which does not have to be constant. So when $\alpha \neq 1$, we recover, and somewhat extend, the behaviour of the differential equation in our discretisation.

In addition, the simplified analysis afforded us by discretising the problem allows us to extend our results to the critical case $\alpha = 1$.

Note that in this critical case, whether or not the solutions of (7.1.1) are a.s. oscillatory depends on the relationship of the decay rate of τ to the intensity of the noise. This relates back to the idea that, regardless of the strength of the returning force in the drift, or the rate of decay of the delay, oscillations cannot persist without some kind of noise perturbation.

The three cases of Theorem 7.2.2 are a summary of the results obtained in Theorems 7.2.10-7.2.9. The proofs of these theorems comprise the remainder of the chapter. Our analysis is broken down into subsections according to the ranges of the parameters α and β as they appear in the statement of the Theorem 7.2.2. We begin with the critical case.

7.2.1 Case 1, part (a). Oscillatory behaviour.

We consider the behaviour of (7.1.1) when $\alpha = 1$. In this case

$$\lim_{t\to\infty} \frac{\log \tau(t)}{\log\log t} = -1, \tag{7.2.3}$$

and

$$\lim_{t\to\infty} \frac{\tau(t)\log t}{(\log\log t)^2} = \beta \in (\frac{1}{2\sigma^2}, \infty]. \tag{7.2.4}$$

As a consequence of (7.2.4),

$$\lim_{t\to\infty} \tau(t)\log t = \infty. \tag{7.2.5}$$

Lemma 7.2.4. *Suppose that τ is a continuous function satisfying (4.3.1), (7.2.4), and (7.2.3). Let $\{\xi_n\}$ be a sequence of independent standard Normal random variables and define two sequences of random variables $\{\widetilde{W}_n\}$ and $\{W_n\}$ by*

$$\widetilde{W}_n = \log \tau(t_n) + |\sigma|\sqrt{\tau(t_n)}\xi_n, \tag{7.2.6}$$

and

$$W_n = \log \tau(t_{n+1}) + |\sigma|\sqrt{\tau(t_n)}\xi_n. \tag{7.2.7}$$

Then

$$\mathbb{P}[W_n > c] \geq \mathbb{P}[\widetilde{W}_{n+1} > c], \quad \textit{for all } c > 0, \ \textit{and } n \in \mathbb{N}.$$

Proof. The statement of the lemma is true if and only if

$$\mathbb{P}\left[\xi_n > \frac{c - \log \tau(t_{n+1})}{|\sigma|\sqrt{\tau(t_n)}}\right] \geq \mathbb{P}\left[\xi_n > \frac{c - \log \tau(t_{n+1})}{|\sigma|\sqrt{\tau(t_{n+1})}}\right].$$

This holds since, by (4.3.1), $\tau(t_n) > \tau(t_{n+1})$ for all n, and each ξ_n is a standard Normal random variable. □

Lemma 7.2.5. *Let $b < 0$. Suppose that τ is a continuous function satisfying (4.3.1), (7.2.3), and (7.2.4). If $B(t)$ is a standard Brownian motion and $M_\tau = \{t_n\}_{n\geq 0}$ is the mesh defined in Section 7.1.1, then*

$$\limsup_{n\to\infty} \tau(t_n) e^{|\sigma|(B(t_n)-B(t_{n-1}))} = \infty, \quad a.s.$$

Proof. We consider the case where $\beta \in (\frac{1}{2\sigma^2}, \infty)$ first. Consider the random variable

$$\tau(t_n)e^{-\sigma(B(t_n)-B(t_{n-1}))}. \tag{7.2.8}$$

For every $n > 0$, (7.2.8) has the same distribution as the random variable Z_n, defined as

$$Z_n = \tau(t_n)e^{|\sigma|\sqrt{\tau(t_n)}\xi_n}$$

where $\{\xi_n\}$ is a sequence of independent standard Normal random variables. It is in fact sufficient to prove that $\limsup_{n\to\infty} \log Z_n = \infty$, a.s. For any $\varepsilon > 0$,

$$\begin{aligned}
\log Z_n &= \log \tau(t_n) + |\sigma|\sqrt{\tau(t_n)}\xi_n \\
&= \log \tau(t_n) + (1-\varepsilon)|\sigma|\sqrt{\tau(t_n)}\sqrt{2\log n} \\
&\quad + |\sigma|\sqrt{2\tau(t_n)\log n}\left(\frac{\xi_n}{\sqrt{2\log n}} - (1-\varepsilon)]\right).
\end{aligned}$$

By (4.3.1), there exists an $N_0(\varepsilon) \in \mathbb{N}$ large enough that for all $n > N_0(\varepsilon)$, $t_n < n$, and therefore that $\tau(t_n) \geq \tau(n)$. So, by (7.2.5), and since

$$\limsup_{n\to\infty} \frac{\xi_n}{\sqrt{2\log n}} = 1, \qquad \text{a.s.},$$

it follows that, for each fixed $\varepsilon > 0$,

$$\limsup_{n\to\infty}\left\{|\sigma|\sqrt{2\tau(t_n)\log n}\left(\frac{\xi_n}{\sqrt{2\log n}} - (1-\varepsilon)]\right)\right\} = \infty, \quad \text{a.s.} \tag{7.2.9}$$

It remains to quantify the large deviations of

$$\log\tau(t_n) + (1-\varepsilon)|\sigma|\sqrt{\tau(t_n)}\sqrt{2\log n}.$$

Since, by (7.2.4),

$$\lim_{n\to\infty}\frac{\tau(n)\log n}{(\log\log n)^2} = \beta > \frac{1}{2\sigma^2},$$

we have that for every $\varepsilon \in (0,1)$, there is an $N_1(\varepsilon) \in \mathbb{N}$ such that $n > N_1(\varepsilon)$ implies

$$\frac{\tau(t_n)\log n}{(\log\log n)^2} > \beta(1-\varepsilon).$$

Hence

$$\tau(t_n) > \frac{\beta(1-\varepsilon)(\log\log n)^2}{\log n}, \quad n > N(\varepsilon).$$

Thus, for $n > N_0(\varepsilon) \vee N_1(\varepsilon)$,

$$\begin{aligned}
&\log\tau(t_n) + (1-\varepsilon)|\sigma|\sqrt{\tau(t_n)}\sqrt{2\log n} \\
>\ & \log\left(\frac{\beta(1-\varepsilon)(\log\log n)^2}{\log n}\right) + (1-\varepsilon)|\sigma|\sqrt{2\log n}\sqrt{\beta(1-\varepsilon)}\frac{\log\log n}{\sqrt{\log n}}, \\
=\ & \log\log n\left\{\frac{\log\beta(1-\varepsilon)}{\log\log n} + \frac{2\log\log\log n}{\log\log n} + \sqrt{2}|\sigma|(1-\varepsilon)\sqrt{\beta(1-\varepsilon)} - 1\right\}.
\end{aligned}$$

By (7.2.4), $\sqrt{2}|\sigma|(1-\varepsilon)\sqrt{\beta(1-\varepsilon)} - 1 > 0$, for all $\varepsilon > 0$. So

$$\limsup_{n\to\infty}\{\log\tau(t_n) + (1-\varepsilon)|\sigma|\sqrt{\tau(t_n)}\sqrt{2\log n}\} = \infty, \quad \text{a.s.} \tag{7.2.10}$$

for all $\varepsilon > 0$. Combining this with (7.2.9) allows us to conclude that

$$\limsup_{n\to\infty}\log Z_n = \infty, \quad \text{a.s.}$$

Therefore, the statement of the lemma holds when $\beta \in (\frac{1}{2\sigma^2}, \infty)$.

The statement of the lemma also holds in the case where $\beta = \infty$, since (7.2.9) is independent of β, and (7.2.10) is still true. This can be seen by noting that for every $\tilde{\beta} > \frac{1}{2\sigma^2}$, there is an $N(\tilde{\beta}) \in \mathbb{N}$ such that $n > N(\tilde{\beta})$ implies that $\frac{\tau(t_n)\log n}{(\log\log n)^2} > \tilde{\beta}$. □

Theorem 7.2.6. *Let $b < 0$. Suppose that τ is a continuous function satisfying (4.3.1), (7.2.4), and (7.2.3). If $\beta > \frac{1}{2\sigma^2}$, then all solutions of equation*

$$\begin{aligned} Y_{n+1} &= Y_n - \Delta_{n+1} P_n Y_{n-1}, \quad n \geq 0 \\ (Y_0, Y_{-1}) &\in \mathbb{R}^+ \times \mathbb{R}^+, \end{aligned}$$

evolving on the mesh M_τ are a.s. oscillatory.

Proof. Assume first, without loss of generality, that $\sigma < 0$. We proceed by contradiction. Assume that there exists $N_1(\omega) \in \mathbb{N}$ such that the event

$$D = \left\{ \omega : \frac{Y_{n+1}}{Y_n}(\omega) > 0 \text{ for all } n > N_1 \right\}$$

has probability greater than zero. h_n is positive for all n and P_n is a.s. positive for all n. So for every $n \geq N_1 + 2$, and $\omega \in D$,

$$1 - \frac{Y_{n-1}}{Y_n}(\omega) < 0 \tag{7.2.11}$$

We can rewrite (7.1.1) as

$$\frac{Y_{n+1}}{Y_n} = (1 - \Delta_{n+1}P_n) + \Delta_{n+1}P_n\left(1 - \frac{Y_{n-1}}{Y_n}\right).$$

By (7.2.11), we can elicit the required contradiction, and therefore show that $\mathbb{P}[D] = 0$, if we can show that (5.2.3) implies

$$\mathbb{P}[1 - \Delta_{n+1}P_n < 0 \text{ i.o.}] = 1. \tag{7.2.12}$$

for all $b < 0$. It is sufficient to show that

$$\limsup_{n\to\infty} \tau(t_{n+1}) e^{|\sigma|(B(t_n) - B(t_{n-1}))} = \infty, \quad \text{a.s.} \tag{7.2.13}$$

Consider W_n and $\widetilde{W}_n$ as defined in (7.2.6) and (7.2.7). Lemma 7.2.5 implies that $\sum_{n=0}^{\infty}\mathbb{P}[\widetilde{W}_n > c] = \infty$ for all $c > 0$. Therefore, by Lemma 7.2.4, $\sum_{n=0}^{\infty}\mathbb{P}[W_n > c] = \infty$ for all $c > 0$. Therefore, (7.2.13) holds, and the proof is complete. □

7.2.2 Case 1, part (b). Nonoscillatory behaviour.

We again consider the behaviour of (7.1.1) when $\alpha = 1$. However, in this case

$$\lim_{t\to\infty}\frac{\tau(t)\log t}{(\log\log t)^2} = \beta \in [0, \frac{1}{2\sigma^2}). \tag{7.2.14}$$

As a consequence of (7.2.14),

$$\lim_{t\to\infty}\tau(t)\log t = \infty.$$

Lemma 7.2.7. *Let $b < 0$. Suppose that τ is a continuous function satisfying (4.3.1), $\alpha =$ 1, and (7.2.4). If $\beta \in (0, \frac{1}{2\sigma^2})$ then there exists $\widetilde{N} \in \mathbb{N}$ such that, for every $N > \widetilde{N}$, there is an $\varepsilon(N) \in (0,1)$ and a set $\Omega_N \in \mathcal{F}^B(\infty)$ with $\mathbb{P}[\Omega_N] \geq 1 - \varepsilon(N)$ and $\lim_{N\to\infty}\mathbb{P}[\Omega_N] = 1$, such that for each $\omega \in \Omega_N$, almost all solutions of*

$$\begin{aligned} Y_{n+1} &= Y_n - \Delta_{n+1}P_n(\omega)Y_{n-1}, \quad n \geq N, \\ (Y_N, Y_{N-1}) &\in \mathbb{R}^+ \times \mathbb{R}^+, \end{aligned}$$

evolving on M_τ, are positive.

Proof. By Lemma 7.2.1, the statement of the theorem is true if, for every $N > \widetilde{N}$, there exists $\varepsilon(N) \in (0,1)$, and a set $\Omega_N \in \mathcal{F}^B(\infty)$, with $\mathbb{P}[\Omega_N] \geq 1-\varepsilon(N)$, and $\lim_{N\to\infty}\mathbb{P}[\Omega_N] =$ 1, such that for each $\omega \in \Omega_N$,

$$\Delta_{n+1}P_n(\omega) \leq \frac{1}{4} \text{ for all } n \geq N.$$

Without loss of generality, set $\sigma < 0$. Let $\tau_0 = \tau(0)$ and τ_1 be given by $\tau_1 - \tau(\tau_1) = 0$. Define $\gamma > 0$ by

$$\gamma = \frac{1}{|\sigma|\sqrt{2\beta}} - 1.$$

Now consider the function

$$H(t) = |b|\tau(t)e^{|\lambda|\tau_0+|\sigma|(1+\gamma)\sqrt{2\tau(t)\log t}}.$$

Since $\alpha = 1$, and since $\lim_{t\to\infty} \frac{\tau(t)\log t}{(\log\log t)^2} = \frac{1}{2\sigma^2(1+\gamma)^2}$,

$$\begin{aligned}\lim_{n\to\infty} \frac{\log H(t_n)}{\log\log t_n} &= \lim_{n\to\infty}\left\{\frac{\log |b|e^{|\lambda|\tau_0}}{\log\log t_n} + \frac{\log\tau(t_n)}{\log\log t_n} + \frac{|\sigma|\sqrt{2\tau(t_n)\log t_n}}{\log\log t_n}\right\} \\ &< 0.\end{aligned}$$

and therefore $\lim_{n\to\infty} H(t_n) = 0$. Choose $\widetilde{N}$ large enough that $H(t_n) < \frac{1}{4}$ for all $n > \widetilde{N}$.

Now, for $t > \tau_1$, we can define the $\mathcal{F}^B(\infty)$–measurable random variable

$$C(t) = \sup_{s\geq t} \frac{|B(s) - B(s-\tau(s))|}{\sqrt{2\tau(s)\log s}}.$$

$C(t)$ is well defined on a set of probability one as, by Corollary 4.3.3,

$$\limsup_{s\to\infty} \frac{|B(s) - B(s-\tau(s))|}{\sqrt{2\tau(s)\log s}} = 1, \quad a.s. \tag{7.2.15}$$

Therefore, for every $N > \widetilde{N}$, there is an $\varepsilon(N) \in (0,1)$ such that $n \geq \widetilde{N}$ implies that

$$\mathbb{P}[C(t_n) > 1+\gamma] \leq \varepsilon(N).$$

Therefore we can define the set

$$\Omega_N = \left\{\omega \in \Omega : \sup_{n\geq N} \frac{(B(t_n) - B(t_{n-1}))(\omega)}{\sqrt{2\tau(t_n)\log t_n}} \leq 1+\gamma\right\}$$

satisfying $\mathbb{P}[\Omega_N] \geq 1 - \varepsilon(N)$, since

$$|B(t_n) - B(t_{n-1})| \geq B(t_n) - B(t_{n-1}) \tag{7.2.16}$$

for all n. By (7.2.15), $\mathbb{P}[\lim_{N\to\infty} C(t_N) = 1] = 1$. Because a.s. convergence implies convergence in probability,

$$\lim_{N\to\infty} \mathbb{P}[|C(t_N) - 1| \leq \gamma] = 1.$$

Since $C(t)$ is nonincreasing for all $t > \tau_1$, and $\lim_{t\to\infty} C(t) = 1$ a.s., $C(t) \geq 1$ for all $t > \tau_1$.

Consequently,

$$\lim_{N\to\infty} \mathbb{P}[C(t_N) \leq 1+\gamma] = 1.$$

By (7.2.16),

$$\lim_{N\to\infty} \mathbb{P}\left[\sup_{n\geq N} \frac{B(t_n) - B(t_{n-1})}{\sqrt{2\tau(t_n)\log t_n}} \leq 1+\gamma\right] = 1,$$

and therefore $\lim_{N\to\infty}\mathbb{P}[\Omega_N] = 1$. So for all $N > \widetilde{N}$, there is a set Ω_N with $\mathbb{P}[\Omega_N] \geq 1-\varepsilon(N)$ and $\lim_{N\to\infty}\mathbb{P}[\Omega_N] = 1$ such that for all $n \geq N$ and $\omega \in \Omega_N$,

$$\begin{aligned} h_{n+1}P_n(\omega) &= |b|\tau(t_n)\frac{\tau(t_{n+1})}{\tau(t_n)}e^{-\lambda\tau(t_n)}e^{|\sigma|(B(t_n)-B(t_{n-1}))} \\ &\leq |b|\tau(t_n)e^{|\lambda|\tau_0}e^{|\sigma|(1+\gamma)\sqrt{2\tau(t_n)\log t_n}} \\ &< \frac{1}{4} \end{aligned}$$

as required. □

Lemma 7.2.8. *Let $b < 0$. Suppose that τ is a continuous function satisfying (4.3.1), $\alpha = 1$, and (7.2.4). If $\beta = 0$ then there exists $\widetilde{N} \in \mathbb{N}$ such that, for every $N > \widetilde{N}$, there is an $\varepsilon(N) \in (0,1)$ and a set $\Omega_N \in \mathcal{F}^B(\infty)$ with $\mathbb{P}[\Omega_N] \geq 1-\varepsilon(N)$ and $\lim_{N\to\infty}\mathbb{P}[\Omega_N] = 1$, such that for each $\omega \in \Omega_N$, almost all solutions of*

$$\begin{aligned} Y_{n+1} &= Y_n - \Delta_{n+1}P_n(\omega)Y_{n-1}, \quad n \geq N, \\ (Y_N, Y_{N-1}) &\in \mathbb{R}^+ \times \mathbb{R}^+, \end{aligned}$$

evolving on M_τ, are positive.

Proof. Since $\lim_{n\to\infty} e^{\lambda\tau(t_n)} = 1$, and because $\frac{\tau(t_{n+1})}{\tau(t_n)} < 1$ for all n it is sufficient to show that

$$\lim_{n\to\infty} \tau(t_n)e^{|\sigma|(B(t_n)-B(t_{n-1}))} = 0. \tag{7.2.17}$$

Because $\alpha = 1$, and by Lemma 4.3.4,

$$\lim_{t\to\infty}\{\log\tau(t) + |\sigma|(B(t) - B(t-\tau(t)))\}$$
$$= \lim_{t\to\infty}\left\{\log\log t\left(\frac{\log\tau(t)}{\log\log t} + |\sigma|\frac{B(t) - B(t-\tau(t))}{\log\log t}\right)\right\} = -\infty.$$

Therefore (7.2.17) holds. The rest of the proof follows in the manner of Lemma 7.2.7. □

Theorem 7.2.9. *Let $b < 0$. Suppose that τ is a continuous function satisfying (4.3.1), $\alpha = 1$ and (7.2.4), with $\beta \in [0, \frac{1}{2\sigma^2})$. Then all solutions of (7.1.1) evolving on M_τ are a.s. nonoscillatory.*

Proof. Define the set D_m by

$$D_m = \{\omega \in \Omega : \Upsilon(\omega) = m\}$$

for all m, where Υ is as defined in (6.2.1). By Lemmata 7.1.2, 7.2.7, and 7.2.8,

$$\lim_{m\to\infty}\mathbb{P}[D_m] = 1.$$

Now for every $\delta > 0$, there is an $M(\delta) > 0$ such that $m > M(\delta)$ implies that $\mathbb{P}[\overline{D_m}] < \delta$. Then, for $m > M(\delta)$,

$$\mathbb{P}\left[\overline{\bigcup_{m=1}^{\infty} D_m}\right] = \mathbb{P}\left[\bigcap_{m=1}^{\infty}\overline{D_m}\right] \le \mathbb{P}[\overline{D_m}] < \delta.$$

So $\mathbb{P}[\overline{\bigcup_{m=1}^{\infty} D_m}] = 0$, and therefore $\mathbb{P}[\bigcup_{m=1}^{\infty} D_m] = 1$. So,

$$\mathbb{P}[\Upsilon < \infty] = 1$$

and therefore the solution of (7.1.1), evolving on M_τ, is a.s. nonoscillatory. □

7.2.3 Case 2. Oscillatory behaviour.

We consider the behaviour of (7.1.1) when $\alpha < 1$. This implies that $\beta = \infty$.

Theorem 7.2.10. *Suppose τ is a continuous function satisfying (4.3.1) and (5.2.3). Then all solutions of (7.1.1) are a.s. oscillatory.*

Proof. Because $\alpha < 1$ and $\beta = \infty$, the results stated in Lemmata 7.2.4 and 7.2.5 hold. Therefore the proof of Theorem 7.2.6 is valid in this case. □

7.2.4 Case 3. Nonoscillatory behaviour.

We consider the behaviour of (7.1.1) when $\alpha > 1$. This implies that (5.2.5) holds.

Lemma 7.2.11. *Let $b < 0$. Suppose that τ is a continuous function satisfying (4.3.1) and (5.2.5). Then, for every $N > 0$, there is an $\varepsilon(N) \in (0,1)$ and a set $\Omega_N \in \mathcal{F}^B(\infty)$, with $\mathbb{P}[\Omega_N] \geq 1 - \varepsilon(N)$, and $\lim_{N\to\infty} \mathbb{P}[\Omega_N] = 1$, such that for each $\omega \in \Omega_N$, almost all solutions of*

$$\begin{aligned} Y_{n+1} &= Y_n - \Delta_{n+1} P_n(\omega) Y_{n-1}, \quad n \geq N, \\ (Y_N, Y_{N-1}) &\in \mathbb{R}^+ \times \mathbb{R}^+, \end{aligned}$$

evolving on M_τ, are positive.

Proof. By Lemma 7.2.1, the statement of the theorem is true if, for all $N > 0$, there exists $\varepsilon(N) \in (0,1)$, and a set $\Omega_N \in \mathcal{F}^B(\infty)$, with $\mathbb{P}[\Omega_N] \geq 1 - \varepsilon(N)$, and $\lim_{N\to\infty} \mathbb{P}[\Omega_N] = 1$, such that for each $\omega \in \Omega_N$,

$$\Delta_{n+1} P_n(\omega) \leq \frac{1}{4} \text{ for all } n \geq N.$$

Without loss of generality, set $\sigma < 0$. Let $\tau_0 = \tau(0)$ and τ_1 be given by $\tau_1 - \tau(\tau_1) = 0$. For $t > \tau_1$ we can define the $\mathcal{F}^B(\infty)$–measurable random variable

$$C(t) = \sup_{s \geq t} |B(s) - B(s - \tau(s))|.$$

This is well defined on a set of probability one as, by Lemma 1 in [2], (5.2.5) implies that

$$\lim_{t\to\infty} B(t) - B(t - \tau(t)) = 0, \quad a.s. \tag{7.2.18}$$

Thus $\lim_{t\to\infty} C(t) = 0$, a.s. Now define the deterministic constant

$$\eta = \frac{1}{|\sigma|}\left\{\ln\left(\frac{1}{4|b|\tau_0}\right) - |\lambda|\tau_0\right\}.$$

For every $N > 0$, there is an $\varepsilon(N) \in (0, 1)$ such that $n \geq N$ implies that

$$\mathbb{P}[C(t_n) > \eta] \leq \varepsilon(N).$$

Therefore we can define the set

$$\Omega_N = \{\omega \in \Omega : \sup_{n\geq N} |B(t_n, \omega) - B(t_{n-1}, \omega)| \leq \eta\}$$

which has $\mathbb{P}[\Omega_N] \geq 1 - \varepsilon(N)$. By (7.2.18),

$$\mathbb{P}[\lim_{N\to\infty} C(t_N) = 0] = 1.$$

Because a.s. convergence implies convergence in probability,

$$\lim_{N\to\infty} \mathbb{P}[C(t_N) \leq \eta] = 1$$

and therefore $\lim_{N\to\infty} \mathbb{P}[\Omega_N] = 1$. Now,

$$\Delta_{n+1}P_n = |b|\tau(t_n)\frac{\tau(t_{n+1})}{\tau(t_n)}e^{-\lambda\tau(t_n)}e^{|\sigma|(B(t_n)-B(t_{n-1}))}.$$

Since $\tau(t)$ is strictly decreasing, $\frac{\tau(t_{n+1})}{\tau(t_n)} < 1$ for all n. Also, for all $\omega \in \Omega_N$ and $n \geq N$,

$$\Delta_{n+1}P_n \leq |b|\tau_0 e^{|\lambda|\tau_0+|\sigma|\eta} < \frac{1}{4},$$

as required. □

Theorem 7.2.12. *Let $b < 0$. Suppose that τ is a continuous function satisfying (4.3.1) and (5.2.5). Then all solutions of (7.1.1) evolving on M_τ are a.s. nonoscillatory.*

Proof. Using Lemmata 7.1.2 and 7.2.11, the proof follows in a similar manner to that of Theorem 7.2.9. □

Theorems 7.2.6, 7.2.9, 7.2.10, and 7.2.12 together comprise the statement of Theorem 7.2.2.

Chapter 8

Summary of Findings

In this final chapter, we gather together the principal results of the thesis.

8.1 Global existence and uniqueness.

For the deterministic equation

$$x'(t) = g_{(\tau,\overline{\tau})}(t, x_t), \quad t \geq 0, \tag{8.1.1a}$$

$$x(t) = \psi(t), \quad t \in [-\overline{\tau}, 0] \tag{8.1.1b}$$

with delay structure as described in (2.1.1), (2.1.2), (2.1.3), and (2.1.4) we merely require that $g_{(\tau,\overline{\tau})}$ be continuous to ensure a unique global solution. The stochastic perturbation of (8.1.1) embodied in

$$\begin{aligned} dX(t) &= g_{(\tau,\overline{\tau})}(t, X_t)\, dt + h(X(t))\, dB(t), \\ X(t) &= \psi(t), \quad t \in [-\overline{\tau}, 0]. \end{aligned}$$

can be shown to have a unique global solution if h is locally Lipschitz continuous. Thus, in this case, the addition of noise does not prevent the solution from existing globally. In fact if some of the feedback in the drift is instantaneous, as given by

$$\begin{aligned} dX(t) &= \left(f(X(t)) + g_{(\tau,\overline{\tau})}(t, X_t)\right) dt + h(X(t))\, dB(t), \\ X(t) &= \psi(t), \quad t \in [-\overline{\tau}, 0], \end{aligned}$$

it is possible to suppress an explosion in the solution of the deterministic equation with a carefully chosen noise term. Specifically,

$$\sup_{x \in \mathbb{R}} \frac{xf(x) - \frac{1}{2}h(x)^2}{1 + |x|^2} < \infty,$$

gives a sufficient condition on the relative intensity of the diffusion coefficient for a given drift coefficient. It is interesting to note that the delayed feedback component of the drift appears to play no role whatsoever in either causing or suppressing an explosion in such equations. A generalisation of this result to finite dimensions can be found in Appendix A.

8.2 The nonlinear stochastic equation with fixed delay.

Consider the deterministic equation

$$x'(t) = -g(x(t-\tau)), \quad t > 0, \tag{8.2.1a}$$

$$x(t) = \psi(t), \quad t \in [-\overline{\tau}, 0] \tag{8.2.1b}$$

where g is a continuous function satisfying $xg(x) > 0$ for all $x \in \mathbb{R}$. Additionally $g(0) = 0$ guarantees the existence of an equilibrium solution.

Oscillatory solutions exist if the drift is sublinear, in the sense that there is $\infty > L > 0$ such that

$$\lim_{x\to 0} \frac{g(x)}{x} = L.$$

Oscillation can be guaranteed for every solution by also ensuring that $\tau L > \frac{1}{e}$.

If g provides a weaker action towards equilibrium, in the sense that

$$\lim_{x\to 0} \frac{|g(x)|}{|x|^\gamma} = L \tag{8.2.2}$$

for some $\gamma > 1$ and $L > 0$, solutions of (8.2.1) do not have to oscillate. In fact, it is always possible to choose a sufficiently small scaling factor for the initial data function ψ which guarantees the nonoscillation of the solution of (8.2.1a).

The addition of a stochastic perturbation removes the requirement that τ have a minimum length in order to guarantee oscillation. Consider the stochastic equation

$$dX(t) = -g(X(t-\tau))dt + \sigma h(X(t))dB(t), \tag{8.2.3a}$$

$$X(t) = \psi(t), \quad t \in [-\tau, 0], \tag{8.2.3b}$$

where h is locally Lipschitz continuous, $h(0) = 0$ preserves the equilibrium solution and there exists $0 < \underline{h} \leq 1 \leq \overline{h}$ such that

$$\underline{h}|x|^2 \leq xh(x) \leq \overline{h}|x|^2, \tag{8.2.4}$$

and

$$\lim_{x \to 0} \frac{h(x)}{x} = 1. \tag{8.2.5}$$

If the drift coefficient is sublinear at zero, in the sense that (8.2.2) holds, then all solutions of (8.2.3) are a.s. oscillatory.

If the drift coefficient is superlinear, in the sense that there exists $\gamma > 1$ and $0 < L \leq \overline{L}$ such that

$$\lim_{x \to 0} \frac{|g(x)|}{|x|^\gamma} = L,$$

and

$$|g(x)| \leq \overline{L}|x|^\gamma, \quad x \in \mathbb{R},$$

then the solutions of (8.2.3) are nonoscillatory with positive probability. This probability can be made arbitrarily close to one by choosing an appropriate initial data function.

8.3 The linear stochastic equation with vanishing delay.

Finally, we look at the properties of the solutions of

$$\begin{aligned} dX(t) &= (aX(t) + bX(t - \tau(t)))\,dt + \sigma X(t)\,dB(t), && (8.3.1a) \\ X(t) &= \psi(t), \quad -\overline{\tau} \leq t \leq 0, && (8.3.1b) \end{aligned}$$

where $b < 0$, and τ is a continuous function which vanishes as $t \to \infty$. To begin with it is useful to consider the properties of the corresponding limiting equations.

Consider the stochastically unperturbed equation

$$\begin{aligned} x'(t) &= ax(t) + bx(t - \tau(t)), \quad t > 0, && (8.3.2a) \\ x(t) &= \psi(t), \quad [-\overline{\tau}, 0]. && (8.3.2b) \end{aligned}$$

If the delay function τ is integrable, then there exists a nonoscillatory solution of (8.3.2).

Consider also the stochastic equation with zero delay

$$\begin{aligned} dX(t) &= (aX(t) + bX(t))\,dt + \sigma X(t)\,dB(t), && (8.3.3a) \\ X(0) &\in \mathbb{R}^+. && (8.3.3b) \end{aligned}$$

This is geometric Brownian motion, and thus all solutions of (8.3.3) are nonoscillatory.

The combination of a vanishing feedback delay and a stochastic perturbation, however, allows for the possibility of oscillatory behaviour. We can parameterise the rate of decay of τ as follows. Suppose that τ obeys

$$\lim_{t\to\infty}\frac{\log\tau(t)}{\log\log t}=-\alpha,\ \ \alpha\in(0,\infty] \tag{8.3.4}$$

and, when $\alpha=1$,

$$\lim_{t\to\infty}\frac{\tau(t)\log t}{(\log\log t)^2}=\beta,\ \ \beta\in[0,\infty]. \tag{8.3.5}$$

When $\alpha<1$, all solutions of (8.3.1) are a.s. oscillatory. When $\alpha>1$, by keeping the initial data constant, and scaling the delay function τ appropriately, we can construct an equation related to (8.3.1) that has a nonoscillatory solution with probability arbitrarily close to one.

By discretising the auxiliary form of (8.3.1) over a nonuniform mesh, we can build a more complete picture of this behaviour. Define the sequence $M_\tau=\{t_n\}_{n\geq 0}$ by

$$t_0=0,\quad t_{n+1}=\inf\{t>0:t-\tau(t)=t_n\}.$$

Now let $\Delta_n=\tau(t_n)=t_n-t_{n-1}$. Consider the two–step Euler difference equation

$$Y_{n+1}\ =\ Y_n-\Delta_{n+1}P_n(\omega)Y_{n-1},\ \ n\geq 0, \tag{8.3.6a}$$

$$(Y_{-1},Y_0)\ \in\ \mathbb{R}^+\times\mathbb{R}^+. \tag{8.3.6b}$$

Then the oscillatory behaviour of (8.3.6) can be classified in a more complete fashion.

If $\alpha<1$, then all solutions of (8.3.6) evolving on M_τ, are a.s. oscillatory. This result coincides with the behaviour of the solutions of (8.3.1) in the same parameter region.

If $\alpha>1$, then all solutions of (8.3.6) evolving on M_τ, are a.s. nonoscillatory. This result coincides with, and extends, the equivalent result for the behaviour of solutions of (8.3.1).

Finally, we can describe the dynamical behaviour of the solutions of (8.3.6) in the critical region around the switch from a.s. oscillation to a.s. nonoscillation. If $\alpha=1$, and $\beta>\frac{1}{2\sigma^2}$, then all solutions of (8.3.6) evolving on M_τ, are a.s. oscillatory. If $\beta<\frac{1}{2\sigma^2}$ then all solutions of (8.3.6) evolving on M_τ, are a.s. nonoscillatory. The significance of the noise intensity σ in this parameter region is noteworthy in this instance.

The evidence suggests that this behaviour is also characteristic of the solutions of (8.3.1).

Existence and Uniqueness in Finite Dimensions

Theorem 2.4.2 can be generalised to cover the global existence of solutions of a class of finite dimensional stochastic equations. Such equations find application in modelling the population dynamics of several interacting species. See Mao, Marion and Renshaw [31], for example.

Theorem A.0.1. *Let h_0 be a locally Lipschitz continuous function from $\mathbb{R}^d$ to $\mathbb{R}$, and $\mathbf{h}(\mathbf{x}) = h_0(\mathbf{x})\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^d$. Suppose that $\mathbf{g}_{(\tau,\overline{\tau})}$ is continuous with delay structure $(\tau, \overline{\tau})$ obeying (2.1.1), (2.1.2), and (2.1.4). If $\psi \in C([-\overline{\tau}, 0]; \mathbb{R}^d)$, $\mathbf{f}$ and $\mathbf{h}$ are locally Lipschitz continuous, and*

$$\sup_{\mathbf{x}\in\mathbb{R}^d} \frac{\langle \mathbf{x}, \mathbf{f}(\mathbf{x})\rangle - \frac{1}{2}\|\mathbf{h}(\mathbf{x})\|^2}{1 + \|\mathbf{x}\|^2} < \infty, \tag{A.0.1}$$

then there is a unique, continuous, adapted $\mathbb{R}^d$-valued process which is a strong solution of

$$d\mathbf{X}(t) = \big(\mathbf{f}(\mathbf{X}(t)) + \mathbf{g}_{(\tau,\overline{\tau})}(t, \mathbf{X}_t)\big)\, dt + \mathbf{h}(\mathbf{X}(t))\, dB(t), \tag{A.0.2a}$$

$$\mathbf{X}(t) = \psi(t), \quad t \in [-\overline{\tau}, 0]. \tag{A.0.2b}$$

on $[0, \infty)$.

Note that the condition (A.0.1) reduces to (2.4.7) when ($d = 1$).

Proof of Theorem A.0.1. Let V_s be the scalar function defined by (2.3.13). Define $V \in C(\mathbb{R}^d; \mathbb{R})$ by $V(\mathbf{x}) = V_s(1 + \|\mathbf{x}\|^2)$, $\mathbf{x} \in \mathbb{R}^d$. We redefine the explosion time given in (2.3.4) for finite dimensional processes as

$$\tau_e^{\psi} = \inf\{t > 0 : \lim_{s\to t^-} |\mathbf{X}_i(s, [-\overline{\tau}, 0], \psi)| = \infty \text{ for some } i = 1, \ldots, d\},$$

where $\mathbf{X}_i = \langle \mathbf{X}, \mathbf{e}_i \rangle$, and $\mathbf{e}_i$ is the vector with 1 in the i-th component, and zeros elsewhere. Let $k^* \in \mathbb{R}$ be sufficiently large that the initial data of each component of $\mathbf{X}$ lies within $[-k^*, k^*]$. Then for every integer $k > k^*$, we can define the stopping time

$$\tau_k^{\psi} = \inf\{t \in [0, \tau_e^{\psi}) : \mathbf{X}_i(t, [-\overline{\tau}, 0], \psi) \notin (-k, k), \text{ for some } i = 1, \ldots, d\}.$$

Clearly, τ_k^{ψ} is an increasing sequence. Set $\tau_\infty^{\psi} = \lim_{k\to\infty} \tau_k^{\psi}$, as before. We show that (A.0.1) is sufficient to ensure that $\tau_\infty^{\psi} = \infty$, and therefore that $\tau_e^{\psi} = \infty$, by assuming the converse. That is, for some $\psi \in C([-\overline{\tau}, 0]; \mathbb{R}^d)$ there exists a pair of constants $T > 0$ and $\varepsilon \in (0,1)$ such that $\mathbb{P}[\tau_\infty^{\psi} \leq T] > \varepsilon$. Hence there is an integer $k_1 > k^*$ such that

$$\mathbb{P}[\tau_k^{\psi} \leq T] \geq \varepsilon, \text{ for all } k \geq k_1.$$

We define a sequence of times $\{t_n\}_{n\geq 0}$ depending on the delay structure $(\tau, \overline{\tau})$ as in (2.2.1). Up to the truncated stopping time $\tau_k^{\psi} \wedge t_1$, Itô's rule shows that the semimartingale decomposition of $V(\mathbf{X}(t))$ is given by

$$V(\mathbf{X}(\tau_k^{\psi} \wedge t_1)) = V(\mathbf{X}(0)) + \int_0^{\tau_k^{\psi}\wedge t_1} F(\mathbf{X}(s), \mathbf{g}_{(\tau,\overline{\tau})}(s, \mathbf{X}_s))\, ds + \int_0^{\tau_k^{\psi}\wedge t_1} G(\mathbf{X}(s))\, dB(s)$$

where $F(\mathbf{x}, \boldsymbol{\mu}) = L(\mathbf{x}) + 2V_s'(\|\mathbf{x}\|^2)\langle \mathbf{x}, \boldsymbol{\mu}\rangle$, $G(\mathbf{x}) = 2V_s'(\|\mathbf{x}\|)^2\langle \mathbf{x}, \mathbf{h}(\mathbf{x})\rangle$, and

$$L(\mathbf{x}) = 2\left(\frac{\langle \mathbf{x}, f(\mathbf{x})\rangle - \frac{1}{2}\|\mathbf{h}(\mathbf{x})\|^2}{1 + \|\mathbf{x}\|^2} \right).$$

L is a globally bounded function by virtue of condition (A.0.1). Therefore (A.0.1) guarantees that $\sup_{\mathbf{x}\in\mathbb{R}^d} |F(\mathbf{x}, \boldsymbol{\mu})| \leq C_1\|\boldsymbol{\mu}\| + C_2$ where C_1, C_2 are positive and $\boldsymbol{\mu}$-independent constants. Letting $|k| \to \infty$ delivers the contradiction required to show that τ_∞^{ψ} is not on $(0, t_1]$. We can continue this proof, as before, on successive intervals $(t_{n-1}, t_n]$. Thus, by induction, condition (A.0.1) ensures that a unique, strong solution to (A.0.2) exists on $[-\overline{\tau}, \infty)$. $\square$

Bibliography

[1] J. A. D. Appleby and E. Buckwar. Noise induced oscillation in solutions of delay differential equations. *Dynam. Systems Appl.*, 2005. To appear.

[2] J. A. D. Appleby and C. Kelly. Asymptotic and oscillatory properties of linear stochastic delay differential equations with vanishing delay. *Funct. Differ. Equ.*, 11(3–4):235–265, 2004.

[3] J. A. D. Appleby and C. Kelly. Oscillation and non-oscillation in solutions of nonlinear stochastic delay differential equations. *Electron. Comm. Probab.*, 9:106–118 (electronic), 2004.

[4] J. A. D. Appleby and C. Kelly. Prevention of explosion in solutions of functional differential equations by noise perturbation. *Dynam. Systems Appl.*, 2005. To appear.

[5] M. A. Berger and V. J. Mizel. Volterra equations with Itô integrals I. *J. Integral Equations*, 2(3):187–245, 1980.

[6] S. Castillo and M. Pinto. L^p perturbations in delay differential equations. *Electron. J. Differential Equations*, pages No. 55, 11 pp. (electronic), 2001.

[7] D. Dufresne. The distribution of a perpetuity, with applications to risk theory and pension funding. *Scand. Actuarial J.*, 1-2:39–79, 1990.

[8] V. Lakshmikantham G. Ladas and J. S. Papadakis. Oscillations of higher-order retarded differential equations generated by the retarded argument. In *Delay and functional differential equations and their applications, K. Schmitt, editor*, pages 219–231. Academic Press, 1972.

[9] V. Lakshmikantham G. S. Ladde and B. G. Zhang. *Oscillation theory of differential equations with deviating arguments*, volume 110 of *Monographs and Textbooks in Pure and Applied Mathematics.* Marcel Dekker, New York, 1987.

[10] Ĭ. Ī. Gīhman and A. V. Skorohod. *The theory of stochastic processes. I*, volume 210 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, english edition, 1980. Translated from the Russian by Samuel Kotz.

[11] K. Gopalsamy. *Stability and Oscillations in Delay Differential Equations of Population Dynamics.* Mathematics and its applications. Kluwer Academic Publishers, Dordrecht, 1992.

[12] A. A. Gushchin and U. Küchler. On oscillations of the geometric Brownian motion with time delayed drift. *Statist. Probab. Lett.*, 70(1):19–24, 2004.

[13] I. Györi. Sharp conditions for existence of nontrivial invariant cones of nonnegative initial values of difference equations. *Appl. Math. Comput.*, 36:89–111, 1990.

[14] F. Hartung I. Györi and J. Turi. Numerical approximations for a class of differential equations with time- and state-dependent delays. *Appl. Math. Lett.*, 8(6):19–24, 1995.

[15] K. Itô and M. Nisio. On stationary solutions of a stochastic differential equation. *J. Math. Kyoto Univ.*, 4:1–75, 1964.

[16] J. Jacod and P. Protter. *Probability Essentials.* Springer-Verlag., Berlin, 2003.

[17] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus*, volume 113 of *Graduate Texts in Mathematics.* Springer, New York, 1991.

[18] A. Karoui and R. Vaillancourt. A numerical method for vanishing-lag delay differential equations. *Appl. Numer. Math.*, 17(4):383–395, 1995.

[19] Fima C. Klebaner. *Introduction to stochastic calculus with applications.* Imperial College Press, London, 1999. Reprint of the 1998 original.

[20] V. Kolmanovskii and A. Myshkis. *Introduction to the theory and applications of Functional Differential Equations*, volume 463 of *Mathematics and its applications.* Kluwer Academic, Dordrecht, 1999.

[21] R. Koplatadze. Oscillation of linear difference equations with deviating arguments. *Comput. Math. Appl.*, 42(3–5):477–486, 2001.

[22] G. S. Ladde. Stability and oscillation in a single species process with past memory. *Int. J. System. Sci.*, 10:621–647, 1979.

[23] D. Lamberton and B. Lapeyre. *Introduction to stochastic calculus applied to finance.* Chapman & Hall, London, 1996. Translated from the 1991 French original by N. Rabeau and F. Mantion.

[24] X. Mao. Stochastic differential delay equations. In *Lecture notes for Spring School in Stochastic Delay Differential Equations, Humboldt University Berlin, March 2003.*

[25] X. Mao. *Stochastic differential equations and their applications.* Horwood Publishing Limited, Chichester, 1997.

[26] S. Mohamad and K. Gopalsamy. Dynamics of a class of discrete-time neural networks and their continuous-time counterparts. *Math. Comput. Simulation.*, 53(1–2):1–39, 2000.

[27] B. Øksendal. *Stochastic differential equations.* Universitext. Springer-Verlag, Berlin, fifth edition, 1998. An introduction with applications.

[28] P. E. Protter. On the existence, uniqueness, convergence and explosion of solutions of systems of stochastic integral equations. *Ann. Probab.*, 5:243–261, 1977.

[29] W. Shreve. Oscillation in first order nonlinear retarded argument differential equations. *Proc. Amer. Math. Soc.*, 41:565–568, 1973.

[30] V. A. Staïkos and I. Stavroulakis. Bounded oscillations under the effect of retardations for differential equations of arbitrary order. *Proc. Roy. Soc. Edinburgh Sect. A*, 77(1-2):129–136, 1977.

[31] G. Marion X. Mao and E. Renshaw. Environmental Brownian noise suppresses explosions in population dynamics. *Stochastic Process. Appl.*, 97(1):95–110, 2002.