Asian and Australian Options: A Common Perspective

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Abstract

We show that Australian options are equivalent to fixed or floating strike Asian options and consequently that by studying Asian options from the Australian perspective and vice versa, much can be gained. One specific application of this "Australian Approach" leads to a natural dimension reduction for the pricing PDE of Asian options, with or without stochastic volatility, featuring time independent coefficients. Another application lies in the improvement of Monte Carlo schemes, where the "Australian Approach" results in a path-independent method. We also show how the Milevsky and Posner (1998) result on the reciprocal Γ-approximation for Asian options can be quickly obtained by using the connection to Australian options. Further, we present an analytical (exact) pricing formula for Australian options and adapt a result of Carr, Ewald and Xiao (2008) to show that the price of an Australian call option is increasing in the volatility and by doing this answering a standing question by Moreno and Navas (2008).

Keywords: Asset pricing; Derivatives; Asian Options; Quanto Options; Dollar cost averaging (DCA); Numerical Methods

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1 Introduction

Australian options are options, whose payoffs depend on the quotient of the average stock price over a specified time interval and the stock price at maturity. The average can be taken in various ways, most prominently are the arithmetic or geometric average. Referring to Australian options in the following, we mean arithmetic Australian options with payoff of type $h\left(\frac{\int_0^T S(t)\,dt}{S(T)}\right)$. If we refer to the geometric case, we state this explicitly. A fundamental question is how to price such options and how these option fit into the existing spectrum of other better known options.

Australian options occur as special types of variable purchase options (VPOs) and have been traded on the Australian stock exchange since 1992. Depending on an average, these options have attached an Asian feature, and are in fact one of very few examples of Asian type options which are traded on an institutionalized option market. Almost all Asian type options are traded OTC, which makes an empirical analysis of their prices almost impossible. For financial research, the existence of exchange traded Australian options is extremely valuable, as "academic" pricing formulas can then be verified and models calibrated at market prices.

Australian options are related to Quanto options and dollar cost averaging (DCA). A Quanto option is a derivative, where the underlying is denoted in one currency, but settlement is in a different currency (see for instance Reiner (1992)). If the underlying of a Quanto option is of average type, the product is closely related to an Australian option. This analogy comes to light, if the underlying $\frac{\int_0^T S(t)\,dt}{S(T)}$ is interpreted as the value of the average $\int_0^T S(t)$ in units of $S(T)$, while the payoff is denoted in units of discounted money.[1] For further information on quanto options we refer to Dimitroff et al. (2009),

[1]A practical example for this particular type of option, where $S(t)$ is interpreted as an exchange rate, is the following: A company which exports its products and sells overseas accumulates revenues in a foreign currency. To save on transaction costs, these revenues are held in a foreign bank account and transferred back annually to a domestic bank account. Assuming for simplicity that the domestic price and quantity are normalized to one, then the revenue of this company at the end of the year is $\frac{\int_0^T S(t)dt}{S(T)}$ and to protect revenues from downside risks, the company may in fact purchase an option of the Australian type discussed in this article.

Kwok and Wong (2005) as well as Martzoukos (2001). The inverse of the underlying of an Australian option on the other side resembles the underlying of a dollar cost averaging strategy. Under a DCA strategy an investor makes gradual investments in the stock market. Assuming continuous time and that the instantaneous investment rate is a constant proportion of total initial wealth, the payoff of a DCA strategy is proportional to $\int_0^T \frac{S(T)}{S(t)} dt$, compare for example Vanduffel et al. (2012) or Constantinides (1979).

Obviously Australian options formally differ from Asian options, where the underlying is simply the average, and not a quotient, and it is unclear, whether the prices of exchange traded Australian options can be used for research on Asian options as well. As we show in this paper, Australian options are in fact truly Asian, that is, after the interest rate is adjusted appropriately. Virtually all data available for Australian options, can therefore be applied to study Asian options.

In the literature, Australian options have been considered in Handley (2000), Handley (2003), and Moreno and Navas (2008). Moreno and Navas have considered various pricing techniques, including the application of Milevsky and Posner (1998). However, they fail to give an explanation as to why this result can be applied, and simply match the first two moments of the Australian underlying to the reciprocal Γ-distribution. In this paper, the Milevsky and Posner result for Australian options will be derived from scratch, using that the inverse of the Australian option underlying, is in fact a geometric mean reversion process, and that the equilibrium distribution of this process is the Γ-distribution. This is far more direct and shorter than the derivation given in Milevsky and Posner (1998). Knowing that Australian options and Asian options are practically the same thing, our result carries over to the classical Asian case, and in fact includes Milevsky and Posner's (1998) Theorem 1. We point out though, that because of a coefficient restriction which appears in Milevsky and Posner (1998) and which inverts in the Australian case, the application of the reciprocal Γ-distribution as an appropriate approximating distribution to evaluate Australian options, is slightly limited.

For this reason, we also study the pricing of Australian options via PDE methods. We derive PDE's for the price of an Australian option under both the Black-Scholes assumption and the assumption of stochastic volatility. The interesting thing here is, that the PDEs derived feature a "natural" dimension reduction by one, similar as in Vecer (2001), with the advantage of having time independent coefficients but also a disadvantage of potential instabilities in the numerical solution for small volatilities (which is due to the relationship to the PDE by Rogers and Shi (1995)).

In the Black-Scholes case, Moreno and Navas (1998) questioned, whether the price of an Australian option is increasing in the volatility parameter σ. When the average is geometric, this is in fact not the case and Moreno and Navas give examples for this. However, if the average is arithmetic, we can use our PDE and draw on similar techniques as used in Carr, Ewald, and Xiao (2008) in order to conclude, that Australian call options are indeed increasing in price with increasing volatility.

Finally, we study the implications of our "Australian" approach on the pricing of Asian options via Monte Carlo methods. Under the assumption of stochastic volatility, we show that using the Australian underlying, which transforms the path-dependent Asian option into a path-independent option, Monte Carlo performs smoother and better.

While there are only a few research papers on Australian options, the pricing of Asian options has been studied extensively. For instance, Kemna and Vorst (1990) developed a Monte Carlo scheme for pricing Asian options, Turnbull and Wakeman (1991) introduced a quick algorithm for pricing Asian options, Geman and Yor (1993 and 1996) derived an integral representation, Rogers and Shi (1995) reduced the dimension of the PDE and derived upper and lower bounds for the prices of Asian options, and Carr and Schröder (2004) give an overview of deriving Asian option prices via Laplace transforms. For a more comprehensive literature review we refer to the excellent survey paper of Boyle and Potapchik (2008).[2]

[2]With the exception of Boyle and Potapchik (2008) all of the authors above address exclusively the pricing aspect, but not the hedging aspect. Hedging strategies of Asian options are studied by Albrecher et al. (2005) for incomplete markets and Jacques (1995) for Black-Scholes markets. However, these hedging strategies only present approximations. Recently, Yang et al. (2011) and Vecer (2011) developed exact hedging strategies under

The remainder of the paper is organized as follows. In Section 2, we show that Australian options are equivalent to both fixed strike and floating strike Asian options, using two different approaches, time inversion and change of measure. We provide a quick proof of the Milevsky and Posner result in Section 3, while in Section 4 we provide an approximate and an analytic pricing formula for Australian options. We derive a pricing PDE for Australian options under the assumptions of constant volatility and study the effect of volatility changes on the price in Section 5. Then, in Section 6, we consider the case of stochastic volatility, followed by a discussion of numerical aspects and results in Sections 7 and 8. The main conclusions are summarized in Section 9.

2 The relationship between Australian and Asian options

In this section, we show how Australian options with payoff structure

$$h\left(\frac{\int_0^T S(t)\,dt}{S(T)}\right) \tag{1}$$

as introduced in Section 1, and classical Asian options with payoff structure

$$h\left(\int_0^T S(t)\,dt\right) \tag{2}$$

are related to each other and what consequences we can infer from this relationship. We start by considering the Black-Scholes case of constant volatility, where

$$dS(t) = S(t)\left[r\,dt + \sigma\,dW(t)\right]\,. \tag{3}$$

and r denotes the market interest rate. The case of stochastic volatility is considered in Section 6.

Black-Scholes assumptions.

Proposition 2.1. *Setting $\tilde{r} = \sigma^2 - r$, we have the following equivalence in distribution*[3]

$$\left\{ \begin{array}{c} \frac{\int_0^T S(t)\,dt}{S(T)} \\ dS(t) = S(t)\,[r\,dt + \sigma\,dW(t)] \end{array} \right\} \sim \left\{ \begin{array}{c} \int_0^T S(t)\,dt \\ dS(t) = S(t)\,[\tilde{r}\,dt + \sigma\,dW(t)] \\ S(0) = 1 \end{array} \right\}. \tag{4}$$

Proof. The key to the proof of Proposition 2.1. is a well known result on time inversion, which states that if $W(t)$ is a Brownian motion on $[0,T]$, the process

$$B(t) = W(T-t) - W(T) \tag{5}$$

is also a Brownian motion on $[0,T]$. The expression on the left hand side of (4) can now be written as

$$\begin{aligned} \int_0^T \frac{S(t)}{S(T)}\,dt &= \int_0^T \exp\left(\left(r - \frac{1}{2}\sigma^2\right)(t-T) + \sigma(W(t) - W(T))\right)\,dt \\ &= \int_0^T \exp\left(\left(r - \frac{1}{2}\sigma^2\right)((T-t)-T) + \sigma(W(T-t) - W(T))\right)\,dt \\ &= \int_0^T \exp\left(-\left(r - \frac{1}{2}\sigma^2\right)t + \sigma B(t)\right)\,dt\,, \end{aligned}$$

where for the second equality we used the substitution $t \mapsto T-t$ and for the third equality Equation (5). Clearly, the integrand on the right hand side satisfies the stochastic differential equation

$$dS(t) = S(t)\left[\left(\sigma^2 - r\right)\,dt + \sigma\,dB(t)\right]\,,$$

which is the same as on the right hand side of (4). □

This relationship means that virtually any result obtained for the case of Asian options can also be applied to the case of Australian options and vice versa. In particular, the prices of Asian and Australian options coincide,

[3]If the objective is to price an Asian option and to use the Australian approach, then consider $\tilde{r}$ as the market interest rate, and $r = \sigma^2 - \tilde{r}$ is a hypothetical interest rate used in the Australian approach.

when taking account of the interest rate transformation $\tilde{r} = \sigma^2 - r$ and allowing for a different discount factor. Care needs to be taken though, where the application of analytical results requires conditions on the parameters. This is often the case, and these conditions need to be checked on a case by case basis, taking the transformation $\tilde{r} = \sigma^2 - r$ into account.

In the following, we present a more general derivation, which extends the case above and also includes the case of floating strike Asian options. We assume that the tradeable asset is paying a dividend q and follows the dynamic

$$dX(t) = (r - q)X(t)dt + \sigma X(t)dW(t).$$

under the risk neutral measure. We observe that the price of a more general type of Australian option[4], can be computed as

$$AusPrice = \frac{e^{-r(T-t)}}{T}\mathbb{E}\left[\frac{1}{X(T)}\left(\int_0^T X(u)du - k_1T - k_2TX(T)\right)^+ \middle| \mathcal{F}_t\right].$$

We introduce an equivalent probability measure $\mathbb{Q}$ defined via the Radon-Nikodym derivative,

$$\begin{aligned}\frac{d\mathbb{Q}}{d\mathbb{P}} &= \frac{X(0)e^{(r-q-\sigma^2)T}}{X(T)} \\ &= \exp\left(-\frac{1}{2}\int_0^T \sigma^2\, du - \int_0^T \sigma\, dW_u\right).\end{aligned}$$

It follows from the Girsanov theorem, that

$$W(t)^{\mathbb{Q}} = W(t) + \sigma t$$

is a Brownian motion under the measure $\mathbb{Q}$. Given σ is a constant, Novikov's condition is thus satisfied, and the Radon-Nikodym derivative is valid. Using Lemma 8.9.2 in Kuo (2006), the Australian option price at time t can then

[4]The payoff for this option is $\left(\frac{x}{T} - k_1 - k_2y\right)^+$, where $x = \frac{\int_0^T X(u)du}{X(T)}$ and $y = X(T)$. The $\frac{1}{T}$ factor in the first term is chosen so that this type of Australian option can be related to the standard Asian option problem.

be computed as,[5]

$$
\begin{aligned}
&AusPrice(r)\\
&= \frac{e^{-r(T-t)}}{X(0)e^{(r-q-\sigma^2)T}}\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\left(\frac{1}{T}\int_0^T X(u)\,du - k_1 - k_2X(T)\right)^+ \Bigg| \mathcal{F}_t\right]\\
&= \frac{e^{-r(T-t)}}{X(0)e^{(r-q-\sigma^2)T}}\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\Bigg|\mathcal{F}_t\right]\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T X(u)\,du - k_1 - k_2X(T)\right)^+ \Bigg| \mathcal{F}_t\right]\\
&= \frac{e^{-(2r-q-\frac{1}{2}\sigma^2)(T-t)}}{X_t}\mathbb{E}\left[e^{-\sigma W_{T-t}}\right]\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T X(u)\,du - k_1 - k_2X(T)\right)^+ \Bigg|\mathcal{F}_t\right]\\
&= \frac{e^{-(r-q)(T-t)}}{X(t)}\; e^{-(r-\sigma^2)(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{1}{T}\int_0^T X(u)du - k_1 - k_2X(T)\right)^+ \Bigg| \mathcal{F}_t\right]\\
&= \frac{e^{-(r-q)(T-t)}}{X(t)}\; AsianPrice(r-\sigma^2),
\end{aligned}
$$

where under $\mathbb{Q}$ the process $X(t)$ satisfies

$$dX(t) = (r - \sigma^2 - q)X(t)dt + \sigma X(t)dW(t)^{\mathbb{Q}}.$$

The last expectation above is equal to a general Asian option, but with the risk-free rate now being $r-\sigma^2$.[6] The case $k_2 = 0$ corresponds to a fixed strike Asian option, while the case $k_1 = 0$ corresponds to a floating strike Asian option.

Using the equivalence for Australian options as Asian options, setting r to $r + \sigma^2$ and rearranging the terms, gives the result for Asian options as Australian options. This can also be worked out from first principles by similar arguments, but using the change of measure $\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{X(T)}{X(0)e^{(r-q)T}}$. We

[5] An argument a in $AsianPrice(a)$ or $AusPrice(a)$ indicates that the option price is computed using a discount rate of a. All prices are computed at time t. Compare footnote 3.

[6] Note that the first equality in the sequence above relies on the multiplicative linearity of the payoff function and that this line of arguments would fail for pay-off functions featuring non-linearities, e.g. power options.

then obtain for an Asian option, the formula

$$
\begin{aligned}
AsianPrice(r) &= X(t)e^{-q(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{AU(T)}{T}-k_2\right)^+ \middle| \mathcal{F}_t\right] \\
&= X(t)e^{(r+\sigma^2-q)(T-t)}AusPrice(r+\sigma^2),
\end{aligned}
\tag{6}
$$

with

$$
dAU(t) = (q-r)AU(t)dt + dt - \sigma AU(t)dW(t)^{\mathbb{Q}}.
$$

As demonstrated above, Asian and Australian options are hence equivalent after the drift rate is adjusted appropriately. This of course is very relevant for the pricing of Australian options. But beyond this, we will see in the following sections, that a lot can be learned about Asian options, from looking at Asian options from what we call the ”Australian Perspective”, thus this equivalence can be used for the mutual benefit of pricing both Asian and Australian options.

3 A simplified proof of the Milevsky and Posner reciprocal Γ-approximation formula

We have shown in the previous section that under the assumption of constant volatility Asian and Australian options are in principle the same thing.[7] A particular conclusion from this is that the approximative pricing formula obtained by Milevsky and Posner (1998) also holds for Australian options. This was indicated in Moreno and Navas (2008) but not executed. Milevsky and Posner's result was obtained by using results and methodology for the computation of distributions of first hitting times of stochastic processes. Here we will go the other way round. Using information about the equilibrium distribution of geometric mean reversion, we will derive an approximate option pricing formula for Australian options, and from this and Proposition 2.1.

[7] Using the same line of arguments as presented in Section 6, where the case of stochastic volatility is considered, this equivalence extends to deterministic time dependent coefficients.

conclude the Milevsky and Posner result. In order to do this, we consider the Australian process

$$AU(t) = \frac{\int_0^t S(u)\,du}{S(t)} \tag{7}$$

and its inverse, the New Zealander

$$N\!Z(t) = \frac{S(t)}{\int_0^t S(u)\,du}. \tag{8}$$

Note that $N\!Z(t)$ is only defined for $t > 0$, but this is not of belong here. Applying the Itô formula to $N\!Z(t)$ gives

$$\begin{aligned} dN\!Z(t) &= d\left(\frac{S(t)}{\int_0^t S(u)\,du}\right) \\ &= S(t)\,d\left(\frac{1}{\int_0^t S(u)\,du}\right) + \left(\frac{1}{\int_0^t S(u)\,du}\right) dS(t) \\ &= -\left(\frac{S(t)}{\int_0^t S(u)\,du}\right)^2 dt + \left(\frac{S(t)}{\int_0^t S(u)\,du}\right)(r\,dt + \sigma\,dW(t)) \\ &= -N\!Z(t)^2\,dt + N\!Z(t) r\,dt + N\!Z(t)\sigma\,dW(t) \\ &= N\!Z(t)\left[(r - N\!Z(t))\;dt + \sigma\,dW(t)\right]\,. \end{aligned}$$

This shows that $N\!Z(t)$ is a geometric mean reversion process, with mean reversion level r and mean reversion speed 1. As is well known, geometric mean reversion admits an equilibrium distribution, if certain conditions on the coefficients apply. These conditions translate in the case of $N\!Z(t)$ to $2r > \sigma^2$. The equilibrium distribution, intuitively the distribution of $N\!Z(\infty)$, is then given by the Γ-distribution:

$$N\!Z(\infty) \sim \Gamma(k, \delta)$$

with $k = \frac{2r}{\sigma^2} - 1$, $\delta = \frac{\sigma^2}{2}$ and density

$$\pi(k,\delta)(x) = \left\{ \begin{array}{cc} x^{k-1}\frac{e^{-x/\delta}}{\delta^k \Gamma(k)} & x > 0 \\ 0 & x \leq 0 \end{array} \right\}. \tag{9}$$

The condition $2r > \sigma^2$ imposes some serious constraints on interest rate and volatility, and care needs to be taken here. Using the same notation $g_R(\cdot|\alpha,\beta)$ for the reciprocal Γ-distribution, we conclude the analogue to Milevsky and Posner's result for Australian options:

Proposition 3.1. *Under the condition $r - \frac{1}{2}\sigma^2 > 0$, we have that $AU(\infty)$ is reciprocal Γ-distributed, i.e.*

$$AU(\infty) \sim g_R(\cdot|\alpha,\beta) \tag{10}$$

with $\alpha = \frac{2r}{\sigma^2} - 1 > 0$ and $\beta = \frac{\sigma^2}{2}$.

We will derive an explicit approximate option pricing formula for Australian options from this later on, but before this draw the following Corollary from Proposition 3.1. and Proposition 2.1. using $\tilde{r} = \sigma^2 - r$:

Corollary 3.1. *Under the condition $\tilde{r} - \frac{1}{2}\sigma^2 < 0$, we have that $I(\infty) = \int_0^\infty S(u)\,du$ is reciprocal Γ-distributed, i.e.*

$$I(\infty) \sim g_R(\cdot|\alpha,\beta) \tag{11}$$

with $\alpha = 1 - \frac{2\tilde{r}}{\sigma^2} > 0$ and $\beta = \frac{\sigma^2}{2}$.

Obviously, Corollary 3.1 is identical with Theorem 1 in Milevsky and Posner (1998), but its proof has essentially be shortened to a couple of lines.

4 Pricing formulas for Australian options

Using that $AU(\infty)^{-1} = NZ(\infty)$ is Γ-distributed, we obtain that

$$\begin{aligned}
\mathbb{E}\left[(AU(\infty) - K)^+\right] &= \mathbb{E}\left[\left(\frac{1}{NZ(\infty)} - K\right)^+\right] \\
&= \int_0^{1/K} \left(\frac{1}{x} - K\right) d\Gamma(k,\delta)(x) \\
&= \int_0^{1/K} \frac{1}{x}\, d\Gamma(k,\delta)(x) - K \cdot \Gamma(k,\delta)(1/K)\,,
\end{aligned}$$

with k and δ as above. We use that

$$\frac{1}{x}\pi(k,\delta)(x) = \frac{1}{\delta(k-1)}\pi(k-1,\delta)(x)$$

to compute the remaining integral and obtain

$$\mathbb{E}\left[(AU(\infty)-K)^+\right] = \frac{\Gamma\left(k-1,\delta,\frac{1}{K}\right)}{\delta\,(k-1)} - K\cdot\Gamma\left(k,\delta,\frac{1}{K}\right). \tag{12}$$

The following proposition is now simply obtained by expressing k and δ in terms of the original variables and by discounting to the corresponding time.

Proposition 4.1. *Under the condition $r > \sigma^2$, an approximate option pricing formula in the sense of Milevsky and Posner for an Australian call option*

$$\left(\frac{\int_0^T S(u)\,du}{S(T)} - K\right)^+ \tag{13}$$

is given by

$$\Pi_{AU} \approx e^{-rT}\left\{\frac{\Gamma\left(\frac{2r}{\sigma^2}-2,\frac{\sigma^2}{2},\frac{1}{K}\right)}{r-\sigma^2} - K\cdot\Gamma\left(\frac{2r}{\sigma^2}-1,\frac{\sigma^2}{2},\frac{1}{K}\right)\right\}, \tag{14}$$

where $\Pi_{AU} = e^{-rT}\mathbb{E}\left[(AU(T)-K)^+\right]$ denotes the current price of the Australian call option.

This formula is a good approximation in the sense that at least in theory, the distribution of the underlying approaches the distribution used for computing the option price in (14) for $T\to\infty$. In practice, there are many problems. The assumption that $r > \sigma^2$ restricts the application significantly. Even, when the assumption is satisfied, we experience quite slow convergence. The way how Milevsky and Posner (1998) promote to apply their result is of course slightly different from the above. They propose to match the first two moments of the underlying, which in case of a classical Asian option and $I(t) = \int_0^t S(u)\,du$ are known, to the first two moments of the reciprocal Γ-distribution. For the case of an Australian option, Moreno and Navas (2008) computed expectation and variance of the underlying:

$$\begin{aligned}\mathbb{E}\left[AU(T)\right] &= \frac{1-e^{-(r-\sigma^2)T}}{r-\sigma^2} \\ \mathrm{Var}\left[AU(T)\right] &= e^{-(2r-3\sigma^2)T}\left(2\frac{\phi(2r-3\sigma^2)-\phi(r-2\sigma^2)}{r-\sigma^2} - e^{-\sigma^2 T}\phi^2(r-\sigma^2)\right)\end{aligned} \tag{15}$$

with $\phi(x) = \frac{e^{xT}-1}{x}$ for $x \neq 0$ and $\phi(0) = T$. We point out that in order to have that the equilibrium distribution of $AU(t)$ has finite expectation and variance, one actually needs to restrict the interest rate further to satisfy $r > \sigma^2$ for the expectation, and $r > 2\sigma^2$ for the variance. Empirical evidence would generally point out, that in many cases the latter condition is not satisfied. Nevertheless, matching these two to the expectation $E_R = \frac{1}{\delta(k-1)}$ and variance $V_R = \frac{1}{\delta^2(k-1)^2(k-2)}$ of the reciprocal Γ-distribution and using the obtained parameters and formula (12) rather than formula (14), one can in theory obtain reasonably accurate results. Moreno and Navas (2008) (table 2, page 82) demonstrate this, choosing an interest rate of $r = 10\%$. For lower interest rates however, we experienced significant errors in applying the reciprocal Γ-distribution to price Australian options. Analog formulas to (12) and (14) can be obtained for the case where the underlying is $NZ(t)$, as well as for the corresponding put option.

We have already indicated, that for low interest rates or high volatility the formulas derived above run into trouble. To have an alternative at hand, we provide in the remainder of this section an analytic formula for the price of an Australian option, and in the next two sections, present PDE based approaches.

Proposition 4.2. *The actual density function of $NZ(t)$ is given by*

$$\begin{aligned}p_t(x) &= \frac{\sigma^2}{4x^2}\exp\left(-\frac{(\frac{\sigma^2}{2}-r)^2 t}{2\sigma^2}\right)\times \\ &\int_{-\infty}^{\infty}\exp\left(2\left(1-\frac{\frac{\sigma^2}{2}-r}{\sigma^2}\right)z\right) f_{\frac{\sigma^2 t}{4}}\left(\frac{\sigma^2\exp(2z)}{4\kappa x}, z\right) dz\end{aligned}$$

for $y>0$ *and* $p_t(y)=0$ *for* $y\le 0$*. Here* $f_t(x,y)=0$ *for* $x\le 0$*, and*

$$f_t(x,y)=\rho_t(x,y)\int_0^\infty \exp\left(-\frac{z^2}{2t}-\frac{\exp(y)}{x}\cosh(z)\right)\sinh(z)\sin\left(\frac{\pi z}{t}\right)\,dz \tag{16}$$

for $x>0$*, where*

$$\rho_t(x,y)=\left(x^2\sqrt{2\pi^3 t}\right)^{-1}\exp\left(\frac{2xyt+\pi^2 x-t-t\exp(2y)}{2xt}\right).$$

Proof. This is a direct consequence of Yang and Ewald (2010). □

Using this density function, we obtain the following result:

Proposition 4.3. *The price of an Australian call option with strike* K *and maturity* T *is given by*

$$\begin{aligned}\Pi_{AU} \quad &= \quad \int_0^{1/K}\left(\frac{\sigma^2}{4x^2}\exp\left(-\frac{(\frac{\sigma^2}{2}-r)^2 t}{2\sigma^2}\right)\times\right.\\ &\left.\int_{-\infty}^{\infty}\exp\left(2\left(1-\frac{\frac{\sigma^2}{2}-r}{\sigma^2}\right)z\right)f_{\sigma^2 t/4}\left(\frac{\sigma^2\exp(2z)}{4\kappa x},z\right)dz\right)(x-K)\,dx\,.\end{aligned}$$

The price of an Australian put option with payoff $(K-AU(T))^+$ can be calculated using the following Australian put–call parity:

$$\begin{aligned}&e^{-r(T-t)}\mathbb{E}\left[(K-AU(T))^+\middle|AU(t)=\eta\right] \qquad (17)\\ &=e^{-r(T-t)}\mathbb{E}\left[(AU(T)-K)^+\middle|AU(t)=\eta\right]+Ke^{-r(T-t)}+\\ &\quad -e^{-r(T-t)}\mathbb{E}\left[AU(T)\middle|AU(t)=\eta\right],\end{aligned}$$

where the last term is the price of an Australian forward, which can be easily calculated as

$$\mathbb{E}\left[AU\left(T_1\right)\middle|AU(t)=\eta\right]=\begin{cases}\eta e^{\left(\sigma^2-r+q\right)(T_1-t)} \\ \quad +\frac{e^{\left(\sigma^2-r+q\right)(T_1-t)}-1}{(\sigma^2-r+q)} & \text{if } \sigma^2-r+q\neq 0, \\ (T_1-t)+\eta & \text{if } \sigma^2-r+q=0,\end{cases} \tag{18}$$

for $t \leq T_1 \leq T$. We will further discuss the implications of the Australian put-call parity in the next section.

5 A pricing PDE and volatility effects

Within the Black-Scholes framework Moreno and Navas (2008) show that the price of a geometric Australian call option can be decreasing in volatility, but leave the question open if this can happen for an arithmetic Australian call option. In fact, the analogue question for the case of an arithmetic Asian option, was long unanswered until Carr, Ewald, and Xiao (2008), who showed (by using the Pontryagin maximum principle), that the price of an arithmetic Asian option is indeed increasing with volatility. Motivated by this result, Baker and Yor (2009) later gave a martingale based proof, and in fact extended the result. In this section, we derive a pricing PDE for an Australian option under the assumption of constant volatility and by using the Pontryagin maximum principle show that the price of an arithmetic Australian call option is indeed increasing with the volatility parameter.

Deriving the pricing PDE for an Australian option is very interesting from the perspective that, when using the process $AU(t)$ as an underlying (compare e.g. Vecer (2011)), the pricing PDE naturally features one state variable less than it would normally do. This achievement carries over to Asian options from the general correspondence between Australian and Asian options discussed in Section 2. The Australian perspective on Asian options hence naturally leads to a dimension reduction.

Applying stochastic partial integration and the Itô formula, we obtain

$$
\begin{aligned}
d\left(\frac{\int_0^t S(u)\,du}{S(t)}\right) &= \left(\int_0^t S(u)\,du\right) d\left(\frac{1}{S(t)}\right) + \frac{1}{S(t)}\, d\left(\int_0^t S(u)\,du\right) \\
&= \left(\int_0^t S(u)\,du\right)\left(-\frac{1}{S(t)^2}\,dS(t) + \frac{1}{2}\frac{2}{S(t)^3}\,(dS(t))^2\right) + dt \\
&= \left(\frac{\int_0^t S(u)\,du}{S(t)}\right)\left(-r dt - \sigma\, dW(t) + \sigma^2\, dt\right) + dt
\end{aligned}
$$

and with $AU(t) = \left(\frac{\int_0^t S(u)\,du}{S(t)}\right)$, we conclude that

$$
dAU(t) = -AU(t)\sigma\, dW(t) + AU(t)\left(\sigma^2 - r\right)\, dt + dt. \tag{19}
$$

This can of course also be concluded from our previous result by using that $NZ(t)$ is geometric mean reversion and by applying the Itô formula to $AU(t) = NZ(t)^{-1}$. However, this is not much shorter, and less instructive. We can now apply the Feynman-Kac theorem to price the Australian option: The price function

$$
v(x,t) = \mathbb{E}\left[e^{-r(T-t)}\left(AU(T) - K\right)^+ | AU(t) = x\right] \tag{20}
$$

satisfies the PDE

$$
\begin{aligned}
v_t + \left(\left(\sigma^2 - r\right)x + 1\right)v_x + \frac{1}{2}\sigma^2 x^2 v_{xx} - rv &= 0 \\
v(x,T) &= (x-K)^+.
\end{aligned} \tag{21}
$$

Note that this PDE has time independent coefficients. This is a significant advantage as compared to Vecer (2001). However, there are also some disadvantages. In the case of discrete monitoring times (a case which is not considered in this paper) the process $AU(t)$ jumps and the set of partial differential equations replacing (19) needs to be pasted together at the monitoring times, which results in extra computational effort. Further, it is known that the related Rogers and Shi (1995) PDE can lead to instabilities in the numerical solutions for small volatilities. While Vecer's (2001) PDE does not suffer from this problem, Dewynne and Shaw (2008) have shown that low volatility asymptotic techniques are able to provide highly accurate approximations to Asian options when using a PDE with time independent coefficients.[8] A similar approach is described in Ting and Ewald (2011), showing that the technique is applicable to the PDE in (19). Dewynne and Shaw (2008) relied on transforming the PDE of the Asian option with two spatial dimensions, by using ratios of the option price over the stock price. What is done here, effectively leads to the same result, but is more direct and provides more insights as to what is actually being priced. Furthermore, the payoff functions considered in Dewynne and Shaw (2008) are either fixed or floating strike, and not a combination of both. Rogers and Shi (1995) also considered dimension reduction through means of probability arguments, similar to what is presented here. However, like Dewynne and Shaw (2008), they too lack the unified nature of Vecer and Xu (2004) in considering general Asian payoffs.

Let us now come to the question whether the price of an Australian call option is increasing in σ. Making use of the Pontryagin maximum principle we can then prove the following result (see Appendix A.1 for more details):

Proposition 5.1. *Denote by* $\Pi_{AU}(\sigma) = e^{-rT}\mathbb{E}\left[(AU(T) - K)^{+}\right]$ *the price of an Australian call option as a function of the volatility parameter* $\sigma > 0$. *Then* $\Pi_{AU}(\sigma)$ *is increasing in* σ.

Figure 1 shows fixed strike Australian call option prices for various strikes K and various volatilities σ. The prices are calculated using Crank-Nicholson and PDE (21) with parameters $r = 0.1$, $T = 1$ and initial stock price $S_0 = 100$.[9] This figure confirms the statement of Proposition 5.1 — that is, the Australian call option price is increasing in the volatility parameter σ.

Let us note at this point that the proof of Proposition 5.1 would also go through for the Australian forward which can be obtained from the Australian call option by setting $K = 0$. In consequence, the Australian forward

[8] Dewynne and Shaw (2008) present a PDE for Asian options with one spatial variable and time independent coefficients. It is not difficult to show that their PDE is equivalent to the PDE obtained when applying the Feymann Kac theorem to Equation (6).

[9] Note that since these are fixed strike Australian call options, their prices are actually independent of S_0. We observe spurious oscillation for some grid sizes, a phenomenon described in Duffy (2004).

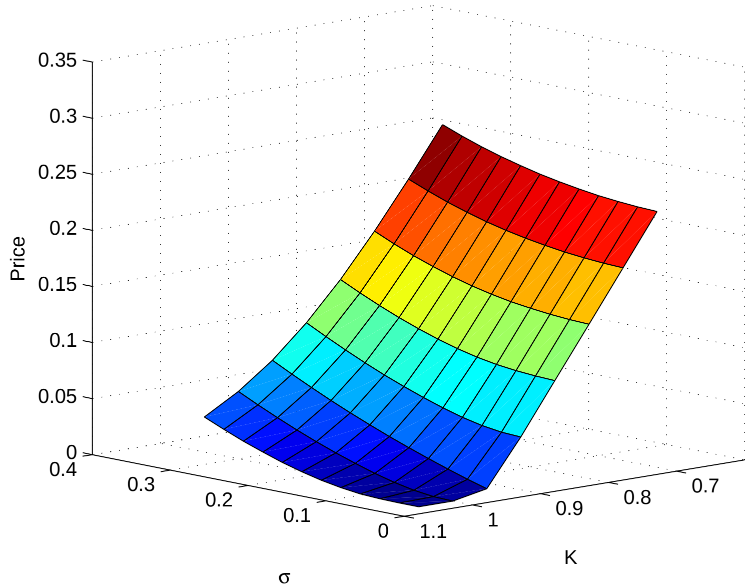

Figure 1: Fixed Strike Australian Call Option Prices as a Function of the Strike Price K and the Volatility σ.

is also increasing in σ. This fact can also be derived directly from the expectation in Equation (15) by verifying that the derivative of the right hand side with respect to σ is always non-negative. A more interesting case, however, is the case of an Australian put. The price of the Australian put can be computed by the put-call parity (17). As both, the Australian call and the Australian forward are increasing in σ and appear with opposite signs on the right hand side of (17), it is not clear whether the Australian put is increasing in σ. In fact, as Figure 2 below shows, the price of an Australian put is not monotonic in σ and can fall, when volatility increases. The same parameters as for Figure 1 have been used here.

This shows that the relationship between Australian calls and puts is very different from European calls and puts, where in the latter case, the forward price is independent of σ and the (European) put-call parity implies the same monotonic behavior for European put and call.[10]

[10]Note that the proof of Proposition 5.1 hinges on the fact that the pay-off function is monotonic increasing, which is not the case for the Australian put.

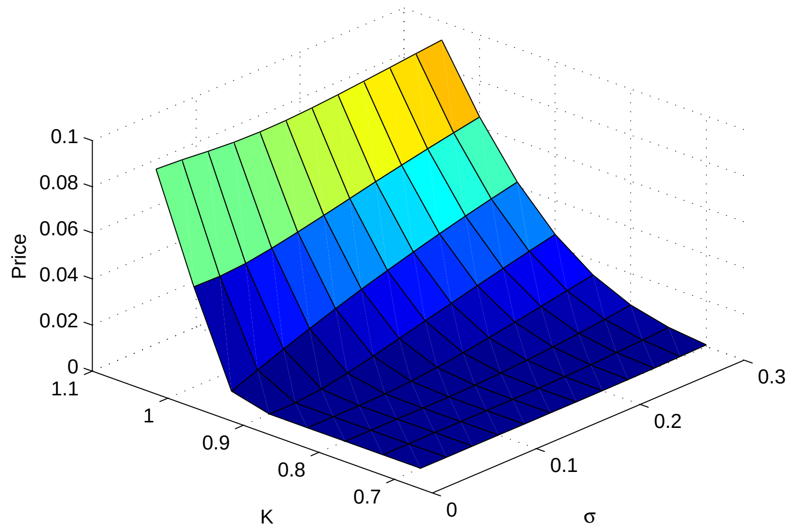

Figure 2: Fixed Strike Australian Put Option Prices as a Function of the Strike Price K and the Volatility σ.

6 Pricing under stochastic volatility

In this section, we consider Australian and Asian options under the assumption of stochastic volatility. We choose a specific functional form, the Heston model, but the results presented carry over to other stochastic volatility models without much effort. We generalize the discussion presented in section 2 on the equivalence of these two type of options.

Let the stock price $X(t)$ and variance process $Y(t)$ under the measure $\mathbb{P}$ be given via the SDEs,

$$dX(t) = (r-q)X(t)dt + \sqrt{Y(t)}X(t)dW(t) \tag{22}$$

$$dY(t) = \alpha\left(m - Y(t)\right)dt + \beta\sqrt{Y(t)}dZ(t)\,, \tag{23}$$

where r and q are the risk-free interest and dividend yield of the stock, α, m, and β are the mean reverting rate, mean reverting level, and volatility of volatility of the variance process, respectively. Further, $W(t)$ and $Z(t)$ are Brownian motions with correlation ρ. We can write $Z(t) = \rho W(t) +$

$\sqrt{1-\rho^2}\tilde{Z}(t)$, where $\tilde{Z}(t)$ is a Brownian motion independent of $W(t)$.

The solution of the stock price, given $\mathcal{F}_t$, can be written in integral form as

$$X(T) = X(t)\exp\left((r-q)(T-t) - \frac{1}{2}\int_t^T Y(u)du + \int_t^T \sqrt{Y(u)}dW_u\right),$$

and the price of a general Asian call option with fixed strike k_1 and floating strike k_2 is given as,

$$AsianPrice \quad = e^{-r(T-t)}\mathbb{E}\left[\left(\tfrac{1}{T}\int_0^T X(u)du - k_1 - k_2X(T)\right)^+ \middle| \mathcal{F}_t\right]. \quad (24)$$

Similar as in section 2, we define an equivalent measure $\mathbb{Q}$ via the Radon-Nikodym derivative,

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{X(T)}{X(0)e^{(r-q)T}}.$$

With this, the process defined by

$$W^*(t) = W(t) - \int_0^t \sqrt{Y(u)}du,$$

is according to the Girsanov theorem a Brownian motion under the measure $\mathbb{Q}$. The Novikov condition is satisfied as

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T Y(u)\,du\right)\right] < \infty,$$

if $\alpha > \beta$. The proof of this can be found in Wong and Heyde (2008). Furthermore, the stock price dynamics $X(t)$ and the variance dynamics $Y(t)$ satisfy the following SDEs under $\mathbb{Q}$

$$\begin{aligned} dX(t) &= [r-q+Y(t)]\,X(t)\,dt + \sqrt{Y(t)}X(t)\,dW^*(t) \\ dY(t) &= \alpha^*[m^* - Y(t)]\,dt + \beta\sqrt{Y(t)}\left[\rho\,dW^*(t) + \sqrt{1-\rho^2}\,dZ_t\right], \quad (25) \end{aligned}$$

where $\alpha^* = \alpha - \beta\rho$ and $m^* = \alpha m/\alpha^*$, and the price of the Asian option

(Equation (24)) can be obtained as,

$$AsianPrice = \frac{X(t)e^{-q(T-t)}}{T}\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{\int_0^T X(u)\,du - k_1T}{X(T)} - k_2T\right)^+ \,\middle|\, \mathcal{F}_t\right], \quad (26)$$

where the expectation is taken under the measure $\mathbb{Q}$. We define

$$AU(t) = \frac{\int_0^t X(u)\,du - k_1T}{X(t)},$$

which is only a minor variation of the Australian state variable considered before. Using Ito's lemma, we conclude that $AU(t)$ satisfies

$$dAU(t) \quad = (q-r)\,AU(t)\,dt + dt - \sqrt{Y(t)}AU(t)\,dW^*(t)\,. \quad (27)$$

The above extends the equivalence established in Section 2 between Asian and Australian options. However note that (25) features the equivalent of a stochastic interest rate[11] as compared with (22) which features a constant interest rate, so that the two models used are not fully equivalent.

7 Numerical methods under stochastic volatility

We will now use the ideas presented in the previous section to derive a pricing PDE for Asian options under stochastic volatility which is reduced in dimension and features time independent coefficients. Moreover, we will indicate how the Australian perspective can be useful for the implementation of Monte Carlo (MC) methods to price Asian options under stochastic volatility. Numerical results are presented and discussed in the following section.

[11] The stochastic interest rate applied here is in fact $\tilde{r}(t) = r + Y(t)$, compare this with the analogue transformation in Section 2. Note that in Section 2, the analogue transformation is applied to an Australian option, while in Section 6 it is applied to an Asian option, which explains the different sign by which the variance enters.

7.1 The PDE approach

From Equations (27), (25), and (26), the price of an Asian option can be computed using the PDE method. Starting with Equation (26), define the expectation as

$$
\begin{aligned}
u(t,x,y) &= \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{\int_0^T X(u)du - k_1 T}{X(T)} - k_2 T\right)^+ \,\middle|\, \mathcal{F}_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[(AU(T) - k_2 T)^+ \,\middle|\, AU(t) = x, Y(t) = y\right].
\end{aligned}
$$

We conclude from the Feynman Kac theorem, that $u(t,x,y)$ satisfies the PDE

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \frac{1}{2}x^2 y \frac{\partial^2 u}{\partial x^2} + \left((q-r)\,x + 1\right)\frac{\partial u}{\partial x} + & \qquad\qquad (28)\\
+\frac{1}{2}\beta^2 y \frac{\partial^2 u}{\partial y^2} + \alpha^*\left(m^* - y\right)\frac{\partial u}{\partial y} - \rho\beta x y \frac{\partial^2 u}{\partial x \partial y} &= 0\,, \quad \text{with} \\
u(T,x,y) &= (x - k_2 T)^+\,. \qquad (29)
\end{aligned}
$$

The PDE is a three-dimensional PDE with two spatial variables and one time variable, and is an extension of the Rogers and Shi (1995) type PDE for Asian options, but under stochastic volatility. The coefficients of the PDE are also time independent constants. Alternatively, it is possible to derive a PDE to price Asian options naively. This PDE has four dimensions, with three spatial variables. The first two are to accommodate the asset's value and variance, while the last one is reserved for the running sum of the asset's value. From this, it is already clear that the PDE derived through the Australian perspective, has an advantage over the naively obtained PDE.

Dimension reduction is not an entirely new concept and has been discussed within the Asian option context for example by Benhamou and Duguet (2003). Vecer and Xu (2004) presented a non path-dependent method[12], for pricing Asian options under a more general semi-martingale model. Their method is based on the dimension reduction results first presented by Vecer

[12] The Australian approach also leads to a non-path dependent method as discussed below.

(2001), and extended to stochastic volatility by Fouque and Han (2004). The variable of interest in this case is again a quotient of the integral of the asset value over time to the final asset value price. However, this ratio also contains some time dependent functions and as such, the associated SDE and the resulting PDE also have time dependent coefficients. The addition of these time dependent coefficients makes it numerically more difficult to solve the three-dimensional case.

In order to solve Equation (28) above numerically, we need to add the following boundary conditions to Equation (29)

$$
\begin{aligned}
u(t,-\infty,y) &= 0\,,\\
\frac{\partial u(t,\infty,y)}{\partial x} &= 1\,,\\
\frac{\partial u(t,x,0)}{\partial t} + [(q-r)x+1]\,\frac{\partial u(t,x,0)}{\partial x} + \alpha^* m^* \frac{\partial u(t,x,0)}{\partial y} &= 0\,,\\
u(t,x,\infty) &= x\,.
\end{aligned}
$$

The two boundary conditions for x are straightforward, when x approaches infinity, it is reasonable to assume that the option will finish extremely in-the-money such that its partial derivative with respect to x approaches 1. Similarly, when x approaches minus[13] infinity, the option will finish extremely out-of-the-money such that it will be worthless. The two boundary conditions for y are not as straightforward. For the $y = 0$ boundary, it makes intuitive sense to consider the PDE at $y = 0$ as its boundary condition. For when y approaches infinity the boundary is chosen to be the value x (for a call option). The latter boundary condition is similar to the choice made when solving the call option problem under a Heston model using Finite Difference Methods (FDM), see In't Hout and Foulon (2010).

The FDM used to solve Equation (28) will be that of the alternating direction implicit (ADI) method. ADI methods are methods that reduce multi-dimensional PDEs into a series of one-dimensional steps, which explains the origins of its name. In't Hout and Foulon (2010) covers this topic

[13]It is possible for x to be negative if $k_1 > 0$.

quite thoroughly for the Heston model, while Lin (2008), provides working MATLAB codes implementing the ADI scheme for a call option under the Heston model. Given the similarity between the call option PDE and the Asian-Australian option PDE, it is easy to modify the code to solve for the Asian option problem, using the Asian-Australian equivalence.

7.2 The Monte Carlo approach

As is well known, MC methods often provide powerful alternatives to numerical PDE methods. In the following, we comment briefly on the issue of pricing Asian options under stochastic volatility with MC methods, and show how the Australian approach can be very helpful here. In transforming the fixed strike Asian call option to an Australian-like call option, we have essentially reduced the problem from being path-dependent to being path-independent. The significant advantage for doing so will be outlined below.

Unlike in the case of constant volatility, the system of stochastic differential equations (22) – (23) can not be solved explicitly and an appropriate numerical scheme needs to be used. The most popular such schemes are the Euler-Maruyama and Milstein schemes.

It is well understood that in one-dimensional problems, for numerical results, the Euler-Maruyama scheme shows weak and strong convergence of order 1 and 1/2, respectively, while the Milstein scheme shows convergence of orders 1 and 1, respectively (see e.g. Kloeden and Platen (1992) for a proof, or Higham (2001)). For a path dependent option, the strong convergence order determines how good the MC method performs. Hence, the Milstein scheme is often advocated when dealing with Asian options. For path independent options the performance of the Euler-Maruyama scheme and the Milstein scheme are about equal. A disadvantage of the Milstein scheme, however, is that in the multi-dimensional case it becomes rather difficult to implement, with serious consequences on its performance. The difficulties lie in the computation of the double integral, known as the Levy area, involving the multiple Brownian motions, see Higham (2001).

When pricing an Asian or Australian option under stochastic volatility, we face a multi-dimensional problem. Poklewski-Koziell (2009) comments that the Milstein scheme does not perform well for the Heston model because the drift and diffusion coefficients are not "sufficiently smooth, real-valued functions satisfying a linear growth bound". Thus in our case, strong convergence order of one may actually not hold anymore for the Milstein scheme, even if the technical difficulties for its implementation are ignored. Additionally, available MATLAB implementations of the Milstein scheme are rather slow. Poklewski-Koziell also noted that the Euler-Maruyama scheme is a robust enough scheme to handle the pricing under the Heston model, see also Deelstra and Delbaen (1998). From the Australian perspective, applying the Euler scheme (27) and (25) in order to compute (26), and hence price the Asian option in a path independent manner comes naturally. Further to this, Lord, Koekkoek and Van Dijk (2010), showed that their modifications to the normal Euler scheme, called the full truncation scheme, reduces bias in MC simulations for the Heston model. Their modification deals with issues found at the zero boundary for the variance path, much like the usual absorption and reflection corrections, but expanding on this work. It is shown below that this scheme and the path independent approach show improved numerical performance if compared to the standard path dependent approach.

In addition to the Euler-Maruyama scheme with full truncation modifications we also test the second order Taylor 2.0 scheme. This scheme is derived by extending the Euler-Maruyama scheme to the next term in the expansion, by considering the double stochastic integrals in the Itô-Taylor expansion, see Kloeden and Platen (1992). When the usual smoothness and boundedness conditions hold, this scheme has a weak order of convergence of 2.0. More details for the Taylor 2.0 scheme can be found in Kloeden and Platen (1992). Note, that because we have transformed the path dependent option into a path independent option, the weak convergence order is really what matters here. Therefore this approach appears to be very promising.

8 Numerical results

8.1 Within the Black-Scholes framework

While the option price formula for Australian options given in Proposition 4.3 looks very promising, it turns out that it is not feasible to calculate the triple integral given in the formula (see section 6.1 Estimating $U(t, x)$ in Yang et al. (2011)). However, Yang at al. (2011) bypass this problem by using Monte Carlo simulation to calculate specific parts of their pricing and hedging formulas. Because of the above developed relationship between Asian and Australian options, it is straightforward to apply the methods used in Yang et al. (2011) to the case considered here.

8.2 Within the stochastic volatility framework

In this subsection, numerical results for Asian options under stochastic volatility are presented. We will compare the results obtained via FDM and Monte Carlo simulations.

We use the following sets of parameters found in Table 1 to implement the Heston model. Parameter set 1, has been previously used by Poulsen, Schenk-Hoppe and Ewald (2009) in the Heston model. Parameter sets 2 and 3 feature variations around this benchmark set.

Set	α	m	β	ρ	r	q	T	k_1	k_2	x_0	y_0
1	4.75	0.0483	0.550	-0.569	0.04	0.00	1	100	0	100	0.0483
2	3	0.1	0.2	0.7	0.1	0.00	1	100	0	100	0.1
3	7	0.05	0.3	-0.4	0.07	0.00	1	100	0	100	0.075

Table 1: Parameter Sets

Using the three-dimensional PDE developed in Section 6.1, we choose the domain for the state and variance variable as $[-3, 3]$ and $[0, 3]$, respectively. We use an equidistant partition of the x, y and t variable with the grid size in each variable being $1,000$. Using the above parameters, the alternating direction implicit (ADI) scheme, based on the modifications of the work by Lin (2008), returned a price of 5.852839, 9.323261, 7.122597, for parameter sets 1, 2 and 3, respectively. The time taken to compute the price using the ADI method was approximately 17 minutes. When testing the performance of the various MC settings below, we interpret these solutions as the exact solution.

We begin the MC analysis by first examining the results for parameter set 1. The first MC test will be to test the rate at which the option price converges to the ADI produced price as a function of the number of paths used. For this test, the number of time-steps is fixed to $1,000$, while the number of simulated paths used varies. Figures 3 and 4 show the convergence of the price and the RMSE of those prices as a function of number of simulated paths, respectively. The convergence of the price in using the Asian Euler and Australian Euler (in short Aus Euler) methods both follow a similar pattern, with the Aus Euler method reaching closer to the ADI solution with less simulations. The Aus Taylor 2.0 method reaches the ADI solution much quicker than the other two methods, and then hovers near the ADI solution. Lord, Koekkoek and Van Dijk's (2010) full truncation method has been used here.

For added reference, the Milstein method applied to the regular Asian options method produced a price and RMSE of 5.8298 and 7.4649, respectively. This simulation used $100,000$ paths, with $1,000$ time-steps each and took approximately 27 minutes to compute, which is significantly more than even the ADI method. The code for this scheme can be found in Poklewski-Koziell (2009). The Aus Taylor 2.0 and the Euler solutions, with the same number of simulated paths and time steps took 30 and 8 seconds, respectively. We also note that for the Aus Taylor 2.0 method, $25,000$ paths took approximately 8 seconds to compute. All computations were done using MATLAB 2010a on an Intel Core 2 Quad 3.6Ghz PC with 8Gb of RAM, running Windows 7. We will shortly come back to the relevance of these computational times.

The second MC test considered the rate of convergence as a function of the number of time-steps, while fixing the total number of simulated paths. The test consists of simulating $100,000$ paths at each time-step, with the number of time-steps chosen to be 100, 200, $\ldots$, $1,000$. Figures 5 and 6 show the convergence of price and the RMSE of this test, respectively. As evident in

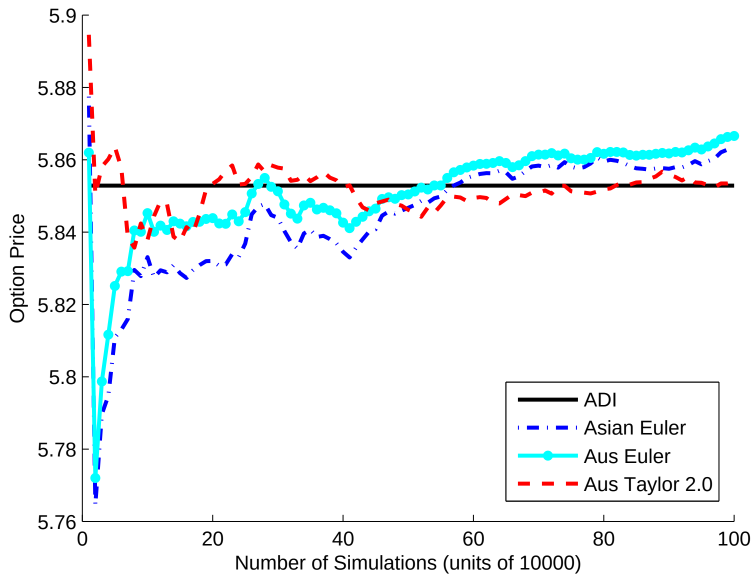

Figure 3: Convergence of Price as a Function of the Number of Simulation - The Aus Taylor 2.0 solution approaches the ADI solution more rapidly in this simulation run.

Figure 5, the Aus Taylor 2.0 method increases in performance as the number of time-steps per simulation increases. This result is not observed for the Euler methods, however in these particular simulations, for a small number of time-steps the Euler methods performed better than the Aus Taylor 2.0 method. Looking in terms of the RMSE in Figure 6, it can be seen that the Asian-Australian equivalence method once again provides a lower RMSE across all numbers of time-steps.

So far, the results demonstrate that the Asian-Australian equivalence method does seem to have advantages over the classical naive method in dealing with MC methods. However, the results thus far have not shown whether it is worthwhile to implement the simplified Order 2.0 Weak Taylor method and whether the results are due to particular seeds used in the simulation. To answer this question, the two previous tests are repeated 100 times, with different seeds at each run. The methodology for testing these repetition tests can be found in Appendix A.2.

Figures 7 and 6 show the results of the two repetition tests. From both

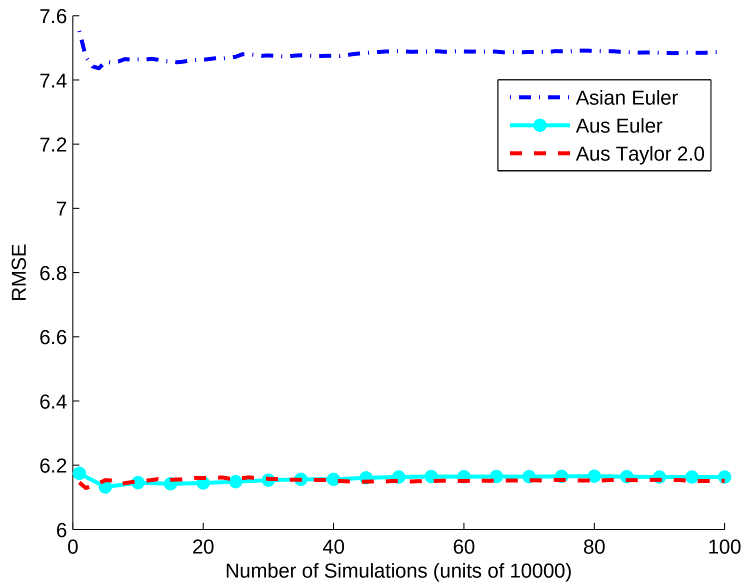

Figure 4: RMSE as a Function of Number of Simulation Paths - The standard Asian Euler method produces a higher RMSE than the Asian-Australian equivalence methods.

figures, the Euler method applied to both the naive and Asian-Australian equivalence method shows a very similar result. It is hard to distinguish with certainty which of the two methods is better. However, when taking the simplified Order 2.0 Weak Taylor method into consideration, it shows that the Asian-Australian equivalence method provides a superior result with a much lower RMSE across both number of simulations and number of time-steps.

As stated earlier, if we fix the time-step sizes and only vary the number of simulations, then the Aus Taylor 2.0 method increases computational time by a factor of 4 approximately. Thus, for a fairer comparison, we need to distinguish whether it is computationally efficient to use the Aus Taylor 2.0 method over the Euler method, even if both methods are using the Asian-Australian equivalence. From Figure 7, if we consider using $800,000$ and $200,000$ simulated paths for the Aus Euler and Aus Taylor 2.0 methods, respectively, for each of the 100 runs, they should take roughly the same

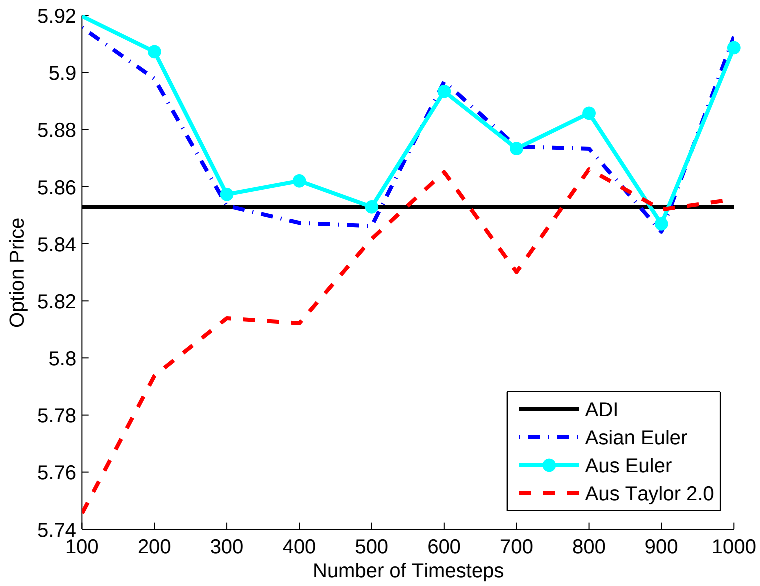

Figure 5: Option Price as a Function of the Number of Time-steps - Time-step size does not influence the convergence of the option price.

amount of time to compute. However, the Aus Taylor 2.0 method in this case returns a lower RMSE than the Aus Euler results. Of course, one could argue that taking 400, 000 paths for the Aus Euler and 100, 000 paths for the Aus Taylor 2.0 methods, respectively, gives better results for the Aus Euler. However, the fact is, if one demands the RMSE to be below a certain point, for example around 0.018 in this case, then clearly it is possible to achieve this by using the Aus Taylor 2.0 method, which in this case is computationally more efficient time-wise. The interesting point here is that the Aus Taylor 2.0 method is capable (on average) of returning a better estimate, even if we limit ourselves to using the same amount of time in the computation. This is indeed an advantage of using the Aus Taylor 2.0 method for pricing Asian options.

The repetition tests for parameter sets 2 and 3 are shown in Figure 9. The two graphs at the top are repetition test 1 and 2 (number of simulated path and number of time-steps), for parameter set 2, while the bottom two are for parameter set 3. The repetition tests shows that the Asian-Australian

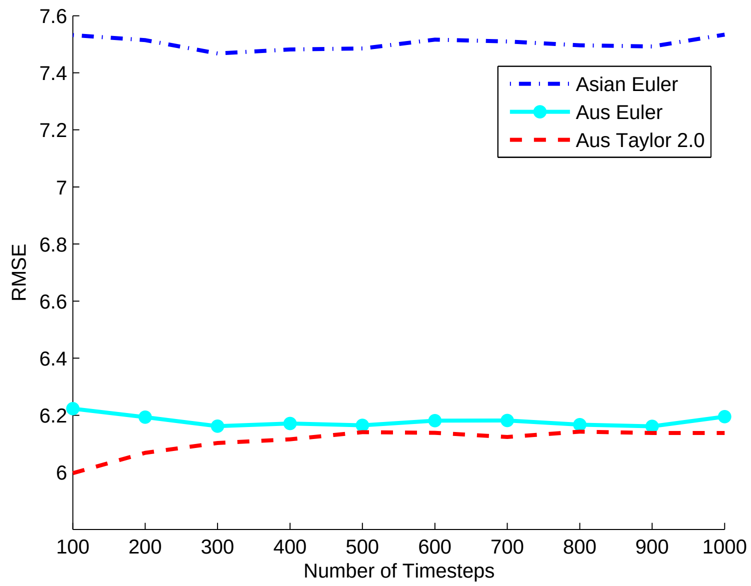

Figure 6: RMSE as a Function of Number of Time-steps - The standard Asian Euler method produces a higher RMSE than the Asian-Australian equivalence method.

equivalence method returns a lower RMSE than the naive method across all number of simulated paths and most of the number of time-steps considered. For the first repetition tests, the computational efficiency outlined above, of using the Aus Taylor 2.0 method was not observed for parameter set 2. In parameter set 3, if one considers the RMSE from the $200,000$ and $800,000$ simulated paths, for the Aus Taylor 2.0 and Aus Euler method, respectively, then the RMSE are quite similar. Nevertheless, if computational time is not of concern, then the Aus Taylor 2.0 method returns a lower RMSE than the Euler methods when using the same number of simulated paths. If computational time is of concern, then there are still advantages in using the Aus Euler method over the Asian Euler method. The results for repetition test 2 are similar to the results obtained earlier.

A summary of the ADI solutions along with the MC solutions, and their RMSE for repetition test 1 is shown in table 2. The MC solutions are obtained by averaging across the 100 repetitions with $1,000,000$ simulated

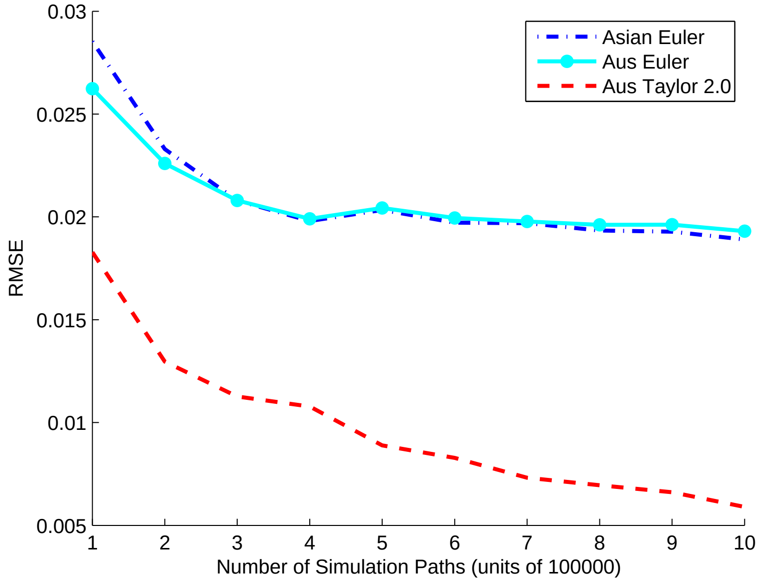

Figure 7: Repetition Test 1 - When the first test is repeated 100 times, the Aus Taylor 2.0 method returns a lower RMSE than the other two methods as the number of simulated paths increases.

paths. Whilst the Aus Euler MC solutions returned a slightly poorer estimate for the Asian option price compared to the Asian Euler method, there are some improvements to the RMSEs. Across all three parameter sets, the Aus Taylor 2.0 provided the closest price to the ADI solution with the lowest RMSE.

	ADI	Asian Euler		Aus Euler		Aus Taylor 2.0	
Set No.	Price	Price	RMSE	Price	RMSE	Price	RMSE
1	5.853	5.870	0.0190	5.871	0.0193	5.851	0.0059
2	9.323	9.333	0.0168	9.334	0.0140	9.321	0.0108
3	7.123	7.135	0.0156	7.136	0.0152	7.123	0.0078

Table 2: Repetition Test 1 Results; Asian Options

For added reference, the same analysis was repeated for Australian options using parameter sets 1, 2 and 3. Unlike the constant volatility case where pricing one is equivalent to pricing the other, due to the two models not being fully equivalent, we cannot obtain Australian option prices directly

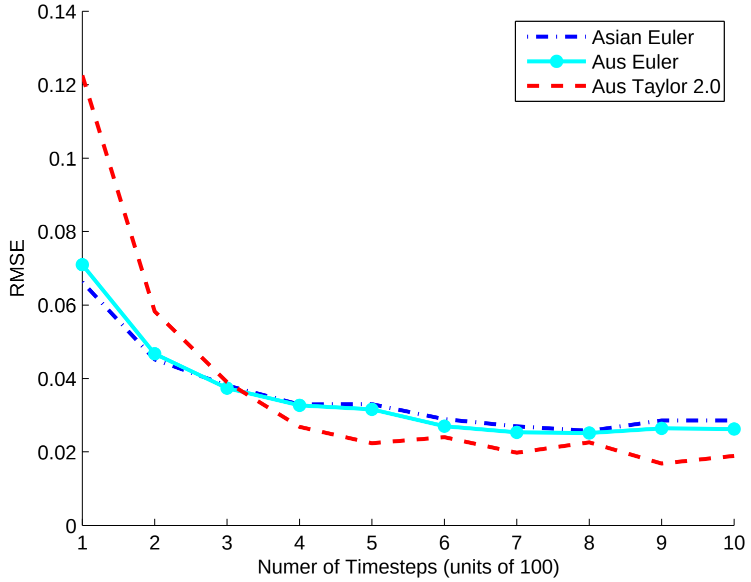

Figure 8: Repetition Test 2 - When the second test is repeated 100 times, the Aus Taylor 2.0 method returns a lower RMSE than the other two methods. Also the time-step size does not affect the Aus Taylor 2.0 method as much as the other two methods.

from Asian option prices (or vice versa). The results of these tests can be found in table 3. The results show that the Aus Taylor 2.0 method provides better MC simulation results than the Aus Euler method, which is to be expected as it is of higher order of convergence.

	ADI	Aus Euler		Aus Taylor 2.0	
Set No.	Price	Price	RMSE	Price	RMSE
1	0.04745	0.04759	1.5024×10^{-4}	0.04740	8.2574×10^{-5}
2	0.06426	0.06430	9.4181×10^{-5}	0.06423	8.9252×10^{-5}
3	0.05485	0.05494	1.1000×10^{-4}	0.05483	6.3888×10^{-5}

Table 3: Repetition Test 1 Results; Australian Options

Overall, the numerical results are promising in using the path-independent Asian-Australian equivalence to price Asian options under stochastic volatility. Whilst we have only tested the Weak Taylor 2.0 scheme, it is possible to look at higher order schemes for path-independent simulations, and com-

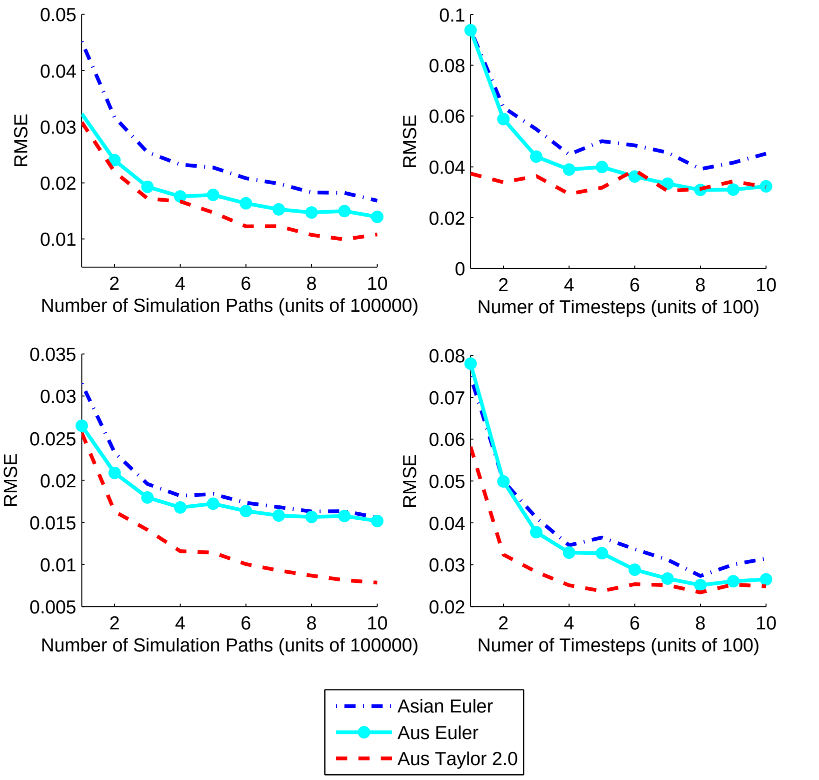

Figure 9: Repetition Tests - The top and bottom graphs are for parameter set 2 and 3 respectively.

pare this to equivalent higher order path-dependent schemes. Furthermore, easily implementable numerical algorithms such as the ADI method, can be applied to the Australian equivalent PDE with the time independent coefficients to produce solutions for Asian option prices. These advantages makes it worthwhile to consider Asian options following the proposed Australian approach.

9 Conclusion

We have shown that Australian options are equivalent to fixed or floating strike Asian options and consequently showed that by studying Asian options via Australian options, much can be gained. One specific application of this "Australian Approach" leads to a natural dimension reduction for the pricing PDE of Asian options, with or without stochastic volatility, featuring time independent coefficients. Another application lies in the improvement of Monte Carlo schemes. We also showed how the Milevsky and Posner result on the reciprocal Γ-approximation for Asian options can be quickly obtained by using the connection to Australian options, and in fact presented an analytical (exact) pricing formula for Australian options. Finally we discussed the Australian put-call parity and the qualitative dependence of Australian option prices on the level of volatility. Overall, we think that it is very useful in a conceptual way, to think about Asian options in the way presented here, and expect further progress on the study of Asian options as a result of this approach.

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A Appendix

A.1 Sketch of the proof of Proposition 5.1

Note that because the discount factor $e^{-r(T-t)}$ does not depend on σ, we only need to show that

$$\tilde{v}(x,t) = \mathbb{E}\left[(AU(T) - K)^+ \mid AU(t) = x\right] \tag{30}$$

satisfies

$$\mathcal{V}(x,t) := \tilde{v}_\sigma(t,x) = \frac{\partial}{\partial\sigma}\tilde{v}(x,t) > 0 \tag{31}$$

for $\sigma > 0$. The PDE for $\tilde{v}(x,t)$ is given by

$$\begin{aligned} \tilde{v}_t + \left(\left(\sigma^2 - r\right)x + 1\right)\tilde{v}_x + \frac{1}{2}\sigma^2 x^2 \tilde{v}_{xx} &= 0 \\ \tilde{v}(x,T) &= (x-K)^+. \end{aligned} \tag{32}$$

Differentiating this with regards to σ gives the following:

$$\mathcal{V}_t + \left(\left(\sigma^2 - r\right)x + 1\right)\mathcal{V}_x + \frac{1}{2}\sigma^2 x^2 \mathcal{V}_{xx} = -2\sigma x \tilde{v}_x - \sigma x^2 \tilde{v}_{xx} \tag{33}$$

and $\mathcal{V}$ vanishes on $[0,\infty) \times \{T\}$. Since (33) is a parabolic PDE, it follows in exactly the same way as in Carr, Ewald and Xiao (2008) from the Pontryagin maximum principle (see e.g. Theorem 3.1.1 in Stroock and Varadhan (1996)), that $\mathcal{V}(x,t) > 0$ for all (x,t), if it can be shown that the right hand side of (33) is always negative. Since we are only considering the case $x \geq 0$ the right hand side of (33) is negative, if it can be shown that $\tilde{v}_x$ and $\tilde{v}_{xx}$ are positive. This is intuitive, but not a priori clear. For $\tilde{v}_x$, we may consider the first variation process $\frac{\partial AU(t)}{\partial AU(0)}$ of $AU(t)$, which (following Protter (2003), Chapter 5.7) is given by the solution of

$$d\left(\frac{\partial AU(t)}{\partial AU(0)}\right) = \left(\frac{\partial AU(t)}{\partial AU(0)}\right)\sigma\, dW(t) + \left(\frac{\partial AU(t)}{\partial AU(0)}\right)\left(\sigma^2 + r\right)\, dt. \tag{34}$$

This means that the first variation process associated to $AU(t)$ is a geometric Brownian motion starting at 1, in particular it is always positive

$$\frac{\partial AU(t)}{\partial AU(0)} > 0 \tag{35}$$

and does not further depend on $AU(0)$. We conclude that for any increasing, convex, and two-times continuously differentiable function

$$\frac{\partial}{\partial x}\left(\mathbb{E}\left[h\left(AU(T)|AU(0)=x\right)\right]\right) = \mathbb{E}\left[h'(AU(T))\frac{\partial AU(T)}{\partial AU(0)}\right] > 0$$

and since $\frac{\partial AU(T)}{\partial AU(0)}$ does not depend on $AU(0)$, that is

$$\frac{\partial^2}{\partial x^2}\left(\mathbb{E}\left[h\left(AU(T)|AU(0)=x\right)\right]\right) = \mathbb{E}\left[h''(AU(T))\left(\frac{\partial AU(T)}{\partial AU(0)}\right)^2\right] > 0.$$

The payoff function $h(x) = (x-K)^+$ is sufficiently regular as to be approximated by a sequence of continuously differentiable functions as above, from which by a limit argument the desired positivity of $\tilde{v}_x$ and $\tilde{v}_{xx}$ can now be concluded. This is similar as in Carr, Ewald, and Xiao (2008).

A.2 Repetition test

The repetition of the first test is carried out as follows;

1. Simulate $100,000$ paths using $1,000$ time-steps and calculate the corresponding price.
2. Repeat step 1 a 100 times, with new seeds for each run.
3. Using the 100 prices obtained from step 2, calculate the RMSE of the mean prices.
4. Repeat all the above steps with $200,000$, $300,000$, $\ldots$, $1,000,000$ paths.

The repetition of the second test is carried out as follows;

1. Simulate $100,000$ paths using 100 time-steps and calculate the corresponding price.

2. Repeat step 1 a 100 times, with new seeds for each run.

3. Using the 100 prices obtained from step 2, calculate the RMSE of the mean prices.

4. Repeat all the above steps with 200, 300, $\ldots$, $1,000$ time steps.

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