# Spacelike Surfaces of Constant Mean Curvature having Contınuous Internal Symmetry <br> in Mınkowskı three space 

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I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Master of Science in Applied Mathematical Sciences is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work


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## Remark

I wish to thank the external examıner Prof D Simms for making following suggestions which lead to simplification and unification in the presentation of chapter two

As given on page 9 , embedd $\mathbb{R}^{3}{ }^{1}$ n $\mathbb{C}^{3}$ vis

$$
\dagger \mathbb{R}^{3} \rightarrow \mathbb{C}^{3} \quad x=\left(\begin{array}{c}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right) \mapsto x^{\dagger}=\left(\begin{array}{c}
x_{1} \\
x_{2} \\
\imath x_{3}
\end{array}\right) \text { then }
$$

$1\langle,\rangle_{1}=\left\langle x^{\dagger}, y^{\dagger}\right\rangle$
$2 x^{\dagger} \times y^{\dagger}=\imath(x * y)^{\dagger}$, where $\times$ is the usual crossproduct and $*$ is defined as on page 7
$3 \dagger$ leads to the embedding of $G L(3, \mathbb{R}) \hookrightarrow G L(3, \mathbb{C})$ by the commutative diagram

$$
\begin{array}{cc} 
& \mathbb{R}^{3} \xrightarrow{L} \mathbb{R}^{3} \\
\dagger \downarrow & \downarrow \dagger \\
\mathbb{C}^{3} \xrightarrow{L^{\dagger}} \mathbb{C}^{3}
\end{array}
$$

1 e $L^{\dagger} x^{\dagger}=(L x)^{\dagger}$ So if all linear maps are represented relatıve to the basıs

$$
e_{1}=\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right], e_{2}=\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right], e_{3}=\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right]
$$

(both in $\mathbb{R}^{3}$ and $\mathbb{C}^{3}$ ) then in matrix notation

$$
\left(\begin{array}{ccc}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array}\right)^{\dagger}=\left(\begin{array}{ccc}
a_{11} & a_{12} & -\imath a_{13} \\
a_{21} & a_{22} & -\imath a_{23} \\
\imath a_{31} & \imath a_{32} & a_{33}
\end{array}\right)
$$

as given on page 9
$4 S O(2,1)$ is then replaced by $S O(2,1)^{\dagger} \subset S O(3, \mathbb{C})$
5 If we now identıfy $\mathbb{R}^{3}, S O(2,1)$ etc with their embedded images, 1 e write $x$ to mean $x^{\dagger}, A$ to mean $A^{T}$ etc then the adjoint of $A$ is just $A^{\dagger}$ So the Lie algebra condition for $\mathcal{S O}(2,1)$ in $\mathcal{S O}(3, \mathbb{C})$ is just

$$
A+A^{T}=0
$$

from which lemma 24 follows immediately

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#### Abstract

The purpose of this thesis is to give a characterization of all spacelike constant mean curvature surfaces which have contınuous internal symmetry in Minkowskı three space The properties that these surfaces must satısfy leads to a system of partial differential equations and every solution of this system results in a desired surface Further examination of this system leads to a differential relation that the metric must satisfy The behaviour of the solution to this relation is investigated to determine if the resulting surface is complete


## Chapter 1

## Introduction

In 1841 Delaunay [2] charterized constant mean curvature surfaces of revolution (in Euclidean three space) as those whose profile curve is the roulette of a conic

## Theorem 1.1 Delaunay

A curve $\gamma$ in the $x-y$ plane generates a surface of constant mean curvature when rotated about the $x$-axis if and only of $\gamma$ is a prece of the roulette of a convc $\imath e$ the locus of the focus of a conic in this plane as it is rolled along the $x$-axis

These surfaces admit a one parameter group of internal isometries This was generalized by Smyth in [6] and the result was

## Theorem 12 Smyth

For each integer $m \geq 0$ there exists a one-parmeter family of conformal immersions

$$
f_{m} \quad \mathbb{C} \rightarrow \mathbb{R}^{3}
$$

with constant mean curvature 1 , such that the induced metric is complete and invariant by the group of rotations about 0 Moreover 0 is an umbilic of index -m/2, only powers of the rotation through $2 \pi /(m+2)$ about 0 extend to motions of $\mathbb{R}^{3}$ and the associates of $f_{m}$ are given by $\left(f_{m}\right)_{\theta}=f_{m} \circ e^{-2 \theta} \quad$ Conversly any complete surface of constant mean curvature 1 admitting a one parameter group of isometries us, to within assocıates, congruent exther to such an $f_{m}$ or to a Delaunay surface

In this classification it is assumed that $M$ is simply connected and it is shown that $(M, g)$ is conformally equivalent to the region in $\mathbb{C} a<\operatorname{Re}(z)<b$ where $a$ and $b$ are constant (either finite or infinite ) and the metric $g=e^{\phi}|d z|^{2}$ is invariant by translations in the $y$-directions An alternative characterızation to Smyth's was given by Burns and Clancy [1] whose result is as follows

Theorem,1.3 Burns and Clancy

If $M=\{z \in \mathbb{C} \quad a<$ Re $z<b\}$ and of $g=\lambda^{2}\left(d x^{2}+d y^{2}\right)$, then $f(M, g) \rightarrow \mathbb{R}^{3}$ us an isometric immersion of constant mean curvature $H$ if and only if $f$ satisfies the following system of $p d e$ 's

$$
\begin{aligned}
& f_{x x}=-\alpha f_{x}-(\mathbf{E}-\alpha H f) \times f_{y}+2 H f_{x} \times f_{y} \\
& \tilde{f}_{x y}=-\alpha f_{y}+(\mathbf{E}-\alpha H f) \times f_{x} \\
& f_{y y}=\alpha f_{x}+(\mathbf{E}-\alpha H f) \times f_{y}
\end{aligned}
$$

with inatıal conditions

$$
\left\|f_{x}\left(x_{0}, x_{0}\right)\right\|=\left\|f_{y}\left(x_{0}, x_{0}\right)\right\|=\lambda\left(x_{0}\right) \quad \text { and } \quad\left\langle f_{x}\left(x_{0}, y_{0}\right), f_{y}\left(x_{0}, y_{0}\right)\right\rangle=0
$$

where $\times$ denotes the usual cross product in $\mathbb{R}^{3}, \alpha$ is an arbitary (non-negative) real constant and $\mathbf{E}$ is an arbitary constant vector in $\mathbb{R}^{3}$

In this thesis we follow the arguments set out by Burns and Clancy to find a classification of spacelike constant mean curvature surfaces with continuous internal symmetry in Minkowskı three space We note that Minkowskı three space is just $\mathbb{R}^{3}$ with the scalar product, $\langle,\rangle_{1}$, between two vectors $x=\left(x_{1}, x_{2}, x_{3}\right)^{T}$ and $y=\left(y_{1}, y_{2}, y_{3}\right)^{T}$ being defined as

$$
\langle x, y\rangle_{1}=x_{1} y_{1}+x_{2} y_{2}-x_{3} y_{3}
$$

The mann theorems are as follows

## Theorem 1.4

Let $M=\left\{\mathbb{C} \mid x_{1}<\operatorname{Re} z<x_{2}\right\}$ where $x_{1}, x_{2}$ are constants (either finite or infinite) and $g=e^{\phi(x)}\left(d x^{2}+d y^{2}\right) \quad$ Then every immersıon $f_{1}(M, g) \rightarrow\left(\mathbb{R}^{3},\langle,\rangle_{1}\right)$ which satisfies either conditions 1 or 2 given below represents a spacelake surface in $\mathbb{R}^{3}$ with constant mean curvature $H \neq 0$ and the maps $\Psi_{t}(x, y) \rightarrow(x, y+t)$ form a one-parameter group of internal symmetries Conversely every spacelike surface in $\mathbb{R}^{3}$ with continuous internal symmetry and constant mean curvature $H \neq 0$ areses in thes way

1

$$
\begin{aligned}
f_{x y} & =f_{y}+H f * f_{x} \\
f_{y y} & =-f_{x}+H f * f_{y} \\
f_{x x}+f_{y y} & =2 H f_{x} * f_{y}
\end{aligned}
$$

2

$$
\begin{aligned}
f_{y} & =\mathbf{E} * f \\
\xi_{y} & =\mathbf{E} * \xi \\
f_{x x}+f_{y y} & =2 H f_{x} * f_{y}
\end{aligned}
$$

where $\mathbf{E} \in \mathbb{R}^{3}$

Some work has been covered with regard to the completeness of these surfaces but only partial results were found These can be seen in chapter 12

## Theorem 15

Every Spacelike Minimal Surface ( $\imath$ e spacelike surface of constant mean curvature equal to zero) with continuous internal symmetry in Minkowskı three space is, up to a hyperbolic motzon, one of the following

1

$$
f(x, y)=\frac{l r_{1}}{6}\left(x+r_{2}\right)\left(\begin{array}{c}
3 / r_{1}^{2}+\left(x+r_{2}\right)^{2}-3 y^{2}  \tag{11}\\
-6 / r y \\
-3 / r_{1}^{2}+\left(x+r_{2}\right)^{2}-3 y^{2}
\end{array}\right)
$$

where $l, r_{1}, r_{2} \in \mathbb{R}$ 2
where $r_{1}, r_{2} \in \mathbb{R}$
3

$$
f(x, y)=-r_{1}\left(\begin{array}{c}
\sinh \left(\epsilon x+r_{2}\right) \cos y  \tag{13}\\
-\sinh \left(\epsilon x+r_{2}\right) \sin y \\
-\left(\epsilon x+r_{2}\right)
\end{array}\right)
$$

where $r_{1}, r_{2} \in \mathbb{R}$

4

$$
f(x, y)=r_{1} e^{-\alpha x}\left(\begin{array}{c}
\frac{a_{1}}{r_{1} \epsilon} \cos \alpha y \\
\cos \alpha y \cosh \epsilon y \sin \left(\epsilon x+r_{2}\right)+\sin \alpha y \sinh \epsilon y \sin \left(\epsilon x-r_{2}\right) \\
\cos \alpha y \sinh \epsilon y \sin \left(\epsilon x+r_{2}\right)+\sin \alpha y \cosh \epsilon y \sin \left(\epsilon x-r_{2}\right)
\end{array}\right)
$$

where $a_{1}, r_{1}, r_{2} \in \mathbb{R}$
5

$$
f(x, y)=\frac{a_{1} r_{1}}{2} e^{-\alpha x}\left(\begin{array}{c}
{\left[\frac{1}{r_{1}^{2}}+\frac{1}{\alpha^{2}}+\left(x+r_{2}\right)^{2}-y^{2}\right] \cos \alpha y+2 y\left(x+r_{2}\right) \sin \alpha y} \\
\frac{2}{r_{1}}\left(-y \cos \alpha y+\left(x+r_{2}\right) \sin \alpha y\right) \\
{\left[\frac{-1}{r_{1}^{2}}+\frac{1}{\alpha^{2}}+\left(x+r_{2}\right)^{2}-y^{2}\right] \cos \alpha y+2 y\left(x+r_{2}\right) \sin \alpha y}
\end{array}\right)
$$

where $a_{1}, r_{1}, r_{2} \in \mathbb{R}$
6

$$
f(x, y)=\left(\begin{array}{c}
r_{1} \cos (-\alpha+\epsilon) y e^{(-\alpha+\epsilon) x}+r_{2} \cos (\alpha+\epsilon) y e^{(-\alpha-\epsilon) x} \\
-r_{1} \sin (-\alpha+\epsilon) y e^{(-\alpha+\epsilon) x}-r_{2} \sin (\alpha+\epsilon) y e^{(-\alpha-\epsilon) x} \\
-\frac{a_{1}}{\epsilon} e^{-\alpha x} \cos \alpha y
\end{array}\right)
$$

where $a_{1}, r_{1}, r_{2} \in \mathbb{R}$
7

$$
f(x, y)=\left(\begin{array}{c}
r_{2} e^{-2 \alpha x} \cos (2 \alpha y)-\frac{r_{1}}{2 \alpha} x \\
-r_{2} e^{-2 \alpha x} \sin (2 \alpha y)+\frac{r_{1} y}{2 \alpha} y \\
-\frac{a_{1}}{\alpha} e^{-\alpha x} \cos (\alpha y)
\end{array}\right)
$$

where $a_{1}, r_{1}, r_{2} \in \mathbb{R}$

## Chapter ${ }^{2}$

## The Lie Group SO $(2,1)$

We begin with some prelımınary remarks on indefinite scalar product spaces (see, for example Nomızu [5])

## Theorem 2.1

Suppose that $\langle,\rangle_{1}$ is a bulnnear form on a real vector space $\mathbf{V}$ of dimension $n$ Then there exists a basis $\left\{u_{1}, u_{2}, \quad, u_{n}\right\}$ of $\mathbf{V}$ such that

$$
\begin{aligned}
& 1\left\langle u_{\imath}, u_{\jmath}\right\rangle_{1}=0 \text { for } \imath \neq \jmath \\
& \text { 2 }\left\langle u_{\imath}, u_{\imath}\right\rangle_{1}=1 \text { for } 1 \leq \imath \leq p \\
& 3\left\langle u_{\imath}, u_{\imath}\right\rangle_{1}=-1 \text { for } p+1 \leq \imath \leq r \\
& 4\left\langle u_{\imath}, u_{\imath}\right\rangle_{1}=0 \text { for } r+1 \leq \imath \leq n
\end{aligned}
$$

The numbers $r$ and $p$ are determined solely by the bilnear form, $r$ is called the rank, $r-p$ is called the index, and the ordered pair $(p, r-p)$ is called the signature The theorem shows that any two spaces of the same dimension with bilinear forms of the same signature are isometrically isomorphic By a scalar product we mean a nondegenerate bilinear form, 1 e , a form with rank equal to the dimension of $\mathbf{V}$ We let $U^{\perp}$ denote the orthogonal complement of a subspace $U$ with respect to the given scalar product

## Theorem 2.2

Suppose that $\langle,\rangle_{1}$ is a scalar product on a finite dimensional real vector space $V$ and suppose that $U$ is a subspace of $V$, then

$$
1\left(U^{\perp}\right)^{\perp}=U \text { and } \operatorname{dım} U+\operatorname{dım} U^{\perp}=\operatorname{dım} V
$$

2 The form $\langle,\rangle_{1}$ is nondegenerate on $U$ iff it is nondegenerate on $U^{\perp}$ and when it is nondegenerate on $U$, then $V=U \oplus U^{\perp}$ (the direct sum of $U$ and $U^{\perp}$ )

3 If $V$ is the orthogonal direct sum of two subspaces $U$ and $W$, then the form is nondegenerate on $U$ and $W$, and $W=U^{\perp}$

Let $\langle,\rangle_{1}$ be the indefinte scalar product on $\mathbb{R}^{n}$ defined by

$$
\langle x, y\rangle_{1}=x_{1} y_{1}+x_{2} y_{2}+\quad+x_{n-1} y_{n-1}-x_{n} y_{n}
$$

where $x=\left(x_{1}, x_{2}, \quad, x_{n}\right)^{T} \in \mathbb{R}^{n}$ and $y=\left(y_{1}, y_{2}, \quad, y_{n}\right)^{T} \in \mathbb{R}^{n}$ We call this space Minkowskı $n$-space and the scalar product $\langle,\rangle_{1}$ shall be called the Mınkowskı metric A vector $x$ is sand to be spacelike, timelıke, or lightlike depending on whether $\langle x, x\rangle_{1}$ 1s positıve, negitıve or zero, respectıvely In Mınkowskı n-space the set of all lightlıke vectors, given by the equation

$$
x_{1}^{2}+x_{2}^{2}+\quad+x_{n-1}^{2}=x_{n}^{2},
$$

forms a cone of revolution, called the light cone Timelike vectors are "inside the cone" and spacelike vectors are "outside the cone"

If $x$ is a non-zero vector, let $x^{\perp}$ denote the orthogonal complement of $x$ with respect to the Minkowskı metric If $x$ is timelike, then the metric restricts to a positive definite form on $x^{\perp}$, and $x^{\perp}$ intersects the light cone only at the origin If $x$ is spacelıke, then the metric has signature $(n-1,1)$ on $x^{\perp}$, and $x^{\perp}$ intersects the cone in a cone of one dimension less If $x$ is lightlike, then $x^{\perp}$ is tangent to the cone along the line through the origin determined by $x$ The metric has signature $(n-1,0)$ on this $n$-1-dimensional plane

Now, for all $x, y \in \mathbb{R}^{3}$ we define

$$
x * y=\left(\begin{array}{cc}
x_{3} y_{2}-x_{2} y_{3} \\
x_{1} y_{3} & -x_{3} y_{1} \\
x_{1} y_{2} & -x_{2} y_{1}
\end{array}\right)
$$

## Remark

$x * y$ is just the usual cross product $x \times y$ with the first two components negated It can easily be verified that the following conditons hold

- $x * y=-y * x$
- $x *(y+z)=x * y+x * z$
- $(x+y) * z=x * z+y * z$
- For every $r \in \mathbb{R}, r x * y=(r x) * y=x *(r y)$
- $x *(y * z)+y *(z * x)+z *(x * y)=0$

So that $\left(\mathbb{R}^{3}, *\right)$ is a Lie Algebra with bracket product

$$
[x, y]=x * y
$$

As usual for all $x \in \mathbb{R}^{3}$ we define $a d x \in \mathcal{E} n d\left(\mathbb{R}^{3}\right)$, the endomorphısms of $\mathbb{R}^{3}$, by

$$
(a d x) y=[x, y]
$$

Then the matrix representation of $a d x$ relative to the standard basis for $\mathbb{R}^{3}$ is

$$
a d x=\left(\begin{array}{ccc}
0 & x_{3} & -x_{2} \\
-x_{3} & 0 & x_{1} \\
-x_{2} & x_{1} & 0
\end{array}\right)
$$

## Orthogonal Groups

## Definition :

$$
\begin{aligned}
O(2,1) & =\left\{\text { linear } \Theta \mathbb{R}^{3} \rightarrow \mathbb{R}^{3} \mid\langle\Theta x, \Theta y\rangle_{1}=\langle x, y\rangle_{1} \quad \forall x, y \in \mathbb{R}^{3}\right\} \\
S O(2,1) & =\text { connected component to the identity of } O(2,1) \\
& =\left\{\Theta \in O(2,1) \mid \operatorname{det} \Theta=1 \text { and } \operatorname{s\imath gn}\left\langle e_{3}, \Theta e_{3}\right\rangle_{1}=-1\right\}
\end{aligned}
$$

These are Lie Groups in the usual way The Lie Algebra of $S O(2,1)$ is given by

$$
\mathcal{S O}(2,1)=\left\{3 \times 3 \text { matrices } A \mid e^{A} \in S O(2,1)\right\}
$$

where $e^{A}$ is the usual exponential of the matrix $A$ To obtain a more explicit description of $\mathcal{S O}(2,1)$ observe that $A \in \mathcal{S O}(2,1)$

$$
\begin{align*}
\Rightarrow \quad\left\langle e^{t A} x, e^{t A} y\right\rangle_{1} & =\langle x, y\rangle_{1} \quad \forall x, y \in \mathbb{R}^{3}, t \in \mathbf{R} \\
\left.\frac{d}{d t}\left\langle e^{t A} x, e^{t A} y\right\rangle_{1}\right|_{t=0} & =0 \\
\langle A x, y\rangle_{1}+\langle x, A y\rangle_{1} & =0 \tag{21}
\end{align*}
$$

That is, $A$ is skew-symmetric relative to $\langle,\rangle_{1}$ Thus, it is clear that $A$ being skewsymmetric is a necessary and sufficient conditon for $A \in \mathcal{S O}(2,1)$

For any $x=\left(x_{1}, x_{2}, x_{3}\right)^{T} \in \mathbb{R}^{3}$, define $x^{\dagger}=\left(x_{1}, x_{2}, 2 x_{3}\right)^{T} \in \mathbb{C}^{3}$ where $\imath=\sqrt{-1}$ and $T$ denotes transpose, then

$$
\langle x, y\rangle_{1}=x_{1} y_{1}+x_{2} y_{2}-x_{3} y_{3}=\left(x^{\dagger}\right)^{T} y^{\dagger}
$$

For

$$
A=\left(\begin{array}{lll}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array}\right)
$$

define

$$
A^{\dagger}=\left(\begin{array}{ccc}
a_{11} & a_{12} & -\imath a_{13} \\
a_{21} & a_{22} & -\imath a_{23} \\
\imath a_{31} & \imath a_{32} & a_{33}
\end{array}\right),
$$

then

$$
A^{\dagger} x^{\dagger}=(A x)^{\dagger}
$$

and

$$
\left(A^{\dagger}\right)^{T}=\left(\left(A^{\dagger \dagger}\right)^{T}\right)^{\dagger}
$$

At this stage it is worth alerting the reader to the remark made on page 11

## Lemma 23

If $A$ is a $3 \times 3$ matrix and $x, y \in \mathbb{R}^{3}$, then

$$
\langle A x, y\rangle_{1}=\left\langle x,\left(A^{\dagger \dagger}\right)^{T} y\right\rangle_{1}
$$

## Proof :

$$
\begin{aligned}
\langle A x, y\rangle_{1} & =\left((A x)^{\dagger}\right)^{T} y^{\dagger}=\left(A^{\dagger} x^{\dagger}\right)^{T} y^{\dagger} \\
& =\left(x^{\dagger}\right)^{T}\left(A^{\dagger}\right)^{T} y^{\dagger}=\left(x^{\dagger}\right)^{T}\left(A^{\dagger}\right)^{T} y^{\dagger} \\
& =\left(x^{\dagger}\right)^{T}\left(\left(A^{\dagger \dagger}\right)^{T}\right)^{\dagger} y^{\dagger}=\left(x^{\dagger}\right)^{T}\left(\left(A^{\dagger \dagger}\right)^{T} y\right)^{\dagger} \\
& =\left\langle x,\left(A^{\dagger \dagger}\right)^{T} y\right\rangle_{1}
\end{aligned}
$$

## Lemma 2.4

$\mathcal{S O}(2,1)$ is the set of all $3 \times 3$ real matrices $A$ of the form

$$
A=\left(\begin{array}{ccc}
0 & a_{3} & -a_{2} \\
-a_{3} & 0 & a_{1} \\
-a_{2} & a_{1} & 0
\end{array}\right)
$$

## Proof .

Using Lemma 23 and (21) we see that of $A \in \mathcal{S O}(2,1)$ then

$$
\begin{aligned}
& A=-\left(A^{\dagger \dagger}\right)^{T} \\
&\left(\begin{array}{lll}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array}\right)=\left(\begin{array}{rrr}
-a_{11} & -a_{21} & a_{31} \\
-a_{12} & -a_{22} & a_{32} \\
a_{13} & a_{23} & -a_{33}
\end{array}\right)
\end{aligned}
$$

and hence we have that

$$
a_{11}=a_{22}=a_{33}=0
$$

and

$$
a_{21}=-a_{12}, \quad a_{31}=a_{13}, \quad a_{32}=a_{23}
$$

Thus of $A \in \mathcal{S O}(2,1)$

$$
A=\left(\begin{array}{ccc}
0 & a_{3} & -a_{2} \\
-a_{3} & 0 & a_{1} \\
-a_{2} & a_{1} & 0
\end{array}\right) \text { for some } a=\left(\begin{array}{c}
a_{1} \\
a_{2} \\
a_{3}
\end{array}\right) \in \mathbb{R}^{3}
$$

which is the matrix representation of $a d a$, for some $a$ relative to the standard basis for $\mathbb{R}^{3}$ Thus $a d \mathbb{R}^{3} \rightarrow \mathcal{S O}(2,1) \quad x \rightarrow a d x$ is a Lie Algebra isomorphısm We note that

$$
\langle(a d n) x, y\rangle_{1}=-\langle x,(a d n) y\rangle_{1}
$$

and

$$
\begin{aligned}
\langle x * y, z\rangle_{1} & =\langle-y * x, z\rangle_{1}=\langle-(a d y) x, z\rangle_{1} \\
& =\langle x,(a d y) z\rangle_{1}=\langle x, y * z\rangle_{1}
\end{aligned}
$$

## Remark

Both $x$ and $y$ are orthogonal to $x * y$, because

$$
\langle x * y, y\rangle_{1}=\langle x, y * y\rangle_{1}=0
$$

and sımılarly

$$
\langle x * y, x\rangle_{1}=0
$$

## Lemma 2.5

For $x, y, z \in \mathbb{R}^{3}$

$$
x *(y * z)=\langle x, y\rangle_{1} z-\langle x, z\rangle_{1} y
$$

## Proof :

If $y$ and $z$ are linearly dependent then the result is trivial as both sides are zero We now fix $y$ and $z$ and assume they are linearly independent The vector $x *(y * z)$ is orthogonal to $y * z$ and therefore lies in the plane spanned by $y$ and $z$, accordingly,

$$
\begin{equation*}
x *(y * z)=\alpha_{y z}(x) y+\beta_{y z}(x) z \tag{22}
\end{equation*}
$$

for some $\alpha_{y z}(x), \beta_{y z}(x) \in \mathbf{R} \quad$ Also the map $\mathbb{R}^{3} \rightarrow \mathbb{R}^{3} \quad x \rightarrow x *(y * z)$ is linear, hence the maps

$$
\begin{aligned}
& \mathbb{R}^{3} \rightarrow \mathbf{R} \quad x \rightarrow \alpha_{y z}(x) \\
& \mathbb{R}^{3} \rightarrow \mathbf{R} \quad x \rightarrow \beta_{y z}(x)
\end{aligned}
$$

are linear, so there exists an $\mathrm{A}, \mathrm{B} \in \mathbb{R}^{3}$, which depend on $y$ and $z$ but not $x$, such that

$$
\alpha_{y z}(x)=\langle A, x\rangle_{1} \text { and } \beta_{y z}(x)=\langle B, x\rangle_{1}
$$

and so

$$
\begin{equation*}
x *(y * z)=\langle A, x\rangle_{1} y+\langle B, x\rangle_{1} z \forall x \in \mathbb{R}^{3} \tag{23}
\end{equation*}
$$

Setting $x=y * z$ in (23), we obtaın

$$
0=(y * z) *(y * z)=\langle A, y * z\rangle_{1} y+\langle B, y * z\rangle_{1} z
$$

Hence $\langle A, y * z\rangle_{1}=\langle B, y * z\rangle_{1}=0$ and we can write

$$
\begin{aligned}
& A=a_{1} y+a_{2} z \\
& B=b_{1} y+b_{2} z
\end{aligned}
$$

for some $a_{\imath}, b_{2} \in \mathbf{R}, 1 \leq \imath \leq 2$ which may depend on $y$ and $z$ Substıtution into (23) gives

$$
\begin{equation*}
x *(y * z)=\left\langle a_{1} y+a_{2} z, x\right\rangle_{1} y+\left\langle b_{1} y+b_{2} z, x\right\rangle_{1} z \tag{24}
\end{equation*}
$$

and it follows that

$$
\begin{equation*}
0=\langle x *(y * z), x\rangle_{1}=\left\langle a_{1} y+a_{2} z, x\right\rangle_{1}\langle y, x\rangle_{1}+\left\langle b_{1} y+b_{2} z, x\right\rangle_{1}\langle z, x\rangle_{1} \tag{25}
\end{equation*}
$$

now choose $x \perp y, x \nsucceq z$ in (25), where $\perp$ means "orthogonal to", then

$$
\left\langle b_{1} y+b_{2} z, x\right\rangle_{1}=0 \Rightarrow b_{2}\langle z, x\rangle_{1}=0 \Rightarrow b_{2}=0
$$

choose $x \not \nvdash y, x \perp z$, in (2 5), then

$$
\begin{align*}
& \left\langle a_{1} y+a_{2} z, x\right\rangle_{1}=0 \Rightarrow a_{1}\langle y, x\rangle_{1}=0 \Rightarrow a_{1}=0 \\
& \Rightarrow \quad 0=\left(a_{2}+b_{1}\right)\langle z, x\rangle_{1}\langle y, x\rangle_{1} \forall x \in \mathbb{R}^{3} \tag{26}
\end{align*}
$$

If we now choose an $x$ which is $\perp$ to netther y nor $z$ in (26), we obtain

$$
0=\left(a_{2}+b_{1}\right) \Rightarrow b_{1}=-a_{2}
$$

and substituting this result into (24) we obtian

$$
\begin{equation*}
x *(y * z)=a_{2}\left[\langle z, x\rangle_{1} y-\langle y, x\rangle_{1} z\right] \forall x \in \mathbb{R}^{3} \tag{27}
\end{equation*}
$$

Now let $x=z$ and substitute this into (27)

$$
\begin{aligned}
z *(y * z) & =a_{2}\left[\|z\|^{2} y-\langle y, z\rangle_{1} z\right] \\
\langle y, z *(y * z)\rangle_{1} & =a_{2}\left[\|z\|^{2}\|y\|^{2}-\langle y, z\rangle_{1}^{2}\right] \\
\langle(y * z),(y * z)\rangle_{1} & =-a_{2}\left[\langle y, z\rangle_{1}^{2}-\|z\|^{2}\|y\|^{2}\right]
\end{aligned}
$$

finally, expanding componentwise we see

$$
\begin{equation*}
\langle(y * z),(y * z)\rangle_{1}=\|(y * z)\|^{2}=\langle y, z\rangle_{1}^{2}-\|z\|^{2}\|y\|^{2} \tag{28}
\end{equation*}
$$

hence $a_{2}=-1$ and (27) gives the required result

## Lemma 2.6

If the spacellke vectors $x, y \in \mathbb{R}^{3}$ satisfy

$$
\langle x, y\rangle_{1}=0 \quad \text { and } \quad\langle x, x\rangle_{1}=\langle y, y\rangle_{1}=e^{\phi} \quad, \text { for some } \phi \in \mathbb{R}
$$

and $\imath f$ we define $\xi \in \mathbb{R}^{3}$ by

$$
\xi=\frac{1}{e^{\phi}} x * y=e^{-\phi}\left(\begin{array}{lll}
x_{3} y_{2} & -x_{2} y_{3} \\
x_{1} y_{3} & -x_{3} y_{1} \\
x_{1} y_{2} & -x_{2} y_{1}
\end{array}\right)
$$

then the following statements hold
$1\langle\xi, \xi\rangle_{1}=-1$
2 the matrix $A=\left[\begin{array}{lll}e^{-\phi / 2} x & e^{-\phi / 2} y & \xi\end{array}\right] \in O(2,1)$
$3 \xi * x=-y$ and $\xi * y=x$

## Proof -

1

$$
\begin{aligned}
\langle\xi, \xi\rangle_{1} & =e^{-2 \phi}\langle x * y, x * y\rangle_{1} \\
& =e^{-2 \phi}\left(\langle x, y\rangle_{1}^{2}-\|x\|^{2}\|y\|^{2}\right) \quad \text { by }(28) \\
& =-e^{-2 \phi} e^{\phi} e^{\phi} \\
& =-1
\end{aligned}
$$

2 Since

$$
A=e^{-\phi / 2}\left(\begin{array}{ccc}
x_{1} & y_{1} & e^{\phi / 2} \xi_{1} \\
x_{2} & y_{2} & e^{\phi / 2} \xi_{2} \\
x_{3} & y_{3} & e^{\phi / 2} \xi_{3}
\end{array}\right)
$$

we have

$$
\left(A^{\dagger \dagger}\right)^{T}=e^{-\phi / 2}\left(\begin{array}{ccc}
x_{1} & x_{2} & -x_{3} \\
y_{1} & y_{2} & -y_{3} \\
-e^{\phi / 2} \xi_{1} & -e^{\phi / 2} \xi_{2} & e^{\phi / 2} \xi_{3}
\end{array}\right)
$$

and hence

$$
\begin{aligned}
\left(A^{\dagger \dagger}\right)^{T} A & =e^{-\phi}\left(\begin{array}{ccc}
x_{1} & x_{2} & -x_{3} \\
y_{1} & y_{2} & -y_{3} \\
-e^{\phi / 2} \xi_{1} & -e^{\phi / 2} \xi_{2} & e^{\phi / 2} \xi_{3}
\end{array}\right)\left(\begin{array}{lll}
x_{1} & y_{1} & e^{\phi / 2} \xi_{1} \\
x_{2} & y_{2} & e^{\phi / 2} \xi_{2} \\
x_{3} & y_{3} & e^{\phi / 2} \xi_{3}
\end{array}\right) \\
& =\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right) \\
& =
\end{aligned}
$$

For any $u, v \in \mathbb{R}^{3}$

$$
\langle A u, A v\rangle_{1}=\left\langle u,\left(A^{\dagger \dagger}\right)^{T} A v\right\rangle_{1}=\langle u, v\rangle_{\mathbf{1}}
$$

and hence $A \in O(2,1)$

$$
\text { Note } \quad \operatorname{det}(A)=-e^{-\phi}\langle x * y, \xi\rangle_{1}=-\langle\xi, \xi\rangle_{1}=1
$$

Furthermore

$$
\left\langle A e_{3}, A e_{3}\right\rangle_{1}=\left\langle e_{3}, e_{3}\right\rangle_{1}=-1
$$

so that the $x_{3}$-component of $A e_{3}$ cannot be zero Hence $\left\langle e_{3}, A e_{3}\right\rangle_{1} \neq 0$ Now

$$
\operatorname{s\imath gn}\left\langle e_{3}, A e_{3}\right\rangle_{1}=\operatorname{s\imath gn}\left(\left\langle e_{3}, \xi\right\rangle_{1}\right)=\operatorname{s\imath gn}\left(-\xi_{3}\right)=-\operatorname{s\imath gn}\left(\xi_{3}\right)
$$

Thus $A$ is in the connected component of the identity (or equivalently $A \in$ $S O(2,1)$ ) providing $\xi_{3}>0$ Therefore either

$$
\left[\begin{array}{lll}
e^{-\phi / 2} x & e^{-\phi / 2} y & x * y
\end{array}\right] \in S O(2,1)
$$

or

$$
\left[\begin{array}{lll}
e^{-\phi / 2} y & e^{-\phi / 2} x & y * x
\end{array}\right] \in S O(2,1)
$$

3

$$
\begin{aligned}
\xi * x & =e^{-\phi}(x * y) * x \\
& =-e^{-\phi} x *(x * y) \\
& =-e^{-\phi}\left(\langle x, x\rangle_{1} y-\langle x, y\rangle_{1} x\right) \\
& =-e^{-\phi}\left(e^{\phi} y\right) \quad\left(a s\langle x, y\rangle_{1}=0\right) \\
& =-y
\end{aligned}
$$

Similảrly,

$$
\begin{aligned}
\xi * y & =e^{-\phi}(x * y) * y \\
& =-e^{-\phi} y *(x * y) \\
& =-e^{-\phi}\left(\langle y, x\rangle_{1} y-\langle y, y\rangle_{1} x\right) \\
& =-e^{-\phi}\left(-e^{\phi} x\right) \quad\left(a s\langle x, y\rangle_{1}=0\right) \\
& =x
\end{aligned}
$$

## Lemma 27

Let $\Theta(t) \in S O(2,1)$ for all $t \in \mathbb{R}$ Then

$$
\Theta\left(\Theta^{\dagger \dagger}\right)^{T}=a d \eta(t)
$$

where $\eta(t) \in \mathbb{R}^{3}$ for all $t \in \mathbb{R}$

## Proof .

We need to show that

$$
\left(\left\{\Theta\left(\Theta^{\dagger \dagger}\right)^{T}\right\}^{\dagger \dagger}\right)^{T}=-\Theta\left(\Theta^{\dagger \dagger}\right)^{T}
$$

We first note that for all $3 \times 3$ matrices $X, Y$ we have

$$
(X Y)^{\dagger}=X^{\dagger} Y^{\dagger}
$$

and

$$
\left(X^{T}\right)^{\dagger \dagger}=\left(X^{\dagger \dagger}\right)^{T}
$$

so that

$$
\begin{aligned}
{\left[\left\{\Theta\left(\Theta^{\dagger \dagger}\right)^{T}\right\}^{\dagger \dagger}\right]^{T} } & =\left[\Theta^{\dagger \dagger}\left(\left(\Theta^{\dagger \dagger}\right)^{T}\right)^{\dagger \dagger}\right]^{T} \\
& =\left[\Theta^{\dagger \dagger}\left(\left(\Theta^{\dagger \dagger}\right)^{\dagger \dagger}\right)^{T}\right]^{T} \\
& =\left[\Theta^{\dagger \dagger} \Theta^{T}\right]^{T} \\
& =\Theta\left(\Theta^{\dagger \dagger}\right)^{T}
\end{aligned}
$$

As $\Theta \in S O(2,1)$ we have by lemma 23 that $\Theta\left(\Theta^{\dagger \dagger}\right)^{T}=I$ so that $\Theta\left(\Theta^{\dagger \dagger}\right)^{T}+$ $\Theta\left(\Theta^{\mathrm{tt}}\right)^{T}=01 \mathrm{e}$

$$
\Theta\left(\Theta^{\dagger \dagger}\right)^{T}=-\Theta\left(\Theta^{\dagger \dagger}\right)^{T}
$$

Hence

$$
\left(\left\{\Theta\left(\Theta^{\dagger \dagger}\right)^{T}\right\}^{\dagger \dagger}\right)^{T}=-\Theta\left(\Theta^{\dagger \dagger}\right)^{T}
$$

and the lemma is proved

## Chapter 3

## The Gauss-Weingarten equations in Minkowski space

As we are isometrically mapping a two dımensional manıfold into Mınkowskı three space as opposed to Euclidean three space we find that there are some subtle changes in the preliminary stages

Let $M$ denote a simply connected oriented 2 dimensional manıfold and let $f \quad M \rightarrow$ $\mathbb{R}^{3}$ be a space-like immersion, that 1 for all $p \in M$ and $X_{p} \in T_{p} M, \operatorname{ker}\left(f_{*}\right)_{p}=\{0\}$ and $\left(f_{*}\right)_{p} X$ is space-like (where subscript * means the derivative) Thus $f$ induces a Riemannian metric $g$ on $M$, the pull back $g=(f)^{*}\left(\langle,\rangle_{1}\right)$, of the scalar product $\langle,\rangle_{1}$ in Mınkowskı three space, that is $g(X, Y)=\left\langle f_{*} X, f_{*} Y\right\rangle$ Hence $M$ is a Riemannian manifold and $f(M, g) \rightarrow\left(\mathbb{R}^{3},\langle,\rangle_{1}\right)$ is an isometric immersion We note that $g$ is positive definite since $f$ is spacelike

There exists local coordınates ( $x, y$ ) on $M$ called isothermal coordınates which satısfy the conditions

$$
g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right)=g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right)>0 \quad \text { and } g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)=0
$$

With isothermal coordinates $(x, y)$ defined in some neighbourhood of $p \in M$ we may write $g=e^{\phi(x, y)}\left(d x^{2}+d y^{2}\right)$ for some positive function $e^{\phi}$ For each $p \in M$ there is a local mappıng $\xi \quad M \rightarrow \mathbb{R}^{3}$, defined by

$$
\xi(p)=e^{-\phi(p)} f_{x}(p) * f_{y}(p)
$$

that satisfies the following equations

$$
\left\langle f_{x}, \xi\right\rangle_{1}=\left\langle f_{y}, \xi\right\rangle_{1}=0
$$

$$
\operatorname{det}\left[f_{x}, f_{y}, \xi\right]>0
$$

and

$$
\langle\xi, \xi\rangle_{1}=-1
$$

where $f_{x}, f_{y}$ represent the partial derivative of $f$ with respect to $x$ and $y$ respectively As defined $\xi$ is called the gauss map

For all smooth vector fields $Y$ on $M$ and $X_{p} \in T_{p} M$ we define

$$
\begin{gather*}
\nabla_{X_{p}} Y \in T_{p} M \quad \text { and } \quad I I_{p}\left(X_{p}, Y\right) \in \mathbb{R}^{3} \text { by } \\
X_{p}(Y f)=\left(f_{*}\right)_{p}\left(\nabla_{X_{p}} Y\right)+I I_{p}\left(X_{p}, Y\right) \xi \tag{31}
\end{gather*}
$$

Suppose X and Y are smooth vector fields on $M$ then

$$
\begin{gathered}
X_{p}(Y f)=\left(f_{*}\right)_{p}\left(\nabla_{X_{p}} Y\right)+I I_{p}\left(X_{p}, Y\right) \xi \\
Y_{p}(X f)=\left(f_{*}\right)_{p}\left(\nabla_{Y_{p}} X\right)+I I_{p}\left(Y_{p}, X\right) \xi
\end{gathered}
$$

and subtracting we find

$$
f_{*}[X, Y]=\left(X_{p} Y-Y_{p} X\right) f=\left(f_{*}\right)_{p}\left(\nabla_{X_{p}} Y-\nabla_{Y_{p}} X\right)+\left(I I_{p}\left(X_{p}, Y\right)-I I_{p}\left(Y_{p}, X\right)\right) \xi
$$

thus

$$
\begin{aligned}
\nabla_{X_{p}} Y-\nabla_{Y_{p}} X & =[X, Y]_{p} \\
I I_{p}\left(X_{p}, Y\right) & =I I_{p}\left(Y_{p}, X\right)
\end{aligned}
$$

Accordıngly we obtaın a symmetric bilınear form $I I_{p} \quad T_{p} M \times T_{p} M \rightarrow \mathbf{R} \quad\left(X_{p}, Y_{p}\right) \rightarrow$ $I I_{p}\left(X_{p}, Y_{p}\right)$ where $I I_{p}\left(X_{p}, Y_{p}\right)=I I_{p}\left(X_{p}, Y\right)$ for all smooth vector fields $Y$ with value $Y_{p}$ at $p \quad I I_{p}($,$) is called the second fundamental form on M$ at $p$

Locally we have $\xi$ satisfying the following

$$
\begin{aligned}
\langle\xi, \xi\rangle_{1} & =-1 \\
X_{p}\langle\xi, \xi\rangle_{1} & =0 \\
2\left\langle X_{p} \xi, \xi\right\rangle_{1} & =0
\end{aligned}
$$

thus $X_{p} \xi$ is orthogonal to $\xi$ implying $X_{p} \xi$ is tangent to $f(M)$ at $f(p)$,so

$$
\begin{equation*}
\xi_{*}(X)=X_{p} \xi=\left(f_{*}\right)_{p}\left(A_{p} X_{p}\right) \quad \text { for some } A_{p} X_{p} \in T_{p} M \tag{3}
\end{equation*}
$$

Note $\xi_{*}$ and $f_{*}$ are linear maps and therefore $A_{p} T_{p} M \rightarrow T_{p} M \quad X \rightarrow A_{p} X$ is also linear We note the absence of the minus sıgn as is usual in the case of immersions into Euclıdean space

For all smooth vector fields $Y$ on $M$ we have

$$
\begin{aligned}
0 & =\langle Y f, \xi\rangle_{\mathbf{1}} \\
0 & =X_{p}\langle Y f, \xi\rangle_{1} \\
& =\left\langle X_{p}(Y f), \xi\right\rangle_{1}+\left\langle Y_{p} f, X_{p} \xi\right\rangle_{1} \\
& =\left\langle\left(f_{*}\right)_{p}\left(\nabla_{X_{p}} Y\right)+I I_{p}\left(X_{p}, Y_{p}\right) \xi, \xi\right\rangle_{\mathbf{l}}+\left\langle Y_{p} f,\left(f_{*}\right)_{p}\left(A_{p} X_{p}\right)\right\rangle_{1} \\
& =I I_{p}\left(X_{p}, Y_{p}\right)\langle\xi, \xi\rangle_{1}+\left\langle\left(f_{*}\right)_{p} Y_{p},\left(f_{*}\right)_{p}\left(A_{p} X_{p}\right)\right\rangle_{1} \\
& =-I I_{p}\left(X_{p}, Y_{p}\right)+g_{p}\left(Y_{p}, A_{p} X_{p}\right)
\end{aligned}
$$

Thus we have

$$
I I_{p}\left(X_{p}, Y_{p}\right)=g_{p}\left(Y_{p}, A_{p} X_{p}\right)
$$

but $I_{p}($,$) is symmetric and bilnear thus g_{p}(A X, Y)=g_{p}(X, A Y)$ showing that $A$ is symmetric with respect to $g$ Therefore from (31) we have

$$
\begin{equation*}
X_{p}(Y f)=\left(f_{*}\right)_{p}\left(\nabla_{X_{p}} Y\right)+g_{p}\left(Y_{p}, A_{p} X_{p}\right) \xi \tag{33}
\end{equation*}
$$

(3 3) and (32) are called the Gauss-Wemgarten equations

## Remark:

In addition to

$$
\begin{equation*}
\nabla_{X_{p}} Y-\nabla_{Y_{p}} X=[X, Y]_{p} \tag{3}
\end{equation*}
$$

it is easy to check that $\nabla$ satısfies
$1 \nabla_{\alpha X_{p}+\beta Y_{p}} Z=\alpha \nabla_{X_{p}} Z+\beta \nabla_{Y_{p}} Z \quad \forall \alpha, \beta \in \mathbf{R}, \quad X_{p}, Y_{p} \in T_{p} M$, and Z a smooth vector field
$2 \nabla_{X_{p}}(Y f)=\left(X_{p} f\right) Y+f(p) \nabla_{X_{p}} Y \quad \forall X_{p} \in T_{p} M, f \quad M \rightarrow \mathbf{R}_{1}^{3}, \quad$ and Y a smooth vector field
$3 \quad \nabla_{X_{p}}(Y+Z)=\nabla_{X_{p}} Y+\nabla_{X_{p}} Z \quad \forall X_{p} \in T_{p} M, \quad$ and $\mathrm{Y}, \mathrm{Z}$ smooth vector fields as well as

$$
\begin{equation*}
Z g(X, Y)=g\left(\nabla_{Z} X, Y\right)+g\left(X, \nabla_{Z} Y\right) \tag{35}
\end{equation*}
$$

Therefore $\nabla$ is the unique Levi-Civita connection determined by the Riemannian metric $g$

Using the Gauss-Weingarten equations we establish the following results, the proof which can be seen in Appendıx A
$1 g_{p}\left(R\left(X_{p}, Y_{p}\right) Y_{p}, X_{p}\right)=-\operatorname{det}\left(A_{p}\right)$ where $R(X, Y) Z$ is the curvature tensor, defined as

$$
R(X, Y) Z=\nabla_{X} \nabla_{Y} Z-\nabla_{Y} \nabla_{X} Z-\nabla_{[X, Y]} Z
$$

Note the mınus sıgn (1 e $-\operatorname{det} A$ ) which does not appear when mapping into Euclıdean space
$2\left(\nabla_{X_{p}} A\right) Y=\left(\nabla_{Y_{p}} A\right) X$ called Codazzı's equation where

$$
\left(\nabla_{X} A\right) Y=\nabla_{X}(A Y)-A \nabla_{X} Y
$$

There exists functıons $\Gamma_{\imath j}^{k}$ defined near $p \in M$ so that $\nabla_{\left(\frac{\partial}{\partial x_{1}}\right)} \frac{\partial}{\partial x_{j}} \in T_{p} M$ can be expressed as

$$
\nabla_{\left(\frac{\partial}{\partial x_{t}}\right)_{p}} \frac{\partial}{\partial x_{\jmath}}=\sum_{k=1}^{2} \Gamma_{\imath \jmath}^{k}(p)\left(\frac{\partial}{\partial x_{k}}\right)_{p}, \imath, \jmath=1,2
$$

That is the $\Gamma_{\imath \jmath}^{k}$ 's are just the components of $\nabla_{\left(\frac{\partial}{\partial x_{1}}\right)} \frac{\partial}{\partial x_{j}}$ relative to the coordınates $\left(\frac{\partial}{\partial x_{1}}\right)_{p},\left(\frac{\partial}{\partial x_{2}}\right)_{p}$ for $T_{p} M$ These $\Gamma_{\imath \jmath}^{k}$ 's are called the Christoffel symbols

In the case when $M$ has locally defined isothermal coordinates $x_{1}, x_{2} 1 \mathrm{e}$

$$
\begin{gathered}
g\left(\frac{\partial}{\partial x_{1}}, \frac{\partial}{\partial x_{1}}\right)=g\left(\frac{\partial}{\partial x_{2}}, \frac{\partial}{\partial x_{2}}\right)=e^{\phi(x, y)} \\
g\left(\frac{\partial}{\partial x_{1}}, \frac{\partial}{\partial x_{2}}\right)=0
\end{gathered}
$$

Then

$$
\begin{aligned}
& \Gamma_{11}^{1}=\frac{1}{2} \phi_{x}, \quad \Gamma_{22}^{1}=-\frac{1}{2} \phi_{x}, \quad \Gamma_{12}^{1}=\frac{1}{2} \phi_{y}, \quad \Gamma_{21}^{1}=\frac{1}{2} \phi_{y} \\
& \Gamma_{11}^{2}=-\frac{1}{2} \phi_{y}, \quad \Gamma_{22}^{2}=\frac{1}{2} \phi_{y}, \quad \Gamma_{12}^{2}=\frac{1}{2} \phi_{x}, \quad \Gamma_{21}^{2}=\frac{1}{2} \phi_{x}
\end{aligned}
$$

Proof : see Appendix B

## Chapter 4

## Isometric deformations and the drehriss

Let $M$ be defined as in the chapter 3 and let $f^{t}$ be a one parameter group of isometric ımmersions mappıng $M$ into $\mathbb{R}^{3}$ Then for each $t$, we may assume that $f^{t}$ induces the same Riemanmian metric $g$ on $M, 1$ e $g=\left(f^{t}\right)^{*}\left(\langle,\rangle_{1}\right)$ is independent of $t$ Hence for each $t, f^{t}\left(M, g^{t}\right) \rightarrow\left(\mathbb{R}^{3},\langle,\rangle_{1}\right)$ is an isometric immersion

We now follow the approach of Burns and Clancy [1] with the appropriate changes Fix a $p \in M$ and consider the moving frame

$$
\Omega_{p}^{t}=\left[f_{x}^{t}(p), f_{y}^{t}(p), \xi^{t}(p)\right]
$$

along the curve $t \mapsto f^{t}(p) \in \mathbb{R}^{3}$ where

$$
\xi^{t}(p)=e^{-\phi} f_{x}^{t}(p) * f_{y}^{t}(p)
$$

For each $t, f^{t}$ is an isometric immersion so there exists $\Theta_{p}^{t} \in S O(2,1)$, depending smoothly on $t$ such that $\Omega_{p}^{t}=\Theta_{p}^{t} \Omega_{p}^{0}$ and, therefore, takıng the $t$-derivatıve with $(=d / d t),\left(\Theta_{p}^{t}\right)^{T}$ denoting the transpose of $\Theta_{p}^{t}$ and $\left(\Theta_{p}^{t}\right)^{\dagger}$ as defined in the prelımınaries we have using lemma 27 that

$$
\begin{aligned}
\Omega_{p}^{t} & =\Theta_{p}^{t} \Omega_{p}^{0}=\Theta_{p}^{t}\left[\left(\left(\Theta_{p}^{t}\right)^{\dagger \dagger}\right)^{T} \Theta_{p}^{t}\right] \Omega_{p}^{0} \\
& =\left[\Theta_{p}^{t}\left(\left(\Theta_{p}^{t}\right)^{\dagger \dagger}\right)^{T}\right] \Omega_{p}^{t}=\left(\operatorname{ad} \eta^{t}(p)\right) \Omega_{p}^{t}
\end{aligned}
$$

for some unıquely determıned $\eta^{t}(p) \in \mathbb{R}^{3}$ So for each " $\imath \imath m e$ " $t$ we obtaın a map

$$
M \rightarrow \mathbb{R}^{3} \quad p \mapsto \eta^{t}(p)
$$

called the drehriss of the deformation $f^{t}$ at time $t$

Now if we express the formula $\Omega_{p}^{t}=\left(\operatorname{ad} \eta^{t}(p)\right) \Omega_{p}^{t} \mathrm{~m}$ component form we obtann the fundamental equations

$$
\begin{align*}
f_{x}^{t}(p) & =\eta^{t}(p) * f_{x}^{t}(p) \\
f_{y}^{t}(p) & =\eta^{t}(p) * f_{y}^{t}(p)  \tag{array}\\
\xi^{t}(p) & =\eta^{t}(p) * \xi^{t}(p)
\end{align*}
$$

Furthermore, since $f_{x y}=f_{y x}$ it follows from equations (41) that

$$
\eta_{x} * f_{y}=\eta_{y} * f_{x}
$$

and consequently

$$
\begin{array}{rlcc}
\left\langle\eta_{y}, \xi\right\rangle_{1} e^{\phi} & =\left\langle\eta_{y}, f_{x} * f_{y}\right\rangle_{1} & =\left\langle\eta_{y} * f_{x}, f_{y}\right\rangle_{1} \\
& =\left\langle\eta_{x} * f_{y}, f_{y}\right\rangle_{1} & = & 0
\end{array}
$$

Thus, with a sımılar argument apphed to $\left\langle\eta_{x}, \xi\right\rangle_{1}$, we obtain

$$
\begin{equation*}
\left\langle\eta_{x}, \xi\right\rangle_{1}=\left\langle\eta_{y}, \xi\right\rangle_{1}=0 \tag{42}
\end{equation*}
$$

## Proposition 4.1

If $J$ is the complex structure on $M$ which is compatzble with the metric $g$ and the orzentation, then

$$
\eta_{*}=-f_{*} \circ J \circ A
$$

## Proof:

If we put $\left(x^{1}, x^{2}\right)=(x, y), f_{1}=f_{x}, f_{2}=f_{y}$ and we use the summation convention,
then at $p$ we have

$$
\begin{aligned}
\xi_{\jmath} & =\frac{\partial}{\partial t} \frac{\partial \xi}{\partial x^{3}} \\
& =\frac{\partial}{\partial t} f_{*}\left(A \frac{\partial}{\partial x^{\jmath}}\right) \\
& =\frac{\partial}{\partial t}\left(a_{\jmath}^{2}\right) \frac{\partial f}{\partial x^{2}} \\
& =a_{3}^{2} f_{\imath}+a_{\jmath}^{2} f_{2} \\
& =f_{*}\left(a_{\jmath}^{2} \frac{\partial}{\partial x^{2}}\right)+a_{\jmath}^{2} f_{\imath}
\end{aligned}
$$

On the other hand, using equations (41) we have

$$
\begin{aligned}
\xi_{\jmath} & =\frac{\partial}{\partial x^{3}}(\eta * \xi) \\
& =\eta_{\jmath} * \xi+\eta * \xi_{\jmath} \\
& =-\xi * \eta_{\jmath}+\eta * a_{\jmath}^{2} f_{\imath} \\
& =-\xi * \eta_{\jmath}+a_{\jmath}^{2} f_{\imath}
\end{aligned}
$$

Comparıng these two expressions for $\xi_{j}$ we obtain

$$
\xi * \eta_{j}=-f_{*}\left(A \frac{\partial}{\partial x^{j}}\right)
$$

and, hence

$$
\xi *\left(\xi * \eta_{\jmath}\right)=-\xi * f_{*}\left(A \frac{\partial}{\partial x^{\jmath}}\right)
$$

Recall from the prelıminaries that $\xi * f_{x}=-f_{y}$ and $\xi * f_{y}=f_{x}$ and so

$$
\xi *\left(\xi * \eta_{\jmath}\right)=f_{*}\left(J A \frac{\partial}{\partial x^{3}}\right)
$$

since $f$ is an isometric immersion preserving the orientation it now follows from
lemma 25 that

$$
\langle\xi, \xi\rangle_{1} \eta_{3}-\left\langle\xi, \eta_{3}\right\rangle_{1} \xi=f_{*}\left(J A \frac{\partial}{\partial x^{j}}\right)
$$

so using equation (42) and the fact that $\langle\xi, \xi\rangle_{1}=-1$ we obtain

$$
\eta_{*} \frac{\partial}{\partial x^{3}}=-\left(f_{*} \circ J \circ A\right) \frac{\partial}{\partial x^{3}}, \quad \jmath=1,2
$$

which proves the proposition

## Chapter 5

## Deformations Preserving Mean

## Curvature

In this section we list some of the results found by Burns and Clancy [1] and include them for completeness only Suppose that $f^{t} \quad M \rightarrow \mathbb{R}^{3}$ is an isometric deformation which (as $t$ varies) preserves mean curvature at $p \in M$ The elgenvalues $\lambda_{1}(p) \leq$ $\lambda_{2}(p)$ of $A^{t}$ the second fundamental form, are just the roots of the equation

$$
\lambda^{2}-2 H(p) \lambda-K(p)=0
$$

where $H(p)=(1 / 2) \operatorname{Tr} A_{p}^{t}$ is the mean curvature of $f^{t}$ at $p$ and $K(p)$ is the gauss curvature at $p$ These eigenvalues are independent of $t$ and therefore, when $p$ is not an umbilic (1 e $\lambda_{1}(p) \neq \lambda_{2}(p)$ ), there exists a unıque $\Theta(p, t) \in[0,2 \pi)$ such that

$$
A_{p}^{t}=e^{\Theta(p, t) J} A_{p}^{0} e^{-\Theta(p, t) J}
$$

where J is as described in proposition 41 Also, when $p$ is an umbilic we can choose $\Theta(p, t)$ arbitrarıly Furthermore, since $A_{p}^{0}$ is symmetric and $J$ is skew-symmetric with respect to $g$

$$
\begin{equation*}
\left(A^{t}-H I\right)_{p}=e^{2 \Theta(p, t) J}\left(A^{0}-H I\right)_{p} \tag{array}
\end{equation*}
$$

## Proposition 51

Let $f^{0} \quad M \rightarrow\left(\mathbb{R}^{3},\langle,\rangle_{1}\right)$ be an isometric immersion having constant mean curvature $H$ If $f^{t} \quad(M, g) \rightarrow \mathbb{R}^{3}$ is an usometric deformation of $f^{0}$ which preserves this constant mean curvature, then there exists a smooth function $t \mapsto \psi(t)$, depending on $t$ only, such that

$$
\left(A^{t}-H I\right)_{p}=e^{\psi(t) J}\left(A^{0}-H I\right)_{p}
$$

for every $p \in M$ and the drehriss for this deformation is given by

$$
\eta^{t}(p)=\psi(t)\left[\xi^{t}(p)-H f^{t}(p)\right]+\mathbf{E}^{t}
$$

where $\mathbf{E}^{t} \in \mathbb{R}^{3}$ does not depend on $p$

## Proof :

Given that each immersion $f^{t}$ is of constant mean curvature $H$, $\mathrm{e} \operatorname{Tr} A^{t}=2 H$ for all $t$ and $g=e^{\phi(x, y)}\left(d x^{2}+d y^{2}\right)$, we may write $A^{t}$ in terms of local isothermal coordinates $(x, y)$ on $M$ as follows

$$
A^{t}=\left(\begin{array}{cc}
a_{11}^{t} & a_{12}^{t} \\
a_{12}^{t} & a_{22}^{t}
\end{array}\right)
$$

It can now be shown that Codazzı's equation for $A^{t}$ is equivalent to the CauchyRiemann equations for the complex function

$$
\Psi^{t}(x+\imath y)=\left(\left(a_{11}^{t}-a_{22}^{t}\right)-2 \imath a_{12}^{t}\right) e^{\phi}
$$

This was first observed by H Hopf and a proof can be seen in Appendix C We note that the umbilics of the immersion $f^{t}$ (which are the zeros of $\Psi^{t}$ ) are isolated Note, we rule out the case $\Psi^{t} \equiv 0$ since this corresponds to the immersed surface being a portion of the standard hyperbolic sphere in $\mathbb{R}^{3}$ Therefore, the function $\Theta(p, t)$ in equation (51) is a smooth function defined for all $p \in M$ except for these isolated umbilics If we fix $t$ and recall that $\nabla J=0$, then by applying Codazzı's equation to both sides of equation (51) we find that since

$$
\left(\nabla_{\frac{\partial}{\partial x}}\left(A^{t}-H I\right)_{p}\right) \frac{\partial}{\partial y}=\left(\nabla_{\frac{\theta}{\partial y}}\left(A^{t}-H I\right)_{p}\right) \frac{\partial}{\partial x}
$$

we must have

$$
\left(\nabla_{\frac{\partial}{\partial x}} e^{2 \Theta(p, t) J}\left(A^{0}-H I\right)_{p}\right) \frac{\partial}{\partial y}=\left(\nabla_{\frac{\theta}{\partial y}} e^{2 \Theta(p, t) J}\left(A^{0}-H I\right)_{p}\right) \frac{\partial}{\partial x}
$$

and so

$$
\begin{gathered}
\nabla_{\frac{\theta}{\partial x}}\left(e^{2 \Theta(p, t) J}\left(A^{0}-H I\right)_{p} \frac{\partial}{\partial y}\right)-e^{2 \Theta(p, t) J}\left(A^{0}-H I\right)_{p} \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y}= \\
\nabla_{\frac{\partial}{\partial y}}\left(e^{2 \Theta(p, t) J}\left(A^{0}-H I\right)_{p} \frac{\partial}{\partial x}\right)-e^{2 \Theta(p, t) J}\left(A^{0}-H I\right)_{p} \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x}
\end{gathered}
$$

ımplying

$$
\begin{aligned}
& 2 J \Theta_{x} e^{2 \Theta(p, t) J}\left(A^{0}-H I\right)_{p} \frac{\partial}{\partial y}+e^{2 \Theta(p, t) J} \nabla_{\frac{\partial}{\partial x}}\left(A^{0}-H I\right)_{p} \frac{\partial}{\partial y}= \\
& 2 J \Theta_{y} e^{2 \Theta(p, t) J}\left(A^{0}-H I\right)_{p} \frac{\partial}{\partial x}+e^{2 \Theta(p, t) J} \nabla_{\frac{\Theta}{\partial y}}\left(A^{0}-H I\right)_{p} \frac{\partial}{\partial x}
\end{aligned}
$$

Hence $\Theta_{x}=\Theta_{y}=0$ 1e $\Theta$ is independent of $p$ Accordingly, the first statement follows if we put $\psi(t)=2 \Theta(p, t) \quad$ For the second statement we take the $t$-derivative (with $p$ fixed) across the equation

$$
\left(A^{t}-H I\right)_{p}=e^{\psi(t) J}\left(A^{0}-H I\right)_{p}
$$

to obtain

$$
A_{p}^{t}=\psi(t) J e^{\psi(t) J}\left(A^{0}-H I\right)_{p}=\psi(t) J\left(A^{t}-H I\right)_{p}
$$

and from proposition 41 we have

$$
\begin{aligned}
\eta_{*} & =-f_{*} \circ J A=-f_{*} \circ J(\psi J(A-H I)) \\
& =\psi f_{*} \circ(A-H I)=\psi\left(f_{*} \circ A-H f_{*}\right) \\
& =\psi\left(\xi_{*}-H f_{*}\right)=(\psi(\xi-H f))_{*}
\end{aligned}
$$

from which the second statement follows

## Chapter 6

## The manifold

Throughout this section we will assume that the simply-connected Riemann Surface $M$ with metric $g$ admıts a 1-parameter group of isometries

A one parameter group of isometries of $M$ is a family $\left\{\Phi_{t} \quad M \rightarrow M \mid t \in \mathbb{R}\right\}$ of mappings with the following three properties
a $\left\{\Phi_{t} \quad M \rightarrow M \mid t \in \mathbb{R}\right\}$ is a group under composition 1 e

$$
\Phi_{t} \circ \Phi_{s}=\Phi_{t+s} \text { for all } s, t \in \mathbb{R} \text { and } \Phi_{0}=\imath d
$$

b For all $p \in M$ the mapping $t \mapsto \Phi_{t}(p)$ is differentiable
c For all $t \in \mathbb{R}, \Phi_{t} \quad M \rightarrow M$ is an isometry

The uniformization theorem states that every simply connected Riemann surface is conformally equivalent to just one of

1 The Riemann Sphere $S^{2}$
11 The Complex Plane C
111 The Open Unit Disk $\mathcal{U}$
hence there exısts a conformal diffeomorphism $F \quad \Delta \rightarrow M$ where $\Delta=S^{2}, \mathbf{C}$, or $\mathcal{U}$, each equipped with its standard metric

Let $\tilde{g}=F^{*}(g)$ be the metric induced on $\Delta$ by $F$ Then $F \quad(\Delta, \tilde{g}) \rightarrow(M, g)$ is isometric The one parameter group $\left\{\Phi_{t} \quad M \rightarrow M \mid t \in \mathbb{R}\right\}$ of isometries of $M$ induces a one parameter group $\left\{\psi_{t} \quad \Delta \rightarrow \Delta \mid t \in \mathbb{R}\right\}$ of isometries of $\Delta$ Hence we let $\Delta$ equal $S^{2}, \mathrm{C}$ and $\mathcal{U}$ and investıgate all one parameter groups of isometries

If $\Delta=S^{2}$ then $f$ immerses $M$ into a plece of the hyperbolic 2 -sphere, a proof of this is found in Appendix E If $\Delta \neq S^{2}$ then we may assume (see smyth [6] for example) that after a conformal change in the model, the group $\psi_{t}$ is one of the following
(a1) all rotations about a fixed point of $\mathcal{U}$ ( the origin 0 , say )
(a2) all automorphisms of $\mathcal{U}$ fixing one boundary point ( $z=1$, say)
(a3) all automorphısms of $\mathcal{U}$ fixıng two boundary points ( $z= \pm 1$, say)
(b1) all translations in a fixed direction of $\mathbf{C}$ (the $y$-axis, say)
(b2) all rotations about a fixed point of $\mathbf{C}$ (the orıgın 0 ,say )

Now under the covering map $z \rightarrow e^{z}$, the regions in (a1) and (b2) correspond to the half-plane $\operatorname{Re}(z)<0$ and $\mathbf{C}$, respectively, and the group of translations parallel to the y -axis If we transform the disk $\mathcal{U}$ into the half plane $\operatorname{Re}(z)<0$ so that 1 is transformed to $\infty$ then the group in (a2) must transform into the group of translations parallel to the $y$-axis If we transform the disk $\mathcal{U}$ into the strip $a<\operatorname{Re}(z)<b$ so that $\pm 1$ are transformed to $y= \pm \infty$ then the group in (a3) must transform into the group of translations parallel to the $y$-axis

Let $V=\{z \in \mathbf{C} \mid \sigma<\operatorname{Re}(z)<\tau\}$ Then for each of the regions $\mathcal{R}$ occurmg in the cases above there is a conformal mapping $\pi \quad V \rightarrow \mathcal{R}$ such that $\psi_{t}$ pulls back under $\pi$ to the group of vertical translations of $V$

The relevant information for the five cases is as follows
(a1) $(\sigma, \tau)=(-\infty, 1), \quad \pi$ is a coverıng map onto $D-\{0\}$ and $\pi\{x=-\infty\}=0$
(a2) $(\sigma, \tau)=(-\infty, 1), \quad \pi$ is a diffeomorphism
(a3) $(\sigma, \tau)=(a, b), \quad \pi$ is a diffeomorphism and $a, b$ are any fixed constants of our choosing
(b1) $(\sigma, \tau)=(-\infty, \infty), \pi$ is a diffeomorphism
(b2) $(\sigma, \tau)=(-\infty, \infty), \pi$ is a coverıng map onto $\mathbf{C}-\{0\}$ and $\pi\{x=-\infty\}=0$

The quantities arısing on $V$ from the induced immersion $f \circ \pi$ are denoted by the same letters as before Thus $g=e^{\phi}|d z|^{2}$, the function $\phi$ depends only on $x$ and we assume $M=\{z \in \mathbf{C} \mid \sigma<\operatorname{Re}(z)<\tau\}$ with $\psi_{t}$ the group of vertical translations of M

## Chapter 7

## CMC-Surfaces with Internal Symmetry

Throughout this section we will assume that $M$ is a simply-connected surface with Riemannian metric $g$ which admits a 1-parameter group of isometries

$$
\psi^{t} \quad(M, g) \rightarrow(M, g)
$$

We wish to classify all isometric immersions

$$
f \quad(M, g) \rightarrow\left(\mathbb{R}^{3},\langle,\rangle_{\mathbf{1}}\right)
$$

which have constant mean curvature To this end observe that the 1-parameter famıly of immersions

$$
f^{t} \quad(M, g) \rightarrow \mathbb{R}^{3} \quad p \mapsto f\left(\psi^{t}(p)\right)
$$

is an isometric deformation of $f=f^{0}$ which preserves the constant mean curvature $H$ Now, since we may assume that conformally

$$
(M, g)=\left(\left\{(x, y) \in \mathbf{R}^{2} \quad a<x<b\right\} \quad \text { for some } a, b \in \mathbf{R}, e^{\phi(x)}\left(d x^{2}+d y^{2}\right)\right)
$$

and the isometry $\psi^{t}(x, y)=(x, y+t)$ for all $t$ therefore for all $x, y$ and $t$ we have

$$
\begin{aligned}
f^{t}(x, y) & =f^{0}(x, y+t) \\
\xi^{t}(x, y) & =\xi^{0}(x, y+t) \\
A^{t}(x, y) & =A^{0}(x, y+t)
\end{aligned}
$$

In partıcular, $A^{t}(x, 0)=A^{0}(x, t) \quad \forall(x, t)$ and combining this with proposition 51 we obtain

$$
\left(A^{0}-H I\right)_{(x, t)}=\left(A^{t}-H I\right)_{(x, 0)}=e^{\psi(t) J}\left(A^{0}-H I\right)_{(x, 0)}
$$

If we replace $t$ by $y$, then

$$
\left(A^{0}-H I\right)_{(x, y)}=e^{\psi(y) J}\left(A^{0}-H I\right)_{(x, 0)}
$$

and applyıng Codazzı's equation we find

$$
\psi(y)=-\alpha y+\beta \text { for some constants } \alpha, \beta \in \mathbf{R}
$$

Thus, $\psi(t) \equiv-\alpha$ and, from proposition 51 , we see that the drehriss of this deformation is

$$
\eta^{t}(x, y)=-\alpha \xi^{t}(x, y)+\left(\mathbf{E}^{t}+\alpha H f^{t}(x, y)\right)
$$

For later use we also observe that $\beta=\psi(0)=2 \pi n$ for some integer $n$, which we may choose to be zero Therefore, the constant $\alpha$ is unıquely determıned by the condition

$$
\begin{equation*}
\left(A^{0}-H I\right)_{(x, y)}=e^{-\alpha y J}\left(A^{0}-H I\right)_{(x, 0)} \tag{71}
\end{equation*}
$$

Now, from the identity $f^{t}(x, y) \equiv f^{0}(x, y+t)$ it follows that

$$
\begin{aligned}
& f^{t}(x, y)=f_{2}^{0}(x, y+t) \\
& f_{1}^{t}(x, y)=f_{21}^{0}(x, y+t) \\
& f_{2}^{t}(x, y)=f_{22}^{0}(x, y+t) \\
& \xi^{t}(x, y)=\xi_{2}^{0}(x, y+t)
\end{aligned}
$$

where the subscripts 1 and 2 denote partıal derivatıves with respect to the first and second variables, respectıvely

From now on we set $t=0$ and drop the superscript " 0 " from all functions so that using the fundamental equations (41) we obtain

$$
\begin{aligned}
& f_{x y}(x, y)=f_{x}(x, y)=\eta(x, y) * f_{x}(x, y) \\
& f_{y y}(x, y)=f_{y}(x, y)=\eta(x, y) * f_{y}(x, y) \\
& \xi_{y}(x, y)=\xi(x, y)=\eta(x, y) * \xi(x, y)
\end{aligned}
$$

where $\eta(x, y)=-\alpha \xi(x, y)+(\mathbf{E}+\alpha H f(x, y))$ for some constant vector $\mathbf{E} \in \mathbb{R}^{3}$ Therefore,

$$
\begin{align*}
f_{x y} & =-\alpha \xi * f_{x}+(\mathbf{E}+\alpha H f) * f_{x} \\
f_{y y} & =-\alpha \xi * f_{y}+(\mathbf{E}+\alpha H f) * f_{y} \\
\xi_{y} & =(\mathbf{E}+\alpha H f) * \xi \tag{72}
\end{align*}
$$

However, the coordinate system $(x, y)$ is isothermal and $\xi$ was defined so that $\xi * f_{x}=$ $-f_{y}$ and $\xi * f_{y}=f_{x}$ Therefore, we obtain the equations

$$
\begin{align*}
f_{x y} & =\alpha f_{y}+(\mathbf{E}+\alpha H f) * f_{x}  \tag{73}\\
f_{y y} & =-\alpha f_{x}+(\mathbf{E}+\alpha H f) * f_{y} \tag{74}
\end{align*}
$$

When $\alpha \neq 0$ one obtains from the integrability conditions, $f_{x y y}=f_{y y x}$ and $f_{x y}=f_{y x}$, the additional equation

$$
\begin{equation*}
f_{x x}+f_{y y}=2 H f_{x} * f_{y} \tag{75}
\end{equation*}
$$

furthermore when $\alpha \neq 0$ and $H \neq 0$ equations (73), (74) and (75) can be reduced
to

$$
\begin{align*}
f_{x y} & =\alpha f_{y}+\alpha H f * f_{x}  \tag{76}\\
f_{y y} & =-\alpha f_{x}+\alpha H f * f_{y}  \tag{77}\\
f_{x x}+f_{y y} & =2 H f_{x} * f_{y} \tag{78}
\end{align*}
$$

by replacing $f(x, y)$ with $f(x, y)-\frac{1}{\alpha H} \mathbf{E}, 1 \mathrm{e}$ a simple translation and we note for future reference that (72) simplifies to

$$
\begin{equation*}
\xi_{y}=\alpha H f * \xi \tag{79}
\end{equation*}
$$

Hence the above transformation allows us to assume that $\mathbf{E}=(0,0,0)^{T}$ in the orıginal equations The differential equations (76),(77) and (78) can be further simplified by replacing $f(x, y)$ by $H f(x, y)$ to give

$$
\begin{align*}
f_{x y} & =\alpha f_{y}+\alpha f * f_{x}  \tag{710}\\
f_{y y} & =-\alpha f_{x}+\alpha f * f_{y} \\
f_{x x}+f_{y y} & =2 f_{x} * f_{y} \tag{712}
\end{align*}
$$

1 e we may further assume that $H=1$

When $\alpha=0$ equations (73), (74) and (75)reduce to the form

$$
\begin{aligned}
f_{x y} & =\mathbf{E} * f_{x} \\
f_{y y} & =\mathbf{E} * f_{y} \\
\xi_{y} & =\mathbf{E} * \xi
\end{aligned}
$$

We now determine a necessary and sufficient condition in terms of isothermal coordınates $(x, y)$ for a spacelıke immersion $f$ into Mınkowskı three space to be of constant mean curvature $H$ As

$$
\begin{align*}
& \left\langle f_{x}, f_{x}\right\rangle_{1}=\left\langle f_{y}, f_{y}\right\rangle_{1}>0  \tag{713}\\
& \left\langle f_{x}, f_{y}\right\rangle_{1}=0 \tag{714}
\end{align*}
$$

we conclude from (713) that

$$
\left\langle f_{x x}, f_{x}\right\rangle_{1}=\left\langle f_{x y}, f_{y}\right\rangle_{1}
$$

and by (714)

$$
\begin{equation*}
\left\langle f_{x y}, f_{y}\right\rangle_{1}+\left\langle f_{x}, f_{y y}\right\rangle_{1}=0 \tag{716}
\end{equation*}
$$

Using (715) and (716) we see that

$$
\left\langle f_{x x}+f_{y y}, f_{x}\right\rangle_{\mathbf{1}}=0
$$

and sımılarly one can show that

$$
\begin{equation*}
\left\langle f_{x x}+f_{y y}, f_{y}\right\rangle_{1}=0 \tag{718}
\end{equation*}
$$

1mplying that $f_{x x}+f_{y y}=\vartheta(x, y) \xi$ for some function $\vartheta(x, y)$

Now

$$
\begin{aligned}
\left\langle f_{x x}+f_{y y}, \xi\right\rangle_{1} & =\left\langle f_{x x}, \xi\right\rangle_{1}+\left\langle f_{y y}, \xi\right\rangle_{1} \\
& =\left(\left\langle f_{x}, \xi\right\rangle_{1}\right)_{x}-\left\langle f_{x}, \xi_{x}\right\rangle_{1}+\left(\left\langle f_{y}, \xi\right\rangle_{1}\right)_{y}-\left\langle f_{y}, \xi_{y}\right\rangle_{1} \\
& =-\left\langle f_{x}, a_{11} f_{x}+a_{12} f_{y}\right\rangle_{1}-\left\langle f_{y}, a_{21} f_{x}+a_{22} f_{y}\right\rangle_{1} \\
& =-2 H e^{\phi}
\end{aligned}
$$

and since $\langle\vartheta(x, y) \xi, \xi\rangle_{1}=-\vartheta(x, y)$ we have that

$$
f_{x x}+f_{y y}=2 H f_{x} * f_{y}
$$

That is the integrability condition (75) (in isothermal coordinates) is a necessary and sufficient condition for the immersion to be of constant mean curvature and so it holds whether $\alpha=0$ or not

## Chapter 8

## The Associates

With the notation of the previous sections, let $f(M, g) \rightarrow \mathbb{R}^{3}$ be an isometric immersion of constant mean curvature $H$ If $A^{0}$ denotes the second fundamental form of $f$, then, by definition the associates of $f$ are the isometric immersions in the 1-parameter famıly $f^{t} \quad(M, g) \rightarrow \mathbb{R}^{3}$ which have their second fundamental forms $A^{t}$ determıned by

$$
\begin{equation*}
\left(A^{t}-H I\right)_{p}=e^{t J}\left(A^{0}-H I\right)_{p} \quad \forall p \in M \tag{81}
\end{equation*}
$$

These immersions have the same constant mean curvature as $f$ If we begin with one of the immersions $f$ as determıned by in chapter 7 , that is, the constants $H, e^{\phi\left(x_{0}\right)}, \alpha$ and $\mathbf{E}$ are specified, then the question arises how should these parameters be varied to obtain the associates of $f$ ? At once we see that $H$ remains constant and so also does $\lambda\left(x_{0}\right)$ since it determınes the metric $g$ at $\left(x_{0}, y_{0}\right)$ which is the same for all associates Now, let $\alpha(t)$ and $\mathbf{E}^{t}$ denote the remaining parameters which correspond to the associate $f^{t}$ From the previous section, see equation (71), $\alpha(t)$ is uniquely determined by the following

$$
\begin{aligned}
e^{\alpha(t) y J}\left(A^{t}-H I\right)_{(x, 0)} & =\left(A^{t}-H I\right)_{(x, y)} \\
& =e^{t J}\left(A^{0}-H I\right)_{(x, y)} \\
& =e^{t J} e^{\alpha(0) y J}\left(A^{0}-H I\right)_{(x, 0)} \\
& =e^{\alpha(0) y J} e^{t J}\left(A^{0}-H I\right)_{(x, 0)} \\
& =e^{\alpha(0) y J}\left(A^{t}-H I\right)_{(x, 0)}
\end{aligned}
$$

Therefore, $e^{\alpha(t) y J} \equiv e^{\alpha(0) y J}$ and $\alpha(t)=\alpha(0)+2 n \pi$ where $n$ is some integer

## Remark.

If as in the previous section $\psi^{s}(x, y)=(x, y+s) \quad s \in \mathbb{R}$ denotes the 1-parameter group of internal isometries, then $f \circ \psi^{s}$ has second fundamental form $e^{\alpha s J}\left(A^{0}-\right.$ $H I)+H I$ while the associate $f^{t}$ has second fundamental form $e^{t J}\left(A^{0}-H I\right)+H I$ Therefore up to a Hyperbolic motion we have

$$
f^{t}=f \circ \psi^{t / \alpha} \quad \forall \alpha \neq 0
$$

That is when $\alpha \neq 0$, the associates no longer generate "new surfaces" but rather (up to a Hyperbolic motion) correspond to the flow of the internal symmetry along $f$

## Chapter 9

## The second fundamental form

## Lemma 91

When $\alpha=0$ we have

$$
\langle\mathbf{E}, f\rangle_{1}=c_{1} y+c_{2} x+c_{3}+H \int e^{\phi} d x
$$

for some $c_{1}, c_{2}, c_{3} \in \mathbb{R}$ and the second fundamental form, $A$, satısfies

$$
A-H I=e^{-\phi}\left(\begin{array}{rr}
c_{2} & c_{1}  \tag{91}\\
c_{1} & -c_{2}
\end{array}\right)
$$

## Proof

When $\alpha=0$ we have that every constant mean curvature surface which has contınuous internal symmetry satısfies the following differential equations

$$
\begin{align*}
f_{x y} & =\mathbf{E} * f_{x}  \tag{92}\\
f_{y y} & =\mathbf{E} * f_{y}  \tag{93}\\
\xi_{y} & =\mathbf{E} * \xi  \tag{94}\\
f_{x x}+f_{y y} & =2 H f_{x} * f_{y} \tag{95}
\end{align*}
$$

and hence

$$
\left\langle E, f_{x y}\right\rangle_{1}=\left\langle E, f_{y y}\right\rangle_{1}=0
$$

which implies

$$
\left\langle E, f_{y}\right\rangle_{1}=c_{1}
$$

$$
\begin{aligned}
\left\langle E, f_{x x}\right\rangle_{1} & =\left\langle E, f_{x x}\right\rangle_{1}+\left\langle E, f_{y y}\right\rangle_{1} \\
& =2 H\left\langle E, f_{x} * f_{y}\right\rangle_{1} \\
& =2 H\left\langle E * f_{x}, f_{y}\right\rangle_{1} \\
& =2 H\left\langle f_{x y}, f_{y}\right\rangle_{1} \\
& =H\left(\left\langle f_{y}, f_{y}\right\rangle_{1}\right)_{x} \\
& =H\left(e^{\phi}\right)_{x} \\
& =\left(H e^{\phi}\right)_{x}
\end{aligned}
$$

Hence

$$
\left\langle E, f_{x}\right\rangle_{1}=H e^{\phi}+c_{2}(y)
$$

but

$$
\left\langle E, f_{x y}\right\rangle_{\mathbf{1}}=0
$$

and so

$$
\begin{aligned}
\left\langle E, f_{x}\right\rangle_{1} & =H e^{\phi}+c_{2} \\
\xi_{y} & =E * \xi \\
\frac{\partial}{\partial y} \xi & =e^{-\phi} E *\left(f_{x} * f_{y}\right) \\
f_{*}\left(A \frac{\partial}{\partial y}\right) & =e^{-\phi}\left(\left\langle E, f_{x}\right\rangle_{1} f_{y}-\left\langle E, f_{y}\right\rangle_{1} f_{x}\right) \\
f_{*}\left(a_{21} \frac{\partial}{\partial x}+a_{22} \frac{\partial}{\partial y}\right) & =e^{-\phi}\left(\left(H e^{\phi}+c_{2}\right) f_{y}-c_{1} f_{x}\right) \\
a_{21} f_{x}+a_{22} f_{y} & =\left(H+c_{2} e^{-\phi}\right) f_{y}-c_{1} e^{-\phi} f_{x}
\end{aligned}
$$

and hence

$$
\begin{align*}
& a_{22}=H+c_{2} e^{-\phi}  \tag{96}\\
& a_{21}=-c_{1} e^{-\phi} \tag{97}
\end{align*}
$$

we recall that $a_{12}=a_{21}$ and $a_{11}+a_{22}=2 H$ and so

$$
A=H I+e^{-\phi}\left(\begin{array}{rr}
c_{2} & c_{1}  \tag{98}\\
c_{1} & -c_{2}
\end{array}\right)
$$

this completes the proof Following the approach of Burns and Clancy [1] we also have the following lemma

## Lemma 9.2

When $\alpha \neq 0$

$$
\left\langle\mathbf{E}+\frac{\alpha H f}{2}, f\right\rangle_{1}=a \cos (\alpha y+t) e^{\alpha x}+H \int e^{\phi} d x
$$

for some $a \in \mathbb{R}$ and the second fundamental form, $A$, satısfies

$$
A-H I=a \alpha e^{-\phi+\alpha x}\left(\begin{array}{rr}
\cos (\alpha y+t) & \sin (\alpha y+t) \\
\sin (\alpha y+t) & -\cos (\alpha y+t)
\end{array}\right)
$$

## Proof.

We have

$$
\begin{equation*}
\left\langle\mathbf{E}+\alpha H f, f_{x y}\right\rangle_{1}=\left(\left\langle\mathbf{E}+\alpha H f, f_{y}\right\rangle_{1}\right)_{x}=\left(\left\langle\mathbf{E}+\frac{\alpha H f}{2}, f\right\rangle_{1}\right)_{x y} \tag{99}
\end{equation*}
$$

$$
\begin{align*}
\left\langle\mathbf{E}+\alpha H f, f_{y y}\right\rangle_{1} & =\left(\left\langle\mathbf{E}+\alpha H f, f_{y}\right\rangle_{1}\right)_{y}-\alpha H\left\langle f_{y}, f_{y}\right\rangle_{1} \\
& =\left(\left\langle\mathbf{E}+\frac{\alpha H f}{2}, f\right\rangle_{1}\right)_{y y}-\alpha H e^{\phi}  \tag{910}\\
\left\langle\mathbf{E}+\alpha H f, f_{x x}\right\rangle_{1} & =\left(\left\langle\mathbf{E}+\alpha H f, f_{x}\right\rangle_{1}\right)_{x}-\alpha H\left\langle f_{x}, f_{x}\right\rangle_{1} \\
& =\left(\left\langle\mathbf{E}+\frac{\alpha H f}{2}, f\right\rangle_{1}\right)_{x x}-\alpha H e^{\phi} \tag{911}
\end{align*}
$$

Using (9 9) we have

$$
\begin{align*}
\left(\left\langle\mathbf{E}+\frac{\alpha H f}{2}, f\right\rangle_{1}\right)_{x y} & =\left\langle\mathbf{E}+\alpha H f, f_{x y}\right\rangle_{1} \\
& =\left\langle\mathbf{E}+\alpha H f, \alpha f_{y}+(\mathbf{E}+\alpha H f) * f_{x}\right\rangle_{1} \quad \text { by }(73) \\
& =\alpha\left\langle\mathbf{E}+\alpha H f, f_{y}\right\rangle_{1} \\
& =\alpha\left(\left\langle\mathbf{E}+\frac{\alpha H f}{2}, f\right\rangle_{1}\right)_{y} \tag{912}
\end{align*}
$$

so that

$$
\begin{equation*}
\left(\left\langle\mathbf{E}+\frac{\alpha H f}{2}, f\right\rangle_{1}\right)_{y}=\Gamma(y) e^{\alpha x} \tag{913}
\end{equation*}
$$

for some function $\Gamma(y)$ Differentiating both sides of this equation by $y$ we have that

$$
\begin{aligned}
\Gamma^{\prime \prime}(y) e^{\alpha x} & =\left(\left\langle\mathbf{E}+\frac{\alpha H f}{2}, f\right\rangle_{1}\right)_{y y y} \\
& =\left(\left\langle\mathbf{E}+\alpha H f, f_{y y}\right\rangle_{1}+\alpha H e^{\phi}\right)_{y} \quad \text { by }(910) \\
& =\left(\left\langle\mathbf{E}+\alpha H f, f_{y y}\right\rangle_{1}\right)_{y} \\
& =\left(\left\langle\mathbf{E}+\alpha H f,-f_{x x}+2 H e^{\phi} \xi\right\rangle_{1}\right)_{y} \quad \text { by }(75)
\end{aligned}
$$

$$
\begin{aligned}
& =-\left(\left\langle\mathbf{E}+\alpha H f, f_{x x}\right\rangle_{1}\right)_{y}+2 H e^{\phi}\left(\langle\mathbf{E}+\alpha H f, \xi\rangle_{1}\right)_{y} \\
& =-\left(\left\langle\mathbf{E}+\alpha H f, f_{x x}\right\rangle_{1}\right)_{y}+2 H e^{\phi}\left(\alpha H\left\langle f_{y}, \xi\right\rangle_{1}+\left\langle\mathbf{E}+\alpha H f, \xi_{y}\right\rangle_{1}\right) \\
& =-\left(\left\langle\mathbf{E}+\alpha H f, f_{x x}\right\rangle_{1}\right)_{y} \quad \text { by }(72) \\
& =-\left(\left\langle\mathbf{E}+\frac{\alpha H f}{2}, f\right\rangle_{1}\right)_{x x y} \\
& =-\left(\Gamma(y) e^{\alpha x}\right)_{x x} \text { by }(913) \\
& =-\Gamma(y) \alpha^{2} e^{\alpha x}
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
\Gamma^{\prime \prime}(y) & =-\alpha^{2} \Gamma(y) \\
\Gamma(y) & =-a \alpha \sin (\alpha y+t) \text { for some } a, t \in \mathbb{R} \\
\left(\left\langle\mathbf{E}+\frac{\alpha H f}{2}, f\right\rangle_{1}\right)_{y} & =-a \alpha \sin (\alpha y+t) e^{\alpha x} \\
\left\langle\mathbf{E}+\frac{\alpha H f}{2}, f\right\rangle_{1} & =a \cos (\alpha y+t) e^{\alpha x}+\chi(x)
\end{aligned}
$$

Also

$$
\begin{aligned}
\chi^{\prime}(x)+a \alpha \cos (\alpha y+t) e^{\alpha x} & =\left(\left\langle\mathbf{E}+\frac{\alpha H f}{2}, f\right\rangle_{1}\right)_{x} \\
& =\left\langle\mathbf{E}+\alpha H f, f_{x}\right\rangle_{1} \\
& =\left\langle\mathbf{E}+\alpha H f,-\frac{1}{\alpha} f_{y y}+\frac{1}{\alpha}(\mathbf{E}+\alpha H f) * f_{y}\right\rangle_{1} \text { by }(74) \\
& =-\frac{1}{\alpha}\left\langle\mathbf{E}+\alpha H f, f_{y y}\right\rangle_{1} \\
& =-\frac{1}{\alpha}\left(\left(\left\langle\mathbf{E}+\frac{\alpha H f}{2}, f\right\rangle_{1}\right)_{y y}-\alpha H e^{\phi}\right) \text { by }(910) \\
& =-\frac{1}{\alpha}\left(-a \alpha^{2} \cos (\alpha y+t) e^{\alpha x}-\alpha H e^{\phi}\right) \\
\chi^{\prime}(x) & =H e^{\phi} \\
& \\
\left\langle\mathbf{E}+\frac{\alpha H f}{2}\right. & , f\rangle_{1}=a \cos (\alpha y+t) e^{\alpha x}+H \int e^{\phi} d x
\end{aligned}
$$

proving the first part

Now

$$
\begin{aligned}
\xi_{y} & =(\mathbf{E}+\alpha H f) * \xi \\
f_{*}\left(A \frac{\partial}{\partial y}\right) & =e^{-\phi}(\mathbf{E}+\alpha H f) *\left(f_{x} * f_{y}\right) \\
f_{*}\left(a_{21} \frac{\partial}{\partial x}+a_{22} \frac{\partial}{\partial y}\right) & =e^{-\phi}\left(\left\langle\mathbf{E}+\alpha H f, f_{x}\right\rangle_{1} f_{y}-\left\langle\mathbf{E}+\alpha H f, f_{y}\right\rangle_{1} f_{x}\right) \\
a_{21} \frac{\partial f}{\partial x}+a_{22} \frac{\partial f}{\partial y} & =e^{-\phi}\left(\left(\left\langle\mathbf{E}+\frac{\alpha H f}{2}, f\right\rangle_{1}\right)_{x} f_{y}-\left(\left\langle\mathbf{E}+\frac{\alpha H f}{2}, f\right\rangle_{1}\right)_{y} f_{x}\right) \\
a_{21} f_{x}+a_{22} f_{y} & =e^{-\phi}\left(\left(a \alpha \cos (\alpha y+t) e^{\alpha x}+H e^{\phi}\right) f_{y}+a \alpha \sin (\alpha y+t) e^{\alpha x} f_{x}\right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
a_{12}=a_{21} & =a \alpha e^{-\phi+\alpha x} \sin (\alpha y+t) \\
a_{22} & =H+a \alpha e^{-\phi+\alpha x} \cos (\alpha y+t) \\
a_{11}=2 H-a_{22} & =H-a \alpha e^{-\phi+\alpha x} \cos (\alpha y+t) \\
A-H I & =a \alpha e^{-\phi+\alpha x}\left(\begin{array}{rr}
-\cos (\alpha y+t) & \sin (\alpha y+t) \\
\sin (\alpha y+t) & \cos (\alpha y+t)
\end{array}\right)
\end{aligned}
$$

Thus proving the lemma

## Lemma 93

For every $\alpha \in \mathbb{R}$ we may assume without loss of generality that the second fundamental form A may be written as

$$
A=H I-c e^{-\phi+\alpha x}\left(\begin{array}{rr}
-\cos \alpha y & \sin \alpha y  \tag{914}\\
\sin \alpha y & \cos \alpha y
\end{array}\right)
$$

where $c \in \mathbb{R}, c>0$

## Proof

We recall when $\alpha \neq 0$ that the associates, $f_{\theta}$, of the immersion $f$ do not generate new surfaces and that the corresponding second fundamental form $A_{\theta}$ is given by

$$
A_{\theta}=e^{\theta J}(A-H I)+H I
$$

When $\alpha \neq 0$ we then have

$$
A_{\theta}-H I=a \alpha e^{-\phi+\alpha x}\left(\begin{array}{rr}
-\cos (\alpha y+t-\theta) & \sin (\alpha y+t-\theta)  \tag{915}\\
\sin (\alpha y+t-\theta) & \cos (\alpha y+t-\theta)
\end{array}\right)
$$

By choosing $\theta=t$ or $\theta=t+\pi$ if necessary we have

$$
A_{\theta}=H I-c e^{-\phi+\alpha x}\left(\begin{array}{rr}
-\cos \alpha y & \sin \alpha y  \tag{916}\\
\sin \alpha y & \cos \alpha y
\end{array}\right)
$$

where $c=|a \alpha|>0$

When $\alpha=0$ we have

$$
A_{\theta}-H I=e^{-\phi}\left(\begin{array}{rr}
c_{2} \cos \theta-c_{1} \sin \theta & c_{2} \cos \theta+c_{1} \sin \theta  \tag{917}\\
c_{2} \cos \theta+c_{1} \sin \theta & -c_{2} \cos \theta+c_{1} \sin \theta
\end{array}\right)
$$

Choosing $\theta=\tan ^{-1}\left(-\frac{c_{1}}{c_{2}}\right)$ or $\theta=\tan ^{-1}\left(-\frac{c_{1}}{c_{2}}\right)+\pi$ we may assume $c_{1}=0$ and $c_{2}>0$ giving

$$
A_{\theta}=H I-c e^{-\phi}\left(\begin{array}{rr}
-1 & 0  \tag{918}\\
0 & 1
\end{array}\right)
$$

where $c=c_{2}$ Hence we may assume

$$
A=H I-c e^{-\phi+\alpha x}\left(\begin{array}{rr}
-\cos \alpha y & \sin \alpha y \\
\sin \alpha y & \cos \alpha y
\end{array}\right)
$$

for every $\alpha \in \mathbb{R}$ where $c>0$ This completes the proof of lemma 93

## Lemma 9.4

When $\alpha=0$ we may assume without loss of generality that the set of differential equations (9 2), (9 3), (94) and (95) reduce to three

$$
\begin{align*}
f_{y} & =\mathbf{E} * f  \tag{920}\\
\xi_{y} & =\mathbf{E} * \xi  \tag{921}\\
f_{x x}+f_{y y} & =2 H f_{x} * f_{y} \tag{922}
\end{align*}
$$

## Proof-

By (9 2) and (9 3) we have

$$
\begin{align*}
f_{x y} & =\mathbf{E} * f_{x}  \tag{923}\\
f_{y y} & =\mathbf{E} * f_{y} \tag{924}
\end{align*}
$$

which simplıfy to

$$
f_{y}=\mathbf{E} * f+v
$$

for some $v \in \mathbb{R}^{3}$ Using lemma 91 and lemma 93 we may assume $\left\langle E, f_{y}\right\rangle_{1}=0$
and hence we may assume $\langle\mathbf{E}, v\rangle_{1}=0_{1 \mathrm{e}}^{\mathrm{e}} \quad v \in \mathbf{E}^{\perp} \quad$ Thus $v=\mathbf{E} * b$ for some $b \in \mathbb{R}^{3}$ Replacing $f(x, y)$ with $f(x, y)+b_{1} \mathrm{e}$ a translation we find we may assume $v=(0,0,0)^{T}$ This concludes the proof

## Chapter 10

## Conformal transformations

With the conformal structure determined by $g$ and the given orientation, we have that $M$ is a Riemann surface and in terms of local conformal coordınate $z=x+\imath y$ we write $g=e^{\phi}|d z|^{2}$, where $\phi$ is a real function of $z$ and $A=\left(a_{2 \jmath}\right)$ with respect to the coordınate field $\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$

Let $h \quad U \rightarrow h(U) \subset M$ and $\tilde{h} \quad \tilde{U} \rightarrow \tilde{h}(\tilde{U}) \subset M$ be two positively oriented conformal parameterisations of $M$ about a point $p \in M$ with $h(U)=\tilde{h}(\tilde{U})$ So that $\vartheta=h^{-1} \circ \tilde{h} \quad \tilde{U} \rightarrow U$ is a bıjectıve holomorphıc mappıng Also let $z=x+\imath y$ be the local coordınates on $h(U)$ and $w=\tilde{x}+\imath \tilde{y}$ be the local coordınates on $\tilde{h}(\tilde{U})$

Now

$$
g=e^{\phi}|d z|^{2}=e^{\phi}\left|\frac{\partial z}{\partial w} d w\right|^{2}=e^{\phi}\left|\frac{\partial z}{\partial w}\right|^{2}|d w|^{2}=e^{\tilde{\phi}}|d w|^{2}
$$

and hence

$$
\begin{equation*}
e^{\tilde{\phi}}=\left|\frac{\partial z}{\partial w}\right|^{2} e^{\phi} \tag{101}
\end{equation*}
$$

From chapter 5 we have

$$
\begin{equation*}
\Psi=\left\{\left(a_{11}-a_{22}\right)-2 \imath a_{12}\right\} e^{\phi} \tag{102}
\end{equation*}
$$

is a holomorphic function It can easily be shown that in terms of local coordinates

$$
\Psi=4 g\left(A \frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right)
$$

Since

$$
\frac{\partial}{\partial z}=\frac{\partial w}{\partial z} \frac{\partial}{\partial w} \quad \text { and } \quad d z=\frac{\partial z}{\partial w} d w
$$

we see $\Psi d z^{2}$ is a holomorphic quadratic differential independent of the coordinate
system Now

$$
\Psi=4 g\left(A \frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right)=4 g\left(A \frac{\partial w}{\partial z} \frac{\partial}{\partial w}, \frac{\partial w}{\partial z}, \frac{\partial}{\partial z}\right)=\left|\frac{\partial w}{\partial z}\right|^{2} 4 g\left(A \frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right)=\left(\frac{\partial w}{\partial z}\right)^{2} \tilde{\Psi}
$$

and hence

$$
\begin{equation*}
\tilde{\Psi}=\left(\frac{\partial z}{\partial w}\right)^{2} \Psi \tag{103}
\end{equation*}
$$

By lemma 93 we may assume

$$
A=\left(\begin{array}{rr}
H+c e^{-\phi+\alpha x} \cos \alpha y & -c e^{-\phi+\alpha x} \sin \alpha y \\
-c e^{-\phi+\alpha x} \sin \alpha y & H-c e^{-\phi+\alpha x} \cos \alpha y
\end{array}\right)
$$

and so from (10 2) we have

$$
\begin{aligned}
\Psi & =\left\{\left(a_{11}-a_{22}\right)-2 \imath a_{12}\right\} e^{\phi} \\
& =2 c e^{\alpha x} \cos \alpha y+2 \imath c e^{\alpha x} \sin \alpha y
\end{aligned}
$$

or more simply

$$
\begin{equation*}
\Psi(z)=2 c e^{\alpha z} \tag{104}
\end{equation*}
$$

this along with (10 3) gives

$$
\begin{equation*}
\Psi(w)=\left(\frac{\partial z(w)}{\partial w}\right)^{2} 2 c e^{\alpha z(w)} \tag{105}
\end{equation*}
$$

If $\alpha \neq 0$ we let $z=\frac{1}{\alpha}\left(w+\log \frac{\alpha^{2}}{c}\right)$ then by (10 1)

$$
e^{\bar{\phi}}=\left|z^{\prime}(w)\right|^{2} e^{\phi(\operatorname{Re}(z(w)))}=\frac{1}{\alpha^{2}} e^{\phi\left(\frac{1}{\alpha}\left(\tilde{x}+\log \frac{\alpha^{2}}{c}\right)\right)}
$$

which is still just some function of $\tilde{x}$ and using (105) we have

$$
\tilde{\Psi}(w)=\left(z^{\prime}(w)\right)^{2} 2 c e^{\alpha z(w)}=\frac{1}{\alpha^{2}} 2 c e^{w+\log \frac{\alpha^{2}}{c}}=2 e^{w}
$$

thus comparing this to (104) we may assume that $\alpha=1$ and $c=1$

If $\alpha=0$ we let $z=\frac{1}{\sqrt{c}} w$ then by (10 1 )

$$
e^{\tilde{\phi}}=\left|z^{\prime}(w)\right|^{2} e^{\phi(R e(z(w)))}=e^{\left.\phi\left(\frac{1}{\sqrt{c}} \tilde{x}\right)-\log c\right)}
$$

which is again just some function of $\tilde{x}$ and using (105) we have

$$
\tilde{\Psi}=\left(z^{\prime}(w)\right)^{2} 2 c=2
$$

thus comparıng this to (104) when $\alpha=0$ we may assume $c=1$

## Chapter 11

## The metric

As shown in the prelımınaries the Gauss curvature $K(p)$ is given by

$$
K(p)=g_{p}\left(R\left(X_{p}, Y_{p}\right) Y_{p}, X_{p}\right)=-\operatorname{det}\left(A_{p}\right)
$$

computing the left hand side locally with $X=\frac{\partial}{\partial x}$ and $Y=\frac{\partial}{\partial y}$ we find

$$
K=-\frac{1}{2} e^{-\phi} \Delta \phi
$$

relative to the local coordinates $(x, y)$ A proof of this is given in Appendix D Since $\phi$ depends on $x$ only we have

$$
\Delta \phi=\frac{\partial^{2} \phi}{\partial x^{2}}
$$

From the previous chapter

$$
A=H I-e^{-\phi+\alpha x}\left(\begin{array}{rr}
-\cos \alpha y & \sin \alpha y \\
\sin \alpha y & \cos \alpha y
\end{array}\right)
$$

where $\alpha=0$ or 1 Hence

$$
\operatorname{det} A=H^{2}-e^{-2 \phi+2 \alpha x}
$$

and so when $H \neq 0$

$$
\begin{aligned}
\frac{\partial^{2} \phi}{\partial x^{2}} & =2 e^{\phi}\left(H^{2}-e^{-2 \phi+2 \alpha x}\right) \\
& =2 H^{2} e^{\phi}-2 e^{-\phi+2 \alpha x} \\
& =2|H| e^{\alpha x}\left(|H| e^{\phi-\alpha x}-\frac{1}{|H|} e^{-\phi+\alpha x}\right) \\
& =2|H| e^{\alpha x}\left(e^{\phi-\alpha x+\log |H|}-e^{-\phi+\alpha x-\log |H|}\right)
\end{aligned}
$$

Letting $\eta(x)=\phi-\alpha x+\log |H|$ we find that $\eta(x)$ satisfies the differential equation

$$
\begin{equation*}
\eta^{\prime \prime}(x)=e^{\alpha x+\log |4 H|} \sinh \eta(x) \tag{array}
\end{equation*}
$$

and $\alpha=0$ or 1 We note that this differential equation appears in Smyth [6] but with a minus sign Using (73) we have

$$
\begin{aligned}
f_{x y} & =\alpha f_{y}+(\mathbf{E}+\alpha H f) * f_{x} \\
\left\langle f_{x y}, f_{y}\right\rangle_{1} & =\alpha\left\langle f_{y}, f_{y}\right\rangle_{1}+\left\langle(\mathbf{E}+\alpha H f) * f_{x}, f_{y}\right\rangle_{1} \\
\frac{1}{2}\left(\left\langle f_{y}, f_{y}\right\rangle_{1}\right)_{x} & =\alpha e^{\phi}+\left\langle(\mathbf{E}+\alpha H f), f_{x} * f_{y}\right\rangle_{1} \\
\frac{1}{2}\left(e^{\phi}\right)_{x} & =\alpha e^{\phi}+e^{\phi}\left\langle(\mathbf{E}+\alpha H f), e^{-\phi} f_{x} * f_{y}\right\rangle_{1} \\
\frac{1}{2} e^{\phi} \phi_{x} & =\alpha e^{\phi}+e^{\phi}\langle(\mathbf{E}+\alpha H f), \xi\rangle_{1} \\
\phi_{x} & =2 \alpha+2\langle(\mathbf{E}+\alpha H f), \xi\rangle_{1}
\end{aligned}
$$

and hence

$$
\eta^{\prime}(x)=\alpha+2\langle(\mathbf{E}+\alpha H f), \xi\rangle_{1}
$$

## Chapter 12

## Completeness

We recall from chapter 6 that we may assume $M$ is a strip on the complex plane given by $x_{1}<x<x_{2}$ where $\left(x_{1}, x_{2}\right)=(-\infty, 1),(-\pi / 2, \pi / 2)$ or $(-\infty, \infty)$ and hence $\phi(x)$ must be defined on all of this interval For every $y$ let $\gamma_{y}(t) \mathbb{R} \rightarrow \mathbf{C}$ be given by $\gamma_{y}(t)=t+\imath y$ We are interested in the length of this curve, $L\left(\left.\gamma_{y}\right|_{\left(x, x_{0}\right)}\right)$, from any point $t=x$ to any other $t=x_{0}$ To this end we have

$$
\gamma_{y}(t)=d \gamma_{y}\left(\frac{\partial}{\partial t}\right)=\left.\frac{\partial}{\partial x}\right|_{t+\imath y}
$$

and hence the length is given by

$$
L\left(\left.\gamma_{y}\right|_{\left(x, x_{0}\right)}\right)=\int_{x}^{x_{0}} \sqrt{g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right)} d t=\int_{x}^{x_{0}} e^{\phi / 2} d t
$$

in order for the metric to be complete we should have that the length from any point in $M$ to the boundary of $M$ should be infinite and so $L\left(\left.\gamma_{y}\right|_{\left(x_{1}, x\right)}\right)$ and $L\left(\left.\gamma_{y}\right|_{\left(x, x_{2}\right)}\right)$ should be infinite for any finite $x \in\left(x_{1}, x_{2}\right)$

To get any further we first must analyze the differential equation (12 1) given below which we obtained in chapter 11

## Problem

Classify the solutions to the differential equation (12 1)

$$
\begin{equation*}
\eta^{\prime \prime}(x)=e^{\alpha x+2 \beta} \sinh \eta(x) \tag{121}
\end{equation*}
$$

where $\alpha$ equals 0 or 1 and $\beta \in \mathbb{R}$

In all that follows we shall assume $\eta(x)$ is a solution of (12 1)

## Lemma 12.1

If $\eta(x)$ is not the trivial solution then it can have at most one critical point Moreover a cratical point must either be a positive minımum or a negative maximum

## Proof :

Consıder the case when $x=c$ is a critical point of $\eta(x)$ ı $\eta^{\prime}(c)=0$ We must have $\eta(c)$ satisfying one of the following
(1) $\eta(c)=0$, $\eta(x) \equiv 0$ due to unıqueness of solutions
(ii.) $\eta(c)>0$,
$\eta^{\prime \prime}(c)=e^{\alpha x+2 \beta} \sinh (\eta(c))>0$, and therefore $\eta(c)$ is a local mınımum of $\eta(x)$ It follows $\eta$ cannot have another critical point and consequently $\eta$ has a global minımum at $c$
(111.) $\eta(c)<0$,
$\eta^{\prime \prime}(c)=e^{\alpha x+2 \beta} \sinh (\eta(c))<0$, and therefore $\eta(c)$ is a local maxımum of $\eta$ and as in (11) must be a global maxımum

## Lemma 12.2

If $x_{0}$ us a point at whuch $\eta$ is defined with $\eta\left(x_{0}\right)>0$ and $\eta^{\prime}\left(x_{0}\right)>0$ then there exists a finite number $b>x_{0}$ such that $\eta(x) \rightarrow \infty$ as $x \rightarrow b$

## Proof :

Since $\eta\left(x_{0}\right)>0$ and $\eta^{\prime}\left(x_{0}\right)>0$ it follows from lemma 121 that $\eta(x)>0$ for all $x>x_{0}$ at which $\eta(x)$ is defined

For every $x>x_{0}$ at which $\eta(x)$ is defined we shall examine $\eta^{\prime \prime}(x)$

$$
\begin{aligned}
\eta^{\prime \prime}(x) & =e^{\alpha x+2 \beta} \sinh \eta(x) \\
& =e^{\alpha x+2 \beta}\left(\eta(x)+\eta^{3}(x) / 3^{\prime}+\eta^{5}(x) / 5^{\prime}+\quad\right) \\
& >e^{2 \beta_{1}} \eta^{3}(x) \quad(\text { as } \eta(x)>0)
\end{aligned}
$$

where $\beta_{1}=\alpha x_{0} / 2+\beta+\ln (6) / 2$ Choose a positive real number $r$ such that $0<r<\min \left\{\sqrt{\eta\left(x_{0}\right)}, \sqrt[4]{e^{-\beta_{1}} \sqrt{2} \eta^{\prime}\left(x_{0}\right)}\right\}$ Then

$$
0<r^{2}<\eta\left(x_{0}\right) \text { and } 0<r^{4}\left(\frac{e^{\beta_{1}}}{\sqrt{2}}\right)<\eta^{\prime}\left(x_{0}\right)
$$

Now let $g(x)$ be the solution to the differential equation

$$
\begin{equation*}
g^{\prime \prime}(x)=e^{2 \beta_{1}} g^{3}(x) \tag{122}
\end{equation*}
$$

with initial conditions

$$
g\left(x_{0}\right)=r^{2} \quad \text { and } \quad g^{\prime}\left(x_{0}\right)=\frac{r^{4} e^{\beta_{1}}}{\sqrt{2}}
$$

From Appendix F we can see that the solution is

$$
g(x)=\frac{\sqrt{2}}{e^{\beta_{1}}(d-x)}, \quad d=x_{0}+\frac{\sqrt{2}}{e^{\beta_{1}} r^{2}}
$$

We note that $d$ is finite and that $g(x)>0$ for all $x<d$ Moreover $g(x) \rightarrow \infty$ as $x \rightarrow d$ Comparing the two problems we see that

$$
\eta^{\prime \prime}(x)>g^{\prime \prime}(x) \quad \eta^{\prime}\left(x_{0}\right)>g^{\prime}\left(x_{0}\right) \quad \eta\left(x_{0}\right)>g\left(x_{0}\right)
$$

and consequently $\eta(x)>g(x)$ for all $x>x_{0}$ Since $g(x) \rightarrow \infty$ as $x \rightarrow d$ it is obvious that $\eta(x) \rightarrow \infty$ as $x \rightarrow b$ where $x_{0}<b \leq d$ This concludes the proof of the lemma

From here on we consider the cases when $\alpha=0$ and $\alpha=1$ separately

Lemma $12.3(\alpha=0)$

Let $x_{0} \in \mathbb{R}$ be a point at which $\eta$ is defined with $\eta\left(x_{0}\right)>0$ Then there exısts unıque real numbers $\gamma_{l}<0<\gamma_{u}$ such that

- ${ }^{2} \eta^{\prime}\left(x_{0}\right)>\gamma_{u}$
$\eta(x)$ is a strictly increasing function defined on a finite interval ( $a, b$ )
- $\quad$ if $\eta^{\prime}\left(x_{0}\right)=\gamma_{u}$
$\eta(x)$ is a strictly increasing function defined on the semı-infinite interval $(-\infty, b), b \in \mathbb{R}$ with $\eta(x) \rightarrow 0$ as $x \rightarrow-\infty$
- ${ }^{\circ} f \gamma_{l}<\eta^{\prime}\left(x_{0}\right)<\gamma_{u}$
$\eta(x)$ has one critucal point - a positive minimum and is defined on a finite interval ( $a, b$ )
- ${ }^{\text {f }} \eta^{\prime}\left(x_{0}\right)=\gamma_{l}$
$\eta(x)$ is a strictly decreasing function defined on a semı-fintte interval $(a, \infty), a \in \mathbb{R}$ weth $\eta(x) \rightarrow 0$ as $x \rightarrow \infty$
- ${ }^{2} \eta^{\prime}\left(x_{0}\right)<\gamma_{l}$
$\eta(x)$ is strictly decreasing defined on a finte interval $(a, b)$


## Proof:

We first divide the solutions of (12 1) into three different categories
(a) $\eta(x)$ has critical points
(b) $\eta(x)$ has no critical points but there exists a point $x_{1} \in \mathbb{R}$ such that $\eta\left(x_{1}\right)=0$
(c) $\eta(x)$ has no critical points and there is no point $x_{1} \in \mathbb{R}$ such that $\eta\left(x_{1}\right)=0$
(a) By lemma 121 we know $\eta(x)$ has at most one critical point and it is either a positive mınımum or a negative maxımum Since $\eta\left(x_{0}\right)>0$ the critial point must be a positive minımum, say it occurs at $x=x_{1}$ Let $h_{1}(x)=\eta\left(x+x_{1}\right)$ then

$$
h_{1}^{\prime \prime}(x)=\eta^{\prime \prime}\left(x+x_{1}\right)=e^{2 \beta} \sinh \eta\left(x+x_{1}\right)=e^{2 \beta} \sinh h_{1}(x)
$$

1 e $h_{1}(x)$ satisfies the exact same differential equation with

$$
h_{1}(0)=\eta\left(x_{1}\right) \quad \text { and } \quad h_{1}^{\prime}(0)=\eta^{\prime}\left(x_{1}\right)=0
$$

hence by replacing $\eta(x)$ with $\eta\left(x+x_{1}\right)$ we may assume that the minımum occurs at $x=0$ For any $x>0$ at which $\eta(x)$ is defined it is obvious that $\eta(x)>0$ and $\eta^{\prime}(x)>0$ and hence by lemma 123 there exists some finite number $b$ with $\eta(x) \rightarrow \infty$ as $x \rightarrow b$ Now let $h_{2}(x)=\eta(-x)$ then

$$
h_{2}^{\prime \prime}(x)=\eta^{\prime \prime}(-x)=e^{2 \beta} \sinh \eta(-x)=e^{2 \beta} \sinh h_{2}(x)
$$

and again $h_{2}(x)$ satisfies the exact same differential equation with

$$
h_{2}(0)=\eta(0) \quad \text { and } \quad h_{2}^{\prime}(0)=\eta^{\prime}(0)=0
$$

since we are guaranteed unıqueness of solutions we know $h_{2}(x) \equiv \eta(x)$ Hence $\eta(x)$ is symmetric about the $y$-axis and hence $\eta(x) \rightarrow \infty$ as $x \rightarrow-b$ In the general case $\eta(x)$ is symmetric about the line $x=x_{1}$, its minımum, and has singularities at $x=x_{1} \pm b \quad \triangle$
(b) $\eta(x)$ has no critical points but there exists a point $x_{1} \in \mathbb{R}$ such that $\eta\left(x_{1}\right)=0$ As in part (a) if we replace $\eta(x)$ with $\eta\left(x+x_{1}\right)$ we may assume that $x_{1}=0$ ı e $\eta(0)=0$ Since $\eta(x)$ has no critical points it must be either strictly increasing or strictly decreasing Replacing $\eta(x)$ with $\eta(-x)$ If necessary we may assume that the function is strictly increasing Hence, at any $x>0$ where $\eta$ is defined we must have that $\eta(x)>0$ and also that $\eta^{\prime}(x)>0$ and by lemma 122 there must exist a finite number $b$ such that $\eta(x) \rightarrow \infty$ as $x \rightarrow b$ Letting $h_{4}(x)=-\eta(-x)$ we see

$$
h_{4}^{\prime \prime}(x)=-\eta^{\prime \prime}(-x)=-e^{2 \beta} \sinh \eta(-x)=e^{2 \beta} \sinh h_{4}(x)
$$

with

$$
h_{4}(0)=-\eta(0)=0 \quad \text { and } \quad h_{4}^{\prime}(0)=\eta^{\prime}(0)
$$

By unıqueness we must have $h_{4}(x) \equiv \eta(x)$ and hence we find that $\eta(x) \rightarrow-\infty$ as $x \rightarrow-b$ Hence, in the general case if $\eta$ is a strictly increasing function then $\eta(x)$ is symmetric about $x_{1}$ with $\eta(x) \rightarrow \infty$ as $x \rightarrow x_{1}+b$ and $\eta(x) \rightarrow-\infty$ as $x \rightarrow x_{1}-b$ If $\eta$ is a strictly decreasing function then $\eta(x)$ is symmetric about $x_{1}$ with $\eta(x) \rightarrow \infty$ as $x \rightarrow x_{1}-b$ and $\eta(x) \rightarrow-\infty$ as $x \rightarrow x_{1}+b$

## $\triangle$

(c) $\eta(x)$ has no critical points and there is no point $x_{\mathbf{1}} \in \mathbb{R}$ such that $\eta\left(x_{1}\right)=0$ Having no point $x_{1}$ at which $\eta\left(x_{1}\right)=0$ means that the function is strictly positive or strictly negative and since $\eta\left(x_{0}\right)>0$ we must have that $\eta$ is strictly positive Having no critical points means $\eta(x)$ is either strictly increasing or strictly decreasing Let $x_{2}$ be a point at which $\eta$ is defined Replacing $\eta(x)$ with $\eta\left(x+x_{2}\right)$ we may assmue that $x_{2}=0$ Then replacing $\eta(x)$ with $\eta(-x)$ if nessecary we may further assume that $\eta$ is strictly decreasing Since $\eta$ is both strictly positive and strictly decreasing it must exist over the entire interval $[0, \infty)$

If $\lim _{x \rightarrow \infty} \eta(x)=c \in \mathbf{R} /\{0\}$ then $\lim _{x \rightarrow \infty} \eta^{\prime \prime}(x)=\lim _{x \rightarrow \infty} \sinh (\eta(x))=d \neq 0$ Therefore, $\eta^{\prime}(x) \rightarrow \pm \infty$ as $x \rightarrow \infty$ and consequently $\eta(x) \rightarrow \pm \infty$ as $x \rightarrow \infty$ which is a contadıction Hence $\eta(x) \rightarrow 0$ or $\pm \infty$ as $x$ tends to infinity As we may assume that $\eta$ is both strictly positive and strictly decreasing we must have $\operatorname{lım}_{x \rightarrow \infty} \eta(x)=0$

If we now let $u(x)=\eta(1 / x)$ then,

$$
\begin{aligned}
u^{\prime}(x) & =\eta^{\prime}\left(\frac{1}{x}\right)\left(-x^{-2}\right) \\
& =-\frac{1}{x^{2}} \eta^{\prime}\left(\frac{1}{x}\right) \\
u^{\prime \prime}(x) & =2 x^{-3} \eta^{\prime}\left(\frac{1}{x}\right)-\frac{1}{x^{2}} \eta^{\prime \prime}\left(\frac{1}{x}\right)\left(-x^{-2}\right) \\
& =-\frac{2}{x}\left[-\frac{1}{x^{2}} \eta^{\prime}\left(\frac{1}{x}\right)\right]+\frac{1}{x^{4}}\left[\eta^{\prime \prime}\left(\frac{1}{x}\right)\right] \\
& =-\frac{2}{x} u^{\prime}(x)+\frac{1}{x^{4}} e^{2 \beta} \sinh \eta\left(\frac{1}{x}\right) \\
& =-\frac{2}{x} u^{\prime}(x)+\frac{1}{x^{4}} e^{2 \beta} \sinh u(x)
\end{aligned}
$$

thus

$$
\begin{aligned}
u^{\prime \prime}(x)+\frac{2}{x} u^{\prime}(x) & =\frac{1}{x^{4}} e^{2 \beta} \sinh u(x) \\
x^{2} u^{\prime \prime}(x)+2 x u^{\prime}(x) & =\frac{1}{x^{2}} e^{2 \beta} \sinh u(x) \\
\left(x^{2} u^{\prime}(x)\right)^{\prime} & =\frac{1}{x^{2}} e^{2 \beta} \sinh u(x)
\end{aligned}
$$

$$
\begin{aligned}
2 x^{2} u^{\prime}(x)\left(x^{2} u^{\prime}(x)\right)^{\prime} & =2 u^{\prime}(x) e^{2 \beta} \operatorname{smh} u(x) \\
\left(\left(x^{2} u^{\prime}(x)\right)^{2}\right)^{\prime} & =\left(2 e^{2 \beta} \cosh u(x)\right)^{\prime}
\end{aligned}
$$

$$
\left(x^{2} u^{\prime}(x)\right)^{2}=2 e^{2 \beta} \cosh u(x)+c_{1}, \quad c_{1} \in \mathbf{R}
$$

Now $u(x) \rightarrow 0$ as $x \downarrow 0$ because $\eta(x) \rightarrow 0$ as $x \rightarrow \infty$ Furthermore $x^{2} u^{\prime}(x)=$ $-\eta^{\prime}(1 / x) \rightarrow 0$ as $x \downarrow 0$ because $\eta^{\prime}(x) \rightarrow 0$ as $x \rightarrow \infty$ Hence $c_{1}=-2 e^{2 \beta}$ and so we have

$$
\begin{aligned}
{\left[x^{2} u^{\prime}(x)\right]^{2} } & =e^{2 \beta}(2 \cosh (u(x))-2) \\
& =e^{2 \beta}\left(e^{u(x)}+e^{-u(x)}-2\right) \\
& =e^{2 \beta}\left[e^{u(x) / 2}-e^{-u(x) / 2}\right]^{2} \\
& =\left[2 e^{\beta} \sinh (u(x) / 2)\right]^{2}
\end{aligned}
$$

giving

$$
x^{2} u^{\prime}(x)= \pm 2 e^{\beta} \sinh (u(x) / 2)
$$

Now $\eta(x)$ and $\eta^{\prime}(x)$ have opposite signs as $\eta$ is a positive function decreasing to 0 Thus $u(x)$ and $u^{\prime}(x)$ have the same sign as $u(x)=\eta(1 / x)$ and $u^{\prime}(x)=$ $-1 / x^{-2} \eta^{\prime}(1 / x)$ Thus

$$
x^{2} u^{\prime}(x)=2 e^{\beta} \sinh (u(x) / 2)
$$

and

$$
\frac{1}{2} \int \frac{1}{\sinh (u / 2)} d u=e^{\beta} \int \frac{1}{x^{2}} d x
$$

giving

$$
\ln \left|\frac{\sinh (u(x) / 4)}{\cosh (u(x) / 4)}\right|=e^{\beta}\left(-\frac{1}{x}+c_{2}\right)
$$

As $\eta$ is positive

$$
\tanh (u(x) / 4)=e^{e^{\beta}\left(-1 / x+c_{2}\right)}
$$

thus

$$
u(x)=4 \tanh ^{-1}\left(e^{e^{\beta}\left(c_{2}-1 / x\right)}\right)
$$

and

$$
\eta(x)=4 \tanh ^{-1}\left(e^{e^{\beta}\left(c_{2}-x\right)}\right)
$$

which has a singularıty at $x=c_{2}<0$ Replacıng $\eta(x)$ with $\eta\left(x-x_{2}\right)$ we arrıve back at the general case when $\eta$ is strictly decreasing and so

$$
\begin{equation*}
\eta(x)=4 \tanh ^{-1}\left(e^{e^{\beta}(c-x)}\right) \tag{123}
\end{equation*}
$$

for some $c \in \mathbb{R}$ We note that $\eta$ is defined only on the interval $(c, \infty)$ Also replacing $\eta(x)$ with $\eta(-x)$ we have

$$
\begin{equation*}
\eta(x)=4 \tanh ^{-1}\left(e^{e^{\beta}(x-c)}\right) \tag{124}
\end{equation*}
$$

the general case for when $\eta$ is strictly increasing We note that here $\eta$ is defined only on the interval $(-\infty, c)$
$\triangle$

To summarise, for any initial condition $\eta\left(x_{0}\right)=\nu>0$ there is exactly one solution to (12 1) with $\alpha=0$ of the form (124) call this $\eta_{1}(x)$ and one solution to (12 1) with $\alpha=0$ of the form (123) call this $\eta_{2}(x)$ We note that these are the only solutions to (121) with $\alpha=0$ of type (c) mentioned above Let $\gamma_{u}=\eta_{1}^{\prime}\left(x_{2}\right)$ and $\gamma_{l}=\eta_{2}^{\prime}\left(x_{2}\right)$ Obviously $\gamma_{l}<0<\gamma_{u}$

If $\eta(x)$ is another solution to (121) with $\alpha=0$ and $\eta\left(x_{0}\right)=\nu$ then the solution must be of the form (a) or (b) defined above and hence one of the following must hold
$1 \eta^{\prime}\left(x_{0}\right)>\gamma_{u}$ in which case, by lemma 124 given below, $\eta(x)>\eta_{1}(x)$ for all $x>x_{0}$ and $\eta(x)<\eta_{1}(x)$ for all $x<x_{0}$ Therefore $\eta(x)$ is of the form b defined above
$2 \eta^{\prime}\left(x_{0}\right)<\gamma_{l}$ in which case, by lemma 124 given below, $\eta(x)<\eta_{2}(x)$ for all $x>x_{0}$ and $\eta(x)>\eta_{2}(x)$ for all $x<x_{0}$ Therefore $\eta(x)$ is also of the form b above
$3 \gamma_{u}>\eta^{\prime}\left(x_{0}\right)>\gamma_{l}$ in which case $\eta(x)>\eta_{2}(x)$ for all $x>x_{0}$ and $\eta(x)>\eta_{1}(x)$ for all $x<x_{0}$ by lemma 124 and hence is of the form (a) above 1 e it has a positive minımum

This concludes the proof of the lemma

## Lemma 12.4

Let $x_{0}, a, b, c \in \mathbb{R}$ with $b>c$ and let $\eta_{1}(x)$ and $\eta_{2}(x)$ be solutions to the differentral equation

$$
\eta^{\prime \prime}=e^{2 \beta} \sinh \eta
$$

with

$$
\eta_{1}\left(x_{0}\right)=a \quad \eta_{1}^{\prime}\left(x_{0}\right)=b
$$

and

$$
\eta_{2}\left(x_{0}\right)=a \quad \eta_{2}^{\prime}\left(x_{0}\right)=c
$$

then $\eta_{1}(x)>\eta_{2}(x)$ for every $x>x_{0}$ at which both $\eta_{1}$ and $\eta_{2}$ are defined and $\eta_{1}(x)<$ $\eta_{2}(x)$ for every $x<x_{0}$ at whuch both $\eta_{1}$ and $\eta_{2}$ are defined

## Proof:

Since $\eta_{1}\left(x_{0}\right)=\eta_{2}\left(x_{0}\right)$ and $\eta_{1}^{\prime}\left(x_{0}\right)>\eta_{2}^{\prime}\left(x_{0}\right)$ there must exist some interval $\left(x_{0}, x_{1}\right)$ in which in which both $\eta_{1}(x)>\eta_{2}(x)$ and $\eta_{1}^{\prime}(x)>\eta_{2}^{\prime}(x)$ Hence $\eta_{1}^{\prime \prime}(x)=e^{2 \beta} \sinh \eta_{1}(x)$ $>e^{2 \beta} \sinh \eta_{2}(x)=\eta_{2}^{\prime \prime}(x)$ for every $x \in\left(x_{0}, x_{1}\right)$ and so $\eta_{1}\left(x_{1}\right)>\eta_{2}\left(x_{1}\right), \eta_{1}^{\prime}\left(x_{1}\right)>$ $\eta_{2}^{\prime}\left(x_{1}\right)$ and $\eta_{1}^{\prime \prime}\left(x_{1}\right)>\eta_{2}^{\prime \prime}\left(x_{1}\right)$ (assuming $\eta_{2}(x)$ exists at $x_{1}$, if not then we have already completed the proof) We then may extend beyond $x_{1}$ to another interval ( $x_{1}, x_{2}$ ) in which $\eta_{1}(x)>\eta_{2}(x), \eta_{1}^{\prime}(x)>\eta_{2}^{\prime}(x)$ and $\eta_{1}^{\prime \prime}(x)>\eta_{2}^{\prime \prime}(x)$ and the process can contınue so long as $\eta_{1}(x)$ and $\eta_{2}(x)$ are defined We use the same approach to prove $\eta_{1}(x)<\eta_{2}(x)$ for all $x$ less than $x_{0}$ in which $\eta_{1}(x)$ and $\eta_{2}(x)$ are defined

Lemma $12.5 \quad(\alpha=0)$

Let $x_{0} \in \mathbb{R}$ be a point at which $\eta$ is defined with $\eta\left(x_{0}\right)<0$ Then there exısts unıque real numbers $\gamma_{l}<0<\gamma_{u}$ such that

- ${ }^{\circ} \eta^{\prime}\left(x_{0}\right)>\gamma_{u}$
$\eta(x)$ is a strictly increasing function defined on a finite interval $(a, b)$
- ${ }^{\text {f }} \eta^{\prime}\left(x_{0}\right)=\gamma_{u}$
$\eta(x)$ is strictly increasing function defined on the semi-infinite interval $(-\infty, b), b \in \mathbb{R}$ with $\eta(x) \rightarrow 0$ as $x \rightarrow-\infty$
- if $\gamma_{l}<\eta^{\prime}\left(x_{0}\right)<\gamma_{u}$
$\eta(x)$ has only one critical point - a negative maximum and is defined on the finite interval $(a, b)$
- ${ }^{2} f \eta^{\prime}\left(x_{0}\right)=\gamma_{l}$
$\eta(x)$ is a strictly decreasing functıon defined on a semı-ınfinte strıp $(a, \infty), a \in$ $\mathbb{R}$ with $\eta(x) \rightarrow 0$ as $x \rightarrow \infty$
- ${ }^{2} \eta^{\prime}\left(x_{0}\right)<\gamma_{l}$
$\eta(x)$ is strictly decreasing and is defined on the finite $\operatorname{strip}(a, b)$


## Proof:

The proof follows by replacing $y(x)$ with $-y(x)$ and then using lemma 123

We now examıne the completeness of the metric when $\alpha=0$ Here $\phi(x)=\eta(x)-$ $\ln |H|$ and $\eta$ satisfies the differential equation

$$
\eta^{\prime \prime}(x)=4|H| \sinh \eta(x)
$$

Using lemmas 123 and 125 we can see that other than the solution $\phi(x)=-\ln |H|$, $\phi$ never exists over the entire real line Again using lemmas 123 and 125 we see there are exactly two solutions which exist over the interval $(-\infty, 1)$ which are given by

$$
\phi(x)= \pm 4 \tanh ^{-1}\left(e^{2 \sqrt{|H|}(x-1)}\right)-\ln |H|
$$

When

$$
\phi(x)=-4 \tanh ^{-1}\left(e^{2 \sqrt{|H|} \mid(x-1)}\right)-\ln |H|
$$

we see $\phi \rightarrow-\infty$ as $x \uparrow 1$ and so $e^{\phi / 2} \rightarrow 0$ as $x \uparrow 1$ hence the metric is not complete On the other hand when

$$
\phi(x)=4 \tanh ^{-1}\left(e^{2 \sqrt{|H|}(x-1)}\right)-\ln |H|
$$

$\phi(x) \rightarrow \infty$ as $x \uparrow 1$ and furthermore

$$
e^{\phi(x) / 2}=\frac{1}{\sqrt{|H|}} e^{2 \tanh ^{-1} e^{2 \sqrt{|H|}(x-1)}}>\frac{1}{\sqrt{|H|}} e^{2 \tanh ^{-1} e^{-2 \sqrt{|H|}}} \frac{1}{1-x}
$$

for all $0<x<1$ so that $\int e^{\phi / 2} d x \rightarrow \infty$ as $x \uparrow 1$ As $x \rightarrow-\infty$ we have that $\eta(x) \rightarrow 0$ and hence $e^{\phi / 2} \rightarrow \frac{1}{\sqrt{|H|}}$ It follows that $\int e^{\phi / 2} d x \rightarrow \infty$ as $x \rightarrow-\infty$ and hence the metric is complete

This leaves us with the case when $\phi$ is defined on the interval $(a, b)$ As we are looking for the metric to be complete we want $\phi \rightarrow \infty$ as $x \rightarrow a$ and $\phi \rightarrow \infty$ as $x \rightarrow b$ From lemmas 123 and 125 we see that there is only one possibility that $\eta$ has a positive mınımum From the proof of lemma 123 we recall that if $\eta$ has a positive mınımum then $\eta$ is symmetric about 1 ts mınımum Hence in order that $\eta(x) \rightarrow \infty$ as $x \rightarrow a$ and $b$ we must have $\eta^{\prime}\left(\frac{a+b}{2}\right)=0$ The question now arises as to whether these solutions give rise to $\phi$ being complete Hence let $\eta_{u}$ be a solution to the differential equation

$$
\eta^{\prime \prime}(x)=4|H| \sinh \eta(x)
$$

which has a positive minimum and singularities at some values $x=a$ and $x=b$ Also let $\eta_{l}(x)$, be the solution to the differential equation

$$
\eta^{\prime \prime}(x)=4|H| \sinh \eta
$$

with

$$
\eta(x) \rightarrow 0 \text { as } x \rightarrow-\infty \text { and } \eta(x) \rightarrow \infty \text { as } x \rightarrow b
$$

1 e let

$$
\eta_{l}(x)=4 \tanh ^{-1} e^{2 \sqrt{|F|}(x-b)}
$$

Now $\eta_{u}(x)$ and $\eta_{l}(x)$ both tend to infinity as $x$ tends to $b$ and $\eta_{u}^{\prime}\left(\frac{a+b}{2}\right)=0<\eta_{l}^{\prime}\left(\frac{a+b}{2}\right)$ so that $\eta_{u}\left(\frac{a+b}{2}\right)>\eta_{l}\left(\frac{a+b}{2}\right)$ It follows that $\eta_{u}(x)>\eta_{l}(x)$ for all $x>\frac{a+b}{2}$ and hence $\int e^{\phi / 2} d x \rightarrow \infty$ as $x \rightarrow b$ Now $\phi$ is symmetric about $\frac{a+b}{2}$ since $\eta_{u}(x)$ is and so $\int e^{\phi / 2} d x \rightarrow \infty$ as $x \rightarrow a$ Hence, the metric is complete

We now study the differential equation (12 1) with $\alpha=1$ Letting $g(x)=\eta(x-2 c)$ we observe that $g^{\prime}(x)=\eta^{\prime}(x-2 c)$ and

$$
\begin{aligned}
g^{\prime \prime}(x) & =\eta^{\prime \prime}(x-2 c)=e^{x-2 c+2 \beta} \sinh \eta(x-2 c) \\
& =e^{x+2(\beta-c)} \sinh g(x)
\end{aligned}
$$

So that $g(x)$ satisfies the same differential equation as $\eta$ with only the value of $\beta$ changıng Hence from here on we shall assume that $\eta$ satisfies the differential equation

$$
\eta^{\prime \prime}(x)=4 e^{x} \sinh \eta
$$

Lemma $12.6(\alpha=1)$

Let $x_{0}, a, b, c \in \mathbb{R}$ with $a>c$ and let $\eta_{1}(x)$ and $\eta_{2}(x)$ be solutions to the differential equation

$$
\eta^{\prime \prime}=4 e^{x} \sinh \eta
$$

with

$$
\eta_{1}\left(x_{0}\right)=a \quad \eta_{1}^{\prime}\left(x_{0}\right)=b
$$

and

$$
\eta_{2}\left(x_{0}\right)=c \quad \eta_{2}^{\prime}\left(x_{0}\right)=b
$$

then $\eta_{1}(x)>\eta_{2}(x)$ for every $x$ at which both $\eta_{1}$ and $\eta_{2}$ are defined

## Proof.

Since $\eta_{1}^{\prime \prime}\left(x_{0}\right)=4 \boldsymbol{e}^{x_{0}} \sinh \eta_{1}\left(x_{0}\right)>4 e^{x_{0}} \sinh \eta_{2}\left(x_{0}\right)=\eta_{2}^{\prime \prime}\left(x_{0}\right)$ and $\eta_{1}\left(x_{0}\right)>\eta_{2}\left(x_{0}\right)$ there must exist some interval $\left(x_{0}, x_{1}\right)$ in which both $\eta_{1}(x)>\eta_{2}(x)$ and $\eta_{1}^{\prime \prime}(x)>\eta_{2}^{\prime \prime}(x)$ and since $\eta_{1}^{\prime}\left(x_{0}\right)=\eta_{2}^{\prime}\left(x_{0}\right)$ we must have that $\eta_{1}^{\prime}(x)>\eta_{2}^{\prime}(x)$ in this interval Hence $\eta_{1}\left(x_{1}\right)>\eta_{2}\left(x_{1}\right), \eta_{1}^{\prime}\left(x_{1}\right)>\eta_{2}^{\prime}\left(x_{1}\right)$ and $\eta_{1}^{\prime \prime}\left(x_{1}\right)>\eta_{2}^{\prime \prime}\left(x_{1}\right)$ (assuming both $\eta_{1}(x)$ and $\eta_{2}(x)$ exist at $x_{1}$, if not then we have already completed the proof) We then may extend beyond $x_{1}$ to another interval $\left(x_{1}, x_{2}\right)$ in which $\eta_{1}(x)>\eta_{2}(x), \eta_{1}^{\prime}(x)>\eta_{2}^{\prime}(x)$ and $\eta_{1}^{\prime \prime}(x)>\eta_{2}^{\prime \prime}(x)$ and the process can contınue so long as both $\eta_{1}(x)$ and $\eta_{2}(x)$ are defined We use the same approach to prove $\eta_{1}(x)>\eta_{2}(x)$ for all $x$ less than $x_{0}$ in which both $\eta_{1}(x)$ and $\eta_{2}(x)$ are defined

## Lemma 127

Let $x_{0}, a, b, c \in \mathbb{R}$ with $b>c$ and let $\eta_{1}(x)$ and $\eta_{2}(x)$ be solutions to the defferentral equation

$$
\eta^{\prime \prime}=4 e^{x} \sinh \eta
$$

with

$$
\eta_{1}\left(x_{0}\right)=a \quad \eta_{1}^{\prime}\left(x_{0}\right)=b
$$

and

$$
\eta_{2}\left(x_{0}\right)=a \quad \eta_{2}^{\prime}\left(x_{0}\right)=c
$$

then $\eta_{1}(x)>\eta_{2}(x)$ for every $x>x_{0}$ at which both $\eta_{1}$ and $\eta_{2}$ are defined and $\eta_{1}(x)<$ $\eta_{2}(x)$ for every $x<x_{0}$ at which both $\eta_{1}$ and $\eta_{2}$ are defined

## Proof.

Since $\eta_{1}\left(x_{0}\right)=\eta_{2}\left(x_{0}\right)$ and $\eta_{1}^{\prime}\left(x_{0}\right)>\eta_{2}^{\prime}\left(x_{0}\right)$ there must exist some interval $\left(x_{0}, x_{1}\right)$ in which in which both $\eta_{1}(x)>\eta_{2}(x)$ and $\eta_{1}^{\prime}(x)>\eta_{2}^{\prime}(x)$ Hence $\eta_{1}^{\prime \prime}(x)=4 e^{x} \sinh \eta_{1}(x)$ $>4 e^{x} \sinh \eta_{2}(x)=\eta_{2}^{\prime \prime}(x)$ and so $\eta_{1}\left(x_{1}\right)>\eta_{2}\left(x_{1}\right), \eta_{1}^{\prime}\left(x_{1}\right)>\eta_{2}^{\prime}\left(x_{1}\right)$ and $\eta_{1}^{\prime \prime}\left(x_{1}\right)>\eta_{2}^{\prime \prime}\left(x_{1}\right)$ (assuming $\eta_{2}(x)$ exists at $x_{1}$, if not then we have already completed the proof) We
then may extend beyond $x_{1}$ to another interval $\left(x_{1}, x_{2}\right)$ in which $\eta_{1}(x)>\eta_{2}(x)$, $\eta_{1}^{\prime}(x)>\eta_{2}^{\prime}(x)$ and $\eta_{1}^{\prime \prime}(x)>\eta_{2}^{\prime \prime}(x)$ and the process can continue so long as $\eta_{1}(x)$ and $\eta_{2}(x)$ are defined We use the same approach to prove $\eta_{1}(x)<\eta_{2}(x)$ for all $x$ less than $x_{0}$ in which $\eta_{1}(x)$ and $\eta_{2}(x)$ are defined

## Lemma 128

If $\eta_{1}(x)$ ıs a solution of

$$
\eta^{\prime \prime}=4 e^{x} \sinh \eta
$$

with

$$
\eta\left(x_{0}\right)=a \quad \eta^{\prime}\left(x_{0}\right)=b
$$

and $\eta_{1}(x)$ satısfies all the following propertites

- has one critical point - a positive minimum
- has a singularity at some finite value $x_{1}<0$
then there is no solution to the dufferential equation with $\eta\left(x_{0}\right) \geq a$ which exists over the whole real line

Note any solution to the differential equation which has the same properties as $\eta_{1}$ shall be called a solution of type $U$ Also by lemma 122 every solution of type $U$ has a singularity at some point $x_{2}>0$

## Proof:

We first show that there is no solution, $\eta_{2}(x)$ to the differential equation (12 1) with $\eta_{2}\left(x_{0}\right)=a$ which exists over the whole real line

If $\eta_{2}^{\prime}\left(x_{0}\right)=b$ then $\eta_{2}(x) \equiv \eta_{1}(x)$ by uniqueness and hence does not exist over the whole real line If $\eta_{2}^{\prime}\left(x_{0}\right)>b$ then by lemma $127 \eta_{2}(x)>\eta_{1}(x)$ for all $x$ greater than $x_{0}$ and since $\eta_{1}(x)$ has a singularity at some $x$ greater than zero so must $\eta_{2}(x)$ Simılarly when $\eta_{2}^{\prime}\left(x_{0}\right)<b$ we have that $\eta_{2}(x)$ has a singularıly at some $x<0$

To prove the statement when $\eta_{2}\left(x_{0}\right)>a$ First let $\eta_{3}(x)$ be another solution to the differential equation with $\eta_{3}\left(x_{0}\right)=\eta_{2}\left(x_{0}\right)>a$ and $\eta_{3}^{\prime}\left(x_{0}\right)=b$ then by lemma $127 \eta_{3}(x)$ satisifies the same three conditions that $\eta_{1}(x)$ satisfies Then using the approach used in the previous paragraph we can show that $\eta_{2}(x)$ cannot exist over the entire real line This completes the proof

We now study more closely the properties of the solutions to the differential equation

We now note that $\eta(x) \equiv 0$ is a solution of the differential equation and from here on we shall assume $\eta$ is non-trivial Letting $\tilde{\eta}=-\eta(x)$ we note that $\tilde{\eta}^{\prime}(x)=-\eta^{\prime}(x)$ and $\tilde{\eta}^{\prime \prime}=-\eta^{\prime \prime}=-4 e^{x} \sinh \eta=4 e^{x} \sinh -\eta=4 e^{x} \sinh \tilde{\eta} 1$ e $\tilde{\eta}$ is also a solution of the differential equation Thus by replacing $\eta(x)$ with $-\eta(x)$ if necessary we may assume in all that follows that $\eta\left(x_{0}\right) \geq 0$ As we are dealing with non-trivial solutions we may further assume $\eta\left(x_{0}\right)>0$

In view of lemma 121 we remark that each solution $\eta(x)$ must satisfy one of the following conditions
a $\eta(x)$ has a positive mınımum
b $\eta(x)$ has no critical points but there exists a point $x_{1} \in \mathbb{R}$ such that $\eta\left(x_{1}\right)=0$
c $\eta(x)$ has no critical points and there is no point $x_{1} \in \mathbb{R}$ such that $\eta\left(x_{1}\right)=0$

Solutions of type a and b exist for every initial condition $\eta\left(x_{0}\right)>0$ and by lemma 122 these solutions will tend to $\infty$ as $x$ tends to some finite number $b>x_{0}$ Hence if a solution exists on an interval $\left[x_{0}, \infty\right)$ then it must be of type c above 1 e it must be either strıctly increasıng or strıctly decreasing and given $\eta\left(x_{0}\right)>0$ it must also be strıctly positive Agan by lemma 122 a strictly increasing solution would tend to infinity as $x$ tended to some finite $b>x_{0}$ hence we have that $\eta$ is stricly decreasıng If $\lim _{n \rightarrow \infty} \eta(x)=d>0$ then $\eta^{\prime \prime}(x)=4 e^{x} \sinh \eta(x)$ tends to infinty as $x$ tends to infinity which in turn implys $\eta^{\prime}(x)$ and $\eta(x)$ would tend to infinity as
$x$ tends to infinity - a contradiction Hence $\lim _{n \rightarrow \infty} \eta(x)=0$ In summary, every solution of type $c$ which exists over the interval $\left[x_{0}, \infty\right)$ must be a strictly positive and strictly decreasing and tend to zero as $x$ tends to infinity

## Lemma 129

If $x_{1}$ us a point at which $\eta$ is defined wath $\eta\left(x_{1}\right)>0$ and $\eta^{\prime}\left(x_{1}\right)<0$ then $\eta(x) \rightarrow \infty$ as $x \rightarrow b$ where $b=-\infty$ or is some finte number less than $x_{1}$

## Proof

Let $\eta$ be defined on the region $\left(b, x_{1}\right)$ where $b=-\infty$ or is just some finite number less than $x_{1}$ As $\eta$ is of type a b or c above - it is obvious that $\eta(x)>0$ for all $x \in\left(b, x_{1}\right)$ Now let $g(x)$ be the solution to the differential equation

$$
g^{\prime \prime}(x)=0 \quad g^{\prime}\left(x_{1}\right)=\eta^{\prime}\left(x_{1}\right) \quad g\left(x_{1}\right)=\eta\left(x_{1}\right)
$$

so that $g(x)=\eta^{\prime}\left(x_{1}\right) x+\eta\left(x_{1}\right)$ Comparing $\eta(x)$ to $g(x)$ we see $\eta^{\prime \prime}(x)>g^{\prime \prime}(x)$ for all $x \in\left(b, x_{1}\right), \eta^{\prime}(x)=g^{\prime}\left(x_{1}\right)$ and $\eta\left(x_{1}\right)=g\left(x_{1}\right)$ so we must have $\eta(x)>g(x)$ for all $x \in\left(b, x_{1}\right)$ Since $g(x)$ tends to $\infty$ as $x$ tends to $\infty$ we must have $\eta(x)$ tending to $\infty$ as $x$ tends to $b$

We now examine the completeness of these solutions
a $\eta(x)$ has a positive minımum Thus $\eta$ is defined on the region $(a, b)$ where b is some finite number and $a \in \mathbb{R}$ or $a=-\infty$ Also $\eta(x)$ tends to infinity as $x$ tends to both $a$ and $b$ Hence $e^{\eta(x)+x+c}$ tends to infinity as $x$ tends to both $a$ and $b$ Thus depending on the initial conditions - the surfaces could be complete
b $\quad \eta(x)$ has no critical points but there exists a point $x_{1} \in \mathbb{R}$ such that $\eta\left(x_{1}\right)=0$ Let $\eta$ be defined on the region $(a, b)$ where $a \in \mathbb{R}$ or $a=-\infty$ and $b \in \mathbb{R}$ or $b=\infty$ If we assume $\eta$ is strictly decreasing then by lemma $122 b$ would be finite and
$\eta(x) \rightarrow-\infty$ as $x \rightarrow b$ Hence $e^{\eta+x+c} \rightarrow 0$ as $x \rightarrow b$ and the solution would not be complete If we assume that $\eta(x)$ is strictly increasing then $\eta(x)$ is negative for all $x \in\left(a, x_{1}\right)$ and hence $e^{\eta+x+c}$ tends to zero as $x$ tends to $a$ Hence if the solution to the differential equation is of type $b$ then the surface will not be complete
c $\quad \eta(x)$ has no critical points and there is no point $x_{1} \in \mathbb{R}$ such that $\eta\left(x_{1}\right)=0$ Let $\eta$ be defined on the region $(a, b)$ where $a \in \mathbb{R}$ or $a=-\infty$ and $b \in \mathbb{R}$ or $b=\infty$ Since we may assume $\eta\left(x_{0}\right)>0$ we have that $\eta$ is stricly positive If it is strictly increasing then $\lim _{x \rightarrow a} \eta(x)=d \in \mathbb{R}$ and hence $e^{\eta+x+c} \rightarrow 0$ as $x \rightarrow a$ 1 e the surface is not complete On the other hand if $\eta$ is strictly decreasing we know it must exist on the interval $\left[x_{0}, \infty\right)$ and $\eta(x) \rightarrow 0$ as $x \rightarrow \infty$ Also by lemma 129 we have that $\eta(x) \rightarrow \infty$ as $x \rightarrow a$ and $a$ may be finte or equal $-\infty$ We recall from chapter 6 that we are only interested in solutions which exist over the intervals $(-\infty, \infty),(-\infty, 1)$ or ( $c, d$ ) where $c$ and $d$ are finite So if a solution exists on the interval $\left[x_{0}, \infty\right)$ it is clear that the it would be finite at $x=1$ or $x=d$ ( assuming it exists there) and hence would lead to the surface not being complete So we are only interested in solutions with $a=-\infty$ Hence if a solution of type comes rise to a complete surface we must have that the solution exists over the entire real line with $\eta(x) \rightarrow 0$ as $x \rightarrow \infty$ and $\eta(x) \rightarrow \infty$ as $x \rightarrow-\infty$ If such a solution exists it is clear that $e^{\eta+x+c} \rightarrow \infty$ as $x \rightarrow \pm \infty$

In summary, any solution to the differential equation (12 1) which gives rise to a complete metric must either

1 have a positive mınımum and is defined on a finite interval $(a, b)$

11 have a positive mınımum and is defined on a semı-mfinite interval $(-\infty, b)$

111 be a strictly positive, strictly decreasing function which tends to 0 as $x$ tends to $\infty$ and to $\infty$ as $x$ tends to $-\infty$

We note that the existence of solutions of type 11 and 111 have not been proved

We do however recall lemma 128 which states that if a solution of type 1 occurs with initial condition $\eta\left(x_{0}\right)=\gamma>0$ then a solution of type 111 cannot occur with initial condtion $\eta\left(x_{0}\right)=\gamma$, the converse of this is also true I would conjecture that no solutions of type 111 occur Letting $g(x)=\eta(1 / x)$ this conjecture is equivalent to stating that the following differential equation

$$
\left(x^{2} g^{\prime}(x)\right)^{2}=\frac{4}{x^{2}} e^{1 / x} \sinh g(x)
$$

with the mixed boundary conditons

$$
g(1)>0 \quad \lim _{x \rightarrow 0} g(x)=0
$$

has no solutions

## Chapter 13

## Minimal Surfaces

From here on we shall drop the subscript 1 from the inner product symbol 1 e

$$
\langle x, y\rangle=\langle x, y\rangle_{1}
$$

and subscript 1 will ımply partial differentiation with respect to $x$, sımılarly subscript 2 will imply differentiation with respect to $y$

We first study the mınımal surfaces when $\alpha \neq 0$ Replacing $\alpha$ with $-\alpha$ we have from equations (73) (74) and (75) that $f$ satisfies the following

$$
\begin{align*}
f_{12} & =-\alpha f_{2}+\mathbf{E} * f_{1}  \tag{array}\\
f_{22} & =\alpha f_{1}+\mathbf{E} * f_{2}  \tag{132}\\
f_{11} & =-f_{22} \tag{133}
\end{align*}
$$

If we assume that $\mathbf{E}$ is the zero vector then these equations reduce to

$$
\begin{align*}
f_{12} & =-\alpha f_{2}  \tag{134}\\
f_{22} & =\alpha f_{1}  \tag{135}\\
f_{11} & =-\alpha f_{1} \tag{136}
\end{align*}
$$

(134) and (136) imply

$$
f_{1}=-\alpha f+v_{1}
$$

for some $v_{1} \in \mathbb{R}^{3}$ This has solution

$$
f(x, y)=e^{-\alpha x} v_{2}(y)+\frac{v_{1}}{\alpha}
$$

(135) then implies

$$
v_{2}(y)=v_{3} \cos (\alpha y)+v_{4} \sin (\alpha y)
$$

for some $v_{3}, v_{4} \in \mathbb{R}^{3}$, and so $f$ is planar Hence from here on we shall assume $\mathbf{E}$ is not the zero vector We let $\epsilon_{1}=\langle\mathbf{E}, \mathbf{E}\rangle$ We recall from lemma 92 that

$$
\langle\mathbf{E}, f\rangle=a \cos (\alpha y+t) e^{-\alpha x}
$$

By switching to an associate if necessary we may assume $t=0$ and hence that

$$
\begin{equation*}
\left\langle\mathbf{E}, f_{y}(0,0)\right\rangle=0 \tag{137}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle\mathbf{E}, f_{x}(0,0)\right\rangle=-\alpha a \tag{138}
\end{equation*}
$$

## Lemma 13.1

After a translation orthogonal to $\mathbf{E}$ we may assume without loss of generality that

$$
\mathbf{E} * f_{2}=\alpha f_{1}+\left(\alpha^{2}+\epsilon_{1}\right) f-\langle\mathbf{E}, f\rangle \mathbf{E}+c
$$

where $c \in \mathbb{R}^{3}$ and $\langle\mathbf{E}, c\rangle=0 \quad$ Futhermore we may assume $c=(0,0,0)^{T}$ if $\epsilon_{1} \neq-\alpha^{2}$
(0L \& L)


(6 \& I$)$
pur


$$
\begin{aligned}
& { }^{\tau}(f * \boldsymbol{H}) * \boldsymbol{G}+{ }^{\tau} f_{z^{x}}+{ }^{\tau 1} f_{n}=
\end{aligned}
$$

$$
\begin{aligned}
& { }^{\imath} f * G+{ }^{\mathrm{L}} \mathrm{f} \partial=\quad{ }^{\mathrm{z}} \mathrm{f} f \Leftarrow(Z, \mathrm{I})
\end{aligned}
$$

osIV

$$
\begin{aligned}
& { }^{\mathrm{I}}\left(f^{\prime} * \boldsymbol{H}\right) * \text { 宜 }+{ }^{\mathrm{L}} f_{z^{0}}+{ }^{\mathrm{I}} f^{0}= \\
& \left({ }^{I} f * G\right) * G+\left({ }^{I} f O-\tau z_{f}\right) O-=
\end{aligned}
$$

$$
{ }^{I} f * G+{ }^{z} f o-\quad=\quad{ }^{\chi I} f \Leftarrow(I \varepsilon I)
$$

and we have

$$
\begin{array}{rlr}
0 & =\left\langle\mathbf{E}, \mathbf{E} * f_{2}\right\rangle \\
& =\alpha\left\langle\mathbf{E}, f_{1}\right\rangle+\alpha^{2}\langle\mathbf{E}, f\rangle+\langle\mathbf{E}, \mathbf{E} *(\mathbf{E} * f)\rangle+\langle\mathbf{E}, c\rangle & \text { by }(139) \\
& =\langle\mathbf{E}, c\rangle & \tag{1310}
\end{array}
$$

that is, $c$ is orthogonal to $\mathbf{E}$ and

$$
\mathbf{E} *(\mathbf{E} * c)=\langle\mathbf{E}, \mathbf{E}\rangle c-\langle\mathbf{E}, c\rangle \mathbf{E}=\epsilon_{1} c
$$

Now if $\epsilon_{1} \neq-\alpha^{2}$ replace $f$ by $\tilde{f}=f+\frac{c}{\alpha^{2}+\epsilon_{1}}$ then

$$
\begin{aligned}
(139) \Rightarrow \mathbf{E} * \tilde{f}_{2} & =\mathbf{E} * f_{2} \\
& =\alpha f_{1}+\alpha^{2} f+\mathbf{E} *(\mathbf{E} * f)+c \\
& =\alpha \tilde{f}_{1}+\alpha^{2} \tilde{f}-\frac{\alpha^{2} c}{\alpha^{2}+\epsilon_{1}}+\mathbf{E} *(\mathbf{E} * \tilde{f})-\mathbf{E} *\left(\mathbf{E} * \frac{c}{\alpha^{2}+c_{1}}\right)+c \\
& =\alpha \tilde{f}_{1}+\alpha^{2} \tilde{f}+\mathbf{E} *(\mathbf{E} * \tilde{f})-\frac{\alpha^{2} c}{\alpha^{2}+\epsilon_{1}}-\frac{1}{\alpha^{2}+\epsilon_{1}} \epsilon_{1} c+c \\
& =\alpha \tilde{f}_{1}+\alpha^{2} \tilde{f}+\mathbf{E} *(\mathbf{E} * \tilde{f})
\end{aligned}
$$

Thus we have now shown that if $\epsilon_{1} \neq-\alpha^{2}$ then we may assume $c=(0,0,0)^{T}$ Hence after a translation orthogonal to $\mathbf{E}$ we may assume

$$
\begin{aligned}
f & =\alpha f_{1}+\alpha^{2} f+\mathbf{E} *(\mathbf{E} * f)+c \\
& =\alpha f_{1}+\alpha^{2} f+\langle\mathbf{E}, \mathbf{E}\rangle f-\langle\mathbf{E}, f\rangle \mathbf{E}+c
\end{aligned}
$$

proving the lemma

We shall now divide our analysis into four cases

| 1 | $\\|\mathbf{E}\\|^{2}=\epsilon^{2}$ |
| :--- | :--- |
| 2 | $\\|\mathbf{E}\\|^{2}=-\epsilon^{2} \neq-\alpha^{2}$ |
| 3 | $\\|\mathbf{E}\\|^{2}=-\alpha^{2}$ |
| 4 | $\\|\mathbf{E}\\|^{2}=0$ |

for some $\epsilon>0$

## Lemma 132

If $\|\mathbf{E}\|^{2}=\epsilon^{2}>0$ then we may assume

$$
f(x, y)=e^{-\alpha x}\left(\mathbf{U}(y) \cos (\epsilon x)+\mathbf{V}(y) \sin (\epsilon x)+\frac{a}{\epsilon^{2}} \cos (\alpha y+t) \mathbf{E}\right)
$$

where $\langle\mathbf{U}(y), \mathbf{E}\rangle=\langle\mathbf{V}(y), \mathbf{E}\rangle=0$
and $f\|\mathbf{E}\|^{2}=-\epsilon^{2}<0$ where $\epsilon^{2} \neq \alpha^{2}$ then after a translation in $\mathbb{R}^{3}$ we may assume

$$
f(x, y)=\mathbf{U}(y) e^{(-\alpha+\epsilon) x}+\mathbf{V}(y) e^{(-\alpha-\epsilon) x}-\frac{a}{\epsilon^{2}} e^{-\alpha x} \cos (\alpha y+t) \mathbf{E}
$$

where $\langle\mathbf{U}(y), \mathbf{E}\rangle=\langle\mathbf{V}(y), \mathbf{E}\rangle=0$

## Proof :

In both cases we have using lemma 131 that

$$
\begin{aligned}
(133) \Rightarrow f_{11} & =-f_{22} \\
& =-\left(\alpha f_{1}+\mathbf{E} * f_{2}\right) \\
& =-\left(2 \alpha f_{1}+\left(\alpha^{2}+\epsilon_{1}\right) f-\langle\mathbf{E}, f\rangle \mathbf{E}\right)
\end{aligned}
$$

therefore

$$
f_{11}+2 \alpha f_{1}+\left(\alpha^{2}+\epsilon_{1}\right) f=a \cos (\alpha y+t) e^{-\alpha x} \mathbf{E}
$$

for a partıcular solution of (1311) we try

$$
P(x, y)=\mathbf{W}(y) e^{-\alpha x}
$$

and substituting this in, we get

$$
\begin{gathered}
\alpha^{2} \mathbf{W}(y) e^{-\alpha x}-2 \alpha^{2} \mathbf{W}(y) e^{-\alpha x}+\left(\alpha^{2}+\epsilon_{1}\right) \mathbf{W}(y) e^{-\alpha x}=a e^{-\alpha x} \cos (\alpha y+t) \mathbf{E} \\
\Rightarrow \mathbf{W}(y)=\frac{a}{\epsilon_{1}} \cos (\alpha y+t) \mathbf{E}
\end{gathered}
$$

The homogenous equation

$$
f_{11}+2 \alpha f_{1}+\left(\alpha^{2}+\epsilon_{1}\right) f=0
$$

has characterıstic equation

$$
\lambda^{2}+2 \alpha \lambda+\left(\alpha^{2}+\epsilon_{1}\right)=0
$$

which has roots

$$
\begin{aligned}
\lambda & =\frac{-2 \alpha \pm \sqrt{4 \alpha^{2}-4(1)\left(\alpha^{2}+\epsilon_{1}\right)}}{2} \\
& =\frac{-2 \alpha \pm \sqrt{-4 \epsilon_{1}}}{2} \\
& =-\alpha \pm \sqrt{-\epsilon_{1}}
\end{aligned}
$$

So in the case when $\langle E, E\rangle=\epsilon^{2}>0$ we have

$$
f(x, y)=\mathbf{U}(y) e^{-\alpha x} \cos (\epsilon x)+\mathbf{V}(y) e^{-\alpha x} \sin (\epsilon x)+(\text { a partıcular solution })
$$

and so

$$
f(x, y)=\left(\mathbf{U}(y) \cos \epsilon x+\mathbf{V}(y) \sin \epsilon x+\frac{a}{\epsilon^{2}} \cos (\alpha y+t) \mathbf{E}\right) e^{-\alpha x}
$$

Now take the innerproduct with $\mathbf{E}$ across this equation to give

$$
\langle\mathbf{E}, f\rangle=(\langle\mathbf{U}(y), \mathbf{E}\rangle \cos (\epsilon x)+\langle\mathbf{V}(y), \mathbf{E}\rangle \sin (\epsilon x)+a \cos (\alpha y+t)) e^{-\alpha x}
$$

Note from lemma $1\langle\mathbf{E}, f\rangle=a e^{-\alpha x} \cos (\alpha y+t)$

$$
\begin{gathered}
0=\langle\mathbf{U}(y), \mathbf{E}\rangle \cos (\epsilon x)+\langle\mathbf{V}(y), \mathbf{E}\rangle \sin (\epsilon x) \\
\Rightarrow\langle\mathbf{U}(y), \mathbf{E}\rangle \equiv\langle\mathbf{V}(y), \mathbf{E}\rangle \equiv 0
\end{gathered}
$$

proving the lemma when $\langle\mathbf{E}, \mathbf{E}\rangle>0$ When $\langle E, E\rangle=-\epsilon^{2}<0, \epsilon^{2} \neq \alpha^{2}$ we have

$$
f(x, y)=\mathbf{U}(y) e^{(-\alpha+\epsilon) x}+\mathbf{V}(y) e^{(-\alpha-\epsilon) x}+(\text { a partıcular solution })
$$

and so

$$
f(x, y)=\mathbf{U}(y) e^{(-\alpha+\epsilon) x}+\mathbf{V}(y) e^{(-\alpha-\epsilon) x}-\frac{a}{\epsilon^{2}} e^{-\alpha x} \cos (\alpha y+t) \mathbf{E}
$$

again taking the innerproduct with $\mathbf{E}$ across this equation to give

$$
\langle\mathbf{E}, f\rangle=\langle\mathbf{U}(y), \mathbf{E}\rangle e^{(-\alpha+\epsilon) x}+\langle\mathbf{V}(y), \mathbf{E}\rangle e^{(-\alpha-\epsilon) x}+a e^{-\alpha x} \cos (\alpha y+t)
$$

again from lemma $1\langle\mathbf{E}, f\rangle=a e^{-\alpha x} \cos (\alpha y+t)$

$$
\begin{gathered}
0=\langle\mathbf{U}(y), \mathbf{E}\rangle e^{(-\alpha+\epsilon) x}+\langle\mathbf{V}(y), \mathbf{E}\rangle e^{(-\alpha-\epsilon) x} \\
\Rightarrow\langle\mathbf{U}(y), \mathbf{E}\rangle \equiv\langle\mathbf{V}(y), \mathbf{E}\rangle \equiv 0
\end{gathered}
$$

proving lemma 132

## Lemma 13.3

If $\|\mathbf{E}\|^{2}=\epsilon^{2}>0$ then we may assume

$$
f(x, y)=r_{1} e^{-\alpha x}\left(\begin{array}{c}
\frac{a_{1}}{r_{1} \epsilon} \cos \alpha y \\
\cos \alpha y \cosh \epsilon y \sin \left(\epsilon x+r_{2}\right)+\sin \alpha y \sinh \epsilon y \sin \left(\epsilon x-r_{2}\right) \\
\cos \alpha y \sinh \epsilon y \sin \left(\epsilon x+r_{2}\right)+\sin \alpha y \cosh \epsilon y \operatorname{sm}\left(\epsilon x-r_{2}\right)
\end{array}\right)
$$

where $r_{1}, r_{2} \in \mathbb{R}$

## Proof :

We shall first use the fact that

$$
a \cos (\alpha y+t)=a_{1} \cos \alpha y+a_{2} \sin \alpha y
$$

for some $a_{1}, a_{2} \in \mathbb{R}$ and hence

$$
\langle\mathbf{E}, f\rangle=\left(a_{1} \cos \epsilon y+a_{2} \sin \epsilon y\right) e^{-\alpha x}
$$

recall from Lemma 132 that

$$
f(x, y)=e^{-\alpha x}\left(\mathbf{U}(y) \cos (\epsilon x)+\mathbf{V}(y) \sin (\epsilon x)+\frac{1}{\epsilon^{2}}\left(a_{1} \cos \alpha y+a_{2} \sin \alpha y\right) \mathbf{E}\right)
$$

so that

$$
\begin{aligned}
& f_{1}=\{\cos \epsilon x[-\alpha \mathbf{U}(y)+\epsilon \mathbf{V}(y)]+\sin \epsilon x[-\alpha \mathbf{V}(y)-\epsilon \mathbf{U}(y)] \\
&\left.-\frac{\alpha}{\epsilon^{2}}\left(a_{1} \cos \alpha y+a_{2} \sin \alpha y\right) \mathbf{E}\right\} e^{-\alpha x} \\
& f_{2}=\left\{\mathbf{U}^{\prime}(y) \cos \epsilon x+\mathbf{V}^{\prime}(y) \sin \epsilon x+\frac{\alpha}{\epsilon^{2}}\left(a_{2} \cos \alpha y-a_{1} \sin \alpha y\right) \mathbf{E}\right\} e^{-\alpha x}
\end{aligned}
$$

and

$$
\begin{gathered}
f_{11}=\left\{\cos \epsilon x\left[\alpha^{2} \mathbf{U}(y)-2 \alpha \epsilon \mathbf{V}(y)-\epsilon^{2} \mathbf{U}(y)\right]\right. \\
+\sin \epsilon x\left[\alpha^{2} \mathbf{V}(y)+2 \alpha \epsilon \mathbf{U}(y)-\epsilon^{2} \mathbf{V}(y)\right] \\
\left.+\frac{\alpha^{2}}{\epsilon^{2}}\left(a_{1} \cos \alpha y+a_{2} \sin \alpha y\right) \mathbf{E}\right\} e^{-\alpha x} \\
f_{22}=\left\{\mathbf{U}^{\prime \prime}(y) \cos \epsilon x+\mathbf{V}^{\prime \prime}(y) \sin \epsilon x-\frac{\alpha^{2}}{\epsilon^{2}}\left(a_{1} \cos \alpha y+a_{2} \sin \alpha y\right) \mathbf{E}\right\} e^{-\alpha x}
\end{gathered}
$$

Therefore the fact that

$$
f_{11}+f_{22}=0 \Rightarrow\left\{\begin{array}{l}
\mathbf{U}^{\prime \prime}(y)+\left(\alpha^{2}-\epsilon^{2}\right) \mathbf{U}(y)=2 \alpha \epsilon \mathbf{V}(y) \\
\mathbf{V}^{\prime \prime}(y)+\left(\alpha^{2}-\epsilon^{2}\right) \mathbf{V}(y)=-2 \alpha \epsilon \mathbf{U}(y)
\end{array}\right.
$$

which in turn imply

$$
\begin{aligned}
& \mathbf{U}(y)=e^{\epsilon y}\left(c_{1} \cos \alpha y+c_{3} \sin \alpha y\right)+e^{-\epsilon y}\left(c_{2} \cos \alpha y+c_{4} \sin \alpha y\right) \\
& \mathbf{V}(y)=e^{\epsilon y}\left(c_{3} \cos \alpha y-c_{1} \sin \alpha y\right)-e^{-\epsilon y}\left(c_{4} \cos \alpha y-c_{2} \sin \alpha y\right)
\end{aligned}
$$

for some constant vectors $c_{1}, c_{2}, c_{3}, c_{4} \in \mathbb{R}^{3}$ The fact that $\left\langle f_{1}, f_{2}\right\rangle=0$ imphes

$$
\begin{aligned}
& \left\langle c_{2}, c_{4}\right\rangle=\left\langle c_{1}, c_{3}\right\rangle=0 \\
& \left\langle c_{3}, c_{3}\right\rangle=\left\langle c_{1}, c_{1}\right\rangle \\
& \left\langle c_{4}, c_{4}\right\rangle=\left\langle c_{2}, c_{2}\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
& \left\langle c_{2}, c_{3}\right\rangle+\left\langle c_{1}, c_{4}\right\rangle=\frac{\alpha^{2}}{\epsilon^{2}\left(\alpha^{2}+\epsilon^{2}\right)}\left(-a_{1} a_{2}\right) \\
& \left\langle c_{1}, c_{2}\right\rangle+\left\langle c_{3}, c_{4}\right\rangle=\frac{\alpha^{2}}{\epsilon^{2}\left(\alpha^{2}+\epsilon^{2}\right)}\left(\frac{a_{2}^{2}-a_{1}^{2}}{2}\right)
\end{aligned}
$$

We now recall the fact that

$$
\langle\mathbf{U}(y), \mathbf{E}\rangle=\langle\mathbf{V}(y), \mathbf{E}\rangle=0
$$

which implies

$$
\left\langle c_{\imath}, \mathbf{E}\right\rangle=0, \quad \forall \imath=1,2,3,4
$$

once these are satısfied we find all other conditions are automatically satisfied including the fact that $\left\langle f_{1}, f_{1}\right\rangle=\left\langle f_{2}, f_{2}\right\rangle$ So we have that

$$
\begin{array}{r}
\left\langle c_{1}, \mathbf{E}\right\rangle=0, \quad\left\langle c_{2}, \mathbf{E}\right\rangle=0 \\
\left\langle c_{1}, c_{1}\right\rangle=\left\langle c_{3}, c_{3}\right\rangle \quad\left\langle c_{1}, c_{3}\right\rangle=0
\end{array}
$$

As $\langle\mathbf{E}, \mathbf{E}\rangle=\epsilon^{2}>0$ from the prelimınaries we know that $\mathbf{E}^{\perp}$ is a plane with metric $(-1,1)$ and hence $\left\|c_{2}\right\|^{2}, r=1,2,3,4$ may be positive negative or zero Let us assume $\left\|c_{1}\right\|^{2}{ }_{\text {is }}$ positive, since $\left\langle c_{3}, \mathbf{E}\right\rangle=0$ and $\left\langle c_{3}, c_{1}\right\rangle=0$ we must have that $\left\|c_{3}\right\|^{2}$ is negative, but $\left\langle c_{1}, c_{1}\right\rangle=\left\langle c_{3}, c_{3}\right\rangle$ hence this is a contradiction A simılar argument shows that $\left\|c_{1}\right\|^{2}$ cannot be negative Hence

$$
\begin{equation*}
\left\langle c_{1}, c_{1}\right\rangle=\left\langle c_{3}, c_{3}\right\rangle=0 \tag{1312}
\end{equation*}
$$

as $c_{2}$ and $c_{4}$ have simular conditions imposed we also have

$$
\begin{equation*}
\left\langle c_{2}, c_{2}\right\rangle=\left\langle c_{4}, c_{4}\right\rangle=0 \tag{1313}
\end{equation*}
$$

Thus

$$
\begin{gathered}
\left\langle c_{2}, \mathbf{E}\right\rangle=0, \quad \forall \imath=1,2,3,4 \\
\left\langle c_{1}, c_{1}\right\rangle=\left\langle c_{3}, c_{3}\right\rangle=\left\langle c_{1}, c_{3}\right\rangle=0 \\
\left\langle c_{2}, c_{2}\right\rangle=\left\langle c_{4}, c_{4}\right\rangle=\left\langle c_{2}, c_{4}\right\rangle=0 \\
\left\langle c_{2}, c_{3}\right\rangle+\left\langle c_{1}, c_{4}\right\rangle=\frac{\alpha^{2}}{\epsilon^{2}\left(\alpha^{2}+\epsilon^{2}\right)}\left(-a_{1} a_{2}\right) \\
\left\langle c_{1}, c_{2}\right\rangle+\left\langle c_{3}, c_{4}\right\rangle=\frac{\alpha^{2}}{\epsilon^{2}\left(\alpha^{2}+\epsilon^{2}\right)}\left(\frac{a_{2}^{2}-a_{1}^{2}}{2}\right)
\end{gathered}
$$

After a hyperbolic motion we may assume $\mathbf{E}=(\epsilon, 0,0)^{T}$ and $f_{y}(0,0)=\left(s_{1}, 0, s_{2}\right)^{T}$, where $s_{1}, s_{2} \in \mathbb{R}$ By switchıng to an associate we may also assume that $a_{2}=0$ and hence $f_{y}(0,0)=(0,0, s)$

As $\left\langle c_{\imath}, \mathbf{E}\right\rangle=0$ and $\left\langle c_{\imath}, c_{\imath}\right\rangle=0$ for each $\imath=1,2,3,4$, we have

$$
\begin{array}{ll}
c_{1}=\left(0, d_{1}, \jmath_{1} d_{1}\right)^{T} & c_{2}=\left(0, d_{2}, \jmath_{2} d_{2}\right)^{T} \\
c_{3}=\left(0, d_{3}, \jmath_{3} d_{3}\right)^{T} & c_{4}=\left(0, d_{4}, \jmath_{4} d_{4}\right)^{T}
\end{array}
$$

where $\jmath_{\imath}= \pm 1, \imath=1,2,3,4$ Given $\left\langle c_{1}, c_{3}\right\rangle=0$ and $\left\langle c_{2}, c_{4}\right\rangle=0$ we have $\jmath_{1}=\jmath_{3}$ and $\jmath_{2}=\jmath_{4} \quad$ If $\jmath_{1}=\jmath_{2}$ then $f$ would be planar hence letting $\jmath=\jmath_{1}$ we have $\jmath=\jmath_{1}=\jmath_{3}=-\jmath_{2}=-\jmath_{4}$ Now $\left\langle c_{2}, c_{3}\right\rangle+\left\langle c_{1}, c_{4}\right\rangle=0$ so

$$
d_{2} d_{3}+d_{1} d_{4}=0
$$

and as $f_{y}(0,0)=(0,0, s)^{T}$ we also have

$$
\epsilon d_{2}=\epsilon d_{1}-\alpha d_{3}-\alpha d_{4}
$$

hence etther $d_{4}=-d_{3}$ and $d_{2}=d_{1}$ or $\alpha d_{3}=-\epsilon d_{1}$ and $\epsilon d_{2}=\alpha d_{4}$ The second of these conditions leads to $f$ being planar Hence we now have

$$
\begin{array}{ll}
c_{1}=\left(0, d_{1}, \jmath d_{1}\right)^{T} & c_{2}=\left(0, d_{1},-\jmath d_{1}\right)^{T} \\
c_{3}=\left(0, d_{3}, \jmath d_{3}\right)^{T} & c_{4}=\left(0,-d_{3}, \jmath d_{3}\right)^{T}
\end{array}
$$

and so

$$
f(x, y)=\left(\begin{array}{c}
\frac{a_{1}}{\epsilon} \cos \alpha y e^{-\alpha x} \\
\\
\left(\left(d_{1} \cos \alpha y \cosh \epsilon y-d_{3} \sin \alpha y \sinh \epsilon y\right) \cos \epsilon x\right. \\
\left.+\left(d_{3} \cos \alpha y \cosh \epsilon y+d_{1} \sin \alpha y \sinh \epsilon y\right) \sin \epsilon x\right) e^{-\alpha x} \\
\\
\jmath\left(\left(d_{1} \cos \alpha y \sinh \epsilon y-d_{3} \sin \alpha y \cosh \epsilon y\right) \cos \epsilon x\right. \\
\left.+\left(d_{3} \cos \alpha y \sinh \epsilon y+d_{1} \sin \alpha y \cosh \epsilon y\right) \sin \epsilon x\right) e^{-\alpha x}
\end{array}\right)
$$

Checking that $f$ now satisfies the original differential equation results in $\jmath=1$ With these conditions imposed we find all other condtions are automatically satisfied Lettıng $r_{1}=\sqrt{d_{1}^{2}+d_{3}^{2}}$ and $r_{2}=\arctan \left(\frac{d_{1}}{d_{3}}\right)$ we find these simplıfy to

$$
f(x, y)=\left(\begin{array}{c}
\frac{a_{1}}{\epsilon} \cos \alpha y e^{-\alpha x} \\
r_{1} e^{-\alpha x}\left(\cos \alpha y \cosh \epsilon y \sin \left(\epsilon x+r_{2}\right)+\sin \alpha y \sinh \epsilon y \operatorname{sm}\left(\epsilon x-r_{2}\right)\right) \\
r_{1} e^{-\alpha x}\left(\cos \alpha y \sinh \epsilon y \operatorname{sm}\left(\epsilon x+r_{2}\right)+\sin \alpha y \cosh \epsilon y \sin \left(\epsilon x-r_{2}\right)\right)
\end{array}\right)
$$

and hence the lemma is proved

## Lemma 13.4

If $\|\mathbf{E}\|^{2}=-\epsilon^{2}<0, \epsilon^{2} \neq \alpha^{2}$ then we may assume

$$
f(x, y)=\left(\begin{array}{c}
r_{1} \cos (-\alpha+\epsilon) y e^{(-\alpha+\epsilon) x}+r_{2} \cos (\alpha+\epsilon) y e^{(-\alpha-\epsilon) x} \\
-r_{1} \sin (-\alpha+\epsilon) y e^{(-\alpha+\epsilon) x}-r_{2} \sin (\alpha+\epsilon) y e^{(-\alpha-\epsilon) x} \\
-\frac{a_{1}}{\epsilon} e^{-\alpha x} \cos \alpha y
\end{array}\right)
$$

where $r_{1}, r_{2} \in \mathbb{R}$

## Proof $\cdot$

From lemma 132 we have

$$
\begin{gathered}
f_{11}=\mathbf{U}(y)(-\alpha+\epsilon)^{2} e^{(-\alpha+\epsilon) x}+\mathbf{V}(y)(-\alpha-\epsilon)^{2} e^{(-\alpha-\epsilon) x}+\frac{a \alpha^{2}}{-\epsilon^{2}} e^{-\alpha x} \cos (\alpha y+t) \mathbf{E} \\
f_{22}=\mathbf{U}^{\prime \prime}(y) e^{(-\alpha+\epsilon) x}+\mathbf{V}^{\prime \prime}(y) e^{(-\alpha-\epsilon) x}-\frac{a \alpha^{2}}{-\epsilon^{2}} e^{-\alpha x} \cos (\alpha y+t) \mathbf{E}
\end{gathered}
$$

Therefore

$$
f_{11}+f_{22}=0 \Rightarrow\left\{\begin{array}{l}
\mathbf{U}^{\prime \prime}(y)+(-\alpha+\epsilon)^{2} \mathbf{U}(y)=0 \\
\mathbf{V}^{\prime \prime}(y)+(-\alpha-\epsilon)^{2} \mathbf{V}(y)=0
\end{array}\right.
$$

Hence

$$
\begin{align*}
& \mathbf{U}(y)=v_{1} \cos (-\alpha+\epsilon) y+v_{2} \sin (-\alpha+\epsilon) y  \tag{1314}\\
& \mathbf{V}(y)=v_{3} \cos (\alpha+\epsilon) y+v_{4} \sin (\alpha+\epsilon) y \tag{1315}
\end{align*}
$$

for some $v_{1}, v_{2}, v_{3}, v_{4} \in \mathbb{R}^{3}$ Also with the ald of the fact that

$$
\begin{gathered}
\langle\mathbf{U}(y), \mathbf{E}\rangle \equiv\langle\mathbf{V}(y), \mathbf{E}\rangle \equiv 0 \\
\left\langle\mathbf{U}^{\prime}(y), \mathbf{E}\right\rangle \equiv\left\langle\mathbf{V}^{\prime}(y), \mathbf{E}\right\rangle \equiv 0 \\
\left\langle E, v_{\imath}\right\rangle=0 \quad \imath=1,, 4
\end{gathered}
$$

We now have

$$
\begin{aligned}
f(x, y)= & \left(v_{1} \cos (-\alpha+\epsilon) y+v_{2} \sin (-\alpha+\epsilon) y\right) e^{(-\alpha+\epsilon) x} \\
& +\left(v_{3} \cos (\alpha+\epsilon) y+v_{4} \sin (\alpha+\epsilon) y\right) e^{(-\alpha-\epsilon) x} \\
& -\frac{a}{\epsilon^{2}} e^{-\alpha x} \cos (\alpha y+t) \mathbf{E} \\
f_{1}(x, y)= & (-\alpha+\epsilon)\left(v_{1} \cos (-\alpha+\epsilon) y+v_{2} \sin (-\alpha+\epsilon) y\right) e^{(-\alpha+\epsilon) x} \\
& +(-\alpha-\epsilon)\left(v_{3} \cos (\alpha+\epsilon) y+v_{4} \sin (\alpha+\epsilon) y\right) e^{(-\alpha-\epsilon) x} \\
& +\alpha \frac{a}{\epsilon^{2}} e^{-\alpha x} \cos (\alpha y+t) \mathbf{E} \\
f_{2}(x, y)= & (-\alpha+\epsilon)\left(-v_{1} \sin (-\alpha+\epsilon) y+v_{2} \cos (-\alpha+\epsilon) y\right) e^{(-\alpha+\epsilon) x} \\
& +(\alpha+\epsilon)\left(-v_{3} \sin (\alpha+\epsilon) y+v_{4} \cos (\alpha+\epsilon) y\right) e^{(-\alpha-\epsilon) x} \\
& +\alpha \frac{a}{\epsilon^{2}} e^{-\alpha x} \sin (\alpha y+t) \mathbf{E}
\end{aligned}
$$

Using the fact that $\left\langle f_{1}, f_{1}\right\rangle=\left\langle f_{2}, f_{2}\right\rangle^{\prime}$ we have

$$
\begin{align*}
(-\alpha+\epsilon)^{2}\langle\mathbf{U}(y), \mathbf{U}(y)\rangle & =\left\langle\mathbf{U}^{\prime}(y), \mathbf{U}^{\prime}(y)\right\rangle  \tag{1316}\\
(\alpha+\epsilon)^{2}\langle\mathbf{V}(y), \mathbf{V}(y)\rangle & =\left\langle\mathbf{V}^{\prime}(y), \mathbf{V}^{\prime}(y)\right\rangle  \tag{1317}\\
\langle\mathbf{U}(y), \mathbf{V}(y)\rangle\left(\alpha^{2}-\epsilon^{2}\right)-\left\langle\mathbf{U}^{\prime}(y), \mathbf{V}^{\prime}(y)\right\rangle & =-\frac{a^{2} \alpha^{2}}{\epsilon^{2}} \cos 2(\alpha y+t) \tag{1318}
\end{align*}
$$

Substıtutıng (13 14) and (13 15) into (13 16), (13 17) and (13 18) and evaluatıng at $y=0$ results in

$$
\begin{aligned}
& \left\langle c_{1}, c_{1}\right\rangle=\left\langle c_{2}, c_{2}\right\rangle \quad\left\langle c_{3}, c_{3}\right\rangle=\left\langle c_{4}, c_{4}\right\rangle \\
& \left\langle c_{1}, c_{3}\right\rangle+\left\langle c_{2}, c_{4}\right\rangle=\frac{\alpha^{2}}{2 \epsilon^{2}\left(\alpha^{2}-\epsilon^{2}\right)}\left(a_{2}^{2}-a_{1}^{2}\right)
\end{aligned}
$$

and on substituting these back we find

$$
\begin{aligned}
\left\langle c_{1}, c_{2}\right\rangle=0 & \left\langle c_{3}, c_{4}\right\rangle=0 \\
\left\langle c_{1}, c_{4}\right\rangle-\left\langle c_{2}, c_{3}\right\rangle= & \frac{\alpha^{2}}{2 \epsilon^{2}\left(\alpha^{2}-\epsilon^{2}\right)}\left(2 a_{1} a_{2}\right)
\end{aligned}
$$

After a hyperbolic motion we may assume $\mathbf{E}=(0,0, \epsilon)^{T}$ and $f_{y}(0,0)=\left(0, r_{1}, r_{2}\right)^{T}$ Then after switching to an associate we may assume $a_{2}=0$ and hence $f_{y}(0,0)=$ $(0, r, 0)^{T}$

As $\left\langle c_{1}, c_{1}\right\rangle=\left\langle c_{2}, c_{2}\right\rangle,\left\langle c_{1}, c_{2}\right\rangle=0$ and $\left\langle\mathbf{E}, v_{\imath}\right\rangle=0, \imath=1,2$ we have

$$
\begin{aligned}
& c_{1}=\left(r_{1} \cos t_{1}, r_{1} \sin t_{1}, 0\right)^{T} \\
& c_{2}=\left(-\jmath_{1} r_{1} \sin t_{1}, \jmath_{1} r_{1} \cos t_{1}, 0\right)^{T}
\end{aligned}
$$

simılarly

$$
\begin{aligned}
& c_{3}=\left(r_{2} \cos t_{2}, r_{2} \sin t_{2}, 0\right)^{T} \\
& c_{4}=\left(-\jmath_{2} d_{2} \sin t_{2}, \jmath_{2} r_{2} \cos t_{2}, 0\right)^{T}
\end{aligned}
$$

where $r_{1}, r_{2}, t_{1}, t_{2} \in \mathbb{R}$ and $\jmath_{\imath}= \pm 1, \imath=1,2$ Now $\left\langle c_{1}, c_{4}\right\rangle=\left\langle c_{2}, c_{3}\right\rangle$ implying $-\left(\jmath_{1}+\right.$ $\left.\jmath_{2}\right) \sin (t 1-t 2)=0$ so that etther $\jmath_{1}=-\jmath_{2}$ or $\jmath_{1}=\jmath_{2}$ and $t_{2}=t_{1}+n \pi$ for some
integer $n$ The first of these result in $f$ being planar hence

$$
\begin{aligned}
& c_{1}=\left(r_{1} \cos t_{1}, r_{1} \sin t_{1}, 0\right)^{T} \\
& c_{2}=\left(-\jmath_{1} r_{1} \sin t_{1}, \jmath_{1} r_{1} \cos t_{1}, 0\right)^{T} \\
& c_{3}=\left(\jmath_{3} r_{2} \cos t_{1}, \jmath_{3} r_{2} \sin t_{1}, 0\right)^{T} \\
& c_{4}=\left(-\jmath_{3} \jmath_{1} r_{2} \sin t_{1}, \jmath_{3} \jmath_{1} r_{2} \cos t_{1}, 0\right)^{T}
\end{aligned}
$$

where $\jmath_{3}= \pm 1$ As $f_{y}(0,0)=(0, r, 0)^{T}$ we may assume $t_{1}=n \pi / 2$ where $n$ is an integer Checking $f$ satısfies the orıgınal differential equatıons result in $\jmath_{1}=-1$ By replacing $\jmath_{3} r_{2}$ with $r_{2}$ we may assume $\jmath_{3}=1$ Finally replacing $r_{1}$ with $-r_{1}$ and $r_{2}$ with $-r_{2}$ if necessary we assume $t_{1}=0$ and hence

$$
f(x, y)=\left(\begin{array}{c}
r_{1} \cos (-\alpha+\epsilon) y e^{(-\alpha+\epsilon) x}+r_{2} \cos (\alpha+\epsilon) y e^{(-\alpha-\epsilon) x} \\
-r_{1} \sin (-\alpha+\epsilon) y e^{(-\alpha+\epsilon) x}-r_{2} \sin (\alpha+\epsilon) y e^{(-\alpha-\epsilon) x} \\
-\frac{a_{1}}{\epsilon} e^{-\alpha x} \cos \alpha y
\end{array}\right)
$$

the lemma is proved

## Lemma 13.5

If $\langle\mathbf{E}, \mathbf{E}\rangle=-\alpha^{2}$ we may assume

$$
f(x, y)=\left(\begin{array}{c}
r_{2} e^{-2 \alpha x} \cos (2 \alpha y)-\frac{r_{1}}{2 \alpha} x \\
-r_{2} e^{-2 \alpha x} \sin (2 \alpha y)+\frac{r_{1}}{2 \alpha} y \\
-\frac{a_{1}}{\alpha} e^{-\alpha x} \cos (\alpha y)
\end{array}\right)
$$

where $r_{1}, r_{2} \in \mathbb{R}$

## Proof :

Recall

$$
\begin{aligned}
(133) \Rightarrow f_{11} & =-f_{22} \\
& =-\left(\alpha f_{1}+\mathbf{E} * f_{2}\right) \\
& =-\left(2 \alpha f_{1}-\langle\mathbf{E}, f\rangle \mathbf{E}+c\right)
\end{aligned}
$$

therefore

$$
\begin{equation*}
f_{11}+2 \alpha f_{1}=a \cos (\alpha y+t) e^{-\alpha x} \mathbf{E}-\mathbf{c} \tag{1319}
\end{equation*}
$$

To find a homogenous solution we look at

$$
f_{11}=-2 \alpha f_{1}
$$

which may be written

$$
f_{1}=-2 \alpha f+\mathbf{V}
$$

which in turn has solution

$$
f=\mathbf{U} e^{-2 \alpha x}+\frac{\mathbf{V}}{2 \alpha}
$$

for a partıcular solution of (13 19) we try

$$
P(x, y)=\mathbf{W}_{1}(y) a \cos (\alpha y+t) e^{-\alpha x} \mathbf{E}+c x \mathbf{W}_{2}
$$

and substituting this in, we find

$$
\mathbf{W}_{1}=-1 / \alpha^{2} \quad \text { and } \quad \mathbf{W}_{2}=-1 /(2 \alpha)
$$

and hence

$$
f(x, y)=\mathbf{U}(y) e^{-2 \alpha x}+\frac{1}{2 \alpha} \mathbf{V}(y)-\frac{a}{\alpha^{2}} \cos (\alpha y+t) e^{-\alpha x} \mathbf{E}-\frac{1}{2 \alpha} c x
$$

Now take the innerproduct with $\mathbf{E}$ across this equation to give

$$
\langle\mathbf{U}(y), \mathbf{E}\rangle \equiv\langle\mathbf{V}(y), \mathbf{E}\rangle \equiv 0
$$

Recallıng $f_{11}+f_{22}=0$ we have

$$
f_{11}=4 \alpha^{2} \mathbf{U}(y) e^{-2 \alpha x}-a e^{-\alpha x} \cos (\alpha y+t) \mathbf{E}
$$

and

$$
f_{22}=\mathbf{U}^{\prime \prime}(y) e^{-2 \alpha x}+\frac{1}{2 \alpha} \mathbf{V}^{\prime \prime}(y)+a e^{-\alpha x} \cos (\alpha y+t) \mathbf{E}
$$

hence

$$
\begin{array}{ll}
\mathbf{V}^{\prime \prime}(y) & =0 \\
\mathbf{U}^{\prime \prime}(y)+4 \alpha^{2} \mathbf{U}(y) & =0
\end{array}
$$

and so

$$
\begin{aligned}
\mathrm{V}(y) & =v_{1} y+v_{2} \\
\mathrm{U}(y) & =v_{3} \cos (2 \alpha y)+v_{4} \sin (2 \alpha y)
\end{aligned}
$$

for some $v_{1}, v_{2}, v_{3}, v_{4} \in \mathbb{R}^{3}$ Also with the and of the fact that

$$
\begin{aligned}
& \langle\mathbf{U}(y), \mathbf{E}\rangle \equiv\langle\mathbf{V}(y), \mathbf{E}\rangle \equiv 0 \\
& \left\langle\mathbf{U}^{\prime}(y), \mathbf{E}\right\rangle \equiv\left\langle\mathbf{V}^{\prime}(y), \mathbf{E}\right\rangle \equiv 0
\end{aligned}
$$

we get

$$
\left\langle\mathbf{E}, v_{\imath}\right\rangle=0 \quad \imath=1,, 4
$$

Now $a \cos (\alpha y+t)=a_{1} \cos \alpha y+b \sin \alpha y$ for some $a_{1}, a_{2} \in \mathbb{R}$ and we let $v_{5}=-c$

$$
\begin{aligned}
f(x, y)= & \left(v_{3} \cos (2 \alpha y)+v_{4} \sin (2 \alpha y)\right) e^{-2 \alpha x}+\frac{1}{2 \alpha}\left(v_{5} x+v_{1} y+v_{2}\right) \\
& -\frac{1}{\alpha^{2}} e^{-\alpha x}\left(a_{1} \cos \alpha y+a_{2} \sin \alpha y\right) \mathbf{E} \\
f_{1}(x, y)= & -2 \alpha\left(v_{3} \cos (2 \alpha y)+v_{4} \sin (2 \alpha y)\right) e^{-2 \alpha x}+\frac{1}{2 \alpha} v_{5} \\
& +\frac{1}{\alpha} e^{-\alpha x}\left(-a_{1} \sin \alpha y-a_{2} \cos \alpha y\right) \mathbf{E} \\
f_{2}(x, y)= & -2 \alpha\left(v_{3} \cos (2 \alpha y)-v_{4} \sin (2 \alpha y)\right) e^{-2 \alpha x}+\frac{1}{2 \alpha} v_{1} \\
& +\frac{1}{\alpha} e^{-\alpha x}\left(-a_{1} \sin \alpha+a_{2} \cos \alpha y\right) \mathbf{E}
\end{aligned}
$$

after checking $\left\langle f_{1}, f_{2}\right\rangle=0$ we find

$$
\begin{aligned}
\left\langle v_{3}, v_{3}\right\rangle & =\left\langle v_{4}, v_{4}\right\rangle \\
\left\langle v_{3}, v_{4}\right\rangle & =0 \\
\left\langle v_{5}, v_{1}\right\rangle & =0 \\
\left\langle v_{4}, v_{5}\right\rangle-\left\langle v_{1}, v_{3}\right\rangle & =-a_{1} a_{2} \\
\left\langle v_{3}, v_{5}\right\rangle+\left\langle v_{1}, v_{4}\right\rangle & =\frac{a_{2}^{2}-a_{1}^{2}}{2}
\end{aligned}
$$

we also check $\left\langle f_{1}, f_{1}\right\rangle=\left\langle f_{2}, f_{2}\right\rangle$ and so

$$
\left\langle v_{5}, v_{5}\right\rangle=\left\langle v_{1}, v_{1}\right\rangle
$$

hence

$$
\begin{aligned}
f(x, y)= & \left(v_{3} \cos (2 \alpha y)+v_{4} \sin (2 \alpha y)\right) e^{-2 \alpha x}+\frac{1}{2 \alpha}\left(v_{5} x+v_{1} y+v_{2}\right) \\
& -\frac{1}{\alpha^{2}} e^{-\alpha x}\left(a_{1} \cos \alpha y+a_{2} \sin \alpha y\right) \mathbf{E}
\end{aligned}
$$

and

$$
\begin{aligned}
\left\langle v_{3}, v_{3}\right\rangle=\left\langle v_{4}, v_{4}\right\rangle & \left\langle v_{3}, v_{4}\right\rangle=0 \\
\left\langle v_{5}, v_{5}\right\rangle=\left\langle v_{1}, v_{1}\right\rangle & \left\langle v_{5}, v_{1}\right\rangle=0 \\
\left\langle v_{4}, v_{5}\right\rangle-\left\langle v_{1}, v_{3}\right\rangle=-a_{1} a_{2} & \left\langle v_{3}, v_{5}\right\rangle+\left\langle v_{1}, v_{4}\right\rangle=\frac{a_{2}^{2}-a_{1}^{2}}{2} \\
\left\langle E, v_{\imath}\right\rangle=0 & \forall \imath=1,2,3,4,5
\end{aligned}
$$

After a hyperbolic motion we may assume $\mathbf{E}=(0,0, s)^{T}$ and $f_{y}(0,0)=(0, \jmath s, 0)^{T}$ for some $s \in \mathbb{R}, \jmath= \pm 1$, By switching to an associate we may also assume $a_{2}=0$ Now if $f$ is a solution to the original differential equations then so is $f+v$ where $v \in \mathbb{R}^{3}$ and hence we may assume $v_{2}=(0,0,0)^{T}$ The conditions

$$
\begin{aligned}
\left\langle v_{3}, v_{3}\right\rangle=\left\langle v_{4}, v_{4}\right\rangle & \left\langle v_{3}, v_{4}\right\rangle=0 \\
\left\langle v_{5}, v_{5}\right\rangle=\left\langle v_{1}, v_{1}\right\rangle & \left\langle v_{5}, v_{1}\right\rangle=0 \\
\left\langle E, v_{\imath}\right\rangle=0 & \forall \imath=1,2,3,4,5
\end{aligned}
$$

1mply

$$
\begin{aligned}
& v_{1}=\left(r_{1} \cos t_{1}, r_{1} \sin t_{1}, 0\right)^{T} \\
& v_{5}=\left(-\jmath_{1} r_{1} \sin t_{1}, \jmath_{1} r_{1} \cos t_{1}, 0\right)^{T} \\
& v_{3}=\left(r_{2} \cos t_{2}, r_{2} \sin t_{2}, 0\right)^{T} \\
& v_{4}=\left(-\jmath_{2} r_{2} \sin t_{2}, \jmath_{2} r_{2} \cos t_{2}, 0\right)^{T}
\end{aligned}
$$

where $r_{1}, r_{2}, t_{1}, t_{2} \in \mathbb{R}$ and $\jmath_{1}, \jmath_{2}= \pm 1$
The conditions $\left\langle v_{3}, v_{5}\right\rangle=\left\langle v 1, v_{3}\right\rangle$ and $\left\langle v_{3}, v_{5}\right\rangle+\left\langle v_{1}, v_{4}\right\rangle=0$ together imply $\jmath_{3}=-\jmath_{1}$ and $t_{2}=t_{1}+n \pi / 2$ where $n$ is an odd integer Hence we may assume

$$
\begin{aligned}
& v_{1}=\left(r_{1} \cos t_{1}, r_{1} \sin t_{1}, 0\right)^{T} \\
& v_{5}=\left(-\jmath_{1} r_{1} \sin t_{1}, \jmath_{1} r_{1} \cos t_{1}, 0\right)^{T} \\
& v_{3}=\left(\jmath_{3} r_{2} \sin t_{2}, \jmath_{3} r_{2} \cos t_{2}, 0\right)^{T} \\
& v_{4}=\left(\jmath_{3} \jmath_{1} r_{2} \cos t_{2},-\jmath_{3} \jmath_{1} r_{2} \sin t_{2}, 0\right)^{T}
\end{aligned}
$$

where $\jmath_{3}= \pm 1$ As we may assume the first coordınate of $f_{1}(0,0)$ is 0 we have $t_{1}= \pm \pi / 2$ By replacing $\jmath_{3} r_{2}$ with $r_{2}$ we may assume $\jmath_{3}=1$ Checking that $f$ now satisfies the original differential equations results in $\jmath_{3}=1$ Finally replacing $r_{1}$ with $-r_{1}$ and $r_{2}$ with $-r_{2}$ if necessary we may assume $t=\pi / 2$ and hence

$$
f(x, y)=\left(\begin{array}{c}
r_{2} e^{-2 \alpha x} \cos (2 \alpha y)-\frac{r_{1}}{2 \alpha} x \\
-r_{2} e^{-2 \alpha x} \sin (2 \alpha y)+\frac{r_{1}}{2 \alpha} y \\
-\frac{a_{1}}{\alpha} e^{-\alpha x} \cos (\alpha y)
\end{array}\right)
$$

proving the lemma

## Lemma 136

If $\|\mathbf{E}\|^{2}=0$ then after a translation on $\mathbb{R}^{3}$ we may assume

$$
f(x, y)=\left(\mathbf{U}(y)+x \mathbf{V}(y)+\frac{a}{2} x^{2} \cos (\alpha y+t) \mathbf{E}\right) e^{-\alpha x}
$$

where $\langle\mathbf{V}(y), \mathbf{E}\rangle=0$ and $\langle\mathbf{U}(y), \mathbf{E}\rangle=a \cos (\alpha y+t)$

## Proof :

$$
\begin{aligned}
f_{11} & =-f_{22} \\
& =-\left(\alpha f_{1}+\mathbf{E} * f_{2}\right) \\
& =-\left(2 \alpha f_{1}+\alpha^{2} f-\langle\mathbf{E}, f\rangle \mathbf{E}\right)
\end{aligned}
$$

therefore

$$
\begin{equation*}
f_{11}+2 \alpha f_{1}+\alpha^{2} f=\grave{a} \cos (\alpha y+t) e^{-\alpha x} \mathbf{E} \tag{1320}
\end{equation*}
$$

The homogenous equation

$$
f_{11}+2 \alpha f_{1}+\alpha^{2} f=0
$$

has characterıstic equation

$$
\lambda^{2}+2 \alpha \lambda+\alpha^{2}=0
$$

which has a single root

$$
\lambda=-\alpha
$$

Thus we have

$$
f(x, y)=(\mathbf{U}(y)+x \mathbf{V}(y)) e^{-\alpha x}+(\text { a particular solution })
$$

for a particular solution of (1320) we try

$$
P(x, y)=\mathbf{W}(y) \frac{x^{2}}{2} e^{-\alpha x}
$$

and substituting in we get

$$
\mathbf{W}(y)\left(1-2 \alpha x+\alpha^{2} \frac{x^{2}}{2}+2 \alpha x-\alpha^{2} x^{2}+\alpha^{2} \frac{x^{2}}{2}\right) e^{-\alpha x}=a \cos (\alpha y+t) e^{-\alpha x} \mathbf{E}
$$

and sımplifying we get

$$
\mathbf{W}(y)=a \cos (\alpha y+t) \mathbf{E}
$$

Thus

$$
f(x, y)=\left(\mathbf{U}(y)+x \mathbf{V}(y)+\frac{a}{2} x^{2} \cos (\alpha y+t) \mathbf{E}\right) e^{-\alpha x}
$$

Now take the innerproduct with $\mathbf{E}$ across this equation to give

$$
\langle\mathbf{E}, f\rangle=\left(\langle\mathbf{U}(y), \mathbf{E}\rangle+x\langle\mathbf{V}(y), \mathbf{E}\rangle+\frac{a}{2} \cos (\alpha y+t)\langle\mathbf{E}, \mathbf{E}\rangle\right) e^{-\alpha x}
$$

from lemma $1\langle\mathbf{E}, f\rangle=a e^{-\alpha x} \cos (\alpha y+t)$

$$
\begin{aligned}
& a \cos (\alpha y+t) e^{-\alpha x}=(\langle\mathbf{U}(y), \mathbf{E}\rangle+x\langle\mathbf{V}(y), \mathbf{E}\rangle) e^{-\alpha x} \\
& \Rightarrow\langle\mathbf{U}(y), \mathbf{E}\rangle=a \cos (\alpha y+t) \text { and }\langle\mathbf{V}(y), \mathbf{E}\rangle \equiv 0
\end{aligned}
$$

proving the lemma

## Lemma 13.7

If $\|\mathbf{E}\|^{2}=0$ then we may assume

$$
f(x, y)=\frac{a_{1} r_{1}}{2} e^{-\alpha x}\left(\begin{array}{c}
{\left[\frac{1}{r_{1}^{2}}+\frac{1}{\alpha^{2}}+\left(x+r_{2}\right)^{2}-y^{2}\right] \cos \alpha y+2 y\left(x+r_{2}\right) \sin \alpha y} \\
\frac{2}{r_{1}}\left(-y \cos \alpha y+\left(x+r_{2}\right) \sin \alpha y\right) \\
{\left[\frac{-1}{r_{1}^{2}}+\frac{1}{\alpha^{2}}+\left(x+r_{2}\right)^{2}-y^{2}\right] \cos \alpha y+2 y\left(x+r_{2}\right) \sin \alpha y}
\end{array}\right)
$$

for some $r_{1}, r_{2} \in \mathbb{R}$

## Proof :

We shall first use the fact that

$$
a \cos (\alpha y+t)=a_{1} \cos \alpha y+a_{2} \sin \alpha y
$$

for some $a_{1}, a_{2} \in \mathbb{R}$ and hence

$$
\langle\mathbf{E}, f\rangle=\left(a_{1} \cos \epsilon y+a_{2} \sin \epsilon y\right) e^{-\alpha x}
$$

Now

$$
f(x, y)=\left(\mathbf{U}(y)+x \mathbf{V}(y)+\frac{x^{2}}{2}\left(a_{1} \cos \alpha y+a_{2} \sin \alpha y\right) \mathbf{E}\right) e^{-\alpha x}
$$

and hence

$$
\begin{aligned}
& f_{1}=\left(-\alpha \mathbf{U}+(1-\alpha x) \mathbf{V}+\left(x-\frac{\alpha x^{2}}{2}\right)\left(a_{1} \cos \alpha y+a_{2} \sin \alpha y\right) E\right) e^{-\alpha x} \\
& f_{2}=\left(\mathbf{U}^{\prime}(y)+x \mathbf{V}^{\prime}(y)-\frac{\alpha x^{2}}{2}\left(-a_{2} \cos \alpha y+a_{1} \sin \alpha y\right) \mathbf{E}\right) e^{-\alpha x}
\end{aligned}
$$

and

$$
\begin{aligned}
f_{11} & =\left(\alpha^{2} \mathbf{U}+\left(\alpha^{2} x-2 \alpha\right) \mathbf{V}+\left(\frac{\alpha^{2} x^{2}}{2}-2 \alpha x+1\right)\left(a_{1} \cos \alpha y+a_{2} \sin \alpha y\right) E\right) e^{-\alpha x} \\
f_{22} & =\left(\mathbf{U}^{\prime \prime}(y)+x \mathbf{V}^{\prime \prime}(y)-\frac{\alpha^{2} x^{2}}{2}\left(a_{1} \cos \alpha y+a_{2} \sin \alpha y\right) \mathbf{E}\right) e^{-\alpha x}
\end{aligned}
$$

Since $f_{11}+f_{22}=0$ we have

$$
\left(\mathbf{U}^{\prime \prime}(y)+\alpha^{2} \mathbf{U}-2 \alpha \mathbf{V}\right)+\left(x \mathbf{V}^{\prime \prime}(y)+\alpha^{2} x \mathbf{V}\right)=(2 \alpha x-1)\left(a_{1} \cos \alpha y+a_{2} \sin \alpha y\right) \mathbf{E}
$$

therefore

$$
\begin{aligned}
& \mathbf{V}^{\prime \prime}(y)+\alpha^{2} \mathbf{V}(y)=2 \alpha\left(a_{1} \cos \alpha y+a_{2} \sin \alpha y\right) \mathbf{E} \\
& \mathbf{U}^{\prime \prime}(y)+\alpha^{2} \mathbf{U}(y)=2 \alpha \mathbf{V}(y)-\left(a_{1} \cos \alpha y+a_{2} \sin \alpha y\right) \mathbf{E}
\end{aligned}
$$

ımplyıng

$$
\begin{aligned}
\mathbf{V}(y)= & \left(c_{1}+\frac{1}{2 \alpha}\left(a_{1}(\cos 2 \alpha y-1)+a_{2} \sin 2 \alpha y\right) \mathbf{E}\right) \cos \alpha y \\
& +\left(c_{2}+\frac{1}{2 \alpha}\left(a_{2}(1-\cos 2 \alpha y)+a_{1} \sin 2 \alpha y\right) \mathbf{E}\right) \sin \alpha y \\
& +y\left[-a_{2} \cos \alpha y+a_{1} \sin \alpha y\right] \mathbf{E}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{U}(y)= & \left(c_{3}+\frac{1}{2 \alpha}\left(c_{1}(\cos 2 \alpha y-1)+\left(c_{2}+\frac{a_{2}}{\alpha} \mathbf{E}\right) \sin 2 \alpha y\right)\right) \cos \alpha y \\
& +\left(c_{4}+\frac{1}{2 \alpha}\left(c_{2}+\frac{a_{2}}{\alpha} \mathbf{E}-\left(c_{2}+\frac{a_{2}}{\alpha} \mathbf{E}\right) \cos 2 \alpha y+c_{1} \sin 2 \alpha y\right) \sin \alpha y\right. \\
& +y\left[\left(-c_{2}+\frac{1}{2 \alpha}\left(a_{1} \sin 2 \alpha y-a_{2}(1+\cos 2 \alpha y)\right) \mathbf{E}\right) \cos \alpha y\right. \\
& \left.+\left(c_{1}+\frac{1}{2 \alpha}\left(-a_{1}(1+\cos 2 \alpha y)-a_{2} \sin 2 \alpha y\right) \mathbf{E}\right) \sin \alpha y\right] \\
& -\frac{y^{2}}{2}\left[a_{1} \cos \alpha y+a_{2} \sin \alpha y\right] \mathbf{E}
\end{aligned}
$$

for some $c_{1}, c_{2}, c_{3}, c_{4} \in \mathbb{R}^{3}$ As

$$
\begin{aligned}
& \langle\mathbf{E}, \mathbf{U}(y)\rangle=a_{1} \cos \epsilon y+a_{2} \sin \epsilon y \\
& \langle\mathbf{E}, \mathbf{V}(y)\rangle=0
\end{aligned}
$$

we see that

$$
\begin{gathered}
\left\langle\mathbf{E}, c_{1}\right\rangle=0 \quad\left\langle\mathbf{E}, c_{2}\right\rangle=0 \\
\left\langle\mathbf{E}, c_{3}\right\rangle=a_{1} \quad\left\langle\mathbf{E}, c_{4}\right\rangle=a_{2}
\end{gathered}
$$

The condition $\left\langle f_{1}, f_{2}\right\rangle=0$ results in the following equalities

$$
\begin{aligned}
& \left\langle c_{1}, c_{2}\right\rangle=-a_{1} a_{2} \\
& \left\langle c_{2}, c_{2}\right\rangle=\left\langle c_{1}, c_{1}\right\rangle+a_{1}^{2}-a_{2}^{2} \\
& \left\langle c_{1}, c_{4}\right\rangle=-\left\langle c_{2}, c_{3}\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
& \left\langle c_{2}, c_{4}\right\rangle=\left\langle c_{1}, c_{3}\right\rangle-\frac{1}{\alpha}\left(a_{1}^{2}+\left\langle c_{1}, c_{1}\right\rangle\right) \\
& \left\langle c_{3}, c_{4}\right\rangle=-\frac{1}{\alpha}\left\langle c_{2}, c_{3}\right\rangle \\
& \left\langle c_{4}, c_{4}\right\rangle=\frac{1}{\alpha^{2}}\left\langle c_{1}, c_{1}\right\rangle-\frac{2}{\alpha}\left\langle c_{1}, c_{3}\right\rangle+\left\langle c_{3}, c_{3}\right\rangle
\end{aligned}
$$

these also satısfy the condition $\left\langle f_{1}, f_{1}\right\rangle=\left\langle f_{2}, f_{2}\right\rangle$ and so we have

$$
\begin{aligned}
f(x, y)= & \left(c_{3}+\frac{1}{2 \alpha}\left(c_{1}(\cos 2 \alpha y-1)+\left(c_{2}+\frac{a_{2}}{\alpha} \mathbf{E}\right) \sin 2 \alpha y\right)\right) \cos \alpha y \\
& +\left(c_{4}+\frac{1}{2 \alpha}\left(c_{2}+\frac{a_{2}}{\alpha} \mathbf{E}-\left(c_{2}+\frac{a_{2}}{\alpha} \mathbf{E}\right) \cos 2 \alpha y+c_{1} \sin 2 \alpha y\right) \sin \alpha y\right. \\
& +x\left[\left(c_{1}+\frac{1}{2 \alpha}\left(a_{1}(\cos 2 \alpha y-1)+a_{2} \sin 2 \alpha y\right) \mathbf{E}\right) \cos \alpha y\right. \\
& \left.+\left(c_{2}+\frac{1}{2 \alpha}\left(a_{2}(1-\cos 2 \alpha y)+a_{1} \sin 2 \alpha y\right) \mathbf{E}\right) \sin \alpha y\right] \\
& +y\left[\left(-c_{2}+\frac{1}{2 \alpha}\left(a_{1} \sin 2 \alpha y-a_{2}(1+\cos 2 \alpha y)\right) \mathbf{E}\right) \cos \alpha y\right. \\
& \left.+\left(c_{1}+\frac{1}{2 \alpha}\left(-a_{1}(1+\cos 2 \alpha y)-a_{2} \sin 2 \alpha y\right) \mathbf{E}\right) \sin \alpha y\right] \\
& \left.\left\{\frac{x^{2}-y^{2}}{2}\left(a_{1} \cos \alpha y+a_{2} \sin \alpha y\right)+x y\left(-a_{2} \cos \alpha y+a_{1} \sin \alpha y\right)\right\} \mathbf{E}\right] e^{-\alpha x}
\end{aligned}
$$

for some constants $a_{1}, a_{2} \in \mathbb{R}$ and some constant vectors $c_{1}, c_{2}, c_{3}, c_{4} \in \mathbb{R}^{3}$ with the following conditions

$$
\begin{array}{cl}
\left\langle\mathbf{E}, c_{1}\right\rangle=0 & \left\langle\mathbf{E}, c_{2}\right\rangle=0 \\
\left\langle c_{1}, c_{2}\right\rangle=-a_{1} a_{2} & \left\langle c_{2}, c_{2}\right\rangle=\left\langle c_{1}, c_{1}\right\rangle+a_{1}^{2}-a_{2}^{2} \\
\left\langle\mathbf{E}, c_{3}\right\rangle=a_{1}, & \left\langle\mathbf{E}, c_{4}\right\rangle=a_{2} \\
\left\langle c_{1}, c_{4}\right\rangle=-\left\langle c_{2}, c_{3}\right\rangle & \left\langle c_{3}, c_{4}\right\rangle=-\frac{1}{\alpha}\left\langle c_{2}, c_{3}\right\rangle \\
\left\langle c_{2}, c_{4}\right\rangle= & \left\langle c_{1}, c_{3}\right\rangle-\frac{1}{\alpha}\left\langle a_{1}^{2}+\left\langle c_{1}, c_{1}\right\rangle\right) \\
\left\langle c_{4}, c_{4}\right\rangle= & \frac{1}{\alpha^{2}}\left\langle c_{1}, c_{1}\right\rangle-\frac{2}{\alpha}\left\langle c_{1}, c_{3}\right\rangle+\left\langle c_{3}, c_{3}\right\rangle
\end{array}
$$

After a hyperbolic motion we may assume the third component of $f_{y}(0,0)$ is zero Then after a further hyperbolic motion we may also assume $\mathbf{E}=\left(r_{1}, 0, r_{1}\right)^{T}$ for some $r \in \mathbb{R}$ Using (137) we have that the first component of $f_{y}(0,0)$ must also be zero and hence we may assume

$$
\mathbf{E}=\left(r_{1}, 0, r_{1}\right)^{T} \quad \text { and } \quad f_{y}(0,0)=(0, b, 0)^{T}
$$

for some $r_{1}, b \in \mathbb{R}$ Now $\left\langle f_{x}(0,0), f_{y}(0,0)\right\rangle=0$ and so the second component of $f_{x}(0,0)$ is zero Also from (138) we have $\left\langle\mathbf{E}, f_{x}(0,0)\right\rangle=-\alpha a$ and hence

$$
f_{x}(0,0)=\left(c, 0, c+\frac{\alpha a}{r_{1}}\right)^{T}
$$

for some $c \in \mathbb{R} \quad$ Finally $\left\langle f_{x}(0,0), f_{x}(0,0)\right\rangle=\left\langle f_{y}(0,0), f_{y}(0,0)\right\rangle$ and hence $c=$ $\frac{r_{1}}{2 \alpha a}\left(-b^{2}-\frac{\alpha^{2} a^{2}}{r_{1}^{2}}\right)$ and so

$$
f_{x}(0,0)=\left(-\frac{b}{2}\left(\frac{r_{1} b}{\alpha a}+\frac{\alpha a}{r_{1} b}\right), 0,-\frac{b}{2}\left(\frac{r_{1} b}{\alpha a}-\frac{\alpha a}{r_{1} b}\right)\right)^{T}
$$

From above we know the value of $\left\langle\mathbf{E}, c_{\imath}\right\rangle$ for $\imath=1,, 4$ and given $\mathbf{E}=\left(r_{1}, 0, r_{1}\right)^{T}$ we have

$$
\begin{aligned}
& c_{1}=\left(d_{1}, d_{2}, d_{1}\right)^{T} \\
& c_{2}=\left(d_{3}, d_{4}, d_{3}\right)^{T} \\
& c_{3}=\left(d_{5}+a_{1} / r_{1}, d_{6}, d_{5}\right)^{T} \\
& c_{4}=\left(d_{7}+a_{2} / r_{1}, d_{8}, d_{7}\right)^{T}
\end{aligned}
$$

Since $\left\langle c_{1}, c_{2}\right\rangle=0$ then $d_{2} d_{4}=0{ }_{1} \mathrm{e}$ ether $d_{2}=0$ or $d_{4}=0 \quad$ Also $\left\langle c_{2}, c_{2}\right\rangle=\left\langle c_{1}, c_{1}\right\rangle+$ $a_{1}^{2}-a_{2}^{2}$ hence $d_{4}^{2}=d_{2}^{2}+a_{1}^{2}$ Now if $d_{4}=0$ we would have to have $a_{1}=0$ makıng $f$ planar thus $d_{2}=0$ and $d_{4}=\jmath a_{1}$ where $\jmath= \pm 1$ Checkıng $\left\langle c_{1}, c_{4}\right\rangle+\left\langle c_{2}, c_{3}\right\rangle=0$ leads to $d_{3}=-\jmath r_{1} d_{6}$, where $\jmath= \pm 1$ and the condition $\left\langle c_{1}, c_{4}\right\rangle+\frac{1}{\alpha}\left\langle c_{2}, c_{3}\right\rangle=0$ imples $d_{7}=-r_{1} / a_{1} d_{6} d_{8}$ Given $\left\langle c_{2}, c_{4}\right\rangle=\left\langle c_{1}, c_{3}\right\rangle-\frac{1}{\alpha}\left(a_{1}^{2}+\left\langle c_{1}, c_{1}\right\rangle\right)$ we have $d_{1}=$ $a_{1} r_{1} / \alpha+d_{8} J r_{1}$ Finally checking $\left\langle c_{4}, c_{4}\right\rangle+2 / \alpha\left\langle c_{1}, c_{3}\right\rangle=\left\langle c_{3}, c_{3}\right\rangle+1 / \alpha^{2}\left\langle c_{1}, c_{1}\right\rangle$ results in $d_{5}=r_{1} /\left(2 a_{1}\right)\left(2 a_{1}^{2} / a^{2}-d_{6}^{2}+d_{8}^{2}+2 a_{1} d_{8} \jmath / \alpha-a_{1}^{2} / r_{1}^{2}\right)$ Now as

$$
f_{x}(0,0)=\left(-\frac{b}{2}\left(\frac{r_{1} b}{\alpha a}+\frac{\alpha a}{r_{1} b}\right), 0,-\frac{b}{2}\left(\frac{r_{1} b}{\alpha a}-\frac{\alpha a}{r_{1} b}\right)\right)^{T}
$$

we must have $d_{6}=0$ and since $f_{y}(0,0)=(0, b, 0)^{T}$ we also have $d_{8}=b / \alpha$ Checking that $f$ satısfies the original differential equations imphes $\jmath=1$ Thus on simplifying we have

$$
f(x, y)=\frac{a_{1} r_{1}}{2} e^{-\alpha x}\left(\begin{array}{c}
{\left[\frac{1}{r_{1}^{2}}+\frac{1}{\alpha^{2}}+\left(x+r_{2}\right)^{2}-y^{2}\right] \cos \alpha y+2 y\left(x+r_{2}\right) \sin \alpha y} \\
\frac{2}{r_{1}}\left(-y \cos \alpha y+\left(x+r_{2}\right) \sin \alpha y\right) \\
{\left[\frac{-1}{r_{1}^{2}}+\frac{1}{\alpha^{2}}+\left(x+r_{2}\right)^{2}-y^{2}\right] \cos \alpha y+2 y\left(x+r_{2}\right) \sin \alpha y}
\end{array}\right)
$$

where $r_{2}=\frac{b+\alpha}{a_{1} \alpha}$ and so the lemma is proved

We now study the Minımal Surfaces when $\alpha=0$

The differential equations (920) and (922) in this case reduce to the form

$$
\begin{align*}
f_{2} & =E * f  \tag{1321}\\
f_{11}+f_{22} & =0 \tag{1322}
\end{align*}
$$

We note that $\tilde{f}=f+s E, s \in \mathbb{R}$ satisfies the differential equation once $f$ does Now (1321) and (13 22) imply

$$
\left\langle E, f_{2}\right\rangle=\left\langle E, f_{11}\right\rangle=0
$$

and so

$$
\langle E, f\rangle=-c(x+d)
$$

for some $c, d \in \mathbb{R}$

$$
\begin{aligned}
f_{22} & =E *(E * f) \\
& =\langle E, E\rangle f-\langle E, f\rangle E \\
& =\langle E, E\rangle f+c(x+d) E
\end{aligned}
$$

and so

$$
\begin{equation*}
f_{22}-\|E\|^{2} f=c(x+d) E \tag{1323}
\end{equation*}
$$

At this point we shall split the analysis into three cases

- $\langle E, E\rangle=0$
- $\langle E, E\rangle<0$
- $\langle E, E\rangle>0$
- $\langle E, E\rangle=0$

$$
\begin{aligned}
f_{22} & =\mathbf{c}(x+d) E \\
f_{2} & =c(x y+d y) E+g_{1}(x) \quad \text { where } \mathrm{g}_{1}(\mathrm{x}) \in \mathbb{R}^{3} \\
f & =\frac{c}{2}\left(x y^{2}+d y^{2}\right) E+g_{1}(x) y+g_{2}(x) \quad \text { where } \mathrm{g}_{2}(\mathrm{x}) \in \mathbb{R}^{3} \\
f_{1} & =\frac{c}{2} y^{2} E+g_{1}^{\prime}(x) y+g_{2}^{\prime}(x) \\
f_{11} & =g_{1}^{\prime \prime}(x) y+g_{2}^{\prime \prime}(x) \\
& =-f_{22}=-c(x+d) E
\end{aligned}
$$

Hence $g_{1}^{\prime \prime}(x)=0$ and $g_{2}^{\prime \prime}(x)=-c(x+d) E, 1 \mathrm{e}$

$$
g_{1}(x)=v_{2} x+v_{3}
$$

and

$$
g_{2}(x)=-c\left(\frac{1}{6} x^{3}+\frac{1}{2} d x^{2}\right) E+v_{4} x+v_{5}
$$

resulting in

$$
f=\frac{c}{6}\left(-x^{3}+3 x y^{2}+3 d y^{2}-3 d x^{2}\right) E+v_{2} x y+v_{3} y+v_{4} x+v_{5}
$$

and so

$$
\begin{aligned}
E * f & =E * v_{2} x y+E * v_{3} y+E * v_{4} x+E * v_{5} \\
& =f_{2} \\
& =c x y E+c d y E+v_{2} x+v_{3}
\end{aligned}
$$

g1ving

$$
\begin{aligned}
& E * v_{2}=c E \\
& E * v_{3}=c d E \\
& E * v_{4}=v_{2} \\
& E * v_{5}=v_{3}
\end{aligned}
$$

Now after a hyperbolic rotation and a stretchıng we may assume that $E=(r, 0, r)^{T}$ and hence we have

$$
\begin{aligned}
v_{2} & =\left(s_{2}, c, s_{2}\right)^{T} \\
v_{3} & =\left(s_{3}, c d, s_{3}\right)^{T} \\
v_{4} & =\left(s_{4}, \frac{s_{2}}{r}, s_{4}+\frac{c}{r}\right)^{T} \\
v_{5} & =\left(s_{5}, \frac{s_{3}}{r}, s_{5}+\frac{c d}{r}\right)^{T}
\end{aligned}
$$

where $s_{2}, s_{3}, s_{4}, s_{5} \in \mathbb{R}$ We may also assume $f_{y}(0,0)=(0,-b l, 0)^{T}$ and $f_{x}(0,0)=$ $\left(\left(b^{2} r^{2}+1\right) l /(2 r), 0,\left(b^{2} r^{2}-1\right) l /(2 r)\right)^{T}$ for some $b, l \in \mathbb{R}$ and hence

$$
\begin{aligned}
c & =-l \\
d & =b \\
s_{3} & =0 \\
s_{2} & =0 \\
s_{4} & =l b / 2(r b+1 /(r b))
\end{aligned}
$$

resulting in

$$
f(x, y)=\frac{l r}{6}(x+b)\left(\begin{array}{c}
3 / r^{2}+(x+b)^{2}-3 y^{2} \\
-6 / r y \\
-3 / r^{2}+(x+b)^{2}-3 y^{2}
\end{array}\right)+\left(\begin{array}{c}
s_{6}+c d /(2 r)+c d^{3} r / 6 \\
0 \\
s_{6}+c d /(2 r)+c d^{3} r / 6
\end{array}\right)
$$

for some $s_{6} \in \mathbb{R}$

It is worth noting that $\left\langle f_{y}, f_{y}\right\rangle=\left\langle f_{x}, f_{x}\right\rangle$ and $\left\langle f_{x}, f_{y}\right\rangle=0$ for all $x, y \in \mathbb{R}$ and so $f$ satısfies all the conditions for mınımality

Recall that if $f$ is a solution of the differential equations then so is $\tilde{f}=f+s E$ hence we may assume $s_{6}=-c d /(2 r)-c d^{3} r / 6$ and hence

$$
f(x, y)=\frac{l r}{6}(x+b)\left(\begin{array}{c}
3 / r^{2}+(x+b)^{2}-3 y^{2}  \tag{1324}\\
-6 / r y \\
-3 / r^{2}+(x+b)^{2}-3 y^{2}
\end{array}\right)
$$

This is Enneper's surface of the second kind, which is a mınımal spacelike surface of revolution

- $\langle E, E\rangle=e^{2}>0$

Hence we need to solve

$$
f_{22}-e^{2} f=c(x+d) E
$$

Solution of Homogenous Equation

$$
g_{1}(x, y)=v_{2}(x) \sinh e y+v_{3}(x) \cosh e y
$$

for some $v_{2}(x), v_{3}(x) \in \mathbb{R}^{3}$ A particular solution is

$$
g_{2}(x, y)=-\frac{c}{e^{2}}(x+d) E
$$

Therefore the solution of this equation is

$$
\begin{aligned}
f & =v_{2}(x) \sinh e y+v_{3}(x) \cosh e y-c e^{-2}(x+d) E \\
E * f & =\sinh e y E * v_{2}(x)+\cosh e y E * v_{3}(x) \\
& =f_{2} \\
& =e v_{2}(x) \cosh e y+e v_{3}(x) \sinh e y
\end{aligned}
$$

hence $e v_{3}(x)=E * v_{2}(x)$ and $e v_{2}(x)=E * v_{3}(x)$

$$
\begin{aligned}
-f_{22} & =-e^{2} v_{2}(x) \sinh e y-e^{2} v_{3}(x) \cosh e y \\
f_{11} & =v_{2}^{\prime \prime}(x) \sinh e y+v_{3}^{\prime \prime}(x) \cosh e y
\end{aligned}
$$

hence

$$
\begin{aligned}
& v_{2}(x)=v_{4} \cos e x+v_{5} \sin e x \\
& v_{3}(x)=v_{6} \cos e x+v_{7} \sin e x
\end{aligned}
$$

where $v_{4}, v_{5}, v_{6}, v_{7} \in \mathbb{R}^{3}$

Hence
$f=\left(v_{4} \cos e x+v_{5} \operatorname{sm} e x\right) \sinh e y+\left(v_{6} \cos e x+v_{7} \sin e x\right) \cosh e y-c e^{-2}(x+d) E$
and

$$
\begin{aligned}
& E * v_{4}=e v_{6} \\
& E * v_{5}=e v_{7} \\
& E * v_{6}=e v_{4} \\
& E * v_{7}=e v_{5}
\end{aligned}
$$

Now after a hyperbolic motion and a strecthing we may assume $E=(0, \epsilon, 0)^{T}$ and hence

$$
\begin{aligned}
& v_{4}=\left(s_{1}, 0, s_{2}\right)^{T} \\
& v_{5}=\left(s_{3}, 0, s_{4}\right)^{T} \\
& v_{6}=\left(-s_{2}, 0,-s_{1}\right)^{T} \\
& v_{7}=\left(-s_{4}, 0,-s_{3}\right)^{T}
\end{aligned}
$$

and we may also assume $f_{y}(0,0)=(r \cos b, 0,0)^{T}$ and $f_{x}(0,0)=(0, r, r \sin b)^{T}$ resulting in

$$
\begin{aligned}
s_{4}=0 & s_{2}=0 \\
s_{\mathbf{1}}=\frac{r}{e} \cos b & s_{3}=-\frac{r}{e} \sin b
\end{aligned}
$$

and $c=-\frac{r}{\epsilon} \epsilon$, hence

$$
f(x, y)=r\left(\begin{array}{c}
\sinh (-\epsilon y) \cos (\epsilon x+\tan b) \\
\epsilon(x+d) \\
\cosh (-\epsilon y) \cos (\epsilon x+\tan b)
\end{array}\right)
$$

and recalling that if $f$ is a solution of the differential equation then so is $\tilde{f}=f+s E$ allows us assume $d=\tan b 1 \mathrm{e}$

$$
f(x, y)=r_{1}\left(\begin{array}{c}
\sinh (-\epsilon y) \cos \left(\epsilon x+r_{2}\right)  \tag{1325}\\
\epsilon\left(x+r_{2}\right) \\
\cosh (-\epsilon y) \cos \left(\epsilon x+r_{2}\right)
\end{array}\right)
$$

where $r_{1}=\frac{r}{\epsilon}$ and $r_{2}=\tan b$ This surface is the catenold of the $2^{\text {nd }}$ kind, which is a minımal surface of revolution

- $\langle E, E\rangle=-e^{2}<0$

Hence we need to solve

$$
f_{22}+e^{2} f=c(x+d)
$$

Solution of Homogenous Equation

$$
g_{1}(x, y)=v_{2}(x) \sin e y+v_{3}(x) \cos e y
$$

Partıcular Solution

$$
g_{2}(x, y)=\frac{c}{e^{2}}(x+d) E
$$

Therefore the solution of this equation is

$$
\begin{aligned}
f & =v_{2}(x) \sin e y+v_{3}(x) \cos e y+c e^{-2}(x+d) E \\
E * f & =\sin e y E * v_{2}(x)+\cos e y E * v_{3}(x) \\
& =f_{2} \\
& =e v_{2}(x) \cos e y-e v_{3}(x) \sin e y
\end{aligned}
$$

hence $e v_{2}(x)=E * v_{3}(x)$ and $-e v_{3}(x)=E * v_{2}(x)$

$$
\begin{aligned}
-f_{22} & =e^{2} v_{2}(x) \sin e y+e^{2} v_{3}(x) \cos e y \\
f_{11} & =v_{2}^{\prime \prime}(x) \sin e y+v_{3}^{\prime \prime}(x) \cos e y
\end{aligned}
$$

Hence

$$
\begin{aligned}
& v_{2}(x)=v_{4} \sinh e x+v_{5} \cosh e x \\
& v_{3}(x)=v_{6} \sinh e x+v_{7} \cosh e x
\end{aligned}
$$

for some $v_{4}, v_{5}, v_{6}, v_{7} \in \mathbb{R}^{3}$

Recall $e v_{2}(x)=E * v_{3}(x)$ and $-e v_{3}(x)=E * v_{2}(x)$, hence
$f=\left(v_{4} \sinh e x+v_{5} \cosh e x\right) \sin e y+\left(v_{6} \sinh e x+v_{7} \cosh e x\right) \cos e y+c e^{-2}(x+d) E$
with

$$
\begin{align*}
& e v_{4}=E * v_{6}  \tag{1326}\\
& e v_{5}=E * v_{7}  \tag{1327}\\
& e v_{6}=-E * v_{4}  \tag{1328}\\
& e v_{7}=-E * v_{5} \tag{1329}
\end{align*}
$$

Now after a hyperbolic motion we may assume $E=(0,0, \epsilon)^{T}$ hence

$$
\begin{aligned}
& v_{4}=\left(s_{1},-s_{2}, 0\right)^{T} \\
& v_{5}=\left(s_{3},-s_{4}, 0\right)^{T} \\
& v_{6}=\left(s_{2}, s_{1}, 0\right)^{T} \\
& v_{7}=\left(s_{4}, s_{3}, 0\right)^{T}
\end{aligned}
$$

we may also assume $f_{x}(0,0)=(r \cosh b, 0,-r)^{T}$ and $f_{y}(0,0)=(0,-r \sinh b, 0)^{T}$ and hence

$$
f(x, y)=r\left(\begin{array}{c}
\sinh (\epsilon x+b) \cos \epsilon y \\
-\sinh (\epsilon x+b) \sin \epsilon y \\
-\epsilon x+d
\end{array}\right)
$$

finally recalling that if $f$ is a solution of the differential equation then so is $\tilde{f}=f+s E$ lets us assume that $d=-b$ and so

$$
f(x, y)=-r\left(\begin{array}{c}
\sinh (\epsilon x+b) \cos y  \tag{1330}\\
-\sinh (\epsilon x+b) \sin y \\
-(\epsilon x+b)
\end{array}\right)
$$

this surface is the catenord of the $1^{s t}$ kind which is a mınımal surface of revolution

## Appendux A

## Determination of $\nabla_{X_{p}} Y$

Proof that given $g($,$) so that equations (A 1) and (A 2) (given below) hold for all$ $X_{p} \in T_{p} M$ and for all smooth vector fields $Y$, that $\nabla_{X_{p}} Y$ is completely determined

$$
\begin{gather*}
Z g(X, Y)=g\left(\nabla_{Z} X, Y\right)+g\left(X, \nabla_{Z} Y\right)  \tag{A1}\\
\nabla_{X_{p}} Y-\nabla_{Y_{p}} X=[X, Y]_{p} \tag{A2}
\end{gather*}
$$

from (A 1)

$$
\begin{align*}
& X g(Y, Z)=g\left(\nabla_{X} Y, Z\right)+g\left(Y, \nabla_{X} Z\right)  \tag{A3}\\
& Y g(Z, X)=g\left(\nabla_{Y} Z, X\right)+g\left(Z, \nabla_{Y} X\right)  \tag{A4}\\
& Z g(X, Y)=g\left(\nabla_{Z} X, Y\right)+g\left(X, \nabla_{Z} Y\right) \tag{A5}
\end{align*}
$$

If we examıne (A 3) + (A 4) - (A 5) we see

$$
\begin{aligned}
X g(Y, Z)+Y g(Z, X)-Z g(X, Y)= & g\left(\nabla_{X} Y, Z\right)+g\left(\nabla_{X} Z-\nabla_{Z} X, Y\right) \\
& +g\left(\nabla_{Y} X, Z\right)+g\left(\nabla_{Y} Z-\nabla_{Z} Y, X\right) \\
= & g\left(\nabla_{X} Y, Z\right)+g([X, Z], Y) \\
& +g\left(\nabla_{X} Y-[X, Y], Z\right)+g([Y, Z], X) \\
= & 2 g\left(\nabla_{X} Y, Z\right)+g([X, Z], Y) \\
& -g([X, Y], Z)+g([Y, Z], X)
\end{aligned}
$$

Therefore

$$
\begin{align*}
2 g\left(\nabla_{X} Y, Z\right)= & X g(Y, Z)+Y g(Z, X)-Z g(X, Y) \\
& +g([X, Y], Z)-g([X, Z], Y)-g([Y, Z], X) \tag{A6}
\end{align*}
$$

The right hand side of (A 6) in independent of $\nabla$ Suppose $Z_{1}, Z_{2}$ form orthonormal basis for $T_{p} M$ then

$$
\nabla_{X} Y=g\left(\nabla_{X} Y, Z_{1}\right) Z_{1}+g\left(\nabla_{X} Y, Z_{2}\right) Z_{2}
$$

and the right hand side of this equation is determined from above

Using the Gauss-Weingarten equations we can arrive at the following results
$1 g_{p}\left(R\left(X_{p}, Y_{p}\right) Y_{p}, X_{p}\right)=-\operatorname{det}\left(A_{p}\right)$ where $R(X, Y) Z$ is the cuvature tensor, defined by

$$
R(X, Y) Z=\nabla_{X} \nabla_{Y} Z-\nabla_{Y} \nabla_{X} Z-\nabla_{[X, Y]} Z
$$

$2\left(\nabla_{X_{P}} A\right) Y=\left(\nabla_{Y_{p}} A\right) X$ called Codazzı's equation

## Proof

For all smooth vector fields $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ we have,

$$
\begin{align*}
Y(Z f)= & \left(f_{*}\right)\left(\nabla_{Y} Z\right)-g(A Y, Z) \xi \\
X(Y(Z f))= & X\left[\left(\nabla_{Y} Z\right) f\right]-[X g(A Y, Z)] \xi-g(A Y, Z) X \xi \\
= & \left(f_{*}\right)\left(\nabla_{X} \nabla_{Y} Z\right)-g\left(A X, \nabla_{Y} Z\right) \xi \\
& -[X g(A Y, Z)] \xi+g(A Y, Z)\left(f_{*}\right)(A X) \\
& \\
& -\left[g\left(A X, \nabla_{Y} Z\right)+X g(A Y, Z)\right] \xi  \tag{A7}\\
X Y(Z f)= & \left(f_{*}\right)\left(\nabla_{X} \nabla_{Y} Z+g(A Y, Z)(A X)\right) \\
&  \tag{A8}\\
& -\left[g\left(A Y, \nabla_{X} Z\right)+Y g(A X, Z)\right] \xi  \tag{A9}\\
Y X(Z f)= & \left(f_{*}\right)\left(\nabla_{Y} \nabla_{X} Z+g(A X, Z)(A Y)\right) \\
{[X, Y](Z f)=} & \left(f_{*}\right)\left(\nabla_{[X, Y]} Z\right)-g(A[X, Y], Z) \xi
\end{align*}
$$

but $X Y-Y X=[X, Y]$ thus $(\mathrm{A} 7)-(\mathrm{A} 8)=(\mathrm{A} 9)$ or $0=(\mathrm{A} 7)-(\mathrm{A} 8)-(\mathrm{A} 9) 1 \mathrm{e}$

$$
\begin{aligned}
& 0=\left(f_{*}\right)\left[\nabla_{X} \nabla_{Y} Z-\nabla_{Y} \nabla_{X} Z-\nabla_{[X, Y]} Z+g(A Y, Z)(A X)-g(A X, Z)(A Y)\right] \\
&+\left[-g\left(A X, \nabla_{Y} Z\right)-X g(A Y, Z)\right. \\
&\left.+g\left(A Y, \nabla_{X} Z\right)+Y g(A X, Z)+g(A[X, Y], Z)\right] \xi
\end{aligned}
$$

Thus,

$$
\nabla_{X} \nabla_{Y} Z-\nabla_{Y} \nabla_{X} Z-\nabla_{[X, Y]} Z=g(A X, Z)(A Y)-g(A Y, Z)(A X)
$$

and

$$
g\left(A X, \nabla_{Y} Z\right)-g\left(A Y, \nabla_{X} Z\right)+X g(A Y, Z)-Y g(A X, Z)-g(A[X, Y], Z)=0
$$

$R(X, Y) Z=\nabla_{X} \nabla_{Y} Z-\nabla_{Y} \nabla_{X} Z-\nabla_{[X, Y]} Z$, is called the curvature tensor So we have

$$
R(X, Y) Z=g(A X, Z)(A Y)-g(A Y, Z)(A X)
$$

Suppose $X, Y$ are smooth vector fields on $M$ such that $X_{p}, Y_{p}$ form an orthonormal basis for $T_{p} M$ with respect to $g_{p}($,$) , then$

$$
\begin{aligned}
g(R(X, Y) Y, X) & =g([g(A X, Y)(A Y)-g(A Y, Y)(A X)], X) \\
& =g(A X, Y) g(A Y, X)-g(A Y, Y) g(A X, X) \\
& =-\operatorname{det}\left(\begin{array}{ll}
g(A X, X) & g(A Y, X) \\
g(A X, Y) & g(A Y, Y)
\end{array}\right) \\
& =-\operatorname{det}\left(\text { matrix representation of the linear map } \mathrm{A}_{\mathrm{p}}\right) \\
& =-\operatorname{det} A_{p}
\end{aligned}
$$

From the normal component we have

$$
\begin{aligned}
0= & g\left(A X, \nabla_{Y} Z\right)-g\left(\nabla_{X} Y-\nabla_{Y} X, Z\right)+g\left(\nabla_{X}(A Y), Z\right) \\
& +g\left(A Y, \nabla_{X} Z\right)-g\left(\nabla_{Y} A X, Z\right)-g\left(A X, \nabla_{Y} Z\right)
\end{aligned}
$$

Therefore,
$g\left(\nabla_{X}(A Y)-A\left(\nabla_{X} Y\right), Z\right)=g\left(\nabla_{Y}(A X)-A\left(\nabla_{Y} X\right), Z\right)$ for all smooth vector fields $Z$
and so,

$$
\nabla_{X}(A Y)-A\left(\nabla_{X} Y\right)=\nabla_{Y}(A X)-A\left(\nabla_{Y} X\right)
$$

Define,

$$
\left(\nabla_{X_{p}} A\right) Y=\nabla_{X_{p}}(A Y)-A\left(\nabla_{X_{p}} Y\right)
$$

then we have

$$
\left(\nabla_{X_{p}} A\right) Y=\left(\nabla_{Y_{p}} A\right) X
$$

## Appendıx $B$

## The Christoffel symbols

In the case when $M$ has locally defined isothermal coordinates $x_{1}, x_{2}{ }_{1} \mathrm{e}$

$$
\begin{gathered}
g\left(\frac{\partial}{\partial x_{1}}, \frac{\partial}{\partial x_{1}}\right)=g\left(\frac{\partial}{\partial x_{2}}, \frac{\partial}{\partial x_{2}}\right)=e^{\phi} \\
g\left(\frac{\partial}{\partial x_{1}}, \frac{\partial}{\partial x_{2}}\right)=0
\end{gathered}
$$

Then

$$
\begin{aligned}
& \Gamma_{11}^{1}=\frac{1}{2} \phi_{x}, \quad \Gamma_{22}^{1}=-\frac{1}{2} \phi_{x}, \quad \Gamma_{12}^{1}=\frac{1}{2} \phi_{y}, \quad \Gamma_{21}^{1}=\frac{1}{2} \phi_{y} \\
& \Gamma_{11}^{2}=-\frac{1}{2} \phi_{y}, \quad \Gamma_{22}^{2}=\frac{1}{2} \phi_{y}, \quad \Gamma_{12}^{2}=\frac{1}{2} \phi_{x}, \quad \Gamma_{21}^{2}=\frac{1}{2} \phi_{x}
\end{aligned}
$$

Proof .

$$
\begin{gather*}
\frac{\partial}{\partial y} g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right)=\phi_{y} e^{\phi} \\
2 g\left(\nabla \frac{\partial}{\partial y} \frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right)=\phi_{y} e^{\phi} \\
2 g\left(\Gamma_{21}^{1} \frac{\partial}{\partial x}+\Gamma_{21}^{2} \frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right)=\phi_{y} e^{\phi} \\
2 \Gamma_{21}^{1} e^{\phi}=\phi_{y} e^{\phi} \\
\Gamma_{21}^{1}=\frac{1}{2} \phi_{y} \tag{B1}
\end{gather*}
$$

Simılarly

$$
\begin{gathered}
g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right)=e^{\phi} \\
\frac{\partial}{\partial x} g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right)=\phi_{x} e^{\phi} \\
2 g\left(\nabla \frac{\partial}{\partial x} \frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right)=\phi_{x} e^{\phi} \\
2 g\left(\Gamma_{12}^{1} \frac{\partial}{\partial x}+\Gamma_{12}^{2} \frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right)=\phi_{x} e^{\phi}
\end{gathered}
$$

$$
\begin{align*}
2 \Gamma_{12}^{2} e^{\phi} & =\phi_{x} e^{\phi} \\
\Gamma_{12}^{2} & =\frac{1}{2} \phi_{x} \tag{B2}
\end{align*}
$$

and

$$
\begin{gathered}
{\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]=0} \\
\nabla \frac{\partial}{\partial x} \frac{\partial}{\partial y}-\nabla \frac{\partial}{\partial y} \frac{\partial}{\partial x}=0 \\
\Gamma_{12}^{1} \frac{\partial}{\partial x}+\Gamma_{12}^{2} \frac{\partial}{\partial y}-\Gamma_{21}^{1} \frac{\partial}{\partial x}-\Gamma_{21}^{2} \frac{\partial}{\partial y}=0
\end{gathered}
$$

hence

$$
\Gamma_{12}^{1}=\Gamma_{21}^{1} \text { and } \Gamma_{12}^{2}=\Gamma_{21}^{2}
$$

and the others are proved by following a simılar argument

## Appendxx $C$

## Codazzi's equation

To show that Codazzı's equation

$$
\left(\nabla_{X}\right) Y=\left(\nabla_{Y}\right) X
$$

is equivalent to $\psi$ being holomorphic where

$$
\psi=\left\{\left(a_{11}-a_{22}\right)-2 \imath a_{12}\right\} e^{\phi}
$$

## Proof :

Codazzı's equation

$$
\left(\nabla_{X}\right) Y=\left(\nabla_{Y}\right) X
$$

in local coordinates with $X=\frac{\partial}{\partial x}$ and $Y=\frac{\partial}{\partial y}$ is

$$
\nabla_{\frac{\theta}{\partial x}}\left(A \frac{\partial}{\partial y}\right)-A\left(\nabla_{\frac{\theta}{\partial x}} \frac{\partial}{\partial y}\right)=\nabla_{\frac{\partial}{\partial y}}\left(A \frac{\partial}{\partial x}\right)-A\left(\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x}\right)
$$

Recall $\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]=\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y}-\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x}=0$, thus

$$
\begin{gathered}
\nabla_{\frac{\partial}{\partial x}}\left(A \frac{\partial}{\partial y}\right)=\nabla_{\frac{\partial}{\partial y}}\left(A \frac{\partial}{\partial x}\right) \\
\nabla_{\frac{\partial}{\partial x}}\left(a_{21} \frac{\partial}{\partial x}+a_{22} \frac{\partial}{\partial y}\right)=\nabla_{\frac{\partial}{\partial y}}\left(a_{11} \frac{\partial}{\partial x}+a_{12} \frac{\partial}{\partial y}\right) \\
\left(a_{21}\right)_{x} \frac{\partial}{\partial x}+a_{12} \nabla_{\frac{\theta}{\partial x}} \frac{\partial}{\partial x}+\left(a_{22}\right)_{x} \frac{\partial}{\partial y}+a_{22} \nabla_{\frac{\theta}{\partial x}} \frac{\partial}{\partial y} \\
=\left(a_{11}\right)_{y} \frac{\partial}{\partial x}+a_{11} \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x}+\left(a_{12}\right)_{y} \frac{\partial}{\partial y}+a_{12} \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y}
\end{gathered}
$$

Hence as $a_{21}=a_{12}$ and $\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x}=\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y}$ we have

$$
\begin{aligned}
& \left\{\left(a_{12}\right)_{x}-\left(a_{11}\right)_{y}\right\} \frac{\partial}{\partial x}+\left\{\left(a_{22}\right)_{x}-\left(a_{21}\right)_{y}\right\} \frac{\partial}{\partial y} \\
& \quad=a_{12}\left(\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y}-\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x}\right)+\left(a_{11}-a_{22}\right)\left(\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y}\right) \\
& \quad=a_{12}\left\{\left(\Gamma_{22}^{1}-\Gamma_{11}^{1}\right) \frac{\partial}{\partial x}+\left(\Gamma_{22}^{2}-\Gamma_{11}^{2}\right) \frac{\partial}{\partial y}\right\}+\left(a_{11}-a_{22}\right)\left(\Gamma_{12}^{1} \frac{\partial}{\partial x}+\Gamma_{12}^{2} \frac{\partial}{\partial y}\right)
\end{aligned}
$$

equating $\frac{\partial}{\partial y}$ 's we have

$$
\left(a_{11}\right)_{x}+\left(a_{12}\right)_{y}+a_{12}\left(\Gamma_{22}^{2}-\Gamma_{11}^{2}\right)+2\left(a_{11}-H\right) \Gamma_{12}^{2}=0
$$

and equating $\frac{\partial}{\partial x}$ 's we have

$$
-\left(a_{12}\right)_{x}+\left(a_{11}\right)_{y}+a_{12}\left(\Gamma_{22}^{1}-\Gamma_{11}^{1}\right)+2\left(a_{11}-H\right) \Gamma_{12}^{1}=0
$$

resulting in

$$
\left(a_{11}\right)_{x}+\left(a_{11}-H\right) \phi_{x}+\left(a_{12}\right)_{y}+a_{12} \phi_{y}=0
$$

and

$$
\left(a_{11}\right)_{y}+\left(a_{11}-H\right) \phi_{y}+\left(a_{12}\right)_{x}+a_{12} \phi_{x}=0
$$

multiplying by $2 e^{\phi}$ and rearranging we have

$$
2\left(a_{11}\right)_{x} e^{\phi}+\left(2 a_{11}-2 H\right) \phi_{x} e^{\phi}=-2\left(a_{12}\right)_{y} e^{\phi}-2 a_{12} \phi_{y} e^{\phi}
$$

and

$$
2\left(a_{11}\right)_{y} e^{\phi}+\left(2 a_{11}-2 H\right) \phi_{y} e^{\phi}=-2\left(a_{12}\right)_{x} e^{\phi}-2 a_{12} \phi_{x} e^{\phi}
$$

Now $2\left(a_{11}\right)_{x}=\left(2 a_{11}-2 H\right)_{x}=\left(a_{11}-a_{22}\right)_{x}$ and sımılarly $2\left(a_{11}\right)_{y}=\left(a_{11}-a_{22}\right)_{y}$, hence

$$
\left(a_{11}-a_{22}\right)_{x} e^{\phi}+\left(a_{11}-a_{22}\right) \phi_{x} e^{\phi}=-2\left(a_{12}\right)_{y} e^{\phi}-2 a_{12} \phi_{y} e^{\phi}
$$

and

$$
\left(a_{11}-a_{22}\right)_{y} e^{\phi}+\left(a_{11}-a_{22}\right) \phi_{y} e^{\phi}=2\left(a_{12}\right)_{x} e^{\phi} 2 a_{12} \phi_{x} e^{\phi}
$$

or simlarıly

$$
\left[\left(a_{11}-a_{22}\right) e^{\phi}\right]_{x}=\left[-2 a_{12} e^{\phi}\right]_{y}
$$

and

$$
\left[\left(a_{11}-a_{22}\right) e^{\phi}\right]_{y}=\left[2 a_{12} e^{\phi}\right]_{x}
$$

letting

$$
u(x, y)=\left(a_{11}-a_{22}\right) e^{\phi} \quad \text { and } \quad v(x, y)=-2 a_{12} e^{\phi}
$$

it is obvious that these are simply the Cauchy Riemann Equations for the function

$$
\psi=\left\{\left(a_{11}-a_{22}\right)-2 \imath a_{12}\right\} e^{\phi}
$$

It is clear that if we start with the assumption that $\psi$ is holomorphic then we can get back to Codazzı's eqation by reversing the order of these steps

Appendıx D

## The Gauss Curvature

From the prelımınaries we have

$$
K=\frac{g(R(X, Y) Y, X)}{g(X, X) g(Y, Y)-g(X, Y)^{2}}
$$

and letting $X=\frac{\partial}{\partial x}$ and $Y=\frac{\partial}{\partial y}$ we have

$$
\begin{aligned}
R\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \frac{\partial}{\partial y}= & \nabla_{\frac{\partial}{\partial x}} \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y}-\nabla_{\frac{\partial}{\partial y}} \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y}-\nabla_{\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]} \frac{\partial}{\partial y} \\
= & \nabla_{\frac{\partial}{\partial x}}\left\{\Gamma_{22}^{1} \frac{\partial}{\partial x}+\Gamma_{22}^{2} \frac{\partial}{\partial y}\right\}-\nabla_{\frac{\partial}{\partial y}}\left\{\Gamma_{12}^{1} \frac{\partial}{\partial x}+\Gamma_{12}^{2} \frac{\partial}{\partial y}\right\} \\
= & \nabla_{\frac{\partial}{\partial x}}\left\{-\frac{\phi_{x}}{2} \frac{\partial}{\partial x}+\frac{\phi_{y}}{2} \frac{\partial}{\partial y}\right\}-\nabla_{\frac{\partial}{\partial y}}\left\{\frac{\phi_{y}}{2} \frac{\partial}{\partial x}+\frac{\phi_{x}}{2} \frac{\partial}{\partial y}\right\} \\
= & -\frac{\phi_{x x}}{2} \frac{\partial}{\partial x}-\frac{\phi_{x}}{2} \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x}+\frac{\phi_{x y}}{2} \frac{\partial}{\partial y}+\frac{\phi_{y}}{2} \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} \\
& -\left[\frac{\phi_{y y}}{2} \frac{\partial}{\partial x}+\frac{\phi_{y}}{2} \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x}+\frac{\phi_{x y}}{2} \frac{\partial}{\partial y}+\frac{\phi_{x}}{2} \nabla_{\frac{\partial}{\partial y}}^{\partial y} \frac{\partial}{\partial y}\right] \\
= & -\frac{1}{2}\left(\phi_{x x}+\phi_{y y}\right) \frac{\partial}{\partial x}-\frac{\phi_{x}}{2}\left(\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x}+\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y}\right) \\
= & -\frac{1}{2}\left(\phi_{x x}+\phi_{y y}\right) \frac{\partial}{\partial x}
\end{aligned}
$$

thus

$$
\begin{aligned}
K & =\frac{g\left(-\frac{1}{2}\left(\phi_{x x}+\phi_{y y}\right) \frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right)}{e^{\phi} e^{\phi}-0} \\
& =-\frac{1}{2} e^{-2 \phi} \Delta \phi g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) \\
& =-\frac{1}{2} e^{-\phi} \Delta \phi
\end{aligned}
$$

## Appendux $E$

## Immersion into hyperbolic 2-sphere

$\Delta=\mathbf{S}^{2}$

In this case we find that $f(M)$ is conformally a prece of the hyperbolic 2 -sphere $S_{1}^{2}$

## Proof

It is easy to see that when $\Delta$ is just a sphere, the shape operator $A$ is simply a multiple of of the 1 dentity $I$ Hence

$$
A=\lambda I
$$

for some function $\lambda=\lambda(x, y)$ Codazzı's equation implies that the function $\lambda$ is just a constant and infact

$$
A=\frac{1}{r} I
$$

where $r$ is the radius of the sphere Recall

$$
X \xi=f_{*}(A X)
$$

letting $X=\frac{\partial}{\partial x}$ we have

$$
\xi_{x}=f_{x}
$$

simularly with $X=\frac{\partial}{\partial y}$ we have

$$
\xi_{y}=f_{y}
$$

thus

$$
\xi=f-c
$$

for some $c \in \mathbb{R}^{3}$ Hence

$$
f=\xi+c
$$

1 e $f(M)$ is a plece of the hyperbolic 2 -sphere $S_{1}^{2}$

## Appendıx $F$

## A differential equation

To examine solutions of the differential equations

$$
\begin{equation*}
f^{\prime \prime}(x)=2 c^{2} f^{3}(x) \tag{F1}
\end{equation*}
$$

where $c \in \mathbf{R}$ and initial conditions

$$
\begin{array}{llll}
1 & f\left(x_{0}\right)=r^{2} & \text { and } & f^{\prime}\left(x_{0}\right)=c r^{4} \\
& & & \\
& f\left(x_{0}\right)=-r^{2} & \text { and } & f^{\prime}\left(x_{0}\right)=-c r^{4}
\end{array}
$$

## Results:

(F 1) with initial conditions 1 gives the solution

$$
f(x)=\frac{1}{c(d-x)}, \quad d=x_{0}+\frac{1}{c r^{2}}
$$

which is defined on the semı-infinite line $(-\infty, d)$ or $(d, \infty)$ depending on whether $c$ is positive of negative repectıvely
(F 1) with mitial conditions il gives the solution

$$
f(x)=\frac{-1}{c(d-x)}, \quad d=x_{0}+\frac{1}{c r^{2}}
$$

which is defined on the semı-nfinite line $(-\infty, d)$ or $(d, \infty)$ again depending on whether $c$ is positive of negative repectively

Proof of (F 1) with initial conditions 1

$$
\begin{aligned}
(F 1) \Rightarrow 2 f^{\prime}(x) f^{\prime \prime}(x) & =4 c^{2} f^{3}(x) f^{\prime}(x) \\
\left(f^{\prime}(x)^{2}\right)^{\prime} & =\left(c^{2} f^{4}(x)\right)^{\prime}
\end{aligned}
$$

$$
f^{\prime}(x)^{2}=c^{2} f^{4}(x)+c_{1}
$$

$$
f^{\prime}\left(x_{0}\right)^{2}=c^{2} f^{4}\left(x_{0}\right)+c_{1} \Rightarrow c_{1}=0
$$

$$
f^{\prime}(x)^{2}=\left[c f^{2}(x)\right]^{2}
$$

$$
f^{\prime}(x)= \pm c f^{2}(x)
$$

$$
f^{\prime}\left(x_{0}\right)= \pm c f^{2}\left(x_{0}\right) \Rightarrow \pm=+
$$

$$
f^{\prime}(x)=c f^{2}(x)
$$

$$
\int \frac{1}{f^{2}} d f=c \int d x
$$

$$
\frac{-1}{f(x)}=c(x-d)
$$

$$
f(x)=\frac{-1}{c(x-d)}
$$

$$
f\left(x_{0}\right)=\frac{-1}{c\left(x_{0}-d\right)}=r^{2} \Rightarrow d=x_{0}+\frac{1}{c r^{2}}
$$

and (F 1) with initial conditions 11 is proved in a similar manner

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## Examples of Minimal Surfaces

Theorem 15 , equation 1 with $r_{1}=1, r_{2}=0$ and $l=6$


This is the surface of Enneper of the second kind, which is a minimal spacelike surface of revolution

Theorem 15 , equation 2 with $r_{1}=1, r_{2}=0$ and $\epsilon=1$


This is the catenord of the first kind, which is a minımal surface of revolution

Theorem 15 , equation 3 with $r_{1}=-1, r_{2}=0$ and $\epsilon=-1$


This is the catenord of the first kind, which is a minımal surface of revolution

Theorem 15 , equation 6 with $r_{1}=2, r_{2}=4, \alpha=-2$ and $\epsilon=-1$


Theorem 15 , equation 7 , with $r_{1}=2, r_{2}=-1$ and $\alpha=\frac{1}{2}$


