# Dynamic Buckling of Linear Viscoelastic Rods 

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I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Doctor of Philosophy in Applied Mathematical Sciences is entirely my own work and has not. been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.


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#### Abstract

This thesis examines the planar bending of a viscoelastic rod subject to a uniaxial load $\lambda(t)$. The rod is assumed to be inextensible, and the torsion and shear of the sections are ignored. The bending moment across a section is assumed to depend on the curvature through a linear hereditary law of Boltzmann type. The rod is composed of a solid material, so the creep function remains bounded for all time. Thus a viscoelastic solid rod in simple extension eventually approaches an equilibrium state. This is equivalent to $G(\infty):=\lim _{t \rightarrow \infty} G(t)>0$, where $G(t)$ is the material specific relaxation function. The exact nonlinear dynamic problem can be linearised about the straight equilibrium position to yield an integro-differential equation. It is this linear problem which is investigated here. The initial history of the deflection is allowed to be nontrivial. Usually this initial history is prescribed, but we also consider the problem without this assumption.

For constant loads, Laplace transform techniques can be employed to show that solutions decay if $\lambda<\lambda_{1} G(\infty) / G(0)$, and grow exponentially if $\lambda>\lambda_{1} G(\infty) / G(0)$, where $\lambda_{1}>0$ is the Euler critical load calculated using the instantaneous elasticity $G(0)$. For the standard viscoelastic material, we derive necessary and sufficient conditions on the material parameters which ensure that the solution is oscillatory.

For time-varying loads, the evolution equation for the initial history problem generates a semigroup, and has a unique solution which depends continuously on the initial data. This is in contrast to the corresponding results in the quasi-static theory. The Volterra-Graffi energy is used to construct a suitable Lyapunov function, which can be used to demonstrate that the zero solution is stable for a large class of loads satisfying $0<\lambda(t)<\lambda_{1} G(\infty) / G(0)$.

Multiple scale methods are also used to determine various approximate solutions. For a standard viscoelastic material with long relaxation time, the elastic and creep effects occur on different time scales. If $\lambda>\lambda_{1} G(\infty) / G(0)$, an approximate solution is determined and is used to investigate the effect of the different types of initial disturbance on the growth rate of the solution. Also if a standard viscoelastic material is subject to a periodic load $\lambda(t)$, an approximate stability region in parameter space is found when the parametric excitation is near the principal resonance.


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## Notation

$G_{0} \quad$ Instantaneous modulus of elasticity, $G_{0}=G(0)$.
$G_{\infty} \quad$ Long-term (Equilibrium) modulus of elasticity, $G_{\infty}=\lim _{t \rightarrow \infty} G(t)$.
$\lambda_{n} \quad$ Instantaneous buckling load, $\lambda_{n}=n^{2} \pi^{2} G_{0}$ ( $=\pi^{2} n^{2}$ scaled).
$\gamma_{n} \quad \gamma_{n}=\left(1+\sigma n^{2} \pi^{2}\right)^{-1}$.
$y^{l}(\tau) \quad$ History of $y$ up to time $t, y^{t}(\tau)=y(t-\tau)$ for $\tau \geq 0$.
$\bar{\mu} \quad$ Laplace-Stieltjes transform of a measure $\mu, \bar{\mu}(p)=\int_{0}^{\infty} \mathrm{e}^{-p t} \mathrm{~d} \mu(t)$ if $\mu$ is a measure on $\mathbb{R}^{+}$, and $\bar{\mu}(p)=\int_{-\infty}^{\infty} \mathrm{e}^{-p t} \mathrm{~d} \mu(t)$ if $\mu$ is a measure on $\mathbb{R}$.
$\bar{a} \quad$ Laplace transform of a function $a, \bar{a}(p)=\int_{0}^{\infty} \mathrm{e}^{-p t} a(l) \mathrm{d} t$ if $\mu$ is defined on $\mathbb{R}^{+}$, and $\bar{a}(p)=\int_{-\infty}^{\infty} \mathrm{e}^{-\overline{p t}} a(t) \mathrm{d} t$ if $a$ is defined on $\mathbb{R}$.
$V \quad V=\left\{y \in H^{4}: y(0)=y(1)=0, y_{s,}(0)=y_{s s}(1)=0\right\}$.
$H \quad H^{2} \cap H_{0}^{\perp}$.
$L^{p}, \quad$ Measurable functions with finite norm $\left\{|x(t)|^{p} \mathrm{~d} t\right\}^{1 / p}$. $p \in[1, \infty)$
$L^{\infty} \quad$ Measurable functions with finite norm ess sup $|x(t)|$.
$L_{\alpha}^{2} \quad$ Denotes the $\alpha$-weighted $L^{2}$ space of functions $\int$ with $\|f\|_{L_{\alpha}^{2}}=\left(\int_{0}^{\infty} \alpha(t)|f(t)|^{2} \mathrm{~d} t\right)^{1 / 2}<\infty$.

M Finite measures on $\mathbb{R}^{+}$and total variation norm.
$B C \quad$ Bounded continuous functions; sup-norm.
$B C \quad$ Bounded continuous functions tend to zero at infinity; sup-norm.
BUC Bounded uniformly continuous functions; sup-norm.

## Chapter 1

## Introduction

This thesis examines the dynamic planar flexure of a thin, inextensible, uniform viscoelastic rod. Sections are labelled by their arc-length $s$ along the central axis from the left end. For simplicity we assume the rod has unit length. Both ends are pinned and on the same horizontal level, (cf. Figure 1.1). Let $y(s, t)$ be the vertical displacement of the centroid of section $s$ at time $t$. The left end is held fixed, and a load $\lambda(t)$ applied to the right.


Figure 1.1: Buckling of a thin perfect rod under end loading.

The rod is assumed to be viscoelastic in the sense that the bending moment $M(s, t)$ across a section satisfies the linear constitutive equation

$$
M(s, t)=G(0) y_{s s}(s, t)+\int_{0}^{\infty} \dot{G}(\tau) y_{s s}(s, t-\tau) \mathrm{d} \tau
$$

where $G(t)$ is the relaxation function of the material. This form of constitutive equation is quite general and includes viscoelastic materials that can be modelled by rheological structures consisting of configurations of springs and dash-pots. The rod is assumed to be solid rather than fluid. Hence we require that the creep function $J(t)$ corresponding to $G(t)$, which is defined in (2.13), remains bounded. Thus a constant applied stress causes a bounded strain. The boundedness of $J(t)$ is implied by $G(\infty)>0$. An important example of such a model is the standard viscoelastic material which obeys

$$
G(t)=G_{\infty}+\left(G_{0}-G_{\infty}\right) \mathrm{e}^{-a t}
$$

for some $a>0$.

The linearised dynamic equation of motion of the rod is

$$
\begin{equation*}
y_{t t}-\sigma y_{s s t t}=-y_{s s s s}-\int_{0}^{\infty} \dot{G}(\tau) y_{s s s s}(t-\tau) \mathrm{d} \tau-\lambda(t) y_{s s}, \tag{1.1}
\end{equation*}
$$

for $0<s<1$. Here $\lambda(t)$ is continuous and $\sigma>0$. The term $\sigma y_{s s t t}$ is due to rotatory inertia. We impose the boundary conditions

$$
\begin{equation*}
0=y(0, t)=y_{s s}(0, t), \quad 0=y(1, t)=y_{s s}(1, t) \tag{1.2}
\end{equation*}
$$

which correspond to the rod being pinned at each end. The results of this thesis are true with obvious modifications, if the ends are both clamped, one end is clamped and the other free or one end is clamped and the other pinned. In an initialhistory problem we require that a solution $y(s, t)$ satisfy (1.1) and (1.2) for $(s, t) \in$ $[0,1] \times[0, \infty)$, and that $y(s, t)$ is known for $(s, t) \in[0,1] \times(-\infty, 0]$ with (1.2) holding for $-\infty<t \leq 0$. For a problem without initial history we require that a solution satisfy (1.1) and (1.2) for $(s, t) \in[0,1] \times \mathbb{R}$.

The initial history problem (1.1) and (1.2) has been studied by Dost \& Glockner [14], Szyszkowski \& Glockner [39] and Spena [38] under the simplifying assumptions that the load is constant and the initial history of $y$ is trivial. Indeed these papers confine their analysis to rods composed of standard viscoelastic material. However in [14], equations (1.1) and (1.2) are derived and an equation for the Laplace transform of the solution found. Spena derives the equations for a more general class of ageing viscoelastic materials. Cederbaum \& Mond [8] consider the dynamic problem (1.1) and (1.2) with $\lambda(t)$ periodic and the initial history trivial. They analyse a retarded version of this problem using multiple scale methods. Dall'Asta \& Menditto [10] also examine the dynamic problem (1.1) and (1.2) for trivial initial history. A variational principle is derived, which is then used to calculate numerical solutions when $\lambda(t)$ is constant and periodic. Rotatory inertia is also ignored in these papers.

The quasi-static approximation of (1.1) neglects the inertia terms, and therefore assumes that the bending moment is constant at each instant across all sections.

The problem reduces to

$$
\lambda(t) y(t)+G(0) y_{s s}(t)+\int_{0}^{\infty} \dot{G}(\tau) y_{s s}(t-\tau) \mathrm{d} \tau=0
$$

subject to

$$
y(0, t)=y(1, t)=0 .
$$

For constant loads, this problem has been considered by Distefano [11, 12, 13]. Some of the Russian work is presented in Rabotnov [35]. The problem with $\lambda(t)$ non-constant has been studied by Gurtin [19], Gurtin, Mizel \& Reynolds [20] and Reynolds [36].

A detailed discussion of the derivation of the equations (1.1) and (1.2) is given in Chapter 2. It is important to appreciate that, though the viscoelastic rod dissipates energy in the sense that part of the energy supplied is generally not converted to kinetic nor free energy, the system is not necessarily dissipative. The terminal load is doing work, and hence energy is being supplied to the system. It is shown in Chapter 2 that $E(t)$, the sum of the kinetic and Graffi-Volterra free energies, satisfies the inequality

$$
\frac{\mathrm{d} E}{\mathrm{~d} t} \leq \lambda(t) \frac{\mathrm{d}}{\mathrm{~d} t} \int_{0}^{1} y_{s}^{2}(s, t) \mathrm{d} s
$$

The case of constant loads is investigated in Chapter 3. We use Laplace transform techniques to show that solutions of the initial history problem exist and are unique. We also find that the load $\lambda(t)$ determines the stability of the rod. Assuming that the initial history satisfies certain mild conditions, if the load is less than the critical value of $\pi^{2} G(\infty)$ the solution is bounded, integrable and tends to zero. For loads exceeding this critical value, the solution can be decomposed into a sum of an exponentially increasing term and a bounded integrable function which tends to zero.

In the special case of the standard viscoelastic material we can obtain more precise information. The form of its solution depends on the location in the complex plane of the roots of a cubic equation. Necessary and sufficient conditions for all these roots to lie in the left-half complex plane are derived in Chapter 3. These
conditions are of course equivalent to the stability conditions applying in the general case. We also find necessary and sufficient conditions for a pair of these roots to be complex, implying an oscillatory motion for the rod. These conditions do not agree with those given by Dost \& Glockner [39], who developed an approximate series solution.

In Chapter 3 we also consider solutions of the dynamic problem with no initial history. Mild conditions on histories are specified just to ensure that the bending moment always exists. We use work by Titchmarsh [40] involving Fourier integrals to show that the trivial solution is unique for constant loads $\lambda<\pi^{2} G(\infty)$. Furthermore, it is shown that there exists an exponential increasing solution if the load $\lambda>\pi^{2} G(\infty)$. Virga and Capriz [42] use a contraction mapping argument to investigate the standard displacement problem in linear viscoelasticity with Cauchy data. We apply a similar procedure and show that, for loads satisfying $\lambda<\pi^{2} G(\infty), y=0$ is the only solution in a suitable weighted space of continuous functions.

Dafermos in [9] used a semigroup approach to prove existence and uniqueness, as well as asymptotic stability of the trivial solution, for the standard displacement problem of linear viscoelasticity, which is autonomous. In Chapter 4 we consider time-dependent loads, and extend the method of Dafermos to this nonautonomous initial history problem. We construct a suitable Hilbert space in which the norm of the solution is defined to be the energy of the rod. The evolution equation is shown to generate a. $C_{0}$ contraction semigroup on this Hilbert space, and have a unique solution which depends continuously on the initial data. This is in contrast to the results in the quasi-static theory, for which Reynolds [36] showed that uniqueness is ensured only if $\max _{t \geq 0}|\lambda(t)|<\pi^{2} G(\infty)$. The energy of the rod is also used in Chapter 4 to construct a Lyapunov function, which is used to demonstrate that the zero solution is stable for a large class of loads satisfying $0<\lambda(t)<\pi^{2} G(\infty)$.

Multiple scale methods are used in Chapter 5 to determine approximate solutions to various problems. In Section 5.3, we consider the rod to be of a standard viscoelastic material with long relaxation time $a^{-1}$. The elastic and creep effects occur on different time scales. If the load is constant and $\lambda<\pi^{2} n^{2} G(0)$, we obtain an approximate solution valid for $0 \leq t \leq O\left(a^{-3}\right)$ for the $n^{\text {th }}$ Fourier mode. The ex-
act solution must eventually grow if $\pi^{2} n^{2} G(\infty)<\lambda<\pi^{2} n^{2} G(0)$. The approximate solution allows an investigation of the effect of the different types of initial disturbance on the growth rate. The retarded problem is investigated in Section 5.4 for general viscoelastic materials. This entails replacing the relaxation function $G$ by its retardation $G_{\varepsilon}$, where $G_{\varepsilon}(t)=G(\varepsilon t)$. The retardation parameter $\varepsilon$ is small, so that the elastic and creep responses occur on different time scales in the new problem. The leading order and the first order terms are calculated using a procedure similar to that used by Angell \& Olmstead [1], [2] in their work on singularly perturbed Volterra equations. Surprisingly if we set $\varepsilon=1$, in this approximate solution of the retarded problem, we get a function which agrees exactly with the approximate solution found in Section 5.3 if the rod is of standard viscoelastic material. In Section 5.5 we consider a generalisation of the results described in Section 5.3 by constructing a multiple scale approximation for the $n^{\text {th }}$ Fourier mode under the assumption of a slowly varying load. Finally in Section 5.6 we consider periodic loads of the form $\lambda(t)=P_{0}+P_{1} \cos \Omega t$, where $P_{0}<\pi^{2} n^{2} G(\infty)$, and $P_{1}$ is small. The principal parametric resonance occurs when $\Omega \approx 2 \omega, \omega$ being the natural frequency of the rod. If the rod is elastic, the corresponding dynamic equation reduces to a family of Mathieu equations. For $\Omega \approx 2 \omega$, we determine an approximate stability region in parameter space. These regions agree well with numerical calculations, but differ qualitatively from the stability regions obtained by Cederbaum \& Mond [8] in their analysis of the retarded problem.

Finally, the derivation of the standard viscoelastic material from models involving springs and dash-pots is presented in Appendix A. Also the algorithms used to calculate numerical solutions to the problems considered in this work are outlined in Appendix B.

## Chapter 2

## Buckling Problem

### 2.1 Introduction

In this chapter we briefly show how the system of dynamic equations (1.1) were derived from the general dynamic theory of Cosserat rods. The material independent mechanical equations are augmented with a constitutive equation that is assumed to be of Boltzmann type which depends on a material specific relaxation function. The geometrically exact problem is then linearised assuming small transverse displacement. Also we define the Volterra-Graffi energy of the rod and show that it satisfies an inequality which expresses the dissipative nature of the problem.

### 2.2 Derivation of the Mathematical Problem

This is based on the procedure described in Antman [3, 4]. We model as a rod, an axially loaded, slender, structural element, whose length is large compared to the greatest linear dimension of the cross-section. We assume that every configuration of the rod is determined once the centroid and orientation of each normal crosssection are specified. The torsion and shear of the cross-sections are assumed to be negligible and ignored. Thus, cross-sections that are normal to the line of centroids in the unstressed configuration remain planar and normal to the line of centroids in any deformed configuration. The length in the reference configuration from the fixed end, along the axis of the rod, to the centroid of a cross-section is denoted by $s$, and is used to label the section.

The motion of the rod of length $a$ is given by a triplet of vector-valued functions

$$
[0, a] \times \mathbb{R} \ni(s, t) \mapsto(\mathbf{r}(s, t), \mathbf{b}(s, t), \mathbf{c}(s, t)) \in \mathbb{R}^{3}
$$

where $\mathbf{b}$ and $\mathbf{c}$ are orthonormal. $\mathbf{r}(s, t)$ is the position of the particle at the centroid of
the cross-section $s$, while $\mathbf{b}(s, t)$ and $\mathbf{c}(s, t)$ represent the orientation of the section $s$ at time $t$. In particular, $\mathbf{b}(s, t)$ and $\mathbf{c}(s, t)$ may be regarded as tangents to curvilinear co-ordinate curves which are the images of plane cartesian co-ordinate axes in the reference description of section $s$. The orientation of the section can be deformed in general by shear, torsion and compression. A third director is defined by $\mathbf{a}=\mathbf{b} \times \mathbf{c}$. In the absence of shear, this director is the unit-length and is tangent to the strained axis of centroids, (cf. Pigure 2.1).


Figure 2.1: Configuration of a deformed Cosseral rod.

Let $\mathbf{n}(s, t)$ be the resultant force and $\mathbf{m}(s, t)$ the resultant moment across the section $s$ at time $l$. We assume that there are no body forces or body couples acting on the rod; in particular, we ignore gravitational effects. Conservation of linear momentum over the interval $\left(s_{1}, s_{2}\right) \subset(0, a)$ requires that

$$
\frac{\mathrm{d}}{\mathrm{~d} t} \int_{s_{1}}^{s_{2}} \rho A \mathbf{r}_{t} \mathrm{~d} s=\mathbf{n}\left(s_{2}, t\right)-\mathbf{n}\left(s_{1}, t\right),
$$

where $\rho(s) \Lambda(s)$ is the mass density per unit length at $s$. Differentiating with respect to $s$ gives

$$
\begin{equation*}
\rho A \mathbf{r}_{t t}=\mathbf{n}_{s} . \tag{2.1}
\end{equation*}
$$

The conservation of angular momentum can be expressed in the following form

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{~d} t} \int_{s_{1}}^{s_{2}} \rho\left(I \mathbf{b} \times \mathbf{b}_{t}+J \mathbf{c} \times \mathbf{c}_{t}+A \mathbf{r} \times \mathbf{r}_{t}\right) \mathrm{d} s & = \\
& \mathbf{m}\left(s_{2}, t\right)-\mathbf{m}\left(s_{1}, t\right)+\mathbf{r}\left(s_{2}, t\right) \times \mathbf{n}\left(s_{2}, t\right)-\mathbf{r}\left(s_{1}, t\right) \times \mathbf{n}\left(s_{1}, t\right) .
\end{aligned}
$$

Here $\rho(s) I(s)$ and $\rho(s) J(s)$ denote the principal mass moments of inertia of crosssection $s$ about lines through the centroid in the directions $\mathbf{c}$ and $\mathbf{b}$ respectively. Differentiating with respect to $s$ and simplifying using (2.1) gives

$$
\begin{equation*}
\rho I \mathrm{~b} \times \mathrm{b}_{t t}+\rho J \mathrm{c} \times \mathbf{c}_{t t}+\rho A \mathbf{r} \times \mathbf{r}_{t t}=\mathrm{m}_{s}+\mathbf{r}_{s} \times \mathbf{n} . \tag{2.2}
\end{equation*}
$$

The motion of the rod is assumed to be planar, and hence

$$
\begin{equation*}
\mathbf{r}=x(s, t) \mathbf{i}+y(s, t) \mathbf{j}, \quad \mathbf{c}(s, t)=-\mathbf{k} \tag{2.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbf{n}=-\Lambda(s, t) \mathbf{i}+\mathrm{N}(s, t) \mathbf{j}, \quad \mathrm{m}=M(s, t) \mathbf{k}, \tag{2.4}
\end{equation*}
$$

where $\{\mathbf{i}, \mathbf{j}, \mathrm{k}\}$ is a fixed orthonormal basis with vector $\mathbf{j}$ pointing upwards. $M$ is the bending moment about an axis parallel to k , while $\Lambda$ and N are the horizontal and vertical components of the resultant force respectively. The rod is also assumed to be inextensible; hence $\left|\mathbf{r}_{s}\right|=1$. Let $\theta(s, t)$ be the angle between the tangent vector $\mathrm{r}_{s}(s, t)$ and the horizontal vector i , so that, (cf. Figure 2.2),

$$
\begin{equation*}
x_{s}=\cos \theta, \quad y_{s}=\sin \theta . \tag{2.5}
\end{equation*}
$$

The directors $\mathbf{a}$ and $\mathbf{b}$ satisfy

$$
\begin{equation*}
\mathbf{a}=\cos \theta \mathbf{i}+\sin \theta \mathbf{j}, \quad \mathbf{b}=-\sin \theta \mathbf{i}+\cos \theta \mathbf{j} . \tag{2.6}
\end{equation*}
$$

The substitution of (2.3)-(2.6) into the conservation equations (2.1) and (2.2) yields


Figure 2.2: Planar buckling of a rod.
the dynamic equations, (c[. [5, 3]),

$$
\begin{aligned}
\rho A x_{t t} & =-\Lambda_{s} \\
\rho A y_{t t} & =\mathrm{N}_{s} \\
\rho I \theta_{t t} & =M_{s}+\Lambda \sin \theta+\mathrm{N} \cos \theta
\end{aligned}
$$

For simplicity, we also assume that the rod has uniform cross-section, i.e., both $\rho A$ and $\rho I$ are constant.

These dynamic equations of motion must be augmented by a constitutive assumption. Here the bending moment $M$ at the present time is assumed to depend on the history of the curvature of the axis of the column, through a linear hereditary law of Boltzmann type. More precisely, we suppose that there is a constant $\beta>0$, and a function $\alpha:[0, \infty) \rightarrow[0, \infty)$, such that

$$
M(s, t)=\beta \theta_{s}(s, t)-\int_{0}^{\infty} \alpha(\tau) \theta_{s}(s, t-r) \mathrm{d} \tau
$$

The moment-curvaiure relaxation function is given by

$$
t \mapsto \beta-\int_{0}^{t} \alpha(\tau) \mathrm{d} \tau
$$

It is convenient to work with non-dimensional variables. Therefore we set $s=$ $s^{*} a, x=x^{*} a, y=y^{*} a, t=t^{*} a^{2} \sqrt{\rho A / \beta}$ and

$$
\begin{aligned}
\theta^{*}\left(s^{*}, t^{*}\right) & =\theta(s, t), \\
\Lambda^{*}\left(s^{*}, t^{*}\right) & =\frac{a^{2}}{\beta} \Lambda(s, t), \\
\mathrm{N}^{*}\left(s^{*}, t^{*}\right) & =\frac{a^{2}}{\beta} \mathrm{~N}(s, t), \\
\sigma & =\frac{\rho I}{a^{2} \rho A}, \\
\alpha^{*}\left(t^{*}\right) & =a^{2} \sqrt{\frac{\rho A}{\beta^{3}}} \alpha(t), \\
M^{*}\left(s^{*}, t^{*}\right) & =\frac{a}{\beta} M(s, t) .
\end{aligned}
$$

For notational convenience, we only use these non-dimensional variables and neglect to write the stars.

The problem is then to find solutions of

$$
\begin{align*}
x_{t t} & =-\Lambda_{s}  \tag{2.7}\\
y_{t t} & =\mathrm{N}_{s}  \tag{2.8}\\
\sigma \theta_{t t} & =M_{s}+\Lambda \sin \theta+\mathrm{N} \cos \theta \tag{2.9}
\end{align*}
$$

for $(s, t) \in(0,1) \times \mathbb{R}$. Here $x(s, t), y(s, t), 0(s, t)$ and $M(s, t)$ are related by

$$
\begin{equation*}
x_{s}=\cos \theta, \quad y_{s}=\sin \theta \tag{2.10}
\end{equation*}
$$

and

$$
\begin{equation*}
M(s, t)=\theta_{s}(s, t)-\int_{0}^{\infty} \alpha(\tau) \theta_{s}(s, t-\tau) \mathrm{d} \tau \tag{2.11}
\end{equation*}
$$

[^0]We introduce the relaxation function $G:[0, \infty) \rightarrow \mathbb{R}^{+}$by

$$
G(t)=1-\int_{0}^{t} \alpha(\tau) \mathrm{d} \tau
$$

so that equation (2.11) becomes

$$
\begin{equation*}
M(s, t)=G_{0} \theta_{s}(s, t)+\int_{0}^{\infty} \dot{G}(\tau) \theta_{s}(s, t-\tau) \mathrm{d} \tau \tag{2.12}
\end{equation*}
$$

with $G_{0}:=G(0)=1$. The creep function $J:[0, \infty) \rightarrow \mathbb{R}^{+}$is the absolutely continuous solution of

$$
\begin{equation*}
\int_{0}^{t} G(\tau) J(t-\tau) \mathrm{d} \tau=t, \quad t \geq 0 \tag{2.13}
\end{equation*}
$$

It is well known that $J(t)$ is bounded if $G_{\infty}:=\lim _{t \rightarrow \infty} G(t)>0$, but unbounded if $G_{\infty}=0$. If $G_{\infty}=0$, an arbitrarily small constant moment would cause the curvature to become unbounded in time. Hence we assume that

$$
G_{\infty}>0 .
$$

It is easily shown that

$$
\begin{equation*}
\theta_{s}(s, t)=M(s, t)+\int_{0}^{\infty} \dot{J}(\tau) M(s, t-\tau) \mathrm{d} \tau, \quad t \in \mathbb{R} \tag{2.14}
\end{equation*}
$$

While for the remainder of this work we are mainly interested in the solutions of the dynamic viscoelastic flexure problem with a general relaxation function, we note an important example of a viscoelastic material, the standard viscoelastic material (or three element model) for which

$$
\begin{equation*}
G(t)=G_{\infty}+\left(G_{0}-G_{\infty}\right) \mathrm{e}^{-a t}, \tag{2.15}
\end{equation*}
$$

where $G_{0}, G_{\infty}$ and $a$ are material parameters. The derivation of (2.15) and its properties are described in Appendix A.

Next we specify the boundary conditions associated with a rod with pinned ends
and a known axial thrust ${ }^{2}$. The end $s=0$ is fixed and the end $s=1$ is constrained to be level with it, so that

$$
\begin{equation*}
x(0, t)=y(0, t)=0, \quad y(1, t)=0, \quad t \in \mathbb{R} \tag{2.16a}
\end{equation*}
$$

Since the ends are pinned,

$$
\begin{equation*}
M(0, t)=M(1, t)=0, \quad t \in \mathbb{R} \tag{2.16~b}
\end{equation*}
$$

Due to (2.14), this implies that

$$
\begin{equation*}
\theta_{s}(0, t)=\theta_{s}(1, t)=0, \quad t \in \mathbb{R} \tag{2.16c}
\end{equation*}
$$

Since the axial thrust applied at $s=1$ is known, $\Lambda(1, t)=\lambda(t)$ is prescribed on $\mathbb{R}$.
The dynamic counterpart of the Euler elastica is obtained by letting $\alpha=0$, to give the problem

$$
\begin{align*}
x_{t t} & =-\Lambda_{s}  \tag{2.17}\\
y_{t t} & =\mathrm{N}_{s}  \tag{2.18}\\
\sigma \theta_{t t} & =\theta_{s s}+\Lambda \sin \theta+\mathrm{N} \cos \theta \tag{2.19}
\end{align*}
$$

subject to (2.16a)-(2.16c). This has been studied in $[7]^{3}$.

### 2.3 Linearisation of Mathematical Problem

We now consider the linearisation of (2.7)-(2.11) about the solution $x(s, t)=s$, $y(s, t)=0, \theta(s, t)=0, \Lambda(s, t)=\lambda(t)$ and $N(s, t)=0$. Clearly the solution of the linearised problem satisfies $x_{s}=1, y_{s}=\theta$, and $\Lambda(s, t)=\lambda(t)$. Equation (2.9)

[^1]becomes
$$
\sigma \theta_{t t}=\theta_{s s}-\int_{0}^{\infty} \alpha x(\tau) \theta_{s s}(t-\tau) \mathrm{d} \tau+\lambda(t) \theta+\mathrm{N} .
$$

Therefore

$$
\mathrm{N}_{s}-\sigma \theta_{s t t}=-\theta_{s, s}+\int_{0}^{\infty} \alpha(\tau) \theta_{s s s}(t-\tau) \mathrm{d} \tau-\lambda(l) \theta_{s_{2}}
$$

or, using (2.8),

$$
\begin{equation*}
y_{t t}-\sigma y_{s s t t}=-y_{s s s s}+\int_{0}^{\infty} \alpha(\tau) y_{s s s s}(t-\tau) \mathrm{d} \tau-\lambda(t) y_{s s}, \tag{2.20}
\end{equation*}
$$

for $(s, t) \in(0,1) \times$ 㵊. This integro-diferential equation is augmented by the boundary conditions

$$
\begin{equation*}
y(0, t)=y(1, t)=0, \quad y_{s s}(0, t)=y_{s s}(1, t)=0, \quad t \in \mathbb{R} . \tag{2.21}
\end{equation*}
$$

The quasi-static approximation ignores the inertia terms and the linearised problem reduces to

$$
\lambda(t) y(t)+G_{0} y_{s s}(t)+\int_{0}^{\infty} \dot{G}(\tau) y_{s s}(t-\tau) \mathrm{d} \tau=0,
$$

subject to

$$
y(0, t)=y(1, t)=0 .
$$

This has been studied in [19], [20] and [36] for varying load and in [11, 12, 1.3] for constant loads.

### 2.4 Mechanical Considerations

The power, or rate of working, of the terminal load $\Lambda(1, t)=\lambda(t)$, acting on the rod is

$$
\begin{aligned}
-\lambda(t) x_{t}(1, t) & =-\lambda(t) \int_{0}^{1} x_{s t}(s, t) \mathrm{d} s \\
& =-\lambda(t) \frac{\mathrm{d}}{\mathrm{~d} t} \int_{0}^{1} \cos \theta(s, t) \mathrm{d} s
\end{aligned}
$$

Since we shall work within the linearised theory, a guadratic approximation to the power is required. Since

$$
\begin{aligned}
-\lambda(t) x_{t}(1, t) & =\lambda(t) \frac{\mathrm{d}}{\mathrm{~d} t} \int_{0}^{1} \frac{1}{2} 0^{2}(s, t) \mathrm{d} s+\cdots \\
& =\frac{1}{2} \lambda(t) \frac{\mathrm{d}}{\mathrm{~d} t} \int_{0}^{1} y_{s}^{2}(s, t) \mathrm{d} s+\cdots
\end{aligned}
$$

we define $P(t)$, the (quadratic approximation to the) power, to be

$$
\begin{equation*}
P(t)=\frac{1}{2} \lambda(t) \frac{\mathrm{d}}{\mathrm{~d} t} \int_{0}^{1} y_{s}^{2}(s, t) \mathrm{d} s \tag{2.22}
\end{equation*}
$$

Also we define the kinetic energy $K(t)$ by

$$
2 K(t)=\int_{0}^{1}\left(y_{t}^{2}+\sigma y_{t s}^{2}\right) \mathrm{d} s
$$

and the Volterra-Grafifi free energy ${ }^{4}$ by

$$
2 \Psi(t)=\int_{0}^{1}\left(G_{\infty} y_{s s}^{2}-\int_{0}^{\infty} \dot{G}(\tau)\left[y_{s s}(t)-y_{s s}(t-\tau)\right]^{2} \mathrm{~d} \tau\right) \mathrm{d} s
$$

The total energy of the rod is given by

$$
\begin{equation*}
E(t)=K(t)+\Psi(t) . \tag{2.23}
\end{equation*}
$$

We can now prove the following bound on the rate of change of energy of the rod, which expresses the fact that the work done by the external load $\lambda(t)$ is partially dissipated.

Proposition 2.1 Suppose that the relaxation function $G(t)$ satisfies the conditions $G(t)>0, \dot{G}(t) \leq 0$ and $\ddot{G}(t) \geq 0$ for all $t \geq 0$. Then

$$
\dot{E}(t) \leq P^{\prime}(t) .
$$

[^2]Proof. Rearranging equation (2.20) we have

$$
\begin{equation*}
y_{t t}=\sigma y_{s s t t}-G_{\infty} y_{s s s s}+\int_{0}^{\infty} \dot{G}(\tau)\left[y_{s s s s}(t)-y_{s s s s}(t-\tau)\right] \mathrm{d} \tau-\lambda(t) y_{s s} \tag{2.24}
\end{equation*}
$$

The substitution of (2.24) into the derivative of the total energy yields

$$
\begin{aligned}
\dot{E}(t)= & \dot{K}(t)+\dot{\Psi}(t) \\
= & \int_{0}^{1}\left(y_{t} y_{t t}+\sigma y_{t s} y_{t t s}+G_{\infty} y_{s s} y_{t s s}\right. \\
& \left.-\int_{0}^{\infty} \dot{G}(\tau)\left[y_{s s}(t)-y_{s s}(t-\tau)\right]\left[y_{t s s}(t)-y_{t s s}(t-\tau)\right] \mathrm{d} \tau\right) \mathrm{d} s \\
= & \int_{0}^{1}\left(y_{t}\left(\sigma y_{s s t t}-G_{\infty} y_{s s s s}+\int_{0}^{\infty} \dot{G}(\tau)\left[y_{s s s s}(t)-y_{s s s s}(t-\tau)\right] \mathrm{d} \tau-\lambda(t) y_{s s}\right)\right. \\
& +\sigma y_{t s} y_{t t s}+G_{\infty} y_{s s} y_{t s s} \\
& \left.-\int_{0}^{\infty} \dot{G}(\tau)\left[y_{s s}(t)-y_{s s}(t-\tau)\right]\left[y_{t s s}(t)-y_{t s s}(t-\tau)\right] \mathrm{d} \tau\right) \mathrm{d} s
\end{aligned}
$$

Integrating by parts using boundary conditions (2.21) yields

$$
\begin{aligned}
\dot{E}(t)= & \int_{0}^{1}\left(y_{t s s} \int_{0}^{\infty} \dot{G}(\tau)\left[y_{s s}(t)-y_{s s}(t-\tau)\right] \mathrm{d} \tau+\lambda(t) y_{t s} y_{s}\right. \\
& \left.-\int_{0}^{\infty} \dot{G}(\tau)\left[y_{s s}(t)-y_{s s}(t-\tau)\right]\left[y_{t s s}(t)-y_{t s s}(t-\tau)\right] \mathrm{d} \tau\right) \mathrm{d} s \\
= & \int_{0}^{1} \int_{0}^{\infty} \dot{G}(\tau)\left[y_{s s}(t)-y_{s s}(t-\tau)\right] y_{t s s}(t-\tau) \mathrm{d} \tau \mathrm{~d} s+\int_{0}^{1} \lambda(t) y_{t s} y_{s} \mathrm{~d} s \\
= & -\int_{0}^{1} \int_{0}^{\infty} \dot{G}(\tau)\left[y_{s s}(t)-y_{s s}(t-\tau)\right] y_{\tau s s}(t-\tau) \mathrm{d} \tau \mathrm{~d} s+\int_{0}^{1} \lambda(t) y_{t s} y_{s} \mathrm{~d} s
\end{aligned}
$$

Integrate by parts over $\tau$ and using the specified properties of the relaxation function we get

$$
\begin{align*}
\dot{E}(t) & =-\int_{0}^{1} \int_{0}^{\infty} \ddot{G}(\tau)\left[y_{s s}(t)-y_{s s}(t-\tau)\right]^{2} \mathrm{~d} \tau \mathrm{~d} s+\int_{0}^{1} \lambda(t) y_{t s} y_{s} \mathrm{~d} s \\
& \leq \frac{1}{2} \lambda(t) \frac{\mathrm{d}}{\mathrm{~d} t} \int_{0}^{1} y_{s}^{2} \mathrm{~d} s=P(t) . \tag{2.25}
\end{align*}
$$

Calculations of these kinds are well known in viscoelasticity. See for example Dafermos [9]. Gurtin and Reynolds proved similar results for the nonlinear quasi-
static problem in [19].
The second law of thermodynamics only requires that a non-constant relaxation function satislies, (cf. [15], Sec 3.2),

$$
\begin{equation*}
\int_{0}^{\infty} \dot{G}(t) \sin (\omega t) \mathrm{d} t<0, \quad \text { for all } \omega>0 \tag{2.26}
\end{equation*}
$$

While our restrictions on the relaxation function are stronger, experimental evidence suggests that the stronger monoticity conditions we assume are reasonable, (cf. [15] Sec 4.2).

Throughout this work it is assumed that the relaxation function belongs to $C^{1}$ and is non-constant, nonnegative and nonincreasing. In terrms of $\alpha(t)=-\dot{G}(t)$ this implies $\alpha(t)$ is continuous, nonnegative and nonincreasing on $\mathbb{R}^{+}$. Additional conditions on the relaxation function or $\alpha(t)$ are specified where required.

## Chapter 3

## Constant Load Problem

### 3.1 Introduction

In this chapter we examine the dynamic linearised viscoelastic flexure problem under the assumption of a constant load. Note that for constant loads the dynamic equation is autonomous. We formulate the dynamic problem as an abstract Volterra integro-differential equation which can be diagonalised using a Fourier Sine series. We consider two types of problems:

In the initial history problem, the history up to some initial time $t_{0}$, of the deflection is assumed to be known and the dynamic problem is solved for $t>t_{0}$. We only need to consider the initial time of $t_{0}=0$, as the dynamic equation is autonomous. Laplace transforms are used to show existence, uniqueness and asymptotic properties of the solution for $t>0$. Section 3.6 deals with the special case of the standard viscoelastic material, for which more detailed results are available. The initial history problem for the standard viscoelastic material has been studied in [39] and [14], assuming that the initial history is zero and the rod is statically and/or dynamically disturbed at time $t=0$. These papers do not include the term due to the rotatory inertia of the rod.

Also we consider solutions of the dynamic problem for all time. The initial history is not specified but conditions must be put on it to ensure the bending moment always exists. Whole line Laplace transforms are used to show uniqueness of the trivial solution for stable loads and the existence of a non-zero solution for loads greater than $\lambda_{n} G_{\infty}$. In Section 3.5 an alternative procedure is used to examine the uniqueness of the solution of the dynamic problem when the history of the displacement is not specified but is assumed to decay exponentially in the past. This procedure was used by Virga and Capriz [42] to investigate the standard displacement problem in linear viscoelasticity with Cauchy data.

### 3.2 Abstract Formulation and Diagonalisation

In this section we formulate the equation of motion (2.20) subject to the boundary conditions (2.21) as a retarded functional differential equation. This equation can be diagonalised into a family of Volterra integro-differential equations which, for constant loads, can be analysed using Laplace transforms.

We define the operator $L_{\sigma}: H \rightarrow L^{2}$, for any $\sigma>0$, by

$$
\begin{equation*}
L_{a} y=y-\sigma y_{s s}, \tag{3.1}
\end{equation*}
$$

where $H=H^{2}(0,1) \cap H_{0}^{1}(0,1) . H$ is a Hilbert space when given the inner product

$$
\left\langle y_{1}, y_{2}\right\rangle_{H}=\int_{0}^{1} y_{1}^{\prime \prime}(s) y_{2}^{\prime \prime}(s)+y_{1}^{\prime}(s) y_{2}^{\prime}(s) \mathrm{d} s
$$

We denote the inverse of $L_{\sigma}$ by $K_{\sigma}: L^{2} \rightarrow H$, i.e.,

$$
K_{\sigma} L_{\sigma} y=y, \quad \forall y \in H
$$

and

$$
L_{\sigma} K_{\sigma} y=y, \quad \forall y \in L^{2}
$$

Note that $L_{\sigma}$ is formally self-adjoint as

$$
\begin{equation*}
\int_{0}^{1} y_{1}(s)\left(L_{\sigma} y_{2}\right)(s) \mathrm{d} s=\int_{0}^{1} y_{2}(s)\left(L_{\sigma} y_{1}\right)(s) \mathrm{d} s, \quad \forall y_{1}, y_{2} \in \Pi . \tag{3.2}
\end{equation*}
$$

The equation of motion (2.20) can be written as

$$
L_{\sigma} y_{t t}=-y_{s s s s}+\int_{0}^{\infty} \alpha(\tau) y_{s s s s}(t-\tau) \mathrm{d} \tau-\lambda(t) y_{s s},
$$

or

$$
\begin{equation*}
y_{n}=-K_{\sigma} y_{s s s s}+\int_{0}^{\infty} \alpha(\tau) K_{\sigma} y_{s s s s,}(t-\tau) \mathrm{d} \tau-\lambda(t) K_{\sigma} y_{s s}, \tag{3.3}
\end{equation*}
$$

subject to the boundary conditions (2.21). Set

$$
V=\left\{y \in H^{4}(0,1) ; y(0)=y(1)=y_{s s}(0)=y_{s s}(1)=0\right\}
$$

with inner product

$$
\left\langle y_{1}, y_{2}\right\rangle_{v}=\int_{0}^{1} y_{1}^{(4)}(s) y_{2}^{(4)}(s)+y_{1}^{(3)}(s) y_{2}^{(3)}(s) \mathrm{d} s+\left\langle y_{1}, y_{2}\right\rangle_{H}
$$

Then $V \hookrightarrow H$ and $K_{\sigma} y_{s s s s} \in H$ for all $y \in V$.
It is convenient to write the problem as a retarded functional differential equation. Let $\mathbf{u}=(y, v)$ where $v=y_{t}$. Then we seek

$$
\begin{equation*}
\dot{\mathbf{u}}(t)+A(l) \mathbf{u}(t)+\int_{0}^{\infty} B(\tau) \mathbf{u}(t-\tau) \mathrm{d} \tau=0 \tag{3.4}
\end{equation*}
$$

where $A$ and $B$ are bounded linear operators mapping $V \times I$ into $H \times H$, defined by

$$
A(t):=\left(\begin{array}{cc}
0 & -1 \\
K_{\sigma}^{\prime}\left(\partial_{s}^{4}-\lambda(t) \partial_{s}^{2}\right) & 0
\end{array}\right), \quad B(t):=\left(\begin{array}{cc}
0 & 0 \\
-\alpha(t) K_{\sigma}^{\prime} \partial_{s}^{4} & 0
\end{array}\right) .
$$

The operators in (3.4) are easily diagonalised. If

$$
\mathbf{u}(s, t)=\sum_{n=1}^{\infty} \mathbf{u}_{n}(t) \sin n \pi s,
$$

where $\mathbf{u}_{n}(t)=\left(y_{n}(t), v_{n}(t)\right)$, then

$$
\begin{equation*}
A(t) \mathbf{u}(t)=\sum_{n=1}^{\infty} A_{n}(t) \mathbf{u}_{n}(t) \sin \pi \pi s \tag{3.5}
\end{equation*}
$$

and

$$
\begin{equation*}
B(\tau) \mathbf{u}(t)=\sum_{n=1}^{\infty} B_{n}(\tau) \mathbf{u}_{n}(t) \sin n \pi s, \tag{3.6}
\end{equation*}
$$

where

$$
A_{n}(t)=\left(\begin{array}{cc}
0 & -1 \\
\lambda_{n} \gamma_{n}\left(\lambda_{n}-\lambda(t)\right) & 0
\end{array}\right), \quad B_{n}(t)=\left(\begin{array}{cc}
0 & 0 \\
-\lambda_{n}^{2} \gamma_{n} \alpha(t) & 0
\end{array}\right)
$$

with

$$
\lambda_{n}=n^{2} \pi^{2}, \quad \gamma_{n}=\frac{1}{1+\sigma n^{2} \pi^{2}}
$$

Substitution of (3.5) and (3.6) into (3.4) yields

$$
\begin{equation*}
\dot{\mathbf{u}}_{n}(t)+A_{n}(t) \mathbf{u}_{n k}(t)+\int_{0}^{\infty} B_{n}(\tau) \mathbf{u}_{n}(t-\tau) \mathrm{d} \tau=0 \tag{3.7}
\end{equation*}
$$

In this chapter we shall assume that the terminal load is constant. This allows us to apply Laplace transforms to (3.7). In order to utilise the many known results concerning the resolvents of Volterra integro-differential equations we rewrite (3.7) as

$$
\begin{equation*}
\dot{\mathbf{u}}_{n}(t)+\int_{0}^{\infty} \mathrm{d} \mu_{n}(\tau) \mathbf{u}_{n}(t-\tau)=0 \tag{3.8}
\end{equation*}
$$

where $\mu_{n}$ is the $2 \times 2$ matrix of measures

$$
\mu_{n}=\left(\begin{array}{cc}
0 & -\delta_{0} \\
\lambda_{n} \gamma_{n}\left(\lambda_{n}-\lambda\right) \delta_{0}-\lambda_{n}^{2} \gamma_{n} \mathrm{~d} G & 0
\end{array}\right)
$$

and $\delta_{0}$ is the Dirac measure concentrated at 0 .

### 3.3 Problem with Initial History

The initial history problem corresponds to solving equation (3.8) on $\mathbb{R}^{+}$with a specified history of the displacement up to time $t=0$. We can decompose the
integral in (3.8) into a convolution ${ }^{1}$ and a forcing function which depends on the known initial history. Thus,

$$
\begin{equation*}
\dot{\mathbf{u}}_{n}(t)+\left(\mu_{n} * \mathbf{u}_{n}\right)(t)=\mathbf{f}_{n}(t), \quad t>0 \tag{0}
\end{equation*}
$$

and $\mathbf{u}_{n}(0)$ is prescribed.

$$
\mathrm{f}_{n}(t)=-\int_{t}^{\infty} \mathrm{d} \mu_{n}(\tau) \mathbf{u}_{n}(t-\tau)=\binom{0}{\lambda_{n}^{2} \gamma_{n} \int_{t}^{\infty} \alpha(\tau) y_{n}(t-\tau) \mathrm{d} \tau}, \quad t \geq 0
$$

Through the following theorem we show that for mild conditions on the initial histories the differential equation $\left(V_{0}\right)$ has a unique, absolutely continuous solution. Moreover, for a suitable forcing function $\mathbf{f}_{n}(t)$, this solution possesses the same asymptotic properties as $\mathbf{f}_{n}(t)$ if the load is less than the critical value of $\lambda_{n} G_{\infty}$. When the load is greater than this critical value the solution is in general unbounded. This is a generalisation of the work in Miller [31] where the initial history was assumed to be continuous.

Theorem 3.1 Let the relaxation function $G(t)$ be an arbitrary function in $C^{1}\left(\mathbb{R}^{+}\right)$. If the constant load $\lambda$ is finite and the forcing function $\mathbf{f}_{n}(t)$ belongs to $L_{l o c}^{1}\left(\mathbb{R}^{+} ; \mathbb{R}^{2}\right)$, then $\left(V_{0}\right)$ has a unique, locally absolutely continuous solution in $L_{l o c}^{1}\left(\mathbb{R}^{+} ; \mathbb{R}^{2}\right)$.
Furthermore, let the relaxation function $G(t)$, be non-constant, nonnegative, nonincreasing and convex and let $\mathbf{f}_{n}(t) \in S$, where $S$ is one of the following function spaces ${ }^{2}$ :
(i) $L^{p}\left(\mathbb{R}^{+} ; \mathbb{R}^{2}\right), p \in[1, \infty]$,
(ii) $B C\left(\mathbb{R}^{+} ; \mathbb{R}^{2}\right), B U C\left(\mathbb{R}^{+} ; \mathbb{R}^{2}\right)$ or $B C_{0}\left(\mathbb{R}^{+} ; \mathbb{R}^{2}\right)$,

[^3]Then, if

1. $\lambda<\lambda_{n} G_{\infty}$ the solution $\mathbf{u}_{n} \in S$ and $\mathbf{u}_{n}$ depends continuously on $\mathbf{f}_{n}$ in the norm of $S$. Also, if $\mathrm{f}_{n}$ is continuous then $\dot{\mathbf{u}}_{n} \in S$.
2. $\lambda=\lambda_{n} G_{\infty}$ and the second moment of $\dot{G}$ exists, the solution can be decomposed into

$$
\mathbf{u}_{n}(t)=\mathbf{c}_{n}+\mathbf{w}_{n}(t),
$$

where $\mathbf{c}_{n}$ is a constant vector that depends on the initial history up to time $t=0$ and $\mathrm{w}_{n} \in S$.
3. $\lambda>\lambda_{n} G_{\infty}$ the solution can be decomposed into

$$
\mathbf{u}_{n}(t)=\mathbf{c}_{n} \mathrm{e}^{p_{n}^{*} t}+\mathbf{w}_{n}(t),
$$

where $\mathbf{c}_{n}$ is a constant vector that depends on the initial history up to time $t=0, p_{n}^{*}$ is a positive real number and $\mathrm{w}_{n} \in S$.

Proof. It is known that ${ }^{3}$ there is a unique absolutely continuous matrix-valued function $r_{n}$ defined on $\mathbb{R}^{+}$satisfying

$$
\begin{equation*}
\dot{r}_{n}(t)+\left(r_{n} * \mu_{n}\right)(t)=\mathbf{0}, \quad r_{n}(0)=\mathbf{I} \tag{3.10}
\end{equation*}
$$

for almost all $t \in \mathbb{R}^{+}$. $r_{n}$ is called the differential resolvent of $\mu_{n}$. The unique absolutely continuous solution of $\left(V_{0}\right)$ is then given by the variation of constants formula ${ }^{4}$

$$
\begin{equation*}
\mathbf{u}_{n}(t)=r_{n}(t) \mathbf{u}_{n}(0)+\left(r_{n} * \mathbf{f}_{n}\right)(t) . \tag{3.11}
\end{equation*}
$$

The resolvent $r_{n}$ is particularly useful for investigating the asymptotic behaviour of the solution. A result due to Miller [31] shows that the condition $r_{n} \in L^{1}\left(\mathbb{R}^{+}\right)$is equivalent to the uniform asymptotic stability of the trivial solution of equation ( $V_{0}$ )

[^4]in $C\left(\mathbb{R}^{+}\right)$if the initial history is continuous. Since the measure $\mu_{n} \in M\left(\mathbb{R}^{+} ; \mathbb{R}^{2 \times 2}\right)$, its Laplace transform ${ }^{5}$ is defined on the right-half complex plane and we can use the following Paley-Wiener type result for integro-differential equations ${ }^{6}$.

Lemma 3.2 Let $\mu_{n} \in M\left(\mathbb{R}^{+} ; \mathbb{R}^{2 \times 2}\right)$. Then the differential resolvent $r_{n}$, of $\mu_{n}$ satisfies

$$
r_{n} \in L^{1}\left(\mathbb{R}^{+} ; \mathbb{R}^{2 \times 2}\right)
$$

if and only if the characteristic equation

$$
\begin{equation*}
\Delta_{n}(p)=\operatorname{det}\left[p \mathbf{I}+\bar{\mu}_{n}(p)\right] \neq 0, \quad \Re p \geq 0 \tag{3.12}
\end{equation*}
$$

Calculating the characteristic equation for $\left(V_{0}\right)$ we have

$$
\begin{equation*}
\Delta_{n}(p)=p^{2}+\gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda-\lambda_{n} \bar{\alpha}(p)\right), \tag{3.13}
\end{equation*}
$$

defined on the right-half complex plane. The relationship between the location of roots of this equation and the magnitude of the constant load is described by the following lemma.

Lemma 3.3 If $G(t)$ is non-constant, nonnegative, nonincreasing and convex then the characteristic equation (3.13) has:

1. No solution for $\lambda<\lambda_{n} G_{\infty}$.
2. A root at the origin when $\lambda=\lambda_{n} G_{\infty}$. This root is simple if the first moment of $\dot{G}$ exists.
3. A simple positive real root $p_{n}^{*}$, when $\lambda>\lambda_{n} G_{\infty}$.
[^5]Proof. From the specified properties of $G(t)$ and the definition of $\alpha(t)$, we can see that $\alpha(t)$ is non-constant, nonnegative and nonincreasing. This implies that ${ }^{7}$ $\Im p \Im \bar{\alpha}(p)<0$ for $\Im p \neq 0$. Examining the imaginary part of (3.13) this implies

$$
y \Im \Delta_{n}(p)=2 x y^{2}-\lambda_{n}^{2} \gamma_{n} y \Im \bar{\alpha}(p)>0, \quad \text { for } y \neq 0, x \geq 0,
$$

where $p=x+i y$. Hence, if a root exists, it must lie on the positive real axis. Now looking at (3.13) on the positive real axis we have

$$
\begin{equation*}
\Delta_{n}(x)=x^{2}+\gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda-\lambda_{n} \bar{\alpha}(x)\right), \quad \text { for } x \geq 0 \tag{3.14}
\end{equation*}
$$

Since $\alpha(t)$ is non-constant, nonnegative and nonincreasing

$$
\frac{\mathrm{d} \bar{\alpha}(x)}{\mathrm{d} x}=-\int_{0}^{\infty} \tau \mathrm{e}^{-x \tau} \alpha(\tau) \mathrm{d} \tau
$$

is a well-defined, negative valued, function for $x>0$. Hence equation (3.14) is an increasing function in $x$ and has at most one root for $x \geq 0$. But

$$
\begin{aligned}
\Delta_{n}(0) & =\gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda-\lambda_{n} \alpha(0)\right) \\
& =\gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda-\lambda_{n}\left(1-G_{\infty}\right)\right) \\
& =\gamma_{n} \lambda_{n}\left(\lambda_{n} G_{\infty}-\lambda\right)
\end{aligned}
$$

Therefore, if $\lambda<\lambda_{n} G_{\infty}$ the characteristic equation $\Delta_{n}(p)$ has no roots in the right half complex plane $\Re p \geq 0$, but if $\lambda \geq \lambda_{n} G_{\infty}$ there exists one real root.

Looking at $\Delta_{n}(p)$ near the root $x=p_{n}^{*}$ we have

$$
\begin{equation*}
\Delta_{n}(p)=\left(p-p_{n}^{*}\right)\left(2 p_{n}^{*}+\gamma_{n} \lambda_{n}^{2} \int_{0}^{\infty} \tau \mathrm{e}^{-p_{n}^{*} \tau} \alpha(\tau) \mathrm{d} \tau\right)+O\left(\left(p-p_{n}^{*}\right)^{2}\right) \tag{3.15}
\end{equation*}
$$

Thus, if $\lambda>\lambda_{n} G_{\infty}$ or if $\lambda=\lambda_{n} G_{\infty}$ and $\alpha$ has a finite first moment the root $p=p_{n}^{*}$ has a multiplicity of one.

[^6]Lemmas 3.2 and 3.3 now imply that the differential resolvent $r_{n}$, is integrable if $\lambda<\lambda_{n} G_{\infty}$. Using Theorem 3.9, Chap 3, [18] assertion (1.) is proved.

For the case $\lambda>\lambda_{n} G_{\infty}$ we can expand the differential resolvent into a function in $L^{1}$ and an exponential term ${ }^{8}$. So we now have

$$
r_{n}(t)=K_{n} \mathrm{e}^{p_{n}^{0} t}+q_{n}(t), \quad \text { for } t \in \mathbb{R}^{+}
$$

where $K_{n}$ is determined by the principal pars of $\left[p I+\bar{\mu}_{n}(p)\right]^{-1}$ in the sense that, in some neighbourhood of $p=p_{n}^{*}$ we have

$$
\left[p I+\bar{\mu}_{n}(p)\right]^{-1}=\frac{K_{n}}{\left(p-p_{n}^{*}\right)}+\bar{q}_{n}(p)
$$

where the resolvent remainder $\bar{q}_{n}(p)$ is analytic near $p=p_{n}^{*}$. Calculating $K_{n}$ we find that for $p \neq p_{n}^{*}$

$$
\left[p I+\bar{\mu}_{n}(p)\right]^{-1}=\left(\begin{array}{cc}
p & -1 \\
\gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda-\lambda_{n} \bar{\alpha}(p)\right) & p
\end{array}\right)^{-1} .
$$

Equation (3.15) now implies

$$
\begin{aligned}
& \left(p-p_{n}^{*}\right)\left[p I+\bar{\mu}_{n}(p)\right]^{-1} \\
& =\frac{1}{2 p_{n}^{*}+\gamma_{n} \lambda_{n}^{2} \int_{0}^{\infty} \tau \mathrm{e}^{-p_{n}^{*} \tau} \alpha(\tau) \mathrm{d} \tau}\left(\begin{array}{cc}
p_{n}^{*} & 1 \\
-\gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda-\lambda_{n} \bar{\alpha}\left(p_{n}^{*}\right)\right) & p_{n}^{*}
\end{array}\right)+O\left(p-p_{n}^{*}\right),
\end{aligned}
$$

for $p$ near $p_{n}^{*}$. Hence

$$
K_{n}=\frac{1}{2 p_{n}^{*}+\gamma_{n} \lambda_{n}^{2} \int_{0}^{\infty} \tau \mathrm{e}^{-p_{n}^{*} \tau} \overline{\alpha(\tau) \mathrm{d} \tau}}\left(\begin{array}{cc}
p_{n}^{*} & 1 \\
\gamma_{n} \lambda_{n}\left(\lambda-\lambda_{n}+\bar{\alpha}\left(p_{n}^{*}\right)\right) & p_{n}^{*}
\end{array}\right) .
$$

[^7]The solution of $\left(V_{0}\right)$ is now given by

$$
\begin{aligned}
\mathbf{u}_{n}(t) & =r_{n}(t) \mathbf{u}_{n}(0)+\left(r_{n} * \mathbf{f}_{n}\right)(t) \\
& =\mathrm{e}^{p_{n}^{*} t} K_{n}^{\prime} \mathbf{u}_{n}(0)+K_{n} \mathrm{e}^{p_{n}^{p} t} \int_{0}^{t} \mathrm{e}^{-p_{n}^{*} \tau} \mathbf{f}_{n}(\tau) \mathrm{d} \tau+q_{n}(t) \mathbf{u}_{n}(0)+\int_{0}^{t} q_{n}(t-\tau) \mathbf{f}_{n}(\tau) \mathrm{d} \tau \\
& =K_{n}^{\prime} \mathrm{e}^{p_{n}^{*} t}\left(\mathbf{u}_{n}(0)+\int_{0}^{t} \mathrm{e}^{-p_{n}^{*} \tau} \mathbf{f}_{n}(\tau) \mathrm{d} \tau\right)+q_{n}(t) \mathbf{u}_{n}(0)+\left(q_{n} * f_{n}\right)(t)
\end{aligned}
$$

By defining $q_{n}(t)=-K_{n} \mathrm{c}^{p_{n}^{*} t}$, for $t<0$, we can write our solution in the form

$$
\begin{equation*}
\mathbf{u}_{n}(t)=\mathbf{c}_{n} \mathrm{c}^{\mathbf{p}_{\mathrm{n}}^{*} t}+\mathbf{w}_{n}(t), \tag{3.16}
\end{equation*}
$$

where constant vector $\mathbf{c}_{n}$ is given by

$$
\begin{aligned}
\mathbf{c}_{n} & =K_{n}\binom{y_{n}(0)}{v_{n}(0)}+K_{n} \int_{0}^{\infty} \mathrm{e}^{-p_{n}^{*} \tau}\binom{0}{\gamma_{n} \lambda_{n}^{2} \int_{\tau}^{\infty} \alpha(\xi) y_{n}(\tau-\xi) \mathrm{d} \xi} \mathrm{~d} \tau \\
& =K_{n}\binom{y_{n}(0)}{v_{n}(0)+\gamma_{n} \lambda_{n}^{2} \int_{0}^{\infty} \mathrm{e}^{-p_{n}^{*} \tau} \int_{\tau}^{\infty} \alpha(\xi) y_{n}(\tau-\xi) \mathrm{d} \xi \mathrm{~d} \tau},
\end{aligned}
$$

and

$$
\mathbf{w}_{n}(t)=q_{n}(t) \mathbf{u}(0)+\int_{0}^{\infty} q_{n}(t-\tau) \mathbf{f}_{n}(\tau) \mathrm{d} \tau
$$

When $\lambda=\lambda_{n} G_{\infty}$ we require additional conditions on the characteristic equation in the neighbourhood of the origin ${ }^{9}$ in order to get a similar expansion for the solution. These smootlness requirements are equivalent to the existence of the first two moments of $\alpha(t)$. Following a similar approach to above, we get an expansion for the solution as

$$
\mathbf{u}_{n}(t)=\mathbf{c}_{n}+\mathbf{w}_{n}(t), \quad \text { for } t \in \mathbb{R}^{+}
$$

where vector $\mathrm{c}_{n}$ is given by

$$
\mathbf{c}_{n}=\binom{y_{n}(0)}{v_{n}(0)+\gamma_{n} \lambda_{n}^{2} \int_{0}^{\infty} \int_{\tau}^{\infty} \alpha(\xi) y_{n}(\tau-\xi) \mathrm{d} \xi \mathrm{~d} \tau}
$$

[^8]and $\mathrm{w}_{n}(t)$ is defined as before.
Note that for critical loads $\lambda \geq \lambda_{n} G_{\infty}$, the component of the solution due to the critical characteristic exponent may vanish. This will occur if and only if
\[

$$
\begin{aligned}
& y_{n}(0)=0 \\
& v_{n}(0)=-\gamma_{n} \lambda_{n}^{2} \int_{0}^{\infty} \mathrm{e}^{-p_{n}^{*} \tau} \int_{\rho_{0}}^{\infty} \alpha(t+\xi) y_{n}(-\xi) \mathrm{d} \xi \mathrm{~d} \tau
\end{aligned}
$$
\]

where $p_{n}^{*} \geq 0$ is the simple root of the characteristic equation. While this solution is bounded in the norm of $S$ it is unstable in the sense that arbitrarily small perturbations in the initial history up to time $t=0$ will result in solutions that do not belong to $S$.

Remark 3.4 For clarity our results in this section are presented in terms of the Fourier modes which solve equation (3.8) rather than solutions of (3.4). Once the Fourier modes $\mathbf{u}_{n}(t)$ have been obtained, the main part in proving that they define a weak solution of (3.4) is to show that

$$
\sum_{k=1}^{\infty} n^{4}\left|\mathbf{u}_{n}(t)\right|^{2}<\infty
$$

uniformly for $0 \leq t \leq T$ for each $T>0$. This is an easy consequence of equation (3.11) and regularity of the initial history. The synthesis of the Fourier modes for a similar problem is discussed in [17].

### 3.4 Problem without Initial History

In considering solutions to (3.8) over the whole real line, we need to impose conditions on the displacement in order that the integral in (3.8) is well defined. Furthermore, we are only interested in solutions that are physically realisable.

Definition 3.5 $A$ solution to (3.8) over $\mathbb{R}$ is admissible if

$$
\mathbf{u}_{n}(t) \in L^{1}\left(\mathbb{R}^{-}\right)
$$

We now use some results on Fourier integrals to show that under suitable conditions on the relaxation function the trivial solution is unique for stable loads and there exists an admissible, nontrivial solution for loads greater than the critical load $\lambda_{n} G_{\infty}$.

Theorem 3.6 Let the relaxation function $G(t)$ be non-constant, nonnegative, nonincreasing and convex and let $\alpha(t)=-\dot{G}(t)$ satisfy $\alpha(t) \mathrm{e}^{c^{\prime} t} \in L\left(\mathbb{R}^{+}\right)$for some constant $c^{\prime}>0$. Let $\mathbf{u}_{n}$ be an admissible solution of (3.8) satisfying

$$
\mathrm{u}_{n} \mathrm{e}^{-\mathrm{c}| | t \mid} \in L^{2}\left(\mathbb{R} ; \mathbb{R}^{2}\right)
$$

and

$$
\lim _{t \rightarrow \pm \infty} \mathrm{e}^{-p|t|} \dot{\mathbf{u}}_{n}(t)=0
$$

with $\Re p \geq c$ where $0<c<c^{\prime}$. Then,

1. if the constant load satisfies $\lambda \leq \lambda_{n} G_{\infty}$,

$$
\mathrm{u}=0
$$

2. if $\lambda>\lambda_{n} G_{\infty}$.

$$
\mathbf{u}_{n}=\mathbf{c}_{n} e^{p_{n}^{*} t}
$$

where $p_{n}^{*}$ is the simple zero of the characteristic equation (3.13) and $\mathbf{c}_{n}$ is a constant vector.

Proof.
We shall make use of the following result from [40].
Lemma 3.7 Let $\phi(p)$ be regular in the strip $a_{1} \leq \Re p \leq a_{2}$, and let $\phi(\eta+i \omega)$ belong to $L^{2}(\mathbb{R})$, and tend to 0 uniformly as $\omega \rightarrow \pm \infty$, for $\eta$ in the above interval. Let $\psi(p)$ have similar properties in $b_{1} \leq \Re p \leq b_{2}$, where $b_{2}<a_{1}$. Let

$$
\begin{equation*}
\lim _{T \rightarrow \infty} \int_{a-i T}^{a+i T} \phi(p) \mathrm{e}^{p t} \mathrm{~d} p+\lim _{T \rightarrow \infty} \int_{b-i T}^{b+i T} \psi(p) \mathrm{e}^{p t} \mathrm{~d} p=0 \tag{3.17}
\end{equation*}
$$

for all $t$, where $a_{1}<a<a_{2}, b_{1}<b<b_{2}$. Then $\phi$ and $\psi$ are regular for $b_{1}<\Re p<a_{2}$, their sum is 0 in this strip, and they lend to 0 as $\Im p \rightarrow \pm \infty$, uniformly in any interior strip.

We extend $\alpha(t)$ over the whole real line by defining $\alpha(t)=0$ for $t<0$. Equation (3.8) is now given by

$$
\begin{equation*}
\dot{\mathrm{u}}_{n}+\int_{-\infty}^{\infty} \mathrm{d} \mu_{n}(\tau) \mathrm{u}_{n}(\ell-\tau)=0, \quad(-\infty<t<\infty) \tag{3.18}
\end{equation*}
$$

Since $\alpha(t) \mathrm{e}^{\mathrm{c}^{\prime}|t|} \in L^{1}(R)$ this implies $\mathrm{e}^{c^{\prime}|t|} \mu_{n} \in \mathrm{M}\left(\mathbb{R}, \mathbb{R}^{2 \times 2}\right)$ and $\bar{\mu}_{n}(p)=\int_{\infty}^{\infty} \mathrm{e}^{-p t} \mathrm{~d} / \mu_{n}(t)$ is defined and analytic on the half-plane $\Re p>-c^{\prime}$. We decompose the solution $\mathrm{u}_{n}(t)$ into

$$
\mathrm{u}_{n}(t)=\mathrm{u}_{-}(t)+\mathrm{u}_{+}(t)
$$

where $\mathbf{u}_{-}(-t)=\mathbf{u}_{+}(t)=0,(t<0)$. Since $\mathbf{u}_{n}(t) \mathrm{e}^{-c|t|} \in L^{2}\left(\mathbb{R} ; R^{2}\right) \overline{\mathbf{u}}_{+}(p)$ is defined and analytic for $\Re p>c$ and $\bar{u}_{-}(p)$ is defined and analytic for $\Re p<-c$. Using integration by parts, it is easily seen that

$$
\int_{0}^{\infty} \dot{\mathrm{u}}_{n}(t) \mathrm{e}^{-p t} \mathrm{~d} t=p \overline{\mathrm{u}}_{+}(p)-\mathbf{u}_{n}(+0), \quad \text { for } \Re p>c
$$

and

$$
\int_{-\infty}^{0} \dot{\mathbf{u}}_{n}(t) \mathrm{e}^{-p t} \mathrm{~d} t=p \overline{\mathbf{u}}_{-}(p)+\mathbf{u}_{n}(-0), \quad \text { for } \Re p<-c
$$

Hence for any $a$ satisfying $c<a$ we have that

$$
\overline{\dot{\mathrm{u}}}_{+}(p)=p \overline{\mathrm{u}}_{+}(p)-\mathbf{u}_{n}(+0), \quad \text { for } \Re p=a
$$

or

$$
\begin{equation*}
\lim _{T \rightarrow \infty} \frac{1}{\overline{2} \pi \imath} \int_{a-i T^{a}}^{a+i T^{\prime}}\left[p \overline{\mathbf{u}}_{+}(p)-\mathbf{u}_{n}(+0)\right] \mathrm{e}^{p t} \mathrm{~d} p=\dot{\mathbf{u}}_{+}(p) \tag{3.19a}
\end{equation*}
$$

in the mean-square sense, and

$$
\begin{equation*}
\lim _{T \rightarrow \infty} \frac{1}{2 \pi i} \int_{a-i T}^{a+i T} \bar{\mu}_{n}(p) \overline{\mathbf{u}}_{+}(p) \mathrm{e}^{p t} \mathrm{~d} p=\int_{-\infty}^{t} \mathrm{~d} \mu_{n}(\tau) \overline{\mathbf{u}}_{+}(t-\tau) \tag{3.19b}
\end{equation*}
$$

in the mean-square sense. Similarly for any $b$ in the interval $-c^{\prime}<b<-c$ we have for $\bar{u}_{-}(p)$

$$
\begin{equation*}
\lim _{T \rightarrow \infty} \frac{1}{2 \pi i} \int_{b-i T}^{b+i T}\left[p \overline{\mathbf{u}}_{-}(p)+\mathbf{u}_{n}(-0)\right] \mathrm{e}^{p t} \mathrm{~d} p=\dot{\mathbf{u}}_{-}(p) \tag{3.19c}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim _{T \rightarrow \infty} \frac{1}{2 \pi i} \int_{b-i T}^{b+i T} \bar{\mu}_{n}(p) \overline{\mathbf{u}}_{-}(p) \mathrm{e}^{p t} \mathrm{~d} p=\int_{t}^{\infty} \mathrm{d} \mu_{n}(\tau) \mathbf{u}_{-}(t-\tau) \tag{3.19d}
\end{equation*}
$$

with equality in the mean-square sense. Adding equations (3.19a)-(3.19d) we get

$$
\begin{equation*}
\lim _{T \rightarrow \infty} \int_{a-i T}^{a+i T}\left[p \mathbf{I}+\bar{\mu}_{n}(p)\right] \mathbf{u}_{+}(p) \mathrm{d} p+\lim _{T \rightarrow \infty} \int_{b-i T}^{b+i T}\left[p \mathbf{I}+\bar{\mu}_{n}(p)\right] \mathbf{u}_{-}(p) \mathrm{d} p=\mathbf{0} \tag{3.20}
\end{equation*}
$$

in the mean-square sense.
Using the results from the previous section the solution $\mathbf{u}_{n}$ will be continuous. Hence $\mathbf{u}_{n}(-0)=\mathbf{u}_{n}(+0)$. Adding equations (3.19a)-(3.19d) we get

$$
\begin{equation*}
\lim _{T \rightarrow \infty} \int_{a-i T}^{a+i T}\left[p \mathbf{I}+\bar{\mu}_{n}(p)\right] \mathbf{u}_{+}(p) \mathrm{d} p+\lim _{T \rightarrow \infty} \int_{b-i T}^{b+i T}\left[p \mathbf{I}+\bar{\mu}_{n}(p)\right] \mathbf{u}_{-}(p) \mathrm{d} p=\mathbf{0} \tag{3.21}
\end{equation*}
$$

in the mean-square sense. Applying Lemma 3.7 we can extend both $\left[p \mathbf{I}+\bar{\mu}_{n}(p)\right] \overline{\mathbf{u}}_{+}(p)$ and $\left[p \mathbf{I}+\bar{\mu}_{n}(p)\right] \overline{\mathbf{u}}_{-}(p)$ throughout the strip $b<\Re p<a$ with $\overline{\mathbf{u}}_{+}(p)=-\overline{\mathbf{u}}_{-}(p)$ in this strip except possibly for poles at the zeros of $\Delta_{n}(p)$. Hence $\bar{u}_{-}(p)$ and $\overline{\mathbf{u}}_{+}(p)$ are regular in the strip except possibly for poles at the zeros of $\Delta_{n}(p)$ for $\Re p>c$.

We can now write

$$
\begin{equation*}
\mathrm{u}_{n}(t)=\lim _{T \rightarrow \infty} \frac{1}{2 \pi i} \int_{a-i T}^{a+i T} \mathrm{u}_{+}(p) \mathrm{e}^{p t} \mathrm{~d} p-\lim _{T \rightarrow \infty} \frac{1}{2 \pi i} \int_{b-i T}^{b+i T} \mathrm{u}_{+}(p) \mathrm{e}^{p t} \mathrm{~d} p, \tag{3.22}
\end{equation*}
$$

and, since $\bar{u}_{+}(\eta+i \omega) \rightarrow 0$ as $\omega \rightarrow \pm \infty$, we can evaluate the right-hand side by calculus of residues in the usual way.

We are only interested in solutions that are in $L^{1}\left(\mathbb{R}^{-}\right)$. Therefore, we are only concerned with zeros of the characteristic equation $\Delta_{n}(p)$ that lie on the right-half complex plane $\Re p>0$. Using Lemma 3.3 we find that:

1. If the load satisfies $\lambda \leq \lambda_{n} G_{\infty}$, then the characteristic equation (3.13) has no zeros on the right-half complex plane $\Re p>0$. Hence $\mathbf{u}_{n}(t)$ given by (3.22) is not admissible and $\mathbf{u}_{n}=\mathbf{0}$ is the unique solution of (3.8) over $\mathbb{R}$.
2. If $\lambda>\lambda_{n} G_{\infty}$, then $\Delta_{n}(p)$ has a simple real root $p_{n}^{*}>0$. Hence (3.8) has a solution given by

$$
\mathbf{u}_{n}=\mathbf{c}_{n} e^{p_{n}^{*} t}
$$

where $\mathbf{c}_{n}$ is a constant vector.
The uniqueness of the zero solution for stable loads can also be shown using the half-line resolvent used in Theorem 3.1.

Remark 3.8 For clarity, our result in this section is also presented in terms of the Fourier modes which solve equation (3.8) rather than solutions of (3.4). Since we find that there are only a finite number of Fourier modes $\mathrm{u}_{n}(t)$ there is no difficulty in showing that we have found solutions to (3.4).

### 3.5 Alternative Uniqueness Result

In this section we use an alternative procedure to examine the uniqueness of the trivial solution to (3.7) for stable loads when the history of the displacement is not specified but is assumed to decay exponentially in the past. This procedure was used by Virga and Capriz [42] whose work is outlined in [15].

Given $\varepsilon>0$, consider the Banach space $U_{\varepsilon}$ of continuous functions $\mathbf{u}_{n}=\left(y_{n}, v_{n}\right)$ defined on $(-\infty, 0]$ with

$$
\left\|u_{n i}\right\|_{\varepsilon}:=\sup _{t \leq 0} \mathrm{e}^{\varepsilon t}\left|y_{n}(t)\right|+\sup _{t \leq 0} \mathrm{e}^{\varepsilon t}\left|v_{n}(t)\right|<\infty .
$$

Theorem 3.9 The trivial solution of (3.7) is unique in $U_{\varepsilon}$ for $0 \leq \lambda \leq \lambda_{n 2} G_{\infty}$, if $\varepsilon>\varepsilon^{*}$, where $\varepsilon^{*}$ is the (single) positive real root of

$$
\begin{equation*}
\varepsilon^{*}=\sqrt{\gamma_{n} \lambda_{n}\left(\lambda_{n}\left(1-\bar{\alpha}\left(\varepsilon^{*}\right)\right)-\lambda\right)} \tag{3.23}
\end{equation*}
$$

Proof. A solution of (3.7) which lies in $U_{\varepsilon}$ satisfies the equation

$$
\begin{equation*}
\mathbf{u}_{n}(t)=-\int_{-\infty}^{t}\left[A_{n} \mathbf{u}_{n}(r)+\int_{0}^{\infty} B_{n}(\tau) \mathbf{u}_{n}(r-\tau) \mathrm{d} \tau\right] \mathrm{d} r, \quad t \in \mathbb{R} \tag{3.24}
\end{equation*}
$$

In order to show that the trivial solution is unique, we define the operator $\mathcal{F}_{n}$ on $U_{\varepsilon}$ by

$$
\mathcal{F}_{n}\left(\mathbf{u}_{n}\right)(t)=-\int_{-\infty}^{t}\left[A_{n} \mathbf{u}_{n}(r)+\int_{0}^{\infty} B_{n}(\tau) \mathbf{u}_{n}(r-\tau) \mathrm{d} \tau\right] \mathrm{d} r
$$

and show that the second iterate of $\mathcal{F}_{n}\left(\mathbf{u}_{n}\right)$ is a contraction for stable loads if $\varepsilon>\varepsilon^{*}$. We need only show that the unique solution of (3.24) on ( $-\infty, 0$ ] is $\mathbf{u}_{n}=\mathbf{0}$. Then we can apply the results of Section 3.3 to show that the solution of (3.7) is zero for all time.

First we show that $\mathcal{F}_{n}\left(\mathbf{u}_{n}\right) \in U_{\varepsilon}$ if $\mathbf{u}_{n} \in U_{\varepsilon}$. Note that for any $l \leq 0$,

$$
\begin{aligned}
\mathcal{F}_{n}\left(\mathbf{u}_{n}\right) & =-\int_{-\infty}^{t} A_{n} \mathbf{u}_{n}(\tau) \mathrm{d} \tau-\int_{-\infty}^{t} \int_{0}^{\infty} B_{n}(\tau) \mathbf{u}_{n}(r-\tau) \mathrm{d} \tau \mathrm{~d} \tau \\
& =-\int_{-\infty}^{t} A_{n} \mathbf{u}_{n}(\tau) \mathrm{d} \tau-\int_{-\infty}^{t} \int_{-\infty}^{r} B_{n}(r-\tau) \mathbf{u}_{n}(\tau) \mathrm{d} \tau \mathrm{~d} \tau \\
& =-\int_{-\infty}^{t} A_{n} \mathbf{u}_{n}(\tau) \mathrm{d} \tau-\int_{-\infty}^{t} \int_{\tau}^{t} B_{n}(r-\tau) \mathrm{d} r \mathbf{u}_{n}(\tau) \mathrm{d} \tau \\
& =-\int_{0}^{\infty}\left[A_{n}+\int_{0}^{\tau} B_{n}(r) \mathrm{d} r\right] \mathbf{u}_{n}(t-\tau) \mathrm{d} \tau \\
& =-\int_{0}^{\infty}\binom{v_{n}(t-\tau)}{\gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda-\lambda_{n} \int_{0}^{\tau} \alpha(r) \mathrm{d} r\right) y_{n}(t-\tau)} \mathrm{d} \tau
\end{aligned}
$$

Hence

$$
\mathrm{e}^{-\varepsilon t} \mathcal{F}_{n}\left(\mathrm{u}_{n}\right)=-\int_{0}^{\infty} \mathrm{e}^{-\varepsilon \tau}\binom{\mathrm{e}^{-\varepsilon(t-\tau)} v_{n}(t-\tau)}{\gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda-\lambda_{n} \int_{0}^{\tau} \alpha(r) \mathrm{d} r\right) \mathrm{e}^{-\varepsilon(t-\tau)} y_{n}(t-\tau)} \mathrm{d} \tau
$$

and for stable loads $0 \leq \lambda \leq \lambda_{n} G_{\infty}$,

$$
\begin{aligned}
\left\|\mathcal{F}_{n}\left(\mathbf{u}_{n}\right)\right\|_{\varepsilon} & <\frac{1}{\varepsilon}\left\|v_{n}\right\|_{\varepsilon}+\gamma_{n} \lambda_{n} \int_{0}^{\infty} \mathrm{e}^{-\varepsilon r}\left(\lambda_{n}-\lambda-\lambda_{n} \int_{0}^{\tau} \alpha(r) \mathrm{d} r\right) \mathrm{d} \tau\left\|y_{n}\right\|_{\varepsilon} \\
& =\frac{1}{\varepsilon}\left\|v_{n}\right\|_{\varepsilon}+\frac{\gamma_{n} \lambda_{n}}{\varepsilon}\left(\lambda_{n}-\lambda-\lambda_{n} \bar{\alpha}(\varepsilon)\right)\left\|y_{n}\right\|_{\varepsilon}
\end{aligned}
$$

Therefore $\mathcal{F}_{n}\left(\mathbf{u}_{n}\right)$ is bounded in $U_{\varepsilon}$ by

$$
\begin{equation*}
\left\|\mathcal{F}_{n}\left(\mathbf{u}_{n}\right)\right\|_{\varepsilon}<\frac{K}{\varepsilon}\left\|\mathbf{u}_{n}\right\|_{\varepsilon}, \tag{3.25}
\end{equation*}
$$

where

$$
K=\max \left\{1, \gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda-\lambda_{n} \bar{\alpha}(\varepsilon)\right)\right\} \leq \max \left\{1, \gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda\right)\right\} .
$$

This proves that $\mathcal{F}_{n}\left(\mathbf{u}_{n}\right)$ is in $U_{\varepsilon}$ for all $\varepsilon>0$.
We consider the second iterate of $\mathcal{F}_{n}$,

$$
\begin{align*}
\mathcal{F}_{n}\left(\mathcal{F}_{n}\left(\mathbf{u}_{n}\right)\right)= & \int_{-\infty}^{t} \int_{-\infty}^{\xi} A_{n}^{2} \mathbf{u}_{n}(\tau) \mathrm{d} \tau \mathrm{~d} \xi+\int_{-\infty}^{t} A_{n} \int_{-\infty}^{\xi} \int_{0}^{\infty} B_{n}(\tau) \mathbf{u}_{n}(r-\tau) \mathrm{d} \tau \mathrm{~d} r \mathrm{~d} \xi \\
& +\int_{-\infty}^{t} \int_{0}^{\infty} B_{n}(\tau) \int_{-\infty}^{\xi-\tau} A_{n} \mathbf{u}_{n}(r) \mathrm{d} r \mathrm{~d} \tau \mathrm{~d} \xi \tag{3.26}
\end{align*}
$$

since $B_{n}(t) B_{n}(\tau)=0$. The integrals in (3.26) can be written as

$$
\int_{-\infty}^{t} \int_{-\infty}^{\xi} A_{n}^{2} \mathbf{u}_{n}(\tau) \mathrm{d} \tau \mathrm{~d} \xi=\int_{0}^{\infty} \int_{0}^{\infty} A_{n}^{2} \mathbf{u}_{n}(t-\tau-\xi) \mathrm{d} \tau \mathrm{~d} \xi,
$$

$$
\begin{aligned}
\int_{-\infty}^{t} A_{n} \int_{-\infty}^{\xi} \int_{0}^{\infty} B_{n}(\tau) \mathbf{u}_{n}(r-\tau) \mathrm{d} \tau \mathrm{~d} \tau \mathrm{~d} \xi & =\int_{-\infty}^{t} \int_{-\infty}^{\xi} \int_{-\infty}^{\sigma} A_{n} B_{n}(r-\tau) \mathbf{u}_{n}(\tau) \mathrm{d} \tau \mathrm{~d} \tau \mathrm{~d} \xi \\
& =\int_{-\infty}^{t} \int_{-\infty}^{\xi} \int_{\tau}^{\xi} A_{n} B_{n}(r-\tau) \mathrm{d} r \mathbf{u}_{n}(\tau) \mathrm{d} \tau \mathrm{~d} \xi \\
& =\int_{-\infty}^{t} \int_{-\infty}^{\xi} \int_{0}^{\xi-\tau} A_{n} B_{n}(r) \mathrm{d} r \mathbf{u}_{n}(\tau) \mathrm{d} \tau \mathrm{~d} \xi \\
& =\int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\tau} A_{n} B_{n}(r) \mathrm{d} r \mathbf{u}_{n}(t-\tau-\xi) \mathrm{d} \tau \mathrm{~d} \xi
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{-\infty}^{t} \int_{0}^{\infty} B_{n}(\tau) \int_{-\infty}^{\xi-\tau} A_{n} \mathbf{u}_{n}(r) \mathrm{d} r \mathrm{~d} \tau \mathrm{~d} \xi & =\int_{-\infty}^{t} \int_{-\infty}^{\xi} B_{n}(\xi-\tau) \int_{-\infty}^{\tau} A_{n} \mathbf{u}_{n}(r) \mathrm{d} r \mathrm{~d} \tau \mathrm{~d} \xi \\
& =\int_{-\infty}^{t} \int_{-\infty}^{\xi} \int_{r}^{\xi} B_{n}(\xi-\tau) A_{n} \mathrm{~d} \tau \mathbf{u}_{n}(r) \mathrm{d} r \mathrm{~d} \xi \\
& =\int_{-\infty}^{t} \int_{-\infty}^{\xi} \int_{0}^{\xi-r} B_{n}(\tau) A_{n} \mathrm{~d} \tau \mathbf{u}_{n}(r) \mathrm{d} r \mathrm{~d} \xi \\
& =\int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{r} B_{n}(\tau) A_{n} \mathrm{~d} \tau \mathbf{u}_{n}(t-\xi-r) \mathrm{d} r \mathrm{~d} \xi
\end{aligned}
$$

Hence we can write

$$
\begin{align*}
\mathcal{F}_{n}\left(\mathcal{F}_{n}\left(\mathbf{u}_{n}\right)\right) & =\int_{0}^{\infty} \int_{0}^{\infty}\left[A_{n}^{2}+\int_{0}^{r} A_{n} B_{n}(\tau)+B_{n}(\tau) A_{n} \mathrm{~d} \tau\right] \mathrm{u}_{n}(t-\xi-r) \mathrm{d} r \mathrm{~d} \xi  \tag{3.27}\\
& =\int_{0}^{\infty} \int_{0}^{\infty} \gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda-\int_{0}^{r} \lambda_{n} \alpha(\tau) \mathrm{d} \tau\right)\binom{y_{n}(t-\xi-r)}{v_{n}(t-\xi-r)} \mathrm{d} r \mathrm{~d} \xi \tag{3.28}
\end{align*}
$$

and

$$
\begin{aligned}
& \mathrm{e}^{-\varepsilon t} \mathcal{F}_{n}\left(\mathcal{F}_{n}\left(\mathrm{u}_{n}\right)\right) \\
& =\int_{0}^{\infty} \int_{0}^{\infty} \gamma_{n} \lambda_{n} \mathrm{e}^{-\varepsilon(\xi+r)}\left(\lambda_{n}-\lambda-\lambda_{n} \int_{0}^{r} \alpha(\tau) \mathrm{d} \tau\right) \mathrm{e}^{-\varepsilon(t-\xi-r)}\binom{y_{n}(t-\xi-r)}{v_{n}(t-\xi-r)} \mathrm{d} \tau \mathrm{~d} \xi
\end{aligned}
$$

Therefore $\left\|\mathcal{F}_{n}\left(\mathcal{F}_{n}\left(\mathbf{u}_{n}\right)\right)\right\|_{\varepsilon}$ is bounded by

$$
\left\|\mathcal{F}_{n}\left(\mathcal{F}_{n}\left(\mathrm{u}_{n}\right)\right)\right\|_{\varepsilon} \leq \gamma_{n} \lambda_{n} \int_{0}^{\infty} \int_{0}^{\infty} \mathrm{e}^{-\varepsilon(\xi+r)}\left|\lambda_{n}-\lambda-\lambda_{n} \int_{j_{0}}^{r} \alpha(\tau) \mathrm{d} \tau\right| \mathrm{d} r d \xi\left\|\mathbf{u}_{n}\right\|_{\varepsilon}
$$

Since $0 \leq \lambda \leq \lambda_{n} G_{\infty}$, the second iterate of $\mathcal{F}_{n}$ is a contraction if

$$
\gamma_{n} \lambda_{n} \int_{0}^{\infty} \int_{0}^{\infty} \mathrm{e}^{-\varepsilon(\xi+r)} \lambda_{n}-\lambda-\lambda_{n} \int_{J_{0}}^{r} \alpha(\tau) \mathrm{d} \tau \mathrm{~d} r \mathrm{~d} \xi=\gamma_{n} \lambda_{n} \frac{\lambda_{n}-\lambda-\lambda_{n} \bar{\alpha}(\varepsilon)}{\varepsilon^{2}}<1 .
$$

This will be true if $\varepsilon>\varepsilon^{*}$. The existence of a unique positive solution can be shown using an argument similar to that used in Lemma 3.3.

A simple upper bound for $\varepsilon^{*}$ is given by

$$
\begin{equation*}
\varepsilon^{*}<\sqrt{\gamma_{n} \lambda_{n}}\left(\lambda_{n}-\lambda\right) . \tag{3.29}
\end{equation*}
$$

Note that for continuous time-dependent loads satisfying $0 \leq \lambda(t) \leq \lambda_{n} G_{\infty}$, equation (3.24) has a unique solution in $U_{\varepsilon}$ if $\varepsilon>\varepsilon^{*}$, with

$$
\varepsilon^{*}=\sqrt{\gamma_{n} \lambda_{n}\left(\lambda_{n}\left(1-\bar{\alpha}\left(\varepsilon^{*}\right)\right)-\lambda_{M}\right)}
$$

where $\lambda_{M}=\inf _{t \leq 0} \lambda(t)$. For constant loads this is not as strong as Theorem 3.6.

### 3.6 Standard Viscoelastic Material

A rod comprising of standard viscoelastic material has a relaxation function of the form

$$
\begin{equation*}
G(t)=G_{\infty}+\left(G_{0}-G_{\infty}\right) \mathrm{e}^{-a t} \tag{3.30}
\end{equation*}
$$

with $a>0$ and $0 \leq G_{\infty}<G_{0}$. This material has been widely used as it possesses many of the observed properties of physical viscoelastic materials. Appendix A contains a description of the standard viscoelastic material in terms of its rheological model and the derivation of (3.30). Dost and Glockner [14] examined the zero initial history problem for the standard viscoelastic material using Laplace transforms.

We will consider the initial history problem for which the history of the displacement at time $t=0$, denoted by, $y_{n}^{0}(\tau)=y_{n}(-\tau)$ is known. After normalising, as in

Chapter 2 we have

$$
\begin{equation*}
\alpha(t)=a\left(1-G_{\infty}\right) \mathrm{e}^{-a t} \tag{3.31}
\end{equation*}
$$

The contribution due to the initial history in (3.7) is now given by

$$
a\left(1-G_{\infty}\right) \int_{t}^{\infty} \mathrm{e}^{-a \tau} y_{n}(t-\tau) \mathrm{d} \tau=a_{3} \mathrm{e}^{-a t}
$$

where $a_{3}:=a\left(1-G_{\infty}\right) \int_{0}^{\infty} \mathrm{e}^{-a \tau} y_{n}^{0}(\tau) \mathrm{d} \tau$.
The dynamic equation (3.7) can be written as a system of three first order, ordinary differential equations. To do this we define $m_{n}(t)$ to be

$$
\begin{aligned}
m_{n}(l) & =-y_{n}(t)+\int_{0}^{t} \alpha(t-\tau) y_{n}(\tau) \mathrm{d} \tau \\
& =-y_{n}(t)+a\left(1-G_{\infty}\right) \int_{0}^{t} \mathrm{e}^{-a(t-\tau)} y_{y_{n}}(\tau) \mathrm{d} \tau
\end{aligned}
$$

By differentiation we find that $m_{n}(t)$ satisfies the differential equation

$$
\begin{aligned}
\dot{m}_{n}(t) & =-\dot{y}_{n}(t)+a\left(1-G_{\infty}\right) y_{n}(t)-a\left(m_{n}(t)+y_{n}(t)\right) \\
& =-a G_{\infty} y_{n}(t)-\dot{y}_{n}(t)-a m_{n}(t)
\end{aligned}
$$

with initial condition $m_{n}(0)=-y_{n}(0)$. Then (3.7) can be expressed as

$$
\frac{\mathrm{d}}{\mathrm{~d} t}\left(\begin{array}{c}
y_{n}  \tag{3.32}\\
v_{n} \\
m_{n}
\end{array}\right)=\left(\begin{array}{ccc}
0 & 1 & 0 \\
\gamma_{n} \lambda_{n} \lambda(l) & 0 & \gamma_{n} \lambda_{n}^{2} \\
-a G_{\infty} & -1 & -a
\end{array}\right)\left(\begin{array}{c}
y_{n} \\
v_{n} \\
m_{n}
\end{array}\right)+\left(\begin{array}{c}
0 \\
f_{n}(t) \\
0
\end{array}\right)
$$

where $v_{n}(t)=\dot{y}_{n}(t)$ and $f_{n}(t)$ is given by

$$
f_{n}(t)=\gamma_{n} \lambda_{n}^{2} a_{3} \mathrm{e}^{-a t}
$$

The initial conditions are

$$
y_{n}(0)=a_{0}, \quad v_{n}(0)=a_{1}, \quad m_{n}(0)=-a_{0}
$$

Alternatively, we can write the dynamic equation as a third order scalar equation by differentiating

$$
\ddot{y}_{n}=-\gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda(t)\right) y_{n}+\gamma_{n} \lambda_{n}^{2} a\left(1-G_{\infty}\right) \int_{0}^{t} \mathrm{e}^{-a \tau} y_{n}(t-\tau) \mathrm{d} \tau+f(t)
$$

to give

$$
\begin{align*}
\dddot{y}_{n}+a \dddot{y}_{n}+\gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda\right) \dot{y}_{n}+\gamma_{n} \lambda_{n} a\left(\lambda_{n} G_{\infty}-\lambda\right) y_{n}-\gamma_{n} \lambda_{n} a \dot{\lambda} y_{n} & =\dot{f}_{n}+a f_{n} \\
& =0 \tag{3.33}
\end{align*}
$$

subject to the initial conditions

$$
y_{n}(0)=a_{0}, \quad \dot{y}_{n}(0)=a_{1}, \quad \ddot{y}_{n}(0)=a_{2},
$$

where

$$
a_{2}=-\gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda(0)\right) a_{0}+f_{n}(0)=\gamma_{n} \lambda_{n}^{2}\left(a_{3}-a_{0}\right)+\gamma_{n} \lambda_{n} \lambda(0) a_{0} .
$$

In this section we will consider (3.32) or (3.33) for constant loads. The characteristic equation is given by

$$
\begin{equation*}
\Delta_{n}(p)=p^{3}+a p^{2}+\gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda\right) p+\gamma_{n} \lambda_{n} a\left(\lambda_{n} G_{\infty}-\lambda\right) \tag{3.34}
\end{equation*}
$$

Our primary interest is in determining the values of $\lambda$ for which the roots of (3.34) have negative real parts.

Theorem 3.10 The characteristic equation (3.34) will have:

- Three roots with negative real parts if $\lambda<\lambda_{n} G_{\infty}$.
- Two roots with negative real parts and a root at the origin if $\lambda=\lambda_{n} G_{\infty}$.
- Two roots with negative real parts and a single real rool greater than zero if $\lambda>\lambda_{n} G_{\infty}$.

Moreover, if $\lambda<\lambda_{n}$ and $0<a<\sqrt{3 \gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda\right)}$ then a complex pair of roots exist for all $G_{\infty} \leq 1$. Otherwise, a complex pair of roots will exist only if $G_{-}<G_{\infty}<G_{+}$ where

$$
\begin{equation*}
G_{ \pm}=\frac{2 \lambda+\lambda_{n}}{3 \lambda_{n}}-\frac{2 a^{3} \mp\left(a^{2}+3 \gamma_{n} \lambda_{n}\left(\lambda-\lambda_{n}\right)\right)^{3 / 2}}{27 \gamma_{n} \lambda_{n}^{2}} . \tag{3.35}
\end{equation*}
$$

Proof. Applying the Routh-Hurwitz criterion or the Liénard-Chipart test to (3.34) we see that the roots of (3.34) have negative (nonpositive) real parts if and only if the determinants

$$
\begin{aligned}
D_{1} & =\left|\begin{array}{cc}
a & 1 \\
a \gamma_{n} \lambda_{n}\left(\lambda_{n} G_{\infty}-\lambda\right) & \gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda\right)
\end{array}\right| \\
& =a \gamma_{n} \lambda_{n}^{2}\left(1-G_{\infty}\right)>0, \\
D_{2} & =\left|\begin{array}{ccc}
a & 1 & 0 \\
a \gamma_{n} \lambda_{n}\left(\lambda_{n} G_{\infty}-\lambda\right) & \gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda\right) & a \\
0 & 0 & a \gamma_{n} \lambda_{n}\left(\lambda_{n} G_{\infty}-\lambda\right)
\end{array}\right| \\
& =a \gamma_{n} \lambda_{n}\left(\lambda_{n} G_{\infty}-\lambda\right) D_{1},
\end{aligned}
$$

are strictly positive (nonnegative). Hence we have that for $\lambda$ less than the equilibrium buckling load of $\lambda_{n} G_{\infty}$, the roots of (3.34) have negative real components.

Applying the Routh-Hurwitz criterion to $\Delta_{n}(-p)$, we find that at least one root has a negative real part for all values of $\lambda$. Since $\Delta_{n}(0)<0$ for $\lambda>\lambda_{n} G_{\infty}$ this implies there exists two roots (possibly complex conjugates) with negative real parts and one positive real root. Finally, it is easily seen from (3.34) that the origin is simple when $\lambda=\lambda_{n} G_{\infty}$. This result agrees with Lemma 3.3 for general relaxation functions.

The existence of a complex pair can be determined by examining the discriminant of the cubic equation (3.34), which is given by

$$
D=\frac{\left(\gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda\right)-a^{2}\right)^{3}}{729}+a^{2} \frac{\left(2 a^{2}-9 \gamma_{n} \lambda_{n}\left(\lambda_{n}\left(1-G_{\infty}\right)-2\left(\lambda_{n} G_{\infty}-\lambda\right)\right)\right)^{2}}{2916}
$$

There exists one real root and a complex pair, three real roots at least two of which coincide or three real distinct roots if the discriminant satisfies $D>0, D=0$ or $D<0$ respectively.

The discriminant is quadratic in $G_{\infty}$ and can be solved in the usual manner to give the expressions for the roots $G_{-}$and $G_{+}$in (3.35). The bounds in (3.35) are complex if $\lambda<\lambda_{n}$, and $a<\sqrt{3 \gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda\right)}$, then the discriminant is strictly positive for all $0 \leq G_{\infty}<1$.

The region in the ( $a, G_{\infty 0}$ )-plane where complex roots exist is shown for various values of the load $\lambda$ in Figure 3.1.

Theorem 3.10 implies that the trivial solution of (3.32) is asymptotically stable, stable or unstable if the constant load $\lambda$ satisfies $\lambda<\lambda_{n} G_{\infty}, \lambda=\lambda_{n} G_{\infty}$ or $\lambda>\lambda_{n} G_{\infty}$ respectively.

While the solution is unbounded when the load is greater than the critical load $\lambda_{n} G_{\infty}$, the rate of growth in the displacement is strongly linked to the ratio between the load and the instantaneous buckling load $\lambda_{n}$. When the load is greater than $\lambda_{n}$ the displacement begins to grow the instant the load is applied. But if the load lies between the instantaneous and equilibrium buckling loads and the relaxation time is large, the initial displacement oscillates with a decreasing amplitude and the instability will only become apparent after a significant length of time (cf. Figure 3.2). Thus, although the solution is unstable it may behave like a damped oscillator initially if the relaxation time is large. The critical time at which the amplitude of the creep term is comparable to the amplitude of the damped oscillations depends on the type of initial disturbance as well as the relaxation time. Szyszkowski \& Glockner [39] called this phenomenon of a time-dependent instability viscoelastic instability. In Chapter 5 we examine the relationship between the type of initial disturbance and the growth rate of the instability term in the solution.

Figure 3.2 contains plots of the solution ( $n=1$ ), for loads $\lambda=0.1 \lambda_{n}<\lambda_{n} G_{\infty}$, $\lambda=0.3 \lambda_{n}>\lambda_{n} G_{\infty}$, and $\lambda=1.01 \lambda_{n}$ respectively for the following material parameter values.

| Parameter | Value |
| :---: | :---: |
| $n$ | 1 |
| $\lambda_{n}$ | 9.8696 |
| $\gamma_{n}$ | 1.0001 |
| $G_{\infty}$ | 0.1 |
| $a$ | 0.1 |

These plots were calculated using the procedure described in Section B.3.1.


Figure 3.1: Regions with complex and real roots for mode $n=1$.

Stable Load : $\lambda<\lambda_{n} G_{\infty}$




Figure 3.2: Stability behaviour for the standard viscoelastic material.

## Chapter 4

## Nonautonomous Problem

### 4.1 Introduction

We now consider the initial history problem with time-dependent loads. We construct a suitable Hilbert space in which the norm of the solution is defined to be the energy of the rod, and extend the method of Dafermos [9] to the nonautonomous initial history problem. Dafermos in [9] used a semigroup approach to prove existence and uniqueness, as well as asymptotic stability of the trivial solution, for the standard displacement problem of linear viscoelasticity. The evolution equation is shown to generate a $C_{0}$ contraction semigroup on this Hilbert space, and has a unique solution which depends continuously on the initial data. This is in contrast to the results in the quasi-static theory, for which Reynolds [36] showed that uniqueness is ensured only if $\max _{t \geq 0}|\lambda(t)|<\lambda_{1} G_{\infty}$. The energy of the rod is used to construct a Lyapunov function, which is used to demonstrate that the zero solution is stable for a large class of loads satisfying $0<\lambda(t)<\lambda_{1} G_{\infty}$.

### 4.2 Existence and Uniqueness

In this section we examine the existence and uniqueness of the initial history problem for time-dependent loads. We formulate the retarded functional equation (3.4) as an evolution equation. The state-space at time $t$ consists of the displacement $y(s, t)$, the momentum $v(s, t)$, and the history of the displacement $w(s, \tau, t):=y^{t}(s, \tau)=$ $y(s, t-\tau)$. The existence of a unique solution is proven using semigroups and the contraction mapping theorem.

Definition 4.1 A triplet $\chi=(y, v, w)$ is in the space $\mathcal{H}$ if $y \in H, v \in H_{0}^{1}$ and $w \in L_{\alpha}^{2}\left(\mathbf{R}^{+}, H^{2} \cap H_{0}^{1}\right)^{1} . \mathcal{H}$ is endowed with inner product

$$
\begin{align*}
& <(y, v, w),(\hat{y}, \hat{v}, \hat{w})>_{\mathcal{H}}= \\
& \int_{0}^{1}\left(v \hat{v}+\sigma v_{s} \hat{v}_{s}+G_{\infty} y_{s s} \hat{y}_{s s}+\int_{0}^{\infty} \alpha(\tau)\left[y_{s s}-w_{s s}(\tau)\right]\left[\hat{y}_{s s}-\hat{w}_{s s}(\tau)\right] \mathrm{d} \tau\right) \mathrm{d} s . \tag{4.1}
\end{align*}
$$

The space $\mathcal{H}$ is complete with respect to inner product (4.1). Note that due to (3.2), the norm on $\mathcal{H}$ is given by

$$
\begin{equation*}
\|\chi\|_{\mathcal{H}}^{2}=\int_{0}^{1}\left(v\left(L_{\sigma} v\right)+G_{\infty} y_{s s}^{2}+\int_{0}^{\infty} \alpha(\tau)\left[y_{s s}-w_{s s}(\tau)\right]^{2} \mathrm{~d} \tau\right) \mathrm{d} s \tag{4.2}
\end{equation*}
$$

The first term on the right-hand side of (4.2) represents the kinetic energy of the rod, and the last two terms its Graffi-Volterra free energy ${ }^{2}$. We define linear operators $\mathcal{A}: D(\mathcal{A}) \rightarrow \mathcal{H}$ and $\mathcal{B}: \mathcal{H} \rightarrow \mathcal{H}$ by

$$
\begin{aligned}
& \mathcal{A}(y, v, w)=\left(v, K_{\sigma}\left(-y_{s s s s}+\int_{0}^{\infty} \alpha(\tau) w_{s s s s}(\tau) \mathrm{d} \tau\right),-w_{\tau}\right), \\
& \mathcal{B}(y, v, w)=\left(0,-K_{\sigma} y_{s s}, 0\right)
\end{aligned}
$$

Here $(y, v, w) \in D(\mathcal{A})$ if and only if $(y, v, w) \in \mathcal{H}, y \in V, v \in H, w(\cdot, \tau) \in V$, $w_{\tau} \in L_{\alpha}^{2}\left(\mathbb{R}^{+}, H\right), w(\cdot, 0)=y$, and $-y_{\text {ssss }}+\int_{0}^{\infty} \alpha(\tau) w_{s s s s}(\tau) \mathrm{d} \tau \in L^{2}$. Using these definitions, the problem of finding solutions to (3.3) becomes that of solving

$$
\begin{equation*}
\frac{\mathrm{d} \chi}{\mathrm{~d} t}=(\mathcal{A}+\lambda(t) \mathcal{B}) \chi, \quad \chi(0)=\chi_{0} . \tag{4.3}
\end{equation*}
$$

Theorem 4.2 Suppose that the relaxation function $G(t)$ is positive, nonincreasing and convex with $G_{\infty}>0$. Let $\lambda:[0, \infty) \rightarrow \mathbb{R}$ be bounded and continuous. Then for each $\chi_{0} \in D(\mathcal{A})$ there exists a unique solution

$$
\begin{equation*}
\chi \in C^{1}\left(\mathbb{R}^{+} ; \mathcal{H}\right) \cap C\left(\mathbb{R}^{+} ; D(\mathcal{A})\right) \tag{4.4}
\end{equation*}
$$

${ }^{1} L_{\alpha}^{2}$ denotes the $\alpha$-weighted $L^{2}$ space of functions $f$ with $\|f\|_{L_{\alpha}^{2}}=\left(\int_{0}^{\infty} \alpha(t)|f(t)|^{2} \mathrm{~d} t\right)^{1 / 2}<\infty$.
${ }^{2}$ Cf. Section 2.4.
of (4.3) satisfying $\chi(0)=\chi_{0}$.

The proof consists of showing that $\mathcal{A}$ generates a contraction semigroup. Then, the solution of (4.3) can be written using a variation of constants formula. This equation can be shown to possess the unique solution satisfying (4.4) if $\mathcal{B}$ is bounded. In order to show that $\mathcal{A}$ generates a contraction semigroup on $\mathcal{H}$ we must show that $\mathcal{A}$ is a maximal, dissipative operator.

Lemma 4.3 Let $G(t)$ satisfy the conditions of Theorem 4.2. Then, the operator $\mathcal{A}$ is dissipative, the domain of $\mathcal{A}$ is dense in $\mathcal{H}$ and the range of $I-\mathcal{A}$ is $\mathcal{H}$. Also $\mathcal{B}$ is bounded.

Proof. To show that operator $A$ is dissipative we must prove that $\langle\mathcal{A} \chi, \chi\rangle_{\mathcal{H}} \leq 0$, for all $\chi \in \mathcal{H}$. By definition of $\mathcal{H}$,

$$
\begin{aligned}
\langle\mathcal{A} \chi, \chi\rangle_{\mathcal{H}}= & \int_{0}^{1}\left\{G_{\infty} y_{s s} v_{s s}+v K_{\sigma}\left(-y_{s s s s}+\int_{0}^{\infty} \alpha(\tau) w_{s s s s}(\tau) \mathrm{d} \tau\right)\right. \\
& +\sigma v_{s}\left(K_{\sigma}\left(-y_{s s s s}+\int_{0}^{\infty} \alpha(\tau) w_{s s s s}(\tau) \mathrm{d} \tau\right)\right)_{s} \\
& \left.+\int_{0}^{\infty} \alpha(\tau)\left[v_{s s}+w_{\tau s s}(\tau)\right]\left[y_{s s}-w_{s s}(\tau)\right] \mathrm{d} \tau\right\} \mathrm{d} s \\
= & \int_{0}^{1}\left\{G_{\infty} y_{s s} v_{s s}+L_{\sigma} v K_{\sigma}\left(-y_{s s s s}+\int_{0}^{\infty} \alpha(\tau) w_{s s s s}(\tau) \mathrm{d} \tau\right)\right. \\
& \left.+\int_{0}^{\infty} v_{s s} y_{s s} \alpha(\tau)-v_{s s} w_{s s}(\tau)+\alpha(\tau) w_{\tau s s}(\tau)\left[y_{s s}-w_{s s}(\tau)\right] \mathrm{d} \tau\right\} \mathrm{d} s
\end{aligned}
$$

Since operator $L_{\sigma}$ is formally self-adjoint, integration by parts shows that

$$
\begin{align*}
\langle\mathcal{A} \chi, \chi\rangle_{\mathcal{H}}= & \int_{0}^{1}\left\{-v\left(y_{s s s s}-\int_{0}^{\infty} \alpha(\tau) w_{s s s s}(\tau) \mathrm{d} \tau\right)\right. \\
& \left.+v_{s s}\left(y_{s s}-\int_{0}^{\infty} \alpha(\tau) w_{s s}(\tau) \mathrm{d} \tau\right)+\int_{0}^{\infty} \alpha(\tau) w_{\tau s s}\left[y_{s s}-w_{s s}(\tau)\right] \mathrm{d} \tau\right\} \mathrm{d} s \\
= & \int_{0}^{1} \int_{0}^{\infty} \alpha(\tau) w_{\tau s s}(\tau)\left[y_{s s}-w_{s s}(\tau)\right] \mathrm{d} \tau \mathrm{~d} s \\
& =\frac{1}{2} \int_{0}^{1} \int_{J_{0}}^{\infty} \dot{\alpha}(\tau)\left[y_{s s}-w_{s s}(\tau)\right]^{2} \mathrm{~d} \tau \mathrm{~d} s \leq 0 \tag{4.5}
\end{align*}
$$

To show that the range of $I-\mathcal{A}$ is $\mathcal{H}$ we let $\hat{\chi}=(\hat{y}, \hat{v}, \hat{w})$ be in $\mathcal{H}$. We show that the solution of the system $(I-\mathcal{A}) \chi=\hat{\chi}$ satisfies $\chi \in \mathcal{H}$.

$$
\begin{align*}
y-v & =\hat{y}  \tag{4.6a}\\
v-K_{\sigma}\left(-y_{s s s s}+\int_{0}^{\infty} \alpha(\tau) w_{s s s s}(\tau) \mathrm{d} \tau\right) & =\hat{v}  \tag{4.6b}\\
w+w_{\tau} & =\hat{w} . \tag{4.6c}
\end{align*}
$$

From equation (4.6c),

$$
\begin{aligned}
w_{s s}^{2} & =\hat{w}_{s s}^{2}-w_{\tau s s}^{2}-2 w_{\tau s s} w_{s s} \\
& \leq \hat{w}_{s s}^{2}-2 w_{\tau s s} w_{s s} .
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
& \int_{0}^{\infty} \int_{0}^{1} \alpha(\tau) w_{s s}^{2}(\tau) \mathrm{d} s \mathrm{~d} \tau \\
& \leq \int_{0}^{\infty} \int_{0}^{1} \alpha(\tau) \hat{w}_{s s}^{2}(\tau) \mathrm{d} s \mathrm{~d} \tau-2 \int_{0}^{\infty} \int_{0}^{1} \alpha(\tau) w_{\tau s s}(\tau) w_{s s}(\tau) \mathrm{d} s \mathrm{~d} \tau \\
& =\int_{0}^{\infty} \int_{0}^{1} \alpha(\tau) \hat{w}_{s s}^{2}(\tau) \mathrm{d} s \mathrm{~d} \tau-\left[\alpha(\tau) \int_{0}^{1} w_{s s}^{2}(\tau) \mathrm{d} s\right]_{\tau=0}^{\tau=\infty}+\int_{0}^{\infty} \int_{0}^{1} \dot{\alpha}(\tau) w_{s s}^{2}(\tau) \mathrm{d} s \mathrm{~d} \tau \\
& \leq \int_{0}^{\infty} \int_{0}^{1} \alpha(\tau) \hat{u}_{s s}^{2}(\tau) \mathrm{d} s \mathrm{~d} \tau+G(0) \int_{0}^{1} y_{s s}^{2} \mathrm{~d} s
\end{aligned}
$$

If $y \in H$, then $w \in L_{\alpha}^{2}\left(\mathbb{R}^{+}, H\right)$ as $\hat{w} \in L_{\alpha}^{2}\left(\mathbb{R}^{+}, H\right)$. Solving equation (4.6c) using variation of constants yields

$$
\begin{equation*}
w(\tau)=\mathrm{e}^{-\tau} y+\int_{0}^{\tau} \mathrm{e}^{\xi-\tau} \hat{w}(\xi) \mathrm{d} \xi \tag{4.7}
\end{equation*}
$$

Using this result for $w$ and equation (4.6c), we see that $w_{\tau} \in L_{\alpha}^{2}\left(\mathbb{R}^{+}, H\right)$. Now to solve for $y$ we substitute $v$ from (4.6a) and $w$ from (4.7) into (4.6b) to give

$$
\begin{gathered}
y+K_{\sigma}\left(y_{s s s s}-\int_{0}^{\infty} \alpha(\tau) w_{s s s s}(\tau) \mathrm{d} \tau\right)=\hat{y}+\vec{v} \\
y+\left(1-\int_{0}^{\infty} \alpha(\tau) \mathrm{e}^{-\tau} \mathrm{d} \tau\right) K_{\sigma} y_{s s s s}=\hat{y}+\hat{v}+K_{\sigma}\left(\int_{0}^{\infty} \alpha(\tau) \int_{0}^{\tau} \mathrm{e}^{\xi-\tau} \hat{w}_{s s s s}(\xi) \mathrm{d} \xi \mathrm{~d} \tau\right) .
\end{gathered}
$$

The coefficient ( $1-\int_{0}^{\infty} \alpha(\tau) \mathrm{e}^{-\tau} \mathrm{d} \tau$ ) is strictly positive as it is bounded below by $G_{\infty}$. Hence there exists a unique solution to the above equation since the operator $K_{\sigma}$ is positive. The solution is in $H$ as the right hand side is in $H^{-2} \cap H_{0}^{-3}$ Finally, $v$ is given by (4.6a) and is in $H_{0}^{1}$

The operator $\mathcal{B}$ is bounded, as can be seen from

$$
\begin{aligned}
\|\mathcal{B} \chi\|_{\mathcal{H}}^{2} & =\int_{0}^{1}\left(K_{\sigma} y_{s s}\right) L_{\sigma}\left(K_{\sigma}\right) y_{s s} \mathrm{~d} s \\
& =\int_{0}^{1} y_{s s}\left(K_{\sigma} y_{s s}\right) \mathrm{d} s \leq\left\|y_{s s}\right\|_{L^{2}}^{2} \leq \frac{1}{G_{\infty}}\|\chi\|_{\mathcal{H}}^{2},
\end{aligned}
$$

since $G_{\infty}>0$.
Proof. (Theorem 4.2)
By the Lumer-Philips theorem ${ }^{3}, \mathcal{A}$ generates a $C_{0}$ contraction semigroup $\mathcal{S}(l)_{t \geq 0}$ on $\mathcal{H}$. Using the variation of parameters formula ${ }^{4},(4.3)$ is equivalent to

$$
\begin{equation*}
\chi(t)=(\mathcal{T} \chi)(t):=\mathcal{S}(t) \chi_{0}+\int_{0}^{t} \lambda(\tau) \mathcal{S}(t-\tau) \mathcal{B} \chi(\tau) \mathrm{d} \tau \tag{4.8}
\end{equation*}
$$

Since $\mathcal{S}(t)$ is a $C_{0}$ semigroup there exist constants $\omega>0$ and $M \geq 1$ such that

$$
\|\mathcal{S}(t)\| \leq M \mathrm{e}^{\omega t}, \quad \text { for } 0 \leq t<\infty
$$

Also, $\lambda(l)$ is bounded and $G_{\infty}>0$. Therefore there exists $\ell^{=}$such that

$$
0<t^{*}<1 / \omega \ln \left(1+\frac{\omega G_{\infty}}{M \max _{t \in \mathbb{R}^{+}}|\lambda(t)|}\right)
$$

[^9]The operator $\mathcal{T}$ restricted to $C\left(\left[0, t^{*}\right], \mathcal{H}\right): \mathcal{H} \rightarrow \mathcal{H}$ is a contraction for $0 \leq t \leq t^{*}$ since for any $\chi, \psi \in \mathcal{H}$ we have

$$
\begin{aligned}
\|(\mathcal{T} \chi)(t)-(\mathcal{T} \psi)(t)\|_{\mathcal{H}} & \leq M\|\mathcal{B}\| \int_{0}^{t} \lambda(\tau) \mathrm{e}^{\omega(t-\tau)}\|\chi(\tau)-\psi(\tau)\|_{\mathcal{H}} \mathrm{d} \tau \\
& \leq M 1 / G_{\infty} \int_{0}^{t} \mathrm{e}^{\omega(t-\tau)} \max _{t \in\left[0, t^{*}\right]} \mid \lambda(t)\|\chi(\tau)-\psi(\tau)\|_{\mathcal{H}} \mathrm{d} \tau \\
& \left.=\frac{M\left(\mathrm{e}^{\omega t}-1\right)}{\omega G_{\infty}} \max _{t \in\left[0, t^{*}\right]} \right\rvert\, \lambda(t)\|\chi(t)-\psi(t)\|_{\mathcal{H}} \\
& <\max _{t \in\left[0, t^{*}\right]}\|\chi(t)-\psi(t)\|_{\mathcal{H}} .
\end{aligned}
$$

Hence, using the contraction mapping theorem, we have a unique solution in the interval $t \in\left[0, t^{*}\right]$. This solution can be extended over the whole real line by repeating the above procedure over intervals of length $t^{*}$. If the load is continuous then differentiation of (4.8) shows that $\chi$ satisfies (4.4).

### 4.3 Lyapunov Stability

In this section we use the energy of the rod to analyse the stability using Lyapunov functions. When the load is constant we get a result analogous to that derived using Laplace transforms in Chapter 3. For time dependent loads we show that the solution is stable for loads satisfying $0 \leq \lambda(t)<\lambda_{1} G_{\infty}$, and describe conditions for which the solution is asymptotically stable.

From the definition of total energy (2.23) we see that

$$
\begin{equation*}
2 E(t)=\|\chi\|_{\mathcal{H}}^{2}=\langle\chi, \chi\rangle_{\mathcal{H}} . \tag{4.9}
\end{equation*}
$$

By differentiating (4.9) we can prove the energy inequality (2.25) using the dissipativity of $\mathcal{A}$.

Proposition 4.4 Let $\chi$ be a solution of (4.3) then

1. $E(t) \geq \lambda_{1} G_{\infty} \int_{0}^{1} y_{s}^{2} \mathrm{~d} s$.
2. $\dot{E}(t) \leq \lambda(t) \frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} t} \int_{0}^{1} y_{s}^{2} \mathrm{~d} s$.

Proof. From the definition of $E(t)$ and the Poincaré inequality,

$$
\begin{aligned}
E(t) & =\int_{0}^{1}\left(v\left(L_{\sigma} v\right)+G_{\infty} y_{s s}^{2}+\int_{0}^{\infty} \alpha(\tau)\left[y_{s s}-w_{s s}(\tau)\right]^{2} \mathrm{~d} \tau\right) \mathrm{d} s \\
& \geq G_{\infty} \int_{0}^{1} y_{s s}^{2} \mathrm{~d} s \\
& \geq \lambda_{1} G_{\infty} \int_{0}^{1} y_{s}^{2} \mathrm{~d} s .
\end{aligned}
$$

Differentiating (4.9) we get the relationship

$$
\begin{aligned}
\dot{E} & =\langle\mathcal{A} \chi, \chi\rangle_{\mathcal{H}}+\lambda(t)\langle\mathcal{B} \chi, \chi\rangle_{\mathcal{H}} \\
& =\frac{1}{2} \int_{0}^{1} \int_{0}^{\infty} \dot{\alpha}(\tau)\left[y_{s s}(t)-w_{s s}(\tau)\right]^{2} \mathrm{~d} \tau \mathrm{~d} s-\lambda(t) \int_{0}^{1} y_{s s} v \mathrm{~d} s \\
& =\frac{1}{2} \int_{0}^{1} \int_{0}^{\infty} \dot{\alpha}(\tau)\left[y_{s s}(t)-w_{s s}(\tau)\right]^{2} \mathrm{~d} \tau \mathrm{~d} s+\lambda(t) \frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} t} \int_{0}^{1} y_{s}^{2} \mathrm{~d} s
\end{aligned}
$$

The desired result now follows the convexity of $G(t)$.
The stability behaviour of the solution of (4.3) can be analysed using a suitable Lyapunov function. Standard results involving Lyapunov functions imply the following result ${ }^{5}$.

Lemma 4.5 Suppose that $V: \mathbb{R} \times \mathcal{H} \rightarrow \mathbb{R}$ is continuous, $V(t, 0)=0$ and there exists positive definite functions ${ }^{6} w$ and $W$ such that

$$
\begin{equation*}
w(\|\chi\|) \leq V(t, \chi) \leq W(\|\chi\|) \tag{4.10}
\end{equation*}
$$

Then, the solution of (4.3) is uniformly stable.
Theorem 4.6 Suppose that the relaxation function $G(t)$ is positive, nonincreasing and convex with $G_{\infty}>0$. Let $\lambda(t)$ belong to $C^{1}\left(\mathbb{R}^{+}\right)$and suppose there exists numbers $\varepsilon_{1}>0$ and $\varepsilon_{2}>0$ such that $\varepsilon_{1}<\lambda(t)<\lambda_{1} G_{\infty}-\varepsilon_{2}$. Then, if the function $F$ defined by

$$
\begin{equation*}
F(t):=\max \left\{\frac{\dot{\lambda}(t)}{\lambda(t)},-\frac{\dot{\lambda}(t)}{\lambda_{1} G_{\infty}-\lambda(t)}\right\} \geq 0 \tag{4.11}
\end{equation*}
$$

[^10]is integrable over $\mathbb{R}^{+}$, the solution of (4.3) is uniformly stable.
We prove the preliminary lemma.
Lemma 4.7 Suppose $\lambda \in C^{1}\left(\mathbb{R}^{+}\right)$and satisfies $0<\lambda(t) \leq \lambda_{1} G_{\infty}$ for all $t \geq 0$. Then, if $F(t)$ is integrable over $\mathbb{R}^{+}$there exists a $p(t) \in C^{1}\left(\mathbb{R}^{+} ; \mathbb{R}^{+}\right)$sulisfying

- $p(t) \geq p_{\infty}>0$.
- $\dot{p} \lambda+p \dot{\lambda} \leq 0$.
- $\dot{p}\left(\lambda_{1} G_{\infty}-\lambda\right)-p \dot{\lambda} \leq 0$.

Proof. Assume $\lambda$ and $F$ are as in the hypothesis of lemma. We define $p(l)$ as

$$
\begin{equation*}
p(l)=\exp \left(-\int_{0}^{t} F(\tau) \mathrm{d} \tau\right) \tag{4.12}
\end{equation*}
$$

Since $F(t)$ is locally integrable and positive, $p(t)$ is positive and nonincreasing. Also

$$
p(t) \geq p_{\infty}:=\exp \left(-\int_{0}^{\infty} F(\tau) \mathrm{d} \tau\right)>0
$$

Taking natural logarithon of (4.12) and differentiating we get the equation

$$
\dot{p}(t)=-p(t) F(t)
$$

Then, from the definition of $F(t)$ we can see that

$$
\dot{p} \leq-p \dot{\lambda} / \lambda \Rightarrow \dot{p} \lambda+p \dot{\lambda} \leq 0
$$

and

$$
\dot{p} \leq p \dot{\lambda} /\left(\lambda_{1} G_{\infty}-\lambda\right) \Rightarrow\left(\lambda_{1} G_{\infty}-\lambda\right) \dot{p}-\dot{\lambda} p \leq 0
$$

since $p$ is positive and $0<\lambda(t)<\lambda_{1} G_{c o}$.
Proof. (Theorem 4.6)
We seek a Lyapunov functional $V(t, \chi) \in C\left(\mathbb{R}^{+} \times \mathcal{H} ; \mathbb{R}^{+}\right)$of the form

$$
V(t, \chi)=p(t)\left[\langle\chi, \chi\rangle_{\mathcal{H}}-\lambda(t) \int_{0}^{1} y_{s}^{2} \mathrm{~d} s\right]
$$

for some suitable function $p(t) \in C^{1}\left(\mathbb{R}^{+} ; \mathbb{R}^{+}\right)$.
Since the load satisfies $\varepsilon_{1}<\lambda(t)<\lambda_{1} G_{\infty}-\varepsilon_{2}$ and $0<p_{\infty} \leq p(t) \leq 1$ we can bound $V(t, \chi)$ between two positive definite functions

$$
p_{\infty} \varepsilon_{2}\langle\chi, \chi\rangle_{\mathcal{H}} \leq V(t, \chi) \leq\langle\chi, \chi\rangle_{\mathcal{H}} .
$$

Differentiating $\dot{V}$ we get

$$
\begin{aligned}
\dot{V}(t, \chi)= & \dot{p}(t)\left[\langle\chi, \chi\rangle_{\mathcal{H}}-\lambda(t) \int_{0}^{1} y_{s}^{2} \mathrm{~d} \tau\right] \\
& +p(t)\left[\int_{0}^{1} \int_{0}^{\infty} \dot{\alpha}(\tau)\left[y_{s s}(t)-w_{s s}(\tau)\right]^{2} \mathrm{~d} \tau \mathrm{~d} s-\dot{\lambda}(t) \int_{0}^{1} y_{s}^{2} \mathrm{~d} s\right] \\
= & p(t)\langle\chi, \chi\rangle_{\mathcal{H}}-[\dot{p}(t) \lambda(t)+p(t) \dot{\lambda}(t)] \int_{0}^{1} y_{s}^{2} \mathrm{~d} s \\
& +p(t) \int_{0}^{1} \int_{0}^{\infty} \dot{\alpha}(\tau)\left[y_{s s}(t)-w_{s s}(\tau)\right]^{2} \mathrm{~d} \tau \mathrm{~d} s .
\end{aligned}
$$

Since $p(t)$ satisfies (H2), we have the inequality

$$
-[\dot{p}(l) \lambda(t)+p(t) \dot{\lambda}(t)] \int_{0}^{1} y_{s}^{2} \mathrm{~d} s \leq-\frac{\dot{p}(t) \lambda(t)+p(t) \dot{\lambda}(t)}{\lambda_{1} G_{\infty}}\langle\chi, \chi\rangle_{\mathcal{H}}
$$

Thus

$$
\begin{aligned}
\dot{V}(t, \chi) \leq & \frac{\langle\chi, \chi\rangle_{\mathcal{H}}}{\lambda_{1} G_{\infty}}\left\{\lambda_{1} G_{\infty} \dot{p}(t)-\dot{p}(t) \lambda(t)-\dot{\lambda}(t) p(t)\right\} \\
& +p(t) \int_{0}^{1} \int_{0}^{\infty} \dot{\alpha}(\tau)\left[y_{s s}(t)-w_{s s}(\tau)\right]^{2} \mathrm{~d} \tau \mathrm{~d} s .
\end{aligned}
$$

Because $p(t)$ satisfies (H3) and the last term is negative we get

$$
\begin{equation*}
\dot{V}(t, \chi) \leq 0 . \tag{4.13}
\end{equation*}
$$

By applying Lemma 4.5 we get the stated result.
We now discuss the meaning of the hypothesis in Theorem 4.6 that $F$ is integrable. The following lemma shows that for loads with a finite number of critical points $F$ is integrable. On the other hand, for certain oscillatory loads whose amplitude does not tend to zero as $t \rightarrow \infty, F$ is not integrable.

Lemma 4.8 Suppose the load $\lambda(t)$ satisfies the conditions of Theorem 4.6. Then,

1. if $\lambda(t)$ is eventually monotonic, $F$ is integrable.
2. if $\lambda(t)$ has an infinite number of critical points $t_{k}, k \in \mathbf{N}$, then a necessary condition for $F(t)$ to be integrable is

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \frac{\lambda\left(t_{2 k}\right)}{\lambda\left(t_{2 k+1}\right)}=1 \tag{4.14}
\end{equation*}
$$

Proof. Suppose there exists $t^{*} \in \mathbb{R}^{+}$such that $\dot{\lambda}(t) \geq 0$ for all $t>t^{*}$. Then

$$
\begin{aligned}
\int_{0}^{\infty} F(\tau) \mathrm{d} \tau & =\int_{0}^{t^{*}} F(\tau) \mathrm{d} \tau+\int_{t^{*}}^{\infty} F(\tau) \mathrm{d} \tau \\
& =\int_{0}^{t^{*}} F(\tau) \mathrm{d} \tau-\left[\ln \left(\lambda_{1} G_{\infty}-\lambda(t)\right)\right]_{t^{*}}^{\infty}<\infty,
\end{aligned}
$$

since $F$ is locally integrable and $\varepsilon_{1}<\lambda(t)<\lambda_{1} G_{\infty}-\varepsilon_{2}$ for some positive numbers $\varepsilon_{1}, \varepsilon_{2}$. The proof is similar for the case $\dot{\lambda}(t)<0$ for all $t>t^{*}$.

Assume the load $\lambda(t)$ has local minima at the points $t_{2 k}$ and local maxima at $t_{2 k+1}$ where $k \in \mathbf{N}$. Then

$$
\begin{align*}
-\int_{t_{0}}^{t_{2 k+2}} F(\tau) \mathrm{d} \tau & =-\sum_{j=0}^{k}\left(\int_{t_{2 j}}^{t_{2 j+1}}+\int_{t_{2 j+1}}^{t_{2 j+2}}\right) F(\tau) \mathrm{d} \tau \\
& =\sum_{j=0}^{k}\left(-\int_{t_{2 j}}^{t_{2 j+1}} \frac{\lambda(\tau)}{\lambda(\tau)} \mathrm{d} \tau+\int_{t_{2 j+1}}^{t_{2 j+2}} \frac{\dot{\lambda}(\tau)}{\lambda_{1} G_{\infty}-\lambda(\tau)} \mathrm{d} \tau\right) \\
& =\sum_{j=0}^{k}\left(\ln \frac{\lambda\left(t_{2 j}\right)}{\lambda\left(t_{2 j+1}\right)}+\ln \frac{\lambda_{1} G_{\infty}-\lambda\left(t_{2 j+1}\right)}{\lambda_{1} G_{\infty}-\lambda\left(t_{2 j+2}\right)}\right) \\
& =\sum_{j=0}^{k}\left(\ln \frac{\lambda\left(t_{2 j}\right)\left(\lambda_{1} G_{\infty}-\lambda\left(t_{2 j+1}\right)\right)}{\lambda\left(t_{2 j+1}\right)\left(\lambda_{1} G_{\infty}-\lambda\left(t_{2 j+2}\right)\right)}\right) . \tag{4.15}
\end{align*}
$$

Note that all the terms in the series (4.15) are negative. Hence, taking the exponential of (4.15) we have

$$
\begin{align*}
\exp \left(\int_{t_{0}}^{t_{2 k+2}} F(\tau) \mathrm{d} \tau\right) & =\prod_{j=0}^{k} \frac{\lambda\left(t_{2 j+1}\right)\left(\lambda_{1} G_{\infty}-\lambda\left(t_{2 j+2}\right)\right)}{\lambda\left(t_{2 j}\right)\left(\lambda_{1} G_{\infty}-\lambda\left(t_{2 j+1}\right)\right)} \\
& =\prod_{j=0}^{k}\left(1+\frac{w_{j}}{\lambda_{1} G_{\infty}-\lambda\left(t_{2 j+1}\right)}\right) \tag{4.16}
\end{align*}
$$

where the $w_{j} \geq 0$, are given by

$$
\begin{align*}
w_{j} & =\frac{\lambda\left(t_{2 j+1}\right)}{\lambda\left(t_{2 j}\right)}\left(\lambda_{1} G_{\infty}-\lambda\left(t_{2 j+2}\right)\right)-\left(\lambda_{1} G_{\infty}-\lambda\left(t_{2 j+1}\right)\right) \\
& =\left(\frac{\lambda\left(t_{2 j+1}\right)}{\lambda\left(t_{2 j}\right)}-1\right) \lambda_{1} G_{\infty}+\frac{\lambda\left(t_{2 j+1}\right)}{\lambda\left(t_{2 j}\right)}\left(\lambda\left(t_{2 j}\right)-\lambda\left(t_{2 j+2}\right)\right) . \tag{4.17}
\end{align*}
$$

The denominator in (4.16) is finite since the load satisfies $\varepsilon_{1}<\lambda(t)<\lambda_{1} G_{\infty}-\varepsilon_{2}$, with $\varepsilon_{1}, \varepsilon_{2}>0$. Hence, a necessary condition for the infinite product to converge is that $w_{j} \rightarrow 0$ as $j \rightarrow \infty$.

In order to prove the condition (4.14) is necessary we assume $w_{j} \rightarrow 0$ as $j \rightarrow \infty$, and

$$
\lim _{j \rightarrow \infty} \frac{\lambda\left(t_{2 j}\right)}{\lambda\left(t_{2, j+1}\right)}=\delta_{0} \neq 1 .
$$

Since

$$
\lambda\left(t_{2 j+2}\right)-\lambda\left(t_{2 j}\right)=-\frac{\lambda\left(t_{2 j}\right)}{\lambda\left(t_{2 j+1}\right)} w_{j}+\lambda_{1} G_{\infty}\left(1-\frac{\lambda\left(t_{2 j}\right)}{\lambda\left(t_{2 j+1}\right)}\right),
$$

we have

$$
\lim _{j \rightarrow \infty} \lambda\left(t_{2 j+2}\right)-\lambda\left(t_{2 j}\right)=\lambda_{1} G_{\infty}\left(1-\frac{1}{\delta_{0}}\right)=\lambda_{1} G_{\infty} \frac{\delta_{0}-1}{\delta_{0}}>0
$$

Hence

$$
\lim _{j \rightarrow \infty} \lambda\left(t_{2 j}\right)=\infty .
$$

This contradicts the boundedness of $\lambda(t)$.
We will now brielly look at some important time-varying loads.

Example 4.9 An important example for which $F$ is not integrable is given by

$$
\lambda(t)=P_{0}+P_{1} \cos (\Omega t),
$$

where $\varepsilon \leq P_{0}-P_{1}, P_{0}+P_{1} \leq \lambda_{1} G_{\infty}-\varepsilon$, with $\varepsilon>0 . F(t)$ is not integrable since

$$
\frac{\lambda\left(t_{2 j}\right)}{\lambda\left(t_{2 j+1}\right)}=\frac{P_{0}-P_{1}}{P_{0}+P_{1}}<1
$$

The behaviour of the $n^{\text {th }}$ Fourier mode is studied in Section 5.6 using this type of
load for the special case of a standard viscoelastic material. Numerical computations suggest however that, even though $F$ is nonintegrable, the solutions are bounded.

Example 4.10 In the case of nonstationary parametric excitation of the form

$$
\begin{equation*}
\lambda(t)=P_{0}+P_{1} \mathrm{e}^{-P_{2} t} \cos (\Omega t), \tag{4.18}
\end{equation*}
$$

it is easily seen that $w_{j} \rightarrow 0$ as $j \rightarrow \infty$. In fact,

$$
\begin{align*}
w_{j} & \leq \lambda_{1} G_{\infty}\left(\frac{P_{0}+P_{1} \mathrm{e}^{-P_{2} t_{2 j+1}}}{P_{0}-P_{1} \mathrm{e}^{-P_{2} t_{2 j}}}-1\right)+\left(P_{0}+P_{1} \mathrm{e}^{-P_{2} t_{2 j+1}}\right)\left(1-\frac{P_{0}-P_{1} \mathrm{e}^{-P_{2} t_{2 j+2}}}{P_{0}+P_{1} \mathrm{e}^{-P_{2} t_{2 j}}}\right) \\
& \leq P_{0} \lambda_{1} G_{\infty} \frac{\mathrm{e}^{-F_{2} t_{2 j+1}}+\mathrm{e}^{-P_{2} t_{2 j}}}{P_{0}-P_{1}}+P_{1}\left(P_{0}+P_{1}\right) \frac{\mathrm{e}^{-P_{2} t_{2 j}}+\mathrm{e}^{-P_{2} t_{2 j+2}}}{P_{0}} \\
& \leq 2\left[\frac{P_{0} \lambda_{1} G_{\infty}}{P_{0}-P_{1}}+\frac{P_{1}\left(P_{0}+P_{1}\right)}{P_{0}}\right] \mathrm{e}^{-P_{2} t_{2 j}}, \tag{4.19}
\end{align*}
$$

where

$$
t_{j}=-1 / \Omega \arctan \left(P_{2} / \omega\right)+\pi(j+1) / \Omega, \quad j=0,1, \ldots
$$

In order to show that $F(t)$ is integrable, we use the fact that the infinite product (4.16) converges if and only if the series

$$
\begin{equation*}
\sum_{j=0}^{\infty} w_{j} \tag{4.20}
\end{equation*}
$$

converges. Using the bound (4.19) it is easily seen that the series (4.20) converges by comparison with a geometric series. Hence, $F(t)$ is integrable. Figure 4.1 shows the behaviour a load given by (4.18) and the integral of $F(t)$ calculated up to $t_{20}$. These figures were generated using the following data:

| Parameter | $P_{0}$ | $P_{1}$ | $P_{2}$ | $\Omega$ |
| :---: | :---: | :---: | :---: | :---: |
| Value | 1.1 | 1.0 | 0.1 | 1.0 |



Figure 4.1: Nonstationary periodic load (4.18) and the integral of $F(t)$.

## Chapter 5

## Multiple Scale Approach

### 5.1 Introduction

In this chapter we examine the dynamics of the viscoelastic rod using various multiple scale expansions. A brief outline of the procedure is discussed in the next section.

In Section 5.3 we study the autonomous problem for the standard viscoelastic material. If the relaxation time is large, the elastic and creep effects should occur on different time scales. This expectation is supported by the work in Chapter 3 on constant loads $\lambda<\lambda_{n}$. For stable loads $\lambda<\lambda_{n} G_{\infty}$, the deflection modes behave like a damped linear oscillator with an exponentially decaying amplitude. If $\lambda_{n} G_{\infty}<\lambda<\lambda_{n}$, the rod's initial oscillatory behaviour is damped, but the effect of a slowly increasing term, due to creep, results in the rod buckling. This instability is discernible after a critical time ${ }^{1}$ that is an order of magnitude greater than the period of the oscillations.

The retarded problem is investigated in Section 5.4 for general viscoelastic materials. This entails replacing the relaxation function $G$ by its retardation $G_{\varepsilon}$, where $G_{\epsilon}(t)=G(\varepsilon t)$ and retarding the known initial history. For small values of the retardation parameter $\varepsilon$ the elastic and creep response occur on different time scales. The leading order and the first order terms are calculated using a procedure similar to that used in $[1,2]$. When we set the retardation parameter $\varepsilon=1$ in this approximate solution of the retarded problem, it happens that we get a function which agrees exactly with the approximate solution found in Section 5.3, if the rod is of standard viscoelastic material.

In Section 5.5 we consider the effect of a slowly varying load. In particular, we

[^11]examine the behaviour as the load slowly crosses the equilibrium buckling load.
In Section 5.6 we consider periodic loads of the form $\lambda(t)=P_{0}+P_{1} \cos \Omega t$, where $P_{0}<\lambda_{n} G_{\infty}$. We know from the work in Chapter 3 that the trivial solution is asymptotically stable if $P_{1}=0$. We use multiple scales to examine the instability produced by principal parametric resonance when $\left|P_{1}\right|$ is small. This approach allows us to calculate the transition curves that decompose the parameter space into stable and unstable regions and also to calculate the form of the solution in the neighbourhood of these curves.

Note that in order to simplify notation, we will use $\varepsilon$ to represent different quantities in the separate problems discussed in Sections 5.3-5.6. Also, the variables $t_{k}, k=0,1,2, \ldots$ are used to represent different time scales throughout this chapter.

### 5.2 Multiple Scales

In Smith [37] second order differential equations of the form

$$
\begin{equation*}
\frac{\mathrm{d}^{2} y}{\mathrm{~d} t^{2}}+y=\varepsilon f\left(\varepsilon, t, y, \frac{\mathrm{~d} y}{\mathrm{~d} t}\right), \quad t>0 \tag{5.1}
\end{equation*}
$$

subject to the initial conditions $y(0)=a, \dot{y}(0)=b$ are studied for suitable functions $f$. In each case, an approximation solution was found by considering an ansatz of the form

$$
\begin{equation*}
y(t ; \varepsilon)=\sum_{k=0}^{N} \varepsilon^{k} Y_{k}(t, \varepsilon t)+\varepsilon^{N+1} R_{N}(t, \varepsilon), \tag{5.2}
\end{equation*}
$$

where each term $Y_{k}\left(t, t_{1}\right)$ is assumed to depend on the fast time $t$, and the slow time $t_{1}=\varepsilon t$. This is similar to the Lindstedt-Poincaré procedure, but in this case the fast time is 'detuned' and given by $t^{+}=t \cdot\left(1+c_{1} \varepsilon+c_{2} \varepsilon^{2}+\cdots\right)$.

For a number of functions $f$ in (5.1), the two-time expansion in (5.2) has been shown to be uniform in the sense that the quantity $R_{N}(t, \varepsilon)$ is uniformly bounded (as $\varepsilon \rightarrow 0^{+}$) for all $t$ on an expanding interval $0 \leq t \leq T / \varepsilon$. For example, see Kollett [25] or Smith [37].

As we shall discuss later, we need to use a more general form of the expansion in equation (5.2). The two-time expansion of (5.2) is extended by introducing the time scales,

$$
\begin{equation*}
t_{k}=\varepsilon^{k} t, \quad \text { for } k=0,1,2, \ldots \tag{5.3}
\end{equation*}
$$

Using the chain rule, it follows that the derivatives with respect to $t$ can be written in terms of partial derivatives with respect to the new independent variables $t_{k}$, i.e.,

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t}=D_{0}+\varepsilon D_{1}+\cdots, \quad \frac{\mathrm{d}^{2}}{\mathrm{~d} t^{2}}=D_{0}^{2}+2 \varepsilon D_{0} D_{1}+\varepsilon^{2}\left(D_{1}^{2}+2 D_{0} D_{2}\right)+\cdots, \tag{5.4}
\end{equation*}
$$

etc.. Here we denote the partial derivatives with respect to the new time scales by $D_{k}=\frac{\partial}{\partial t_{k}}$. A partial differential equation in the $M$ independent variables $t_{0}, \ldots, t_{M}$ is obtained by substituting (5.4) into (5.1).

Let us assume that the solution of this partial differential equation can be represented by a multiple scale expansion of the form

$$
\begin{equation*}
Y\left(t_{0}, \ldots, t_{M} ; \varepsilon\right) \sim \sum_{k=0}^{\infty} \varepsilon^{k} Y_{k}\left(t_{0}, \ldots, t_{M}\right), \quad \text { as } \varepsilon \rightarrow 0 \tag{5.5}
\end{equation*}
$$

where each term in the expansion, $Y_{k}$, depends on $t$ and $\varepsilon$ in such a way that the result is uniformly valid over the time scales under consideration. In other words, we require for each $N$ that

$$
\begin{equation*}
\sup _{0 \leq t \leq T / \varepsilon^{M}}\left|\left(y(t ; \varepsilon)-\sum_{k=0}^{N} \varepsilon^{k} Y_{k}\left(t_{0}, \ldots, t_{M}\right)\right) / \varepsilon^{N+1}\right| \tag{5.6}
\end{equation*}
$$

be uniformly bounded for all small $\varepsilon$. In general, the number $M$ of slow time scales required will be no greater than the order $N$ to which the expansion is carried out. The necessity of more than two time scales for higher order equations was recognised by Kabakow [23] for systems of coupled oscillators ${ }^{2}$.

[^12]
### 5.3 Decomposition of Elastic and Creep Effects

It can be inferred from the linearised dynamic equation for the standard viscoelastic linear material that $y(s, t)=u(t) \sin n \pi s$ is a solution, if $u(t)$ satisfies

$$
\begin{equation*}
\ddot{u}=-\gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda(t)\right) u+\gamma_{n} \lambda_{n}^{2} a\left(1-G_{\infty}\right) \int_{-\infty}^{t} \mathrm{e}^{-a(t-\tau)} u(\tau) \mathrm{d} \tau, \quad t>0, \tag{5.7}
\end{equation*}
$$

subject to the initial conditions $u(0)=a_{0}, \dot{u}_{n}(0)=a_{1}$ and a prescribed initial history. The physical parameter $a$, which represents the reciprocal of the relaxation time is small for many viscoelastic materials. In the limit as the relaxation period approaches infinity we get the dynamic equation corresponding to an elastic material. In this section we wish to construct a uniformly valid expansion as $a \rightarrow 0$ for the solution $u(t)$, thereby decomposing the elastic and creep effects. We shall examine this problem for constant loads less than the instantaneous buckling load $\lambda_{n}$. In particular, we shall investigate the effect which the type of initial disturbance has on the critical time in the case of viscoelastic instability.

In order to use a multiple scale expansion, we first need to convert (5.7) into an ordinary differential equation. The integral in equation (5.7) can be removed if we define a function $c(t)$ by

$$
\begin{equation*}
c(t)=\gamma_{n} \lambda_{n}^{2}\left(1-G_{\infty}\right) \int_{-\infty}^{t} \mathrm{e}^{-a(t-\tau)} u(\tau) \mathrm{d} \tau, \quad t>0 \tag{5.8}
\end{equation*}
$$

In this chapter we will only consider the typical case of an integrable initial history. Hence $c(t)$ is well defined as $a \rightarrow 0^{+}$and satisfies the differential equation

$$
\dot{c}=\gamma_{n} \lambda_{n}^{2}\left(1-G_{\infty}\right) u-a c, \quad c(0)=\gamma_{n} \lambda_{n}^{2}\left(1-G_{\infty}\right) \int_{-\infty}^{0} \mathrm{e}^{a(\tau)} u(\tau) \mathrm{d} \tau
$$

The value of $c$ at time $t=0$ depends on the initial history of $u$. Hence, as $a \rightarrow 0^{+}$ we have

$$
\begin{equation*}
c(0)=\sum_{i=0}^{\infty} a^{i} c_{i}:=\gamma_{n} \lambda_{n}^{2}\left(1-G_{\infty}\right) \int_{-\infty}^{0} \mathrm{e}^{a(\tau)} u(\tau) \mathrm{d} \tau . \tag{5.9}
\end{equation*}
$$

In order to simplify expressions we introduce the parameters ${ }^{3}$

$$
\omega=\sqrt{\gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda\right)}, \quad \mu=\gamma_{n} \lambda_{n}^{2}\left(1-G_{\infty}\right)
$$

The dynamic problem for the $n^{\text {th }}$ mode can be expressed as

$$
\begin{equation*}
\ddot{u}(t)+\omega^{2} u(t)=a c(t), \quad \dot{c}(t)=\mu u(t)-a c(t), \tag{5.10}
\end{equation*}
$$

subject to the initial conditions $u(0)=a_{0}, \dot{u}(0)=a_{1}$ and equation (5.9). Let $\varepsilon=a$. We would like to obtain a multiple scale expansion for $u(t)$ and $c(t)$, uniformly valid on the interval $0 \leq t \leq T / \varepsilon^{2}$ as $\varepsilon \rightarrow 0$. A two-time expansion of the form (5.2) or with a detuning of the fast time as in the Lindstedt-Poincaré procedure is unsuitable for this problem, due to the length of the time interval required to investigate the relationship between the type of initial disturbance and the critical time. A twotime expansion over $0 \leq t \leq T / \varepsilon^{2}$ produces secular terms. Therefore, we assume that $u(t)$ and $c(t)$ depend on the three time scales $t_{k}=\varepsilon^{k} t, k=0,1,2$ such that

$$
u(t ; \varepsilon)=U\left(t_{0}, t_{1}, t_{2} ; \varepsilon\right), \quad c(t ; \varepsilon)=C\left(t_{0}, t_{1}, t_{2} ; \varepsilon\right)
$$

$U\left(t_{0}, t_{1}, t_{2} ; \varepsilon\right)$ and $C\left(t_{0}, t_{1}, t_{2} ; \varepsilon\right)$ satisfy the system of partial differential equations

$$
\begin{gather*}
\left(D_{0}+\varepsilon D_{1}+\varepsilon^{2} D_{2}\right)^{2} U\left(t_{0}, t_{1}, t_{2} ; \varepsilon\right)+\omega^{2} U\left(t_{0}, t_{1}, t_{2} ; \varepsilon\right)=\varepsilon C\left(t_{0}, t_{1}, t_{2} ; \varepsilon\right)  \tag{5.11a}\\
\quad\left(D_{0}+\varepsilon D_{1}+\varepsilon^{2} D_{2}\right) C\left(t_{0}, t_{1}, t_{2} ; \varepsilon\right)=\mu U\left(t_{0}, t_{1}, t_{2} ; \varepsilon\right)-\varepsilon C\left(t_{0}, t_{1}, t_{2} ; \varepsilon\right) \tag{5.11b}
\end{gather*}
$$

We assume that $U\left(t_{0}, t_{1} ; \varepsilon\right)$ and $C\left(t_{0}, t_{1} ; \varepsilon\right)$ satisfy the asymptotic expansions

$$
\begin{align*}
U\left(t_{0}, t_{1} ; \varepsilon\right) & \sim U_{0}\left(t_{0}, t_{1}\right)+\varepsilon U_{1}\left(t_{0}, t_{1}\right)+\ldots,  \tag{5.12a}\\
C\left(t_{0}, t_{1} ; \varepsilon\right) & \sim C_{0}\left(t_{0}, t_{1}\right)+\varepsilon C_{1}\left(t_{0}, t_{1}\right)+\ldots, \tag{5.12b}
\end{align*}
$$

[^13]as $\varepsilon \rightarrow 0$.
We now successively determine $U_{k}$ and $C_{k}$ by substituting (5.12a)-(5.12b) into (5.11a)-(5.11b). As the technique used here is standard we shall omit the details and refer the reader to [24] or [22]. The $O(1), O(\varepsilon)$ and $O\left(\varepsilon^{2}\right)$ solutions' dependence on the fast time and the two slow times can be shown to be of the form
\[

$$
\begin{align*}
U_{k}\left(t_{0}, t_{1}, t_{2}\right)= & A_{k} \mathrm{e}^{-\left(\mu / 2 \omega^{2}\right) t_{1}} \cos \left(\omega t_{0}+\frac{\mu^{2}}{8 \omega^{5}} t_{2}\right)+B_{k} \mathrm{e}^{-\left(\mu / 2 \omega^{2}\right) t_{1}} \sin \left(\omega t_{0}+\frac{\mu^{2}}{8 \omega^{5}} t_{2}\right) \\
& +F_{k-1} \mathrm{e}^{\left(\mu / \omega^{2}-1\right) t_{1}} \tag{5.13}
\end{align*}
$$
\]

for $k=0,1,2$. Using the initial conditions we can calculate the constant coefficients $A_{k}, B_{k}$ and $F_{k}$. The $O(1)$ and $O(\varepsilon)$ coefficients are given by

$$
\begin{aligned}
& A_{0}=a_{0}, \quad B_{0}=a_{1} / \omega, \quad F_{0}=c_{0}+a_{1} \mu / \omega \\
& A_{1}=-c_{0} / \omega^{2}-a_{1} \mu / \omega^{3}, \quad B_{1}=a_{0} \mu /\left(2 \omega^{3}\right), \\
& F_{1}=c_{1}-a_{0} 2 \mu\left(2 \omega^{3}-\mu \omega-1\right) / \omega^{5} .
\end{aligned}
$$

The $O\left(\varepsilon^{2}\right)$ coefficients used in determining $U_{2}$ are given by

$$
\begin{gathered}
A_{2}=-c_{1} / \omega^{2}+a_{0} 2 \mu\left(2 \omega^{3}-\mu \omega-1\right) / \omega^{7}, \\
B_{2}=c_{0}\left(\mu-2 \omega^{2}\right) /\left(2 \omega^{5}\right)-a_{1} \mu\left(\mu-4 \mu \omega+8 \omega^{3}\right) /\left(8 \omega^{7}\right) .
\end{gathered}
$$

Remark 5.1 Note that over the interval $0 \leq t \leq O\left(\varepsilon^{-1}\right)$ there may be a reordering in the first three terms of the multiple scale expansion. This will occur for loads in the range $\lambda_{n} G_{\infty}<\lambda<\lambda_{n}$ since the $O(1)$ solution does not contain a creep term with the type of initial conditions assumed here. A creep term will appear in the $O(\varepsilon)$ or $O\left(\varepsilon^{2}\right)$ solutions depending on the initial conditions.

Hence the type of initial disturbance has a large effect on the size of the creep component in the solution. This effect has also been discussed in [39] where the static and dynamic initial disturbances for the zero history problem were compared. For constant loads that result in viscoelastic instability, i.e., if $\lambda \in\left(\lambda_{n} G_{\infty}, \lambda_{n}\right)$, this relationship between the type of initial disturbance and the magnitude of the creep response component influences the critical time of the rod. We consider the
multiple scale approximation and determine the critical time for which the creep component is of the same order as the decaying elastic response for various types of initial disturbances.

In the case of a static disturbance of size $a_{0}$, with zero initial velocity ( $a_{1}=0$ ) and ( $c_{0}=c_{1}=0$ ), the creep term becomes dominant when

$$
O\left(\left(A_{0}+\varepsilon B_{1}\right) \mathrm{e}^{-\mu /\left(2 \omega^{2}\right) t_{1}}\right)=O\left(\varepsilon^{2} F_{1} \mathrm{e}^{\left(\mu / \omega^{2}-1\right) t_{1}}\right)
$$

Hence the static critical time is given by

$$
t_{\mathrm{cr}}=\frac{1}{\varepsilon} \frac{2 \omega^{2}-3 \mu}{2 \omega^{2}} \ln \left|\frac{\varepsilon^{2} 2 \mu\left(2 \omega^{2}-\mu \omega-1\right) / \omega^{3}}{\omega^{2}+\varepsilon \mu / 2}\right|
$$

For loads in the interval $\left(\lambda_{n} G_{\infty}, \lambda_{n}\right)$ we have $2 \omega^{2}-3 \mu<0$. Note also that the critical time is independent of the size of the initial static disturbance.

The critical time for the rod with zero static disturbance ( $a_{0}=0$ ), an initial history with ( $c_{0}=0$ ), and an initial dynamic disturbance of size $a_{1}$, is approximated by

$$
O\left(\left(B_{0}+\varepsilon A_{1}\right) \mathrm{e}^{-\mu /\left(2 \omega^{2}\right) t_{1}}\right)=O\left(\varepsilon F_{0} \mathrm{e}^{\left(\mu / \omega^{2}-1\right) t_{1}}\right),
$$

for small $\varepsilon$. Hence the dynamic critical time is given by

$$
t_{\mathrm{cr}}=\frac{1}{\varepsilon} \frac{2 \omega^{2}-3 \mu}{2 \omega^{2}} \ln \left|\frac{\varepsilon \mu \omega}{\omega+\varepsilon \mu}\right| .
$$

This relationship can also be seen in the Figures (5.1)-(5.2), which were generated using the following data.

| Parameter | $n$ | $\lambda_{n}$ | $\gamma_{n}$ | $G_{\infty}$ | $\varepsilon$ | $\lambda$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Value | 1 | 9.8696 | 1.0001 | 0.1 | 0.6 | 1.9739 | $\left(=2 \lambda_{n} G_{\infty}\right)$ |

In each case the numerical solution (calculated using the procedure discussed in Section B.3.1) is not plotted as it is indistinguishable from the multiple scale approximation of $y_{n}$ to within the thickness of the curve.


Figure 5.1: Static initial disturbance with $a_{1}=c_{0}=c_{1}=0$.


Figure 5.2: Dynamic initial disturbance with with $a_{0}=c_{0}=0$.

### 5.4 Retarded General Dynamic Equation

In this section we examine the retarded problem for the general autonomous dynamic equation. The retarded problem is formed by replacing the relaxation function $G(t)$ by $G_{\varepsilon}(t):=G(\varepsilon t)$, and the known initial history $u(t)$ by $u(\varepsilon t)$. where $\varepsilon$ is a small positive parameter. In the limit as $\varepsilon \rightarrow 0^{+}$, we have the dynamic equation corresponding to an elastic material. The motivation for this approximation is that some relaxation functions are slowly varying. We examine this problem using multiple scales and determine the leading order terms. Also by setting $\varepsilon=1$, we get a function, which we can compare with the series solution found in the previous section for a rod of standard viscoelastic material.

The $n^{\text {th }}$ Fourier mode $u(t)$, of the general autonomous initial history problem is given by $\ddot{u}(t)+\gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda\right) u(t)=-\gamma_{n} \lambda_{n}^{2} \int_{-\infty}^{t} \dot{G}(t-\tau) u(\tau) \mathrm{d} \tau$, with initial conditions $u(0)=a_{0}, \dot{u}(0)=a_{1}$ and $u(t)$ known for $t \leq 0$.

We will consider the retarded problem with a relaxation function of the form $G_{\varepsilon}(t):=G(\varepsilon t)$, where $0<\varepsilon<1$. So now defining $\alpha(\varepsilon t)=-\varepsilon \frac{\mathrm{d} G}{\mathrm{~d}(\varepsilon t)}(\varepsilon t)$, we can write the retarded dynamic equation as

$$
\begin{equation*}
\ddot{U}(t ; \varepsilon)+\gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda\right) U(t ; \varepsilon)=\gamma_{n} \lambda_{n}^{2} \varepsilon \int_{j_{0}}^{t} \alpha(\varepsilon(t-\tau)) U(\tau ; \varepsilon) \mathrm{d} \tau+\gamma_{n} \lambda_{n}^{2} f(\varepsilon t), \tag{5.14}
\end{equation*}
$$

where the initial history function $f(t)$ depends on the retarded initial history $u(\varepsilon t)$. Setting $\varepsilon=1$ we recover the original dynamic equation and the $n^{\text {th }}$ Fourier mode is given by $u(t)=U(t ; 1)$. In order to simplify notation, we denote the natural frequency of the corresponding elastic system by $\omega=\sqrt{\gamma_{n} \lambda\left(\lambda_{n}-\lambda\right)}$.

We introduce two times $t_{0}=t$ and $t_{1}=\varepsilon t$. Expanding the time derivatives in (5.14) in terms of the new time scales we obtain a partial differential equation for $U\left(t_{0}, t_{1} ; \varepsilon\right)$. We propose as an ansatz

$$
\begin{equation*}
U(t ; \varepsilon)=U\left(t_{0}, t_{1} ; \varepsilon\right) \sim \sum_{k=0}^{\infty} \varepsilon^{k} U_{n}\left(t_{0}, t_{1}\right), \quad \text { as } \varepsilon \rightarrow 0 . \tag{5.15}
\end{equation*}
$$

Since $U\left(t_{0}, t_{1} ; \varepsilon\right)$ is the solution of an integro-partial differential equation, we
cannot apply the multiple scale approach in the same manner as in the previous section. In order to overcome this difficulty we adapt the procedure described in work performed by Angell and Olmstead [1, 2].

The leading order term is determined by substituting $U(t ; \varepsilon)=U_{0}\left(t_{0}, t_{1}\right)$ into equation (5.14). Hence $U_{0}\left(t_{0}, t_{1}\right)$ satisfies

$$
\begin{equation*}
\left(D_{0}^{2}+\omega^{2}\right) U_{0}\left(t_{0}, t_{1}\right)=\gamma_{n} \lambda_{n}^{2} \varepsilon \int_{0}^{t_{0}} \alpha\left(\varepsilon\left(t_{0}-\tau\right)\right) U_{0}(\tau, \varepsilon \tau) \mathrm{d} \tau+\gamma \lambda_{n} f\left(t_{1}\right)+O(\varepsilon) \tag{5.16}
\end{equation*}
$$

Following the approach in [1] we write the integral in terms of the fast time, $t_{0}$, and take the limit as $\varepsilon \rightarrow 0^{+}$. Hence

$$
\begin{equation*}
U_{0}\left(t_{0}, t_{1}\right)=A_{0}\left(t_{1}\right) \cos \left(\omega t_{0}\right)+B_{0}\left(t_{1}\right) \sin \left(\omega t_{0}\right)+F_{0}\left(t_{1}\right) . \tag{5.17}
\end{equation*}
$$

Now in order to ensure that we have a uniform expansion in $t$ we substitute (5.17) into equation (5.14) and rewrite this equation in terms of the slow variable $t_{1}$. Taking the limit as $\varepsilon \rightarrow 0^{+}$while holding $t_{1}$ fixed yields an equation for $F_{0}\left(t_{1}\right)$ which can be solved using Laplace transforms to give

$$
\begin{equation*}
F_{0}(p)=\frac{\gamma_{n} \lambda_{n}^{2} \bar{f}(p)}{\omega^{2}-\gamma_{n} \lambda_{n}^{2} \bar{\alpha}(p)}=\frac{\lambda_{n} \bar{f}(p)}{\lambda_{n}(1-\bar{\alpha}(p))-\lambda} \tag{5.18}
\end{equation*}
$$

where $\bar{F}_{0}(p)$ represents the Laplace transform of $F_{0}\left(t_{1}\right)$, etc..
The leading term for the creep, $F_{0}\left(t_{1}\right)$, can be shown to be in $L^{1}\left(\mathbb{R}^{+}\right)$if $\lambda<$ $\lambda_{n} G_{\infty}$. When $\lambda_{n} G_{\infty} \leq \lambda<\lambda_{n}$ the creep term can be separated into an exponential increasing term and a remainder which is in $L^{1}\left(\mathbb{R}^{+}\right)$.

It remains for us to determine the $t_{1}$ dependence in the coefficients $A_{0}\left(t_{1}\right)$ and $B_{0}\left(t_{1}\right)$. To do this we look at the right hand side of the first order equation. Using $U(t ; \varepsilon)=U_{0}\left(t_{0}, t_{1}\right)+\varepsilon U_{1}\left(t_{0}, t_{1}\right)$ in (5.14) gives

$$
\begin{align*}
& \left(D_{0}^{2}+\omega^{2}\right) U_{1}-\gamma_{n} \lambda_{n}^{2} \varepsilon \int_{0}^{t_{0}} \alpha\left(\varepsilon\left(t_{0}-\tau\right)\right) U_{1}(\tau, \varepsilon \tau) \mathrm{d} \tau= \\
- & 2 D_{0} D_{1} U_{0}-1 / \varepsilon\left(\left(D_{0}^{2}+\omega^{2}\right) U_{0}-f\left(\varepsilon t_{0}\right)\right)+\gamma_{n} \lambda_{n}^{2} \int_{0}^{t_{0}} \alpha\left(\varepsilon\left(t_{0}-\tau\right)\right) U_{0}(\tau, \varepsilon \tau) \mathrm{d} \tau+O(\varepsilon) \tag{5.19}
\end{align*}
$$

Expanding the integral on the right hand side of equation (5.19) using integration by parts we find that the terms involving $\cos \left(\omega t_{0}\right)$ and $\sin \left(\omega t_{0}\right)$ will produce secular terms in the particular solution for $U_{1}$. Hence we have

$$
A_{0}\left(t_{1}\right)=A_{0} \exp \left(-\frac{\gamma_{n} \lambda_{n}^{2} \alpha(0)}{2 \omega^{2}} t_{1}\right), \quad B_{0}\left(t_{1}\right)=B_{0} \exp \left(-\frac{\gamma_{n} \lambda_{n}^{2} \alpha(0)}{2 \omega^{2}} t_{1}\right) .
$$

We solve for $U_{1}\left(t_{0}, t_{1}\right)$ and $F_{1}\left(t_{1}\right)$ in the usual manner to give

$$
\begin{equation*}
\bar{F}_{1}(p)=\frac{\gamma_{n} \lambda_{n}^{2} \bar{\alpha}(p)}{\omega^{2}\left(\omega^{2}-\gamma_{n} \lambda_{n}^{2} \bar{\alpha}(p)\right)}, \quad F_{1}(0)=\frac{\gamma_{n} \lambda_{n}^{2}}{\omega^{2}} \alpha(0) a_{1}, \tag{5.20}
\end{equation*}
$$

and

$$
U_{1}\left(t_{0}, t_{1}\right)=A_{1}\left(t_{1}\right) \cos \left(\omega t_{0}\right)+B_{1}\left(t_{1}\right) \sin \left(\omega t_{0}\right)+F_{1}\left(t_{1}\right)
$$

We now have the leading order terms and the first order correction term. This procedure can be continued to get the slow time behaviour of the first order correction term. Finally we use the initial conditions to determine $A_{0}, B_{0}$, etc.. Hence

$$
\begin{align*}
& U(t ; \varepsilon)=F_{0}\left(t_{1}\right)+\varepsilon F_{1}\left(t_{1}\right)+O\left(\varepsilon^{2}\right) \\
& +\left(A_{0}+\varepsilon A_{1}\right) \exp \left(-\frac{\gamma_{n} \lambda_{n}^{2} \alpha(0)}{2 \omega^{2}} t_{1}\right) \cos (\omega t)+\left(B_{0}+\varepsilon B_{1}\right) \exp \left(-\frac{\gamma_{n} \lambda_{n}^{2} \alpha(0)}{2 \omega^{2}} t_{1}\right) \sin (\omega t), \tag{5.21}
\end{align*}
$$

where $F_{0}\left(t_{1}\right)$ and $F_{1}\left(t_{1}\right)$ are given by (5.18) and (5.20).
Using the initial conditions at time $t_{0}=0$, we have

$$
A_{0}=a_{0}-F_{0}(0)=a_{0}-\varepsilon \frac{\gamma_{n} \lambda_{n}^{2}}{\omega^{2}} f(0), \quad A_{1}=F_{1}(0)=-a_{1} \frac{\gamma \lambda_{n}}{\omega^{2}} \alpha(0)
$$

and

$$
B_{0}=a_{1}, \quad B_{1}=\left(a_{0}-F_{0}(0)\right) \frac{\omega^{2}-\beta}{2 \omega^{3}}-\frac{\gamma_{n} \lambda_{n}^{2}}{\omega^{2}} \dot{f}(0)-\frac{\left(\gamma_{n} \lambda_{n}^{2}\right)^{2}}{\omega^{5}} \alpha(0) f(0)
$$

An approximate solution to the retarded problem is now given by (5.21). Setting $\varepsilon=1$ in (5.21) we get a candidate for an approximate solution for the dynamic equation. Comparing this result with that calculated in the previous section for the standard viscoelastic material we see that they happen to agree up to $O\left(\varepsilon^{2}\right)$.
material, while setting $\rho<1$ results in the effects of the slowly varying load being masked by the viscoelastic effects. Hence we set $\rho=1$. We look for solutions which satisfy

$$
U\left(t_{0}, t_{1}, t_{2} ; \varepsilon\right) \sim U_{0}\left(t_{0}, t_{1}, t_{2}\right)+\varepsilon U_{1}\left(t_{0}, t_{1}, t_{2}\right)+\ldots, \quad \text { as } \varepsilon \rightarrow 0
$$

and a similar asymptotic expansion for $C\left(t_{0}, t_{1}, t_{2} ; \varepsilon\right)$. In order for the oscillations to appear for as $\varepsilon \rightarrow 0$ it turns out that $f_{t}=k(\varepsilon t)$. Hence

$$
t_{0}=f(t ; \varepsilon)=\int_{0}^{t} k(\varepsilon \tau) \mathrm{d} \tau=\frac{1}{\varepsilon} \int_{0}^{t_{1}} k(\tau) \mathrm{d} \tau .
$$

The general solution to the $O(1)$ problem is

$$
\begin{aligned}
& U_{0}\left(t_{0}, t_{1}\right)=A_{0}\left(t_{1}\right) \cos t_{0}+B_{0}\left(t_{1}\right) \sin t_{0} \\
& C_{0}\left(t_{0}, t_{1}\right)=\mu / k A_{0}\left(t_{1}\right) \sin t_{0}-\mu / k B_{0}\left(t_{1}\right) \cos t_{0}+F_{0}\left(t_{1}\right) .
\end{aligned}
$$

Using the $O(1)$ solution we have

$$
\begin{aligned}
k^{2}\left(D_{0}^{2}+1\right) U_{1}= & \left(\alpha_{0} \mu / k A_{0}\left(t_{1}\right)+k^{\prime} A_{0}\left(t_{1}\right)+2 D_{1} A_{0}\left(t_{1}\right)\right) \sin t_{0} \\
& -\left(\alpha_{0} \mu / k B_{0}\left(l_{1}\right)+k^{\prime} B_{0}\left(t_{1}\right)+2 D_{1} B_{0}\left(t_{1}\right)\right) \cos t_{0}+\alpha_{0} F_{0}\left(l_{1}\right)
\end{aligned}
$$

Removing secular terms over the interval $0<t<O\left(\varepsilon^{-t}\right)$ implies

$$
A_{0}\left(t_{1}\right)=\Lambda_{0} \mathrm{e}^{-\alpha_{0} \mu / 2 K\left(t_{1}\right)} \sqrt{k(0) / k\left(t_{1}\right)}, \quad B_{0}\left(t_{1}\right)=B_{0} \mathrm{e}^{-\alpha_{0} \mu / 2 K\left(t_{1}\right)} \sqrt{k(0) / k\left(t_{1}\right)},
$$

where $K\left(t_{1}\right):=\int_{0}^{\ell_{3}} \frac{1}{k(\tau)^{2}} \mathrm{~d} \tau$.
I'herefore, the fast time dependence of $U_{1}$ is

$$
U_{1}\left(t_{0}, t_{1}\right)=A_{1}\left(t_{1}\right) \cos t_{0}+B_{1}\left(t_{1}\right) \sin t_{0}+\alpha_{0} / k^{2} F_{0} .
$$

Removing the secular terms in the equation for $C_{1}$ implies $F_{0}\left(t_{1}\right)=F_{0} \mathrm{e}^{\alpha_{0}\left(\mu K\left(t_{1}\right)-t_{1}\right)}$. This is the behaviour of the creep response, over the interval $0<t_{1}<1$, due to
the slowly varying load. The dependence of $C_{1}$ on the fast time is easily found by integration. Using the $O(1)$ and $O(\varepsilon)$ solutions we can expand the equation determining the fast time dependence of $U_{2}$ and remove secular terms over the interval $0<t<O\left(\varepsilon^{-1}\right)$. Hence we determine the slow time dependence of the $O(\varepsilon)$ coefficients as

$$
\begin{aligned}
A_{1}\left(t_{1}\right)= & \mathrm{e}^{-\alpha_{0} \mu / 2 K\left(t_{1}\right)} \sqrt{k(0) / k\left(t_{1}\right)}\left(A_{1}\right. \\
& \left.-\int_{0}^{t_{1}} \alpha_{1} \mu / k(\tau) A_{0}-\alpha_{0} \mu\left(\alpha_{0} \mu-2 \alpha_{0} k(\tau)^{2}+3 k^{\prime}(\tau) k(\tau)\right) /\left(2 k(\tau)^{4}\right) B_{0} \mathrm{~d} \tau\right) \\
B_{1}\left(t_{1}\right)= & \mathrm{e}^{-\alpha_{0} \mu / 2 K\left(t_{1}\right)} \sqrt{k(0) / k\left(t_{1}\right)}\left(B_{1}\right. \\
& \left.-\int_{0}^{t_{1}} \alpha_{1} \mu / k(\tau) B_{0}+\alpha_{0} \mu\left(\alpha_{0} \mu-2 \alpha_{0} k(\tau)^{2}+3 k^{\prime}(\tau) k(\tau)\right) /\left(2 k(\tau)^{4}\right) A_{0} \mathrm{~d} \tau\right)
\end{aligned}
$$

Note that $A_{1}\left(t_{1}\right)$ and $B_{1}\left(t_{1}\right)$ will, in general, contain terms which are secular over the interval $O(1)<t_{1}<O\left(\varepsilon^{-1}\right)$. While these terms can be removed if we use a second slow time scale, the algebra is not straightforward.

Finally, using the initial conditions we have $A_{0}=a_{0}, B_{0}=a_{1}, F_{0}=c_{0}+a_{1} \mu / k(0)$, $A_{1}=-\alpha_{0}\left(k(0) c_{0}+a_{1} \mu\right) / k(0)^{3}$ and $B_{1}=a_{0}\left(k^{\prime}(0) k(0)+\alpha_{0} \mu\right) /\left(2 k(0)^{2}\right)$.

Figures (5.3)-(5.5) shows the behaviour of the numerical solution and the multiple scale approximation for a number of slowly varying loads. In Figures (5.3) and (5.4) we examine the behaviour for a slowly varying periodic load. The term which determines the stability behaviour is given by the exponent of the creep term. If $\lim \sup _{t \rightarrow \infty} \mu K\left(t_{1}\right) / t_{1}<1$ then the multiple scales approximation is stable otherwise it is unstable if $\lim _{\inf } f_{t \rightarrow \infty} \mu K\left(t_{1}\right) / t_{1}>1$. In Figures (5.3) and (5.4) the load slowly oscillates around the long term critical value. In Figure (5.3) the solution is stable while in Figure (5.4) it is unstable. In Figure (5.5) the load changes from a stable value to an unstable value at time $t_{1}=10$.

| Parameter | n | $\lambda_{n}$ | $\gamma_{n}$ | $G_{\infty}$ | $\alpha_{0}$ | $k\left(t_{1}\right)$ | $\varepsilon$ | $a_{0}$ | $a_{1}$ | $c_{0}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Value | 1 | 9.8696 | 1.001 | 0.5 | 0.1 | $7.5-0.9 \cos \left(t_{1}\right)$ | 0.1 | 1.0 | 0.0 | 0.1 |



Plot of slowly varying load and multiple scales creep exponent.


Figure 5.3: Numerical solution with multiple scale bound on amplitude.

| Parameter | n | $\lambda_{n}$ | $\gamma_{n}$ | $G_{\infty}$ | $\alpha_{0}$ | $k\left(t_{1}\right)$ | $\varepsilon$ | $a_{0}$ | $a_{1}$ | $c_{0}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Value | 1 | 9.8696 | 1.001 | 0.5 | 1.0 | $7.2+\cos (\varepsilon t)$ | 0.1 | 1.0 | 0.0 | 0.1 |



Plot of slowly varying load and multiple scales creep exponent.


Figure 5.4: Numerical solution with multiple scale bound on amplitude.

| Parameter | n | $\lambda_{n t}$ | $\gamma_{n}$ | $G_{\infty}$ | $\alpha_{0}$ | $k\left(t_{1}\right)$ | $\varepsilon$ | $a_{0}$ | $a_{1}$ | $c_{0}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Value | 1 | 9.8696 | 1.001 | 0.5 | 1.0 | $7 .+.2 \tanh \left(t_{1}-10\right)$ | 0.1 | 1.0 | 0.0 | 0.1 |



Plot of slowly varying load and multiple scales creep exponent.


Figure 5.5: Numerical solution with multiple scale bound on amplitude.

### 5.6 Parametrically Excited Rod

In this section we examine the stability of a rod of standard viscoelastic material for time-dependent terminal loads of the form

$$
\begin{equation*}
\lambda(t)=P_{0}+P_{1} \cos \Omega t \tag{5.24}
\end{equation*}
$$

where the amplitude $P_{1}$ of the $\cos \Omega t$ term is small.
The dynamic equation for the amplitude of the $n^{\text {th }}$ Fourier mode is given by

$$
\begin{gather*}
\ddot{u}(t)+\lambda_{n} \gamma_{n}\left(\lambda_{n}-P_{0}-P_{1} \cos (\Omega t)\right) u(t)=a c(t),  \tag{5.25a}\\
\dot{c}(t)=\mu u(t)-a c(t), \tag{5.25b}
\end{gather*}
$$

where $c(t)$ is defined by (5.8) and $\mu=\lambda_{n}^{2} \gamma_{n}\left(1-G_{\circ}\right)$. The behaviour of a general linear viscoelastic rod with zero initial history was studied by Cederbaum \& Mond [8] by examining the retarded equation using an approach similar to that used in Section 5.4. We shall use a multiple scale expansion to determine approximations to the boundaries between the regions in a suitable parameter space for which all solutions to (5.25a)-(5.25b) are stable, and the regions in which there are unstable solutions. From Chapter 3 we know that, when $P_{1}=0$, the general solution of (5.25a)-(5.25b) is asymptotically stable if $P_{0}<\lambda_{n} G_{\infty}$. Here we examine the phenomenon of instability due to principal parametric resonance when the constant component of the load satisfies $P_{0}<\lambda_{n} G_{\infty}$.

In order to simplify our exposition we rescale the dynamic equation. We set $\Omega t=2 \tau$, and define $\tilde{u}(\tau):=u\left(\frac{2 \tau}{\Omega}\right)$, and $\tilde{c}(\tau):=\frac{2}{\Omega} c\left(\frac{2 \tau}{\Omega}\right)$. Hence equations (5.25a)(5.25b) can be expressed in the form

$$
\begin{equation*}
\ddot{\tilde{u}}+\delta \tilde{u}=\alpha \tilde{c}+2 \varepsilon \tilde{u} \cos (2 \tau), \quad \dot{\tilde{c}}=\tilde{\mu} \tilde{u}-\alpha \tilde{c}, \tag{5.26}
\end{equation*}
$$

where

$$
\alpha=\frac{2 a}{\Omega}, \quad \omega^{2}=\gamma_{n} \lambda_{n}\left(\lambda_{n}-P_{0}\right), \quad \tilde{\mu}=\frac{4 \mu}{\Omega^{2}}, \quad \delta=\frac{4 \omega^{2}}{\Omega^{2}}, \quad \varepsilon=\frac{2 \gamma_{n} \lambda_{n} P_{1}}{\Omega^{2}} .
$$



Figure 5.6: Stable ( $\Sigma_{s}$ ) and unstable $\left(\Sigma_{u}\right)$ regions in the parameter plane for the Mathieu equation.

Now, in order to clarify the presentation, we will drop the notation and replace $\tau$ by $t$ in equations (5.26).

If we take the limit as the relaxation time goes to infinity, $\left(\alpha \rightarrow 0^{+}\right)$, then equation (5.26) reduces to the Mathieu equation,

$$
\begin{equation*}
\ddot{u}+(\delta-2 \varepsilon \cos 2 t) u=0 . \tag{5.27}
\end{equation*}
$$

This is the dynamic equation for a parametrically excited elastic rod ${ }^{4}$. The transition curves in the $(\delta, \varepsilon)$ plane which divide stable and unstable regions are well known. Floquet theory considers systems of linear periodic ordinary differential equations (see, for example, Hale [21], Magnus \& Winkler [30]). Using basic Floquet theory it is easily shown, for example in $\S 5.5$ of [30], that the transition curves intersect $\varepsilon=0$ at the critical points $\delta=k^{2}$, for $k=0,1,2, \ldots$ Thus, at these critical points the natural frequency $\omega$, is a nonnegative integer multiple of half the forcing frequency $\Omega$. Figure 5.6 shows the behaviour of the transition curves for $|\varepsilon|$ small and $k \leq 2$.

We look for a uniformly valid approximation on the interval $0 \leq t \leq T / \varepsilon$, such that the first term in the asymptotic expansion shows the transition from stability to instability. Hence, we assume that $u(t)$ and $c(t)$ depend on two time scales $t_{0}=t$ and $t_{1}$ such that $u(t ; \varepsilon)=U\left(t_{0}, t_{1} ; \varepsilon\right)$ and $c(t ; \varepsilon)=C\left(t_{0}, t_{1} ; \varepsilon\right)$.

[^14]The actual choice for the slow time scale $t_{1}$ depends on the parameter region under consideration. In particular the slow time scale is chosen so that the instability of the solution due to resonance is determined only by the unboundedness with respect to $t_{1}$ and not $t_{0} . U\left(t_{0}, t_{1} ; \varepsilon\right)$ and $C\left(t_{0}, t_{1} ; \varepsilon\right)$ will now satisfy the system of partial differential equations

$$
\begin{gather*}
\left(D_{0}+\frac{\mathrm{d} t_{1}}{\mathrm{~d} t_{0}} D_{1}\right)^{2} U\left(t_{0}, t_{1} ; \varepsilon\right)+\delta U\left(t_{0}, t_{1} ; \varepsilon\right)=2 \varepsilon U\left(t_{0}, t_{1} ; \varepsilon\right) \cos 2 t_{0}+\alpha C\left(t_{0}, t_{1} ; \varepsilon\right)  \tag{5.28}\\
\left(D_{0}+\frac{\mathrm{d} t_{1}}{\mathrm{~d} t_{0}} D_{1}\right) U\left(t_{0}, t_{1} ; \varepsilon\right)=\mu U\left(t_{0}, t_{1} ; \varepsilon\right)-\alpha C\left(t_{0}, t_{1} ; \varepsilon\right) \tag{5.29}
\end{gather*}
$$

We are not concerned with the solution behaviour in a neighbourhood of the critical point corresponding to $k=0$, since it can only occur for loads $P_{0}>\lambda_{n} G_{\infty}$. Hence, in this region, the solution to the unperturbed problem is unstable due to the creep component.

In order to determine the stability boundary near the $\mathrm{k}^{\text {th }}$ critical point we expand $\delta$ in powers of $\varepsilon$ as

$$
\begin{equation*}
\delta=k^{2}+\varepsilon \delta_{1}+\varepsilon^{2} \delta_{2}+\ldots . \tag{5.30}
\end{equation*}
$$

If the relaxation time is small ( $\alpha$ large) then the motion is completely stabilised by the viscoelastic behaviour of the rod. Hence we set

$$
\begin{equation*}
\alpha=\varepsilon^{\rho}\left(\alpha_{0}+\varepsilon \alpha_{1}+\cdots\right), \quad \text { for some } \rho>0 \tag{5.31}
\end{equation*}
$$

The choice used for $\rho$ will be so that there is a balance between the viscoelastic effect and the resonance due to the parametric excitation. Also we assume that

$$
U\left(t_{0}, t_{1} ; \varepsilon\right) \sim U_{0}\left(t_{0}, t_{1}\right)+\varepsilon U_{1}\left(t_{0}, t_{1}\right)+\ldots, \quad \text { as } \varepsilon \rightarrow 0
$$

and a similar asymptotic expansion for $C\left(t_{0}, t_{1} ; \varepsilon\right)$.
We shall now discuss in detail the derivation of the leading behaviour in the neighbourhood of the critical points $k=1$ and $k=2$.

CASE $k=1$
We let $t_{1}=\varepsilon$. If we set $\rho>1$, we get the aforementioned behaviour of the Mathieu equation. On the other hand, if $\rho<1$ then the behaviour of the zero order term is determined by the viscoelastic behaviour of the solution. Hence we set $\rho=1$, and the $O(1)$ problem has a general solution of the form

$$
\begin{gathered}
U_{0}\left(t_{0}, t_{1}\right)=A_{0}\left(t_{1}\right) \cos t_{0}+B_{0}\left(t_{1}\right) \sin t_{0} \\
C_{0}\left(t_{0}, t_{1}\right)=F_{0}\left(t_{1}\right)+\mu A_{0}\left(t_{1}\right) \sin t_{0}-\mu B_{0}\left(t_{1}\right) \cos t_{0}
\end{gathered}
$$

We now consider the $O(\varepsilon)$ equations to determine the dependence of $A_{0}\left(t_{1}\right), B_{0}\left(t_{1}\right)$ and $F_{0}\left(t_{1}\right)$ on the slow time $t_{1}$. The equation determining $U_{1}\left(t_{0}, t_{1}\right)$ is given by

$$
\begin{aligned}
\left(D_{0}^{2}+1\right) U_{1}= & \alpha_{0} A_{0}\left(t_{1}\right) \cos 3 t_{0}+\alpha_{0} B_{0}\left(t_{1}\right) \sin 3 t_{0}+\alpha_{0} F_{0}\left(t_{1}\right) \\
& +2\left\{2 D_{1} A_{0}\left(t_{1}\right)+\alpha_{0} \mu A_{0}\left(t_{1}\right)-\left(\delta_{1}+1\right) B_{0}\left(t_{1}\right)\right\} \sin t_{0} \\
& -2\left\{2 D_{1} B_{0}\left(t_{1}\right)+\alpha_{0} \mu B_{0}\left(t_{1}\right)+\left(\delta_{1}-1\right) A_{0}\left(t_{1}\right)\right\} \cos t_{0}
\end{aligned}
$$

The first order solution $U_{1}$ will contain terms that are not uniformly valid over the time scale under consideration unless the coefficients of $\cos \left(t_{0}\right)$ and $\sin \left(t_{0}\right)$ vanish. Thus, we require that

$$
\frac{\mathrm{d}}{\mathrm{~d} t_{1}}\binom{A_{0}}{B_{0}}=\underbrace{\frac{1}{2}\left(\begin{array}{cc}
-\alpha_{0} \mu & 1+\delta_{1} \\
1-\delta_{1} & -\alpha_{0} \mu
\end{array}\right)}_{M}\binom{A_{0}}{B_{0}}
$$

We require that the eigenvalues $\eta_{j}, j=1,2$, of $M$ have nonpositive real parts in order that $A_{0}\left(t_{1}\right)$ and $B_{0}\left(t_{1}\right)$ are bounded. The characteristic exponents are given by

$$
\eta_{j}=\frac{-\alpha_{0} \mu \pm \sqrt{1-\delta_{1}^{2}}}{2}
$$

If $\alpha_{0} \mu>1$ then the motion is completely stabilised to the first order by the dissipative behaviour of the viscoelastic material. On the other hand, if $\alpha_{0} \mu<1$, then one of the characteristic exponents has a positive real part when $\left|\delta_{1}\right|<\sqrt{1-\alpha_{0}^{2} \mu^{2}}$.

This bound on $\delta_{1}$ determines the region of instability of the zero order solution. If $\left|\delta_{1}\right|>\sqrt{1-\alpha_{0}^{2} \mu^{2}}$ both exponents lie in the left-half complex plane so the zero order solution is bounded and has the form of damped modulated oscillations.

The transition curves separating the region where the zero order solution is stable from the regions where it is unstable are given by

$$
\begin{equation*}
\delta=1 \pm \sqrt{\varepsilon^{2}-\alpha^{2} \mu^{2}}+\cdots \tag{5.32}
\end{equation*}
$$

Therefore, the viscoelastic behaviour has a stabilising effect, in that, the instability region is narrowed and is raised from the $\delta$-axis. This gives the following critical amplitude, by which we mean the minimum amplitude at which the rod can become unstable due to resonance,

$$
\begin{equation*}
\varepsilon_{c}=\alpha \mu=a \sqrt{\frac{\gamma_{n} \lambda_{n}^{3}\left(G_{\infty}-1\right)^{2}}{\lambda_{n}-P_{0}}} \tag{5.33}
\end{equation*}
$$

For smaller values of $\alpha$ these effects become less pronounced. This behaviour can be seen in Figure 5.7 for various values of $\alpha$. The $(\delta, \varepsilon)$-parameter space is divided into two regions $\Sigma_{s}$ and $\Sigma_{u}$ in which the leading order term in the solution is stable or unstable respectively. The broken lines represent the corresponding transition curves for the Mathieu equation.

Figure 5.8 displays the solution of the leading order term for loads with stable and unstable amplitudes using the following data.

| Parameter | $n$ | $\lambda_{n}$ | $\gamma_{n}$ | $G_{\infty}$ | a | $P_{0}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Value | 1 | 9.8696 | 1.0001 | 0.1 | 0.02 | 0.0 |

Using (5.33) we find that the critical amplitude is given by $\varepsilon_{c}=0.177653$. In Figure 5.8(a) the amplitude of the parametric excitation is $\varepsilon=0.1$. Hence the zero order solution is stable since this amplitude is less than the critical value.

The amplitude used in Figure 5.8(b) and Figure $5.8(\mathrm{c})$ is $\varepsilon=0.4$, which is greater than the critical value of 0.177653 . However, only the zero order solution in Figure 5.8(b) is unbounded since the value of $\delta_{1}=0.95$ used in Figure 5.8(c) does not satisfy $\left|\delta_{1}\right|<\sqrt{1-\alpha_{0}^{2} \mu^{2}}$.

Transition Curve for $\alpha \mu=0.6$


Transition Curve for $\alpha \mu=0.3$



Figure 5.7: Transition curves in the $(\delta, \varepsilon)$-plane with $\alpha$ fixed. $(k=1,2)$

(b) Multiple scale solution with an instable amplitude ( $\delta_{1}=0$ ).

(c) Multiple scale solution with an instable amplitude ( $\delta_{1}=.95$ ).


Figure 5.8: Behaviour of the $O(1)$ multiple scale approximation under parametric excitation (near critical point $k=1$ ).

Case $k=2$
Similar considerations apply near the critical point $\delta=4$. However, in this case the slow time scale must be chosen by $t_{1}=\varepsilon^{2} t$. We also need to set $\rho=2$ in equation (5.31) to obtain the richest equation when $\varepsilon \rightarrow 0$.

The general solution to the $O(1)$ problem is

$$
\begin{gathered}
U_{0}\left(t_{0}, t_{1}\right)=A_{0}\left(t_{1}\right) \cos 2 t_{0}+B_{0}\left(t_{1}\right) \sin 2 t_{0}, \\
C_{0}\left(t_{0}, t_{1}\right)=F_{0}\left(t_{1}\right)+\mu / 2 A_{0}\left(t_{1}\right) \sin 2 t_{0}-\mu / 2 B_{0}\left(t_{1}\right) \cos 2 t_{0} .
\end{gathered}
$$

Using the $O(1)$ solution the equation determining $U_{1}\left(t_{0}, t_{1}\right)$ is given by $\left(D_{0}^{2}+4\right) U_{1}=A_{0}\left(t_{1}\right)\left(1+\cos 4 t_{0}\right)+B_{0}\left(t_{1}\right) \sin 4 t_{0}-\delta_{1} A_{0}\left(t_{1}\right) \cos 2 t_{0}-\delta_{1} B_{0}\left(t_{1}\right) \sin 2 t_{0}$.

Hence $\delta_{1}=0$, otherwise the $O(\varepsilon)$ solution would contain secular terms. Therefore,

$$
\begin{aligned}
U_{1}\left(t_{0}, t_{1}\right)= & A_{1}\left(t_{1}\right) \cos 2 t_{0}+B_{1}\left(t_{1}\right) \sin 2 t_{0} \\
& +\sin ^{2} t_{0}\left(2 A_{0}\left(t_{1}\right)+A_{0}\left(t_{1}\right) \cos 2 t_{0}+B_{0}\left(t_{1}\right) \sin 2 t_{0}\right) / 3
\end{aligned}
$$

We need to examine the $O\left(\varepsilon^{2}\right)$ equations in order to determine the slow time dependence of $A_{0}\left(t_{1}\right)$ and $B_{0}\left(t_{1}\right)$. Note also that we do not need to calculate the fast time dependence of $C_{1}\left(t_{0}, t_{1}\right)$. This is because it not used in calculating $U_{2}$ since it does not appear on the right hand side of (5.28) until the $O\left(\varepsilon^{3}\right)$ terms.

In order to remove the secular terms in the $O\left(\varepsilon^{2}\right)$ equation determining fast time dependence of $U_{2}$, we require that the coefficients of $\cos \left(2 t_{0}\right)$ and $\sin \left(2 t_{0}\right)$ must vanish. That is

$$
\frac{\mathrm{d}}{\mathrm{~d} t_{1}}\binom{A_{0}}{B_{0}}=\underbrace{\frac{1}{4}\left(\begin{array}{cc}
-\alpha_{0} \mu / 2 & \frac{1}{12}+\delta_{2} \\
\frac{5}{12}-\delta_{2} & -\alpha_{0} \mu / 2
\end{array}\right)}_{M}\binom{A_{0}}{B_{0}} .
$$

This system of equations is of the same form as that studied for the critical point
$k=1$. The eigenvalues $\eta_{j}, j=1,2$, of the matrix $M$ are given by

$$
\eta_{j}=\frac{-\alpha_{0} \mu \pm 2 \sqrt{\left(\frac{1}{12}+\delta_{2}\right)\left(\frac{5}{12}-\delta_{2}\right)}}{2}
$$

If $\alpha_{0} \mu<1 / 2$, then a region of instability exists since one of the characteristic exponents has a positive real part when

$$
\begin{equation*}
\left|\delta_{2}-1 / 6\right|<1 / 4 \sqrt{1-4 \alpha_{0}^{2} \mu^{2}} \tag{5.34}
\end{equation*}
$$

The transition curves near the critical point $\delta=4$ are given by

$$
\begin{equation*}
\delta=4+\varepsilon^{2} / 6+\varepsilon / 4 \sqrt{\varepsilon^{2}-4 \alpha^{2} \mu^{2}}+\cdots \tag{5.35}
\end{equation*}
$$

Therefore, there exists a non-zero critical amplitude given by $\varepsilon_{c}=2 \alpha_{0} \mu$, due to the stabilising effects of the viscoelastic damping. This behaviour can be seen in Figure 5.7 for various values of $\alpha$.

Case $k=3$
While the process is similar for all the critical points with $k \geq 1$, the amount of algebra required becomes lengthier as $k$ increases. Hence we shall simply state the leading behaviour for transitions curves and the critical amplitudes.

The slow time scale is $t_{1}=\varepsilon^{k}$ and $\rho=k$ in order to achieve a balance between the resonance and viscoelastic effects.

The critical amplitude is $\varepsilon_{c}=64 / 3 \alpha \mu$ with transition curve given by

$$
\delta=9+\varepsilon^{2} / 16 \pm \varepsilon^{2} \sqrt{\varepsilon^{2} 2^{-12}-\alpha^{2} \mu^{2} / 9}+\cdots
$$

Case $k=4$
The critical amplitude is $\varepsilon_{c}=1152 / 2 \alpha \mu$ with transition curve given by

$$
\delta=4+\varepsilon^{2} / 30+\varepsilon^{4} \frac{2.9}{432000} \pm \varepsilon^{3} \sqrt{\varepsilon^{2} / 1152^{2}-\alpha^{2} \mu^{2} / 4}+\cdots
$$

## Chapter 6

## Conclusions

The topic of this work is to study the bending of thin perfect viscoelastic rods subject to a longitudinal load, $\lambda(t)$, close to the straight equilibrium position.

For constant loads, it has been shown that the initial history problem possesses a unique, absolutely continuous solution. Furthermore, this solution is asymptotically stable for loads $\lambda<\lambda_{1} G_{\infty}$. When the constant load exceeds this critical value the solution can be decomposed into a sum of exponential terms and a function in $L^{1}$. Results like this had only been proved for the standard viscoelastic material. This part of the thesis fills a gap in the literature, and leaves no important unsolved problems behind.

The trivial solution was shown to be unique for the problem without initial history when $\lambda<\lambda_{1} G_{\infty}$. If $\lambda_{n} G_{\infty} \leq \lambda<\lambda_{n+1} G_{\infty}$ for some $n \geq 1$, then the solution will consist of a sum of $n$ exponentials whose exponents are the simple positive real roots of the characteristic equations associated with the first $n$ Fourier modes. This approach is important, because it suggests buckling solutions are attracted to these exponentially increasing solutions rather than the zero solution if the load exceeds the critical value $\lambda_{1} G_{\infty}$.

Semigroup techniques were used to show global existence and uniqueness for continuous time-dependent loads. This is in contrast to the corresponding results for the quasi-static problem, suggesting that the nonuniqueness that can arise on such problems is spurious, caused by the omission of the inertia terms in the equation of motion.

Using the energy of the rod a uniform stability result was obtained for a wide class of loads satisfying $\varepsilon \leq \lambda(t) \leq \lambda_{1} G_{\infty}-\varepsilon$. Numerical calculations suggest the existence of a stronger result. I conjecture the zero solution is uniformly asymptotically stable if the function $F(t)$, given by equation (4.11), is integrable.

We also examined the behaviour of the $n^{\text {th }}$ Fourier mode using multiple scale techniques. For the special case of the standard viscoelastic material with a large
relaxation time we were able to decompose the elastic and creep effects for constant loads satisfying $\lambda<\lambda_{n}$. This decomposition allowed us to examine the relationship between the growth of the creep term and the type of initial disturbance. The solution obtained was exceedingly accurate, and almost indistinguishable from the numerical solution.

For general, slowly varying relaxation functions we examined the retarded equation and using the method of Angel \& Olmstead [1, 2] we were able to decompose the creep and elastic effects as the retardation parameter tends to zero.

We considered a generalisation of the results described in Section 5.3 by constructing a multiple scale approximation for the $n^{\text {th }}$ Fourier mode under the assumption of a slowly varying load. In particular, we examine the behaviour as the load slowly crosses the equilibrium buckling load.

Finally, we considered the important special case of time dependent loading of $\lambda(t)=P_{0}+P_{1} \cos (\Omega t)$ with $P_{1}$ small. In particular, the case of the forcing frequency being close to the primary resonant frequency of twice the natural frequency was studied for standard viscoelastic materials. The zero order solution in a multiple scale expansion can be unstable, but in smaller regions of parameter space than for elastic rods. These regions of instability in fact vanish if the relaxation time is small, as the viscoelasticity completely damps the motion. The behaviour of the Fourier modes can be analysed for other types of time dependent loads. Important examples would include a slowly varying load $\lambda$ passing through the instantaneous critical load $\lambda_{n}$ or loads with non-stationary oscillations.

Finally, it is worth noting some other possible directions along which this work could develop. Namely, the effect of imperfections in the rod, nonuniform crosssection and slight extensibility could be considered. Also the geometrically exact nonlinear model should examined.

## Standard Viscoelastic Material

## A. 1 Introduction

While this work is primarily devoted to general linear viscoelastic rods, rods of standard viscoelastic material are used to corroborate the analytic results and to find approximations to the solution under special conditions. We shall now give a brief description of standard viscoelastic materials in terms of its rheological structure and its mechanical properties. We show that the differential and integral form of the stress-strain relationship can be derived from the rheological model. A more comprehensive description of rheological models, and in particular standard viscoelastic materials, can be found in Chapter 5 of [16] or Chapter 3 of [35].

## A. 2 Rheological Model

The theory of viscoelasticity is a natural generalisation of the theory of elasticity for solid materials and the mechanics of viscous fluids. Elastic properties exemplifed by linear springs and viscous properties exemplifed by dash-pots can be combined in various ways to construct rheological models of hypothetical media. The models correspond in behaviour, at least qualitatively, to real substances that are intermediate between solids and liquids.

An initial attempt at combining these properties would be to consider a material that behaves like a spring and a dash-pot connected in series (Maxwell material) or in parallel (Voigt material). In the linear spring the stress $\sigma$ and the strain $\epsilon$ are related by Hooke's law

$$
\sigma=E \epsilon,
$$

where $E$ represents the spring constant or Young's modulus. In a viscous liquid the
stress and the strain rate are related by Newton's law of viscosity, namely,

$$
\sigma=\eta \dot{\epsilon}
$$

where $\eta$ is the coefficient of viscosity. Although the Maxwell and Voigt models have been used to describe the properties of various solids they both display behaviour which is unrealistic - the Maxwell material does not possess a finite creep limit under constant stress and shows no time-dependent recovery while the Voigt material does not exhibit time-independent strain on loading or unloading.

The standard viscoelastic material is described using a three element model; a spring and a dash-pot connected in parallel and this is connected to a second spring in series (cf. Figure A.1). The strain of the spring in series $\epsilon_{1}$ on application of a


Figure A.1: Rheological model of standard viscoelastic material.
force $\sigma$ is

$$
\begin{equation*}
\epsilon_{1}=\frac{\sigma}{E_{1}}, \tag{A.1}
\end{equation*}
$$

where $E_{1}$ is the spring stiffness. The strain of the spring in parallel $\epsilon_{2}$ is given by the formula for the Voigt material, namely,

$$
\begin{equation*}
\sigma=E_{2} \epsilon_{2}+\eta \dot{\epsilon}_{2} \tag{A.2}
\end{equation*}
$$

where $E_{2}$ is the spring stiffness of the second spring. The total strain is given by $\epsilon=\epsilon_{1}+\epsilon_{2}$. Combining equations (A.1) and (A.2) and eliminating $\epsilon_{1}$ and $\epsilon_{2}$, we
obtain

$$
\begin{equation*}
\dot{\sigma}+a \sigma=G_{0} \dot{\epsilon}+a G_{\infty} \epsilon, \tag{A.3}
\end{equation*}
$$

where

$$
G_{0}=E_{1}, \quad a=\frac{E_{1}+E_{2}}{\eta}, \quad G_{\infty}=\frac{E_{1} E_{2}}{\eta} .
$$

Using (A.3) the standard viscoelastic material can be shown to possess the following properties (cf. Figure A.2) :

- Instantaneous elastic deformation and recovery. The material exhibits a timeindependent elastic deformation of size $\sigma_{0} / G_{0}$ on loading or unloading of a stress of size $\sigma_{0}$.
- Finite creep limit. Under constant stress $\sigma_{0}$ the material creeps, with a decreasing rate which is a characteristic of primary creep, to the finite limit $\sigma_{0} / G_{\infty}$.
- Instantaneous and time-dependent recovery. On removing a constant load $\sigma_{0}$ the material instantaneously recovers the elastic component of the deformation and displays an exponential rate of decay of the creep deformation.
- Exponential relaxation. If the material is subjected to a stress in order to maintain a constant strain, the material will relax, i.e., the stress required to maintain a constant strain will decrease over time. For the standard viscoelastic material the required stress decays like a negative exponential.

Equation (A.3) can be solved for the stress (or strain) using an integrating factor to give the integral representation of the stress-strain relationship

$$
\begin{equation*}
\sigma(t)=G_{0} \epsilon(t)+a\left(G_{\infty}-G_{0}\right) \int_{-\infty}^{t} \epsilon(\tau) \mathrm{e}^{-a(t-\tau)} \mathrm{d} \tau \tag{A.4}
\end{equation*}
$$

The relaxation function (2.15) is derived from (A.4) by solving for the stress when


Figure A.2: Creep and recovery behaviour of the standard viscoelastic material. the strain is given by the unit step function, i.e.,

$$
\begin{aligned}
G(t) & =G_{0}+a\left\{G_{\infty}-G_{0}\right) \int_{0}^{t} \mathrm{e}^{-a(t-\tau)} \mathrm{d} \tau \\
& =G_{0}+\left(G_{\infty}-G_{0}\right)\left(1-\mathrm{e}^{-a t}\right) \\
& =G_{\infty}+\left(G_{0}-G_{\infty}\right) \mathrm{e}^{-a t} .
\end{aligned}
$$

## Appendix $B$

## Numerical Results

## B. 1 Introduction

In this thesis we have considered 3 problems which have been solved numerically. Namely, the initial history problem, the Volterra integro-differential equation for the Fourier modes and the solution for the standard viscoelastic material. In this appendix we give an outline of the algorithms used.

In Chapter 2 it was shown that the linearisation of the nonlinear flexure about the straight equilibrium solution yields the following system

$$
\begin{align*}
y_{t} & =v,  \tag{B.1}\\
L_{\sigma} v_{t} & =-y_{s s s s}+\int_{0}^{t} \alpha(\tau) y_{s s s s}(t-\tau) \mathrm{d} \tau-\lambda(t) y_{s s}+f(t, s), \tag{B.2}
\end{align*}
$$

where the operator $L_{\sigma}$ is given by (3.1). The initial history up to time $t=0$ determines $f$ through

$$
f(s, t)=\int_{0}^{\infty} \alpha(t+\tau) y_{s s s s}(s,-\tau) \mathrm{d} \tau .
$$

The boundary conditions are

$$
y(s, t)=y_{s s}(s, t)=v(s, t)=v_{s s}(s, t)=0 \text { at } s=0,1 \text { for all } t .
$$

And initial conditions

$$
\begin{aligned}
y(s, 0) & =y^{0}(s), \\
y_{t}(s, 0) & =y^{1}(s) .
\end{aligned}
$$

In order to simplify notation, we will use $\dot{y}$ and $y^{\prime}$ to represent the derivatives with
respect to $t$ and $s$ respectively.

## B. 2 Solution using Finite Differencing

In order to solve the above problem we discretise over a uniform grid. We represent the functions $y(s, t)$ and $v(s, t)$ by their values at a discrete set of points given by

$$
\begin{array}{ll}
s_{j}=(j-1) h, & j=1, \ldots, m, \\
t_{k}=k \varepsilon, & k=0,1, \ldots
\end{array}
$$

where $h$ and $\varepsilon$ are the step sizes along the spatial and time axes respectively. We shall denote the furction $y(s, t)$ at these points using $y_{j, k}=y\left(s_{j}, t_{k}\right)=y((j-1) h, k \varepsilon)$, and similarly for functions $v$ and $f$. Also for functions independent of $s$ we have $\lambda_{k}=\lambda\left(t_{k}\right)$ and $\alpha_{k}=\alpha\left(t_{k}\right)$.

The time derivatives in equations (B.1) and (B.2) are approximated using

$$
\begin{equation*}
\left.\frac{\partial y}{\partial t}\right|_{j, k}=\dot{y}_{j, k}=\frac{y_{j, k+1}-y_{j, k}}{\varepsilon} \tag{B.3}
\end{equation*}
$$

and similarly for $v$

$$
\begin{equation*}
\left.\frac{\partial v}{\partial t}\right|_{j, k}=\dot{y}_{j, k}=\frac{v_{j, k+1}-v_{j, k}}{\varepsilon} . \tag{B.4}
\end{equation*}
$$

The spatial derivatives in $y$ are replaced by

$$
\begin{equation*}
\left.\frac{\partial^{2} y}{\partial s^{2}}\right|_{j, k}=u_{j, k}^{\prime \prime}=\frac{y_{j-1, k}-2 y_{j, k}+y_{j+1, k}}{h^{2}}, \tag{13.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\left.\frac{\partial^{4} y}{\partial s^{4}}\right|_{j, k}=u_{j, k}^{(i v)}=\frac{y_{j-2, k}-4 y_{j-1, k}+6 y_{j, k}-4 y_{j+1, k}+y_{j+2, k}}{h^{4}} \tag{B.6}
\end{equation*}
$$

In order to simplify the following discussion, we shall ignore the dependence on $s$ and use $y_{k}$ to represent $y\left(\cdot, t_{k}\right)$ and again similarly for the function $v$.

Equation (B.1) can now be approximated at the mesh points by

$$
\frac{y_{k+1}-y_{k}}{\varepsilon}=\beta v_{k}+(1-\beta) v_{k+1}, \text { for } k=1,2, \ldots
$$

where the parameter $\beta \in[0,1]$ is used to combine an explicit and an implicit scheme for stepping forward in time. A fully implicit scheme with $\beta=1$, tends to diminish small-scale features to their equilibrium form but is only first order accurate in time. On the other hand, a fully explicit scheme with $\beta=0$, is only conditionally stable. By taking $\beta=1 / 2$, we get a Crank-Nicolson iteration scheme which is unconditionally stable and second order accurate in time ${ }^{1}$.

The rod is divided into $m-1$ segments giving $m$ spatial points $y_{j, k}$ for all $k$. Using equation (B.5) as an approximation for the second derivative with respect to $s$ and the boundary conditions in $y$ we have

$$
\begin{align*}
y_{1, k} & =y_{m, k}=0, \\
y_{0, k} & =-y_{1, k}  \tag{B.7}\\
y_{m+1, k} & =-y_{m-1, k}, \text { for all } k .
\end{align*}
$$

Similarly for $v$ we have

$$
\begin{align*}
v_{1, k} & =v_{m, k}=0, \\
v_{0, k} & =-v_{1, k}  \tag{B.8}\\
v_{m+1, k} & =-v_{m-1, k}, \text { for all } k .
\end{align*}
$$

Equation (B.1) can now be represented by

$$
\begin{aligned}
y_{1, k+1} & =1, \\
y_{j, k+1}-\varepsilon(1-\beta) v_{j, k+1} & =y_{j, k}+\varepsilon \beta v_{j, k}, \quad j=2, \ldots, m-1, \\
y_{m, k+1} & =1,
\end{aligned}
$$

[^15]for $k=1,2, \ldots$, and $y_{j, 0}=y_{j}^{0}$ for $k=0$.
Turning to equation (B.2) we first look at the convolution term. This is approximated using a trapezoidal rule giving
$$
\int_{0}^{t_{k}} \alpha(\tau) y^{(i v)}(t-\tau) d \tau=\varepsilon / 2\left(\alpha_{0} y_{k}^{(i v)}+\alpha_{k} y_{0}^{(i v)}\right)+\varepsilon \sum_{k k=1}^{k-1} \alpha_{k k} y_{k-k k}^{(i v)}
$$

Equation (B.2) is now given by

$$
\begin{aligned}
& 1 / \varepsilon\left[\left(v_{k+1}-\sigma v_{k+1}^{\prime \prime}\right)-\left(v_{k}-\sigma v_{k}^{\prime \prime}\right)\right]= \\
& -(1-\beta)\left[y_{k+1}^{(i v)}+\lambda_{k+1} y_{k+1}^{\prime \prime}\right]-\beta\left[y_{k}^{(\mathrm{iv})}+\lambda_{k} y_{k}^{\prime \prime}\right]+(1-\beta) f_{k+1}+\beta f_{k} \\
& +\varepsilon / 2 \alpha_{0}\left[(1-\beta) y_{k+1}^{(i v)}+\beta y_{k}^{(i v)}\right]+\varepsilon / 2 y_{0}^{(\mathrm{iv})}\left[(1-\beta) \alpha_{k+1}+\beta \alpha_{k}\right] \\
& +\varepsilon \sum_{k k=1}^{k-1} \alpha_{k k}\left[(1-\beta) y_{k+1-k k}^{(i v)}+\beta y_{k-k k}^{(i v)}\right] .
\end{aligned}
$$

Bringing $u$ and $v$ terms involving $t_{k+1}$ together we have

$$
\begin{align*}
& 1 / \varepsilon\left(v_{k+1}-\sigma v_{k+1}^{\prime \prime}\right)+(1-\beta)\left[y_{k+1}^{(i v)}+\lambda_{k+1} y_{k+1}^{\prime \prime}\right]-\varepsilon / 2 \alpha_{0}(1-\beta) y_{k+1}^{(i v)}= \\
& 1 / \varepsilon\left(v_{k}-\sigma v_{k}^{\prime \prime}\right)-\beta\left[y_{k}^{(i v)}+\lambda_{k} y_{k}^{\prime \prime}\right]+(1-\beta) f_{k+1}+\beta f_{k} \\
& +\varepsilon / 2 \alpha_{0} \beta y_{k}^{(i v)}+\varepsilon / 2 y_{0}^{(i v)}\left[(1-\beta) \alpha_{k+1}+\beta \alpha_{k}\right] \\
& +\varepsilon \sum_{k k=1}^{k-1} \alpha_{k k}\left[(1-\beta) y_{k+1-k k}^{(i v)}+\beta y_{k-k k}^{(i v)}\right] . \tag{B.9}
\end{align*}
$$

Now taking into account the $s$ dependence, we expand the spatial derivatives using
approximations (B.5) and (B.6) and multiplying by factor $\varepsilon h^{4}$ to get

$$
\begin{align*}
&\left(h^{4}+\sigma h^{2}\right) v_{j, k+1}-\sigma h^{2}\left[v_{j-1, k+1}+v_{j+1, k+1}\right] \\
&+\varepsilon(1-\beta)\left(1-\varepsilon / 2 \alpha_{0}\right)\left[y_{j-2, k+1}-4 y_{j-1, k+1}+6 y_{j, k+1}-4 y_{j+1, k+1}+y_{j+2, k+1}\right] \\
&+\varepsilon(1-\beta) \lambda_{k+1} h^{2}\left[y_{j-1, k+1}-2 y_{j, k+1}+y_{j+1, k+1}\right] \\
&=\left(h^{4}+\sigma h^{2}\right) v_{j, k}-\sigma h^{2}\left[v_{j-1, k}+v_{j+1, k}\right]-\varepsilon \beta \lambda_{k} h^{2}\left[y_{j-1, k}-2 y_{j, k}+y_{j+1, k}\right] \\
&+\varepsilon h^{4}(1-\beta) f_{k+1}+\varepsilon h^{4} \beta f_{k} \\
&+\varepsilon \beta\left(\varepsilon / 2 \alpha_{0}-1\right)\left[y_{j-2, k}-4 y_{j-1, k}+6 y_{j, k}-4 y_{j+1, k}+y_{j+2, k}\right] \\
&+\varepsilon^{2} / 2\left((1-\beta) \alpha_{k+1}+\beta \alpha_{k}\right)\left[y_{j-2,0}-4 y_{j-1,0}+6 y_{j, 0}-4 y_{j+1,0}+y_{j+2,0}\right] \\
&+\varepsilon^{2} \sum_{k k=1}^{k-1} \alpha_{k k}\{(1-\beta) \times \\
& {\left[y_{j-2, k+1-k k}-4 y_{j-1, k+1-k k}+6 y_{j, k+1-k k}-4 y_{j+1, k+1-k k}+y_{j+2, k+1-k k}\right] } \\
&\left.+\beta\left[y_{j-2, k-k k}-4 y_{j-1, k-k k}+6 y_{j, k-k k}-4 y_{j+1, k-k k}+y_{j+2, k-k k}\right]\right\} \tag{B.10}
\end{align*}
$$

for $k=1,2, \ldots$, and for $k=0$, we have $y_{j, 0}=y_{j}^{0}$, and $v_{j, 0}=y_{j}^{1}$. Equation (B.10) is only valid for $4 \leq j \leq m-3$. For $j$ outside this range we use the boundary conditions given in equations (B.7) and (B.8). Hence for the fourth derivative of $y_{j, k}$ with respect to $s$ we would use instead

$$
\begin{array}{cl}
y_{2, k}^{(i v)}=\left(5 y_{2, k}-4 y_{3, k}+y_{4, k}\right) / h^{4}, & \text { for } j=2, \\
y_{3, k}^{(i v)}=\left(-4 y_{2, k}+6 y_{3, k}-4 y_{4, k}+y_{5, k}\right) / h^{4}, & \text { for } j=3, \\
y_{m-2, k}^{(i v)}=\left(y_{m-4, k}-4 y_{m-3, k}+6 y_{m-2, k}-4 y_{m-1, k}\right) / h^{4}, & \text { for } j=m-2, \\
y_{m-1, k}^{(i v)}=\left(y_{m-3, k}-4 y_{m-2, k}+5 y_{m-1, k}\right) / h^{4}, & \text { for } j=m-1,
\end{array}
$$

and similarly for the second derivatives.
The finite difference approximation to the linearised problem is now easily solved. At each time point $t_{k}$ the solution at the next time point $t_{k+1}$ is given by the solution
of the following $(4 m-4) \times(4 m-4)$ linear system

$$
\begin{equation*}
A_{k+1} \mathbf{u}_{k+1}=\mathcal{F}\left(\mathbf{u}_{k}, t_{k+1}\right) \tag{B.11}
\end{equation*}
$$

The function $\mathcal{F}\left(\mathrm{u}_{k}, t_{k+1}\right)$ depends on u at all time points up to time $t_{k}$. It is given by the right hand side of equation (B.10). The first $m-2$ entries of the vector $\mathrm{u}_{k}$ contain the values of $y_{j, k}$ at the interior points $y_{j, k}$ for $j=2, \ldots, m-1$ and the second $m-2$ entries contain the values of $v_{j, k}$.

The matrix $A_{k+1}$ is given by

where

$$
\begin{aligned}
& c_{1}=-(1-\beta) \varepsilon, \\
& c_{2}=\varepsilon(1-\beta)\left(1-\varepsilon / 2 \alpha_{0}\right), \\
& c_{3}=\varepsilon(1-\beta) \lambda_{k+1} h^{2}, \\
& c_{4}=h^{4}+2 \sigma h^{2}, \\
& c_{5}=-\sigma h^{2} .
\end{aligned}
$$

For constant loads and equal step sizes the matrix $A_{k+1}$ will be independent of $k$ and hence a significant time improvement can be achieved by generating a LU decomposition of $A$ and using this in solving the linear equation (B.11). Finally we address the problem of numerically approximating the initial history term $f_{j, k}$. The improper integral

$$
f_{j, k}=\int_{0}^{\infty} \alpha\left(t_{k}+\tau\right) y^{(i \nu)}\left(s_{j},-\tau\right) \mathrm{d} \tau
$$

is first converted using change of variable $\omega=\mathrm{e}^{-\tau}$, or $\tau=-\log \omega$, giving

$$
f_{j, k}=\int_{0}^{1} \frac{\alpha\left(t_{k}-\log \omega\right) y^{(i v)}\left(s_{j},+\log \omega\right)}{\omega} \mathrm{d} \omega .
$$

This can now be solved using an open type quadrature rule. The extended midpoint rule was used with the refinement level decreasing as time increases since the influence of the initial history term will diminish for larger time values.

## B. 3 Numerical Solution of Fourier Modes

The linear problem (B.1) and (B.2) has been solved using the Fourier series expansion for $y$ and $v$

$$
y(s, t)=\sum_{n=1}^{\infty} y_{n}(t) \sin n \pi s
$$

etc. The functions $f_{n}$ are then given by

$$
f_{n}(t)=\int_{0}^{\infty} \alpha(t+\tau) y_{n}(-\tau) \mathrm{d} \tau
$$

for $n=1,2, \ldots$ The problem then reduces to the systems

$$
\begin{align*}
& \dot{y}_{n}=v_{n}(t),  \tag{B.12}\\
& \dot{v}_{n}=-\left(\gamma_{n} \lambda_{n}^{2}-\gamma_{n} \lambda_{n} \lambda(t)\right) y_{n}+\gamma_{n} \lambda_{n}^{2} \int_{0}^{t} \alpha(\tau) y_{n}(t-\tau) \mathrm{d} \tau+\gamma_{n} \lambda_{n}^{2} f_{n}, \tag{B.13}
\end{align*}
$$

for $n=1,2, \ldots$. The procedure used to numerically solve the integro-differential system (B.12) and (B.13) involves reducing it to a system of Volterra equations ${ }^{2}$. Define function $z_{n}(t)$ by

$$
\begin{equation*}
z_{n}=\gamma_{n} \lambda_{n}^{2} \int_{0}^{t} \alpha(\tau) y_{n}(t-\tau) d \tau+\gamma_{n} \lambda_{n}^{2} f_{n} \tag{B.14}
\end{equation*}
$$

Equation (B.13) now implies that

$$
\dot{v}_{n}=-\gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda(t)\right) y_{n}+z_{n} .
$$

Integrating over t and using initial condition $v_{n}(0)=a_{n 1}$ gives

$$
v_{n}=\int_{0}^{t}-\gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda(\tau)\right) y_{n}(\tau)+z_{n}(\tau) \mathrm{d} \tau+a_{n 1}
$$

Substituting this result into equation (13.12), integrating over $t$ and using initial condition $y_{n}(0)=a_{n 0}$ and $v_{n}(0)=a_{n 1}$ we get, after changing order of integration,

$$
\begin{align*}
& y_{n}=-\gamma_{n} \lambda_{n} \int_{0}^{t}(t-\tau)\left(\lambda_{n}-\lambda(\tau)\right) y_{n}(\tau) \mathrm{d} \tau \\
& +\int_{0}^{t}(t-\tau) z_{n}(\tau) \mathrm{d} \tau+t a_{n 1}+a_{n 0} . \tag{B.15}
\end{align*}
$$

We can now solve for function $y_{n}$ and $z_{n}$ using equations (B.15) and (B.14) using a standard Volterra system solver. The problem of approximating the initial history term in the forcing function for equation (B.14) was solved using a scheme similar to that used in the previous section.

[^16]
## B.3.1 Solution using Fourier Series and Laplace Transform

After reducing the linear problem (B.1) and (B.2) using a. l'ourier series expansion as described in the previous section, the resulting integro-differential Volterra system can be analytically solved using Laplace transforms under the assumption of a constant load $\lambda$. Except for a small class of kernel functions $\alpha(t)$ and initial history configurations, getting a closed form for the inverse Laplace transform is non-trivial. 'Taking the kernel related to relaxation function of the standard viscoelastic material, we get the following equation for the Japlace transform of $y_{n}$

$$
\begin{equation*}
\bar{y}_{n}(p)=\frac{a_{n 0} p^{2}+\left(a_{n 1}+a_{n 0} \alpha\right) p+a a_{n 1}}{p^{3}+a / p^{2}+\gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda\right) p+a \gamma_{n} \lambda_{n}\left(\lambda_{n} G_{\infty}-\lambda\right)} . \tag{B.16}
\end{equation*}
$$

In order to use the inverse Laplace transform we must first determine the position of the roots in the complex plane of the characteristic equation

$$
f(p)=p^{3}+a p^{2}+\gamma_{n} \lambda_{n}\left(\lambda_{n}-\lambda\right) p-a \gamma_{n} \lambda_{n}\left(\lambda_{n} G_{\infty}-\lambda\right) .
$$

'The roots can casily be found numerically ${ }^{3}$ and the solution $y_{n}(t)$ can then be readily calculated.

[^17]
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[^0]:    ${ }^{1}$ Note the dimensions of the following physical variables and constants: $[\rho A]=M L^{-1},[\Lambda]=$ $M L T^{-2},[\mathrm{~N}]=M L T^{-2},[\rho I]=M L,[\beta]=M L^{3} T^{-2},[\alpha]=M L^{3} T^{-3}$.

[^1]:    ${ }^{2}$ Other boundary conditions are treated similarly.
    ${ }^{3}$ This paper does not require the density and moment of inertia to be constant, but does assume that $\lambda(t)$ is independent of time.

[^2]:    ${ }^{4}$ The free energy is not unigue within a multiplicative constant.

[^3]:    ${ }^{1}$ The convolution $\mu_{n} * \mathbf{u}_{n}$ of a measure $\mu_{n} \in \mathrm{M}\left(\mathbb{R}^{+}\right)$, where $\mathrm{M}\left(\mathbb{R}^{+}\right)$is the space of finite measures on $\mathbb{R}^{+}$, and a Lebesgue measurable function $\mathbf{u}_{n}$ defined on $\mathbb{R}^{+}$, is the function

    $$
    \begin{equation*}
    \left(\mu_{n} * \mathbf{u}_{n}\right)(t)=\int_{0}^{t} \mathrm{~d} \mu_{n}(\tau) \mathbf{u}_{n}(t-\tau) \tag{3.9}
    \end{equation*}
    $$

    defined for those $t$ for which the function $\tau \rightarrow \mathbf{u}_{n}(t-\tau)$ is $\left|\mu_{n}\right|$-integrable.
    ${ }^{2}$ The notation is described in page iii.

[^4]:    ${ }^{3}$ Cf. Theorem 3.1 of Ch. 3 of [18].
    ${ }^{4} \mathrm{Cf}$. Theorem 3.3 of Ch. 3 of [18].

[^5]:    ${ }^{5}$ The Laplace transform $\bar{\mu}_{n}(p)$ of a measure $\mu_{n} \in \mathrm{M}\left(\mathbb{R}^{+}\right)$is the function

    $$
    \bar{\mu}_{n}(p)=\int_{0}^{\infty} e^{-p t} \mathrm{~d} \mu_{n}(t)
    $$

    defined for those $p \in \mathbb{C}$ for which the integral converges absolutely.
    ${ }^{6} \mathrm{Cf}$. Theorem 3.5 of Ch. 3 of [18].

[^6]:    ${ }^{7}$ Cf. [27] page 245.

[^7]:    ${ }^{8} \mathrm{Cf}$. Theorem 2.1 of Ch. 7 of [18].

[^8]:    ${ }^{9} \mathrm{Cf}$. Theorem 3.7 of Ch. 7 of [18].

[^9]:    ${ }^{3}$ Cf. §1.4 of [33].
    ${ }^{4}$ Cf. $\S 4.2$ of [33].

[^10]:    ${ }^{5}$ Cf. $\S 10.13$ of [21] or Chapter 5 of [41].
    ${ }^{6} \mathrm{~A}$ function $W: \mathbb{R} \rightarrow \mathbb{R}$ is positive definite on $\mathbb{R}$ if it is continuous, $W(0)=0$, and $W(t)>0$, for $t \neq 0$.

[^11]:    ${ }^{1}$ The term critical time is used in various senses in the literature on creep buckling. The definition used here is not standard.

[^12]:    ${ }^{2}$ Cf. [24], Sec. 3.5.

[^13]:    ${ }^{3}$ Note that $\omega$ represents the natural frequency for the corresponding elastic rod and is always positive for the range of loads examined in this section, i.e., $\lambda<\lambda_{n}$.

[^14]:    ${ }^{4}$ A brief history of parametric resonance in an elastic rod due to an axial time-varying load is contained in [32], Sec 5.1.4.

[^15]:    ${ }^{1}$ Cf. [34] sections 19.0-19.2

[^16]:    ${ }^{2}$ Cr. Link [28] section 11.5, [6]

[^17]:    ${ }^{3}$ Cf. [34] section 5.6

