Exponential Asymptotics and Spectral Theory for Optical Tunnelling.

Ph.D. thesis

Fiona Lawless

Supervisor Prof A. D. Wood,
School of Mathematical Sciences,
Dublin City University.

" I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of **PhD** *is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work."*

Signed: Fiona Lawless. **ID No.** 89700147

Date: February 24th , 1993

To Dara, with love
Thanks for the memories.

Seed growing smaller into wet soils,
Scent of dampness, toughing earth skins,
Soft enlosened smile, making moving pictures
of Sunbeamed beaches

Feeling cornered into lifeless Solitude,
Darkness files lonely layers of confusion,
As your candle dims into a tasteless stream of
HARDNESS

[Dara Lawless, Sept. 1991.]

Acknowledgements.

I owe so much to so many !

To my very caring family (much extended) all I have achieved is due to you, all of you Thanks also to my many friends, your friendship and support, particularly over the last year, was much needed and greatly appreciated

Quite often I have encountered people who have shown me kindness and understanding beyond reason Not least among these was my supervisor, Prof A D Wood Alastair, not only do I owe to you my enthusiam for and love of mathematics, my knowledge of and introduction into the world of mathematics, and various extremely exciting and terrifying excursions to mathematical conferences, but the completion of this thesis must largely be attributed to you Only through your patience and encouragement have I made it this far Thank you for believing in me!

I thank also the staff and post-grads of the sums department for their companionship over the last seven years

Finally I would like to thank Dr Declan Fitzpatrick and Dr Justine McCarthy-Woods for their time and patience

Abstract.

Mathematically this thesis involves an investigation of the non-self-adjoint Sturm-Liouville problem comprising the differential equation, $y''(x) + (\lambda + \varepsilon x^2)y(x) = 0$ with a linear homogeneous boundary condition at $x = 0$ and an "outgoing wave " condition as $x \to \infty$, in a number of different settings The purpose of such an investigation is to obtain an accurate estimate for the imaginary part of the eigenvalue λ

Physically, this singular eigenproblem arises in the mathematical modelling of radiation losses in bent fibre-optic waveguides, with the imaginary part of the desired eigenvalue providing a measure of the magnitude of loss due to bending The imaginary part of the desired eigenvalue turns out to be of much smaller order $[O(e^{-\frac{1}{\sqrt{\varepsilon}}}), \varepsilon \to 0+]$ than the perturbation of the real part $[O(\varepsilon), \varepsilon \to 0+]$ To overcome the resulting computational difficulties we appeal to the area of *exponential asymptotics* and become involved in the *smoothing of Stokes discontinuities* A number of exponentially improved approximations are required for proper estimation of $Im\lambda$ and these are obtained either directly from the literature or by application of recent results

The non-self-adjoint nature of the above tunnelling problem results from the unusual condition at infinity While we investigate this problem directly, using special functions and variational techniques, and obtain an accurate estimate for imaginary part of the desired eigenvalue, an alternative setting is also found This more abstract approach involves the theory of "resonance poles" in quantum mechanics We show that under certain conditions, satisfied by the tunnelling problem being considered, the "eigenvalue" of a non-self-adjoint problem corresponds to a pole in the Titchmarsh-Weyl function $m(\lambda)$ for a related but formally self-adjoint problem

Contents

Chapter 1

Mathematical prerequisites.

1.1 Introduction

In mathematical terms this thesis concerns the accurate computation of the imaginary part of the "eigenvalue" of the non-self-adjoint Sturm-Liouville problem comprising the ordinary differential equation

$$y''(x) + \{\lambda + \varepsilon x^2\}y(x) = 0, \qquad x \in (0, \infty) \tag{1 1}$$

with initial condition

$$y'(0) + hy(0) = 0, \qquad h > 0 \tag{1 2}$$

and the "outgoing wave" condition

$$as\ x \to \infty, \quad y(x)\ has\ controlling\ factor \quad e^{ip(x)}, \tag{1 3}$$

[$p(x)$, a positive function of x]

The "outgoing wave" condition arises from consideration of the physical problem being modelled [§1 2] The unique function $p(x)$ may be determined by the *Liouville-Green* approximation, which uses the substitution $y(x) = e^{ip(x)}$ to identify the controlling factor of the solution as $x \to +\infty$, and for the present problem yields $p(x) = \frac{1}{2}\varepsilon^{\frac{1}{2}}x^2$
This boundary condition may also be written in a Wronskian format more familiar in spectral theory as

$$\lim_{x \to \infty} W[y(x), e^{i\frac{1}{2}\varepsilon^{\frac{1}{2}}x^2}] = 0$$

It is straightforward to show that the model problem has an eigenvalue at $\lambda = -h^2$ when $\varepsilon = 0$, in fact the unperturbed differential equation, $y''(x) = -\lambda y(x)$, has linearly independent solutions $y_\pm = e^{\pm i\sqrt{\lambda}x}$ The boundary condition at $x = 0$ is satisfied only if λ takes the value $-h^2$, $\lambda = -h^2$ is therefore the only possible eigenvalue of problem [(1 1)-(1 3)] with $\varepsilon = 0$ The perturbed problem however is rendered non-self-adjoint by the form of the boundary condition at infinity

To see this, denote by L the formal differential operator

$$Ly = -y'' - \varepsilon x^2 y$$

and let $<,>$ denote the usual inner-product in $L^2(0,\infty)$, that is $< u, v >= \int_0^\infty u(x)\overline{v(x)}dx$ for all $u, v \in L^2(0,\infty)$ Then the operator L is self-adjoint iff

$$< Lu, v >=< u, Lv >,$$

where u, v are any functions which satisfy the boundary conditions, (1 2) and (1 3) Using integration by parts we find

$$< Lu, v >= -[u'(x)\overline{v(x)}]_0^\infty + \int_0^\infty u'(x)\overline{v'(x)}dx - \varepsilon \int_0^\infty x^2 u(x)\overline{v(x)}dx,$$

$$< u, Lv >= -[u(x)\overline{v'(x)}]_0^\infty + \int_0^\infty u'(x)\overline{v'(x)}dx - \varepsilon \int_0^\infty x^2 u(x)\overline{v(x)}dx$$

and the operator is self-adjoint provided

$$[u'(x)\overline{v(x)}]_0^\infty = [u(x)\overline{v'x)}]_0^\infty$$

By virtue of the boundary condition at the origin, this is equivalent to the condition that

$$\lim_{x\to\infty}[u'(x)\overline{v(x)}] = \lim_{x\to\infty}[u(x)\overline{v'(x)}]$$

Imposing the "outgoing-wave" condition, that is , $u(x), v(x) \sim e^{ip(x)}$, as $x \to \infty$, we find that as $x \to \infty$ $[u'(x)\overline{v(x)}] \sim ip'(x)$ while $[u(x)\overline{v'(x)}] \sim -ip'(x)$ and the problem is **non-self-adjoint**

It is thus possible for the eigenvalue of the perturbed problem to be non-real, and we shall see that the desired eigenvalue is in fact complex The imaginary part of this eigenvalue turns out to be of much smaller order $[O(e^{-1/\sqrt{\varepsilon}}), \varepsilon \to 0+]$ than the perturbation of the real part $[O(\varepsilon), \varepsilon \to 0+]$, with consequent computational difficulties Correct computation of $Im\lambda$ leads us into the deep new area of *Smoothing of Stokes discontinuities* or *Asymptotics beyond all orders* or *Exponential improvement of asymptotic expansions* We shall show in §1 3 that these three ideas, which have emerged over the past three years, are broadly equivalent

The physical problem being treated is essentially one of radiation damping, a difficult area of transcendental asymptotics which has yet to be given a satisfactory general treatment, even for ordinary differential equations It may be considered as a special case of the problems treated by Lozano and Meyer in their broader analysis of radiation damping While they consider ordinary differential equations too nasty for any application of special functions and become involved with the details of connecting WKB solutions across transition points, our rigorous result relies on finding exact solutions in terms of special functions The exponential asymptotics of such special functions may be constructed from known results

Liu and Wood, [24], consider a generalisation of the present problem, which results from replacing the differential equation (1 1) by the more general equation

$$y''(x) + (\lambda + \varepsilon x^n)y(x) = 0, \qquad n \in \mathcal{Z}^+$$

When $n > 2$ no special function solutions are available, and they rely on the method of matched asymptotic expansions Their entirely formal approach is similar to that of Lozano and Meyer in that they match WKB solutions near transition points Obtaining the approximate WKB solution for large x, which satisfies the outgoing wave condition, they proceed to match this to the Airy function approximate solution valid near the turning point $x = (-\lambda/\varepsilon)^n$ This turning point will be exponentially close to the axis and tend to it as $\varepsilon \rightarrow 0+$ The resulting combination of Airy functions is in turn matched to the WKB solution valid to the left of the turning point This approximate solution is then substituted into the boundary condition (1 2) to yield the desired eigen-relation

This manuscript is made up essentially of two parts The first uses special functions, and develops various exponentially improved approximations to these functions, to obtain a good approximation to the desired eigenvalue of the non-self-adjoint problem The exponentially improved asymptotics for the gamma function are generalised to those for solutions of a class of first order difference equations The second relates this result to the theory of resonance poles in quantum mechanics

We shall start by giving a brief history of the mathematical development of this fibre-optics problem, indicating what led us to the current investigation Exponential asymptotics are used to overcome the computational difficulties arising from the small magnitude of the imaginary part of the desired eigenvalue Thus, because of the very recent developments in this area, of *Asymptotics beyond all orders* , and the related *smoothing of Stokes' discontinuities* we shall give an overview of recent advancements, and the major contributions from mathematicians/physicists such as Berry, Olver, Kruskal and Segur, McLeod, Byatt-Smith, Paris and Wood

Two very different but equivalent approaches shall be taken to this problem The first, that of Chapter 2, which we term the *direct method*, involves the estimation of the desired eigenvalue of problem [(1 1)-(1 3)] using special functions and exponential asymptotics This approach was first taken up by Neal Brazel, following the work of Paris and Wood on a similar problem However his investigation of the problem occurred before many of the very recent results on Stokes' phenomenon/Exponentially-improved asymptotics were available In fact, it was not until our investigation of the problem using the second, more abstract, approach revealed inaccuracies in his results that the need for exponentially improved approximations to the gamma function was realised In Chapter 3, using the work of Batchelder on difference equations we shall obtain the required exponentially improved approximations directly from the relevant difference equation

We shall verify the result of the above *direct method* using the more classical operator theory / J-function technique By extending the ideas just used in establishing the non-self-adjoint nature of our problem we obtain a representation of $Im\lambda$ Then using the known asymptotics of Miller's parabolic cylinder functions, which result from a particular transformation of problem [(1 1)-(1 3)], we obtain the desired estimate for $Im\lambda$ thus verifying our results

The complex number λ, whose imaginary part we are trying to estimate, is an eigenvalue in the sense that the differential equation has a non-trivial solution which satisfies both boundary conditions when that particular value of λ is taken It has no direct interpretation in linear operator theory To provide an abstract setting we must go to quantum mechanics and the theory of resonances as found in the opening chapters of Reed and Simon [40] and paralleled in Titchmarsh's treatment of "pseudo-eigenvalues" [48], although the term "resonance" is never used by Titchmarsh Thus our second approach to the optical tunnelling problem, involves setting it up as a self-adjoint problem, in which the "outgoing wave" condition is replaced by an $L^2(0,\infty)$ condition, and trying to estimate its resonance poles In Chapter 4 we shall describe the general theory of Titchmarsh-Weyl [47], [48] and indicate how it can be adapted to the current problem. We shall then proceed to establish the equivalence of these two very different approaches to such problems, under certain conditions on the potential functions involved

1.2 Mathematical development of the fibre-optics problem.

The Sturm-Liouville eigenproblem with which this thesis is concerned arises in the area of fibre-optics, or more specifically, in the mathematical modelling of optical tunnelling. Its origins lie in an investigation by Kath and Kreigsmann [21] into radiation losses in bent fibre-optic waveguides.

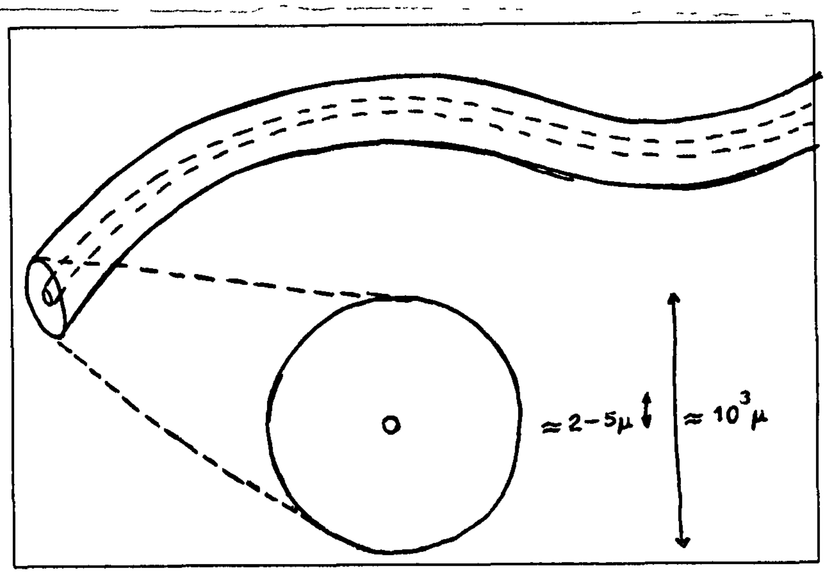


Figure 1.1: Geometry of a bent optical fibre.

Briefly, while the design of optical waveguides is based on the principle that a flat interface between a vacuum and a material with negative dielectric constant can support electromagnetic waves, these waves become unbounded when the interface is not flat. It follows that a bent optical waveguide does not trap light perfectly Instead of travelling within the core and decaying away from it, the electromagnetic wave tunnels into the cladding and radiates away It is in the estimation of this loss, due to bending of the fibre, that we are interested

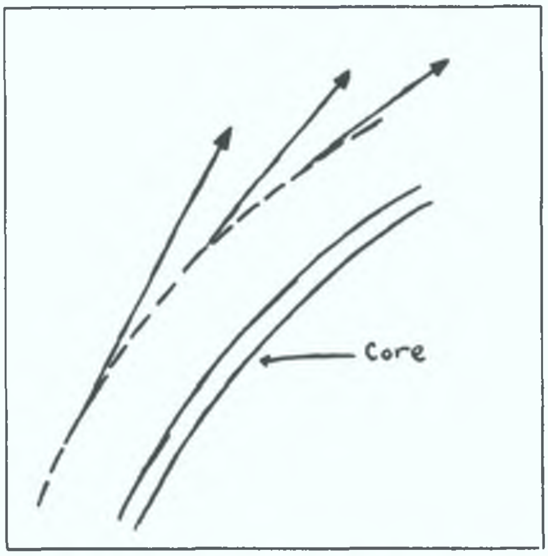


Figure 1.2: Energy leakage out of the core region.

In their consideration of radiation loss in bent optical waveguides Kath and Kreigsmann [21] make no restriction on the type of deformation allowed. They do, however, in the interest of simplicity and clarity, make several reasonable approximations. The radius of curvature is assumed to be large compared to the wavelength of light used; the fibre is "gently bent". In addition, the fibre is taken to be "weakly guiding", that is, deviations from the mean refractive index are assumed small. This allows for the replacement of Maxwell's equations, which govern the behaviour of electromagnetic waves, by a scalar wave equation, as it implies that the main direction of light propagation is along the axis of the fibre. By constructing a suitable co-ordinate system, one which follows the centre-line of the fibre, whilst taking curvature into account, they derive the equation

$$\nabla^2 y + f(\xi,\eta)y + \lambda y + \varepsilon\alpha(\xi,\eta)y = 0. \tag{1.4}$$

(Here ξ, η are axes of the co-ordinate system chosen, $f(\xi,\eta)$ is the scaled difference in the index of refraction in the core, ϵ is a small positive number which measures the curvature of the fibre and $\alpha(\xi,\eta)$ is a function linear in ξ,η).
This equation (1.4), together with appropriate boundary conditions, constitutes an eigenvalue problem in the parameter λ; an estimate for the energy loss can be obtained from the imaginary part of this eigenvalue. Difficulties arise in the solution of this problem due to its singular nature and it is hard to put any such analysis on a rigorous footing.

In the hope of obtaining explicit solutions to, and gaining a better understanding of the mechanics of, such eigenproblems Paris and Wood [34] considered the model equation,

$$i\phi_t = -\phi_{xx} - \varepsilon S(x)\phi, \tag{1.5}$$

with general linear homogeneous boundary condition

$$\phi_x(0,t) + h\phi(0,t) = 0, \tag{1.6}$$

h a positive constant.
Their justification for considering this model lies in the fact that they are interested

in the behaviour of solutions in the cladding region where the perturbation $f(\xi, \eta)$ in the refractive index is zero While having essentially the same structure as (1 4) in the cladding, explicit solutions exist for certain choices of $S(x)$ allowing for a more in-depth analysis and consequently a better understanding of this generalisation of the optical tunnelling problem The case $S(x) = x$, considered by Paris and Wood [34], possesses a solution in terms of Hankel and Airy functions while, as we shall see, the problem with $S(x) = x^2$ can be solved in terms of parabolic cylinder functions For $S(x) = x^n$, $n > 2$, Liu and Wood [24] found it necessary to employ asymptotic matching, there being no obvious special function representation

By means of separation of variables Paris and Wood [34] reduce problem [(1 5)-(1 6)] to the model o d e eigenvalue problem

$$y''(x) + \{\lambda + \varepsilon S(x)\}y(x) = 0, \qquad x \in (0, \infty) \tag{1 7}$$

$$y'(0) + hy(0) = 0, \qquad h > 0 \tag{1 8}$$

with $y(x)$ possessing a controlling behaviour of the form $e^{ip(x)}$ as $x \to \infty$
The Sturm-Liouville eigenproblem considered in this thesis is a specific case of the Paris and Wood model problem The unusual boundary condition at infinity, results from the fact that beyond a certain point, the transition point, solutions change from being evanescent to propagating and thus take on the form of an "outgoing-wave"

It should be noted that Burzlaff and Wood in their paper *"Optical Tunnelling from Square Well Potentials"*, [12], consider more realistic, although closely related, one-dimensional models for optical tunnelling with a refractive index in the shape of a square well They too find it necessary to obtain exponentially improved approximations, in this instance to the Airy function $Bi(x)$

This work of Paris and Wood [34] was carried out before the publication of Berry [1989] on *smoothing of Stokes phenomenon* Their paper pointed out the need for a result which would justify *averaging across Stokes lines* While they were able to overcome this difficulty in the case $S(x) = x$ by an ingenious analysis using special functions, it was not until the paper of Berry that a generic form of smoothing was given This was subsequently established rigorously by Olver for a wide range of special functions in the same year [33]

1.3 Review of Exponential Asymptotics.

As already mentioned correct computation of $Im\lambda$ leads us into the areas of *Stokes Phenomenon* and *Asymptotics beyond all orders*, due to Kruskal and Segur

The whole area of *Asymptotics beyond all orders* began with an investigation by Kruskal and Segur into *crystal growth* They found that exponentially small phenomena play an important role in such an investigation They came up with a method for treating problems in which the governing ordinary differential equation involves a small parameter The basis of the Kruskal-Segur method is the extension of the problem into the whole complex-plane since, as shall been seen, asymptotic expansions do not remain uniformly valid throughout the complex plane

There are numerous examples of this phenomenon in quantum mechanics, starting with the earliest work of Povroskii and Khalatnikov [39] on tunnelling through potential barriers, Bender and Wu on the anharmonic oscillator [4], and Simon and others on resonances [40], [41] In fact the optical tunnelling problem is just one of a class of applications to asymptotics, having arisen recently, which require information concerning a subdominant, usually transcendentally small, term For an additional application the reader is referred to the work of Lozano and Meyer in the theory of surface waves trapped by round islands with small seabed slope [25] Other problems involving such exponential asymptotics include the *Saffman-Taylor finger problem* [1958, 1986], *viscous boundary layers* [Byatt-Smith, 1991 , Hooper and Grimshaw, 1985, 1991], and *solitary waves with surface tension* [Beale, 1991, Byatt-Smith, 1991, Sun and Shen, 1991]

Before delving into the area of exponential asymptotics it is advisable to introduce some notation specific to asymptotics

$<<$ The notation

$$f(z) << g(z), \quad z \to z_0,$$

which reads "$f(z)$ is much smaller than $g(z)$ as z tends to z_0", means

$$\lim_{z \to z_0} f(z)/g(z) = 0$$

$\sim$ The notation

$$f(z) \sim g(z), \quad z \to z_0,$$

which reads "$f(z)$ is asymptotic to $g(z)$ as z tends to z_0", means the relative error between f and g goes to zero as $z \to z_0$ That is,

$$f(z) - g(z) << g(z), \quad z \to z_0$$

Two other symbols which are used quite frequently in asymptotics are $o()$ and $O()$

$o()$ The notation

$$f(z) = o(g(z)), \quad z \to z_0,$$

means that f(z) is of order less that g(z) as z tends to z_0, that is $f(z)/g(z) \to 0$, as $z \to z_0$

$O()$ The notation

$$f(z) = O(g(z)), \quad z \to z_0,$$

means that f(z) is of order not exceeding g(z) as z tends to z_0, that is $|f(z)/g(z)|$ is bounded as $z \to z_0$

Poincaré [1886] defined an asymptotic series as follows Let $f(z)$ be a function of the real or complex variable z, and $\sum_{n=0}^{\infty} a_n z^{-n}$ a formal power series (convergent or divergent) Then the series $\sum_{s=0}^{\infty} a_s z^{-s}$ is an asymptotic expansion for $f(z)$, in a given region R, written

$$f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \qquad (z \to \infty \; in \; R),$$

if $R_n(z) = O(z^{-n})$ for each fixed value of n, $R_n(z)$ being the difference between $f(z)$ and the n^{th} partial sum of the series By Poincaré definition we neglect all terms in an asymptotic expansion which are exponentially small compared to other terms, whereas in the *complete sense of Watson* such terms are retained whenever they have numerical significance It follows that regions of validity differ with definition

Although the asymptotics of special functions given in books such as Abramowitz and Stegun [1] are correct in the sense of Poincaré, they give no information about such subdominant exponential terms While the definition of Poincaré was useful in that it set asymptotic analysis on a rigorous footing, it was also restrictive in that it excluded certain applications and cases where neglected terms are of computational importance

Typically, an asymptotic series depending on a large parameter k and variable z has form

$$y(z,k) = M(z,k)\exp[k\phi_+(z)]\sum_{r=0}^{\infty} Y_r(z,k),$$

here $Y_0 = 1$ and $Y_r \propto k^{-r}$, it should be noted that quite often k will be $|z|$, $z \in C$ In general the series is divergent and as such is meaningless when interpreted conventionally The usual asymptotics is the study of the series truncated at fixed order, $r = N$ According to the Poincaré definition the series is then asymptotic if the resulting error is of order k^{-N-1} However, as was known to Stokes nearly half a century before Poincaré, much more accurate approximations can be obtained by truncation not at fixed order but at the least term, which typically increases with k It is common to achieve errors of order e^{-k} with such optimal truncation

The *Stokes phenomenon* concerns the behaviour of small exponentials while hidden behind larger ones This phenomenon is said to lie at the very heart of asymptotics On the other hand, as remarked by Berry [5], it is impossible to study Stokes phenomenon in the framework of the Poincaré definition of an asymptotic expansion The Poincaré definition is inadequate in that it captures only the asymptotics of y(z,k) to power law accuracy whereas understanding Stokes multipliers requires exponential accuracy Thus the areas of Stokes phenomenon and exponential asymptotics are closely related and finding correct multipliers for these recessive terms depend crucially on an understanding of Stokes phenomenon

1.3.1 Stokes Phenomenon.

Consider the function

$$g(z) = \sinh(1/z) = (e^{1/z} - e^{-1/z})/2$$

which has leading behaviours

$$g(z) \sim e^{1/z}/2, \qquad z \to 0 \qquad |argz| < \pi/2,$$

$$g(z) \sim e^{-1/z}/2, \qquad z \to 0 \qquad \pi/2 < argz < 3\pi/2$$

Clearly the asymptotic behaviour of $g(z)$ in the complex plane depends upon the path along which the irregular singular point, $z_0 = 0$, is approached Asymptotic relations in the complex plane must necessarily involve the concept of a sector of validity For example, consider functions $f(z)$ and $g(z)$ such that

$$f(z) \sim g(z), \qquad z \to z_0,$$

in a certain sector S of the complex z-plane Then writing

$$f(z) = g(z) + [f(z) - g(z)]$$

we see that what is meant by "$f(z) \sim g(z), \quad z \to z_0$ in S", is that the term $[f(z) - g(z)]$ is small compared to $g(z)$ in S That is, $g(z)$ is dominant while $[f(z) - g(z)]$ is subdominant in S On the boundary of S both terms are of the same magnitude and upon crossing the boundary dominant and subdominant terms interchange, with $[f(z) - g(z)]$ becoming the dominant term while $g(z)$ becomes subdominant This occurrence is known as *Stokes phenomenon*

Stokes lines are those asymptotes of the curves in the complex plane upon which the difference between dominant and subdominant terms is of greatest magnitude In the case of linear differential equations, these curves may often be straight lines Similarly, anti-Stokes lines are those asymptotes upon which dominant and subdominant terms are of equal magnitude anti-Stokes lines thus correspond to the boundaries of the sector of validity, S In the case of just two exponentials, say, $e^{\phi_+(z)}$ and $e^{\phi_-(z)}$ as $z \to z_0$ the Stokes lines are asymptotic to the curves $Im[\phi_+(z) - \phi_-(z)] = 0$ while the anti-Stokes are given by the asymptotes to the curves $Real[\phi_+(z) - \phi_-(z)] = 0$

A more relevant example of the occurrence of Stokes phenomenon arises in the investigation of the behaviour of the parabolic cylinder functions for large z Consider the parabolic cylinder equation

$$y''(z) + (\nu + \frac{1}{2} - z^2/4)y(z) = 0$$

The Liouville-Green method provides a means of obtaining the leading behaviour of a solution near an irregular singular point of the differential equation It involves a substitution of the form $y = e^{S(z)}$, where $S'' << (S')^2$ near the irregular singular point combined with the method of dominant balance Applying this method to the parabolic cylinder equation shows that leading behaviours to its solutions are of the form,

$$y(z) \sim c_1 z^{-\nu-1} e^{z^2/4}, \quad z \to \infty \tag{1 9}$$

and

$$y(z) \sim c_2 z^{\nu} e^{-z^2/4}, \quad z \to \infty \tag{1 10}$$

The parabolic cylinder function $D_\nu(z)$ is conventionally defined as the solution whose asymptotic behaviour is given by (1 10) with $c_2 = 1$ Clearly then the Stokes lines occur at arg $z = \pi, |z| \to \infty$ while the anti-Stokes lines occur at arg $z = \pm 3\pi/4, |z| \to \infty$

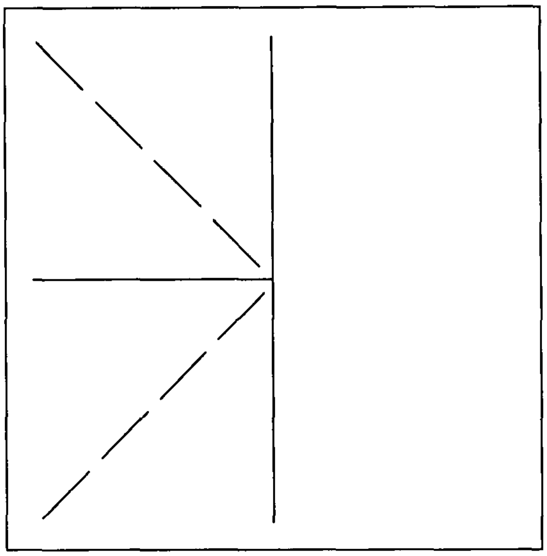


Figure 1 3 Stokes and anti-Stokes lines for D_ν

Stokes discovered that if we are approximating a function $y(z)$ by a linear combination of two independent solutions $y_+(z)$ and $y_-(z)$, the multiplier of the subdominant solution $y_-(z)$ jumps by i times the multiplier of the dominant exponential $y_+(z)$ across each of the associated Stokes lines

Such jumps," Stokes phenomenon", are necessary to achieve agreement between different asymptotic representations valid in different regions of the complex plane However, the need for such discontinuities in the representation of analytic and smooth functions has always been hard to accept Since its discovery in 1857 Stokes phenomenon has always had an air of mystery attached This was in no way lessened by Stokes' own account of the phenomenon

" *the inferior term enters as it were into a mist is lost for a while from view and comes out with the coefficient changed The range during which the inferior term remains in a mist decreases indefinitely as [the asymptotic parameter] increases indefinitely* "

STOKES 1819-1903

Sir George Gabriel Stokes (born Skreen, County Sligo August 13th 1819). Stokes' theoretical and experimental investigations covered the entire realm of natural philosophy. He concentrated on the physical importance of problems making the mathematical analyses subservient to physical requirements. His few excursions into pure mathematics were prompted either by a need to develop methods to solve a specific physical problem or by a desire to establish the validity of mathematics he had already employed. Stokes was universally honoured with degrees and medals and was knighted in 1889.

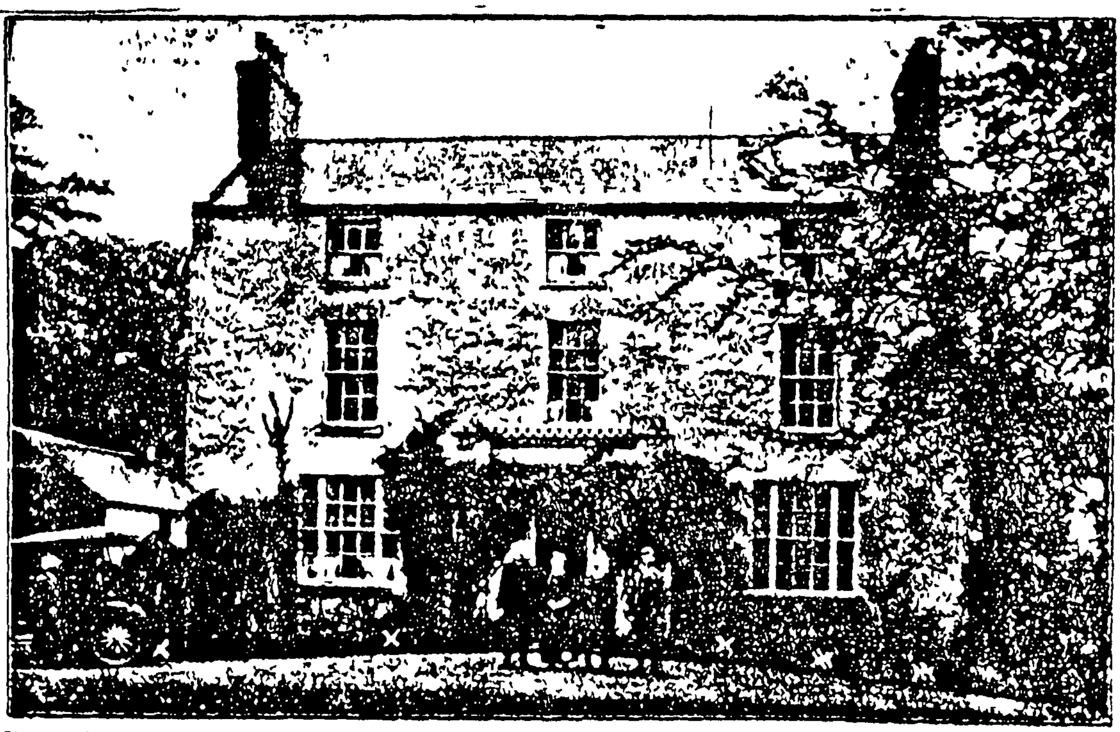


Photo by] SKREEN RECTORY. *[Slater, Ballina*

It was not until the results of Berry in 1989 that the phenomenon was somewhat demystified Berry showed that the change in form of a compound asymptotic expansion as a Stokes line is crossed, although very rapid is in fact continuous, with a generic error function dependency, which he did using Dingle's theory of terminants

In his theory of terminants Dingle developed a systematic theory for interpreting asymptotic expansions beyond their least term He illustrated, for a range of functions, the common pattern

function = first n terms of asymptotic series + n^{th} term × terminant,
in which the terminant is expressible in terms of "basic terminants", whose properties are known

While producing correct answers Berry's methods are quite formal, and later in 1989 at the Winnipeg conference in honour of his 65^{th} birthday, Olver provided a new analysis to place the theories of Berry on rigorous mathematical foundations He constructed uniform exponentially improved asymptotics for a class of functions defined by Laplace integrals These expansions possess the greater accuracy required in modern applications

Independently of Berry and Olver, Jones[20] obtained similar results A related rigorous method has been provided by Boyd[10] using Stieltjes transforms
More recently, Paris [38] described an alternative theory of this smoothing using Mellin-Barnes integrals These are of the form

$$I(z) = \frac{1}{2\pi i} \int_C g(s) z^{-s} ds,$$

where $g(s)$ is typically a product or quotient of gamma functions, perhaps with trignometric functions C is an appropriate path in the complex z-plane The domains of convergence of such integrals are discussed in Chapter 2 of Paris and Wood [37] The flexibility of such integrals and their domains of convergence allow uniform exponentially improved expansions to be readily constructed

Due to the importance of exponential asymptotics for the present problem we shall proceed to detail Berry's formal *smoothing of Stokes discontinuities* and Olver's rigorous analysis of these results The applications which we make in Chapter 3 will be based on the Paris and Wood Mellin-Barnes integral representation of Olver's terminant functions, and it is essential for the reader to have an understanding of these

1.3.2 Smoothing of Stokes discontinuities.

Berry in his paper " *Uniform asymptotic smoothing of Stokes's discontinuities*" [5], shows that with sufficient resolution the change in the subdominant multiplier is continuous near a Stokes line His work is based on two ideas due to Dingle

Firstly, an asymptotic series may be viewed as "a compact encoding of a function" and its divergence as a source of information in that it indicates the presence of exponentially small terms This explains why interpretation of the divergent tail as occurs in *Asymptotics beyond all orders* can yield exponential accuracy As a consequence of this, the late terms of the asymptotic series associated with one exponential (dominant) are frequently related by resurgence to the early terms of a second asymptotic series associated with the subdominant exponential Resurgence has been defined as the process by which a subdominant exponential is born from the tail of the dominant asymptotic series.

Secondly, he considers only functions with asymptotic series representations whose late terms, those of the divergent tail, follow a common pattern factorial divided by a power This common pattern allows for the use of powerful resummation techniques, due also to Dingle, which enable the asymptotics to be decoded

Stokes phenomenon, being associated with exponentials, frequently arises in the asymptotic approximation of functions $y(z,k)$ defined by integrals or differential equations and dependent on a large parameter k In the simplest case there are just two exponentials, one dominant the other subdominant, and we may write

$$y(z,k) \approx M_+(z,k)\exp[k\phi_+(z)] + iS(z,k)M_-(z,k)\exp[k\phi_-(z)], \qquad (1\ 11)$$

with $Real\phi_+(z) > Real\phi_-(z)$ and the dominant/subdominant contributions labelled +/- The prefactors $M_\pm(z,k)$ are simple powers of k and they vary slowly with z $S(z,k)$ is the Stokes multiplier function which weights the subdominant exponential and varies rapidly near the Stokes line of $y(z,k)$

In order for (1 11) to maintain its validity throughout the complex z-plane it is essential to retain both the exponentially small term, despite it numerical insignificance, and $S(z,k)$, its weight function To see this more clearly consider the complex error function integral,

$$y(z,k) = \int_{-i\infty}^{Z} dt \exp(kt^2), \qquad Z = X_1 + iX_2 \qquad (1\ 12)$$

For **Z near the positive real axis** the dominant contribution to y as $k \to \infty$ comes from $t = Z$ and

$$y \sim (2kZ)^{-1}\exp(kZ^2)$$

Near the **positive imaginary axis** this asymptotic representation for y would seem to suggest that y is exponentially small However from (1 12), we see that the integral is then dominated by the stationary point $t = 0$ and thus in this region

$$y \sim i(\pi/k)^{1/2}$$

These two representations would suggest the asymptotics

$$y \sim (2kZ)^{-1}\exp(kZ^2) + i(\pi/k)^{1/2} \qquad (1\ 13)$$

[The Stokes line is then $X_2 = 0$ and the anti-Stokes lines are $X_2 = \pm X_1$]
However, from the integral representation we see that y is exponentially small near the negative imaginary axis while (1 13) would suggest that $y \sim i(\pi/k)^{1/2}$ Thus we need a weight function for the subdominant term which will vary between 0 and 1 as Z varies between the two anti-stokes lines $X_2 = X_1$ and $X_2 = -X_1$, The representation of y valid in the three regions is then

$$y \sim (2kZ)^{-1}\exp(kZ^2) + iS(z,k)(\pi/k)^{1/2} \qquad (1\ 14)$$

The conventional view has been that the change in S is discontinuous and localised at the Stokes line, on one side of this line S takes a value, say S_- , on the other side $S = S_- + 1$ while on the line itself $S = S_- + 1/2$

The conclusions of first Stokes and then Dingle/Berry, although quite different, stem from an analysis of the dominant asymptotic expansion

$$y(z,k) \approx M_+(z,k)\exp[k\phi_+(z)]\sum_{r=0}^{\infty} Y_r \qquad (1\ 15)$$

(As shall become evident, there is no need to specify the subdominant term as it will be born out of the divergent tail of the dominant series)
Observe that away from the Stokes line the argument, or phase, of the Y_r terms vary resulting in a degree of cancellation thus allowing resummation of the divergent tail However, on the Stokes line the Y_r terms all have the same phase

Stokes concluded that the divergence is incurable and that after summing down to the least term the asymptotic expansion specifies $y(z,k)$ only up to an "irremovable vagueness" This vagueness is just sufficient to permit the discontinuous emergence of the small exponential

In contrast, Dingle viewed this divergent series as a coded representation for y(z,k), which can be reconstructed exactly (in principle and sometimes in practice) by proper interpretation of the late terms Berry's derivation of the functional form of S across a Stokes line is a simple development in Dingle's theory

It is thus the presence of the subdominant exponential term that prevents the series (1 15) from converging However, Dingle observes that, the existence of such a subdominant exponential results in a remarkable universality of the late terms, Y_r for $r >> 1$ In fact Dingle shows that

$$Y_r \rightarrow \frac{M_-(r-\beta)!}{2\pi M_+ F^{r-\beta+1}}, \tag{1 16}$$

as $r \rightarrow \infty$ with $\beta = O(1)$, taking specific values and F denoting the singulant, which he defines by $F = k(\phi_+ - \phi_-)$

Using this result for the late terms of the series in (1 15) yields

$$y(z,k) \approx M_+(z,k)\exp[k\phi_+(z)]\sum_{r=0}^{n-1} Y_r + iS_n(F)M_-(z,k)\exp[k\phi_-(z)],$$

where

$$S_n(F) = \frac{-i}{2\pi}\exp(F)\sum_{r=n}^{\infty}\frac{(r-\beta)!}{F^{r-\beta+1}}$$

Then applying Borel summation to $S_n(F)$, that is writing the factorial as an integral and interchanging summation and integration, $S_n(F)$ reduces to

$$\begin{aligned} S_n(F) &= \frac{-i}{2\pi F^{1-\beta}}\exp(F)\int_0^{\infty} ds\exp(-s)s^{-\beta}\sum_{r=n}^{\infty}(s/F)^r \\ &= \frac{-i}{2\pi}\int_0^{\infty} dt\frac{t^{n-\beta}\exp[F(1-t)]}{1-t} \end{aligned}$$

It should be noted at this point, that this interchanging of integration and summation presupposes uniform convergence which is absent This is one of the reasons why Berry's methods are considered to be purely formal For complete interpretation it is necessary to specify the contour relative to the pole at $t = 1$ In Berry's specification the contour must pass above $t = 1$ so that

$$S_n(F) = 1/2 - \frac{-i}{2\pi}\int_0^{\infty} dt\frac{t^{n-\beta}\exp[F(1-t)]}{1-t},$$

with the principal value of the integral being taken, we are in the case where $S_- = 0$

The problem now reduces to determining the dominant asymptotics of $S_n(F)$, the Stokes multiplier Using optimal truncation Berry forces the stationary point and pole, at t=1, to coincide Then by expanding the integrand about $x = t-1 = 0$, whose neighbourhood dominates, and retaining only dominant terms Berry shows that

$$S_n(F) \sim \frac{1}{\sqrt{\pi}} \int_{-\infty}^{ImF/\sqrt{2RealF}} dt \exp[-t^2],$$

where $ImF << RealF$ and $RealF >> 1$ Thus the Stokes multiplier is given by

$$S(\sigma) \sim \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\sigma} dt \exp[-t^2], \tag{1 17}$$

with $\sigma = ImF/\sqrt{2RealF}$ being termed the natural variable, or the Stokes variable

The importance of Berry's result lies in the fact that it quantifies the change in the Stokes multiplier, $S(z,k)$, as ph z varies with $|z|$ fixed $S(z,k)$ varies smoothly from S_- to $S_- + 1$ across a Stokes line, the functional dependence on the natural variable being that of the error function

Clearly, the replacement of the late terms , Y_r for $r >> 1$, by an approximation of the form

$$\frac{M_-\Gamma(r-\beta+1)}{M_+F^{r-\beta+1}},$$

due to Dingle, is central to Berry's work Indeed it is surprising that Dingle himself did not make the additional step Perhaps he did not see the need for it ? It is an essential requirement if Berry's method is to succeed for a given asymptotic expansion When this condition is fulfilled Dingle's theories are applicable and yield an integral representation for the Stokes multiplier from which approximations in the form of the error function follow

Olver [33] observed that mathematically there are two major problems in putting this formal analysis on a rigorous footing -

(1) Justification of Dingle's theory used in constructing an appropriate integral representation for the Stokes multiplier

(2) The approximation to this multiplier in terms of the error function

Central to Olver's justification of this formal procedure is the function $T_\nu(z)$,

$$T_\nu(z) = \frac{e^{\pi\nu i}\Gamma(\nu)}{2\pi i}\frac{E_\nu(z)}{z^{\nu-1}} \tag{1 18}$$

where E_ν is the generalised exponential function An integral representation for $E_\nu(z)$ is given by

$$E_\nu(z) = \frac{z^{\nu-1}e^{-z}}{\Gamma(z)} \int_0^\infty \frac{e^{-zt}t^{\nu-1}}{1+t}dt, \quad |argz| < \pi/2,$$

valid in the half-plane $Real(\nu) > 0$ with $z^{\nu-1}$and $t^{\nu-1}$ taking their principal values This function forms the basis for the analysis of many of those authors mentioned previously and as such it is of extreme importance Much of Olver's analysis is based on the idea of converging factors introduced in Chapter 14 of his book [31]

In his paper " *On Stokes phenomenon and converging factors*" [33], Olver obtained the asymptotic properties of $T_\nu(z)$ for large $|z|$, where $z = |z|e^{i\phi}$. Using the Laplace integral representation

$$T_\nu(z) = \frac{e^{\pi i\nu}}{2\pi i}e^{-z}\int_0^\infty \frac{e^{-zt}t^{\nu-1}}{1+t}dt, \tag{1.19}$$

with $Real(\nu) > 0$, valid when $|argz| < \pi/2$ and by analytic continuation elsewhere, he investigates the behaviour of $T_\nu(z)$ for large $|z|$, with $|z| \sim |\nu|$ (optimal truncation). He shows, under these circumstances, that $T_\nu(z)$ possesses the asymptotic behaviour

$$T_\nu(z) \sim -\frac{e^{(\pi-\phi)\nu i}}{1+e^{-i\phi}}\frac{e^{-z-|z|}}{\sqrt{2\pi|z|}}[1+O(1/z)], \quad |\phi| \leq \pi - \varepsilon$$

$$T_\nu(z) \sim \frac{1}{2}+\frac{1}{2}erf[c(\phi)\sqrt{|z|/2}] + O(\frac{e^{-|z|c^2(\phi)/2}}{\sqrt{2\pi|z|}}), \quad -\pi+\varepsilon \leq \phi \leq 3\pi - \varepsilon.$$

With conjugate behaviour valid in the sector $-3\pi + \varepsilon \leq \phi \leq \pi - \varepsilon$. The quantity $c(\phi)$ is defined by

$$\frac{1}{2}c^2(\phi) = 1 + c(\phi - \pi) - e^{i(\phi-\pi)}$$

and corresponds to the branch of $c(\phi)$ which behaves like

$$c(\phi) = \phi - \pi + \frac{1}{6}i(\phi-\pi)^2 - \frac{1}{36}(\phi-\pi)^3 + ...$$

in the neighbourhood of $\phi = \pi$.

This result, the relationship between $T_\nu(z)$ and the error function, solves the second problem of Berry's formal procedure. For the class of problems considered by Olver, the Stokes multiplier function can be written in terms of $T_\nu(z)$, which from the above is related to the error function.

To overcome the problem of constructing an appropriate integral representation for the Stokes multiplier Olver considers the n^{th} remainder term, $R_n(z)$, resulting from optimal truncation of the dominant asymptotic series. He constructs a double integral representation for $R_n(z)$ and by a change of integration variable derives a uniform asymptotic expansion for $R_n(z)$ as a series of generalised exponential integrals. Retaining only the dominant term of this series, and noting (1.18) , Olver employs the large $|z|$ asymptotics of $T_\nu(z)$ to approximate $R_n(z)$ in terms of the error function.

Olver uses the confluent hypergeometric function, $U(a, a-b+1, z)$, to illustrate this procedure, in this instance using Cauchy's integral formula for the remainder term in Taylor's theorem to obtain the appropriate double integral representation for $R_n(z)$. However, Olver's method is quite general and indeed is applicable in the same circumstances that the methods of Dingle and Berry can be used.

Subsequently, several authors have obtained exponentially-improved approximations for a variety of specific "special functions". While Olver, [33], using integral representation of the solutions, provided rigorous smoothing of Stokes phenomenon for certain differential equations, McLeod [27] carried out such smoothing directly for a class of differential equations. Generalisations of the results of McLeod by Olver are in press. Berry himself [9] gave an alternative discussion for asymptotic series arising from second order differential equations Paris [38] also provides an extension of this rigorous smoothing He establishes the smooth transition of a Stokes multiplier across a Stokes line for a certain type of differential equation of arbitrary order n There is however, still a need for a reappraisal by mathematicians of the formal methods of Dingle, and the smoothing due to Berry, with a view to establishing a rigorous theory of exponentially improved asymptotics for a wide class of functions

1.3.3 Hyperasymptotics.

Berry's findings extend Dingle's work in two ways

The first, as we have seen, results in the smoothing of Stokes phenomenon Together with Olver's justification of this smoothing it goes a long way to lessen the *"aroma of paradox and audacity"* [McLeod] that has hung about the whole subject of divergent series, connection formulae in WKB theory and related areas

The second result is a technique termed *"hyperasymptotics"* Consider the general divergent series

$$Y = \sum_{r=0}^{\infty} (-)^r Y_r(F), \qquad Y_0 = 1, \;\; Y_r \propto k^{-r}$$

By systematically and repeatedly applying optimal truncation, resurgence and Borel summation to this divergent series, Berry obtains a nested sequence of asymptotic series truncated at their least term Namely,

$$Y = S_0 + S_1 + S_2 +$$

where

$$S_n = \sum_{r=0}^{N_n - 1} (-)^r Y_r K_{rn},$$

with $K_{r0} = 1$ and

$$\begin{aligned} K_{rn} &= K_{rn}(F, N_0, \;\; , N_{n-1}) \\ &= \frac{1}{(2\pi)^n F^{N_0 - r}} \prod_{i=0}^{n} \int_0^\infty d\xi_i \frac{e^{-\xi_i} \xi_i^{N_{i-1} - N_i - 1} (-)^{N_{i-1}}}{[1 + xi_i / xi_{i-1}]}, \end{aligned}$$

($\xi_0 = F, \xi_i = N_{i-1} - N_i - 1$) *Hyperasymptotics* is then defined as the systematic study of these approximations to the small exponential error left by truncation of the main series

Clearly, each *"hyperseries"*, S_n, involves a term of the original asymptotic series for the particular function being approximated, together with terminant integrals that are of universal form In general, the resummation generates asymptotic series which themselves require resummation Each series is half the length of its predecessor and the process terminates naturally with the last hyperseries containing just one term In fact, using Stirling's approximation, optimal truncation is seen to be achieved when $N_n = Int[|F|/2^n]$ Then each *hyperseries* indeed contains half as many terms as the preceding series and termination occurs after $n_{max} = Int[\log_2|F|]$ stages with $N_{n_{max}} = 1$

As we have already seen, optimal truncation of the original asymptotic series results in an error of order e^{-k} This procedure, *Asymptotics beyond all order*, is just the zeroth stage in *hyperasymptotics* and corresponds to what Berry terms *"superasymptotics"* *Hyperasymptotics* goes *" beyond asymptotics beyond all orders"* in that it systematically reduces this exponentially small error The error at the last stage is of order $\exp[-(1 + 2\ln 2)k]$, less than the square of the error at the superasymptotic stage

Chapter 2

Direct and variational methods for the optical tunnelling equation.

2.1 Introduction and Prerequisites.

In this chapter, using a knowledge of special functions, we shall obtain the leading asymptotic behaviour, as $\varepsilon \to 0+$, of the exponentially small imaginary part of the "eigenvalue" for the perturbed problem [(1 1)-(1 3)] , comprising the differential equation $y''(x) + (\lambda + \varepsilon x^2)y(x) = 0$ with a linear homogeneous boundary condition at $x = 0$ and an "outgoing wave" condition as $x \to +\infty$ It makes sense to talk of "complex" eigenvalues since the above problem is non-self-adjoint, the non-self-adjointness has been shown to arise from the condition at infinity

In their treatment of the problem with $S(x) = x$ Paris and Wood, [34], encountered certain technical difficulties with Stokes discontinuities since the imaginary part of the eigenvalue was found to be

$$Im\lambda \sim \frac{-2h^2}{e} \exp\left[\frac{-4h^3}{3\varepsilon}\right] \quad as\ \varepsilon \to 0+$$

compared to the eigenvalue itself

$$\lambda \sim -h^2 - \frac{\varepsilon}{2h} - \frac{\varepsilon^2}{8h^4} + O(\varepsilon^3) \quad as\ \varepsilon \to 0+ \tag{2 1}$$

Such difficulties also occur in the x^2-case However, our exponential asymptotics arise in evaluating the boundary condition at 0 and not, as was the case with the x-problem, in the solution itself We shall illustrate the existence of such an exponentially small phenomena for problem (1 1) and show that

$$Im\lambda \sim -2h^2 exp\left[\frac{-h^2\pi}{2\sqrt{\varepsilon}}\right] \quad as\ \varepsilon \to 0+ \tag{2 2}$$

A knowledge of the asymptotics of solutions to the parabolic cylinder differential equation enables us to construct a linear combination of such solutions which satisfies the outgoing wave condition at infinity Insertion of this combination in the boundary condition at the origin, if manipulated with due respect for the exponentially small terms, yields the expression (2.2). The complex number λ obtained by the direct method is an eigenvalue in the sense that the differential equation (1 1) possesses a non-trivial solution satisfying both boundary conditions when this value of λ is taken

Before tackling the optical tunnelling problem we shall obtain an exponentially improved asymptotic expansion for the quotient $\Gamma(z+\frac{1}{4})/\Gamma(z+\frac{3}{4})$, as $z \to \infty$ close to the negative imaginary axis Since the result (5 02) of Olver [32] contains only algebraic terms it is insufficient for our purposes and an improved approximation is essential if the direct method is to yield an accurate estimate for the imaginary part of the desired eigenvalue We shall derive the result for this required special case $\Gamma(\frac{1}{4}+z)/\Gamma(\frac{3}{4}+z)$, although the method, which resulted from discussions with Paris and Wood and is contained in their paper [36], holds for $\Gamma(\alpha+z)/\Gamma(\beta+z)$ with $\alpha, \beta \in \mathcal{C}$ such that $\alpha+\beta=1$

By the reflection formula, we may rearrange the quotient as

$$\frac{\Gamma(\frac{1}{4}+z)}{\Gamma(\frac{3}{4}+z)} = \frac{1}{\pi}\Gamma(\frac{1}{4}+z)\Gamma(\frac{1}{4}-z)\sin\pi(\frac{1}{4}-z)$$

Writing $z = iy,\ \ y \in \mathcal{R}$, and using the result $\Gamma(\overline{z}) = \overline{\Gamma(z)}$,

$$\begin{aligned}\frac{\Gamma(\frac{1}{4}+z)}{\Gamma(\frac{3}{4}+z)} &= \frac{1}{\pi}|\Gamma(\frac{1}{4}+iy)|^2 \sin\pi(\frac{1}{4}-iy) \\ &\sim -i|y|^{\frac{-1}{2}}e^{-\pi|y|}\left\{e^{i\pi(\frac{1}{4}-iy)} - e^{-i\pi(\frac{1}{4}-iy)}\right\} \quad as\ |y| \to \infty\end{aligned}$$

on employing Stirling's approximation For $y = |y|\ \ (z = i|y|)$ we find that

$$\frac{\Gamma(\frac{1}{4}+z)}{\Gamma(\frac{3}{4}+z)} \sim z^{-\frac{1}{2}}\left(1+ie^{2i\pi z}\right) \tag{2 3}$$

while for $y = -|y|\ \ (z = -i|y|)$

$$\frac{\Gamma(\frac{1}{4}+z)}{\Gamma(\frac{3}{4}+z)} \sim z^{\frac{-1}{2}}\left(1-ie^{-2i\pi z}\right) \tag{2 4}$$

Higher order real and imaginary terms could be obtained by using the full series expansion in Stirling's formula

This result is also valid in a small sector to the right of the imaginary axis To see this let $z = iy+\delta$ where $\delta = O(e^{-2\pi|y|})$ Using Taylor's expansion about $\delta = 0$, we have

$$\frac{\Gamma(\frac{1}{4}+(z+\delta))}{\Gamma(\frac{3}{4}+(z+\delta))} = \frac{\Gamma(\frac{1}{4}+z)}{\Gamma(\frac{3}{4}+z)}\left[1+\delta\left\{\psi(\frac{1}{4}+z)-\psi(\frac{3}{4}+z)\right\}+O(\delta^2)\right] \tag{2 5}$$

where, as usual, ψ denotes the logarithmic derivative of Γ By means of the expansion [Abramowitz and Stegun, [1] p 259]

$$\psi(z) \sim \log z - \frac{1}{2z} - \qquad as\ \ z \to \infty \quad in\ |argz| < \pi$$

Then

$$\begin{aligned}\psi(1/4+z)-\psi(3/4+z) &\sim \log\left[\frac{1/4+z}{3/4+z}\right] - \frac{1}{2(z+1/4)} + \frac{1}{2(3/4+z)} + \\ &= \log\left[1-\frac{1}{2(z+3/4)}\right] - \frac{1}{2(z+1/4)} + \frac{1}{2(3/4+z)} + \\ &= -\frac{1}{2(z+3/4)} + \ldots - \frac{1}{2(z+1/4)} + \frac{1}{2(3/4+z)} +\end{aligned}$$

and

$$\psi(1/4 + z) - \psi(3/4 + z) \sim -\frac{1}{2z} + O(z^{-2})$$

Thus (2 5) reduces to

$$\frac{\Gamma(\frac{1}{4} + (z+\delta))}{\Gamma(\frac{3}{4} + (z+\delta))} = \frac{\Gamma(\frac{1}{4} + z)}{\Gamma(\frac{3}{4} + z)}\left[1 + \frac{\imath\delta}{2y} + O(\delta^2)\right]$$

But $\frac{\imath\delta}{2y}$ is $O(\frac{e^{-2\pi|y|}}{|y|})$ which may be neglected in (2 4) with $|y| \to \infty$ Thus (2 4) remains valid when z is replaced by $z + \delta$ for sufficiently small δ This will be adequate for our purposes in this chapter In Chapter 3 a more general result for such a quotient will be obtained directly from the appropriate difference equation, which results from a generalisation of the result $\Gamma(z + 1) = z\Gamma(z)$

2.2 The direct method for the non-self-adjoint problem

We are now in a position to consider the non-self adjoint model for the optical tunnelling problem, given by [(1 1)-(1 3)] Despite the singular nature of the problem we start by using regular perturbation techniques to obtain an estimate for λ Such methods yield a valid estimate for $Real\lambda$ and give some indication of the magnitude of $Im\lambda$ We start with a trial solution of the form

$$y(x) = \sum_{n=0}^{\infty} y_n(x)\varepsilon^n,$$

$$\lambda = \sum_{n=0}^{\infty} \lambda_n \varepsilon^n$$

Upon substituting into the differential equation (1 1) we obtain

$$\sum_{n=0}^{\infty} y_n''(x)\varepsilon^n + \left(\sum_{n=0}^{\infty} \lambda_n\varepsilon^n + \varepsilon x^2\right) \sum_{n=0}^{\infty} y_n(x)\varepsilon^n = 0$$

and

$$\sum_{n=0}^{\infty}(y_n'' + \lambda_n y_0 + \lambda_{n-1} y_1 + \quad + \lambda_0 y_n)\varepsilon^n + \sum_{n=0}^{\infty} y_n x^2 \varepsilon^{n+1} = 0$$

Then equating like powers of ε,

$$\varepsilon^0 \qquad y_0'' + \lambda_0 y_0 = 0,$$

with solutions $y_0 = \exp[\pm\imath\sqrt{\lambda_0}x]$ Taking the positive branch, in the hope of satisfying the condition at infinity, we find that this solution satisfies the boundary condition at the origin provided $\lambda_0 = -h^2$, then $y_0(x) = e^{-hx}$ Continuing in this fashion

$$\varepsilon^1 \qquad y_1'' + \lambda_0 y_1 + (\lambda_1 + x^2)y_0 = 0$$

we then multıply across by $y_0 = e^{-hx}$, notıng that $y_0'' = h^2 y_0$ Integratıng over $(0, \infty)$ leads to

$$\int_0^{\infty} e^{-hx} y_1'' dx - h^2 \int_0^{\infty} e^{-hx} y_1 dx + \int_0^{\infty} (\lambda_1 + x^2) e^{-2hx} dx = 0 \qquad (2\ 6)$$

Applyıng ıntegratıon by parts twıce we obtaın

$$\int_0^{\infty} e^{-hx} y_1'' dx = e^{-hx}[y_1' + hy_1]|_0^{\infty} + h^2 \int_0^{\infty} e^{-hx} y_1 dx$$

The first term can be seen to vanısh due to the boundary condıtıon at the orıgın (1 2), and also the fact that h ıs a posıtıve constant Then (2 6) reduces to

$$\int_0^{\infty} (\lambda_1 + x^2) e^{-2hx} dx = 0$$

By means of further ıntegratıon by parts, thıs ımplıes that $\lambda_1 = -1/(2h^2)$ After a number of such ıteratıons we obtaın

$$\lambda = -h^2 - \frac{\varepsilon}{2h^2} - \frac{7\varepsilon^2}{8h^6} - \frac{121\varepsilon^3}{16h^{10}} + O(\varepsilon^4) \qquad (2\ 7)$$

and the coefficıent of every power $\varepsilon^n, n \in \mathcal{N}$ ıs real We conclude that $Im\lambda << \varepsilon^n$, for all $n \in \mathcal{N}$ and hence $Im\lambda$ must be transcendentally small as $\varepsilon \to 0+$

We shall now appeal to the theory of specıal functıons, or more specıfically to the known propertıes of the parabolıc cylınder functıons, ın the hope of obtaınıng an estımate for $Im\lambda$

Usıng the transformatıon $z = e^{\frac{\imath\pi}{4}} 2^{\frac{1}{2}} \varepsilon^{\frac{1}{4}} x$ and $a = \frac{1}{2} \varepsilon^{\frac{-1}{2}} \imath\lambda$ the optıcal tunnellıng problem reduces to

$$\frac{d^2 y}{dz^2} = (a + \frac{1}{4} z^2)\, y \qquad (2\ 8)$$

wıth boundary condıtıon

$$e^{\frac{\imath\pi}{4}} 2^{\frac{1}{2}} \varepsilon^{\frac{1}{4}} \frac{dy(0)}{dz} + hy(0) = 0 \qquad (2\ 9)$$

and the same "outgoıng wave" condıtıon at ınfinıty (but note that $z \to \infty e^{\frac{\imath\pi}{4}}$ as $x \to +\infty$) Equatıon (2 8) ıs Weber's parabolıc cylınder equatıon whıch has lınearly ındependent solutıons, $U(a, z), U(-a, -\imath z)$ Other solutıons to thıs equatıon are gıven by $U(a, -z)$ and $U(-a, \imath z)$, as may be seen dırectly from the dıfferentıal equatıon

As ındıcated we shall start by constructıng an "outgoıng wave" solutıon from these fundamental solutıons The resultıng "outgoıng wave" solutıon of the transformed dıfferentıal equatıon (2 8) shall then be forced to satısfy the the transformed boundary condıtıon at the orıgın (2 9) thus yıeldıng the requıred eıgenrelatıon For fixed a and large $|z|$ we see from Olver [32] that

$$U(a, z) \sim z^{-a-\frac{1}{2}} e^{-\frac{z^2}{4}} [1 + O(|z|^{-2})], \qquad |argz| < \frac{3\pi}{4}, \qquad (2\ 10)$$

also

$$\begin{aligned} U(a, -z) \sim\ & e^{\imath\pi(a+\frac{1}{2})} z^{-a-\frac{1}{2}} e^{\frac{-z^2}{4}} [1 + O(|z|^{-2})] \\ + & \frac{\sqrt{2\pi}}{\Gamma(\frac{1}{2} + a)} z^{a-\frac{1}{2}} e^{\frac{z^2}{4}} [1 + O(|z|^{-2})], \qquad 0 < argz < \frac{\pi}{2} \qquad (2\ 11) \end{aligned}$$

Using the connection formula

$$U(-a,-\imath z) = (2\pi)^{-1/2}\Gamma(a+\frac{1}{2})\left[e^{-\imath\pi(a/2-1/4)}U(a,-z) + e^{\imath\pi(a/2-1/4)}U(a,z)\right],$$

the asymptotics for $U(-a,-\imath z)$ can be seen to be given by

$$U(-a,-\imath z) \sim z^{a-1/2}e^{z^2/4}, \quad z\to\infty, \quad 0 < arg z < \pi/2 \tag{2 12}$$

In particular, this is true for $arg z = \pi/4$ So our outgoing wave solution is given by a constant multiple of $U(-a,-\imath z)$, since its large z asymptotics involve only terms of the form $e^{z^2/4}$

$$\begin{aligned} y(z) &= c_0 U(-a,-\imath z) \\ &= c_0\left[e^{-\imath\pi(a/2-1/4)}U(a,-z) + e^{\imath\pi(a/2-1/4)}U(a,z)\right] \\ &= c_1\left[U(a,-z) - \imath e^{\imath\pi a}U(a,z)\right] \end{aligned}$$

We observe that the equation with potential εx^2 differs from that with potential εx considered by Paris and Wood [34] in that it is not necessary to use the results of Berry [5] and Olver [33] in the neighbourhood of Stokes lines at this stage

Substituting the outgoing wave solution into the transformed boundary condition (2 9) gives

$$e^{\frac{\imath\pi}{4}}2^{\frac{1}{2}}\varepsilon^{\frac{1}{4}}\left\{-U'(a,0) - \imath e^{\imath a\pi}U'(a,0)\right\} + h\left\{U(a,0) - \imath e^{\imath a\pi}U(a,0)\right\} = 0 \tag{2 13}$$

which we rearrange as

$$\frac{U(a,0)}{U'(a,0)} = \frac{e^{\frac{\imath\pi}{4}}2^{\frac{1}{2}}\varepsilon^{\frac{1}{4}}\left(1+\imath e^{\imath a\pi}\right)}{h\left(1-\imath e^{\imath a\pi}\right)} \tag{2 14}$$

This becomes

$$\frac{\Gamma(\frac{1}{4}+\frac{a}{2})}{\Gamma(\frac{3}{4}+\frac{a}{2})} = -\frac{2e^{\frac{\imath\pi}{4}}\varepsilon^{\frac{1}{4}}\left(1+\imath e^{\imath a\pi}\right)}{h\left(1-\imath e^{\imath a\pi}\right)} \tag{2 15}$$

on inserting the values of the parabolic cylinder function and its derivatives at the origin given in (19 3 5) of Abramowitz and Stegun [1] Recall that $a = \frac{1}{2}\varepsilon^{\frac{-1}{2}}\imath\lambda$ It then follows from our estimate for λ, (2 7), that $a \to -\imath\infty$ *as* $\varepsilon \to 0+$, in such a way as to be able to apply the exponentially improved approximation (2 4) to the left-hand side of (2 15)

We find that

$$\frac{\Gamma(\frac{1}{4}+\frac{a}{2})}{\Gamma(\frac{3}{4}+\frac{a}{2})} \sim \left(\frac{a}{2}\right)^{\frac{-1}{2}}\left(1-\imath e^{-\imath a\pi}\right) \quad as\ \varepsilon\to 0+$$

Hence

$$\left(\frac{a}{2}\right)^{\frac{-1}{2}}\left(1-\imath e^{-\imath a\pi}\right) \sim -\frac{2}{h}e^{\frac{\imath\pi}{4}}\varepsilon^{\frac{1}{4}}\frac{1+\imath e^{\imath a\pi}}{1-\imath e^{\imath a\pi}}$$

which, on factoring out the exponentially large term $e^{\imath a\pi}$, gives

$$\left(\frac{a}{2}\right)^{\frac{-1}{2}} \sim \frac{-2e^{\frac{i\pi}{4}}\varepsilon^{\frac{1}{4}}}{h\,(1 + ie^{-ia\pi})},$$

and

$$a \sim \frac{h^2(1 + ie^{-ia\pi})^2}{2i\sqrt{\varepsilon}}$$

Remembering that $a = \frac{1}{2}\varepsilon^{\frac{-1}{2}} i\lambda$, where λ is approximated by (2 7), yields

$$\begin{aligned} \lambda &\sim -h^2\left\{1 + 2i\exp\left[\frac{\pi\lambda}{2\sqrt{\varepsilon}}\right]\right\} \\ &\sim -h^2\left\{1 + 2i\exp\left[\frac{-\pi h^2}{2\sqrt{\varepsilon}}\right]\right\} \quad as\ \varepsilon \to 0+ \end{aligned}$$

Thus

$$Im\lambda \sim -2h^2 \exp\left[\frac{-\pi h^2}{2\sqrt{\varepsilon}}\right] \tag{2 16}$$

As anticipated the imaginary part of the desired eigenvalue is exponentially small This illustrates the need for exponentially-improved approximations to ensure accurate estimation of the desired eigenvalue

2.3 Variational methods applied to optical tunnelling.

In this section we shall use the well-established *Variational principles* to verify the result already obtained, namely

$$Im\lambda \sim -2h^2 \exp\left[\frac{-h^2\pi}{2\sqrt{\varepsilon}}\right], \qquad \varepsilon \to 0+$$

As we have just illustrated the transformation $a = i\lambda/2\sqrt{\varepsilon}$ and $z = e^{i\pi/4}2^{1/2}\varepsilon^{1/4}x$ reduces the optical tunnelling differential equation,

$$y''(x) + (\lambda + \varepsilon x^2)y(x) = 0 \qquad x \in (0, \infty)$$

to the standard form for Weber's parabolic cylinder equation, with basic solution $U(a, z)$ A lot of time and effort was spent in trying to successfully apply the variational method to this transformed problem, as complex z led to all kinds of complications with *Stokes lines* for the *Airy function* $Ai(z)$, and Olver's 1959 paper [32]

Observe that the parabolic cylinder differential equation has two **real and distinct** forms

$$y'' - (\frac{x^2}{4} + a)y(x) = 0, \tag{2 17}$$

$$y'' + (\frac{x^2}{4} - a)y(x) = 0 \tag{2 18}$$

We eventually decided that a better approach would be to work with the second of these two equations, and not the first equation which had been previously used in its complex extension, as it avoids us moving off the real axis for what is essentially a real variable problem

Letting $w = \sqrt{2}\varepsilon^{1/4}x$, $A = \frac{-\lambda}{2\sqrt{\varepsilon}}$ and noting that $iA = -a$, we observe that the transformed variable w is real Also since $\lambda = -h^2 + O(\varepsilon)$ as $\varepsilon \to 0+$, $A \sim \frac{h^2}{2\sqrt{\varepsilon}} + O(\varepsilon^{1/2})$ can be taken as approximately real and positive, and $A \to +\infty$ as $\varepsilon \to 0+$ The transformed problem is then given by,

$$\begin{aligned} & y''(w) + (\frac{w^2}{4} - A)y(w) = 0 && w \in (0,\infty) \\ & 2^{1/2}\varepsilon^{1/4}y'(0) + hy(0) = 0, && h > 0 \\ & y(w)\ \textit{an outgoing wave at}\ +\infty, && \end{aligned} \qquad (2\ 19)$$

Instead of $U(a,z)$ with related argument, we now work with Miller's standard solution $W(A,w)$ defined in Abramowitz and Stegun , (19 17 1), as

$$W(A,w) = 2^{-3/4}\left(\sqrt{\frac{|\Gamma(1/4 + iA/2)|}{|\Gamma(3/4 + iA/2)|}}y_1 - \sqrt{\frac{2|\Gamma(3/4 + iA/2)|}{|\Gamma(1/4 + iA/2)|}}y_2\right), \qquad (2\ 20)$$

where

$$y_1 = 1 + A\frac{x^2}{2!} + (A^2 - 1/2)\frac{x^4}{4!} + (A^3 - 7A/2)\frac{x^6}{6!} + \quad ,$$

and

$$y_2 = x + A\frac{x^3}{3!} + (A^2 - 3/2)\frac{x^5}{5!} + (A^3 - 13A/2)\frac{x^7}{7!} +$$

Observe that $W(A,-w)$, $W(-A,iw)$ and $W(-A,-iw)$ are also solutions

Using our estimate for λ, which implies that $A \sim \frac{h^2}{2\sqrt{\varepsilon}}$ as $\varepsilon \to 0+$ and therefore is approximately real, we wish to obtain an estimate of the **exponentially small** imaginary part of A In doing so we shall approximate $Im\lambda$ Multiplying the transformed differential equation by $\overline{y(w)}$ and subtracting from it the conjugate equation, we obtain

$$\begin{aligned} & \overline{y(w)}[y''(w) + (\frac{w^2}{4} - A)y(w)] = 0 \\ (-) \quad & y(w)[\overline{y''(w)} + (\frac{w^2}{4} - \overline{A})\overline{y(w)}] = 0 \\ \hline & \overline{y(w)}y''(w) - y(w)\overline{y''(w)} = (A - \overline{A})y(w)\overline{y(w)} \end{aligned}$$

We then integrate over [0,X], where X is chosen such that $X^2 << A$, that is $X << \varepsilon^{-1/2}$. To be specific choose $X = O(\varepsilon^{-1/4})$ This gives us

$$2iImA\int_0^X y(w)\overline{y(w)}dw = \int_0^X \left[\overline{y(w)}y''(w) - y(w)\overline{y''(w)}\right]dw$$

Applying integration by parts to the right-hand side yields

$$ImA = -\frac{[J(X) - J(0)]}{2i\int_0^X y(w)\overline{y(w)}dw}, \qquad (2\ 21)$$

with $J(X) = \overline{y'(X)}y(X) - y'(X)\overline{y(X)}$ Then $J(w)$ is simply the Wronskian of the solution evaluated at w, that is

$$\begin{aligned} J(w) &= \overline{y'(w)}y(w) - \overline{y(w)}y'(w) \\ &= W(y,\overline{y})(w) \end{aligned}$$

Clearly as we move through the interval $[0,\infty)$ w passes through three regions, $w^2 << A$, $w^2 \sim A$ to $w^2 >> A$ respectively The justification for our choice of X, $X = O(\varepsilon^{-1/4})$, which incorporates only the first of these regions, lies in the fact that it does not effect the evaluation of (2 21) We shall illustrate that not only is the evaluation of both $J(X)$ and $J(0)$ independent of X, but also, as was the case for the x-problem, that the integral $\int_0^X |y(w)|^2 dw$ is dominated by the contribution in the neighbourhood of zero That is,

$$\int_0^X |y(w)|^2 dw \sim \int_0^\delta |y(w)|^2 dw, \qquad \varepsilon \to 0+ \tag{2 22}$$

Here δ is a small positive constant

We therefore need to obtain a representation for our solution $y(w)$ in terms of Miller's standard functions Consider the complex solution

$$E(A,w) = k^{-1/2}W(A,w) + ik^{1/2}W(A,-w),$$

where $k = \sqrt{1+e^{2\pi A}} - e^{\pi A}$ and $k^{-1} = \sqrt{1+e^{2\pi A}} + e^{\pi A}$ Noting that, by Abramowitz and Stegun [1] (19 17 9),

$$E(A,w) = \sqrt{2}e^{[A\pi/4 + i\pi/6 + i\phi_2/2]}U(iA, e^{-i\pi/4}w),$$

$E(A,w)$ corresponds to the outgoing wave solution, $U(-a,-iz)$ of the previous section, and the appropriate solution is clearly a multiple of $E(A,w)$ Here $\phi_2 = arg\Gamma(iA + 1/2)$

In the region $w^2 >> A$ the asymptotics of $E(A,w)$ are those for large w and moderate A, in Abramowitz and Stegun [1] (19 21 1)

$$E(A,w) = \sqrt{2/w}\exp\left[i\left(\frac{w^2}{4} - A\ln w + \frac{\phi_2}{2} + \frac{\pi}{4}\right)\right]\left(1 + O(w^{-2})\right), \tag{2 23}$$

as $w \to +\infty$, A moderate This representation confirms that $E(A,w)$ is the outgoing wave solution Also note that since $|E(A,w)|^2 \sim \frac{2}{w}$ in this region its contribution to the required integral would be negligible, compared to the exponential contribution which will be seen to come from the first region Similarly by investigating the asymptotics of the differential equation in the neighbourhood of the turning point, $w = 2\sqrt{A}$ we see that the contribution from the middle region $w^2 \sim A$ is also negligible The differential equation,

$$y''(w) + (\frac{w^2}{4} - A)y(w) = 0$$

can be written as

$$\frac{1}{4}y''(t) + t(t + 2\sqrt{A})y(t) = 0,$$

with $t = w/2 - \sqrt{A}$ Then in the neighbourhood of the turning point, $t = 0$, the differential equation is approximated by

$$y''(t) + 8\sqrt{A}ty(t) = 0,$$

which, through the substitution $T = 2A^{3/2}t$, is transformed into

$$y''(T) + Ty(T) = 0$$

This is the standard form for the Airy equation with solutions $Ai(-T)$ and $Bi(-T)$ and it follows that for $w \sim 2\sqrt{A}$, $y(w)$ reduces to a combination of Airy functions Then from the asymptotics of the Airy functions $y(w)$ is necessarily $O(1)$ when w is near the turning point

Thus the approximation given in (2 22) is justified On applying the result [1] (19 18 2) on Wronskians of the complex solution and its conjugate

$$\begin{aligned} J(X) &= c\bar{c}W(E(A,w),\overline{E(A,w)})(X) \\ &= -2ic\bar{c}, \end{aligned}$$

where c is a non-zero constant, which from §2 2 is given by $c_0 e^{-[A\pi/4 + i\pi/6 + i\phi_2/2]}/\sqrt{2}$ From the boundary condition at the origin we can evaluate $J(0)$ quite simply,

$$\begin{aligned} J(0) &= \overline{y'(0)}y(0) - \overline{y(0)}y'(0), \\ &= \left(\frac{-h}{2\sqrt{\varepsilon}}\overline{y(0)}\right)y(0) - \left(\frac{-h}{2\sqrt{\varepsilon}}y(0)\right)\overline{y(0)}, \\ &= \left(\frac{-h}{2\sqrt{\varepsilon}}\right)[\overline{y(0)}y(0) - y(0)\overline{y(0)}], \end{aligned}$$

and thus $J(0)$ vanishes identically Then (2 21) reduces to

$$ImA = \frac{c\bar{c}}{\int_0^X y(w)\overline{y(w)}dw}, \tag{2 24}$$

and all depends on the denominator

To evaluate $\int_0^X |E(A,w)|^2 dw$, we therefore require the asymptotics of $E(A,w)$ throughout the interval [0,X], when $A >> w^2$ The appropriate asymptotics are those valid for A large and w moderate, namely,

$$W(A,w) \sim W(A,0)\exp\left[-\sqrt{A}w + \frac{(w/2)^3}{3\sqrt{A}}\right], \quad \varepsilon \to 0^+$$

$$W(A,-w) \sim W(A,0)\exp\left[\sqrt{A}w - \frac{(w/2)^3}{3\sqrt{A}}\right], \quad \varepsilon \to 0^+$$

Hence

$$E(A,w) \sim W(A,0)\left\{k^{-1/2}\exp\left[-\sqrt{A}w + \frac{(w/2)^3}{3\sqrt{A}}\right] + ik^{1/2}\exp\left[\sqrt{A}w - \frac{(w/2)^3}{3\sqrt{A}}\right]\right\} \tag{2 25}$$

Recall that $k = \sqrt{1+e^{2\pi A}} - e^{\pi A}$ then employing the binomial theorem we reduce k as follows,

$$\begin{aligned} k &= \sqrt{1+e^{2\pi A}} - e^{\pi A} \\ &= e^{\pi A}[1+e^{-2\pi A}]^{1/2} - e^{\pi A} \\ &= e^{\pi A}[1+\frac{1}{2}e^{-2\pi A} - \quad] - e^{\pi A} \\ &\sim \frac{1}{2}e^{-\pi A}, \qquad\qquad A \to \infty \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{1}{k} &= \sqrt{1+e^{2\pi A}} + e^{\pi A} \\ &= e^{\pi A}[1+\frac{1}{2}e^{-2\pi A} - \quad] + e^{\pi A} \\ &\sim 2e^{\pi A}, \qquad\qquad A \to \infty \end{aligned}$$

Also, by (19 17 4) in [1],

$$W(A,0) = \frac{1}{2^{3/4}}\left(\frac{|\Gamma(1/4+iA/2)|}{|\Gamma(3/4+iA/2)|}\right)^{1/2}$$

This quotient of gamma functions can now be evaluated for large A by the results in §2 1 Using (2 3)

$$\begin{aligned} \frac{\Gamma(1/4+iA/2)}{\Gamma(3/4+iA/2)} &\sim \left(\frac{iA}{2}\right)^{-1/2}\left(1+ie^{-\pi A}\right), \\ \frac{|\Gamma(1/4+iA/2)|}{|\Gamma(3/4+iA/2)|} &\sim \sqrt{\frac{2}{A}}\left(1+e^{-2\pi A}\right)^{1/2}, \\ &\sim \sqrt{\frac{2}{A}}\left(1+\frac{1}{2}e^{-2\pi A}\right) \end{aligned}$$

Then

$$\begin{aligned} \left(\frac{|\Gamma(1/4+iA/2)|}{|\Gamma(3/4+iA/2)|}\right)^{1/2} &\sim \left(\frac{2}{A}\right)^{1/4}\left(1+\frac{1}{2}e^{-\pi A}\right)^{1/2}, \\ &\sim \left(\frac{2}{A}\right)^{1/4}\left(1+\frac{1}{4}e^{-2\pi A}\right) \end{aligned}$$

Therefore,

$$W(A,0) \sim \frac{1}{\sqrt{2}}A^{-1/4}\left(1+\frac{1}{4}e^{-2\pi A}\right)$$

Substituting into (2 25) and retaining only the dominant contribution we obtain,

$$E(A,w) \sim \frac{1}{\sqrt{2}}A^{-1/4}\left(1+\frac{1}{4}e^{-2\pi A}\right)\sqrt{2}e^{\pi A/2}e^{-\sqrt{A}w}$$

Thus

$$E(A,w) \sim A^{-1/4}e^{\pi A/2}e^{-\sqrt{A}w},$$

as $A \to \infty$ for moderate w This representation is valid over the interval of integration Then

$$\begin{aligned}
\int_0^X |y(w)|^2 dw &\sim c\bar{c}A^{-1/2}e^{\pi A}\int_0^\delta e^{-2\sqrt{A}w}dw, \qquad \varepsilon \to 0+, \\
&\sim c\bar{c}A^{-1/2}e^{\pi A}\int_0^\infty e^{-2\sqrt{A}w}dw, \qquad \varepsilon \to 0+, \\
&= c\bar{c}A^{-1/2}e^{\pi A}\left[\frac{e^{-2\sqrt{A}w}}{-2\sqrt{A}}\right]_0^\infty \\
&= c\bar{c}\frac{A^{-1/2}e^{\pi A}}{2\sqrt{A}} \\
&= c\bar{c}\frac{e^{\pi A}}{2A}
\end{aligned}$$

Putting this into (2 24) yields

$$ImA \sim 2Ae^{-\pi A}$$

Finally recalling that $A = -\frac{\lambda}{2\sqrt{\varepsilon}} \sim \frac{h^2}{2\sqrt{\varepsilon}}$, we obtain the desired approximation for $Im\lambda$, namely

$$Im\lambda \sim -2h^2 \exp\left[\frac{-h^2\pi}{2\sqrt{\varepsilon}}\right], \qquad \varepsilon \to 0+$$

This argument is essentially the *J-function* method adopted by Kath and Kriegsmann in [23] While Kath and Kriegsmann were forced to apply the method to *WKBJ* approximate solutions we, like Paris and Wood in [34], had at our disposal an exact outgoing wave solution As a result our findings are more dependable, in that we do not have to justify averaging across *Stokes lines* as no such averaging is required Also since we have available full asymptotic expansions for our solution we can justify any neglect of subdominant terms

Chapter 3

Difference equations.

3.1 Introduction

In the previous chapter we derived an exponentially improved approximation for the quotient of gamma functions $\Gamma(z+\frac{1}{4})/\Gamma(z+\frac{3}{4})$ as $z \to \infty$ close to the negative imaginary axis While adequate for our purposes this result is of very limited scope We shall now generalise this result to solutions of a class of first order difference equations We shall illustrate how, using the theory of Batchelder [2], Whittaker and Watson [50] and Paris and Wood [36], exponentially improved approximations to certain functions can be obtained directly from the appropriate difference equation

Exponentially improved approximations to the solution of the difference equation

$$y(z+1) = r(z)y(z),$$

with $r(z)$ a rational function

$$r(z) = c_0 \frac{(z-\alpha_1)\ (z-\alpha_q)}{(z-\beta_1)\ (z-\beta_p)},$$

shall be obtained In contrast to the functions considered in §1 3, *Review of exponential asymptotics*, these solutions shall be seen to possess an infinite number of exponential behaviours This result is not surprising when we note that solutions to such difference equations are known to be of the form

$$y(z) = c_0^z \frac{\Gamma(z-\alpha_1)\ \Gamma(z-\alpha_q)}{\Gamma(z-\beta_1)\ \Gamma(z-\beta_p)}$$

Unlike most special functions, eg $Ai(z)$, $D_\nu(z)$, which are solutions of 2^{nd} order linear ordinary differential equations and therefore have at most 2 distinct exponential type controlling behaviours, the gamma function does not satisfy any such differential equation Paris showed that n^{th} order differential equations have up to n exponential smoothings and since 1^{st} order difference equations correspond to infinite order differential equations, the existence of infinitely many exponential behaviours is not unreasonable

Using the relation between the *shift operator* E, defined by $Ey(z) = y(z+1)$, and the differential operator $\exp(d/dn)$, namely

$$\begin{aligned} Ey(z) &= y(z) + \frac{dy(z)}{dz} + \frac{1}{2!}\frac{d^2y(z)}{dz^2} + \\ &= \exp(d/dz)y(z), \end{aligned}$$

the above difference equation reduces to the infinite order differential equation

$$\exp[d/dz]y(z) = r(z)y(z)$$

The results of Berry [7] and, Paris and Wood [36] on the infinitely many smoothings of the gamma function verify our findings By expressing the error term, resulting from optimal truncation of an asymptotic series representation for $\log\Gamma(z)$, in terms of a Mellin-Barnes integral, Paris and Wood obtain an exponentially-improved asymptotic expansion for $\Gamma(z)$ as $|z| \to \infty$ In the process of obtaining this result they illustrate the existence of infinitely many subdominant exponential terms Their smoothing involves the first of these subdominant exponentials, further treatment of the more recessive exponential terms enters the domain of "Hyperasymptotics" and this is indeed the technique used by Berry to obtain infinitely many smoothings for the gamma function in [7] In fact the results of Berry may be obtained from the Paris and Wood [36] paper by interchanging summation and integration at an earlier stage, thus overcoming the problem of different numbers of terms in each of the truncated series The following is very closely related to the work of Paris and Wood in " Exponentially-improved asymptotics for the gamma function "

In light of these infinitely many exponentials, the fact that the gamma function does not satisfy any algebraic differential equation [Holder] seems to add substance to the statement " *the transcendental functions defined by difference equations are of an essentially different type from those defined by differential equations* "

3.2 First order difference equations- Exponential asymptotics.

We start by investigating the behaviour of the solutions of the difference equation

$$y(z+1) = r(z)y(z), \tag{3 1}$$

where $r(z)$ is a rational function, which we shall write as

$$r(z) = \frac{c_0(z-\alpha_1)(z-\alpha_2)\ \ (z-\alpha_q)}{(z-\beta_1)(z-\beta_2)\ \ (z-\beta_p)},$$

or equivalently,

$$r(z) = z^{q-p}\left[c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \quad \right]$$

with $p, q \in \mathcal{Z}^+, c_0 \neq 0$ and $-1 \leq \alpha_i, \beta_j < 0$ for $i = 1,2, \ \ ,q$, $j = 1,2, \ \ ,p$

Following the theory of Batchelder, (3 1) is satisfied formally by the series

$$S(z) = {c_0}^z z^{(q-p)z} e^{-(q-p)z} z^{\frac{c_1}{c_0} - \frac{(q-p)}{2}} \left(s_0 + \frac{s_1}{z} + \quad \right)$$

$S(z)$ is in general divergent, but it plays an important role in the following theory and represents the principal solutions asymptotically for large z, in certain sectors of the plane

For fixed $k \in \mathcal{N}$ let $T(z)$ be the sum of the first k terms in $S(z)$ Batchelder shows that there also exist two analytic solutions, given by

$$\begin{aligned} h(z) &= \lim_{n\to\infty} \frac{1}{r(z)} \frac{1}{r(z+1)} \quad \frac{1}{r(z+n-1)} T(z+n), \\ g(z) &= \lim_{n\to\infty} r(z-1) r(z-2) \quad r(z-n) T(z-n), \end{aligned}$$

with $h(z)$ being represented asymptotically by $S(z)$ in the sector $-\pi < argz < \pi$, while $g(z)$ is similarly represented by $S(z)$ in the region $0 < argz < 2\pi$

Since it is shown in Batchelder [2] that as $n \to \infty$ the solutions are independent of the value of k, let k be unity and take $T(z) = {c_0}^z z^{(q-p)z} e^{-(q-p)z} z^{\frac{c_1}{c_0} - \frac{(q-p)}{2}} s_0$ We then obtain

$$h(z) = \lim_{n\to\infty} \left[\left(\frac{(z-\beta_1)}{(z-\alpha_1)} \frac{(z-\beta_p)}{(z-\alpha_q)} \right) \left(\frac{(z+n-1-\beta_1)}{(z+n-1-\alpha_1)} \frac{(z+n-1-\beta_p)}{(z+n-1-\alpha_q)} \right) \right] \times ({c_0}^{z+n} + (z+n)^{(q-p)(z+n-\frac{1}{2})+c_1/c_0} e^{-(q-p)(z+n)} s_0)$$

Taking y(z) to be the solution satisfying the initial condition $y(0) = y_0$, we find

$$\frac{y(z)}{y_0} = \frac{\lim_{n\to\infty} \left[\left(\frac{(z-\beta_1)}{(z-\alpha_1)} \frac{(z-\beta_p)}{(z-\alpha_q)} \right) \left(\frac{(z+n-1-\beta_1)}{(z+n-1-\alpha_1)} \frac{(z+n-1-\beta_p)}{(z+n-1-\alpha_q)} \right) \right]}{\lim_{n\to\infty} \left[\left(\frac{(-\beta_1)}{(-\alpha_1)} \frac{(-\beta_p)}{(-\alpha_q)} \right) \left(\frac{(n-1-\beta_1)}{(n-1-\alpha_1)} \frac{(n-1-\beta_p)}{(n-1-\alpha_q)} \right) \right]} \times \frac{\lim_{n\to\infty} ({c_0}^{z+n} (z+n)^{(q-p)(z+n-\frac{1}{2})+\frac{c_1}{c_0}} e^{-(q-p)(z+n)} s_0)}{\lim_{n\to\infty} ({c_0}^n n^{(q-p)(n-\frac{1}{2})+\frac{c_1}{c_0}} e^{-(q-p)n} s_0)}$$

which reduces to

$$\frac{y(z)}{y_0} = \frac{\lim_{n\to\infty} \left[\left(\frac{(z-\beta_1)}{(-\beta_1)} \frac{(z-\beta_p)}{(-\beta_p)} \right) \left(\frac{(z+n-1-\beta_1)}{(n-1-\beta_1)} \frac{(z+n-1-\beta_p)}{(n-1-\beta_p)} \right) \right]}{\lim_{n\to\infty} \left[\left(\frac{(z-\alpha_1)}{(-\alpha_1)} \frac{(z-\alpha_q)}{(-\alpha_q)} \right) \left(\frac{(z+n-1-\alpha_1)}{(n-1-\alpha_1)} \frac{(z+n-1-\alpha_q)}{(n-1-\alpha_q)} \right) \right]} \times \lim_{n\to\infty} \left({c_0}^z (\tfrac{z+n}{n})^{(q-p)(n-\frac{1}{2})+\frac{c_1}{c_0}} (z+n)^{(q-p)z} e^{-(q-p)z} \right)$$

Observing that

$$\begin{aligned} \lim_{n\to\infty} \left(\frac{z+n}{n} \right)^{an+b} &= \lim_{n\to\infty} exp[(an+b) ln(1+z/n)] \\ &= \lim_{n\to\infty} exp[(an+b)[z/n + O(n^{-2})]] \\ &= \lim_{n\to\infty} exp[az + O(n^{-1})] \\ &= \lim_{n\to\infty} e^{az}, \end{aligned}$$

we note the following limits

$$\lim_{n\to\infty}\left(\frac{z+n}{n}\right)^{(q-p)(n-\frac{1}{2})+\frac{c_1}{c_0}} = e^{(q-p)z},$$
$$\lim_{n\to\infty}(z+n)^{(q-p)z} = n^{(q-p)z}$$

The above representation for $y(z)$ can then be rearranged to give

$$\begin{aligned}\frac{y(z)}{y_0} &= {c_0}^z\left(\frac{(z-\beta_1)\ (z-\beta_p)}{(-\beta_1)\ (-\beta_p)}\right)\left(\frac{(-\alpha_1)\ (-\alpha_q)}{(z-\alpha_1)\ (z-\alpha_q)}\right)\\ &\times \frac{\lim_{n\to\infty}\left[\left(\frac{(z+1-\beta_1)\ (z+1-\beta_p)}{(1-\beta_1)\ (1-\beta_p)}\right)\ \left(\frac{(z+n-1-\beta_1)\ (z+n-1-\beta_p)}{(n-1-\beta_1)\ (n-1-\beta_p)}\right)n^{-pz}\right]}{\lim_{n\to\infty}\left[\left(\frac{(z+1-\alpha_1)\ (z+1-\alpha_q)}{(1-\alpha_1)\ (1-\alpha_q)}\right)\ \left(\frac{(z+n-1-\alpha_1)\ (z+n-1-\alpha_q)}{(n-1-\alpha_1)\ (n-1-\alpha_q)}\right)n^{-qz}\right]}\end{aligned}$$

Using Euler's constant $\gamma = \lim_{n\to\infty}\left[\sum_{i=1}^{n-1}\frac{1}{i} - \log n\right]$, this limit becomes

$$\begin{aligned}\frac{y(z)}{y_0} &= {c_0}^z e^{-(q-p)z\gamma}\left(\frac{(z-\beta_1)\ (z-\beta_p)}{(-\beta_1)\ (-\beta_p)}\right)\left(\frac{(-\alpha_1)\ (-\alpha_q)}{(z-\alpha_1)\ (z-\alpha_q)}\right)\\ &\times \frac{\lim_{n\to\infty}\left[\left(\frac{(z+1-\beta_1)\ (z+1-\beta_p)}{(1-\beta_1)\ (1-\beta_p)}e^{\frac{-pz}{1}}\right)\ \left(\frac{(z+n-1-\beta_1)\ (z+n-1-\beta_p)}{(n-1-\beta_1)\ (n-1-\beta_p)}e^{\frac{-pz}{n-1}}\right)\right]}{\lim_{n\to\infty}\left[\left(\frac{(z+1-\alpha_1)\ .(z+1-\alpha_q)}{(1-\alpha_1)\ (1-\alpha_q)}e^{\frac{-qz}{1}}\right)\ \left(\frac{(z+n-1-\alpha_1)\ (z+n-1-\alpha_q)}{(n-1-\alpha_1)\ (n-1-\alpha_q)}e^{\frac{-qz}{n-1}}\right)\right]}\end{aligned}$$

Then, evaluating the limit as $n\to\infty$ we obtain the infinite product

$$\frac{y(z)}{y_0} = {c_0}^z e^{-(q-p)z\gamma}\frac{\left(\frac{(z-\beta_1)\ (z-\beta_p)}{(-\beta_1)\ (-\beta_p)}\right)\prod_{n=1}^{\infty}\left[\frac{(z+n-\beta_1)\ (z+n-\beta_p)}{(n-\beta_1)\ (n-\beta_p)}e^{\frac{-pz}{n}}\right]}{\left(\frac{(z-\alpha_1)\ (z-\alpha_q)}{(-\alpha_1)\ (-\alpha_q)}\right)\prod_{n=1}^{\infty}\left[\frac{(z+n-\alpha_1)\ (z+n-\alpha_q)}{(n-\alpha_1)\ (n-\alpha_q)}e^{\frac{-qz}{n}}\right]}$$

Recalling Euler's infinite product formula for the gamma function, namely,

$$\Gamma(z) = \frac{1}{z}e^{-\gamma z}\prod_{n=1}^{\infty}e^{\frac{z}{n}}\frac{n}{z+n}$$

we see that the solution of (3 1) is given by

$$y(z) = {c_0}^z\frac{\Gamma(z-\alpha_1)\ \Gamma(z-\alpha_q)}{\Gamma(z-\beta_1)\ \Gamma(z-\beta_p)}, \qquad (3\ 2)$$

where $y_0 = y(0) = [\Gamma(-\alpha_1)\ \Gamma(-\alpha_q)]/[\Gamma(-\beta_1)\ \Gamma(-\beta_p)]$

Taking the principal value of the logarithm yields

$$\begin{aligned}\log y(z) &= \log y_0 + z\log c_0 - (q-p)\gamma z\\ &+ \sum_{i=1}^{p}\left[\log\left(\frac{z-\beta_i}{-\beta_i}\right) + \sum_{n=1}^{\infty}\left\{\log\left(\frac{z+n-\beta_i}{n-\beta_i}\right) - \frac{z}{n}\right\}\right]\\ &- \sum_{j=1}^{q}\left[\log\left(\frac{z-\alpha_j}{-\alpha_j}\right) + \sum_{n=1}^{\infty}\left\{\log\left(\frac{z+n-\alpha_j}{n-\alpha_j}\right) - \frac{z}{n}\right\}\right].\end{aligned}$$

Then

$$\begin{aligned}\log y(z) &= \log y_0 + z\log c_0 - (q-p)z\gamma \\ &+ \sum_{i=1}^{p}\left[\log\left(1+\frac{z}{-\beta_i}\right)+\sum_{n=1}^{\infty}\left\{\log\left(1+\frac{z}{n-\beta_i}\right)-\frac{z}{n}\right\}\right] \\ &- \sum_{j=1}^{q}\left[\log\left(1+\frac{z}{-\alpha_j}\right)+\sum_{n=1}^{\infty}\left\{\log\left(1+\frac{z}{n-\alpha_j}\right)-\frac{z}{n}\right\}\right]\end{aligned}$$

and introducing the Taylor series expansion for $\log(1+z)$ we find

$$\begin{aligned}\log y(z) &= \log y_0 + z\log c_0 - (q-p)z\gamma \\ &+ \sum_{i=1}^{p}\left[\sum_{m=1}^{\infty}\frac{(-)^{m-1}z^m}{m(-\beta_i)^m}+\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{(-)^{m-1}z^m}{m(n-\beta_i)^m}-\frac{z}{n}\right)\right] \\ &- \sum_{j=1}^{q}\left[\sum_{m=1}^{\infty}\frac{(-)^{m-1}z^m}{m(-\alpha_j)^m}+\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{(-)^{m-1}z^m}{m(n-\alpha_j)^m}-\frac{z}{n}\right)\right]\end{aligned}$$

Manipulating these sums

$$\begin{aligned}\log y(z) &= \log y_0 + z\log c_0 - (q-p)z\gamma \\ &+ \sum_{i=1}^{p}\left[\left(\frac{z}{-\beta_i}\right)+\sum_{m=2}^{\infty}\frac{(-)^{m-1}z^m}{m(-\beta_i)^m}+\sum_{n=1}^{\infty}\left(\left(\frac{z}{n-\beta_i}\right)-\frac{z}{n}+\sum_{m=2}^{\infty}\frac{(-)^{m-1}z^m}{m(n-\beta_i)^m}\right)\right] \\ &- \sum_{j=1}^{q}\left[\left(\frac{z}{-\alpha_j}\right)+\sum_{m=2}^{\infty}\frac{(-)^{m-1}z^m}{m(-\alpha_j)^m}+\sum_{n=1}^{\infty}\left(\left(\frac{z}{n-\alpha_j}\right)-\frac{z}{n}+\sum_{m=2}^{\infty}\frac{(-)^{m-1}z^m}{m(n-\alpha_j)^m}\right)\right]\end{aligned}$$

and pulling together terms involving the index m, yields

$$\begin{aligned}\log y(z) &= \log y_0 + z\log c_0 - (q-p)z\gamma \\ &+ \sum_{i=1}^{p}\left[\left(\frac{z}{-\beta_i}\right)+\sum_{n=1}^{\infty}\left(\frac{\beta_i z}{n(n-\beta_i)}\right)+\sum_{m=2}^{\infty}\left(\frac{(-)^{m-1}z^m}{m(-\beta_i)^m}+\sum_{n=1}^{\infty}\frac{(-)^{m-1}z^m}{m(n-\beta_i)^m}\right)\right] \\ &- \sum_{j=1}^{q}\left[\left(\frac{z}{-\alpha_j}\right)-\sum_{n=1}^{\infty}\left(\frac{\alpha_j z}{n(n-\alpha_j)}\right)+\sum_{m=2}^{\infty}\left(\frac{(-)^{m-1}z^m}{m(-\alpha_j)^m}+\sum_{n=1}^{\infty}\frac{(-)^{m-1}z^m}{m(n-\alpha_j)^m}\right)\right],\end{aligned}$$

which we rearrange as

$$\begin{aligned}\log y(z) &= \log y_0 + z\log c_o - (q-p)z\gamma \\ &+ \sum_{i=1}^{p}\left[\left(\frac{z}{-\beta_i}\right)+\sum_{n=1}^{\infty}\left(\frac{\beta_i z}{n(n-\beta_i)}\right)+\sum_{m=2}^{\infty}\left(\frac{(-)^{m-1}z^m}{m}\sum_{n=0}^{\infty}\frac{1}{(n-\beta_i)^m}\right)\right] \\ &- \sum_{j=1}^{q}\left[\left(\frac{z}{-\alpha_j}\right)+\sum_{n=1}^{\infty}\left(\frac{\alpha_j z}{n(n-\alpha_j)}\right)+\sum_{m=2}^{\infty}\left(\frac{(-)^{m-1}z^m}{m}\sum_{n=0}^{\infty}\frac{1}{(n-\alpha_j)^m}\right)\right]\end{aligned}$$

From the definition of the generalised zeta function,

$$\zeta(s,a)=\sum_{m=0}^{\infty}\frac{1}{(m+a)^s}, \quad \textit{with} \quad \zeta(s,1)=\zeta(s),$$

valid for $Re(s) > 0$ with $0 < a \leq 1$, we obtain the representation

$$\begin{aligned}\log y(z) &= \log y_0 + z\log c_0 - (q-p)z\gamma \\ &+ \sum_{i=1}^{p}\left[\left(\frac{z}{-\beta_i}\right) + \sum_{n=1}^{\infty}\left(\frac{\beta_i z}{n(n-\beta_i)}\right) + \sum_{m=2}^{\infty}\left(\frac{(-)^{m-1}z^m}{m}\zeta(m,-\beta_i)\right)\right] \\ &- \sum_{j=1}^{q}\left[\left(\frac{z}{-\alpha_j}\right) + \sum_{n=1}^{\infty}\left(\frac{\alpha_j z}{n(n-\alpha_j)}\right) + \sum_{m=2}^{\infty}\left(\frac{(-)^{m-1}z^m}{m}\zeta(m,-\alpha_j)\right)\right].\end{aligned}$$

As we shall see through investigating the sum $\sum_{m=2}^{\infty}\left[\frac{(-)^{m-1}z^m}{m}\zeta(m,a)\right]$, this representation for $\log y(z)$ facilitates the use of contour integrals in determining the exponential asymptotics of $y(z)$ In fact, considering the integral

$$-\frac{1}{2\pi i}\int_C \frac{\pi z^s}{s\sin(\pi s)}\zeta(s,a)ds$$

where C is a Hankel-type contour enclosing the points $s = 2,3,4..$ [see Fig3 1], we see that

$$\begin{aligned}\frac{-1}{2\pi i}\int_C \frac{\pi z^s}{s\sin(\pi s)}\zeta(s,a)ds &= \sum_{m=2}^{\infty} Residues|_{s=m} \\ &= \sum_{m=2}^{\infty}\left[\frac{(-)^{m-1}z^m}{m}\zeta(m,a)\right]\end{aligned}$$

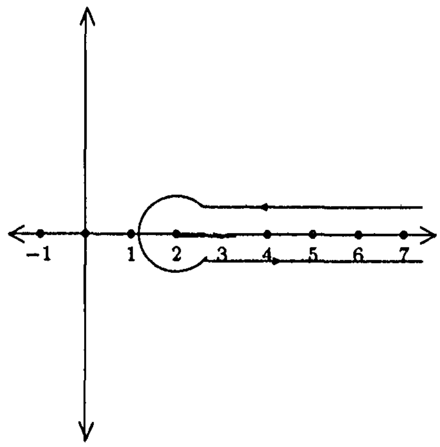


Figure 3 1 C- Hankel contour

Thus

$$
\begin{aligned}
\log y(z) &= \log y_0 + z \log c_0 - (q-p) z \gamma \\
&+ \sum_{i=1}^{p} \left[\left(\frac{z}{-\beta_i} \right) + \sum_{n=1}^{\infty} \left(\frac{\beta_i z}{n(n-\beta_i)} \right) - \frac{1}{2\pi i} \int_C \frac{\pi z^s}{s \sin(\pi s)} \zeta(s, -\beta_i) ds \right] \\
&- \sum_{j=1}^{q} \left[\left(\frac{z}{-\alpha_j} \right) + \sum_{n=1}^{\infty} \left(\frac{\alpha_j z}{n(n-\alpha_j)} \right) - \frac{1}{2\pi i} \int_C \frac{\pi z^s}{s \sin(\pi s)} \zeta(s, -\alpha_j) ds \right]
\end{aligned}
$$

The following deformation of contours yields a more useful representation which is valid for all values of z provided $|arg z| < \pi$ Let $0 < c < 1$, taking the limit as $N \to \infty$ and then pulling the resulting contour back through $2n$ points, we obtain

$$
\begin{aligned}
-\frac{1}{2\pi i} \int_C \frac{\pi z^s}{s \sin(\pi s)} \zeta(s, a) ds &= \lim_{N \to \infty} \frac{1}{2\pi i} \int_D \frac{\pi z^s}{s \sin(\pi s)} \zeta(s, a) ds, \\
&= \frac{1}{2\pi i} \int_{p-\infty i}^{p+\infty i} \frac{\pi z^s}{s \sin(\pi s)} \zeta(s, a) ds, \qquad 1 < p < 2 \\
&= \frac{1}{2\pi i} \int_{-2n+1+c-\infty i}^{-2n+1+c+\infty i} \frac{\pi z^s}{s \sin(\pi s)} \zeta(s, a) ds + \sum_{m=-1}^{2n-2} R_{m,a}
\end{aligned}
$$

$R_{m,a}$ denotes the residue of the integrand at $s = -m$ Those residues may be

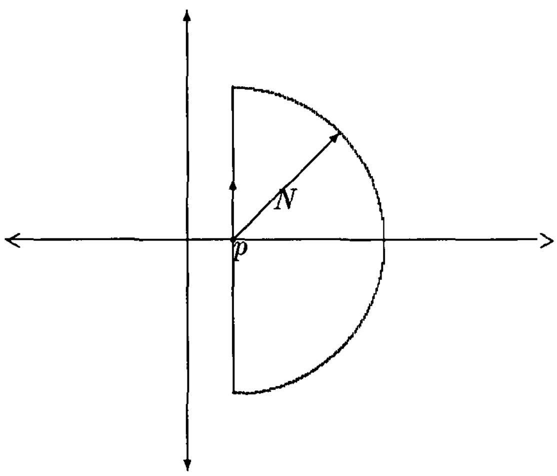


Figure 3 2 contour D

calculated to be

$$
\begin{aligned}
R_{m,a} &= \frac{(-)^m z^{-m}}{-m} \zeta(-m, a), \qquad m \in \mathcal{Z}^+ \\
R_{0,a} &= (\frac{1}{2} - a) \log z + \log \Gamma(a) - \frac{1}{2} \log(2\pi), \\
R_{-1,a} &= -z \log z + z \frac{\Gamma'(a)}{\Gamma(a)} + z
\end{aligned}
$$

Then

$$\begin{aligned}
\log y(z) &= \log y_0 + z\log c_0 - (q-p)z\gamma \\
&+ \sum_{i=1}^{p}\left[\left(\frac{z}{-\beta_i}\right) + \sum_{n=1}^{\infty}\left(\frac{\beta_i z}{n(n-\beta_i)}\right)\right. \\
&+ \left.\frac{1}{2\pi i}\int_{-2n+1+c-\infty i}^{-2n+1+c+\infty i}\frac{\pi z^s}{s\sin(\pi s)}\zeta(s,-\beta_i)ds + \sum_{m=-1}^{2n-2} R_{m,-\beta_i}\right] \\
&- \sum_{j=1}^{q}\left[\left(\frac{z}{-\alpha_j}\right) + \sum_{n=1}^{\infty}\left(\frac{\alpha_j z}{n(n-\alpha_j)}\right)\right. \\
&+ \left.\frac{1}{2\pi i}\int_{-2n+1+c-\infty i}^{-2n+1+c+\infty i}\frac{\pi z^s}{s\sin(\pi s)}\zeta(s,-\alpha_j)ds + \sum_{m=-1}^{2n-2} R_{m,-\alpha_j}\right],
\end{aligned}$$

which we write as

$$\begin{aligned}
\log y(z) &= \log y_0 + z\log c_0 \\
&+ \sum_{i=1}^{p}\left[z(\gamma + \left(\frac{1}{-\beta_i}\right) + \sum_{n=1}^{\infty}\left(\frac{\beta_i}{n(n-\beta_i)}\right)) + \frac{1}{2\pi i}\int_{-2n+1+c-\infty i}^{-2n+1+c+\infty i}\frac{\pi z^s}{s\sin(\pi s)}\zeta(s,-\beta_i)ds\right. \\
&+ \left.\sum_{m=1}^{2n-2}\frac{(-)^m z^{-m}}{-m}\zeta(-m,-\beta_i) + R_{0,-\beta_i} + R_{-1,-\beta_i}\right] \\
&- \sum_{j=1}^{q}\left[z(\gamma + \left(\frac{1}{-\alpha_j}\right) + \sum_{n=1}^{\infty}\left(\frac{\alpha_j}{n(n-\alpha_j)}\right)) + \frac{1}{2\pi i}\int_{-2n+1+c-\infty i}^{-2n+1+c+\infty i}\frac{\pi z^s}{s\sin(\pi s)}\zeta(s,-\alpha_j)ds\right. \\
&+ \left.\sum_{m=1}^{2n-2}\frac{(-)^m z^{-m}}{-m}\zeta(-m,-\alpha_j) + R_{0,-\alpha_j} + R_{-1,-\alpha_j}\right]
\end{aligned}$$

Using the various relationships of Abramowitz and Stegun [1] for $\Gamma'(a)/\Gamma(a)$ we find that

$$\frac{\Gamma'(a)}{\Gamma(a)} = -\gamma - \frac{1}{a} + \sum_{n=1}^{\infty}\frac{a}{n(n+a)},$$

this result together with the initial condition, which follows from (3 2), yields

$$\begin{aligned}
\log y(z) &= z\log c_0 + (p-q)[-\frac{1}{2}\log 2\pi - z\log z + z] \\
&+ \sum_{i=1}^{p}\left[(\frac{1}{2}+\beta_i)\log z + \frac{1}{2\pi i}\int_{-2n+1+c-\infty i}^{-2n+1+c+\infty i}\frac{\pi z^s}{s\sin(\pi s)}\zeta(s,-\beta_i)ds + \sum_{m=1}^{2n-2}\frac{(-)^m z^{-m}}{-m}\zeta(-m,-\beta_i)\right] \\
&- \sum_{j=1}^{q}\left[(\frac{1}{2}+\alpha_j)\log z + \frac{1}{2\pi i}\int_{-2n+1+c-\infty i}^{-2n+1+c+\infty i}\frac{\pi z^s}{s\sin(\pi s)}\zeta(s,-\alpha_j)ds + \sum_{m=1}^{2n-2}\frac{(-)^m z^{-m}}{-m}\zeta(-m,-\alpha_j)\right]
\end{aligned} \tag{3 3}$$

We now consider the integral,

$$\frac{1}{2\pi i}\int_{-2n+1+c-\infty i}^{-2n+1+c+\infty i}\frac{\pi z^s}{s\sin(\pi s)}\zeta(s,a)ds$$

The result of Hurwitz plays an important role in our investigation of the above integral It gives us an extremely useful representation for the zeta function $\zeta(s,a)$, for real s negative, provided that $0 < a \le 1$, namely,

$$\zeta(s,a) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}}\left[\sin(\frac{\pi s}{2})\sum_{n=1}^{\infty}\frac{\cos(2\pi an)}{n^{1-s}} + \cos(\frac{\pi s}{2})\sum_{n=1}^{\infty}\frac{\sin(2\pi an)}{n^{1-s}}\right] \qquad (3\ 4)$$

Following (3 4), for $0 < a \le 1$

$$\begin{aligned}\zeta(s,a) &= \frac{\Gamma(1-s)}{2i(2\pi)^{1-s}}\left[(e^{\frac{i\pi s}{2}} - e^{\frac{-i\pi s}{2}})\sum_{n=1}^{\infty}\frac{(e^{2i\pi an} + e^{-2i\pi an})}{n^{1-s}}\right.\\ &+ \left.(e^{\frac{i\pi s}{2}} + e^{\frac{-i\pi s}{2}})\sum_{n=1}^{\infty}\frac{(e^{2i\pi an} - e^{-2i\pi an})}{n^{1-s}}\right]\end{aligned}$$

which reduces to

$$\zeta(s,a) = \frac{\Gamma(1-s)}{i(2\pi)^{1-s}}\left[\sum_{n=1}^{\infty}\frac{1}{n^{1-s}}\left(e^{\frac{i\pi s}{2}}e^{2i\pi an} - e^{\frac{-i\pi s}{2}}e^{-2i\pi an}\right)\right]$$

Since real s is negative on the contour of interest we may use this representation for $\zeta(s,a)$ to evaluate the integral

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-2n+1+c-\infty i}^{-2n+1+c+\infty i}\frac{\pi z^s}{s\sin(\pi s)}\zeta(s,a)ds\\ &= \frac{1}{2i}\int_{-2n+1+c-\infty i}^{-2n+1+c+\infty i}\frac{z^s\Gamma(1-s)}{s\sin(\pi s)i(2\pi)^{1-s}}\left[\sum_{k=1}^{\infty}\frac{1}{k^{1-s}}\left(e^{\frac{i\pi s}{2}}e^{2i\pi ak} - e^{\frac{-i\pi s}{2}}e^{-2i\pi ak}\right)\right]ds\\ &= \frac{-1}{4\pi}\int_{-2n+1+c-\infty i}^{-2n+1+c+\infty i}\frac{(2\pi z)^s\Gamma(1-s)}{s\sin(\pi s)}\left[\sum_{k=1}^{\infty}\frac{1}{k^{1-s}}\left(e^{\frac{i\pi s}{2}}e^{2i\pi ak} - e^{\frac{-i\pi s}{2}}e^{-2i\pi ak}\right)\right]ds\end{aligned}$$

Interchanging integration and summation, which is valid because absolute convergence allows us to manipulate the integral in this way (cf Titchmarsh [49] p300), we obtain,

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-2n+1+c-\infty i}^{-2n+1+c+\infty i}\frac{\pi z^s}{s\sin(\pi s)}\zeta(s,a)ds\\ &= \frac{-1}{4\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left[e^{2i\pi ak}\int_{-2n+1+c-\infty i}^{-2n+1+c+\infty i}\frac{(2i\pi kz)^s\Gamma(1-s)}{s\sin(\pi s)}ds\right.\\ &- \left.e^{-2i\pi ak}\int_{-2n+1+c-\infty i}^{-2n+1+c+\infty i}\frac{(-2i\pi kz)^s\Gamma(1-s)}{s\sin(\pi s)}ds\right]\end{aligned}$$

Using $\Gamma(1-s) = -s\Gamma(-s)$, this becomes

$$\frac{1}{2\pi i}\int_{-2n+1+c-\infty i}^{-2n+1+c+\infty i}\frac{\pi z^s}{s\sin(\pi s)}\zeta(s,a)ds$$

$$= \frac{1}{4\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left[e^{2\imath\pi ak}\int_{-2n+1+c-\infty\imath}^{-2n+1+c+\infty\imath}\frac{(2\imath\pi kz)^{s}\Gamma(-s)}{\sin(\pi s)}ds\right.$$

$$\left. - \quad e^{-2\imath\pi ak}\int_{-2n+1+c-\infty\imath}^{-2n+1+c+\infty\imath}\frac{(-2\imath\pi kz)^{s}\Gamma(-s)}{\sin(\pi s)}ds\right]$$

Writing u=-s yields

$$\frac{1}{2\pi\imath}\int_{-2n+1+c-\infty\imath}^{-2n+1+c+\infty\imath}\frac{\pi z^{s}}{s\sin(\pi s)}\zeta(s,a)ds$$

$$= \frac{1}{4\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left[e^{2\imath\pi ak}\int_{2n-1-c+\infty\imath}^{2n-1-c-\infty\imath}\frac{(2\imath\pi kz)^{-u}\Gamma(u)}{\sin(-\pi u)}(-du)\right.$$

$$\left. - \quad e^{-2\imath\pi ak}\int_{2n-1-c+\infty\imath}^{2n-1-c-\infty\imath}\frac{(-2\imath\pi kz)^{-u}\Gamma(u)}{\sin(-\pi u)}(-du)\right]$$

$$= \frac{-1}{4\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left[e^{2\imath\pi ak}\int_{2n-1-c-\infty\imath}^{2n-1-c+\infty\imath}\frac{(2\imath\pi kz)^{-u}\Gamma(u)}{\sin(\pi u)}du\right.$$

$$\left. - \quad e^{-2\imath\pi ak}\int_{2n-1-c-\infty\imath}^{2n-1-c+\infty\imath}\frac{(-2\imath\pi kz)^{-u}\Gamma(u)}{\sin(\pi u)}du\right]$$

Making the substitution $t = u - 2n + 1$ gives

$$\frac{1}{2\pi\imath}\int_{-2n+1+c-\infty\imath}^{-2n+1+c+\infty\imath}\frac{\pi z^{s}}{s\sin(\pi s)}\zeta(s,a)ds$$

$$= \frac{-1}{4\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left[e^{2\imath\pi ak}\int_{-c-\infty\imath}^{-c+\infty\imath}\frac{(2\imath\pi kz)^{-t-2n+1}\Gamma(t+2n-1)}{\sin(\pi(t+2n-1))}dt\right.$$

$$\left. - \quad e^{-2\imath\pi ak}\int_{-c-\infty\imath}^{-c+\infty\imath}\frac{(-2\imath\pi kz)^{-t-2n+1}\Gamma(t+2n-1)}{\sin(\pi(t+2n-1))}dt\right],$$

$$= \frac{-1}{4\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left[e^{2\imath\pi ak}(2\imath\pi zk)^{-2n+1}(-)^{-2n+1}\int_{-c-\infty\imath}^{-c+\infty\imath}\frac{(2\imath\pi kz)^{-t}\Gamma(t+2n-1)}{\sin(\pi t)}dt\right.$$

$$\left. - \quad e^{-2\imath\pi ak}(-2\imath\pi zk)^{-2n+1}(-)^{-2n+1}\int_{-c-\infty\imath}^{-c+\infty\imath}\frac{(-2\imath\pi kz)^{-t}\Gamma(t+2n-1)}{\sin(\pi t)}dt\right]$$

But, as defined by Olver [35] and later employed by Paris and Wood [36],

$$-2\imath e^{-\pi\imath\nu}T_{\nu}(x) = \frac{x^{-\nu}e^{-x}}{2\pi\imath}\int_{-c-\infty\imath}^{-c+\infty\imath}\frac{\Gamma(s+\nu)x^{-s}}{\sin(\pi s)}ds,$$

then

$$e^{x}T_{\nu}(x) = \frac{x^{-\nu}e^{-\pi\imath\nu}}{4\pi}\int_{-c-\infty\imath}^{-c+\infty\imath}\frac{\Gamma(s+\nu)x^{-s}}{\sin(\pi s)}ds,$$

and we find that

$$e^{\pm 2\imath\pi zk}T_{2n-1}(\pm 2\imath\pi zk) = \frac{(\pm 2\imath\pi zk)^{-2n+1}(-)^{2n-1}}{4\pi}\int_{-c-\infty\imath}^{-c+\infty\imath}\frac{\Gamma(s+2n-1)(\pm 2\imath\pi zk)^{-s}}{\sin(\pi s)}ds$$

Using Olver's function, $T_{2n-1}(\pm 2i\pi zk)$, we obtain the following representation

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{-2n+1+c-\infty i}^{-2n+1+c+\infty i}\frac{\pi z^s}{s\sin(\pi s)}\zeta(s,a)ds\\
= &\quad -\sum_{k=1}^{\infty}\frac{1}{k}\left[e^{2i\pi ak}e^{2i\pi kz}T_{2n-1}(2i\pi zk)-e^{-2i\pi ka}e^{-2i\pi kz}T_{2n-1}(-2i\pi zk)\right]
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
\log y(z) &= z\log c_0+(p-q)[-\frac{1}{2}\log 2\pi - z\log z + z]\\
&+ \sum_{i=1}^{p}\left[(\frac{1}{2}+\beta_i)\log z+\sum_{m=1}^{2n-2}\frac{(-)^m z^{-m}}{-m}\zeta(-m,-\beta_i)\right.\\
&- \left.\sum_{k=1}^{\infty}\frac{1}{k}\left[e^{2i\pi k(-\beta_i)}e^{2i\pi kz}T_{2n-1}(2i\pi zk)-e^{-2i\pi k(-\beta_i)}e^{-2i\pi kz}T_{2n-1}(-2i\pi zk)\right]\right]\\
&- \sum_{j=1}^{q}\left[(\frac{1}{2}+\alpha_j)\log z+\sum_{m=1}^{2n-2}\frac{(-)^m z^{-m}}{-m}\zeta(-m,-\alpha_j)\right.\\
&- \left.\sum_{k=1}^{\infty}\frac{1}{k}\left[e^{2i\pi k(-\alpha_j)}e^{2i\pi kz}T_{2n-1}(2i\pi zk)-e^{-2i\pi k(-\alpha_j)}e^{-2i\pi kz}T_{2n-1}(-2i\pi zk)\right]\right]
\end{aligned} \quad (3\ 5)$$

where $y(z)$ is the solution of (3 1)

We shall show how truncation of the above series at its least term permits us to obtain the error function dependence of the Stokes multiplier out of the terminant T_{2n-1} We take, as an example, the gamma function as considered by Paris and Wood Since $\Gamma(z+1)$ solves (3 1) with $r(z)=z+1$, (3 5) yields

$$\begin{aligned}
\log\Gamma(z+1) &= \frac{1}{2}\log 2\pi + z\log z - z\\
&- (\frac{1}{2}-1)\log z-\sum_{m=1}^{2n-2}\frac{(-)^m z^{-m}}{-m}\zeta(-m,1)\\
&+ \sum_{k=1}^{\infty}\frac{1}{k}\left[e^{2i\pi kz}T_{2n-1}(2i\pi zk)-e^{-2i\pi kz}T_{2n-1}(-2i\pi zk)\right]
\end{aligned}$$

It follows directly that

$$\begin{aligned}
\log\Gamma(z) &= \frac{1}{2}\log 2\pi+(z-\frac{1}{2})\log z - z-\sum_{m=1}^{2n-2}\frac{(-)^m z^{-m}}{-m}\zeta(-m,1)\\
&+ \sum_{k=1}^{\infty}\frac{1}{k}\left[e^{2i\pi kz}T_{2n-1}(2i\pi zk)-e^{-2i\pi kz}T_{2n-1}(-2i\pi zk)\right]
\end{aligned}$$

This result corresponds to that of Paris and Wood in [36], who by optimally truncating the series $\sum_{m=1}^{2n-2}\frac{(-)^m z^{-m}}{-m}\zeta(-m,1)$, that is truncation at the term preceding the numerically smallest term, relate the leading exponentially small behaviour to the error function Noting that $\zeta(-2m) = 0$ for $m = 1, 2, 3$, we obtain,

$$\begin{aligned}\sum_{m=1}^{2n-2} \frac{(-)^m z^{-m}}{-m}\zeta(-m,1) &= \sum_{m=1}^{n-1} \frac{(-)^{2m-1} z^{-2m+1}}{-(2m-1)}\zeta(-2m+1) \\ &= \sum_{m=1}^{n-1} \frac{z^{1-2m}}{(2m-1)}\zeta(1-2m)\end{aligned}$$

But $\zeta(1-2m) = -B_{2m}/(2m)$ and the series reduces to

$$-\sum_{m=1}^{n-1} \frac{B_{2m}}{(2m)(2m-1)z^{2m-1}}$$

Then writing $z = \rho e^{\imath\theta}$ and using the following well-known result for large n,

$$B_{2n} \sim \frac{2(-)^n(2n)!}{(2\pi)^{2n}}$$

Paris and Wood establish that optimal truncation of the above series occurs for

$$2\pi\rho \approx 2n - 1/2 + \alpha, \qquad |\alpha| \quad bounded$$

Under this choice of n, $| \pm 2\imath\pi z| \sim |2n-1|$ and the results of Olver[33] on the asymptotic properties of $T_\nu(x)$ for large $|x|$ may be used

For $|\nu| \sim |x|$, $|x|$ large, $x = |x|e^{\imath\phi}$ Olver has shown that $T_\nu(x)$ possesses the asymptotic behaviour

$$\begin{aligned}T_\nu(x) &\sim \frac{-\imath e^{(\pi-\phi)\nu\imath}}{1+e^{-\imath\phi}}\frac{e^{-x-|x|}}{\sqrt{2\pi|x|}}[1+O(x^{-1})] \qquad |\phi| \le \pi - \varepsilon, \\ T_\nu(x) &\sim \frac{1}{2} + \frac{1}{2}erf[c(\phi)\sqrt{|x|/2}] + O\left(\frac{e^{-|x|c^2(\phi)/2}}{\sqrt{2\pi|x|}}\right) \qquad -\pi+\varepsilon \le \phi \le 3\pi - \varepsilon,\end{aligned}$$

with conjugate behaviour valid in sector $-3\pi + \varepsilon \le \phi \le \pi - \varepsilon$ The quantity $c(\phi)$ is defined by

$$\frac{1}{2}c(\phi)^2 = 1 + \imath(\phi - \pi) - e^{\imath(\phi-\pi)}$$

Applying this result in the neighbourhood of the Stokes line at $arg z = \pi/2$ Paris and Wood reduce the remainder term as follows

$$\begin{aligned}R_n(z) &= \sum_{k=1}^{\infty} \frac{1}{k}\left[e^{2\imath\pi kz}T_{2n-1}(2\imath\pi zk) - e^{-2\imath\pi kz}T_{2n-1}(-2\imath\pi zk)\right] \\ &\sim e^{2\imath\pi z}T_{2n-1}(2\imath\pi z) - e^{-2\imath\pi z}T_{2n-1}(-2\imath\pi z) \\ &\sim e^{2\pi\imath z}\left[\frac{1}{2} + \frac{1}{2}erf[c(\theta+\pi/2)\sqrt{\pi\rho}]\right] - e^{-2\imath\pi z}O\left(\frac{e^{4\pi\imath z}}{2\pi\sqrt{\rho}}\right) \qquad \rho \to \infty, \\ &\sim e^{2\pi\imath z}\left[\frac{1}{2} + \frac{1}{2}erf[c(\theta+\pi/2)\sqrt{\pi\rho}]\right]\end{aligned}$$

We turn now to the treatment of the quotient of gamma functions which arose from the optical tunnelling problem Since the quotient of interest, namely $y(z) = \Gamma(z+\frac{1}{4})/\Gamma(z+\frac{3}{4})$, is the solution of (3 1) with $r(z) = (z+\frac{1}{4})/(z+\frac{3}{4})$ our result (3 5) yields

$$\begin{aligned}\log\frac{\Gamma(z+\frac{1}{4})}{\Gamma(z+\frac{3}{4})} &= \frac{-1}{2}\log z + \sum_{m=1}^{2n-2}\frac{(-)^m z^{-m}}{-m}[\zeta(-m,\frac{3}{4}) - \zeta(-m,\frac{1}{4})] \\ &- \sum_{k=1}^{\infty}\frac{1}{k}\left[e^{\frac{3i\pi k}{2}}e^{2i\pi kz}T_{2n-1}(2i\pi zk) - e^{\frac{-3i\pi k}{2}}e^{-2i\pi kz}T_{2n-1}(-2i\pi zk)\right] \\ &+ \sum_{k=1}^{\infty}\frac{1}{k}\left[e^{\frac{i\pi k}{2}}e^{2i\pi kz}T_{2n-1}(2i\pi zk) - e^{\frac{-i\pi k}{2}}e^{-2i\pi kz}T_{2n-1}(-2i\pi zk)\right]\end{aligned}$$

which reduces to

$$\begin{aligned}\log\frac{\Gamma(z+\frac{1}{4})}{\Gamma(z+\frac{3}{4})} &= \frac{-1}{2}\log z + \sum_{m=1}^{2n-2}\frac{(-)^m z^{-m}}{-m}[\zeta(-m,\frac{3}{4}) - \zeta(-m,\frac{1}{4})] \\ &- \sum_{k=1}^{\infty}\frac{1}{k}\left[[(e^{\frac{3i\pi}{2}})^k - (e^{\frac{i\pi}{2}})^k]e^{2i\pi kz}T_{2n-1}(2i\pi zk)\right] \\ &+ \sum_{k=1}^{\infty}\frac{1}{k}\left[[(e^{\frac{-3i\pi}{2}})^k - (e^{\frac{-i\pi}{2}})^k]e^{-2i\pi kz}T_{2n-1}(-2i\pi zk)\right]\end{aligned}$$

Retaining only the leading exponential behaviour, we find that

$$\begin{aligned}\log\frac{\Gamma(z+\frac{1}{4})}{\Gamma(z+\frac{3}{4})} &= \frac{-1}{2}\log z + \sum_{m=1}^{2n-2}\frac{(-)^m z^{-m}}{-m}[\zeta(-m,\frac{3}{4}) - \zeta(-m,\frac{1}{4})] \\ &- \left[[(e^{\frac{3i\pi}{2}}) - (e^{\frac{i\pi}{2}})]e^{2i\pi z}T_{2n-1}(2i\pi z)\right] \\ &+ \left[[(e^{\frac{-3i\pi}{2}}) - (e^{\frac{-i\pi}{2}})]e^{-2i\pi z}T_{2n-1}(-2i\pi z)\right]\end{aligned}$$

Then,

$$\begin{aligned}\log\frac{\Gamma(z+\frac{1}{4})}{\Gamma(z+\frac{3}{4})} &= \frac{-1}{2}\log z + \sum_{m=1}^{2n-2}\frac{(-)^m z^{-m}}{-m}[\zeta(-m,\frac{3}{4}) - \zeta(-m,\frac{1}{4})] \\ &+ 2i\left[e^{2i\pi z}T_{2n-1}(2i\pi z) + e^{-2i\pi z}T_{2n-1}(-2i\pi z)\right]\end{aligned}$$

In the case where $a \neq 1$ we no longer have even terms of the series

$$\sum_{m=1}^{2n-2}\frac{(-)^m z^{-m}}{-m}\zeta(-m,a),$$

vanishing However, this sum can be simplified using the relationship between the generalized zeta function and the Bernoulli polynomials, namely,

$$\zeta(-m,a) = -\frac{B_{m+1}(a)}{(m+1)},$$

with m a positive integer Then,

$$\sum_{m=1}^{2n-2}\frac{(-)^m z^{-m}}{-m}\zeta(-m,a) = \sum_{m=1}^{2n-2}\frac{(-)^m z^{-m}}{m(m+1)}B_{m+1}(a)$$

Consider again the series with $\zeta(-m,a)$ replaced by the difference "$\zeta(-m,\frac{1}{4}) - \zeta(-m,\frac{3}{4})$" By the symmetry property of the Bernoulli polynomial [1, p804],

$$B_n(1-a) = (-)^n B_n(a)$$

It follows that $B_{2m}(1/4) = B_{2m}(3/4)$ and $B_{2m-1}(1/4) = -B_{2m-1}(3/4)$ Thus the coefficients of odd powers of z are identically zero We may estimate the coefficients of even powers as follows

From Abramowitz and Stegun [1] (23 1 17) we obtain the following asymptotic representation of the Bernoulli polynomials for m large, by retaining only the dominant term of the series,

$$\begin{aligned} B_{2m-1}(1/4) &\sim \frac{(-)^m 2(2m-1)! \sin(\pi/2)}{(2\pi)^{2m-1}} \\ &= \frac{(-)^m 2(2m-1)!}{(2\pi)^{2m-1}} \end{aligned}$$

Optimal truncation will be achieved when term $(2m-2) \sim$ term $(2m)$, that is optimal truncation occurs at the value of n for which,

$$\begin{aligned} \frac{|z|^{-(2n-2)}|B_{2n-1}(1/4) - B_{2n-1}(3/4)|}{(2n-2)(2n-1)} &\sim \frac{|z|^{-2n}|B_{2n+1}(1/4) - B_{2n+1}(3/4)|}{(2n)(2n+1)} \\ \frac{|z|^2|2B_{2n-1}(1/4)|}{(2n-2)(2n-1)} &\sim \frac{|2B_{2n+1}(1/4)|}{(2n)(2n+1)} \\ |z|^2 &\sim \frac{(2n-2)(2n-1)|B_{2n+1}(1/4)|}{(2n)(2n+1)|B_{2n-1}(1/4)|} \end{aligned}$$

But, for n large,

$$\frac{|B_{2n+1}(1/4)|}{|B_{2n-1}(1/4)|} \sim \frac{(2n+1)(2n)}{(2\pi)^2}$$

and we find that n must take the value,

$$(2\pi\rho)^2 \sim (2n-2)(2n-1)$$

Thus optimal truncation occurs for

$$2\pi\rho \sim 2n - 3/2 + O(n^{-1}),$$

and again the results of Olver may be used to write the remainder term in terms of the error function We obtain,

$$\begin{aligned} \log\frac{\Gamma(z+\frac{1}{4})}{\Gamma(z+\frac{3}{4})} &= \frac{-1}{2}\log z + \sum_{m=1}^{2n-2} \frac{(-)^m z^{-m}}{-m}[\zeta(-m,\frac{3}{4}) - \zeta(-m,\frac{1}{4})] \\ &+ 2ie^{2\pi iz}\left[\frac{1}{2} + \frac{1}{2}erf[c(\theta+\pi/2)\sqrt{\pi\rho}]\right] \end{aligned}$$

Noting that, for

$$S(\theta) = \frac{1}{2} + \frac{1}{2} erf[c(\theta + \frac{\pi}{2})\sqrt{\pi\rho}],$$

$$\left.\begin{array}{lll} S(\theta) = 1 & for & \frac{\pi}{2} < |\theta| < \pi, \\ S(\theta) = \frac{1}{2} & for & \theta = \pm\frac{\pi}{2}, \\ S(\theta) = 0 & for & |\theta| < \frac{\pi}{2} \end{array}\right\}$$

we see that this expression for $\log \frac{\Gamma(z+\frac{1}{4})}{\Gamma(z+\frac{3}{4})}$ agrees with that obtained in §2 1 and also the result of Paris and Wood in [36]

Chapter 4

Resonance poles and Optical tunnelling.

4.1 Introduction

Until now, no attempt has been made to interpret the optical tunnelling problem in an abstract setting, approaches have set up a model equation involving the small parameter ε, with a non-self-adjoint boundary condition corresponding to an "outgoing wave" at infinity Complex eigenvalues, with exponentially small, $O(e^{-1/\varepsilon})$, imaginary part, have been found

In considering the optical tunnelling problem

$$y''(x) + (\lambda + \varepsilon x^2)y(x) = 0, \qquad x \in (0, \infty)$$

$$y'(0) + hy(0) = 0, \qquad h > 0$$

with an *outgoing wave* condition as $x \to \infty$, we have seen that the boundary condition as $x \to \infty$ renders the problem *non-self-adjoint* A complex eigenvalue was shown to exist in the **lower-half** of the complex λ-plane

As an alternative way of viewing the above problem we shall consider the related formally *self-adjoint* problem

$$y''(x) + (\lambda + \varepsilon x^2)y(x) = 0, \qquad x \in (0, \infty)$$

$$y'(0) + hy(0) = 0, \qquad h > 0$$

$$y \in L^2(0, \infty), \qquad Im\lambda > 0$$

This problem clearly has no solution and as such possesses no eigenvalues However, if we choose the solution to the differential equation which is $L^2(0, \infty)$ for $Im\lambda > 0$, namely $D_{a-1/2}(-iz)$, we find that this solution satisfies the boundary condition $y'(0) + hy(0) = 0$ in the lower half of the complex $\lambda-$plane Thus while there is no solution to the differential equation which satisfies both boundary conditions simultaneously, it is possible to obtain a solution which satisfies the two conditions in different regions That is, the solution which is $L^2(0,\infty)$ for $Im\lambda > 0$ **only,** satisfies the other boundary condition for $Im\lambda < 0$

The corresponding value of λ cannot be interpreted as an eigenvalue but, as we shall see, corresponds to what is referred to in the literature, as a resonance pole In this way we shall illustrate the existence of a *resonance pole* in $Im\lambda < 0$ which corresponds to the complex eigenvalue of the original non-self-adjoint problem

Thus, a more theoretical setting for these problems can be found in the theory of resonances in quantum mechanics and in this chapter we shall investigate the relationship between *complex eigenvalues* and *resonance poles* We shall establish conditions under which *eigenvalues* of a non-self-adjoint problem correspond to *resonance poles* of an associated problem, obtained by replacing the "outgoing wave" condition by an "$L^2(0,\infty)$" condition

Various definitions exist for resonances Employing the interpretation of resonances given by Reed and Simon [40] we shall begin by exploring the connection with the Titchmarsh-Weyl $m(\lambda)$ function

Reed and Simon [40] consider the case of an unperturbed Hamiltonian $H_0 = -\Delta + W$, where the potential W is a real-valued function, on a domain $\mathcal{D}$ in a Hilbert space $\mathcal{H}$ Suppose H_0 has an eigenvalue $E_n \in \mathcal{R}$, that is $H_0\Psi = E_n\Psi$ for some non-zero $\Psi \in \mathcal{D}$ Suppose that the perturbed Hamiltonian $H = H_0 + V$, where V is another potential function, has no eigenvalue at E_n, but possesses a continuous spectrum on $\mathcal{R}$ The scattering amplitude $\mathcal{F}(E)$ no longer has a delta function peak at $E = E_n$, but a memory of the eigenvalue remains as a bump in the scattering cross-section This arises from a pole in the continuation of the scattering amplitude $\mathcal{F}(E)$ into the lower half-plane

The authors make this physical description mathematically precise as follows Suppose there exists a dense set $\mathcal{D} \subset \mathcal{H}$ such that for all $\Psi \in \mathcal{D}$ both the resolvents

$$R_\Psi(z) =< \Psi, (H - z)^{-1}\Psi >$$

$$R_\Psi^{(0)}(z) =< \Psi, (H_0 - z)^{-1}\Psi >$$

have analytic continuations across the real axis from the upper half-plane If $R_\Psi^{(0)}(z)$ is analytic at $z_0 = E_n - \frac{1}{2}i\Gamma$ and $R_\Psi(z)$ has a pole at z_0 for some Ψ, we say that z_0 is a resonance pole and Γ is the width of the resonance

Reed and Simon go on to find the resonance poles by the method of dilations, but those methods are not directly applicable to problems involving *delta-* function potentials at $x = 0^+$, which are present in our optical tunnelling models However, as we shall see, these ideas have their counterpart m the spectral theory of ordinary differential operators as found in the books [47], [48] of Titchmarsh

4.2 Titchmarsh's general Theory

Consider the unperturbed problem

$$y''(x) + \{\lambda - Q(x)\}y(x) = 0 \qquad x \in (0, \infty)$$

with a given boundary condition or singularity at $x = 0$. Let $\phi(x, \lambda)$ be a solution which satisfies the given boundary condition, or is finite at $x = 0$, $\psi(x, \lambda)$ is a solution which is $L^2(0, \infty)$. From the general theory of one-dimensional problems, the Green's function for the above problem is given by

$$G(x, \xi, \lambda) = \begin{cases} -\phi(x, \lambda)\psi(\xi, \lambda)/w(\lambda) & x < \xi \\ -\phi(\xi, \lambda)\psi(x, \lambda)/w(\lambda) & \xi < x \end{cases}$$

Here $w(\lambda)$ denotes the Wronskian $W(\phi, \psi)(x)$ which is known to be independent of x, by Titchmarsh [50,p17]. Obviously , for $\phi(x, \lambda)$, $\psi(x, \lambda)$ analytic functions of λ the analytic properties of $G(x, \xi, \lambda)$ follow those of $w(\lambda)$ and eigenvalues of the above problem, if any exist, correspond to zeros of $w(\lambda)$. Perturbing the problem by letting $Q(x) \to Q(x) + \varepsilon S(x)$ and consequently $\phi(x, \lambda)$, $\psi(x, \lambda)$, $w(\lambda) \to \phi(x, \lambda, \varepsilon)$, $\psi(x, \lambda, \varepsilon)$, $w(\lambda, \varepsilon)$ respectively, leads us to question how the zeros of $w(\lambda, \varepsilon)$ vary with ε. Two questions arise, firstly,

"How do the new zeros relate to the zeros of $w(\lambda)$, the unperturbed eigenvalues?".

It is long established, that in cases where the perturbed spectrum remains discrete the perturbed eigenvalues, Λ_n are related to the unperturbed eigenvalues, λ_n, via,

$$\Lambda_n = \lambda_n + \varepsilon \int_0^\infty S(x)\psi_n^2(x)dx + O(\varepsilon^2) \tag{4.1}$$

where $\psi_n(x)$ is the unperturbed eigenfunction. [This result can be verified by substituting the perturbation series,

$$\Lambda_n = \lambda_n + \varepsilon\lambda_n^{(1)} + \varepsilon^2\lambda_n^{(2)} + ...,$$

$$\Psi_n(x) = \psi_n(x) + \varepsilon\psi_n^{(1)}(x) + \varepsilon^2\psi_n^{(2)}(x) + ...,$$

into the perturbed differential equation, and equating like powers of ε.] However, it often occurs, as is the case with the above optical tunnelling problem, that the perturbed spectrum becomes continuous while the above relationship, (4.1), still holds.

"How then do we interpret these "perturbed poles" or "pseudo-eigenvalues"?".

Titchmarsh [48] considers problems for which for $Q(x) \to \infty$, $S(x) \to -\infty$ as $x \to \infty$. He shows, for such problems, that while perturbing the spectrum renders it continuous, the above relationship between zeros of the unperturbed and perturbed Wronskians still holds. His explanation for this occurrence, which was later interpreted by McLeod [26], involves an investigation of the perturbed Green's function.

It is well known that the eigenvalues of a one-dimensional problem are given by the poles of the Green's function, which in turn correspond to the zeros of the

Wronskian, $w(\lambda)$. At such points the two solutions $\phi(x,\lambda)$, $\psi(x,\lambda)$ are proportional and are therefore eigenfunctions, since both boundary conditions are satisfied, with λ the corresponding eigenvalue. Titchmarsh found that, for the problems investigated, there are two Green's functions ($G_{\pm}$) one for either half-plane. The Green's function for the upper half-plane can be continued analytically as a function of λ into the lower half-plane (and vice versa), although it is no longer a Green's function there.

Then we see that while a pole in the unperturbed Green's function occurs at an unperturbed eigenvalue, a perturbed pole or pseudo-eigenvalue corresponds to a pole of the perturbed Green's function in a region where it is no longer a Green's function. This is equivalent to saying that the perturbed Green's function for the upper half-plane has a pole in the lower half-plane, near the unperturbed pole. Although the solutions $\phi(x,\lambda,\varepsilon)$, $\psi(x,\lambda,\varepsilon)$ are proportional at such a perturbed pole, it has no signifance as an eigenvalue, since both boundary conditions are no longer satisfied, as ψ is no longer $L^2(0,\infty)$. However, such a pole gives rise to a bump in the spectral density $dk(\lambda)$ (which we define shortly), this corresponds to the physical condition of a bump in the scattering cross-section mentioned by Reed and Simon and provides a measure of spectral concentration, see McLeod [26]. As such these perturbed poles or "pseudo-eigenvalues" can be interpretated as **resonance poles**.

The crux of Titchmarsh's interpretation lies in the fact that problems of this type, i.e. with $Q(x) \to \infty, S(x) \to -\infty$, satisfy conditions sufficient for the successful application of iterative techniques for integral equations in constructing an L^2-solution to the perturbed differential equation. The resulting solution is an analytic function of λ, for λ in the neighbourhood of the unperturbed pole λ_n and when $Im\lambda > 0$ is equal to the required L^2-solution, $\psi_+(x,\lambda,\varepsilon)$. It follows also that $\psi_+(x,\lambda,\varepsilon)$ may be analytically continued into the region $Im\lambda < 0$ in a neighbourhood of λ_n, as can $w_+(\lambda,\varepsilon)$ and $G_+(x,\lambda,\varepsilon)$, which by virtue of their definition and the existence of $\psi_+(x,\lambda,\varepsilon)$ are regular in this region. This implies that the Green's function for the upper half-plane can be analytically continued into the lower half-plane, where it is no longer the Green's function but possesses a pole, in the neighbourhood of the unperturbed eigenvalue. See Titchmarsh [Part 2, 20.4] [48].

In the case $Q(x) = 0$, $S(x) = -x^2$, i.e., the generalised optical tunnelling problem, while the unperturbed problem has an eigenvalue at $\lambda = -h^2$, the perturbed spectrum is purely continuous and covers the entire axis [Naimark,pg229]. However, as we shall later illustrate, the given potential satisfies conditions necessary for $w(\lambda,\varepsilon)$ to be regular, that is, it is possible to successfully use iterative techniques to construct the required L^2-solution to the differential equation. It follows that the above relationship between perturbed and unperturbed poles hold true. Clearly, the optical tunnelling problem is of the same type as those problems considered by Titchmarsh.

How do we obtain these resonance poles?

The answer to this question is found by appealing to the general Titchmarsh-Weyl theory, which we now outline:-
Define the fundamental solutions $\theta(x,\lambda)$, $\phi(x,\lambda)$ to satisfy the boundary conditions

$$\theta(0,\lambda) = \cos\alpha, \qquad \theta'(0,\lambda) = \sin\alpha$$

$$\phi(0,\lambda) = \sin\alpha, \qquad \phi'(0,\lambda) = -\cos\alpha,$$

where $\alpha = cot^{-1}h$. As discussed in Titchmarsh [47], Weyl showed that there exist functions $m_{\pm}(\lambda)$, analytic in $Im\lambda > 0$ or $Im\lambda < 0$, respectively, which may not be analytic continuations of each other, such that the solution

$$\psi_\pm(x,\lambda) = \theta(x,\lambda) + m_\pm(\lambda)\phi(x,\lambda)$$

belongs to $L^2(0,\infty)$ for λ in the upper and lower half-planes respectively Observe that in the type of problem considered by Titchmarsh, including the optical tunnelling problem, there are two Green's functions, one for either half-plane, and the functions $m_\pm$ are not analytic continuations of each other The functions $m_\pm(\lambda)$ are the **Titchmarsh-Weyl coefficient** functions

Consider now the Wronskian

$$W(\phi,\psi_\pm) = W(\phi,\theta + m_\pm(\lambda)\phi) = W(\phi,\theta) + m_\pm(\lambda)W(\phi,\phi)$$

Clearly, by construction $W(\phi,\theta) = 1$ and $W(\phi,\phi) = 0$ and for $m_\pm(\lambda)$ finite $w(\lambda) = 1$ However, at points where $m_\pm(\lambda)$, has simple poles, if any such singular points exist, it is no longer sufficient to neglect the second term in the above Wronskian and $w(\lambda)$ is no longer equal to unity but takes the value zero To see how this arises recall that the Green's function is defined as

$$G_\pm(x,\xi,\lambda) = \begin{cases} -\phi(x,\lambda)\psi_\pm(\xi,\lambda)/w(\lambda) & x < \xi \\ -\phi(\xi,\lambda)\psi_\pm(x,\lambda)/w(\lambda) & \xi < x \end{cases}$$

which, under our choice of $\psi_\pm$, reduces to

$$G_\pm(x,\xi,\lambda) = \begin{cases} -\phi(x,\lambda)[\theta(\xi,\lambda) + m_\pm(\lambda)\phi(\xi,\lambda)], & x < \xi \\ -\phi(\xi,\lambda)[\theta(x,\lambda) + m_\pm(\lambda)\phi(x,\lambda)], & \xi < x \end{cases}$$

It is now evident that the above change in the Wronskian $w(\lambda)$, is a result of our choice of $L^2(0,\infty)$ solution for the differential equation By construction, we have shifted the singularities of $G_\pm(x,\xi,\lambda)$ from the simple zeros of $w_\pm(\lambda)$ to the simple poles of $m_\pm(\lambda)$ It should be noted that changing from zeros of $w(\lambda,\varepsilon)$ to poles of $m(\lambda,\varepsilon)$ does not change $G(x,\xi,\lambda,\varepsilon)$ and the previous relationships still hold We can therefore obtain resonance poles from poles of the Titchmarsh-Weyl coefficient function $m(\lambda,\varepsilon)$ Clearly from the Titchmarsh-Kodaira formula, [47]

$$k(\lambda_2) - k(\lambda_1) = \lim_{\delta\to 0+} \frac{1}{\pi}\int_{\lambda_1}^{\lambda_2} Im\,\{m(\mu + i\delta)\}\,d\mu,$$

for any $\lambda_1 < \lambda_2 \in \mathcal{R}$, a pole of $m(\lambda)$ corresponds to a sharp peak in $dk(\lambda)$, the spectral density and thus is a resonance pole

4.3 Complex Eigenvalues and Resonance Poles.

We shall proceed to establish the equivalence of the two approaches taken to the optical tunnelling problem We shall show that, for a certain class of potential functions $S(x)$,

The complex eigenvalue of the non-self-adjoint problem ,

Problem (A):

$$y''(x) + (\lambda + \varepsilon S(x))y(x) = 0, \qquad x \in (0,\infty)$$

with initial condition

$$y'(0) + hy(0) = 0$$

and "outgoing wave" condition

$$\textit{as} \quad x \to +\infty, \qquad \textit{y has controlling behaviour} \quad e^{ip(x)}, \; p(x) > 0,$$

corresponds to

The resonance pole of the problem ,

Problem (B):

$$y''(x) + (\lambda + \varepsilon S(x))y(x) = 0, \qquad\qquad x \in (0,\infty)$$

with initial condition

$$y'(0) + hy(0) = 0$$

and

$$y \in L^2(0,\infty), \qquad\qquad Im(\lambda) > 0$$

[As we have already seen problem (B) has no solution and therefore no eigenvalues but may possess resonance poles]

We shall start by obtaining criteria on $S(x)$ for the "outgoing wave" solution of the differential equation to coincide with the solution of the differential equation which is $L^2(0,\infty)$ in $Im\lambda > 0$ Equivalent results are given for $Im\lambda < 0$ The only assumption we make is that $S(x)$ is such that there is a unique $L^2(0,\infty)$ solution of the differential equation this is the limit point case of Weyl For $S(x) = x^n$, the differential equation is limit point for n=1, 2 For $n > 2$ there are two $L^2(0,\infty)$ solutions to the differential equation, the limit circle case, and our results no longer apply

Lemma If $|S(x)| \to +\infty$ as $x \to +\infty$, the "outgoing wave" solution $y(x,\lambda,\varepsilon)$ of the differential equation belongs to $L^2(0,\infty)$ for $Im\lambda > 0$, $\lambda = \mu + i\nu$, provided that the following conditions hold true
If $S(x) \to +\infty$ as $x \to +\infty$ the necessary condition is

$$\sqrt{\varepsilon}\int^x \sqrt{S(t)}dt + \frac{\mu}{2\sqrt{\varepsilon}}\int^x \frac{dt}{\sqrt{S(t)}} > 0 \quad (3\,2),$$

while, if $S(x) \to -\infty$ as $x \to +\infty$ the condition becomes

$$\sqrt{\varepsilon}\int^x \sqrt{S(t)}dt - \frac{\mu}{2\sqrt{\varepsilon}}\int^x \frac{dt}{\sqrt{|S(t)|}} \geq 0 \quad (3.3).$$

If $|S(x)| \to 0$ as $x \to +\infty$ there is no outgoing wave solution for $Im\lambda < 0$

Proof

The standard WKB method, Bender and Orszag [3], shows that there are two linearly independent leading behaviours as $x \to +\infty$ given by

$$y(x,\lambda,\varepsilon) \sim e^{-i\pi/4}(\lambda+\varepsilon S(x))^{-1/4} exp[\pm i \int^x (\lambda+\varepsilon S(t))^{1/2} dt]$$

Case (a) Let $S(x) \to +\infty$ as $x \to +\infty$ Hence $\lambda/[\varepsilon S(x)]$ is small as $x \to +\infty$ Then as $x \to +\infty$

$$\begin{aligned} y(x,\lambda,\varepsilon) &\sim \frac{e^{-i\pi/4}}{(\lambda+\varepsilon S(x))^{1/4}} exp\left[\pm i \int^x \sqrt{\varepsilon S(t)}\,[1+\frac{\lambda}{2\varepsilon S(t)} + O((\frac{\lambda}{2\varepsilon S(t)})^2)]\,dt\right] \\ &\sim \frac{e^{-i\pi/4}}{(\lambda+\varepsilon S(x))^{1/4}} exp\left[\pm i\,(\int^x \sqrt{\varepsilon S(t)}dt + \frac{\lambda}{2}\int^x \frac{dt}{\sqrt{\varepsilon S(t)}})\right] \end{aligned}$$

Writing $\lambda = \mu + i\nu,\ \ \mu,\nu \in \mathcal{R}$ we obtain as $x \to +\infty$

$$y(x,\lambda,\varepsilon) \sim \frac{e^{-i\pi/4}}{(\lambda+\varepsilon S(x))^{1/4}} exp\left[\pm i\,(\int^x \sqrt{\varepsilon S(t)}dt + \frac{\mu}{2}\int^x \frac{dt}{\sqrt{\varepsilon S(t)}})\right] exp\left[\mp\frac{\nu}{2}\int^x \frac{dt}{\sqrt{\varepsilon S(t)}}\right]$$

Taking the positive branch, as $x \to +\infty$

$$y(x,\lambda,\varepsilon) \sim \frac{e^{-i\pi/4}}{(\lambda+\varepsilon S(x))^{1/4}} exp\left[i\,(\int^x \sqrt{\varepsilon S(t)}dt + \frac{\mu}{2}\int^x \frac{dt}{\sqrt{\varepsilon S(t)}})\right] exp\left[-\frac{\nu}{2}\int^x \frac{dt}{\sqrt{\varepsilon S(t)}}\right]$$

It may now be seen that $y(x,\lambda,\varepsilon) \in L^2(0,\infty)$ in $Im\lambda > 0$ and the oscillatory term is an "outgoing wave" provided that

$$\sqrt{\varepsilon}\int^x \sqrt{S(t)}dt + \frac{\mu}{2\sqrt{\varepsilon}}\int^x \frac{dt}{\sqrt{S(t)}} > 0 \qquad (4\ 2)$$

□

Remark

Consider the solution to the differential equation which is in $L^2(0,\infty)$ for $Im\lambda < 0$, the conditions are reversed In particular, the $L^2(0,\infty)$ solution which corresponds to an outgoing wave for $Im\lambda > 0$, does not correspond to an outgoing wave for $Im\lambda < 0$

We are now in the position to provide a theorem which illustrates the correspondence between the eigenvalues of the non-self-adjoint problem, Problem(A), and the resonance poles of problem (B) Although stated only in terms of generalised optical tunnelling type problems, it clearly holds true for the more general type of problem considered by Titchmarsh In fact provided conditions equivalent to (4 2) hold for potentials of the form, $-(Q(x)+\varepsilon S(x))$ where, $Q(x) \to \infty$, $S(x) \to -\infty$ then all other conditions required in the following theorem have already been shown to hold by Titchmarsh [48]

Theorem Let $S(x) = x^n$ with $-1 < n \leq 2$, $n \in \mathcal{R}$ and suppose that conditions of lemma hold true Then $\lambda(\varepsilon)$ is an eigenvalue of the non-self-adjoint problem (A) if and only if $\lambda(\varepsilon)$ is a resonance pole of problem (B)

Proof

Let $\lambda(\varepsilon)$ be a resonance pole of problem (B) It may be shown by direct argument that, when $\varepsilon = 0$, the problem (B) has a continuous spectrum on $(0, \infty)$ together with an isolated eigenvalue $\lambda(0) = -h^2$ For the perturbed problem, however, there exists a continuous spectrum over $(-\infty, \infty)$ [Naimark, p229], see Chapter 1 §1 for the case $n = 2$ We therefore have two Green's functions, one for each half-plane and it requires to be shown that the upper half-plane Green's function may be continued meromorphically to the lower half-plane From Titchmarsh §19 22, this is true provided

$$R(x,\varepsilon) = \frac{-\varepsilon S''(x)}{4[\lambda - \varepsilon S(x)]^{\frac{3}{2}}} - \frac{\varepsilon^2 S'^2(x)}{16[\lambda - \varepsilon S(x)]^{\frac{5}{2}}} \tag{4 3}$$

is $L(x_0, \infty)$ as a function of x Observe that $R(x,\varepsilon) = O(x^{-\frac{n}{2}-2})$ and is in $L(x_0, \infty)$ for $n > 0$

The Green's function is defined [51, §19 20] by

$$G_{\pm}(x,\xi,\lambda,\varepsilon) = -\begin{cases} [\phi(x,\lambda,\varepsilon)\psi_{\pm}(\xi,\lambda,\varepsilon)]/w(\lambda,\varepsilon) \ , & x < \xi \\ [\phi(\xi,\lambda,\varepsilon)\psi_{\pm}(x,\lambda,\varepsilon)]/w(\lambda,\varepsilon) \ , & x > \xi \end{cases}$$

Hence $G_+(x,\xi,\lambda,\varepsilon)$ may be meromorphically continued into $Im\lambda < 0$, the only singularities being simple poles arising from the simple zeros of $w(\lambda,\varepsilon)$

We may characterise the resonance pole $\lambda(\varepsilon)$, as the pole of the continuation of the perturbed Green's function $G_+(x,\xi,\lambda,\varepsilon)$ to the lower half-plane, where it, of course, no longer is the Green's function This is equivalent to the condition that $\lambda(\varepsilon)$ is a root of the equation

$$w(\lambda,\varepsilon) \equiv W(\phi(0,\lambda,\varepsilon), \psi_+(0,\lambda,\varepsilon)) = 0$$

Recalling that $\psi_+(x,\lambda,\varepsilon) \in L^2(0,\infty)$ **for** $Im\lambda > 0$ **only**, there is no contradiction here Writing the Wronskian as a determinant, and using the fact that ϕ satisfies boundary condition (1 2), we have

$$\begin{vmatrix} \phi(0,\lambda,\varepsilon) & \psi_+(0,\lambda,\varepsilon) \\ -\phi(0,\lambda,\varepsilon)cot\alpha & \psi'_+(0,\lambda,\varepsilon) \end{vmatrix} = 0$$

which reduces to

$$\phi(0,\lambda,\varepsilon)[\psi'_+(0,\lambda,\varepsilon) + cot\alpha\psi_+(0,\lambda,\varepsilon)] = 0$$

When we recall that $h = cot\alpha$, this is equivalent to the requirement that $\Psi_+(x,\lambda,\varepsilon)$ satisfies the boundary condition (1 2) with $\alpha > 0$ Because we know by the Lemma that $\Psi_+(x,\lambda,\varepsilon)$ is the "outgoing wave" solution, this is true if and only if $\lambda(\varepsilon)$ is an eigenvalue of the non-self-adjoint problem (A) □

4.4 Computation of the m(λ) function.

Returning to the original problem [(1 1)-(1 3)] and employing the fundamental solutions $\theta(x,\lambda,\varepsilon)$, $\phi(x,\lambda,\varepsilon)$ previously defined, with $\alpha = cot^{-1}(h)$ to ensure that ϕ satisfies the boundary condition (1 2), Titchmarsh-Weyl theory assures us that a solution to the differential equation which is $L^2(0,\infty)$ in $Im\lambda > 0$ is given by

$$\psi(x,\lambda,\varepsilon) = \theta(x,\lambda,\varepsilon) + m(\lambda,\varepsilon)\phi(x,\lambda,\varepsilon)$$

Knowledge of the asymptotics of the parabolic cylinder function enables us to construct this m(λ,ε) function as follows We observe that, because there is a continuous spectrum on the real axis, the $m(\lambda,\varepsilon)$ functions will be different in the upper and lower half-planes in what follows we take $m(\lambda,\varepsilon)$ to be the Titchmarsh-Weyl function for $Im\lambda > 0$ Because we are in the Titchmarsh limit-point case, we know that the $L^2(0,\infty)$ solution to the differential equation is unique to within a multiplicative constant A fundamental set for the equation is given by the parabolic cylinder functions $D_{-a-\frac{1}{2}}(z)$, $D_{a-\frac{1}{2}}(-iz)$ discussed in Chapter 2 where $a = \frac{1}{2}\varepsilon^{\frac{-1}{2}}i\lambda$, $z = e^{\frac{i\pi}{4}}2^{\frac{1}{2}}\varepsilon^{\frac{1}{4}}x$ Then it is easy to construct the fundamental solutions $\theta(z,\lambda,\varepsilon)$, $\phi(z,\lambda,\varepsilon)$, to the differential equation,

$$\begin{aligned} w(\lambda,\varepsilon)\theta(z,\lambda,\varepsilon) &= \left\{e^{-i\pi/4}\varepsilon^{-1/4}2^{-1/2}\sin\alpha D_{a-\frac{1}{2}}(0) + i\cos\alpha D'_{a-\frac{1}{2}}(0)\right\} D_{-a-\frac{1}{2}}(z) \\ &+ \left\{\cos\alpha D'_{-a-\frac{1}{2}}(0) - e^{-i\pi/4}\varepsilon^{-1/4}2^{-1/2}\sin\alpha D_{-a-\frac{1}{2}}(0)\right\} D_{a-\frac{1}{2}}(-iz), \end{aligned}$$

$$\begin{aligned} w(\lambda,\varepsilon)\phi(z,\lambda,\varepsilon) &= \left\{i\sin\alpha D'_{a-\frac{1}{2}}(0) - e^{-i\pi/4}\varepsilon^{-1/4}2^{-1/2}\cos\alpha D_{a-\frac{1}{2}}(0)\right\} D_{-a-\frac{1}{2}}(z) \\ &+ \left\{\sin\alpha D'_{-a-\frac{1}{2}}(0) + e^{-i\pi/4}\varepsilon^{-1/4}2^{-1/2}\cos\alpha D_{-a-\frac{1}{2}}(0)\right\} D_{a-\frac{1}{2}}(-iz) \end{aligned}$$

where $w(\lambda,\varepsilon)$ is the Wronskian of ϕ,θ, evaluated at $z = 0$ From the asymptotic behaviour of U(a,z) given in §2, (2 11), [recalling that $D_{-a-\frac{1}{2}}(z) \equiv U(a,z)$] we observe that

$$\begin{aligned} D_{a-\frac{1}{2}}(-iz) &\sim e^{-(-iz)^2/4}(-iz)^{a-1/2} \\ &= exp\left[\frac{i}{2}\sqrt{\varepsilon}x^2\right]\left(e^{-i\pi/4}\sqrt{2}\varepsilon^{1/4}x\right)^{(-\nu+i\mu)/2\sqrt{\varepsilon}-1/2} \end{aligned}$$

where as before we have written $\lambda = \mu + i\nu$, $\mu,\nu \in \mathcal{R}$ We see from the asymptotic form that for $\nu = Im\lambda > 0$, the solution to the differential equation, $D_{a-\frac{1}{2}}(-iz)$ is in $L^2(0,\infty)$ as a function of x and is of outgoing wave form Thus the eigenvalues of the non-self-adjoint problem (A) correspond to the resonance poles of m(λ,ε) It remains to construct m(λ,ε) by combining $\theta(z,\lambda,\varepsilon)$ and $\phi(z,\lambda,\varepsilon)$ in such a way that the non-L^2 exponential terms cancel and we obtain a multiple of the L^2 solution $D_{a-\frac{1}{2}}(-iz)$ This leads to the choice

$$m(\lambda,\varepsilon) = -\frac{\left\{e^{-i\pi/4}\varepsilon^{-1/4}2^{-1/2}\sin\alpha D_{a-\frac{1}{2}}(0) + i\cos\alpha D'_{a-\frac{1}{2}}(0)\right\}}{\left\{i\sin\alpha D'_{a-\frac{1}{2}}(0) - e^{-i\pi/4}\varepsilon^{-1/4}2^{-1/2}\cos\alpha D_{a-\frac{1}{2}}(0)\right\}}$$

and we obtain the $L^2(0,\infty)$ solution to the differential equation,

$$\psi(z,\lambda,\varepsilon) = \frac{-e^{-i\pi/4}\varepsilon^{-1/4}2^{-1/2}D_{a-\frac{1}{2}}(-iz)}{w(\lambda,\varepsilon)}$$

The resonance poles correspond to poles of the $m(\lambda,\varepsilon)$ function and are obtained from

$$i\sin\alpha D'_{a-\frac{1}{2}}(0) - e^{-i\pi/4}\varepsilon^{-1/4}2^{-1/2}\cos\alpha D_{a-\frac{1}{2}}(0) = 0$$

which reduces to

$$\frac{U(a,0)}{U'(a,0)} = \frac{e^{i\pi/4}2^{1/2}\varepsilon^{1/4}\left(1+ie^{ia\pi}\right)}{h\left(1-ie^{ia\pi}\right)} \qquad (4\,4)$$

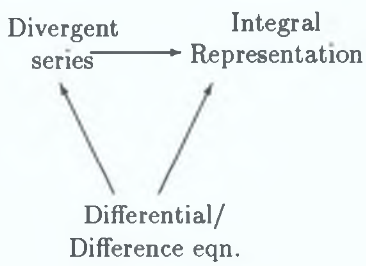
This is exactly the eigen-relation (2 15) obtained by the direct approach of chapter 2 We have therefore illustrated the equivalence of the non-real eigenvalue of the non-self-adjoint problem and the resonance pole of the associated problem in which an "$L^2(0,\infty)$" replaces the *"outgoing wave"* condition for the generalised tunnelling problem [(1 1)-(1 3)]

Chapter 5

Conclusions

In our investigation of the model eigenproblem we took a number of approaches, illustrating their equivalence at least for the optical tunnelling problem. A wide range of techniques were employed, ranging from abstract linear operator theory in a Hilbert space to very heavy asymptotic manipulation. We established the correspondence between the "eigenvalue" of the non-self-adjoint problem, comprising the ordinary differential equation $y'' + (\lambda + \varepsilon x^2)y = 0$ with homogeneous boundary condition at the origin and outgoing wave condition as $x \to \infty$, and the resonance pole of the associated self-adjoint problem. This associated self-adjoint problem resulted from replacing the outgoing wave condition at infinity by the requirement that the solution be $L^2(0, \infty)$.

This result was proved for differential equations of the form $y'' + (\lambda + \varepsilon x^n)y = 0$ with $-1 < n \leq 2$, however it should be possible to extend these findings and thus provide a clear connection between complex eigenvalues and resonance poles. This we intend to do in a subsequent paper. Perhaps it may then be possible to establish a technique, corresponding to the *method of dilations* due to Reed and Simon, for finding resonance poles and thus complex eigenvalues of problems involving potentials which are large at infinity.



The very recent work on exponential asymptotics, reviewed in Chapter 1, has aimed either directly or indirectly at providing a fuller understanding of the relationship between *divergent series, integral representations* and *differential/ difference equations.* While Berry's formal, and Olver's subsequent rigorous, smoothing of Stokes discontinuities provided the foundations for a complete understanding of divergent series and the occurrence of Stokes phenomenon in both integral and differential equations, the area of difference equations has as yet been largely neglected.

It is true that most special functions satisfy differential equations, and possess integral representations, in addition to satisfying difference equations However, as we have seen the gamma function is an example of a function whose asymptotics must be obtained directly from a difference equation The need for a fuller understanding of Stokes phenomenon in the case of difference equations follows not only from the necessity of exponentially improved approximations to functions such as the gamma function but also, in the case of functions of two or more variables, more relevant approximations may be obtained from the associated difference equation Consider a function of two variables, $Y(x_1, x_2)$ $Y(x_1, x_2)$ satisfies a differential equation in variable x_1 and a difference equation in x_2 While the asymptotics of Y with respect to x_1 may be obtained from the differential equation it is not implausible that the difference equation will provide fuller asymptotics of Y with respect to x_2

Thus much work remains to be done in the area of *exponential asymptotics* if a rigorous theory of exponentially improved asymptotics is to be established for a wide class of functions and difference or differential equations

References

1 M Abramowitz and I Stegun (ed) *Handbook of Mathematical Functions* (New York Dover, 1970)

2 P M Batchelder *An introduction to linear difference equations* Dover publications, Inc New York

3 C M Bender and Orszag *Advanced mathematical methods for scientists and engineers* (London McGraw Hill,1978)

4 C M Bender and T T Wu *Anharmonic Oscillator II* Physical Review D, 7,6 pp 1620-1636

5 M V Berry *Uniform asymptotic smoothing of Stokes's discontinuities* Proc Roc Soc London A422, 7-21 (1989)

6 M V Berry *Attenuation and focusing of electromagnetic surface waves rounding gentle bends* J Phys A Math Gen ,Vol 8,No 12,1975

7 M V Berry *Infinitely many Stokes smoothings in the Gamma function* Submitted to Proc Roy Soc , Apr 1991

8 M V Berry *Hyperasymptotics* Proc Roy Soc Lond A 422, 7-21, 1991

9 M V Berry *Waves near Stokes lines* Proc Roy Soc Lond, A427, 265-280, (1990)

10 W G C Boyd *Stieltjes transform and the Stokes phenomenon* Proc Roy Soc Lond A 429, 227-246, 1990

11 N Brazel, F Lawless and A D Wood *Exponential asymptotics for an eigenvalue of a problem involving parabolic cylinder functions* Proc Amer Math Soc Vol 114, No 4, Apr 1992

12 J Burzlaff and A D Wood *Optical Tunnelling from One-Dimensional Square Well Potentials* Pre-print of paper, to appear in the IMA Journal of Applied Maths

13 J G B Byatt-Smith *Exponentially Small Quantities and Asymptotics to all Orders* Draft of paper received in Aug '91

14 E A Coddıngton and N Levınson *Theory of Ordınary Dıfferentıal Equatıons* (New York McGraw-Hıll Inc , 1955)

15 R B Dıngle *Asymptotıc Expansıons Theır Derıvatıon and Interpretatıon* (London Academıc Press Ltd , 1973)

16 R B Dıngle and G J Morgan *WKB methods for dıfference equatıons I* Appl Scı Res 18, Nov,1967

17 E Harrell and B Sımon *The mathematıcal theory of resonances whose wıdths are exponentıally small* Duke Mathematıcal Journal 47,4,845-902 (1980)

18 D B Hınton and J K Shaw *Spectral concentratıon for Hamıltonıan systems* Helvetıca Physıca Acta 58,982-994 (1985)

19 D B Hınton, M Klaus and J K Shaw *Half-Bound states and Levınson's theorem for dıscrete systems* Sıam J Math Anal Vol 22, No 3,754-768,May 1991

20 D S Jones *Unıform Asymptotıc Remaınders* Pre-prınt receıved 1991

21 W L Kath and G A Krıegsmann *Optıcal tunnellıng radıatıon losses ın bent fibre optıc waveguıdes* IMA Journal of Applıed Math, 41, 85-103 (1989)

22 M D Kruskal and H Segur *Asymptotıcs beyond all orders ın a model of Crystal Growth* Studıes ın Applıed Mathematıcs 85 129-181 (1991)

23 F R Lawless and A D Wood *Resonances and optıcal tunnellıng* Draft

24 J Lıu and A D Wood *Matched Asymptotıcs for a generalısatıon of a model equatıon for optıcal tunnellıng* European Journal of Applıed Mathematıcs, 2, 223-231 (1991)

25 C Lozano and R E Meyer *Leakage and response of waves trapped by round ıslands* Physıcs of fluıds, 19, 8, 1075-1088 (1976)

26 J B McLeod *Spectral Concentratıon 1 - the one dımensıonal Schrodınger Operator*, ın Perturbatıon Theory and Applıcatıonsm Quantum Mechanıcs (ed C H Wılcox),Vol 16 (New York Wıley, 1966)

27 J B McLeod *Smoothıng of Stokes dıscontınuıtıes* Proc Roy Soc Lond A(1992)

28 D Marcuse *Lıght transmıssıon Optıcs* , 2nd edıtıon, Van Nostrand Reınhold, New York

29 R E Meyer *Exponential asymptotics* SIAM Review 22, 2, 213-224 (1980)

30 R E Meyer and J F Painter *Connection for Wave Modulation* Siam J Math Anal Vol 14, No 3, May 1983

31 F W J Olver *Asymptotics and special functions* (New York Academic Press, 1974)

32 F W J Olver *Uniform asymptotic expansions for Weber parabolic cylinder functions of large order* Journ Res of Nat Bureau of standards B Math and Math Phy 63B, 2

33 F W J Olver *On Stokes' phenomenon and converging factors,* in Asymptotic and Computational Analysis (New York Marcel Dekker, 1990) p 329-355

34 R B Paris and A D Wood *A model equation for optical tunnelling* IMA Journal of Applied Math 43, 273-284 (1989)

35 R B Paris and A D Wood *On eigenvalues with exponentially small imaginary part* Proceedings of the International Symposium on Asymptotics and Computional Analysis, Winnipeg 1989

36 R B Paris and A D Wood *Exponentially-improved asymptotics for the gamma function* J Comp Appl Math, 41, 135-143 (1992)

37 R B Paris and A D Wood *Asymptotics of higher order differential equations* Longman Scientific and Technical, 1986

38 R B Paris *Smoothing of the Stokes phenomenon for higher order differential equations* Vol A436, 165-186 (1992)

39 V L Povrovskii and I M Khalatnikov *On the problem of above barrier reflection of high-energy particles* Soviet Physics JETP 13, 6 1207-1210, (1961)

40 M Reed and B Simon *Methods of modern mathematical physics Vol 4 Analysis of operators* (San Diego Academic Press Inc , 1978)

41 B Simon *Resonances and Complex Scaling A Rigorous Overview* Int Journal Quan Chem Vol 14, 529-542, 1978

42 A W Snyder and J D Love *Optical Waveguide Theory* (London Chapman and Hall, 1983)

43 R Spigler and M Vianello *WKBJ-type approximations for finite moments perturbations of the differential equation* $y'' = 0$ *and the analogous difference equation* To appear in J Math Anal Appl in 1991

44 R Spigler and Marco Vianello *Liouville-Green approximations for a class of linear oscillatory difference equations of the second order* To appear in J Comput Applied Maths in 1991

45 E C Titchmarsh *Some theorems on perturbation theory, III* Proc Roy Soc A,207,321-8

46 E C Titchmarsh *Some theorems on perturbation theory, IV* Proc Roy Soc A,210,30-47

47 E C Titchmarsh *Eigenfunction Expansions Part 1* (Oxford Claredon Press, 1946)

48 E C Titchmarsh *Eigenfunction Expansions Part 2* (Oxford Claredon Press, 1958)

49 E C Titchmarsh *The Theory of Functions* (Oxford University Press, 1985)

50 E T Whittaker and G N Watson *A course in modern analysis* Fourth edition (New York Cambridge University Press, 1990)

51 A D Wood *Exponential asymptotics and spectral theory for curved optical waveguides* Draft of talk for San Diego '91