Dublin City University

School of Mathematical Sciences

MSc Thesis

A Model Equation for the Optical Tunnelling Problem Using Parabolic Cylinder Functions

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This thesis is based on the candidate's own work

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Finally , I would like to thank my family , and in particular my mother , for providing me with the support (both financial and emotional) which enabled me to further my education

To my Family

For their continuous encouragement and support throughout my education

Abstract

The fundamental purpose of this thesis is to estimate the exponentially small imaginary part of the eigenvalue of a second order ordinary differential equation subject to certain stated boundary conditions This problem is modelled on a partial differential equation which arises when examining wave losses in bent fibre optic waveguides

In Chapter 1 we provide an overview of the thesis and introduce the area of mathematics known as exponential asymptotics In Chapter 2 we investigate the physical background to the problem of energy losses due to optical tunnelling in fibre optic waveguides We then derive the partial differential equation upon which we base our model. In Chapter 3 we commence by manipulating the partial differential equation into a more convenient form We then outline the model problem we shall consider and obtain a preliminary estimate for the eigenvalue of this problem In Chapter 4 we introduce the special function known as the parabolic cylinder function and derive its asymptotic behaviour We also examine its connection with Stokes phenomenon and deduce its Stokes and anti-Stokes lines In Chapter 5 , we finally solve the model problem by transforming it into one form of Weber's parabolic cylinder equation We then use the boundary conditions of the problem together with properties of parabolic cylinder functions to obtain a valid estimate for the imaginary part of the eigenvalue In Chapter 6 we conclude the thesis by commenting on this result and indicating future developments in this area

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CHAPTER 1

Introduction

This Chapter is partitioned into two sections In the first section , we present a condensed outline of the layout of the thesis and its origins In the second section , we briefly discuss some of the published papers which concern themselves with the world of exponential asymptotics

1 1 The thesis.

The inspiration for this thesis lies in work carried out by R Paris and A. Wood (See Reference [14]) who concern themselves with the model problem given below with $g(x) = x$

$$\iota\phi_t = -\phi_{xx} - \varepsilon g(x)\phi \qquad [1\ 1\ 1]$$

with the general linear homogeneous boundary condition

$$\phi_x(0,t) + h\phi(0,t) = 0 \qquad [1\ 1\ 1a]$$

and for physical reasons , any solution ϕ is constrained to be an outgoing wave beyond the turning point (See Chapter 3) [In the above h is a positive constant which is essentially a matching parameter] . They in turn were motivated by the

I M A lecture entitled " Mathematics in Industry and the prevalence of the free boundary problems " given by Dr J Ockendon at the Differential Equations meeting at the National Institute of Higher Education , Dublin (now Dublin City University) on 29th May , 1987 In this lecture , Dr Ockendon queried the validity of methods used by W Kath and G Kriegsmann in a forthcoming paper (See Reference [6]) In this paper, the authors attempt to estimate the energy loss in a fibre optic waveguide due to curvature in the fibre This requires estimating the imaginary part of an eigenvalue which is extremely small R Paris and A Wood successfully solved the model equation [1 1 1] with g(x) = x The fundamental purpose of this thesis is to successfully treat the case $g(x) = x^2$

In Chapter 2 , we examine the physical problem of energy loss in fibre optic waveguides We construct a suitable coordinate system which follows the centreline of the waveguide whilst taking into account the curvature of said waveguide. We use this system together with Maxwell's equations to derive the following partial differential equation

$$2\iota \mathbf{A}_{\sigma} + \mathbf{A}_{\xi\xi} + \mathbf{A}_{\eta\eta} + f(\xi,\eta)\mathbf{A} + 2k^2\delta K_1 \alpha \mathbf{A} + O(k^2\delta , \delta , 1/k^2) = 0 \qquad [1\ 1\ 2]$$

We shall base our mathematical model upon this equation.

In Chapter 3 , we manipulate the above equation to a more suitable form We then proceed to specify our model problem and justify its validity Carrying out a separation of variables on this problem finally leads us to the following model problem

$$y''(x) + (\lambda + \varepsilon x^2)y = 0 \text{ on } (0,\infty)$$
$$y'(0) + hy(0) = 0 \qquad [1\ 1\ 3]$$

$y(x)$ has controlling behaviour $e^{\iota p(x)}$, $x \longrightarrow +\infty$

where p(x) is a positive function of x and h is as before

We then use a regular perturbation expansion to obtain a preliminary estimate for the eigenvalue λ and indicate why this method cannot produce an estimate for Im λ

In Chapter 4 , we shall assemble the mathematical tools required to solve the model problem In particular , we shall study the asymptotic behaviour of the parabolic cylinder function U(a,z) In the process we shall introduce the concept of Stokes phenomenon and calculate the Stokes and anti-Stokes lines for U(a,z) Finally , we deduce the asymptotic behaviour for U(a,-z) using the connection formulae which exist for the parabolic cylinder functions of differing arguments

In Chapter 5, we use the properties of the parabolic cylinder function to solve the model problem [1.1.3] We first transform the problem into one form of Weber's parabolic cylinder equation , of which a combination of parabolic cylinder functions provides a solution We then modify this solution to take into account the required outgoing wave condition Finally , we use the boundary condition at the origin to obtain our final result That is ,

$$\mathrm{Im}\,\lambda \sim -4h^2\exp\left\{ \frac{-h^2\pi}{2\varepsilon^{1/2}} \right\} \quad , \quad \varepsilon \longrightarrow 0^+ \qquad [1\ 1\ 4]$$

We conclude the thesis by commenting on this result for Im λ and briefly indicating future developments in this area

1 2 The world of exponential asymptotics

The difficulties associated with calculating exponentially small values were first indicated in a paper by V L Povrovski and I M Khalatnikov (See Reference [15]) who were interested in calculating the amplitude for above barrier reflection of a particle from a one-dimensional potential barrier They used properties of the potential in the complex plane when dealing with the Schrodinger equation

$$\frac{d^2\psi}{dx^2} + \rho^2\psi = 0$$

finding that the reflection coefficient R is

$$R = |F(\lambda)|^2\exp\{ 4\iota\mathrm{Im}\int_{-\infty}^{x_1} \rho\ dx \}$$

Thus the exponentially small behaviour of R is revealed

- Chapter 1 -

In his paper on exponential asymptotics , R E Meyer (See Reference [10]) details three examples illustrating the need for high precision asymptotics of an unusual kind He examines the difficulties encountered in dealing with certain situations involving the modulation of an oscillator , wave reflection and wave trapping He further outlines a method for dealing with such difficulties In a later paper he also encounters the same complexities (See Reference [9]) as does F B Hanson when examining certain mathematical models for population dynamics (See Reference [5]) H Segur and M Kruskal also encounter eigenvalues with exponentially small imaginary part in their paper on the non-existence of small amplitude breather solutions (See Reference [7]) A Wood and R Paris in their paper on eigenvalues with exponentially small imaginary part (See Reference [20]) cite an example given by F W J Olver (See Reference [11] p 76) where neglect of an exponentially small error term in calculating the integral

$$I(\varepsilon) = \int_0^{\pi} \frac{\cos(t/\varepsilon)}{1 + t^2} \, dt$$

results in a large relative error when compared to exact solutions He indicates that the perturbed differential equations where exponentially small behaviour arises are of the type known as singular perturbations Singular perturbations are characterised by an abrupt change in the nature of the solutions to the problem as $\varepsilon \longrightarrow 0$ Since the model we shall consider is singular in nature , we are not suprised by the final result for Im λ given by [1 1 4]

In Chapter 4 , we introduce the concept of Stokes phenomenon which is intimately linked with the appearance of these eigenvalues with exponentially small imaginary parts Associated with Stokes phenomenon are the Stokes multipliers (or constants) whose property of changing value as one crosses a Stokes line has resulted in much controversy The controversy stems from the unknown behaviour of the multipliers as they cross the Stokes line George Stokes' opinion was that the change was discontinuous He wrote (See Reference [19])

" *the inferior term* [subdominant term] *enters as it were into a mist , is hidden for a little from view , and comes out with its coefficient changed The range during which the inferior term remains in a mist decreases indefinitely as the* [asymptotic parameter] *increases indefinitely* "

In recent work M V Berry (See Reference [4]) has proposed that this change occurs continuously as one approaches the Stokes line and that the value of the Stokes multiplier on this line is precisely the average of its values on either side of said line F W J. Olver recently has put this supposition on firmer mathematical footing (See Reference [13]) We believe his work finally puts to rest this most perplexing problem

The above discussion indicates the broad area in which exponential asymptotics appears Although we shall deal with one particular problem we cannot emphasise enough the scope of this stimulating area of asymptotics

CHAPTER 2

The Physical Problem

In this chapter , we shall investigate the physical background to the problem of energy losses due to optical tunnelling in fibre optic waveguides We will first establish a suitable coordinate system and use this system to derive the partial differential equation upon which we base our mathematical model

2 1 Formulation of the coordinate system

We begin by describing the position of the centre of the fibre as a function of arc length

$$\mathbf{x} = \mathbf{x}_0(s) \qquad [2\ 1\ 1]$$

This function contains a system of local coordinates which naturally follows the fibre, that is , the unit tangent $\hat{\mathbf{t}}$, normal $\hat{\mathbf{n}}$ and binormal $\hat{\mathbf{b}}$ vectors defined by the Frenet-Serret Formulae (See Reference [17] p 57) as follows [See figure 2 1].

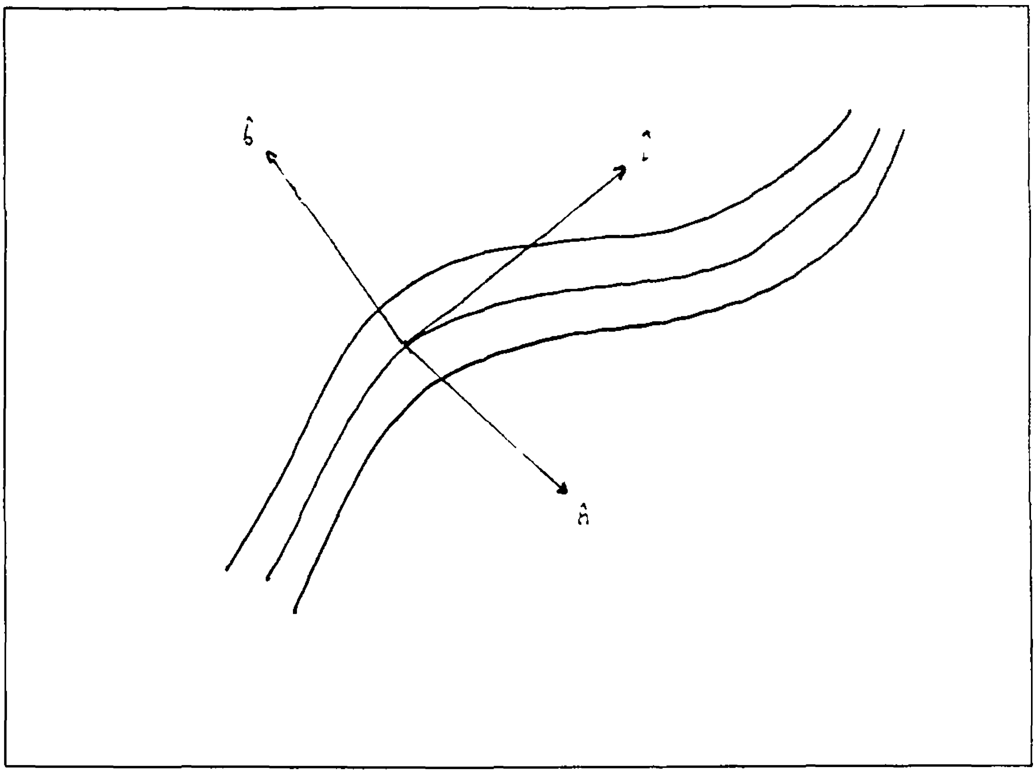

Figure 2.1: The Frenet Serret coordinate system along the centreline of the optical fibre

$$\frac{d\mathbf{x}_0}{ds} = \hat{\mathbf{t}}$$

$$\frac{d\hat{\mathbf{t}}}{ds} = K\,\hat{\mathbf{n}}$$

$$\frac{d\hat{\mathbf{n}}}{ds} = \tau\hat{\mathbf{b}} - K\hat{\mathbf{t}}$$

$$\frac{d\hat{\mathbf{b}}}{ds} = -\tau\hat{\mathbf{n}}$$

where

K is the curvilinear curvature,

τ is the curvilinear torsion,

and $$\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}} \quad .$$

To change the equations into dimensionless form , we note that [2 1 1] is given in dimensionless form as

$$\mathbf{x}_0(s) = \ell\mathbf{X}(s/\ell)$$

Where ℓ is a typical length scale for the bent centreline such as a characteristic size for the radius of curvature

Let $s = as'$, $\mathbf{x} = a\mathbf{x}'$, $K = K'/a$, $\tau = \tau'/a$ etc

and $\delta = a/\ell \ll 1$ where a is the radius of the fibre core [See Figure 2 2] Typical values of a are $a \approx 2 - 5\ \mu m$

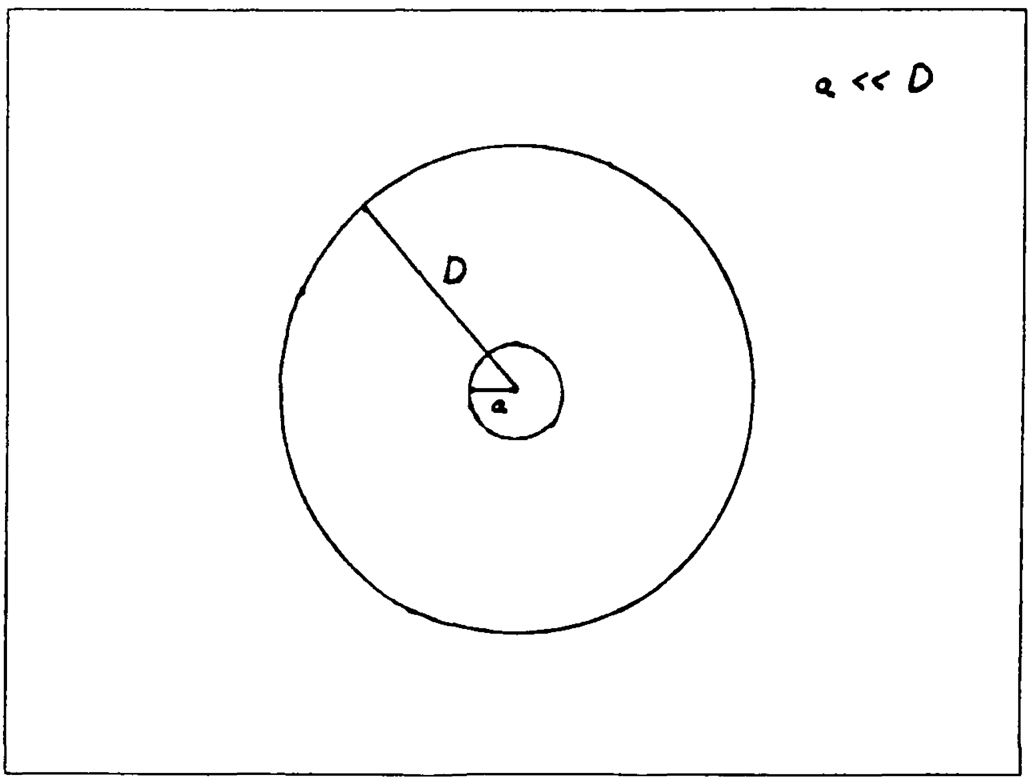

Figure 2 2 : Cross sectional view of the fibre optic waveguide

The dimensionless Frenet-Serret formula along the centreline are

$$\mathbf{x}' = \frac{1}{\delta}\,\mathbf{X}(\delta s')$$

$$\frac{d\mathbf{x}'}{ds} = \frac{d}{ds'}\left(\frac{1}{\delta}\,\mathbf{X}(\delta s')\right) = \hat{\mathbf{t}}(\delta s')$$

$$\frac{d\hat{\mathbf{t}}}{ds} = K'\hat{\mathbf{n}}(\delta s') \qquad [2\ 1\ 4]$$

$$\frac{d\hat{\mathbf{n}}}{ds} = \tau'\hat{\mathbf{b}}(\delta s') - K'\hat{\mathbf{t}}\,(\delta s')$$

$$\frac{d\hat{\mathbf{b}}}{ds} = -\tau'\hat{\mathbf{n}}(\delta s')$$

The function $\mathbf{X}$ is dependent on $\delta s'$ and hence in these scaled coordinates the position of the centreline and all functions resulting from it are slowly varying

For convenience, we shall drop the primes but it is understood that all distances remain in dimensionless form

On examining equations [2 1 4], we see that the left hand sides are $O(\delta)$, thus the dimensionless curvature and torsion must also be of this order

Hence, we can rescale them as follows :

$$K = \delta K_1(\delta s) \text{ and } \tau = \delta\tau_1(\delta s) \qquad [2\ 1\ 5]$$

where K_1 and τ_1 are both assumed to be of order 1

The Frenet-Serret formulae provide a natural co-ordinate system for following the fibre if distances are measured along the fibre in terms of the dimensionless arc length and distances perpendicular to the fibre in terms of the dimensionless distances along the unit normal and binormal We denote this new system by (s,α,β) defined by the transformation

$$\mathbf{x} = \frac{1}{\delta}\,\mathbf{X}(\delta s) - \alpha\hat{\mathbf{n}}(\delta s) - \beta\hat{\mathbf{b}}(\delta s) \qquad [2\ 1\ 6]$$

where the negative signs are placed for convenience only However, the Frenet-Serret frame is determined only from the position of the centreline not the entire fibre Hence the twisting or torsion of the fibre is not fully accounted for in this frame Thus a coordinate system which more accurately follows the fibre is one in which the torsion τ is removed by rotating the above system.

The transformation to this new set of coordinates (s,ξ,η) is

$$\mathbf{x} = \frac{1}{\delta}\,\mathbf{X}(\delta s) + \xi[\hat{\mathbf{n}}\cos\upsilon + \hat{\mathbf{b}}\sin\upsilon] + \eta[\hat{\mathbf{b}}\cos\upsilon - \hat{\mathbf{n}}\sin\upsilon] \qquad [2\ 1\ 7a]$$

where $\qquad \frac{d\upsilon}{ds} = -\delta\tau_1(\delta s)$

Defining the new vectors

$$\hat{\mathbf{u}} = \hat{\mathbf{n}}\cos\upsilon + \hat{\mathbf{b}}\sin\upsilon$$

$$\hat{\mathbf{v}} = \hat{\mathbf{b}}\cos\upsilon - \hat{\mathbf{n}}\sin\upsilon$$

the transformation becomes

$$\mathbf{x} = \frac{1}{\delta}\,\mathbf{X}(\delta s) + \xi\hat{\mathbf{u}}(\delta s) + \eta\hat{\mathbf{v}}(\delta s)\ . \qquad [2\ 1\ 7b]$$

This new coordinate system follows more closely the orientation of the fibre

Moreover,

$$\mathbf{dx.dx} = h_1^2 ds^2 + d\xi^2 + d\eta^2$$

where $h_1 = 1 - \delta K_1(\xi\cos\upsilon - \eta\sin\upsilon)$

$= 1 + \delta K_1\alpha$

Thus (s,ξ,η) define orthogonal curvilinear coordinates , giving the added advantage that it is easier to transform the equations into the new coordinates

2 2 Formulation of the problem

As stated by D Marcuse (See Reference [8] p 339)for weakly guiding fibres a scalar theory is a reasonable approximation This scalar approximation is obtained as follows

The curl version of the time-harmonic wave equation for the electric field (assumed to be in dimensionless form) is

$$\nabla \times (\nabla \times \mathbf{E}) - n^2k^2\mathbf{E} = \mathbf{0} \qquad [2\ 2\ 1]$$

The magnetic field is easily calculated once the electric field is known

Here $k = k_0 a n_c$ is a dimensionless wave no

k_0 = physical wave no

n_c = refractive index of the cladding

$n = n_0/n_c$ is the normalised index of refraction

The weakly guiding approximation is made by assuming that the refractive index of the cladding and the core differ only slightly.

[Typical values for a monomode fibre are k ≈ 15-40 where $k_0 \approx 6 \times 10^4 cm^{-1}$, a ≈ 2-5 μm , $n_c \approx 1\ 3$](See Reference [18])

Typical values of n^2 suggest that the correct scaling should be

$$n^2 = 1 + \frac{f(\xi,\eta)}{k^2} \qquad [2\ 2\ 2]$$

where $f(\xi,\eta)$ is O(1) and is non-zero only in the core region [See Figure 2 3 below]

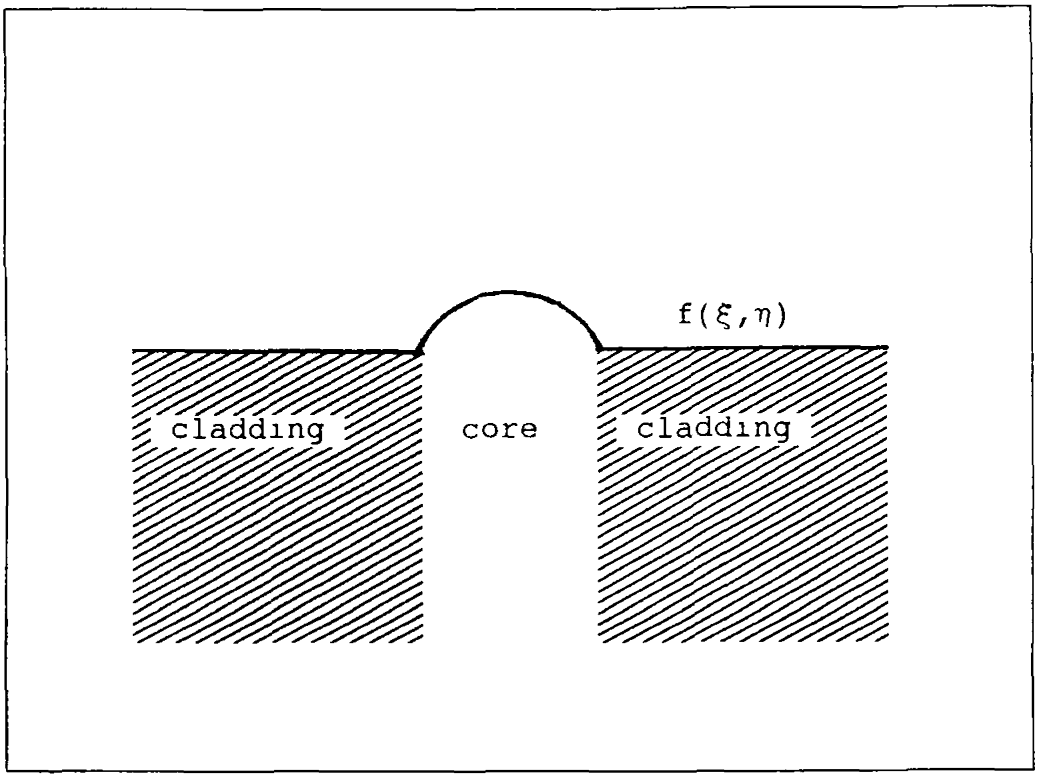

Figure 2.3 : Schematic of the behaviour of $f(\xi,\eta)$

We now make the paraxial approximation (See Reference [8])

$$\mathbf{E} = \mathbf{A}(\sigma,\xi,\eta)\ e^{\iota ks} \quad , \ \sigma = s/k \qquad [2\ 2\ 3]$$

i e we say that the main propagation direction of the electromagnetic energy is along the length of the fibre Substituting into Maxwell's equations [2 2 1] using the coordinate system (s,ξ,η) where in this system

$$\mathbf{A} = A_1\hat{\mathbf{t}} + A_2\hat{\mathbf{u}} + A_3\hat{\mathbf{v}} \qquad [2\ 2\ 4],$$

we find

$$A_1 = \frac{\iota}{k}\ (\ A_{2\xi} + A_{3\eta}\ ) + O(1/k^3,\delta/k) \qquad [2\ 2\ 5]$$

Thus the longitudinal electric field component is smaller than the others by a factor of 1/k, so that the field is mainly transverse Both components of the transverse field obey the same equation, namely

$$2\iota A_{j\sigma} + A_{\xi\xi} + A_{j\eta\eta} + f(\xi,\eta)A_j + 2k^2\delta K_1\alpha A_j + O(k^2\delta,\delta,1/k^2) = 0\ , \qquad [2\ 2\ 6]$$

$j = 2,3$

Finally, if we assume that the curvature produces an effect comparable with the scaled index of refraction difference $f(\xi,\eta)$ then this means we should take $k^2\delta = 1$ Combining this with the range of reasonable values of k, i e $k \approx 15\text{-}40$, gives a dimensional radius of curvature of the order of a few millimetres, which is too small.

Therefore , assuming that

$$\delta = 1/k^3 \qquad [2\ 2\ 7]$$

(giving a radius of curvature in the range of a few centimetres to a few tens of centimetres) is a more logical choice for δ

With this choice of δ and neglecting all of the small terms, $O(1/k^2)$ and smaller, we then obtain the equation

$$2\iota A_{\sigma} + A_{\eta\eta} + A_{\xi\xi} + f(\xi,\eta)A + \frac{1}{k} 2K_1\alpha A = 0 \qquad [2\ 2\ 8]$$

where again $\alpha = \xi\cos\upsilon - \eta\sin\upsilon$ and in these new co-ordinates

$$\frac{d\upsilon}{d\sigma} = \frac{1}{k^2} \tau_1(\sigma/k^2) \qquad [2\ 2\ 9]$$

2 3 Physical Explanation

From equation [2 2.8] we see that, after the approximations have been made , the only effect of the curvature is to introduce a perturbation into the index of refraction, which is small in the core but not in the cladding where α is large. We can explain this curvature perturbation by viewing the situation in normal cartesian coordinates [See Figure 2 4]

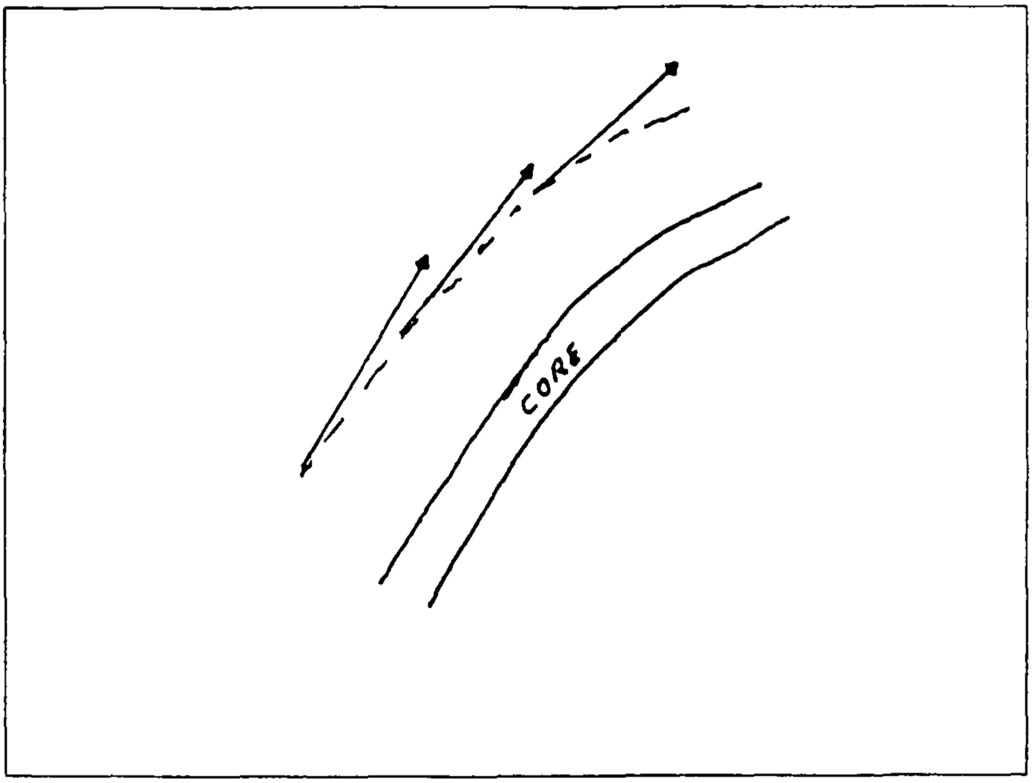

Figure 2 4 . Energy loss out of the core region

In this co-ordinate system , we see that energy for large positive values of α (out in the evanescent tail of the mode) must travel further than energy propagating in the core region. On transforming to the local coordinate system following the fibre the influence of this extra distance is changed to an effective slowing of the wave via an increased index of refraction.

The loss of energy in the mode can be explained as follows As one moves away from the core eventually a point is reached where the energy propagating in the evanescent tail cannot keep up with the main part of the wave propagating in the core and thereby changes from an evanescent to a propagating wave The energy is then shed as it radiates away into the cladding Of course, because this happens in the evanescent part of the mode the energy loss is not dramatic but over a lengthy run can be significant

This thesis is primarily an attempt to understand mathematically how this small but important energy loss occurs We set out to achieve this aim by examining a model equation which essentially exhibits the same behaviour as equation [2 2 8]

CHAPTER 3

The Model Problem

In this chapter , we begin by manipulating the partial differential equation [2 2 8] to a more convenient form We then outline the model problem we shall consider and finally obtain a preliminary estimate for the eigenvalue of the model problem

3 1 Preliminaries

To obtain an estimate for the small, but important , energy loss caused by bending the fibre optic waveguide, we must examine equation [2.2 8] reproduced below for convenience

$$2\iota \mathbf{A}_{\sigma} + \mathbf{A}_{\xi\xi} + \mathbf{A}_{\eta\eta} + f(\xi,\eta)\mathbf{A} + \frac{1}{k} K_1 \alpha \mathbf{A} = 0 \qquad [3\ 1\ 1]$$

It should be noted that α is a linear combination of ξ and η (see equation [2 1 7]).

Kath and Kriegsmann (See Reference [6]) use a variation of the following procedure

Set $\mathbf{A}(\sigma,\xi,\eta) = y(\xi,\eta)\ e^{-\iota\Lambda\sigma}$

which gives

$$\nabla^2 y + f(\xi,\eta)y - 2\Lambda y + \frac{2K_1\alpha}{k} y = 0 \qquad [3\ 1\ 2]$$

Note that Λ is basically the difference between the propagation constant of the mode and k The decay rate is Im Λ which must be positive

We can simplify the form of equation [3 1 2] slightly by making the substitutions

$$\varepsilon = \frac{2K_1}{k} \qquad [3\ 1\ 3a]$$

$$\lambda = -2\Lambda \qquad [3\ 1\ 3b]$$

Observe that since Im Λ must be positive then Im λ **must be negative** Equation [3 1 2] thus becomes

$$\nabla^2 y + f(\xi,\eta)y + \lambda y + \varepsilon\alpha y = 0 \qquad [3\ 1\ 4]$$

A regular perturbation expansion of the form

$$\begin{aligned} y &= y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \varepsilon^3 y_3 + \ldots \\ \lambda &= \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \varepsilon^3\lambda_3 + \ldots \end{aligned} \qquad [3\ 1\ 5]$$

fails to yield any information on Im λ (See section 3 3)

Indeed Kath and Kriegsmann show that using this method ,

Im $\lambda_n = 0 \quad \forall\ n = 0,1,2,\ldots$

However by assuming y and λ are of the form [3 1 5] we are stating that the changes caused by the perturbation are all small which is true in the core , but in the cladding region , the perturbation $\varepsilon\alpha$ is not small Thus we must use alternative methods for finding Im λ

3 2 Derivation of the model problem

In order to gain more information on the problem of estimating the small, but crucial, imaginary part of the eigenvalue in equation [3 1 4] above ,the following one dimensional problem will be examined

In the original problem , for small ε, we are in the cladding region where the perturbation $f(\xi,\eta)$ in the refractive index is zero We are interested in the neighbourhood of a turning point which is situated well into the cladding region Therefore , we feel justified in considering the following model problem ·

$$\iota\phi_t = - \phi_{xx} - \varepsilon\, g(x)\phi \qquad [3\ 2\ 1]$$

with the general linear homogenous boundary condition

$$\phi_x(0,t) + \phi(0,t) = 0 \qquad [3\ 2\ 1a]$$

where the positive constant h is essentially a matching parameter

The case $g(x) = x$ has been successfully dealt with by Paris and Wood (See Reference [14]) We examine here the case when the perturbation in the refractive index can be described by $g(x) = x^2$.

Equation [3 2 1] has the same structure as equation [3 1 1] yet has an exact solution in terms of parabolic cylinder functions which permits a rigorous mathematical analysis

Making the same separation of variables

$$\phi(x,t) = e^{-\iota\lambda t}y(x) \text{ with } \operatorname{Im}\lambda < 0 \quad ,$$

then Equation [3 2 1] becomes

$$e^{-\iota\lambda t}\lambda y(x) = -e^{-\iota\lambda t}y''(x) - \varepsilon x^2 e^{-\iota\lambda t}y(x)$$

Hence,

$$y''(x) + (\lambda + \varepsilon x^2)y(x) = 0$$

with the boundary condition becoming

$$e^{-\iota\lambda t}y'(0) + e^{-\iota\lambda t}hy(0) = 0$$

that is,

$$y'(0) + hy(0) = 0$$

The physical discussion in Chapter 2 indicates that the solution must be an outgoing wave beyond the turning point at $x = \sqrt{\frac{-\lambda}{\varepsilon}}$

We express this condition by constraining any solution y to have controlling behaviour of the form $e^{\iota p(x)}$, where p(x) is a positive function in x as $x \longrightarrow +\infty$.

Thus our model problem is of the form

$$y''(x) + (\lambda + \varepsilon x^2)y(x) = 0 \quad [3\ 2\ 2]$$

$$y'(0) + hy(0) = 0 \quad [3\ 2\ 2a]$$

$y(x)$ has controlling behaviour $e^{\iota p(x)}, x \longrightarrow +\infty$ [3 2 2b]

where $p(x)$ is a positive function in x, h is a positive constant and $\varepsilon > 0$

3 3 A "trial" solution using regular perturbation methods

We first attempt a trial solution of the form

$$y(x) = \sum_{n=0}^{\infty} y_n(x)\ \varepsilon^n$$

$$\lambda = \sum_{n=0}^{\infty} \lambda_n\ \varepsilon^n \quad [3\ 3\ 1]$$

Substituting into equation [3 2 2] we find

$$\sum_{n=0}^{\infty} y_n''(x)\ \varepsilon^n + \sum_{n=0}^{\infty} (\lambda_n y_0 + \lambda_{n-1} y_1 + \quad +\lambda_0 y_n)\varepsilon^n + x^2 \sum_{n=0}^{\infty} y_n(x)\ \varepsilon^{n+1} = 0 \quad [3\ 3\ 1]$$

Comparing powers of ε we see

ε^0

$$y_0'' + \lambda_0 y_0 = 0$$

Choosing $y_0(x) = e^{\iota\sqrt{\lambda_0}\,x}$

then from equation [3 2 2a] we have

$$\iota\sqrt{\lambda_0} + h = 0$$

$$\Rightarrow \qquad \lambda_0 = -h^2$$

$$\Rightarrow \qquad y_0(x) = e^{-hx}$$

ε^1

$$y_1'' + \lambda_0 y_1 + \lambda_1 y_0 + x^2 y_0 = 0 \qquad [3\ 3\ 3]$$

$$y_1'' + \lambda_0 y_1 + (\lambda_1 + x^2) y_0 = 0$$

$$y_0 y_1'' + \lambda_0 y_1 y_0 + (\lambda_1 + x^2) y_0^2 = 0$$

But $y_0'' + \lambda_0 y_0 = 0$

hence,

$$y_0 y_1'' - y_1 y_0'' + (\lambda_1 + x^2) y_0^2 = 0 \qquad [3\ 3\ 4]$$

thus,

$$\int_0^\infty y_0 y_1'' dx - \int_0^\infty y_1 y_0'' dx + \int_0^\infty (\lambda_1 + x^2) y_0^2 dx = 0$$

But $\int_0^\infty y_0 y_1'' dx = \int_0^\infty e^{-hx} y_1'' dx$

On integrating by parts twice and using the boundary condition at the origin for the zeroth order equation , we find

$$\int_0^{\infty} y_0 y_1'' dx = h^2 \int_0^{\infty} e^{-hx} y_1 dx$$

Equation [3 3 4] then becomes

$$h^2 \int_0^{\infty} y_1 e^{-hx} dx \quad -h^2 \int_0^{\infty} y_1 e^{-hx} dx + \int_0^{\infty} (\lambda_1 + x^2) e^{-2hx} dx = 0$$

Thus,

$$\int_0^{\infty} (\lambda_1 + x^2) e^{-2hx} dx = 0$$

That is ,

$$\lambda_1 = \frac{\int_0^{\infty} x^2 e^{-2hx} dx}{\int_0^{\infty} e^{-2hx} dx}$$

Straightforward integration provides

$$\lambda_1 = \frac{-1}{2h^2}$$

Substituting this result into equation [3 3 3] we find

$$y_1'' = \lambda_0 y_1 - [\frac{1}{2h^2} - x^2] y_0 = 0$$

$$\Rightarrow y_1'' - h^2 y_1 = [\frac{1}{2h^2} - x^2] e^{-hx}$$

We shall solve this inhomogeneous second order differential equation as follows The associated homogeneous equation is

$$y_{1p}'' - h^2 y_{1p} = 0$$

We choose $y_{1p} = e^{-hx}$ as a solution

Assume $y_{1c} = [Ax^3 + Bx^2 + Cx] e^{-hx}$

then $y_{1c}' = [3Ax^2 + 2Bx + C - hAx^3 - hBx^2 - hCx] e^{-hx}$

$$\Rightarrow \qquad y'_{1c} = [\ -hAx^3 + (3A-hB)x^2\ (2B-hC)X]e^{-hx}$$

$$\Rightarrow \qquad y''_{1c} = \{\ h^2Ax^3 + (-3Ah + h^2B)x^2 + (h^2C-2hB)x\ -hC$$

$$-3Ahx^2 + (6A-2hB)x + 2B-2C\ \}\ e^{-hx}$$

$$= \{\ h^2Ax^3 + (h^2B-6Ah)x^2 + (h^2C+6A-4hB)x +$$

$$(2B-2hC)\}e^{-hx}$$

Now $\quad -h^2y_{1c} = [\ -h^2Ax^3\ -h^2Bx^2\ -h^2Cx\ ]e^{-hx}$

Thus,

$$y''_{1c}\ -h^2y_{1c} = [\ -6Ah^2 + (6A-4hB)x + (2b-2hC)\ ]e^{-hx}$$

$$= [\ \frac{1}{2h^2} - x^2\ ]e^{-hx}$$

$$\Rightarrow \quad [\ -6Ah^2 + (6A-4hB)x + (2b-2hC)\ ] = [\ \frac{1}{2h^2} - x^2\ ]e^{-hx}$$

Equating powers of x gives

$$-6Ah = -1 \qquad \Rightarrow \qquad A = \frac{1}{6h}$$

$$6A - 4hB = 0 \qquad \Rightarrow B = \frac{1}{4h^2}$$

$$2B - 2hC = 0 \qquad \Rightarrow C = 0$$

Thus

$$y_{1c} = \frac{(2hx^3 + 3x^2)}{12h^2}\ e^{-hx}$$

and

$$y_1 = e^{-hx} + \frac{(2hx^3 + 3x^2)}{12h^2}\ e^{-hx}$$

ε^2

$$y''_2 + \lambda_2 y_0 + \lambda_1 y_1 + \lambda_0 y_2 + x^2 y_1 = 0$$

$$\Rightarrow \qquad y_2'' + \lambda_2 y_0 + (\lambda_1 + x^2)y_1 + \lambda_0 y_2 = 0$$

$$\Rightarrow \qquad y_0 y_2'' + \lambda_2 y_0^2 + (\lambda_1 + x^2)y_1 y_0 + \lambda_0 y_0 y_2 = 0$$

But $y_0'' + \lambda_0 y_0 = 0$

$$\Rightarrow \quad y_0 y_2'' + \lambda_2 y_0^2 + (\lambda_1 + x^2)y_1 y_0 - y_0'' y_2 = 0 \qquad [3\ 3\ 5]$$

$$\Rightarrow \int_0^\infty y_0 y_2''dx - \int_0^\infty y_0'' y_2\, dx + \int_0^\infty \lambda_2 y_0^2\, dx + \int_0^\infty (\lambda_1 + x^2)y_1 y_0\, dx = 0$$

Now $\int_0^\infty y_0 y_2''dx = \int_0^\infty e^{-hx} y_2''dx$

On integrating by parts twice and using the boundary condition at the origin for the zeroth order equation this becomes

$$\int_0^\infty y_0 y_2''dx = h^2 \int_0^\infty y_0 y_2 dx$$

Also

$$\int_0^\infty y_2 y_0''dx = h^2 \int_0^\infty y_0 y_2 dx$$

Substituting these results into equation [3 3 5] we find

$$\int_0^\infty \lambda_2 y_0^2\, dx + \int_0^\infty (\lambda_1 + x^2)y_1 y_0\, dx = 0$$

$$\Rightarrow \qquad \lambda_2 = \frac{\int_0^\infty (\lambda_1 + x^2)y_1 y_0\, dx}{\int_0^\infty y_0^2\, dx} . \qquad [3\ 3\ 6]$$

But,

$$\int_0^\infty (\lambda_1 + x^2)y_1 y_0\, dx$$

$$= \int_0^\infty \left(\frac{-1}{2h^2} + x^2\right)e^{-hx}\left[e^{-hx} + e^{-hx}\left(\frac{2hx^3 + 3x^2}{12h^2}\right)dx\right.$$

$$= \frac{-1}{2h^2}\int_0^\infty e^{-2hx}dx - \frac{1}{2h^2}\int_0^\infty e^{-2hx}\left(\frac{2hx^3 + 3x^2}{12h^2}\right)dx$$

$$+ \int_0^\infty e^{-2hx}x^2dx + \int_0^\infty e^{-2hx}\left(\frac{2hx^5 + 3x^4}{12h^2}\right)dx$$

$$= \frac{-1}{2h^2}\int_0^\infty e^{-2hx}dx - \frac{1}{12h^3}\int_0^\infty e^{-2hx}x^3dx - \frac{1}{8h^4}\int_0^\infty e^{-2hx}x^2dx$$

$$+ \int_0^\infty e^{-2hx}x^2dx + \frac{1}{6h}\int_0^\infty e^{-2hx}x^5dx + \frac{1}{4h^2}\int_0^\infty e^{-2hx}x^4dx$$

$$= \frac{-1}{2h^2}\left(\frac{1}{2h}\right) - \frac{1}{12h^3}\left(\frac{3}{8h^4}\right) - \frac{1}{8h^4}\left(\frac{1}{4h^3}\right)$$

$$+ \frac{1}{4h^3} + \frac{1}{6h}\left(\frac{15}{8h^6}\right) + \frac{1}{4h^2}\left(\frac{3}{4h^5}\right)$$

$$= \frac{7}{16h^7}$$

And $\int_0^\infty y_0^2\,dx = \int_0^\infty e^{-2hx}\,dx$

$$= \frac{1}{2h}$$

Thus

$$\lambda_2 = -\frac{7}{8h^6}$$

Equation [3 3 5] then becomes

$$y_2'' - \frac{7}{8h^6}e^{-hx} - \frac{1}{2h^2}\left(1 + \frac{2hx^3 + 3x^2}{12h^2}\right)e^{-hx} - h^2y_2$$

$$+ \; x^2\left(1 + \frac{2hx^3 + 3x^2}{12h^2}\right) e^{-hx} \quad = \quad 0$$

$$\Rightarrow y_2'' - h^2 y_2 + \left\{ \left(1 + \frac{2hx^3 + 3x^2}{12h^2}\right)\left(x^2 - \frac{1}{2h^2} \right) - \frac{7}{8h^6} \right\} e^{-hx}$$

$$= 0$$

Using the MACSYMA© computer package , the solution y_2 is found to be

$$y_2 = \frac{1}{1440h^6}\left[20h^4x^6 + 96h^3x^5 + 2\text{-}25h^4x^4 + (240h^5 + 420h)x^3 + (360h^4 + 630)x^2 \right] e^{-hx} + e^{-hx}$$

| ε^3 |

$$y_3'' + \lambda_3 y_3 + \lambda_2 y_1 + \lambda_1 y_2 + \lambda_0 y_3 + x^2 y_2 = 0$$

$$\Rightarrow \quad y_0 y_3'' + \lambda_3 y_0^2 + \lambda_2 y_1 y_0 + \lambda_1 y_2 y_0 + \lambda_0 y_3 y_0 + x^2 y_2 y_0 = 0$$

Again $\lambda_0 y_0 = -y_0''$

$$\Rightarrow \quad y_0 y_3'' - y_0'' y_3 + \lambda_3 y_0^2 + \lambda_2 y_1 y_0 + \lambda_1 y_2 y_0 + x^2 y_2 y_0 = 0$$

$$\Rightarrow \int_0^\infty y_0 y_3'' dx - \int_0^\infty y_0'' y_3 dx + \int_0^\infty \lambda_3 y_0^2 \; dx \; \int_0^\infty \lambda_2 y_1 y_0 dx + \int_0^\infty \lambda_1 y_2 y_0 dx$$

$$+ \int_0^\infty x^2 y_2 y_0 dx = 0$$

And as before

© Computations reported in this paper (when indicated) were achieved with the aid of MACSYMA, a large symbolic manipulation program developed at the MIT Laboratory for Computer Science and supported from 1975 to 1983 by the national Aeronautics and Space Administration under grant N00014-77-C-0641, by the U S Department of Energy under grant F49620-79-C-020, and since 1982 by Symbolics Inc of Cambridge Mass Macsyma is a Trademark of Symbolics, Inc

$$\lambda_3 = - \frac{\int_0^\infty \lambda_2 y_1 y_0 dx + \int_0^\infty \lambda_1 y_2 y_0 dx + \int_0^\infty x^2 y_2 y_0 dx}{\int_0^\infty y_0^2 \, dx}$$

Using MACSYMA , this turned out to be

$$\lambda_3 = \frac{-1}{128h^{10}} \left\{ -112h^4+28-64h^8-16h^4-69+64h^8+128h^4+1065 \right\} ,$$

that is,

$$\lambda_3 = - \frac{121}{16h^{10}}$$

Thus a regular perturbation expansion yields

$$\lambda = -h^2 - \frac{\varepsilon}{2h^2} - \frac{7\varepsilon^2}{8h^6} - \frac{121\varepsilon^3}{16h^{10}} + O(\varepsilon^4) \qquad [3\ 3\ 8]$$

It is apparent that this method yields no information on Im λ This is not suprising since it can be seen that the components of $y(x)$ above fail to satisfy the outgoing wave condition [3 2 2b]

In fact , it is this condition that makes the problem singular in nature and hence regular perturbative methods are destined to fail It should be noted however that [3.3 8] above is a valid estimate for Re λ However, since it conveys no information on Im λ we must resort to more subtle methods in order to calculate this small but crucial component of the eigenvalue.

CHAPTER 4

Mathematical Prerequisites

In this chapter we shall derive the asymptotic expansion for the parabolic cylinder function U(a,z) and discuss its various properties We shall also examine its connection with Stokes phenomenon and derive the Stokes and anti-Stokes lines for U(a,z)

4.1 The asymptotic behaviour of U(a,z)

We first consider the following second order ordinary differential equation which is one form of Weber's equation

$$\frac{d^2y}{dx^2} = \left(\frac{1}{4} x^2 + a \right) y \qquad . \qquad [4\ 1\ 1]$$

Using the Liouville Green approximation (Reference [3] Chapter 3) we first assume

$$y = e^{S(x)} \qquad [4\ 1\ 2]$$

Substituting this into [4.1 1] we obtain

$$S''(x) + (S'(x))^2 = \frac{1}{4} x^2 + a \qquad [4\ 1\ 3]$$

Making the approximations

$$S''(x) \ll (S'(x))^2 \ , \ a \ll \frac{1}{4} x^2 \qquad , \ x \longrightarrow +\infty \qquad [4\ 1\ 4]$$

gives the asymptotic differential equation

$$(S')^2 \sim \tfrac{1}{4}x^2 \quad , \; x \longrightarrow +\infty \qquad [4\ 1\ 5]$$

whose solutions are

$$S(x) \sim \pm \tfrac{1}{4}x^2 \; , x \longrightarrow +\infty$$

We have now determined that the possible controlling factors of the leading behaviour of y(x) are

$$e^{S_1(x)} = e^{x^2/4} \qquad [4\ 1\ 6]$$

and $$e^{S_2(x)} = e^{-x^2/4}$$

To derive the leading behaviour of y(x) for large x, we re-examine equation [4 1 5]

If $$(S')^2 \sim \tfrac{1}{4}x^2 \quad , \; x \longrightarrow +\infty$$

then $$S(x) \sim \pm \tfrac{1}{4}x^2 + C(x) \; , x \longrightarrow +\infty$$

where $$C(x) \ll \tfrac{1}{4}x^2 \text{ as } x \longrightarrow +\infty$$

Substituting $S_1(x) = \frac{1}{4}x^2 + C(x)$, $x \longrightarrow +\infty$ into equation [4 1 3] we find

$$\tfrac{1}{2} + C''(x) + [(\tfrac{1}{2}x + C'(x)]^2 = \tfrac{1}{4}x^2 + a$$

$$\Rightarrow \quad \tfrac{1}{2} + C''(x) + [C'(x)]^2 + \tfrac{1}{4}x^2 + xC'(x) = \tfrac{1}{4}x^2 + a$$

$$\Rightarrow \quad \tfrac{1}{2} + C''(x) + [C'(x)]^2 + xC'(x) = a \qquad [4\ 1\ 7]$$

Making the approximations $C''(x) \ll \frac{1}{2}$, $x \longrightarrow +\infty$ and $(C'(x))^2 \ll xC'(x)$, $x \longrightarrow +\infty$ { we note that these approximations follow from $C(x) \ll \frac{1}{4}x^2$, $x \longrightarrow +\infty$ } we find that equation [4.1 7] becomes

$$\frac{1}{2} + xC'(x) \sim a , \qquad x \longrightarrow +\infty$$

Thus $$C'(x) \sim \frac{a-1/2}{x} , \; x \longrightarrow +\infty$$

and $$C(x) \sim (a-1/2)\ln x , x \longrightarrow +\infty$$

Therefore , we find that

$$y_1(x) \sim c_1 x^{a-1/2} e^{x^2/4} , \; x \longrightarrow +\infty \qquad [4\;1'\;8]$$

On substituting $S_2(x) = -\frac{1}{4}x^2 + C(x)$, $x \longrightarrow +\infty$ into equation [4 1 3] we find

$$-\frac{1}{2} + C''(x) + [(-\frac{1}{2}x + C'(x)]^2 = \frac{1}{4}x^2 + a$$

$$\Rightarrow \; -\frac{1}{2} + C''(x) + [C'(x)]^2 - xC'(x) = a$$

Making the same approximations as before, we find

$$- xC'(x) \sim a + 1/2 , \; x \longrightarrow +\infty$$

$$\Rightarrow \qquad C'(x) \sim \frac{(-a - 1/2)}{x} , \quad x \longrightarrow +\infty$$

$$\Rightarrow \qquad C'(x) \sim (-a-1/2)\ln x , \; x \longrightarrow +\infty$$

Therefore, we see that

$$y_2(x) \sim c_2 x^{-a-1/2} e^{-x^2/4} , \; x \longrightarrow +\infty \qquad [4\;1\;9]$$

It is conventional to define the parabolic cylinder function U(a,x) to be that solution of equation [4 1 1] whose asymptotic behaviour is given by [4 1.9] with $c_2 = 1$ This means that $c_1 = 0$ because we observe that U(a,x) is subdominant on the positive x-axis and for it to be a solution we must eliminate the dominant behaviour given by [4 1.6]

In general, the principal solution U(a,z) to the equation

$$\frac{d^2w}{dz^2} = \left(\frac{1}{4} z^2 + a \right) w \qquad [4\ 1\ 10]$$

is determined by

$$U(a,z) = z^{-a-1/2} e^{-z^2/4} \text{ as } z \longrightarrow \infty \qquad [4\ 1\ 11]$$

Other solutions to [4 1 10] are $U(-a,\pm iz)$ and $U(a,-z)$ We shall refrain from discussing these solutions and their connection formulae until we have introduced Stokes phenomenon

Figure 4 1 overleaf illustrates typical level curves of U(a,x) where x and a are both real

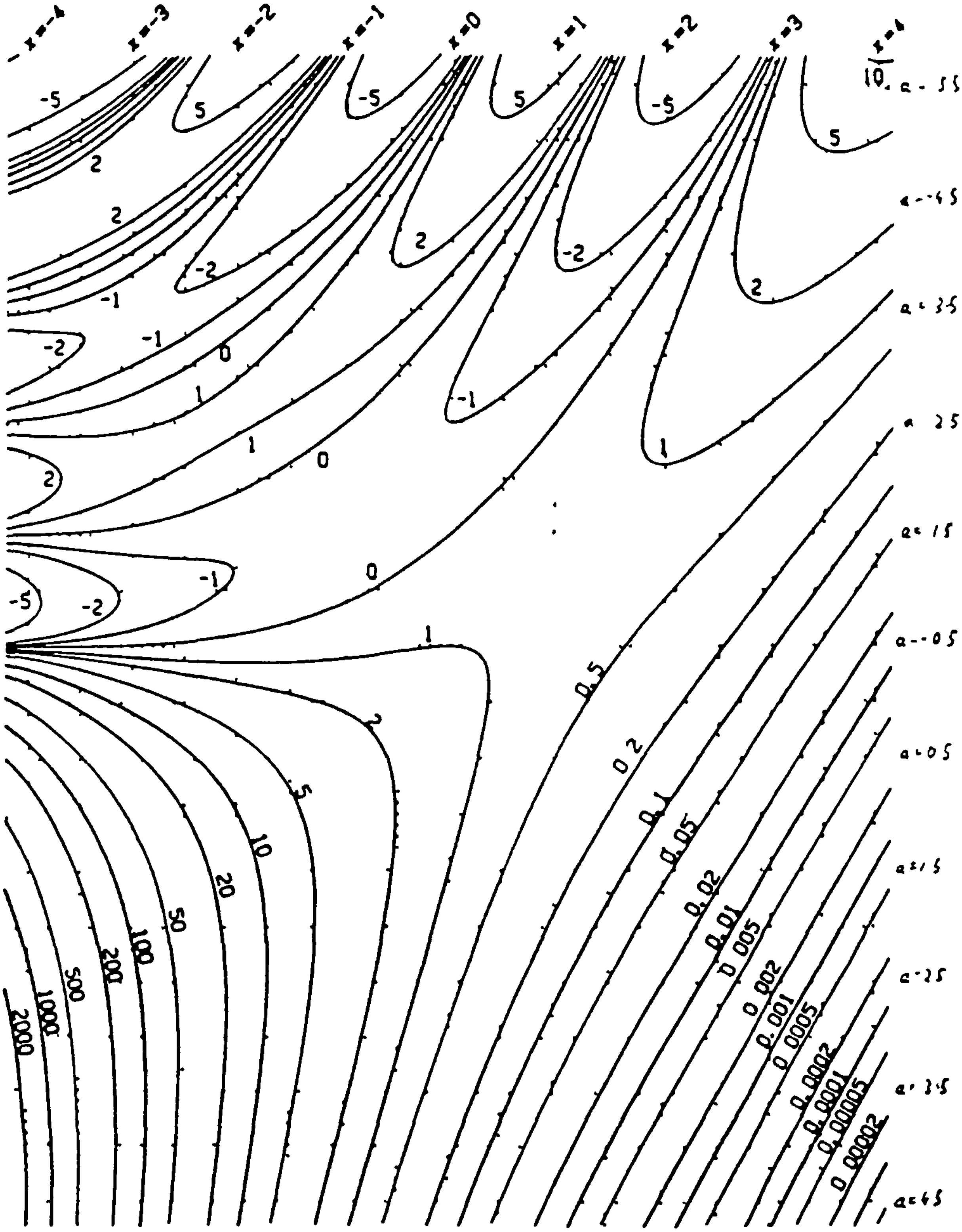

Figure 4.1 : Level curves of the parabolic cylinder function U(a,x).

4 2 Stokes Phenomenon

We have seen in [4 1.11] that the solution U(a,z) of the differential equation [4 1 10] satisfies

$$U(a,z) = z^{-a-1/2} e^{-z^2/4} \text{ as } z \longrightarrow +\infty$$

as z tends to infinity along the positive real axis We now consider what happens when z is allowed to approach infinity along any path in the complex plane There are two difficulties with this The first is that U(a,z) is an analytic function of z, defined by a convergent series for $|z| < \infty$, while the right-hand-side of the asymptotic equivalence is a multi-valued function of the complex variable z , with branch points at $z = 0,\infty$ We cannot sensibly define an analytic function to be asymptotic to a non-analytic function The second difficulty arises from the fact that the leading asymptotic behaviour of U(a,z) as $z \longrightarrow \infty$ along the negative real axis can be shown to be different from that along the positive real axis A simpler example of the same behaviour arises in the function

$$g(z) = \sinh(\tfrac{1}{z}) = \tfrac{1}{2} (e^{1/z} - e^{-1/z})$$

which has leading behaviours

$$g(z) \sim \tfrac{1}{2} e^{1/z} \quad \text{as } z \longrightarrow 0 \text{ in } |\arg z| < \tfrac{\pi}{2}$$

$$g(z) \sim \tfrac{1}{2} e^{-1/z} \quad \text{as } z \longrightarrow 0 \text{ in } \tfrac{\pi}{2} < |\arg z| < \tfrac{3\pi}{2}$$

It is clear from these examples that asymptotic behaviour in the complex plane depends on the path along which the irregular singular point z_0 is approached We cannot say $f(z) \sim g(z)$ as $z \longrightarrow z_0$ because , since a function can take infinitely many values in the neighbourhood of an essential singularity , the limit of $f(z)/g(z)$ as $z \longrightarrow z_0$ need not exist This discussion suggests that asymptotic relations in the complex plane must involve the concept of a sector of validity with vertex at the singular point

For example, given f(z) and g(z) as before such that

$$f(z) \sim g(z) \text{ as } z \longrightarrow z_0$$

in some sector D of the complex plane Then if we write

$$f(z) = g(z) + [f(z) - g(z)]$$

then what we are saying by writing $f(z) \sim g(z)$ as $z \longrightarrow z_0$ in D is that $f(z) - g(z)$ is small (or subdominant) in D as compared with g(z) (which is dominant)
On the boundary of D , both $f(z) - g(z)$ and g(z) are of equal magnitude and as we cross this line , the characteristics of $f(z) - g(z)$ and g(z) change while $f(z) - g(z)$ becomes dominant, g(z) becomes subdominant This occurrence is known as Stokes phenomenon

We define Stokes lines to be those asymptotes in the complex plane upon which the difference between the dominant and subdominant terms is greatest in magnitude

Similarly , we define anti-Stokes lines to be those asymptotes in the complex plane upon which the "dominant" and "subdominant" terms are of equal magnitude †

If the controlling behaviour of solutions to a second order differential equation are given by $e^{S_1(x)}$ and $e^{S_2(x)}$ as $z \longrightarrow z_0$, then the Stokes lines are the asymptotes as $z \longrightarrow z_0$ of the curves

$$\text{Im} [S_1(x) - S_2(x)] = 0 \qquad [4\ 2\ 1]$$

while the anti-Stokes lines are the asymptotes as $z \longrightarrow z_0$ of the curves

$$\text{Re} [S_1(x) - S_2(x)] = 0 \qquad [4\ 2.2]$$

4 3 Stokes phenomenon and U(a,z)

We have seen [Section 4 1] that for the parabolic cylinder function U(a,z) we have

$$S_1(z) = \tfrac{1}{4} z^2$$

$$S_2(z) = - \tfrac{1}{4} z^2$$

Thus, the Stokes lines for U(a,z) are the lines

† It should be noted that many text books define " Stoke s lines" to be what we would term "anti-Stokes lines" and vice versa We prefer to abide by the terminology used by George Stokes in his original work

$$\text{Im}\left[\ \frac{1}{4}z^2 + \frac{1}{4}z^2\ \right] = 0$$

$$\Rightarrow \quad \text{Im}\ (\ e^{2\iota\theta}) = 0$$

where we have written $z = Re^{\iota\theta}$.

Thus

$$\sin 2\theta = 0$$

$$\Rightarrow \quad \theta = 0\ ,\ \frac{\pi}{2}\ ,\ \pi\ ,\ \frac{3\pi}{2}$$

i e Stokes lines occur when arg $z = \theta = \frac{\pi}{2}\ ,\ \pi\ ,\ \frac{3\pi}{2}$ †

Anti-Stokes lines occur when

$$\text{Re}\left[\ \frac{1}{4}z^2 + \frac{1}{4}z^2\ \right] = 0$$

$$\Rightarrow \quad \text{Re}\ (\ e^{2\iota\theta}) = 0$$

$$\Rightarrow \quad \cos 2\theta = 0$$

$$\Rightarrow \quad \theta = \pm\frac{\pi}{4}\ ,\ \frac{3\pi}{4}$$

i e anti-Stokes lines occur when arg $z = \pm\frac{\pi}{4}\ ,\ \pm\frac{3\pi}{4}$

The Stokes and anti-Stokes lines for U(a,z) are illustrated in figure 4 2 overleaf.

† The reason why arg z = 0 is not a Stokes line for U(a,z) is apparent from its definition In defining U(a,z) we set its stokes multipliers to be 0 and 1 respectively in the sector $|\arg z| < \pi/4$ Thus the fluctuation of the multipliers as we cross the positive real axis is not deemed possible (See dicussion at end of this section)

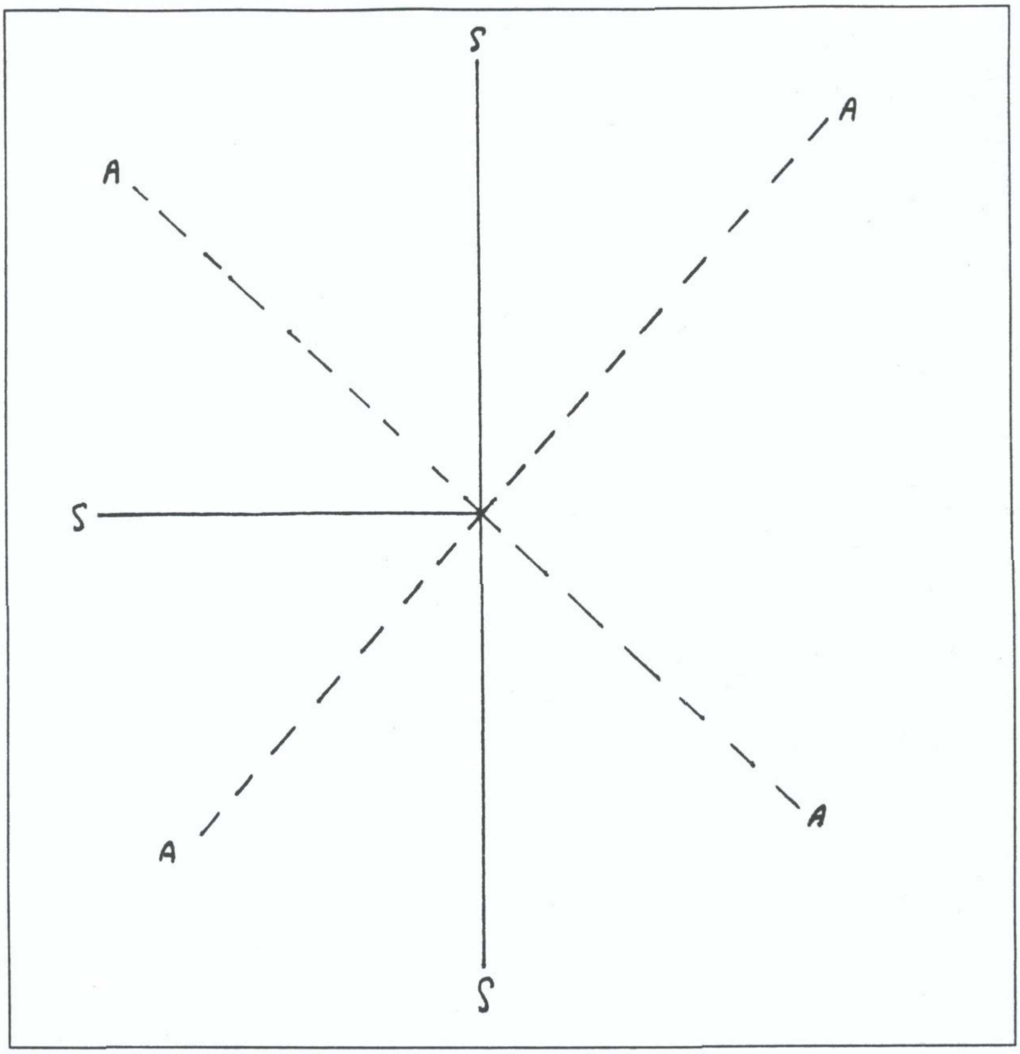

Figure 4.2 : Stokes and anti-Stokes lines for U(a,z)

We have seen that U(a,z) is defined as the subdominant solution to equation [4.1.1] along the positive x-axis. As a rule, that solution which decays most rapidly along the positive real axis grows as z is rotated through the anti-Stokes line nearest the positive real axis. For this kind of solution , it is correct to continue analytically the leading asymptotic behaviour up to the anti-Stokes lines beyond the ones nearest the real axis

Thus the leading asymptotic behaviour of U(a,z) given by $z^{-a-1/2}e^{-z^2/4}$ is valid up to the anti-Stokes lines at $\arg z = \pm \frac{3\pi}{4}$

In fact, it can be shown by carrying out repeated corrections to the leading behaviour of U(a,z) that

for fixed a and large $|z|$,

$$U(a,z) = z^{-a-1/2}e^{-z^2/4} \{ 1 + O(|z|^{-2}) \} , \quad |\arg z| < \frac{3\pi}{4} \qquad [4\ 3\ 1]$$

where the O is uniform with respect to arg z.
(See Reference [12])

For any solution of a second order differential equation, we must have

$$y(z) = c_1 y_1(z) + c_2 y_2(z)$$

We have seen that for Weber's parabolic cylinder equation [4 1 10]

$$y_1(z) \sim c_1 z^{a-1/2} \; e^{z^2/4} \quad , \; z \longrightarrow \infty ,$$

$$y_2(z) \sim c_2 z^{-a-1/2} \; e^{-z^2/4} \quad , \; z \longrightarrow \infty ,$$

and hence

$$y(z) \sim c_1 z^{a-1/2} \; e^{z^2/4} + c_2 z^{-a-1/2} \; e^{-z^2/4} \quad , \; z \longrightarrow \infty$$

c_1 and c_2 are called Stokes multipliers and have the remarkable property that they fluctuate as one travels from sector to sector through the Stokes lines

By the very definition of U(a,z) we have

$$\left.\begin{array}{l} c_1 = 0 \\ c_2 = 1 \end{array}\right\} \text{ when } |\arg z| < \frac{\pi}{4} \quad .$$

The Stokes constants c_1 and c_2 can be calculated in other sectors by using connection formulae (See Reference [3] Chapter 3)

We are particularily interested in the asymptotic behaviour of U(a,-z) as $z \longrightarrow \infty$ with $0 < \arg z < \frac{\pi}{2}$ In the next section we shall derive this behaviour and in the process, ascertain its Stokes multipliers in this sector

4 4 The asymptotic behaviour of U(a,-z)

We have seen that $U(a,\pm z)$, $U(-a,\pm\iota z)$ are solutions to Weber's parabolic cylinder equation [4 1 10] The connection formula between them are [See Reference [12] p 133]

$$U(-a, \pm\iota z) = (2\pi)^{-1/2}\Gamma(\tfrac{1}{2} + a)\Big[\exp\Big\{-\iota\pi(\tfrac{1}{2}a - \tfrac{1}{4})\Big\} U(a,\pm z) + \Big\{\iota\pi(\tfrac{1}{2}a - \tfrac{1}{4})\Big\} U(a,\mp z)\Big] \qquad [4\ 4\ 1]$$

$$U(a, \pm z) = (2\pi)^{-1/2}\Gamma(\tfrac{1}{2} - a)\Big[\exp\Big\{-\iota\pi(\tfrac{1}{2}a + \tfrac{1}{4})\Big\} U(-a,\pm\iota z) + \Big\{\iota\pi(\tfrac{1}{2}a + \tfrac{1}{4})\Big\} U(a,\mp\iota z)\Big] \quad . \qquad [4\ 4.2]$$

From [4 4 1],

$$U(-a,\ -\iota z) = (2\pi)^{-1/2}\,\Gamma(\tfrac{1}{2} + a)\left[\exp\left\{-\iota\pi(\tfrac{1}{2}a - \tfrac{1}{4})\right\} U(a,-z) + \exp\left\{\iota\pi(\tfrac{1}{2}a - \tfrac{1}{4})\right\} U(a,+z)\right]$$

and therefore

$$U(a,-z) = \frac{(2\pi)^{1/2}}{\Gamma(\frac{1}{2} + a)} \exp\left\{\iota\pi(\tfrac{1}{2}a - \tfrac{1}{4})\right\} U(-a,-\iota z) - \exp\left\{2\iota\pi(\tfrac{1}{2}a - \tfrac{1}{4})\right\} U(a,z) \qquad [4\ 4\ 3]$$

Now when $0 < \arg z < \frac{\pi}{2}$ then $-\frac{\pi}{2} < \arg(-\iota z) < 0$

and we can use the asymptotic representation [4 3 1] for $U(a,z)$ and $U(-a,-\iota z)$

Accordingly, $U(a,-z) =$

$$\frac{(2\pi)^{1/2}}{\Gamma(\frac{1}{2} + a)} \exp\left\{\iota\pi(\tfrac{1}{2}a - \tfrac{1}{4})\right\}\left[(-\iota z)^{a-1/2}\{1 + O(|z|^{-2})\}\right]e^{z^2/4}$$

$$- \exp\left\{2\iota\pi(\tfrac{1}{2}a - \tfrac{1}{4})\right\}\left[\ z^{-a-1/2}\{1 + O(|z|^{-2})\}\right]e^{-z^2/4}$$

but

$$(-\iota)^{a-1/2} = \exp(a-1/2)\ln(-\iota) = \exp(-\pi\left\{\tfrac{1}{2}a - \tfrac{1}{4}\right\}\ \iota\ )$$

and

$$-\exp\left\{2\iota\pi(\tfrac{1}{2}a - \tfrac{1}{4})\right\} = \exp\left\{\iota\pi(a + \tfrac{1}{2})\right\}$$

Consequently,

$$U(a,-z) = \frac{(2\pi)^{1/2}}{\Gamma(\frac{1}{2} + a)} z^{a-1/2} e^{z^2/4} \left\{ 1 + O(|z|^{-2})\right\}$$
$$+ e^{\iota\pi(a+1/2)} z^{-a-1/2} e^{-z^2/4} \left\{ 1 + O(|z|^{-2})\right\} \qquad [4\ 4\ 4]$$

as $z \longrightarrow \infty$ in $0 < \arg z < \frac{\pi}{2}$

In the next chapter we shall use the above relations to derive a solution to the model problem outlined in Chapter 3

CHAPTER 5

The Solution of the Model Problem

In this chapter we shall derive a solution to the model problem by first transforming it into the parabolic cylinder equation and then using the associated parabolic cylinder function solutions to obtain a valid estimate for Im λ We shall first find a combination of parabolic cylinder functions to satisfy the boundary condition at infinity We then substitute this combination into the boundary condition at the origin to yield the eigenvalue relation

5 1 Transforming the Model Problem

We shall restate the problem here for clarity

$$y''(x) + (\lambda + \varepsilon x^2)y(x) = 0 \quad \text{on} \quad (0,\infty) \qquad [5.1\ 1]$$

with,

$$y'(0) + hy(0) = 0 \qquad [5.1\ 2]$$

and

$$y(x) \text{ has controlling behaviour } e^{\iota p(x)} \text{ as } x \longrightarrow +\infty \qquad [5\ 1.3]$$

h is a positive constant and p(x) is a positive function in x.

If we let

$$z = e^{\iota\pi/4} 2^{1/2} \varepsilon^{1/4} x \qquad [5\ 1\ 4]$$

then

$$\frac{dy}{dx} = e^{\iota\pi/4}\ 2^{1/2}\ \varepsilon^{1/4}\ \frac{dy}{dz}$$

and

$$\frac{d^2y}{dz^2} = 2\iota\varepsilon^{1/2}\ \frac{d^2y}{dz^2}$$

Substituting these into equations [5 1 1] and [5 1 2] we find

$$\frac{d^2y}{dz^2} = (\tfrac{1}{2}\ \varepsilon^{-1/2}\iota\lambda\ + \tfrac{1}{4}\ z^2\ )\ y$$

If we let

$$a = \tfrac{1}{2}\ \varepsilon^{-1/2}\ \iota\ \lambda \qquad [5\ 1\ 5]$$

then we have

$$\frac{d^2y}{dz^2} = (\ a + \tfrac{1}{4}\ z^2\ )\ y$$

The boundary condition at the origin (equation [5 1 2]) becomes

$$(\ e^{\iota\pi/4}\ 2^{1/2}\ \varepsilon^{1/4}\ )\ \frac{dy}{dz}(0)\ +\ h\ y(0)\ = 0$$

Thus the transformed problem [where the dash now denotes differentiation with respect to z] becomes

$$y''(z) = \left(\tfrac{1}{4} z^2 + a \right) y(z) \quad \text{on } (0,\infty) \qquad [5\ 1\ 6]$$

$$e^{\pi i/4}\, 2^{1/2}\, \varepsilon^{1/4}\, y'(0) + h y(0) = 0 \qquad [5\ 1\ 7]$$

$y(z)$ is an outgoing wave as $z \longrightarrow \infty$ [5 1 8]

5.2 Solution of the Transformed Problem

We have seen (See Chapter 4) that equation [5 1 6] above is one form of Weber's parabolic cylinder equation This equation has general solution (See Reference [12])

$$y(z) = C_1\, U(a,z) + C_2\, V(a,z) \qquad [5\ 2\ 1]$$

It is worth noting that arg $z = \pi/4$ for future reference

$U(a,z)$ is the parabolic cylinder function defined as the solution to [5 1 6] determined by [See Chapter 4]

$$U(a,z) \sim z^{-a-1/2}\, e^{-z^2/4} \qquad \text{as } z \longrightarrow \infty$$

$V(a,z)$ is defined as

$$V(a,z) = \frac{1}{\pi}\, \Gamma\!\left(\tfrac{1}{2} + a\right) \{ \sin \pi a\, U(a,z) + U(a,-z) \} \qquad [5\ 2\ 2]$$

For fixed a and large $|z|$,

$$U(a,z) = z^{-a-1/2}\, e^{-z^2/4} \{ 1 + O(|z|^{-2}) \} , \quad |\arg z| < 3\pi/4 \qquad [5\ 2.3]$$

$$U(a,-z) = e^{i\pi(a+1/2)} z^{-a-1/2} e^{-z^2/4} \{1 + O(|z|^{-2})\} +$$

$$\frac{(2\pi)^{1/2}}{\Gamma(\frac{1}{2} + a)} \{ z^{a-1/2} e^{z^2/4} \{1 + O(|z|^{-2})\}$$

where $0 < (\arg z) < \pi/2$ [5 2 4]

Thus

$$V(a,z) = (\frac{1}{\pi}) \Gamma(\frac{1}{2} + a)\{\sin \pi a \; z^{-a-1/2} e^{-z^2/4} \{1 + O(|z|^{-2})\}$$

$$+ e^{i\pi(a+1/2)} z^{-a-1/2} e^{-z^2/4} \{1 + O(|z|^{2})\} +$$

$$\frac{(2\pi)^{1/2}}{\Gamma(\frac{1}{2} + a)} z^{a-1/2} e^{z^2/4} \{1 + O(|z|^{-2})\}$$

$$\text{in } 0 < (\arg z) < \frac{\pi}{2}$$

The outgoing wave condition requires the exclusion of the incoming wave associated with the term involving $e^{-z^2/4}$

We resolve this condition by choosing

$C_2 = 1$ and $C_1 = -(\frac{1}{\pi}) \Gamma(\frac{1}{2} + a)\{\sin \pi a + e^{i\pi(a+1/2)}\}$

Now $\quad C_1 = -(\frac{1}{\pi}) \Gamma(\frac{1}{2} + a)\{\sin \pi a + ie^{ia\pi}\}$

$$= -(\frac{1}{\pi}) \Gamma(\frac{1}{2} + a)\{\sin \pi a + i\cos \pi a - \sin \pi a\}$$

$$= -(\frac{1}{\pi}) \Gamma(\frac{1}{2} + a)\{ i\cos \pi a \}$$

Accordingly, our solution satisfying the outgoing wave condition is

$$y(z) = V(a,z) - \left(\frac{\iota}{\pi} \right) \Gamma(\tfrac{1}{2} + a) \cos \pi a \; U(a,z) \qquad [5\ 2\ 6]$$

Using equation [5 2.2] this becomes

$$y(z) = \frac{\Gamma(a+1/2)}{\pi} \{U(a,-z) - \iota e^{\iota a\pi} U(a,z)\} \qquad [5\ 2\ 7]$$

Although this solution ensures that the leading terms associated with $e^{-z^2/4}$ cancel , we cannot be confident that lower order terms also cancel We resolve these doubts by manipulating equation [4 4 3] in Chapter 4 into the following form ·

$$-\iota e^{\iota a\pi} U(a,z) = - U(a,-z) + \frac{(2\pi)^{1/2}}{\Gamma(a+1/2)} e^{\iota\pi(a-1/2)/2} U(-a,-\iota z)$$

If we substitute this into our solution , we obtain

$$y(z) = (2/\pi)^{1/2} e^{\iota\pi(a-1/2)/2} U(-a,-\iota z)$$

As illustrated in Chapter 4 , the asymptotics of $U(-a,-\iota z)$ contains terms involving only $e^{z^2/4}$ as $|z| \longrightarrow \infty$

5 3 Calculation of Im λ

Equation [5 2 7] defines the solution satisfying the outgoing wave condition We must now use this solution to estimate Im λ

Substituting equation [5 2 7] into the boundary condition at the origin (equation [5 1 7]) we find

$$e^{\iota\pi/4}\, 2^{1/2}\, \varepsilon^{1/4} \left[-U'(a,o) - \iota e^{\iota a\pi}\, U'(a,0) \right] + h\left[U(a,0) - \iota e^{\iota a\pi}\, U(a,0) \right] = 0$$

Therefore,

$$-\varepsilon^{1/4} e^{\iota\pi/4}\, 2^{1/2} \left[1 + \iota e^{\iota a\pi}\right] U'(a,0) + h\left[1 - \iota e^{\iota a\pi}\right] U(a,0) = 0$$

Thus,

$$\frac{U(a,0)}{U'(a,0)} = \frac{e^{\iota\pi/4}\, 2^{1/2}\, \varepsilon^{1/4}}{h} \frac{\left[1 + \iota e^{\iota a\pi}\right]}{\left[1 - \iota e^{\iota a\pi}\right]} \qquad [5\ 3\ 1]$$

But (See Reference [1] p 687),

$$U(a,0) = \frac{(\pi)^{1/2}}{2^{(1/2)a + 1/4}\, \Gamma\left(\frac{3}{4} + \frac{1}{2}a\right)}$$

and

$$U'(a,0) = \frac{-(\pi)^{1/2}}{2^{(1/2)a+1/4}\,\Gamma(\frac{1}{4}+\frac{1}{2}a)}$$

Therefore

$$\frac{U(a,0)}{U'(a,0)} = -\frac{2^{(1/2)a-1/4}\,\Gamma(\frac{1}{4}+\frac{1}{2}a)}{2^{(1/2)a+1/4}\,\Gamma(\frac{3}{4}+\frac{1}{2}a)}$$

Therefore ,equation [5 3 1] becomes

$$\frac{U(a,0)}{U'(a,0)} = \frac{-2^{-1/2}\,\Gamma(\frac{1}{4}+\frac{1}{2}a)}{\Gamma(\frac{3}{4}+\frac{1}{2}a)}$$

which implies,

$$\frac{\Gamma(\frac{1}{4}+\frac{1}{2}a)}{\Gamma(\frac{3}{4}+\frac{1}{2}a)} = \frac{-2e^{\iota\pi/4}\varepsilon^{1/4}}{h}\,\frac{[\,1+\iota e^{\iota a\pi}\,]}{[\,1-\iota e^{\iota a\pi}\,]} \qquad [5\ 3\ 2]$$

But we know (See Reference [11] p 118)

$$\frac{\Gamma(w+p)}{\Gamma(w+q)} = \left\{ 1 + \frac{(p-q)(p+q-1)}{w} + O(w^{-2}) \right\} \qquad [5\ 3.3]$$

as $|w| \longrightarrow \infty$ in $|\arg w| < \pi$

Bearing in mind that $a = \frac{1}{2}\varepsilon^{-1/2}\iota\lambda$ and $\operatorname{Im}\lambda < 0$ (see Chapter 4)

then

as $\varepsilon \longrightarrow 0^{+}$, $|a| \longrightarrow \infty$.

Hence the criteria for the relation [5 3 3] are satisfied and

$$\frac{\Gamma(\frac{1}{4} + \frac{1}{2}a)}{\Gamma(\frac{3}{4} + \frac{1}{2}a)} \sim (\tfrac{1}{2}a)^{-1/2} \qquad , \varepsilon \longrightarrow 0^+$$

Examining $\dfrac{[1 + \iota e^{\iota a\pi}]}{[1 - \iota e^{\iota a\pi}]}$

we see $$\frac{[1 + \iota e^{\iota a\pi}]}{[1 - \iota e^{\iota a\pi}]} = - \frac{[1 - \iota e^{-\iota a\pi}]}{[1 + \iota e^{-\iota a\pi}]}$$

we observe that since Im $\lambda < 0$ and Re $\lambda < 0$ then we can say $\lambda = - H^2$ where $|\arg H| < \pi/2$

and

$$-\iota a\pi = -(\tfrac{1}{2})\varepsilon^{-1/2}H^2$$

which implies that $e^{-\iota a\pi} << 1$ as $\varepsilon \longrightarrow 0^+$

Accordingly, we can expand $[1 + \iota e^{-\iota a\pi}]^{-1}$ in a Binomial series

$[1 + \iota e^{-\iota a\pi}]^{-1} = [1 - \iota e^{-\iota a\pi} + O(e^{-2\iota a\pi})]$ as $\varepsilon \longrightarrow 0^+$

Thus,

$$- \frac{[1 - \iota e^{-\iota a\pi}]}{[1 + \iota e^{\iota a\pi}]} = - [1 - \iota e^{-\iota a\pi}][1 - \iota e^{-\iota a\pi} + O(e^{-2\iota a\pi})]$$

as $\varepsilon \longrightarrow 0^+$

and therefore

$$\frac{[1 + \iota e^{\iota a\pi}]}{[1 - \iota e^{\iota a\pi}]} = - [1 - 2\iota e^{\iota a\pi} + O(e^{2\iota a\pi})] \ , \ \varepsilon \longrightarrow 0^+$$

[5 3 5]

Inserting Equations [5 3.4] and [5 3 5] into [5 3 2] we observe that

$$(a/2)^{-1/2} \sim \frac{-[1 - 2\iota e^{\iota a\pi} + O(e^{2\iota a\pi})][2e^{\iota\pi/4}\varepsilon^{1/4}]}{h} \ ,\varepsilon \longrightarrow 0^+$$

$$2/a \sim \frac{[1 - 4\iota e^{\iota a\pi}] \ 4\iota\varepsilon^{1/2}}{h^2} \ ,\varepsilon \longrightarrow 0^+$$

$$a \sim \frac{-[1 + 4\iota e^{\iota a\pi}]\iota h^2}{2\varepsilon^{1/2}} \ ,\varepsilon \longrightarrow 0^+$$

But from equation [5.1 5] $a = (\frac{1}{2})\varepsilon^{-1/2}\iota\lambda$

So ,

$$\frac{1}{2}\varepsilon^{-1/2}\iota\lambda \sim \frac{-[1 + 4\iota e^{\iota a\pi}]\iota h^2}{2\varepsilon^{1/2}} \ ,\varepsilon \longrightarrow 0^+$$

$$\lambda \sim -h^2[1 + 4\iota e^{\lambda\pi/2\varepsilon^{1/2}}] \ ,\varepsilon \longrightarrow 0^+$$

From our perturbative analysis we know that a good approximation to λ is (See equation [3.3.8])

$$\lambda = -h^2 - \frac{\varepsilon}{2h^2} \qquad +$$

and as $\varepsilon \longrightarrow 0^+$, Re $\lambda \approx -h^2$

Therefore ,

$$\text{Im } \lambda \sim -4h^2 \exp \{ \frac{\pi}{\varepsilon^{1/2}} \}\{-h^2 - \varepsilon/2h^2 - \quad ..\} \ ,\varepsilon \longrightarrow 0^+$$

Hence,

$$\text{Im } \lambda \sim -4h^2 e^{-h^2\pi/2\varepsilon^{1/2}} \qquad ,\varepsilon \longrightarrow 0^+ \qquad [5\ 3\ 6]$$

Thus it is not suprising that we were unable to pick up any information on Im λ with our perturbative expansion due to its small size since it can be seen that Im λ is $o(\varepsilon^n)$ for $n \in \mathbb{N}$ (provided $n \neq 0$)

CHAPTER 6

Conclusion

Chapter 5 provided us with our final result for Im λ That is

$$\text{Im } \lambda \sim -4h^2\exp(-h^2\pi/2\varepsilon^{1/2}) \quad , \ \varepsilon \longrightarrow 0^+. \qquad [6\ 1]$$

We observe that we have entered the area now known as exponential asymptotics as discussed in Chapter 1. The problem , being singular in nature , was destined to produce such a result Regular perturbation methods provided us with our first estimate for Im λ , (See Chapter 3)

$$\lambda \sim -h^2 - \frac{\varepsilon}{2h^2} - \frac{7\varepsilon^2}{8h^6} - \frac{121\varepsilon^3}{10h^{10}} + O(\varepsilon^4) \qquad [6\ 2]$$

As $\varepsilon \longrightarrow 0^+$, we find $\lambda \sim -h^2$ from equation [6 2] above and Im λ tends to zero since $\exp[-h^2\pi/2\varepsilon^{1/2}]$ tends to zero (See equation [6.1]) . Equation [6 1] above indicates why the regular perturbation expansion fails to convey any information on Im λ as it is " swamped " by the comparatively large size of Re λ As we indicated in Chapter 1 , the exponential nature of Im λ is not suprising since the problem is singular in nature.

In their paper examining the case g(x)=x , R. Paris and A

Wood find that

$$\text{Im}\,\lambda \sim -\frac{2h^2}{e}\exp\left\{-\frac{4h^3}{3\varepsilon}\right\} \quad , \varepsilon \longrightarrow 0^+ \qquad [6\ 3]$$

Comparison of the relations [6 1] and [6.3] leads us to speculate that for $g(x) = x^n$ in the model problem ,

$$\text{Im}\,\lambda \sim -A\exp[-B\varepsilon^{-1/n}] \qquad , n = 1 , 2 , 3 ,$$

where A and B are positive real constants Indeed it is to the task of finding the behaviour of Im λ for $g(x) = x^n$ that Mr Liu Jing Song , under the supervision of Prof. A Wood , has applied himself

The minute size of Im λ in [6 1] can be seen clearly if we set the matching parameter h equal to 1 and evaluate Im λ for small values of ε. This is accomplished in table 6 1 below

ε	Imλ
0.1	$-6\ 96\times10^{-5}$
0 01	-1.51×10^{-7}
0 001	$-2\ 67\times10^{-22}$
0 0001	$-6\ 04\times10^{-69}$

Table 6.1 : Values of Imλ for several values of ε.

We observe that for ε = 0.0001 , Im λ ≈ 6.04×10^{-69} The vast majority of computers (and their associated operating systems) would not have sufficient precision to accurately represent a number of this size It results in underflow i e the computer treats the number as zero Thus , in this area "analytics" triumph over "numerics"

Finally we wish to emphasise again the existence of an intimate relationship between exponential asymptotics and Stokes phenomenon In treating this problem with g(x) = x , R Paris and A Wood are confronted with Stokes phenomenon directly because their solution was situated on a Stokes line for the Hankel function Thus , they were obliged to consider the problem of averaging across a Stokes line and the validity of said averaging This task was successfully accomplished In our case , (i e the model with $g(x) = x^2$) the solution requires asymptotics only along the anti-Stokes lines for U(a,z) In conclusion we state that without a basic awareness of the pitfalls associated with neglecting sub-dominant terms in asymptotic relations , one cannot be assured that consequent results are entirely valid

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