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M.Sc. THESIS

Special Solutions of the Optical Soliton Eigenvalue Problem

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ABSTRACT

Electromagnetic pulse propagation in optical fibres is described by the non-linear Schrodinger equation. The solutions, or solitons, remain completely unchanged as they propagate along the fibre. The question we are concerned with is, given an initial input pulse, does it contain solitons, and if so, how many.

Answering this question means solving the non-linear Schrodinger equation and this is done by using the Inverse Scattering Method. This method utilises several linear problems which are comparitively easier to solve - in this work we focus on the linear eigenvalue problem since it gives all the information about solitons

In Chapter 1, we first show that pulse propagation in optical fibres is described by the non-linear Schrodinger equation. Chapter 2 deals with the Inverse Scattering method and, in particular, how it is used to solve the non-linear Schrodinger equation.

In Chapters 3, 4 and 5, the eigenvalue problem is exactly solved for three special families of input pulses. We show for the three cases that the soliton number depends upon the area of the pulse only, regardless of the pulse's shape.

Finally, in Chapter 6, the eigenvalue problem is discussed for the super-Gaussian pulse, the type of pulse produced by semiconductor lasers. Formal solutions are obtained in terms of an infinite series of functions. To calculate the exact solutions, numerical computations are required. We present the working software code and suggestions for tackling this problem.

CONTENTS

PREFACE

The non-linear Schrodinger equation (N.L S.) belongs to a class of equations which are of much interest in applied mathematics. Aside from their complexity, these equations are interesting because their solutions are highly stable and remain unchanged in shape and form through time. The N.L.S is of particular interest because its solutions have been shown to represent pulses which propagate along optical fibres.

The stable solutions of the N.L.S. are called solitons and are considered to have a potentially important role to play in fibre optical data transmission. Unlike conventional systems, a fibre optical system that uses solitons requires no repeaters and the soliton pulses can be made to travel at different speeds. Already, a rate length product for distortionless transmission of 11,000 GHz-km has been achieved, and this represents a significant improvement on current systems.

In this work, we focus on a further aspect of this subject. We ask the question, given an input pulse, are solitons produced, and if so, under what conditions do they arise. From our mathematical viewpoint, we must solve the N.L.S. for actual pulses as produced by semiconductor lasers. This work sets out to solve this problem.

We solve the N.L.S. exactly for three pulses and have developed software for numerically calculating the solutions for a fourth, more realistic pulse. In all three exactly solvable cases, the same simple equation is found which gives the soliton number in terms of the pulse's area. This result is very important: - It says that despite the shape of any pulse we choose to inject into a fibre, the number of solitons born from it will depend only on the pulse area.

The fourth pulse we study is the super-Gaussian pulse, which models the kind of pulses generated by semiconductor lasers. Unlike the other pulses studied, we can only find the solution in the form of infinite series and so a computer needs to be used to generate them numerically. We include the software to generate these solutions. In order to generate these solutions to any desired degree of accuracy however, a detailed numerical analysis has to be added.

It still remains to see whether or not the area rule for the soliton number holds in this case. For this question to be answered satisfactorily, an analysis like that for the three preceeding pulses is required.

Kevin Breen.

CHAPTER ONE

NONLINEAR PULSE PROPAGATION IN A

MONOMODE DIELECTRIC WAVEGUIDE

In this section, we present the derivation of a non-linear wave equation which describes the propagation of an electromagnetic wave's envelope function in a monomode dielectric waveguide. Up to now, derivations of such a wave equation have been based on very general arguments and are for this reason non rigorous. We present one such derivation and then proceed to the recent derivation of the non-linear Schrodinger equation [1]. The non-linear Schrodinger equation is the equation which describes the propagation of solitons in optical fibres. A non rigorous derivation of the non-linear Schrodinger equation is as follows : -

Consider the wave equation

$$\left[c^2 \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} \right] \Psi(z,t) = 0 \qquad (1.1)$$

For solutions $\Psi(z,t)$ of eq(1.1) of the form $\Psi(z,t)=\exp[i(kz-\omega t)]$, which propagate along the linear fibre, the linear relation

$$k = \pm \frac{\omega}{c} \qquad (1.2)$$

can be shown to hold. Here, c is the speed of the electromagnetic waves, ω is the frequency and k is the wave number. If we now consider a monomode fibre which supports a pulse with carrier frequency ω_1, wavenumber k_1 and slowly varying amplitude q(z,t), we can expand k and obtain to low order

$$k - k_1 = \left.\frac{\partial k}{\partial \omega_1}\right|_{(\omega_1,0)} (\omega - \omega_1) + \frac{1}{2}\left.\frac{\partial^2 k}{\partial \omega_1^2}\right|_{(\omega_1,0)} (\omega - \omega_1)^2 + \left.\frac{\partial k}{\partial |E|^2}\right|_{(\omega_1,0)} |E|^2 \tag{1.3}$$

where the coefficients are evaluated at $\omega = \omega_1$ and $|E| = 0$. We now look for an equation for q which reproduces eq.(1.3) for the special case given by

$$q(z,t) \approx e^{i\left[(k-k_1)z - (\omega - \omega_1)t\right]} \tag{1.4}$$

Such an equation is

$$i\left(\frac{\partial q}{\partial z} + k'\frac{\partial q}{\partial t}\right) - \frac{1}{2}k''\frac{\partial^2 q}{\partial t^2} + \left.\frac{\partial k}{\partial |E|^2}\right|_{(\omega_1,0)} |q|^2 q = 0 \tag{1.5}$$

where $k' = \left.\partial k/\partial \omega_1\right|_{(\omega_1,0)}$ and $k'' = \left.\partial^2 k/\partial \omega_1^2\right|_{(\omega_1,0)}$ and normalisation is such that $|E|^2 = |q|^2$. If we now consider the case where $k'' < 0$ (anomalous dispersion), a change of variables

$$\gamma = x, \quad \tau = \sqrt{\frac{2}{-k''}}\,(t - k'z), \quad u = \sqrt{\left.\frac{\partial k}{\partial |E|^2}\right|_{(\omega_1,0)}}\; q$$

leads to the equation

$$i\frac{\partial u}{\partial \gamma} + \frac{\partial^2 u}{\partial \tau^2} + |u|^2 u = 0 \tag{1.6}$$

- which is the non-linear Schrodinger equation. Note: Because these arguments are very general, one would expect the non-linear Schrodinger equation to be relevant in many different areas which is indeed the case. □

We now present the rigorous derivation of the non-linear Schrodinger equation from the Maxwell equations -

$$\nabla \times \vec{H} = \frac{1}{\epsilon_0 c} \frac{\partial \vec{D}}{\partial t}, \qquad \nabla \times \vec{E} = \frac{-1}{c} \frac{\partial \vec{B}}{\partial t} \tag{1.7}$$

where ϵ_0 is the dielectric constant. If the magnetic field $\vec{H}$ is assumed to be equal to the magnetic induction $\vec{B}$ for the fibres considered, then equations (1.7) can be further written as

$$\nabla \times \nabla \times \vec{E} = \frac{-1}{\epsilon_0 c^2} \frac{\partial^2}{\partial t^2} \vec{D} \tag{1.8}$$

The $\vec{D}$ in (1.8) is called the dielectric displacement vector and, for a cubic nonlinear medium, $\vec{D}$ is given to third order by $D = \epsilon_0 \tilde{\chi} * \vec{E}$, where $\epsilon_0 \tilde{\chi} * \vec{E}$ is defined by

$$\epsilon_0 \tilde{\chi} * \vec{E} := \epsilon_0 \int_{-\infty}^{t} dt_1 \, \tilde{\chi}^{(0)}(t-t_1) E(t_1)$$
$$+ \epsilon_0 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 \int_{-\infty}^{t} dt_3 \, \tilde{\chi}^{(2)}(t-t_1, t-t_2, t-t_3) \cdot \left[\vec{E}(t_1) \cdot \vec{E}(t_2) \right] \vec{E}(t_3) \tag{1.9}$$

$\tilde{\chi}^{(0)}$ and $\tilde{\chi}^{(2)}$ are linear and nonlinear dielectric tensors respectively and are 3x3 matrices. The triple integral in eq.(1.9) describes the nonlinear response of the fibre itself to the propagating electromagnetic disturbance due to two effects, the Raman effect and the Kerr effect.

The Raman effect is due to the passing electromagnetic wave and can be most easily understood as the emission of radiation by the carrier medium by the excitation and subsequent dropping back of electrons to their original states. The Kerr effect results from alignment of the fibres anisotropic molecules due to the propagating wave. These alignments in turn affect the propagation. In eq.(1.9), the upper limits of integration show that it is the electric field at previous times only which contributes to the displacement. This follows from causality.

We now wish to recast eq.(1.8) in matrix form. To do this, we first use the identity $\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$ to obtain

$$\nabla^2 \vec{E} - \frac{1}{\epsilon_0 c^2} \frac{\partial^2}{\partial t^2} D = \nabla(\nabla \cdot \vec{E}) \qquad (1.10)$$

from eq.(1.8). Secondly, we replace the cartesian coordinates x, y and z with the cylindrical coordinates r, θ and z and give the components of $\vec{E}$, the electric field, by $E_r = E_x \cos\theta + E_y \sin\theta$, $E_\theta = -E_x \sin\theta + E_y \cos\theta$ and E_z instead of E_x, E_y and E_z. From this, we can then write

$$\nabla_T^2 \vec{E} = O_1 L_A \begin{bmatrix} E_r \\ E_\theta \\ E_z \end{bmatrix} = O_1 L_A O_1^{-1} \vec{E} \qquad (1.11.a)$$

where (1.11.b)

$$\nabla_T^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

and

$$D_3 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L_a = \begin{bmatrix} \nabla_T^2 - \frac{1}{r^2} & -\frac{2}{r^2}\frac{\partial}{\partial\theta} & 0 \\ \frac{2}{r^2}\frac{\partial}{\partial\theta} & \nabla_T^2 - \frac{1}{r^2} & 0 \\ 0 & 0 & \nabla_T^2 \end{bmatrix} \quad (1.11.c)$$

If we now define $\chi^{(i)}$ ($i = 0, 2$) as

$$\tilde{\chi}^{(i)} \vec{E} = D_3 \chi^{(i)} D_3^{-1} \vec{E} \quad (1.12)$$

then eq.(1.11.a) can be written in the form

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\tilde{\chi}_*\right)\vec{E} = D_3\left(L_a + L_b\right) D_3^{-1}\vec{E} \quad (1.13)$$

with L_b defined as

$$L_b = \left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\chi_*\right)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (1.14)$$

Note: $\vec{E}(t1)\cdot\vec{E}(t2) = D_3\vec{E}\, D_3^{-1}\vec{E}$. Finally, we use

$$\nabla \cdot E = \frac{\partial E_r}{\partial r} + \frac{1}{r} E_r + \frac{1}{r}\frac{\partial E_\theta}{\partial \theta} + \frac{\partial E_z}{\partial z} \qquad (1.15)$$

to show that $\nabla(\nabla \vec{E}) = D_3 L D_3^{-1} \vec{E}$, where

$$L_c = \begin{bmatrix} \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} r & \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial \theta} & \frac{\partial^2}{\partial r \partial z} \\ \frac{1}{r^2}\frac{\partial^2}{\partial r \partial \theta} r & \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} & \frac{1}{r}\frac{\partial^2}{\partial \theta \partial z} \\ \frac{1}{r}\frac{\partial^2}{\partial r \partial z} r & \frac{1}{r}\frac{\partial^2}{\partial \theta \partial z} & \frac{\partial^2}{\partial z^2} \end{bmatrix} \qquad (1.16)$$

If, from now on, we denote $(E_r, E_\theta, E_z) = D_3^{-1}$ as $\vec{E}$, the equation

$$L\vec{E} := (L_a + L_b + L_c)\vec{E} = 0 \qquad (1.17)$$

is an equation equivalent to eq. (1.10). Equation (1.17) is Maxwell's equation in cylindrical coordinates. We now assume that the electric field is a nearly monochromatic wave propagating along the z axis with a wavenumber k and an angular frequency of ω, i.e., the electric field is assumed to be of the form

$$\vec{E}(r,\theta,z,t) = \sum_{l=-\infty}^{\infty} \vec{E}_l(r,\theta,\gamma,\tau,\varepsilon)\, e^{i(k_l z - \omega_l t)} \qquad (1.18)$$

where

$$\gamma = \varepsilon^2 z \quad , \quad \tau = \varepsilon\left(t - \frac{z}{V_g}\right), \quad |\varepsilon| \ll 1 \qquad (1.19)$$

are slow variables and $E_{-l} = E_{+l}^*$, $k_l = lk_1$ and $\omega_l = l\omega_1$.

With this change of variable, E changes slowly in z and t. To lowest order in ϵ eq.(1.18) is of the form

$$\vec{E}(r,\theta,z,t) = Re\left\{\epsilon \vec{E}_1^{(1)}(r,\theta,\gamma,\tau)\exp i[(k_1 z - \omega_1 t)]\right\} \quad (1.20)$$

The E_l, for $l \geq 2$, are generated by the nonlinear response which is due to the Raman and Kerr effects. If we expand $E_l(t_1)$ as

$$E_l(t_1) = E_l(t) + \frac{\partial E}{\partial t}(t - t_1) + \frac{1}{2}\frac{\partial^2 E}{\partial t^2}(t_1 - t)^2 + \cdots \quad (1.21)$$

we then obtain, up to third order in ϵ-

$$\chi * \vec{E} = \sum_{l=-\infty}^{\infty}\left| \chi_l^{(0)}\vec{E}_l + i\epsilon \frac{\partial \chi_l^{(0)}}{\partial \omega_l}\frac{\partial \vec{E}_l}{\partial \tau} - \frac{\epsilon^2}{2}\frac{\partial^2 \chi_l^{(0)}}{\partial \omega_l^2}\frac{\partial^2 \vec{E}_l}{\partial \tau^2}\right|$$

$$\cdot\ e^{i(k_l z - \omega_l t)} + \sum_{l_1 + l_2 + l_3 = l}\chi^{(2)}\left|(\vec{E}_{l_1}\cdot\vec{E}_{l_2})\vec{E}_{l_3}\right| e^{i(k_l z - \omega_l t)} \quad (1.22)$$

where

$$\chi_l^{(0)} = \int_0^\infty dt_1\ \chi_l^{(0)}(t_1)\ e^{i\omega_l t_1}$$

and:

$$\chi_{l_1 l_2 l_3}^{(2)} = \int_0^\infty dt_1 \int_0^\infty dt_2 \int_0^\infty dt_3\ \chi^{(2)}(t_1,t_2,t_3)\, e^{i(\omega_1 t_1 + \omega_2 t_2 + \omega_3 t_3)} \quad (1.23)$$

We now use equation(1.22) to write out equation (1.17). If we keep only the terms which contribute to order ϵ less than three, we can write

$$\sum_{l=-\infty}^{\infty} L\vec{E}_l\, e^{i(k_l z - \omega_l t)} = \sum_{l=-\infty}^{\infty}\left\{(L_a \vec{E}_l) - k_l^2 \vec{E}_l + 2ik_l\frac{\partial \vec{E}_l}{\partial z} + \frac{\partial^2 \vec{E}_l}{\partial z^2}\right.$$

$$+\frac{\omega_\ell^2}{c^2}\left[\chi_\ell^{(0)}\vec{E}_\ell + i\varepsilon\frac{\partial\chi_\ell^{(0)}}{\partial\omega_\ell}\frac{\partial\vec{E}_\ell}{\partial\tau} - \frac{\varepsilon^2}{2}\frac{\partial^2\chi_\ell^{(0)}}{\partial\omega_\ell^2}\frac{\partial^2\vec{E}_\ell}{\partial\tau^2}\right]$$

$$+\frac{2i\omega_\ell}{c^2}\left[\chi_\ell^{(0)}\frac{\partial\vec{E}_\ell}{\partial t} - i\varepsilon\frac{\partial\chi_\ell^{(0)}}{\partial\omega_\ell}\frac{\partial^2\vec{E}_\ell}{\partial t^2}\right] - \frac{1}{c^2}\chi_\ell^{(0)}\frac{\partial^2\vec{E}_\ell}{\partial t^2}$$

$$\left. - \hat{L}_\ell\vec{E}_\ell - \hat{\hat{L}}_\ell\vec{E}_\ell - \hat{\hat{\hat{L}}}_\ell\vec{E}_\ell\right\} e^{i(k_\ell z - \omega_\ell t)}$$

$$+\frac{\omega_\ell^2}{c^2}\sum_{\ell_1+\ell_2+\ell_3=\ell}\left\{\chi^{(2)}_{\ell_1\ell_2\ell_3}\left(\vec{E}_{\ell_1}\cdot\vec{E}_{\ell_2}\right)\vec{E}_{\ell_3}\right\} e^{i(k_\ell z - \omega_\ell t)} = 0 \tag{1.24}$$

$\hat{L}_\ell$, $\hat{\hat{L}}_\ell$ and $\hat{\hat{\hat{L}}}_\ell$ are 3x3 matrices and are given by

$$\hat{L}_\ell = \begin{bmatrix} \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}r & \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta} & ik_\ell\frac{\partial}{\partial r} \\ \frac{1}{r^2}\frac{\partial^2}{\partial r\partial\theta}r & \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} & \frac{ik_\ell}{r}\frac{\partial}{\partial\theta} \\ \frac{ik_\ell}{r}\frac{\partial}{\partial r}r & \frac{ik_\ell}{r}\frac{\partial}{\partial\theta} & -k_\ell^2 \end{bmatrix}$$

$$\hat{\hat{L}}_\ell \begin{bmatrix} 0 & 0 & \frac{\partial^2}{\partial r\partial z} \\ 0 & 0 & \frac{1}{r}\frac{\partial^2}{\partial\theta\partial z} \\ \frac{1}{r}\frac{\partial^2}{\partial r\partial z}r & \frac{1}{r}\frac{\partial^2}{\partial\theta\partial z} & 2ik_\ell\frac{\partial}{\partial z} \end{bmatrix} \hat{\hat{\hat{L}}}_\ell = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\partial^2}{\partial z^2} \end{bmatrix} \tag{1.25}$$

If we now expand $\vec{E}_\ell$ in terms of ε as

$$\vec{E}_\ell(r,\theta,\gamma,\tau,\varepsilon) = \sum_{n=1}^{\infty}\varepsilon^n E_\ell^{(n)}(r,\theta,\gamma,\tau) \tag{1.26}$$

and equate in (1.24) and (1.26) terms of equal order in ϵ, we obtain

$$L_\ell \vec{E}_\ell^{(1)} = 0 \tag{1.27}$$

where

$$L_\ell = L_a - k_\ell^2 + \frac{\omega_\ell^2}{c^2} \chi_\ell^{(0)} - \hat{L}_\ell \tag{1.28}$$

Note that L_ℓ is $L = L_a + L_b + L_c$ in eq.(1.17) where d/dz, d/dt and X have been replaced with ik_ℓ, $i\omega_\ell$ and $\chi_\ell^{(0)}$ respectively. Furthermore, it can be shown that L_ℓ is hermitian, i.e.,

$$(\vec{U}, L_\ell \vec{V}) = \int U^* \cdot L_\ell \vec{V} \, ds = \int (L_\ell \vec{U}) \vec{V} \, ds = (L_\ell \vec{U}, \vec{V})$$

where ds is the volume element rdrdθ and all surface terms vanish. We write the solution of eq.(1.27) in the form

$$E_\ell^{(1)}(r,\theta,\gamma,\tau) = \begin{cases} q(\gamma,\tau)\, U(r,\theta) & \text{for } \ell = \pm 1 \\ 0 & \text{for } \ell \neq \pm 1 \end{cases} \tag{1.29}$$

That $E_\ell^{(1)}$ is zero in eq.(1.29) for $l \neq \pm 1$ is due to the fact that the fibres in question support only one mode. Equation (1.27) implies that $L_1 \vec{U} = 0$ and therefore implies that $(\vec{U}, L_1 \vec{U}) = 0$, from which we obtain the following dispersion relation, $k = k_\ell(\omega_1)$:

$$\frac{1}{4} k_1^2 S_0 = \frac{\omega_1^2}{c^2} (\vec{U}, n_0^2 \vec{U}) + (\vec{U}, L_0 \vec{U})$$
$$+ ik_\ell \int (U_z \nabla_T \cdot U^* - U_z^* \nabla_T \cdot U) \, ds \tag{1.30}$$

where $S_0 \; 4\int (|U_r|^2 + |U_\theta|^2) \, ds$, $\nabla_\perp \left(\frac{1}{r} \frac{\partial}{\partial r} r, \frac{1}{r} \frac{\partial}{\partial \theta}, 0 \right)$

and $\chi_1^{(0)}$ has been replaced with n_0^2.

If we use

$$\frac{\partial}{\partial z} = \varepsilon^2 \frac{\partial}{\partial r} - \frac{\varepsilon}{V_g}\frac{\partial}{\partial \tau} \tag{1.31}$$

$$\frac{\partial L_l}{\partial \omega_l} = \frac{2\omega_l}{c^2}\chi_l^{(0)} + \frac{\omega_l^2}{c^2}\frac{\partial \chi_l^{(0)}}{\partial \omega_l} + \frac{\partial}{\partial \omega_l}\left(L_l - \frac{\omega_l^2}{c^2}\chi_l^{(0)}\right) \tag{1.32}$$

and

$$\frac{\partial}{\partial k_l}\left(L_l - \frac{\omega_l^2}{c^2}\chi_l^{(0)}\right) = -2k_l - i\tilde{\tilde{L}}_l \tag{1.33}$$

where $\tilde{\tilde{L}}_l$ is $\hat{\hat{L}}_l$ with $\frac{\partial}{\partial z}$ replaced by 1, we obtain, at order ε^2 -

$$L_l \vec{E}_l^{(2)} = i\left\{-\frac{\partial L_l}{\partial \omega_l} - \left(\frac{1}{V_g} - \frac{\partial k_l}{\partial \omega_l}\right)\frac{\partial}{\partial k_l}\left(L_l - \frac{\omega_l^2}{c^2}\chi_l^{(0)}\right)\right\}\frac{\partial \vec{E}_l^{(1)}}{\partial \tau} \tag{1.34}$$

from which we find

$$E_l^{(2)} = 0 \tag{1.35}$$

for $l \neq 1$. Now, since L_1 is hermitian and since $L_1 \cdot \vec{U} = 0$, we can write

$$\left(\vec{U}, L_1 \vec{E}_1^{(2)}\right) = 0 \tag{1.36}$$

This condition implies

$$\frac{1}{V_g} = \frac{\partial k_1}{\partial \omega_1} \tag{1.37}$$

where Vg is the group velocity. Here we have used

$$\left(\vec{U}, \frac{\partial L_1}{\partial \omega_1}\frac{\partial \vec{E}_1^{(1)}}{\partial \tau}\right) = \frac{\partial}{\partial \omega_1}\left(\vec{U}, L_1 \frac{\partial \vec{E}_1^{(1)}}{\partial \tau}\right) = 0 \tag{1.38}$$

$$\text{and} \quad \left(\vec{U}, \frac{\partial}{\partial k}\left(L_l - \frac{\omega_l^2}{c^2}\chi_l^{(0)}\right)\frac{\partial \vec{E}_l^{(1)}}{\partial \tau}\right) \neq 0 \tag{1 39}$$

For l = 1, equation (1.34) becomes

$$L_1 \vec{E}_1^{(2)} = -i\frac{\partial L_1}{\partial \omega_1}\frac{\partial \vec{E}_1^{(1)}}{\partial \tau} = -i\frac{\partial L_1}{\partial \omega_1}\frac{\partial q}{\partial \tau}\vec{U} \tag{1.40}$$

The corresponding equation to equation (1.34) for order ϵ^3 is

$$L_1 E_1^{(3)} = i\frac{\partial L_1}{\partial \omega_1}\frac{\partial \vec{E}_1^{(2)}}{\partial \tau} + \frac{1}{2}\frac{\partial^2 L_1}{\partial \omega_1^2}\frac{\partial^2 \vec{E}_1^{(1)}}{\partial \tau^2}$$
$$+ \left(\frac{i\partial q}{\partial \tau} - \frac{1}{2}\frac{\partial^2 k_1}{\partial \omega_1^2}\frac{\partial^2 q}{\partial \tau^2}\right)\left[\frac{\partial}{\partial k_1}\left(L_1 - \frac{\omega_1^2 n_0^2}{c^2}\right)\right]\vec{U}$$
$$- |q|^2 q \frac{\omega_1^2}{c^2}\vec{F}\left(\vec{U}, \vec{U}^*, \chi^{(2)}\right) \tag{1.41}$$

where $\vec{F}$ is defined as

$$\vec{F} = \chi^{(2)}_{-111}\left(\vec{U}^*, \vec{U}\right)\vec{U} + \chi^{(2)}_{1-11}\left(\vec{U}, \vec{U}^*\right)\vec{U}$$
$$+ \chi^{(2)}_{11-1}\left(\vec{U}, \vec{U}\right)\vec{U}^* \tag{1.42}$$

Equation (1.36), for order ϵ^3 is $(\vec{U}, L_1 E_1^{(3)}) = 0$. This condition implies

$$i\frac{\partial q}{\partial \tau} - \frac{1}{2}\frac{\partial^2 k_1}{\partial \omega_1^2}\frac{\partial^2 q}{\partial \tau^2} + \nu|q|^2 q = 0 \tag{1.43}$$

$$\text{where} \quad \nu = \frac{2\omega_1^2}{\tilde{k}_1 c^2 S_0}\left(\vec{U}, \vec{F}\right), \tag{1.44}$$

$$\tilde{k}_1 = k_1 - \frac{2i}{S_0}\int\left(\vec{U}_z \nabla_\perp \cdot \vec{U}^* - \vec{U}_z^* \nabla_\perp \cdot \vec{U}\right)ds \tag{1.45}$$

Equation (1.43) is the equation which describes nonlinear pulse propagation in a monomode dielectric guide.

CHAPTER TWO

THE INVERSE SCATTERING METHOD

The non-linear Schrodinger equation is one of the many non-linear wave equations which is solved by means of the Inverse Scattering method. This method bypasses the direct solution of a non-linear equation and instead enables one to solve it by means of a series of linear problems which are easier to solve. A detailed discussion of the Inverse Scattering method can be found in [2] In this chapter, we describe the Inverse Scattering method for the non-linear Schrodinger equation and we concentrate in particular on the equation's associated linear eigenvalue problem.

To begin with the Inverse Scattering method, one must first find a Lax pair of differential operators L and B which satisfy the linear equations

$$L\psi = \lambda\psi \qquad (2.1.a)$$

$$i\frac{\partial\psi}{\partial t} = B\psi \qquad (2.1.b)$$

where λ is a time independant eigenvalue and L and B depend on a function u(x,t). L and B are chosen so that the consistancy condition of eqs.(2.1) -

$$i\frac{\partial L}{\partial t} + [L, B] = 0 \qquad (2.2)$$

leads to an equation

$$K(u) = 0 \qquad (2\ 3)$$

for u(x,t) and K a non-linear operator. If a Lax pair can be found such that eq.(2.3) is the non-linear Schrodinger equation, the Inverse Scattering method can solve the initial value problem for eq.(2.3) as follows -

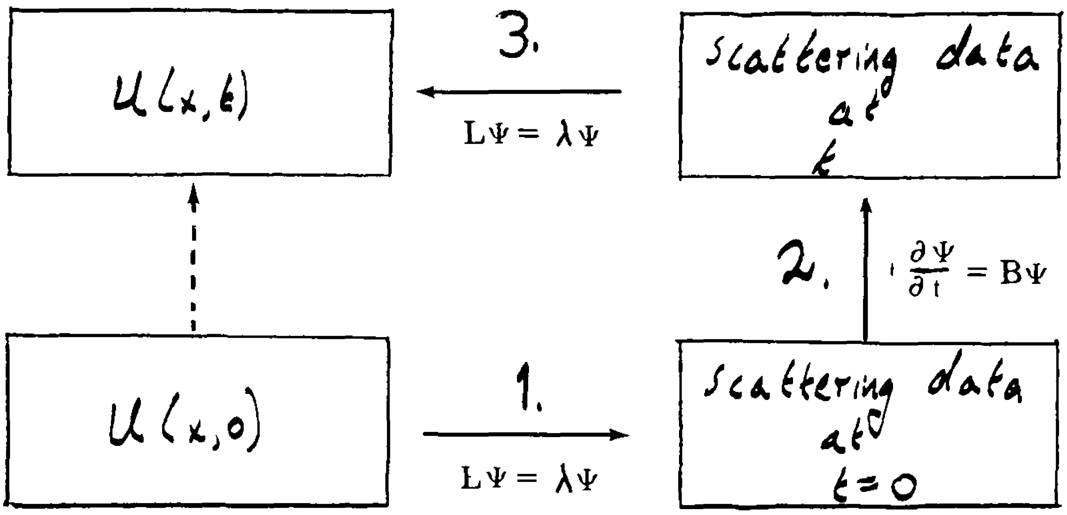


FIGURE (2.1)

The Inverse Scattering method is subdivided into three steps

STEP 1: For fixed t, the linear eigenvalue problem $L\Psi = \lambda\Psi$ is solved for both bound state and scattering solutions, and the set $S = \{N, \varkappa_n, c_n$ (for $n=1,,,N)$, $R(k)$, $T(k).(0<k<\infty)\}$ of scattering data is found. Here, the R(k) and T(k) are the reflection and transmission coefficients respectively and are obtained from the scattering solutions. The number N gives the number of bound state solutions to each of which there is associated an eigenvalue λ.

STEP 2: The linear equation $i\frac{\partial \Psi}{\partial t} = B\Psi$ is used to calculate the scattering data at any time t in the future.

STEP 3: Using the equation $L\Psi = B\Psi$ and the scattering data at time t, the potential $u(x,t)$ is reconstructed. This step is called the Inverse Problem.

For the non-linear Schrodinger equation, a Lax pair is provided by

$$L = \begin{bmatrix} i(1+\rho)\frac{\partial}{\partial x} & u^* \\ u & i(1-\rho)\frac{\partial}{\partial x} \end{bmatrix} \qquad \varrho = \frac{2}{1-\rho^2}$$

$$0 < \rho < 1. \tag{2.4}$$

$$B = \begin{bmatrix} -\rho\frac{\partial^2}{\partial x^2} + \frac{|u|^2}{1+\rho} & \frac{i\partial u^*}{\partial x} \\ -\frac{i\partial u}{\partial x} & -\rho\frac{\partial^2}{\partial x^2} - \frac{|u|^2}{1-\rho} \end{bmatrix} \tag{2.5}$$

and a straight forward calculation shows that

$$\frac{i\partial L}{\partial t} + [L, B] = \begin{bmatrix} 0 & -K(u^*) \\ K(u) & 0 \end{bmatrix} \tag{2.6}$$

where $K(u) = 0$ is the non-linear Schrodinger equation. The Ψ in eq.(2.1.a) is $(\Psi 1,\Psi 2)^T$. and if we define

$$\Psi_1 = \sqrt{1-\rho}\; exp\left\{-i\frac{\lambda}{1-\rho^2}x\right\} V_2 \qquad (2.7.a)$$

$$\Psi_2 = \sqrt{1+\rho}\; exp\left\{-i\frac{\lambda}{1-\rho^2}x\right\} V_1 \qquad (2.7.b)$$

- the eigenvalue problem (2.1.a) becomes

$$V_2' - i\frac{\lambda\rho}{1-\rho^2} = \frac{iu^*}{\sqrt{1-\rho^2}} V_1 \qquad (2.8\ a)$$

$$V_1' + i\frac{\lambda\rho}{1-\rho^2} = \frac{-iu}{\sqrt{1-\rho^2}} V_2 \qquad (2.8.b)$$

or, equivalently

$$V_1' + i\xi V_1 = q V_2 \qquad (2.9.a)$$

$$V_2' - i\xi V_2 = -q^* V_1 \qquad (2.9\ b)$$

where q and ξ are such that

$$q = \frac{iu}{\sqrt{1-\rho^2}}, \qquad \xi = \frac{\lambda\rho}{1-\rho^2} \qquad (2.10)$$

Equations (2.9.a) and (2.9.b) are the eigenvalue problem (2.1.a) in a more convenient form.

(hereafter all eigenvalues will be denoted by λ)

To proceed with the method of Inverse

Scattering, we must find the scattering solutions to this eigenvalue problem, i.e., we must find the solutions of eqs.(2.9) with the following asymptotic behaviour

$$e^{-i\lambda x}\begin{pmatrix}1\\0\end{pmatrix}\xleftarrow[x\to-\infty]{}\begin{pmatrix}v_1\\v_2\end{pmatrix}\xrightarrow[x\to+\infty]{} a(\lambda)\,e^{-i\lambda x}\begin{pmatrix}1\\0\end{pmatrix}+b(\lambda)\,e^{i\lambda x}\begin{pmatrix}0\\1\end{pmatrix} \qquad (2.11)$$

Here, the eigenvalues λ of $a(\lambda)$ can be analytically continued to the complex upper half plane. At the zeros λ_n of a, the asymptotic behaviour (2.11) reads

$$e^{-i\lambda_n x}\begin{pmatrix}1\\0\end{pmatrix}\xleftarrow[x\to-\infty]{}\begin{pmatrix}v_1\\v_2\end{pmatrix}\xrightarrow[x\to\infty]{} C_n e^{i\lambda_n x}\begin{pmatrix}0\\1\end{pmatrix} \qquad (2.12)$$

and the solutions are square integrable. The λ_n, C_n and $b(\lambda)$ are the scattering data and the eigenvalues are time independant. To find out how C_n and $b(\lambda)$ evolve in time we use eq.(2.1.b) for large x. For large x we can write

$$B = -\rho\,\frac{\partial^2}{\partial x^2}\,I \qquad (2.13)$$

and obtain

$$i\frac{\partial v_j}{\partial t} = -\rho\frac{\partial^2 v_j}{\partial x^2} + 2i\lambda\frac{\partial v_j}{\partial x} + \frac{1}{\rho}\lambda^2 v_j \qquad (2.14)$$

If we insert into eq.(2.14) the asymptotic form for $x\to+\infty$

$$\begin{bmatrix} a(\lambda,t)\,e^{-i\lambda x}\\ b(\lambda,t)\,e^{i\lambda x}\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}0\\ C_n e^{i\lambda_n x}\end{bmatrix} \qquad (2.15)$$

we find

$$i\frac{\partial a}{\partial t} = \lambda^2\left(p + 2 + \frac{1}{p}\right)a \qquad (2.16.a)$$

$$i\frac{\partial b}{\partial t} = \lambda^2\left(p - 2 + \frac{1}{p}\right)b \qquad (2.16.b)$$

$$i\frac{\partial c_n}{\partial t} = \lambda^2\left(p - 2 + \frac{1}{p}\right)b \qquad (2.16.c)$$

Without loss of generality, we can replace B with B+CoI with Co = $-\lambda^2(p+2+1/p)$. Then the equations

$$a(\lambda,t) = a(\lambda,0) \qquad (2.17.a)$$

$$b(\lambda,t) = b(\lambda,0)e^{4i\lambda^2 t} \qquad (2.17.b)$$

$$c_n(t) = c_n e^{4i\lambda^2 t} \qquad (2.17.c)$$

follow. Equations (2.17.a,b,c) give the time development of the scattering data.

Having found the evolution of the scattering data, the last and most difficult step is to reconstruct u(x,t) from the scattering data. We sketch this procedure below:

Defining $\phi(x,t)$ as

$$\phi(x,\lambda) = \begin{cases} \phi(x,\lambda)e^{i\lambda x}/a(\lambda) & \text{Im}\,\lambda > 0 \\ \begin{pmatrix} \psi_2^*(x,\lambda^*) \\ -\psi_1^*(x,\lambda^*) \end{pmatrix} e^{i\lambda x} & \text{Im}\,\lambda < 0 \end{cases} \qquad (2.18)$$

where ϕ and Ψ are the solutions with asymptotic behaviour

$$\phi \xrightarrow[x \to -\infty]{} e^{-i\lambda x} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Psi \xrightarrow[x \to \infty]{} e^{i\lambda x} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (2.19)$$

- it can be shown that $\phi(x,\lambda)$ has a discontinuity given by

$$\sigma(x,\lambda) = \phi(x,\lambda + i0) - \phi(x,\lambda - i0) = \frac{b(\lambda)}{a(\lambda)} \Psi(x,\lambda) e^{i\lambda x} \qquad (2.20)$$

and poles at the zeros λ_n of $a(\lambda)$. $\phi(x,\lambda)$ is otherwise analytic and approaches $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at infinity. One now uses Cauchy's formula

$$\Psi(x,\lambda) = \int_{\gamma} \frac{1}{2\pi i} \frac{\Psi(x,\lambda')}{\lambda' - \lambda} \, d\lambda' \qquad (2.21)$$

where $\operatorname{Im}\lambda \neq 0$ and γ is a contour containing a small neighbourhood of λ If γ is distorted to a circle of infinite radius, we pick up the residues and the contribution of the discontinuity:

$$\phi(x,\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{k=1}^{N} \frac{e^{i\lambda_k x}}{\lambda - \lambda_k} \frac{\Psi(x,\lambda_k)}{a'(\lambda_k)}$$

$$+ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\sigma(x,\lambda')}{\lambda' - \lambda} \, d\lambda' \qquad (2.22)$$

Equation (2.22) can be evaluated for $\lambda = \varkappa - i0$, $\varkappa \in \mathbb{R}$ and at the poles $\lambda = \lambda_n$. This yields

$$\sigma_2^*(x,\lambda) + c^*(x,\lambda)\frac{1-J}{2}\sigma_1 = c^*(x,\lambda) + c^*(x,\lambda)\sum_{k=1}^{N}\frac{e^{i\lambda_k x}}{\lambda-\lambda_k}\frac{c_k}{a'(\lambda_k)}\psi_1(x,\lambda_k) \qquad (2.23)$$

$$\sigma_1 - c\cdot\frac{1+J}{2}\sigma_2^* = -c\sum_{k=1}^{N}\frac{e^{-i\lambda_k^* x}}{\lambda-\lambda^*}\left(\frac{c_k}{a'(\lambda_k)}\right)^{*}\psi_2^*(x,\lambda_k) \qquad (2.24)$$

where

$$c(x,\lambda) = \frac{b(\lambda)}{a(\lambda)}e^{2i\lambda x}, \qquad (2.25)$$

$$\psi_1(x,\lambda_j)e^{-i\lambda x} + \sum_{k=1}^{N}\frac{e^{-i\lambda_k^* x}}{\lambda_n-\lambda_k^*}\left(\frac{c_k}{a'(\lambda_k)}\right)^{*}\psi_2^*(x,\lambda_n) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\sigma_2^*(x,\lambda)\,d\lambda}{\lambda-\lambda_n}, \qquad (2.26)$$

and $$\psi_2^*(x,\lambda_n)e^{i\lambda_n^* x} - \sum_{k=1}^{N}\frac{e^{i\lambda_k x}}{\lambda_n^*-\lambda_k}\frac{c_k}{a'(\lambda_k)}\psi_1(x,\lambda_k) = 1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\sigma_1(x,\lambda)\,d\lambda}{\lambda-\lambda_n^*} \qquad (2.27)$$

These equations determine $\psi(x,\lambda_k)$ and $\sigma(\lambda)$.

Finally, to recover q(x,t) from the scattering data, we evaluate eq.(2,26) for large λ:

$$\begin{pmatrix}\psi_2(x,\lambda)\\ -\psi_1(x,\lambda)\end{pmatrix}e^{-i\lambda x} = \begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\lambda}\left[\sum_{k=1}^{N}\left(\frac{c_k}{a'(\lambda_k)}\right)^{*}e^{-i\lambda_k^* x}\psi^*(x,\lambda_k) + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\sigma^*(\lambda)\,d\lambda\right] + O\left(\frac{1}{\lambda^2}\right) \qquad (2.28)$$

The eigenvalue problem (2.9), yields

$$\psi_1(x,\lambda)\, e^{-i\lambda x} = \frac{1}{2i\lambda}\, q(x) \qquad (2.29)$$

Therefore

$$q(x) = -2i\sum_{k=1}^{N}\left(\frac{c_k}{a'(\lambda_k)}\right)^{*} e^{-i\lambda_k^{*}x}\, \psi_2^{*}(x,\lambda_k) - \frac{1}{\pi}\int_{-\infty}^{\infty}\sigma_2^{*}(\lambda)\, d\lambda \qquad (2\ 30)$$

from which, using eq.(2.10), we can easily find u(x,t).

The solution of the initial value problem for the non-linear Schrodinger equation which we have just sketched is very complicated. However, the method involves linear calculations only and this allows us to derive some interesting conclusions. In particular, it can be shown that the L^2 integrable solutions of the eigenvalue problem (2.1.a) correspond to pulses which are stable. We call these stable pulses optical solitons.

As an example, we solve eqs.(2.9.a,b) for b(λ,t)=0, i.e., σ =c=0. If we define

$$\psi_{ij} = \sqrt{\frac{c_k}{a'(\lambda_k)}}\;\psi_i(x,\lambda_j) \qquad (2.31)$$

then equations(2.6,2.7) read

$$\psi_{ij} + \sum_{k=1}^{N}\frac{\lambda_j\,\lambda_k^{*}}{\lambda_j-\lambda_k^{*}}\,\psi_{2k}^{*} = 0 \qquad (2.32)$$

$$-\sum_{k=1}^{N}\frac{\lambda_k\lambda_j^{*}}{\lambda_j^{*}-\lambda_k}\,\psi_{1k} + \psi_{2j}^{*} = \lambda_j^{*} \qquad (2\ 33)$$

and equation (2.30) reduces to

$$q(x) = -2i \sum_{k=1}^{N} \lambda_k^* \psi_{2k}^* \qquad (2.34)$$

For N = 1, the solution is

$$u(x,t) = \sqrt{2x}\,\eta \frac{e^{i[-4(\varkappa^2-\eta^2)t - 2\varkappa x + \phi]}}{\cosh[2\eta(x-x_0) + 8\eta\varkappa t]} \qquad (2.35)$$

In physical space time coordinates, equation (2.35) describes a stable pulse whose amplitude is determined by Im $\lambda = \eta$ and whose velocity is determined by Re $\lambda = \varkappa$.

Due to their stable nature, such pulses have a potentially important role in data communication. It is therefore interesting to discover which initial pulse contains solitons. Using the Inverse Scattering method, this amounts to solving eq.(2.1.a) for initial pulses u(x,0).

CHAPTER THREE

PULSES OF HYPERBOLIC SECANT FORM

In order to make this study of the non-linear Schrodinger equation's eigenvalue problem complete, we include the results of Satsuma and Yajima [3] in which was studied the initial value problem for the pulse

$$u(x,0) = A\,\mathrm{sech}(x) \qquad (3.1)$$

The eigenvalue problem for the non-linear Schrodinger equation is, from chapter 3,

$$v_1'' - \frac{u'}{u} v_1' + \left[-i\lambda \frac{u'}{u} + \lambda^2 + |u|^2 \right] v_1 = 0 \qquad (3.2)$$

If we change the independant variable, x, into s, where s is defined as s: = [1 - tanh(x)]/2, then we find

$$v_1' = \frac{dv_1}{dx} = \frac{dv_1}{ds}\frac{ds}{dx} = -\frac{1}{2}\mathrm{sech}^2(x)\frac{dv_1}{ds} \qquad (3.3)$$

$$v_1'' = \frac{d^2 v_1}{dx^2} = \frac{dv_1}{ds}\frac{d^2 s}{dx^2} + \left(\frac{ds}{dx}\right)^2 \frac{d^2 v_1}{ds^2}$$
$$= \mathrm{sech}^2(x)\tanh(x)\frac{dv_1}{ds} + \frac{\mathrm{sech}^4(x)}{4}\frac{d^2 v_1}{ds^2} \qquad (3.4)$$

With these derivatives, equation (3.2) then becomes

$$\frac{\operatorname{sech}^4(x)}{4}\frac{d^2 v_1}{ds^2} + \frac{\operatorname{sech}^2(x)\tanh(x)}{2}\frac{dv_1}{ds} + \left[i\lambda\tanh(x) + \lambda^2 + A^2\operatorname{sech}^2(x)\right]v_1 = 0 \quad (3.5)$$

Since $s = \frac{1}{2}[1 - \tanh(x)]$, we have $\tanh(x) = (1 - 2s)$ and also $\operatorname{sech}^2(x) = 4s(1 - s)$. Hence

$$s(1-s)\frac{d^2 v_1}{ds^2} + (1/2 - s)\frac{dv_1}{ds} + \left[A^2 + \frac{\lambda^2 + i\lambda(1-2s)}{4s(1-s)}\right]v_1 = 0 \quad (3.6)$$

If we further transform the dependant variable, v1, into $s^{\frac{i\lambda}{2}}(1 - s)^{-\frac{i\lambda}{2}}w1$, then dv1/ds and dv1/ds then become

$$\frac{dv_1}{ds} = \frac{i\lambda}{2} s^{i\lambda/2 - 1}(1-s)^{-i\lambda/2 - 1}\cdot\omega_1 + s^{i\lambda/2}(1-s)^{-i\lambda/2}\cdot\frac{d\omega_1}{ds} \quad (3.7)$$

$$\frac{d^2 v_1}{ds^2} = s^{\frac{i\lambda}{2}}(1-s)^{-\frac{i\lambda}{2}}\frac{d^2\omega_1}{ds^2} + \frac{i\lambda}{s(1-s)} s^{\frac{i\lambda}{2}}(1-s)^{-\frac{i\lambda}{2}}\frac{d\omega_1}{ds} + \frac{i\lambda}{2(s)^2(1-s)^2}\left(\frac{i\lambda}{2} - 1 + 2s\right)s^{\frac{i\lambda}{2}}(1-s)^{-\frac{i\lambda}{2}}\omega_1 \quad (3.8)$$

Substituting these expressions into eq.(3.6) yields

$$s(s-1)\frac{d^2\omega_1}{ds^2} + (s - 1/2 - i\lambda)\frac{d\omega_1}{ds} - A^2\omega_1 = 0 \quad (3.9)$$

Comparing to the Hypergeometric Differential Equation -

$$s(s-1)\frac{d^2\omega}{ds^2} + \left[(\alpha+\beta+1)s - \gamma\right]\frac{d\omega}{ds} + \alpha\beta\omega = 0 \quad (3.10)$$

it is clear that if we let $\gamma = i\lambda + 1/2$, $\alpha = -A$ and $\beta = A$, then eq.(3 9) is the Hypergeometric equation (3.10).

Using the solutions F to the hypergeometric equation, we find the two following solutions

$$v_1^{(1)}(s) = s^{i\lambda/2}(1-s)^{-i\lambda/2} F(-A, A, i\lambda + 1/2 ; s) \quad (3.11.a)$$

$$v_1^{(2)}(s) = s^{1/2 - i\lambda/2}(1-s)^{-i\lambda/2} F(1/2 - i\lambda + A, 1/2 - i\lambda - A, 3/2 - i\lambda ; s) \quad (3.11.b)$$

The equation for v2 differs from eq.(3.2) only in the sign of λ. The solution v2 (s) and it's linearly independant companion v2 (s) are therefore

$$v_2^{(1)}(s) = s^{-i\lambda/2}(1-s)^{i\lambda/2} F(-A, A, -i\lambda + 1/2 ; s) \quad (3.11.c)$$

$$v_2^{(2)}(s) = s^{1/2 + i\lambda/2}(1-s)^{i\lambda/2} F(1/2 + i\lambda + A, 1/2 + i\lambda - A, 3/2 + i\lambda ; s) \quad (3.11.d)$$

We must now look at the behaviour of these functions as x approaches minus infinity.

Using the definition of s, we find

$$x \to -\infty , \quad s \approx 1 - e^{2x} \approx 1 \quad (3.12)$$

For s = 1, the hypergeometric function can be written as

$$F(a, b, c ; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (3.13)$$

This result, together with the asymptotic behaviour of s as given in eq.(3.12) allows us to write

$x \to -\infty$, $(s \to 1)$:

$$V_1^{(1)} \approx e^{-i\lambda x} \frac{\Gamma^2(i\lambda + 1/2)}{\Gamma(i\lambda + 1/2 + A)\Gamma(i\lambda + 1/2 - A)} \quad (3\ 14.a)$$

$$v_1^{(2)} \approx e^{-i\lambda x} \frac{\Gamma(3/2 - i\lambda)\Gamma(i\lambda + 1/2)}{\Gamma(1+A)\ \Gamma(1-A)} \qquad (3.14\ b)$$

$$v_2^{(1)} \approx e^{i\lambda x} \frac{\Gamma^2(-i\lambda + 1/2)}{\Gamma(-i\lambda + 1/2 + A)\Gamma(-i\lambda + 1/2 - A)} \qquad (3.14.c)$$

$$v_2^{(2)} \approx e^{i\lambda x} \frac{\Gamma(3/2 + i\lambda)\Gamma(1/2 - i\lambda)}{\Gamma(1-A)\Gamma(1+A)} \qquad (3.14.d)$$

As $x \to +\infty$, the asymptotic behaviour of s is given by

$$x \to +\infty, \quad s \approx e^{-2x}, \quad s \to 0 \qquad (3.15)$$

and the hypergeometric function satisfies

$$F(a, b, c; 0) = 1 \qquad (3.16)$$

This allows us to write

$$v_1^{(1)} \approx e^{-i\lambda x}, \quad v_1^{(2)} \approx e^{i\lambda x - x} \qquad (3.17.a,b)$$

$$v_2^{(1)} \approx e^{i\lambda x}, \quad v_2^{(2)} \approx e^{-i\lambda x - x} \qquad (3.17.c,d)$$

Using the properties of the hypergeometric function, one can then show that

$$i\frac{d}{dx} v_1^{(2)} - \lambda v_1^{(2)} = -u \frac{\lambda + i/2}{A} v_2^{(1)} \qquad (3.18.a)$$

and

$$i\frac{d}{dx}v_1^{(1)} - \lambda v_1^{(1)} = u\frac{A}{\lambda - i/2} v_2^{(2)} \qquad (3.18.b)$$

This implies that

$$\begin{pmatrix} \frac{A}{\lambda + i/2} \cdot v_1^{(2)} \\ v_2^{(1)} \end{pmatrix} \xrightarrow[x \to \infty]{} e^{i\lambda x}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (3.19.a)$$

$$\begin{pmatrix} v_1^{(1)} \\ \frac{-A}{\lambda - i/2} \cdot v_2^{(2)} \end{pmatrix} \xrightarrow[x \to \infty]{} e^{-i\lambda x}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (3.19.b)$$

are solutions with the given asymptotic behaviour at infinity. The solution $\vec{v}$ can therefore be written as

$$\vec{v} = a(\lambda)\begin{pmatrix} v_1^{(1)} \\ \frac{-A}{\lambda - i/2} v_2^{(2)} \end{pmatrix} + b(\lambda)\begin{pmatrix} \frac{A}{\lambda + i/2} v_1^{(2)} \\ v_2^{(1)} \end{pmatrix} \qquad (3.20)$$

From the asymptotic behaviour (3.14) and the asymptotic condition at minus infinity, we obtain

$$a(\lambda)\begin{pmatrix} \frac{\Gamma^2(i\lambda + 1/2)}{\Gamma(i\lambda + 1/2 + A)\Gamma(i\lambda + 1/2 - A)} e^{-i\lambda x} \\ \frac{-A}{\lambda - i/2}\frac{\Gamma(3/2 + i\lambda)\Gamma(1/2 - i\lambda)}{\Gamma(1 + A)\Gamma(1 - A)} e^{i\lambda x} \end{pmatrix} + b(\lambda)\begin{pmatrix} \frac{A}{\lambda + i/2}\frac{\Gamma(3/2 - i\lambda)\Gamma(i\lambda + 1/2)}{\Gamma(1 + A)\Gamma(1 - A)} e^{-i\lambda x} \\ \frac{\Gamma^2(-i\lambda + 1/2)}{i\Gamma(-i\lambda + 1/2 - A)\Gamma(-i\lambda + 1/2 - A)} e^{i\lambda x} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\lambda x} \qquad (3.21)$$

Rewriting this equation as

$$\begin{pmatrix} \dfrac{\Gamma^2(i\lambda+\frac12)}{\Gamma(i\lambda+\frac12+A)\Gamma(i\lambda+\frac12-A)} & \dfrac{A}{\lambda+\frac12}\,\dfrac{\Gamma(\frac32-i\lambda)\Gamma(i\lambda+\frac12)}{\Gamma(1+A)\,\Gamma(1-A)} \\ \dfrac{-A}{\lambda-\frac12}\,\dfrac{\Gamma(\frac32+i\lambda)\Gamma(\frac12-i\lambda)}{\Gamma(1+A)\Gamma(1-A)} & \dfrac{\Gamma^2(-i\lambda+\frac12)}{\Gamma(-i\lambda+\frac12+A)\Gamma(-i\lambda+\frac12-A)} \end{pmatrix} \begin{pmatrix} a(\lambda) \\ b(\lambda) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.22}$$

and using the relationships:

$$\Gamma(1+z) = z\,\Gamma(z)$$

$$\Gamma(\tfrac12 - i\lambda) = \Gamma^*(\tfrac12 + i\lambda) \tag{3.23}$$

the system (3 22) reduces to

$$\begin{pmatrix} \dfrac{\Gamma^2(i\lambda+\frac12)}{\Gamma(i\lambda+\frac12+A)\Gamma(i\lambda+\frac12-A)} & \dfrac{-i\left|\Gamma(\frac12+i\lambda)\right|^2}{\Gamma(A)\,\Gamma(1-A)} \\ \dfrac{i\left|\Gamma(\frac12-i\lambda)\right|^2}{\Gamma(A)\Gamma(1-A)} & \dfrac{\Gamma^2(-i\lambda+\frac12)}{\Gamma(-i\lambda+\frac12+A)\Gamma(-i\lambda+\frac12-A)} \end{pmatrix} \begin{pmatrix} a(\lambda) \\ b(\lambda) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{3.24}$$

which is readily solved for $a(\lambda)$ and $b(\lambda)$. We find

$$a(\lambda) = \frac{1}{D}\,\frac{\Gamma^2(-i\lambda+\frac12)}{\Gamma(-i\lambda+\frac12+A)\Gamma(-i\lambda+\frac12-A)} \tag{3.25}$$

$$b(\lambda) = \frac{1}{D}\,\frac{i\left|\Gamma(\frac12+i\lambda)\right|^2}{\Gamma(A)\Gamma(1-A)} \tag{3.26}$$

where D is given by

$$D = \frac{\left|\Gamma(\frac12+i\lambda)\right|^4\left[\Gamma^2(1-A)\Gamma^2(A) - \left|\Gamma(\frac12+i\lambda+A)\right|^2\left|\Gamma(\frac12+i\lambda-A)\right|^2\right]}{\left|\Gamma(\frac12+i\lambda+A)\right|^2\left|\Gamma(\frac12+i\lambda-A)\right|^2\Gamma^2(1-A)\,\Gamma^2(A)}$$

It remains to find the number of zeros of $a(\lambda)$. The eigenvalues occur at the zeros of $a(\lambda)$ To find the zeros of $a(\lambda)$, after analytic continuation in the upper half plane, we note that $a(\lambda) = 0$ only at the poles of $F(-i\lambda - A + 1/2)$ in its denominator. The poles of the gamma function occur when the argument z = 0,1,2.. [4] or for values of r satisfying

$$r = z + 1 = 1, 2, 3, \ldots \quad (3.27)$$

This means

$$-i\gamma_r - A - 1/2 = -r \quad (3.28)$$

Im $\lambda > 0$ implies

$$\gamma_r = i(A + 1/2 - r) > 0 \quad (3.29)$$

which can only be satisfied for a finite number of r values, r = 1,2,..,N. If we define F to be

$$F = \int_{-\infty}^{\infty} u(x,0)\, dx \quad (3.30)$$

then

$$F = \int_{-\infty}^{\infty} A \operatorname{sech}(x)\, dx = A\pi \quad (3.31)$$

Hence the formula

$$N = \left\langle \frac{F}{\pi} + \frac{1}{2} \right\rangle \quad (3.32)$$

gives N, the number of solitons contained in the initial pulse where $\langle .. \rangle$ denotes the greatest integer less than the argument.

We have found a simple relation between the number of solitons N, contained in the initial pulse and the area of the pulse itself.

CHAPTER FOUR

PULSES OF RECTANGULAR FORM

In this chapter, we solve the eigenvalue problem

$$\begin{aligned} v_1' + i\lambda v_1 &= q v_2 \\ v_2' - i\lambda v_2 &= -q^* v_1 \end{aligned} \tag{4.1}$$

for the special family of input pulses given by

$$q(x) = \begin{cases} 0 & \text{for } |x| > d/2 \\ \beta & \text{for } |x| \leq d/2 \end{cases} \tag{4.2}$$

and look for solutions v1 and v2 which exhibit the asymptotic behaviour given by eq.(2.11) In chapter two, we saw that solving $a(\lambda) = 0$ means finding the eigenvalues of the eigenvalue problem (2.1.a).It is clear from eq.(2.11) that the asymptotic behaviour whose coefficient is $a(\lambda)$ prevents the solutions from being square integrable. If this term can be eliminated, then only the right kind of asymptotic behaviour will remain - We do this by finding the zeros of $a(\lambda)$.

Solving eq.(4.1) in the regions defined by the boundaries of q(x), we obtain the following solutions

$$\begin{aligned} x < d/2 \quad &: \quad v_1 = e^{-i\lambda x}, \quad v_2 = 0 \\ x > d/2 \quad &: \quad v_1 = 0, \quad v_2 = e^{i\lambda x} \end{aligned} \tag{4.3.a}$$

$$v_1 = -\frac{1}{\beta}\left[\left(-i\lambda A - B\sqrt{\beta^2+\lambda^2}\right)\sin\sqrt{\beta^2+\lambda^2}\,x + \left(-i\lambda B + A\sqrt{\beta^2+\lambda^2}\right)\cos\sqrt{\beta^2+\lambda^2}\,x\right]$$

(4.3.b)

$$v_2 = A\sin\sqrt{\beta^2+\lambda^2}\,x + B\cos\sqrt{\beta^2+\lambda^2}\,x \quad , |x| \le \alpha/2$$

If we match these solutions at the boundaries of q(x), the input pulse, then for the asymptotic behaviour given by eq.(2.1.a) we obtain, at $x = -\alpha/2$, the following equations

$$v_1 = -\frac{1}{\beta}\left[\left(+i\lambda A + B\sqrt{\beta^2+\lambda^2}\right)\sin\sqrt{\beta^2+\lambda^2}\,\frac{\alpha}{2} + \left(-i\lambda B + A\sqrt{\beta^2+\lambda^2}\right)\cos\sqrt{\beta^2+\lambda^2}\,\frac{\alpha}{2}\right] = e^{-i\lambda x} \qquad (4.4\ a)$$

$$-A\sin\sqrt{\beta^2+\lambda^2}\,\frac{\alpha}{2} + B\cos\sqrt{\beta^2+\lambda^2}\,\frac{\alpha}{2} = 0 \qquad (4.4.b)$$

For $x = +\alpha/2$ we find

$$A\sin\sqrt{\beta^2+\lambda^2}\,\frac{\alpha}{2} + B\cos\sqrt{\beta^2+\lambda^2}\,\alpha/2 = b(\lambda)e^{i\lambda x} \qquad (4.5.a)$$

$$-\frac{1}{\beta}\left[\left(-i\lambda A - B\sqrt{\beta^2+\lambda^2}\right)\sin\sqrt{\beta^2+\lambda^2}\,\frac{\alpha}{2} + \left(-i\lambda B + A\sqrt{\beta^2+\lambda^2}\right)\cos\sqrt{\beta^2+\lambda^2}\,\frac{\alpha}{2}\right] = a(\lambda)e^{-i\lambda x} \qquad (4.5.b)$$

These four equations, linear in A, B, $a(\lambda)$ and $b(\lambda)$ can then be solved and we obtain, for $a(\lambda)$ and $b(\lambda)$

$$a(\lambda) = \frac{e^{-i\lambda\alpha}}{\sqrt{\beta^2+\lambda^2}}\left(i\lambda\sin\sqrt{\beta^2+\lambda^2}\,\alpha - \sqrt{\beta^2+\lambda^2}\cos\sqrt{\beta^2+\lambda^2}\,\alpha\right) \qquad (4.6.a)$$

$$b(\lambda) = \frac{\beta}{\sqrt{\beta^2+\lambda^2}}\cdot\sin\sqrt{\beta^2+\lambda^2}\,\alpha \qquad (4.6.b)$$

The zeros of $a(\lambda)$, which give the eigenvalues, are then found to occur when

$$\frac{-i\lambda}{\sqrt{\beta^2+\lambda^2}} = \operatorname{cotan}\sqrt{\beta^2+\lambda^2}\,\alpha \qquad (4.7)$$

Having found the eigenvalues, we can further show that they are purely imaginary. We do this as follows :

From eq.(4.7), we obtain

$$\beta \sin\sqrt{\beta^2+\lambda^2}\,\alpha = \mp\sqrt{\beta^2+\lambda^2} \qquad (4.8.a)$$

and
$$\beta \cos\sqrt{\beta^2+\lambda^2}\,\alpha = \mp i\lambda \qquad (4\ 8\ b)$$

If we now define ρ as

$$\rho = i\sqrt{\beta^2+\lambda^2}\,\alpha = \rho_1 + i\rho_2 \qquad (4.9)$$

then we can write, using eq.(4.8.a)

$$\frac{\rho_1 + i\rho_2}{\alpha\beta} = \mp i \sin\sqrt{\beta^2+\lambda^2}\,\alpha \qquad (4.10.a)$$

$$= \mp \sinh(\rho_1)\cos(\rho_2) + i\cosh(\rho_1)\sin(\rho_2) \qquad (4.10.b)$$

and hence we find $\rho 1$ and $\rho 2$ to be

$$\rho_1 = \mp\alpha\beta\sinh(\rho_1)\cos(\rho_2) \qquad (4.11.a)$$

$$\rho_2 = \mp\alpha\beta\cosh(\rho_1)\sin(\rho_2) \qquad (4.11.b)$$

Using eq.(4.8.b), we can also write

$$\frac{i\kappa - \eta}{\beta} = \mp\cos\sqrt{\beta^2+\lambda^2}\,\alpha \qquad (4.12\ a)$$

$$= \mp\cosh(\rho_1)\cos(\rho_2) + i\sinh(\rho_1)\sin(\rho_2) \qquad (4.12.b)$$

and these equations allow us to write

$$\eta = \pm \beta \cosh(\rho 1) \cos(\rho 2) \tag{4.13.a}$$

$$\chi = \mp \beta \sinh(\rho 1) \sin(\rho 2) \tag{4.13.b}$$

Now, if $\rho 1 \neq 0$, we have

$$\cos(\rho 2) = \mp \frac{\rho 1}{\alpha \beta \sinh(\rho 1)} \lessgtr 0 \tag{4.14.a}$$

$$\cos(\rho 2) = \pm \frac{\eta}{\beta \cosh(\rho 1)} \gtrless 0 \tag{4.14.b}$$

which are contradictory for $\alpha > 0$ and $\eta > 0$. We conclude that there are no zeros of $a(\lambda)$ with non-zero real part, i.e., the eigenvalues are purely imaginary

$$\lambda = i\eta \quad , \quad \chi = 0 \tag{4 15}$$

It now remains to solve eq.(4 7). For purely imaginary eigenvalues η, this equation looks like

$$\eta = \sqrt{\beta^2 - \eta^2} \; \operatorname{cotan}\left(\sqrt{\beta^2 - \eta^2}\, \alpha\right) \tag{4.16}$$

If we define ρ as

$$\rho = \sqrt{\beta^2 - \eta^2} \; \alpha \tag{4.17}$$

then eq.(4.17) can be expressed as two equations given by

$$\eta = -\frac{\rho}{\alpha} \cot(\rho) \tag{4.18 a}$$

and (4 18 b)

$$\rho^2 = \beta^2\alpha^2 - \eta^2\alpha^2$$

- the solutions of which are given by the points of intersection of the family of ellipses given by (4.18.b) and the cotan function given by (4.18.a). A computer plot of these functions is given below in figure (4.1).

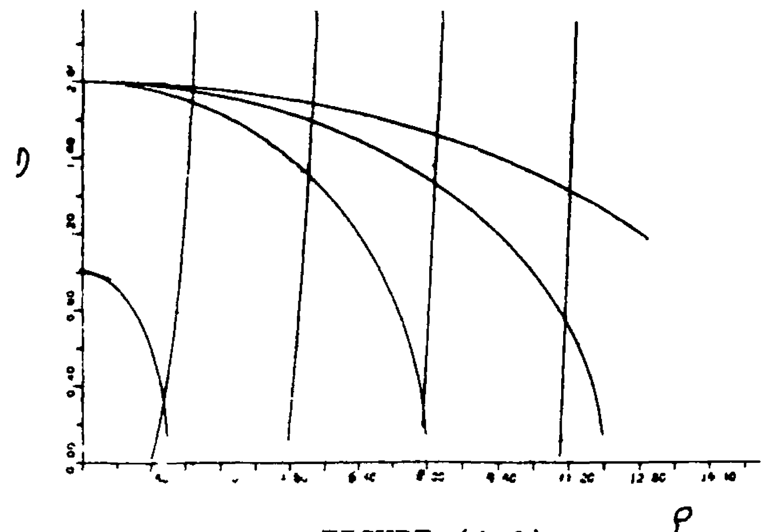


FIGURE (4.1)

As can be seen in the diagram, increasing values of the pulse parameters α and β result in larger ellipses and thus give rise to more and more intersections. Hence, it can be seen that the size of the pulse determines the number of eigenvalues, thus determining the number of solitons.

The eigenvalues were obtained from the numerical solution of the equation

$$\frac{\rho}{\alpha} \cot(\rho) - \left(\beta^2 - \frac{\rho^2}{\alpha^2}\right)^{1/2} = 0 \tag{4.19}$$

and these eigenvalues are given in table (4.1) for three different sets of pulse parameters. Figures (4.2) - (4.4) are computer plots of the eigenfunctions obtained from the eigenvalues in table (4.1).

$\alpha=2,\ \beta=1$	$\alpha=3,\ \beta=2$	$\alpha=4,\ \beta=2$
$\eta=0.391022$	$\eta=0.982587$	$\eta=0.206312$
	$\eta=1.789603$	$\eta=1.447273$
		$\eta=1.874812$

TABLE (4.1)

Figure (4.1) shows that the number of intersections, N, which is the number of eigenvalues, is given by the equation

$$N = \left\langle \frac{1}{2} + \frac{\alpha\beta}{\pi} \right\rangle \tag{4.20}$$

- where $\langle..\rangle$ denotes the greatest integer less than the argument.

That eq.(4.20) holds can be easily seen if one considers the ellipse which intersects the x-axis at the point $\rho = \pi/2$. For this value of ρ, $\eta = 0$ and so we have $\rho^2 = \alpha^2\beta^2$ from which we deduce $\alpha\beta/\pi = 1/2$. The parameters α and β are just large enough for this ellipse to cross the cotan function once Equation (4.20) gives $N = \langle 1/2 + \alpha\beta/\pi \rangle = 1$ which is the same number.

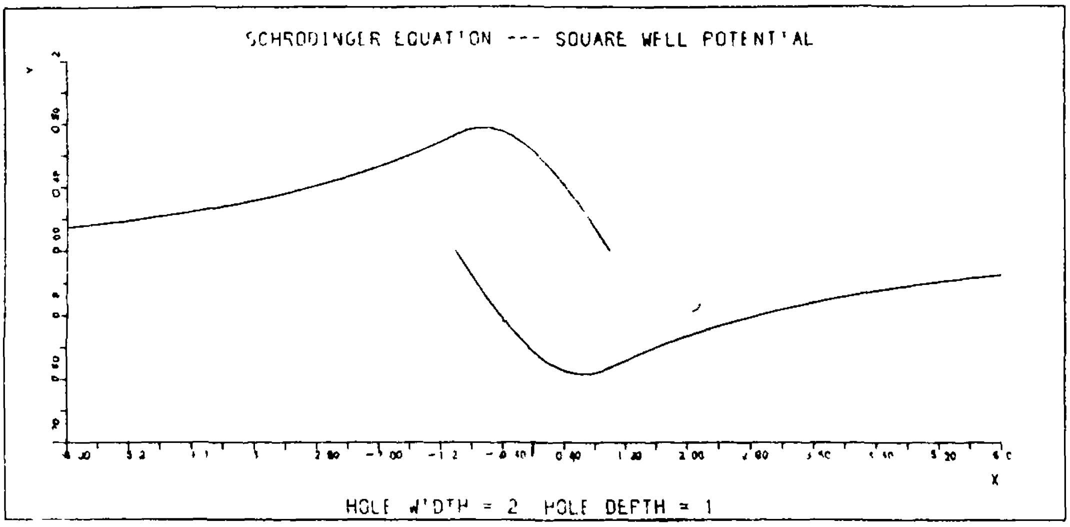


FIGURE (4.2) : Pulse parameters: $\alpha = 2$, $\beta = 1$.

The pulse parameters $\alpha=2$ and $\beta=1$ define an ellipse which is just large enough to intersect with the cotan function once only and hence a pulse of this size results in only one eigenvalue, the eigenfunction for which is given in figure (4.2) above.

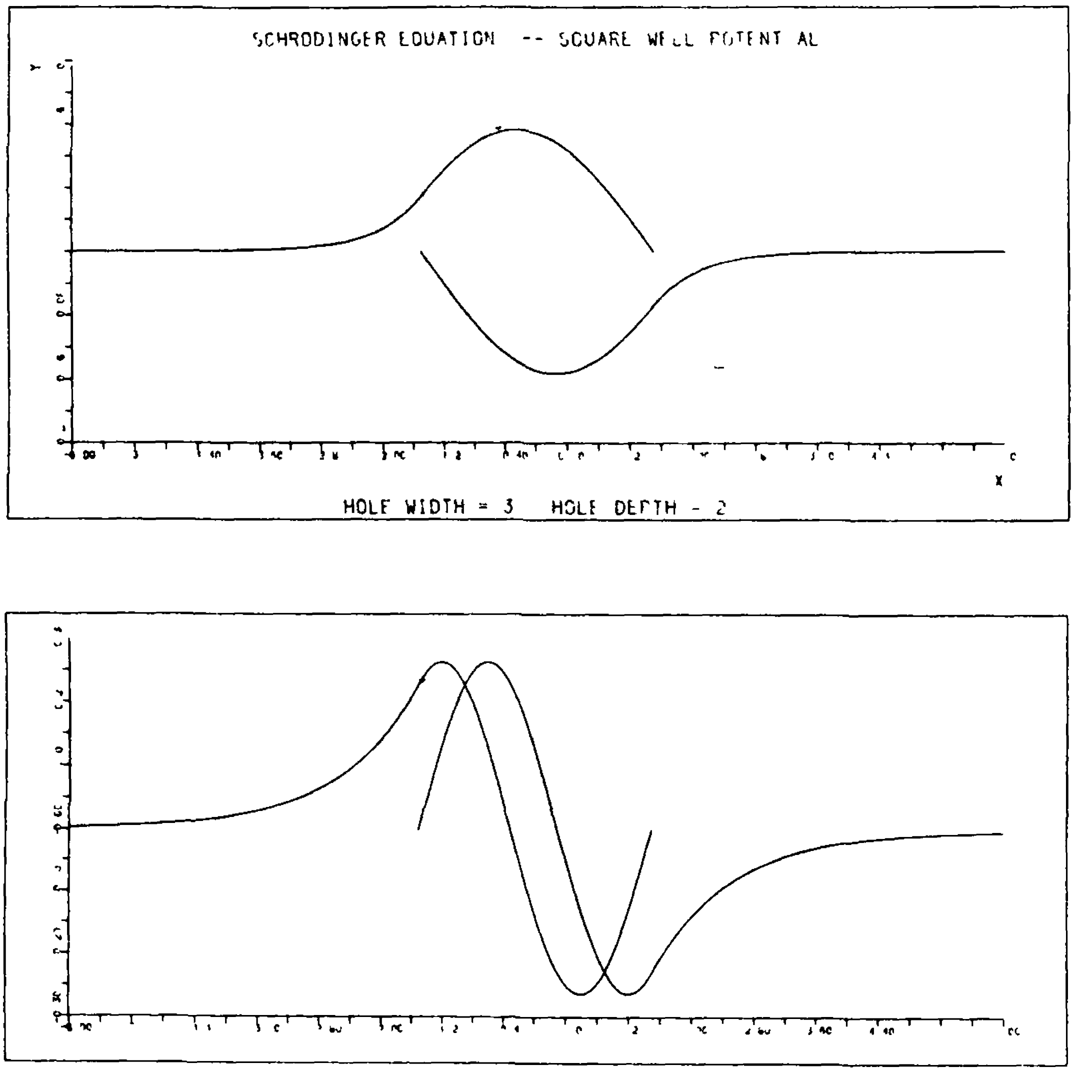


FIGURE (4.3) : pulse parameters: α = 3, β = 2.

For larger pulse parameters, $\alpha=3$ and $\beta=2$, the resulting ellipse intersects with the cotan function twice admitting two eigenvalues. The two eigenfunctions are given in figure (4.3). Figure (4.4) contains the eigenfunctions of the three allowable eigenstates for the pulse parameters α =4 and β =2.

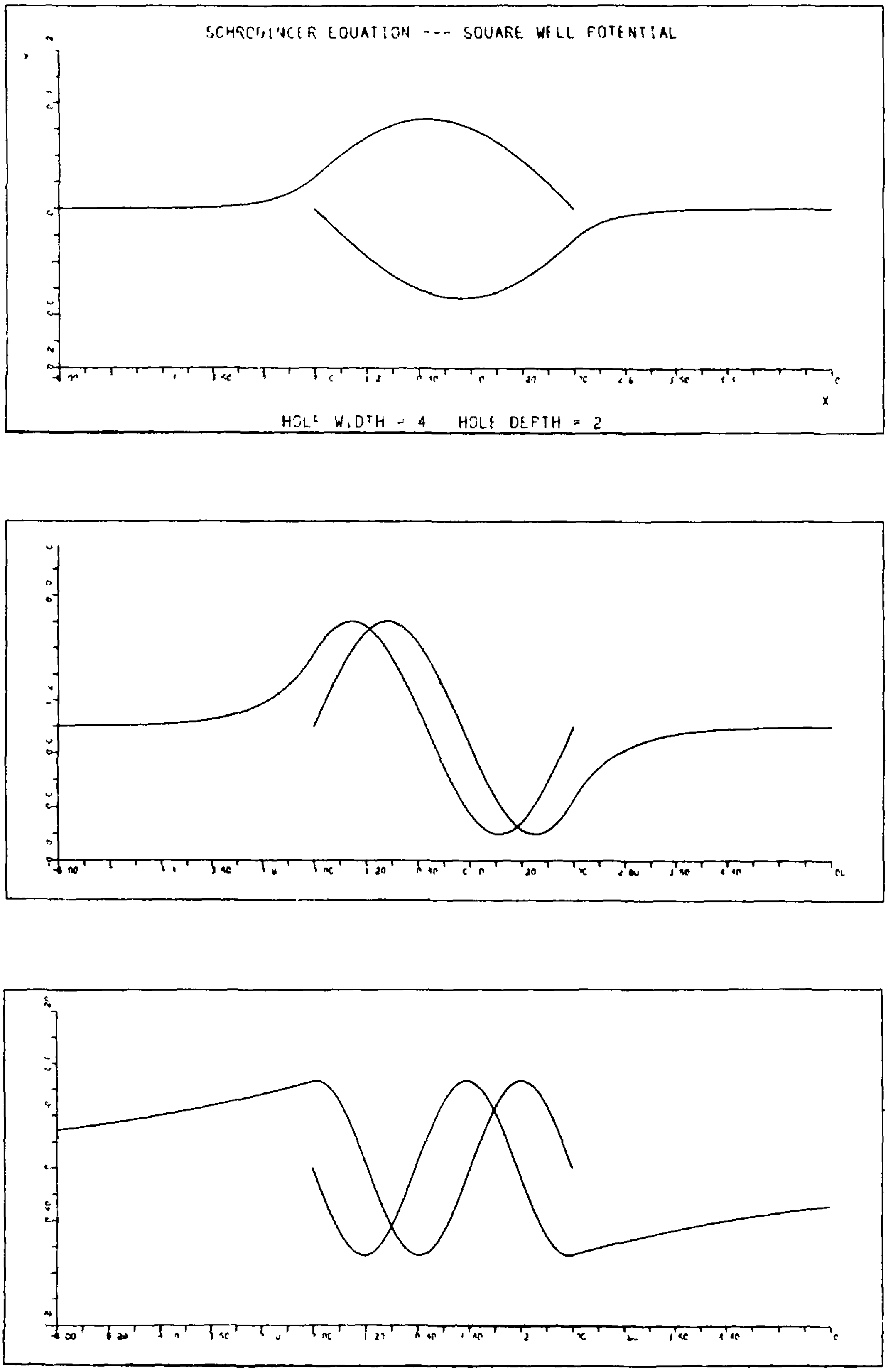


FIGURE (4.4) : Pulse parameters: α = 4, β = 2.

In every case, the number of solitons contained in a given pulse q(x) is the same as the number of eigenvalues of the underlying wave equations eigenvalue problem.

We conclude that the number of solitons contained in any rectangular pulse is given by the simple equation, eq.(4.20), which gives the soliton number in terms of the pulse area. This is the result obtained in chapter 3 where pulses of sech(x) form were studied. This chapter completes the analysis [5] of rectangular pulses where only purely imaginary eigenvalues had been considered.

CHAPTER FIVE

EXPONENTIAL PEAK PULSES

In this chapter, we solve the eigenvalue problem

$$v_1' + i\lambda v_1 = q v_2 \tag{5.1.a}$$

$$v_2' - i\lambda v_2 = -q^* v_1 \tag{5.1.b}$$

where q is a family of input pulses given by

$$q(x) = \beta \exp(-\alpha |x|) \tag{5.2}$$

Equations (5.1.a,b) imply

$$v_1'' - \frac{u'}{u} v_1' + \left[\lambda^2 - i\lambda \frac{u'}{u} + |u|^2\right] v_1 = 0 \tag{5.3.a}$$

$$v_2'' - \frac{(u^*)'}{u} v_2' + \left[\lambda^2 + i\lambda \frac{(u^*)'}{u^*} + |u|^2\right] v_2 = 0 \tag{5.3.b}$$

where q(x) = iu(x). If, for x > 0, we make the following change of variables:

$$x > 0 : \quad s = \frac{\beta}{\alpha} \exp(-\alpha x), \quad \psi = \frac{v_2}{\sqrt{s}} \tag{5.4}$$

then eq.(5.3.b) can be transformed to Bessel's equation as outlined below:

Equation (5.4) implies

$$\frac{dV_2}{dx} = -\alpha s \frac{dV_2}{ds} \qquad (5.5.a)$$

$$\frac{d^2 V_2}{dx^2} = \alpha^2 s \frac{dV_2}{ds} + \alpha^2 s^2 \frac{d^2 V_2}{ds^2} \qquad (5\ 5.b)$$

Substituting these derivatives into eq.(5.3.b) gives

$$\alpha^2 s^2 \frac{d^2 V_2}{ds^2} + \left[\lambda^2 - i\lambda\alpha + \alpha^2 s^2\right] V_2 = 0 \qquad (5.6)$$

If we now use $\psi = v2/\sqrt{s}$, we find

$$\frac{d\psi}{ds} = \frac{1}{s^{1/2}}\frac{dV_2}{ds} - \frac{1}{2}\frac{1}{s^{3/2}} V_2 \qquad (5.7.a)$$

$$\frac{d^2\psi}{ds^2} = -\frac{1}{2}\frac{1}{s^{3/2}}\frac{dV_2}{ds} + \frac{1}{s^{1/2}}\frac{d^2 V_2}{ds^2} + \frac{3}{4}\frac{1}{s^{5/2}} V_2 - \frac{1}{2}\frac{1}{s^{3/2}}\frac{dV_2}{ds} \qquad (5.7.b)$$

Using eqs(5.7) and defining

$$\nu = \left(-\frac{i\lambda}{\alpha} - \frac{1}{2}\right) \qquad (5.8)$$

allows us to write eq.(5.6) as Bessel's equation of order ν

$$\frac{d^2\psi}{ds^2} + \frac{1}{s}\frac{d\psi}{ds} + \left(1 - \frac{\nu^2}{s^2}\right)\psi = 0 \qquad (5.9)$$

For $x \leqslant 0$, the change of variables

$$x < 0, \quad s = \frac{\beta}{\alpha}\exp(\alpha x), \quad \psi = \frac{V_1}{\sqrt{s}} \qquad (5\ 10)$$

enables us to obtain the following

$$\frac{dV_1}{dx} = \alpha s \frac{dV_1}{ds} \qquad (5\ 11\ a)$$

$$\frac{d^2 V_1}{dx^2} = \alpha^2 s \frac{dV_1}{ds} + \alpha^2 s^2 \frac{d^2 V_1}{ds^2} \qquad (5\ 11\ b)$$

Substituting eqs(5.11) into eq.(5.3.a), we find

$$\alpha^2 s^2 \frac{d^2 V_1}{ds^2} + \left[\lambda^2 - i\lambda\alpha + \alpha^2 s^2 \right] V_1 = 0 \qquad (5.12)$$

Now, using $\psi = v1/\sqrt{s}$, we find

$$\frac{d\psi}{ds} = \frac{1}{s^{1/2}} \frac{dV_1}{ds} - \frac{1}{2} \frac{1}{s^{3/2}} V_1 \qquad (5.13.a)$$

$$\frac{d^2\psi}{ds^2} = \frac{1}{s^{1/2}} \frac{d^2 V_1}{ds^2} - \frac{1}{s^{3/2}} \frac{dV_1}{ds} + \frac{3}{4} \frac{1}{s^{5/2}} V_1 \qquad (5\ 13.b)$$

again defining ν as

$$\nu = \left(\frac{-i\lambda}{\alpha} - \frac{1}{2} \right) \qquad (5.14)$$

it is easily seen that the Bessel equation of order is the same equation as equation (5 12).

For x < 0, Bessel's equation, (5.9), has the solution

$$V_1 = = c_1 \sqrt{s}\, J_\nu(s) + c_2 \sqrt{s}\, J_{-\nu}(s) \qquad (5.15)$$

where c1 and c2 are constants to be determined and $J_\nu(s)$ is the Bessel function of order ν given by

$$J_\nu(s) = \frac{1}{\Gamma(\nu+1)} \left[\left(\frac{s}{2}\right)^\nu - \frac{1}{\nu+1} \left(\frac{s}{2}\right)^{\nu+2} + \ \cdot\ \cdot\ \cdot \right] \qquad (5.16)$$

Using this v1, we can find v2 by using eq.(5.1.a). We find:

$$v_2 = \frac{d}{ds} v_1 + \frac{i\lambda}{\alpha s} v_1 \tag{5.17}$$

For the first component of v1, $c1\sqrt{s}J_\nu(s)$, v2 is

$$v_1 = \sqrt{s}J_{\nu}(s) : v_2 = \frac{1}{2\sqrt{s}} J_\nu(s) + \sqrt{s}\frac{d}{ds} J_\nu(s) + \frac{i\lambda}{\alpha\sqrt{s}} J_\nu(s)$$
$$= \sqrt{s}\left[\frac{d}{ds} J_\nu(s) - \frac{\nu}{s} J_\nu(s)\right] = -\sqrt{s} J_{\nu+1}(s) \tag{5.18}$$

For the second component, $c2\sqrt{s}J_{-\nu}(s)$, we find

$$v_1 = \sqrt{s}J_{-\nu}(s) : v_2 = \sqrt{s}\left[\frac{d}{ds} J_{-\nu}(s) - \frac{\nu}{s} J_{-\nu}(s)\right]$$
$$= \sqrt{s} J_{-\nu-1}(s) \tag{5.19}$$

- giving the complete solution for $x < 0$ as

$$v_1 = c_1\sqrt{s}J_\nu(s) + c_2\sqrt{s}J_{-\nu}(s)$$
$$v_2 = -c_1\sqrt{s}J_{\nu+1}(s) + c_2\sqrt{s}J_{-\nu-1}(s) \tag{5.20}$$

These solutions must have the asymptotic behaviour as given by eq.(2.11) for $x \to -\infty$ Expressing v1 and v2 in terms of x we obtain

$$\sqrt{s}\begin{pmatrix} J_\nu(s) \\ -J_{\nu+1}(s)\end{pmatrix} \xrightarrow[s\to 0]{x\to\infty} \begin{pmatrix} \frac{1}{\Gamma(\nu+1)}\left(\frac{\beta}{\alpha}\right)^{-\frac{i\lambda}{2}}\left(\frac{1}{2}\right)^{\nu} e^{-i\lambda x} \\ \frac{-1}{\Gamma(\nu+2)}\left(\frac{\beta}{\alpha}\right)^{1-\frac{i\lambda}{2}}\left(\frac{1}{2}\right)^{\nu+1} e^{-i\lambda x+\alpha x}\end{pmatrix} \tag{5.21}$$

$$\sqrt{s}\begin{pmatrix} J_{-\nu}(s) \\ J_{-\nu-1}(s)\end{pmatrix} \xrightarrow[s\to 0]{x\to\infty} \begin{pmatrix} \frac{1}{\Gamma(-\nu+1)}\left(\frac{\beta}{\alpha}\right)^{1+\frac{i\lambda}{\alpha}}\left(\frac{1}{2}\right)^{-\nu} e^{i\lambda x+\alpha x} \\ \frac{1}{\Gamma(-\nu)}\left(\frac{\beta}{\alpha}\right)^{i\lambda/2}\left(\frac{1}{2}\right)^{-\nu-1} e^{i\lambda x}\end{pmatrix} \tag{5.22}$$

In order to make the coefficients of $\exp(-i\lambda x)$ one and zero for v1 and v2 respectively, we first make use of the fact that for v1, the $J_{-\nu}(s)$ function becomes infinite as $x \to -\infty$. (That this is so is obvious from eq.(5.16)). This implies that c2 must be zero. That v1 must approach $\exp(-i\lambda x)$ as $x \to -\infty$ then implies that $c1 = 2^{\nu}(\alpha/\beta)^{\frac{-i\lambda}{\alpha}}\Gamma(\nu+1)$.

For $x > 0$, the solution of Bessel's equation is given in terms of v2 as

$$V_2 = c3\sqrt{s}\,J_{\nu}(s) + c4\sqrt{s}\,J_{-\nu}(s) \qquad (5.23)$$

for constants c3 and c4. Using (5.1.b), the corresponding solution, v1, can then be found in a similar way to finding v2 from v1 for $x < 0$. The complete solution for $x > 0$ is then

$$\begin{aligned} V_1 &= -c3\sqrt{s}\,J_{\nu+1}(s) + c4\sqrt{s}\,J_{-\nu-1}(s) \\ V_2 &= c3\sqrt{s}\,J_{\nu}(s) + c4\sqrt{s}\,J_{-\nu}(s) \end{aligned} \qquad (5.24)$$

Expressing these solutions in terms of x, for $x > 0$, we can write

$$\sqrt{s}\begin{pmatrix} -J_{\nu+1}(s) \\ J_{\nu}(s) \end{pmatrix} \xrightarrow[s\to 0]{x\to\infty} \begin{pmatrix} \frac{-1}{\Gamma(\nu+2)}\left(\frac{\beta}{\alpha}\right)^{1-\frac{i\lambda}{\alpha}}\left(\frac{1}{2}\right)^{\nu+1} e^{i\lambda x-\alpha x} \\ \frac{1}{\Gamma(\nu+1)}\left(\frac{\beta}{\alpha}\right)^{\frac{-i\lambda}{\alpha}}\left(\frac{1}{2}\right)^{\nu} e^{i\lambda x} \end{pmatrix} \qquad (5.25)$$

$$\sqrt{s}\begin{pmatrix} J_{-\nu-1}(s) \\ J_{-\nu}(s) \end{pmatrix} \xrightarrow[s\to 0]{x\to\infty} \begin{pmatrix} \frac{1}{\Gamma(-\nu)}\left(\frac{\beta}{\alpha}\right)^{\frac{i\lambda}{\alpha}}\left(\frac{1}{2}\right)^{-\nu-1} e^{-i\lambda x} \\ \frac{1}{\Gamma(-\nu-1)}\left(\frac{\beta}{\alpha}\right)^{1+\frac{i\lambda}{\alpha}}\left(\frac{1}{2}\right)^{-\nu} e^{-i\lambda x-\alpha x} \end{pmatrix} \qquad (5.26)$$

If we compare these equations with the asymptotic behaviour of $\vec{v}$ as given in eq (2 11), we find c3 and c4

$$c_3 = b(\lambda)\, 2^{\nu} \left(\frac{\alpha}{\beta}\right)^{-i\lambda/\alpha} \Gamma(\nu+1)$$
$$c_4 = a(\lambda)\, 2^{-\nu-1} \left(\frac{\alpha}{\beta}\right)^{i\lambda/\alpha} \Gamma(-\nu) \tag{5.27}$$

The complete solution, $\vec{v}$ for $x > 0$ is therefore given by

$$\vec{v} = -a(\lambda) 2^{-\nu-1} \left(\frac{\alpha}{\beta}\right)^{\frac{i\lambda}{\alpha}} \Gamma(-\nu) \sqrt{s} \begin{pmatrix} J_{-\nu-1} \\ J_{-\nu} \end{pmatrix} + b(\lambda) 2^{\nu} \left(\frac{\alpha}{\beta}\right)^{-\frac{i\lambda}{\alpha}} \Gamma(\nu+1) \sqrt{s} \begin{pmatrix} -J_{\nu+1} \\ J_{\nu} \end{pmatrix} \tag{5.28}$$

It now remains to calculate the coefficients $a(\lambda)$ and $b(\lambda)$. To do so, we note that the two solutions obtained above, i.e., for positive and negative x, must be the same at $x = 0$. This matching condition then gives, for $s = (\beta/\alpha)$

$$2^{\nu} \left(\frac{\alpha}{\beta}\right)^{-\frac{i\lambda}{\alpha}} \Gamma(\nu+1) \sqrt{\frac{\beta}{\alpha}} \begin{pmatrix} J_{\nu}(\beta/\alpha) \\ -J_{\nu+1}(\beta/\alpha) \end{pmatrix} = -a(\lambda) 2^{-\nu-1}$$

$$\times \left(\frac{\alpha}{\beta}\right)^{\frac{i\lambda}{\alpha}} \Gamma(-\nu) \sqrt{\frac{\beta}{\alpha}} \begin{pmatrix} J_{-\nu-1}(\beta/\alpha) \\ J_{-\nu}(\beta/\alpha) \end{pmatrix} + b(\lambda) 2^{\nu} \left(\frac{\alpha}{\beta}\right)^{-\frac{i\lambda}{\alpha}} \Gamma(\nu+1) \sqrt{\frac{\beta}{\alpha}} \begin{pmatrix} -J_{\nu+1}(\beta/\alpha) \\ J_{\nu}(\beta/\alpha) \end{pmatrix} \tag{5.29}$$

If, for convenience, we define $\tilde{a}(\lambda)$ as

$$\tilde{a}(\lambda) = -a(\lambda) 2^{-2\nu-1} \left(\frac{\alpha}{\beta}\right)^{\frac{2i\lambda}{\alpha}} \left(\frac{\Gamma(-\nu)}{\Gamma(\nu+1)}\right) \tag{5 30}$$

then eq(5 29) can be written in the following form

$$\begin{pmatrix} J_{\nu}(\beta/\alpha) \\ -J_{\nu+1}(\beta/\alpha) \end{pmatrix} = \tilde{a}(\lambda) \begin{pmatrix} J_{-\nu-1}(\beta/\alpha) \\ J_{-\nu}(\beta/\alpha) \end{pmatrix} + b(\lambda) \begin{pmatrix} -J_{\nu+1}(\beta/\alpha) \\ J_{\nu}(\beta/\alpha) \end{pmatrix} \tag{5.31}$$

Hence

$$\begin{pmatrix} J_{-\nu-1}(\beta/\alpha) & -J_{\nu+1}(\beta/\alpha) \\ J_{-\nu}(\beta/\alpha) & J_{\nu}(\beta/\alpha) \end{pmatrix} \begin{pmatrix} \tilde{a}(\lambda) \\ b(\lambda) \end{pmatrix} = \begin{pmatrix} J_{\nu}(\beta/\alpha) \\ -J_{\nu+1}(\beta/\alpha) \end{pmatrix} \tag{5.32}$$

Solving for $\tilde{a}(\lambda)$ and $b(\lambda)$, we obtain

$$\tilde{a}(\lambda) = \frac{J_\nu^2(\rho/\lambda) - J_{\nu+1}^2(\rho/\lambda)}{J_{-\nu-1}(\rho/\lambda)\, J_\nu(\rho/\lambda) + J_{-\nu}(\rho/\lambda)\, J_{\nu+1}(\rho/\lambda)} \tag{5.33}$$

$$b(\lambda) = \frac{-J_{\nu+1}(\rho/\lambda)\, J_{-\nu-1}(\rho/\lambda) - J_{-\nu}(\rho/\lambda)\, J_\nu(\rho/\lambda)}{J_{-\nu-1}(\rho/\lambda)\, J_\nu(\rho/\lambda) + J_{-\nu}(\rho/\lambda)\, J_{\nu+1}(\rho/\lambda)} \tag{5.34}$$

We have found $\tilde{a}(\lambda)$ for real values of λ. Real values of λ imply that $\nu \notin N \cup \{0\}$ Analytically continuing into the complex upper half plane, in which $\mathrm{Im}\,\lambda > 0$, then allows that ν to belong to the set $N \cup \{0\}$.

Note: $a(\lambda)$ and $b(\lambda)$ which are given in terms of the linearly independant pair J_ν and $J_{-\nu}$ can also be expressed [5] in terms of J_ν and Y_ν In any case, the condition on the number of zeros is the same.

The zeros of $\tilde{a}(\lambda)$ are the eigenvalues of eq.(5.1) and are the solutions of

$$J_{\nu+1}(\rho/\lambda) = \pm J_\nu(\rho/\lambda) \tag{5.35}$$

It can be further shown that there are no solutions of eq.(5.35) for non-real ν with $\mathrm{Re}\,\nu > -1/2$ The proof runs as follows[6]· We assume that $J_\nu \pm J_{\nu+1}$ has a real zero s for non-real ν . Then, using the Mittag-Leffler expansion [7,p497]

$$1 \pm \sum_{n=1}^{\infty} \frac{2s}{j_{\nu n}^2 - s^2} = 0 \tag{5.36}$$

and therefore $$\sum_{n=1}^{\infty} \frac{\mathrm{Re}\, j_{\nu n}\, \mathrm{Im}\, j_{\nu n}}{|j_{\nu n}^2 - s^2|^2} = 0 \tag{5.37}$$

follows, where $j_{\nu n}$ are the zeros of $s^{-\nu}J_\nu(s)$. This equation cannot hold because Re $j_{\nu n}$ /Im $j_{\nu n}$ $\gtrless$ 0 for $\kappa \gtrless 0$ for all n with Im $j_{\nu n}$ $\neq 0$ [8], and because there are $j_{\nu n}$ with Re $j_{\nu n} \neq 0$ and Im $j_{\nu n}$ $\neq 0$ This leaves with the problem of finding and studying the points of intersection of J_ν and $J_{\nu+1}$. We will denote these intersection points as $s_n(\nu)$ for real order $\nu = \beta/\alpha - 1/2$ $> -1/2$. Showing that labelling these points makes sense is easy because, if ν changes, the number of points of intersection stays the same, and s_n changes continuously with ν. Furthermore, $s_n \rightarrow \infty$ for $n \rightarrow \infty$ and for $\nu \rightarrow \infty$, and s_n increases monotonically with ν [6; consequences of Lemmas 2.3 and 2.5 in ref. 9]. This implies that $s_n(-1/2) = (2n-1)\pi/2$ determines the number of solitons.

Solutions of eq(5.35) were obtained numerically using Newton's method Using the values of the intersection points obtained, a graph of (β/α) against ν was plotted, (figure(5.1)). As can be seen from the diagram, increasing values of (β/α) give rise to more and more intersection points and hence more eigenvalues. As in the case of rectangular pulses, see chapter 4, an increasing pulse size means an increasing capacity for eigenstates and hence solitons.

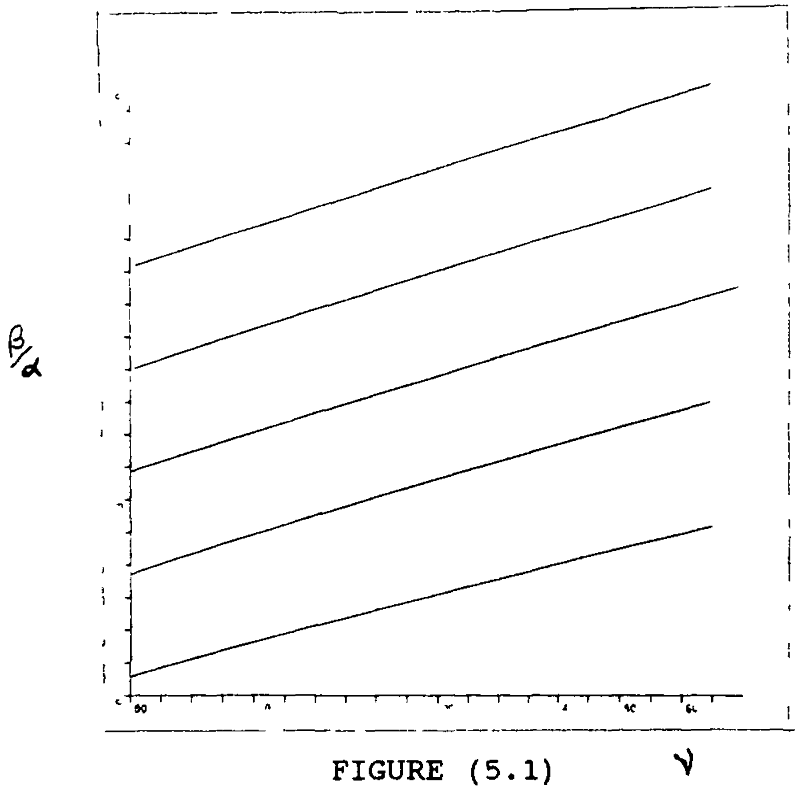


FIGURE (5.1)

Table (5.2) contains several of the eigenvalue solutions to equation (5.35) for their corresponding pulse parameters α and β The associated eigenfunctions are given in figures (5.3) and (5.4). The initial pulse for the parameters given in figure (5.3) is just large enough to support one eigenstate. The pulse for the parameters in figure (5.4) is large enough for two eigenvalues and the two associated eigenfunctions are shown.

α = 1.0 β = 2.043	α = 1.0 β = 2.514
η = 1.0	η = 1.4 η = 0.1

TABLE (5.2)

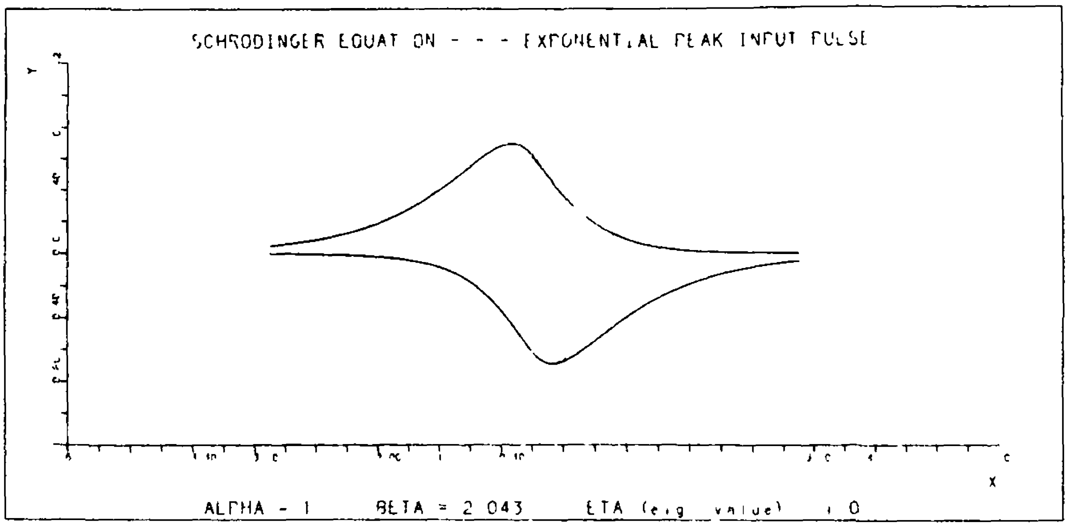
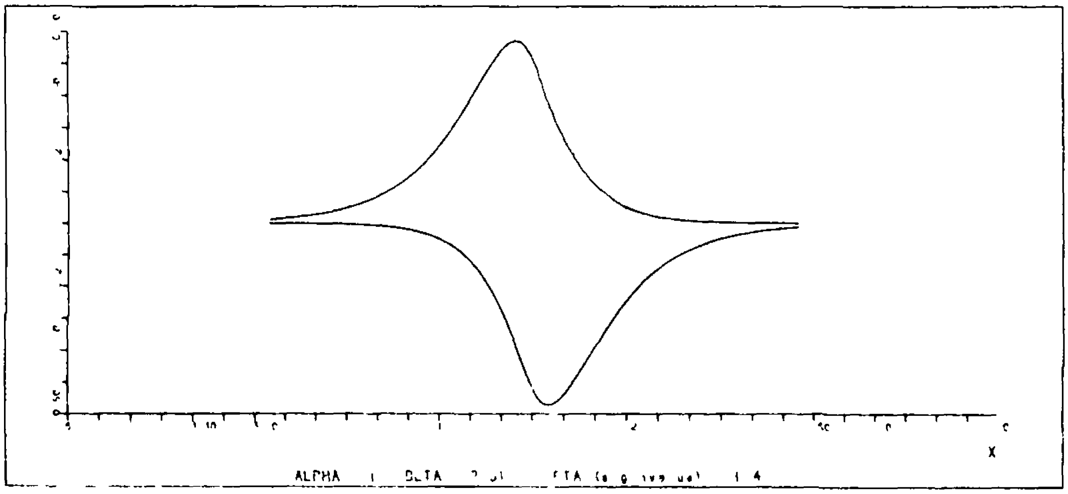


FIGURE (5.3) . Pulse parameters: $\alpha = 1, \beta = 2.043$.



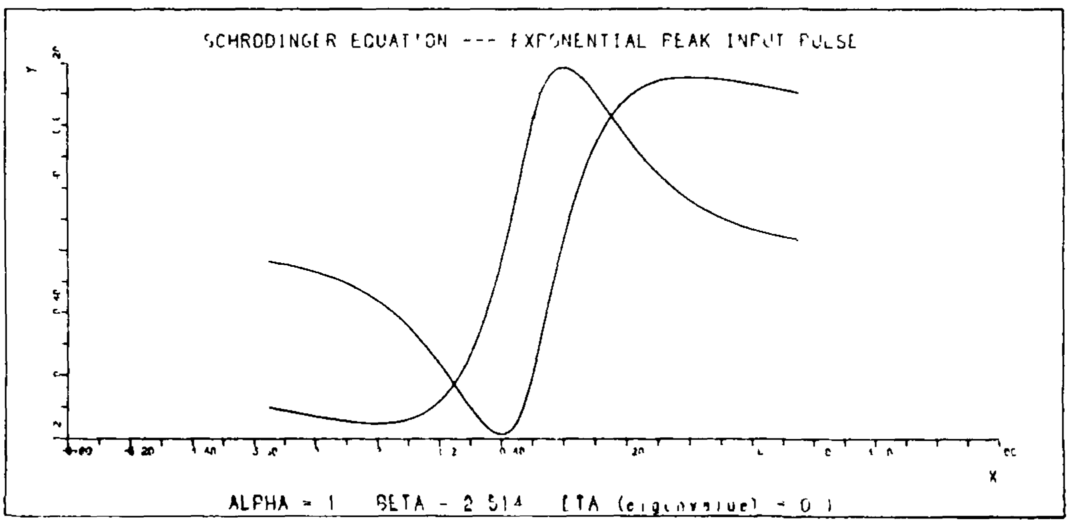


FIGURE (5.4) Pulse parameters: $\alpha = 1$, $\beta = 2.514$

If we define F as follows

$$F = \int_{-\infty}^{\infty} u(x,0)\, dx \tag{5.38}$$

then the number of solitons contained in a given pulse with parameters α and β is N where

$$N = \left\langle \frac{1}{2} + \frac{F}{\pi} \right\rangle \tag{5.39}$$

and $\langle .. \rangle$ denotes the integer less than the argument.

That this is so can most readily be seen if one considers the points of intersection of $J_{\nu+1}(\beta/\alpha)$ and $\pm J_\nu(\beta/\alpha)$ when $\nu=-1/2$. For this particular value of ν, $J_{\nu+1}(\beta/\alpha) = \sqrt{2/\pi}\,\sqrt{(\alpha/\beta)}.\sin(\beta/\alpha)$ and $J_\nu(\beta/\alpha) = \sqrt{2/\pi}\sqrt{(\alpha/\beta)}$ $.\cos(\beta/\alpha)$. The sine and +cosine functions intersect at intervals of 2π with the first intersection occurring at $(\beta/\alpha) = \pi/4$, while the sine and -cosine functions first intersect at the value $(\beta/\alpha) = 3\pi/4$. For any value of (β/α) that we take, the number of intersections of $J_{\nu+1}(\beta/\alpha)$ with $J_\nu(\beta/\alpha)$ up to this value of (β/α) is n, where n is the greatest integer such that

$$\left[\frac{(2n-1)}{4}\right]\pi \leq \beta/\alpha \tag{5.40}$$

For this value of (β/α), then, it is clear that

$$\frac{1}{2} + \frac{F}{\pi} = \frac{1}{2} + \frac{2\beta}{\alpha\pi} \geq n \tag{5.41}$$

Equation (5.39) then gives the number of intersections as N. This equation is the equation for the number of solitons contained in the initial pulse. It relates the soliton number to the area of the pulse and is the same result we obtained when we studied rectangular pulses and pulses of sech(x) form.

CHAPTER SIX

THE SUPER-GAUSSIAN PULSE

We want to solve the linear eigenvalue problem (2.1.a) for the special case

$$u(x,0) = A_0 \exp\left[-\frac{1}{2}(1-i\alpha)(x/\sigma)^{2m}\right] \qquad (6.1)$$

The pulse in eq.(6.1) represents a typical laser pulse and is for this reason a very important one. From eq.(6.1), we obtain

$$u' = -m(1-i\alpha)\frac{x^{2m-1}}{\sigma^{2m}} \cdot u \qquad (6.2)$$

Substituting (6.2) into the eigenvalue equation

$$v_1'' - \frac{u'}{u} v_1' + \left[\lambda^2 - i\lambda \frac{u'}{u} + |u|^2\right] v_1 = 0 \qquad (6.3)$$

we obtain

$$v_1'' + m(1-i\alpha)\frac{x^{2m-1}}{\sigma^{2m}} v_1'$$
$$+ \left[\lambda^2 + i\lambda m(1-i\alpha)\frac{x^{2m-1}}{\sigma^{2m}} + A_0^2 e^{-(x/\sigma)^{2m}}\right] v_1 = 0 \qquad (6.4)$$

First, we try to find a solution in terms of a series for $x < 0$, i.e.,

$$V_1 = e^{-i\lambda x} \sum_{n=0}^{\infty} U_n \; , \quad U_0 = 1 \; , \; x < 0 \qquad (6\ 5)$$

and using eq.(6.5) and eq.(6.3) we obtain

$$\sum_{n=0}^{\infty} \left[U_n'' + \left[-2i\lambda + m(1-i\alpha)\frac{x^{2m-1}}{\sigma^{2m}} \right] U_n' \right] = -A_0^2 \sum_{n=0}^{\infty} e^{-(x/\sigma)^{2m}} U_n \qquad (6.6)$$

If for convenience we let

$$\gamma(x) = -2i\lambda + m(1-i\alpha)\frac{x^{2m-1}}{\sigma^{2m}} \qquad (6\ 7)$$

then eq.(6.6) becomes

$$\sum_{n=1}^{\infty} U_n'' + \gamma(x)\sum_{n=1}^{\infty} U_n' = -A_0^2 e^{-(x/\sigma)^{2m}} \sum_{n=1}^{\infty} U_{n-1} \qquad (6.8)$$

Solving for U_n' we find

$$U_n'(x) = -A_0^2 e^{2i\lambda x' - \frac{1}{2}(1-i\alpha)(x/\sigma)^{2m}} \int_{-\infty}^{x} e^{-2i\lambda x' - \frac{1}{2}(1+i\alpha)(x'/\sigma)^{2m}} U_{n-1}(x')\,dx' \qquad (6.9)$$

whence

$$U_n(x) = -A_0^2 \int_{-\infty}^{x} e^{2i\lambda x' - \frac{1}{2}(1-i\alpha)(x'/\sigma)^{2m}} \int_{-\infty}^{x'} e^{-2i\lambda x'' - \frac{1}{2}(1+i\alpha)(x''/\sigma)^{2m}} U_{n-1}(x'')\,dx''\,dx' \qquad (6.10)$$

Having now found v1(x) in terms of an infinite series, it is easy to show that

$$V_2 = \frac{-i}{A_0} e^{-i\lambda x + \frac{1}{2}(1-i\alpha)(x/\sigma)^{2m}} \sum_{n=1}^{\infty} U_n' \qquad (6.11)$$

Equations (6.11) and (6.5) are the formal solutions to the eigenvalue problem (2.1.a). We must show convergence before we can proceed any further. To do this, we use the Weirstrasse M-test to show that the series of Un for Un given by eq.(6.10) is uniformly convergent, i.e., we must find constants Mn, n = 1,2,..,∞ such that

$$|U_n(x)| \leq M_n \tag{6.12}$$

with the constants Mn satisfying

$$\sum_{n=1}^{\infty} M_n < \infty \tag{6.13}$$

Equations (6.12) and (6.13) then imply uniform convergence. We find the constants Mn as follows: Using

$$U_n'(x) = \frac{A_0^2}{2i\lambda} e^{-\frac{1}{2}(1-i\alpha)(x/\sigma)^{2m}} \int_{-\infty}^{x} e^{-\frac{1}{2}(1+i\alpha)(x'/\sigma)^{2m}} U_{n-1}(x')\,dx' \tag{6.14}$$

we can write

$$|U_1'(x)| \leq \frac{A_0^2}{2|\lambda|} e^{-\frac{1}{2}(x/\sigma)^{2m}} \int_{-\infty}^{x} e^{-\frac{1}{2}(x'/\sigma)^{2m}} dx' \tag{6.15}$$

Choosing a suitable ε such that

$$\left|\frac{x'}{\varepsilon}\right| \geq \left|\frac{x}{\varepsilon}\right| \geq 1 \quad , \varepsilon > 0 \tag{6.16}$$

- the equation

$$|U_1'(x)| \leq \frac{A_0^2}{2|\lambda|} e^{-\frac{1}{2}(x/\sigma)^{2m}} \int_{-\infty}^{x} e^{-\frac{1}{2}(\varepsilon/\sigma)^{2m}|x'/\varepsilon|} dx' \tag{6.17}$$

follows for m $\geq$ 1/2. This then implies

$$|u_1| \leq \frac{A_0^2 \sigma^{2m}}{\varepsilon^{2m+1}|\lambda|} \int_{-\infty}^{x} e^{-\frac{1}{2}(x'/\sigma)^{2m} - \frac{1}{2}(\varepsilon/\sigma)^{2m}|x'/\varepsilon|} dx' \quad (6.18)$$

and therefore

$$|u_1| \leq \frac{A_0^2 \sigma^{4m}}{\varepsilon^{2(2m+1)}|\lambda|} e^{-(\varepsilon/\sigma)^{2m}|x/\varepsilon|} \quad (6.19)$$

For n = 2, we find a similar formula, and in general

$$u_n(x) \leq \frac{\sqrt{\pi} A_0^{2n} \sigma^{4nm} e^{-n(\varepsilon/\sigma)^{2m}|x/\varepsilon|}}{2^n n! \Gamma(\frac{2n-1}{2}) |\lambda|^n \varepsilon^{2n(2m-1)}} \quad (6.20)$$

holds for $x \in (-\infty, -\varepsilon)$, $\varepsilon > 0$. Using the Weirstrasse M-test it can be shown that the series (6.5) and (6.11) are convergent on this interval. Analagously, we can find v2 and v1 in the form of convergent series on (ε, ∞).

Letting the eigenvalue λ be of the general complex form -

$$\lambda = x + i\eta \quad (6.21)$$

we can now split up Un(x) in equation (6.14) into its real and imaginary parts. Combining these with (6.5) and (6.11), we obtain the real and imaginary parts of v1(x) and v2(x). These simple but tedious calculations yield the following:

$$Re\{V_1(x)\} = \frac{A_0^2 e^{\eta x}}{2(x^2+\eta^2)} \left[\begin{aligned} &(-\eta \cos(xx) - x \sin(xx))(I_1(x) + I_2(x)) \\ &+ (\eta \sin(xx) - x \cos(xx))(I_3 - I_4(x)) \end{aligned} \right] \quad (6.22)$$

$$\mathrm{Im}\{U_1(x)\} = \frac{-A_0^2 e^{\eta x}}{2(\varkappa^2+\eta^2)} \begin{bmatrix} \left(-\eta \sin(\varkappa x) - \varkappa \cos(\varkappa x)\right)\left(I_1(x) + I_2(x)\right) \\ +\left(\eta \cos(\varkappa x) - \varkappa \sin(\varkappa x)\right)\left(I_3(x) - I_4(x)\right) \end{bmatrix} \quad (6.23)$$

$$\mathrm{Re}\{U_2(x)\} = \frac{-A_0}{2(\varkappa^2+\eta^2)} e^{\eta x + k_2 (x/\sigma)^{2m}} \begin{bmatrix} \left(-\eta P_1(x) - \varkappa P_2(x)\right)\left(T_1(x) + T_2(x)\right) \\ +\left(\eta P_2(x) - \varkappa P_1(x)\right)\left(T_3(x) - T_4(x)\right) \end{bmatrix} \quad (6.24)$$

$$\mathrm{Im}\{U_2(x)\} = \frac{-A_0}{2(\varkappa^2+\eta^2)} e^{\eta x + k_2 (x/\sigma)^{2m}} \begin{bmatrix} \left(-\eta P_2(x) - \varkappa P_1(x)\right)\left(T_1(x) + T_2(x)\right) \\ +\left(\eta P_1(x) - \varkappa P_2(x)\right)\left(T_3(x) - T_4(x)\right) \end{bmatrix} \quad (6.25)$$

where

$$I_1(x) = \sum_{n=1}^{\infty} \int_{-\infty}^{x} e^{-\frac{1}{2}(x'/\sigma)^{2m}} \cos\left[(\alpha/2)(x'/\sigma)^{2m}\right] \int_{-\infty}^{x'} e^{-\frac{1}{2}(x''/\sigma)^{2m}} \cos\left[(\alpha/2)(x''/\sigma)^{2m}\right] u_{n-1}(x'')\,dx''\,dx' \quad (6.26.a)$$

$$I_2(x) = \sum_{n=1}^{\infty} \int_{-\infty}^{x} e^{-\frac{1}{2}(x'/\sigma)^{2m}} \sin\left[(\alpha/2)(x'/\sigma)^{2m}\right] \int_{-\infty}^{x'} e^{-\frac{1}{2}(x''/\sigma)^{2m}} \sin\left[(\alpha/2)(x''/\sigma)^{2m}\right] u_{n-1}(x'')\,dx''\,dx' \quad (6.26.b)$$

$$I_3(x) = \sum_{n=1}^{\infty} \int_{-\infty}^{x} e^{-\frac{1}{2}(x'/\sigma)^{2m}} \cos\left[(\alpha/2)(x'/\sigma)^{2m}\right] \int_{-\infty}^{x'} e^{-\frac{1}{2}(x''/\sigma)^{2m}} \sin\left[(\alpha/2)(x''/\sigma)^{2m}\right] u_{n-1}(x'')\,dx''\,dx' \quad (6.26.c)$$

$$I_4(x) = \sum_{n=1}^{\infty} \int_{-\infty}^{x} e^{-\frac{1}{2}(x'/\sigma)^{2m}} \sin\left[(\alpha/2)(x'/\sigma)^{2m}\right] \int_{-\infty}^{x'} e^{-\frac{1}{2}(x''/\sigma)^{2m}} \cos\left[(\alpha/2)(x''/\sigma)^{2m}\right] u_{n-1}(x'')\,dx''\,dx' \quad (6.26.d)$$

$$T_j(x) = \frac{d}{dx} I_j(x), \quad j = 1,\dots,4. \quad (6.27)$$

$$P_1(x) = \cos(\varkappa x) . \sin\left[(\alpha/2)(x/\sigma)^{2m}\right] + \sin(\varkappa x) . \cos\left[(\alpha/2)(x/\sigma)^{2m}\right] \quad (6.28.a)$$

$$P_2(x) = \cos(\varkappa x) . \cos\left[(\alpha/2)(x/\sigma)^{2m}\right] - \sin(\varkappa x) \sin\left[(\alpha/2)(x/\sigma)^{2m}\right] \quad (6.28.b)$$

What remains is to match these solutions for $x < 0$ and $x > 0$ and thus determine the eigenvalues This, of course, can only be done with the help of a computer and a VAX 11/785 mainframe was used.

In order to numerically calculate the solutions v1 and v2, several approximations must be found for the infinite series given by eqs. (6.5) and (6.11). In particular, the number of terms in the series given by (6.10) must be finite, for computational reasons, and the choice of the integration limits for the Un and the Un must be calculated.

The choice for N, the number of terms summed in the series of the U_n' must be such that the ratio of the sum of the number of terms discarded to the number of terms kept is small. In a similar manner, the choice of the lower limit of integration must be an accurate approximation to minus infinity. Choosing such a point x1 for minus infinity would mean , for each U, excluding an integration from minus infinity to x1 and only considering the integral from x1 to x'. The point x1 would have to be chosen in order to ensure that the sum of these N omitted integrals is negligible when compared to those N kept.

Appendix 1 contains working software code necessary for computing the real parts of v1 and v2 given by equations (6.22) and (6.24) for this problem. The programs for obtaining the imaginary parts, given by equations (6.23) and (6.25), can be readily obtained through small changes to the subroutines FCN and SCN A full description of the code is given in the appendix.

Much work remains to be done in order to solve completely the eigenvalue problem for this input pulse. In particular, the solution v must be matched at x = 0 for realistic values of the parameters A_0, σ, α and m. Matching the solutions like this will determine the eigenvalues λ. We hope that we have opened the way for obtaining the solution for this problem.

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APPENDIX

FORTRAN code for the evaluation of the solutions to the eigenvalue problem for the super-Gaussian pulse

```
c--------------------------------------------------------------
c
c       Author         .     Kevin Breen
c       File                 REV21 FOR
c       Date                 12-June-1988
c       Version        :     3 4
c       Computer             DEC VAX 11/785
c
c       Description          this program computes a numerical
c                            approximation to the following integral
c
```

$$\sum_{n=1}^{\infty}\int_{-\infty}^{x} e^{-\frac{1}{2}\left(\frac{x'}{\sigma}\right)^{2m}} F(x') \int_{-\infty}^{x'} e^{-\frac{1}{2}\left(\frac{x''}{\sigma}\right)^{2m}} G(x'')\, u_{n-1}(x'')\, dx''\, dx'$$

```
c
c       Auxillary
c       routines             the program calls the following NAG
c                            subroutines
c
c                            D01AJF
c                            E02ADF
c                            E02AEF
c
c                            and the following function subprograms
c
c                            TCN
c                            OFCN
c                            UFCN
c
c       User defined
c       routines             The user must define the following
c                            functions
c
c                            FCN
c                            SCN
c
c       References           NAG FORTRAN Library Manual
c                            Volumes 1 and 2
c
c                            Press,Flannery,Teukolsky,Vetterling
c                            Numerical Recipes
c                            Cambridge University Press
c
c                            Gerald, Curtis (1978)
c                            Applied Numerical Analysis
c                            Addison-Wesley Publishing Company
c
c--------------------------------------------------------------
```

```
c-------------------------------------------------------------------
c
         program integrate
c
c-------------------------------------------------------------------
        parameter ( liw=1001,nx=50,ilim=10 )
        implicit double precision (a-h,o-z)
        dimension x(nx),y(nx)
        dimension work1(liw),work2(liw)
        dimension iwork(liw),rms(nx)
        dimension weight(nx)
        dimension u(nx,ilim),v(nx,ilim)
        dimension ustore(nx),vstore(nx)
        dimension utemp10(nx),vtemp10(nx)
        dimension t1(nx),t2(nx),t3(nx),t4(nx),t5(nx),t6(nx)
        dimension t7(nx),factor(nx),temp5(nx)
        dimension temp(nx),temp2(nx),temp3(nx),temp4(nx)
        common /block01/ delta,m,a0,eta,alpha,tcheb(nx,nx)
        common /block02/ xmax,xmin,coeff(nx)
        external fcn,scn,tcn,ufcn,ofcn
c------------------------------------------------------------------
c
c       Open the data file and read the following variables
c
c       a                  lower integration limit
c       b          .       upper      "          "
c       delta      .       used in integrand
c       m                   "    "    "
c       a0                  "    "    "
c       eta        :       imaginary part of eigenvalue (lambda)
c       alpha      .       used in integrand
c       nprint     :       output unit number
c       akap               real part of eigenvalue (lambda)
c
c------------------------------------------------------------------
        open(unit=10,file='suzy.dat',status='old')
        read(10,10000)a
        read(10,10000)b
        read(10,10000)delta
        read(10,10010)m
        read(10,10000)a0
        read(10,10000)eta
        read(10,10000)alpha
        read(10,10010)nprint
        read(10,10000)akap
        close(unit=10,status='keep')
```

```
c-----------------------------------------------------------------
c
c       print file header in the output file
c
c-----------------------------------------------------------------
        write(nprint,20000)
        write(nprint,10030)a
        write(nprint,10031)delta
        write(nprint,10032)m
        write(nprint,10033)a0
        write(nprint,10034)eta
        write(nprint,10036)alpha
        write(nprint,20000)
c-----------------------------------------------------------------
c
c       compute the set of inner integration points
c
c-----------------------------------------------------------------
        delx=dabs( (b-a)/dfloat(nx-1) )
        do 205 k=1,nx
        x(k)=a+dfloat(k-1)*delx
205     continue
c-----------------------------------------------------------------
c
c       set tolerances for use in the NAG routines
c
c-----------------------------------------------------------------
        atol=0 1d-5
        rtol=0 1d-5
c-----------------------------------------------------------------
c
c      D01AJF calculate an approximation to the integral of a
c      function over a finite interval [a,b] , here we compute
c      the integral of the user supplied function FCN and store
c      the result in the y array
c
c-----------------------------------------------------------------
        do 20 k=1,nx
                ifail=0
                call d01ajf(fcn,a,x(k),atol,rtol,result,err,
     *          work1,liw,iwork,liw,ifail)
                if(ifail .ne. 0)stop
                y(k)=result
20      continue
        write(*,*) ' inner integral evaluated '
```

```
c-----------------------------------------------------------------
c
c       E02ADF computes weighted least squares polynomial
c       approximations to an arbitrary set of data points , we
c       now fit such a polynomial to the discrete set of points
c       generated by D01AJF
c
c-----------------------------------------------------------------
        xmin = x(1)
        xmax = x(nx)
        weight(1)=0 1d1
        do 21 k=2,nx
        weight(k)=dabs(1 0d0/y(k))
21      continue
        call e02adf(nx,nx,nx,x,y,weight,work1,work2,tcheb,rms,
     *              ifail)
        do 961 k=1,nx
        coeff(k)=tcheb(nx,k)
961     continue
        write(*,*) ' tchebyshev polynomial calculated '
c-----------------------------------------------------------------
c
c       The outer integral is now computed using D01AJF and we
c       store this approximation in u(k,1)
c
c-----------------------------------------------------------------
        do 725 k=1,nx
                v(k,1)=tcn(x(k))
                ifail=0
                call d01ajf(tcn,a,x(k),atol,rtol,result,err,
     *          work1,liw,iwork,liw,ifail)
                u(k,1)=result
                if(ifail  ne  0)stop
725     continue
        write(*,*) ' u(1) computed '
c-----------------------------------------------------------------
c
c       Compute the remaining sequence of integrals
c
c-----------------------------------------------------------------
        do 4000 iterm = 2,6
              write(*,*)' iterm = ',iterm
c-----------------------------------------------------------------
c
c       Fit Chebyshev polynomials to previously computed value
c
c-----------------------------------------------------------------
        do 221 k=2,nx
              weight(k)=1.0d0
221     continue
```

```
             do 233 k=1,nx
             ustore(k)=u(k,iterm-1)
233      continue
         call e02adf(nx,nx,nx,x,ustore,weight,work1,work2,tcheb,
     *                 rms,ifail)
         do 962 k=1,nx
             coeff(k)=tcheb(nx,k)
962      continue
c----------------------------------------------------------------
c
c        Now compute the inner integral for all x values
c
c----------------------------------------------------------------
             do 605 k=1,nx
               ifail=0
               call d01ajf(ufcn,a,x(k),atol,rtol,result,err,
     *         work1,liw,iwork,liw,ifail)
               if(ifail  ne  0)stop
               temp(k)=result
605          continue
c----------------------------------------------------------------
c
c        The inner integral has been computed and the result
c        stored in the temp array  The next loop computes the
c        approximations to the outer integrals by first fitting
c        a polynomial to the inner points and then calling the
c        D01AJF integration routine
c
c----------------------------------------------------------------
             call e02adf(nx,nx,nx,x,temp,weight,work1,work2,
     *                      tcheb,rms,ifail)
             do 963 k=1,nx
                   coeff(k)=tcheb(nx,k)
963          continue
             do 606 k=1,nx
               v(k,iterm)=ofcn(x(k))
               ifail=0
               call d01ajf(ofcn,a,x(k),atol,rtol,result,err,
     *         work1,liw,iwork,liw,ifail)
               if(ifail  ne. 0)stop
               u(k,iterm)=result
606          continue
4000     continue
c----------------------------------------------------------------
c
c        Open the output file and print results
c
c----------------------------------------------------------------
         open(unit=22,file='v21 dat',status='new')
         do 5000 k=1,nx
             sumu=0 0d0
             sumv=0.0d0
```

```
               do 6000 iterm=1,ilim
                    sumu=sumu+u(k,iterm)
                    sumv=sumv+v(k,iterm)
6000           continue
               ustore(k)=0 1d1+sumu
               vstore(k)=0 1d1+sumv
               utemp10(k)=ustore(k)*dcos(akap*x(k))*dexp(eta
     *                   *x(k))
               t1(k)=0 5d0*((x(k)/delta)**(2*m))
               t2(k)=alpha*t1(k)
               t3(k)=dexp(t1(k))
               t4(k)=dcos(akap*x(k))
               t5(k)=dsin(akap*x(k))
               t6(k)=dcos(t2(k))
               t7(k)=dsin(t2(k))
               factor(k)=-(1 0d0/a0)*dsin((x(k)*akap)+t2(k))
               vtemp10(k)=factor(k)*vstore(k)
               write(22,826)x(k),vtemp10(k)
               write(56,826)x(k),utemp10(k)
5000     continue
         close(unit=22,status='keep')
         write(nprint,20001)
         write(nprint,20000)
826      format(1h ,2d16.6)
300      format(1h ,d16 6)
900      format(1h ,2d16 6)
10010    format(i6)
10000    format(d16 6)
98000    format(3d16.6)
20000    format(1h,50h ----------------------------------------,
     *   20h--------------------)
20001    format(1h,50h ----------------------------------------,
     *   20h--------------------)
10020    format(1h ,2x,8hb      = ,d10 4,2x,
     *   8hans    = ,d16 6,2x,6herr = ,d16 6)
10030    format(1h ,2x,8ha      = ,d16 6)
10031    format(1h ,2x,8hdelta  = ,d16 6)
10032    format(1h ,2x,8hm      = ,i6)
10033    format(1h ,2x,8ha0     = ,d16 6)
10034    format(1h ,2x,8heta    = ,d16.6)
10035    format(1h ,2x,8han     = ,d16 6)
10036    format(1h ,2x,8halpha  = ,d16 6)
         end
```

```
c-----------------------------------------------------------------
        function fcn(x)
c-----------------------------------------------------------------
c
c       User supplied function to define the f(x) integrand
c
c-----------------------------------------------------------------
        implicit double precision (a-h,o-z)
        parameter ( liw=1001,nx=50,ilim=10 )
        common /block01/ delta,m,a0,eta,alpha,tcheb(nx,nx)
        term1=(x/delta)
        m2=2*m
        term2=-0.5d0*(term1**m2)
        term3=dexp(term2)
        term4=((alpha/2.0d0)*(term1**m2))
        term5=-eta*dcos(term4)
        term6=-akap*dsin(term4)
        fcn=(term3*(term5+term6))
        return
        end
c-----------------------------------------------------------------
        function scn(x)
c-----------------------------------------------------------------
c
c       User supplied function to define the g(x) integrand
c
c-----------------------------------------------------------------
        implicit real*8(a-h,o-z)
        parameter ( liw=1001,nx=50,ilim=10 )
        common /block01/ delta,m,a0,eta,alpha,tcheb(nx,nx)
        term11=x/delta
        m2=2*m
        term22=-0.5d0*(term11**m2)
        term33=dexp(term22)
        term44=((alpha/2.0d0)*(term11**m2))
        term55=dcos(term44)
        scn=(term33*term55)
        return
        end
```

```
c-----------------------------------------------------------------
      function tcn(x)
c-----------------------------------------------------------------
c
c     E02AEF evaluates a polynomial from it's Chebyshev
c     series representation
c
c-----------------------------------------------------------------
      implicit double precision (a-h,o-z)
      parameter ( liw=1001,nx=50,ilim=10 )
      common /block02/ xmax,xmin,coeff(nx)
      xbar=((x-xmin)-(xmax-x))/(xmax-xmin)
      ifail=0
      call e02aef(nx,coeff,xbar,poly,ifail)
      if(ifail ne 0)stop
      tcn=scn(x)*poly
      return
      end
c-----------------------------------------------------------------
      function ofcn(x)
c-----------------------------------------------------------------
      implicit double precision (a-h,o-z)
      parameter ( liw=1001,nx=50,ilim=10 )
      common /block02/ xmax,xmin,coeff(nx)
      xbar=((x-xmin)-(xmax-x))/(xmax-xmin)
      ifail=0
      call e02aef(nx,coeff,xbar,poly,ifail)
      if(ifail.ne.0)stop
      ofcn=scn(x)*poly
      return
      end
c-----------------------------------------------------------------
      function ufcn(x)
c-----------------------------------------------------------------
      implicit double precision (a-h,o-z)
      parameter ( liw=1001,nx=50,ilim=10 )
      common /block02/ xmax,xmin,coeff(nx)
      xbar=((x-xmin)-(xmax-x))/(xmax-xmin)
      ifail=0
      call e02aef(nx,coeff,xbar,poly,ifail)
      if(ifail ne.0)stop
      ufcn=fcn(x)*poly
      return
      end
```