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MSc THESIS

The spectrum of the simplified magnetohydrodynamic ballooning equation by computation of the Titchmarsh-Weyl coefficient

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ABSTRACT

The boundary value problem involving the magnetohydrodynamic ballooning equation,

$$(1 + x^2)d^2y/dx^2 + 2xdy/dx - (\lambda + \gamma^2(1 + x^2) - \mu^2/(1 + x^2))y = 0, \quad x \in R,$$

with boundary conditions $y \to 0$ as $x \to \pm\infty$, arises from a study of the stability of disturbances in a magnetically confined plasma. In order to facilitate the analysis, the equation is transformed to Schrodinger form,

$$Y'' + [\lambda - q(x)]Y = 0$$

$q(x) \in R$, with the same boundary conditions and the results of Titchmarsh applied. Two cases are examined, The first with $\gamma^2 = 0$ (modified Legendre equation) and the second with $\gamma^2 > 0$. In both cases the Titchmarsh-Weyl coefficients $m(\lambda)$ are constructed and their singularities examined. The singularities yield expressions for the eigenvalues of the problem. In the second case the expressions involve the joining factors of the spheroidal equation, $K_{\nu}^{\mu}(\lambda)$ and $K_{-\nu-1}^{\mu}(\lambda)$, which join solutions at infinity with solutions about ± 1. For the case $\gamma^2 > 0$, the limit of $m(\lambda)$ as $\gamma^2 \to 0$, is shown to be the $m(\lambda)$ coefficient in the case $\gamma^2 = 0$.

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CHAPTER 1

THE STABILITY OF MAGNETICALLY CONFINED PLASMAS THE DERIVATION OF AN EXPRESSION FOR THE GROWTH RATE OF A DISTURBANCE

INTRODUCTION

This thesis concerns itself with a problem which arises physically as a result of the attempt to contain an extremely hot ionised gas using magnetic forces. The ionised gas or plasma is hydrogen, heated to a few million degrees celsius. If such a plasma is compressed sufficiently, it will undergo nuclear fusion to produce helium and enormous quantities of energy. This energy, in the form of heat, can be used to produce steam which in turn drives generators, thus producing electricity. This is a relatively clean source of energy and the fuel, hydrogen, is abundant in sea water.

Because of the high temperature of the plasma, no material could contain it without being itself vapourised. However the plasma is composed of charged particles (electrons and positively charged ions) and therefore feels the effects of electromagnetic force fields. In practice, magnetic forces are used to contain the plasma in a "magnetic bottle", for sufficient time to allow the plasma to be heated so that fusion can take place. Practical devices have been built which try to achieve these conditions, but unfortunately, they are all subject to limitations, due to instabilities in the plasma, which grow with increased power input. The instabilities lead to a breakdown of the confining field and escape of the plasma. This limits the power output of such devices to levels which are uneconomical for commercial purposes. The origin and properties of these instabilities are thus key questions to be answered in assessing the viability of a controlled, nuclear fusion device.

In order to decribe the instabilities a model is required. The model is in the form of a plasma slab of infinite extent. This is an idealisation of practical devices, which typically have a toroidal geometry (the Tokamaks). In the case of the Tokamaks, the magnetic "walls" develop weaknesses similar to the ballooning experienced in rubber inner tubes or to aneurysms in blood vessels. A radial (centripetal) force is required in this case to constrain the plasma to a circular path. This force can be simulated in the plane slab model, by a uniform gravitational force field acting perpendicularly to the plane surfaces of the slab and to the applied magnetic field.

THE BASIC EQUATIONS

The basic equations of resistive Magnetohydrodynamics, under the conditions of low plasma density and over time intervals where there is assumed to be no significant heat transfer, are given in [11] to be

$$\rho \frac{d\underset{\sim}{v}}{dt} = -\nabla p + \frac{\underset{\sim}{J} \times \underset{\sim}{B}}{c} + \rho \underset{\sim}{g} \tag{1.1}$$

$$\frac{d\rho}{dt} + \rho \nabla . \underset{\sim}{v} = 0 \tag{1.2}$$

$$\frac{dp}{dt} + \gamma p \nabla . \underset{\sim}{v} = 0 \tag{1.3}$$

The symbols have their usual meaning and $d/dt = \partial/\partial t + \underset{\sim}{v}.\nabla$

Ohm's Law has the form

$$E + \frac{1}{c}(\underset{\sim}{v} \times \underset{\sim}{B}) = \frac{\eta}{c^2}\underset{\sim}{j} \tag{1.4}$$

and the truncated Maxwell's equations are

$$\begin{aligned} \underset{\sim}{j} &= \frac{c}{4\pi}\nabla \times \underset{\sim}{B} \quad , \\ \frac{\partial \underset{\sim}{B}}{\partial t} &= -c\nabla \times \underset{\sim}{E} \quad , \\ \nabla . \underset{\sim}{B} &= 0 \quad . \end{aligned} \tag{1.5}$$

Combining (1.4) and (1.5) yields the induction equation, which describes the rate of change of $\underset{\sim}{B}$ due to the motion of the plasma and resistive diffusion.

$$\begin{aligned} \frac{\partial \underset{\sim}{B}}{\partial t} &= \nabla \times (\underset{\sim}{v} \times \underset{\sim}{B}) - \nabla \times (\frac{\eta}{4\pi}\nabla \times \underset{\sim}{B}) \\ &= \nabla \times (\underset{\sim}{v} \times \underset{\sim}{B}) + \frac{\eta}{4\pi}\nabla^2 \underset{\sim}{B} \end{aligned} \tag{1.6}$$

η Constant.

The first term measures the effect on the magnetic field $\underset{\sim}{B}$ of convective motion in the plasma. The second term measures the effects of diffusion due to the resistivity of the plasma. The similarity between (1.6) and equations in the hydrodynamics of viscous and incompressible fluids allows one to use many of the conclusions of hydrodynamics, in particular one can define a magnetic Reynolds number, analogous to the Reynolds number in hydrodynamics which measures the ratio of the inertial (convective) forces to the viscous forces. For (1.6) the ratio of the magnitude of the two terms, one convective, one resitive defines the magnetic Reynolds number.

$$\frac{|\nabla \times (\underset{\sim}{v} \times \underset{\sim}{B})|}{|\frac{\eta}{4\pi} \nabla^2 \underset{\sim}{B}|} \sim \frac{4\pi}{\eta} V a = S$$

V and a are a characteristic velocity and dimension of the particular problem.

If η = 0 (the case of an infinitely conducting plasma) then $s \to \infty$ and the first term in (1.6) dominates. The equation now represents the case of Flux-freezing, where the magnetic field lines are constrained to move with the plasma. As a result of this, they possess an effective mass and behave like stretched elastic string, allowing the propagation of transverse waves along the lines of force. The speed at which these waves travel is found to be

$$V_A = B/(4\pi\rho)^{\frac{1}{2}}$$

and is known as the Alfven wave speed, see [1] [3] and [11]. This allows us to define a basic time scale (The hydrodynamic time scale)

$$\tau_H = a(4\pi\rho)^{\frac{1}{2}}/B$$

on which disturbances in the plasma propagate along the field. τ_H can vary enormously, depending on the problem considered, but in Laboratory fusion plasmas $\tau_H \sim 1\mu sec$.

This shortness implies that disturbances will propagate very rapidly through the plasma and if inclined to grow, would do so very quickly. This is partly the reason why Magnetohydrodynamic instabilities impose such a severe limitation on thermonuclear fusion plasma confinement. If $s \ll 1$ then (1.6) is basically the resistive diffusion equation

$$\frac{\partial \underset{\sim}{B}}{\partial t} \approx \frac{\eta}{4\pi} \nabla^2 \underset{\sim}{B}$$

This has a characteristic diffusion time $\tau_R = 4\pi a^2/\eta$.
This time scale again varies considerably, depending on the problem under consideration, for laboratory thermonuclear fusion plasmas $\tau_R \sim 1$ - 10 msecs.
The ratio of the two time scales

$$S = \tau_R / \tau_H = \frac{4\pi}{\eta} . V_A a$$

is the magnetic Reynolds number with V_A the Alfven wave speed. It is an important quantity in resistive instability theory. In the case of plasmas generated in laboratory fusion experiments, S is large -in the range 10^3 - 10^7. The significance of this is that resistive diffusion effects are small and the condition of flux-freezing considered to hold to good approximation.
If one layer of plasma with density ρ , is supported against gravity by another layer, whose density is less than ρ , then another instability analogous to the Rayleigh-Taylor instability in an incompressible fluid can occur. The Rayleigh-Taylor instability arises in the case where a heavier liquid is supported against gravity by a lighter liquid. In the Magnetohydrodynamic case a characteristic time scale τ_g exists where $\tau_g = ((-\rho_0'/\rho_0)g)^{-\frac{1}{2}}$ and ρ_0' is the adverse density gradient. In a highly

conducting plasma, confined by a sheared, magnetic field in the presence of a gravitational acceleration, acting normal to the magnetic surfaces, the instabilities, called the tearing, rippling and resistive g-modes will be characterised by time scales which depend on the three characteristic time scales, τ_R , τ_H and τ_g of the problem. The time scale τ on which these instabilities grow in the linear domain, is found to scale like

$$\tau \sim \tau_R^a \tau_H^b \tau_g^c$$

where a, b, c are fractional powers and $|a|,|b| < 1$.
As can be seen from the above a disturbance can propagate by a number of different means through the plasma at different rates and all of it depending on the equilibrium state of the plasma.
The next step in the stability analysis of the model, upon deciding on the equilibrium state, is to perturb the state and derive an expression for the growth rate of the perturbation. Analysis of this expression will reveal the conditions for unfavourable growth rates and the characteristic time for their propagation. The expression typically is a boundary value problem and in an important set of examples the boundary value problem turns out to be a form of the oblate spheroidal equation whose properties are the concern of the remaining chapters of this work.

AN EXAMPLE LEADING TO THE SPHEROIDAL EQUATION

We will choose as our model the plane slab with an infinitely conducting plasma ($\eta \equiv 0$). The equilibrium magnetic field $\underset{\sim}{B}_o = [B_{ox}(y), 0, B_{oz}(y)]$. The equilibrium density $\rho_0(y)$ varies in the vertical y-direction only. The gravitational acceleration is $\underset{\sim}{g} = g\underset{\sim}{e}_y$. Assume a stationary equilibrium ($v_0 \equiv 0$). Perturbed quantities will be written as $f = f_0 + f_1$ where f_0 is the equilibrium value and f_1 the perturbing value. The equations being perturbed are (1.1) (use (1.5) to eliminate j), (1.2) and (1.6). Terms up to the first order in the perturbing quantities are taken, i.e a standard linearisation process, see [3, Ch.1]. We then look for incompressible modes i.e. $\nabla . \underset{\sim}{v} = 0$. The linearised form of (1.1) is

$$\rho_0 \frac{\partial \underset{\sim}{v}_1}{\partial t} = -\nabla p_1 + \left\{ \frac{1}{4\pi}\left((\underset{\sim}{B}_1 . \nabla)\underset{\sim}{B}_0 + (\underset{\sim}{B}_0 . \nabla)\underset{\sim}{B}_1 \right) + \rho_1 \underset{\sim}{g} \right\}$$

$$+ \left\{ -\nabla p_0 + \left(\frac{1}{4\pi}(\underset{\sim}{B}_0 . \nabla)\underset{\sim}{B}_0 + \rho_0 \underset{\sim}{g} \right) \right\}$$

The third term is the equilibrium term and equals zero. Taking the curl then eliminates the pressure term giving

$$\nabla \times \rho_0 \frac{\partial \underset{\sim}{v}_1}{\partial t} = \nabla \times \left\{ \frac{1}{4\pi}\left((\underset{\sim}{B}_1 . \nabla)\underset{\sim}{B}_0 + (\underset{\sim}{B}_0 . \nabla)\underset{\sim}{B}_1 \right) + \rho_1 \underset{\sim}{g} \right\} \quad (1.7)$$

(1.2) becomes:

$$\frac{\partial \rho_1}{\partial t} = -\rho_0' v_{y1} \quad (1.8)$$

(where prime indicates diff. w.r.t. y) and (1.6) with $\eta = 0$ becomes

$$\frac{\partial \underset{\sim}{B}_1}{\partial t} = \nabla \times [\underset{\sim}{v}_1 \times \underset{\sim}{B}_0]$$

$$= (\underset{\sim}{B}_0 . \nabla)\underset{\sim}{v}_1 - (\underset{\sim}{v}_1 . \nabla)\underset{\sim}{B}_0 \quad (1.9)$$

In addition of course $\nabla . \underset{\sim}{B}_1 = 0$.

Because the coefficients of the linearised equations above, are functions of y only (independent of x, z and t) we can represent the perturbed quantities by their Fourier transforms which have the form

$$f_1(x, y, z, t) = f_1(y)\exp(ik_x x + ik_z z + \omega t), \quad (1.10)$$

See description in [3, Ch.1] and [11]

Here f_1 represents any perturbed quantity.

$k = [k_x, 0, k_z]$ is the wave vector in the x - z plane and ω is the growth rate of the disturbance. The object of the exercise is to derive an expression for the growth rate ω, from which the stability of the equilibrium state may be determined. Broadly speaking, the system is said to be stable if the real part of ω is negative for all k. If it turned out to be positive for even one value of k, then it would be unstable. The disturbance growing with time. The imaginary part of ω, will determine whether the system is subject to oscillations. Non-zero values of Im(ω) implying oscillations.

Note: the operator $(\underset{\sim}{B}_0 . \nabla) f_1 = i(\underset{\sim}{k} . \underset{\sim}{B}_0) f_1$ and in this case

$$\underset{\sim}{k} . \underset{\sim}{B}_0 = k_x B_{0x}(y) + k_z B_{0z}(y) \quad . \quad (1.11)$$

Equation (1.8) now reduces to

$$\omega \rho_1 = -\rho_0' v_{y1} \quad , \quad (1.12a)$$

and the y-component of (1.9) is

$$\begin{aligned} \omega B_{1y} &= i(k_x B_{0x}(y) + k_z B_{0z}(y)) v_{1y} \\ &= i(\underset{\sim}{k} . \underset{\sim}{B}_0) v_{1y} \quad . \end{aligned} \quad (1.12b)$$

Proceeding as in [11], we obtain the equation

$$0 = \frac{d}{dy}\left\{\left(\rho_0\omega^2 + \frac{(\underset{\sim}{k} . \underset{\sim}{B}_0)^2}{4\pi}\right)\frac{dv_{y1}}{dy}\right\} - k^2\left(\rho_0\omega^2 + \frac{(\underset{\sim}{k} . \underset{\sim}{B}_0)^2}{4\pi} + \rho_0' g\right)v_{y1} \quad (1.13)$$

We can simplify this equation by making an approximation. We assume the density inhomogeneity to be weak and make the Boussinesq approximation (see [3, Ch.2] and [11, II]), which assumes $\rho_0(y)$ to be constant, while still retaining ρ_0' in the gravitational term on the right. (1.13) then becomes

$$\frac{d}{d\mu}\left\{\left((\omega\tau_H)^2+\alpha^2F^2\right)\frac{dv_{y1}}{d\mu}\right\}-\alpha^2\left((\omega\tau_H)^2+\alpha^2F^2-G\right)v_{y1} = 0 \qquad (1.14)$$

where quantities have been normalised as follows

$$\mu = y/a\,,\quad k^2 = k_x^2+k_z^2\,,\quad \alpha = |k|a\,,\quad F = \underset{\sim}{k}\cdot\underset{\sim}{B}_0/|\underset{\sim}{k}||\underset{\sim}{B}_0|\,,$$

$$\tau_H = a(4\pi\rho_0)^{\frac{1}{2}}/|\underset{\sim}{B}_0|\,,\quad G = (-\rho_0'/\rho_0)g\tau_H^2\;.$$

Here a represents a characteristic dimension of the current layer.

From [11] it is known that the most dangerous modes will be localised about $\underset{\sim}{k}\cdot\underset{\sim}{B}_0 = 0$ (known as the resonant surface). In the neighbourhood of this surface, field lines can interchange without undergoing significant distortion and as a result, shear tends to localise the instability about this surface. The resonant surface corresponds to $F(\mu) = 0$ where we assume $F(\mu) = 0$ when $\mu = 0$. Considering only localised modes about $\mu = 0$ (where the amplitude decays rapidly away from $\mu = 0$) then $F(\mu) \simeq F'(0)\mu$ (Taylor expansion about $\mu = 0$) and we obtain the equation

$$\frac{d}{d\tilde{\mu}}\left\{(1+\tilde{\mu}^2)\frac{dv_{y1}}{d\tilde{\mu}}\right\}+\left\{-A(1+\tilde{\mu}^2)+\frac{G}{(F')^2}\right\}v_{y1} = 0 \qquad (1.15)$$

where

$$A = \frac{(\omega \tau_H)^2}{(F')^2} \quad , \qquad \tilde{\mu} = \frac{\alpha \mu}{A^{1/2}} \quad .$$

The boundary conditions $v_{y1} \to 0$ as $\tilde{\mu} \to \pm\infty$ then constitute an eigenvalue problem for the eigenvalue A as a function of the gravitational-shear term $G/(F')^2$.

An equation of the same type arises in a more complicated way in ideal ballooning, (See Los Alamos technical report L A -9055- M S, by R. B. Paris and R.Y. Dagazian, 1982). The authors take as their model the case of a magetised plasma slab equilibrium with magnetic shear. The equilibrium magnetic field is

$$\underset{\sim}{B}_0 = [0, B_{y0}(x), B_{0z}(x)] \; .$$

They simulate toroidal curvature effects by introducing a gravitational acceleration modulated along a given line of force.

$$\underset{\sim}{G} = G_0\left[-\alpha + \cos\frac{2\pi}{L}z\right]\hat{e}_x \quad , \quad G_0 \equiv \beta_0 \frac{a}{R_c}$$

They then consider perturbations of the form

$$f_1(x,y,z,t) = f(\delta x, z)\exp\left[ik(B_{y0}z - B_{z0}y) + \omega t\right] ,$$

This is the crucial form of a ballooning perturbation. $k \gg 1$ $\exp(ik(B_y z - B_z y))$ describes fast dependence across field lines, while $f(z)$ describes slow dependence along the field lines. Neglecting the variation of perturbed quantities as limits $\delta \to 0$, $k \to \infty$ are taken, then taking the divergence of the equation of motion, we arrive at the equation describing ideal ballooning

$$B_{0z}^2 \frac{\partial}{\partial z}(1 + s^2z^2)\frac{\partial}{\partial z} v_{1x} - \omega^2(1 + s^2z^2)v_{1x} + 2\gamma\Gamma G(z)v_{1x} = 0.$$

By letting $\hat{z} = sz$ we obtain

$$\frac{\partial}{\partial \hat{z}}(1+\hat{z}^2)\frac{\partial}{\partial \hat{z}}v_x - [\hat{\lambda} + \hat{\gamma}^2(1+\hat{z}^2)]v_x = 0 \tag{1.15a}$$

If we compare (1.15a) with the oblate spheroidal equation with imaginary argument i.e.

$$\frac{d}{dx}(1+x^2)\frac{dy}{dx} - \left[\lambda + \gamma^2(1+x^2) - \frac{\mu^2}{1+x^2}\right]y = 0 \tag{1.16}$$

we see

$$\hat{\lambda} \equiv \frac{2\hat{G}_0}{s^2 B_{0z}^2}\left[1 - \alpha + \frac{1}{2}\left(\frac{2\pi}{sL}\right)^2\right], \qquad \hat{\gamma}^2 \equiv \frac{\omega^2 + \hat{G}_0\left(\frac{2\pi}{sL}\right)^2}{s^2 B_{0z}^2}.$$

This is a Sturm-Liouville-type boundary value problem and to determine the eigenvalues, we employ the Titchmarsh-Weyl theory, which had its origins in the famous 1910 paper of Hermann Weyl where he showed that the differential equation

$$-(py')' + qy = \lambda y \qquad \text{on } [0, \infty)$$

when λ is not real, always has a non-trivial solution which is absolutely square-integrable on the interval $[0, \infty)$. Given two solutions θ and ϕ which satisfy the initial conditions

$$\theta(0,\lambda) = 1 \qquad \theta'(0,\lambda) = 0$$

$$\phi(0,\lambda) = 0 \qquad \phi'(0,\lambda) = 1$$

he demonstrated the existence of a coefficient $m(\lambda)$ (the Titchmarsh-Weyl coefficient) regular in the two half planes of C created by the real line, the solution being

$$\psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\phi(x,\lambda) \qquad x \in [0,\infty)$$

where

$$\int_0^\infty |\psi(x,\lambda)|^2 dx < \infty, \quad (\lambda \in C \text{ and } \operatorname{im}\lambda \neq 0).$$

The poles and zeros of $m(\lambda)$ yielding the eigenvalues of $\theta(x,\lambda)$ and $\phi(x,\lambda)$ respectively.

THE LAYOUT OF THE THESIS

The layout of the remaining chapters is as follows:
Chapter 2 is a review of spectral theory by the Titchmarsh method. It begins with the regular Sturm-Liouville case on $[a,b]$, then examines the singular case leading to a series expansion, firstly on the interval $[0,\infty)$ and then on $(-\infty,\infty)$. This last case is of particular relevance to this work, especially the case where the potential function $q(x)$ is even. The Titchmarsh-Weyl coefficient, $m(\lambda)$ is defined and the relationship between the poles and zeros of $m(\lambda)$ and the eigenvalues of the boundary value problem established. Finally the general singular case is examined and the spectral function $k(\lambda)$ defined. The extension to the interval $(-\infty,\infty)$ follows along with definitions of the functions $\xi(\lambda)$, $\eta(\lambda)$ and $\zeta(\lambda)$. The general classification of spectra are described. Chapter 3 consists of examples illustrating the theory of Chapter 2. Some theorems on the dependence of the spectrum of

$$\psi'' + (\lambda - q(x))\psi = 0$$

on $q(x)$ are listed and discussed. Finally the potential functions of the Magnetohydrodynamic ballooning equation for different values of the parameters μ^2 and γ^2 are examined in detail, properties of the spectrum for the various values of the parameters are elucidated using the listed theorems.

Chapter 4 examines the special case of the Magnetohydrodynamic equation when $\gamma^2 = 0$, where it reduces to the associated Legendre equation with imaginary argument. Expressions for the eigenvalues and eigenfunctions are derived using two different methods, one, that of truncated hypergeometric series, the second using the Titchmarsh-Weyl coefficient $m(\lambda)$. Chapter 5 looks at the properties of the spheroidal functions which are solutions of the Magnetohydrodynamic ballooning equation. The case $\gamma^2 > 0$ only is considered (real growth rate ω). The spheroidal functions $Ps_\nu^\mu(x,\gamma)$ and $Qs_\nu^\mu(x,\gamma)$ can be expressed as infinite series of Legendre functions about the equations singularities ± 1, while they can be expressed as infinite series of Bessel functions as $|x| \to \infty$. The joining factors $K_\nu^\mu(\lambda)$ linking the two types of series are derived, and a number of methods for evaluating the characteristic exponent ν described. Chapter 6 begins by constructing the even and odd solutions θ and ϕ in terms of $Ps_\nu^\mu(x,\gamma)$ and $Qs_\nu^\mu(x,\gamma)$ and hence in terms of ${}_2F_1$ hypergeometric functions. The two solutions θ and ϕ are continued as infinite series of Bessel functions $S_\nu^{\mu(3)}(x,\gamma)$, using the joining factors. $m(\lambda)$ is then calculated by comparing the solution

$$\psi = \theta + m\phi$$

to $S_\nu^{\mu(3)}(x,\gamma)$. A relation for the eigenvalues is then derived. Finally, by using an approximation for $m_\gamma(\lambda)$ for small γ^2, we show that $m_\gamma(\lambda) \to m_0(\lambda)$ corresponds to the $m(\lambda)$ obtained in the case $\gamma^2 = 0$ of Chapter 4.

CHAPTER 2

REVIEW OF THE SPECTRAL THEORY BY TITCHMARSH'S METHOD

In this chapter we will outline the conditions under which an arbitrary function f(x) may be expressed in terms of the eigenfunctions of a singular self-adjoint boundary value problem.

$$y'' + (\lambda - q(x))y = 0 \tag{2.1}$$

where y(x) is $L^2(-\infty, \infty)$.

We will proceed as follows:

(1) The regular case

(2) The singular case with series expansion on $[0, \infty)$

(3) The singular case with series and integral terms on $[0, \infty)$

(4) The singular case over the full interval $(-\infty, \infty)$

Firstly then, we review the regular case, that of the formally self-adjoint boundary value problem on a finite interval [a,b].

$$y'' + (\lambda - q(x))y = 0$$

with boundary conditions:

$$\begin{aligned} y(a)\cos\alpha + y'(a)\sin\alpha &= 0 \\ y(b)\cos\beta + y'(b)\sin\beta &= 0 \end{aligned} \tag{2.2}$$

In all cases λ is a complex parameter, α and β are given real numbers. In this instance also q(x) is assumed continuous on [a,b]. The methods described are those given in [15]. For an alternative method see [4]. The problem (2.2) above is the classical Sturm-Liouville one. The expansion of f(x) in this case involves Cauchy's residue theorem and integration around a large circular contour in the λ - plane. In [15, Ch.1] the proof involves asymptotic expansions of the solutions of (2.2) for large $|\lambda|$ and the point-wise convergence of the series is established by comparison with the Fourier series i.e. the case $q(x) = 0$.

We will now outline the proof in the regular case:
Let $\phi(x,\lambda)$ and $\chi(x,\lambda)$ be solutions of (2.2) such that

$$\phi(a,\lambda) = \operatorname{Sin}\alpha \qquad \phi'(a,\lambda) = -\operatorname{Cos}\alpha$$

$$\chi(b,\lambda) = \operatorname{Sin}\beta \qquad \chi'(b,\lambda) = -\operatorname{Cos}\beta$$

Let $\lambda = s^2$ where s is real and positive when λ is real and positive and $s = \sigma + i\tau$. Then, as $|s| \to \infty$ it is proved in [15, p. 10] that, assuming $\operatorname{Sin}\alpha \neq 0$ ($\operatorname{Sin}\beta \neq 0$).

$$\phi(x,\lambda) = \operatorname{Cos} s(x-a) \operatorname{Sin}\alpha + O(|s|^{-1}.\, e^{|\tau|(x-a)}),$$

$$\phi'(x,\lambda) = -s \operatorname{Sin} s(x-a) \operatorname{Sin}\alpha + O(e^{|\tau|(x-a)}),$$

$$\chi(x,\lambda) = \operatorname{Cos} s(b-x) \operatorname{Sin}\beta + O(|s|^{-1}.\, e^{|\tau|(b-x)}),$$

$$\chi'(x,\lambda) = s \operatorname{Sin} s(b-x) \operatorname{Sin}\beta + O(e^{|\tau|(b-x)}) .$$

The Wronskian of $\phi(x,\lambda)$ and $\chi(x,\lambda)$, $\omega(\lambda)$ is independent of x and it follows from above that

$$\omega(\lambda) = s \operatorname{Sin} s(b-a) \operatorname{Sin}\alpha \operatorname{Sin}\beta + O(e^{|\tau|(b-a)}).$$

If $\operatorname{Sin}\alpha = 0$ ($\operatorname{Sin}\beta = 0$) similar results can be obtained. We assume though that $\operatorname{Sin}\alpha$ (and $\operatorname{Sin}\beta$) $\neq 0$. We will outline the proof that when f(x) is of bounded variation in the neighbourhood of x, it may be expanded in terms of the eigenfunctions of (2.2).

Let $\Gamma^*_{s,n}$ be the positively oriented rectangular contour in the s-plane with vertices $\pm R_n$, $\pm R_n + iR_n$ where $R = (n + 1/2)\pi/(b - a)$. Let $\Gamma_{s,n} = \Gamma^*_{s,n} \setminus (-R_n, R_n)$ (the contour less the real line portion), also let $\Gamma_{\lambda,n}$ be the arc in the λ-plane corresponding to $\Gamma_{s,n}$. We note that on $\Gamma_{s,n}$ and inside $\Gamma^*_{s,n}$, $|\sin s(b - a)| > Ae^{\tau(b-a)}$ for some constant A, hence

$$\frac{1}{\omega(\lambda)} = \frac{1 + O(|s|^{-1})}{s \sin s(b-a) \sin\alpha \sin\beta} .$$

Then for $x > t$ and on $\Gamma_{s,n}$ and inside $\Gamma^*_{s,n}$

$$\frac{\chi(x,\lambda)\phi(t,\lambda)}{\omega(\lambda)} = \frac{\cos s(b-x)\cos s(t-a)}{s \sin s(b-a)} + O\left(\frac{e^{\tau(t-x)}}{|s|^2}\right).$$

Writing the trigonometric functions as exponentials, the first term becomes on re-arranging.

$$-\tfrac{1}{2}ie^{-is(b-x)}.e^{-is(t-a)}.e^{is(b-a)}.s^{-1}[1+e^{2is(b-x)}][1+e^{2is(t-a)}][1-e^{2is(b-a)}]^{-1}.$$

Since $|e^{2is(b-a)}| = e^{-2\tau(b-a)} < 1$, we can use the binomial expansion on the last term to give

$$\frac{\chi(x,\lambda)\phi(t,\lambda)}{\omega(\lambda)} = \frac{-ie^{is(x-t)}}{2s}[1+E^{-}(s,x,t)] + F^{-}(s,x,t) \qquad (2.3)$$

where $E^{-}(s,x,t)$ is a sum of terms of the form $\exp[2mis(\zeta a + \eta b + \kappa x + \xi t)]$, with $m = 1,2,\ldots.$ ζ, η, κ, ξ real numbers with the property that

$$2m\zeta a + 2m\eta b + (2m\kappa + 1)x + (2m\xi - 1)t = 0$$

for t outside [a, x] and where

$$F^{-}(s,x,t) = O(|s|^{-2} e^{\tau(t-x)}) \quad \text{as } |s| \to \infty .$$

Similarly for $t \in [x, b]$

$$\frac{\chi(t,\lambda)\phi(x,\lambda)}{\omega(\lambda)} = \frac{-ie^{is(t-x)}}{2s}\left[1 + E^{+}(s,x,t)\right] + F^{+}(s,x,t) \tag{2.4}$$

where again the exponents in $e^{is(t-x)}E^{+}(s,x,t)$ vanish only for t outside [x,b] and

$$F^{+}(s,x,t) = O(|s|^{-2} e^{\tau(x-t)}) \quad \text{as } |s| \to \infty$$

It is proved in [15, Ch.1] that when f(x) is of bounded variation

$$\lim_{n\to\infty} \frac{1}{2\pi i}\int_{\Gamma_{\lambda,n}} \left\{\frac{\chi(x,\lambda)}{\omega(\lambda)}\int_a^x \phi(t,\lambda)f(t)dt + \frac{\phi(x,\lambda)}{\omega(\lambda)}\int_x^b \chi(t,\lambda)f(t)dt\right\}d\lambda$$

$$= \tfrac{1}{2}\left[f(x+0) + f(x-0)\right]. \tag{2.5}$$

If we let $\Phi(x,\lambda)$ equal the quantity in the braces above, then we can show by differentiating, that it is a solution of the non-homogeneous boundary value problem.

$$\Phi'' + (\lambda - q(x))\Phi = f(x)$$

with the same boundary conditions as (2.2) for all λ. Also it is proved in [15, Ch.1] that all the zeros of $\omega(\lambda)$ must be simple and on the real axis (say $\lambda_0, \lambda_1, \lambda_2 \ldots$). Then at λ_n the Wronskian of $\phi(x, \lambda_n)$ and $\chi(x, \lambda_n)$ is zero and so ϕ_n and χ_n are linearly dependent i.e.

$$\chi(x,\lambda_n) = k_n \phi(x,\lambda_n) .$$

Now from the boundary conditions, k_n is neither 0 nor ∞ hence at $\lambda = \lambda_n$, $\phi(x,\lambda)$ has the residue

$$\frac{k_n}{\omega'(\lambda_n)}\phi(x,\lambda_n)\int_a^b \phi(t,\lambda_n)f(t)\,dt \, .$$

Recalling the result at (2.5) and the fact that

$$\frac{1}{2\pi i}\int \phi(x,\lambda)\,d\lambda = \sum(\text{residues of } \phi(x,\lambda) \text{ inside contour})$$

Then,

$$f(x) = \sum_{n=0}^{\infty}\frac{k_n}{\omega'(\lambda_n)}\phi(x,\lambda_n)\int_a^b \phi(t,\lambda_n)f(t)\,dt \tag{2.6}$$

This is the Sturm-Liouville expansion, see also [14] If the Wronskian $\omega(\lambda)$ has a branch point, then this introduces a continuous spectrum component to the expansion of f(x). It appears as an additional integral term on the right hand side of (2.6). It is possible that the series in (2.6) may have only a finite number of terms. See [15, Ch. 1 and Ch. 4].

THE SINGULAR CASE

We now take a look at the results pertaining to the singular case. That is the case where q(x) has a discontinuity at one end of the interval or both, or the interval goes to infinity in one direction or both. We will examine first a special case, that one where the expansion is always a series. This case is somewhat simpler than the most general one (which will be dealt with next) but it is the one relevant to the problem which will be examined later.

The particular case then, to be considered is the one on the interval $(0, \infty)$ where $q(x)$ is continuous on that interval.

The finite interval case with a singularity at one end is analagous. The problem, over the whole real line or the similar finite interval with a singularity at both ends, can be dealt with by breaking the interval up into two parts at some regular point of $q(x)$. We would then deal with each interval separately.

Let $F(x)$ and $G(x)$ satisfy the differential equations

$$\frac{d^2y}{dx^2} + (\lambda - q(x))y = 0 \tag{2.7}$$

and the corresponding equation with λ' instead of λ respectively, then

$$\begin{aligned}
&(\lambda' - \lambda)\int_a^b F(x)\,G(x)\,dx \\
&= \int_a^b \left[F(x)\left\{q(x)G(x) - G''(x)\right\} - G(x)\left\{q(x)F(x) - F''(x)\right\}\right] dx \\
&= -\int_a^b \left\{F(x)G''(x) - G(x)F''(x)\right\} dx \\
&= W_a(F,G) - W_b(F,G)\,.
\end{aligned} \tag{2.8}$$

If $\lambda = \mu + i\nu$, $\lambda' = \bar{\lambda} = \mu - i\nu$, $a = 0$ & $G = \bar{F}$ this gives

$$2\nu\int_0^b |F(x)|^2 dx = iW_0(F\bar{F}) - iW_b(F,\bar{F}) \tag{2.9}$$

Now let $\phi(x) = \phi(x;\lambda)$, $\theta(x) = \theta(x;\lambda)$ be the solutions of (2.7) such that

$$\begin{aligned} \phi(0) &= \sin\alpha & \phi'(0) &= -\cos\alpha \\ \theta(0) &= \cos\alpha & \theta'(0) &= \sin\alpha \end{aligned} \tag{2.10}$$

$\alpha \in \mathbb{R}$. Then

$$W_x(\phi,\theta) = W_0(\phi,\theta) = \sin^2\alpha + \cos^2\alpha = 1 .$$

The general solution of (2.7) can be written in the form $\theta(x) + l\,\phi(x)$. Consider the solutions which satisfy a real boundary condition at x = b, say

$$\{\theta(b) + l\phi(b)\}\cos\beta + \{\theta'(b) + l\phi'(b)\}\sin\beta = 0 ,$$

$\beta \in \mathbb{R}$. Rearranging it gives

$$l = l(\lambda) = \frac{\theta(b)\cot\beta + \theta'(b)}{\phi(b)\cot\beta + \phi'(b)} \tag{2.11}$$

For each b as $\cot\beta$ varies, l describes a circle in the complex plane, say c_b. Replacing $\cot\beta$ by a complex variable z, we obtain

$$l = l(\lambda) = \frac{\theta(b)z + \theta'(b)}{\phi(b)z + \phi'(b)}$$

When $z = -\phi'(b)/\phi(b)$, $l = \infty$. Hence the centre of the circle c_b corresponds to the conjugate $z = -\overline{\phi'}(b)/\overline{\phi}(b)$:

it is therefore

$$l = \frac{W_b(\theta,\overline{\phi})}{W_b(\phi,\overline{\phi})} .$$

Also

$$\operatorname{im}\left\{-\frac{\phi'(b)}{\phi(b)}\right\} = \frac{1}{2}i\left\{\frac{\phi'(b)}{\phi(b)} - \frac{\bar{\phi}'(b)}{\bar{\phi}(b)}\right\} = -\frac{1}{2}i\,\frac{W_b(\phi,\bar{\phi})}{|\phi(b)|^2} ,$$

which has the same sign as ν by (2.9) with $F = \phi$,

since

$$W_0(\phi,\bar{\phi}) = 0 .$$

Hence, if $\nu > 0$, the exterior of c_b corresponds to the upper half of the z - plane.

Because $-\theta'(b)/\phi'(b)$ is on c_b (z = 0) the radius r_b of c_b is

$$r_b = \left|\frac{\theta'(b)}{\phi'(b)} - \frac{W_b(\theta,\bar{\phi})}{W_b(\phi,\bar{\phi})}\right| = \left|\frac{W_b(\theta,\phi)}{W_b(\phi,\bar{\phi})}\right|$$

$$= 1\Big/2\nu\int_0^b|\phi(x)|^2dx \qquad (2.12)$$

by (2.9) and (2.10).

λ is inside c_b if $\operatorname{im} z < 0$, if $i(z - \bar{z}) > 0$,

that is if

$$i\left\{-\frac{\lambda\phi'(b)+\theta'(b)}{\lambda\phi(b)+\theta(b)} + \frac{\bar{\lambda}\bar{\phi}'(b)+\bar{\theta}'(b)}{\bar{\lambda}\bar{\phi}(b)+\bar{\theta}(b)}\right\} > 0 ,$$

i.e. if

$$i\left[|\lambda|^2W_b(\phi,\bar{\phi}) + \lambda W_b(\phi,\bar{\theta}) + \bar{\lambda}W_b(\theta,\bar{\phi}) + W_b(\theta,\bar{\theta})\right] > 0,$$

i.e. if

$$iW_b(\theta+\lambda\phi,\ \bar{\theta}+\bar{\lambda}\bar{\phi}) > 0 ,$$

by (2.9) if

$$2\nu\int_0^b|\theta+\lambda\phi|^2dx < iW_0(\theta+\lambda\phi,\bar{\theta}+\bar{\lambda}\bar{\phi}) .$$

Since $W_0(\phi, \theta) = 1$, $W(\phi, \bar{\phi}) = 0$, etc.,

$$W_0(\theta + l\phi, \bar{\theta} + \bar{l}\bar{\phi}) = l - \bar{l} = 2i\,\mathrm{im}\,l .$$

Therefore l is interior to c_b when $\nu > 0$, and

$$\int_0^b |\theta + l\phi|^2 dx < \frac{-\mathrm{im}\,l}{\nu} \tag{2.13}$$

The same is true if $\nu < 0$. In each case the sign of $\mathrm{im}\,l$ is opposite to that of ν .

It follows that, if l is inside c_b and $0 < b' < b$, then

$$\int_0^{b'} |\theta + l\phi|^2 dx < \int_0^b |\theta + l\phi|^2 dx < \frac{-\mathrm{im}\,l}{\nu} .$$

Hence l is also inside $c_{b'}$. Hence $c_{b'}$ includes c_b if $b' < b$. It follows that, as $b \to \infty$, the circles c_b converge either to a limit-circle or to a limit-point. If $m = m(\lambda)$ is the limit-point, or any point on the limit-circle,

$$\int_0^b |\theta + m\phi|^2 dx \leq \frac{-\mathrm{im}\,m}{\nu} \tag{2.14}$$

for all values of b. Hence

$$\int_0^\infty |\theta + m\phi|^2 dx \leq \frac{-\mathrm{im}\,m}{\nu} \tag{2.15}$$

It follows that, for every value of λ other than real values, (2.7) has a solution

$$\psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\phi(x,\lambda)$$

belonging to $L^2(0, \infty)$. The function $m(\lambda)$ is the Titchmarsh-Weyl coefficient.

In the limit-circle case, r_b tends to a positive limit as $b \to \infty$; hence, by (2.12), $\phi(x)$ is $L^2(0,\infty)$; so in fact, in this case, every solution of (2.7) belongs to $L^2(0,\infty)$.

For a given β, $l = l(\lambda)$, is an analytic function of λ; in fact it is a meromorphic function, regular except for poles on the real axis.

This is because the poles of $l(\lambda)$ are the zeros of

$$\phi(b,\lambda)\cos\beta + \phi'(b,\lambda)\sin\beta$$

and this is just $-\omega(\lambda)$, the Wronskian as denoted earlier. It is proved in [15, Ch. 1] that the zeros of $\omega(\lambda)$ are all real and simple. Also it is proved in [15, Ch. 2] that $l(\lambda)$ is analytic in the upper and lower half--planes. Hence in both the limit-point and limit-circle cases the $m(\lambda)$ function is analytic in both half-planes, with its (simple) poles on the real axis. The functions in the two half-planes are not necessarily analytic continuations of each other. This would be the general case, but in this instance, we make the simplifying assumption that they are. Therefore $m(\lambda)$ is a single meromorphic function of the whole complex plane, whose only singularities are simple poles on the real axis, λ_0, λ_1, λ_2, ..., and the residues at those points may be denoted, r_0, r_1, r_2, ...

From (2.11) it can be seen that $l(\lambda)$ takes conjugate values for conjugate values of λ, and hence so do $m(\lambda)$ and $\Psi(x, \lambda)$.

We will now write down some results for the functions $\Psi(x,\lambda)$ and $\Phi(x,\lambda)$ defined earlier. Firstly though, we need two lemmas. The proofs of these lemmas are to be found in [15, Ch.2].

Lemma 1:

For any fixed complex λ and λ'

$$\lim_{x\to\infty} W(\Psi(x,\lambda), \Psi(x,\lambda')) = 0 \qquad (2.16)$$

Lemma 2:

Let fn(x) be a sequence of functions which converges in mean square to f(x) over any finite interval, while

$$\int_0^\infty |f_n(x)|^2 dx \leq K \qquad \text{for all } n$$

Then f(x) is $L^2(0,\infty)$, and if g(x) belongs to $L^2(0,\infty)$,

$$\lim_{n\to\infty} \int_0^\infty f_n(x) g(x) dx = \int_0^\infty f(x) g(x) dx .$$

We now apply these two results from (2.8)

$$(\lambda-\lambda')\int_0^b \Psi(x,\lambda)\Psi(x,\lambda')dx$$
$$= W_0\{\Psi(x,\lambda), \Psi(x,\lambda')\} - W_b\{\Psi(x,\lambda), \Psi(x,\lambda')\} .$$

The first term on the right is

$\{\cos\alpha + m(\lambda)\sin\alpha\}\{\sin\alpha - m(\lambda)\cos\alpha\} = m(\lambda) - m(\lambda')$ and, if λ and λ' are not real, the second term tends to zero as $b \to \infty$, by lemma 1 (2.16). Hence

$$\int_0^\infty \psi(x,\lambda)\psi(x,\lambda')\,dx = \frac{m(\lambda) - m(\lambda')}{\lambda - \lambda'} \tag{2.17}$$

If we take $\lambda' = \bar{\lambda}$, we obtain

$$\int_0^\infty |\psi(x,\lambda)|^2 dx = -\frac{\mathrm{im}(m(\lambda))}{\nu} \tag{2.18}$$

so that the case of equality holds in (2.15).

Now let λ_n be an eigenvalue, and let $\lambda_n' = \lambda_n + i\nu$, $\nu \to 0$.

Then for any fixed X,

$$\int_0^X |\nu\psi(x,\lambda') + ir_n\phi(x,\lambda_n)|^2 dx \to 0$$

Because the left hand side is

$$\int_0^X |\nu\theta(x,\lambda') + \{\nu m(\lambda') + ir_n\}\phi(x,\lambda') - ir_n\{\phi(x,\lambda') - \phi(x,\lambda_n)\}|^2 dx$$

and each of the three terms contributes zero to the limit.

Also, by (2.18),

$$\int_0^\infty |\nu\psi(x,\lambda')|^2 dx \leq |\nu m(\lambda')| = O(1)$$

as $\nu \to 0$, since the pole of $m(\lambda')$ at λ_n is simple. On multiplying (2.17) by $i\nu/r_n$, making $\nu \to 0$, and using lemma 2 we see that $\phi(x, \lambda_n)$ is $L^2(0, \infty)$, and

$$\int_0^\infty \psi(x,\lambda)\phi(x,\lambda_n)\,dx = \frac{1}{\lambda - \lambda_n} \tag{2.19}$$

If λ is equal to a different eigenvalue λ_m, on multiplying (2.19) by $i\nu/r_n$ and making $\nu \to 0$, gives us

$$\int_0^\infty \phi(x,\lambda_m)\,\phi(x,\lambda_n)\,dx = 0 \quad . \tag{2.20}$$

If λ tends to the same eigenvalue λ_n , it follows similarly that

$$\int_0^\infty |\phi(x,\lambda_n)|^2 dx = \frac{1}{r_n} \tag{2.21}$$

Hence the functions

$$\psi_n(x) = r_n^{\frac{1}{2}}\,\phi(x,\lambda_n) \tag{2.22}$$

form an orthonormal set.

(2.19) can now be written

$$\int_0^\infty \psi(x,\lambda)\,\psi_n(x)\,dx = \frac{r_n^{\frac{1}{2}}}{\lambda - \lambda_n} \quad . \tag{2.23}$$

<u>SOME PROPERTIES OF THE RESOLVENT OPERATOR $\Phi(x,\lambda)$</u>

Let f(y) be $L^2(0,\infty)$, and let

$$\Phi(x,\lambda) = \tag{2.24}$$

$$\psi(x,\lambda)\int_0^x \phi(y,\lambda)f(y)\,dy + \phi(x,\lambda)\int_x^\infty \psi(y,\lambda)f(y)\,dy \; ,$$

where ϕ and ψ are as defined above. Then, for every x, $\Phi(x,\lambda)$ is an analytic function of λ , regular for $\mathrm{im}\lambda > 0$ or $\mathrm{im}\lambda < 0$. Also if f(y) is continuous, $\Phi(y,\lambda)$ satisfies the differential equation

$$\Phi'' + (\lambda - q(x))\,\Phi = f(x) \; . \tag{2.25}$$

and

$$\Phi(0,\lambda) = \phi(0,\lambda)\int_0^\infty \psi(y,\lambda)f(y)\,dy,$$

$$\Phi'(0,\lambda) = \phi'(0,\lambda)\int_0^\infty \psi(y,\lambda)f(y)\,dy,$$

so that $\Phi(x,\lambda)$ also satisfies the boundary condition

$$\Phi(0,\lambda)\cos\alpha + \Phi'(0,\lambda)\sin\alpha = 0. \qquad (2.26)$$

It can be shown [15, Ch.2] that Φ has a simple pole at λ_n and its residue there is

$$r_n\,\phi(x,\lambda_n)\int_0^\infty \phi(y,\lambda_n)f(y)\,dy = \Psi_n(x)\int_0^\infty \Psi_n(y)f(y)\,dy$$

$$= c_n\Psi_n(x), \qquad (2.27)$$

Hence, the c_n are the generalised Fourier co-efficients. Also, it can be shown that

$$\Phi(x,\lambda) = \sum_{n=0}^{\infty}\frac{c_n\Psi_n(x)}{\lambda-\lambda_n}.$$

The Green's function $G(x,y,\lambda)$ is the solution of the basic equation with the right hand side equal to the Dirac delta function centred on $x = y$. It is symmetrical in x and y; a continuous function of x for each y, but such that

$$\left(\frac{\partial G}{\partial x}\right)_{x=y+} - \left(\frac{\partial G}{\partial x}\right)_{x=y-} = -1.$$

It is of integrable squared-modulus with respect to either variable, and satisfies the boundary condition.

$$G(0,y,\lambda)\cos\alpha + G_x(0,y,\lambda)\sin\alpha = 0$$

In terms of the functions defined already

$$G(x,y,\lambda) = -\Psi(x,\lambda)\phi(y,\lambda) \qquad (y \le x),$$

$$G(x,y,\lambda) = -\phi(x,\lambda)\Psi(y,\lambda) \qquad (y > x),$$

$$\Phi(x,\lambda,f) = -\int_0^\infty G(x,y,\lambda)f(y)\,dy .$$

THE INTERVAL $(-\infty,\infty)$

We now consider the case where the interval is $(-\infty,\infty)$, or there is a singularity at both ends of a finite interval. Assume $q(x)$ continuous over $(-\infty,\infty)$.

Let $\phi(x,\lambda)$ and $\theta(x,\lambda)$ be the solutions of

$$y'' + (\lambda - q(x))y = 0$$

such that

$$\phi(0) = 0 \qquad \phi'(0) = -1,$$

$$\theta(0) = 1 \qquad \theta'(0) = 0 .$$

Then $W(\phi,\theta) = 1$

By the previous theory there are functions $m_1(\lambda)$and $m_2(\lambda)$ regular in the upper half plane, such that

$$\Psi_1(x,\lambda) = \theta(x,\lambda) + m_1(\lambda)\phi(x,\lambda) \tag{2.28}$$

is $L^2(-\infty, 0)$ and

$$\Psi_2(x,\lambda) = \theta(x,\lambda) + m_2(\lambda)\phi(x,\lambda) \tag{2.29}$$

is $L^2(0,\infty)$. Then

$$W(\Psi_1,\Psi_2) = m_1(\lambda) - m_2(\lambda) .$$

As in (2.18)

$$\int_{-\infty}^{0} |\psi_1(x,\lambda)|^2 dx = \frac{\operatorname{im} m_1}{\nu}$$
$$\int_{0}^{\infty} |\psi_2(x,\lambda)|^2 dx = -\frac{\operatorname{im} m_2}{\nu} \qquad (2.30)$$

Hence $\operatorname{im}(m_1) > 0$ and $\operatorname{im}(m_2) < 0$, when $\nu > 0$.

The Green's function in this case is

$$G(x,y;\lambda) = \frac{\psi_1(y,\lambda)\psi_2(x,\lambda)}{m_2(\lambda) - m_1(\lambda)} \qquad (y \le x)$$
$$(2.31)$$
$$G(x,y;\lambda) = \frac{\psi_1(x,\lambda)\psi_2(y,\lambda)}{m_2(\lambda) - m_1(\lambda)} \qquad (y > x)$$

and

$$\Phi(x,\lambda) = -\int_{-\infty}^{\infty} G(x,y;\lambda) f(y) dy \ , \qquad (2.32)$$

where f is an arbitrary function to be expanded. The expansion will reduce to a series if both $m_1(\lambda)$ and $m_2(\lambda)$ are meromorphic functions.

We have

$$\Phi(x,\lambda) = \frac{\theta(x,\lambda) + m_2(\lambda)\phi(x,\lambda)}{m_1(\lambda) - m_2(\lambda)} \int_{-\infty}^{x} \big(\theta(y,\lambda) + m_1(\lambda)\phi(y,\lambda)\big) f(y) dy$$
$$+ \frac{\theta(x,\lambda) + m_1(\lambda)\phi(x,\lambda)}{m_1(\lambda) - m_2(\lambda)} \int_{x}^{\infty} \big(\theta(y,\lambda) + m_2(\lambda)\phi(y,\lambda)\big) f(y) dy \ .$$

There are three possibilities:

(i) At an eigenvalue λ_n , $m_1(\lambda_n) = m_2(\lambda_n) = a \neq 0$,

$$m_1(\lambda) - m_2(\lambda) \sim (\lambda - \lambda_n) b \ .$$

Then $\Phi(x,\lambda)$ has the residue

$$\frac{1}{b}\left[\theta(x,\lambda_n)+a\phi(x,\lambda_n)\right]\int_{-\infty}^{\infty}\left[\theta(y,\lambda_n)+a\phi(y,\lambda_n)\right]f(y)\,dy \; . \tag{2.33}$$

(ii) $m_1(\lambda)$ and $m_2(\lambda)$ both have simple zeros

$$m_1(\lambda) \sim a_1(\lambda-\lambda_n) \qquad m_2(\lambda) \sim a_2(\lambda-\lambda_n)$$

Then $\Phi(x,\lambda)$ has the residue

$$\frac{1}{a_1-a_2}\theta(x,\lambda_n)\int_{-\infty}^{\infty}\theta(y,\lambda_n)f(y)\,dy \; . \tag{2.34}$$

(iii) $m_1(\lambda)$ and $m_2(\lambda)$ both have simple poles

$$m_1(\lambda) \sim \frac{a_1}{\lambda-\lambda_n} \qquad m_2(\lambda) \sim \frac{a_2}{\lambda-\lambda_n}$$

Then $\Phi(x,\lambda)$ has the residue

$$\frac{a_1 a_2}{a_1-a_2}\phi(x,\lambda_n)\int_{-\infty}^{\infty}\phi(y,\lambda_n)f(y)\,dy \; . \tag{2.35}$$

The theory in this case is much the same as that already considered. A particular case of relevance in this work is the case where $q(x)$ is an even function. Then $\phi(x,\lambda)$ is an odd function of x and $\theta(x,\lambda)$ is an even function of x. It follows that if $\theta(x,\lambda) + m_2(\lambda)\phi(x,\lambda)$ is $L^2(0,\infty)$ then $\theta(x,\lambda) - m_2(\lambda)\phi(x,\lambda)$ is $L^2(-\infty,0)$. Hence $m_1(\lambda) = -m_2(\lambda)$. All eigenvalues occur under (ii) or (iii) above and each eigenfunction is either odd or even. This result will be particularly useful in the case of the Magnetohydrodynamic ballooning equation.

THE LIMIT POINT AND LIMIT CIRCLE CASES

These may be interpreted in terms of the number of linearly independent $L^2(0,\infty)$ solutions as follows.

The differential equation $y'' + (\lambda - q(x))y = 0$ has two independent solutions $\theta(x, \lambda)$ and $\phi(x, \lambda)$ where

$$\theta(0,\lambda) = 1 \qquad \theta'(0,\lambda) = 0$$

$$\phi(0,\lambda) = 0 \qquad \phi'(0,\lambda) = -1$$

The equation, when λ is not real, always has a solution (non-trivial) which is absolutely square-integrable on the interval $[0, \infty)$

$$\Psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\phi(x,\lambda)$$

$$\int_0^\infty |\Psi(x,\lambda)|^2 dx < \infty \qquad \lambda \in \mathbb{C} \qquad im\lambda \neq 0 \ .$$

The equation can be classified into two mutually exclusive cases, limit-point and limit-circle. From above with the same conditions on λ it follows that either $\theta(x,\lambda)$ and $\phi(x, \lambda)$ are both not integrable-square, or both $\theta(x,\lambda)$ and $\phi(x,\lambda)$ are integrable-square. The first is the limit-point case, the second the limit-circle case. In the limit-circle case, for each λ (real or complex) the differential equation $y'' + (\lambda - q(x))y = 0$, has only integrable-square solutions.

In fact, also, all solutions are integrable-square for real λ. In the limit-point case, there is not a single value of λ (real or complex) for which the equation has two such linearly independent solutions. One of the solutions may be square-integrable for certain real values of λ, but not both at the same point. In addition, if at some point λ ($im\lambda \neq ,0$) either the limit-point or limit-circle case holds, then, it holds for all strictly complex λ.

THE GENERAL SINGULAR CASE

In this case we remove the requirement that the Titchmarsh-Weyl co-efficient $m(\lambda)$ be meromorphic only. We know only that it is an analytic function of λ , regular in the upper half plane, and that $\operatorname{im} m(\lambda) < 0$. A consequence of this is that a continuous spectrum may also arise. We will just list some results, the proofs of which may be found in [15, Ch.3].

THE TITCHMARSH-KODAIRA FUNCTION $k(\lambda)$

Assume the interval to be $[0, \infty]$:

Define $$k(\lambda) = \lim_{\delta\to 0}\int_0^\lambda \{-\operatorname{im} m(u+i\delta)\}\,du \qquad (2.36)$$

then the expansion theorem for f(x) can be written

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\phi(x,\lambda)\,dk(\lambda)\int_0^{\infty}\phi(y,\lambda)f(y)\,dy \qquad (2.37)$$

(see [15]).

When the interval is $(-\infty, \infty)$, we have, corresponding to the three cases, (2.33), (2.34) and (2.35) the following:

$$\xi(\lambda) = \lim_{\delta\to 0}\int_0^\lambda -\operatorname{im}\left\{\frac{1}{m_1(u+i\delta)-m_2(u+i\delta)}\right\}du \qquad (2.38)$$

$$\eta(\lambda) = \lim_{\delta\to 0}\int_0^\lambda -\operatorname{im}\left\{\frac{m_1(u+i\delta)}{m_1(u+i\delta)-m_2(u+i\delta)}\right\}du \qquad (2.39)$$

$$\zeta(\lambda) = \lim_{\delta\to 0}\int_0^\lambda -\operatorname{im}\left\{\frac{m_1(u+i\delta).m_2(u+i\delta)}{m_1(u+i\delta)-m_2(u+i\delta)}\right\}du\,. \qquad (2.40)$$

We then obtain the formal expansion formula

$$\begin{aligned} \pi . f(x) = & \int_{-\infty}^{\infty} \theta(x,\lambda)\, d\xi(\lambda) \int_{-\infty}^{\infty} \theta(y,\lambda) f(y)\, dy \\ & + \int_{-\infty}^{\infty} \theta(x,\lambda)\, d\eta(\lambda) \int_{-\infty}^{\infty} \phi(y,\lambda) f(y)\, dy \\ & + \int_{-\infty}^{\infty} \phi(x,\lambda)\, d\eta(\lambda) \int_{-\infty}^{\infty} \theta(y,\lambda) f(y)\, dy \\ & + \int_{-\infty}^{\infty} \phi(x,\lambda)\, d\zeta(\lambda) \int_{-\infty}^{\infty} \phi(y,\lambda) f(y)\, dy . \end{aligned} \tag{2.41}$$

On the $[0,\infty)$ interval, we can use the $k(\lambda)$ function to determine the spectrum of the equation $y''+(\lambda-q(x))y = 0$. If $k(\lambda)$ is constant over any interval of real values of λ, then such an interval contributes nothing to the expansion formula (2.37). Thus the spectrum can be defined as the complement of the set of open intervals over which $k(\lambda)$ is constant. That is, a point at which $k(\lambda)$ is discontinuous belongs to the spectrum. This set forms the discrete spectrum.

The continuous spectrum is the set of points where $k(\lambda)$ is continuous but in the neighbourhood of which $k(\lambda)$ is not constant. The continuous spectrum may also contain points of the discrete spectrum.

TRANSFORMATION TO SCHRODINGER FORM

Throughout this chapter we have assumed that the equation is in the Schrodinger or normal form $y'' + (\lambda - q(x))y = 0$. There is no loss of generality in this assumption because the more general second order equation may always be transformed as shown below:

Given

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + (\lambda - c(x))y = 0$$

Let

$$w = \int \frac{dx}{\sqrt{a(x)}}$$

then

$$\frac{d^2y}{dw^2} + \beta(w)\frac{dy}{dw} + (\lambda - \gamma(w))y = 0$$

where $\beta(w) = [b(x) - \frac{1}{2}a'(x)]/\sqrt{a(x)}$ $\qquad \gamma(w) = c(x)$

Put $y = r(w)u$ where $r(w) = e^{-\frac{1}{2}\int \beta(w)dw}$, we obtain

$$\frac{d^2u}{dw^2} + \left\{\lambda - \frac{1}{4}\beta^2(w) - \frac{1}{2}\beta'(w) - \gamma(w)\right\}u = 0$$

which is of standard form. We assume $b'(x)$ and $a''(x)$ exist.

CHAPTER 3

EXAMPLES AND SOME PROPERTIES OF THE POTENTIAL FUNCTION q(x)

In this chapter we will illustrate the theory outlined in Chapter 2 with some examples. The examples will cover the three cases which were described there. Then we will examine the way the function q(x) can determine the spectrum of $y''+(\lambda -q(x))y = 0$.

Some theorems will be stated and a qualitative picture of q(x) as a potential function in a Schrodinger type equation will be used, to provide a physical interpretation of the ideas discussed. Examples, using the Legendre and Magneto-hydrodynamic ballooning equations will demonstrate how q(x) may allow one to determine the type of spectrum to be expected and also how that spectrum depends on the various parameters involved in q(x).

The first case we examined was the classical Sturm -Liouville one where q(x) was assumed to be continuous on some finite interval [a,b]. The problem to be solved was

$$y'' + (\lambda - q(x))y = 0$$

with solutions $\phi(x,\lambda)$ and $\chi(x,\lambda)$ where the boundary conditions are

$$\begin{aligned} \phi(a,\lambda) &= \text{Sin}\,\alpha & \phi'(a,\lambda) &= -\text{Cos}\,\alpha \\ \chi(b,\lambda) &= \text{Sin}\,\beta & \chi'(b,\lambda) &= -\text{Cos}\,\beta . \end{aligned} \tag{3.1}$$

We take the simplest case where $q(x) = 0$ on $[a,b]$, this gives us the Fourier equation

$$y'' + \lambda y = 0$$

which has two linearly independent solutions

$$\phi_0(x,\lambda) = \operatorname{Cos}\sqrt{\lambda}\, x$$
$$\chi_0(x,\lambda) = \operatorname{Sin}\sqrt{\lambda}\, x \tag{3.2}$$

and $\quad W_0(\lambda) = \sqrt{\lambda}$

Given our two solutions in (3.2) we wish to construct solutions which satisfy the boundary conditions in (3.1). Two such solutions are

$$\phi(x,\lambda) = A\phi_0(x,\lambda) + B\chi_0(x,\lambda)$$

and $$\chi(x,\lambda) = C\phi_0(x,\lambda) + D\chi_0(x,\lambda)$$

If we now apply the boundary conditions in (3.1) we will obtain the following

$$A\phi_0(a,\lambda) + B\chi_0(a,\lambda) = \operatorname{Sin}\alpha$$

$$A\phi_0'(a,\lambda) + B\chi_0'(a,\lambda) = -\operatorname{Cos}\alpha$$

and

$$C\phi_0(b,\lambda) + D\chi_0(b,\lambda) = \operatorname{Sin}\beta$$

$$C\phi_0'(b,\lambda) + D\chi_0'(b,\lambda) = -\operatorname{Cos}\beta.$$

This is a set of four equations in four unknowns; A, B, C and D. Assume $\alpha = 0$ and $\beta = 0$, then, solving, we obtain the two solutions

$$\chi(x,\lambda) = \frac{\sin\sqrt{\lambda}\,(b-x)}{\sqrt{\lambda}}$$

$$\phi(x,\lambda) = \frac{-\sin\sqrt{\lambda}\,(x-a)}{\sqrt{\lambda}} \tag{3.4}$$

and the Wronskian $\omega(\lambda)$ is

$$\omega(\lambda) = \frac{\sin\sqrt{\lambda}\,(b-a)}{\sqrt{\lambda}} \tag{3.5}$$

The zeros of $\omega(\lambda)$ are λ_n where

$$\lambda_n = \left(\frac{n\pi}{b-a}\right)^2 \qquad n = 1, 2, \ldots \tag{3.6}$$

$\lambda = 0$ is not an eigenvalue in this case because $\omega(0) \neq 0$

$$\omega'(\lambda) = \frac{(b-a)\cos\sqrt{\lambda}\,(b-a)}{2\lambda} - \frac{\sin\sqrt{\lambda}\,(b-a)}{2\lambda^{3/2}},$$

$\lambda = \lambda_n$

$$\omega'(\lambda_n) = \frac{(-1)^n(b-a)}{2\lambda_n} = \frac{(-1)^n(b-a)^3}{2n^2\pi^2}. \tag{3.7}$$

From $$\chi(x,\lambda_n) = k(\lambda_n)\phi(x,\lambda_n),$$

$$k(\lambda_n) = \frac{\chi(x,\lambda_n)}{\phi(x,\lambda_n)},$$

$$k(\lambda_n) = k_n = \frac{-\sin n\pi\left(\frac{b-x}{b-a}\right)}{\sin n\pi\left(\frac{x-a}{b-a}\right)} = (-1)^n. \tag{3.8}$$

Using these results in (2.6) we obtain the Fourier sine series

$$f(x) = \frac{2}{b-a}\sum_{n=1}^{\infty} \sin\left(n\pi \frac{x-a}{b-a}\right)\int_a^b \sin\left(n\pi \frac{y-a}{b-a}\right) f(y)\,dy .$$

If we put $\alpha = \frac{\pi}{2}$ and $b = \frac{\pi}{2}$ we obtain the cosine series

$$f(x) = \frac{1}{b-a}\int_a^b f(y)\,dy + \frac{2}{b-a}\sum_{n=1}^{\infty} \cos\left(n\pi \frac{x-a}{b-a}\right)\int_a^b \cos\left(n\pi \frac{y-a}{b-a}\right) f(y)\,dy .$$

The general Fourier series will be obtained using different boundary conditions, namely if $\Psi(x)$ is a solution of Fourier's equation which satisfies the following periodic conditions

$$\Psi(a) = \Psi(b)$$
$$\Psi'(a) = \Psi'(b) .$$

Another example this time involving Bessel's equation of order ν

$$y'' + \frac{1}{x}y' + \left(\lambda - \frac{\nu^2}{x^2}\right) y = 0 \qquad (3.9)$$

on the interval $a \leq x \leq b < \infty$ where $a > 0$, and boundary conditions as before.

Solutions of Bessel's equation (3.9) are

$J_\nu(\sqrt{\lambda}\,.\,x)$ and $Y_\nu(\sqrt{\lambda}\,.\,x)$.

Letting $y = x^{-1/2}.u$ we obtain the equation

$$u'' + \left(\lambda - \frac{\nu^2 - 1/4}{x^2}\right) u = 0 \qquad (3.10)$$

This equation is of the form (3.1) and its solutions are $\sqrt{x}J_\nu(\sqrt{\lambda}\,.\,x)$ and $\sqrt{x}Y_\nu(\sqrt{\lambda}\,.\,x)$.

Let

$$\phi_0(x,\lambda) = x^{\frac{1}{2}} J_\nu(s.x) , \qquad \chi_0(x,\lambda) = x^{\frac{1}{2}} Y_\nu(s.x)$$

where $s = \sqrt{\lambda}$. Then

$$\omega_0(\lambda) = s\, x\left[J_\nu(s.x) Y_\nu'(s.x) - Y_\nu(s.x) J_\nu'(s.x)\right] = \frac{2}{\pi} \qquad (3.11)$$

let $\alpha = 0$ and $\beta = 0$ then

$$\phi(x,\lambda) = \frac{\pi}{2}(a.x)^{\frac{1}{2}}\left[J_\nu(s.x) Y_\nu(s.a) - Y_\nu(s.x) J_\nu(s.a)\right],$$

$$\chi(x,\lambda) = \frac{\pi}{2}(b.x)^{\frac{1}{2}}\left[J_\nu(s.x) Y_\nu(s.b) - Y_\nu(s.x) J_\nu(s.b)\right],$$

and $\omega(\lambda) = \frac{\pi}{2}(a.b)^{\frac{1}{2}}\left[J_\nu(a.s) Y_\nu(bs) - Y_\nu(a.s) J_\nu(bs)\right]$.

Hence

$$\omega'(\lambda) =$$

$$-\frac{\pi}{2}(a.b)^{\frac{1}{2}}\left\{\frac{a}{2s}\left[J_\nu(bs) Y_\nu'(a.s) - J_\nu'(a.s) Y_\nu(bs)\right] + \frac{b}{2s}\left[J_\nu'(bs) Y_\nu(a.s) - J_\nu(as) Y_\nu'(bs)\right]\right\}$$

Using (3.11) we can substitute for Y_ν' and after some rearranging we obtain

$$\omega'(\lambda) = -\left\{\frac{a}{2s}\frac{J_\nu'(a.s)}{J_\nu(b.s)} + \frac{b}{2s}\frac{J_\nu'(b.s)}{J_\nu(a.s)}\right\}\omega(\lambda)$$

$$-\frac{(a.b)^{\frac{1}{2}}}{2s^2}\left\{\frac{J_\nu(bs)}{J_\nu(as)} - \frac{J_\nu(a.s)}{J_\nu(bs)}\right\}.$$

Hence if the zeros of $\omega(\lambda)$ are λ_n

$$\omega'(\lambda_n) = \frac{(a.b)^{\frac{1}{2}}}{2s_n^2} \cdot \frac{J_\nu^2(bs_n) - J_\nu^2(a.s_n)}{J_\nu(bs_n) J_\nu(a.s_n)}$$

and

$$k_n = \left(\frac{b}{a}\right)^{\frac{1}{2}} J_\nu(b.s_n) / J_\nu(a.s_n) .$$

Again using (2.6) we obtain the Fourier Bessel expansion

$$f(x) = \frac{\pi}{2}^2 \sum_{n=0}^{\infty} \frac{s_n^2 J_\nu^2(b.s_n) \,.\, x^{\frac{1}{2}}}{J_\nu^2(a,s_n) - J_\nu^2(b,s_n)}\left[J_\nu(x.s_n) Y_\nu(a.s_n) - Y_\nu(x,s_n) J_\nu(a.s_n)\right]$$

$$\times \int_a^b y^{\frac{1}{2}}\left[J_\nu(ys_n) Y_\nu(as_n) - Y_\nu(y,s_n) J_\nu(a.s_n)\right] f(y)\, dy . \qquad (3.12)$$

We will now look at an example involving the second case. That is, the case where there is a singularity at one or both ends of the x interval, but the Titchmarsh-Weyl co-efficient $m(\lambda)$ is meromorphic, leading to a purely series expansion of the arbitrary function $f(x)$.

We will again look at Bessel's equation (3.10)

$$y'' + \left(s^2 - \frac{\nu^2 - \frac{1}{4}}{x^2}\right)y = 0$$

but this time on the interval $0 < x < b$, b finite. $\lambda = s^2$ for convenience. The basic solutions of the equation are $\sqrt{x}J_{\nu}(x.s)\sqrt{x}Y_{\nu}(x.s)$ as for the last example. $\phi(x,\lambda)$ and $\theta(x,\lambda)$ are the solutions which satisfy the following boundary conditions.

$$\phi(b) = 0 \qquad \phi'(b) = -1$$

$$\theta(b) = 1 \qquad \theta'(b) = 0$$

The interval (0,b] has the singularity at the point $x = 0$ and when $\nu > \frac{1}{2}$ we are in the limit point case. This is easily shown if we put $\lambda = 0$ in equation (3.9). The solutions of this simpler equation are x^{ν} and $x^{-\nu}$ and these will be $L^2(0,b)$ when $-\frac{1}{2} < \nu < \frac{1}{2}$. Hence we will be in the limit circle case. When $\nu = 0$ the solutions 1 and $\ln x$ are also $L^2(0,b)$ and the limit circle case applies here also.

The limit point case is somewhat easier to deal with because we need to use only the boundary conditions at the regular point b and that the solution $\psi(x,\lambda) = \theta(x,\lambda) + m(\lambda).\phi(x,\lambda)$ be $L^2(0,b)$ (in this case). We will deal only with the limit point case as this will be the only one relevant to later chapters. Some examples of the limit circle case may be found in [14] and [15].

The solutions satisfying the above boundary conditions are

$$\phi(x,\lambda)=\tfrac{\pi}{2}b^{\frac{1}{2}}x^{\frac{1}{2}}[J_\nu(x.s)Y_\nu(b.s)-Y_\nu(x.s)J_\nu(b.s)]$$

and (3.13)

$$\theta(x,\lambda)=\tfrac{\pi}{2}b^{\frac{1}{2}}x^{\frac{1}{2}}s[J_\nu(x.s)Y_\nu'(b.s)-Y_\nu(x.s)J_\nu'(b.s)]+\frac{\phi(x,\lambda)}{2b}.$$

The solution which is $L^2(0,b)$ is $x^{\frac{1}{2}}J_\nu(x.s)$ and so the Titchmarsh-Weyl coefficient $m(\lambda)$ can be found by arranging that it cancel the non-$L^2(0,b)$ part of

$$\psi(x,\lambda)=\theta(x,\lambda)+m(\lambda)\phi(x,\lambda)$$

giving us some constant multiple of $x^{\frac{1}{2}}J_\nu(x.s)$. Upon doing the calculations we find that

$$m(\lambda)=-\sqrt{\lambda}\,\frac{J_\nu'(b.\sqrt{\lambda})}{J_\nu(b.\sqrt{\lambda})}-\frac{1}{2b}. \tag{3.14}$$

The eigenvalues λ_n are the poles of $m(\lambda)$ which are the zeros of $J_\nu(b\sqrt{\lambda})$. If we expand $J_\nu(b\sqrt{\lambda})$ in a Taylor series about λ_n

i.e. $J_\nu(b.\sqrt{\lambda})=J_\nu(b\sqrt{\lambda_n})+\frac{1}{2}b\lambda_n^{-\frac{1}{2}}J_\nu'(b\sqrt{\lambda_n})(\lambda-\lambda_n)+\ldots$

at λ_n $J_\nu(b\sqrt{\lambda_n})=0$. Hence the residue r_n of $m(\lambda)$ at λ_n is

$$r_n=-2\lambda_n/b$$

by inserting the Taylor expansion into (3.14). Also

$$\phi(x,\lambda_n)=\tfrac{\pi}{2}b^{\frac{1}{2}}x^{\frac{1}{2}}.J_\nu(x.\sqrt{\lambda_n})Y_\nu(x\sqrt{\lambda_n})$$

$$=\frac{-x^{\frac{1}{2}}J_\nu(x.\sqrt{\lambda_n})}{\sqrt{\lambda_n}\,b^{\frac{1}{2}}\,J_\nu'(x\sqrt{\lambda_n})}$$

using the Wronskian relation at λ_n. Hence the normalised eigenfunctions are (from (2.2))

$$\left(\frac{2\lambda_n}{b}\right)^{\frac{1}{2}} \frac{x^{\frac{1}{2}} J_\nu(x.\sqrt{\lambda_n})}{\sqrt{\lambda_n}\, b^{\frac{1}{2}} J_\nu'(b\sqrt{\lambda_n})}$$

$$= \frac{2^{\frac{1}{2}} x^{\frac{1}{2}} J_\nu(x.\sqrt{\lambda_n})}{b\, J_\nu'(b.\sqrt{\lambda_n})}$$

The Fourier-Bessel expansion will be

$$f(x) = \frac{2}{b^2}\sum_{n=1}^{\infty} \frac{x^{\frac{1}{2}} J_\nu(x.\sqrt{\lambda_n})}{\left[J_\nu'(b.\sqrt{\lambda_n})\right]^2} \int_0^b y^{\frac{1}{2}} J_\nu(y\sqrt{\lambda_n}) f(y)\, dy \,. \qquad (3.15)$$

The two problems dealt with in later chapters both have Titchmarsh-Weyl coefficients which are of the form described here. The problems are the modified Legendre equation on the interval $[0, \infty]$ and the Magnetohydrodynamic ballooning equation on $(-\infty, \infty)$.

We will now look at some examples illustrating the third case covered in Chapter 2. The Fourier equation again provides a simple example. We have

$$Y'' + \lambda Y = 0$$

on the interval $(-\infty, \infty)$ and boundary conditions at the origin.

$$\phi(0) = \operatorname{Sin}\alpha \qquad \phi'(0) = -\operatorname{Cos}\alpha$$

$$\theta(0) = \operatorname{Cos}\alpha \qquad \theta'(0) = \operatorname{Sin}\alpha$$

Then

$$\theta(x,\lambda) = \cos\alpha \cos(x\sqrt{\lambda}) + \frac{\sin\alpha \sin(x\sqrt{\lambda})}{\sqrt{\lambda}}$$

$$\phi(x,\lambda) = \sin\alpha \cos(x\sqrt{\lambda}) - \frac{\cos\alpha \sin(x\sqrt{\lambda})}{\sqrt{\lambda}}$$

If for example we choose $\alpha = \frac{\pi}{2}$ then

$$\theta(x,\lambda) = \frac{\sin(x\sqrt{\lambda})}{\sqrt{\lambda}}$$

$$\phi(x,\lambda) = \cos(x.\sqrt{\lambda}) \tag{3.16}$$

assuming $\operatorname{im}\lambda > 0$ then $e^{i\sqrt{\lambda}x}$ will be in $L^2(0.\infty)$ while the $e^{-i\sqrt{\lambda}x}$ will be in $L^2(-\infty,0)$.
The solutions $\Psi_1(x,\lambda)$ and $\Psi_2(x,\lambda)$ will be constant multiples of these. Writing

$$\Psi_1(x,\lambda) = \theta(x,\lambda) + m_1(\lambda)\phi(x,\lambda) \quad \epsilon\, L^2(-\infty,0)$$

we find that

$$m_1(\lambda) = \frac{i}{\sqrt{\lambda}} \qquad \Psi_1(x,\lambda) = \frac{i}{\sqrt{\lambda}} e^{-i\sqrt{\lambda}x},$$

similarly with $\Psi_2(x,\lambda)$ e $L^2(0,\infty)$ (3.17)

$$m_2(\lambda) = \frac{-i}{\sqrt{\lambda}} \qquad \Psi_2(x,\lambda) = \frac{-i}{\sqrt{\lambda}} e^{i\sqrt{\lambda}x}.$$

Using the expressions in Chapter 2, (2.38), (2.39) and (2.40) we have

$$\xi'(\lambda) = -\operatorname{im}\left\{\frac{1}{m_1(\lambda) - m_2(\lambda)}\right\} = \frac{\sqrt{\lambda}}{2} \qquad \lambda > 0, \ 0 \ \ \lambda < 0,$$

$$\zeta'(\lambda) = -\operatorname{im}\left\{\frac{m_1(\lambda)\, m_2(\lambda)}{m_1(\lambda) - m_2(\lambda)}\right\} = \frac{1}{2\sqrt{\lambda}} \qquad \lambda > 0, \ 0 \ \ \lambda < 0,$$

$$\eta(\lambda) = 0. \tag{3.18}$$

Hence using (2.41)

$$f(x) = \frac{1}{2\pi}\left\{ \int_0^\infty \frac{\sin(x\sqrt{\lambda})}{\sqrt{\lambda}} d\lambda \int_{-\infty}^\infty \sin(y\sqrt{\lambda}) f(y)\, dy \; + \right.$$

$$\left. + \int_0^\infty \frac{\cos(x\sqrt{\lambda})}{\sqrt{\lambda}} d\lambda \int_{-\infty}^\infty \cos(y\sqrt{\lambda}) f(y)\, dy \right\} . \qquad (3.19)$$

The ordinary form of Fouriers formula.

The case of Bessel's equation on the interval $(0,\infty)$ provides us with our final example. We have already examined the case on the interval $(0,b)$, we must now examine the case (b,∞) before dealing with the full interval $(0,\infty)$. In this latter case of course both end points are singular.

Firstly though the case (b,∞). We will take the same boundary conditions at b as for the previous Bessel's function example i.e. $\alpha = 0$. Thus we will have the same two linearly independent solutions (taking $\lambda = s^2$).

$$\phi(x,\lambda) = \frac{\pi}{2} b^{\frac{1}{2}} x^{\frac{1}{2}} \left[J_\nu(x.s) Y_\nu(b.s) - Y_\nu(x.s) J_\nu(b.s) \right],$$

$$\theta(x,\lambda) = \frac{\pi}{2} b^{\frac{1}{2}} x^{\frac{1}{2}} . s \left[J_\nu(x.s) Y_\nu'(b.s) - Y_\nu(x.s) J_\nu'(b.s) \right] + \frac{\phi(x,\lambda)}{2b} .$$

The only solution which is small as $x \to \infty$ for $\mathrm{im}(s) > 0$, is

$$H_\nu^{(1)}(x.s) = J_\nu(x.s) + i Y_\nu(x.s) .$$

Hence $\Psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\phi(x,\lambda)$ must be a multiple of this; therefore

$$m(\lambda) = -s \frac{J_\nu'(b.s) + i Y_\nu'(b.s)}{J_\nu(b.s) + i Y_\nu(b.s)} - \frac{1}{2b} = -s . \frac{H_\nu^{(1)\prime}(bs)}{H_\nu^{(1)}(bs)} - \frac{1}{2b} . \qquad (3.20)$$

If $\lambda > 0$ i.e. s real then,

$$-\operatorname{im}(\lambda) = \operatorname{im}\left\{ s.\frac{J_\nu'(bs) + iY_\nu'(bs)}{J_\nu(bs) + iY_\nu(bs)} \right\}$$

$$= \frac{J_\nu(bs)Y_\nu'(bs) - Y_\nu(bs)J_\nu'(bs)}{J_\nu^2(bs) + Y_\nu^2(bs)}.s$$

$$= \frac{2}{\pi b} \cdot \frac{1}{J_\nu^2(bs) + Y_\nu^2(b.s)} ,$$

using the Wronskian. A similar formula applies when $\lambda < 0$.

Now to the $(0, \infty)$ case. We break up the interval into the two intervals $(0,b]$ and $[b,\infty)$ and take the point b as our basic point instead of 0 as in Chapter 2. Assuming the same boundary conditions as before we obtain the same solutions i.e.

$$\phi(x,\lambda) = \frac{\pi}{2} b^{\frac{1}{2}} x^{\frac{1}{2}} \left[J_\nu(xs)Y_\nu(bs) - Y_\nu(xs)J_\nu(bs) \right]$$

$$\theta(x,\lambda) = \frac{\pi}{2} b^{\frac{1}{2}} x^{\frac{1}{2}} s\left[J_\nu(xs)Y_\nu'(bs) - Y_\nu(x.s)J_\nu'(bs) \right] + \frac{\phi(x,\lambda)}{2b}$$

where as before $\lambda = s^2$. Taking the limit point case i.e. $\nu > \frac{1}{2}$ then the solution which is $L^2(0, b)$ is $x^{\frac{1}{2}}J_\nu(xs)$ and the solution which is $L^2(b, \infty)$ is $x^{\frac{1}{2}}H_\nu^{(1)}(xs)$. Hence from (3.14)

$$m_1(\lambda) = -s\frac{J_\nu'(b.s)}{J_\nu(b.s)} - \frac{1}{2b}$$

and

$$\psi_1(x,\lambda) = \frac{\pi}{2} a^{\frac{1}{2}} x^{\frac{1}{2}} s\, J_\nu(x.s)\left[Y_\nu'(bs) - Y_\nu(bs)\frac{J_\nu'(bs)}{J_\nu(bs)} \right]$$

$$= \frac{x^{\frac{1}{2}} J_\nu(x.s)}{b^{\frac{1}{2}} J_\nu(b.s)} . \qquad (3.21)$$

and from (3.20)

$$m_2(\lambda) = -s\frac{H_\nu^{(1)\prime}(b.s)}{H_\nu^{(1)}(b.s)} - \frac{1}{2b}$$

and $$\psi_2(x,\lambda) = \left(\frac{x}{b}\right)^{\frac{1}{2}} H_\nu^{(1)}(x,\lambda) \tag{3.22}$$

$$-im\left(\frac{1}{m_1(\lambda) - m_2(\lambda)}\right) = im\left[s\left\{\frac{J_\nu'(bs)}{J_\nu(bs)} - \frac{H_\nu^{(1)'}(bs)}{H_\nu^{(1)}(bs)}\right\}\right]$$

$$= im\left\{-\frac{\pi b}{2i} J_\nu(bs) H_\nu^{(1)}(bs)\right\} = \begin{cases} \frac{\pi}{2} b J_\nu^2(bs), & s > 0 \\ 0 \quad (s = it, \ t > 0) \end{cases},$$

that is when $\lambda > 0$ $s > 0$ and real, when $\lambda < 0$ $s = it$, purely imaginary.

Hence (2.41) gives

$$f(x) = \frac{1}{2}\int_0^\infty x^{\frac{1}{2}} J_\nu(x\sqrt{\lambda})\, d\lambda \int_0^\infty y^{\frac{1}{2}} J_\nu(y\sqrt{\lambda}) f(y)\, dy, \tag{3.23}$$

because in this case $d\eta(u) = m_1(u) d\xi(u)$ and $d\zeta(u) = [m_1(u)]^2 d\xi(u)$, see [15, Ch.3].

THE SPECTRUM OF $y'' + (\lambda - q(x))y = 0$ AND ITS DEPENDENCE ON $q(x)$.

The equation $y'' + (\lambda - q(x))y = 0$ can be considered to be a one dimensional, time independent Schrodinger equation. It is well known from basic quantum mechanics that the type of spectrum obtained for the equation with given boundary conditions, will depend entirely on the function $q(x)$. This function is usually known as the potential function because of its association with the potential energy term in classical and quantum theory. Because most second order linear differential equations can be converted to Schrodinger form using the prescription in Chapter 2, we can obtain a potential function associated with that equation. With this function we can determine the spectrum of the equation.

The necessary theorems and their proofs are to be found in [15, Ch.5]. Here, we shall merely state the theorems, the conditions under which they work and apply them to our equation (the Magnetohydrodynamic ballooning equation).

$$\frac{d}{dx}\left[(1+x^2)\frac{dy}{dx}\right] + \left[\lambda + \gamma^2(1+x^2) - \frac{\mu^2}{1+x^2}\right]y = 0 \qquad (3.24)$$

As the parameters μ and γ are varied (independently) the spectrum will be shown to depend on them and on another parameter ν called the characteristic exponent see Ch. 5 & 6. For some values the spectrum changes its character quite abruptly.

Firstly though, we must obtain the potential function associated with the ballooning equation (3.24).

Using the method described in Chapter 2, we transform the equation (3.24) to

$$y'' + \left[(-\lambda - \tfrac{1}{4}) + (\mu^2 - \tfrac{1}{4})\operatorname{Sech}^2 x - \gamma^2 \operatorname{Cosh}^2 x\right]y = 0 \qquad (3.25)$$

where

$$\Lambda = (-\lambda - \tfrac{1}{4}) \qquad (3.26)$$

and

$$q(x) = \gamma^2 \operatorname{Cosh}^2 x - (\mu^2 - \tfrac{1}{4})\operatorname{Sech}^2 x \qquad (3.27)$$

We shall assume that μ^2 and γ^2 are real quantities.

The theorems determining how the spectrum depends on q(x) follow. (Note: we are dealing with the $(-\infty, \infty)$ interval)

THEOREM (i): If $q(x) \to \infty$ as $x \to \pm\infty$, then both $m_1(\lambda)$ and $m_2(\lambda)$ are meromorphic, hence the functions

$$\frac{1}{m_1(\lambda) - m_2(\lambda)}, \quad \frac{m_1(\lambda)}{m_1(\lambda) - m_2(\lambda)} \quad \frac{m_1(\lambda)m_2(\lambda)}{m_1(\lambda) - m_2(\lambda)}$$

occurring in the $(-\infty, \infty)$ theory, (see Ch.2) are also meromorphic. This implies that the expansion is a series and the spectrum discrete.

THEOREM (ii): If $q(x)$ is $L(-\infty, \infty)$, there is a continuous spectrum on $(0, \infty)$, with a point spectrum (which may be null) on $(-\infty, 0)$. These are proved in [15, Ch. 5].
Before these theorems are applied however to our potential function $q(x)$, it would be useful to know how its behaviour depends on the parameters μ and γ.

$$q(x) = \gamma^2 \operatorname{Cosh}^2 x - (\mu^2 - 1/4) \operatorname{Sech}^2 x \quad .$$

This has stationary points at the origin and at

$$x_0^{\pm} = \ln\left(\left(\frac{1/4 - \mu^2}{\gamma^2}\right)^{1/4} \pm \sqrt{\left(\frac{1/4 - \mu^2}{\gamma^2}\right)^{1/2} - 1} \right) \quad . \tag{3.28}$$

These stationary points will be real when $\gamma^2 > 0$ and $\mu^2 < \frac{1}{4} - \gamma^2$ and when $\gamma^2 < 0$ and $\mu^2 > \frac{1}{4} - \gamma^2$.
For all other values of μ and γ the only stationary point will be at the origin. When $\gamma^2 > 0$ it will be a local minimum and when $\gamma^2 < 0$ it will be a local maximum. Its height above or below the x axis is given by (in all cases)

$$q(0) = \gamma^2 + 1/4 - \mu^2 \quad , \tag{3.29}$$

The various regions described above are illustrated on the following graph of μ^2 against γ^2 (see fig. 1). The shaded regions are where (3.28) has real solutions. Graphs of q(x) for typical values of μ^2 and γ^2 in the various regions follow. They serve to illustrate how q(x) behaves as the two parameters are varied.

We may now investigate the type of spectrum to be expected as μ^2 and γ^2 are varied. The first case we take is that of $\gamma^2 > 0$ and $\mu^2 \geq \frac{1}{4} - \gamma^2$. q(x) has the form illustrated in fig. 2. As one can see this has the shape of a classical "potential well" and so, qualitatively at least, one would expect bound states to occur, i.e. a discrete spectrum. $q(x) = O(e^{2|x|})$ as $x \rightarrow \pm\infty$ and so satisfies the requirements of theorem (i). Hence a discrete spectrum does in fact exist (over the whole real λ line).

The second case we can examine is that when $\gamma^2 = 0$. See fig 3. In this case

$$q(x) = -(\mu^2 - \tfrac{1}{4})\operatorname{Sech}^2(x) \tag{3.30}$$

and the equation reduces to the associated Legendre equation with imaginary argument. It is obvious that q(x) is $L(-\infty,\infty)$ and so theorem (ii) applies. Hence we can conclude that a continuous spectrum exists for $\Lambda = -(\lambda + \frac{1}{4})$ on $(0,\infty)$ and a discrete one bounded below on $(-\infty,0)$. For then, the continuous spectrum exists for $(-\infty, -\frac{1}{4})$ and the finite discrete spectrum exists on the interval $(-\frac{1}{4}, \infty)$. We examine this case ($\gamma^2 = 0$) in detail in Chapter 4.

When $\gamma^2 < 0$ (see fig. 4), $q(x) \rightarrow -\infty$ as $x \rightarrow \pm\infty$ and a "potential well" no longer exists. It seems likely then that there is only a continuous spectrum in this case.

A plot of $\mu**2$ against $\gamma**2$

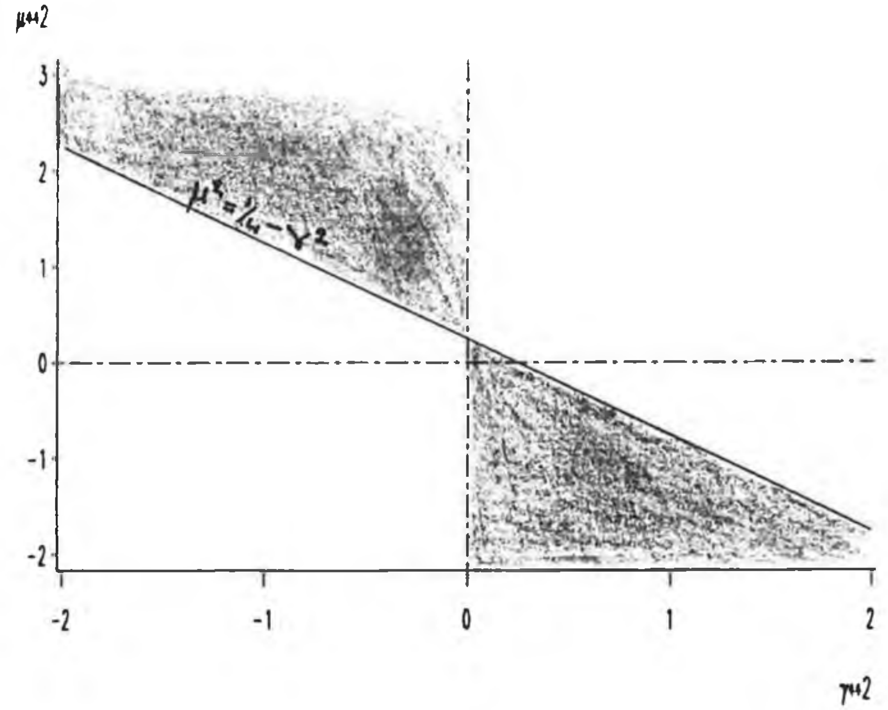

FIG 1

The potential function has two local maxima or minima and a local min. or max. when the parameter values lie in the shaded regions

A plot of the potential function

q(x)= γ**2Cosh(x)**2 −(μ**2−1/4)Sech(x)**2

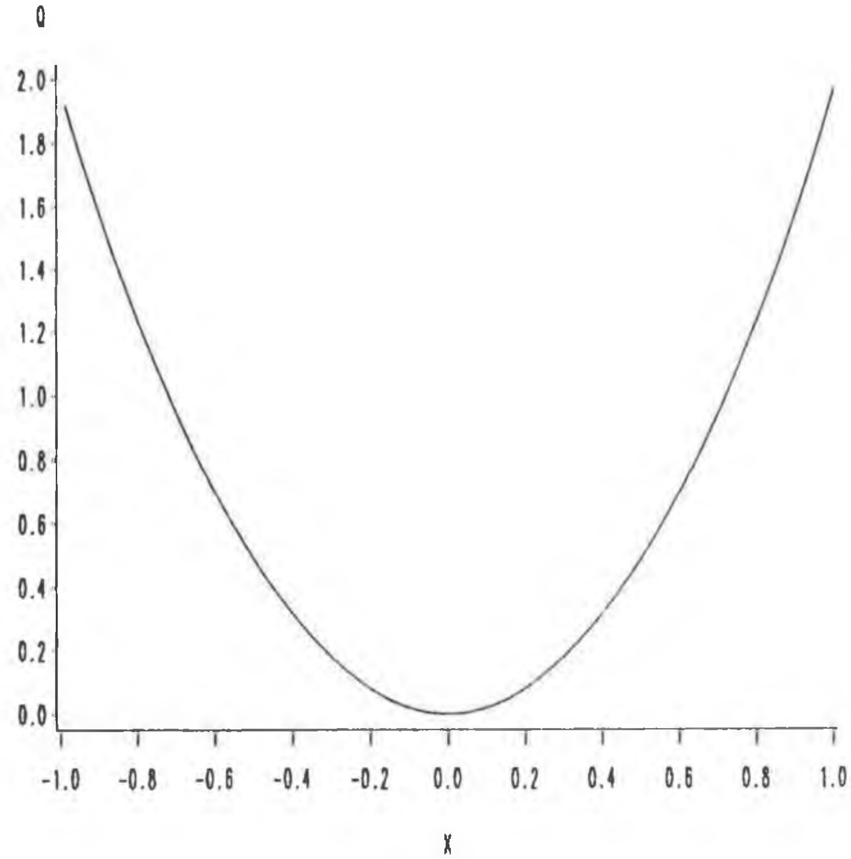

FIG 2

γ**2 = 1 and μ**2 = 1.25

A plot of the potential function
q(x)= γ**2Cosh(x)**2 -(μ**2-1/4)Sech(x)**2

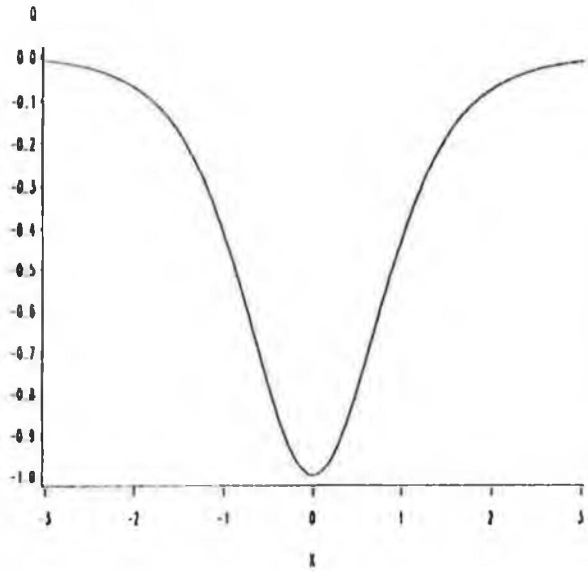

FIG 3

γ**2 = 0 and μ**2 = 1 25

A plot of the potential function
q(x)= γ**2Cosh(x)**2 -(μ**2-1/4)Sech(x)**2

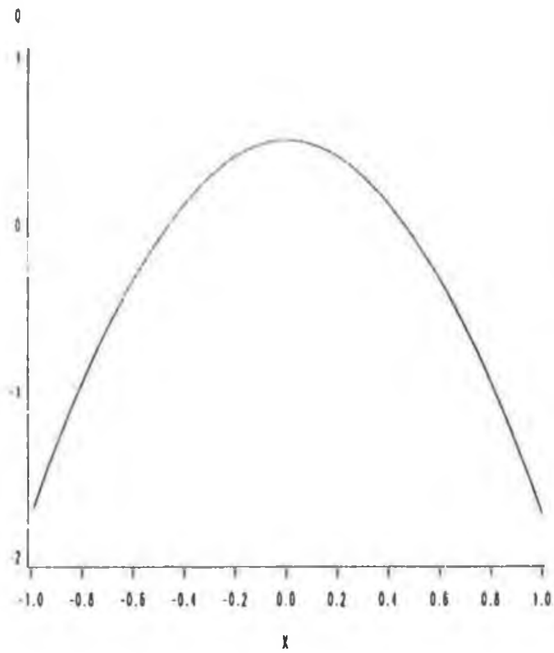

FIG 4

γ**2 = -1 and μ**2 = -1 25

A plot of the potential function

q(x)= γ**2Cosh(x)**2 -(μ**2-1/4)Sech(x)**2

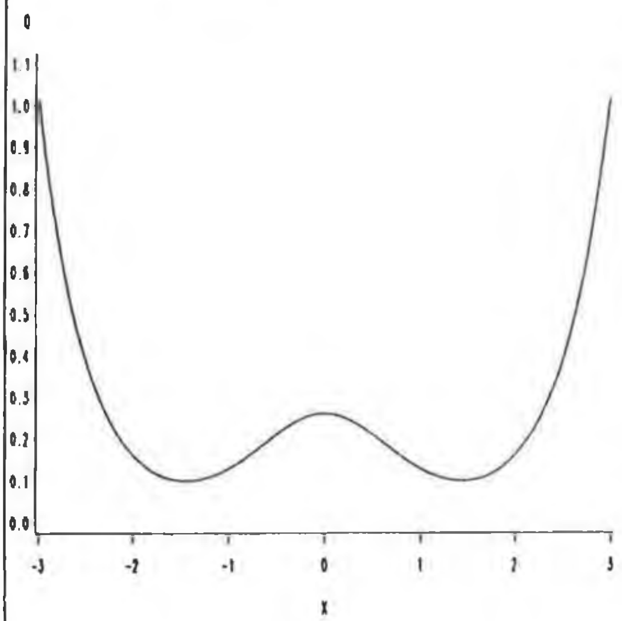

FIG 5

γ**2=.01 and μ**2=0

A plot of the potential function

q(x)= γ**2Cosh(x)**2 -(μ**2-1/4)Sech(x)**2

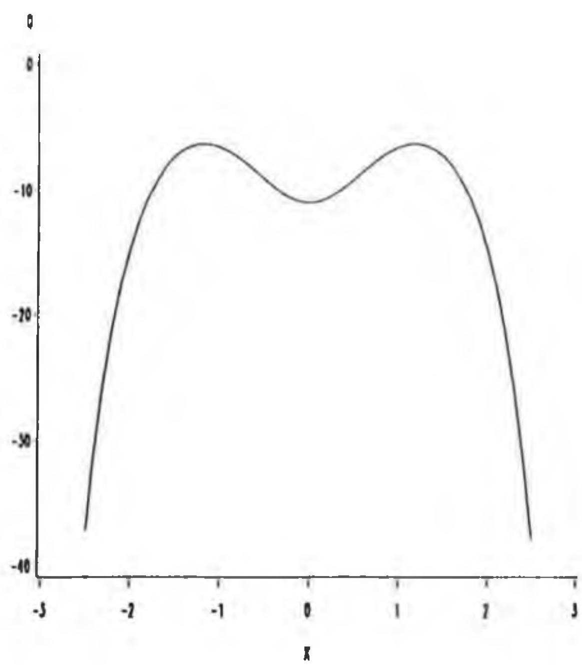

FIG 6

γ**2 = -1 and μ**2 = 10.25

CHAPTER 4

DETERMINATION OF THE EIGENVALUES IN THE CASE $\gamma = 0$

In this chapter the special case where the parameter γ equals zero is examined. The Titchmarsh-Weyl co-efficient will be constructed and from it the eigenvalue and eigen-functions explicitly determined.

Our boundary value problem is:

$$(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - \left[\lambda + \gamma^2(1+x^2) - \frac{\mu^2}{1+x^2}\right]y = 0 \qquad (4.1)$$

$y(x) \to 0$ as $x \to \pm\infty$ $\qquad x \in R$.

With $\gamma = 0$ this reduces to the modified Legendre equation with the same boundary conditions.

$$(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - \left[\lambda - \frac{\mu^2}{1+x^2}\right]y = 0 \qquad (4.2)$$

$y(x) \to 0$ as $x \to \pm\infty$ $\qquad x \in R$.

The properties of Legendre's equation and its solutions have been well documented (see for ex. [5], [9] or [16]). We obtain the modified equation by making the substitution

$$z = ix \qquad x \in R$$

in Legendres equation

$$(1-z^2)\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + \left[\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right]y = 0 \qquad (4.3)$$

where $\lambda = \nu(\nu+1)$ in this case. We will obtain the solutions of (4.3) firstly and then use the above subsitution to find the solutions to the modified equation (4.2).

THE SPECTRUM OF THE MODIFIED LEGENDRE EQUATION

By using the substitution

$$x = \mathrm{Sinh}(u), \qquad u \in R$$
$$y = \Psi / \sqrt{\mathrm{Cosh}(u)},$$

i.e. we can transform equation (4.2) to Schrodinger form, see [15, p. 22], also Chapter 3:

$$\frac{d^2\Psi}{du^2} + [\Lambda - q(u)]\Psi = 0 \tag{4.3a}$$

In the case of the modified Legendre equation

$$\Lambda = -\lambda - 1/4 \tag{4.4}$$

$$q(u) = -(\mu^2 - 1/4)\mathrm{Sech}^2 u \tag{4.5}$$

q(u) in this case is $L(0, \infty)$ and we can conclude that λ will have a continuous spectrum on $(-\infty, -1/4)$ and a finite discrete spectrum on the interval $(-1/4, \infty)$ (see Ch.3). We will see that the position of eigenvalues and their number depends on the parameter μ.

THE SOLUTIONS OF LEGENDRE'S EQUATION

(4.3) is a linear second order differential equation with three regular singular points at ∓ 1 and ∞. They have indices $\pm \mu/2$ at ∓ 1 and $-\nu$ and $\nu+1$ at ∞.

Homogeneous linear equations with three regular singular points are all forms of Riemann's equation (or Papperitz equation), see [5], [8], [9] or [16].

The symbol

$$\Psi = P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ a_1 & a_2 & a_3 & z \\ b_1 & b_2 & b_3 & \end{Bmatrix} \tag{4.6}$$

represents the complete set of solutions of Riemann's equation where the singularities are at $z=z_n (n = 1,2,3)$ and the constants a_n, b_n are the exponents belonging to $z=z_n$.

It was shown by B. Riemann (1857) that

$$\left(\frac{z-z_1}{z-z_2}\right)^{\rho}\cdot\left(\frac{z-z_3}{z-z_2}\right)^{\sigma} P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ a_1 & a_2 & a_3 & z \\ b_1 & b_2 & b_3 & \end{Bmatrix}$$

$$= P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ a_1+\rho & a_2-\rho-\sigma & a_3+\sigma & z \\ b_1+\rho & b_2-\rho-\sigma & b_3+\sigma & \end{Bmatrix} \tag{4.7}$$

and

$$P\begin{Bmatrix} \zeta_1 & \zeta_2 & \zeta_3 & \\ a_1 & a_2 & a_3 & \zeta \\ b_1 & b_2 & b_3 & \end{Bmatrix} = P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ a_1 & a_2 & a_3 & z \\ b_1 & b_2 & b_3 & \end{Bmatrix} \tag{4.8}$$

where

$$\zeta = \frac{Az+B}{Cz+D} \quad , \qquad \zeta_n = \frac{Az_n+B}{Cz_n+D} \tag{4.9}$$

A, B, C, D are constants such that AD-BC $\neq$ 0.

We can combine the expressions (4.7) and (4.8) to get

$$P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ a_1 & a_2 & a_3 & z \\ b_1 & b_2 & b_3 & \end{Bmatrix}$$

$$= \left(\frac{z-z_1}{z-z_2}\right)^{-\rho} \cdot \left(\frac{z-z_3}{z-z_2}\right)^{-\sigma} \cdot P\begin{Bmatrix} \zeta_1 & \zeta_2 & \zeta_3 & \\ a_1+\rho & a_2-\rho-\sigma & a_3+\sigma & \zeta \\ b_1+\rho & b_2-\rho-\sigma & b_3+\sigma & \end{Bmatrix} \tag{4.10}$$

where

$$\zeta = \frac{Az+B}{Cz+D} \quad , \qquad \zeta_n = \frac{Az_n+B}{Cz_n+D}$$

Hence we can express the solutions of any Riemann-type equation in terms of the solutions of any other Riemann -type equation.

If we choose our singularities ζ_n to be at the standard points 0, ∞ and 1, and choose ρ and σ so that two of the indices are zero we can express the solutions of our general equation as follows:

$$P\begin{Bmatrix} z_1 & z_2 & z_3 & \\ a_1 & a_2 & a_3 & z \\ b_1 & b_2 & b_3 & \end{Bmatrix}$$

$$= \left(\frac{z-z_1}{z-z_2}\right)^{a_1} \cdot \left(\frac{z-z_3}{z-z_2}\right)^{a_3} \cdot P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a_1+a_2+a_3 & 0 & \zeta \\ b_1-a_1 & b_2+a_1+a_3 & b_3-a_3 & \end{Bmatrix}. \tag{4.11}$$

Solving for A, B, C and D

$$\frac{Az_1+B}{Cz_1+D} = 0 \Rightarrow Az_1+B = 0 \Rightarrow z_1 = -B/A \quad (i)$$

$$\frac{Az_2+B}{Cz_2+D} = \infty \Rightarrow Cz_2+D = 0 \Rightarrow z_2 = -D/C \quad (ii) \tag{4.12}$$

$$\frac{Az_3+B}{Cz_3+D} = 1 \Rightarrow (C-A)z_3 = B-D \Rightarrow z_3 = \frac{B-D}{C-A} \quad (iii)$$

and choosing A = 1 we find that

$$\zeta = \left(\frac{z-z_1}{z-z_2}\right)\left(\frac{z_3-z_2}{z_3-z_1}\right) \tag{4.13}$$

If in the original equation $z_2 = \infty$, then from (4.12) (ii) C = 0 and again choosing A = 1 we obtain

$$\zeta = \frac{z-z_1}{z_3-z_1} \tag{4.14}$$

Our solutions now take the form:

$$P\begin{Bmatrix} z_1 & \infty & z_3 & \\ a_1 & a_2 & a_3 & z \\ b_1 & b_2 & b_3 & \end{Bmatrix}$$

$$= \zeta^{a_1}(\zeta-1)^{a_3}.P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & a_1+a_2+a_3 & 0 & \zeta \\ b_1-a_1 & b_2+a_1+a_3 & b_3-a_3 & \end{Bmatrix} \tag{4.15}$$

where ζ is as in (4.14)

The equation with its singularities at the standard points is the hypergeometric equation

$$\zeta(\zeta-1)F'' + [(a+b+1)\zeta - c]F' + abF = 0 . \qquad (4.16)$$

It is quite usual therefore, to express solutions of the Riemann equation in terms of the solutions of (4.16) which are

$$F = P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0 & \zeta \\ 1-c & b & c-a-b & \end{matrix}\right\} \qquad (4.17)$$

The two independent solutions at the origin are

$$u_1 = F(a, b; |c| \zeta)$$

and

$$u_2 = \zeta^{1-c} F(b-c+1, a-c+1; |2-c| \zeta)$$

where F is the gauss hypergeometric function defined by the series

$$F(a,b|c|x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n x^n}{n!(c)_n} \qquad (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$$

$$\text{in } |x| < 1 .$$

The solutions of (4.3) can be expressed in Riemann form as follows:

$$y = P\left\{\begin{matrix} 1 & \infty & -1 & \\ \mu/2 & -\nu & \mu/2 & z \\ -\mu/2 & \nu+1 & -\mu/2 & \end{matrix}\right\} . \qquad (4.18)$$

Using (4.15) we can express these solutions in terms of the hypergeometric equations solutions. Thus

$$y_1 = 2^{-\mu}(z-1)^{\mu/2}.(z+1)^{\mu/2} P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & \mu-\nu & 0 & \tfrac{1}{2}(1-z) \\ -\mu & \mu+\nu+1 & -\mu & \end{Bmatrix} \qquad (4.19)$$

Comparing the p equation in (4.19) with that in (4.17) we find that one solution of Legendres equation at +1 is defined to be [9, page 170]

$$P_\nu^{-\mu}(z) = 2^{-\mu}(z^2-1)^{\mu/2}.F\left(\mu-\nu,\ 1+\mu+\nu \,\middle|\, 1+\mu \,\middle|\, \tfrac{1}{2}(1-z)\right) \qquad (4.20)$$

where $a = \mu - \nu$, $b = \mu + \nu + 1$, $c = 1 + \mu$. The second independent solution at +1 is

$$P_\nu^{\mu}(z) = (z+1)^{\mu/2}.(z-1)^{-\mu/2}.F\left(-\nu,\ \nu+1, \,\middle|\, 1-\mu \,\middle|\, \tfrac{1}{2}(1-z)\right) \quad (4.21)$$

for μ not an integer. From (4.20) it is also obvious that

$$P_\nu^{\mu}(z) = 2^{\mu}(z^2-1)^{-\mu/2}.F\left(-\mu-\nu,\ 1+\nu-\mu \,\middle|\, 1-\mu \,\middle|\, \tfrac{1}{2}(1-z)\right). \quad (4.22)$$

These solutions are known as Legendres function of the first kind.

The Legendre function of the second kind is defined from one of the solutions about the point at infinity

$$Q_\nu^{\mu}(z) = 2^{\nu}\Gamma(\nu+1)\frac{(z+1)^{\mu/2}}{(z-1)^{\mu/2+\nu+1}}F\left(\nu+1,\ 1+\nu+\mu \,\middle|\, 2\nu+2 \,\middle|\, \frac{2}{1-z}\right) \quad (4.23)$$

See [9, page 170, 12.07]

The properties of the Legendre functions are well documented. Some of these are listed in later chapters of this work where necessary. These properties and many more are to be found in the references already quoted.
Our objective is to construct the Titchmarsh-Weyl co-efficient. In order to do this we need solutions to Legendre's equation $\theta(z)$ and $\phi(z)$, which satisfy the boundary conditions.

$$\theta(0) = 1 \qquad \theta'(0) = 0$$

$$\phi(0) = 0 \qquad \phi'(0) = -1$$

Such that

$$\psi_2(z) = \theta(z) + m_2(\lambda)\phi(z)$$

is $L^2(0,\infty)$. $m_2(\lambda)$ is the Titchmarsh-Weyl co-efficient. This is because q(u) is an even function of u (see (4.5)) and hence $m_1(\lambda) = -m_2(\lambda)$. Also $\theta(z)$ is an even function of z while $\phi(z)$ is an odd function, (see Chapter 2). Neither of the two solutions defined so far satisfy the boundary conditions above. But from the analytic continuation of the hypergeometric functions [5, Sn 2.10] we can express $P_\nu^\mu(z)$ in terms of two solutions which do satisfy the boundary conditions, namely

$$P_\nu^\mu(ix) = (1+x^2)^{-\mu/2}\left[A F_1(-x^2) + ixB F_2(-x^2)\right], \qquad (4.24)$$

where A & B are constants and z = ix, also

$$F_1(-x^2) = F(-\tfrac{1}{2}\nu - \tfrac{1}{2}\mu, \tfrac{1}{2} + \tfrac{1}{2}\nu - \tfrac{1}{2}\mu \,|\, \tfrac{1}{2} \,|\, -x^2)$$

$$F_2(-x^2) = F(\tfrac{1}{2} - \tfrac{1}{2}\nu - \tfrac{1}{2}\mu, 1 + \tfrac{1}{2}\nu - \tfrac{1}{2}\mu \,|\, \tfrac{3}{2} \,|\, -x^2) \tag{4.25}$$

[5, Sn 3.2 (22)]. Applying the boundary conditions to (4.24) gives us:

$$\text{(a)}\quad \Theta(x) = (1+x^2)^{-\mu/2} . F(-\tfrac{1}{2}\nu - \tfrac{1}{2}\mu, \tfrac{1}{2} + \tfrac{1}{2}\nu - \tfrac{1}{2}\mu \,|\, \tfrac{1}{2} \,|\, -x^2)$$

$$\text{(b)}\quad \phi(x) = -x(1+x^2)^{-\mu/2} . F(\tfrac{1}{2} - \tfrac{1}{2}\nu - \tfrac{1}{2}\mu, 1 + \tfrac{1}{2}\nu - \tfrac{1}{2}\mu \,|\, \tfrac{3}{2} \,|\, -x^2) \tag{4.26}$$

Note that $\Theta(x)$ is an even function of x and $\phi(x)$ is odd as expected. These two functions are expressed in terms of hypergeometric functions which converge as $x \to 0$.
In order to construct the Titchmarsh-Weyl co-efficient we must express $\Theta(x)$ and $\phi(x)$ in terms of hypergeometric functions which coverge as $x \to \infty$.
The analytic continuation of $F(a, b|c|z)$ as $z \to \infty$ is

$$F(a,b|c|z) = B_1(-z)^{-a} F(a, 1-c+a \,|\, 1-b+a \,|\, 1/z)$$

$$+ B_2(-z)^{-b} F(b, 1-c+b \,|\, 1-a+b \,|\, 1/z) ,$$

$$B_1 = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} , \qquad B_2 = \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}$$

[5, 2.10, (2)]. Hence

$$F_1(x) = B_1(-x)^{\nu+\mu} F\left(-\tfrac{1}{2}\nu - \tfrac{1}{2}\mu, \tfrac{1}{2} - \tfrac{1}{2}\nu - \tfrac{1}{2}\mu \,\middle|\, \tfrac{1}{2} - \nu \,\middle|\, \tfrac{-1}{x^2}\right)$$

$$+ B_2(-x)^{-1-\nu+\mu} F\left(\tfrac{1}{2} + \tfrac{1}{2}\nu - \tfrac{1}{2}\mu, 1 + \tfrac{1}{2}\nu - \tfrac{1}{2}\mu \,\middle|\, \tfrac{3}{2} + \nu \,\middle|\, \tfrac{-1}{x^2}\right)$$

$$B_1 = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\nu)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu)}, \tag{4.27}$$

$$B_2 = \frac{\Gamma(\frac{1}{2})\Gamma(-\frac{1}{2}-\nu)}{\Gamma(-\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(\frac{1}{2}\nu-\frac{1}{2}\mu)} .$$

Therefore as $x \to \infty$

$$\theta(x) \sim \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\nu)(-x)^{\nu}}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu)} + O(x^{\nu-2})$$

$$+ \frac{\Gamma(\frac{1}{2})\Gamma(-\frac{1}{2}-\nu)(-x)^{-1-\nu}}{\Gamma(-\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(\frac{1}{2}\mu-\frac{1}{2}\nu)} + O(-x^{-\nu-3}). \tag{4.28}$$

Similarly, using the same transformation:

$$\phi(x) \sim -\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2}+\nu)(-x)^{\nu}}{\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(1+\frac{1}{2}\nu+\frac{1}{2}\mu)} + O(-x^{\nu-2})$$

$$- \frac{\Gamma(\frac{3}{2})\Gamma(-\frac{1}{2}-\nu)(-x)^{-1-\nu}}{\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(\frac{1}{2}-\frac{1}{2}\nu+\frac{1}{2}\mu)} + O(-x^{-\nu-3}). \tag{4.29}$$

For a discrete spectrum $\lambda > -\frac{1}{4}$ and that implies that $\nu > -\frac{1}{2}$. In that case the x^{ν} term is not $L^2(0,\infty)$. Hence in the expression for $\Psi_2(x)$, which is $L^2(0,\infty)$, the terms containing x^{ν} in $\phi(x)$ and $\Theta(x)$ must cancel each other. We have therefore:

$$\Psi_2(x) = \theta(x) + m_2(\lambda)\phi(x)$$

where we have eliminated the non - L^2 terms by choosing

$$m_2(\lambda) = \frac{2\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(1+\frac{1}{2}\nu+\frac{1}{2}\mu)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu)} \quad . \tag{4.30}$$

From the zeros and poles of $m_2(\lambda)$ we can find the eigen-values of $\Theta(x)$ and $\phi(x)$ respectively.

THE EIGENVALUES AND EIGENFUNCTIONS OF $\Theta(x)$

We will denote $\Theta(x)$ by $\Theta_\nu^\mu(x)$. The eigenvalues of $\Theta_\nu^\mu(x)$ are those values of ν which make $m(\lambda)$ zero. We observe that $m(\lambda)$ is zero at the poles of $\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu)$ or $\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu)$.

The poles of $\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu)$ occur where

$$\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu = -n \qquad n = 0, 1, 2, ...$$

or $$\nu = -2n-\mu-1$$

But when $\nu > -\frac{1}{2}$ that implies that

$$-2n-\mu-1 > -\frac{1}{2}$$

or $n \leq \left[\frac{-1}{2}(\mu+\frac{1}{2})\right]$, where [ ] denotes the integer part. For n to be positive $\mu < -\frac{1}{2}$. If we assume $\mu > 0$, then this gamma function will not have any poles. On the other hand the second gamma function will have poles when $\mu > \frac{1}{2}$. Poles occur when

$$\nu = -2n+\mu-1 \ , \qquad n = 0, 1, 2, ...$$

or $$n \leq \left[\frac{1}{2}(\mu-\frac{1}{2})\right] ,$$

where square brackets again denote the integer part. We see therefore that the even eigenvalues λ_n are given by the expression

$$\lambda_n = (|\mu| - 2n)(|\mu| - (2n+1)) \quad . \tag{4.31}$$

We observe from the foregoing analysis that when $|\mu| < \frac{1}{2}$ there is no discrete spectrum. When $|\mu| > \frac{1}{2}$ $\Theta^{\mu}_{\nu}(x)$ will have j even eigenvalues if

$$\frac{4j+1}{2} < \mu < \frac{4j+5}{2} \qquad j = 0,\ 1,\ 2, \ldots .$$

The case where $\mu = \pm\frac{1}{2}$ reduces equation (4.3a) to Fourier's equation. We know from the previous chapter that this equation has a continuous spectrum over the whole real λ axis.

THE EIGENFUNCTIONS $\Theta^{\mu}_{n}(x)$:

We note that the n subscript denotes the eigenfunction associated with the n^{th} eigenvalue which are known to be simple. We observe from the previous section that the n^{th} eigenfunction is given by

$$\Theta^{\mu}_{n}(x) = (1+x^2)^{-\mu/2} . F(n-\mu+\tfrac{1}{2}, -n \,|\, \tfrac{1}{2} \,|\, -x^2) \tag{4.32}$$

now

$$F(n-\mu+\tfrac{1}{2}, -n \,|\, \tfrac{1}{2} \,|\, -x^2) = \sum_{k=0}^{\infty} \frac{(n-\mu+\frac{1}{2})_k (-n)_k (-x^2)^k}{k!\, (\frac{1}{2})_k} \tag{4.33}$$

where as usual, $(a)_k = \Gamma(a + k)/\Gamma(a)$. What is the limit of $(a)_k$ as $a \to -n$ (n an integer)? There are three cases; $n < k$, $n = k$ and $n > k$. When $n < k$ it is easy to see that $(-n)_k = 0$.

When $n = k$ we have using [5, Sn 1.2, (3)]

$$(-k)_k = (-1)^k k!$$

When $n > k$ we see by putting $a = -n$ in the expression

$$\Gamma(a+k)/\Gamma(a) = (a+k)(a+k-1)(a+k-2)\ldots(a+2)(a+1)$$

that $(-n)_k = (-1)^k n!/k!$

We conclude from this, that the infinite series in (4.33) will be truncated after the n^{th} term. Hence $\Theta_n^\mu(x)$ is a polynomial of degree $2n$, multiplied by the factor $(1 + x^2)^{-\mu/2}$.

We give some examples:-

$n = 0$ $\qquad \lambda_0 = \mu(\mu - 1)$

$$\Theta_0^\mu(x) = (1 + x^2)^{-\mu/2}$$

$n = 1$ $\qquad \lambda_1 = (\mu - 2)(\mu - 3)$

$$\Theta_1^\mu(x) = (1 + x^2)^{-\mu/2} \cdot (1 - (2\mu - 1)x^2) .$$

THE EIGENVALUES AND EIGENFUNCTIONS OF $\phi(x)$

We let $\phi(x) = \phi_\nu^\mu(x)$ as in the previous section. The eigenvalues of $\phi_\nu^\mu(x)$ are those values of ν which make $m_2(\lambda)$ singular. The poles of $m_2(\lambda)$ occur at the poles of $\Gamma(1 + \frac{1}{2}\nu - \frac{1}{2}\mu)$ or $\Gamma(1 + \frac{1}{2}\nu + \frac{1}{2}\mu)$.

The poles of $\Gamma(1+\frac{1}{2}\nu+\frac{1}{2}\mu)$ occur where

$$\nu = -\mu - 2n - 2 \qquad n = 0, 1, 2, \ldots.$$

and for a discrete spectrum $\nu > -\frac{1}{2}$. Hence

$$n \leq \left[\tfrac{1}{2}(\mu + 3/2)\right]$$

and poles will exist when $\mu < -3/2$.

The poles of $\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu)$ occur where

$$\nu = \mu - 2n - 2 \qquad n = 0, 1, 2, \ldots.$$

and so $n \leq [\frac{1}{2}(\mu - 3/2)]$.

Thus poles will exist when $\mu > 3/2$.

Hence from the expression $\lambda = \nu(\nu+1)$ we obtain the odd eigenvalues.

$$\lambda_n = (|\mu| - (2n+1))(|\mu| - (2n+2)) \qquad n = 0, 1, 2, \ldots [\tfrac{1}{2}(|\mu| - \tfrac{1}{2})].$$

We note that there will be no odd eigenvalues when $|\mu| < 3/2$ and that there will be j odd eigenvalues when

$$\frac{4j+3}{2} < \mu < \frac{4j+7}{2} \qquad j = 0, 1, 2, \ldots.$$

We can conclude that the discrete spectrum of (4.2) will be null when $|\mu| \leq \frac{1}{2}$.

THE ODD EIGENFUNCTIONS $\phi_n^\mu(x)$

$\phi_n^\mu(x)$ denotes the n^{th} eigenfunction. From (4.26) the eigenfunctions are:

$$\phi_\nu^\mu(x) = -x(1+x^2)^{-\mu/2} \cdot F(n+1-\mu, -n | 3/2 | -x^2). \qquad (4.33)$$

Following the same argument as for $\Theta_n^\mu(x)$, the hypergeometric function will be truncated to a polynomial of even power. Hence $\phi_n^\mu(x)$ will be a polynomial of degree 2n+1, multiplied by the factor $(1 + x^2)^{-\mu/2}$.

Again we give some examples:-

$n = 0$ $\qquad \lambda_0 = (\mu-1)(\mu-2)$,

$$\phi_0^\mu(x) = -x(1+x^2)^{-\mu/2},$$

$n = 1$ $\qquad \lambda_1 = (\mu-3)(\mu-4)$,

$$\phi_1^\mu(x) = -(1+x^2)^{-\mu/2}\left(x - \tfrac{2}{3}(\mu-2)x^3\right).$$

AN ALTERNATIVE METHOD FOR FINDING THE EIGENVALUES OF Θ_ν^μ AND ϕ_ν^μ

If we transform $F_1(-x^2)$ and $F_2(-x^2)$ of (4.25) using the following transformation

$$F(a,b,|c|z) = (1-z)^{-b}.\, F(b, c-a|c|\tfrac{z}{z-1}) \qquad z \in \mathbb{C}$$

and then let $z = ix$, $x \in R$ [3, Sn. 2.10 (6)], we will obtain the following expressions:

$$F_1(-x^2) = (1+x^2)^{-\frac{1}{2}(1+\nu-\mu)} F(\tfrac{1}{2}+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu, \tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu|\tfrac{1}{2}|\tfrac{x^2}{1+x^2}),$$

$$F_2(-x^2) = (1+x^2)^{-\frac{1}{2}(2+\nu-\mu)} F(1+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu, 1+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu|\tfrac{3}{2}|\tfrac{x^2}{1+x^2}). \qquad (4.34)$$

As $x \to \infty$, $\frac{x^2}{1+x^2} \to 1$ in both hypergeometric functions on the right hand side of (4.34).

We know that F(a,b, c 1) will converge absolutely if $\mathrm{Re}(c-a-b) > 0$ for (4.34) if $\nu < -\frac{1}{2}$ (assuming ν to be real). But, for a discrete spectrum, $\nu > -\frac{1}{2}$, which means that both hypergeometric functions in (4.34) will diverge, unless, these infinite series terminate. We can find the eigenvalues of $\Theta_\nu^\mu(x)$ as follows:-

$$F\left(\tfrac{1}{2}+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu,\ \tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu \,\middle|\, \tfrac{1}{2} \,\middle|\, \frac{x^2}{1+x^2}\right) =$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu\right)_n . \left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu\right)_n \left(\frac{x^2}{1+x^2}\right)^n}{\left(\frac{1}{2}\right)_n . n!} \quad . \qquad (4.35)$$

Now this series will terminate if

(i) $\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu\right)_n = 0$

(ii) $\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu\right)_n = 0$

In other words if

(i) $$\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu+n\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu\right)} = 0$$

(ii) $$\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu+n\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu\right)} = 0$$

The left hand side of (i) equals:

$$\frac{\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu+n\right)\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu+n-1\right)\ldots\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu\right)}$$

and this will be equal to zero if

$$\nu = \mu-1,\ \mu-3,\ \ldots,\ \mu-2n-1 \qquad , \quad n \text{ integer}.$$

Then $\nu > -\frac{1}{2}$ implies that

$$n \leq \left[\tfrac{1}{2}(\mu-\tfrac{1}{2})\right] \quad , \quad \mu > \tfrac{1}{2} \quad .$$

Similarly from (ii):

$$\nu = -\mu-1,\ \mu-3,\ \ldots,\ -\mu-2n-1 \quad .$$

$\nu > -\frac{1}{2}$ implies that

$$n \leq -\left[\tfrac{1}{2}(\mu+\tfrac{1}{2})\right] \quad , \quad \mu < -\tfrac{1}{2} \quad .$$

Hence the even eigenvalues are:-

$$\lambda_n = (|\mu|-2n)(|\mu|-2n-1) \quad , \quad |\mu| > \tfrac{1}{2} \, .$$

The eigenvalues of $\Phi_\nu^\mu(x)$ are found similarly from

$$F\left(1+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu,\ 1+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu \,\middle|\, 3/2 \,\middle|\, \frac{x^2}{1+x^2}\right) =$$

$$= \sum_{n=0}^{\infty} \frac{(1+\frac{1}{2}\nu-\frac{1}{2}\mu)_n (1+\frac{1}{2}\nu+\frac{1}{2}\mu)_n \left(\frac{x^2}{1+x^2}\right)^n}{(3/2)_n \, n!} \qquad (4.36)$$

The series terminates if

(i) $$\frac{\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu+n)}{\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu)} = 0 \quad ,$$

(ii) $$\frac{\Gamma(1+\frac{1}{2}\nu+\frac{1}{2}\mu+n)}{\Gamma(1+\frac{1}{2}\nu+\frac{1}{2}\mu)} = 0 \quad .$$

(i) is zero when

$$\nu = \mu - 2n - 2 \qquad \text{n a positive integer}$$

and with $\nu > -\frac{1}{2}$ then

$$n < \tfrac{1}{2}(\mu - \tfrac{3}{2}) \; .$$

Therefore n = 0, 1, 2... [$\frac{1}{2}(\mu - 3/2)$], $\mu > 3/2$.

(ii) is zero when

$$\nu = -\mu - 2n - 2$$

and

$\nu > -\frac{1}{2}$ implies $n < -\frac{1}{2}(\mu + 3/2)$.

Therefore n = 0, 1, 2,.... -[$\frac{1}{2}(\mu + 3/2)$], $\mu < -3/2$.

Hence the odd eigenvalues are

$$\lambda_n = (|\mu| - 2n - 1)(|\mu| - 2n - 2) \; , \; |\mu| > 3/2$$

These results are identical to the results already obtained by locating the zeros and poles of the Titchmarsh-Weyl co-efficient.

CHAPTER 5

JOINING FACTORS FOR LEGENDRE AND SPHEROIDAL FUNCTIONS

The equations to be examined in this chapter arise when the wave equation

$$\nabla^2\Psi + k\Psi = 0 \tag{5.1}$$

is solved by separation of variables in certain systems of curvilinear coordinates, namely The prolate and oblate spheroidal coordinates.

PROLATE SPHEROIDAL COORDINATES:-

These coordinates u, v, ϕ are introduced by means of the equations

$$x = c\operatorname{Sinh}(u)\operatorname{Sin}(v)\operatorname{Cos}(\phi), \quad y = c\operatorname{Sinh}(u)\operatorname{Sin}(v)\operatorname{Sin}(\phi),$$
$$z = c\operatorname{Cosh}(u)\operatorname{Cos}(v), \qquad c \text{ constant} > 0. \tag{5.2}$$

The surfaces u = constant form a confocal family of prolate spheroids, and the surfaces v = constant a confocal system of two sheeted hyperboloids, the foci of the confocal system being the points $x = y = 0$, $Z = \pm c$. The respective ranges of u, v, and ϕ are:

$$0 \le u < \infty \quad , \quad 0 \le v \le \pi \; , \quad 0 \le \phi < 2\pi \quad .$$

The surfaces ϕ = constant are meridian planes $\phi = 0$, and $\phi = 2\pi$ being the same. $u = 0$ is a degenerate ellipsoid which reduces to the segment $x = y = 0$ $-c \le z \le c$, while

$v = 0$ and $v = \pi$ are the two halves of the degenerate hyperboloid of the system reducing respectively to

$x = y = 0,\ z \geqslant c$, and $x = y = 0,\ z \leqslant -c$.

Thus, the entire axis of revolution (z axis) is a singular line of the coordinate system.

Using the equations introduced by (5.2)

$$\nabla^2\Psi + K\Psi =$$

$$\frac{c^{-2}}{\cosh^2 u - \cos^2 v}\left(\frac{\partial^2\Psi}{\partial u^2} + \frac{\partial^2\Psi}{\partial v^2} + \operatorname{Coth} u\frac{\partial\Psi}{\partial u} + \operatorname{Cot} v\frac{\partial\Psi}{\partial v}\right) \tag{5.3}$$

$$+ \frac{1}{(c\operatorname{Sinh} u\operatorname{Sin} v)^2}\frac{\partial^2\Psi}{\partial\phi^2} + K^2\Psi = 0 \ .$$

If there are normal solutions of the form

$$\Psi = U(u)V(v)e^{\pm i\mu\phi} \tag{5.4}$$

Then the functions U, V must satisfy the ordinary differential equations

$$\frac{d^2U}{du^2} + \operatorname{Coth} u\frac{dU}{du} - \left[\lambda - \gamma^2\operatorname{Sinh}^2 u + \mu^2\operatorname{Cosech}^2 u\right]U = 0 \tag{5.5}$$

$$\frac{d^2V}{dv^2} + \operatorname{Cot} v\frac{dV}{dv} + \left[\lambda + \gamma^2\operatorname{Sin}^2 v - \mu^2\operatorname{Cosec}^2 v\right]V = 0 \tag{5.6}$$

where λ and μ are separation constants. and $\gamma = kc$.

(5.6) is the trigonometric form of the Spheroidal equation. and (5.5) may be reduced to (5.6) by the change of variable $u = iv$.

For a wave function Ψ which is continuous inside, or outside a spheroid u = constant, Ψ must be a periodic function of ϕ with period 2π, and hence μ in (5.4) must be an integer. Also, Ψ must be bounded on ellipsoids u = constant i.e. V(v) must be a solution of (5.6) which is bounded for $0 \leq v \leq \pi$.

As in the case of Legendres eqn. [5, 3.1(2)], to which (5.6) reduces when k = 0, such solutions exist only for certain characteristic values of λ. The bounded solutions of (5.6) are called Spheroidal wave functions. If Ψ is to be continuous inside a spheroid u = constant then it must be bounded on the degenerate spheroid u = 0; this determines the choice of U(u) and indicates that U(u) is a constant multiple of V(iv), that is U(u) is a modified spheroidal wave function of the first kind. On the other hand, if Ψ is a wave function regular outside a spheroid u = constant then usually its behaviour at infinity is prescribed to be asymptotically that of

$$r^{-1}\exp(ikr)$$

where

$$r=(x^2+ y^2+ z^2)^{\frac{1}{2}}= c[(\mathrm{Sinh}(u)\mathrm{Sin}(v))^2 + (\mathrm{Cosh}(u)\mathrm{Cos}(v))^2]^{\frac{1}{2}}$$

is approximately $\frac{1}{2}ce^u$ when u is large. The solutions of (5.5) determined by their behavior at infinity are called modified spheroidal wave functions of the third kind.

OBLATE SPHEROIDAL COORDINATES:-

These coordinates u, v, ϕ are introduced by means of the equations.

$$x = c\cosh(u)\sin(v)\cos(\phi), \quad y = c\cosh(u)\sin(v)\sin(\phi),$$

$$z = c\sinh(u)\cos(v), \qquad c \text{ constant} > 0. \tag{5.7}$$

The surfaces u = constant form a confocal family of oblate spheroids, the surfaces v = constant a confocal system of one-sheeted hyperboloids, and the surfaces ϕ = constant are meridian planes. The focal circle of the confocal system is the circle $x^2 + y^2 = c^2$, $z = 0$.

The ranges of u, v, ϕ are respectively :

$$0 \leq u < \infty, \quad 0 \leq v \leq \pi, \quad 0 \leq \phi \leq 2\pi,$$

$\phi = 0$ and $\phi = 2\pi$ being the same meridian plane.

$u = 0$ is a degenerate spheroid which covers the area inside the focal circle twice. $v = 0$, and $v = \pi$ are two halves of a degenerate hyperboloid reducing respectively to the positive, and negative z-axis, and $v = \frac{\pi}{2}$ is a degenerate hyperboloid which lies in the plane $z = 0$, and covers the area outside the focal circle twice. Thus the entire x,y-plane is a singular surface of the coordinate system.

Using the equations introduced by (5.7)

$$\nabla^2\Psi + K^2\Psi =$$

$$\frac{c^{-2}}{\cosh^2 u - \sin^2 v}\left[\frac{\partial^2\Psi}{\partial u^2} + \frac{\partial^2\Psi}{\partial v^2} + \tanh u\frac{\partial\Psi}{\partial u} + \cot v\frac{\partial\Psi}{\partial v}\right] \tag{5.8}$$

$$+ \frac{1}{(c\cosh u \sin v)^2}\frac{\partial^2\Psi}{\partial\phi^2} + K^2\Psi = 0 .$$

If there are normal solutions of the form

$$\Psi = U(u)V(v)e^{\pm i\mu\phi} \tag{5.9}$$

the functions U and V must satisfy the ordinary differential equations:

$$\frac{d^2U}{du^2} + \operatorname{Tanh} u \frac{dU}{du} - \left[\lambda - \gamma^2 \operatorname{Cosh}^2 u - \mu^2 \operatorname{Sech}^2 u\right] U = 0 \tag{5.10}$$

$$\frac{d^2V}{dv^2} + \operatorname{Cot} v \frac{dV}{dv} + \left[\lambda - \gamma^2 \operatorname{Sin} v - \mu^2 \operatorname{Cosec}^2 v\right] V = 0 \; . \tag{5.11}$$

Here λ and μ are separation constants and $\gamma = kc$.
Note: In (5.11) γ^2 is replaced with $-\gamma^2$ (compare with (5.6)).
(5.10) may be transformed to (5.11) by the substitution

$$u = i(v - \pi/2) \; .$$

For the spheroidal wave functions μ must again be an integer. The solutions U(u) are as for the prolate spheroidal case, with slight modifications: for details see [5b, Page 96].

In the case of the ballooning eigenvalue problem the equation is

$$(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - \left[\lambda + \gamma^2(1+x^2) - \frac{\mu^2}{1+x^2}\right] y = 0 \; . \tag{5.12}$$

This can be obtained from (5.6) by the substitution

$$v = \operatorname{Cos}^{-1} z \qquad\qquad z \in C$$

which gives the algebraic form of the Spheroidal equation

$$(1-z^2)\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + \left[\lambda + \gamma^2(1-z^2) - \frac{\mu^2}{1-z^2}\right] y = 0 \; . \tag{5.13}$$

Now the substitution $z = -ix \quad x \in R$, transforms (5.13) to (5.12). Assuming μ^2 and γ are real, and $y(x) \to 0$ as $x \to \pm \infty$ are the boundary conditions, then (5.12) subject to these conditions is a symmetric Sturm-Liouville problem. Hence the spectrum is a subset of the real λ-axis. In particular the eigenvalues λ_n are real, and the eigenfunctions $y_n(x;\lambda_n)$ form an orthogonal set on $(-\infty, \infty)$ $(n = 0, 1, 2...)$. Unlike the Spheroidal wave functions where μ is restricted to integer values ($\mu = m$), in the ballooning equation case it takes arbitrary real values.
The characteristic exponent ν which depends on γ, μ and λ will generally take complex values in the ballooning equation case, whereas for the spheroidal wave functions it was restricted to integer values ($\nu = n \geq m$).

In this chapter we will examine the properties of the solution to (5.13), bearing in mind that these are the solutions to the ballooning eigenvalue problem (5.12) with imaginary argument. We observe that equation (5.13) has regular singular points at $z = \pm 1$, with exponents $\pm \mu/2$ at both 1 and -1. These can be found using standard techniques to be seen in [2] or [13]. There is an irregular singularity at ∞, and two solutions of (5.13) behave at ∞ like z^{ν} times a single valued function, and $z^{-\nu-1}$ times a single valued function. Here ν is called the characteristic exponent of (5.13). In general it is a function of λ, μ and γ.

It is more usual though and more convenient to represent λ as a function of ν, μ and γ. This dependence is denoted by the notation $\lambda_\nu^\mu(\gamma)$ as in [5b], [10] and [12].

SOLUTIONS OF THE SPHEROIDAL EQUATION

When $\gamma^2 = 0$ (5.13) reduces to Legendres Equation as described in Chapter 4, while for $|z^2| \gg 1$ (5.13) may be approximated by Bessels equation. This suggests that solutions of the Spheroidal equation may be found as infinite series of Legendre or Bessel functions. The Legendre Series solutions (solutions of the first type) are valid close to ± 1. While the Bessel series solutions (solutions of the second type) are valid close to the irregular point at ∞. The two types of solution are analytically connected by joining factors which ensure matching in the domain where both are valid. These joining factors will be derived in a later section.

SOLUTIONS OF THE FIRST TYPE

The Spheroidal equation (5.13) is a second order differential equation, and therefore has two linearly independent solutions. Near ± 1 we take these to be

$$\left.\begin{aligned} Ps_\nu^\mu(z;\gamma) &= \sum_{r=-\infty}^{\infty} (-1)^r a_{\nu,2r}^\mu(\gamma^2)\, P_{\nu+2r}^\mu(z), \\ Qs_\nu^\mu(z;\gamma) &= \sum_{r=-\infty}^{\infty} (-1)^r a_{\nu,2r}^\mu(\gamma^2)\, Q_{\nu+2r}^\mu(z), \end{aligned}\right\} \quad z \in \mathbb{C} \qquad (5.14)$$

$$\left.\begin{aligned} \mathrm{Ps}_\nu^\mu(x;\gamma) &= \sum_{r=-\infty}^{\infty} (-1)^r a_{\nu,2r}^\mu(\gamma^2)\, \mathrm{P}_{\nu+2r}^\mu(x), \\ \mathrm{Qs}_\nu^\mu(x;\gamma) &= \sum_{r=-\infty}^{\infty} (-1)^r a_{\nu,2r}^\mu(\gamma^2)\, \mathrm{Q}_{\nu+2r}^\mu(x), \end{aligned}\right\} \quad x \in (-1,1) \qquad (5.15)$$

where we make the restriction $\mu+\nu \neq 0, \pm 1, \pm 2, \ldots$ because $Q^{\mu}_{\nu+2r}(z)$ is not defined for these values.

For (5.14), z is in the complex plane cut along the real axis from $-\infty$ to $+1$ and $|\mathrm{Arg}(z+1)| < \pi$: See [5, 3.2 (6)]. For (5.15) x is on the cut (-1,1): See [5, 3.4]. We will deal with the solutions in (5.14).

Other solutions valid in a neighbourhood of ± 1 are

$$Ps^{-\mu}_{\nu}(z;\gamma)\ ,\ Q^{-\mu}_{\nu}(z;\gamma)\ ,\ Ps^{\pm\mu}_{-\nu-1}(z;\gamma) \text{ and } Qs^{\pm\mu}_{-\nu-1}(z;\gamma)$$

Relationships between these various solutions can be found using the properties of the Legendre functions, and the expansion coefficients $a^{\mu}_{\nu,2r}(\gamma^2)$.

THE RECURRENCE RELATION FOR $a^{\mu}_{\nu,2r}(\gamma^2)$

Substitution of one of the solutions in (5.14), or (5.15) leads to the following three-term recurrence relation:

$$\frac{\gamma^2}{4}\left\{\frac{(\nu+2r-\mu)(\nu+2r-\mu-1)}{(\nu+2r-3/2)(\nu+2r-\frac{1}{2})}\, a_{2r-2} + \frac{(\nu+2r+\mu+2)(\nu+2r+\mu+1)}{(\nu+2r+3/2)(\nu+2r+5/2)}\right\} a_{2r+2}$$

$$+\left\{\lambda-(\nu+2r)(\nu+2r+1)+\frac{\gamma^2}{2}\,\frac{(\nu+2r)(\nu+2r+1)+\mu^2-1}{(\nu+2r-\frac{1}{2})(\nu+2r+3/2)}\right\} a_{2r}$$

$$= 0\ , \qquad (5.16)$$

where $a_{2r} = a^{\mu}_{\nu,2r}(\gamma^2)$ and $r = 0, \pm 1, \pm 2, \ldots$.

Using (5.16), the coefficients $a^{\pm\mu}_{\nu,2r}$ and $a^{\pm\mu}_{-\nu-1,2r}$ satisfy the following relations:

$$a^{\pm\mu}_{\nu,2r} = a^{\pm\mu}_{-\nu-1,2r}$$

$$a^{-\mu}_{\nu,2r} = \frac{\Gamma(\nu-\mu+1)\Gamma(\nu+\mu+2r+1)}{\Gamma(\nu+\mu+1)\Gamma(\nu-\mu+2r+1)}\, a^{\mu}_{\nu,2r}\,. \tag{5.17}$$

SOME PROPERTIES OF THE SOLUTIONS OF THE FIRST TYPE

From the properties of Legendre functions [5, 3.3.1], we can establish the following identities:-

$$Ps^{\mu}_{-\nu-1}(z;\gamma) = Ps^{\mu}_{\nu}(z;\gamma)\,, \tag{5.18}$$

$$\begin{aligned} &\sin\pi(\nu-\mu)\,Qs^{\mu}_{-\nu-1}(z;\gamma) = \\ &= \sin\pi(\nu+\mu)\,Qs^{\mu}_{\nu}(z;\gamma) - \pi e^{\pi\mu i}\cos\pi\nu \,.\, Ps^{\mu}_{\nu}(z;\gamma)\,, \end{aligned} \tag{5.19}$$

$$Ps^{-\mu}_{\nu}(z;\gamma) = \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\left\{Ps^{\mu}_{\nu}(z;\gamma) - \frac{2}{\pi}e^{-\pi\mu i}\sin\pi\mu\, Qs^{\mu}_{\nu}(z;\gamma)\right\} \tag{5.20}$$

$$Qs^{-\mu}_{\nu}(z;\gamma) = e^{-2\pi\mu i}\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\, Qs^{\mu}_{\nu}(z;\gamma)\,. \tag{5.21}$$

THE CIRCUIT RELATIONS:

$$Qs^{\mu}_{\nu}(ze^{\pi i};\gamma) = e^{-i\pi(\nu+1)}\, Qs^{\mu}_{\nu}(z;\gamma)\,. \tag{5.22}$$

THE CHARACTERISTIC EXPONENT ν

The exponent arises in the following way: A solution of (5.13) valid near +1 or -1 is expressible in terms of an infinite series of Legendre functions (5.14a).

$$Qs_\nu^\mu(z;\gamma) = \sum_{r=-\infty}^{\infty} (-1)^r a_{\nu,2r}^\mu(\gamma^2)\, Q_{\nu+2r}^\mu(z) .$$

Since (5.13) remains unchanged when z is replaced by -z then $Qs_\nu^\mu(-z;\gamma)$ is also a solution. Using the circuit relations for the Legendre function [5, 3.3.1]

$$Q_{\nu+2r}^\mu(z.e^{\pi i}) = e^{-\pi i(\nu+1)}\, Q_{\nu+2r}^\mu(z) . \tag{5.23}$$

Then the two solutions of (5.13) $Qs_\nu^\mu(z;\gamma)$ and $Qs_\nu^\mu(ze^{\pi i};\gamma)$ are proportional i.e. (5.22)

$$Qs_\nu^\mu(ze^{\pi i};\gamma) = e^{-\pi i(\nu+1)}\, Qs_\nu^\mu(z;\gamma) .$$

We can obtain an expression for ν as follows:

To avoid complications with the branch points we let $z_0 > 1$. Take as a fundamental set the solutions $y_1(z)$ and $y_2(z)$ of (5.13), satisfying

$y_1(z_0) = 1, y'_1(z_0) = 0$ and $y_2(z_0) = 0$,

$y'_2(z_0) = 1$. Then a general solution of (5.13) is

$$y(z) = Ay_1(z) + By_2(z) , \qquad \text{A and B constants.}$$

Since y(-z) must also be a solution, then the circuit relation condition

$$y(z_0 e^{\pi i}) = k\, y(z_0) \qquad \text{k constant.}$$

implies

$$A y_1(z_0 e^{\pi i}) + B y_2(z_0 e^{\pi i})$$

$$= A k y_1(z_0) + B k y_2(z_0) \ .$$

using the initial conditions at z_0 we find that

$$A y_1(z_0 e^{\pi i}) + B y_2(z_0 e^{\pi i}) = A k$$

and
$$e^{\pi i} A y_1'(z_0 e^{\pi i}) + e^{\pi i} B y_2'(z_0 e^{\pi i})$$

$$= A k y_1'(z_0) + B k y_2'(z_0) \ .$$

therefore

$$A y_1'(z_0 e^{\pi i}) + B y_2'(z_0 e^{\pi i}) = -B k \ .$$

it follows that

$$\begin{aligned} A(y_1(z_0 e^{\pi i}) - k) + B y_2(z_0 e^{\pi i}) &= 0 \\ A y_1'(z_0 e^{\pi i}) + B(y_2'(z_0 e^{\pi i}) + k) &= 0 \end{aligned} \qquad (5.24)$$

This is a pair of homogeneous linear equations in A and B. For a non-trivial solution the determinant must equal zero, that is

$$(y_1(z_0 e^{\pi i}) - k)(y_2'(z_0 e^{\pi i}) + k) - y_1'(z_0 e^{\pi i}) y_2(z_0 e^{\pi i}) = 0$$

or

$$k^2 + k y_2'(z_0 e^{\pi i}) - k y_1(z_0 e^{\pi i}) - y_2'(z_0 e^{\pi i}) y_1(z_0 e^{\pi i})$$
$$- y_1'(z_0 e^{\pi i}) y_2(z_0 e^{\pi i}) = 0$$

which implies

$$k^2 + k(y_2'(z_0 e^{\pi i}) - y_1(z_0 e^{\pi i})) - W(z_0 e^{\pi i}) = 0 \ ,$$

where $W(z_0 e^{\pi i})$ is the Wronskian which is known to equal 1, at z_0, from the boundary conditions. This gives the condition on k:

$$k^2 + k(y_2'(z_0 e^{\pi i}) - y_1(z_0 e^{\pi i})) - 1 = 0$$

Now putting $k = e^{\pi i \nu}$ and $M = y'_2(z_0 e^{\pi i}) - y_1(z_0 e^{\pi i})$ we obtain

$$e^{2\pi i \nu} + e^{\pi i \nu} M - 1 = 0$$

which implies

$$-2i \sin \pi\nu = M$$

or (5.25)

$$2i \sin \pi\nu = y_1(z_0 e^{\pi i}) - y_2'(z_0 e^{\pi i})$$

which has solutions, ν , $\nu+2n$, and $-1-\nu$, and $-1-\nu+2n$ where $n = 0, \pm 1, \pm 2, \ldots$. Hence the two solutions satisfy the circuit relations about ± 1: thus

$$y_1(z e^{\pi i}) = e^{\pi \nu i} y_1(z)$$

$$y_2(z e^{\pi i}) = e^{-\pi i(\nu+1)} y_2(z)$$

The two solutions will have Laurent expansions convergent in $1<|z|<\infty$ which have the form

$$y_1(z) = z^{\nu} \sum_{r=-\infty}^{\infty} b_{2r}^{\nu} z^{2r}$$

(5.26)

$$y_2(z) = z^{-1-\nu} \sum_{r=-\infty}^{\infty} b_{2r}^{-1-\nu} z^{2r}$$

These two solutions satisfy the above circuit relations. In addition they will be linearly independent solutions as long as $\nu + \frac{1}{2} \neq n$ (n an integer), because in that case $\nu = -1-\nu$ and the two solutions are proportional to each other.

When $\nu+\frac{1}{2}=n$ the second independent solution will contain a logarithmic term. It will not be necessary to study this case in detail.

SOLUTIONS OF THE SECOND TYPE

These are solutions which are convergent for large z.
They have the form:

$$S_\nu^{\mu(j)}(z;\gamma)=(1-z^{-2})^{-\frac{1}{2}\mu}\left[A_\nu^\mu(\gamma^2)\right]^{-1}\sum_{r=-\infty}^{\infty}a^\mu_{\nu,2r}(\gamma)\Psi^{(j)}_{\nu+2r}(z;\gamma) \tag{5.27}$$

$$j=1,2,3,4 .$$

Where, writing $\zeta=\gamma z$,

$$\begin{aligned}
\Psi_\nu^{(1)}(\zeta)&=\left(\frac{\pi}{2\zeta}\right)^{\frac{1}{2}}J_{\nu+\frac{1}{2}}(\zeta)\\
\Psi_\nu^{(2)}(\zeta)&=\left(\frac{\pi}{2\zeta}\right)^{\frac{1}{2}}Y_{\nu+\frac{1}{2}}(\zeta)\\
\Psi_\nu^{(3)}(\zeta)&=\left(\frac{\pi}{2\zeta}\right)^{\frac{1}{2}}H^{(1)}_{\nu+\frac{1}{2}}(\zeta)\\
\Psi_\nu^{(4)}(\zeta)&=\left(\frac{\pi}{2\zeta}\right)^{\frac{1}{2}}H^{(2)}_{\nu+\frac{1}{2}}(\zeta)
\end{aligned} \tag{5.28}$$

We recall that J and Y oscillate at infinity on the real positive ζ axis, while $H^{(1)}$ and $H^{(2)}$ are exponentially small or large on the imaginary ζ axis.
See [5b, 16.9, (5) and (6)] for details. The $a^\mu_{\nu,2r}(\gamma)$ are the same as for the solutions of the first kind, and thus are governed by the same recurrence relation (5.16). Also

$$A_\nu^\mu(\gamma^2)=\sum_{r=-\infty}^{\infty}(-1)^r a^\mu_{\nu,2r}(\gamma^2)$$

is a normalising factor. It is chosen so that as $z \to \infty$ in $|\mathrm{Arg}(\gamma z)| < \pi$

$$\Psi^{(1)}_{\nu+2r} / \Psi^{(1)}_{\nu} \to (-1)^r$$

Then $S^{\mu(1)}_{\nu}(z;\gamma)$ possesses the asymptotic behaviour

$$S^{\mu(1)}_{\nu}(z;\gamma) \sim \Psi^{(1)}_{\nu}(z\gamma) \tag{5.29}$$

as $z \to \infty$ in $|\mathrm{Arg}(\gamma z)| < \pi$.

$\Psi^{\mu(1)}_{\nu+2r}$, and hence $S^{\mu(1)}_{\nu}$, is of the form z^{ν} times a function which is single valued near ∞

[5b]: $S^{\mu(1)}_{\nu}$ is called a solution of the first kind. $S^{\mu(2)}_{\nu}$ is called a solution of the second kind (associated with the $-1-\nu$ exponent) and from (5.28), (5.29) it is seen that $S^{\mu(3)}_{\nu}$ and $S^{\mu(4)}_{\nu}$ vanish exponentially as $z \to \infty$ in the half planes $\mathrm{Im}(\gamma z) > 0$, and $\mathrm{Im}(\gamma z) < 0$ respectively. Thus $S^{\mu(3,4)}_{\nu}$ are solutions of the third kind. Besides the four solutions $S^{\mu(j)}_{\nu}$ there are the twelve other solutions $S^{-\mu(j)}_{\nu}$ and $S^{\pm\mu(j)}_{-\nu-1}$ ($j = 1,2,3,4$). Numerous relations exist between these various solutions as a result of (5.29), or (5.17) and identities between Bessel functions.

A few which we shall use later are listed below.

$$S_\nu^{\mu(j)} = S_\nu^{-\mu(j)} \qquad (5.30)$$

$$S_\nu^{\mu(3)} = S_\nu^{\mu(1)} + i S_\nu^{\mu(2)} = e^{-i\pi(\nu+\frac{1}{2})} S_{-\nu-1}^{\mu(3)}$$

$$S_\nu^{\mu(4)} = S_\nu^{\mu(1)} - i S_\nu^{\mu(2)} = e^{i\pi(\nu+\frac{1}{2})} S_{-\nu-1}^{\mu(4)} \qquad (5.31)$$

$$S_\nu^{\mu(2)} = -(\cos\pi\nu)^{-1}\left[S_\nu^{\mu(1)} \sin\nu\pi + S_{-\nu-1}^{\mu(1)}\right]$$

$$S_\nu^{\mu(3)} = \left[i\cos\nu\pi\right]^{-1}\left[S_{-\nu-1}^{\mu(1)} - S_\nu^{\mu(1)} \cdot e^{-i\pi(\nu+\frac{1}{2})}\right] \qquad (5.32)$$

$$S_\nu^{\mu(4)} = \left[i\cos\nu\pi\right]^{-1}\left[S_\nu^{\mu(1)} e^{i\pi(\nu+\frac{1}{2})} - S_{-\nu-1}^{\mu(1)}\right] .$$

References for these relationships can be found in [5b, Page 138]. As in the case of Bessel functions, it turns out that any two of the four solutions $S_\nu^{\mu(j)}$ are linearly independent as long as $\nu + \frac{1}{2} \neq n$ an integer.

THE JOINING FACTOR $K_\nu^\mu(\gamma)$

From (5.22) with the substitution $-\nu-1$ for ν we see that $Qs_{-\nu-1}^\mu(z;\gamma)$, is a solution of the first kind, as is $S_\nu^{\mu(1)}(z;\gamma)$, i.e. they are both associated with the exponent ν at ∞. This means that they are proportional to each other and from [5b, 16.9, (28)] we can write:

$$S_\nu^{\mu(1)}(z;\gamma) = \pi^{-1}\sin\pi(\nu-\mu)\, e^{-\pi i(\nu+\mu+1)} K_\nu^\mu(\gamma)\, Qs_{-\nu-1}^\mu(z;\gamma) \qquad (5.33)$$

Where $K_\nu^\mu(\gamma)$ is a parameter dependent constant called the joining factor. $K_\nu^\mu(\gamma)$ can be determined by expanding $S_\nu^{\mu(1)}(z,\gamma)$ and $Qs_{-\nu-1}^\mu(z,\gamma)$ as Laurent double series in powers of z^2, and equating coefficients of like powers of z^2 [9, Ch. 4] and [5b, 16.9].

From (5.14b) and [5, 3.2, (41)] we have:

$$z^{-\nu}(1-z^{-2})^{-\frac{1}{2}\mu} e^{i\mu\pi} Qs_{-\nu-1}^\mu(z;\gamma) =$$

$$= 2^\nu \pi^{\frac{1}{2}} \sum_{r=-\infty}^{\infty} (-1)^r a_{-\nu-1,2r}^\mu (2z)^{-2r} \frac{\Gamma(\mu-\nu+2r)}{\Gamma(\frac{1}{2}-\nu+2r)} F(z;\mu,\nu,r)$$

$$F(z;\mu,\nu,r) = F\left(\tfrac{1}{2}-\tfrac{1}{2}\nu+\tfrac{1}{2}\mu+r, \tfrac{1}{2}\mu-\tfrac{1}{2}\nu+r \,\middle|\, \tfrac{1}{2}-\nu+2r \,\middle|\, z^2\right)$$

Using the Legendre duplication formula in the F function and rearranging we obtain:

$$e^{-i\mu\pi} z^{-\nu}(1-z^{-2})^{-\mu/2} Qs_{-\nu-1}^\mu(z;\gamma) =$$

$$= 2^\nu \pi^{\frac{1}{2}} \sum_{r=-\infty}^{\infty} \sum_{t=0}^{\infty} (-1)^r a_{-\nu-1,2r}^\mu (2z)^{-2(r+t)} \frac{\Gamma(\mu-\nu+2r+2t)}{t!\,\Gamma(\frac{1}{2}-\nu+2r+t)} \tag{5.34}$$

Similarly from [5a , 7.2, (2)] and (5.27) and (5.28) we have

$$z^{-\nu}(1-z^{-2})^{\mu/2} S_\nu^{\mu(1)}(z,\gamma) =$$

$$= \tfrac{1}{2}\pi^{\frac{1}{2}} \left[A_\nu^\mu(\gamma^2)\right]^{-1} \sum_{r=-\infty}^{\infty} \sum_{t=0}^{\infty} (-1)^t a_{\nu,2r}^\mu \frac{\left(\frac{\gamma}{2}\right)^{\nu+2(r+t)} z^{2(r+t)}}{t!\,\Gamma(\nu+3/2+2r+t)} \tag{5.35}$$

In (5.35) $S_\nu^{\mu(1)}(z,\gamma)$ is multiplied by the factor $(1-z^{-2})^{\mu/2}$ while in (5.34) $Qs_{-\nu-1}^\mu(z,\gamma)$ is multiplied by $(1-z^{-2})^{-\mu/2}$. This problem can be resolved using (5.30)

$$S_{-\nu}^{\mu(1)} = S_\nu^{-\mu(1)}$$

Hence $$z^{-\nu}(1-z^{-2})^{-\frac{1}{2}\mu} S_\nu^{-\mu(1)}(z;\gamma) =$$

$$= \tfrac{1}{2}\pi^{\frac{1}{2}}\left(\frac{\gamma}{2}\right)^{\nu}\left[A_\nu^{-\mu}(\gamma^2)\right]^{-1}\sum_{r=-\infty}^{\infty}\sum_{t=0}^{\infty}(-1)^t a_{\nu,2r}^{-\mu}\frac{\left(\frac{\gamma}{2}\right)^{2(r+t)} z^{2(r+t)}}{t!\,\Gamma(\nu+3/2+2r+t)} \qquad (5.36)$$

If now, both sides of (5.33) are multiplied by $z^{-\nu}(1-z^{-2})^{-\mu/2}$. We can equate the R.H.S. of (5.34) and (5.36) and obtain:

$$\frac{\pi}{2}\left(\frac{\gamma}{4}\right)^{\nu} e^{\pi i(\nu+1)}\left[A_\nu^{-\mu}(\gamma^2)\right]^{-1}\sum_{r=-\infty}^{\infty}\sum_{t=0}^{\infty}(-1)^t a_{\nu,2r}^{-\mu}\frac{\left(\frac{\gamma}{2}\right)^{2(r+t)} z^{2(r+t)}}{t!\,\Gamma(\nu+3/2+2r+t)} =$$

$$= \sin\pi(\nu-\mu) K_\nu^{\mu}(\gamma)\sum_{r=-\infty}^{\infty}\sum_{t=0}^{\infty}(-1)^r a_{-\nu-1,2r}^{\mu}(2z)^{-2(r+t)}\frac{\Gamma(\mu-\nu+2r+2t)}{t!\,\Gamma(\frac{1}{2}-\nu+2r+t)}$$

(5.37)

We see now that the problem is essentially that of finding k when

$$M\sum_{r=-\infty}^{\infty}\sum_{t=0}^{\infty} c_{r,t}\, z^{2(r+t)} = K\sum_{r=-\infty}^{\infty}\sum_{t=0}^{\infty} d_{r,t}\, z^{-2(r+t)}$$

Where

$$M = \frac{\frac{\pi}{2}\left(\frac{\gamma}{4}\right)^{\nu} e^{\pi i(\nu+1)}\left[A_\nu^{-\mu}(\gamma^2)\right]^{-1}}{\sin\pi(\nu-\mu)},$$

$$c_{r,t} = (-1)^t . a_{\nu,2r}^{-\mu} . \left(\frac{\gamma}{2}\right)^{2(r+t)} . \frac{1}{t!\,\Gamma(\nu+3/2+2r+t)}$$

and

$$d_{r,t} = (-1)^r a_{-\nu-1,2r}^{\mu}\, 2^{-2(r+t)}\frac{\Gamma(\mu-\nu+2r+2t)}{t!\,\Gamma(\frac{1}{2}-\nu+2r+t)} .$$

For a particular t let j = r+t, then t = j-r

Hence

$$\sum_{r=-\infty}^{\infty}\sum_{t=0}^{\infty} = \sum_{r=-\infty}^{\infty}\sum_{j=r}^{\infty} = \sum_{j=-\infty}^{\infty}\sum_{r=-\infty}^{j}$$

Therefore

$$\sum_{r=-\infty}^{\infty}\sum_{t=0}^{\infty} c_{r,t}\, z^{2(r+t)} = \sum_{j=-\infty}^{\infty}\sum_{r=-\infty}^{j} c_{r,j-r}\, z^{2j} = \sum_{j=-\infty}^{\infty} z^{2j}\sum_{r=-\infty}^{j} c_{r,j-r}$$

Similarly

$$\sum_{r=-\infty}^{\infty}\sum_{t=0}^{\infty} d_{r,t}\, z^{-2(r+t)} = \sum_{j=-\infty}^{\infty} z^{-2j}\sum_{r=-\infty}^{j} d_{r,j-r} .$$

Now equating powers of z^2

$$(z^2)^j = (z^2)^{-j}$$

$$j = -j$$

implies $j = 0$ and that implies $r = -t$

Hence

$$M\sum_{r=-\infty}^{0} c_{r,-r} = K_\nu^\mu(\gamma)\sum_{r=-\infty}^{0} d_{r,-r}$$

or

$$M\sum_{r=0}^{\infty} c_{-r,r} = K_\nu^\mu(\gamma)\sum_{r=0}^{\infty} d_{-r,r}$$

We see therefore that:

$$K_\nu^\mu(\gamma) = M\frac{\sum_{r=0}^{\infty} c_{-r,r}}{\sum_{r=0}^{\infty} d_{-r,r}}$$

$$= \frac{\frac{\pi}{2}\left(\frac{\gamma}{4}\right)^\nu e^{\pi i(\nu+1)}}{\left[A_\nu^{-\mu}(\gamma^2)\right]\sin\pi(\nu-\mu)} \cdot \frac{\sum_{r=0}^{\infty}(-1)^r \frac{a^{-\mu}_{\nu,-2r}}{r!\,\Gamma(\nu+3/2-r)}}{\sum_{r=0}^{\infty}(-1)^r \frac{a^{\mu}_{-\nu-1,-2r}\,\Gamma(\mu-\nu)}{r!\,\Gamma(\frac{1}{2}-\nu-r)}} .$$

Using (5.17) we can replace $a^\mu_{-\nu-1,-2r}$ with $a^\mu_{\nu,2r}$, also

$\sin\pi(\nu-\mu) = -\sin\pi(\mu-\nu)$ and

$$\frac{\pi}{\sin\pi(\mu-\nu)} = \Gamma(\mu-\nu)\Gamma(1+\nu-\mu) .$$

Hence we have

$$\frac{2\left(\frac{\gamma}{4}\right)^{-\nu} K_\nu^\mu(\gamma)}{e^{\pi i\nu}\,\Gamma(1+\nu-\mu)} =$$

$$= \left[A_\nu^{-\mu}(\gamma^2)\right]^{-1} \frac{\sum_{r=0}^{\infty}\left[(-1)^r a^{-\mu}_{\nu,2r}(\gamma^2)\right]\left[r!\,\Gamma(\nu+3/2-r)\right]^{-1}}{\sum_{r=0}^{\infty}\left[(-1)^r a^{\mu}_{\nu,2r}(\gamma^2)\right]\left[r!\,\Gamma(\frac{1}{2}-\nu-r)\right]^{-1}} \tag{5.38}$$

All of the $S_\nu^{\mu(j)}(z,\gamma)$ can be expressed in terms of $S_\nu^{\mu(1)}(z,\gamma)$ by (5.32) and $Ps_\nu^\mu(z,\gamma)$ may be expressed in terms of $Qs_\nu^\mu(z,\gamma)$ by for example, rearranging (5.19).

Therefore it is clear that (5.33) suffices to express any one of the Bessel function series in terms of Legendre function series, and vice versa. For further properties of the joining factors, and references see [5b] or [10].

THE EXPRESSION FOR λ IN TERMS OF γ^2, μ AND ν

An expression for $\lambda_\nu^\mu(\gamma^2)$ in powers of γ^2 can be obtained employing the technique of infinite determinants or more conveniently, continued fractions, using the three-term recursion relation (5.16) as a starting point. The continued fraction approach has the advantage of providing expressions for the coefficients $a_{\nu,2r}^\mu(\gamma^2)$ as well. The procedures are analogous to those used in dealing with Mathieus equation, see [8, page 557] or [5b, Ch. 16.2]. For the Spheroidal equations case see also [10] and [12].

As λ is expressed as a function of the variable ν among others ν will have to be determined. The relation for finding ν will be derived in the next chapter and is quite complicated as it involves the joining factors $K_\nu^\mu(\gamma^2)$ and $K_{-\nu-1}^\mu(\gamma^2)$.

In general ν will be complex, but $\mathrm{Re}\,\nu = -1/2$ ensuring that λ is always real.

When $\gamma = 0$ our equation (5.13) reduces to Legendres equation, and (as previously described in Ch.4) in that case

$$\lambda = \nu(\nu+1)$$

ν is then expressed in terms of μ and λ can be explicitly determined. The same principles apply in the case of the Spheroidal equation, but the relations are substantially more difficult and cannot be solved explicitly. We give two methods which are commonly used in these circumstances.

THE INFINITE DETERMINANT METHOD

If we write the recurrence relation (5.16) in the form: (see [10])

$$\theta\left[A_r a_{2r-2} + B_r a_{2r+2}\right] + a_{2r} = 0 \tag{5.39}$$

Where

$$A_r = \frac{1}{C_r}\frac{(\nu+2r-\mu)(\nu+2r-\mu-1)}{(\nu+2r-3/2)(\nu+2r+1/2)}$$

$$B_r = \frac{1}{C_r}\frac{(\nu+2r+\mu+2)(\nu+2r+\mu+1)}{(\nu+2r+3/2)(\nu+2r+5/2)} \tag{5.40}$$

$$C_r = \lambda-(\nu+2r)(\nu+2r+1)+\frac{\gamma^2}{2}\frac{(\nu+2r)(\nu+2r+1)+\mu^2-1}{(\nu+2r-\frac{1}{2})(\nu+2r+3/2)}$$

$$a_{2r} = a^{\mu}_{\nu,2r}(\gamma^2) \quad \text{and} \quad \theta = \frac{\gamma^2}{4}$$

This is a set of linear homogeneous equations

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdots & \theta A_{-2} & 1 & \theta B_{-2} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & \theta A_{-1} & 1 & \theta B_{-1} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \theta A_0 & 1 & \theta B_0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \theta A_1 & 1 & \theta B_1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \vdots \\ a_{-6} \\ a_{-4} \\ a_{-2} \\ a_0 \\ a_2 \\ a_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

and the condition for their non-trivial solution is that the determinant

$$\Delta = \begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \theta A_{-2} & 1 & \theta B_{-2} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & \theta A_{-1} & 1 & \theta B_{-1} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \theta A_0 & 1 & \theta B_0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \theta A_1 & 1 & \theta B_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0 \quad . \tag{5.41}$$

From [16, page 36] we can see that this determinant is absolutely convergent because the product of its diagonal elements is convergent (to one), and the sum of the off-diagonal elements is absolutely convergent. This can be seen from the expressions for A_r , and B_r where $|A_r|$ and $|B_r| \sim r^{-2}$ as $r \to \pm\infty$.

When γ^2 is small enough, the central 3 x 3 determinant gives λ to $O(\gamma^6)$.

i.e.

$$1 - \theta^2 (A_1 B_0 - A_0 B_1) = 0 \quad .$$

Substituting in the values for A_0 A_1, B_0 and B_1 gives us

$$\lambda_\nu^\mu(\gamma^2) \approx \nu(\nu+1) - \frac{1}{2}\left[1 + \frac{(2\mu-1)(2\mu+1)}{(2\nu-1)(2\nu+1)}\right]\gamma^2 +$$

$$+\left[\frac{(\nu-\mu-1)(\nu-\mu)(\nu+\mu-1)(\nu+\mu)}{(2\nu-3)(2\nu-1)^3(2\nu+1)} - \frac{(\nu-\mu+1)(\nu-\mu+2)(\nu+\mu+1)(\nu+\mu+2)}{(2\nu+1)(2\nu+3)^3(2\nu+5)}\right]\frac{\gamma^4}{2}$$

$$+ O(\gamma^6) \, .$$

By taking higher order determinants, better approximations for λ can be found, though the process quickly becomes unwieldly.

CONTINUED FRACTIONS

This is computationally more efficient, and provides expressions for the coefficients $a_{\nu}^{\mu}(\gamma^2)$ as well. If we re-arrange (5.39) we can obtain the following ratios:

$$\frac{a_{2r}}{a_{2r-2}} = \frac{-\Theta A_r}{1+\Theta B_r \frac{a_{2r+2}}{a_{2r}}} = R_r \qquad r = 0, 1, 2 \ldots$$

and

$$\frac{a_{2r}}{a_{2r+2}} = \frac{-\Theta B_r}{1+\Theta B_r \frac{a_{2r+2}}{a_{2r}}} = L_r \qquad r = -1, -2, \ldots \qquad (5.42)$$

Therefore

$$R_r = \frac{-\Theta A_r}{1+\Theta B_r R_{r+1}}$$

$$L_{-r} = \frac{-\Theta B_r}{1+\Theta A_r L_{r-1}}$$

Hence

$$a_{\nu,2r}^{\mu}(\gamma^2) = a_{\nu,0}^{\mu}(\gamma^2) R_1 R_2 \ldots R_r$$

$$(5.43)$$

$$a_{\nu,-2r}^{\mu}(\gamma^2) = a_{\nu,0}^{\mu}(\gamma^2) L_{-1} L_{-2} \ldots L_{-r}$$

$$r = 1, 2, 3, \ldots$$

The expression for λ equivalent to (5.41) can be deduced from

$$L_0 R_1 = 1 \tag{5.44}$$

CHAPTER 6

THE TITCHMARSH-WEYL COEFFICIENTS WHEN $\gamma = 0$

In this chapter we construct the functions $\phi^{\mu}_{\nu}(z;\gamma)$ and $\theta^{\mu}_{\nu}(z;\gamma)$ which satisfy the boundary conditions:

$$\theta(0) = 1\,, \qquad \theta'(0) = 0$$

$$\phi(0) = 0\,, \qquad \phi'(0) = -1$$

We recall from Chapter 2 that an L^2 solution $\psi(x;\lambda)$ of a second order singular ordinary differential equation, may always be constructed from two such solutions i.e.

$$\psi(x;\lambda) = \theta(x;\lambda) + m(\lambda).\phi(x;\lambda)\,.$$

The functions will be constructed from the $Ps^{\mu}_{\nu}(z;\gamma)$ and $Qs^{\mu}_{\nu}(z;\gamma)$, solutions to

$$(1-z^2)y'' - 2zy' + \left(\lambda + \gamma^2(1-z^2) - \frac{\mu^2}{1-z^2}\right)y = 0 \qquad (6.1)$$

obtained in (5.14).

Then the substitution in $\theta(z)$, and $\phi(z)$ of $z = ix$, $x \in R$ provides the solutions to the ballooning equation

$$(1+x^2)y'' + 2xy' - \left(\lambda + \gamma^2(1+x^2) - \frac{\mu^2}{1+x^2}\right)y = 0\,. \qquad (6.2)$$

Using these expressions ($\theta(x)$, and $\phi(x)$) we look for solutions to (6.2).

$$\psi^{\mu}_{1\nu}(x;\gamma) = \theta^{\mu}_{\nu}(x;\gamma) + m^{\mu}_{1\nu}(\lambda;\gamma)\phi^{\mu}_{\nu}(x;\gamma)$$

which is $L^2(-\infty, 0)$ and (6.3)

$$\psi^{\mu}_{2\nu}(x;\gamma) = \theta^{\mu}_{\nu}(x;\gamma) + m^{\mu}_{2\nu}(\lambda;\gamma)\phi^{\mu}_{\nu}(x;\gamma)$$

which is $L^2(0, \infty)$.

This involves constructing the Titchmarsh-Weyl $m(\lambda)$ co-efficient. Because the potential function q(x) in this case is even, from the theory in Chapter 2, we can deduce that $\theta(x)$ and $\phi(x)$ are even and odd functions respectively. This implies, in turn, that $m_1(\lambda) = -m_2(\lambda)$.

It will be shown that the $m(\lambda)$ functions can be expressed in terms of the joining factors $K_\nu^\mu(\gamma)$, and $K_{-\nu-1}^\mu(\gamma)$. The poles and zeros of the $m(\lambda)$ functions then give implicit expressions for the eigenvalues λ_n corresponding to the eigenfunctions $\phi(x;\lambda_n)$ and $\theta(x;\lambda_n)$ respectively.

Finally it will be shown that as $\gamma \to 0$ $m(\lambda,\gamma) \to m(\lambda, 0)$ where

$$m(\lambda,0) = \frac{2\Gamma(1+\frac{1}{2}\nu+\frac{1}{2}\mu)\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu)}$$

is the $m(\lambda)$ function for Legendres equation deduced in (4.30). We observe from (5.14) that the solutions to (6.1) near ± 1 are

$$Ps_\nu^\mu(z;\gamma) = \sum_{r=-\infty}^{\infty} (-1)^r a_{\nu,2r}^\mu(\gamma) P_{\nu+2r}^\mu(z),$$

$$Qs_\nu^\mu(z;\gamma) = \sum_{r=-\infty}^{\infty} (-1)^r a_{\nu,2r}^\mu(\gamma) Q_{\nu+2r}^\mu(z), \qquad (6.4)$$

$\mu+\nu = 0, \pm 1, \pm 2 \ldots$

Where the Legendre functions are defined from [5, Page 126, (22)] as follows:-

$$P_{\nu+2r}^\mu(z) = 2^\mu \pi^{\frac{1}{2}} (z^2-1)^{-\mu/2} . [a_r F_1(z) + b_r z F_2(z)] \qquad (6.5)$$

where F_1 and F_2 are Gauss hypergeometric functions

$$F_1 = F\left(-\tfrac{1}{2}(\nu+\mu+2r),\ \tfrac{1}{2}(1+\nu-\mu+2r)\,\middle|\,\tfrac{1}{2}\,\middle|\,z^2\right)$$

$$F_2 = F\left(\tfrac{1}{2}(1-\nu-\mu-2r),\ \tfrac{1}{2}(2+\nu-\mu+2r)\,\middle|\,\tfrac{3}{2}\,\middle|\,z^2\right). \tag{6.6}$$

Note that both are even functions of z. The constants in (6.5) are given by

$$a_r = \left[\Gamma\left(\tfrac{1}{2}(1-\nu-\mu-2r)\right).\Gamma\left(\tfrac{1}{2}(2+\nu-\mu+2r)\right)\right]^{-1}$$

$$b_r = -2\left[\Gamma\left(\tfrac{1}{2}(1+\nu-\mu+2r)\right).\Gamma\left(-\tfrac{1}{2}(\nu+\mu+2r)\right)\right]^{-1}. \tag{6.7}$$

Also

$$Q^{\mu}_{\nu+2r}(z;\gamma) = 2^{\mu}\pi^{\frac{1}{2}}e^{i\pi\mu}(z^2-1)^{-\mu/2}\left[c_r F_1(z) + d_r z F_2(z)\right] \tag{6.8}$$

where F_1 and F_2 are as in (6.6), while from [5, Page 134, (40)]

$$c_r = \frac{\Gamma\left(\tfrac{1}{2}(1+\nu+\mu+2r)\right)e^{\frac{i\pi}{2}(\mu-\nu-2r-1)}}{2\Gamma\left(\tfrac{1}{2}(2+\nu-\mu+2r)\right)}$$

$$d_r = \frac{\Gamma\left(\tfrac{1}{2}(2+\nu+\mu+2r)\right)e^{\frac{i\pi}{2}(\mu-\nu-2r)}}{\Gamma\left(\tfrac{1}{2}(1+\nu-\mu+2r)\right)} \tag{6.9}$$

We recall that $\Theta(z)$ and $\phi(z)$ are solutions which are even and odd functions respectively, such that their Wronskian $w(\Theta(o), \phi(o)) = 1$.

Hence, there are solutions to (6.1) which are linear combinations of its two solutions $Ps_\nu^\mu(z;\gamma)$ and $Qs_\nu^\mu(z;\gamma)$ satisfying the above conditions. Thus we can write:

$$\Theta_\nu^\mu(z;\gamma) = \mathbb{K}\, Ps_\nu^\mu(z;\gamma) + \mathbb{L}\, Qs_\nu^\mu(z;\gamma)$$

$$\Phi_\nu^\mu(z;\gamma) = \mathbb{M}\, Ps_\nu^\mu(z;\gamma) + \mathbb{N}\, Qs_\nu^\mu(z;\gamma) \tag{6.10}$$

where K, L, M, N are functions of μ, ν only.

THE $\Theta_\nu^\mu(x;\gamma)$ FUNCTION

Using the right hand side of (6.4), (6.5) and (6.8) in (6.10) we obtain:

$$\Theta(z) = 2^\mu \pi^{\frac{1}{2}} (z^2-1)^{-\mu/2} \Big\{ \mathbb{K} \sum_{r=-\infty}^{\infty} (-1)^r a_{\nu,2r}^\mu \big[a_r F_1(z) + b_r z F_2(z) \big]$$

$$+ e^{i\mu\pi} \mathbb{L} \sum_{r=-\infty}^{\infty} (-1)^r a_{\nu,2r}^\mu \big[c_r F_1(z) + d_r z F_2(z) \big] \Big\} . \tag{6.11}$$

We observe that $\Theta(z)$ is an even function, therefore the odd parts of (6.11) must cancel. This implies that:

$$\mathbb{K} = -\frac{e^{i\mu\pi}\mathbb{L}\, d_r}{b_r} .$$

Substituting for d_r and b_r from (6.9) and (6.7) we obtain

$$\mathbb{K} =$$

$$= \frac{e^{i\mu\pi}\mathbb{L}\,\Gamma(1+\frac{1}{2}\nu+\frac{1}{2}\mu+r)\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu+r)\Gamma(-\frac{1}{2}\nu-\frac{1}{2}\mu-r)}{2e^{-\frac{\pi i}{2}(\mu-\nu-2r)}\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu+r)}$$

Using the fact that $\Gamma(z)\Gamma(1-z) = \pi \operatorname{Cosec} \pi z$ we can write

$$\left[\Gamma(1+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu+r)\Gamma(-\tfrac{1}{2}\nu-\tfrac{1}{2}\mu-r)\right]^{-1} =$$

$$= \sin\tfrac{\pi}{2}(-\nu-\mu-2r)/\pi =$$

$$= -(-1)^r \sin\tfrac{\pi}{2}(\nu+\mu)/\pi$$

If we rewrite $e^{\pi i r}$ as $(-1)^r$ we have

$$\mathbb{K} = \frac{-\pi \mathbb{L}\, e^{\frac{\pi i}{2}(3\mu-\nu)}}{2\sin\frac{\pi}{2}(\nu+\mu)} \qquad (6.12)$$

L can be found from the condition $\Theta(o) = 1$. The expression for $\Theta(z)$ can now be written:

$$\Theta(z) = 2^{\mu}\pi^{\frac{1}{2}}(z^2-1)^{-\mu/2}\,\mathbb{L}\sum_{r=-\infty}^{\infty}(-1)^r a^{\mu}_{\nu,2r}\left[\frac{-\pi e^{\frac{\pi i}{2}(3\mu-\nu)}}{2\sin\frac{\pi}{2}(\nu+\mu)}a_r + e^{i\mu\pi}c_r\right]F_1(z).$$

Using (6.7), and (6.9) we observe that

$$\frac{-\pi e^{\frac{\pi}{2}i(3\mu-\nu)}}{2\sin\frac{\pi}{2}(\nu+\mu)}a_r + e^{i\mu\pi}c_r \qquad (6.13)$$

is equal to

$$\frac{-\pi e^{i\frac{\pi}{2}(3\mu-\nu)}}{2\sin\frac{\pi}{2}(\nu+\mu)\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu-r)\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu+r)}$$

$$-\ \frac{i(-1)^r\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu+r)e^{i\frac{\pi}{2}(3\mu-\nu)}}{2\,\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu+r)}$$

$$= \frac{-e^{i\frac{\pi}{2}(3\mu-\nu)}}{2\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu+r)}\left[\frac{\pi}{\sin\frac{\pi}{2}(\nu+\mu)\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu-r)} + i(-1)^r\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu+r)\right]$$

$$= \frac{-e^{\frac{i\pi}{2}(3\mu-\nu)}}{2\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu+r)}\left[\frac{\pi + i(-1)^r \sin\frac{\pi}{2}(\nu+\mu)\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu-r)\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu+r)}{\sin\frac{\pi}{2}(\nu+\mu)\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu-r)}\right].$$

We note, using $\Gamma(\frac{1}{2}+z)\Gamma(\frac{1}{2}-z) = \pi \operatorname{Sec} \pi z$ that

$$\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu+r)\Gamma(\tfrac{1}{2}-\tfrac{1}{2}\nu-\tfrac{1}{2}\mu-r) = \frac{\pi(-1)^r}{\cos\frac{\pi}{2}(\nu+\mu)}$$

which implies that (6.13) is equal to

$$\frac{-e^{\frac{i\pi}{2}(3\mu-\nu)}}{2\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu+r)}\left[\frac{1+i\operatorname{Tan}\frac{\pi}{2}(\nu+\mu)}{\sin\frac{\pi}{2}(\nu+\mu)\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu-r)}\right].$$

We observe that

$$1+i\operatorname{Tan}\frac{\pi}{2}(\nu+\mu) = \frac{e^{\frac{i\pi}{2}(\nu+\mu)}}{\cos\frac{\pi}{2}(\nu+\mu)}$$

Hence (6.13) becomes

$$\frac{\pi e^{i\pi(2\mu+1)}}{\sin\pi(\nu+\mu)\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu+r)\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu-r)},$$

on using the fact that $2\cos z \sin z = \sin 2z$.

Substituting this back into the expression for $\Theta(z)$, and putting z = ix we finally obtain

$$\Theta_\nu^\mu(x;\gamma^2) =$$

$$= \frac{2^\mu \pi^{\frac{3}{2}} e^{\frac{i\pi}{2}(3\mu+2)}(1+x^2)^{-\mu/2}}{\sin\pi(\nu+\mu)} \mathbb{L} \sum_{r=-\infty}^{\infty} \frac{(-1)^r a^\mu_{\nu,2r}(\gamma^2).F_1(x)}{\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu+r)\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu-r)}. \qquad (6.14)$$

THE $\phi_\nu^\mu(x;\gamma)$ FUNCTION

Substituting the right hand side of (6.4), (6.5) and (6.8) into the expression for $\phi(z)$ in (6.10), we obtain

$$\phi_\nu^\mu(z;\gamma)=2^\mu\pi^{\frac{1}{2}}(z^2-1)^{-\mu/2}\Big\{\mathbb{M}\sum_{r=-\infty}^{\infty}(-1)^r a_{\nu,2r}^\mu(\gamma^2)\big[a_r F_1(z)+b_r z F_2(z)\big]$$

$$+e^{i\mu\pi}\mathbb{N}\sum_{r=-\infty}^{\infty}(-1)^r a_{\nu,2r}^\mu(\gamma^2)\big[c_r F_1(z)+d_r z F_2(z)\big]\Big\}\ . \qquad (6.15)$$

Note in this case that $\phi(z)$ is an odd function which implies that the even parts of (6.15) must cancel.

Hence

$$\mathbb{M}a_r + e^{i\mu\pi}\mathbb{N}c_r = 0\ .$$

Substituting the expressions for a_r and c_r in (6.7) and (6.9) into the above equation, yields upon simplifying:

$$\mathbb{M}=-\frac{(-1)^r}{2}\mathbb{N}e^{\frac{i\pi}{2}(3\mu-\nu-1)}\Gamma\left(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu+r\right)\Gamma\left(\tfrac{1}{2}-\tfrac{1}{2}\nu-\tfrac{1}{2}\mu-r\right)$$

$$=\frac{-\pi\mathbb{N}e^{\frac{i\pi}{2}(3\mu-\nu-1)}}{2\cos\frac{\pi}{2}(\nu+\mu)}\ . \qquad (6.16)$$

N can be found from the condition that $\phi'(0) = -1$.

$\phi(z)$ can be written as

$$\phi(z)=$$

$$2^\mu\pi^{\frac{1}{2}}z.(z^2-1)^{-\mu/2}.\sum_{r=-\infty}^{\infty}(-1)^r a_{\nu,2r}^\mu(\gamma^2)\big[\mathbb{M}b_r+\mathbb{N}d_r e^{i\mu\pi}\big]F_2(z).$$

Now using (6.7) and (6.9) we find that

$$\mathbb{M}b_r + \mathbb{N}d_r e^{i\mu\pi}$$

$$\frac{\pi \mathbb{N} e^{\frac{i\pi}{2}(3\mu-\nu-1)}}{\cos\frac{\pi}{2}(\nu+\mu)\,\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu+r)\,\Gamma(-\frac{1}{2}\nu-\frac{1}{2}\mu-r)}\ .$$

After some simplification we obtain the R.H.S. equal to

$$\frac{\pi \mathbb{N} e^{\frac{i\pi}{2}(3\mu-\nu)}}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu+r)}\left[\frac{-i-\cot\frac{\pi}{2}(\nu+\mu)}{\cos\frac{\pi}{2}(\nu+\mu)\,\Gamma(-\frac{1}{2}\nu-\frac{1}{2}\mu-r)}\right]\ .$$

Again we observe the elementary trigonometric identity

$$-i-\cot\frac{\pi}{2}(\nu+\mu) = \frac{-e^{\frac{i\pi}{2}(\nu+\mu)}}{\sin\frac{\pi}{2}(\nu+\mu)}\ .$$

Hence

$$\mathbb{M}b_r + e^{i\mu\pi}\mathbb{N}d_r$$

$$= \frac{2\pi \mathbb{N} e^{i\pi(2\mu+1)}}{\sin\pi(\nu+\mu)\,\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu+r)\,\Gamma(-\frac{1}{2}\nu-\frac{1}{2}\mu-r)}$$

on performing the same manipulations as in the $\Theta(z)$ case. Employing the substitution $z = ix$, we can now write down the expression for $\phi_\nu^\mu(x;\gamma)$.

$$\phi_\nu^\mu(x;\gamma) =$$

$$= \frac{2^{\mu+1}.\pi^{3/2}.x(1+x^2)^{-\mu/2}.e^{\frac{3i\pi}{2}(\mu+1)}}{\sin\pi(\nu+\mu)}\mathbb{N}\sum_{r=-\infty}^{\infty}\frac{(-1)^r a_{\nu,2r}^{\mu}(\gamma^2)F_2(x)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu+r)\,\Gamma(-\frac{1}{2}\nu-\frac{1}{2}\mu-r)} \quad (6.17)$$

THE CONSTRUCTION OF THE $m_\nu^\mu(\lambda,\gamma)$ FUNCTION (FOR $\gamma^2 > 0$)

We are now in a position to construct the two functions

$$\Psi_{1\nu}^{\mu}(x;\gamma) = \Theta_\nu^\mu(x;\gamma) + m_{1\nu}^{\mu}(\gamma^2)\phi_\nu^\mu(x;\gamma)$$

and

$$\Psi_{2\nu}^{\mu}(x;\gamma) = \Theta_\nu^\mu(x;\gamma) + m_{2\nu}^{\mu}(\gamma^2)\phi_\nu^\mu(x;\gamma).$$

Where is $L^2(-\infty,0)$, and is $L^2(0,\infty)$. For this we need to construct the functions $m_{1\nu}^{\mu}(\lambda,\gamma)$, and $m_{2\nu}^{\mu}(\lambda,\gamma)$. Now from Ch. 2 when (6.2) is transformed to standard form [15, page 22], we find that the potential term q(x) is an even function. This implies that $m_{1\nu}^{\mu}(\lambda,\gamma) = -m_{2\nu}^{\mu}(\lambda,\gamma)$. Hence we only have to deal with the $[0,\infty)$ interval i.e. find $m_{2\nu}^{\mu}(\lambda,\gamma)$. As already noted $\Psi_2(x)$ is of integrable square as $x\to\infty$ therefore it must be a multiple of one of the solutions of (6.2) convergent at ∞.

Thus

$$\Psi_{2\nu}^{\mu}(x;\gamma) = hy(x), \tag{6.18}$$

where h is a constant.

When $\gamma^2 > 0$, the solution y(x) associated with the correct asymptotic behaviour as $x\to\infty$ is

$$y(x) = -\pi/2\, e^{\frac{\pi i \nu}{2}} S_\nu^{\mu(3)}(ix;\gamma). \tag{6.19}$$

From (5.27,28) we recall that $S^{(3)}$ is a linear combination of Hankel functions which are small at infinity on the positive imaginary axis, i.e. in this case when x is real. The continuation of $S_\nu^{\mu(3)}(ix,\gamma)$ onto the [0,1) interval can be expressed as follows:-

$$S_\nu^{\mu(3)}(ix;\gamma) = A\,Ps_\nu^\mu(ix;\gamma) + B\,Qs_\nu^\mu(ix;\gamma)$$

therefore

$$y(x) = \left(-\frac{\pi}{2} e^{\frac{\pi i \nu}{2}}\right)\left[A\,Ps_\nu^\mu(ix;\gamma) + B\,Qs_\nu^\mu(ix;\gamma)\right] \tag{6.20}$$

The constants A and B involve the joining factors $K_\nu^\mu(\gamma)$, and $K_{-\nu-1}^\mu(\gamma)$ and are found as follows: From (5.32) we have

$$\cos\pi\nu\, S_\nu^{\mu(3)}(z;\gamma) = e^{-\pi\nu i}\, S_\nu^{\mu(1)}(z;\gamma) - i\, S_{-\nu-1}^{\mu(1)}(z;\gamma)$$

and from (5.33) we know that $S_\nu^{\mu(1)}(z;\gamma)$ is a multiple of $Qs_{-\nu-1}^\mu(z;\gamma)$

$$S_\nu^{\mu(1)}(z;\gamma) = (1/\pi)\sin\pi(\nu-\mu)\, e^{-\pi i(\nu+\mu+1)} K_\nu^\mu(\gamma^2)\, Qs_{-\nu-1}^\mu(z;\gamma).$$

We can find the expression for $S_{-\nu-1}^{\mu(1)}(z;\gamma)$ in terms of $Qs_\nu^\mu(z;\gamma)$ by changing ν to $-\nu-1$, Hence:

$$S_{-\nu-1}^{\mu(1)}(z;\gamma) = (1/\pi)\sin\pi(\nu+\mu)\, e^{\pi i(\nu-\mu)} K_{-\nu-1}^\mu(\gamma^2)\, Qs_{-\nu-1}^\mu(z;\gamma).$$

therefore

$$\cos\pi\nu\, S_\nu^{\mu(3)}(z;\gamma) =$$

$$\frac{1}{\pi}\Big[\sin\pi(\nu-\mu)\, e^{-\pi i(2\nu+\mu+1)} K_\nu^\mu(\gamma^2)\, Qs_{-\nu-1}^\mu(z;\gamma) +$$

$$+\sin\pi(\nu+\mu)\, e^{\pi i(\nu-\mu-\frac{1}{2})} K_{-\nu-1}^\mu(\gamma^2)\, Qs_\nu^\mu(z;\gamma)\Big] \tag{6.21}$$

Also from (5.19) we have the identity

$$\sin\pi(\nu-\mu)\, Qs_{-\nu-1}^\mu(z;\gamma) = \sin\pi(\nu+\mu)\, Qs_\nu^\mu(z;\gamma) - \pi e^{\pi i\mu}\cos\pi\nu\, Ps_\nu^\mu(z;\gamma).$$

hence

$$\cos\pi\nu\, S_\nu^{\mu(3)}(z;\gamma) =$$

$$= \frac{1}{\pi}\Big\{\Big[e^{-\pi i(2\nu+\mu+1)} K_\nu^\mu(\gamma^2) - i\, e^{\pi i(\nu-\mu)} K_{-\nu-1}^\mu(\gamma^2)\Big]\sin\pi(\nu+\mu)\, Qs_\nu^\mu(z;\gamma)$$

$$-\pi e^{\pi i\mu}\, e^{-\pi i(2\nu+\mu+1)} K_\nu^\mu(\gamma^2)\cos\pi\nu\, Ps_\nu^\mu(z;\gamma)\Big\}.$$

Simplifying we find that

$$S_\nu^{\mu(3)}(z;\gamma) =$$

$$= -\frac{e^{-i\pi\mu}}{\pi} \cdot \frac{\sin\pi(\nu+\mu)}{\cos\pi\nu}\left[e^{-2\pi\nu i} K_\nu^\mu(\gamma^2) + i e^{\pi\nu i} K_{-\nu-1}^\mu(\gamma^2)\right] . Qs_\nu^\mu(z;\gamma)$$

$$+ e^{-2\pi\nu i} K_\nu^\mu(\gamma^2) Ps_\nu^\mu(z;\gamma) \quad . \qquad (6.22)$$

$S_\nu^{\mu(3)}(z;\gamma)$ is now in the required form where

$$A = e^{-2\pi\nu i} K_\nu^\mu(\gamma^2)$$

and (6.23)

$$B = -e^{-\pi\mu i} \frac{\sin\pi(\nu+\mu)}{\cos(\pi\nu).\pi}\left[e^{-2\pi\nu i} K_\nu^\mu(\gamma^2) + i e^{\pi\nu i} K_{-\nu-1}^\mu(\gamma^2)\right]$$

From (6.18) and (6.20), writing $m_{2\nu}^\mu(\lambda,\gamma)=m$, we have

$$[\mathbb{K} + m\mathbb{M}]Ps_\nu^\mu(ix;\gamma) + [\mathbb{L} + m\mathbb{N}]Qs_\nu^\mu(ix;\gamma)$$

$$= h(\tfrac{-1}{2}\pi e^{\frac{\pi i\nu}{2}})[A Ps_\nu^\mu(ix;\gamma) + B Qs_\nu^\mu(ix;\gamma)]$$

hence

$$\mathbb{K} + m\mathbb{M} = -\frac{h\pi}{2} e^{\frac{\pi i\nu}{2}} . A \quad , \qquad \text{(i)}$$

$$\mathbb{L} + m\mathbb{N} = -\frac{h\pi}{2} e^{\frac{\pi i\nu}{2}} . B \quad , \qquad \text{(ii)}$$

Multiplying (i) by B and (ii) by A and rearranging, we find

$$m = \frac{\mathbb{A}\mathbb{L} - \mathbb{B}\mathbb{K}}{\mathbb{B}\mathbb{M} - \mathbb{A}\mathbb{N}} \qquad (6.24)$$

We must now look for the zeros, and poles of m

THE POLES OF m

The poles of m occur where

$$\mathbb{B}\mathbb{M} - \mathbb{A}\mathbb{N} = 0 \qquad \text{(assuming } AL - BK \neq 0\text{)}$$

Substituting for the various quantities from (6.18) and (6.23) yields:

$$e^{-2\pi\nu i} K^{\mu}_{\nu}(r^2)$$

$$= \frac{-e^{\frac{i\pi}{2}(\mu-\nu+1)} \sin\pi(\nu+\mu)}{2\cos\frac{\pi}{2}(\nu+\mu)\cos\pi\nu} \left[e^{-2\pi\nu i} K^{\mu}_{\nu}(r^2) + i e^{\pi\nu i} K^{\mu}_{-\nu-1}(r^2)\right],$$

$$\mathbb{N} \neq 0.$$

rearranging we have

$$-\left\{\frac{e^{\frac{i\pi}{2}(\mu-\nu+1)} \sin\frac{\pi}{2}(\nu+\mu)}{\cos\pi\nu} + 1\right\} K^{\mu}_{\nu}(r^2)$$

$$= \frac{e^{\frac{i\pi}{2}(\mu-\nu+1)} \sin\frac{\pi}{2}(\nu+\mu)}{\cos\pi\nu} e^{\pi i(3\nu+\frac{1}{2})} K^{\mu}_{-\nu-1}(r^2) . \qquad (6.25)$$

Writing the trigonometric functions as exponentials we obtain

$$-[1+e^{i\pi(\mu-\nu)}]K_\nu^\mu(\gamma^2) = [e^{i\pi(\mu+\nu)}-1]e^{i\pi(\nu+\frac{1}{2})}K_{-\nu-1}^\mu(\gamma^2).$$

therefore the condition for m to have poles is

$$K_\nu^\mu(\gamma^2) - e^{i\pi(\nu+\frac{1}{2})}\left[\frac{1-e^{i\pi(\mu+\nu)}}{1+e^{i\pi(\mu-\nu)}}\right]K_{-\nu-1}^\mu(\gamma^2) = 0 \quad . \tag{6.26}$$

THE ZEROS OF m

The zeros of m occur where

$$\mathbb{A}\mathbb{L} - \mathbb{B}\mathbb{K} = 0$$

Again substituting in for the various quantities from (6.12) and (6.23) yields

$$e^{-2\pi\nu i}K_\nu^\mu(\gamma^2)$$

$$= \frac{e^{\frac{\pi i}{2}(\mu-\nu)}\cos\frac{\pi}{2}(\nu+\mu)}{\cos\pi\nu}\left[e^{-2\pi\nu i}K_\nu^\mu(\gamma^2) + ie^{\pi\nu i}K_{-\nu-1}^\mu(\gamma^2)\right] \,,$$

$$\mathbb{L} \neq 0 \,,$$

rearranging we have

$$\left[\cos\pi\nu - e^{\frac{\pi i}{2}(\mu-\nu)}\cos\frac{\pi}{2}(\nu+\mu)\right]K_\nu^\mu(\gamma^2) \tag{6.27}$$

$$= e^{\frac{\pi i}{2}(\mu+\nu)}e^{\pi i(2\nu+\frac{1}{2})}\cos\frac{\pi}{2}(\nu+\mu)K_{-\nu-1}^\mu(\gamma^2) \quad .$$

Again writing the cosines as sums of exponentials, we find on simplifying that

$$[e^{i\pi\nu} - e^{i\pi\mu}]K_\nu^\mu(\gamma^2) = e^{i\pi(2\nu+\frac{1}{2})}[e^{i\pi(\nu+\mu)} + 1]K^\mu_{-\nu-1}(\gamma^2) .$$

Therefore the condition for m to have zeros is

$$K_\nu^\mu(\gamma^2) - e^{i\pi(\nu+\frac{1}{2})}\left[\frac{1 + e^{i\pi(\mu+\nu)}}{1 - e^{i\pi(\mu-\nu)}}\right]K^\mu_{-\nu-1}(\gamma^2) = 0 . \tag{6.28}$$

Hence from (6.24) we find that

$$m_{2\nu}^\mu(\lambda,\gamma)$$

$$= \frac{\mathbb{L}}{\mathbb{N}}\left\{\frac{K_\nu^\mu(\gamma^2)(1 - e^{i\pi(\mu-\nu)}) - e^{i\pi(\nu+\frac{1}{2})}(1 + e^{i\pi(\mu+\nu)})K^\mu_{-\nu-1}(\gamma^2)}{K_\nu^\mu(\gamma^2)(1 + e^{i\pi(\mu-\nu)}) - e^{i\pi(\nu+\frac{1}{2})}(1 - e^{i\pi(\mu+\nu)})K^\mu_{-\nu-1}(\gamma^2)}\right\} . \tag{6.29}$$

At $\lambda = \lambda_n$ L and N do not have any zeros or poles because then the eigenfunctions would either vanish, or cease to be L^2. L or N could have branch points, but assuming the theorems of [15, Ch.5] to hold then as $q(x) \to \infty$ there is no continuous spectrum, only a discrete one, bounded below. The condition that (6.29) be zero or infinity corresponds to equation (3.8 of [12]) which was obtained by totally different methods, without appealing to the Titchmarsh--Weyl theory.

THE EXPRESSION FOR $m_{2\nu}^{\mu}(\lambda,\gamma)$ as $\gamma \to 0$

In this section we show that as $\gamma \to 0$, $m_{2\nu}^{\mu}(\lambda,\gamma)$ reduces to the $m(\lambda)$ function obtained for Legendres equation in Chapter 4 (4.30). We consider Legendres equation to be the unperturbed equation, i.e. prior to the application of the $\gamma^2(1+x^2)$ term.

To begin with, we note the following properties of the coefficients $a_{\nu,2r}^{\mu}(\gamma)$, and the normalising constant $A_{\nu}^{\mu}(\gamma^2)$ given in [10]:

from the continued fraction representation

$$a_{\nu,2r}^{\pm\mu}(0) = 0$$

and (6.30)

$$a_{\nu,0}^{\pm\mu}(0) = 1$$

Hence we have

$$A_{\nu}^{\pm\mu}(\gamma^2) = \sum_{r=-\infty}^{\infty} a_{\nu,2r}^{\pm\mu}(\gamma) \longrightarrow 1 \quad \text{as } \gamma \to 0$$

allowing γ to tend to zero in (5.38) yields

$$K_{\nu}^{\mu}(\gamma^2) \sim \tfrac{1}{2}\left(\tfrac{\gamma}{4}\right)^{\nu} \frac{\Gamma(1+\nu-\mu)\,e^{\pi\nu i}\,\Gamma(\frac{1}{2}-\nu)}{\Gamma(\nu+3/2)}$$

similarly

$$K_{-\nu-1}^{\mu}(\gamma^2) \sim 2^{2\nu+1}\,\gamma^{-\nu-1}\,.\,e^{-\pi\nu i}\,\frac{\Gamma(-\nu-\mu)\Gamma(\nu+3/2)}{\Gamma(\frac{1}{2}-\nu)}$$

as $\gamma \to 0$.

hence using (6.29) $m^{\mu}_{2\nu}(\lambda, \gamma) \sim$

$$\frac{\mathbb{L}}{\mathbb{N}}\left\{\frac{\gamma^{\nu}\Gamma(1+\nu-\mu)e^{\pi\nu i}\Gamma(\frac{1}{2}-\nu)[1-e^{i\pi(\mu-\nu)}]}{\Gamma(\nu+3/2)} + \frac{i4^{2\nu+1}\gamma^{-\nu-1}\Gamma(-\nu-\mu)\Gamma(\nu+\frac{3}{2})[1+e^{i\pi(\mu+\nu)}]}{\Gamma(\frac{1}{2}-\nu)}\right\}$$

$$\left\{\frac{\gamma^{\nu}\Gamma(1+\nu-\mu)e^{\pi\nu i}\Gamma(\frac{1}{2}-\nu)[1+e^{i\pi(\mu-\nu)}]}{\Gamma(\nu+3/2)} + \frac{i4^{2\nu+1}\gamma^{-\nu-1}\Gamma(-\nu-\mu)\Gamma(\nu+\frac{3}{2})[1-e^{i\pi(\mu+\nu)}]}{\Gamma(\frac{1}{2}-\nu)}\right\}^{-1}$$

as $\gamma \to 0$. If we factor out $\gamma^{-\nu-1}$ and take the limit as $\gamma \to 0$, we obtain,

$$m^{\mu}_{\nu}(\lambda, 0) = \frac{[1+e^{i\pi(\mu+\nu)}]}{[1-e^{i\pi(\mu+\nu)}]}\frac{\mathbb{L}}{\mathbb{N}}$$

(6.31)

$$= \frac{i\cos\frac{\pi}{2}(\mu+\nu)}{\sin\frac{\pi}{2}(\mu+\nu)}\frac{\mathbb{L}}{\mathbb{N}} .$$

We can find expressions for L and N when γ = 0 from the boundary conditions on Θ and ϕ' at x = 0.

We have

$$\Theta^{\mu}_{\nu}(0,0) = 1 .$$

This implies

$$\frac{2^{\mu}\pi^{3/2}e^{i\frac{\pi}{2}(3\mu+2)}\mathbb{L}}{\sin\pi(\nu+\mu)\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu)} = 1 ,$$

Hence

$$\mathbb{L} = \frac{\sin\pi(\nu+\mu)\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu)}{2^{\mu}\pi^{3/2}e^{i\frac{\pi}{2}(3\mu+2)}} ,$$

$$\gamma^2 = 0 .$$

Also from

$$\phi_\nu^{\mu\prime}(0;0) = -1 ,$$

We have

$$\frac{2^{\mu+1}\pi^{3/2}e^{\frac{3i\pi}{2}(\mu+1)}\mathbb{N}}{\sin\pi(\nu+\mu)\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(-\frac{1}{2}\nu-\frac{1}{2}\mu)} = -1 ,$$

Which gives us

$$\mathbb{N} = \frac{\sin\pi(\nu+\mu)\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(-\frac{1}{2}\nu-\frac{1}{2}\mu)}{2^{\mu+1}\pi^{3/2}e^{\frac{3\pi i}{2}(\mu+1)}} ,$$

$$\gamma^2 = 0 .$$

Therefore

$$\frac{\mathbb{L}}{\mathbb{N}} = \frac{2i\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(-\frac{1}{2}\nu-\frac{1}{2}\mu)}$$

Substituting this expression into (6.31) we obtain

$$M_\nu^\mu(\lambda,0) = -2\frac{\cos\frac{\pi}{2}(\mu+\nu)}{\sin\frac{\pi}{2}(\mu+\nu)}\frac{\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(-\frac{1}{2}\nu-\frac{1}{2}\mu)}$$

If we use the identities

$$\Gamma(-\tfrac{1}{2}\nu-\tfrac{1}{2}\mu)\Gamma(1+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu) = \frac{-\pi}{\sin\frac{\pi}{2}(\nu+\mu)} = \frac{\pi}{\sin\frac{\pi}{2}(-\nu-\mu)}$$

and

$$\frac{1}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu)\Gamma(\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}\mu)} = \frac{\cos\frac{\pi}{2}(\nu+\mu)}{\pi}$$

then upon simplifying we find that

$$m_\nu^\mu(\lambda,0) = \frac{2\Gamma(1+\frac{1}{2}\nu+\frac{1}{2}\mu)\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\mu)}$$

Which agrees with (4.30), as we set out to prove.

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