# The Work of G.G. Stokes <br> in Evaluating the Airy Rainbow <br> Integral and its Ramifications Today 

Anne B. O'Donnell

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The purpose of this thesis is to examine the mathematical work of George Gabriel Stokes arising out of his evaluation of Airy's integral for large values of the variable. The development of the physical understanding of the phenomenon of the rainbow up to the publication of G.B.Airy's work in 1838 is traced. The derivation of the Airy rainbow integral is outlined.

Stokes' work in developing an asymptotic solution, valid at infinity, for the related differential equation is traced using the terminology of modern mathematics. His use of the concept of dominant balance is reported along with his method of finding the asymptotic solution in descending powers, which anticipates the modern WKBJ method. It is shown that the methods Stokes uses to evaluate the original integral for large values of the variable involve the essential elements of the modern saddle point and stationary phase methods.

The origin of "Stokes' phenomenon" is examined in detail with the development of the concepts of domain dependence, dominance/subdominance, discontinuity of arbitrary constants and "Stokes' multipliers". Reference is made to modern developments in the understanding of Stokes phenomenon.

Stokes' pioneering work in the use of divergent series is set out, with examples of his results obtained by truncating the series in the vicinity of the least term and of his estimation of the remainder by resumming the tail of the series. The earliest attempt at determining the asymptotic behaviour of a series of hypergeometric type, which was carried out by Stokes in his 1889 paper is outlined. His method is used to obtain the leading behaviour of the asymptotic solution of the differential equation associated with Airy's integral.

## The Work of G.G. Stokes

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Anne B. O'Donnell

B.A., Dip.Maths.Phys., H.Dip.Ed.

Submitted in fulfilment of the requirements for the degree of Master of Science in Mathematical Sciences.

Dublin City University<br>Supervisor: Prof. A.D.Wood<br>School of Mathematical Sciences.

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of M.Sc. is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

Signed:


ID No.: $927011 / 6$

Date:


Dedicated to the memory of my parents, Kathleen Mc Glinchey and Edward O' Donnell of Clogher, Barnesmore, Co. Donegal.

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## INTRODUCTION

Repeatedly in the history of physical science and of mathematics we find examples of the interdependence of the two fields. Advances in one field give rise to corresponding advances in the other. These advances in turn return to illuminate wider problems in the original field.

One striking example of this is the mathematical discoveries made by George Gabriel Stokes in grappling with the problems presented by developments in physics.

In 1838 Airy [1] employed the newly accepted wave theory of light to explain the well-known natural phenomenon of the rainbow. However, when he attempted to find theoretically the position of the third and later dark bands in a system of supernumerary bows he encountered insuperable mathematical obstacles. Miller [19] had observed 30 such dark bands in a laboratory experiment.

This presented Stokes with a mathematical challenge. Employing his own intuitive methods, he did indeed find the position of 50 such dark bands. But, more importantly, the mathematical understanding he developed of the behaviour in the complex plane of certain functions which are solutions of a broad category of second order differential equations, has been of lasting value to mathematicians.

These functions, of which the Airy function is but one example, have a convergent series solution, about the origin, in ascending powers of $z$, valid for all finite $z$. However, this solution is useless for large values of the variable because of the slow rate of convergence. Stokes developed an asymptotic solution in descending powers of $z$, by means of which the function could be evaluated for large values of the variable.

This representation involves two complex exponential terms each multiplied by an arbitrary constant. It begins by converging rapidly but is ultimately divergent. Stokes realised that the validity of this asymptotic representation is domain dependent, with one exponential being dominant, the other recessive in different sectors of the complex plane.

He developed the concept of what are now called "anti-Stokes lines". These are rays in the complex plane where dominance/recession change over. More significantly, he realised that for the asymptotic solution to be valid in all sectors of the plane the arbitrary constants must be
discontinuous. He defined the positions where the change in the constants can take place. These are along rays now known as "Stokes lines ". He further developed what have become known as "Stokes multipliers" to show precisely how the constants change. In summary, he discovered what is now known as Stokes' phenomenon.

The solutions valid for large values of the variable involves divergent series. This led Stokes to pioneer the use of such series, which Abel in 1828 had described as "the invention of the devil", in solving physical problems. He addressed the problem of managing divergent series by appropriate truncation near the least term and by resumming the divergent tail.

Stokes' work was largely ignored for about 100 years. Mathematicians followed a different path initiated by Poincaré in the 1880 's. But in the 1950's Dingle returned to Stokes' approach and made a major breakthrough in the treatment of divergent series. Although little appreciated at the time, the work of Dingle [12] contains the basic ideas later exploited by Berry[5] and others in the development of modern asymptotics.

The functions, of which the Airy function is an example, whose behaviour in the complex plane was originally elucidated by Stokes, play a very important role in understanding contemporary physical problems. They involve complex exponentials. Stokes showed that as we proceed across the caustic, $z=0$, in the complex plane, a single real exponential function on the dark side of the primary rainbow, suddenly becomes two complex exponential functions on the bright side. According to Berry and Howls [9], the sudden appearance of exponential terms is common in physical applications. It can be used, for example, to describe the escape of a particle from the nucleus in the course of radioactive decay.

In a paper published in 1989 Berry [5] showed that the switching on of a second exponential across the Stokes' line is not sudden but smooth. This has led to further work by mathematical analysts such as: Berry, Howls, Olver, Olde Daalhuis, Boyd, Paris, Wood, and others. This mathematical work has found applications in many diverse areas in the physical sciences from surface tension to quantum mechanics [9]. It has also become important in the very modern field of optical communications [37]. Above all it has found application in more accurately evaluating the Riemann zeta function which is related to deep problems in mathematics through its connection with prime numbers [9].

This thesis traces the work of George Gabriel Stokes arising out of his attempts to evaluate the Airy rainbow integral. A number of key mathematical papers published by Stokes, in which the theory surrounding "Stokes' phenomenon" is developed, are discussed in some detail. The connections between Stokes mathematical methods and the corresponding modern methods are outlined. Where appropriate, the mathematics computer package, Maple, has been used to illustrate the validity of Stokes approximations and intuitive assumptions by numerical examples and graphics.

The thesis begins by discussing the evolution of physical theories of the rainbow (and therefore of light) and the establishment of the Airy rainbow integral based on a full wave theory. The thesis concludes with a summary of modern developments in asymptotics.

## CHAPTER 1

## THE PHYSICS OF THE RAINBOW

### 1.1 Introduction

Much of the discussion in this thesis is concerned with Stokes' work in evaluating the Airy Rainbow Integral and the mathematical developments arising from it. As we shall see, Airy formed his integral in the process of evaluating the intensity of light (which determines the brightness) for a single wavelength (which determines colour) at varying positions across the width of the bow.

Airy based his theory on a wave theory of light which was not at the time fully accepted. Indeed, the success of Airy and Stokes together with the experimental verification carried out by Miller [19], in explaining features of the rainbow which were not understood at the time, dealt an important blow to the old "theory of emissions" or corpuscular theory of light. This theory had been espoused by Newton and according to Boyer [10] was still being vigorously defended by his followers, such as Potter, who was a contemporary of Airy at Cambridge.

### 1.2 The rainbow observed

The development of physical theories of the rainbow is discussed by Boyer [10] and by Nussenveig [22]. As can be seen from Figure 1.1, the rainbow phenomenon as observed by the eye or the camera lens at a fixed position on the earth's surface consists of a number of distinct features. There is a bright primary bow at the smaller angle of observation (lower in the sky). At a greater angle of observation (higher in the sky), there is a secondary bow which is less bright than the primary bow and less commonly observed. The colours change across the width of the bow from red to violet. The red is on the outside (higher side) of the primary bow but on the inside ( lower side ) of the secondary bow.

The space in the sky between the primary and the secondary bow is noticeably darker than other parts of the sky and is known as Alexander's dark band, after Alexander of Aphrodisias, the Greek philosopher who discovered it in 200 A.D.


Figure 1.1: The rainbow (after Nussenveig, [22]). The primary bow, the secondary bow and supernumerary arcs of the primary bow are shown.

Below the primary bow in Figure 1.1, there are a few stripes of pink and green. These are variously described, in the literature, as " spurious rainbows" or "supernumerary arcs". In laboratory experiments, such as those carried out by Miller, many supernumerary arcs of both the primary and secondary bow have been observed. In his 1850 paper to be discussed in Chapter 2, Stokes [38] calculated the position of approximately 50 such supernumeraries of the primary bow.

### 1.3 Newton and Descartes

Attempts to explain the various features of the rainbow have played a major role in the evolution of the theory of light and through the work of Airy and Stokes, an important role in the development of mathematics.

Aristotle attempted to give an explanation of the rainbow on the basis of the reflection of sunlight from clouds. But it was Descartes in the 17th. century who explained why there was a particular brightness at the primary and secondary rainbow angles of observation which Roger Bacon had measured in 1266 at approximately $42^{\circ}$ and $50^{\circ}$ respectively. Descartes accepted and confirmed the work of, Theodoric of Freiburg, a German monk, who showed three centuries earlier that rainbows are produced by reflection of light within drops of water, and that the primary bow corresponds to a single reflection and the secondary bow to two reflections. This explains the lower intensity (brightness) of the secondary bow. But it was Descartes who showed the bunching of rays at the primary and secondary rainbow angles by laboriously drawing ray paths. This process for the case of the primary bow is illustrated in Figure 1.2.

The following simple treatment, which is consistent with the concept of light as the rectilinear propagation of corpuscles, gives the essence of Descartes position. It relies on the well established geometric theory of refraction, which had been put on a quantitative basis by the establishment of Snell's Law in 1621.

In Figure 1.3 the deviation of a ray by a spherical drop is shown for 1 and 2 reflections inside the drop. The angle of deviation, $\delta$, is the angle through which the incoming ray has been rotated when it emerges from the drop. The angle of incidence of the ray at the external drop surface is $i$ and its angle of refraction is $r$.


Figure 1.2: Deviation of rays for a single reflection, primary rainbow ray in red The caustics and the Huygens' wavefront after reflection are shown. (after Nussenveig, [22]).


Figure 1.3: The deviation of a ray by spherical drop.

For $n$ reflections inside the drop the total deviation $\delta$ is given by

$$
\begin{equation*}
\delta=2(i-r)+n\left(180^{\circ}-2 r\right) . \tag{1.1}
\end{equation*}
$$

The rainbow angles are those for which the deviation is a minimum as many rays will have nearly the same deviation at these angles. See Figure 1.2 above.

For a minimum $\frac{d \delta}{d i}=0$. Differentiating (1.1) we have for $n$ reflections inside the drop

$$
\begin{equation*}
\cos i=\sqrt{\frac{\mu^{2}-1}{n(n+2)}} . \tag{1.2}
\end{equation*}
$$

where $\mu=\frac{\sin i}{\sin r}$ is the refractive index from air to water.
Hence for the primary bow at minimum deviation we get

$$
\begin{equation*}
\cos i=\sqrt{\frac{\mu^{2}-1}{3}} . \tag{1.3}
\end{equation*}
$$

The refractive index from air to water was known to be approximately 1.33. Substituting the values of $i$ which give minimum deviation into equation (1.1) gives the approximate rainbow angles.

The use of the newly discovered differential calculus is due to Newton and his contemporaries. It dispensed with the necessity for laborious ray drawing and provided evidence of the usefulness of the new calculus which was still contested at this time.

In his famous prism experiments of 1666, Newton demonstrated not only that white light is a mixture of colours but that the refractive index of light is different for each colour, giving rise to a different degree of bending of each colour on refraction. The white light is dispersed by the medium in which it travels.

If the different measured values of the refractive index of red and violet light are substituted into the above equations, the separate rainbow angles for red and violet light are explained. The angles of observation in Figure (1.4) are given by the respective differences between the angle of minimum deviation and $180^{\circ}$.


Figure 1.4: Geometry of the rainbow from the position of the observer (after Nussenveig,[22]).

According to the theories of Newton and Descartes, the intensity of the light (which determines brightness) is greatest where the rays are bunched or clustered most.

Hence, to a good approximation, the intensity is inversely proportional to $\frac{d \delta}{d i}$. When $\frac{d \delta}{d i}=0$ at rainbow angles, the intensity is infinite. Since $\frac{d \delta}{d i}$ increases as $\delta$ increases, the intensity declines steadily across the width of the bow.

This is illustrated in Fig. IV on the plate which Airy subjoined to his paper [1] and which is shown here as Figure (1.5).

The work of Descartes and Newton had successfully explained the maxima of intensity at the rainbow angles and the differing rainbow angles for the different colours. But, based on this work, there would be no intensity at all in Alexander's dark band due to parallel rays of light internally reflected from drops, as the rainbow angles correspond to a minimum deviation.

The most obvious failure of the Newton\Descartes theory is its inability to provide an explanation for the supernumerary arcs. Laboratory experiments such as those carried out by Miller [19], very definitely showed their existence. Also, an examination of equation (1.2) shows that the rainbow angles of Newtonian theory have no dependence whatever on drop size.

## 1.4 "The imperfect theory of interferences"

In 1803, Thomas Young succeeded in carrying out an experiment in which two beams of light of a single colour (monochromatic) interfered constructively and destructively at different points in space giving rise to bright and dark fringes. The phenomenon of interference was already well known from the study of water waves. Young's experiment represented a major breakthrough for the wave theory of light and a heavy blow to corpuscular theory. Young himself suggested that the supernumerary arcs might be accounted for by constructive and destructive interference of light in the same way as his interference fringes were explained.


Figure 1.5: Plate which Airy subjoined to his paper "On the intensity of light in the neighbourhood of a caustic" [1].

At each angle to the primary rainbow ray, two rays emerge since the primary rainbow ray corresponds to minimum deviation. When the angle is such that the path difference between these rays is a whole number of wavelengths there is constructive interference and when the angle is such that the path difference is an odd number of half wavelengths there is destructive interference. The intensity of light due to a single wavelength is represented by variations on the geometric intensity distribution caused by interference. Like the geometric theory, Young's theory predicted no light in Alexander's dark band. However, in larger drops the path difference between two emerging rays increases more rapidly with increasing angle of incidence than in smaller ones. Therefore, the larger the drops the smaller the angular separation between the supernumerary arcs. Above a certain drop size, the arcs become indistinguishable. Supernumerary arcs are usually seen at the highest point of the rainbow where drops are smaller.

### 1.5 Airy's paper

Although Young's theory gave a qualitative account of most features of the rainbow, there was as yet no qualitative mathematical theory capable of predicting the intensity of the light as a function of drop size and angle of observation. This was provided by George Biddell Airy in his 1838 paper, " On the intensity of light in the neighbourhood of a caustic" $[1]$. Airy applied the now well-known "Huygens' principle" formulated in the late 17th. century by Christian Huygens and the assumptions made by Augustine Jean Fresnel who had recently provided a mathematical theory predicting the intensity of light after passing through a narrow slit. Airy applied diffraction theory which explained that light on passing through an aperture deviated from rectilinear propagation to an increasing extent as the aperture width declined towards a magnitude comparable to the wavelength of the light concerned.

Airy established that the reflected Huygens' wavefront is a cubic curve and assumed that the intensity is constant at all points along it. The Huygens wavefront which is everywhere perpendicular to the rays crossing it, is shown in Fig. III on Airy's plate, Figure 1.5, and in Figure 1.2. In accordance with the Huygens/Fresnel approach Airy derived an expression for the intensity at a point $P$ slightly displaced from the caustic asymptote (primary rainbow ray) at a very great distance from the drop.

Referring to Fig. III on the plate in Figure 1.5, Airy writes

$$
\begin{equation*}
\Delta D=d x \times \sin \frac{2 \pi}{\lambda}(v t-w h o l e ~ p a t h), \tag{1.4}
\end{equation*}
$$

where $\Delta D$ is the displacement at $P$ due to a small element $d x$ of the reflected wavefront, $\lambda$ is the wavelength of the light concerned, $v$ is the velocity of that light in air and "whole path" is the distance from that element to the point $P$.

To find the resultant displacement $D$ at $P$, he sets out to sum the contributions due to all elements $d x$ along the wavefront, by integrating the above expression from $x=-\infty$ to $x=+\infty$.

Firstly he establishes that the whole path is given by

$$
\begin{equation*}
F+\frac{1}{3 a^{2}}\left(x^{3}-3 a^{2} \theta x\right), \tag{1.5}
\end{equation*}
$$

where $F$ is a constant, $a$ is the drop radius, $x$ is the distance along the wavefront from the asymptote and $\theta$ is the angle made by the direction concerned with the caustic asymptote.

The resultant displacement can then be expressed as

$$
\begin{equation*}
D=\left[2 \int_{0}^{\infty} \cos \frac{2 \pi}{\lambda}\left(\frac{1}{3 a^{2}}\right)\left(x^{3}-3 a^{2} \theta x\right) d x\right] \sin \frac{2 \pi}{\lambda}(v t-x) . \tag{1.6}
\end{equation*}
$$

The integral is therefore proportional to the amplitude and intensity is proportional to the square of the amplitude.

$$
\begin{equation*}
I(m) \propto\left[\int_{0}^{\infty} \cos \frac{\pi}{2}\left(w^{3}-m w\right) d w\right]^{2}, \tag{1.7}
\end{equation*}
$$

where $\quad \frac{\pi}{2} w^{2}=\frac{2 \pi}{\lambda}\left(\frac{1}{3 a^{2}}\right) x^{3} \quad$ and $\quad m=\theta\left(\frac{48 a^{2}}{\lambda^{2}}\right)^{\frac{1}{3}}$.

This is the famous Airy integral. The expression for $m$ shows that the intensity distribution can be expected to depend on drop size, $a$, and wavelength of light, $\lambda$.

Though he fails to evaluate the integral analytically, Airy evaluates it approximately by a method of quadratures but only from $m=-4$ to $m=+4$.

The integral is plotted against $m$ in Fig. IV on Airy's plate, Figure 1.5. The intensity as predicted by geometric theory and as predicted by Young's theory are plotted on the same axis.

The essential features of the Airy intensity distribution are
(i) There is light in the dark side of the bow where the intensity declines exponentially with increasing negative $m$.
(ii) The maximum of intensity, which is not infinite, does not occur at the primary rainbow angle, $(m=0)$ but at $m=+1.08$.
(iii) The positions of the supernumerary maxima and of the principal maximum depend on wavelength of light and on drop size.

Airy neatly establishes the position of Young's supernumerary maxima on the graph by using the fact that the rate of change of path length with the point of incidence is zero for two paths which meet near a caustic.

Differentiating the path $F+\frac{\lambda}{4}\left(w^{3}-m w\right)$ with respect to $w$ and setting this equal to zero, he establishes that minima will occur when $\sqrt{\frac{4 m^{3}}{27}}=1,3,5 \ldots$ or $m=1.89,3.93 \ldots$.

The Airy integral places these minima at $m=2.48,4.4 \ldots$.
In the paper published by the experimentalist, Miller (1841) [19], the author claims to have detected light in Alexander's dark band and that the position of supernumerary maxima accords closely with Airy's predictions.

### 1.6 Interface between theories

Berry and Howls[9] consider Airy's results in the context of perturbation theory. They point out that one might expect that Airy's theory would reduce smoothly to the geometric theory when $\frac{\lambda}{a} \rightarrow 0$, that is when drop size is very large in comparison to wavelength. Such a smooth transition takes place from special relativity to Newtonian mechanics as particle velocity gets very small in comparison to the velocity of light.

But such a smooth transition does not take place here. The intensity on the bright side of the bow is oscillatory and might be roughly approximated by fringes whose intensity is a cosine squared function of path difference, as in Young's classical interference experiment. Thus

$$
\begin{equation*}
I=I_{0} \cos ^{2}\left(\frac{2 \pi}{\lambda} \times \frac{\lambda}{2} \sqrt{\frac{4 m^{2}}{27}}\right) \tag{1.8}
\end{equation*}
$$

Recalling that $m=\theta\left(\frac{48 a^{2}}{\lambda^{2}}\right)^{\frac{1}{3}}$, it is clear that a singularity occurs as $\frac{\lambda}{a} \rightarrow 0$ and there is no smooth transition to the geometric case. When the more precise expressions for the intensity established by Stokes as solutions to the Airy integral are discussed in Chapter 3, these singularity problems will be equally evident.

Considering the amplitude rather than the intensity, Berry and Howls [9], following Stokes, interpret the transition across the caustic from the dark side to the bright side as a change from a single exponential to the two complex exponentials required to give a sinusoidal function. The birth of the second exponential is an example of the Stokes' phenomenon. Berry and Howls interpret the second exponential as representing the second ray in Young's interfering pair. They point out that smooth transitions between theories in physics are the exception rather than the rule and cite a number of examples. These include the transition between quantum mechanics and Newtonian mechanics involved in the tunnelling of a particle out of the nucleus in radio-active decay as mentioned in the introduction. Thus Stokes' work is now recognised as being of the greatest importance for modern physics.

## CHAPTER 2

## STOKES' 1850 PAPER

### 2.1 Introduction

In this chapter Stokes' first approach to the evaluation of the Airy integral, far from the caustic where the supernumerary arcs occur, is discussed. The discussion centres on his paper [38], "On the numerical calculation of a class of definite integrals and infinite series ".

It is necessary before proceeding to make some comments about notation. An effort has been made, throughout this thesis, to adhere to Stokes' notation. However there are instances where this would lead to such confusion for modern readers that departures are unavoidable. We note immediately two particular peculiarities. Stokes uses the term " imaginary" number or variable in the sense that the word "complex" is now used. Also, he uses the term " amplitude" to denote what is now known as the " phase" or " argument" of a complex number. In both these cases, the modern term is used here.

In his earlier papers, Stokes does not have a symbol for $\sqrt{-1}$ which is itself written into his expressions. This is replaced by $t$ in his 1868 paper and in later papers. Throughout his papers he uses $i$ as the running variable and $n$ is commonly used as the variable in differential equations. We have made changes in notation in these instances. We have, however, adhered to Stokes notation in all other cases including the use of $x$ to denote a complex variable, although $z$ is more commonly used in this case today.

### 2.2 Asymptotic solution

In this paper Stokes begins with the rainbow integral in the form in which it was established by Airy, as outlined in the previous chapter,

$$
\begin{equation*}
W(m)=\quad \int_{0}^{\infty} \cos \frac{\pi}{2}\left(w^{3}-m w\right) d w . \tag{2.1}
\end{equation*}
$$

Airy had established that the square of this integral is proportional to the intensity of light in the neighbourhood of a caustic.

Using the method of quadratures Airy [1] had calculated this integral for values of $m$ from - 4 to +4 only, at intervals of 0.2 . He was later able to extend the range for which numerical calculations could be made to $m=-5.6 \ldots+5.6$, using a method involving a series in ascending powers of $m$ suggested to him by De Morgan. These values are reproduced by Airy in a later paper [3] which also contains De Morgan's suggestions. The results obtained by these two methods were in close agreement.

Stokes points out that the series solution is convergent for all values of $m$ but that the calculations become very cumbersome when $m$ is large. He comments, for example, that even for values of $m$ around 5.6 it was necessary for Airy to employ 10 -figure logarithms and that even with these he was unable to evaluate the integral for higher values of $m$.

Furthermore only the first two roots of the equation $W=0$, lie in the range of values of $m$ for which Airy calculated the integral.

These roots correspond to the theoretical positions of the first two dark bands in a system of spurious rainbows. Miller [19] had physically observed 30 such dark bands. Accordingly it should be possible to find at least 30 such roots.

Stokes realises it would not be possible to find more than two roots by means of ascending series because of the extent of the numerical calculations involved. He therefore seeks to develop a method of evaluating the integral with ease when $m$ is large and thus of finding further roots of $W=0$.

He uses the following method. The cosine function in the Airy integral is replaced by the associated complex exponential. By differentiating under the integral sign it is shown that

$$
\begin{equation*}
\frac{d^{2} u}{d x^{2}}+\frac{x}{3} u=\frac{1}{3} i \tag{2.2}
\end{equation*}
$$

where $W=\left[\left(\frac{\pi}{2}\right)^{-1 / 3}\right] \operatorname{Re}(u)$ and $x$ is proportional to $m$.

Writing $u=U+i V$, he therefore seeks a solution of

$$
\begin{equation*}
\frac{d^{2} U}{d x^{2}}+\frac{x}{3} U=0 \tag{2.3}
\end{equation*}
$$

which is the real part of (2.2).
The method he uses to find an approximate solution anticipates the modern method of dominant balance and the WKBJ method, as outlined by Bender and Orszag [4].

Stokes argues that if $x$ is very large it can be regarded as a constant in the second term while in the first term it can be replaced by a small increment $\delta x$. We must take it that he regards the equation to be of the form:

$$
\begin{equation*}
\frac{d^{2} U}{d(\delta x)^{2}}+\frac{x}{3} U=0 \tag{2.4}
\end{equation*}
$$

The solution is therefore

$$
\begin{equation*}
U=N \cos \left(\sqrt{\frac{x}{3}} \delta x\right)+M \sin \left(\sqrt{\frac{x}{3}} \delta x\right) \tag{2.5}
\end{equation*}
$$

where $M$ and $N$ are constants.

This suggests a solution of the original differential equation (2.3) of the form

$$
\begin{equation*}
U=N \cos \{f(x)\}+M \sin \{f(x)\} \tag{2.6}
\end{equation*}
$$

Putting $\delta(f(x))=\sqrt{\frac{x}{3}} \delta x$, we have $f(x)=\frac{2}{3} \sqrt{\frac{x^{3}}{3}}$ and the above solution becomes

$$
\begin{equation*}
U=N \cos \left(\frac{2}{3} \sqrt{\frac{x^{3}}{3}}\right)+M \sin \left(\frac{2}{3} \sqrt{\frac{x^{3}}{3}}\right) \tag{2.7}
\end{equation*}
$$

When $x$ is negative, equal to $-x^{\prime}$, the same process gives a solution of the form

$$
\begin{equation*}
U=A \exp \left(\frac{2}{3} \sqrt{\frac{\left(x^{\prime}\right)^{3}}{3}}\right)+B \exp \left(-\frac{2}{3} \sqrt{\frac{\left(x^{\prime}\right)^{3}}{3}}\right) . \tag{2.8}
\end{equation*}
$$

Although both the above terms appear in the general solution, the first is unbounded and therefore physically unsuitable for the solution sought. The solution for negative $x$ is therefore

$$
\begin{equation*}
U=B \exp \left(-\frac{2}{3} \sqrt{\frac{\left(x^{\prime}\right)^{3}}{3}}\right) \tag{2.9}
\end{equation*}
$$

Stokes has, in fact, arrived at a solution which is oscillatory in form when $x$ is positive and represents an exponential decline when $x$ is negative. This is broadly in line with Airy's solution for small values of $x$.

However, these are but approximate solutions. To find closer agreement with the Airy/De Morgan solution in ascending powers for small values of $x$ and to describe more fully the physical reality reported by Miller, better approximate solutions were required.

From consideration of the form of the approximate solution for large positive $x$, Stokes intuitively postulates a general complex solution multiplied by a series in descending powers of $x$.

$$
\begin{equation*}
U=e^{\frac{2}{3} \sqrt{-\frac{x^{3}}{3}}}\left(A^{\prime} x^{\alpha}+B^{\prime} x^{\beta}+C^{\prime} x^{\gamma}+\ldots . .\right) \tag{2.10}
\end{equation*}
$$

where $A^{\prime}, B^{\prime}, C^{\prime}, \ldots . . \alpha, \beta, \gamma, \ldots .$. are constants to be determined.
He acknowledges that he took the idea of "multiplying the circular functions by a series according to descending powers of $x$ " from Cauchy's method of evaluating Fresnel's integrals. It is significant that Stokes, in this case, realised that the powers of $x$ need not necessarily be integers or decrease by 1 from term to term.

By making the appropriate substitution in the differential equation and requiring the powers of $x$ to descend, he finds in fact that the powers descend by $3 / 2$ from term to term. A relationship between successive constants in the series is found in such a way as to reduce the number of constants to one. The solution is then given by

$$
\begin{align*}
& U=A x^{-\frac{1}{4}} e^{\frac{2}{3}} \sqrt[-\frac{x^{3}}{3}]{\{ }\left\{1-\frac{1.5}{1}\left(\frac{i}{16\left(3 x^{3}\right)^{\frac{1}{2}}}\right)+\frac{1.5 .7 \cdot 11}{1.2}\left(\frac{i}{16\left(3 x^{3}\right)^{\frac{1}{2}}}\right)^{2}\right. \\
&\left.-\frac{1.5 .7 \cdot 11.13 .17}{1.2 .3}\left(\frac{i}{16\left(3 x^{3}\right)^{\frac{1}{2}}}\right)^{3}+\ldots .\right\} \tag{2.11}
\end{align*}
$$

He then deduces that if $i$ is replaced everywhere by $-i$ and a different arbitrary constant is used this would also be a solution of the differential equation. The complete solution is then written as

$$
\begin{align*}
& U=A x^{-\frac{1}{4}} e^{i \frac{2}{3} \sqrt{\frac{x^{3}}{3}}}\left\{1-\frac{1.5}{1}\left(\frac{i}{16\left(3 x^{3}\right)^{\frac{1}{2}}}\right) \cdots\right\} \\
&\left.+B x^{-\frac{1}{4}} e^{-i}\right)  \tag{2.12}\\
& \frac{2}{3 \sqrt{x^{3}}}\left\{1+\frac{1.5}{1}\left(\frac{i}{16\left(3 x^{3}\right)^{\frac{1}{2}}}\right)\right]
\end{align*}
$$

which has two arbitrary constants as required by the form of the original differential equation.

The solution will have different forms depending on whether $x$ is positive or negative. In the case of $x$ negative and equal to $-x^{\prime}$, the solution is already real and if the unbounded positive exponential term is discarded, it is given as

$$
\begin{equation*}
U=C\left(x^{\prime}\right)^{-\frac{1}{4}} e^{-\frac{2}{3}{\sqrt{(x)^{3}}}^{3}}\left\{1-\frac{1.5}{1.16\left(3 x^{3}\right)^{\frac{1}{2}}}+\frac{1 \cdot 5 \cdot 7 \cdot 11}{1 \cdot 2 \cdot 16^{2} \cdot 3 x^{3}}-\ldots\right\} \tag{2.13}
\end{equation*}
$$

This equation is seen to be a refinement of the rougher approximation given by (2.9) for the light intensity on the dark side of the rainbow.

To find a real solution when $x$ is positive he notes that the complete solution is of the form

$$
U=A(P+i Q)+B(P-i Q)
$$

where $A$ and $B$ are complex and $P$ and $Q$ are real.
For the solution to be real it must have the form

$$
U=N_{1} P+N_{2} Q
$$

where the arbitrary constants $N_{1}$ and $N_{2}$ are real.
Thus the real solution for positive $x$ is

$$
\begin{equation*}
U=A x^{-\frac{1}{4}}\left[R \cos \left(\frac{2}{3} \sqrt{\frac{x^{3}}{3}}\right)+S \sin \left(\frac{2}{3} \sqrt{\frac{x^{3}}{3}}\right)\right] \tag{2.14}
\end{equation*}
$$

$$
+B x^{-\frac{1}{4}}\left[R \sin \left(\frac{2}{3} \sqrt{\frac{x^{3}}{3}}\right)-S \cos \left(\frac{2}{3} \sqrt{\frac{x^{3}}{3}}\right)\right]
$$

where

$$
\begin{aligned}
& R=1-\frac{1 \cdot 5 \cdot 7 \cdot 11}{1 \cdot 2 \cdot 16^{2} 3 x^{3}}+\frac{1 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19.23}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 16^{4} 3^{2} x^{6}} \ldots, \\
& S=\frac{1.5}{1.16\left(3 x^{3}\right)^{\frac{1}{2}}}-\frac{1 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17}{1 \cdot 2 \cdot 3 \cdot 16^{3}\left(3 x^{3}\right)^{\frac{3}{2}}} \ldots
\end{aligned}
$$

The two series, for $x$ positive and negative respectively, though they begin by converging rapidly, are ultimately divergent.

Airy had used convergent series to explain the position of the first two dark bands in a system of supernumerary rainbows. Remarkably, Stokes used a divergent series method to find the position of the first 50 dark bands in the system. The use of divergent series for practical calculations was a major step in the year 1850. Recall that Abel in 1828 had described such series as "the invention of the devil".

Stokes truncates the divergent series at a point where the terms become very small, in the vicinity of the least term before the series begins to diverge. This work is a major contribution to making divergent series manageable. He shows that the series truncated in this way is a very good approximate solution to the differential equation (2.3).

He argues that as the constants $A^{\prime}, B^{\prime}, C^{\prime}, \ldots . . \alpha, \beta, \gamma, \ldots \ldots$ were determined by substituting an infinite series into the original differential equation, the substitution of a series truncated in the above way into the equation would give a right hand side that differs from zero by a negligible amount for large $x$. Intuitively, this is because the terms are alternatively positive and negative, giving rise to cancellation and also because they are very small near truncation.

To determine the arbitrary constants he returns to the original integral, which he approximates for large values of $x$. This gives him an expression for the leading term of $u$. By comparing this result with the asymptotic solution of the differential equation, he finds the values of the arbitrary constants, $A$ and $B$,

$$
\begin{equation*}
A=B=\frac{\pi^{\frac{l}{2}}}{2^{\frac{l}{2}} 3^{\frac{1}{4}}} \tag{2.15}
\end{equation*}
$$

### 2.3 Saddle point method

The method of integration in the complex plane which Stokes uses here has since been generalised and is now known as the saddle point method. Stokes work in this area is of such historical importance that we set it out below, in full, together with a modern treatment.

By treating the cosine as the real part of its associated exponential he writes the integral given in (2.1) as

$$
\begin{equation*}
Q(q)=\int_{0}^{\infty} e^{-x^{3}+3 q^{2} x} d x, \quad|q / \rightarrow \infty, \quad| \arg (q) \left\lvert\,<\frac{\pi}{6} .\right. \tag{2.16}
\end{equation*}
$$

Stokes gives a very detailed description of and justification for the procedure he employs in evaluating this integral. Although he does not state it explicitly, his method involves a demonstration that the main contribution to this integral comes from the neighbourhood of a saddle point.

Although Stokes makes no mention of saddle point we will show later that the integrand in (2.16) has saddle points at $x=q$ and $x=-q$, with the dominant one occurring at $x=q$.

Stokes makes the substitution

$$
x=q+y,
$$

which gives an integral where the dominant saddle point occurs at the origin

$$
\begin{equation*}
Q=e^{2 q^{3}} \int_{-q}^{\infty} e^{-y^{3}-3 q y^{2}} d y . \tag{2.17}
\end{equation*}
$$

He then scales the variable of integration, by making the substitution $y=$ $(3 q)^{-\frac{1}{2}} t$, where $(3 q)^{\frac{-1}{2}}$ is to take the principal value of its argument, i.e. the value which is real and positive when $q$ is real and positive. For the remainder of the discussion he takes $q=c e^{i \alpha}$ and $t=\rho e^{i \theta}$. His task is now to evaluate the integral

$$
\begin{equation*}
Q(q)=(3 q)^{-\frac{1}{2}} e^{2 q^{3}} I(q), \tag{2.18}
\end{equation*}
$$

where

$$
\begin{equation*}
I(q)=\int_{-\sqrt{3} q^{3 / 2}}^{\infty 0^{\operatorname{ta} / 2}} e^{-(3 q)^{-\frac{3}{2} t^{3}-t^{2}}} d t \tag{2.19}
\end{equation*}
$$

and $\alpha$ is the argument of $q$.

Proceeding heuristically, the original path of integration along the positive real axis is deformed into one which runs from a complex quantity $-\sqrt{3} q^{\frac{3}{2}}$ to another complex quantity with infinite modulus and argument equal to $\alpha / 2$.

The integral $I(q)$ is then treated as the sum of four integrals,

$$
I(q)=T_{1}+T_{2}+T_{3}+T_{4} .
$$

The path of integration is, in effect, divided into 4 separate portions as shown in Figure 2.1. Integral $T_{l}$ is evaluated along the path joining $-\sqrt{3} q^{\frac{3}{2}}$ to $-a$ on the negative real axis; $T_{2}$ is evaluated from $-a$ to $b$ on the positive real axis; $T_{3}$ is evaluated from $b$ to $\infty$ and $T_{4}$ is evaluated along the circular arc from the real axis to the ray $\rho e^{i \alpha / 2}$ at some point $\rho>b$ which is then allowed to approach $\infty$.

In the integration we first let $c$, the modulus of $q$, become infinite while $a$ and $b$ remain constant and finally we let $a$ and $b$ become infinite.


Figure 2.1: Paths of integration for the integral (2.19).

Stokes first evaluates the integral (2.19) along the arc shown as $T_{4}$, where $\rho$, the modulus of $t$, is held constant while $\theta$, the argument of $t$, varies from 0 to $\alpha / 2$. The limit of this integral as $\rho \rightarrow \infty$ is then taken so that

$$
\left.\begin{array}{rl}
T_{4} & =\lim _{\rho \rightarrow \infty}\{i \rho  \tag{2.20}\\
\int_{0}^{\frac{\alpha}{2}} e^{f(\theta)} d \theta
\end{array}\right\}, ~ \begin{aligned}
& \text { where } f(\theta)
\end{aligned}=-(3 q)^{-\frac{3}{2}} \rho^{3} e^{3 i \theta}-\rho^{2} e^{2 i \theta}+i \theta . .
$$

He then uses the fact that the value of $T_{4}<\lim _{\rho \rightarrow \infty} \int e^{f(\theta)} \mid d \theta$ to show that $T_{4} \rightarrow 0$ as $\rho \rightarrow \infty$.

In considering the integral along the contour $T_{l}$, he first changes the sign of $t$ and the order of integration and takes

$$
\begin{equation*}
T_{I}=\quad \int_{a}^{\sqrt{3} c^{\frac{3}{2}}} e^{(3 q)^{-\frac{3}{2} t^{3}-t^{2}}} d t \tag{2.21}
\end{equation*}
$$

Stokes explains that this can be evaluated by integrating first along the path from $\rho=a$ to $\rho=\sqrt{3} c^{\frac{3}{2}}$, while $\theta$ remains constant and equal to 0 and then along the arc from $\theta=0$ to $\theta=\frac{3 \alpha}{2}$, holding $\rho$ constant and equal to $\sqrt{3} c^{\frac{3}{2}}$. He then shows that both these integrals vanish, the first when $a$ becomes infinite, the second when $c$ becomes infinite. Recall that $c$ is the modulus of $q$ and he is evaluating the integral for large values of $q$.

The integral

$$
\begin{equation*}
T_{3}=\int_{b}^{\infty} e^{-(3 q)^{-\frac{3}{2} t^{3}}-t^{2}} d t \tag{2.22}
\end{equation*}
$$

also vanishes when $c$ becomes infinite because then

$$
T_{3} \approx \int_{b}^{\infty} e^{-t^{2}} d t
$$

which is equal to zero when $b$ becomes infinite.
Therefore the only contribution to the integral along the contour comes from $\mathrm{T}_{2}$, that is, the section from $-a$ to $b$ on the real axis which is the
neighbourhood of the saddle point at $t=0$. Also, as we show later, the real axis is $a$ path of steepest descent through the saddle point.

To find the contribution from $T_{2}$ we first make $c$ infinite to obtain

$$
\begin{equation*}
T_{2}=\int_{-a}^{b} e^{-t^{2}} d t \tag{2.23}
\end{equation*}
$$

which, when $a$ and $b$ become infinite, gives

$$
\begin{equation*}
T_{2}=\int_{-\infty}^{\infty} e^{-t^{2}} d t=\sqrt{\pi} \tag{2.24}
\end{equation*}
$$

From (2.18) we now have

$$
\begin{equation*}
Q=\sqrt{\frac{\pi}{3 q}} e^{2 q^{3}} \tag{2.25}
\end{equation*}
$$

Paris [34] has already pointed out that in evaluating the integral, Stokes was in effect employing the saddle point method in the complex plane more than a decade before Riemann, whose fragmentary manuscript on the subject was dated October 1863.

In his attempt to justify mathematically physical observations of the rainbow, Stokes has again here developed a mathematical method which has been found to have general applications. Stokes writing in 1850 did not point to the general applicability of the method.

A modern approach to evaluating the integral (2.16) using the saddle point method would involve the following procedures as outlined by such authors as Bender and Orszag[4] and Nayfeh [21]. It should be noted that neither of the above works provide rigorous treatments. Such treatments are given in Olver [27] and Wong [46].

We must first write the integral given in (2.16) in the form

$$
\begin{equation*}
Q(q)=\int_{C} h(s) e^{q f(s)} d s \quad \text { as } q \rightarrow \infty \tag{2.26}
\end{equation*}
$$

This is achieved by making the substitution $x=q s$ in (2.16)

$$
\begin{equation*}
Q(q)=\int_{0}^{\infty} e^{-q^{3}\left(s^{3}-3 s\right)} d s \quad \text { as } q \rightarrow \infty \tag{2.27}
\end{equation*}
$$

If we write $q^{3}=X, f(s)=-s^{3}+3 s$, then

$$
\begin{equation*}
Q(q)=X^{\frac{1}{3}} \quad \int_{0}^{\infty} e^{X f(s)} d s \tag{2.28}
\end{equation*}
$$

Thus using the method described by Bender and Orszag [4], we find the saddle points occur when $\frac{d f}{d s}=0$, that is when $s=1,-1$.

When $s=-1, \quad\left|e^{X f(s)}\right|=\left|e^{-2 q^{3}}\right|$, when $s=1, \quad\left|e^{X f(s)}\right|=\left|e^{2 q^{3}}\right|$.
Clearly, the contribution to $Q(q)$ at $s=-1$ is exponentially small compared with the contribution at $s=1$, as $q \rightarrow \infty$. Therefore, we need only concern ourselves with the dominant saddle point at $s=1$, which corresponds to $x=q$ in the integral in (2.16).

We now need to find the steepest paths, which are the constant phase contours through the saddle point.

Let $s=u+i v, u$ and $v$ real, then

$$
\begin{equation*}
f(s)=\phi(u, v)+i \psi(u, v)=\left\{3 u\left(1+v^{2}\right)-u^{3}\right\}+i\left(3 v\left(1-u^{2}\right)+v^{3}\right) \tag{2.29}
\end{equation*}
$$

At the saddle point, $s=1, \mathrm{f}(\mathrm{s})$ has imaginary part equal to zero. Therefore $\psi(u, v)$ must equal zero at all points along a constant phase contour through the point $s=1$.

So the equations of the steepest paths are given by $\psi(u, v)=0$, or

$$
\begin{equation*}
v=0, \quad v= \pm \sqrt{3\left(u^{2}-1\right)} \tag{2.30}
\end{equation*}
$$

A diagram showing the steepest paths $\psi(u, v)=0$ is given in Figure 2.2 below. When $v=0, f(s)=3 u-u^{3}$ and $\left|e^{X f(s)}\right|$ decreases as $s \rightarrow \infty$ along this path. Therefore $v=0$ is a path of steepest descent from the saddle. When $v= \pm \sqrt{3\left(u^{2}-1\right)}, f(s)=8 u^{3}-6 u$ and $\left|e^{X S(s)}\right|$ increases as $s \rightarrow \infty$. These, therefore, represent paths of steepest ascent.


Figure 2.2: Maple contourplot showing constant phase contours for $\psi(u, v)$, including the steepest paths through the saddle point at $u=1$. The arrows indicate the direction in which $\left|e^{x f(s)}\right|$ decreases.

We have shown that for the integral (2.16) which Stokes evaluated, the dominant saddle point occurred at $x=q$ and the real axis is a path of steepest descent through the saddle.

Finally, to complete the approximation of the integral we can use the procedure outlined by Nayfeh [21], (Olver[27] and Wong[46] ).
Returning to the integral as represented in (2.28) we see that along the real axis, where $v=0$, the maximum attained by $f(s)$ is 2 . Therefore we can let

$$
\begin{equation*}
X\left(3 s-s^{3}\right)=2 X-\delta^{2}, \quad \text { for some small } \delta \tag{2.31}
\end{equation*}
$$

Expanding around $s=1$, we get

$$
\begin{equation*}
2 X-3 X(s-1)^{2}-X(s-1)^{3}=2 X-\delta^{2} \tag{2.32}
\end{equation*}
$$

Therefore for small $\delta$ and $s$ close to 1 , we can make the approximation

$$
\begin{equation*}
3 X(s-1)^{2} \cong \delta^{2} \tag{2.33}
\end{equation*}
$$

which gives

$$
\begin{equation*}
s \cong 1+\frac{\delta}{\sqrt{3 X}} . \tag{2.34}
\end{equation*}
$$

We can thus write the integral, for $\delta \ll 1$ and $s$ in the neighbourhood of 1 as

$$
\begin{equation*}
Q(X)=\frac{X^{\frac{1}{3}}}{\sqrt{3 X}} \int_{-\infty} e^{2 X-\delta^{2}} d \delta=\frac{e^{2 X} X^{\frac{1}{3}}}{\sqrt{3 X}} \sqrt{\pi}, \tag{2.35}
\end{equation*}
$$

which gives, since $X=q^{3}$,

$$
\begin{equation*}
Q=\sqrt{\frac{\pi}{3 q}} e^{2 q^{3}} . \tag{2.36}
\end{equation*}
$$

This is precisely the result obtained by Stokes in (2.25) above.

### 2.4 Stokes' Phenomenon

In attempting to find the value of the coefficient $C$, multiplying the expression (2.13), valid for negative $x$, Stokes confronts a key problem to which he was to propose an historic solution. This is where we get his first insight into what is now known as the Stokes' phenomenon. He
begins by considering the solution in the neighbourhood of $x=0$. He points out that the solution in descending powers of $x$ will become indeterminate at $x=0$, whereas this problem does not arise in the original differential equation when $x=0$, given in (2.3) or in the series solution in ascending powers.

Recall from Chapter 1, equation (1.7), that as $\frac{\lambda}{a} \rightarrow 0, x \rightarrow 0$, where $\lambda$ is the wavelength of the light and a is the drop radius. The intensity distribution in the primary rainbow, determined by Airy on the basis of the wave theory of light, might be expected to smoothly go over to that predicted by the laws of geometric optics as $\frac{\lambda}{a} \rightarrow 0$. Clearly, this is not the case here and a singularity occurs at $x=0$. Berry and Howls [9] point out that smooth transitions between theories are the exception rather than the rule in physics. Stokes' mathematical treatment of this phenomenon was, therefore, to have a significance going far beyond the rainbow problem.

Stokes, in addressing this problem, took the key step of considering $x$ to be a complex variable. The degree of innovation involved in this can be appreciated from the following extract from a letter by Stokes to the mathematician Cayley, dated 29th Oct. 1849 and recently published by Wilson [45]:
> " Thomson (Lord Kelvin) and I are at present writing to each other about potentials. I think that potentials may throw light on the interpretation of $\mathrm{f}(x+\sqrt{-1} y)$. How horrible you would think it to prove, even in one's own mind, a proposition in pure mathematics by means of physics."

This was written less than a year before the publication of the paper by Stokes under discussion here.

Stokes in this paper considers a general complex solution. The asymptotic solution is of the form

$$
\begin{equation*}
U=A x^{-\frac{1}{4}} N+B x^{-\frac{1}{4}} N^{\prime} \tag{2.37}
\end{equation*}
$$

where $N$ and $N$ 'are series involving a complex exponential multiplying descending integral powers of $x$. The solution in this form raises an immediate problem when $x$ is complex. It will not return to its original value until the argument of $x$ is increased by $8 \pi$ where clearly from the
original differential equation the value of $U$ returns to its original value when the argument of $x$ is increased by $2 \pi$.

To ensure that the complex solution returns to its original value when the argument of $x$ is increased by $2 \pi$, Stokes concludes that the value of the coefficients $A$ and $B$ must change discontinuously at some value(s) of the argument between 0 and $2 \pi$. More generally, as explained by Paris and Wood [37], the solution and the differential equation have different multivalued structures in the complex plane.

To find the value of the coefficient $C$ in the solution for negative $x$, (2.13), Stokes replaces $x$ by $x e^{i \pi}$ in the solution, (2.14), for positive $x$. To make the result compatible with the solution (2.13) for negative $x$ he concludes that $C$ must be given by

$$
\begin{equation*}
C=\frac{\pi^{\frac{1}{2}}}{2.3^{\frac{1}{4}}} . \tag{2.38}
\end{equation*}
$$

Stokes professes himself unhappy with this method of determining the constant $C$ and he returns to this problem in his 1857 paper, which shall be discussed in the next chapter, section 3.5 .

### 2.5 Method of Stationary Phase

As a second example of finding a solution valid at infinity for an integral which has a solution in ascending powers valid at the origin, Stokes takes

$$
\begin{equation*}
u(x)=\frac{2}{\pi} \int_{0}^{\pi / 2} \cos (x \cos \theta) d \theta \tag{2.39}
\end{equation*}
$$

which, Stokes says, occurs in many physical investigations and which we now know is related to the Bessel function of order zero, $J_{0}(x)$. In particular it had posed a problem for Airy[2] in computing the intensity of light diffracted by an annular aperture.

Stokes uses the same procedure as in dealing with the rainbow integral, in arriving at the asymptotic solution. However, in this case, when
evaluating the definite integral for large values of $x$, he uses a method which is a fore-runner of the method of stationary phase later described by Kelvin[15] in 1887.

Stokes splits the integrand into three functions, two of which approximately destroy each other throughout the range of integration including at the end-points. In this way he reduces the integral to one which is easily evaluated by standard methods for large values of $x$. To this end he makes the substitution $\cos \theta=1-\mu$ which gives

$$
\mathrm{d} \theta=\frac{d \mu}{\sqrt{2 \mu-\mu^{2}}} .
$$

Now writing the latter as

$$
d \theta=\frac{d \mu}{\sqrt{2 \mu-\mu^{2}}}+\frac{d \mu}{\sqrt{2 \mu}}-\frac{d \mu}{\sqrt{2 \mu}},
$$

the integral in (2.39) becomes

$$
\begin{align*}
u(x)= & \frac{2}{\pi} \int_{0}^{l} \cos \{x(1-\mu)\} \frac{d \mu}{\sqrt{2 \mu}}+ \\
& \frac{2}{\pi} \int_{0}^{l} \cos \{x(1-\mu)\} \frac{d \mu}{\sqrt{2 \mu-\mu^{2}}}+  \tag{2.40}\\
& \frac{2}{\pi} \int_{0}^{l} \cos \{x(1-\mu)\} \frac{d \mu}{-\sqrt{2 \mu}} .
\end{align*}
$$

Stokes, insightfully, argues that, for large values of $x$, the second and third integrals in (2.40) do not make any contribution to the leading term of $u$. This is because they approximately cancel each other out as described in the heuristic interpretation offered by Bender and Orszag[4] on page 278. This cancellation is demonstrated in Figure 2.3, Figure 2.4, Figure 2.5 and Figure 2.6 below using the mathematical computer package, Maple.


Figure 2.3: Plot of $\frac{\cos [x(1-\mu)]}{\sqrt{2 \mu-\mu^{2}}}$ for $x=100$.


Figure 2.4: Plot of $\frac{\cos [x(1-\mu)]}{-\sqrt{2 \mu}}$ for $x=100$.


Figure 2.5: The functions displayed in Figure 2.3 and Figure 2.4 shown on one graph to highlight the tendency of the two functions to cancel each other out.


Figure 2.6: Plot of $\frac{\cos [x(1-\mu)]}{\sqrt{2 \mu-\mu^{2}}}+\frac{\cos [x(1-\mu)]}{-\sqrt{2 \mu}}$ for $x=100$.

Finally, we see from Figure 2.6, representing the sum of the second and third terms, successive positive and negative areas cancel under the integration process.

Having thus dispensed with the second and third integrals in (2.40) Stokes turns his attention to evaluating the first integral for large values of $x$. This integral can be written as

$$
\begin{equation*}
u(x) \approx \frac{\sqrt{2}}{\pi} \int_{0}^{l} \cos \{x(l-\mu)\} \frac{d \mu}{\sqrt{\mu}} . \tag{2.41}
\end{equation*}
$$

It should be noted that Stokes uses the equality sign in the above expression, and for the remainder of his calculations, although it is clear that approximation is intended.

He makes the substitution $\mu x=v$ to obtain

$$
\begin{equation*}
u(x) \approx \frac{\sqrt{2} \cos (x)}{\pi \sqrt{x}} \int_{0}^{x} \cos (v) \frac{d v}{\sqrt{v}}+\frac{\sqrt{2} \sin (x)}{\pi \sqrt{x}} \int_{0}^{x} \sin (v) \frac{d v}{\sqrt{v}} . \tag{2.42}
\end{equation*}
$$

Since we are concerned with large values of $x$ we can take the limits in the above integrals to be from 0 to $\infty$.

Stokes makes use of what he describes elsewhere as "the known formulae",

$$
\begin{equation*}
\int_{-\infty}^{\infty} \cos \left(x^{2}\right) d x=\int_{-\infty}^{\infty} \sin \left(x^{2}\right) d x=\sqrt{\frac{\pi}{2}} \tag{2.43}
\end{equation*}
$$

to deduce

$$
\begin{equation*}
\int_{0}^{\infty} \cos (v) \frac{d v}{\sqrt{v}}=\int_{0}^{\infty} \sin (v) \frac{d v}{\sqrt{v}}=\sqrt{\frac{\pi}{2}} . \tag{2.44}
\end{equation*}
$$

Thus, for large values of $x$, he arrives at

$$
\begin{equation*}
u(x) \approx \frac{1}{\sqrt{\pi x}}\{\cos (x)+\sin (x)\} . \tag{2.45}
\end{equation*}
$$

We will show later that this result can be obtained by using the method of stationary phase as described, more explicitly, by Lord Kelvin in [15]. This paper was published some thirty-seven years after Stokes' work.

Kelvin considers, for large positive $x$, the integral

$$
\begin{equation*}
u(x, t)=\frac{1}{2 \pi} \int_{0}^{\infty} \cos [m\{x-t f(m)\}] d m \tag{2.46}
\end{equation*}
$$

which represents the effect at a given place and time $(x, t)$ of an infinitely intense disturbance at place and time $(0,0)$. Here, $f(m)$ denotes the velocity of propagation of water waves with wavelength, $\frac{2 \pi}{m}$.

His method of dealing with the integral is based on the principle of interference and had been suggested to him by the work of Stokes and Rayleigh in their treatment of group-velocity and wave-velocity. His basic argument is that for large values of $x-t f(m)$ only the neighbourhoods of the stationary points contribute to the value of the integral, the rest cancels owing to "annulling interference" or destructive interference as it is known to-day.

We first illustrate this principle graphically using Airy's integral

$$
u(q)=\int_{0}^{\infty} \cos \left(x^{3}-q x\right) d x .
$$

Note that the $m$ used in Stokes work has been replaced here by $q$. A plot of the integrand for $q=10$ is shown in Figure 2.7. We note that $x^{3}-10 x$ has a stationary point at $x=1.8$. The plot of this oscillatory function
clearly shows that, away from the stationary point, the contributions to the integral tend to cancel each other out.


Figure 2.7: Plot of the integrand of the Airy integral, showing a stationary value of the phase, $\left(x^{3}-10 x\right)$, at $x=\sqrt{\frac{10}{3}}=1.82 \ldots$.

We now return to the general case given in Kelvin's paper. In evaluating the integral given in (2.46), he denotes the stationary value or values of $m\{x-t f(m)\}$ by $\mu$. In other words $\mu=m$ is the solution of

$$
\begin{equation*}
\frac{d}{d m}[m\{x-t f(m)\}]=0 \tag{2.47}
\end{equation*}
$$

which leads to

$$
\begin{equation*}
x=t\left\{f(\mu)+\mu f^{\prime}(\mu)\right\} \tag{2.48}
\end{equation*}
$$

His next step is to write the Taylor expansion of $m\{x-t f(m)\}$ about $m=\mu$, where $m-\mu$ is very small. This gives

$$
\begin{equation*}
m[x-t f(m)]=\mu[x-t f(\mu)]-t\left[\mu f^{\prime \prime}(\mu)+2 f^{\prime}(\mu)\right] 1 / 2(m-\mu)^{2} \tag{2.49}
\end{equation*}
$$

Combined with equation (2.48) this becomes

$$
\begin{equation*}
m[x-t f(m)]=t\left\{\mu^{2} f^{\prime}(\mu)-\left[\mu f^{\prime \prime}(\mu)+2 f^{\prime}(\mu)\right] 1 / 2(m-\mu)^{2}\right\} \tag{2.50}
\end{equation*}
$$

With the substitution

$$
\begin{equation*}
m-\mu=\frac{\sigma \sqrt{2}}{t^{1 / 2}\left[-\mu f^{\prime \prime}(\mu)-2 f^{\prime}(\mu)\right]^{1 / 2}} \tag{2.51}
\end{equation*}
$$

the integral in (2.46) becomes

$$
\begin{equation*}
u(\mu)=\frac{\sqrt{2} \int_{-\infty}^{\infty} \cos \left[t \mu^{2} f^{\prime}(\mu)+\sigma^{2}\right] d \sigma}{2 \pi t^{1 / 2}\left[-\mu f^{\prime \prime}(\mu)-2 f^{\prime}(\mu)\right]^{1 / 2}} \tag{2.52}
\end{equation*}
$$

where the limits are now from $-\infty$ to $\infty$.
Using the formulae in (2.43) above which Stokes had earlier used he finally obtains

$$
\begin{aligned}
u(\mu) & =\frac{\cos \left[t \mu^{2} f^{\prime}(\mu)\right]-\sin \left[t \mu^{2} f^{\prime}(\mu)\right]}{2 \pi t^{1 / 2}\left[-\mu f^{\prime \prime}(\mu)-2 f^{\prime}(\mu)\right]^{1 / 2}} \\
& =\frac{\sqrt{2} \cos \left[t \mu^{2} f^{\prime}(\mu)+\pi / 4\right]}{2 \pi t^{1 / 2}\left[-\mu f^{\prime \prime}(\mu)-2 f^{\prime}(\mu)\right]^{1 / 2}} .
\end{aligned}
$$

We have discussed earlier the method employed by Stokes in evaluating the Airy integral

$$
\begin{equation*}
u(q)=\int_{0}^{\infty} \cos \left(x^{3}-q x\right) d x \tag{2.54}
\end{equation*}
$$

for large values of $x$. However, in a footnote he gives a very brief description of an alternative method whereby the same result could be obtained by concentrating on the region about the point where the phase, $x^{3}-q x$, is stationary. It is clear, that in its essentials, the method of approximating the integral suggested by Stokes in this footnote is similar to that elaborated by Kelvin 37 years later, although we have been unable to find any reference to this in the letters[45] from Stokes to Kelvin.

We now show how the same result can be obtained along precisely the lines described by Stokes and the lines described by Kelvin.

As can be seen from the plot of the integrand displayed in Figure 2.7, in which $q=10$, the main contribution to the integral comes from the neighbourhood of $x=\sqrt{\frac{q}{2}}$. We can therefore approximate $x^{3}-q x$ by the first two terms of its $\sqrt{\frac{q}{3}}$ ylor series about $x=\sqrt{\frac{q}{3}}$,

$$
\begin{equation*}
\left(x^{3}-q x\right) \approx\left(\frac{q}{3}\right)^{3 / 2}+\sqrt{3 q} x^{2}-2 q x \tag{2.55}
\end{equation*}
$$

and evaluate the integral

$$
\begin{equation*}
u(q)=\int_{-\infty}^{\infty} \cos \left\{\left(\frac{q}{3}\right)^{3 / 2}+\sqrt{3 q} x^{2}-2 q x\right\} d x \tag{2.56}
\end{equation*}
$$

where the limits can now be taken from $-\infty$ to $\infty$.

Once again using the formulae in (2.43), we get

$$
u(q)=\frac{\sqrt{\pi} 3^{1 / 4}\left[\cos \left\{2\left(\frac{q}{3}\right)^{3 / 2}\right\}+\sin \left\{2\left(\frac{q}{3}\right)^{3 / 2}\right\}\right]}{q^{1 / 4}}
$$

$$
\begin{equation*}
=\frac{\sqrt{\pi}\left[\cos \left\{2\left(\frac{q}{3}\right)^{3 / 2}-\frac{\pi}{4}\right\}\right]}{(3 q)^{1 / 4}} \tag{2.57}
\end{equation*}
$$

The above result agrees with that already obtained by Stokes.
We now return to consider the integral

$$
u(x)=\frac{2}{\pi} \int_{0}^{\pi / 2} \cos (x \cos \theta) d \theta
$$

which is the integral evaluated by Stokes by splitting the integrand into three functions as described at the beginning of the section. This time we shall apply the method as described by Kelvin.

The phase function $x(\cos (\theta))$ attains its maximum value at $\theta=0$. Therefore the main contribution to the integral within the given limits occurs in the neighbourhood of the origin. The first two terms of the Taylor expansion about $\theta=0$ are $x-x \theta^{2} / 2$.

We then take the limits from $-\infty$ to $\infty$ and evaluate the integral

$$
\begin{aligned}
u(x) & =\frac{1}{2}\left\{\frac{2}{\pi} \int_{-\infty}^{\infty} \cos \left(x-\frac{x \theta^{2}}{2}\right) d \theta\right\} \\
& =\frac{1}{\pi}\left\{\begin{array}{c}
\cos (x) \int_{-\infty}^{\infty} \cos \left(\frac{x \theta^{2}}{2}\right) d \theta \\
+\sin (x) \\
\int_{-\infty}^{\infty} \cos \left(\frac{x \theta^{2}}{2}\right) d \theta
\end{array}\right\} .
\end{aligned}
$$

Once more making use of the formulae in (2.43), we get

$$
\begin{equation*}
u(x) \approx \frac{1}{\sqrt{\pi x}}\{\cos (x)+\sin (x)\} \tag{2.59}
\end{equation*}
$$

which is the result obtained by Stokes.

### 2.6 Historical Note

Throughout their careers there was a vast correspondence between Stokes and Kelvin. This extended over the period from 1846 up to Stokes' death in 1903 and covered a wide range of topics in physics and mathematics. This correspondence, which has recently been published by David Wilson [45], reflects the major developments in the sciences during that period.

It is surprising, therefore, that no discussion can be found in the correspondence of Kelvin's paper outlining the method of stationary phase. This paper was read to the Society on 3 Feb. 1887 together with another paper by Kelvin[16] on the motion of a solid through a fluid. It is evident from the correspondence that it was this latter paper that was the subject of contention between the two at the time. It would appear that both Stokes and Kelvin under-estimated the importance of their own mathematical work in this area.

## CHAPTER 3

## STOKES' 1857 PAPER

### 3.1 Introduction

Having established in his 1850 paper that the arbitrary constants in the solution of a certain class of linear differential equations must change discontinuously, Stokes returns in this paper to the problem of determining the exact location and nature of the discontinuity. The paper[39] is entitled "On the discontinuity of arbitrary constants which appear in divergent developments."

### 3.2 Example of methods

He starts with a simple example through which he demonstrates methods applicable to more complicated functions. He first considers the integral

$$
\begin{equation*}
u(a)=2 \int_{b}^{\infty} \exp \left(-x^{2}\right) \sin (2 a x) d x \tag{3.1}
\end{equation*}
$$

which has a power series expansion

$$
\begin{equation*}
u=2 a-\frac{(2 a)^{3}}{2.3}+{\frac{(2 a)^{5}}{3.4 .5}}^{5}-\ldots \tag{3.2}
\end{equation*}
$$

Both the integral and the series in ascending powers are convergent for all finite complex values of $a$.

Either of the above expressions for $u$, satisfy the differential equation

$$
\begin{equation*}
\frac{d u}{d a}+2 a u=2 \tag{3.3}
\end{equation*}
$$

We take the initial condition, $u(0)=0$, which gives as a solution for this equation

$$
u=2 \exp \left(-a^{2}\right) \int_{b}^{a} \exp \left(a^{2}\right) d a
$$

$$
\begin{equation*}
=2 \exp \left(-a^{2}\right)\left\{a+\frac{a^{3}}{1.3}+\frac{a^{5}}{1.2 \cdot 5}+\frac{a^{7}}{1 \cdot 2 \cdot 3 \cdot 7}+\ldots . \cdot\right\} \tag{3.4}
\end{equation*}
$$

Both series given in (3.2) and (3.4) give a unique (and determinate) value for $u$ for all values of $a$, real or imaginary. Both are also ultimately convergent but begin by diverging rapidly when $a$ is large. Stokes therefore sets himself the task of finding a solution in descending powers of $a$ which can be used for large values of $a$.

From (3.3) he deduces that when the real part of $a^{2}$ is positive, the most important terms are $2 a u$ and 2. This means that the leading term of the series representing the differential equation is $\frac{1}{a}$. In this example Stokes is making explicit use of the procedure now referred to as the method of dominant balance. He concludes that it is therefore " readily apparent" that the series is given by

$$
\begin{equation*}
u=\frac{1}{a}+\frac{1}{2 a^{3}}+\frac{1.3}{2^{2} a^{5}}+\ldots \tag{3.5}
\end{equation*}
$$

In his 1850 paper, he had described a method of arriving at a series in descending powers for the Airy function [see (2.10)]. It can therefore be deduced that equation (3.5) was arrived at using the same procedure. This procedure is set out below.

Write $u$ as a series in descending powers of $a$ multiplied by the leading term $a^{-1}$.

$$
\begin{align*}
& u=a^{-1}\left(A^{\prime} a^{\alpha}+B^{\prime} a^{\beta}+C^{\prime} a^{\gamma}+\ldots \ldots .\right)  \tag{3.6}\\
& \text { where } A^{\prime}, B^{\prime}, C^{\prime}, \ldots \quad \alpha, \beta, \gamma \ldots \text { are constants to be determined. }
\end{align*}
$$

Differentiating this expression and substituting into the differential equation (3.3) gives

$$
\begin{align*}
& \alpha A^{\prime} a^{\alpha-2}+\beta B^{\prime} a^{\beta-2}+\gamma C^{\prime} a^{\gamma-2}+\ldots-A^{\prime} a^{\alpha-2}-B^{\prime} a^{\beta-2}-C^{\prime} a^{\gamma-2}-  \tag{3.7}\\
& \ldots . .+2 A^{\prime} a^{\alpha}+2 B^{\prime} a^{\beta}+2 C^{\prime} a^{\gamma}+\ldots=2 .
\end{align*}
$$

In order that the powers of $a$ will descend, we must have $\alpha=0, \beta=$ $\alpha-2, \gamma=\beta-2, \ldots$. So from the series in (3.6) we obtain

$$
\begin{align*}
& 2 A^{\prime}+2 B^{\prime} a^{-2}+2 C^{\prime} a^{-4}-\left(\alpha A^{\prime}+A^{\prime}\right) a^{-2}-\left(2 \beta B^{\prime}+B^{\prime}\right) a^{-4}  \tag{3.8}\\
& -\left(4 \gamma C^{\prime}-C^{\prime}\right) a^{-6}+\ldots=2 .
\end{align*}
$$

When we equate the coefficients of terms of like powers this gives

$$
\begin{equation*}
A^{\prime}=1, \quad B^{\prime}=1 / 2, \quad C^{\prime}=3 / 4, \ldots \tag{3.9}
\end{equation*}
$$

Substituting these values for $A^{\prime}, B^{\prime}, C^{\prime}, \ldots \quad \alpha, \beta, \gamma, \ldots$ in equation (3.6) gives the series in descending powers, as given by Stokes, set out in (3.5).

This solution in (3.5) is only a particular solution of the differential equation (3.3). To get the complete solution we must add any solution of

$$
\begin{equation*}
\frac{d u}{d a}+2 a u=0 \tag{3.10}
\end{equation*}
$$

so that the complete solution of the differential equation (3.3) is given by

$$
\begin{equation*}
u=C\left(\exp \left(-a^{2}\right)\right)+\frac{1}{a}+\frac{1}{2 a^{3}}+\frac{1.3}{2^{2} a^{5}}+\ldots \tag{3.11}
\end{equation*}
$$

Having arrived at this solution in descending powers, Stokes now explains why the constant $C$ must be discontinuous.

In the solutions given by the equations (3.2) and (3.4), the function $u$ is shown to be odd, changing sign with $a$. However in the solution given (3.11), $u$ is expressed as the sum of two functions, the first even and the
second odd. The even term is $C\left(\exp \left(-a^{2}\right)\right)$. If we let $C$ equal zero then the even term would disappear and we would be left with the desired odd function. However, as we shall see later, $C$ does in fact have a value different from zero. Therefore, in order that the solution in (3.11) may represent an odd function, $C$ must change sign with $a$.

To investigate how the arbitrary constant $C$ can change with $a$, Stokes considers the general complex form of $a$

$$
a=r(\cos \theta+i \sin \theta) .
$$

Since the divergent series in (3.11) is valid when $r$ is large we must suppose that the sign of $a$ is changed by increasing $\theta$ by $\pm \pi$.
Therefore if we knew the value of $C$ for say, $\alpha \leq \theta \leq \alpha+\pi$, then we would also know its value for $\alpha+\pi<\theta<\alpha+2 \pi$.

This is sufficient since we can always add $\pm n \pi$ to $a$ without changing the value of $u$ as can be seen from the ascending power series (3.2) and (3.4).

When $r$ is large the series in (3.11) begins by converging rapidly but ends, no matter how large $r$ is, by diverging with increasing rapidity. However, it may be used for calculations provided we truncate it before the terms get large again. To show the legitimacy of this Stokes observes that if we stop at the nth term

$$
u_{n}=\frac{1.3 .5 \cdot .(2 n-1)}{2^{n} a^{2 n+1}}
$$

then the value of $u$ obtained will satisfy not (3.3) but

$$
\begin{equation*}
\frac{d u_{n}}{d a}+2 a u_{n}=2-A_{n}, \tag{3.12}
\end{equation*}
$$

where $A_{n}=\frac{1.3 .5 \cdot .(2 n+1)}{2^{n} a^{2 n+2}}$.

It is important to note that Stokes here recognises that the error on truncation is of the order of the last retained term.

He concludes that if $a$ is very large and if we stop where the moduli of the terms have a minimum, then the modulus of $A_{n}$ will be very small in comparison with the other terms in (3.3). Therefore, in general $A_{n}$ can be neglected and we can use the series given in (3.11), when truncated in the vicinity of the least term as a near approximation to $u$.

But to this we must add an important restriction. We can use the series in (3.11) with the arbitrary constant $C$ for $0<\theta<\pi$, and we can use (3.11) multiplied by a different constant $C^{*}$ for $\pi<\theta<2 \pi$, but we must not pass from one sector to the other retaining the same constant. As we have seen, the constant changes sign when $\theta$ is increased by $\pi$ and therefore $C=-C^{*}$. Also, since the value of $u$ is unchanged when $\theta$ is increased by $2 n \pi$, Stokes concludes that in order to make (3.11) generally applicable, it is necessary only to change the sign of the constant when $\theta$ passes through 0 or a multiple of $\pi$.

### 3.3 Evaluation of the least term

Stokes then proceeds to find the location and magnitude of the least term. To this end he expresses the modulus of the general term of (3.11) in terms of the gamma function as

$$
\begin{equation*}
\mu_{n}=\frac{\Gamma(n+1 / 2)}{\Gamma(1 / 2) r^{2 n+1}} . \tag{3.13}
\end{equation*}
$$

Using Stirling's formula, valid for large values of $x$,

$$
\Gamma(x+1) \cong \sqrt{2 \pi x}\left(\frac{x}{e}\right)^{x},
$$

he derives the following approximate expression for $\mu_{n}$, the modulus of the nth term

$$
\begin{equation*}
\mu_{n}=\sqrt{2}\left(n-\frac{1}{2}\right)^{n} e^{-n+1 / 2} r^{-2 n-1} . \tag{3.14}
\end{equation*}
$$

Then using the further approximation, valid for large $n$,

$$
(n+c)^{n} \cong n^{n} e^{c},
$$

he obtains

$$
\begin{equation*}
\mu_{n} \cong \sqrt{2} n^{n} e^{-n} r^{-2 n-1} \tag{3.15}
\end{equation*}
$$

which gives the ratio of consecutive terms as

$$
\begin{equation*}
\frac{\mu_{n+1}}{\mu_{n}} \cong \frac{n}{r^{2}} \tag{3.16}
\end{equation*}
$$

as is obvious (and noted by Stokes) from (3.13).
When $r$ is large this ratio becomes very close to $l$ for a great number of terms in the neighbourhood of where the modulus is a minimum
i.e. in the neighbourhood of the least term. Since $\frac{n}{r^{2}} \cong 1$ at the least term we can find where in the series this term occurs by letting

$$
\begin{equation*}
n=r^{2} . \tag{3.17}
\end{equation*}
$$

in the expression for the general term given by (3.15) we find that the modulus of the least term, $\mu$, is given by

$$
\begin{equation*}
\mu \cong \frac{\sqrt{2}}{r\left(\exp \left(r^{2}\right)\right)} . \tag{3.18}
\end{equation*}
$$

These remarkably simple formulae, deduced by Stokes, for finding the position of and the magnitude of the least term of an asymptotic series yield a very high level of accuracy for large or even moderately large values of the variable. We now verify this accuracy with the help of the symbolic manipulation computer package, Maple.

Let us consider the series whose general term is expressed by (3.13). Recall that r denotes the modulus of a . If we let $r=5$, then according to Stokes' equation (3.17), the least term should occur when $n=25$.

The least term of a series occurs when the derivative, with respect to $n$, of the general term, $\mu_{n}$, equals zero. Using this method with the assistance of Maple gives the minimum value of $n$ as $n=24.99833363$.

Furthermore, when the terms of the series are listed with $r=5$, we find that the 25 th. term is in fact the term with the minimum modulus and is of magnitude $3.921562936 \times 1 \boldsymbol{0}^{-12}$, as can be seen in Table 3.1 below, where the terms have been rounded off to 2 decimal places.

It is interesting to compare this with Stokes' formula (3.20) above, which gives the magnitude of the least term as $\mu=3.928103712 \times 10^{-12}$.

| $\mathbf{n}$ |  | $\mathbf{n}$ |  | $\mathbf{n}$ |  | $\mathbf{n}$ |  | $\mathbf{n}$ |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | $4.00 \times 10^{-3}$ | 21 | $5.50 \times 10^{-12}$ | 41 | $2.84 \times 10^{-10}$ | 61 | $3.89 \times 10^{-4}$ | 81 | $4.24 \times 10^{5}$ |
| 2 | $2.40 \times 10^{-4}$ | 22 | $4.73 \times 10^{-12}$ | 42 | $4.72 \times 10^{-10}$ | 62 | $9.57 \times 10^{-4}$ | 82 | $1.38 \times 10^{6}$ |
| 3 | $2.40 \times 10^{-5}$ | 23 | $4.26 \times 10^{-12}$ | 43 | $8.02 \times 10^{-10}$ | 63 | $2.39 \times 10^{-3}$ | 83 | $4.56 \times 10^{6}$ |
| 4 | $3.36 \times 10^{-6}$ | 24 | $4.00 \times 10^{-12}$ | 44 | $1.40 \times 10^{-9}$ | 64 | $6.08 \times 10^{-3}$ | 84 | $1.52 \times 10^{7}$ |
| 5 | $6.05 \times 10^{-7}$ | $\mathbf{2 5}$ | $\mathbf{3 . 9 2 \times 1 0 ^ { - 1 2 }}$ | 45 | $2.48 \times 10^{-9}$ | 65 | $1.57 \times 10^{-2}$ | 85 | $5.15 \times 10^{7}$ |
| 6 | $1.33 \times 10^{-7}$ | 26 | $4.00 \times 10^{-12}$ | 46 | $4.52 \times 10^{-9}$ | 66 | $4.11 \times 10^{-2}$ | 86 | $1.76 \times 10^{8}$ |
| 7 | $3.46 \times 10^{-8}$ | 27 | $4.24 \times 10^{-12}$ | 47 | $8.41 \times 10^{-9}$ | 67 | $1.09 \times 10^{-1}$ | 87 | $6.10 \times 10^{8}$ |
| 8 | $1.04 \times 10^{-8}$ | 28 | $4.66 \times 10^{-12}$ | 48 | $1.60 \times 10^{-8}$ | 68 | $2.95 \times 10^{-1}$ | 88 | $2.13 \times 10^{9}$ |
| 9 | $3.53 \times 10^{-9}$ | 29 | $5.32 \times 10^{-12}$ | 49 | $3.10 \times 10^{-8}$ | 69 | $8.09 \times 10^{-1}$ | 89 | $7.55 \times 10^{9}$ |
| 10 | $1.34 \times 10^{-9}$ | 30 | $6.27 \times 10^{-12}$ | 50 | $6.14 \times 10^{-8}$ | 70 | 2.25 | 90 | $2.70 \times 10^{10}$ |
| 11 | $5.63 \times 10^{-10}$ | 31 | $7.65 \times 10^{-12}$ | 51 | $1.24 \times 10^{-7}$ | 71 | 6.34 | 91 | $9.79 \times 10^{10}$ |
| 12 | $2.60 \times 10^{-10}$ | 32 | $9.64 \times 10^{-12}$ | 52 | $2.55 \times 10^{-7}$ | 72 | $1.81 \times 10^{1}$ | 92 | $3.58 \times 10^{11}$ |
| 13 | $1.30 \times 10^{-10}$ | 33 | $1.25 \times 10^{-11}$ | 53 | $5.36 \times 10^{-7}$ | 73 | $5.26 \times 10^{1}$ | 93 | $1.33 \times 10^{12}$ |
| 14 | $7.00 \times 10^{-11}$ | 34 | $1.68 \times 10^{-11}$ | 54 | $1.15 \times 10^{-6}$ | 74 | $1.55 \times 10^{2}$ | 94 | $4.96 \times 10^{12}$ |
| 15 | $4.06 \times 10^{-11}$ | 35 | $2.32 \times 10^{-11}$ | 55 | $2.50 \times 10^{-6}$ | 75 | $4.61 \times 10^{2}$ | 95 | $1.87 \times 10^{13}$ |
| 16 | $2.52 \times 10^{-11}$ | 36 | $3.30 \times 10^{-11}$ | 56 | $5.55 \times 10^{-6}$ | 76 | $1.39 \times 10^{3}$ | 96 | $7.16 \times 10^{13}$ |
| 17 | $1.66 \times 10^{-11}$ | 37 | $4.81 \times 10^{-11}$ | 57 | $1.26 \times 10^{-5}$ | 77 | $4.26 \times 10^{3}$ | 97 | $2.76 \times 10^{14}$ |
| 18 | $1.16 \times 10^{-11}$ | 38 | $7.21 \times 10^{-11}$ | 58 | $2.89 \times 10^{-5}$ | 78 | $1.32 \times 10^{4}$ | 98 | $1.08 \times 10^{15}$ |
| 19 | $8.60 \times 10^{-12}$ | 39 | $1.11 \times 10^{-10}$ | 59 | $6.76 \times 10^{-5}$ | 79 | $4.14 \times 10^{4}$ | 99 | $4.25 \times 10^{15}$ |
| 20 | $6.71 \times 10^{-12}$ | 40 | $1.75 \times 10^{-10}$ | 60 | $1.61 \times 10^{-4}$ | 80 | $1.32 \times 10^{5}$ | 100 | $1.69 \times 10^{17}$ |

Table 3.1: The first 100 terms of the series (3.13) for $r=5$, computed using Maple.

Stokes points out that if we knew precisely at which term it was best to stop, his expression for $\mu$ would be " a measure of the uncertainty to which we were liable " in using the divergent series, (3.11), with truncation. Modern day analysts would call this the truncation error. It is clear that we must stop somewhere around the least term in order to get the best possible approximation. However, the number of terms close in magnitude to this least term will increase as $n$ increases.

Stokes observes that $m$, the number of terms of nearly equal magnitude in the vicinity of the least term, seems to increase as $n$ increases but not as fast. The following Maple plots demonstrate this. We see in Figure 3.1 that when $r=2$, the least term occurs at $n=4$ as expected.


Figure 3.1: Maple plot for the first 10 terms of the series (3.11), with $r=2$.

We see from Figure 3.2 that when $r$ is increased to 4 , the least term occurs at $n=16$ and the number of terms of almost equal magnitude has also increased.


Figure 3.2: Maple plot for the terms $5-30$ of the series (3.11), with $r=4$, where $7 \mathrm{e}-006$ means $7 \times 10^{-6} \mathrm{etc}$.

Figure 3.3 shows a plot of 20 terms around the least term, $n=100$, for $r=10$. In this case the magnitude of the terms in this vicinity is so small that it is necessary to show them on a logarithmic plot.


Figure 3.3: Maple logplot for the terms $90-110$ of the series (3.11) when $r=10$, where $1 \mathrm{e}-43$ means $1 \times 10^{-43} \mathrm{etc}$.

### 3.4 Resumming the tail of the divergent series

Stokes uses this behaviour of the divergent series in the vicinity of the least term to find an expression for the remainder on truncation using the difference operator $\Delta$. Suppose we sum the series (3.11) as far as the term $u_{n-1}$. The remainder of the series can then be written as

$$
\begin{equation*}
R_{n}=\exp \{(2 n+1) i \theta\}\left[u_{n}+\exp (-2 i \theta) u_{n+1}+\exp (-4 i \theta) u_{n+2}+\ldots\right] \tag{3.19}
\end{equation*}
$$

Letting $D$ or $1+\Delta$ represent the shift operator which takes $u_{n}$ to $u_{n+1}$, then

$$
R_{n}=\exp \{(2 n+1) i \theta\}\left[\begin{array}{l}
1+D\{\exp (-2 i \theta)\}  \tag{3.20}\\
+D^{2}\{\exp (-4 i \theta)\}+\ldots .
\end{array}\right] u_{n}
$$

which, using the binomial theorem, may be written as

$$
\begin{equation*}
R_{n}=\exp \{(2 n+1) i \theta\}[1-D\{\exp (-2 i \theta)\}]^{-1} u_{n} \tag{3.21}
\end{equation*}
$$

and since $D=1+\Delta$, this becomes

$$
\begin{equation*}
R_{n}=\exp \{(2 n+1) i \theta\}[1-\exp (-2 i \theta)-\Delta \exp (-2 i \theta)]^{-1} u_{n} . \tag{3.22}
\end{equation*}
$$

Now since $1-\exp (-2 i \theta)=2 \sin \left\{\exp i\left(\frac{\pi}{2}-\theta\right)\right\}$ we have

$$
R_{n}=\exp \{(2 n+l) i \theta\}\left[\begin{array}{r}
2 \sin \left\{\exp i\left(\frac{\pi}{2}-\theta\right)\right\}-  \tag{3.23}\\
\Delta \exp (-2 i \theta)
\end{array}\right]^{-1} u_{n} .
$$

The term in square brackets can be rearranged as

$$
\begin{equation*}
(2 \sin \theta)^{-1} \operatorname{expi}\left(\theta-\frac{\pi}{2}\left\{1-(2 \sin \theta)^{-1} \exp \left(-i\left(\frac{\pi}{2}+\theta\right)\right) \Delta\right\}\right. \tag{3.24}
\end{equation*}
$$

which on writing $q$ for $(2 \sin \theta)^{-1}$ becomes

$$
\begin{equation*}
q\left(\operatorname{expi}\left(\theta-\frac{\pi}{2}\right)\right)\left\{1-q\left(\exp \left(-i\left(\frac{\pi}{2}+\theta\right)\right)\right) \Delta\right\} . \tag{3.25}
\end{equation*}
$$

Finally, using the binomial theorem, we obtain from (3.23)

$$
R_{n}=\exp \{(2 n+1) i \theta\}\left[\begin{array}{c}
q \exp i\left(\theta-\frac{\pi}{2}\right) u_{n}+q^{2} \exp (-\pi i) \Delta u_{n}  \tag{3.26}\\
+q^{3} \exp \left(-\left(\theta+\frac{3 \pi}{2}\right) i\right) \Delta^{2} u_{n}+\ldots
\end{array}\right] .
$$

Here, it should be noted, Stokes is using Euler's method for summing slowly convergent series.

If $r$ is very large and if $u_{n}$ belongs to that part of the series where successive terms are almost equal, $\Delta u_{n}, \Delta^{2} u_{n}$ etc. will decrease with great rapidity. Thus Stokes has succeeded in transforming the series, at the point where it begins to diverge into one which is at first rapidly convergent.

We note from (3.15) that $\mu_{n}$ has controlling behaviour $\exp \left(-r^{2}\right)$, where $r$ is the modulus of $a$. Therefore, the expression for the remainder given by (3.26) shows that the error on truncation will become very small compared to the least term of the series and much smaller still in comparison with $\exp \left(-a^{2}\right)$, the expression which is multiplied by the arbitrary constant $C$.

We stress that Stokes has proved the remarkable result that if the divergent series is truncated at the least term, the error is not merely of the order of an inverse power of $r$, but exponentially small like $\exp \left(-r^{2}\right)$. This is in contrast to the Poincaré asymptotics which can only inform us that the error is algebraically small, in fact $O\left(r^{-2 n-1}\right)$.

### 3.5 Application to Airy's integral

Stokes now returns to the differential equation derived from Airy's integral and proceeds to apply the methods established above to its solution.

Writing $-3 x$ for $x$ in the related differential given in (2.3), he obtains

$$
\begin{equation*}
\frac{d^{2} U}{d x^{2}}-9 x U=0 \tag{3.27}
\end{equation*}
$$

Due to this substitution, the solution for positive $x$ will now apply to the dark side of the caustic and the solution for negative $x$ to the bright side. The general solution in ascending powers is given by

$$
\begin{aligned}
U= & A\left\{1+\frac{9 x^{3}}{2 \cdot 3}+\frac{9^{2} x^{6}}{2 \cdot 3 \cdot 5 \cdot 6}+\frac{9^{3} x^{9}}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}+\ldots .\right\} \\
& +B\left\{x+\frac{9 x^{4}}{3.4}+\frac{9^{2} x^{7}}{3 \cdot 4 \cdot 6 \cdot 7}+\frac{9^{3} x^{10}}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10}+\ldots . .\right\} .
\end{aligned}
$$

Both series multiplying the constants above are convergent and give a unique solution for all finite values of $x$. On the other hand the asymptotic solution which has the form

$$
\begin{align*}
U & =C x^{-\frac{1}{4}} e^{-2 x^{\frac{3}{2}}}\left\{1-\frac{1 \cdot 5}{1 \cdot 144 x^{\frac{3}{2}}}+\frac{1 \cdot 5 \cdot 7 \cdot 11}{1 \cdot 2 \cdot 144^{2} x^{3}}-\frac{1 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17}{1 \cdot 2 \cdot 3 \cdot 144^{3} x^{\frac{9}{2}}}+\ldots\right\} \\
& +D x^{-\frac{1}{4}} e^{2 x^{\frac{3}{2}}}\left\{1+\frac{1 \cdot 5}{1 \cdot 144 x^{\frac{3}{2}}}+\frac{1 \cdot 5 \cdot 7 \cdot 11}{1 \cdot 2 \cdot 144^{2} x^{3}}+\frac{1 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17}{1 \cdot 2 \cdot 3 \cdot 144^{3} x^{\frac{9}{2}}}+\ldots\right\} \tag{3.29}
\end{align*}
$$

does not have unique values in the complex plane if the constants $C$ and $D$ remain unchanged. If the argument of $x$ is increased by $2 \pi$ the value of $U$ does not recur as it does in the solution in ascending powers. To ensure that the values taken by $U$ at $\arg x=0,2 \pi$ are the same as they are in practice, the constants must change.

Stokes sets out to specify the change and to find $C, D$ in terms of $A, B$. He writes the asymptotic solution (3.29) in the form

$$
\begin{equation*}
U=C x^{-\frac{1}{4}} f_{1}(x)+D x^{-\frac{1}{4}} f_{2}(x) \tag{3.30}
\end{equation*}
$$

Then taking the modulus of $x$ as constant, the solution can be written as a function of the argument of $x$, which Stokes denotes by $\theta$, as

$$
\begin{equation*}
U=C x^{-\frac{1}{4}} F_{1}(\theta)+D x^{-\frac{1}{4}} F_{2}(\theta) \tag{3.31}
\end{equation*}
$$

From (3.29) it is clear that

$$
\begin{equation*}
F_{l}\left(\theta+\frac{2 \pi}{3}\right)=F_{2}(\theta) \quad \text { and } \quad F_{2}\left(\theta+\frac{2 \pi}{3}\right)=F_{1}(\theta) . \tag{3.32}
\end{equation*}
$$

He defines as "superior" that term in (3.30) in which the real part of the exponential is positive. The term in which the real part of the exponential is negative is called "inferior".

Stokes represents the variation of $f_{1}(x)$ and $f_{2}(x)$ with the argument $\theta$ on an ingenious circular diagram, shown here in Figure 3.4. At any angle $\theta$, the point representing the "superior" function is found by measuring a distance outwards along the radius vector from the circumference of the dotted circle. This distance is proportional to $\cos (3 \theta / 2)$, the real part of the index of the exponential. The point representing the "inferior" function at the same angle, $\theta$, is found by measuring the same distance inwards along the same radius from the same point on the circumference. The locus of these points constitute paths representing the functions. Both paths cut the circle at the same point at intervals of $120^{\circ}$.


Figure 3.4: Stokes' diagram showing how the functions $f_{l}(x)$ and $f_{2}(x)$ vary with $\theta$.

Each function "runs into itself" after two revolutions and becomes the other function after one revolution. When the corresponding curve is inside the circle it is "inferior" (subdominant) and when the corresponding curve is outside the circle it is "superior" or dominant. It is clear from the diagram that maximum dominance/ subdominance occurs at the rays corresponding to $\theta=0, \frac{2 \pi}{3}, \frac{4 \pi}{3}$. These rays are now known as Stokes' lines. It is also clear from the diagram that the transitions from superiority to inferiority occur at $\theta=\frac{\pi}{3}, \pi,-\frac{\pi}{3}$, now referred to as antiStokes' lines.

In order to find out in what situations it is possible for the constants $C$ or $D$ to alter discontinuously Stokes examines what happens if we truncate in the vicinity of the least term.

He writes the modulus of the general term, of the series multiplying the exponential, in terms of gamma functions, with $\rho$ representing the modulus of $x$

$$
\frac{\Gamma\left(k+\frac{l}{6}\right) \Gamma\left(k+\frac{5}{6}\right)}{\Gamma\left(\frac{l}{6}\right) \Gamma\left(\frac{5}{6}\right) \Gamma(k+l)\left(4 \rho^{\frac{3}{2}}\right)^{k}}
$$

Then using Stirling's formula to approximate the gamma function for large values of the variable, he finds that the modulus, $\mu$, of the least term of the series is given approximately by

$$
\begin{equation*}
\mu \approx(2 \pi k)^{-\frac{1}{2}} \frac{k^{k}}{\left(4 \rho^{\frac{3}{2}} e\right)^{k}} \tag{3.34}
\end{equation*}
$$

and for large values of $k$, the modulus $\mu_{k}$ of the $k$ th term satisfies

$$
\begin{equation*}
\frac{\mu_{k+1}}{\mu_{k}} \approx \frac{k}{4 \rho^{\frac{3}{2}}} . \tag{3.35}
\end{equation*}
$$

Stokes here shows a great insight into the nature of the series with which he is working. For large values of $\rho$, at the part of the series where the least term occurs, several terms are of very nearly equal magnitude. This allows him to estimate that the least term will occur at approximately when

$$
\begin{equation*}
k=4 \rho^{\frac{3}{2}} \tag{3.36}
\end{equation*}
$$

which in turn gives the modulus of the least term as

$$
\begin{equation*}
\mu \approx(2 \pi k)^{-\frac{1}{2}} e^{-4 \rho^{\frac{3}{2}}} . \tag{3.37}
\end{equation*}
$$

When $\mu$ is multiplied by the modulus of the exponential in the superior term, we will get

$$
\begin{equation*}
(2 \pi k)^{-\frac{1}{2}} \exp \left[-\left(4 \mp 2 \cos \frac{3 \theta}{2}\right)^{\frac{3}{2}} \rho^{\frac{3}{2}}\right] \tag{3.38}
\end{equation*}
$$

where - is to be taken when $\cos \left(\frac{3 \theta}{2}\right)$ is positive, + to be taken when it is negative.

Stokes has earlier shown that the remainder on truncation is of the order of the last retained term. This now enables him to show that when the series is truncated in the neighbourhood of the least term, the magnitude of the inferior term is less than the error on truncation of the superior term. The dominance of the superior term is greatest when all its terms are positive. This occurs when $\pm \cos \frac{3 \theta}{2}=1$, which defines the positions of the what are now called "Stokes' lines". Stokes concludes, therefore, that the coefficient of the inferior term can change discontinuously at these rays, and only there, without affecting the solution of the differential equation. Moreover he argues that the coefficient of the other term cannot change as long as it remains superior. He now writes the full asymptotic solution, $U$, with sectors of validity included as

$$
\begin{align*}
& \left(-\frac{4 \pi}{3} t o \frac{4 \pi}{3}\right) C x^{-\frac{1}{4}} e^{-2 x^{\frac{3}{2}}}\left\{\begin{array}{l}
1-\frac{1 \cdot 5}{1 \cdot 144 x^{3 / 2}}+\frac{1 \cdot 5 \cdot 7 \cdot 11}{1 \cdot 2 \cdot 144^{2} x^{3}} \\
-\frac{1 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17}{1 \cdot 2 \cdot 3 \cdot 144^{3} x^{9 / 2}}+\ldots
\end{array}\right\} \\
& +\left(-\frac{2 \pi}{3} t o 2 \pi\right) D x^{-\frac{1}{4}} e^{2 x^{\frac{3}{2}}\left\{\begin{array}{l}
1+\frac{1 \cdot 5}{1 \cdot 144 x^{3 / 2}}+\frac{1 \cdot 5 \cdot 7 \cdot 11}{1 \cdot 2 \cdot 144^{2} x^{3}} \\
+\frac{1.5 \cdot 7 \cdot 11 \cdot 13 \cdot 17}{1 \cdot 2 \cdot 3 \cdot 144^{3} x^{9 / 2}}+\ldots
\end{array}\right\}} . \tag{3.39}
\end{align*}
$$

Stokes explains that this notation is to be interpreted in the following way. The expression written after a sector is to be taken whenever $\theta \pm 2 n \pi$, $n=0,1,2,3 \ldots$ falls within these limits, where $\theta$ is the argument of $x$. This will happen once or twice depending on the value of the angle chosen. He points out that if the $D$, in (3.39) is put equal to zero, the solution $U$ will be "equal to "Airy's integral multiplied by an arbitrary constant where $x=-\frac{\pi}{2}^{\frac{2}{3}} \frac{m}{2}$.

When $\theta=0$, we have the exponentially declining solution belonging to the dark side of the caustic and when $\theta=\pi$ we have the oscillatory solution belonging to the bright side of the caustic. When $\theta=0$, the angle $\theta \pm 2 n \pi$ occurs only once in the sector of validity for equation (3.39). However, when $\theta=\pi$ the angle $\theta \pm 2 n \pi$ occurs twice in the same sector. Hence there is but one term in the solution to Airy's integral corresponding to the dark side of the caustic and two terms, which must be added together corresponding to the bright side. Thus we have an exponentially declining amplitude on the dark side and an oscillatory amplitude on the bright side. When $0<\theta<\frac{2 \pi}{3}$, the angle $\theta \pm 2 n \pi$ occurs only once in the sector of validity for equation (3.39), but as $\theta$ just exceeds $\frac{2 \pi}{3}$ the angle $\theta \pm 2 n \pi$ occurs twice, thus we have the birth of the second exponential as the ray at $\theta=\frac{2 \pi}{3}$ is crossed .

A Maple plot of the now well established Airy function is shown below in Figure 3.5. The plot of the intensity of light, which is proportional to the square of the amplitude is shown in Fig. IV on Airy's original plate in Figure 1.5.


Figure 3.5: Maple plot of the Airy function from $\mathrm{x}=-10$ to $\mathrm{x}=10$.

Hence, in his 1850 paper Stokes had shown that if the argument of $x$ is changed by $\pi$ in the solution for positive values of $x$ (bright side), and the unbounded term rejected, the expression for negative values of the parameter (dark side) is found. By his work in this paper he is satisfied that the above procedure is fully justified.

He next finds the constants $C$ and $D$ in terms of $A$ and $B$ which are the constants in the solution in ascending powers of $x$. This he does by expressing the solution of the differential equation as the sum of two definite integrals which he approximates for large values of the parameter, again by what is essentially the saddle point method. By comparing the result thus obtained with the leading behaviour of the asymptotic series and with the solution in ascending powers he is able to arrive at the following relationship

$$
\begin{align*}
& A=\pi^{-\frac{1}{2}} \Gamma\left(\frac{1}{3}\right)\left\{C+e^{-i \frac{\pi}{6}} D\right\} \\
& B=3 \pi^{-\frac{1}{2}} \Gamma\left(\frac{2}{3}\right)\left\{-C+e^{i \frac{\pi}{6}} D\right\} \tag{3.40}
\end{align*}
$$

In his 1868 paper, we shall see that, he develops a method of connecting the constants without having to evaluate the integrals.

### 3.6 Stokes' numerical example

In an appendix to [39], Stokes provides an example to verify the validity of the above process. The particular example he takes is the solution of Airy's integral, in which case $D=0$ in (3.39), because the unbounded term must be discarded on physical grounds. He takes two values of $x$ having the same modulus, 2, but occurring on either side of the Stokes' line at $120^{\circ}$, one at $90^{\circ}$ and the other at $150^{\circ}$. From (3.39) he deduces that when $\theta=90^{\circ}$ there will be no inferior term but when $\theta=150^{\circ}$ there will be an inferior and superior term. In the latter case $\theta \pm 2 n \pi$, ( $n=0,1,2,3 \ldots$ ) occurs twice within the prescribed limits $-\frac{4 \pi}{3}$ to $\frac{4 \pi}{3}$.
He compares the results for these two values of x with the corresponding results for the convergent series in ascending powers. These calculations are shown in Table 3.3 and Table 3.4 below.

In the case of $\theta=90^{\circ}$ he shows that by taking only the superior term in the asymptotic solution the results are in agreement up to the fifth place of decimals, whereas in the case of $\theta=150^{\circ}$ this would give rise to an error in the second place of decimals. Hence the arbitrary constant in the inferior term can be considered to be zero when $\theta=90^{\circ}$, but must be significant when $\theta=150^{\circ}$. He concludes, therefore, that the arbitrary constant in the inferior term must be allowed to be discontinuous.

In order to arrive at the close agreement achieved between the two forms of the solution he truncates the asymptotic series at the least term and he also finds the remainder on truncation by resumming the divergent tail by the method already described.

$$
\underline{x=2 .} \quad \theta=90^{\circ}
$$

| Convergent, ascending series: | $-14.98520+43.81047 \mathrm{i}$ |
| :--- | :--- |
| Asvmptotic series, truncated at least term: |  |
| Superior term | $-14.98520+43.81046 \mathrm{i}$ |
| Inferior term | No inferior term |

Table 3.3: Stokes' calculations for the Airy function with $x=2, \theta=90^{\circ}$, comparing the results obtained from the convergent series and from the asymptotic series.

$$
x=2 . \quad \theta=150^{\circ}
$$

| Convergent, ascending series: | $-45.44882-8.92867 \mathrm{i}$ |
| :--- | :--- |
| Asymptotic series. truncated at least term: |  |
| Superior term | $-45.43360-8.92767 \mathrm{i}$ |
| Inferior term | $-0.01524-0.00100 \mathrm{i}$ |
| Sum of series | $-45.44884-8.92867 \mathrm{i}$ |

Table 3.4: Stokes' calculations for the Airy function with $x=2, \theta=150^{\circ}$, comparing the results obtained from the convergent series and from the asymptotic series.

## CHAPTER 4

## STOKES' 1868 PAPER

### 4.1 Introduction

In this paper, "Supplement to a paper on the discontinuity of arbitrary constants which appear in divergent developments"[40], Stokes applies the asymptotic method developed in previous papers to the solution of a more general class of second order ordinary differential equations that frequently arise in physical problems. Their solutions can be expressed either by an ascending or descending series, the former valid for small, the latter valid for large values of the variable. The analytic determination of the constants multiplying the divergent series is the main difficulty in this process.

The constants multiplying the descending series have been shown to be discontinuous, their values changing suddenly when certain rays in the complex plane are traversed. As he stated in an earlier paper it is not essential to find an analytic relationship between the constants. A numerical relationship can be found by summing the series numerically for different values of the variable in the ascending and descending series separately and equating the results.

In the case of the second order differential equations with which he is concerned, there are two arbitrary constants regarded as known and the two unknown constants are to be expressed in terms of these. This would involve evaluating the independent variable for two different values of the dependent variable in both series and equating the results. The unknown constant can then be found by solving the resulting system of simultaneous equations.

One major consideration in this paper is to find a more efficient means of connecting the coefficients. His achievement is to be able to connect them by calculating the dependent variable for only one value of the independent variable. Also there is no longer a need to solve a simultaneous equation. According to Stokes, this saving of numerical calculations "is not to be despised seeing that the coefficients involved are complex imaginaries." This numerical method will be of use if an analytic relationship between the constants in the convergent and divergent series cannot be found.

### 4.2 Solution of differential equation in general form

The differential equations considered by Stokes in this series of papers have mainly been of the general form

$$
\begin{equation*}
\frac{d^{2} y}{d x^{2}}+\frac{A}{x} \frac{d y}{d x}+\frac{B}{x^{2}} y+C x^{a} y=0 \tag{4.1}
\end{equation*}
$$

which, by rescaling the independent variable, he reduces to the form

$$
\begin{equation*}
\frac{d^{2} y}{d x^{2}}+\frac{1}{x} \frac{d y}{d x}-\frac{v^{2}}{x^{2}} y=y \tag{4.2}
\end{equation*}
$$

where $v$ is a complex number. However, he asserts that in all the physical applications of which he is aware, $v$ is either zero or a finite real quantity. He therefore decides to concentrate mainly on the case when $v$ is real.

The complete solution, in ascending powers, of (4.2) is

$$
\begin{aligned}
y=A x^{v}\{1 & +\frac{x^{2}}{2(2+2 v)}+\frac{x^{4}}{2.4(2+2 v)(4+2 v)} \\
& \left.+\frac{x^{6}}{2.4 .6(2+2 v)(4+2 v)(6+2 v)}+\ldots\right\}
\end{aligned}
$$

$$
\begin{align*}
+B x^{v}\{1 & +\frac{x^{2}}{2(2-2 v)}+\frac{x^{4}}{2.4(2-2 v)(4-2 v)}  \tag{4.3}\\
& \left.+\frac{x^{6}}{2.4 .6(2-2 v)(4-2 v)(6-2 v)}+\ldots\right\} .
\end{align*}
$$

The solution in descending series is

$$
\begin{aligned}
y=C x^{-1 / 2} e^{x}\{1 & +\frac{l^{2}-4 v^{2}}{1.8 x}+\frac{\left(1^{2}-4 v^{2}\right)\left(3^{2}-4 v^{2}\right)}{1.2(8 x)^{2}} \\
& \left.+\frac{\left(1^{2}-4 v^{2}\right)\left(3^{2}-4 v^{2}\right)\left(5^{2}-4 v^{2}\right)}{1.2 .3(8 x)^{3}}+\ldots\right\}
\end{aligned}
$$

$$
\begin{align*}
&+D x^{-1 / 2} e^{-x}\left\{1-\frac{1^{2}-4 v^{2}}{1.8 x}+\frac{\left(1^{2}-4 v^{2}\right)\left(3^{2}-4 v^{2}\right)}{1.2(8 x)^{2}}\right.  \tag{4.4}\\
&\left.-\frac{\left(1^{2}-4 v^{2}\right)\left(3^{2}-4 v^{2}\right)\left(5^{2}-4 v^{2}\right)}{1.2 \cdot 3(8 x)^{3}}+\ldots\right\} .
\end{align*}
$$

Stokes points out that the latter expression takes no "peculiar form" when $v$ is an integer. However both series terminate when $2 v$ is an odd integer given by $2 k+1$, where $k$ is an integer. In this particular case the differential equation, (4.2), is of the form

$$
\frac{d^{2} y}{d x^{2}}+\frac{2}{x} \frac{d y}{d x}-\frac{k(k+1)}{x^{2}} y=y
$$

and has an exact finite solution.

Before continuing, it is necessary at this point to comment on some of the notation used by Stokes in this paper. It is here that he first uses $i$ for $\sqrt{-1}$. Recall, also, that $v$ here refers to the parameter in the differential equation given in (4.2) and $k$ is an integer. In order that $x^{ \pm \nu}, x^{-1 / 2}$ are single valued he specifies that if $x$ is taken as $\rho(\cos \theta+\mathrm{i} \sin \theta)$, then $x^{v}$ will be taken as $\rho^{v}(\cos (v \theta)+i \sin (v \theta))$ and not as $\rho^{v}(\cos (v \theta+2 k v \pi)+i \sin (v \theta+2 k v \pi))$.

### 4.3 Linking the constants

Stokes' aim now is to find an improvement in his previous method of connecting the coefficients of the ascending and descending series. He writes the ascending series in the form

$$
\begin{equation*}
y=A U+B V \tag{4.5}
\end{equation*}
$$

and the descending series solution as

$$
\begin{equation*}
y=C u+D v . \tag{4.6}
\end{equation*}
$$

He now considers what happens to the constants multiplying the functions $U, V, u, v$ as $\theta$ is increased by multiples of $\pi$. This has significance for understanding the dominance/subdominance of the terms of the asymptotic solution in the complex plane.

As a result, in linking up the constants in the ascending and the descending series he proves that the coefficient of the inferior term must remain unchanged if there is no superior term present.

This is arrived at as follows. It has already been shown in previous papers that the constants, $C$ and $D$, multiplying the series in the asymptotic solution are discontinuous. $C$ changes its value when $\theta$, the argument of $x$, passes through an odd multiple of $\pi$ and $D$ changes when $\theta$ passes through an even multiple of $\pi$. Let $C^{\prime}, C^{\prime \prime} \ldots$ denote what $C$ becomes and $D^{\prime}, D^{\prime \prime} \ldots$ denote what $D$ becomes. For a particular value $\theta^{\prime}$ between 0 and $\pi$ let the ascending series be given by (4.5) and the descending series by (4.6). When $\theta$ is increased by $\pi$ the functions multiplied by $A$ and $B$ are reproduced but with constant multipliers, $\mathrm{e}^{v \pi i}$ and $\mathrm{e}^{-v \pi \mathrm{i}}$, introduced. The functions multiplied by $C$ and $D$ reproduce each other with a multiplier $e^{-\pi i / 2}$ introduced.

Suppose $\theta=\theta^{\prime}+k \pi$, where $k$ is an integer, then we get the following relationships.
(4.7) for $0<\theta<\pi, \quad y=A U+B V=C u+D v$,

$$
\begin{array}{ll}
\text { for } \pi<\theta<2 \pi, & y=A e^{v \pi i} U+B e^{-v \pi i} V=C^{\prime} e^{-\pi i / 2} v+D e^{-\pi i / 2} u, \\
\text { for } 2 \pi<\theta<3 \pi, & y=A e^{2 v \pi i} U+B e^{-2 v \pi i} V=C^{\prime} e^{-\pi i} u+D^{\prime} e^{-\pi i} v .
\end{array}
$$

This process can be continued indefinitely with $C$ and $D$ changing alternately.

Because of the linearity of the differential equation in (4.2), $C$ and $D$ must be linear combinations of $A$ and $B$. They can therefore be written as

$$
\begin{align*}
& C=p A+q B  \tag{4.8}\\
& D=r A+s B .
\end{align*}
$$

Therefore it follows from the relationships in (4.7) that, $\mathrm{De}^{-\pi i / 2}, \mathrm{C}^{\prime} \mathrm{e}^{-\pi i / 2}$ are composed of $A e^{v \pi i}, B e^{-v \pi i}$ in the same way as $C, D$ are composed of $A$ and $B$. Similarly $C^{\prime} e^{-\pi i}, \quad D^{\prime} e^{-\pi i}$ are composed of $A e^{2 v \pi i}, B e^{-2 v \pi i}$ as $C, D$ are of $A, B$ and so on. This leads to the further set of relationships

$$
\begin{align*}
& D e^{-\pi i / 2}=p A e^{v \pi i}+q B e^{-v \pi i},  \tag{4.9}\\
& C^{\prime} e^{-\pi i / 2}=r A e^{v \pi i}+s B e^{-v \pi i}, \\
& C^{\prime} e^{-\pi i}=p A e^{2 v \pi i}+q B e^{-2 v \pi i}, \\
& D^{\prime} e^{-\pi i}=r A e^{2 v \pi i}+s B e^{-2 v \pi i}
\end{align*}
$$

and so on.

Now using (4.8) we can write

$$
\begin{equation*}
D=p A e^{(1 / 2+v) \pi i}+q B e^{(1 / 2-v) \pi i}=r A+s B . \tag{4.10}
\end{equation*}
$$

Since $A$ and $B$ are independent it follows from the latter equation that

$$
\begin{equation*}
r=p e^{(I / 2+v) \pi i}, \quad s=q e^{(1 / 2-v) \pi i} \tag{4.11}
\end{equation*}
$$

The constants $C, D, C^{\prime}, D^{\prime}$ etc. can now be expressed in terms of $A, B, p$, and $q$ in the following manner

$$
\begin{align*}
& C=p A+q B  \tag{4.12}\\
& D=p A e^{(1 / 2+v) \pi i}+q B e^{(1 / 2-v) \pi i}, \\
& C^{\prime}=p A e^{(l+2 v) \pi i}+q B e^{(1-2 v) \pi i}, \\
& D^{\prime}=p A e^{(3 / 2+3 v) \pi i}+q B e^{(3 / 2-3 v) \pi i}
\end{align*}
$$

and this can be continued indefinitely with an additional factor $e^{(l+2 v) \pi i}$ or $e^{(1-2 v) \pi i}$ whenever $C$ or $D$ changes. Recall that $C$ changes whenever $\theta$ increases by an odd multiple of $\pi$ and $D$ changes whenever $\theta$ increases by an even multiple of $\pi$. The above pattern also holds when $\theta$ decreases by multiples of $\pi$, with the factor $e^{-(1+2 v) \pi i}$ or $e^{-(1-2 v) \pi i}$ being introduced in this case.

Stokes has thus arrived at a means of determining the constants, $C, C^{\prime}$, $C^{\prime \prime} \ldots D, D^{\prime}, D^{\prime \prime} \ldots$, which appear in the asymptotic solution in terms of the constants $A$ and $B$ which multiply the convergent series. This now has
been reduced to finding the two constants $p$ and $q$ which depend on $v$ only.

As has already been pointed out when $2 v$ is an odd integer we get an exact solution which will have no discontinuity in the constants $C$ and $D$. This is consistent with the above results because in this case the factor $e^{ \pm(I \pm 2 v) p i}$ is equal to $l$ and so $C=C^{\prime}=C^{\prime \prime}=\ldots$ and $D=D^{\prime}=D^{\prime \prime} \ldots$.

Above all, the equations in (4.12) illustrate an extremely important aspect of Stokes' theory of the discontinuity of constants in the solution represented by the divergent series. The main problem for him was to reconcile the discontinuity of the constants with the requirement of a continuous solution to the differential equation. He has previously shown that the coefficient which changes is the one multiplying the series which is subdominant for the value of the argument where the change takes place. The complete solution can be approximated at this point by the dominant series, truncated at the least term. The error involved in this is comparable, according to Stokes, with the whole value of the subdominant series. Therefore, if there is no accompanying dominant series, the constant multiplying the subdominant series, which is liable to change cannot do so.

This can be deduced from equation (4.12), from which we have

$$
\begin{equation*}
C^{\prime}-C=2 i \cos (v \pi) D . \tag{4.13}
\end{equation*}
$$

It follows that if $D=0$, then $C=C^{\prime}$ and there is no change in the multiplier of the subdominant series. Stokes also points out the interesting fact that if $D$ is real then the discontinuity in $C$ as $\theta$ is increased by $\pi$ affects only its imaginary part.

This equation (4.13) is of further significance as it gives, in terms of $C$ and $D$, the new value of $C$ when $\theta$ is increased by $\pi$

$$
\begin{equation*}
C^{\prime}=C+2 i \cos (v \pi) D . \tag{4.14}
\end{equation*}
$$

The quantity $2 i \cos (v \pi)$ is, in modern terminology, known as Stokes' constant for the Bessel function according to Heading [14].

Stokes is now in a position to derive what he considers to be a very important relationship and the main object of this work. Using the fact that the constants $A$ and $B$ are independent he derives from the first equation in (4.7) and the first two in (4.12)

$$
\left.\begin{array}{l}
U=p\left(u+e^{(1 / 2+v) \pi i} v\right) \\
V=q\left(u+e^{(1 / 2-v) \pi i} v\right) \tag{4.15}
\end{array}\right\}
$$

This is important because it enables us to write the convergent series $U$ and $V$ from the ascending solution in terms of the divergent series $u$ and $v$ from the descending solution prior to determining the constants $C$ and $D$ which multiply the latter series. Moreover the complete determination of these arbitrary constants $C$ and $D$ now depends on $p$ and $q$, which are given separately by the two equations in (4.15). Therefore the arbitrary constants multiplying the descending series can be determined by calculating the four functions $U, V, u, v$ for just one value of the variable $x$ and equating the results. The necessity of elimination between two simultaneous equations is therefore avoided.

This is a very important result for Stokes because it means a significant reduction in the amount of numerical calculation involved in connecting the arbitrary constants of the two solutions, (4.3) which is valid at the origin and (4.4) which is valid at infinity.

## CHAPTER 5

## STOKES 1889 PAPER

### 5.1 Introduction

This paper is entitled " Note on the determination of arbitrary constants which appear as multipliers of semi-convergent series"[41]. In it Stokes returns, after a lapse of some twenty years, to consider the relationship between the arbitrary constants which appear respectively in the ascending and descending series solutions to the class of differential equations considered in his 1850,1857 and 1868 papers. In his previous work in this regard the connection between the arbitrary constants was made by getting a third solution in the form of a definite integral and relating each series solution to it. If such a solution cannot be found a numerical method must be used.

Here he points out that the numerical method is inelegant and laborious. He sets out to connect the arbitrary constants analytically, "by means of known transcendents", simply by considering the series themselves. Thus, the solution by definite integral and the numerical method are no longer required. Effectively, by considering the series solution in ascending powers in the neighbourhood of its numerically greatest term, he extracts information about the behaviour of the solution for large values of $x$. We shall give, later on, numerical examples of the acceleration of the rate of convergence which is inherent in Stokes' method.

### 5.2 The general asymptotic expression

The general term of the ascending series, after division by a certain power of the variable, can be given as

$$
\begin{equation*}
u_{n}=\frac{\Gamma(n+a) \Gamma(n+b) \ldots}{\Gamma(n+h) \Gamma(n+k) \ldots} X^{n} . \tag{5.1}
\end{equation*}
$$

where $X=\rho \exp (i \theta)$.
In the above expression there will always be one more gamma function in the denominator than in the numerator because the series is convergent.

In a previous work, the connection between the constants in the descending and ascending series was made by taking a value of $\theta$ which makes the exponential in the descending series as large as possible. Therefore in the expression given in (5.1) we take $\theta=0$ and accordingly $X$ will be regarded as real throughout.

Stokes argues that for large values of $X$ the series diverges for a great number of terms but we eventually get to a greatest term after which the series begins to converge. Let this greatest term be $u_{n_{1}}$. For a great number of terms in the neighbourhood of $u_{n_{1}}$, consecutive terms are almost equal in magnitude. As $X$ increases indefinitely so does the value of $n_{l}$ and the number of terms of almost equal magnitude in the vicinity of $u_{n_{t}}$ will also increase indefinitely. More precisely, he lets $0<\alpha, \beta \ll 1$. Define integers $n_{\alpha}=\left[(1-\alpha) n_{1}+1\right], n_{\beta}=\left[(1+\beta) n_{1}+1\right]$. He claims without proof that

$$
\sum_{n_{\alpha}}^{n_{B}} u_{n} \approx \sum_{0}^{\infty} u_{n} \quad \text { as } \rho \rightarrow \infty
$$

Therefore, for large values of $X$, the sum of that part of the series in the neighbourhood of the greatest term where the terms are of almost equal magnitude will ultimately give a good approximation to the sum of the entire series from 0 to $\infty$.

Stokes sets out to derive an analytic expression for the general term, $u_{n}$, valid for large values of $X$ from considering the convergent series valid at the origin. Using Stirling's formula for large values of $n$ he obtains

$$
\begin{align*}
\Gamma(n+a) & \approx \sqrt{2 \pi(n+a-1)}\left\{\frac{(n+a-1)^{n+a-1}}{\exp (n+a-1)}\right\} \\
& \approx \sqrt{2 \pi n}\left\{\frac{(n)^{n+a-1}\left(1+\frac{a-1}{n}\right)^{n}}{\exp (n+a-1)}\right\}  \tag{5.2}\\
& \approx \sqrt{2 \pi n}\left\{\frac{n^{n} n^{a-1}}{\exp (n)}\right\}
\end{align*}
$$

Thus, by letting $h+k+\ldots-a-b-\ldots=s$ and denoting by $t$ the excess number of gamma functions in the denominator over those in the numerator, the expression for $u_{n}$ given in (5.1) becomes ultimately

$$
\begin{equation*}
u_{n} \approx \frac{X^{n} n^{-s+1} \exp (t n)}{(2 \pi n)^{t / 2} n^{t n}} \text {, for large } n . \tag{5.3}
\end{equation*}
$$

Also, using the fact that

$$
\frac{\Gamma(n+a+1)}{\Gamma(n+a)}=n+a
$$

we can easily see from (5.1) that the ratio of successive late terms, that is when $n$ is large, is given by

$$
\begin{equation*}
\frac{u_{n+1}}{u_{n}} \approx n^{-t} X \tag{5.4}
\end{equation*}
$$

Since this ratio is approximately equal to 1 at the greatest term, $n=n_{1}$, we get

$$
\begin{equation*}
n_{1}^{l} \approx X . \tag{5.5}
\end{equation*}
$$

Stokes points out that within the portion of the series which he has decided it is sufficient to consider, that is, in the vicinity of the greatest term, we can replace $n$ by $n_{1}$ when we are dealing with any finite power of $n$. Therefore from (5.3) he gets

$$
\begin{equation*}
u_{n} \approx v\left(2 \pi n_{1}\right)^{-t / 2} n_{1}^{-s+t} \tag{5.6}
\end{equation*}
$$

where

$$
\begin{equation*}
v=\frac{X^{n} \exp (t n)}{n^{\prime n}} \tag{5.7}
\end{equation*}
$$

Now using (5.5) we can write

$$
\begin{equation*}
v \approx \frac{n_{1}^{n t} \exp (t n)}{n^{t n}} \tag{5.8}
\end{equation*}
$$

Stokes argues that if we consider $n$ to be continuous, we can write (5.8) as

$$
\begin{equation*}
w \approx t n+\operatorname{tn}\left(\log n_{1}-\log n\right) \tag{5.9}
\end{equation*}
$$

where $w=\log (v)$. When $n_{l}=n$ we get

$$
\begin{equation*}
w \approx t n_{1} \tag{5.10}
\end{equation*}
$$

It follows that

$$
\begin{equation*}
\frac{d w}{d n} \approx t\left(\log n_{1}-\log n\right) \tag{5.11}
\end{equation*}
$$

which vanishes when $n=n_{1}$ and

$$
\begin{equation*}
\frac{d^{2} w}{d n^{2}} \approx \frac{-t}{n}=\frac{-t}{n_{l}} \quad \text { when } n=n_{l} \tag{5.12}
\end{equation*}
$$

Therefore by putting $n=n_{I}+\mu$ for some small quantity $\mu$ we get by Taylor's theorem

$$
\begin{equation*}
w=w\left(n_{1}+\mu\right)=t n_{1}-\frac{t \mu^{2}}{2 n_{1}}+\ldots \tag{5.13}
\end{equation*}
$$

Thus for large values of $n_{1}$

$$
\begin{equation*}
w \approx t n_{l}-\frac{t \mu^{2}}{2 n_{l}}, \tag{5.14}
\end{equation*}
$$

the neglect of higher order terms being justified because $-\alpha<\frac{\mu}{n_{l}}<\beta$ where $\alpha, \beta \ll 1$.

Recall that $w=\log (v)$, thus

$$
\begin{equation*}
v \approx \exp \left(t n_{t}-\frac{t \mu^{2}}{2 n_{l}}\right) \tag{5.15}
\end{equation*}
$$

Since in the vicinity of the greatest term, $n_{1}$, consecutive terms are ultimately equal, Stokes argues intuitively that we can replace $\Sigma v$ by $\int_{v} d n$ or $\int_{-\infty}^{\infty} d \mu$. Therefore, by a discrete analogue of Laplace's method,

$$
\begin{align*}
\sum v \approx \sum_{(1-\alpha) n_{l}}^{(1+\beta)^{2} n_{l}} \approx \int_{-\alpha n_{l}}^{\beta n_{l}} v(\mu) d \mu & \approx \int_{-\infty}^{\infty} v(\mu) d \mu \\
& =\int_{-\infty}^{\infty} \exp \left(t n_{l}-\frac{t \mu^{2}}{2 n_{l}}\right) d \mu  \tag{5.16}\\
& =\exp \left(t n_{l}\right) \sqrt{\frac{2 \pi n_{l}}{t}}
\end{align*}
$$

Finally from (5.6) we have

$$
\begin{equation*}
\sum u_{n}=\exp \left(t n_{l}\right) \sqrt{\frac{2 \pi n_{l}}{t}}\left(2 \pi n_{l}\right)^{-1 / 2} n_{l}^{-s+t} \tag{5.17}
\end{equation*}
$$

$$
=\exp \left(t n_{1}\right)\left(2 \pi n_{1}\right)^{(l-t) / 2} t^{-1 / 2} n_{1}^{-s+t}
$$

Stokes' formula may be written in terms of $X$ by means of the relationship given in (5.5) as

$$
\begin{equation*}
\sum u_{n}=\frac{\exp \left(t X^{l / t}\right)(2 \pi)^{(1-t) / 2} X^{(l+t-2 s) / 2 t}}{\sqrt{t}} \tag{5.18}
\end{equation*}
$$

As a trivial check on this theory, consider the function defined by the series, $\sum \frac{x^{n}}{n!}$. Here, $s=t=1$ and Stokes' results yields $e^{x}$ as expected.

### 5.3 Application to Airy's function

Although Stokes did not do so himself, we now apply this method to the solution of Airy's equation in the form discussed by him in his 1857 paper where he takes

$$
\begin{equation*}
\frac{d^{2} U}{d x^{2}}-9 x U=0 \tag{5.19}
\end{equation*}
$$

The general solution of the above differential equation in ascending powers of $x$, valid at the origin, is given by Stokes as

$$
\begin{equation*}
U=A\left\{1+\frac{9 x^{3}}{2 \cdot 3}+\frac{9^{2} x^{6}}{2 \cdot 3 \cdot 5 \cdot 6}+\frac{9^{3} x^{9}}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}+\ldots .\right\} \tag{5.20}
\end{equation*}
$$

$$
+B\left\{x+\frac{9 x^{4}}{3.4}+\frac{9^{2} x^{7}}{3.4 .6 .7}+\frac{9^{3} x^{10}}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10}+\ldots .\right\} .
$$

The asymptotic solution in descending powers, valid at infinity is given by

$$
U=C x^{-\frac{1}{4}} e^{-2 x^{\frac{3}{2}}}\left\{1-\frac{1 \cdot 5}{1 \cdot 144 x^{3 / 2}}+\frac{1 \cdot 5 \cdot 7 \cdot 11}{1 \cdot 2 \cdot 144^{2} x^{3}}-\frac{1 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17}{1 \cdot 2 \cdot 3 \cdot 144^{3} x^{9 / 2}}+\ldots\right\}
$$

$$
\begin{align*}
& +  \tag{5.21}\\
& D x^{-\frac{1}{4}} e^{2 x^{\frac{3}{2}}}\left\{1+\frac{1 \cdot 5}{1 \cdot 144 x^{3 / 2}}+\frac{1 \cdot 5 \cdot 7 \cdot 11}{1 \cdot 2 \cdot 144^{2} x^{3}}+\frac{1 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17}{1 \cdot 2 \cdot 3 \cdot 144^{3} x^{9 / 2}}+\ldots\right\} .
\end{align*}
$$

We now show how the constants from the two series can be linked by consideration of the series themselves.

In order to apply Stokes' method it is necessary to write the general term of both series in (5.20) in terms of the gamma function. We first consider the series multiplied by the arbitrary constant $A$, the general term of which can be written as

$$
\begin{equation*}
R_{n}=\frac{\left(\prod_{j=1}^{n}(3 j-2)\right) 9^{n} x^{3 n}}{(3 n)!} \tag{5.22}
\end{equation*}
$$

Using the fact that

$$
\begin{aligned}
\prod_{j=1}^{n}(3 j-2) & =\frac{\Gamma(n+1 / 3) 3^{n}}{\Gamma(1 / 3)} \\
& =\frac{\Gamma(n+1 / 3) 3^{n+1 / 2} \Gamma(2 / 3)}{2 \pi}
\end{aligned}
$$

and

$$
\begin{aligned}
(3 n)! & =\Gamma(3 n+1) \\
& =\frac{3^{3 n+1 / 2} \Gamma(n+1 / 3) \Gamma(n+2 / 3) \Gamma(n+1)}{2 \pi}
\end{aligned}
$$

we get for the general term of the first series in (5.20)

$$
\begin{equation*}
R_{n}=\frac{\Gamma(2 / 3) x^{3 n}}{\Gamma(n+2 / 3) \Gamma(n+1)} . \tag{5.23}
\end{equation*}
$$

Similarly the second series in (5.20) which is multiplied by the arbitrary constant $B$ can be written as

$$
\begin{equation*}
S_{n}=\frac{2 \pi x^{3 n+1}}{3 \sqrt{3} \Gamma(2 / 3) \Gamma(n+1) \Gamma(n+4 / 3)} . \tag{5.24}
\end{equation*}
$$

Thus the general solution in convergent series of (5.19) can be written alternatively as

$$
\begin{align*}
U_{n}= & A \sum_{0}^{\infty} \frac{\Gamma(2 / 3) x^{3 n}}{\Gamma(n+1) \Gamma(n+2 / 3)} \\
& +B \sum_{0}^{\infty} \frac{2 \pi x^{3 n+1}}{3 \sqrt{3} \Gamma(2 / 3) \Gamma(n+1) \Gamma(n+4 / 3)} . \tag{5.25}
\end{align*}
$$

We now proceed to apply Stokes' formula to the general term of each series in turn. We note that in order to apply the formula to the first series in the above expression we must set $X=x^{3}$ in equation 5.1. Also, for this function, the excess of gamma functions in the denominator over those in the numerator is 2 , so $t=2$ and $s=1+2 / 3=5 / 3$.

Making the above substitutions for $s, t$ and replacing $X$ by $x^{3}$ in (5.18) we get for the first series of the Airy function, valid for large values of $x$

$$
\begin{equation*}
R_{n} \approx \frac{\Gamma(2 / 3) \exp \left(2 x^{3 / 2}\right)}{2 \sqrt{\pi} x^{1 / 4}} \tag{5.26}
\end{equation*}
$$

For the second series, we have $t=2$ and $s=1+4 / 3=7 / 3$. Again we replace $X$ by $x^{3}$, remembering to multiply by $x$ at the end. When we substitute into (5.17) we get for the second series

$$
\frac{\sqrt{\pi} \exp \left(2 x^{3 / 2}\right)}{3 \sqrt{3} \Gamma(2 / 3) x^{5 / 4}}
$$

which when multiplied by $x$ gives for large values of $x$

$$
\begin{equation*}
S_{n} \approx \frac{\sqrt{\pi} \exp \left(2 x^{3 / 2}\right)}{3 \sqrt{3} \Gamma(2 / 3) x^{1 / 4}} . \tag{5.27}
\end{equation*}
$$

We can therefore write the approximate solution for the Airy function, valid at infinity as

$$
\begin{equation*}
U_{n} \approx\left[A \frac{\Gamma(2 / 3)}{2 \sqrt{\pi}}+B \frac{\sqrt{\pi}}{3 \sqrt{3} \Gamma(2 / 3)}\right] \frac{\exp \left(2 x^{3 / 2}\right)}{x^{1 / 4}} . \tag{5.28}
\end{equation*}
$$

Thus, using Stokes' formula, we have arrived at the leading behaviour of the solution valid at infinity directly from consideration of the solution valid at the origin. This is an extremely important result.

As previously stated, Stokes had set out to find a method of directly linking the constant from the two series without having to use the intermediary stage of evaluating a definite integral. Although he did not give a specific example of this, we here use the Airy function to show how this can be achieved.

We recall that in his 1857 paper he had arrived at the following relationships by making use of a definite integral

$$
A=\frac{\Gamma(1 / 3)}{\sqrt{\pi}}\{C+\exp (-i \pi / 6) D\}
$$

$$
\begin{equation*}
B=\frac{3 \Gamma(2 / 3)}{\sqrt{\pi}}\{-C+\exp (i \pi / 6) D\}, \tag{5.29}
\end{equation*}
$$

where $A, B$ are the arbitrary constants which appear in the convergent series solution and $C, D$ in the descending series solution.

He also points out, in that paper, that an approximate solution of the Airy equation, valid at infinity when $\theta=0$ is given by

$$
\begin{equation*}
u \approx D \frac{\exp \left(2 x^{3 / 2}\right)}{x^{1 / 4}} . \tag{5.30}
\end{equation*}
$$

We can therefore let $C=0$ in (5.29), also, we assume $A$ and $B$ are real. Thus, the relationship between the constant, $D$, multiplying the superior
term in the divergent series and the constants, $A$ and $B$, multiplying the two series in the convergent solution is

$$
\begin{equation*}
D=A \frac{\Gamma(2 / 3)}{2 \sqrt{\pi}}+B \frac{\sqrt{\pi}}{3 \sqrt{3} \Gamma(2 / 3)} . \tag{5.31}
\end{equation*}
$$

This is precisely the same relationship obtained here, by applying Stokes method as outlined in his 1889 paper and given in (5.28). This reinforces the validity of Stokes method of approximating the slowly converging series analytically. Stokes does not appear to have published this result on the connection problem for the Airy integral in any journal.

### 5.4 Plots and numerical calculations

With the help of the Maple Computer Package, we are now in a position to take a closer look at the function with which Stokes was working. In the process we will demonstrate the correctness of Stokes' assumptions and the high level of accuracy involved in his method.

Again taking the form of Airy's function which Stokes dealt with in his 1857 paper we examine in detail the first series in the solution in ascending powers, $R_{n}$ given in (5.23) as $\frac{\Gamma(2 / 3) x^{3 n}}{\Gamma(n+2 / 3) \Gamma(n+1)}$.
The main insight which Stokes displays in this work leading to these remarkable results, is the recognition that for large values of the variable there are many terms in the neighbourhood of the greatest term of $R_{n}$, which are approximately equal and much greater than any of the terms outside this neighbourhood. He is also aware that as $x$ increases the number of terms of comparable magnitude also increases. Graphically, this may be seen from the plot of $u_{n}$ against $n$, as the convex curve, about the "peak", turning into a "plateau" as the independent variable, $x$, increases. This leads Stokes to conclude that he can approximate the sum to infinity of the series by taking the sum of the terms at that part of the series where the terms can be considered to be equal.

The validity of approximating the sum to infinity by considering only the nearly equal terms in the vicinity of the greatest term is demonstrated in Figure 5.1. The figure shows plots of $R_{n}$ for $x=1, x=2, x=4$, and $x=9$. It is clear, from the plots, that the approximation improves as the value of $x$ increases, as Stokes predicted. Also, for large values of $x$, successive terms in the vicinity of the greatest term are nearly equal and these terms constitute an ever larger portion of the sum as $x$ increases.


Figure 5.1: Maple plot for the first series of the convergent series solution of Airy's function, evaluated for $x=1, x=2, x=4, x=9$.

Continuing with Airy's function we now present some numerical results to display the accuracy of Stokes' findings.

Using Maple, the individual terms of the series, $\frac{x^{3 n}}{\Gamma(n+2 / 3) \Gamma(n+1)}$, have been evaluated when $x=9$, and the first 80 terms are shown in Table 5.1 below.

| $\mathbf{n}$ |  | $\mathbf{n}$ |  | $\mathbf{n}$ |  | $\mathbf{n}$ |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 0 | $7.38 \times 10^{-1}$ | 20 | $8.29 \times 10^{20}$ | 40 | $1.66 \times 10^{19}$ | 60 | $3.29 \times 10^{8}$ |
| 1 | $8.07 \times 10^{2}$ | 21 | $1.40 \times 10^{21}$ | 41 | $7.27 \times 10^{18}$ | 61 | $6.48 \times 10^{7}$ |
| 2 | $1.77 \times 10^{5}$ | 22 | $2.13 \times 10^{21}$ | 42 | $3.03 \times 10^{18}$ | 62 | $1.23 \times 10^{7}$ |
| 3 | $1.61 \times 10^{7}$ | 23 | $2.98 \times 10^{21}$ | 43 | $1.20 \times 10^{18}$ | 63 | $2.28 \times 10^{6}$ |
| 4 | $8.00 \times 10^{8}$ | 24 | $3.82 \times 10^{21}$ | 44 | $4.57 \times 10^{17}$ | 64 | $4.08 \times 10^{5}$ |
| 5 | $2.50 \times 10^{10}$ | 25 | $4.52 \times 10^{21}$ | 45 | $1.66 \times 10^{17}$ | 65 | $7.07 \times 10^{4}$ |
| 6 | $5.36 \times 10^{11}$ | 26 | $4.93 \times 10^{21}$ | 46 | $5.75 \times 10^{16}$ | 66 | $1.19 \times 10^{4}$ |
| 7 | $8.37 \times 10^{12}$ | $\underline{27}$ | $5.00 \times 10^{21}$ | 47 | $1.91 \times 10^{16}$ | 67 | $1.94 \times 10^{3}$ |
| 8 | $9.95 \times 10^{13}$ | 28 | $4.70 \times 10^{21}$ | 48 | $6.09 \times 10^{15}$ | 68 | $3.08 \times 10^{2}$ |
| 9 | $9.30 \times 10^{14}$ | 29 | $4.12 \times 10^{21}$ | 49 | $1.86 \times 10^{15}$ | 69 | $4.73 \times 10^{1}$ |
| 10 | $7.01 \times 10^{15}$ | 30 | $3.38 \times 10^{21}$ | 50 | $5.46 \times 10^{14}$ | 70 | 7.08 |
| 11 | $4.36 \times 10^{16}$ | 31 | $2.59 \times 10^{21}$ | 51 | $1.54 \times 10^{14}$ | 71 | 1.03 |
| 12 | $2.27 \times 10^{17}$ | 32 | $1.86 \times 10^{21}$ | 52 | $4.18 \times 10^{13}$ | 72 | $1.45 \times 10^{-1}$ |
| 13 | $1.00 \times 10^{18}$ | 33 | $1.26 \times 10^{21}$ | 53 | $1.10 \times 10^{13}$ | 73 | $2.00 \times 10^{-2}$ |
| 14 | $3.83 \times 10^{18}$ | 34 | $8.02 \times 10^{20}$ | 54 | $2.75 \times 10^{12}$ | 74 | $2.67 \times 10^{-3}$ |
| 15 | $1.27 \times 10^{19}$ | 35 | $4.82 \times 10^{20}$ | 55 | $6.66 \times 10^{11}$ | 75 | $3.48 \times 10^{-4}$ |
| 16 | $3.69 \times 10^{19}$ | 36 | $2.74 \times 10^{20}$ | 56 | $1.56 \times 10^{11}$ | 76 | $4.41 \times 10^{-5}$ |
| 17 | $9.49 \times 10^{19}$ | 37 | $1.47 \times 10^{20}$ | 57 | $3.52 \times 10^{10}$ | 77 | $5.44 \times 10^{-6}$ |
| 18 | $2.18 \times 10^{20}$ | 38 | $7.49 \times 10^{19}$ | 58 | $7.66 \times 10^{9}$ | 78 | $6.55 \times 10^{-7}$ |

Table 5.1: The first 80 terms of the first series of the convergent series solution of Airy's function, when $x=9$.

According to Stokes' analysis in (5.5), the greatest term occurs at $n \approx x^{3 / 2}$ which is at $n=27$ in this case. It is clear from Table 5.1 that the terms at first increase very rapidly, then level out and finally decrease steeply. We can also see from the Table that the greatest term does, indeed, occur at $n=27$ and that there are many terms of roughly the same order of magnitude in the vicinity of the greatest term.

It is of interest to note that although this is a convergent series, the rate of convergence is very slow for large $x$. In fact Maple Computer Package is unable to evaluate the sum to infinity of this series for values of $x$ equal to 9 or greater. However, when $x=9$, the sum of the series is found to converge after 53 terms to $4.616366086 \times 10^{22}$.

Using Stokes' formula as expressed in (5.18) we can easily find the sum of the series for $x=9$ to be $4.610377248 \times 10^{22}$. The relative error involved in this approximation as compared with adding up 53 terms of the convergent series is only .0013 .

If we increase the value of $x$ to 20 , we get, as expected, improved accuracy. In this case the series does not converge until we add up 135 terms. The answer thus obtained is $\mathbf{6 . 5 2 0 2 6 6 3 8 6} \times 10^{76}$. Stokes' formula gives as an approximation $6.517728598 \times 10^{76}$. In this case the relative error is only . 0004.

### 5.5 Conclusion

We have shown that Stokes had established in 1889, a method of obtaining the leading behaviour of the asymptotic solution, valid at infinity, directly from consideration of the solution in ascending powers, valid at the origin. He had provided a valid method of approximating the sum of the convergent series, for large values of $x$.

Today, even with the aid of modern computing methods, it is difficult to find the sum to infinity of the slowly converging series for large values of the variable except by brute force. Paris and Wood[35, page 38] remark that Stokes' 1889 paper was the earliest attempt at determining the asymptotic behaviour of a series of hypergeometric type. They describe his method as a discrete analogue of Laplace's method for the asymptotic approximation of integrals. Although they and other authors have obtained rigorous asymptotics by considering the series as a sum of residues at the poles of a Mellin-Barnes integral, a direct proof of Stokes' simple and direct method remains to be found.

## CHAPTER 6

## MODERN DEVELOPMENTS

### 6.1 Survey of Developments

Stokes recognised that there was an air of mystery surrounding the need for discontinuities in the constants occurring in the representation of analytic functions. He presented, what appears to be, his last mathematical paper [42] in 1902, entitled " On the discontinuities of arbitrary constants that appear as multipliers of semi-convergent series' as part of a collection in commemoration of Abel. In this paper he gave a survey of the problem and it is here we get his famous poetic description of the phenomenon: "As $\theta$ passes through the critical value, the inferior term enters as it were into a mist, is hidden for a little from view, and comes out with its coefficient changed. " These coefficients have come to be known as Stokes' multipliers.

This air of mystery continued to cloud the problem for almost one hundred years until the physicist Michael Berry [5] in 1989, threw new light on Stokes' phenomenon. This resulted in a huge revival of interest in the problem and for the past seven years much progress has been made in the field by mathematical analysts such as Olver, Olde Daalhuis, Paris, Wood, and others.

Berry discovered that rather than the multiplier changing discontinuously, the change in the neighbourhood of a Stokes' line is in fact continuous. He contended that Stokes' discontinuity " is an artefact of poor resolution". This is a reference to the example Stokes gave in his 1857 paper to illustrate the change in the multiplier. We have referred to this example in Chapter 4. Stokes considered the Airy function which has a Stokes' line at $\theta=120^{\circ}$. He evaluated the function for two complex values with a common modulus, but whose arguments are equally spaced on either side of the Stokes' line, at $\theta=90^{\circ}$ and $\theta=150^{\circ}$. In this way he was able to illustrate the jump for the Airy function on crossing the Stokes' line at $\theta=120^{\circ}$.

Berry's paper showed that the arguments of $z$ chosen by Stokes are too far removed from the Stokes' ray to show the true picture. At $\theta=90^{\circ}$, the change has barely begun, while at $\theta=150^{\circ}$ it is virtually complete.

He showed that for constant $|z|$, the multiplier of the subdominant term which is liable to change, is not in fact a constant as Stokes had understood, but a function of the argument of $z$. This multiplier displays a behaviour like the error function, changing smoothly and rapidly from 0 to $l$.

Like Stokes, Berry terminated the series at its least term, but he then used Borel summation to resum the tail of the divergent series. This in turn led to further work by Berry and Howls [7] and resulted in their theories of superasymptotics and hyperasymptotics in 1990. Superasymptotics corresponds to the optimal truncation of the Poincaré expansion, as performed by Stokes, followed by a re-expansion of the (exponentially small) remainder term: this is also called by Olver the "exponential improvement" of the asymptotic series. Hyperasymptotics involves repeated optimal truncation and re-expansion of the successive remainders, picking up terms of exponentially smaller order, until the desired level of accuracy is achieved: see Olde Daalhuis[23].

In a paper published the same year Olver [28] provided a rigorous mathematical basis for Berry's new interpretation of Stokes' phenomenon. He also showed how to improve Berry's approximation for the Stokes' multiplier for large $z$. He illustrated his analysis by means of three functions: Macdonald's modified Bessel function, the generalised exponential integral, and the confluent hypergeometric function. The generalised exponential is particularly important as a building block for the generic error function dependence observed for many other functions. Olver concluded that his method like that of Berry has quite general application.

Following Berry's breakthrough and Olver's placing of his results on a firm mathematical footing there was intense work in the field. Paris and Wood [37] gave a detailed exposition of the problem and pulled together all the various strands of work in the area. The following summary of work in the field was prepared with the help of that paper.

In many cases of interest, solutions of differential equations can be represented as integrals involving saddles. The hyperasymptotics of such integrals have been examined in detail by Berry and Howls in [7] and its sequels. In a paper published in 1992 Paris [32] used Mellin-Barnes integrals to construct exponentially improved asymptotic expansions. He also illustrated his theory by means of the confluent hypergeometric function. His method has the advantage of widening the sector of validity
of the asymptotic representation. Boyd [11] preferred to use Stieltjes integrals.

All of the results described above were obtained either by the integral representation approach or from working directly with the tail of the divergent series. However, since all functions considered were solutions of second order linear differential equations it did appear that an exponentially improved asymptotic solution should be obtainable directly from the differential equation, without prior knowledge of the divergent series or of the integral representation of the solution.

This was done in a non-rigorous manner by Berry [6] and rigorously for a restricted class of equations by McLeod [18]. Again, it was Olver [31] who supplied the rigorous mathematical proof for the confluent hypergeometric equation and he and Olde Daalhuis [25] have generalised the results to equations with irregular singularities of rank one at infinity.

A further generalisation to a class of equations of arbitrary order $n$ was made by Paris [33], again using the Mellin transform approach. He showed that where there are more than two linearly independent solutions the question of dominance/subdominance is not clear. In the case of equations of order $n$, there can be $n$ distinct exponential behaviours at infinity.

Not all special functions which display a Stokes' phenomenon satisfy a linear differential equation. The gamma function is such a function and was the subject of research by Paris and Wood [36]. They obtained the exponentially improved expansion for this function. They found that infinitely many exponential behaviours, each associated with its own Stokes' multiplier, were possible at infinity. They established the smooth error function transition of the leading subdominant exponentials in the neighbourhood of the Stokes' lines. The detailed computation of higher order exponentials was subsequently carried out by Berry [8]. An explanation of this behaviour of the gamma function is that it satisfies a difference equation which can be regarded as a differential equation of infinite order. Lawless [17] has obtained exponentially improved asymptotics for solutions of first order difference equations. In this work, she took the gamma function as a special case.

### 6.2 Recent Developments

We give below in some more detail the recent developments in the field, using the differential equation approach. We concentrate on the work of the main practitioners, Olver and Olde Daalhuis. We start with their paper [24], in which they consider a class of homogeneous second order linear differential equations with an irregular singularity of rank one at infinity. Re-expansions are found for the remainder terms after optimal truncation of the asymptotic series solution, giving rise to exponentially improved solutions.

The second order differential equation

$$
\begin{equation*}
\frac{d^{2} w}{d z^{2}}+f(z) \frac{d w}{d z}+g(z) w=0 \tag{6.1}
\end{equation*}
$$

is considered, taking the case where there is an irregular singularity of rank one at infinity. Therefore, $f(z)$ and $g(z)$ can be written as convergent series expansions of the form

$$
\begin{equation*}
f(z)=\sum_{s=0}^{\infty} \underline{z^{s}}, \quad g(z)=\sum_{s=0}^{\infty} \frac{f_{s}}{z^{s}} \tag{6.2}
\end{equation*}
$$

in an annulus $|z|>a$. The assumption must also be made that not all of $f_{0}, g_{0}$, and $g_{l}$ can equal 0 , otherwise we would have a regular singular point at infinity.

The asymptotic solution of (6.1) has been given by Olver [27] as

$$
w_{l}(z) \approx \exp \left(\lambda_{1} z\right) z^{\mu_{l}} \sum_{s=0}^{\infty} \frac{a_{s, l}}{z^{s}}, \quad z \rightarrow \infty,
$$

$$
\begin{equation*}
\left|\arg \left(\lambda_{2}-\lambda_{1}\right)\right| \leq \frac{3 \pi}{2}-\delta \tag{6.3}
\end{equation*}
$$

and

$$
\begin{aligned}
w_{2}(z) \approx \exp \left(\lambda_{2} z\right) z^{\mu_{2}} \sum_{s=0}^{\infty} \frac{a_{s, 2}}{z^{s}}, & z \rightarrow \infty \\
& \left|\arg \left(\lambda_{I}-\lambda_{2}\right)\right| \leq \frac{3 \pi}{2}-\delta .
\end{aligned}
$$

In the above solution, $\lambda_{1}, \lambda_{2}$ are the roots of the characteristic equation

$$
\lambda^{2}+f_{0} \lambda+g_{0}=0
$$

$\mu_{1}, \mu_{2}, a_{s, 1}, a_{s, 2}$ are found by substitution into (6.1) and $\delta$ is an arbitrary small positive quantity. Any branches for $z^{\mu_{1}}$ and $z^{\mu_{2}}$ may be taken provided they are continuous throughout the appropriate sector.

In has been shown in previous works by Olver [30] and [31] and Paris [32], for the special case of the confluent hypergeometric equation, that that the remainder terms associated with the expansions in (6.3) can be re-expanded to give exponentially improved expansions and also to increase the sector of validity. Here Olver and Olde Daalhuis succeed in applying the method to the general class of equations given in (6.1).

The authors note that $w_{1}[z \exp (-2 \pi i)]$ is also a solution of (6.1). Further, $w_{l}(z)$ and $\exp \left(2 \pi i \mu_{l}\right)\left\{w_{l}[z \exp (-2 \pi i)]\right\}$ are dominant solutions in the sector $\frac{\pi}{2}+\delta \leq \arg (z) \leq \frac{3 \pi}{2}-\delta$ and have exactly the same asymptotic expansion there. Since a second order linear differential equation can have only two linearly independent solutions, it follows by matching dominant terms in a sector that

$$
\begin{equation*}
w_{1}(z)=\exp \left(2 \pi i \mu_{1}\right)\left\{w_{1}[z \exp (-2 \pi i)]\right\}+C_{1} w_{2}(z) \tag{6.4}
\end{equation*}
$$

and similarly

$$
\begin{equation*}
w_{2}(z)=\exp \left(-2 \pi i \mu_{1}\right)\left\{w_{2}[z \exp (2 \pi i)]\right\}+C_{2} w_{1}(z) \tag{6.5}
\end{equation*}
$$

Equations (6.4) and (6.5) are called the connection formulae where $C_{l}$ and $C_{2}$ are known constants, the Stokes' constants.

We give below the main result proved by Olver and Olde Daalhuis in this paper for the remainder term on optimal truncation of the asymptotic series.

Theorem 1. Define $R_{N}^{(1)}(z)$ and $R_{N}^{(2)}(z)$ by

$$
\begin{align*}
& w_{1}(z)=\exp \left(\lambda_{1} z\right) z^{\mu_{1}} \sum_{s=0}^{N-1} a_{s, 1} z^{-s}+R_{N}^{(1)}(z),  \tag{6.6}\\
& w_{2}(z)=\exp \left(\lambda_{2} z\right) z^{\mu_{2}} \sum_{s=0}^{N-1} a_{s, 2} z^{-s}+R_{N}^{(2)}(z),
\end{align*}
$$

where $N=|z|+\alpha, \alpha$ bounded as $|z| \rightarrow \infty$.

Then

$$
\begin{align*}
R_{N}^{(1)}(z)= & (-)^{N-1} i \exp (\mu \pi i) \exp \left(\lambda_{2} z\right) z^{\mu_{2}} \times \\
& \left\{C_{1} \sum_{s=0}^{M-1}(-)^{s} a_{s, 2} z^{-s} F_{N+\mu-s}(z)+R_{M, N}^{(l)}(z)\right\} \tag{6.8}
\end{align*}
$$

where $\mu=\mu_{2}-\mu_{1}, M$ is an arbitrary fixed nonnegative integer, and for large $|z|$
$R_{M, N}^{(I)}(z)=O\left(\exp (-|z|-z) z^{-M}\right)$ for $|\arg z| \leq \pi$, $R_{M, N}^{(l)}(z)=O\left(z^{-m}\right)$ for $\pi \leq|\arg z| \leq \frac{5 \pi}{2}-\delta$
and $F_{N+\mu-s}(z)$ is the generalised exponential integral or incomplete gamma function,

$$
F_{p}(z)=\frac{e^{-z}}{2 \pi} \int_{0}^{\infty} \frac{e^{-z t} t^{p-1}}{1+t} d t .
$$

It is the presence of this function $F_{N+\mu-s}(z)$ that gives the error function dependence which leads to the switching on of the second exponential $\operatorname{across} \arg z=\pi$.

A similar result is obtained for $w_{2}(z)$,

$$
\begin{align*}
R_{N}^{(2)}(z)= & (-)^{N} i \exp (\mu \pi i) \exp \left(\lambda_{1} z\right) z^{\mu_{I}} \times \\
& \left\{C_{2} \sum_{s=0}^{M-1}(-)^{s} a_{s, 1} z^{-s} F_{N-\mu-s}(z \exp (-\pi i))+R_{M, N}^{(2)}(z)\right\} \tag{6.9}
\end{align*}
$$

where
$R_{M, N}^{(2)}(z)=O\left(\exp (-|z|+z) z^{-M}\right)$ for $0 \leq \arg z \leq 2 \pi$, $R_{M, N}^{(2)}(z)=O\left(z^{-m}\right)$ for $\frac{-3 \pi}{2}+\delta \arg z \leq 0$ and $2 \pi \leq \arg z \leq \frac{7 \pi}{2}-\delta$

It is of interest to note that the coefficient $a_{s, 2}$ which occurs in the expansion for $w_{2}(z)$ given in (6.3) above, appears here in the re-expansion of the tail of the series for $w_{l}(z)$. This provides an example of the phenomenon of resurgence.

Having obtained the above expression for the remainder term given in (6.8), Olver and Olde Daalhuis [25] go on to re-expand it to give hyperasymptotics.

In their paper [26] they consider differential equations with singularities of rank $r$ at infinity. This means that coefficients, $f(z)$ and $g(z)$ in equation (6.1) have now got expansions of the form

$$
\begin{equation*}
f(z)=z^{r-1} \sum_{s=0}^{\infty} \frac{f_{s}}{z^{s}}, \quad g(z)=z^{2 r-2} \sum_{s=0}^{\infty} \frac{g_{s}}{z^{s}} . \tag{6.10}
\end{equation*}
$$

The basic result from this case is that, in a sector $S_{k}$,

$$
\begin{align*}
& w_{k}(z) \approx \exp \left[\xi_{1}(z)\right] z^{\mu_{1}} \sum_{s=0}^{\infty} a_{s, 1} z^{-s}, \text { for } k \text { even }  \tag{6.11}\\
& w_{k}(z) \approx \exp \left[\xi_{2}(z)\right] z^{\mu_{2}} \sum_{s=0}^{\infty} a_{s, 2} z^{-s}, \text { for } k \text { odd, as } z \rightarrow \infty
\end{align*}
$$

The sector $S_{k}$ is taken as

$$
\begin{aligned}
& \left(k-\frac{3}{2}\right) \pi-\sigma+\delta \leq|\arg z| \leq\left(k+\frac{3}{2}\right) \pi-\sigma-\delta, \quad k \in Z, \\
& \sigma=\arg \sqrt{\frac{f_{0}^{2}}{4}-g_{0}} \text { and } \xi_{1}, \quad \xi_{2} \text { are polynomials of degree } r .
\end{aligned}
$$

A connection formula of the form $w_{k+1}(z)=C_{k} w_{k}(z)+w_{k-1}(z)$ is given in this work, together with two methods of calculating $\mathrm{C}_{\mathrm{k}}$. These methods have been extended by Murphy and Wood [20] who not only provide exponentially improved expansions but go further and provide hyperasymptotic expansions.

Work in topics related to Stokes' phenomenon continues and is being extended to differential equations of arbitrary order $n$ with singularities of rank 1. Three papers have been completed. In the first, Olver reworks the classical asymptotic theory and obtains new results, including extended regions of validity and error bounds. In the second, Olde Daalhuis extends the theory of the Borel-Laplace transform and applies it to the derivation of hyperasymptotic expansions for the solutions. In the third, Olde Daalhuis and Olver investigate the application of the results of the other two papers to the problem of computing the solutions everywhere in the complex plane; this requires the development of appropriate boundary value methods.

## CHAPTER 7

## Conclusion

Four of the papers which Stokes produced in mathematical analysis are discussed in this thesis. This is but a small fraction of the total of his mathematical and physical contributions which occupy five volumes [43]. Stokes was primarily a mathematical physicist. He was also an accomplished mathematical analyst but he concentrated on those mathematical problems which were needed for his physical investigations.

His 1850 paper was a response to the mathematical difficulty encountered by G.B. Airy in approximating the rainbow integral for large values of the variable. This was necessary in order to establish the position of the third and subsequent dark bands in a system of supernumerary rainbows. The method Stokes developed of finding a solution to the Airy differential equation in the form of a series in descending powers anticipated the modern WKBJ method. He established the leading behaviour of the solution by what is essentially the method of dominant balance, that is, by identifying terms in the differential equation that could be neglected when dealing with large values of the variable. He then multiplied this leading term by a series in descending powers to get further terms. He explained in a footnote that he had taken the idea of multiplying the leading term by a series in descending powers from Cauchy's method of evaluating Fresnel integrals. The new insight that Stokes showed here is that the powers do not have to decrease in multiples of 1 . By forcing this series to satisfy the differential equation he arrived at the full asymptotic series solution, valid at infinity.

Stokes points out that the solution in descending powers of $x$ will become indeterminate when $x=0$, though this is not the case in the original differential equation. In overcoming this difficulty, he developed what is now known as the "Stokes' phenomenon". Firstly, he took the brave step of considering $x$ to be complex. This enabled him to get a single solution, valid for positive and negative $x$ but presented him with further difficulties. The solution must return to its original value when the argument of $x$ is increased by $2 \pi$, in accordance with Airy's differential equation, whereas the complex solution repeats itself only when the argument is increased by $8 \pi$. This led him to the novel conception that
the arbitrary constants in the general asymptotic solution must change discontinuously.

In his attempts to find these arbitrary constants in the asymptotic solution of the Airy differential equation Stokes broke new ground in approximating integrals for large values of the variable, $x$. It has been shown in this thesis that he developed early forms of the method of stationary phase and of the saddle point method. In approximating the Airy integral, although he nowhere mentioned "saddle point", he transformed the integral so that the dominant saddle point occurred at the origin. Proceeding intuitively, he then deformed the original path of integration, along the positive real axis, into one which ran from a finite complex quantity to another complex quantity with infinite modulus and argument equal to half the argument of $x$. This path was then divided into 4 sections. He found that when the integral was approximated, for large values of $x$, along 3 of these paths, the contributions were negligible. He showed that, to a very good approximation, the main contribution to the integral came from the neighbourhood of the saddle point. It has been shown in this thesis that the results he obtained agree with those found using the modern saddle point method.

In a footnote, he gave a brief explanation of a method of evaluating the Airy integral for large values of $x$, by concentrating on the region about the point where the phase of the integrand is stationary. This is similar to the method of stationary phase outlined by Kelvin in 1887. We have also outlined an intuitive method he used in approximating the integral related to the Bessel function of order zero, for large values of $x$. He recognised that the integrand could be split so as to form 3 integrals, two of which approximately cancel each other out, leaving one which could be estimated, with ease, for large $x$.

The concepts introduced by Stokes in his 1850 paper in relation to asymptotic solutions of differential equations were developed in a supplement to this paper which he wrote in 1857. Further progress was made in the management of divergent series. When $x$ is large the asymptotic series begins by converging rapidly but ends, no matter how large $x$ is, by diverging with increasing rapidity. However, Stokes recognised that it may be used for calculations provided it is truncated before the terms begin to get large again. He devised a simple formula for finding the position and magnitude of the least term. Further, by resumming the tail of the divergent series, he proved that the error on truncation in the vicinity of the least term is exponentially small. In this
thesis, the accuracy of his methods has been verified using the computer package, Maple.

Above all, there was a major development in his understanding of the nature and location of the discontinuity of arbitrary constants which appear in asymptotic series solutions. This is where he gave a clear exposition of "Stokes' phenomenon". Writing the asymptotic solution to Airy's differential equation as a function of a complex variable whose modulus remains constant but whose argument can change and requiring that the complete solution repeat itself for each change of $2 \pi$ in that argument, he showed that the asymptotic solution in descending powers is valid only when associated with sectors of validity in the complex plane.

He explained that the dominance/subdominance of the two descending series solutions changes over when certain rays in the complex plane are crossed. Stokes used the expressions superior and inferior. This change takes place at $\theta=\pi / 3$ in the case of the Airy function. These rays are now known as "Anti-Stokes' lines". He showed that if the series is truncated in the vicinity of the least term, the error on truncation of one series (superior) in the solution is greater than the entire second (inferior) series. The value of the argument for which the dominance of the superior term is greatest defines "Stokes' lines". He concluded that the coefficient of the inferior term can change there, and there only, without affecting the solution of the differential equation.

Having discarded, for physical reasons, the unbounded term in the general solution of the Airy differential equation, he showed that when the argument is zero (dark side of the caustic) the solution occurs only once in its sector of validity whereas when the argument is equal to $\pi$ (bright side of the caustic) the solution occurs twice in its sector of validity. Thus we have an exponentially declining solution on the dark side and an oscillatory solution on the bright side. The change from one term to two terms occurs when the "Stokes' line" at $\theta=2 \pi / 3$ is crossed.

The concept of the arbitrary constants in the solution to a differential equation changing for differing values of the argument of the variable was so novel that Stokes felt the need to provide a numerical example to illustrate it. He compared the results from the convergent series and the divergent series for two values of the variable with the same modulus but with arguments on either side of the "Stokes' line". He showed that on one side of the ray, to get good agreement with the values obtained from
the solution in ascending series, the superior term alone, of the solution in descending series, is required. However, on the other side of the ray contributions from both the superior and inferior terms are needed. He argued that this shows that the constant multiplying the inferior term must be allowed to change because it equals zero for one value of the argument but is different from zero at another value.

A major consideration for Stokes in his work in the area of asymptotic solutions was the question of linking the constants multiplying the series valid at the origin with the constant multiplying the series valid at infinity. He accomplished this in his 1857 paper but he returned to it in 1868 in order to elaborate a more efficient method. His further achievement was to be able to connect the constants by calculating the dependent variable for only one value of the independent variable and also to dispense with the need to solve a simultaneous equation. This numerical method is needed if an analytic relationship between the constants is not found.

In this paper he also gave, for a general class of differential equation, an expression for the new value of the constant multiplying the inferior term when a "Stokes' line" is crossed. This relationship which is given in this thesis as (4.13) provided him with further proof that the constant multiplying the subdominant series can change only when it is accompanied by the dominant series.

After an interval of some 20 years Stokes returned to his problem of the relationship between the constants in the ascending and descending series in 1889. In his previous works, the constants had been linked by getting a third solution in the form of a definite integral and relating each series solution to it. In this case, by considering the ascending series solution in the vicinity of its greatest term, he arrived directly at the leading term of the asymptotic solution. The methods he used were intuitive and although it has been shown in this thesis, with the help of the computer package Maple, that his results yield a remarkable degree of accuracy, a rigorous proof of Stokes' simple and direct method remains to be found.

When Stokes [42] presented the summary in 1902 of his results in relation to the discontinuities of arbitrary constants that appear as multipliers of semi-convergent series, he did so in a most apologetic manner. Since his paper was written in honour of Abel, he felt that the work should be of pure mathematics and most of his work had been in applications of mathematics. Stokes, who was then 83 years old, commented that he could hardly be expected to produce something new at his age but added
"There is one thing I think might perhaps do, but it has, I fear been too long before the public to make it suitable."

Stokes' work on this problem spanned almost forty years. The next significant advance on his work did not take place until almost one hundred years later. This was followed by a remarkable period of activity which still continues. The widespread manifestation in modern physical and mathematical theory, of the phenomenon explained by Stokes, renders the content of Stokes' paper in commemoration of Abel more eminently suitable than he could have foreseen.

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