Dynamic Optimal Portfolio Theory

James E. Parnell B.Sc.

School of Mathematical Sciences
Dublin City University

Supervisor : Dr. Emmanuel Buffet

A Dissertation submitted for the Degree of Master of Science

September 30, 1997

Acknowledgements

I would like to express my sincere gratitude and appreciation to my supervisor, Dr. Emmanuel Buffet, for his constant guidance and attention throughout the last two years.

I wish to thank my fellow postgraduates who have made the last two years research a very enjoyable experience. I also thank Anne-Marie for her help and suggestions.

I am also very grateful to Allied Irish Bank Group and Forbairt for sponsoring my research.

Especially, I wish to thank my parents, Charles and Mary and my family for their continuing support and encouragement during my years in college.

Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Master of Science is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

Signed James Parnell ID No. : 95970045

Date : 30/9/97

Contents

Abstract

In this thesis we consider a financial market model consisting of a bond with deterministic growth rate, and d risky assets, governed by Brownian motion with drift. We can shift money from one asset to the other without loss of capital.

Optimal investment and consumption (spending) decisions are examined for different types of investors with various criteria for optimality. An investor's level of satisfaction with any amount of wealth is measured by a utility function. The problem has been solved by Merton [4] and others for the *small* investor with no transaction costs. Here we suppose the investor is *large*, i.e., his strategy has an effect on the asset price evolution.

The approach parallels that taken by Cvitanic and Karatzas [5] for constrained portfolios. The theorems therein are adjusted appropriately to account for the investor's effect on prices *instead of* constraining the portfolios as Cvitanic and Karatzas do.

As in Cvitanic and Karatzas [5], Karatzas et al.[6] and several others duality theory and martingale methods are introduced to prove the existence of the optimal portfolio which maximises the expected final utility. An algorithm is suggested to find this portfolio under certain market conditions.

Glossary

I. General Notation

Let a and b be real numbers. As usual $' :=' $ means *is defined to be.*

$a \wedge b$	$:= \min\{a, b\}$.
a^-	$:= \max\{-a, o\}$.
$ess \sup X$	$:= \inf\{a : P(X \leq a) = 1\}$.
$I_A(x)$	$:= 1$ if $x \in A$, 0 if $x \notin A$.
$\|\cdot\|$	:= The Euclidean norm.

II. Sets and Spaces

$\mathcal{A}(x)$	: The set of all admissible policies.
$\mathcal{C}(x)$	: The set defined in (2.26).
$\mathcal{D}(x)$	: The set of (2.26) with equality.
D_t	: The set of processes $\underline{v}$ for which $\tilde{g}(t, \underline{v})$ is finite.
D	: The set of processes in D_t for all times t.
D'	: The set of processes for which ζ is finite.
$\mathcal{L}(x)$	: The set defined in (2.27).
$\mathcal{M}(x)$	: The set (2.27) with equality.
$\mathcal{P}(x)$	: The set defined in (2.28).
$\mathcal{Q}$	: The set of Rational numbers.
$\mathcal{R}^d$	: d-dimensional Euclidean space ; $\mathcal{R}^1 = \mathcal{R}$.
$\mathcal{S}$	: The set of all $\mathcal{F}_t$-stopping times, τ in $[0, T]$.
$\mathcal{S}_{\rho,\sigma}$	: The set of all $\mathcal{F}_t$-stopping times in $[0, T]$ such that $\rho(\omega) \leq \tau(\omega) \leq \sigma(\omega)$.
Γ	: The Hilbert space defined in (3.53).

III. Probability

E : Expectation operator corresponding to probability P.

$\mathcal{F}_t^X$: $\sigma(X_s : 0 \leq s \leq t)$, the smallest σ-field with respect to which the random variable X_s is measurable $\forall\ s \in [0, t]$.

$\mathcal{F}_{t+}$: $\sigma(\cap_{\epsilon>0}\mathcal{F}_{t+\epsilon})$.

$\mathcal{F}_{t-}$: $\sigma(\cup_{s>t}\mathcal{F}_s)$.

P : Probability measure corresponding to Brownian motion.

W : Standard d-dimensional Brownian motion.

Ω : The sample space on which probability measures can be placed.

ω : A sample point of the sample space.

IV. Functions

f_0 : The market effect function for the bond.

f_i : The market effect function for the i^{th} stock.

g : The function defined in (3.3).

$\tilde{g}$: The function defined in (3.4).

I : The inverse marginal utility.

J : The objective function of (2.59).

$\tilde{J}$: The extended functional defined in (3.54).

u : The utility function.

u' : The marginal utility.

$\tilde{u}$: The Legendre Fenschel transform of definition 2.20.

$V(x)$: The value function of (2.61).

$V(\tau)$: The random variable defined in (3.22).

ζ : The function defined in (2.67).

ψ : The inverse of the function ζ.

Ψ : the function defined in (3.36).

V. Defined Processes

$\underline{b}(\cdot)$: Growth rate vector of the stocks.
$c(\cdot)$: The consumption rate process.
$H(\cdot)$: The deflating process of (2.18).
$M(\cdot)$: The process defined in (2.17).
$N(\cdot)$: The process defined in (2.20).
P_0 : Value process of the bond.
P_i : Value process of the i^{th} stock.
$r(\cdot)$: Short interest rate of the bond.
x : The initial capital endowed.
$X(\cdot)$: The wealth process.
$Z(\cdot)$: Exponential martingale of (2.8).
$\gamma(\cdot)$: Discount process of (2.12).
$\underline{\pi}(\cdot)$: Vector of fractions invested in stocks.
$\sigma(\cdot)$: Volatility matrix of the stocks.
$\underline{\theta}(\cdot)$: Relative risk process of (2.7).

VI. Miscellaneous

τ : Any stopping time.
τ_B : Bankruptcy time defined in (2.24).
B : The random variable representing the contingent claim.
$h(0)$: The minimal hedging price.
$\preceq$: The partial ordering defined in Appendix A.4.
$\lceil \cdot \rceil$: The Hilbert Space norm of (3.53).
$\langle \, , \, \rangle$: The inner product of (3.53).
$\lfloor \, , \, \rfloor$: A stochastic interval.

Chapter 1

Introduction to Optimal Portfolio Theory

1.1 Objectives

In this thesis we are primarily concerned with the *happiness* of an investor in the stock market. More precisely, we have a market consisting of various assets in which an agent can invest his wealth. We attempt to allocate this wealth in such a way as to ensure the agent's maximum satisfaction with the *final return* of the investment portfolio.

This problem is important for several reasons :

- The resulting solution, although merely an estimate, gives an insight into the behaviour of a shrewd investor given different criteria for investment.

- It is of considerable interest to any educated investor in a market to have an appropriate model of that market. But perhaps more important, particularly from any potential investor's point of view, is the need for a probabilistically sound model for the recommended behaviour and preferred investment strategy most likely to yield a high return.

- Economically, the aforementioned model of the stock market and investor behaviour can be examined to see if mathematical solutions are reflected in the overall behaviour within the market and in the evolution of the market itself.

- The approach taken and methodology used also yield a method for pricing financial derivatives and other commodities.

For our part, we will confine our interest, and approach this thesis purely from the potential investor's viewpoint. More precisely, our aim is to optimise the investor's level of satisfaction with the terminal return on his investment portfolio, measured by a utility function; cf Section 2.4.

We remark here that maximising the utility of final wealth is not the only criterion on which one could base portfolio investment. This is most appropriate for an investor who wishes to *get rich.* Alternatively, the methodology of the thesis could be used to maximise the utility (satisfaction) from consumption (spending). This would be more appropriate for an investor whose primary concern is to *enjoy living.* For a small investor, this problem is examined in Karatzas et al. [7], Xu and Shreve [8] and others. The approach is identical to that for utility of final wealth with obvious changes in the objective function. Indeed Karatzas [9] shows that there is an investment strategy that will maximise an objective function of both final wealth and consumption. In this way the investor can *live well* and *get rich.*

1.2 Approach Taken

The thesis studies primarily the problem of an agent who receives a deterministic capital, which he may then invest in a market of assets in order to maximise the expected utility of his wealth at some pre-specified final time T. The market consists of d stocks, driven by a d-dimensional Brownian motion (in the case of a complete market[1]), and a bond. The investor is allowed to spend money at any time via a cumulative consumption process. However he is not allowed to input more capital during the interval $[0, T]$. The market coefficients - the interest rate, the appreciation rate of the stocks and the volatility coefficients of the stocks - are random processes adapted to the full d-dimensional Brownian motion.

The principal result of the paper focuses on the strategy for a *large investor,* i.e., one whose policies affect the asset price evolution. Theorem 3.41 provides conditions under which the expected utility is maximised in this market. We then characterise this optimal strategy in terms of a solution to the Dual optimisation problem. The main mathematical tool, namely the martingale approach to stochastic control, is utilised throughout the thesis.

In Chapter 2 we examine the more elementary problem of utility maximisation faced by a small agent. In §2.1 the standard generalisation of the market model of Merton [4] is introduced. In §2.2 we derive a necessary condition for the investment policy to be admissible, i.e., for the investor to avoid debt at all times t in the interval $[0, T]$. We also prove the extent to which the opposite implications are true. In §2.3 we define the problem of pricing and hedging a contingent claim. We solve the pricing problem and suggest a way to find the hedging policy in feedback form.

[1]The incomplete market case, where the number of driving Brownian motions is greater than the number of stocks, is dealt with in Karatzas et al. [6].

In §2.4 we introduce the concept of utility functions used to measure an investor's degree of satisfaction. In §2.5 the utility maximisation problem of a small investor is defined. We then present the solution which proceeds in several steps. Firstly, using results of §2.2 on the sufficiency of the conditions we reformulate the problem as a standard linear optimisation problem with equality constraints. Secondly, we conjecture an optimal form for the final wealth using elementary duality theory. Thirdly, it is shown that a portfolio can be constructed that attains this most desirable form as its final wealth. This portfolio is optimal.

In Chapter 3 we deal with the case of the *large investor.* We parallel the approach taken by Cvitanic and Karatzas [5] for constrained portfolios. This is summarised as follows. In §3.1 we adjust the old model to account for the effect of the agent's actions on asset behaviour. We introduce a suitably large family of probability measures, each of which defines a linear problem similar to the one solved in Chapter 2. In §3.2 we show that, under appropriate conditions, all contingent claims can be replicated. In fact for a large investor this can be done with zero consumption. The minimal initial capital that makes this replication possible is equal to the supremum of the expected discounted values of the claim under these new probability measures. Also, the existence of a hedging portfolio process is proved and the form of the wealth process is found in Theorem 3.21. It is later shown in Chapter 4 how to find the portfolio process in feedback form for certain utility functions. In §3.3 we approach the utility maximisation problem via the results of the previous section. As before, we specify the problem, reformulate it using the sufficiency of certain conditions and approach it using established duality theory. We use informal arguments to conjecture the optimal form of terminal wealth and prove rigidly that it is optimal. We find conditions under which an optimal solution exists. In §3.4 we ensure these conditions are satisfied and we then use Theorem 3.21 to hedge this optimal form. We thus find the form of the optimal wealth process. The optimal portfolio and wealth processes depend on the solution to a dual problem. The dual problem must then be solved. We show that the dual problem has a solution which is in fact unique and demonstrate the dependence of the optimal processes on the solution to the particular dual problem.

In Chapter 4 we illustrate some applications of the previous theory through examples and show how some of the calculations of dual solutions are performed.

Finally, Chapter 5 gives a brief summary of other works in this area including the problem of transaction costs (brokerage fees). Conclusions are also drawn from the thesis and suggestions for possible further work are made.

Chapter 2

The Small Investor Problem

In this chapter we wish to consider only small investors, agents whose decisions cannot affect the asset prices. The approach taken here is the standard approach taken by Merton [4] and Karatzas et al. [7].

2.1 The Stock Market Model

2.1.1 The Probabilistic Setting

In order to treat the questions being asked in the context of a financial market, we require a financial market model. We begin with the standard assumption of continuous trading. The basic securities consist of $d+1$ assets which include one risk-free asset called the *bond*[1] whose value, $P_0(t)$ is governed by the equation

$$\begin{aligned} dP_0(t) &= P_0(t)r(t)dt, \\ P_0(0) &= 1. \end{aligned} \tag{2.1}$$

The d risky assets are called *stocks* and can be traded continuously. The

[1] This is more commonly known as a zero-coupon bond.

prices of these latter are driven by an equal number of independent standard Brownian motions (see Appendix A.1.16). This is the *complete market* model. These driving Brownian motions model the external exogenous forces of uncertainty that influence the market. The price $P_i(t)$ of the i^{th} stock is modelled by the linear stochastic differential equation

$$\begin{aligned} dP_i(t) &= P_i(t)\Big[b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)\Big], \\ P_i(0) &= p_i, \end{aligned} \tag{2.2}$$

for $i = 1..d$.

The process $W = (W^1....W^d)^\top$ is the standard d-dimensional Wiener process defined on the probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, P)$ as in Appendix A.1.16. P is called the *objective probability measure.* In general we assume that the filtration $(\mathcal{F}_t)$ is the natural filtration (see Appendix A.1.4) generated by the Wiener process W, i.e.,

$$\mathcal{F}_t^W = \sigma(W(s) : 0 \leq s \leq t), \qquad t \in [0, \infty). \tag{2.3}$$

With this interpretation of the stock market :

The process $\{r(t) : t \in [0, \infty)\}$ is the *short rate of interest process* for the bond.

The process $\{\underline{b}(t) = (b^1(t)....b^d(t))^\top : t \in [0, \infty)\}$ is the *appreciation rate vector* for the d stocks.

The *volatility matrix* is given by $\sigma(t) := [\sigma_{ij}(t)]$ where $\{\sigma_{ij}(t) : t \in [0, \infty)\}$ is the *volatility coefficient* and models the instantaneous intensity with which the j^{th} source of uncertainty influences the price of the i^{th} stock.

These are called the *coefficients of the market model.* They are all assumed to be random, uniformly bounded in $(t,\omega) \in [0,\infty) \times \Omega$ and progressively measurable processes (see Appendix A.1.9) with values in $\mathbf{R}, \mathbf{R}^d$ and $\mathbf{R}^{d\times d}$ respectively. They are also adapted (see Appendix A.1.8) to the Brownian filtration of (2.3).

For our purposes we fix, from here onwards, a finite time horizon $[0,T]$[2] on which all our problems will be treated.

The following Standing Assumptions are made :

$$\int_0^T ||\underline{b}(t)||^2 dt < \infty, \tag{2.4}$$

$$\int_0^T |r(t)| dt \leq L, \tag{2.5}$$

for some real constant $L > 0$. We assume also that the square matrix $\sigma(t)$ is of full rank, $\forall\ t \in [0,T]$. Finally, we assume that the *covariance matrix* defined by $a(t) := \sigma(t)\sigma^\top(t)$ is strongly nondegenerate, i.e., $\exists\ \epsilon > 0$ such that

$$\underline{\xi}^\top \sigma(t,\omega)\sigma^\top(t,\omega)\underline{\xi} \geq \epsilon ||\underline{\xi}||^2 \qquad \forall\ \underline{\xi} \in \mathbf{R}^d, (t,\omega) \in [0,\infty) \times \Omega. \tag{2.6}$$

It follows from (2.6) that σ^{-1} and $(\sigma^\top)^{-1}$ exist and are bounded above and below by δ and $1/\delta$ respectively, where δ is some positive real constant.

2.1.2 Auxiliary Probability Measures

In order to utilise martingale theory we require the asset prices to behave like martingales. The nondegeneracy condition allows us to introduce an auxiliary probability measure $\widetilde{P}$, equivalent to P, which will be catalyst to all future developments. Now let us introduce the $\mathbf{R}^d$-valued process

$$\underline{\theta}(t) := (\sigma(t))^{-1}[\underline{b}(t) - r(t)\underline{1}], \tag{2.7}$$

[2]Infinite time horizons are dealt with in Taksar et al. [10] and Morton and Pliska [11]. In these articles the logarithmic growth rate of wealth is maximised.

where $\underline{1}$ is the d-dimensional vector whose every component is 1. This is called the *risk premium vector* or *relative risk process.* By (2.6) and (2.4) it is well-defined and bounded. It is also measurable and adapted due to the assumptions on the market model coefficients.

We also introduce the *exponential martingale* (see Appendix A.1.10)

$$Z(t) := \exp\Big\{ -\int_0^t \underline{\theta}^\top(s)dW(s) - \frac{1}{2}\int_0^t \|\underline{\theta}(s)\|^2 ds \Big\} \tag{2.8}$$

and the *auxiliary probability measure* $\widetilde{P}$ defined on $(\Omega, (\mathcal{F}_T))$ by

$$\widetilde{P}(A) := E[Z(T)I_A]. \tag{2.9}$$

Hence, according to the Girsanov Theorem (see Appendix A.2) the process defined by

$$\widetilde{W}(t) := W(t) + \int_0^t \underline{\theta}(s)ds, \qquad t \in [0, T], \tag{2.10}$$

is an $\mathbf{R}^d$-valued Brownian motion under $\widetilde{P}$. Rewriting (2.2) using (2.7) and (2.10) we obtain

$$dP_i(t) = P_i(t)\Big[r(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)d\widetilde{W}_j(t)\Big]. \tag{2.11}$$

Comparing this with (2.1) we can see that $\widetilde{P}$ equates the appreciation rate of all the stocks to the interest rate of the bond, i.e., $\widetilde{P}$ is the *risk-neutral probability measure* of the market model.

Alternatively, under the *discount factor* defined by :

$$\gamma(t) := \exp\Big\{ -\int_0^t r(s)ds\Big\}, \qquad t \in [0, T], \tag{2.12}$$

we can use Ito's Lemma (see Appendix A.5) to solve the equation (2.11) for the discounted stock prices $\gamma(t)P_i(t)$ given by

$$\gamma(t)P_i(t) = p_i \exp\Big\{ -\int_0^t \sigma_i^\top(s)d\widetilde{W}(s) - \frac{1}{2}\int_0^t \|\sigma_i(s)\|^2 ds\Big\}, \tag{2.13}$$

where $\sigma_i(t) = (\sigma_{i1} \ldots . \sigma_{id})^{\top}$. This is completed in Appendix B.2. Therefore the discounted stock prices are martingales under $\widetilde{P}$.

2.1 *Remark*

The existence of $\widetilde{P}$ with the above properties guarantees that the model is arbitrage free; cf Definition 2.8. This means it is impossible to make riskless profits out of nothing - *no free lunch.*

The uniqueness of $\widetilde{P}$ ensures that all risk in the market can be offset or hedged against by an appropriate trading strategy in the assets.
These properties allow us to solve the contingent claim hedging problem, option pricing and investment problems in the context of the current model.

2.2 The Small Investor's Portfolio

A natural requirement for any investor is the necessity to remain *liquid*. For this reason we must ensure that, at all times, the value of an investor's portfolio is nonnegative. In order to apply duality theory to our problem we require constraints which ensure liquidity. In other words, we must find conditions which will ensure that our portfolio is admissible. To this end, we firstly derive conditions necessary to avoid debt. We then examine to what extent these conditions are sufficient.

2.2.1 Necessary Conditions for Admissibility

We denote by $X(t)$ the *wealth* that the agent has at his disposal at time, t. We have the following definitions :

2.2 Definition. *A* **portfolio process** *is an* $\mathbf{R}^d$*-valued process,*

$$\{\underline{\pi}(t) = (\pi_1(t)....\pi_d(t))^\top : t \in [0,T]\}$$

which is progressively measurable with respect to $\{\mathcal{F}_t\}$ *and satisfies*

$$\int_0^T \|\sigma^\top(t)\underline{\pi}(t)\|^2 dt < \infty \quad a.s.\ P. \tag{2.14}$$

For our purpose $\pi_i(t)$ represents the proportion of wealth invested in the i^{th} stock at time, t. We allow $\pi_i(t)$ to become negative. This is called *short-selling*. Similarly $\sum_{i=1}^d \pi_i(t)$ can exceed 1. This represents *borrowing* at the interest rate $r(t)$ of the bond. The investor is also allowed to spend via the cumulative consumption process[3]. This is defined by :

2.3 Definition. *A* **cumulative consumption process**

$$\{c(t) : t \in [0,T]\},$$

[3]Most of the theory developed to date uses a consumption rate process but the treatment is fundamentally the same.

is a nonnegative, nondecreasing **R***-valued process which is progressively measurable with respect to* $\{\mathcal{F}_t\}$ *and satisfies*

$$c(0) = 0 \text{ and } c(T) < \infty \text{ a.s.} \tag{2.15}$$

The adaptivity of both processes above (their unpredictability) arises because the agent cannot anticipate future values of the prices, i.e., *no insider trading.*

We assume that the agent adopts a *self-financing strategy* - one with no input of capital during $[0, T]$. Under the above notation, for an agent given non-random initial wealth $x > 0$, the wealth $X(t)$ evolves as follows :

$$\begin{aligned} dX(t) &= \sum_{i=1}^{d} \frac{\pi_i(t)X(t)}{P_i(t)} dP_i(t) + \frac{(1 - \underline{\pi}^{\top}(t)\underline{1})X(t)}{P_0(t)} dP_0(t) - dc(t) \\ &= X(t)\left\{ \sum_{i=1}^{d} \pi_i(t) \left[b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij} dW_i^j \right] \right\} \\ &\quad + X(t)(1 - \underline{\pi}^{\top}(t)\underline{1})r(t)dt - dc(t) \\ &= X(t)r(t)dt + X(t)\underline{\pi}^{\top}(t)\sigma(t)[dW(t) + \sigma^{-1}(t)(\underline{b}(t) - r(t)\underline{1})dt] - dc(t). \end{aligned}$$

This yields the *evolution equation* given by

$$\left.\begin{aligned} &dX(t) = X(t)r(t)dt + X(t)\underline{\pi}^{\top}(t)\sigma(t)d\widetilde{W}(t) - dc(t), \\ &X(0) = x > 0. \end{aligned}\right\} \tag{2.16}$$

2.4 Definition. *We define the* **corresponding wealth process** *for portfolio policy* $(\underline{\pi}, c)$ *and initial capital* $x \in (0, \infty)$ *to be the solution* $X(\cdot) \equiv X^{x,\underline{\pi},c}(\cdot)$ *of equation (2.16) above.*

2.5 Definition. *A portfolio policy* $(\underline{\pi}, c)$ *is called* **admissible** *for initial capital* $x \in (0, \infty)$ *if*

$$X^{x,\underline{\pi},c}(t) \geq 0, \quad \forall\, t \in [0,T],$$

holds almost surely.

i.e., if it avoids debt at all times during the interval $[0,T]$.

The set of admissible policies is denoted by

$$\mathcal{A}(x) := \Big\{(\underline{\pi}, c) : X^{x,\underline{\pi},c}(t) \geq 0 \ \forall\, t \in [0,T]\Big\}.$$

By Ito's Rule, the solution of (2.16) satisfies (see Appendix B.3) :

$$\begin{aligned} M(t) &:= \int_0^t \gamma(s)dc(s) + \gamma(t)X(t) \\ &= x + \int_0^t \gamma(s)X(s)\underline{\pi}^\top(s)\sigma(s)d\widetilde{W}(s). \end{aligned} \tag{2.17}$$

The left-hand side consists of the current discounted wealth plus the total discounted consumption to-date. It is a continuous local martingale under the risk-neutral probability measure, $\tilde{P}$.

Also, if we define the process $H(t)$ by

$$H(t) := \gamma(t)Z(t), \tag{2.18}$$

which, by Ito's Lemma, (2.12) and (2.8), satisfies the linear stochastic equation

$$dH(t) = -H(t)[r(t)dt + \theta^\top(t)dW(t)], \tag{2.19}$$

then, analogously, we can solve for the process N(t) defined by

$$\begin{aligned} N(t) &:= \int_0^t H(s)dc(s) + H(t)X(t) \\ &= x + \int_0^t H(s)X(s)[\underline{\pi}^\top(s)\sigma(s) - \theta^\top(s)]dW(s), \end{aligned} \tag{2.20}$$

which is a local martingale under P.

2.6 *Remark*

Note that the process $H(t)$ modifies the *discount factor* $\gamma(t)$ with $Z(t)$ which has been used to equate the growth rates of all assets to that of the bond. In other words we have accounted for the presence of the financial market and multiplication by $H(t)$ *deflates* wealth at time t to the equivalent amount at time 0.

For any admissible $(\underline{\pi}, c) \in \mathcal{A}(x)$ the left-hand side of (2.17) is nonnegative. It follows from an application of Fatou's Lemma (see Appendix A.3) that the process $M(t)$ is a $\widetilde{P}$-supermartingale and consequently, by the optional sampling theorem (see Appendix A.12),

$$\widetilde{E}\Big[\gamma(\tau)X(\tau) + \int_0^\tau \gamma(s)dc(s)\Big] \leq x, \tag{2.21}$$

for every $\tau \in \mathcal{S}_{[0,T]} := \{\text{All } \mathcal{F}_t \text{ stopping times in } [0,T]\}$. This yields the following *necessary conditions for admissibility.*

$$\widetilde{E}\Big[\int_0^T \gamma(s)dc(s)\Big] \leq x, \tag{2.22}$$

$$\widetilde{E}\Big[\gamma(T)X(T)\Big] \leq x. \tag{2.23}$$

This can be stated similarly for the process $N(t)$ with obvious equivalent necessary conditions under expectation E and process $H(T)$.

2.7 *Remark*

For any given $(\underline{\pi}, c) \in \mathcal{A}(x)$ define the *bankruptcy time* as

$$\tau_B := \inf\{t \in [0,T] : X(t) = 0\} \wedge T. \tag{2.24}$$

Since $M(t)$ is a $\widetilde{P}$-supermartingale, then for any $t \in [\tau_B, T]$ on the event $\{\tau_B < T\}$ we have

$$\widetilde{E}\Big[\gamma(t)X(t) + \int_0^t \gamma(s)dc(s)\Big] \leq \widetilde{E}\Big[\gamma(\tau_B)X(\tau_B) + \int_0^{\tau_B} \gamma(s)dc(s),\Big]$$

$$\Rightarrow \quad \widetilde{E}\Big[\gamma(t)X(t)\Big] \leq 0 \text{ a.s.},$$

since

$$\widetilde{E}\Big[\int_0^{\tau_B} \gamma(s)dc(s)\Big] - \widetilde{E}\Big[\int_0^t \gamma(s)dc(s)\Big] \leq 0.$$

Hence for admissible policies, $X(t) \equiv 0, \ \forall \ t \in [\tau_B, T]$ almost surely on $\{\tau_B < T\}$, i.e., if the wealth $X(\cdot)$ becomes equal to zero before the terminal time T it stays there. Further values of $\underline{\pi}$ are irrelevant and are ignored.

2.8 Definition. *An* **arbitrage opportunity** *is a policy* $(\underline{\pi}, c)$ *such that*

(i) $(\underline{\pi}, 0) \in \mathcal{A}(0)$,

(ii) The wealth process $X^{x,\underline{\pi},0}(\cdot)$ *satisfies*

$$P[X(T) > 0] > 0. \tag{2.25}$$

In other words an arbitrage opportunity is the existence of an investment strategy that achieves, with zero initial capital and no intermediate investment, an amount of terminal wealth which is almost surely nonnegative and positive with positive probability. Our model excludes arbitrage by virtue of condition (2.23).

2.2.2 Sufficiency of the Conditions for Admissibility

This section examines the *sufficiency* of conditions (2.22) and (2.23) for admissibility. It turns out that these conditions are *sufficient* in the sense of

Lemmas 2.9 and 2.11. Again the standard approach of Karatzas et al. [7] is followed. Once we can prove their sufficiency they shall acquire the significance of *budget constraints* for the utility maximisation problem; cf Section 2.5. According to conditions (2.22) and (2.23) we define the following, for any $x > 0$:

$$\mathcal{C}(x) := \{\text{All processes } c(t) \text{ satisfying inequality (2.22)}\}, \qquad (2.26)$$

respectively $\mathcal{D}(x)$ for equality in (2.22),

$$\mathcal{L}(x) := \{\text{Nonnegative R.V.'s } X_T \text{ satisfying inequality (2.23)}\}, \qquad (2.27)$$

respectively $\mathcal{M}(x)$ for equality in (2.23). Finally we define

$$\begin{aligned} \mathcal{P}(x) \ := \ & \{\text{All portfolios } \underline{\pi} \text{ such that } (\underline{\pi}, 0) \in \mathcal{A}(x) \\ & \text{and } X_T \in \mathcal{M}(x)\}. \end{aligned} \qquad (2.28)$$

We are primarily interested in $\mathcal{L}(x)$[4]. This set consists of all *attainable* levels of wealth. For any random variable $B \in \mathcal{L}(x)$ an agent can construct a policy $(\underline{\pi}, c) \in \mathcal{A}(x)$ with corresponding wealth process $X(\cdot)$ such that $X(T) = B$ almost surely $\widetilde{P}$. Lemma 2.9 formalises the result.

2.9 Lemma

For every $B \in \mathcal{L}(x)$ there exists a pair $(\underline{\pi}, c) \in \mathcal{A}(x)$ with corresponding wealth $X(\cdot)$ such that $X(T) = B$ almost surely $\widetilde{P}$.

Proof:

Define the processes

$$v(t) := \widetilde{E}[\gamma(T)B \mid \mathcal{F}_t] - \widetilde{E}[\gamma(T)B], \qquad t \in [0, T], \qquad (2.29)$$

[4]We wish to maximise utility of final wealth. The maximisation of utility from consumption is dealt with briefly in Karatzas [9] and Karatzas et al. [7] but is primarily a parallel problem.

which is a $\widetilde{P}$-martingale, and

$$\begin{aligned}\hat{X}(t) &:= \frac{1}{\gamma(t)}\Big[x + v(t) + \widetilde{E}[\gamma(T)B - x]t/T\Big] \qquad (2.30)\\ &= \frac{1}{\gamma(t)}\Big[\widetilde{E}[\gamma(T)B \mid \mathcal{F}_t] + (x - \widetilde{E}[\gamma(T)B])(1 - t/T)\Big].\end{aligned}$$

For all $B \in \mathcal{L}(x)$, this is nonnegative by (2.23) and $\hat{X}(T) = B$, $\widetilde{P}$ almost surely. Applying the Fundamental martingale representation theorem (see Appendix A.10) to $v(t)$ shows it can be written as a stochastic integral

$$v(t) = \int_0^t \phi^\top(s)d\widetilde{W}(s), \qquad (2.31)$$

for some $\mathcal{F}_t$-progressively measurable, $\mathbf{R}^d$-valued process $\phi(t)$ satisfying

$$\int_0^T \|\phi(s)\|^2 ds < \infty \text{ a.s.} \qquad (2.32)$$

Now define the process

$$\hat{\underline{\pi}}(t) := \frac{\gamma^{-1}(t)(\sigma^\top(t))^{-1}\phi(t)}{\hat{X}(t)}. \qquad (2.33)$$

This is a valid portfolio process due to (2.32), Remark 2.7 and the adaptivity and boundedness of $(\sigma^\top)^{-1}$. It also means that $v(t)$ can be represented as

$$v(t) = \int_0^t \hat{X}(s)\hat{\underline{\pi}}^\top(s)\gamma(s)\sigma(s)d\widetilde{W}(s). \qquad (2.34)$$

Now define the process

$$d\hat{c}(t) := \frac{1}{\gamma(t)}\frac{x - \widetilde{E}[\gamma(T)B]}{T}dt. \qquad (2.35)$$

Finally we must show that the corresponding wealth process $X^{x,\hat{\underline{\pi}},\hat{c}}$ for the policy $(\hat{\underline{\pi}}, \hat{c})$ defined by (2.33) and (2.35) is in fact $\hat{X}(\cdot)$. From (2.30) and (2.35) we have

$$\begin{aligned}
\int_0^t \gamma(s)d\hat{c}(s) + \gamma(t)\hat{X}(t) &= \int_0^t (x - \widetilde{E}[\gamma(t)B])/T ds \\
&\quad + x + v(t) + (\widetilde{E}[\gamma(t)B] - x)t/T \\
&= x + v(t) \\
&= x + \int_0^t \hat{X}(s)\hat{\underline{\pi}}^{\top}(s)\gamma(s)\sigma(s)d\widetilde{W}(s),
\end{aligned}$$

from (2.34). This is exactly equation (2.17) so that $\hat{X}$ is the corresponding wealth process for the policy defined and $\hat{X}(T) = B$, $\widetilde{P}$ almost surely.

$\diamond$

2.10 Corollary

For any random variable B in the class $\mathcal{M}(x)$, the policy $(\hat{\underline{\pi}}, \hat{c})$ of Lemma 2.9 is unique and in the class $\mathcal{P}(x)$. Furthermore it has corresponding wealth process given by

$$\gamma(t)X(t) = \widetilde{E}[\gamma(T)B \mid \mathcal{F}_t]. \tag{2.36}$$

Lemma 2.9 and relation (2.22) says that $\mathcal{L}(x)$ consists of precisely those 'levels of terminal wealth' which are attainable from the initial endowment $x \geq 0$, via the choice of some portfolio/consumption pair which avoids debt. However Corollary 2.10 shows that the 'extreme' elements of $\mathcal{L}(x)$ are attainable by strategies that *mandate* zero consumption.

On the other hand $\mathcal{C}(x)$ consists of all *reasonable* consumption processes for which an agent can construct a portfolio $\underline{\pi}$ with corresponding wealth process $X(\cdot) \geq 0$, i.e., one avoiding debt. Lemma 2.11 gives this result.

2.11 Lemma

For every $c(\cdot) \in \mathcal{C}(x)$ *there exists a portfolio process* $\underline{\pi}$ *such that* $(\underline{\pi}, c) \in \mathcal{A}(x)$.

Proof:

The proof is similar to that of Lemma 2.9 and is omitted. **2.12 Corollary**

For any consumption rate process $c(t)$ *in the class* $\mathcal{D}(x)$, *the portfolio* $\hat{\underline{\pi}}$ *of Lemma 2.9 is unique and the corresponding wealth process given by :*

$$\gamma(t)X(t) = \widetilde{E}\Big[\int_t^T \gamma(s)dc(s) \mid \mathcal{F}_t\Big], \qquad t \in [0,T], \tag{2.37}$$

and $X(t) = 0$ *almost surely.*

The final four results of this section characterise the possible levels of wealth attainable and the financable consumption policies. The results are utilised in Sections 2.3 and 2.5. We note here that, althouth the conditions (2.22) and (2.23) are not suffucient to ensure a process $(\underline{\pi}, c)$ avoids debt, they are each sufficient for admissibility in their respective problems. This means that if (2.23) holds we can find a suitable corresponding portfolio to avoid debt. In fact, in the case of those elements of $\mathcal{M}(x)$, the consumption process must be zero. Analogously if (2.22) holds then a suitable corresponding consumption process exists to avoid debt. We note that (2.22) and (2.23) together are not sufficient for admissibility but that we do not require the sufficiency of these conditions to solve the respective problems of maximisation of utility from consumption and utility from final wealth.

2.3 Hedging Contingent Claims

Let's suppose that the investor promises to pay someone a random amount $B(\omega) \geq 0$ at time t = T. What is the value of this promise at time t = 0 ? In other words, how much should the investor charge for selling the contractual obligation that entitles its holder to a payment of size $B(\omega)$ at time T. To solve the problem, suppose the investor sets aside the amount $x > 0$ at time 0. He invests the amount in the market according to the portfolio $\underline{\pi}(t)$ and withdraws funds according to the cumulative consumption process c(t). At time t=T he wants to be certain that he can cover his obligation, i.e.,

$$X^{x,\underline{\pi},c}(T) \geq B \text{ a.s.}$$

His wish is to find the cheapest way to cover his obligation, i.e., the least $x > 0$ for which this hedging is possible. For the investor (seller) this is a fair selling price of the contract at time 0. Any price above this represents an arbitrage opportunity.

2.13 Definition. *A* **contingent claim** *is a nonnegative $\mathcal{F}_t$-measurable random variable B.*

It can be thought of as a contract or agreement that pays B at maturity T. We are interested, not only in the fair price of this contract, but in the hedging portfolio mentioned above. The results of Section 2.2 are particularly important in this respect.

2.14 Definition. *The* **hedging price** *of the contingent claim B is defined by*

$$h(0) := \inf\left[x \in (0, \infty) : \exists(\underline{\pi}, c) \in \mathcal{A}(x) \text{ s.t. } X^{x,\underline{\pi},c}(T) \geq B \text{ a.s.}\right]. \quad (2.38)$$

Corollary 2.10 states that $\mathcal{M}(x)$ is the set of all exactly replicable levels of wealth mandating zero consumption given initial capital x. It also says the corresponding initial wealth is $X(0) = \widetilde{E}[\gamma(t)B]$. This leads us to conjecture that once $\widetilde{E}[\gamma(t)B]$ is finite it will be the minimal hedging price for the claim, B (since no consumption occured). Corollary 2.10 also gives us the form of the wealth process corresponding to the hedging portfolio. The result is given in Theorem 2.15.

2.15 Theorem

The infimum in (2.38) is attained. In particular

$$h(0) = \widetilde{E}[\gamma(t)B]. \tag{2.39}$$

Furthermore there exists a policy $(\hat{\underline{\pi}}, 0)$ *such that* $\hat{X} \equiv X^{h(0),\hat{\underline{\pi}},0}(\cdot)$ *is given by*

$$\hat{X}(t) = \frac{1}{\gamma(t)}\widetilde{E}[\gamma(T)B \mid \mathcal{F}_t], \qquad t \in [0,T]. \tag{2.40}$$

Proof:

Assume $h(0) < \infty$ and equal to x, say. In other words there exists some admissible pair $(\hat{\underline{\pi}}, c) \in \mathcal{A}(x)$ such that $X^{x,\hat{\underline{\pi}},c}(T) \geq B$ almost surely. Then, necessarily from (2.23),

$$u \;:=\; \widetilde{E}[\gamma(T)B] \leq x,$$

$$\Rightarrow \quad u \leq h(0).$$

For the opposite inequality, define the process

$$\hat{X}(t) = \frac{1}{\gamma(t)}\widetilde{E}[\gamma(T)B \mid \mathcal{F}_t], \qquad t \in [0,T], \tag{2.41}$$

with $\hat{X}(0) = u$, $\hat{X}(T) = B$. But since $\widetilde{E}[\gamma(T)B \mid \mathcal{F}_t]$ is a $\widetilde{P}$-martingale then, by the Martingale representation theorem $\hat{X}(t)$ can be represented as

$$\hat{X}(t) = \frac{1}{\gamma(t)}\left[u + \int_0^t \phi^{\mathsf{T}}(s)d\widetilde{W}(s)\right], \tag{2.42}$$

for a suitable $\mathcal{F}_t$-progressively measurable process $\phi(t)$ with values in $\mathbf{R}^d$ and satisfying

$$\int_0^T \|\phi(t)\|^2 dt < \infty \text{ a.s.} \tag{2.43}$$

Then we can define $\hat{\underline{\pi}}(t)$ by

$$\gamma(t)\hat{\underline{\pi}}(t)\hat{X}(t) := \sigma^{-1}(t)\phi(t). \tag{2.44}$$

This gives a well defined portfolio process (recall the boundedness of $\sigma^{-1}(t)$ and $\gamma(t)$ and Remark 2.7). Clearly, from (2.42) and (2.44)

$$\gamma(t)\hat{X}(t) = u + \int_0^t \gamma(s)\hat{X}(s)\hat{\underline{\pi}}^{\top}(s)\sigma(s)d\widetilde{W}(s). \tag{2.45}$$

Comparing (2.45) with (2.17) it is clear that $\hat{X} \equiv X^{u,\hat{\underline{\pi}},0}(\cdot)$. Therefore there exists a portfolio process with initial capital u which always hedges B. This implies that $h(0) \leq u$.

◇

2.4 Utility Functions

To formulate meaningful optimisation problems for the investor we will require the concept of utility functions mentioned in Chapter 1. The utility is a function which quantifies precisely the satisfaction derived from any positive level of wealth.

2.16 Definition. *A function* $u : (0, \infty) \mapsto \mathbf{R}$ *is called a* **utility function** *if it has the following properties :*

(i) u is strictly increasing,
(ii) u is strictly concave,
(iii) u is continuously differentiable,

and can satisfy

$$u'(0+) := \lim_{x \downarrow 0} u'(x) = \infty, \tag{2.46}$$

$$u'(\infty) := \lim_{x \to \infty} u'(x) = 0. \tag{2.47}$$

2.17 *Remark*

Property (i) implies that the investor prefers higher levels of wealth to lower levels. Concavity implies that if the investor with wealth x were offered a *bet* with resultant wealths $x + a$ and $x - a$, each with probability $\frac{1}{2}$, his current utility $u(x)$ would exceed the expected utility from the gamble, $\frac{1}{2}[u(x + a) + u(x - a)]$. The investor is decreasingly risk averse, i.e., his marginal utility is decreasing in x and tends to zero as $x \to \infty$. This is known as the *saturation effect.*

2.18 Definition. *We define the* **marginal utility u'** $: (0, \infty) \mapsto (0, \infty)$ *in the obvious way.*

This function satisfies :

(i) u' is continuous,
(ii) u' is strictly decreasing,
(iii) u' is strictly positive.

These properties follow obviously from (iii),(ii),(i) of Definition 2.16 respectively.

2.19 Definition. *We denote by* $\mathbf{I} : (0,\infty) \mapsto (0,\infty)$ *the* **inverse marginal utility** *such that* I *satisfies* $u'(I(x)) = I(u'(x)) = x$.

$I(x)$ also satisfies :

(i) I is continuous,
(ii) I is strictly decreasing,
(iii) I is strictly positive.

These follow from properties (i),(ii),(iii) of Definition 2.18 respectively. The following conditions may also hold :

$$I(0+) := \lim_{y \downarrow 0} I(y) = \infty, \tag{2.48}$$
$$I(\infty) := \lim_{y \to \infty} I(y) = 0. \tag{2.49}$$

2.20 Definition. *Finally we define the* **Legendre-Fenchel transform** $\tilde{u}$ *of* $-u(x)$ *on* $(0,\infty)$ *as :*

$$\tilde{u}(y) := \max_{x>0}[u(x) - xy], \tag{2.50}$$
$$= u(I(y)) - yI(y). \tag{2.51}$$

This function satisfies :

(i) $\tilde{u}$ is strictly decreasing,

(ii) $\tilde{u}$ is strictly convex, and

(iii)

$$\begin{aligned} u(x) &= \min_{y>0}[\tilde{u}(y) + xy], \qquad x \in (0,\infty), & (2.52) \\ &= \tilde{u}(u'(x)) + xu'(x), \quad x \in (0,\infty). & (2.53) \end{aligned}$$

2.21 *Remark*

We can see that the maximum in (2.50) is achieved at

$$u'(x) - y = 0,$$
$$\Rightarrow x = I(y).$$

Similarly we can find the minimum in (2.52) which is achieved at $y = u'(x)$. Also, properties (i) and (ii) follow from the fact that

$$\tilde{u}(y) = u(I(y)) - yI(y),$$

$$\begin{aligned} \Rightarrow \tilde{u}'(y) &= u'(I(y))I'(y) - yI'(y) - I(y) \\ &= -I(y) < 0, \end{aligned}$$

$$\Rightarrow \tilde{u}''(y) = -I'(y) > 0.$$

Following from (2.50) and (2.52) respectively we have, for all $x, y > 0$, the inequalities

$$\begin{aligned} u(I(y)) &\geq u(x) + y(I(y) - x), & (2.54) \\ \tilde{u}(u'(x)) &\leq \tilde{u}(y) - x(u'(x) - y). & (2.55) \end{aligned}$$

The monotonicity of u and $\tilde{u}$ guarantees the limits

$$u(0+) := \lim_{x \downarrow 0} u(x), \qquad u(\infty) := \lim_{x \to \infty} u(x), \tag{2.56}$$

$$\tilde{u}(0+) := \lim_{y \downarrow 0} \tilde{u}(y), \qquad \tilde{u}(\infty) := \lim_{y \to \infty} \tilde{u}(y), \tag{2.57}$$

exist in the extended real number system.

2.22 Lemma

(i) $u(0) = \tilde{u}(\infty)$,
(ii) $\tilde{u}(0) = u(\infty)$.

Proof:

(i) Firstly note that, from Definition 2.16 and Definition 2.19 (iii), we have for all $y > 0$

$$\begin{aligned} \tilde{u}(y) &= u(I(y)) - yI(y) \\ &\leq u(I(y)), \\ \Rightarrow \lim_{y \to \infty} \tilde{u}(y) &\leq \lim_{y \to \infty} u(I(y)) \\ &= u(0). \end{aligned}$$

Also, from (2.50), we have for all $x > 0$

$$\begin{aligned} \tilde{u}(y) &\geq u(x) - xy \\ &= u(\epsilon/y) - \epsilon, \qquad \forall \epsilon > 0, \\ \Rightarrow \lim_{y \to \infty} \tilde{u}(y) &\geq \lim_{y \to \infty} u(\epsilon/y) - \epsilon, \qquad \forall \epsilon > 0, \end{aligned}$$

$$= \quad u(0) - \epsilon, \qquad \forall \epsilon > 0,$$

whence $\tilde{u}(\infty) = u(0)$.

(ii) Similarly, from Definition 2.20 (iii) and Definition 2.18 (iii), we have for all $x > 0$

$$\begin{aligned} u(x) &= \tilde{u}(u'(x)) + xu'(x) \\ &\geq \tilde{u}(u'(x)), \\ \Rightarrow \lim_{x\to\infty} u(x) &\geq \lim_{x\to\infty} \tilde{u}(u'(x)) \\ &= \tilde{u}(0). \end{aligned}$$

Also from (2.52)

$$\begin{aligned} u(x) &\leq \tilde{u}(y) + xy, && \forall\, y > 0 \\ &= \tilde{u}(\epsilon/x) + \epsilon, && \forall\, \epsilon > 0, \\ \Rightarrow \lim_{x\to\infty} u(x) &\leq \lim_{x\to\infty} \tilde{u}(\epsilon/x) + \epsilon, && \forall\, \epsilon > 0, \\ &= \tilde{u}(0) + \epsilon, && \forall\, \epsilon > 0, \end{aligned}$$

whence $u(\infty) = \tilde{u}(0)$.

$\diamond$

We will have reason to use the following assumptions in the theory to follow, in particular to prove the existence of the dual solution of section 3.4.

2.23 Assumption *$x \mapsto xu'(x)$ is non-decreasing on $(0,\infty)$*

2.24 Assumption *For some $\alpha \in (0,1), \gamma \in (1,\infty)$ we have*

$$\alpha u'(x) \geq u'(\gamma x), \quad \forall\, x \in (0,\infty).$$

2.25 Lemma

If the utility functions satisfy assumptions 2.23 and 2.24, then

(i) $yI(y)$ is non-increasing,
(ii) $x \mapsto \tilde{u}(e^x)$ is convex on $\mathbf{R}$,
(iii) $\forall\, \alpha \in (0,1), \exists\, \gamma > 1$ *such that* $I(\alpha y) \leq \gamma I(y),\ \forall\, y > 0.$

Proof:

(i) By assumption 2.23

$$xu''(x) + u'(x) \geq 0, \qquad \forall\, x \in (0,\infty),$$

$$\Rightarrow I(y)u''(I(y)) + y \geq 0, \qquad \forall\, y \in (0,\infty),$$

by letting $x = I(y)$ and $y = u'(x)$. Also, since $u'(I(y)) = y$ implies that $u''(I(y))I'(y) = 1$ then

$$\Rightarrow I(y)/I'(y) + y \geq 0, \qquad \forall\, y \in (0,\infty),$$

$$\Rightarrow I(y) + yI'(y) \leq 0, \qquad \forall\, y \in (0,\infty),$$

$$\Rightarrow (yI(y))' \leq 0, \qquad \forall\, y \in (0,\infty).$$

Hence $yI(y)$ is non-increasing.

(ii) By Remark 2.21

$$\begin{aligned}
\tilde{u}'(y) &= -I(y), \\
\Rightarrow \frac{d(\tilde{u}(e^x))}{dx} &= \tilde{u}'(e^x)e^x, \\
&= -e^x I(e^x),
\end{aligned}$$

which from part (i) is non-decreasing for all $e^x \in (0,\infty)$ and hence for all $x \in \mathbf{R}$. Therefore $\tilde{u}(e^x)$ is convex on $\mathbf{R}$.

(iii) Again, setting $y = u'(x) \Rightarrow x = I(y)$ we obtain from Assumption 2.24

$$\begin{aligned}
\alpha u'(x) \geq u'(\gamma x), &\qquad \forall\, x \in (0,\infty), \\
\Rightarrow \alpha y \geq u'(\gamma I(y)), &\qquad \forall\, y \in (0,\infty), \\
\Rightarrow I(\alpha y) \leq \gamma I(y), &\qquad \forall\, y \in (0,\infty), \qquad (2.58)
\end{aligned}$$

for some $\alpha \in (0,1), \gamma \in (1,\infty)$ since I is decreasing. Therefore, assuming $\tilde{\alpha} \in [\alpha, 1)$,

$$\begin{aligned}
I(\tilde{\alpha} y) &= I(\alpha(\tilde{\alpha}/\alpha)y) \\
&\leq \gamma I((\tilde{\alpha}/\alpha)y) \\
&\leq \gamma I(y),
\end{aligned}$$

using the property (2.58) above and the decreasing property of I. Now, reiterating (2.58) for $\tilde{\alpha} \in (0,\alpha)$,

$$\begin{aligned}
I(\tilde{\alpha} y) &= I(\alpha(\tilde{\alpha}/\alpha)y) \\
&\leq \gamma I((\tilde{\alpha}/\alpha)y) \\
&\vdots \\
&\leq \gamma^n I((\tilde{\alpha}/\alpha^n)y),
\end{aligned}$$

until $\tilde{\alpha}/\alpha^n \geq \alpha$

$$\Rightarrow I(\tilde{\alpha}y) \quad \leq \quad \gamma^{n+1} I(y).$$

Hence for all $\alpha \in (0,1), \exists$ some $\gamma > 1$ such that $I(\alpha y) \leq \gamma I(y), \ \forall\, y > 0$.

◇

2.26 *Remark*

Assumption 2.23 means

$$-xu''(x)/u'(x) \leq 1,$$

i.e., the well known *Arrow-Pratt measure of risk aversion* does not exceed 1. The function $\ln x$ is the limiting case.

2.5 Utility Maximisation

2.5.1 Defining the Problem

In this section we address the following question. How should an investor endowed with initial capital $x > 0$ choose, at every time t, his stock portfolio $\underline{\pi}$ and his cumulative consumption process $c(t)$, from among all admissible pairs $(\underline{\pi}, c) \in \mathcal{A}(x)$, in order to obtain the maximum expected utility from his terminal wealth. More precisely, consider the utility function of Section 2.4. We want to maximise

$$J(x; \underline{\pi}, c) := E[u(X^{x,\underline{\pi},c}(T))]. \tag{2.59}$$

over the set of admissible policies given in Definition 2.5. We introduce formally :

2.27 Definition. *The* **utility maximisation problem** *is to maximise $J(x; \underline{\pi}, c)$ over the class $\mathcal{A}(x)$ of processes $(\underline{\pi}, c)$ that satisfy*

$$E[u^-(X^{x,\underline{\pi},c}(T))] < \infty. \tag{2.60}$$

Recall that $u^- := \max[-u, 0]$.

We denote by $\mathring{\mathcal{A}}(x,u)$ the set of policies in $\mathcal{A}(x,u)$ which satisfy condition (2.60) above.

2.28 Definition. *The* **value function** *of this problem is defined by*

$$V(x) := \sup_{(\underline{\pi},c) \in \mathring{\mathcal{A}}(x,u)} J(x; \underline{\pi}, c). \tag{2.61}$$

Within this set-up, the investor attempts to maximise utility from final wealth X_T, within the constraints imposed by the level of his initial capital and quantified by the condition of (2.21). We will require the results of Section 2.2. By Lemma 2.9 once $B \in \mathcal{L}(x)$ the existence of a policy $(\underline{\pi}, c) \in \mathcal{A}(x, u)$ such that $X(T) = B$, $\tilde{P}$-a.s. is assured. This gives us the sufficiency of condition (2.23) for any distribution of wealth B to be attained. Therefore we can treat it as a *constraint* in the sense of duality theory as it is necessary and sufficient.

Thus the problem of (2.61) amounts to maximising the expression above over the class $\mathcal{L}(x)$ of nonnegative $\mathcal{F}_t$-measurable random variables. But this problem is straightforward. Since utility is derived solely from terminal wealth, it seems reasonable to increase X_T within the limits allowed by constraint (2.23). In other words we ensure that $X_T \in \mathcal{M}(x)$ and we can then apply Corollary 2.10. This result is given in Theorem 2.29.

2.29 Theorem

For every $x > 0$ we have

$$V(x) = \sup_{\substack{(\underline{\pi},0)\in\hat{\mathcal{A}}(x,u) \\ \underline{\pi}\in\mathcal{P}(x)}} J(x;\underline{\pi},0). \tag{2.62}$$

where $\mathcal{P}(x)$ is the class given by (2.28).

Proof:

For any $(\underline{\pi}, c) \in \hat{\mathcal{A}}(x, u)$ we know that its corresponding final wealth $X(T)$ is in the class $\mathcal{L}(x)$ and therefore the number

$$\delta := \tilde{E}[\gamma(T)X(T)],$$

is in $[0, x]$, by (2.23).

If $\delta > 0$ then the number

$$B := \frac{x}{\delta} X_T,$$

belongs to $\mathcal{M}(x)$. Then, from Lemma 2.9 and Corollary 2.10, there exists a portfolio $\hat{\underline{\pi}} \in \mathcal{P}(x)$ with corresponding wealth $\hat{X}(T) = B \geq X(T)$ almost surely.

Obviously this means that $E[u(\hat{X}_T)] \geq E[u(X_T)]$ which bounds $E[u(\hat{X}_T)]$ from below and since $(\underline{\pi}, c) \in \hat{\mathcal{A}}(x, u)$ then the policy $(\hat{\underline{\pi}}, 0)$ is also in $\hat{\mathcal{A}}(x, u)$.

Hence, for every $(\underline{\pi}, c) \in \hat{\mathcal{A}}(x, u)$, $\exists$ $(\hat{\underline{\pi}}, 0) \in \hat{\mathcal{A}}(x, u)$ with expected utility at least as great. This implies that

$$J(x; \underline{\pi}, c) \leq J(x; \hat{\underline{\pi}}, 0).$$

If $\delta = 0$ then $X(T) = 0$ and we define

$$B := \frac{x}{\widetilde{E}[\gamma(T)]} > 0,$$

belonging to $\mathcal{M}(x)$ and, again applying Corollary 2.10, we obtain as before a portfolio with corresponding wealth $\hat{X}(T) = B > X(T)$ implying $J(x; \underline{\pi}, c) \leq J(x; \hat{\underline{\pi}}, 0)$.

◇

According to Theorem 2.29 we can reformulate the problem of (2.61). Therefore we ignore consumption and our new problem now has the equality constraint given below.

$$V(x) = \sup_{\underline{\pi} \in \mathcal{P}(x)} J(x; \underline{\pi}, 0), \qquad \text{s.t. } \widetilde{E}[\gamma(T) X(T)] = x. \tag{2.63}$$

This can be viewed as a linear optimisation problem in X_T. To conjecture the form of the optimal portffolio we attempt to find the so called *optimal form* - the special form of the final wealth X_T which will guarantee optimality in the problem of (2.63). We apply the theory of Lagrange multipliers to optimise the above with respect to X_T. From the standard Lagrange multiplier optimality conditions we have the following conditions necessary for optimality:

Firstly

$$\tilde{E}[\gamma(T)X(T)] = x. \tag{2.64}$$

Secondly, treating the objective function as a function of X_T only

$$
\begin{aligned}
& \frac{d}{dX_T}\Big\{E[u(X_T)] - y\Big(\tilde{E}[\gamma(T)X(T)] - x\Big)\Big\} = 0, \\
\Rightarrow\quad & \frac{d}{dX_T}\Big\{E[u(X_T) - yH(T)X(T) + xy]\Big\} = 0, \\
\Rightarrow\quad & \frac{d}{dX_T}\Big\{\int_\Omega [u(X_T) - yH(T)X(T) + xy]dP\Big\} = 0, \\
\Rightarrow\quad & \int_\Omega \frac{d}{dX_T}\Big\{u(X_T) - yH(T)X(T) + xy\Big\}dP = 0, \\
\Rightarrow\quad & E[u'(X_T) - yH(T)] = 0,
\end{aligned}
$$

for some suitable Lagrange multiplier $y > 0$, by dominated convergence. We note here that the limit can be taken inside the expectation operator once the expression $[u(X_T + \epsilon) - u(X_T)]/\epsilon$ is bounded above for all $\epsilon > 0$. However we are merely conjecturing the optimal form and we propose that this is

$$X(T) = I(yH(T)) \text{ a.s.}, \tag{2.65}$$

and from (2.64) we obtain :

$$\widetilde{E}[\gamma(T)I(yH(T))] = x \text{ a.s.,} \tag{2.66}$$

which must be solved for $y > 0$.

2.5.2 Formalising the Discussion

We have conjectured the form of the optimal final wealth in a non-rigourous fashion. We must now prove that this form does in fact ensure optimality. Introduce the function

$$\zeta(y) = E[H(T)I(yH(T))], \qquad y \in (0, \infty). \tag{2.67}$$

$\zeta(y) : (0,\infty) \mapsto (0,\infty)$ is continuous and strictly decreasing with

$$\begin{aligned} \zeta(0+) &= \infty, \\ \zeta(\infty) &= 0. \end{aligned}$$

Also, we assume that for all $y \in (0, \infty)$

$$\zeta(y) < \infty. \tag{2.68}$$

This is necessary for a solution to (2.66) to exist. Introduce also the inverse of ζ, denoted by ψ. Fixing the initial capital $x > 0$, the $\mathcal{F}_t$-measurable random variable

$$X(T) = I(\psi(x)H(T)), \tag{2.69}$$

belongs to the class $\mathcal{M}(x)$. Hence from Corollary 2.10 there exists a unique $(\hat{\underline{\pi}}, \hat{c})$ such that $X^{x,\hat{\underline{\pi}},\hat{c}}(T) = I(\psi(x)H(T))$ almost surely. In fact $\hat{c} \equiv 0$ and the corresponding wealth process is given by

$$\begin{aligned} X(t) &= \widetilde{E}[\gamma(t,T)X(T) \mid \mathcal{F}_t] \\ &= x + \int_0^t \gamma(s)\hat{\underline{\pi}}^\top(s)\hat{X}(s)\sigma(s)d\widetilde{W}(s). \end{aligned} \tag{2.70}$$

We have conjectured the form of the optimal final wealth. Theorem 2.30 gives the result.

2.30 Theorem

Assume $\psi(x) < \infty$ and consider the random variable given by $X(T) = I(\psi(x)H(T))$. The pair $(\hat{\pi}, 0)$ of Lemma 2.9 belongs to $\hat{\mathcal{A}}(x,u)$ and is optimal for the problem of (2.61).

Proof:

It suffices to show that X_T of the form given by (2.69) satisfies (2.60) and that for any other $\hat{X}(T) \in \mathcal{L}(x)$ satisfying (2.60) we have

$$E[u(\hat{X}_T)] \leq E[u(X_T)] \text{ a.s.} \tag{2.71}$$

Recall the inequality (2.54) which states that for all $\alpha > 0, y > 0$

$$u(I(y)) \geq u(\alpha) + y[I(y) - \alpha].$$

This implies that

$$u(I(y)) \geq u(\hat{X}_T) + y[I(y) - \hat{X}_T],$$

holds almost surely for any $\hat{X}_T \in \mathcal{L}(x)$ (since $\hat{X}_T$ is nonnegative). With $y = \psi(x)H(T) > 0$ for fixed capital x and wealth from (2.69), we obtain

$$u(X_T) \geq u(\hat{X}_T) + \psi(x)H(T)[X_T - \hat{X}_T] \text{ a.s.} \tag{2.72}$$

Now, with the particular choice of

$$\hat{X}_T = B = \frac{x}{\tilde{E}[\gamma(T)]} > 0,$$

from Theorem 2.29, which is in the class $\mathcal{M}(x)$, we can say that the right-hand side of (2.72) is P-integrable. In fact

$$\begin{aligned} E[u(X_T)] &\geq E[u(\hat{X}_T)] + E\Big[\psi(x)H(T)\big(X_T - x/\tilde{E}[\gamma(T)]\big)\Big] \\ &= E[u(\hat{X}_T)], \end{aligned}$$

since X_T belongs to the class $\mathcal{M}(x)$. Now $E[u(X_T)]$ is bounded below by $E[u(\hat{X}_T)]$ and, since $\hat{X}_T$ is a constant B, condition (2.60) holds. For the inequality (2.71) note that

$$
\begin{aligned}
E[u(X_T)] &\geq E[u(\hat{X}_T)] + E[\psi(x)H(T)(X_T - \hat{X}_T)] \\
&= E[u(\hat{X}_T)] + \psi(x)\underbrace{(x - E[H(T)\hat{X}_T])}_{\geq 0}.
\end{aligned}
$$

The expression above is nonnegative once $\hat{X}_T \in \mathcal{L}(x)$. Therefore for any $\hat{X}_T \in \mathcal{L}(x)$ the inequality (2.71) holds.

◇

We have found the optimal form for the terminal wealth. We can use Ito's lemma to solve for the corresponding wealth process in terms of the market parameters. We then use Corollary 2.10 to obtain a second form for the optimal wealth in terms of $\underline{\pi}$. We compare both forms to find the optimal portfolio. The exact strategy is given in Chapter 4.

Chapter 3

The Large Investor Problem

3.1 The New Model

We consider now the type of investor whose investment policy influences the behaviour of the prices $P_0, \{P_i\}_{1\leq i\leq d}$ of the $d+1$ financial assets. More precisely, these prices evolve according to the adjusted market described by the stochastic differential equations :

$$\begin{aligned} dP_0(t) &= P_0(t)[r(t) + f_0(\underline{\pi}_t)], \\ P_0(0) &= 1, \end{aligned} \tag{3.1}$$

for the bond, and with the stocks given by

$$\begin{aligned} dP_i(t) &= P_i(t)\Big[[b_i(t) + f_i(\underline{\pi}_t)]dt + \sum_{j=1}^{d} \sigma_{ij} dW_t^j\Big], \\ P_i(0) &= p_i, \end{aligned} \tag{3.2}$$

for $i = 1...d$.

The functions $f_i : \mathbf{R}^d \to \mathbf{R}$ for $i = 0...d$ are some given functions describing the effect of the investor's strategy on the asset prices. As before the investor is allowed to invest by way of a portfolio process $\underline{\pi}(t)$ defined as in Definition 2.2 and to spend via the cumulative consumption process $c(t)$ given in Definition 2.3.

Similarly to before the wealth $X(t)$ of the investor evolves according to the ***evolution equation*** given by

$$\left.\begin{aligned} dX(t) &= X(t)g(t,\underline{\pi}_t)dt + X(t)\underline{\pi}^\top(t)\sigma(t)dW(t) - dc(t) \\ X(0) &= x > 0, \end{aligned}\right\} \tag{3.3}$$

where

$$g(t,\underline{\pi}_t) := r(t) + f_0(\underline{\pi}_t) + \sum_{i=1}^{d} \underline{\pi}_i(t)\Big[\big(b_i(t) + f_i(\underline{\pi}_t) - r(t) - f_0(\underline{\pi}_t)\big)\Big].$$

3.1 *Remark*

The impact of the investor's strategy may arise because of size only or merely because other traders believe the large investor has superior information. However we must note that the market described above need not be interpreted as a large investor's market. The interpretation of policy-dependent prices is not the only one. We could just start with any economy whose wealth process is, for whatever reasons, given by (3.3) above and forget about the prices.

3.2 Definition. *Similarly we define the* **corresponding wealth process** *for portfolio policy $(\underline{\pi}, c)$ and initial capital $x \in (0,\infty)$ to be the solution $X(\cdot) \equiv X^{x,\underline{\pi},c}(\cdot)$ of equation (3.3) above.*

As before a portfolio policy $(\underline{\pi}, c)$ is called **admissible** *for initial capital*

$x \in (0,\infty)$ *if* $X^{x,\underline{\pi},c}(t) \geq 0 \; \forall \; t \in [0,T]$ *holds almost surely and the set of admissible policies is denoted by*

$$\mathcal{A}(x) := \Big\{(\underline{\pi}, c) : X^{x,\underline{\pi},c}(t) \geq 0 \; \forall \; t \in [0,T]\Big\}.$$

We wish to introduce a set of auxiliary markets, indexed by $\underline{v}$, and structured in the same way as those markets of Chapter 2. Each market will have an short interest rate, denoted later by $\tilde{g}(t,\underline{v})$, and appreciation rates which are independent of the portfolio process. The volatility matrix $\sigma(t)$ will remain the same. There exists a solution to the utility maximisation problem in each of these markets. We require the appropriate $\underline{v}$ such that the large investor assumptions hold. To this end, we restrict ourselves by imposing the following standing assumptions.

3.3 Standing Assumption The function $g(t,\cdot)$ is concave $\forall \; t \in [0,T]$.

3.4 Standing Assumption The function $g(t,\cdot)$ is also uniformly (w.r.t. t) Lipschitz, i.e.,

$$|g(t,\underline{x}) - g(t,\underline{y})| \leq k\|\underline{x} - \underline{y}\|, \quad \forall \; t \in [0,T],$$

for some $k \in (0,\infty)$.

3.5 Definition. *We now define the* **convex conjugate function** $\tilde{g}$ *of* g *by*

$$\tilde{g}(t,\underline{v}) := \sup_{\underline{\pi}\in\mathbf{R}^d} [g(t,\underline{\pi}) + \underline{\pi}^{\top}\underline{v}]. \tag{3.4}$$

We note also that, by definition, $\tilde{g}(t,\cdot)$ is convex (see Appendix B.1).

3.6 Definition. $\tilde{g}(t,\underline{v})$ *is finite on its* **effective domain**

$$D_t := \Big\{\underline{v} : \tilde{g}(t,\underline{v}) < \infty\Big\},$$

and associated with this is the class of processes D defined by

$$D := \Big\{\underline{v}(t) : \tilde{g}(t, \underline{v}_t) < \infty \ \forall \ t\Big\}.$$

We also make the following assumptions on the set D.

3.7 Standing Assumption We assume D is not empty.

3.8 Standing Assumption We also assume that the function $\tilde{g}(t,\cdot)$ is bounded uniformly in t, on its effective domain D_t, i.e.,

$$\tilde{g}(t,\cdot) \leq M, \quad \forall \ \underline{v} \in D_t, \ \forall \ t.$$

3.9 *Remark*

Assumptions 3.3 and 3.4 imply that the sets D_t are uniformly bounded. We also impose the following conditions on the set D. In the theory to follow we require the set D to satisfy

(i) $\underline{v}$ is $\mathcal{F}_t$-measurable,

(ii) $\underline{v}$ is uniformly bounded,

(iii) $E\Big[\int_0^T \|\underline{v}_t\|^2 dt + \int_0^T \tilde{g}(t, \underline{v}_t) dt\Big] < \infty$.

The set D is convex (see Appendix B.1).

Now for all $\underline{v} \in D$ define the processes

$$\gamma_{\underline{v}}(u,t) \ := \ \exp\Big\{ - \int_u^t \tilde{g}(s, \underline{v}_s) ds \Big\}, \tag{3.5}$$

$$\theta_{\underline{v}}(t) \ := \ -\sigma^{-1}(t)\underline{v}(t), \tag{3.6}$$

$$Z_{\underline{v}}(u,t) \;:=\; \exp\Big\{-\int_u^t \theta_{\underline{v}}(s)dW_s - \frac{1}{2}\int_u^t \|\theta_{\underline{v}}(s)\|^2 ds\Big\}, \tag{3.7}$$

$$W_{\underline{v}}(t) \;:=\; W(t) + \int_0^t \theta_{\underline{v}}(s)ds, \tag{3.8}$$

with $\gamma_{\underline{v}}(t) := \gamma_{\underline{v}}(0,t)$ and $Z_{\underline{v}}(t) := Z_{\underline{v}}(0,t)$. Also define the measure

$$P^{\underline{v}}[A] \;:=\; E^{\underline{v}}[I_A] = E[Z_{\underline{v}}(T)I_A]. \tag{3.9}$$

It is clear that, since the set D is uniformly bounded, $Z_{\underline{v}}(\cdot)$ is a martingale, the measure $P^{\underline{v}}$ is a probability measure and by the Girsanov Theorem (see Appendix A.2), $W_{\underline{v}}$ is a Brownian motion. From (3.5) and (3.7), the stochastic equations

$$d\gamma_{\underline{v}}(t) \;=\; -\tilde{g}(t,\underline{v}_t)\gamma_{\underline{v}}(t)dt, \tag{3.10}$$

$$dZ_{\underline{v}}(t) \;=\; -\theta_{\underline{v}}(t)Z_{\underline{v}}(t)dW_t, \tag{3.11}$$

are satisfied by $\gamma_{\underline{v}}(t), Z_{\underline{v}}(t)$ respectively. Hence, by Ito's Lemma (see Appendix A.5) the process defined by

$$H_{\underline{v}}(t) := Z_{\underline{v}}(t)\gamma_{\underline{v}}(t)dt, \tag{3.12}$$

satisfies the stochastic differential equation

$$dH_{\underline{v}}(t) = H_{\underline{v}}(t)[-\tilde{g}(t,\underline{v}_t)dt - \theta_{\underline{v}}(t)dW_t]. \tag{3.13}$$

Reapplying Ito's Rule and by equations (3.3) and (3.13) we have for all $\underline{v} \in D$

$$d(H_{\underline{v}}(t)X(t)) \;=\; H_{\underline{v}}(t)X(t)[g(t,\underline{\pi}_t)dt + \underline{\pi}^{\top}(t)\sigma(t)dW_t] - H_{\underline{v}}(t)dc(t)$$

$$+H_{\underline{v}}(t)X(t)[-\tilde{g}(t,\underline{v}_t)dt - \theta_{\underline{v}}(t)dW_t]$$

$$+H_{\underline{v}}(t)X(t)\underline{\pi}^{\top}(t)\underline{v}(t)dt$$

$$= H_{\underline{v}}(t)X(t)[\underline{\pi}^{\top}(t)\underline{v}(t) + g(t,\underline{\pi}_t) - \tilde{g}(t,\underline{v}_t)]dt - H_{\underline{v}}(t)dc(t)$$

$$+H_{\underline{v}}(t)X(t)[\underline{\pi}^{\top}(t)\sigma(t) + \sigma^{-1}(t)\underline{v}_t]dW_t. \tag{3.14}$$

Then for all $\underline{v} \in D$

$$H_{\underline{v}}(t)X(t) + \int_0^t H_{\underline{v}}(s)X(s)\underbrace{[\tilde{g}(s,\underline{v}_s) - g(s,\underline{\pi}_s) - \underline{\pi}_s^{\top}\underline{v}_s]}_{\geq 0}ds + \int_0^t \underbrace{H_{\underline{v}}(s)dc(s)}_{\geq 0}$$

$$= x + \int_0^t H_{\underline{v}}(s)X(s)[\underline{\pi}^{\top}(s)\sigma(s) + \sigma^{-1}(s)\underline{v}_s]dW(s). \tag{3.15}$$

Recall that for any admissible policy $(\underline{\pi}, c), X(t) \geq 0$. Hence the expression on the left-hand side above is non-negative. In particular the right-hand side is a non-negative local martingale and hence, by Fatou's lemma, a supermartingale under P. Applying the supermartingale property to the left-hand side we obtain

$$E\Big[H_{\underline{v}}(T)X(T) + \int_0^T H_{\underline{v}}(s)dc(s) \tag{3.16}$$

$$+ \int_0^T H_{\underline{v}}(s)X(s)[\tilde{g}(s,\underline{v}_s) - g(s,\underline{\pi}_s) - \underline{\pi}^{\top}(s)\underline{v}_s]ds\Big] \leq x.$$

This is similar to the process $N(t)$ of (2.17). In Appendix B.5 the same procedure is applied for the analogy of $M(t)$ of (2.20). Since the above expression must hold $\forall\ \underline{v} \in D$, under all admissible policies, we have the following definition.

3.10 Definition. *The* **necessary admissibility condition** *for policy* $(\underline{\pi}, c)$ *is*

$$\sup_{\underline{v}\in D} E\Big[H_{\underline{v}}(T)X(T) + \int_0^T H_{\underline{v}}(s)X(s)[\tilde{g}(s,\underline{v}_s) - g(s,\underline{\pi}_s) - \underline{\pi}^{\mathsf{T}}(s)\underline{v}_s]ds + \int_0^T H_{\underline{v}}(s)dc(s)\Big] \leq x. \qquad (3.17)$$

3.11 *Remark*

The supermartingale property excludes arbitrage opportunity from the market. If $x = 0$ then necessarily $X(t) = 0$ and $c(t) = 0$, $\forall\ t \in [0, T]$ almost surely, i.e., *no free lunch.*

3.12 *Remark*

If $f_i \equiv 0$, $\forall\ i = 0...d$, then $g(t, \underline{\pi}) = r(t) + \underline{\pi}^{\mathsf{T}}(t)[\underline{b}(t) - r(t)\underline{1}]$. Hence D consists of only one process $\underline{v} = r(t)\underline{1} - \underline{b}(t)$ and we are in the standard complete market model with $\theta(t) = \sigma^{-1}[\underline{b} - r(t)\underline{1}]$ and the unique equivalent martingale measure $\widetilde{P}$ is defined as previously.

We have just introduced a set of probability measures under which the discounted stock prices are martingales. This can be shown analogously to the discounted stock prices of Chapter 2 by using Ito's Rule to solve for $\gamma_{\underline{v}}(t)P_i(t)$. The result is similar to that of Appendix B.2. The next section uses the martingale property of each auxiliary market to hedge a claim in each market and then choose the appropriate $\underline{v}$ in D to ensure the large investor assumptions hold. It uses the procedures of Cvitanic and Karatzas [5] for constrained portfolios.

3.2 Hedging Contingent Claims

We now wish to hedge a contingent claim under *large investor* assumptions. We take a similar approach to that of Chapter 2. However, the use of duality theory is complicated by the nonlinearity of the evolution equation of (3.3) with respect to portfolio $\underline{\pi}$.

3.2.1 Auxiliary Markets

We have introduced a set of auxiliary markets, each one corresponding to an element $\underline{v} \in D$. In each of these markets, the discounted stock price is a martingale under the new probability measure $P^{\underline{v}}$. Therefore, in each of the markets the hedging price of a claim can be found. We introduce the following definition :

3.13 Definition. *A contingent claim B is called* **hedgeable** *if it satisfies*

$$\begin{aligned} V(0) &:= \sup_{\underline{v} \in D} E^{\underline{v}}[\gamma_{\underline{v}}(T) B] \\ &< \infty. \end{aligned} \tag{3.18}$$

We conjecture that the fair (selling) price of a claim for a large investor is the supremum of the prices from the auxiliary markets. The definition is justified in the subsequent discussion. In particular it will be shown that for any hedgeable claim B, there exists a pair $(\underline{\pi}, c) \in \mathcal{A}(\mathrm{V}(0))$ such that the *corresponding wealth process* satisfies

$$X^{V(0),\underline{\pi},c}(T) = B \text{ a.s.}, \tag{3.19}$$

and that V(0) is the minimal initial wealth for which this can be achieved.

We now introduce the *hedging price* of a contingent claim B under portfolios confined to satisfy (3.3) for the large investor. We then show that this price coincides with V(0).

3.14 Definition. *The* **hedging price under large investor assumptions** *of a contingent claim B is defined by*

$$h(0) = \begin{cases} \inf\left[x \in (0,\infty) : \exists\ (\underline{\pi}, c) \in \mathcal{A}(x)\ s.t.\ X^{x,\underline{\pi},c}(T) \geq B\ a.s.\right] \\ \\ \infty \quad \text{if the above set is empty.} \end{cases} \tag{3.20}$$

Let us define the following

$$\begin{aligned} \mathcal{S} &:= \{\text{ All } \mathcal{F}_t\text{-stopping times } \tau \text{ with values in } [0,T]\ \}, \\ \mathcal{S}_{\rho,\sigma} &:= \{\text{ All stopping times } \tau \text{ s.t. } \rho(\omega) \leq \tau(\omega) \leq \sigma(\omega)\ \forall\ \omega \in \Omega\ \}, \end{aligned}$$

for any $\rho, \sigma \in \mathcal{S}$ such that $\rho \leq \sigma$ almost surely.

3.15 Definition. *For every $\tau \in \mathcal{S}$ define the $\mathcal{F}_\tau$ -measurable random variable* $\mathbf{V}(\tau)$ *by*

$$V(\tau) := ess \sup_{\underline{v} \in D} E^{\underline{v}}[B\gamma_{\underline{v}}(\tau, T) \mid \mathcal{F}_\tau]. \tag{3.21}$$

3.16 *Remark*

Note that

$$\begin{aligned} V(0) &= \sup_{\underline{v} \in D} E^{\underline{v}}[B\gamma_{\underline{v}}(T)], \\ \\ V(T) &= B \text{ a.s.} \end{aligned}$$

We now show that the minimal hedging price for the claim B under large investor assumptions is given by the supremum of the hedging prices in the auxiliary markets. In order to prove that this minimal hedging price is in fact $V(0)$ we require the following three lemmas. We note that the *restriction* of a random variable to a stochastic interval means we consider the random variable only for times within that interval.

3.17 Lemma

For any contingent claim satisfying $V(0) < \infty$ the family of random variables $\{V(\tau)\}_{\tau \in \mathcal{S}}$ satisfies the following dynamic programming equation

$$V(\tau) = ess \sup_{\underline{v} \in D_{\tau,\theta}} E^{\underline{v}}[V(\theta)\gamma_{\underline{v}}(\tau,\theta) \mid \mathcal{F}_\tau], \qquad \forall\ \theta \in \mathcal{S}_{\tau,T}, \tag{3.22}$$

where $D_{\tau,\theta}$ is the restriction of D to the stochastic interval $\lfloor \tau, \theta \rfloor$.

Proof:

Recall D is the convex, uniformly bounded set of Remark 3.9. Define for any $\theta \in \mathcal{S}$, the random variable

$$J_{\underline{v}}(\theta) \;:=\; E^{\underline{v}}[V(T)\gamma_{\underline{v}}(\theta,T) \mid \mathcal{F}_\theta] \tag{3.23}$$

$$= \; E[Z_{\underline{v}}(\theta,T)V(T)\gamma_{\underline{v}}(\theta,T) \mid \mathcal{F}_\theta],$$

by Bayes Rule (Appendix A.6) with $Z_{\underline{v}}(t)$ the exponential martingale of (3.7). This depends only on the restriction of $\underline{v}$ to the stochastic interval $\lfloor \theta, T \rfloor$. Now let $\underline{\mu}, \underline{v} \in D$ and define

$$A := \Big\{(t,\omega) : J_{\underline{\mu}}(t,\omega) \geq J_{\underline{v}}(t,\omega)\Big\}.$$

Also define the process $\underline{\lambda} := \underline{\mu} I_A + \underline{v} I_{\bar{A}}$. By convexity of D the process $\underline{\lambda}$ is in D and we have almost surely

$$J_{\underline{\lambda}}(\theta) = \max\Big[J_{\underline{\mu}}(\theta), J_{\underline{v}}(\theta)\Big].$$

Therefore the family $\{J_{\underline{v}}(\theta)\}_{\underline{v} \in D}$ is directed upwards (in the sense of $\preceq$,the relation defined in Appendix A.4). From Definition 3.15

$$V(\theta) \;:=\; ess \sup_{\underline{v} \in D} E^{\underline{v}}[V(T)\gamma_{\underline{v}}(\theta,T) \mid \mathcal{F}_\theta]$$

$$= \operatorname{ess\,sup}_{\underline{v} \in D} J_{\underline{v}}(\theta).$$

By Neveu [1] (see Appendix A.4) with $\{J_{\underline{v}}(\theta)\}_{\underline{v} \in D}$ as our family F of real measurable functions we can say that there exists at least one sequence

$$\{\underline{v}_k\}_{k \in N} \subseteq D, \tag{3.24}$$

such that $J_{\underline{v}_k}(\theta)$ is increasing and

$$V(\theta) = \sup_k J_{\underline{v}_k}(\theta).$$

Hence, since the family $\{J_{\underline{v}}(\theta)\}$ is directed upwards

$$V(\theta) = \lim_{k \to \infty} \uparrow J_{\underline{v}_k}(\theta) \text{ a.s.} \tag{3.25}$$

Returning to the proof observe that, using the tower property and *taking out what's known* (see Appendix A.3), for $\tau < \theta$,

$$\begin{aligned}
V(\tau) &:= \operatorname{ess\,sup}_{\underline{v} \in D} E^{\underline{v}}\Big[B\gamma_{\underline{v}}(\tau, T) \mid \mathcal{F}_\tau\Big] \\
&= \operatorname{ess\,sup}_{\underline{v} \in D_{\tau,T}} J_{\underline{v}}(\tau) \\
&= \operatorname{ess\,sup}_{\underline{v} \in D_{\tau,T}} E^{\underline{v}}\Big[E^{\underline{v}}\Big[V(T)\gamma_{\underline{v}}(\tau, T) \mid \mathcal{F}_\theta\Big] \mid \mathcal{F}_\tau\Big] \\
&= \operatorname{ess\,sup}_{\underline{v} \in D_{\tau,T}} E^{\underline{v}}\Big[\gamma_{\underline{v}}(\tau, \theta) E^{\underline{v}}\Big[V(T)\gamma_{\underline{v}}(\theta, T) \mid \mathcal{F}_\theta\Big] \mid \mathcal{F}_\tau\Big] \\
&= \operatorname{ess\,sup}_{\underline{v} \in D_{\tau,T}} E^{\underline{v}}\Big[\gamma_{\underline{v}}(\tau, \theta) J_{\underline{v}}(\theta) \mid \mathcal{F}_\tau\Big] \\
&\leq \operatorname{ess\,sup}_{\underline{v} \in D_{\tau,T}} E^{\underline{v}}\Big[\gamma_{\underline{v}}(\tau, \theta) V(\theta) \mid \mathcal{F}_\tau\Big].
\end{aligned}$$

The last inequality here is by definition of $J_{\underline{v}}(\theta)$ and $V(\theta)$. This completes the first half of the proof.

To prove the opposite inequality it suffices to show that for any $\underline{\mu} \in D$

$$V(\tau) \geq E^{\underline{\mu}}\Big[\gamma_{\underline{\mu}}(\tau,\theta)V(\theta) \mid \mathcal{F}_\tau\Big],$$

holds almost surely. Let $\underline{\mu} \in D$ and define

$$M_{\tau,\theta} := \Big\{ \text{ All processes } \underline{v} \in D \text{ s.t. } \underline{v} \equiv \underline{\mu} \text{ on } \lfloor \tau, \theta \rfloor \Big\},$$

and $M_{\tau,\theta} \subseteq D_{\tau,T}$. By Definition 3.15

$$\begin{aligned} V(\tau) &:= ess \sup_{\underline{v}\in D} E^{\underline{v}}\Big[B\gamma_{\underline{v}}(\tau,T) \mid \mathcal{F}_\tau\Big] \\ &= ess \sup_{\underline{v}\in D_{\tau,T}} E^{\underline{v}}\Big[V(T)\gamma_{\underline{v}}(\tau,T) \mid \mathcal{F}_\tau\Big] \\ &\geq ess \sup_{\underline{v}\in M_{\tau,\theta}} E^{\underline{v}}\Big[V(T)\gamma_{\underline{v}}(\tau,T) \mid \mathcal{F}_\tau\Big]. \end{aligned}$$

By the tower property and *taking out what is known* we have, for $\tau < \theta$,

$$\begin{aligned} V(\tau) &\geq ess \sup_{\underline{v}\in M_{\tau,\theta}} E^{\underline{v}}\Big[E^{\underline{v}}\Big[V(T)\gamma_{\underline{v}}(\tau,T) \mid \mathcal{F}_\theta\Big] \mid \mathcal{F}_\tau\Big] \\ &= ess \sup_{\underline{v}\in M_{\tau,\theta}} E^{\underline{v}}\Big[\gamma_{\underline{v}}(\tau,\theta)E^{\underline{v}}\Big[V(T)\gamma_{\underline{v}}(\theta,T) \mid \mathcal{F}_\theta\Big] \mid \mathcal{F}_\tau\Big] \\ &= ess \sup_{\underline{v}\in M_{\tau,\theta}} E^{\underline{v}}\Big[\gamma_{\underline{v}}(\tau,\theta)J_{\underline{v}}(\theta) \mid \mathcal{F}_\tau\Big]. \end{aligned}$$

Now for every $\underline{v} \in M_{\tau,\theta}$ we have, by Bayes Rule, $\forall\ \tau < \theta$

$$V(\tau) \geq E^{\underline{v}}\Big[\gamma_{\underline{v}}(\tau,\theta)J_{\underline{v}}(\theta) \mid \mathcal{F}_\tau\Big]$$

$$
\begin{aligned}
&= E\Big[\gamma_{\underline{v}}(\tau,\theta)J_{\underline{v}}(\theta)Z_{\underline{v}}(\tau,T) \mid \mathcal{F}_\tau\Big] \\
&= E\Big[E\Big[\gamma_{\underline{v}}(\tau,\theta)J_{\underline{v}}(\theta)Z_{\underline{v}}(\tau,T) \mid \mathcal{F}_\theta\Big] \mid \mathcal{F}_\tau\Big] \\
&= E[J_{\underline{v}}(\theta)\gamma_{\underline{v}}(\tau,\theta)Z_{\underline{v}}(\tau,\theta)E[Z_{\underline{v}}(\theta,T) \mid \mathcal{F}_\theta] \mid \mathcal{F}_\tau] \\
&= E[J_{\underline{v}}(\theta)\gamma_{\underline{v}}(\tau,\theta)Z_{\underline{v}}(\tau,\theta) \mid \mathcal{F}_\tau],
\end{aligned}
$$

since $E[Z_{\underline{v}}(\theta,T) \mid \mathcal{F}_\theta] = 1$ by the martingale property. Therefore, since $\underline{v}$ coincides with $\underline{\mu}$ on the stochastic interval $\lfloor \tau, \theta \rfloor$,

$$
\begin{aligned}
V(\tau) &\geq E[J_{\underline{v}}(\theta)\gamma_{\underline{v}}(\tau,\theta)Z_{\underline{v}}(\tau,\theta) \mid \mathcal{F}_\tau] \\
&= E[J_{\underline{v}}(\theta)\gamma_{\underline{\mu}}(\tau,\theta)Z_{\underline{\mu}}(\tau,\theta) \mid \mathcal{F}_\tau] \\
&= \cdots = \\
&= E^{\underline{\mu}}[\gamma_{\underline{\mu}}(\tau,\theta)J_{\underline{v}}(\theta) \mid \mathcal{F}_\tau],
\end{aligned}
$$

for all $\underline{v} \in M_{\tau,\theta}$ by working backwards through the previous four steps. Note here that all expressions above except $J_{\underline{v}}(\theta)$ depend only on the interval $\lfloor \tau, \theta \rfloor$.

Now $J_{\underline{v}}(\theta)$ depends only on the restriction of $\underline{v}$ to $\lfloor \theta, T \rfloor$, so the sequence $\{J_{\underline{v}_k}(\theta)\}_{k\in N}$ with $\underline{v}_k \in D$ in (3.24) can be taken using $\{\underline{v}_k(\theta)\}_{k\in N} \subseteq M_{\tau,\theta}$ and from the above, for all $\underline{v}_k \in M_{\tau,\theta}$,

$$
\begin{aligned}
V(\tau) &\geq E^{\underline{\mu}}[\gamma_{\underline{\mu}}(\tau,\theta)J_{\underline{v}_k}(\theta) \mid \mathcal{F}_\tau], \\
\Rightarrow V(\tau) &\geq \lim_{k\to\infty} E^{\underline{\mu}}[\gamma_{\underline{\mu}}(\tau,\theta)J_{\underline{v}_k}(\theta) \mid \mathcal{F}_\tau]
\end{aligned}
$$

$$= E^{\underline{\mu}}[\gamma_{\underline{\mu}}(\tau,\theta) \lim_{k\to\infty} \uparrow J_{\underline{v}_k}(\theta) \mid \mathcal{F}_\tau]$$

$$= E^{\underline{\mu}}[\gamma_{\underline{\mu}}(\tau,\theta) V(\theta) \mid \mathcal{F}_\tau] \text{ a.s.,}$$

using (3.25) and Monotone Convergence (see Appendix A.7).

◇

3.18 *Remark*

The immediate significance of this lemma (to be used in Lemma 3.19) is that

$$V(\tau)\gamma_{\underline{v}}(\tau) \geq E^{\underline{v}}[V(\theta)\gamma_{\underline{v}}(\theta) \mid \mathcal{F}_\tau] \tag{3.26}$$

holds almost surely for any given $\tau \in \mathcal{S}, \theta \in \mathcal{S}_{\tau,T}$ and $\underline{v} \in$ D.

3.19 Lemma

Let $\mathcal{F}$ satisfy the usual conditions (see Appendix A.1.7). There exists an RCLL process (see Appendix A.1.18), still denoted by V(t) such that for all $t \in [0,T]$

$$V(t) = ess\sup_{\underline{v}\in D} E^{\underline{v}}[B\gamma_{\underline{v}}(t,T) \mid \mathcal{F}_t].$$

In other words, the process $V = \{V(t), \mathcal{F}_t\}$ can be considered in its RCLL modification.

Furthermore $\{Q_{\underline{v}}(t) := V(t)\gamma_{\underline{v}}(t), \mathcal{F}_t, \ \forall\ t \in [0,T]\}$ is a $P^{\underline{v}}$-supermartingale with RCLL paths.

Also $V(\cdot)$ is the smallest adapted, RCLL process satisfying $Q_{\underline{v}}(t)$ is a $P^{\underline{v}}$-supermartingale and $V(T) = B$ almost surely.

Proof:

(i) Let $\mathcal{S}_T = [0,T] \cap \mathcal{Q}$. Consider $\{V(t,\omega), \mathcal{F}_t\}$ defined on $\mathcal{S}_T$ which is positive and adapted. From Remark 3.18 we have for $t, \theta \in \mathcal{S}_T$,

$$E^{\underline{v}}[V(\theta)\gamma_{\underline{v}}(\theta) \mid \mathcal{F}_t] \leq V(t)\gamma_{\underline{v}}(t),$$

holds almost surely for all $t < \theta$.

Therefore $\{Q_{\underline{v}}(t), \mathcal{F}_t\}$ is a $P^{\underline{v}}$-supermartingale on $\mathcal{S}_T$. Then, from Karatzas and Shreve [2] Proposition 1.3.14 (see Appendix A.8), the positive adapted process $\{V(t), \mathcal{F}_t\}$ defined on $\mathcal{S}_T$ has at each point $t \in \mathcal{S}_T$ almost surely finite limits from the right and from the left. The limits

$$V(t+,\omega) = \begin{cases} \lim\limits_{\substack{s \downarrow t \\ s \in \mathcal{S}_T}} V(s,\omega), & t \in [0,T), \\ V(T,\omega), & t = T, \end{cases}$$

$$V(t-,\omega) = \begin{cases} \lim\limits_{\substack{s \uparrow t \\ s \in \mathcal{S}_T}} V(s,\omega), & t \in (0,T], \\ V(0) & t = 0, \end{cases}$$

are well-defined and finite for every $\omega \in \Omega^* := \{t \in \mathcal{Q}\}$ with $P(\Omega^*) = 1$. The resulting processes

$$\left\{V(t+)\gamma_{\underline{v}}(t), \mathcal{F}_{t+}\right\},$$
$$\left\{V(t-)\gamma_{\underline{v}}(t), \mathcal{F}_{t-}\right\},$$

are adapted due to the right continuity of $\mathcal{F}_t$. Furthermore, by Appendix A.8, the process

$$\{V(t+)\gamma_{\underline{v}}(t), \mathcal{F}_{t+}\} \text{ is a RCLL } P^{\underline{v}}\text{-supermartingale, } \forall\, \underline{v} \in D.$$

By right continuity of $\mathcal{F}$ this implies that for all $\underline{v} \in D$ the process

$$\{V(t+)\gamma_{\underline{v}}(t), \mathcal{F}_t\},$$

is a RCLL $P^{\underline{v}}$-supermartingale. In particular, by the supermartingale property of $V(t+)\gamma_{\underline{v}}(t)$, $\forall\, \underline{v} \in D$

$$E^{\underline{v}}[V(T)\gamma_{\underline{v}}(T) \mid \mathcal{F}_t] \leq V(t+)\gamma_{\underline{v}}(t) \text{ a.s.},$$

$$\Rightarrow \quad E^{\underline{v}}[V(T)\gamma_{\underline{v}}(t,T) \mid \mathcal{F}_t] \leq V(t+) \text{ a.s.},$$

$$\Rightarrow \quad ess\sup_{\underline{v}\in D} E^{\underline{v}}[V(T)\gamma_{\underline{v}}(t,T) \mid \mathcal{F}_t] \leq V(t+) \text{ a.s.},$$

$$\Rightarrow \quad V(t) \leq V(t+) \text{ a.s.}$$

On the other hand, setting $s = t + \frac{1}{n}$ a stopping time, and letting $n \to \infty$ we obtain

$$\begin{aligned} V(t+) &:= \lim_{\substack{s\downarrow t \\ s\in\mathcal{S}_T}} V(s,\omega) \\ &= \lim_{n\to\infty} V\Big(t+\frac{1}{n},\omega\Big)\gamma_{\underline{v}}\Big(t,t+\frac{1}{n}\Big). \end{aligned}$$

Since V(t+) is $\mathcal{F}_{t+}$-measurable

$$\begin{aligned} V(t+) &= E^{\underline{v}}\Big[V(t+) \mid \mathcal{F}_{t+}\Big] \\ &= E^{\underline{v}}\Big[V(t+) \mid \mathcal{F}_t\Big] \\ &= E^{\underline{v}}\Big[\lim_{n\to\infty} V\Big(t+\frac{1}{n}\Big)\gamma_{\underline{v}}\Big(t,t+\frac{1}{n}\Big) \mid \mathcal{F}_t\Big] \\ &\leq \lim_{n\to\infty} E^{\underline{v}}\Big[V\Big(t+\frac{1}{n}\Big)\gamma_{\underline{v}}\Big(t,t+\frac{1}{n}\Big) \mid \mathcal{F}_t\Big], \end{aligned}$$

by Fatou's lemma (see Appendix A.3). The above implies that, with t and $t+\frac{1}{n}$ stopping times and using Remark 3.18

$$\begin{aligned} V(t+) &\leq \lim_{n\to\infty} V(t) \text{ a.s.} \\ &= V(t) \text{ a.s.} \end{aligned}$$

The result follows by taking V(t) equal to the above process V(t+) for all times t.

(ii) This follows automatically from the right continuity of $\mathcal{F}$ and part (i).

(iii) Finally, let $\tilde{V}$ be an adapted RCLL process satisfying $\{\tilde{Q}_{\underline{v}}(t) := \tilde{V}(t)\gamma_{\underline{v}}(t), \mathcal{F}_t\}$ is a $P^{\underline{v}}$-supermartingale and V(T) = B almost surely. Then for all $t \in [0,T]$ and for all $\underline{v} \in$D

$$\begin{aligned} & E^{\underline{v}}[\tilde{V}(T)\gamma_{\underline{v}}(T) \mid \mathcal{F}_t] \leq \tilde{V}(t)\gamma_{\underline{v}}(t), \\ \Rightarrow\ & E^{\underline{v}}[\tilde{V}(T)\gamma_{\underline{v}}(t,T) \mid \mathcal{F}_t]\gamma_{\underline{v}}(t) \leq \tilde{V}(t)\gamma_{\underline{v}}(t), \\ \Rightarrow\ & E^{\underline{v}}[B\gamma_{\underline{v}}(t,T) \mid \mathcal{F}_t] \leq \tilde{V}(t), \\ \Rightarrow\ & ess \sup_{\underline{v}\in D} E^{\underline{v}}[B\gamma_{\underline{v}}(t,T) \mid \mathcal{F}_t] \leq \tilde{V}(t), \\ \Rightarrow\ & V(t) \leq \tilde{V}(t), \end{aligned}$$

holds almost surely and the proof is complete.

◇

3.20 Lemma

$\underline{v}^*$ is optimal, i.e.,

$$V(t) = E^{\underline{v}^*}[B\gamma_{\underline{v}^*}(T) \mid \mathcal{F}_t] \ a.s.,$$

if and only if $Q_{\underline{v}^*}(t)$ is a $P^{\underline{v}^*}$-martingale.

Proof:

Using the tower property twice we obtain

$$\begin{aligned}
E^{\underline{v}}\Big[Q_{\underline{v}}(t) \mid \mathcal{F}_s\Big] &= E^{\underline{v}}\Big[V(t)\gamma_{\underline{v}}(t) \mid \mathcal{F}_s\Big] \\
&= \gamma_{\underline{v}}(s)E^{\underline{v}}\Big[V(t)\gamma_{\underline{v}}(s,t) \mid \mathcal{F}_s\Big] \\
&\geq \gamma_{\underline{v}}(s)E^{\underline{v}}\Big[E^{\underline{v}}\Big[B\gamma_{\underline{v}}(t,T) \mid \mathcal{F}_t\Big]\gamma_{\underline{v}}(s,t) \mid \mathcal{F}_s\Big] \\
&= \gamma_{\underline{v}}(s)E^{\underline{v}}\Big[E^{\underline{v}}\Big[B\gamma_{\underline{v}}(s,T) \mid \mathcal{F}_t\Big] \mid \mathcal{F}_s\Big] \\
&= \gamma_{\underline{v}}(s)E^{\underline{v}}\Big[B\gamma_{\underline{v}}(s,T) \mid \mathcal{F}_s\Big].
\end{aligned}$$

($\Rightarrow$) If there exists some optimal $\underline{v}^*$ then we have equality above and

$$\begin{aligned}
E^{\underline{v}^*}\Big[Q_{\underline{v}^*}(t) \mid \mathcal{F}_s\Big] &= \gamma_{\underline{v}^*}(s)E^{\underline{v}}\Big[B\gamma_{\underline{v}^*}(s,T) \mid \mathcal{F}_s\Big] \\
&= Q_{\underline{v}^*}(s).
\end{aligned}$$

($\Leftarrow$) Also if $Q_{\underline{v}^*}(t)$ is a $P^{\underline{v}^*}$-martingale then

$$\begin{aligned}
E^{\underline{v}^*}\Big[Q_{\underline{v}^*}(t) \mid \mathcal{F}_s\Big] &= Q_{\underline{v}^*}(s) \\
&= \gamma_{\underline{v}^*}(s)V(s).
\end{aligned}$$

But from the definition of $Q_{\underline{v}}(t)$ and taking out what is known, $\forall\ s < t$

$$E^{\underline{v}^*}\Big[Q_{\underline{v}^*}(t) \mid \mathcal{F}_s\Big] = \gamma_{\underline{v}^*}(s) E^{\underline{v}^*}\Big[V(t)\gamma_{\underline{v}^*}(s,t) \mid \mathcal{F}_s\Big].$$

Comparing the previous two expressions implies

$$E^{\underline{v}^*}\Big[V(t)\gamma_{\underline{v}^*}(s,t) \mid \mathcal{F}_s\Big] = V(s),$$

and taking $t = T$ gives us

$$\begin{aligned} V(s) &= E^{\underline{v}^*}\Big[V(T)\gamma_{\underline{v}^*}(s,T) \mid \mathcal{F}_s\Big] \\ &= E^{\underline{v}^*}\Big[B\gamma_{\underline{v}^*}(s,T) \mid \mathcal{F}_s\Big] \quad \text{a.s.} \end{aligned}$$

Hence $\underline{v}^*$ is optimal.

$\diamond$

3.2.2 The Hedging Price

We now prove the main result of this chapter, namely that the minimal hedging price of any claim B is given by $V(0)$. Furthermore it is possible to construct a portfolio to ensure that the claim is covered at the terminal time.

3.21 Theorem

(i) For an arbitrary contingent claim B we have $h(0) = V(0)$.

(ii) Furthermore $\exists$ a pair $(\hat{\underline{\pi}}, \hat{c}) \in \mathcal{A}(V(0))$ such that

$$X^{V(0),\hat{\pi},\hat{c}}(\cdot) = V(\cdot).$$

Proof:

(i) $h(0) \leq V(0)$. Clearly we may assume $V(0) < \infty$. From lemma 3.19 we know that $Q_{\underline{v}}(t) := V(t)\gamma_{\underline{v}}(t)$ is a $P^{\underline{v}}$-supermartingale with RCLL paths. By the Doob-Meyer Decomposition (see Appendix A.9) we can rewrite $Q_{\underline{v}}(\mathrm{t})$ as

$$Q_{\underline{v}}(t) = Q_{\underline{v}}(0) + M_{\underline{v}}(t) - A_{\underline{v}}(t), \tag{3.27}$$

where $M_{\underline{v}}(t)$ is a local martingale and $A_{\underline{v}}(t)$ is a non-decreasing, adapted process with RCLL paths and $A_{\underline{v}}(0) = 0$, $A_{\underline{v}}(T) < \infty$ almost surely.

Applying the Martingale Representation Theorem (see Appendix A.10) to $M_{\underline{v}}(t)$ yields

$$Q_{\underline{v}}(t) = V(0) + \int_0^t \psi_{\underline{v}}^{\top}(s)dW_{\underline{v}}(s) - A_{\underline{v}}(t), \tag{3.28}$$

where $\psi_{\underline{v}}^{\top}(t)$ is an $\mathbf{R}^d$-valued, $\{\mathcal{F}_t\}$-measurable and a.s. square integrable process.

Consider the positive, adapted RCLL process defined for all $\underline{v} \in D$ by

$$\hat{X}(t) := V(t) = \frac{Q_{\underline{v}}(t)}{\gamma_{\underline{v}}(t)}, \qquad t \in [0, T], \tag{3.29}$$

with $\hat{X}(0) = V(0)$ and $\hat{X}(T) = V(T) = B$ almost surely.

The idea is to find any pair $(\hat{\underline{\pi}}, \hat{c}) \in \mathcal{A}(\mathrm{V}(0))$ such that its corresponding wealth process is actually equal to $\hat{X}(\cdot)$. That is, if we can find at least one admissible policy $(\hat{\underline{\pi}}, \hat{c})$ with resulting terminal wealth $\mathrm{X(T)} = \mathrm{B}$ almost surely and initial capital $\mathrm{V}(0)$. This will prove $h(0) \leq V(0)$. Firstly recall from (3.28) that

$$dQ_{\underline{v}}(t) \;\; = \;\; \psi_{\underline{v}}^{\top}(t)dW_{\underline{v}}(t) - dA_{\underline{v}}(t), \qquad \forall\, \underline{v} \in D, \tag{3.30}$$

and, from (2.7) and (2.10) that, for all $\underline{\mu}$, $\underline{v} \in D$,

$$dW_{\underline{v}}(t) \quad = \quad dW_{\underline{\mu}}(t) - \sigma^{-1}(t)[\underline{v}(t) - \underline{\mu}(t)]dt. \tag{3.31}$$

Now, by definition of $Q_{\underline{v}}(t)$, for all $\underline{\mu}$, $\underline{v} \in D$

$$Q_{\underline{\mu}}(t)/\gamma_{\underline{\mu}}(t) = Q_{\underline{v}}(t)/\gamma_{\underline{v}}(t).$$

Hence

$$Q_{\underline{\mu}}(t) = Q_{\underline{v}}(t) \exp\left[\int_0^t \tilde{g}(s, \underline{v}_s) - \tilde{g}(s, \underline{\mu}_s)ds\right].$$

Therefore

$$\begin{aligned} dQ_{\underline{\mu}}(t) &= dQ_{\underline{v}}(t) \exp\left[\int_0^t \tilde{g}(s, \underline{v}_s) - \tilde{g}(s, \underline{\mu}_s)ds\right] + \\ &\quad Q_{\underline{v}}(t) \exp\left[\int_0^t \tilde{g}(s, \underline{v}_s) - \tilde{g}(s, \underline{\mu}_s)ds\right]\left[\tilde{g}(t, \underline{v}_t) - \tilde{g}(t, \underline{\mu}_t)\right]dt \\ &= \exp\left[\int_0^t \tilde{g}(s, \underline{v}_s) - \tilde{g}(s, \underline{\mu}_s)ds\right]\left[dQ_{\underline{v}}(t) + Q_{\underline{v}}(t)(\tilde{g}(t, \underline{v}_t) - \tilde{g}(t, \underline{\mu}_t))dt\right] \\ &= \exp\left[\int_0^t \tilde{g}(s, \underline{v}_s) - \tilde{g}(s, \underline{\mu}_s)ds\right]\left[\psi_{\underline{v}}^{\top}(t)dW_{\underline{v}}(t) - dA_{\underline{v}}(t)\right. \\ &\qquad \left. + Q_{\underline{v}}(t)(\tilde{g}(t, \underline{v}_t) - \tilde{g}(t, \underline{\mu}_t))dt\right]. \end{aligned}$$

Therefore

$$\begin{aligned} dQ_{\underline{\mu}}(t) &= \exp\left[\int_0^t \tilde{g}(s, \underline{v}_s) - \tilde{g}(s, \underline{\mu}_s)ds\right] \cdot \\ &\quad \left[\psi_{\underline{v}}^{\top}(t)dW_{\underline{\mu}}(t) - \psi_{\underline{v}}^{\top}(t)\sigma^{-1}(t)(\underline{v}(t) - \underline{\mu}(t))dt\right. \\ &\quad \left. - dA_{\underline{v}}(t) + \hat{X}(t)\gamma_{\underline{v}}(t)(\tilde{g}(t, \underline{v}_t) - \tilde{g}(t, \underline{\mu}_t))dt\right], \end{aligned} \tag{3.32}$$

using both (3.30) and (3.31). Comparing the above expression (3.32) with

(3.30) for $dQ_{\underline{\mu}}(t)$ and equating random parts we obtain

$$\psi_{\underline{\mu}}^{\top}(t)dW_{\underline{\mu}}(t) = \exp\left[\int_0^t \tilde{g}(s,\underline{v}_s) - \tilde{g}(s,\underline{\mu}_s)ds\right]\psi_{\underline{v}}^{\top}(t)dW_{\underline{v}}(t),$$

$$\Rightarrow \quad \exp\left[\int_0^t \tilde{g}(s,\underline{\mu}_s)ds\right]\psi_{\underline{\mu}}^{\top}(t) = \exp\left[\int_0^t \tilde{g}(s,\underline{v}_s)ds\right]\psi_{\underline{v}}^{\top}(t).$$

Since the expressions are independent of $\underline{\mu}$ and $\underline{v}$ respectively it follows that they are time dependent only. Therefore we can define

$$\exp\left[\int_0^t \tilde{g}(s,\underline{v}_s)ds\right]\psi_{\underline{v}}^{\top}(t) =: \hat{X}(t)\underline{\hat{\pi}}^{\top}(t)\sigma(t), \tag{3.33}$$

for some appropriate $\underline{\hat{\pi}}(\cdot)$ which is $\mathbf{R}^d$-valued and adapted (since $\psi_{\underline{v}}(t)$ is $\mathcal{F}_t$-measurable $\forall\, t$). Now, by equating the deterministic parts of (3.30) and (3.32) we obtain

$$\begin{aligned} dA_{\underline{\mu}}(t) \quad &= \quad \exp\left[\int_0^t \tilde{g}(s,\underline{v}_s) - \tilde{g}(s,\underline{\mu}_s)ds\right]\Big[\psi_{\underline{v}}^{\top}(t)\sigma^{-1}(t)[\underline{v}(t) - \underline{\mu}(t)]dt \\ &\qquad + dA_{\underline{v}}(t) - \hat{X}(t)\gamma_{\underline{v}}(t)(\tilde{g}(t,\underline{v}_t) - \tilde{g}(t,\underline{\mu}_t))dt\Big]. \end{aligned}$$

Therefore, by (3.33)

$$\begin{aligned} dA_{\underline{\mu}}(t)\exp\left[\int_0^t \tilde{g}(s,\underline{\mu}_s)ds\right] \quad &= \quad \hat{X}(t)\underline{\hat{\pi}}^{\top}(t)[\underline{v}(t) - \underline{\mu}(t)]dt \\ &\qquad + \exp\left[\int_0^t \tilde{g}(s,\underline{v}_s)ds\right]dA_{\underline{v}}(t) \\ &\qquad - \hat{X}(t)(\tilde{g}(t,\underline{v}_t) - \tilde{g}(t,\underline{\mu}_t))dt, \end{aligned}$$

and we obtain

$$\begin{aligned} \Rightarrow \quad & dA_{\underline{\mu}}(t)\exp\left[\int_0^t \tilde{g}(s,\underline{\mu}_s)ds\right] - \hat{X}(t)[\tilde{g}(t,\underline{\mu}_t) - \underline{\hat{\pi}}^{\top}(t)\underline{\mu}(t)]dt = \\ & dA_{\underline{v}}(t)\exp\left[\int_0^t \tilde{g}(s,\underline{v}_s)ds\right] - \hat{X}(t)[\tilde{g}(t,\underline{v}_t) - \underline{\hat{\pi}}^{\top}(t)\underline{v}(t)]dt. \end{aligned}$$

As before these expressions are independent of $\underline{\mu}$ and $\underline{v}$ respectively and therefore are time dependent only. We can then define

$$\gamma_{\underline{v}}^{-1}(t)dA_{\underline{v}}(t) - \hat{X}(t)[\tilde{g}(t,\underline{v}_t) - \hat{\underline{\pi}}^{\top}(t)\underline{v}_t]dt =: \mathrm{d}\hat{\mathrm{c}}(\mathrm{t}) - g(t,\hat{\underline{\pi}}_t)\hat{X}(t)dt, \quad (3.34)$$

which depends on t, $\hat{X}(t)$ and $\hat{\underline{\pi}}(t)$. We must now prove that $\hat{X}(t)$ is actually a *corresponding wealth process* for $(\hat{\underline{\pi}}, \hat{c})$ and that the defined processes $\hat{\underline{\pi}}(t)$ and $\hat{c}(t)$ satisfy all admissibility conditions.

Firstly, recall from Appendix B.5 that the discounted wealth process can be written in the form

$$\begin{aligned} d(\gamma_{\underline{v}}(t)X(t)) &= \gamma_{\underline{v}}(t)X(t)[g(t,\hat{\underline{\pi}}_t) - \tilde{g}(t,\underline{v}_t) + \underline{\pi}^{\top}(t)\underline{v}_t]dt - \gamma_{\underline{v}}(t)dc(t) \\ &\quad + \gamma_{\underline{v}}(t)X(t)\underline{\pi}^{\top}(t)\sigma(t)dW_{\underline{v}}(t). \end{aligned}$$

Now, from (3.30) and the defined processes $\hat{c}$ and $\hat{\underline{\pi}}$, we have

$$\begin{aligned} dQ_{\underline{v}}(t) &= d(\gamma_{\underline{v}}(t)\hat{X}(t)) \\ &= \psi_{\underline{v}}^{\top}(t)dW_{\underline{v}}(t) - dA_{\underline{v}}(t) \\ &= \gamma_{\underline{v}}(t)\hat{X}(t)\hat{\underline{\pi}}^{\top}(t)\sigma(t)dW_{\underline{v}}(t) - \gamma_{\underline{v}}(t)d\hat{c}(t) \\ &\quad + \gamma_{\underline{v}}(t)\hat{X}(t)[g(t,\hat{\underline{\pi}}_t) - \tilde{g}(t,\underline{v}_t) + \hat{\underline{\pi}}^{\top}(t)\underline{v}_t]dt, \end{aligned}$$

subject to the boundary conditions

$$\begin{aligned}\hat{X}(0) &= V(0),\\ \hat{X}(T) &= B \text{ a.s.},\end{aligned}$$

which is equation (3.3) for the process $\hat{X}(\cdot)$. This proves that $\hat{X}(\cdot) \equiv X^{V(0),\hat{\underline{\pi}},\hat{c}(\cdot)}$. It remains to show that $\hat{\underline{\pi}}(t)$ and $\hat{c}(t)$ are valid processes.

Firstly, since $\psi_{\underline{v}}^{\top}(t)$ is $\mathbf{R}^d$-valued, $\mathcal{F}_t$-measurable $\forall\, t$ and almost surely square integrable, then $\hat{\underline{\pi}}(t)$ is also $\mathbf{R}^d$-valued and adapted and satisfies

$$\int_0^T \|\sigma^{\top}(t)\hat{\underline{\pi}}(t)\|^2 dt < \infty,$$

and hence all conditions of Definition 2.2 are satisfied. We recall here that if $\hat{X}(t) = 0$ in (3.33) for any $t \in [0,T]$ then $\hat{X}(s) = 0$ and $\hat{\underline{\pi}}(s) \equiv 0,\ \forall\ s > t$.

By Definition 2.3, we require the process $c(t)$ to be nonnegative, nondecreasing, $\mathcal{F}_t$-measurable with RCLL paths and $c(0) = 0$ and $c(T) < \infty$. Under Assumption 3.4, g is concave and uniformly Lipschitz. Therefore, by El Karoui et al. [12] we can say that for every $\hat{\underline{\pi}}$ there exists a $\hat{\underline{v}} \in D$ such that

$$\tilde{g}(t, \underline{v}_t) \equiv g(t, \hat{\underline{\pi}}_t) + \underline{\pi}(t)\underline{v}(t).$$

The required properties of $c(t)$ follow from those of the process $A_{\underline{v}}(t)$.

(ii) $h(0) \geq V(0)$. We can assume $h(0) < \infty$ and is actually equal to x, say. This implies there exists some admissible pair $(\underline{\pi}, c) \in \mathcal{A}(x)$ such that $X^{x,\underline{\pi},c}(T) \geq B$ a.s. But we know from Definition 3.10 that for any admissible $(\underline{\pi}, c)$ the wealth process must satisfy, for all $\underline{v} \in D$,

$$E\Big[H_{\underline{v}}(T)X(T) + \int_0^T H_{\underline{v}}(s)X(s)[\tilde{g}(s,\underline{v}_s) - g(s,\underline{\pi}_s) - \underline{\pi}^{\top}(s)\underline{v}s]ds$$

$$+\int_0^T H_{\underline{v}}(s)dc(s)\Big] \leq x,$$

$$\Rightarrow \quad E[H_{\underline{v}}(T)X(T)] \leq x,$$

$$\Rightarrow \quad E^{\underline{v}}[\gamma_{\underline{v}}(T)B] \leq x,$$

$$\Rightarrow \quad \sup_{\underline{v}\in D} E^{\underline{v}}[\gamma_{\underline{v}}(T)B] \leq x,$$

$$\Rightarrow \quad V(0) \leq h(0).$$

◇

The theorem gives us the minimal hedging price for a claim B, namely $V(0)$.

3.22 *Remark*

In fact, the portfolio process $\hat{\underline{\pi}}(\cdot)$, consumption process $c(\cdot)$ and the wealth process $\hat{X}(\cdot)$ of the theorem are actually the hedging processes for the claim B. So we have characterised the appropriate processes required to hedge a claim.

We now ask if we require consumption to hedge the claim B. Theorem 3.27 gives the answer. We require the following definition.

3.23 Definition. *A contingent claim B is called* **attainable** *if there exist a portfolio process $\hat{\underline{\pi}}$ which is admissible and such that $(\hat{\underline{\pi}}, 0) \in \mathcal{A}(V(0))$ and*

$$X^{V(0),\hat{\underline{\pi}},0}(T) = B \qquad a.s.$$

To prove Theorem 3.27 we will also require the following three lemmas, which give us compactness of the set of martingales $Z_{\underline{v}}(t)$ over the set D.

3.24 Lemma

The set $\Gamma := \{Z_{\underline{v}}(\cdot) : \underline{v} \in D\}$ *is a convex set of real-valued processes defined on* $[0, T]$.

Proof:

For every $\lambda > 0, \mu > 0$ with $\lambda + \mu = 1$, and for every $\underline{v}_1, \underline{v}_2 \in D$ define the two processes

$$\xi(t) := \lambda Z_{\underline{v}_1} + \mu Z_{\underline{v}_2},$$

$$\hat{\underline{v}}(t) = \frac{1}{\xi(t)}[\lambda \underline{v}_1 Z_{\underline{v}_1} + \mu \underline{v}_2 Z_{\underline{v}_2}].$$

Clearly $\hat{\underline{v}} \in D$ by convexity of D and the fact that

$$\frac{\lambda Z_{\underline{v}_1}}{\xi(t)} + \frac{\mu Z_{\underline{v}_2}}{\xi(t)} = 1.$$

Also

$$\begin{aligned}
d\xi(t) &= \lambda dZ_{\underline{v}_1} + \mu dZ_{\underline{v}_2} \\
&= -\lambda Z_{\underline{v}_1}(t)[\underline{\theta}(t) + \sigma^{-1}(t)\underline{v}_1(t)]^\top dW_t \\
&\quad -\mu Z_{\underline{v}_2}(t)[\underline{\theta}(t) + \sigma^{-1}(t)\underline{v}_2(t)]^\top dW_t \\
&= -\xi(t)(t)[\underline{\theta}(t) + \sigma^{-1}(t)\hat{\underline{v}}(t)]^\top dW_t, \\
\xi(0) &= 1.
\end{aligned}$$

Therefore, since they are both solutions of the same stochastic differential equation, $\xi(t) = Z_{\hat{\underline{v}}}(t) \in \Gamma$.

$\diamond$

3.25 Lemma

The set $\Gamma_T := \{Z_{\underline{v}}(T) : \underline{v} \in D\}$ *is bounded in* $\mathcal{L}^2(P)$.

Proof:

Recall that for all bounded $\theta_{\underline{v}}$ the exponential martingale

$$Z_{\underline{v}}(t) \quad := \quad \exp\Big\{ -\int_0^t \theta_{\underline{v}}(s)dW_s - \frac{1}{2}\int_0^t \|\theta_{\underline{v}}(s)\|^2 ds\Big\},$$

satisfies $E[Z_{\underline{v}}(T)] = 1$. Therefore

$$\begin{aligned}
[Z_{\underline{v}}(T)]^2 &= \Big(\exp\Big\{ -\int_0^T \theta_{\underline{v}}(t)dW_t - \frac{1}{2}\int_0^T \|\theta_{\underline{v}}(t)\|^2 dt\Big\}\Big)^2 \\
&= \exp\Big\{ -\int_0^T 2\theta_{\underline{v}}(t)dW_t - \int_0^T \|\theta_{\underline{v}}(t)\|^2 dt\Big\} \\
&= \exp\Big\{ -\int_0^T 2\theta_{\underline{v}}(t)dW_t - \frac{1}{2}\int_0^T \|2\theta_{\underline{v}}(t)\|^2 dt + \int_0^T \|\theta_{\underline{v}}(t)\|^2 dt\Big\} \\
&= \exp\Big\{ -\int_0^T 2\theta_{\underline{v}}(t)dW_t - \frac{1}{2}\int_0^T \|2\theta_{\underline{v}}(t)\|^2 dt\Big\}\exp\Big\{\int_0^T \|\theta_{\underline{v}}(t)\|^2 dt\Big\} \\
&= Z_{\underline{\hat{v}}}(T)\exp\Big\{\int_0^T \|\theta_{\underline{v}}(t)\|^2 dt\Big\}.
\end{aligned}$$

Using the boundedness of $\theta_{\underline{v}}$ and hence of $\theta_{\underline{\hat{v}}}$ this has finite expectation, i.e.,

$$E\Big([Z_{\underline{v}}(T)]\Big)^2 < \infty,$$

$$\Rightarrow \quad Z_{\underline{v}}(T) \in \mathcal{L}^2(P).$$

$\diamond$

3.26 Lemma

The set Γ_T is strongly closed in $\mathcal{L}^2(P)$.

3.27 Theorem

Every hedgeable contingent claim B is attainable, namely the process $\hat{c}(\cdot)$ from Theorem 3.21 is a zero process.

Proof:

Let $\{\underline{v}_n : n \in \mathrm{N}\} \subseteq D$ be a maximising sequence for $V(0)$, that is

$$\lim_{n\to\infty} E^{\underline{v}_n}[B\gamma_{\underline{v}_n}(T)] = V(0). \tag{3.35}$$

We know from Appendix B.5 that a necessary condition for any admissible $(\underline{\pi}, c)$ is, for all $\underline{v} \in D$,

$$E^{\underline{v}}\Big[\gamma_{\underline{v}}(T)X(T) + \int_0^T \gamma_{\underline{v}}(t)dc(t) + \int_0^T \gamma_{\underline{v}}(t)X(t)[\tilde{g}(t,\underline{v}_t) - g(t,\underline{\pi}_t) - \underline{\pi}_t\underline{v}_t]dt\Big] \leq x.$$

The wealth process of Theorem 3.21 corresponding to $(\hat{\underline{\pi}}, \hat{c})$ is given by $\hat{X}(\cdot) = V(\cdot)$ and since $B \leq \hat{X}(T)$ almost surely we have

$$E^{\underline{v}_n}\Big[\gamma_{\underline{v}_n}(T)B + \int_0^T \gamma_{\underline{v}_n}(t)d\hat{c}(t) + \int_0^T \gamma_{\underline{v}_n}(t)\hat{X}(t)\Psi^{\underline{v}_n,\hat{\underline{\pi}}}(t)dt\Big] \leq V(0),$$

where

$$\Psi^{\underline{v}_n,\hat{\underline{\pi}}}(t) := \tilde{g}(t,\underline{v}_n(t)) - g(t,\hat{\underline{\pi}}_t) - \hat{\underline{\pi}}_t\underline{v}_n(t). \tag{3.36}$$

Taking limits of both sides as $n \to \infty$

$$\lim_{n\to\infty} E^{\underline{v}_n}[\gamma_{\underline{v}_n}(T)B] \;+\; \lim_{n\to\infty} E^{\underline{v}_n}\Big[\int_0^T \gamma_{\underline{v}_n}(t)d\hat{c}(t)\Big]$$

$$+\; \lim_{n\to\infty} E^{\underline{v}_n}\Big[\int_0^T \gamma_{\underline{v}_n}(t)\hat{X}(t)\Psi^{\underline{v}_n,\hat{\underline{\pi}}}(t)dt\Big] \le V(0),$$

whence from (3.35)

$$\lim_{n\to\infty} E^{\underline{v}_n}\Big[\int_0^T \gamma_{\underline{v}_n}(t)d\hat{c}(t)\Big] = 0,$$

$$\lim_{n\to\infty} E^{\underline{v}_n}\Big[\int_0^T \gamma_{\underline{v}_n}(t)\hat{X}(t)\Psi^{\underline{v}_n,\hat{\underline{\pi}}}(t)dt\Big] = 0.$$

The processes

$$\Big\{\gamma_{\underline{v}_n}(t) := \exp\Big[-\int_0^t \tilde{g}(s,\underline{v}_n(s))ds\Big] : n \in \mathbf{N}\Big\},$$

are bounded away from zero. Hence

$$\lim_{n\to\infty} E^{\underline{v}_n}\Big[\int_0^T \gamma_{\underline{v}_n}(t)d\hat{c}(t)\Big] = 0,$$

$$\Rightarrow \lim_{n\to\infty} E^{\underline{v}_n}\Big[\int_0^T d\hat{c}(t)\Big] = 0,$$

$$\Rightarrow \lim_{n\to\infty} E^{\underline{v}_n}[\hat{c}(T)] = 0,$$

$$\Rightarrow \lim_{n\to\infty} E[Z_{\underline{v}_n}(t)\hat{c}(T)] = 0.$$

By Lemmas 3.24 to 3.26 the set Γ_T is weakly compact in $\mathcal{L}^2(P)$. Therefore $\exists\, \underline{v} \in D$ and a new (relabelled) sequence $\{\underline{v}_n : n \in \mathrm{N}\}$ such that along this subsequence

$$\lim_{n\to\infty} E[Z_{\underline{v}_n}(t)\hat{c}(T)] = E[Z_{\underline{v}}(t)\hat{c}(T)] = 0.$$

It follows, since $c(t)$ is nondecreasing, that $\hat{c}(t,\omega) = 0$, $\ell \times \mathcal{P}$ almost everywhere.

◇

3.3 The Portfolio Optimisation Problem

3.3.1 Problem Specification

The problem considered is to maximise expected utility from terminal wealth for the large investor. More precisely, we want to maximise

$$J(x; \underline{\pi}, c) := E[u(X^{x,\underline{\pi},c}(T))], \tag{3.37}$$

over the set of admissible policies given in Definition 3.2. We define the following :

3.29 Definition. *The* **utility maximisation problem** *is to maximise $J(x; \underline{\pi}, c)$ over the class $\mathcal{A}(x,u)$ of processes $(\underline{\pi}, c)$ that satisfy*

$$E[u^-(X^{x,\underline{\pi},c}(T))] < \infty. \tag{3.38}$$

We denote by $\hat{\mathcal{A}}(x,u)$ the set of policies in $\mathcal{A}(x,u)$ which satisfy condition (3.38) above. .

3.30 Definition. *The* **value function** *of this problem is defined by*

$$V(x) := \sup_{(\underline{\pi},c)\in\hat{\mathcal{A}}(x,u)} J(x; \underline{\pi}, c). \tag{3.39}$$

We may wish to make the following assumption on utility which is sufficient but not necessary for the value function to be finite.

3.31 Assumption. u satisfies the *growth condition*

$$0 \le u(x) \le \kappa(1 + x^\alpha), \qquad \forall\, x \in (0, \infty), \tag{3.40}$$

for some $\kappa \in (0, \infty)$ and $\alpha \in (0, 1)$. We can characterise the value function by the following three lemmas. The following lemma is stated without proof (the proof is similar to that in Karatzas [9]).

3.32 Lemma

The function $V(x)$ is increasing and concave on $(0,\infty)$.

We will require the following lemma. The proof is trivial and is omitted.

3.33 Lemma

If the utility function satisfies the growth condition (3.40) then $\exists\ c > 0$ such that

$$u^p(x) \leq c(1 + x^{\alpha p}), \qquad \forall\ x \in (0,\infty),$$

for any $p \in (1, \frac{1}{\alpha}]$.

3.34 Lemma

If the utility function satisfies the growth condition then

$$V(x) < \infty, \qquad \forall\ x \in (0,\infty).$$

Proof:

From Assumptions 3.7 and 3.8 we concluded that the sets D_t are uniformly bounded. We assume that for all $\underline{v} \in D$

$$\int_0^T \|\theta_{\underline{v}}(t)\|^2 dt \ \leq \ C,$$

holds for the relative risk process of (3.6) and some suitable constant $C > 0$. From Appendix B.6 the wealth is given by

$$
\begin{aligned}
X(T) &= \Big\{x - \int_0^T \exp\Big[-\int_0^t g(s,\underline{\pi}_s) + \underline{\pi}^\top(s)\underline{v}_s - \frac{1}{2}\|\underline{\pi}(s)\sigma(s)\|^2 ds \\
&\qquad - \int_0^t \underline{\pi}^\top(s)\sigma(s)dW_{\underline{v}}(s)\Big]dc(t)\Big\} \\
&\quad \cdot \exp\Big\{\int_0^T g(s,\underline{\pi}_s) + \underline{\pi}^\top(s)\underline{v}_s ds\Big\} \\
&\quad \cdot \exp\Big\{\int_0^T \underline{\pi}^\top(s)\sigma(s)dW_{\underline{v}}(s) - \frac{1}{2}\int_0^T \|\underline{\pi}(s)\sigma(s)\|^2 ds\Big\}.
\end{aligned}
$$

Since $\tilde{g}$ is bounded we have

$$
\begin{aligned}
X(T) &\leq x\exp\Big\{\underbrace{\int_0^T g(s,\underline{\pi}_s) + \underline{\pi}^\top(s)\underline{v}_s ds}_{\leq L}\Big\} \\
&\quad \cdot \exp\Big\{\int_0^T \underline{\pi}^\top(s)\sigma(s)dW_{\underline{v}}(s) - \frac{1}{2}\int_0^T \|\underline{\pi}(s)\sigma(s)\|^2 ds\Big\}.
\end{aligned}
$$

With $\alpha p \leq 1$ we have

$$
\begin{aligned}
[X(T)]^{\alpha p} &\leq x^{\alpha p}\exp[\alpha p L]\cdot\exp\Big\{-\frac{\alpha p(1-\alpha p)}{2}\int_0^T \|\underline{\pi}(s)\sigma(s)\|^2 ds\Big\} \\
&\quad \cdot\exp\Big\{\alpha p\int_0^T \underline{\pi}^\top(s)\sigma(s)dW_{\underline{v}}(s) - \frac{\alpha^2 p^2}{2}\int_0^T \|\underline{\pi}(s)\sigma(s)\|^2 ds\Big\} \\
&\leq x^{\alpha p}\exp[\alpha p L]\widetilde{Z}(T),
\end{aligned}
$$

where $\widetilde{Z}(T)$ is an exponential martingale since $\alpha p\underline{\pi}^\top(t)\sigma(t)$ is bounded. Taking expectations with respect to $P^{\underline{v}}$ we obtain

$$
\begin{aligned}
E^{\underline{v}}\Big[[X(T)]^{\alpha p}\Big] &\leq (e^L x)^{\alpha p} E^{\underline{v}}\Big[\widetilde{Z}(T)\Big] \\
&= (e^L x)^{\alpha p},
\end{aligned}
$$

from the martingale property. Now

$$\begin{aligned}
E^{\underline{v}}[Z_{\underline{v}}^{-q}(T)] &= E[Z_{\underline{v}}^{1-q}(T)] \\
&= E\Big[\exp\Big\{(q-1)\int_0^T \theta_{\underline{v}}(s)dW_s - \frac{1}{2}(q-1)\int_0^T \|\theta_{\underline{v}}(s)\|^2 ds\Big\}\Big] \\
&= E\Big[\exp\Big\{(q-1)\int_0^T \theta_{\underline{v}}(s)dW_s - \frac{1}{2}(q-1)^2\int_0^T \|\theta_{\underline{v}}(s)\|^2 ds\Big\} \\
&\quad \cdot\exp\Big\{\frac{1}{2}(q-1)(q-2)\int_0^T \|\theta_{\underline{v}}(s)\|^2 ds\Big\}\Big] \\
&\leq \exp\Big\{\frac{1}{2}(q-1)(q-2)C\Big\},
\end{aligned}$$

due to the boundedness of $\theta_{\underline{v}}$. Returning to the proof note that, using Holder's inequality for $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}
E[u(X(T))] &= E^{\underline{v}}[Z_{\underline{v}}^{-1}(T)u(X(T))] \\
&\leq \big(E^{\underline{v}}[Z_{\underline{v}}^{-q}(T)]\big)^{\frac{1}{q}}\big(E^{\underline{v}}[u^p(X_T)]\big)^{\frac{1}{p}} \\
&\leq \exp\Big\{\frac{1}{2q}(q-1)(q-2)C\Big\}\big(E^{\underline{v}}[c(1+X_T^{\alpha p})]\big)^{\frac{1}{p}} \\
&\leq \exp\Big\{\frac{1}{2q}(q-1)(q-2)C\Big\}\big(c(1+(e^L x)^{\alpha p})\big)^{\frac{1}{p}} \\
&< \infty.
\end{aligned}$$

◇

3.3.2 Discussing the Problem

It is intuitive that to maximise his utility from final wealth the large investor would not consume during $[0, T]$. Indeed Appendix B.6 contains a solution of the general form

$$X^{x,\underline{\pi},c}(t) = \Big(x - \int_0^t \exp[h(\underline{\pi}, s, W)]dc(s)\Big) \exp\Big(\int_0^t k(\underline{\pi}, t)dW_t\Big).$$

This suggests setting $c(t) \equiv 0$ to maximise the wealth process. Theorem 3.21 gives us the minimal initial capital $V(0)$ required to ensure the existence of some $(\hat{\underline{\pi}}, \hat{c}) \in \mathcal{A}(V(0))$ such that

$$X^{V(0),\hat{\underline{\pi}},\hat{c}}(t) \geq B \text{ a.s.},$$

and the policy $(\hat{\underline{\pi}}, \hat{c})$ is given in the theorem. However, Theorem 3.27 says that every claim B is attainable, in particular $\hat{c}$ from Theorem 3.21 is zero.

Utility is derived solely from final wealth, X_T so that to maximise the expected utility it makes sense that the investor would ensure the final level of wealth is just within the constraints of the market. We pursue an optimality condition for terminal wealth which ensures that once $X(T)$ is of a certain form, called the *optimal form*, then $E[u(X_T)]$ is maximised over all admissible policies. This optimal form obviously depends on x.

The investor is endowed with initial capital, x and for all policies $(\hat{\underline{\pi}}, \hat{c})$ the necessary admissibility condition of Definition 3.10 places a natural bound on all possible levels of terminal wealth.

We find the *optimal form* of X_T, denoted by *B(x)*. Theorem 3.21 says that there exists an admissible policy to hedge *B(x)*. But, by Theorem 3.27 we can exactly replicate it. Indeed, since *B(x)* is the *optimal form*, we must ensure that

$$X^{x,\hat{\underline{\pi}},0}(T) \equiv B(x) \text{ a.s.},$$

for optimality to be ensured. Consequent to our discussion we wish to solve the problem given by

$$V(x) \quad = \quad \sup_{(\underline{\pi},0)\in\mathring{\mathcal{A}}(x,u)} \quad J(x;\underline{\pi},0), \tag{3.41}$$

subject to the constraint

$$\sup_{\underline{v}\in D} E\Big[H_{\underline{v}}(T)X(T) \quad + \quad \int_0^T H_{\underline{v}}(s)X(s)[\tilde{g}(s,\underline{v}_s) - g(s,\underline{\pi}_s) - \underline{\pi}_s\underline{v}_s]ds\Big] \leq x.$$

3.3.3 Formulation of the Dual Problem

We now introduce a stochastic problem which is dual to the problem of (3.41). We define the Dual, establish its basic properties and explore the relationship between the Primal and the Dual. This methodology was introduced in Xu and Shreve [8] under short-selling prohibition. This section follows loosely their approach.

3.36 Definition. *An* **optimal portfolio process** *is one which attains the supremum in (3.41).*

Because of the strict concavity of u if such a process exists then the corresponding terminal wealth, $X(T)$ is uniquely determined (see Xu [13] Theorem 1.4.5).

3.37 Definition. *A* **dual control process** *is any* $\underline{v} \in D$.

3.38 Definition. *For* $\tilde{u}$ *of Definition 2.20,* $y > 0$ *and* $\underline{v} \in D$ *we define the* **dual objective function** *by*

$$\widetilde{J}(y,\underline{v}) := E[\tilde{u}(yH_{\underline{v}}(T))]. \tag{3.42}$$

3.39 Definition. *The* **dual value function** *is defined for all* $y > 0$ *by*

$$\widetilde{V}(y) := \inf_{\underline{v}\in D} \widetilde{J}(y,\underline{v}). \tag{3.43}$$

3.40 Definition. *An* **optimal dual process** *with initial condition y is a process $\underline{v}_y \in D$ which attains the infimum above.*

Due to the strict convexity of $\tilde{u}$, if such a process exists it is unique (proof similar to Xu [13] Theorem 3.3.1). The following theorem gives the relationship between the Primal and Dual problems. More importantly we have sufficient conditions for optimality in the problem of Definition 3.29. Duality Theory forms the basis for the proof. It is used explicitly in Karatzas et al. [6] and Xu and Shreve [8] for special types of constraints and implicitly in Karatzas et al. [7] and Cox and Huang [14].

3.41 Theorem (Weak Duality)

For any $x > 0, y > 0$, $\underline{\pi} \in \mathcal{A}(x,u)$, $\underline{v} \in D$ the inequality

$$J(x,\underline{\pi}) \leq \tilde{J}(y,\underline{v}) + xy, \tag{3.44}$$

holds. Furthermore, equality holds in (3.44) iff the following three conditions hold :

$$X^{x,\underline{\pi}}(T) = I(yH_{\underline{v}}(T)) \qquad a.s., \tag{3.45}$$

$$\tilde{g}(t,\underline{v}_t) = g(t,\underline{\pi}_t) + \underline{\pi}^{\top}(t)\underline{v}(t) \qquad a.e., \tag{3.46}$$

$$E[H_{\underline{v}}(T)X^{x,\underline{\pi}}(T)] = x. \tag{3.47}$$

Proof:

We know that for any admissible portfolio, $\forall\, \underline{v} \in D$,

$$E\Big[H_{\underline{v}}(T)X^{x,\underline{\pi}}(T) + \int_0^T H_{\underline{v}}(s)X(s)[\tilde{g}(s,\underline{v}_s) - g(s,\underline{\pi}_s) - \underline{\pi}^{\top}(s)\underline{v}(s)]ds\Big] \leq x.$$

Young's inequality states that for all $\alpha \geq 0, \gamma > 0$

$$u(\alpha) \;\leq\; \tilde{u}(\gamma) + \alpha\gamma,$$

with equality $\iff$ $\alpha = I(\gamma)$ from (2.50). Letting $\alpha = X_T$ and $\gamma = yH_{\underline{v}}(T)$ in the above we obtain

$$u(X(T)) \;\leq\; \tilde{u}(yH_{\underline{v}}(T)) + X(T)yH_{\underline{v}}(T),$$

$$\Rightarrow E[u(X(T))] \;\leq\; E[\tilde{u}(yH_{\underline{v}}(T))] + yE[X(T)H_{\underline{v}}(T)]$$

$$\leq\; E[\tilde{u}(yH_{\underline{v}}(T))] + xy,$$

for all $\underline{\pi} \in \mathcal{A}(x,u)$, due to the positivity of $\Psi^{\underline{v},\underline{\pi}}(s)$ of (3.36) for all $t \in [0,T]$. Obviously, equality holds above $\iff$

$$X(T) = I(yH_{\underline{v}}(T)) \qquad \text{a.s.},$$

$$E[H_{\underline{v}}(T)X^{x,\underline{\pi}}(T)] = x,$$

and therefore

$$\tilde{g}(t,\underline{v}_t) = g(t,\underline{\pi}_t) + \underline{\pi}^{\top}(t)\underline{v}(t) \qquad \text{a.e.}$$

◇

3.42 Corollary

For every $x \geq 0,\ y > 0$

$$V(x) \leq \tilde{V}(y) + xy.$$

Furthermore, if $\underline{\pi}_y \in \mathcal{A}(x,u)$ *and* $\underline{v}_y \in D$ *satisfy (3.45) to (3.47) then they are optimal for their respective problems; that is*

$$V(x) = J(x, \underline{\pi}_y),$$
$$\widetilde{V}(y) = \widetilde{J}(y, \underline{v}_y).$$

Proof:

For all $x \geq 0, y > 0,\ \underline{\pi} \in \mathcal{A}(x,u),\ \underline{v} \in D$

$$J(x, \underline{\pi}) \leq \widetilde{J}(y, \underline{v}) + xy,$$

$$\Rightarrow \sup_{\underline{\pi} \in \mathcal{A}(x,u)} J(x, \underline{\pi}) \leq \inf_{\underline{v} \in D} \widetilde{J}(y, \underline{v}) + xy.$$

Therefore

$$J(x, \underline{\pi}) \leq V(x) \leq \widetilde{V}(y) + xy \leq \widetilde{J}(y, \underline{v}) + xy.$$

The result follows immediately from Theorem 3.41.

◇

3.4 Formalising the Discussion

Now introduce the function

$$\zeta_{\underline{v}}(y) := E[H_{\underline{v}}(T)I(yH_{\underline{v}}(T))], \tag{3.48}$$

for all $y \in (0, \infty)$ and the set

$$D' := \left\{\underline{v} \in D : \zeta_{\underline{v}}(y) < \infty \ \forall \ y \in (0, \infty)\right\}. \tag{3.49}$$

3.43 *Remark*

Under Assumption 2.24, Lemma 2.25 (iii) and the decrease of I we know that if $\zeta_{\underline{v}}(y) < \infty$ for some $y \in (0, \infty)$ then it is finite $\forall \ y \in (0, \infty)$ and hence $\underline{v} \in D'$. For every $\underline{v} \in D'$, the function $\zeta_{\underline{v}}(y)$ is continuous and strictly decreasing with

$$\begin{aligned} \zeta_{\underline{v}}(0+) &= \infty, \\ \zeta_{\underline{v}}(\infty) &= 0. \end{aligned}$$

We denote its inverse by $\psi_{\underline{v}}(\cdot)$. Now the optimality condition (3.47) is equivalent to $\underline{v} \in D'$ and $y = \psi_{\underline{v}}(x)$ once (3.45) holds.

A four step strategy

From Theorem 3.41 and its corollary we have three conditions (3.45) to (3.47) which are sufficient for the optimality of $\underline{\pi}$ and $\underline{v}$ in both the Primal and Dual problems. We now devise a strategy which will ensure the existence of two such processes.

- **Step 1 :** We show that, for any initial condition $y > 0$ an optimal dual process exists, i.e., we guarantee the existence of a dual solution $\forall \ y \in (0, \infty)$. This is a necessary condition for (3.45)-(3.47) to hold.

- **Step 2 :** We will then prove that for any initial capital $x > 0$ there is a number $y(x) > 0$ such that

$$E\Big[H_{\underline{v}_{y(x)}}(T)I\big(yH_{\underline{v}_{y(x)}}(T)\big)\Big] = x$$

i.e., given initial capital $x > 0$ we can find a particular Lagrange multiplier $y(x)$, solve the corresponding dual problem by step 1 and ensure (3.47) holds.

- **Step 3 :** Third, with initial condition y, we will use Theorem 3.21 to show the existence of, and to characterise, the portfolio $\underline{\pi}_y$ to hedge any claim given by

$$B_{\underline{v}_y} = I\big(yH_{\underline{v}_y}(T)\big). \tag{3.50}$$

The portfolio will require the initial hedging price

$$h_y(0) := \sup_{\underline{v}\in D} E\Big[H_{\underline{v}}(T)I\big(yH_{\underline{v}_y}(T)\big)\Big]. \tag{3.51}$$

However we will show that this supremum is in fact achieved by the dual solution $\underline{v}_y$ corresponding to y

$$\Rightarrow h_y(0) := E\Big[H_{\underline{v}_y}(T)I\big(yH_{\underline{v}_y}(T)\big)\Big]. \tag{3.52}$$

We can ensure then that our initial capital is actually x by choosing y to be the particular $y(x)$ of Step 2. Hence our final wealth will be of the form (3.45) using initial capital x.

- **Step 4 :** Finally having found $\underline{v}_{y(x)}$ and $\underline{\pi}_{y(x)}$ we must prove that (3.46) holds for these processes.

In such a manner we find $\underline{v}_{y(x)}$ and $\underline{\pi}_{y(x)}$, the optimal dual and primal processes. In the next four sections we deal with the above scheme step by step.

3.4.1 Existence of a Dual Problem solution

We establish here the fundamental existence result required for our strategy. We begin by inroducing Γ, the space of all progressively measurable functions $\underline{v}$ with norm defined by

$$[\underline{v}]^2 := E\Big[\int_0^T \|\underline{v}_s\|^2 ds\Big] < \infty. \tag{3.53}$$

Γ is a Hilbert Space when endowed with inner product

$$\langle \underline{v}_1, \underline{v}_2 \rangle := E\Big[\int_0^T \underline{v}_1^{\top}(t)\underline{v}_2(t)dt\Big].$$

Note that D is a subspace of Γ. For any given $y \in (0,\infty)$ we defined the function $\tilde{J}(y,\underline{v})$ of (3.42) for all $\underline{v} \in D$. We now extend this definition to the entirety of Γ by setting

$$\tilde{J}_y(\underline{v}) := \begin{cases} E[\tilde{u}(yH_{\underline{v}}(T))], & \underline{v} \in D \\ \infty, & \underline{v} \in \Gamma/D. \end{cases} \tag{3.54}$$

3.44 *Remark*

Note that the above definition is motivated by the following :

$$\tilde{J}_y(\underline{v}) := E\Big[\tilde{u}\Big(y\exp\Big\{-\int_0^T \tilde{g}(s,\underline{v}_s)ds - \eta_{\underline{v}}(T)\Big\}\Big)\Big],$$

where

$$\eta_{\underline{v}}(T) := \int_0^T \theta_{\underline{v}}(s)dW(s) + \frac{1}{2}\int_0^T \|\theta_{\underline{v}}(s)\|^2 ds. \tag{3.55}$$

Now, by Jensen's inequality (see Appendix A.3) and the convexity of $\tilde{u}(e^z)$, we have

$$\tilde{J}_y(\underline{v}) \;\geq\; \tilde{u}\Big(y\exp\Big\{E\Big[-\int_0^T \tilde{g}(s,\underline{v}_s)ds - \eta_{\underline{v}}(T)\Big]\Big\}\Big),$$

and, unless $\underline{v} \in D$, the expression within the expectation operator is not bounded from below and we have for all $\underline{v} \in \Gamma/D$

$$\begin{aligned} \tilde{J}_y(\underline{v}) &= \tilde{u}(0+) = u(\infty) \\ &= \infty. \end{aligned}$$

The dual solution exists under certain conditions. To prove the existence of the dual solution for all $y > 0$ we will require the following condition. We will in the future need to assume the existence of some $\underline{v} \in D$ such that

$$\tilde{J}(y, \underline{v}) := E[\tilde{u}(yH_{\underline{v}}(T))] < \infty, \tag{3.56}$$

for all $y \in (0, \infty)$. Condition (3.56) will be required to prove the existence of a dual solution. However, in most circumstances it is easier to check that the growth condition is satisfied by the utility function. Lemma 3.45 gives this useful result.

3.45 Lemma

Under the growth condition, $\exists\ \underline{v} \in D$ such that, for all $y \in (0, \infty)$, (3.56) holds.

Proof:

Under the growth condition

$$\begin{aligned} u(x) &\leq \kappa(1 + x^{\alpha}), \\ \Rightarrow \tilde{u}(y) &:= \max_{x>0}[u(x) - xy] \\ &\leq \max_{x>0}[\kappa(1 + x^{\alpha}) - xy]. \end{aligned}$$

By elementary calculus this maximum occurs at $x^* = (\alpha\kappa)^{\frac{1}{1-\alpha}} y^{-\frac{1}{1-\alpha}}$ and its value is

$$\tilde{u}(y) = \kappa\Big[1 + (\alpha\kappa)^{\frac{\alpha}{1-\alpha}}(1-\alpha)y^{-\frac{\alpha}{1-\alpha}}\Big].$$

By choosing large enough $\tilde{\kappa} \in (0,\infty)$ we obtain

$$\tilde{u}(y) \;\; \leq \;\; \tilde{\kappa}[1 + y^{-\rho}],$$

where $\rho = \frac{\alpha}{1-\alpha}$ and this implies

$$E\Big[\tilde{u}(yH_{\underline{v}}(T))\Big] \;\; \leq \;\; \tilde{\kappa}\Big(1 + y^{-\rho}E[H_{\underline{v}}^{-\rho}(T)]\Big).$$

Now, for arbitrary $y > 0$, choose $\underline{v} \equiv 0$

$$\begin{aligned}
\Rightarrow E\Big[\tilde{u}(yH_{\underline{v}}(T))\Big] \;\; &\leq \;\; \tilde{\kappa}\Big(1 + y^{-\rho}E[H_0^{-\rho}(T)]\Big) \\
&= \;\; \tilde{\kappa}\Big(1 + y^{-\rho}E[H_0^{-\rho}(T)]\Big) \\
&= \;\; \tilde{\kappa}\left(1 + y^{-\rho}E\Big[\exp\Big\{-\rho\int_0^T \tilde{g}(s,0)ds\Big\}\Big]\right) \\
&< \;\; \infty,
\end{aligned}$$

from the boundedness provided in Assumption 3.8.

◇

Theorem 3.48 to follow is the main result giving the existence of a dual solution. We will firstly require the following assumption and lemma.

3.46 Assumption

$$\begin{aligned}
u(0+) \;\; &> \;\; -\infty, \\
u(\infty) \;\; &= \;\; \infty.
\end{aligned}$$

3.47 Lemma

Under Assumptions 2.23, 2.24, 3.46 and the condition (3.56), the functional $\tilde{J}_y(\underline{v}) : \Gamma \mapsto \mathbf{R} \cup \{+\infty\}$ *is*

(i) Convex,

(ii) Coercive, i.e.,

$$\lim_{\lceil \underline{v} \rceil \to \infty} \tilde{J}(y, \underline{v}) = \infty, \tag{3.57}$$

(iii) Lower semi-continuous, i.e., for all $\underline{v} \in \Gamma$ *and any sequence* $\{\underline{v}_n\} \subseteq \Gamma$ *with* $\lceil \underline{v}_n - \underline{v} \rceil \to 0$ *we have*

$$\tilde{J}_y(\underline{v}) \leq \lim_{n \to \infty} \inf \tilde{J}_y(\underline{v}_n). \tag{3.58}$$

Proof:

(i) Convexity : Firstly, we have the convexity of $\tilde{g}$ by Appendix B.1 implying

$$\tilde{g}(t, \lambda \underline{v}_1 + \mu \underline{v}_2) \leq \lambda \tilde{g}(t, \underline{v}_1) + \mu \tilde{g}(t, \underline{v}_2).$$

Secondly, we prove the convexity of $\eta_{\underline{v}}(t)$ of (3.55).

$$\begin{aligned} \eta_{\lambda \underline{v}_1 + \mu \underline{v}_2}(t) &= \int_0^t \theta_{\lambda \underline{v}_1 + \mu \underline{v}_2} dW(s) + \frac{1}{2} \int_0^t \|\theta_{\lambda \underline{v}_1 + \mu \underline{v}_2}\|^2 ds \\ &= \int_0^t \lambda \theta_{\underline{v}_1} + \mu \theta_{\underline{v}_2} dW(s) + \frac{1}{2} \int_0^t \|\lambda \theta_{\underline{v}_1} + \mu \theta_{\underline{v}_2}\|^2 ds, \end{aligned}$$

by (3.6). By the convexity of the Euclidean norm we obtain

$$\begin{aligned} \eta_{\lambda \underline{v}_1 + \mu \underline{v}_2}(t) &\leq \int_0^t \lambda \theta_{\underline{v}_1} + \mu \theta_{\underline{v}_2} dW(s) + \frac{1}{2} \int_0^t \lambda^2 \|\theta_{\underline{v}_1}\|^2 + \mu^2 \|\theta_{\underline{v}_2}\|^2 ds \\ &\leq \lambda \int_0^t \theta_{\underline{v}_1} dW(s) + \frac{1}{2} \lambda \int_0^t \|\theta_{\underline{v}_1}\|^2 ds \end{aligned}$$

$$+\mu\int_0^t \theta_{\underline{v}_2} dW(s) + \frac{1}{2}\mu\int_0^t \|\theta_{\underline{v}_2}\|^2 ds$$

$$= \lambda\eta_{\underline{v}_1}(t) + \mu\eta_{\underline{v}_2}(t),$$

since $\lambda, \mu \in [0,1]$. Returning to the proof itself we have

$$\begin{aligned}
\tilde{J}_y(\lambda\underline{v}_1 + \mu\underline{v}_2) &= E\Big[\tilde{u}\big(yH_{\lambda\underline{v}_1+\mu\underline{v}_2}(T)\big)\Big] \\
&= E\Big[\tilde{u}\Big(y\exp\Big\{-\int_0^T \tilde{g}(t,\lambda\underline{v}_1+\mu\underline{v}_2)dt - \eta_{\lambda\underline{v}_1+\mu\underline{v}_2}(T)\Big\}\Big)\Big] \\
&\leq E\Big[\tilde{u}\Big(y\exp\Big\{-\int_0^T \lambda\tilde{g}(t,\underline{v}_1) + \mu\tilde{g}(t,\underline{v}_2)dt \\
&\qquad -[\lambda\eta_{\underline{v}_1}(T) + \mu\eta_{\underline{v}_2}(T)]\Big\}\Big)\Big],
\end{aligned}$$

by the decrease of $\tilde{u}$ and the convexity of $\tilde{g}(t,\cdot)$ and $\eta_{\cdot}(t)$. But, by the convexity of $\tilde{u}(e^x)$ from Lemma 2.25 under Assumptions 2.23 and 2.24, we obtain

$$\begin{aligned}
\tilde{J}_y(\lambda\underline{v}_1 + \mu\underline{v}_2) &\leq E\Big[\lambda\tilde{u}\Big(y\exp\Big\{-\int_0^T \tilde{g}(t,\underline{v}_1(t))dt - \eta_{\underline{v}_1}(T)\Big\}\Big) \\
&\qquad +\mu\tilde{u}\Big(y\exp\Big\{-\int_0^T \tilde{g}(t,\underline{v}_2(t))dt - \eta_{\underline{v}_2}(T)\Big\}\Big)\Big] \\
&= \lambda\tilde{J}_y(\underline{v}_1) + \mu\tilde{J}_y(\underline{v}_2).
\end{aligned}$$

From Ekeland and Temam [3] (see Appendix A.11), the extended functional $\tilde{J}_y(\underline{v})$ is convex since $\tilde{J}(y,\underline{v})$ is convex over D and the set D is convex.

(ii) Coerciveness : By (3.54) and Jensen's inequality we have

$$\tilde{J}_y(\underline{v}) \geq E\Big[\tilde{u}\Big(y\exp\Big\{-\int_0^T \tilde{g}(t,\underline{v}(t)dt - \eta_{\underline{v}}(T)\Big\}\Big)\Big], \qquad \forall\, \underline{v} \in \Gamma,$$

$$\geq \tilde{u}\Big(y\exp\Big\{\underbrace{E\Big[-\int_0^T \tilde{g}(t,\underline{v}(t))dt}_{*} - \eta_{\underline{v}}(T)\Big]\Big\}\Big), \qquad \forall\, \underline{v} \in D \cup \Gamma/D.$$

By Remark 3.9, for all $\underline{v} \in D$, $E\Big[\int_0^T \tilde{g}(t,\underline{v}_t)dt\Big] < \infty$ so that the expression highlighted is bounded from below. From the definition of $\tilde{J}_y(\underline{v})$ for all $\underline{v} \in \Gamma/D$ the expression on the left-hand side is infinity. Hence for all $\underline{v} \in \Gamma$

$$\begin{aligned}
\tilde{J}_y(\underline{v}) &\geq \tilde{u}\Big(y\exp\Big\{M - E[\eta_{\underline{v}}(T)]\Big\}\Big) \\
&= \tilde{u}\Big(ye^M \exp\Big\{-\frac{1}{2}E\Big[\int_0^T \|\theta_{\underline{v}}(s)\|^2 ds\Big]\Big\}\Big) \\
&= \tilde{u}\Big(ye^M \exp\Big\{-\frac{1}{2}\lceil \sigma^{-1}\underline{v}\rceil^2\Big\}\Big),
\end{aligned}$$

by definition of $\eta_{\underline{v}}(t)$ of (3.55) and the norm defined in (3.53). Since $\|\sigma^{-1}(t)\|$ is bounded below by $1/\delta$ then as $\lceil \underline{v}\rceil \to \infty$ the expression above tends to zero.

$$\begin{aligned}
\lim_{\lceil \underline{v}\rceil \to \infty} \tilde{J}_y(\underline{v}) &= \tilde{u}(0+) = u(\infty) \\
&= \infty,
\end{aligned}$$

from Lemma 2.22 and Assumption 3.46.

(iii) Lower semi-continuity : It suffices to prove that if $\{\underline{v}_n\}_{n\in N}$ is a sequence in Γ which converges in norm to $\underline{v}$, i.e.,

$$\lim_{n\to\infty} \lceil \underline{v}_n - \underline{v}\rceil = 0,$$

then

$$\tilde{J}_y(\underline{v}) \leq \lim_{n\to\infty} \inf \tilde{J}_y(\underline{v}_n).$$

Firstly

$$\lim_{n\to\infty} E\Big[\int_0^T \mid \|\theta_{\underline{v}_n}(t)\|^2 - \|\theta_{\underline{v}}(t)\|^2 \mid dt\Big]$$

$$= \lim_{n\to\infty} E\Big[\int_0^T \mid \Big(\theta_{\underline{v}_n}(t) - \theta_{\underline{v}}(t)\Big)^\top \Big(\theta_{\underline{v}_n}(t) + \theta_{\underline{v}}(t)\Big) \mid dt\Big]$$

$$\leq \lim_{n\to\infty} \lceil \theta_{\underline{v}_n}(t) - \theta_{\underline{v}}(t) \rceil \cdot \lceil \theta_{\underline{v}_n}(t) + \theta_{\underline{v}}(t) \rceil$$

$$= 0.$$

Consequently

$$\lim_{n\to\infty} E\Big[\mid \ln Z_{\underline{v}}(T) - \ln Z_{\underline{v}_n}(T) \mid \Big]$$

$$= \lim_{n\to\infty} E\Big[\mid \int_0^T \theta_{\underline{v}_n}(t) dW(t) + \frac{1}{2}\int_0^T \|\theta_{\underline{v}_n}(t)\|^2 dt - \int_0^T \theta_{\underline{v}}(t) dW(t) - \frac{1}{2}\int_0^T \|\theta_{\underline{v}}(t)\|^2 dt \mid \Big]$$

$$= \lim_{n\to\infty} E\Big[\mid \int_0^T (\theta_{\underline{v}_n}(t) - \theta_{\underline{v}}(t)) dW(t) + \frac{1}{2}\int_0^T \|\theta_{\underline{v}_n}(t)\|^2 - \|\theta_{\underline{v}}(t)\|^2 dt \mid \Big]$$

$$\leq \lim_{n\to\infty} E\Big[\int_0^T \mid (\theta_{\underline{v}_n}(t) - \theta_{\underline{v}}(t)) \mid dW(t)\Big]$$

$$+ \lim_{n\to\infty} E\Big[\frac{1}{2}\int_0^T \mid \|\theta_{\underline{v}_n}(t)\|^2 - \|\theta_{\underline{v}}(t)\|^2 \mid dt\Big]$$

$$= 0,$$

and since the exponential function is continuous everywhere we have

$$\lim_{n\to\infty} Z_{\underline{v}_n}(T) = Z_{\underline{v}}(T),$$

holds almost surely on Ω. Secondly, by Fatou's lemma applied to the sequence $\{\tilde{g}(s, \underline{v}_n(s))\}_{n\in N}$ which is integrable, we have

$$\lim_{n\to\infty} \inf \int_0^t \tilde{g}(s, \underline{v}_n(s)) ds \;\geq\; \int_0^t \lim_{n\to\infty} \inf \tilde{g}(s, \underline{v}_n(s)) ds$$

$$= \int_0^t \tilde{g}(s, \underline{v}(s))ds,$$

by continuity of $\tilde{g}$.

Returning to the proof we have, by definition of $\tilde{J}_y(\underline{v})$

$$\lim_{n\to\infty} \inf \tilde{J}_y(\underline{v}) = \lim_{n\to\infty} \inf E\Big[\tilde{u}\Big(y \exp\Big\{-\int_0^T \tilde{g}(s, \underline{v}_n(s))ds\Big\} Z_{\underline{v}_n}(T)\Big)\Big].$$

By positivity of $\tilde{u}$, we can apply Fatou's lemma (see Appendix A.3) to obtain

$$\lim_{n\to\infty} \inf \tilde{J}_y(\underline{v}) = E\Big[\lim_{n\to\infty} \inf \tilde{u}\Big(y \exp\Big\{-\int_0^T \tilde{g}(s, \underline{v}_n(s))ds\Big\} Z_{\underline{v}_n}(T)\Big)\Big].$$

Now since

$$\lim_{n\to\infty} \inf \int_0^t \tilde{g}(s, \underline{v}_n(s))ds \geq \int_0^t \tilde{g}(s, \underline{v}(s))ds,$$

and the function $\tilde{u}(e^x)$ is decreasing we get

$$\lim_{n\to\infty} \inf \tilde{J}_y(\underline{v}) = E\Big[\lim_{n\to\infty} \inf \tilde{u}\Big(y \exp\Big\{-\int_0^T \tilde{g}(s, \underline{v}(s))ds\Big\} Z_{\underline{v}_n}(T)\Big)\Big].$$

Finally since

$$\lim_{n\to\infty} Z_{\underline{v}_n}(T) = Z_{\underline{v}}(T),$$

we can, by continuity of $\tilde{u}$, bring the limit inside to obtain

$$\begin{aligned}\lim_{n\to\infty} \inf \tilde{J}_y(\underline{v}) &\geq E\Big[\tilde{u}\Big(y \exp\Big\{-\int_0^T \tilde{g}(s, \underline{v}_n(s))ds\Big\} \lim_{n\to\infty} \inf Z_{\underline{v}_n}(T)\Big)\Big] \\ &= \tilde{J}_y(\underline{v}).\end{aligned}$$

◇

We can now prove that, for all initial conditions $y > 0$, the existence of a solution to the Dual problem of (3.43) is assured. This will complete the first step. Indeed, to proceed to the solution of step 2, we will require the additional fact that $\underline{v}_y \in D'$.

3.48 Theorem

Under Assumptions 2.23, 2.24, 3.46 and the condition (3.56) the dual problem of (3.43) admits a solution $\underline{v}_y \in D$ for all $y \in (0, \infty)$. Furthermore the optimal dual process, $\underline{v}_y$ is in the set D'.

Proof:

Fix $y \in (0, \infty)$. From Lemma 3.47, the extended functional $\tilde{J}_y(\underline{v})$ is convex and lower semi-continuous over Γ. Thanks to the coercivity property (3.57) we can use Ekeland and Temam [3], Proposition 2.1.2 (see Appendix A.11) so that for all $y \in (0, \infty)$ the infimum of $\tilde{J}_y(\underline{v})$ is attained at some solution $\underline{v}_y \in \Gamma$ such that

$$\inf_{\underline{v} \in \Gamma} \tilde{J}_y(\underline{v}) = \tilde{J}_y(\underline{v}_y).$$

Since $\tilde{J}_y(\underline{v}) = \infty$ for every $\underline{v} \in \Gamma / D$. Then by Lemma 3.45 we know that there exists a $\underline{v}_y \in D$ such that for all $y \in (0, \infty)$

$$\tilde{J}_y(\underline{v}_y) < \infty.$$

Hence $\underline{v}_y \in D$ and

$$\inf_{\underline{v} \in \Gamma} \tilde{J}_y(\underline{v}) = \tilde{J}_y(\underline{v}_y) < \infty. \tag{3.59}$$

We now show $\underline{v}_y \in D'$. We must prove that $\zeta_{\underline{v}_y}(y) < \infty$. By the decrease of $\tilde{u}$ we know that $\forall\, \alpha \in (0, 1),\ \beta > 0$

$$\tilde{u}(\beta) - \tilde{u}(\infty) \;\; \geq \;\; \tilde{u}(\beta) - \tilde{u}(\beta/\alpha)$$

$$\begin{aligned} &= \int_{\beta/\alpha}^{\beta} \tilde{u}'(\xi)d\xi \\ &= \int_{\beta}^{\beta/\alpha} I(\xi)d\xi \\ &\geq (\beta/\alpha - \beta)I(\beta/\alpha), \end{aligned}$$

from the decrease of $I(\cdot)$. By Assumption 2.24

$$\tilde{u}(\beta) - \tilde{u}(\infty) \geq \beta\Big(\frac{1-\alpha}{\alpha}\Big)\frac{1}{\gamma}I(\beta). \tag{3.60}$$

Now, applying (3.60) to $\beta = yH_{\underline{v}_y}(T)$, we obtain

$$\begin{aligned} \zeta_{\underline{v}_y}(y) &= E\Big[yH_{\underline{v}_y}(T)I\Big(yH_{\underline{v}_y}(T)\Big)\Big] \\ &\leq E\Big[\frac{\alpha}{1-\alpha}\gamma\Big[\tilde{u}(yH_{\underline{v}_y}(T)) - \tilde{u}(\infty)\Big]\Big] \\ &= \frac{\alpha\gamma}{1-\alpha}E\Big[\tilde{u}\Big(yH_{\underline{v}_y}(T)\Big) - \tilde{u}(\infty)\Big] \\ &< \infty, \end{aligned}$$

from (3.59), Lemma 2.22 and Assumption 3.46.

◇

3.4.2 A particular choice of $y(x)$

We will now show that for any $x > 0$, there exists a number $y(x) \in (0, \infty)$ with corresponding dual solution $\underline{v}_{y(x)}$ guaranteed by Theorem 3.48 that satisfies

$$\zeta_{\underline{v}_{y(x)}}(y(x)) := E\Big[H_{\underline{v}_{y(x)}}(T)I\Big(yH_{\underline{v}_{y(x)}}(T)\Big)\Big] = x. \tag{3.61}$$

First, we require the following lemma.

3.49 Lemma

Under Assumptions 2.24, 3.46 and the condition (3.56) and providing the existence of the dual solution corresponding to $y \in (0,\infty)$ *denoted by* $\underline{v}_y$ *we have : the function*

$$G_y(\beta) := \tilde{J}(y\beta; \underline{v}_y), \tag{3.62}$$

is well defined, finite and continuously differentiable at $\beta = 1$. *For all* $y \in (0,\infty)$ *it also satisfies*

$$G'_y(1) = -y\zeta_{\underline{v}_y}(y). \tag{3.63}$$

Proof:

As in the proof of Theorem 3.48 we have for all $y \in (0,\infty)$

$$\begin{aligned} \tilde{u}(y) - \tilde{u}(\infty) &= -\int_y^\infty \tilde{u}'(\xi)d\xi \\ &= \int_y^\infty I(\xi)d\xi, \\ \Rightarrow \tilde{u}(\alpha y) - \tilde{u}(\infty) &= \int_{\alpha y}^\infty I(\xi)d\xi \\ &= \alpha \int_y^\infty I(\alpha\eta)d\eta, \end{aligned}$$

substituting $\eta = \xi/\alpha$. Thus, from Lemma 2.25, for any given $\alpha \in (0,1)$, there exists a suitable constant $\gamma \in (1,\infty)$ such that for all $y \in (0,\infty)$

$$\begin{aligned} \tilde{u}(\alpha y) - \tilde{u}(\infty) &\leq \alpha\gamma \int_y^\infty I(\eta)d\eta \\ &= \alpha\gamma[\tilde{u}(y) - \tilde{u}(\infty)] \end{aligned}$$

$$\Rightarrow \tilde{u}(\alpha y) \leq \alpha\gamma\tilde{u}(y) + (1-\alpha\gamma)\tilde{u}(\infty).$$

Therefore

$$\begin{aligned} E\Big[\tilde{u}(\alpha y H_{\underline{v}_y}(T))\Big] &\leq \alpha\gamma E\Big[\tilde{u}(y H_{\underline{v}_y}(T))\Big] + (1-\alpha\gamma)\tilde{u}(\infty) \\ &\leq \alpha\gamma \tilde{J}_y(\underline{v}_y) + (1-\alpha\gamma)u(0) \\ &< \infty, \end{aligned}$$

since $\underline{v}_y$ is the optimal dual so the first expression on the right-hand side is finite by Theorem 3.48 and the second by Assumption 3.46. Since α can be chosen arbitrarily in $(0,1)$

$$E\Big[\tilde{u}(\beta y H_{\underline{v}_y}(T))\Big] < \infty, \tag{3.64}$$

holds for all $\beta \in (0,1)$. But since $\tilde{u}$ is decreasing, (3.64) holds for all $\beta \geq 1$ and the function $G_y(\beta)$ is well defined and finite. The upper finiteness of $G_y(\beta)$ means we can use dominated convergence (see Appendix A.3) to take the limit inside. Now

$$\begin{aligned} G'_y(\beta) &= \frac{d}{d\beta}E\Big[\tilde{u}(\beta y H_{\underline{v}_y}(T))\Big] \\ &= E\left[\frac{d}{d\beta}\tilde{u}(\beta y H_{\underline{v}_y}(T))\right] \\ &= yE\Big[H_{\underline{v}_y}(T)\tilde{u}'(\beta y H_{\underline{v}_y}(T))\Big], \\ \Rightarrow G'_y(1) &= -yE\Big[H_{\underline{v}_y}(T)I(y H_{\underline{v}_y}(T))\Big] \\ &= -y\zeta_{\underline{v}_y}(y). \end{aligned}$$

◇

The existence of a $y(x)$ is ensured in Theorem 3.50 by the fact that $\underline{v}_y \in D'$. This was proved in Theorem 3.48.

3.50 Theorem

Under Assumptions 2.24, 3.46 and the condition (3.56) and providing the existence of the dual solution corresponding to $y \in (0,\infty)$ denoted by $\underline{v}_y$ we have : for any given $x \in (0,\infty)$ there exists a number $y(x) \in (0,\infty)$ that achieves the infimum in

$$\inf_{y>0}[\tilde{V}(y) + xy]. \tag{3.65}$$

Furthermore this number satisfies condition (3.61) above.

Proof:

Define for all $y \in (0,\infty)$ the convex function

$$h_x(y) := \tilde{V}(y) + xy. \tag{3.66}$$

We wish to show this function attains its infimum on $(0,\infty)$. To do this we show that it satisfies

$$h_x(0+) = \infty = h_x(\infty). \tag{3.67}$$

To this end, the boundedness of $\tilde{g}$ (see Remark 3.9) and the supermartingale property of $Z_{\underline{v}}(t)$ imply that for all $t \in [0,T]$

$$\begin{aligned} \gamma_{\underline{v}}(t) &\leq e^M, \\ \Rightarrow E[H_{\underline{v}}(t)] &= E[Z_{\underline{v}}(t)\gamma_{\underline{v}}(t)] \\ &\leq E[Z_{\underline{v}}(t)e^M] \quad \leq e^M. \end{aligned}$$

By convexity of $\tilde{u}$ and Jensen's Inequality we have

$$\tilde{J}_y(\underline{v}) = E\Big[\tilde{u}(yH_{\underline{v}}(T))\Big]$$

$$\geq \tilde{u}\Big(E[yH_{\underline{v}}(T)]\Big)$$

$$\geq \tilde{u}(ye^M).$$

Then, from Lemma 2.22 and Assumption 3.46 we have

$$\begin{aligned} \tilde{V}(0+) &:= \lim_{y \to 0} \Big[\inf_{\underline{v} \in D} \tilde{J}_y(\underline{v})\Big] \\ &\geq \tilde{u}(0+) = u(\infty) \\ &= \infty. \end{aligned}$$

Hence the function of (3.66) satisfies (3.67) and attains its infimum at some $y(x) \in (0, \infty)$. Define for all $\beta \in (0, \infty)$ the function

$$F_x(\beta) := \beta x y(x) + G_{y(x)}(\beta). \tag{3.68}$$

Now, with the dual solution corresponding to y denoted by $\underline{v}_{y(x)}$, we have

$$\begin{aligned} \inf_{\beta>0} F_x(\beta) &= \inf_{\beta>0}[x\beta y(x) + \tilde{J}(\beta y(x); \underline{v}_{y(x)})] \\ &= \inf_{y>0}[xy + \tilde{J}(y; \underline{v}_{y(x)})] \\ &\geq \inf_{y>0}[xy + \tilde{V}(y)] \\ &= h_x(y(x)) \\ &= xy(x) + \tilde{V}(y(x)). \end{aligned}$$

Hence by Theorem 3.48

$$\inf_{\beta>0} F_x(\beta) \geq xy(x) + \tilde{J}(y(x); \underline{v}_{y(x)})$$

$$= \quad F_x(\beta)|_{\beta=1}.$$

So the function F achieves its infimum over $(0,\infty)$ at $\beta = 1$. Hence the derivative must equal zero there.

$$\begin{aligned}
\Rightarrow 0 &= \frac{d}{d\beta} F_x(\beta)|_{\beta=1} = xy(x) + G'_{y(x)}(\beta)|_{\beta=1} \\
&= xy(x) - y(x)\zeta_{\underline{v}_{y(x)}}(y(x)), \\
\Rightarrow x &= \zeta_{\underline{v}_{y(x)}}(y(x)).
\end{aligned}$$

◇

3.4.3 Hedging the Optimal Form

As explained in the strategy, Theorem 3.21 ensures the existence of an optimal portfolio $\hat{\underline{\pi}}$ for hedging any claim B with an initial capital

$$h(0) = \sup_{\underline{v} \in D} E[H_{\underline{v}}(T)B].$$

So the existence of portfolio $\hat{\underline{\pi}}$ for hedging the claim of (3.50) is guaranteed with initial capital

$$h_y(0) = \sup_{\underline{v} \in D} E\Big[H_{\underline{v}}(T)I(yH_{\underline{v}_y}(T))\Big].$$

To utilise step 2 we require this supremum to be attained at $\underline{v} \equiv \underline{v}_y$. Theorem 3.51 gives this result for any $y > 0$.

3.51 Theorem

Under the Assumption that for all $\underline{v} \in D$, $y > 0$

$$E\Big[H_{\underline{v}_y}(T)I(yH_{\underline{v}_y}(T))\Big] < \infty, \tag{3.69}$$

we have

$$E\Big[H_{\underline{v}}(T)I(yH_{\underline{v}_y}(T))\Big] \le E\Big[H_{\underline{v}_y}(T)I(yH_{\underline{v}_y}(T))\Big], \qquad (3.70)$$

for all $\underline{v} \in D,\ y > 0$.

Proof:

Fix $\delta \in (0,1),\ \underline{v} \in D$ and define the following functions

$$G_\delta(t) := (1-\delta)H_{\underline{v}_y}(t) + \delta H_{\underline{v}}(t), \qquad (3.71)$$

$$\mu_\delta(t) := \frac{1}{G_\delta(t)}\Big[(1-\delta)H_{\underline{v}_y}(t)\underline{v}_y + \delta H_{\underline{v}}(t)\underline{v}\Big], \qquad (3.72)$$

$$\tilde{\mu}_\delta(t) := \frac{1}{G_\delta(t)}\Big[(1-\delta)H_{\underline{v}_y}(t)\tilde{g}(t,\underline{v}_y) + \delta H_{\underline{v}}(t)\tilde{g}(t,\underline{v})\Big]. \qquad (3.73)$$

Note that $\lim_{\delta\to 0} G_\delta = H_{\underline{v}_y}$. The convexity of D and the fact that

$$\frac{(1-\delta)H_{\underline{v}_y}(t)}{G_\delta(t)} + \frac{\delta H_{\underline{v}}(t)}{G_\delta(t)} = 1,$$

imply that $\mu_\delta(t) \in D$. Moreover

$$\begin{aligned} dG_\delta &= (1-\delta)dH_{\underline{v}_y}(t) + \delta dH_{\underline{v}}(t) \\ &= (1-\delta)H_{\underline{v}_y}(t)\Big[-\tilde{g}(t,\underline{v}_y(t))dt - \theta_{\underline{v}_y}(t)dW(t)\Big] \\ &\quad + \delta H_{\underline{v}}(t)\Big[-\tilde{g}(t,\underline{v}(t))dt - \theta_{\underline{v}}(t)dW(t)\Big] \\ &= \Big[(1-\delta)H_{\underline{v}_y}(t)\sigma^{-1}\underline{v}_y + \delta H_{\underline{v}}(t)\sigma^{-1}\underline{v}\Big]dW(t) \\ &\quad - \Big[(1-\delta)H_{\underline{v}_y}(t)\tilde{g}(t,\underline{v}_y(t)) + \delta H_{\underline{v}}(t)\tilde{g}(t,\underline{v}(t))\Big]dt \\ &= \sigma^{-1}G_\delta\mu_\delta dW(t) - G_\delta\tilde{\mu}_\delta dt. \end{aligned}$$

Recall that the convexity of $\tilde{g}$ implies

$$\begin{aligned}\tilde{g}(t,\mu_\delta) &:= \tilde{g}\Big(t, \frac{(1-\delta)H_{\underline{v}_y}(t)}{G_\delta(t)}\underline{v}_y + \frac{\delta H_{\underline{v}}(t)}{G_\delta(t)}\underline{v}\Big) \\ &\leq \frac{(1-\delta)H_{\underline{v}_y}(t)}{G_\delta(t)}\tilde{g}(t,\underline{v}_y) + \frac{\delta H_{\underline{v}}(t)}{G_\delta(t)}\tilde{g}(t,\underline{v}) \\ &=: \tilde{\mu}_\delta,\end{aligned}$$

whence

$$\begin{aligned}dG_\delta &\leq G_\delta[\sigma^{-1}\mu_\delta dW(t) - \tilde{g}(t,\mu_\delta)dt], \\ dH_{\mu_\delta} &= H_{\mu_\delta}[\sigma^{-1}\mu_\delta dW(t) - \tilde{g}(t,\mu_\delta)dt].\end{aligned}$$

Comparing these we see that, since $G_\delta(0) = 1 = H_{\mu_\delta}(0)$,

$$G_\delta(\cdot) \leq H_{\mu_\delta}(\cdot) \text{ a.s.}$$

It follows from the dual optimality of $\underline{v}_y$ and the decrease of $\tilde{u}$ respectively that

$$\begin{aligned}E\Big[\tilde{u}(yH_{\underline{v}_y}(T))\Big] &\leq E\Big[\tilde{u}(yH_{\mu_\delta}(T))\Big] \\ &\leq E\Big[\tilde{u}(yG_\delta(T))\Big].\end{aligned}$$

With the random variable L_δ defined by

$$L_\delta := \tilde{u}(yH_{\underline{v}_y}(T)) - \tilde{u}(yG_\delta(T)),$$

we have

$$\frac{1}{\delta}E[L_\delta] \leq 0.$$

Fix $\omega \in \Omega$ and (suppressing the dependence on ω) assume $H_{\underline{v}_y} \leq H_{\underline{v}}$. Then, by continuity of $\tilde{u}$, we apply the mean value theorem to give

$$\begin{aligned} L_\delta &= \tilde{u}(yH_{\underline{v}_y}(T)) - \tilde{u}(yG_\delta(T)) \\ &= \tilde{u}'(K)y(H_{\underline{v}_y}(T) - G_\delta(T)), \end{aligned}$$

where $K \in [yH_{\underline{v}_y}, yG_\delta]$. By the decrease of I and the definition of G_δ

$$\begin{aligned} \frac{1}{\delta}L_\delta &= yI(K)\frac{1}{\delta}[G_\delta(T) - H_{\underline{v}_y}(T)] \\ &= yI(K)[H_{\underline{v}}(T) - H_{\underline{v}_y}(T)] \\ &\geq yI(yH_{\underline{v}})[H_{\underline{v}}(T) - H_{\underline{v}_y}(T)]. \end{aligned}$$

We get the same result for $H_{\underline{v}} \leq H_{\underline{v}_y}$. Consequently we can apply Fatou's Lemma (see Appendix A.3) to the sequence of positive random variables given by

$$\frac{1}{\delta}L_\delta - yI(yH_{\underline{v}})[H_{\underline{v}}(T) - H_{\underline{v}_y}(T)].$$

By the finiteness of $E[yI(yH_{\underline{v}})(H_{\underline{v}}(T) - H_{\underline{v}_y}(T))]$ we obtain

$$\begin{aligned} 0 &\geq \lim_{\delta\to\infty} E\left[\frac{1}{\delta}L_\delta\right] \\ &\geq E\left[\lim_{\delta\to\infty}\frac{1}{\delta}L_\delta\right] \\ &= E\Big[y(H_{\underline{v}} - H_{\underline{v}_y})I(yH_{\underline{v}_y})\Big]. \end{aligned}$$

The result follows.

3.4.4 The Final Test

All that remains to prove is that the choice of $\hat{\underline{\pi}}_{y(x)}$ and $\underline{v}_{y(x)}$ corresponding to the particular choice of Lagrange multiplier $y(x)$ satisfy the final optimal condition (3.46). Theorem 3.51 states that for the claim $B_{\underline{v}_y}$ of (3.50) the process $\underline{v}_y$ achieves the supremum in

$$V(0) = \sup_{\underline{v} \in D} E\left[H_{\underline{v}}(T) B_{\underline{v}_y}\right]. \tag{3.74}$$

We now show that once this condition holds so does condition (3.46).

3.52 Theorem

For any claim B with corresponding policy $(\hat{\underline{\pi}}, \hat{c})$ from Theorem 3.21 we have : if $\underline{v}^*$ achieves the supremum in (3.74) then

$$\tilde{g}(t, \underline{v}^*) \equiv g(t, \underline{v}^*) + \hat{\underline{\pi}} \underline{v}^*.$$

Proof:

If $\underline{v}^*$ achieves the supremum in (3.74)

$$V(0) \quad = \quad E[H_{\underline{v}^*}(T) B] = E^{\underline{v}^*}[\gamma_{\underline{v}^*}(T) B].$$

Hence from Theorem 3.21

$$\begin{aligned} Q_{\underline{v}^*}(0) = E[H_{\underline{v}^*}(T) B] &= E^{\underline{v}^*}[\gamma_{\underline{v}^*}(T) \hat{X}^{V(0), \hat{\pi}, \hat{c}}(T)] \text{ a.s. } P^{\underline{v}^*} \\ &= E^{\underline{v}^*}[Q_{\underline{v}^*}(T)]. \end{aligned}$$

Hence $Q_{\underline{v}^*}(T)$ is a martingale under $P^{\underline{v}^*}$ and the process $A_{\underline{v}^*}(t)$ of (3.27) is identically zero. Then with $\underline{v} \equiv \underline{v}^*$ in the expression (3.34) by

$$\hat{c}(t) := \underbrace{\int_0^t \gamma_{\underline{v}}^{-1}(s) dA_{\underline{v}}(s)}_{0} - \int_0^t \underbrace{\hat{X}(s)[\tilde{g}(s, \underline{v}_s), g(s, \hat{\underline{\pi}}_s) - \hat{\underline{\pi}}_s \underline{v}_s]}_{\geq 0} ds,$$

$\hat{c}$ will be a negative decreasing process unless

$$\tilde{g}(t, \underline{v}_t^*) \equiv g(t, \underline{v}_t^*) + \hat{\underline{\pi}}_t \underline{v}_t^*.$$

◇

We have completed the four steps of our scheme and proved the existence of an optimal portfolio process $\underline{\pi}_{y(x)}$ corresponding to an endowed amount of initial capital x. We summarise the result in the following conclusion.

Conclusion *Under the Assumptions 2.24, 3.46 and condition (3.56); for any given $x > 0$ there exists a particular Lagrange multiplier $y(x)$ (given by Theorem 3.50) with corresponding dual solution (guaranteed by Theorem 3.48) such that by minimally hedging the claim $B_{\underline{\nu}_{y(x)}}$ of (3.50) with the portfolio process of Theorem 3.21 we ensure the utility maximisation problem of (3.37) is solved.*

In Chapter 4, a strategy based upon this conclusion is derived. It is important to note that we have proved the existence of the optimal portfolio and we can now devise an algorithm to calculate it either explicitly or numerically under certain market assumptions.

Chapter 4

Applications

In this chapter we apply the results of Chapters 2 and 3 to solve the utility maximisation problem for investors with certain utilities. Specifically, the methods are applied to investors with the following utilities :

$$u(x) = \ln x, \tag{4.1}$$

called the logarithmic utility and

$$u(x) = \frac{1}{\alpha} x^{\alpha}, \tag{4.2}$$

for $\alpha \in (0, 1)$, called the power utility. Both belong to the class of utilities known as HARA utilities. As $\alpha \to 0$, (4.2) tends to (4.1) at least in terms of relative risk aversion.

4.1 Small investor : Examples

The main result of Chapter 2 (Theorem 2.30) states that, provided $X(T) = I(\psi(x)H(T))$, the portfolio $\hat{\underline{\pi}}$ of Lemma 2.9 is admissible and is optimal for the small investor problem of (2.61). Indeed this portfolio has corresponding wealth process given by

$$\gamma(t)\hat{X}(t) \quad = \quad \tilde{E}\Big[\gamma(T)\hat{X}(T) \mid \mathcal{F}_t\Big]$$

$$= \quad x + \int_0^t \gamma(s)\hat{\underline{\pi}}^\top(s)\hat{X}(s)\sigma(s)d\widetilde{W}(s).$$

Consequently, the method of solution will follow the four steps below :

1. Given the utility u, find the function I and solve the Lagrange multiplier problem

$$\zeta(y) := E\Big[H(T)I(yH(T))\Big] = x, \tag{4.3}$$

for $\psi(x)$.

2. Find the *value function*

$$V(x) := E\Big[u\Big(I(\psi(x)H(T))\Big)\Big]. \tag{4.4}$$

Note that a general characterisation of this function is given in Karatzas [9]. The value function does not affect the chosen portfolio strategy in any way. However it is necessary to find the expected return on the investment.

3. Apply Lemma 2.9 to the *terminal wealth* $X(T) = I(\psi(x)H(T))$ by solving

$$\gamma(t)X(t) = \widetilde{E}\Big[\gamma(T)X(T) \mid \mathcal{F}_t\Big]. \tag{4.5}$$

Express this in the form of an Ito integral with respect to the Brownian motion $\widetilde{W}$, i.e.,

$$\gamma(t)X(t) = x + \int_0^t \Big[\cdots\Big]d\widetilde{W}(s). \tag{4.6}$$

4. The form above is required for comparison purposes. We obtain the *optimal portfolio* $\hat{\underline{\pi}}$ by comparing it with

$$x + \int_0^t \gamma(s)\hat{\underline{\pi}}^\top(s)\hat{X}(s)\sigma(s)d\widetilde{W}(s).$$

In the following example, the scheme above is applied for the logarithmic function of (4.1).

4.1 Example

Step 1 Since utility $u(x) = \ln x$ we have $u'(x) = 1/x$ and $I(y) = 1/y$. We solve the Lagrange multiplier problem

$$\zeta(y) := E\big[H(T)I(yH(T))\big] = E\left[H(T)\frac{1}{yH(T)}\right] = x,$$

$$\Rightarrow \psi(x) = 1/x.$$

Step 2 The value function is then

$$\begin{aligned} V(x) &:= E\Big[u\Big(I(\psi(x)H(T)\Big)\Big] \\ &= E\left[\ln\left(\frac{x}{H(T)}\right)\right] \\ &= \ln x + E[\ln H^{-1}(T)] \\ &= \ln x + E\left[\int_0^T r(t)dt + \int_0^T \underline{\theta}(t)dW(t) + \frac{1}{2}\int_0^T \|\underline{\theta}(t)\|^2 dt\right] \\ &= \ln x + E\left[\int_0^T r(t) + \frac{1}{2}\|\underline{\theta}(t)\|^2 dt\right]. \end{aligned}$$

Step 3 For the optimal terminal wealth we have

$$\begin{aligned} \gamma(T)X(T) &= \gamma(T)I(\psi(x)H(T)) \\ &= \frac{x\gamma(T)}{H(T)} = xZ^{-1}(T). \end{aligned}$$

From Lemma 2.9 we must solve

$$\gamma(t)X(t) = \tilde{E}\big[xZ^{-1}(T) \mid \mathcal{F}_t\big].$$

By Bayes Rule

$$\begin{aligned}\gamma(t)X(t) &= \frac{E\big[Z(T)xZ^{-1}(T) \mid \mathcal{F}_t\big]}{Z(t)} \\ &= xZ^{-1}(t) \\ &= x\exp\Big\{\int_0^t \underline{\theta}(s)dW(s) + \frac{1}{2}\int_0^t \|\underline{\theta}(s)\|^2 ds\Big\}.\end{aligned}$$

We must now apply Ito's Rule to obtain the comparison form. From the above

$$d[\ln(\gamma(t)X(t))] = \theta^{\top}(t)dW(t) + \frac{1}{2}\|\theta(t)\|^2 dt.$$

We need to find $A(t)$ and $B(t)$ such that

$$\frac{d[\gamma(t)X(t)]}{\gamma(t)X(t)} = A(t)dW(t) + B(t)dt.$$

Now by Ito's Rule

$$\begin{aligned}d[\ln(\gamma(t)X(t))] &= \frac{d[\gamma(t)X(t)]}{\gamma(t)X(t)} + \frac{1}{2}Tr\Big[A^2(t)(\gamma(t)X(t))^2\Big(\frac{-1}{(\gamma(t)X(t))^2}\Big)\Big] \\ &= A(t)dW(t) + B(t)dt - \frac{1}{2}A^2(t)dt.\end{aligned}$$

Equating the above two expressions for $d[\ln(\gamma(t)X(t))]$ gives

$$\begin{aligned}A(t) &= \theta^{\top}(t), \\ B(t) &= \theta^{\top}(t)\theta(t).\end{aligned}$$

Hence we obtain the required comparison form

$$\gamma(t)X(t) = x + \int_0^t \gamma(s)X(s)\underline{\theta}^{\top}(s)d\widetilde{W}(s).$$

Step 4 Comparing this with

$$\gamma(t)X(t) = x + \int_0^t \gamma(s)\underline{\pi}^{\top}(s)\hat{X}(s)\sigma(s)d\widetilde{W}(s),$$

gives the optimal portfolio

$$\underline{\hat{\pi}}^{\top}(t)\sigma(t) = \underline{\theta}^{\top}(t),$$

$$\Rightarrow \quad \underline{\hat{\pi}}(t) = \left(\sigma(t)\sigma^{\top}(t)\right)^{-1}(\underline{b}(t) - r(t)\underline{1}).$$

It is clear then that the proportion of wealth invested in any stock will increase linearly with its growth rate. However this investment will also decrease as the volatility coefficient of the stock increases.

4.2 Example

For the utility function $u(x) = \frac{1}{\alpha}x^{\alpha}, 0 < \alpha < 1$, the methodology above is applied to derive the recommended optimal portfolio given by :

$$\underline{\hat{\pi}}(t) = \frac{1}{1-\alpha}\left(\sigma(t)\sigma^{\top}(t)\right)^{-1}(\underline{b}(t) - r(t)\underline{1}).$$

In the previous example, we found explicit formulas for the optimal processes of wealth $X(t)$ and value function $V(x)$. Similarly, we can find the optimal consumption process for the equivalent problem of utility from consumption; see Karatzas [9].

However, for the optimal portfolio process $\underline{\hat{\pi}}$ the martingale methodology that we have employed so far can ensure only the existence of $\underline{\hat{\pi}}$; except of course in the certain cases such as those above where $\underline{\hat{\pi}}$ can be found in feedback form in terms of the random market coefficients.

In general there is no constructive algorithm or useful characterisation that would lead to its computation. For constant market coefficients $r(t) \equiv r, b(t) \equiv b$ and $\sigma(t) \equiv \sigma$, however it is possible to obtain $\underline{\hat{\pi}}$ in explicit form. This is achieved in Xu [15].

4.2 Large investor : Examples

Similarly to that of Chapter 2 the main result of Chapter 3 tells us that once we choose the appropriate $y(x)$ and hedge the claim given by (3.50) with $y \equiv y(x)$ then the portfolio process $\hat{\underline{\pi}}$ of Theorem 3.21 with corresponding wealth process given by

$$\begin{aligned}\gamma_{\underline{v}_{y(x)}}(t)X(t) \quad &+ \quad \int_0^t \gamma_{\underline{v}_{y(x)}}(s)X(s)[\tilde{g}(s,\underline{v}_{y(x)}(s)) - g(s,\hat{\underline{\pi}}_s) - \hat{\underline{\pi}}(s)\underline{v}_{y(x)}(s)]ds \\ = x \quad &+ \quad \int_0^t \gamma_{\underline{v}_{y(x)}}(s)X(s)\underline{\pi}^{\top}(s)\sigma(s)dW_{\underline{v}}(s),\end{aligned} \tag{4.7}$$

achieves the supremum in (3.41).

Consequently our adjusted method of solution is :

1. Given the utility u, find I and solve the Lagrange multiplier problem such that

$$\zeta_{\underline{v}_{y(x)}}(y(x)) := E\Big[H_{\underline{v}_{y(x)}}(T)I(yH_{\underline{v}_{y(x)}}(T))\Big] = x, \tag{4.8}$$

where $\underline{v}_{y(x)}$ denotes the dual solution corresponding to $y(x)$. This gives us $\psi_{\underline{v}}(x)$.

2. Find the *value function*

$$V(x) := E\Big[u\Big(I(\psi(x)H_{\underline{v}_{y(x)}}(T))\Big)\Big]. \tag{4.9}$$

3. Find the space Γ and the Legendre-Fenchel transform and define the dual problem as

$$\inf_{\underline{v}\in\Gamma} E\Big[\tilde{u}(\psi_{\underline{v}}(x)H_{\underline{v}}(T))\Big].$$

Solve this and denote the solution by $\underline{\lambda}$.

4. Apply Theorem 3.21 to the terminal wealth $X(T) = I(\psi_{\underline{\lambda}}(x)H_{\underline{\lambda}}(T))$ by solving

$$H_{\underline{\lambda}}(t)X(t) = E\big[H_{\underline{\lambda}}(T)X(T) \mid \mathcal{F}_t\big], \tag{4.10}$$

and express this in the form of an Ito integral with respect to the Brownian motion W

$$H(t)X(t) = x + \int_0^t \big[\cdots\big]dW(s), \tag{4.11}$$

for comparison purposes.

5. Compare this with

$$x + \int_0^t H_{\underline{\lambda}}(s)X(s)[\underline{\theta}_{\underline{\lambda}}(s) - \hat{\underline{\pi}}^{\top}(s)\sigma(s)]dW(s),$$

to find the optimal portfolio.

This strategy is illustrated in the following examples. The effect of the investment strategies on the asset prices, modelled by the functions f_i ; $i = 0..d$ are also varied in Section 4.3 to represent the following situations :

(i) Price Pressure; cf Section 4.3.2.

(ii) Different borrowing and lending rates for the bond.

To begin with, we examine the case of the large investor with logarithmic utility.

4.3 Example

Step 1 For utility $u(x) = \ln x$ we have $u'(x) = 1/x$ and hence $I(y) = 1/y$.

We require the Lagrange multiplier $\psi_{\lambda}(x)$, where $\underline{\lambda}$ is the optimal dual process (to be solved for). We must solve

$$\begin{aligned}
\zeta_{\underline{v}_{y(x)}}(y(x)) &:= E\Big[H_{\underline{v}_{y(x)}}(T) I(yH_{\underline{v}_{y(x)}}(T))\Big] = x, \\
&\Rightarrow E\Big[H_{\underline{v}_{y(x)}}(T)\frac{1}{yH_{\underline{v}_{y(x)}}(T)}\Big] = x, \\
&\Rightarrow \psi_{\underline{v}}(x) = 1/x.
\end{aligned}$$

Step 2 The value function is

$$\begin{aligned}
V(x) &:= E\Big[u\Big(I(\psi(x)H_{\underline{\lambda}}(T)\Big)\Big] \\
&= E\Big[\ln\Big(\frac{x}{H_{\underline{\lambda}}(T)}\Big)\Big] \\
&= \ln x + E[\ln H_{\underline{\lambda}}^{-1}(T)] \\
&= \ln x + E\Big[\int_0^T \tilde{g}(t,\underline{\lambda}(t))dt + \int_0^T \underline{\theta}_{\underline{\lambda}}(t)dW(t) + \frac{1}{2}\int_0^T \|\underline{\theta}_{\underline{\lambda}}(t)\|^2 dt\Big] \\
&= \ln x + E\Big[\int_0^T \tilde{g}(t,\underline{\lambda}(t)) + \frac{1}{2}\| - \sigma^{-1}(t)\underline{\lambda}(t)\|^2 dt\Big],
\end{aligned}$$

where $\underline{\lambda}$ is to be solved for in the dual.

Step 3 Since $\psi_{\underline{v}}(x) = 1/x$ this implies that $\Gamma \equiv D'$. The Legendre-Fenchel transform is given by

$$\tilde{u}(y) := u(I(y)) - yI(y) = -(1 + \ln y).$$

Therefore

$$E[\tilde{u}(\psi_v(x)H_{\underline{v}}(T))] = E\Big[-1 - \ln\frac{1}{x} - \ln(H_{\underline{v}}(T)\Big],$$

and the dual problem is to find the $\underline{\lambda}$ at which the infimum below is attained

$$
\begin{aligned}
&\inf_{\underline{v}\in D'} E[\ln(H_{\underline{v}}^{-1}(T))] \\
= &\inf_{\underline{v}\in D'} E\Big[\int_0^T \tilde{g}(s,\underline{v}(s)) + \frac{1}{2}\|\theta_{\underline{v}}(s)\|^2 ds + \int_0^T \theta_{\underline{v}}(s)dW(s)\Big].
\end{aligned}
$$

This is equivalent to the pointwise minimisation of the convex function given by

$$\tilde{g}(t,\underline{v}(t)) + \frac{1}{2}\| - \sigma^{-1}(t)\underline{v}(t)\|^2, \tag{4.12}$$

at each $t \in [0,T]$.

We denote the solution to this problem by

$$\underline{\lambda}(t) = arg\min_{\underline{v}\in D'} \Big[2\tilde{g}(t,\underline{v}(t)) + \| - \sigma^{-1}(t)\underline{v}(t)\|^2\Big].$$

This will be solved in the next section for certain market scenarios.

Step 4 As in example 4.1 we can use Ito's rule to rewrite (4.7) as :

$$
\begin{aligned}
H_\lambda(t)X(t) \quad &+ \quad \int_0^t \gamma_\lambda(s)X(s)[\tilde{g}(s,\underline{v}_s) - g(s,\hat{\underline{\pi}}_s) - \hat{\underline{\pi}}(s)\underline{v}_s]ds \\
&= \quad x + \int_0^t H_\lambda(s)X(s)[\theta_\lambda(s) - \underline{\pi}^\top(s)\sigma(s)]dW(s).
\end{aligned}
\tag{4.13}
$$

However, by theorem 3.52, at the optimal $\underline{\lambda}$ the term $\tilde{g}(s,\underline{v}(s)) - g(s,\underline{\pi}(s)) - \underline{\pi}(s)\underline{v}(s)$ disappears and this implies that at time t

$$H_\lambda(t)X(t) = x + \int_0^t H_\lambda(s)X(s)[\theta_\lambda(s) - \underline{\pi}^\top(s)\sigma(s)]dW(s).$$

The optimal terminal wealth is

$$X(T) = I(\psi_\lambda(x)H_\lambda(T)) = \frac{x}{H_\lambda(T)}, \tag{4.14}$$

and from Theorem 3.21

$$\begin{aligned} H_{\underline{\lambda}}(t)X(t) &= E\Big[H_{\underline{\lambda}}(T)X(T) \mid \mathcal{F}_t\Big] \\ &= x. \end{aligned}$$

Step 4 Comparing this with (4.14) we obtain the optimal portfolio in feedback form given by

$$\hat{\underline{\pi}}(t) = -(\sigma(t)\sigma^\top(t))^{-1}\underline{\lambda}(t), \tag{4.15}$$

dependent on the market coefficients and the dual solution $\underline{\lambda}$.

4.4 Example

The case where the investors utility is the power utility $\frac{1}{\alpha}x^\alpha$ is similar to Example 4.3 above. However, in this case $H_\lambda(T)X(T) \neq x$ and our calculations lead to the optimal portfolio process

$$\hat{\underline{\pi}} = -\frac{(\sigma(t)\sigma^\top(t))^{-1}\underline{\lambda}(t)}{1-\alpha}. \tag{4.16}$$

4.3 Some Market Scenarios

From Section 4.2 we have a dual problem to be solved which is dependent on the utility function u and the function $\tilde{g}$. For $u = \ln x$ this dual problem is to find $\underline{\lambda}$ which minimises the expression below

$$2\tilde{g}(t, \underline{v}(t)) + \| - \sigma^{-1}(t)\underline{v}(t)\|^2,$$

where

$$g(t,\underline{\pi}_t) := r(t) + f_0(\underline{\pi}_t) + \sum_{i=1}^{d} \underline{\pi}_i(t)[(b_i(t) + f_i(\underline{\pi}_t) - r(t) - f_0(\underline{\pi}_t)],$$

and

$$\tilde{g}(t,\underline{v}) := \sup_{\underline{\pi}\in\mathbf{R}^d} [g(t,\underline{\pi}) + \underline{\pi}^\top \underline{v}].$$

This section is devoted to solving the dual problem under varying market situations. We will concentrate on logarithmic utility. The theory applies as above for power utilities. These results are also stated. For comparison purposes, we start with the standard complete market setting.

4.3.1 Standard Setting

This is the setting in which prices are exogenously fixed. It has been examined by Karatzas et al. [7] and Cox and Huang [14]. In this case, the *market effect functions* are given by

$$f_0(\underline{\pi}) \equiv 0,$$

and, for all $i = 1...d$,

$$f_i(\underline{\pi}) \equiv 0.$$

Therefore

$$\begin{aligned} g(t,\underline{\pi}_t) &\equiv r(t) + \sum_{i=1}^{d} \underline{\pi}_i(t)[b_i(t) - r(t)] \\ &= r(t) + \underline{\pi}^\top(t)[\underline{b}(t) - r(t)\underline{1}]. \end{aligned}$$

Hence

$$\tilde{g}(t,\underline{v}) = \sup_{\underline{\pi}\in\mathbf{R}^d} \Big[r(t) + \underline{\pi}^\top(t)[\underline{b}(t) - r(t)\underline{1} + \underline{v}]\Big].$$

Indeed we obtain

$$\tilde{g}(t,\underline{v}) = \begin{cases} r(t) & \text{if } \underline{v} \equiv r(t)\underline{1} - \underline{b}(t) \\ \infty & \text{otherwise,} \end{cases}$$

yielding the set

$$D = \left\{ \underline{v} : \underline{v} \equiv r(t)\underline{1} - \underline{b}(t) \right\},$$

with corresponding trivial dual problem with solution

$$\underline{\lambda} = r(t)\underline{1} - \underline{b}(t).$$

We conclude that the optimal investment strategy for an investor with logarithmic utility is

$$\hat{\underline{\pi}}(t) = \left(\sigma(t)\sigma^{\top}(t)\right)^{-1}\left(\underline{b}(t) - r(t)\underline{1}\right).$$

This coincides with the result of example 4.1. The corresponding strategy for the investor with power utility is

$$\hat{\underline{\pi}}(t) = \frac{1}{1-\alpha}\left(\sigma(t)\sigma^{\top}(t)\right)^{-1}\left(\underline{b}(t) - r(t)\underline{1}\right).$$

4.3.2 Price Pressure

Price Pressure occurs when the purchase of a risky asset decreases its expected return while shortselling a stock increases it expected return. In this case, the *market effect functions* are given by

$$f_0(\underline{\pi}) \equiv 0,$$

and, for all $i = 1...d$,

$$f_i(\underline{\pi}) = \begin{cases} \frac{-a(t)\pi}{|\underline{\pi}|} & \text{if } \underline{\pi} \neq \underline{0} \\ 0 & \text{if } \underline{\pi} = \underline{0}, \end{cases}$$

for some function $a : [0, T] \mapsto \mathbf{R}_+$. Therefore

$$\begin{aligned} g(t, \underline{\pi}_t) &\equiv r(t) + \underline{\pi}^{\top}(t)\left[\underline{b}(t) - a(t)\frac{\pi}{|\underline{\pi}|} - r(t)\underline{1}\right] \\ &= r(t) + \underline{\pi}^{\top}(t)\left[\underline{b}(t) - r(t)\underline{1}\right] - a(t)|\underline{\pi}|, \end{aligned}$$

i.e., the expected return on wealth decreases in a concave fashion with the absolute proportions invested in the risky assets. Therefore we obtain

$$\begin{aligned}\tilde{g}(t,\underline{v}) &= \sup_{\underline{\pi}\in\mathbf{R}^d}\Big[r(t)+\underline{\pi}^\top(t)[\underline{b}(t)-r(t)\underline{1}+\underline{v}]-a(t)|\underline{\pi}(t)|\Big] \\ &= \begin{cases} r(t) & \text{if } \|\underline{v}+\underline{b}(t)-r(t)\underline{1}\|\le a(t) \\ \infty & \text{otherwise,}\end{cases}\end{aligned}$$

yielding the set

$$D=\Big\{\underline{v}:\|\underline{v}+\underline{b}(t)-r(t)\underline{1}\|\le a(t)\Big\},$$

with corresponding dual problem given by

$$\underline{\lambda}=arg\min_{\underline{v}:\|\underline{v}+\underline{b}(t)-r(t)\underline{1}\|\le a(t)} r(t)+\frac{1}{2}\|\sigma_t^{-1}\underline{v}_t\|^2.$$

The minimum is achieved at

$$\underline{\lambda}(t) = \begin{cases} 0 & \text{when } |\underline{b}(t)-r(t)\underline{1}|<a(t) \\ r(t)\underline{1}-\underline{b}(t)+a(t)\underline{1} & \text{when } \underline{b}(t)-r(t)\underline{1}\ge a(t) \\ r(t)\underline{1}-\underline{b}(t)-a(t)\underline{1} & \text{when } \underline{b}(t)-r(t)\underline{1}\le -a(t).\end{cases}$$

We conclude the optimal investment strategy for an investor with logarithmic utility is

$$\hat{\underline{\pi}}(t) = \begin{cases} \Big[\sigma(t)\sigma^\top(t)\Big]^{-1}\Big[\underline{b}(t)-r(t)\underline{1}-a(t)\underline{1}\Big] & \text{if } \underline{b}(t)-r(t)\underline{1}\ge a(t) \\ 0 & \text{if } |\underline{b}(t)-r(t)\underline{1}|<a(t) \\ \Big[\sigma(t)\sigma^\top(t)\Big]^{-1}\Big[\underline{b}(t)-r(t)\underline{1}+a(t)\underline{1}\Big] & \text{if } \underline{b}(t)-r(t)\underline{1}\le -a(t)\end{cases}$$

This demonstrates that the fraction of wealth invested in stock is lower than what it would normally be in the absence of price pressure. Note that the comparisons on the right-hand side above can be made component-wise but are written in vector form.

The more general case is the one in which the *market effect functions* are given by

$$f_i(\underline{\pi}) = \begin{cases} \frac{-A(t)\underline{\pi}}{|\underline{\pi}|} & \text{if } \underline{\pi} \neq \underline{0} \\ 0 & \text{if } \underline{\pi} = \underline{0}, \end{cases}$$

for $i = 1...d$ and for some positive definite matrix $A(t)$. The theory above goes through similarly with optimal portfolio

$$\hat{\underline{\pi}}(t) = \begin{cases} \left[\sigma(t)\sigma^{\top}(t)\right]^{-1}\left[\underline{b}(t) - r(t)\underline{1} - A(t)\underline{1}\right] & A^{-1}(t)(\underline{b}(t) - r(t)\underline{1}) \geq 1 \\ 0 & |A^{-1}(t)(\underline{b}(t) - r(t)\underline{1})| < 1 \\ \left[\sigma(t)\sigma^{\top}(t)\right]^{-1}\left[(\underline{b}(t) - r(t)\underline{1} + A(t)\underline{1})\right] & A^{-1}(t)(\underline{b}(t) - r(t)\underline{1}) \leq -1. \end{cases}$$

4.3.3 Different Borrowing and Lending Rates

We now study the case where the borrowing rate is not necessarily equal to the lending rate but the investor has no effect on asset prices. We denote the borrowing rate by $R(t)$ and the lending rate by $r(t)$ where $r(t) \leq R(t)$. We assume the progressively measurable process $R(t)$ is bounded. Therefore, we restrict ourselves to policies for which the relative amount borrowed at time t is

$$(1-\underline{\pi}^\top \underline{1})^- \;=\; \begin{cases} \underline{\pi}^\top \underline{1} - 1 & \text{if } \pi^\top \underline{1} > 1 \\ 0 & \text{otherwise.} \end{cases}$$

We can model this by setting the *market effect functions* as follows :

$$f_0(\underline{\pi}) = [R(t) - r(t)] I_{\{\pi^\top \underline{1} > 1\}},$$

where

$$I_{\{\pi^\top \underline{1} > 1\}} \;=\; \begin{cases} 1 & \text{if } \pi^\top \underline{1} > 1 \\ 0 & \text{otherwise.} \end{cases}$$

For all $i = 1...d$

$$f_i(\underline{\pi}) \equiv 0.$$

Therefore

$$\begin{aligned} g(t, \underline{\pi}_t) \;&\equiv\; r(t) + (R(t) - r(t)) I_{\{\pi^\top \underline{1} > 1\}} \\ &\quad + \underline{\pi}^\top(t) \Big[\underline{b}(t) - r(t)\underline{1} - (R(t) - r(t)) I_{\{\pi^\top \underline{1} > 1\}} \Big] \\ &= r(t) + \underline{\pi}^\top(t) \Big[\underline{b}(t) - r(t)\underline{1} \Big] + \Big[1 - \underline{\pi}^\top(t)\underline{1} \Big] (R(t) - r(t)) I_{\{\pi^\top \underline{1} > 1\}}, \end{aligned}$$

and as expected the evolution equation of (3.3) becomes

$$dX(t) \;=\; X(t) g(t, \underline{\pi}_t) dt + X(t) \underline{\pi}^\top(t) \sigma(t) dW(t) - dc(t)$$

$$
\begin{aligned}
&= r(t)X(t)dt - dc(t) + X(t)\Big[\underline{\pi}^\top(t)\sigma(t)\Big(dW(t) + \sigma^{-1}(t)(\underline{b} - r(t)\underline{1})dt\Big) \\
&\qquad\qquad - (R(t) - r(t))(1 - \underline{\pi}^\top\underline{1})^- dt\Big] \\
&= r(t)X(t)dt - dc(t) + X(t)\Big[\underline{\pi}^\top(t)\sigma(t)d\widetilde{W}(t) - (R(t) - r(t))(1 - \underline{\pi}^\top\underline{1})^- dt\Big].
\end{aligned}
$$

For illustrative purposes we restrict ourselves to the one-dimensional problem of one risky asset and the bond. We obtain

$$
g(t, \pi_t) = r(t) + \pi(t)\Big[b(t) - r(t)\Big] + \Big[1 - \pi(t)\Big](R(t) - r(t))I_{\{\pi > 1\}}.
$$

Consequently

$$
\begin{aligned}
\tilde{g}(t, v(t)) &= \sup_{\pi \in \mathbf{R}} \Big[g(t, \pi) + \pi v(t)\Big] \\
&= \sup_{\pi \in \mathbf{R}} \Big[r(t) + \pi\Big[b(t) + v(t) - r(t)\Big] + \Big[1 - \pi\Big](R(t) - r(t))I_{\{\pi > 1\}}\Big] \\
&= \sup_{\pi \in \mathbf{R}} \begin{cases} r(t) + \pi\Big[b(t) + v(t) - r(t)\Big] & \text{if } \pi \leq 1, \\ R(t) + \pi\Big[b(t) + v(t) - R(t)\Big] & \text{if } \pi > 1. \end{cases}
\end{aligned}
$$

Fixing t, we get three cases :

(i) The first case is

$$
b + v - R > 0 \Rightarrow b + v - r > 0,
$$

and the slopes on both intervals are positive, so that the supremum over π is infinity.

(ii) The case

$$
b + v - r < 0 \Rightarrow b + v - R < 0,
$$

is very similar. Here, the slopes on both intervals are negative and the supremum over π is also infinity.

(iii) The case

$$b + v - R \leq 0 \text{ and } b + v - r \geq 0,$$

means the slope is nonnegative before $\pi = 1$ and non-positive after. Now the supremum is finite and attained at $\pi = 1$. The function value is $b + v$. Therefore, we obtain

$$\tilde{g}(t, v(t)) = \begin{cases} b(t) + v(t) & \text{if } r(t) \leq b(t) + v(t) \leq R(t), \\ \infty & \text{otherwise,} \end{cases}$$

and the set

$$D_t = \left\{v : r(t) \leq b(t) + v(t) \leq R(t)\right\},$$

with corresponding dual problem given by

$$\underline{\lambda} = arg \min_{v: r(t) \leq b(t) + v(t) \leq R(t)} 2(b(t) + v(t)) + \frac{1}{\sigma^2(t)} v^2(t).$$

The global minimum is achieved at

$$\lambda(t) = -\sigma^2(t).$$

However, confined to the set D_t, the minimum of the dual problem is achieved at

$$\lambda(t) = \begin{cases} R(t) - b(t) & \text{once } b(t) - R(t) \geq \sigma^2(t) \\ -\sigma^2(t) & \text{once } b(t) - R(t) \leq \sigma^2(t) \leq b(t) - r(t) \\ r(t) - b(t) & \text{once } b(t) - r(t) \leq \sigma^2(t). \end{cases}$$

We conclude the optimal investment strategy for an investor with logarithmic utility is

$$\lambda(t) = \begin{cases} \frac{1}{\sigma^2(t)}[b(t) - R(t)] & \text{if } \sigma^{-2}(t)(b(t) - R(t)) \geq 1 \\ 1 & \text{if } \sigma^{-2}(t)(b(t) - R(t)) \leq 1 \leq \sigma^{-2}(t)(b(t) - r(t)) \\ \frac{1}{\sigma^2(t)}[b(t) - r(t)] & \text{if } \sigma^{-2}(t)(b(t) - r(t)) \leq 1. \end{cases}$$

This shows that, as expected, the fraction of wealth invested in the risky asset is lower than it would be without the spread between borrowing and lending rates. The agent will shortsell the risky asset and lend if $b(t) < r(t)$ and will invest in the risky asset if $b(t) > r(t)$. Borrowing will only occur when $b(t) > R(t) + \sigma^2(t)$ but as long as $b(t)$ is in the interval $[r(t) + \sigma^2(t), R(t) + \sigma^2(t)]$ all wealth will be invested in the risky asset without the agent borrowing or lending.

The d-dimensional case is computationally more difficult, but essentially similar. We define

$$\begin{aligned} A(t) &= Tr[\sigma^{-1}(t)^\top \sigma^{-1}(t)], \\ B(t) &= [\sigma^{-1}(t)(\underline{b}(t) - r(t)\underline{1})]^\top \sigma^{-1}(t)\underline{1}. \end{aligned}$$

The minimum of the dual problem is achieved at

$$\underline{\hat{\pi}}(t) = \begin{cases} \left[r(t)\underline{1} - \underline{b}(t) + \frac{B(t)-1}{A(t)}\underline{1}\right] & \text{if } 0 < B(t) - 1 \leq A(t)(R(t) - r(t)) \\ \left[r(t)\underline{1} - \underline{b}(t)\right] & \text{if } B(t) \leq 1 \\ \left[R(t)\underline{1} - \underline{b}(t)\right] & \text{if } B(t) - 1 \geq A(t)(R(t) - r(t)). \end{cases}$$

We conclude that the optimal investment strategy for an investor with logarithmic utility is

$$\underline{\hat{\pi}}(t) = \begin{cases} \left[\sigma(t)\sigma^{\top}(t)\right]^{-1}\left[\underline{b}(t) - r(t)\underline{1} - \frac{B(t)-1}{A(t)}\underline{1}\right] & 0 < B(t) - 1 \leq A(t)(R(t) - r(t)) \\ \left[\sigma(t)\sigma^{\top}(t)\right]^{-1}\left[\underline{b}(t) - r(t)\underline{1}\right] & B(t) \leq 1 \\ \left[\sigma(t)\sigma^{\top}(t)\right]^{-1}\left[\underline{b}(t) - R(t)\underline{1}\right] & B(t) - 1 \geq A(t)(R(t) - r(t)). \end{cases}$$

We note that there are numerous market scenarios that can be modelled using the large investor assumptions. It is possible, by choosing the appropriate market effect functions, to model many current situations. The methodology of solution above yields in all cases a minimisation problem which, in many cases, can be solved analytically.

Chapter 5

Summary

5.1 Conclusions

In Chapter 1 we state that our main objective is to ensure the satisfaction of any investor with the return on his investment. We choose the utility of final wealth as an optimality criterion. Again we stress that this is a suitable choice for the investor who wishes to *get rich*. However it is clear that most investors would prefer to spend during the interval and the utility should therefore account for satisfaction derived in this way. This problem has been researched and indeed solved in Karatzas et al. [7]. It is an essentially similar problem and we have therefore concentrated on the problem of utility from final wealth.

This problem has been extensively researched for a small investor. Indeed, to all intents and purposes, the small investor problem is considered *solved*[1]. In fact, for a small investor whose portfolio is confined to lie within a particular region of $\mathbf{R}^d$, the optimal form of the wealth process is known. However, the optimal trading strategy $\underline{\pi}$ can only be calculated for certain utility functions such as the logarithmic utility and the HARA utility $\frac{1}{\gamma}x^\gamma$.

[1]See Karatzas et al. [6].

We wish to solve the equivalent problem for a large investor, using the established martingale methods. Our work has continued along this line of approach - used by Cvitanic and Karatzas [16] for constrained portfolios. More precisely, we define the convex conjugate function $\tilde{g}(t, \underline{v})$ corresponding to a large investor instead of the particular choice of $\delta(\underline{v})$ used in the constrained portfolio case. We find that under the corresponding probability measure $P^{\underline{v}}$ and discount factor $\gamma_{\underline{v}}(\cdot)$ the discounted stock prices become martingales. This introduces a new set of auxilary markets whose evolution equation for wealth is linear and hence can be solved as in the small investor problem. Our aim is then to find the appropriate $\underline{v}$ to reflect the given market conditions. This is the dual problem, and it depends on the particular choice of market effect functions.

We find that the non-linear large investor problem can be solved using the strategy described in Chapter 4 for certain utilities. The form of both the wealth and the optimal portfolio are known in this case. However, as in the case for the small investor, for general utilities we must make certain assumptions on the market coefficients to solve the portfolio problem. This is done for the small investor in Xu [15]. Only to this extent has the large investor problem been *solved.*

More precisely, as in Chapter 4, we can find the form of the optimal wealth process from Theorem 3.21. We can also find explicitly the optimal portfolio process which yields this wealth process. However this is only possible for those same utility functions for which an explicit solution exists in the small investor case.

The optimal portfolio $\underline{\pi}$ depends on the dual solution $\underline{\lambda}$. In short, any problem solved explictly for the small investor can be solved similarly for the large investor with solution dependent on $\underline{\lambda}$.

Our primary concern then, is how to solve for $\underline{\lambda}$. In most cases this involves a straightforward constrained minimisation problem of a concave function (see Section 4.3). However the problem remains (as in the case of the small investor under constrained portfolios) : namely, the existence of the optimal portfolio has been proved, but an algorithm to find it explictly is not available for general utilities.

In addition, a fair price (selling) is found for a large investor selling any claim B. Anything above this price would represent an arbitrage opportunity. The fact that the hedging portfolio may not always be easy to find explictly is not important to someone pricing the option or future derivative in question. The same methods can be applied to find a fair (purchase) price for the buyer of a claim. This would give a bid-ask spread for the price of a claim.

5.2 Further Work

For most given market conditions the problem of solving for the dual solution is, as stated, a constrained maximisation of a concave function. This should not, in general, present too much difficulty to solve - either explicitly or numerically.

The main problem is that, similarly to the case of the small investor with constrained portfolios, we can only find the optimal hedging portfolio $\underline{\pi}$ for certain utilities. An approach similar to Xu [15] could be undertaken to find the optimal portfolio in explicit form for a general utility under the extra assumption of constant market coefficients $r(t) = r$, $\underline{b}(t) \equiv \underline{b}$ and $\sigma(t) \equiv \sigma$. It would be interesting to see if we could combine the function $\delta(\underline{v})$ used by Cvitanic and Karatzas [16] for constrained portfolios and the function $\tilde{g}(t, \underline{v})$ for the large investor to model the case of a *constrained large investor*. In particular a market controller may wish to limit the behaviour of a large investor to minimise his effect on market prices and stabilise the market.

Once the large investor decides on his preferred strategy the market will immediately be affected by his actions. From a small investor's point of view the following question arises - how should he react to a large investor's strategy ? Does his optimal portfolio change in the presence of a large investor ? Given the advantageous position (in terms of size or knowledge) of the large investor, is he in fact better off or does he have an adverse effect on the prices of the assets in which he trades ? If so, should the small investor adopt the same utility and investment strategy as the large investor ?

Numerous articles were studied examining the effect of brokerage fees on the agent's investment strategy. These include Davis and Norman [17], Morton and Pliska [11] and Atkinson and Wilmott [18]. The general result of all papers is that the small investor attempts to keep the proportion invested in stock within a certain wedge in $\mathbf{R}^d$ instead of a singular value as we found in our examples. It is suspected that the same would be true for the large investor. However, given certain extreme effects that he may have on the stock evolution it should be examined whether or not the investor would adopt a *wedge strategy* under prohibitively large brokerage fees or whether he could afford to maintain an exact previsible portfolio process via continuous trading.

Bibliography

[1] Jacques Neveu. *Discrete Parameter Martingales.* American Elsevier Publishing Company, New York, 1976.

[2] Ioannis Karatzas and Steven E. Shreve. *Brownian Motion and Stochastic Calculus.* Springer-Verlag, New York, 1988.

[3] Ivar Ekeland and Roger Temam. *Convex Analysis and Variational Problems.* American Elsevier Publishing Company, New York, 1976.

[4] Robert C. Merton. Optimum consumption and portfolio rules in a continuous time model. *Journal of Economic Theory*, 3:373 – 413, 1971.

[5] Jaksa Cvitanic and Ioannis Karatzas. Hedging contingent claims with constrained portfolios. *The Annals of Applied Probability*, 3(3):652 – 681, 1993.

[6] Ioannis Karatzas, John P. Lehoczky, Steven E. Shreve, and Gan-Lin Xu. Martingale and duality methods for utility maximisation in an incomplete market. *SIAM Journal of Control and Optimization*, 29(3):702 – 730, May 1991.

[7] Ioannis Karatzas, John P. Lehoczky, and Steven E. Shreve. Optimal investment and consumption decisions for a small investor on a finite horizon. *SIAM Journal of Control and Optimization*, 25(6):1557 – 1586, November 1987.

[8] Gan-Lin Xu and Steven E. Shreve. Duality methods for optimal consumption and investment under short- selling prohibition. i. general market coefficients. *Annals of Applied Probability*, 2(1):87 – 112, 1992.

[9] Ioannis Karatzas. Optimization problems in the theory of continuous trading. *SIAM Journal of Control and Optimization*, 27(6):1221 – 1259, May 1989.

[10] Michael Taksar, Michael J. Klass, and David Assaf. A diffusion model for optimal portfolio selection in the presence of brokerage fees. *Mathematics of Operations Research*, 13(2):277, May 1988.

[11] Andrew J. Morton and Stanley R. Pliska. Optimal portfolio management with fixed transaction costs. *Journal of Mathematical Finance*, 5(4):337 – 356, October 1995.

[12] Nicole El Karoui, Peng, and Marie Claire Quenez. Dynamic programming and pricing of contingent claims in an incomplete market, 1994. Preprint.

[13] Gan-Lin Xu. *A Duality Method for Optimal Consumption and Investment under Short-selling Prohibition*. PhD thesis, Carnegie Mellon University, 1990. Doctoral Dissertation.

[14] John C. Cox and Chi-Fu Huang. Optimal consumption and portfolio policies when asset prices follow a diffusion process. *Journal of Economic Theory*, 49:33 – 83, 1989.

[15] Gan-Lin Xu and Steven E. Shreve. Duality methods for optimal consumption and investment under short- selling prohibition. ii. constant market coefficients. *Annals of Applied Probability*, 2(1):314–328, 1992.

[16] Jaksa Cvitanic and Ioannis Karatzas. Convex duality in constrained portfolio optimization. *The Annals of Applied Probability*, 2(4):767 – 818, 1992.

[17] M. H. A. Davis and A. R. Norman. Portfolio selection with transaction costs. *Mathematics of Operations Research*, 15(4):676 – 713, November 1990.

[18] C. Atkinson and P. Wilmott. Portfolio management with transaction costs : An asymptotic analysis of the Morton and Pliska model. *Journal of Mathematical Finance*, 5(4):357 – 367, October 1995.

[19] John C. Hull. *Options, Futures and Other Derivative Securities.* Prentice Hall International, New Jersey, second edition, 1993.

[20] David Williams. *Probability with Martingales.* Cambridge University Press, 1991.

[21] Michael C. Gemignani. *Elementary Topology.* Dover Publications, New York, 1990.

[22] R. Tyrrell Rockafellar. *Convex Analysis.* Princeton University Press, Princeton, 1970.

Appendix A

Some useful definitions and theorems

The following useful theory is taken from Hull [19], Karatzas and Shreve [2] and Williams [20].

A.1 Introduction

A stochastic process is a mathematical model for the occurence, at each moment after the initial time, of a random phenomenon. The randomness is captured by the introduction of a measurable space $(\Omega, \mathcal{F})$ called the *sample space*, on which probability measures can be placed.

A.1.1 Definition

For our purpose, a **stochastic process** is a collection of d random variables $X = \{X_t, t \geq 0\}$ on $(\Omega, \mathcal{F}, P)$ taking values in the state space $(\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d))$ where $\mathcal{B}(\mathbf{R}^d)$ is the smallest σ-field containing all open sets of $\mathbf{R}^d$.

A.1.2 Definition

X is **measurable** if, for every $A \in \mathcal{B}(\mathbf{R}^d)$, the set

$$\{(t,\omega) : X_t(\omega) \in A\},$$

belongs to the σ-field $\mathcal{B}[0,\infty) \otimes \mathcal{F}$. In other words, X is measurable if the mapping

$$(t,\omega) \mapsto X_t(\omega) : ([0,\infty) \times \Omega, \mathcal{B}[0,\infty) \otimes \mathcal{F}) \mapsto (\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d))$$

is measurable.

A.1.3 Definition

We equip our sample space $(\Omega, \mathcal{F})$ with a **filtration**; a non-decreasing family $\{\mathcal{F}_t; t \geq 0\}$ of sub-σ-fields of $\mathcal{F}$:

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}, \qquad 0 \leq s < t < \infty,$$

where $\mathcal{F}_t$ represents the information known at time t.

A.1.4 Definition

The simplest choice of filtration is that generated by the process itself :

$$\mathcal{F}_t^X := \sigma(X_s : 0 \leq s \leq t).$$

This is the smallest σ-field with respect to which X_s is measurable for every $s \in [0,t]$, and is called the **natural filtration**.

A.1.5 Definition

Correspondingly, we define

$\mathcal{F}_{t-} := \sigma(\bigcup_{s<t} \mathcal{F}_s)$ to be the σ- field of events strictly prior to $t > 0$.

$\mathcal{F}_{t+} := \sigma(\bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon})$ to be the σ-field of events immediately after $t \geq 0$.

A.1.6 Definition

The filtration $\mathcal{F}_t$ is **left(right)-continuous** if $\mathcal{F}_t = \mathcal{F}_{t-}$ (respectively $\mathcal{F}_t = \mathcal{F}_{t+}$) holds for every $t \geq 0$.

A.1.7 Definition

A filtration is said to satisfy the **usual conditions** if it is right continuous and $\mathcal{F}_0$ contains all P-negligible events in $\mathcal{F}$.

A.1.8 Definition

X is **adapted** to the filtration $\{\mathcal{F}_t\}$ if, for each $t \geq 0$, X_t is an $\mathcal{F}_t$-measurable random variable

A.1.9 Definition

X is **progressively measurable** with respect to the filtration $\{\mathcal{F}_t\}$ if, for each $t \geq 0$ and $A \in \mathcal{B}(\mathbf{R}^d)$, the set

$$\{(s,\omega) : X_s(\omega) \in A; 0 \leq s \leq t\},$$

belongs to the σ-field $\mathcal{B}[0,t] \otimes \mathcal{F}_t$. In other words, if the mapping

$$(s,\omega) \mapsto X_s(\omega) : ([0,t] \times \Omega, \mathcal{B}[0,T] \otimes \mathcal{F}_t) \mapsto (\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d)),$$

is measurable for each $t \geq 0$.

Martingales

We shall consider, exclusively, real-valued processes $X = \{X_t, t \geq 0\}$ on a probability space $(\Omega, \mathcal{F}, P)$, adapted to a given filtration $\mathcal{F}_t$ and such that $E|X_t| < \infty$.

A.1.10 Definition

The process $X = \{X_t, \mathcal{F}_t, t \geq 0\}$ is a **martingale** if, for every $0 \leq s < t$, we have

$$E[X_t \mid \mathcal{F}_s] = X_s, \qquad P\text{-a.s.}$$

A.1.11 Definition

It is a **supermartingale** if, for every $0 \leq s < t$ we have

$$E[X_t \mid \mathcal{F}_s] \leq X_s, \qquad P\text{-a.s.}$$

A.1.12 Definition

It is a **submartingale** if, for every $0 \leq s < t$ we have

$$E[X_t \mid \mathcal{F}_s] \geq X_s, \qquad P\text{-a.s.}$$

A.1.13 Definition

If these properties hold only for the processes

$$X^{(n)}(t) = X(t \wedge \tau_n),$$

for each τ_n, where τ_n is a non-decreasing sequence of stopping times converging to infinity, then $X(\cdot)$ is a **local martingale** (respectively supermartingale, submartingale).

A.1.14 Definition

A **random time** τ is an $\mathcal{F}$-measurable random variable with values in $[0, \infty)$.

A.1.15 Definition

A random time τ is a **stopping time** of the filtration if the event $\{\omega : \tau(\omega) \leq t\}$ belongs to the σ-field $\mathcal{F}_t$, for every $t \geq 0$.

A.1.16 Definition

A continuous adapted process $W = \{W_t, \mathcal{F}_t, t \geq 0\}$ defined on $(\Omega, \mathcal{F}, P)$ is a **standard Brownian motion** if

(i) $W(0) = 0$ a.s.,

(ii) $W_t - W_s$ is independent of $\mathcal{F}_s$, $\forall\, t > s$,

(iii) $W_t - W_s$ is normal with mean zero and variance $t - s$.

Brownian motion can be one dimensional or d-dimensional. Brownian motion in **R** is a martingale. A stochastic integral (one with respect to Brownian motion) is also a martingale.

A.1.17 Definition

Consider the class $\mathcal{S}$ of all stopping times τ of the filtration $\{\mathcal{F}_t\}$ which satisfy $P[\tau < \infty] = 1$. The right-continuous process $\{X_t, \mathcal{F}_t : 0 \leq t < \infty\}$ is said to be **of class** D if the family $\{X_\tau\}_{\tau \in \mathcal{S}}$ is uniformly integrable.

Alternatively, if we consider the class $\mathcal{S}_a$ with $P[\tau \leq a] = 1$, then the process

$\{X_t, \mathcal{F}_t : 0 \leq t < \infty\}$ is said to be **of class** DL if the family $\{X_\tau\}_{\tau \in \mathcal{S}_a}$ is uniformly integrable for every $0 < a < \infty$.

A.1.18 Definition

The process $\{X_t, \mathcal{F}_t : 0 \leq t < \infty\}$ is said to be **RCLL** if it is right continuous on $[0, \infty)$ and has finite left-hand limits on $(0, \infty)$.

A.2 Girsanov's Theorem

Given W, the standard d-dimensional Brownian motion, let

$$X = \{\underline{X}_t, \mathcal{F}_t, t \geq 0\},$$

be a vector of measurable, adapted processes satisfying

$$P\Big[\int_0^T (X_t^i)^2 dt < \infty\Big] = 1, \qquad 0 \leq T < \infty,$$

for $i = 1..d$. Define

$$Z_{\underline{X}}(t) := \exp\Big\{\sum_{i=1}^{d} \int_0^t X_s^i dW_s^i - \frac{1}{2}\int_0^t \|\underline{X}_s\|^2 ds\Big\},$$

a continuous local martingale with $Z_{\underline{X}}(0) = 1$.

Under the Novikov condition[1] $E[Z_{\underline{X}}(t)] = 1$ and $Z_{\underline{X}}(t)$ is a P-martingale. Consider then a new probability measure $\widetilde{P}_T$ on $(\Omega, \mathcal{F}_T)$ given by

$$\widetilde{P}_T(A) := Z_{\underline{X}}(t) P[A],$$

and hence

$$\widetilde{E}_T[I_A] = E[I_A Z_{\underline{X}}(t)].$$

Girsanov's Rule

If $Z_{\underline{X}}(t)$ defined above is a martingale, then the process $\widetilde{W} = \{\widetilde{W}_t, \mathcal{F}_t, t \geq 0\}$ given by

$$\widetilde{W}_t^i := W_t^i - \int_0^t X_s^i ds,$$

is a d-dimensional Brownian Motion on $(\Omega, \mathcal{F}_T, \widetilde{P}_T)$.

◇

[1]The Novikov condition states that $Z_{\underline{X}}(t)$ defined above for the process $\underline{X}$ is a martingale once $E[\exp(1/2 \int_0^T \|\underline{X}_s\|^2 ds)] < \infty$.

A.3 Properties of Conditional Expectation

$(\Omega, \mathcal{F}, P)$ is our probability triple, and X a random variable with $E[|X|] < \infty$. Let $\mathcal{G}$ and $\mathcal{H}$ denote sub-σ-algebras of $\mathcal{F}$. The following properties hold :

(i) If X is $\mathcal{G}$-measurable then $E(X \mid \mathcal{G}) = X$ a.s.

(ii) *Linearity :* $E(a_1 X_1 + a_2 X_2 \mid \mathcal{G}) = a_1 E(X_1 \mid \mathcal{G}) + a_2 E(X_1 \mid \mathcal{G})$ a.s.

(iii) *Positivity :* If $X \geq 0$ then $E(X \mid \mathcal{G}) \geq 0$ a.s.

(iv) *Fatou's Lemma :* If we have a sequence of random variables $\{X_n \geq 0; n \in \mathbf{N}\}$ then

$$E\Big[\lim_{n\to\infty} \inf X_n \mid \mathcal{G}\Big] \leq \lim_{n\to\infty} \inf E[X_n \mid \mathcal{G}] \text{ a.s.}$$

(v) *Dominated Convergence :* If the sequence above satisfies $|X_n(\omega)| \leq V(\omega) \quad \forall \quad n$ with $E[V] < \infty$ and $\lim_{n\to\infty} X_n = X$ a.s., then

$$\lim_{n\to\infty} E[X_n \mid \mathcal{G}] = E[X \mid \mathcal{G}] \text{ a.s.}$$

(vi) *Jensen's Inequality :* If $c : \mathbf{R} \to \mathbf{R}$ is convex and $E[|c(X)|] < \infty$, then

$$E[c(X) \mid \mathcal{G}] \geq c\big(E[X \mid \mathcal{G}]\big) \text{ a.s.}$$

(vii) *Tower Property :* If $\mathcal{H}$ is a sub-σ-algebra of $\mathcal{G}$, then

$$E\big[E[X \mid \mathcal{G}] \mid \mathcal{H}\big] = E[X \mid \mathcal{H}] \text{ a.s.}$$

(viii) *Taking out what is known :* If Z is $\mathcal{G}$-measurable and bounded, then

$$E[ZX \mid \mathcal{G}] = ZE[X \mid \mathcal{G}] \text{ a.s.}$$

A.4 Neveu [1]

Gemignani [21]

Firstly, we require some elementary definitions from Gemignani [21] :

A relation $\preceq$ on any set S with the following properties is a ***partial ordering on S***

(i) $x \preceq x, \qquad \forall\, x \in S,$

(ii) $x \preceq y,\ y \preceq x \Rightarrow x = y, \quad \forall\, x, y \in S,$

(iii) $x \leq y,\ y \preceq z \Rightarrow x \preceq z, \quad \forall\, x, y, z \in S.$

The set $(S, \preceq)$ is a partially ordered set.

S is *totally ordered* under $\preceq$ if, given any $s, t \in S$ either $s \preceq t$ or $t \preceq s$.

If I is partially ordered under $\preceq$, then I is said to be an *upward directed set* if, given $i, j \in I$, $\exists\, k \in I$ such that $i \preceq k$ and $j \preceq k$.

Neveu [1]

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A family of random variables $\{\beta_i : i \in I\}$ indexed by the ordered set I is *directed upwards* if :

(i) the ordered set I is directed upwards under $\preceq$,

(ii) the mapping $t \mapsto \beta_t$ is increasing for inclusion, i.e., $\beta_{i_1} \leq \beta_{i_2}$ whenever $i_1 \preceq i_2$.

Proposition 6.1.1. from Neveu [1]

For every family F of real-valued measurable functions $f : \Omega \mapsto \bar{\mathbf{R}}$ defined on the probability space $(\Omega, \mathcal{F}, P)$, there exists one and only one (up to equivalence) measurable function $g : \Omega \mapsto \bar{\mathbf{R}}$ such that

(a) $g \geq f$ a.s., $\forall\, f \in F$,

(b) if h is a measurable function such that $h \geq f$ almost surely for all $f \in F$, then $h \geq g$ a.s.

This function g, which is the least upper bound of family F in the sense of almost sure inequality is denoted by $ess\sup(F)$.

Furthermore there exists at least one sequence $\{f_n : n \in N\}$ from F such that

$$ess\sup(F) = \sup_n f_n \text{a.s.}$$

If F is directed upwards the sequence $\{f_n : n \in N\}$ can be chosen to be increasing almost surely and then

$$ess\sup(F) = \lim_{n\to\infty} \uparrow f_n \text{ a.s.}$$

In our case $\{J_{\underline{v}}(\theta)\}_{\underline{v}\in D}$ is the family of random variables of (3.23) indexed by the set D. By construction, the family $\{J_{\underline{v}}(\theta)\}_{\underline{v}\in D}$ satisfies the two conditions above and hence is directed upwards.

We apply the proposition above to the family $F = \{J_{\underline{v}}(\theta)\}_{\underline{v}\in D}$ to obtain

$$ess\sup_{\underline{v}\in D} J_{\underline{v}}(\theta) = \lim_{n\to\infty} \uparrow J_{\underline{v}_n}(\theta) \text{ a.s.,}$$

for some sequence $\{\underline{v}_k\} \subseteq D$.

A.5 Ito's Lemma

Define $W = \{\underline{W}_t, \mathcal{F}_t, t \geq 0\}$, as a standard d-dimensional Brownian Motion, null at zero and defined on the Probability Space $(\Omega, \mathcal{F}, P)$.

$\underline{X}$ is an **Ito Process** if for some $\underline{x} \in \mathbf{R}^d$

$$\underline{X}_t = \underline{x} + \int_0^t \underline{\mu}(s)ds + \int_0^t \theta_s \cdot dW_s,$$

where $\theta \in \mathbf{R}^{d \times d}, \underline{\mu} \in \mathbf{R}^d$ or, written in differential form

$$\begin{aligned} d\underline{X}_t &= \underline{\mu}_t dt + \theta_t \cdot dW_t, \\ \underline{X}_0 &= \underline{x}. \end{aligned}$$

Ito's Rule:

If $\underline{X}$ is an Ito Process in $\mathbf{R}^d$ and $f \in C^{2,1}(\mathbf{R}^d \times [0, \infty))$.

Then $\{f(\underline{X}_t, t); t \geq 0\}$ is an Ito Process with

$$f(\underline{X}_t, t) = f(\underline{X}_0, 0) + \int_0^t D_{\underline{x}} f(\underline{X}_s, s)ds + \int_0^t f_{\underline{x}}(\underline{X}_s, s)\theta_s dW_s,$$

where

$$D_{\underline{x}} f(\underline{X}_t, t) = f_{\underline{x}}(\underline{X}_t, t) \cdot \underline{\mu}_t + f_t(\underline{X}_t, t) + \frac{1}{2} Tr[\theta_t \theta_t^T f_{xx}(\underline{X}_t, t)].$$

◇

It is easy to show that as a consequence of Ito's Rule, for two processes satisfying

$$\begin{aligned} d\underline{X}_1(t) &= \underline{\mu}_1(t)dt + \theta_1(t) \cdot dW_t, \\ d\underline{X}_2(t) &= \underline{\mu}_2(t)dt + \theta_2(t) \cdot dW_t, \end{aligned}$$

we have

$$d(\underline{X}_1(t)\underline{X}_2(t)) = \underline{X}_1(t)d\underline{X}_2(t) + \underline{X}_2(t)d\underline{X}_1(t) + \theta_1(t)\theta_2(t)dt.$$

This is used throughout the thesis.

A.6 Bayes Rule

Fix $T \geq 0$ and assume that $Z_{\underline{X}}(t)$ as in Appendix A.2 is a martingale. If $0 \leq s \leq t \leq T$ and Y is an $\mathcal{F}_t$-measurable random variable satisfying $\tilde{E}[|Y|] < \infty$ then

$$\tilde{E}_T[Y \mid \mathcal{F}_s] = \frac{E[Y Z_{\underline{X}}(t) \mid \mathcal{F}_s]}{Z_{\underline{X}}(s)} \qquad \text{a.s.}$$

A.7 Monotone Convergence

Let $(\Omega, \mathcal{F}, P)$ be a probability triple, and X be a random variable with $E[|X|] < \infty$. Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$.

If we have a sequence of random variables $\{X_n \geq 0; n \in \mathrm{N}\}$ satisfying

$$\lim_{n\to\infty} \uparrow X_n = X,$$

then

$$\lim_{n\to\infty} \uparrow E[X_n \mid \mathcal{G}] = E[X \mid \mathcal{G}] \text{ a.s.}$$

A.8 Karatzas and Shreve [2]

Theorem 1.3.13

Let $X = \{X_t, \mathcal{F}_t, t \geq 0\}$ be a supermartingale with $\mathcal{F}_t$ satisfying the usual conditions (see Appendix A.1.7). The process X has a right continuous modification $\Leftrightarrow t \mapsto E[X_t] : [0, \infty) \mapsto \mathbf{R}$ is right continuous.

If this right continuous modification exists it can be chosen so as to be right-continuous with left-handed limits (RCLL) and adapted to $\{\mathcal{F}_t\}$, hence a supermartingale with respect to $\{\mathcal{F}_t\}$.

Proposition 1.3.14

If $X = \{X_t, \mathcal{F}_t, t \geq 0\}$ is a supermartingale, we have

(i) $\exists$ an event $\Omega^* \in \mathcal{F}$ with $P[\Omega^*] = 1$ such that for every $\omega \in \Omega^*$ the limits

$$X_{t+}(\omega) := \lim_{\substack{s \downarrow t \\ s \in Q}} X_s(\omega), \quad t \geq 0$$
$$X_{t-}(\omega) := \lim_{\substack{s \uparrow t \\ s \in Q}} X_s(\omega), \quad t > 0$$

exist and (ii) these limits satisfy

$$E[X_{t+} \mid \mathcal{F}_t] \leq X_t \qquad P\text{-a.s.},$$
$$E[X_t \mid \mathcal{F}_{t-}] \leq X_{t-} \qquad P\text{-a.s.}$$

(iii) $\{X_{t+}, \mathcal{F}_{t+}, t \geq 0\}$ is an RCLL supermartingale.

◇

A.9 Doob-Meyer Decomposition

Theorem

If $\mathcal{F}_t$ satisfies the usual conditions and the right continuous submartingale $X = \{X_t, \mathcal{F}_t, t \geq 0\}$ is of class DL then it admits the decomposition

$$X_t = M_t + A_t,$$

where $M = \{M_t, \mathcal{F}_t, t \geq 0\}$ is a right continuous martingale and $A = \{A_t, \mathcal{F}_t, t \geq 0\}$ is an increasing process.

Furthermore if X is of class D, then M is uniformly integrable and A is integrable.

Corollary

A supermartingale X can be uniquely decomposed as

$$X_t = X(0) + M_t - A_t,$$

where M is a local martingale with $M(0) = 0$ and A is an non-decreasing locally natural process with $A(0) = 0$.

If X is also positive and of class DL, then M is a martingale.

◇

A.10 Fundamental Martingale Representation Theorem

If $M(t)$ is an $\mathcal{F}_t$-adapted local martingale with RCLL paths and $M(0) = 0$, then there exists an $\mathbf{R}^d$ valued process $\psi(t)$ with

$$M(t) = \int_0^t \psi(s) dW_s,$$

where ψ satisfies

$$\int_0^T \|\psi(s)\|^2 ds < \infty.$$

and if $E[M^2(T)] < \infty$, then

$$E[\int_0^T \|\psi(s)\|^2 ds] < \infty,$$

and $M(t)$ is a martingale.

If $\tilde{\psi}(t)$ is another such process, then

$$\int_0^T \|\psi(s) - \tilde{\psi}(s)\|^2 ds = 0.$$

A.11 Ekeland and Temam [3]

If F is a mapping from $C \subset V \mapsto \mathbf{R}$ we associate with it the functional $\widehat{F} : V \mapsto \mathbf{R}$ defined throughout V by

$$\widehat{F}(u) = \begin{cases} F(u), & u \in C, \\ +\infty, & u \notin C, \end{cases} \tag{A.1}$$

and $\widehat{F}$ is convex $\iff$ $C \subset V$ is convex and $F : C \mapsto \mathbf{R}$ is convex.

Let V be a reflexive Banach Space (with norm $\lceil \cdot \rceil$) and C a non-empty closed convex subset of V. Take a function $F : C \mapsto \mathbf{R}$ and assume F is convex and lower semi-continuous.

We are concerned with the problem

$$\inf_{u \in C} F(u).$$

This problem can be replaced with the identical problem (with the same infimum and the same set of solutions) given by :

$$\inf_{u \in V} \widehat{F}(u), \tag{A.2}$$

with the functional $\widehat{F}$ defined above.

Proposition Assume that $\widehat{F}$ is convex, lower semi-continuous and coercive over V.

Then the problem (A.2) above has at least one solution. This is unique if $\widehat{F}$ is strictly convex over V.

A.12 The Optional Sampling Theorem

Let $\{X_t, \mathcal{F}_t : 0 \leq t \leq \infty\}$ be a right-continuous supermartingale and let $s \leq \tau$ be two optional stopping times of the filtration $\{\mathcal{F}_t\}$. We have

$$E[X_t \mid \mathcal{F}_{s+}] \leq X_s \ P\text{-a.s.}$$

If s is a stopping time, then $\mathcal{F}_s$ can replace $\mathcal{F}_{s+}$ above. In particular $E[X_T] \leq E[X_0]$.

Appendix B

Some proofs and calculations

B.1 Convexity of g and D

From Rockafellar [22], for a convex function f, the conjugate is defined as

$$\tilde{f} := \sup_{\underline{x} \in \mathbf{R}^d} [\underline{x}^T \underline{v} - f(\underline{x})],$$

on $\mathbf{R}^d$ and is convex.

In our case g is concave, $-g$ is convex with convex conjugate given by

$$\tilde{h} := \sup_{\underline{x} \in \mathbf{R}^d} [\underline{x}^T \underline{v} + g(\underline{x})],$$

denoted by $\tilde{g}(t, \underline{v})$ in our notation.

D has been redefined by Remark 3.9. We check if these properties hold taking $\underline{v} = \lambda \underline{v}_1 + \mu \underline{v}_2$. We can see clearly that

(i) $\underline{v}$ is $\mathcal{F}_t$-measurable,

(ii) $\underline{v}$ is also uniformly bounded.

(iii) Finally

$$E\Big[\int_0^T \|\lambda \underline{v}_1 + \mu \underline{v}_2\|^2 dt + \int_0^T \tilde{g}(t, \lambda \underline{v}_1 + \mu \underline{v}_2) dt\Big]$$

$$\leq \; E\Big[\int_0^T [\lambda^2 \|\underline{v}_1\|^2 + \mu^2 \|\underline{v}_2\|^2 + 2\mu\lambda \|\underline{v}_2\| \|\underline{v}_1\|] dt + \int_0^T [\lambda \tilde{g}(t, \underline{v}_1) + \mu \tilde{g}(t, \underline{v}_2)] dt\Big]$$

$$< \; \infty,$$

due to uniform boundedness of D.

(iv) $\tilde{g}(t, \lambda \underline{v}_1(t)) + \mu \tilde{g}(t, \underline{v}_2(t)) < \lambda \tilde{g}(t, \underline{v}_1(t) + \mu \underline{v}_2(t)) < \infty$ follows from the convexity of $\tilde{g}$

◇

B.2 Discounted stock prices

The following shows how the discounted stock prices are martingales under the auxiliary probability measure P. We know that

$$dP_i(t) = P_i(t)[b_i(t)dt + \sigma_i^{\top}(t)dW(t)],$$

$$d\gamma(t) = -\gamma(t)r(t)dt.$$

From the consequence of Ito's lemma of Appendix A.5 we have

$$\begin{aligned} d(\gamma(t)P_i(t)) &= \gamma(t)dP_i(t) + d\gamma(t)P_i(t) \\ &= \gamma(t)P_i(t)[b_i(t)dt + \sigma_i^{\top}(t)dW(t)] - P_i(t)\gamma(t)r(t)dt, \\ \Rightarrow \frac{d(\gamma(t)P_i(t))}{\gamma(t)P_i(t)} &= (b_i(t) - r(t))dt + \sigma_i^{\top}(t)dW(t). \end{aligned}$$

Now, applying Ito's lemma we obtain

$$\begin{aligned} & d[\ln(\gamma(t)P_i(t))] \\ = \; & \frac{1}{\gamma(t)P_i(t)}d(\gamma(t)P_i(t)) + \frac{1}{2}Tr\left[\left((\gamma(t)P_i(t))^2\sigma_i\sigma_i^{\top}\right)\left(-\frac{1}{(\gamma(t)P_i(t))^2}\right)dt\right] \\ = \; & (b_i(t) - r(t))dt + \sigma_i^{\top}(t)dW(t) - \frac{1}{2}\|\sigma_i\|^2 dt \\ = \; & \sigma_i^{\top}(t)d\widetilde{W}(t) - \frac{1}{2}\|\sigma_i(t)\|^2 dt. \end{aligned}$$

Integrating both sides and taking the exponential gives

$$\gamma(t)P_i(t) = p_i \exp\left\{-\int_0^t \sigma_i^{\top}(s)d\widetilde{W}(s) - \frac{1}{2}\|\sigma_i(s)\|^2 ds\right\}.$$

B.3 The process $M(t)$

From the evolution equation of (2.16) we have

$$\begin{aligned} & d(\gamma(t)X(t)) \\ = \; & d\gamma(t)X(t) + \gamma(t)dX(t) \\ = \; & -\gamma(t)X(t)r(t)dt + \gamma(t)[X(t)r(t)dt + X(t)\underline{\pi}^{\top}(t)\sigma(t)d\widetilde{W}(t) - dc(t)] \\ = \; & \gamma(t)X(t)\underline{\pi}^{\top}(t)\sigma(t)d\widetilde{W}(t) - \gamma(t)dc(t). \end{aligned}$$

Integrating as before we obtain

$$\gamma(t)X(t) + \int_0^t \gamma(s)dc(s) = x + \int_0^t \gamma(s)X(s)\underline{\pi}^{\top}(s)\sigma(s)d\widetilde{W}(s).$$

B.4 The process $N(t)$

Similarly for the process $N(t)$, we use the consequence of Ito's lemma applied to (2.16) and the equation

$$dH(t) = -H(t)[r(t)dt + \theta^{\top}(t)dW(t)].$$

We obtain

$$\begin{aligned} d(H(t)X(t)) &= dH(t)X(t) + H(t)dX(t) + [-H(t)\theta^{\top}(t)X(t)\underline{\pi}^{\top}(t)\sigma(t)]dt \\ &= X(t)[-H(t)r(t)dt - H(t)\theta^{\top}(t)dW(t)] \\ &\quad + H(t)[X(t)r(t)dt + X(t)\underline{\pi}^{\top}(t)\sigma(t)d\widetilde{W}(t) - dc(t)] \end{aligned}$$

$$-H(t)X(t)\theta^{\top}(t)\underline{\pi}^{\top}(t)\sigma(t)dt$$

$$= -X(t)H(t)\theta^{\top}(t)dW(t) + H(t)X(t)\underline{\pi}^{\top}(t)\sigma(t)[dW(t) + \theta(t)dt]$$

$$-H(t)X(t)\theta^{\top}(t)\underline{\pi}^{\top}(t)\sigma(t)dt - H(t)dc(t)$$

$$= X(t)H(t)[\underline{\pi}^{\top}(t)\sigma(t) - \theta^{\top}(t)]dW(t) - H(t)dc(t),$$

yielding, by integration

$$H(t)X(t) + \int_0^t + H(s)dc(s) = \int_0^t X(s)H(s)[\underline{\pi}^{\top}(s)\sigma(s) - \theta^{\top}(s)]dW(s).$$

B.5 The process $M_v(t)$

We know that

$$\begin{aligned}
d(\gamma_{\underline{v}}(t)X(t)) &= \gamma_{\underline{v}}(t)dX(t) + d\gamma(t)X(t) \\
&= \gamma_{\underline{v}}(t)[X(t)g(t,\underline{\pi}_t)dt + X(t)\underline{\pi}^{\top}\sigma(t)dW(t) - dc(t)] \\
&\quad -\tilde{g}(t,\underline{v}_t)\gamma_{\underline{v}}(t)X(t)dt \\
&= \gamma_{\underline{v}}(t)X(t)[g(t,\underline{\pi}_t) - \tilde{g}(t,\underline{v}_t)]dt \\
&\quad +\gamma_{\underline{v}}(t)X(t)\underline{\pi}^{\top}\sigma(t)dW(T) - \gamma_{\underline{v}}(t)dc(t) \\
&= \gamma_{\underline{v}}(t)X(t)[g(t,\underline{\pi}_t) - \tilde{g}(t,\underline{v}_t) + \underline{\pi}^{\top}(t)\underline{v}(t)]dt \qquad \text{(B.1)} \\
&\quad +\gamma_{\underline{v}}(t)X(t)\underline{\pi}^{\top}\sigma(t)dW_{\underline{v}}(t) - \gamma_{\underline{v}}(t)dc(t).
\end{aligned}$$

Therefore, by integration the process

$$\begin{aligned} M_{\underline{v}}(t) &:= \gamma_{\underline{v}}(t)X(t) + \int_0^t \gamma_{\underline{v}}(s)dc(s) \\ &\quad + \int_0^t \gamma_{\underline{v}}(s)X(s)[\tilde{g}(s,\underline{v}_s) - g(s,\underline{\pi}_s) - \underline{\pi}^\top(s)\underline{v}(s)]ds \\ &= x + \int_0^t \gamma_{\underline{v}}(s)X(s)\underline{\pi}^\top(s)\sigma(s)dW_{\underline{v}}(s). \end{aligned}$$

$M_{\underline{v}}(t)$ is a nonnegative $P^{\underline{v}}$–martingale. The same procedure can be followed for the analog of $N(t)$ of Chapter 2.

B.6 Required for Lemma 3.34

We have from (B.1) that

$$\begin{aligned} d(\gamma_{\underline{v}}(t)X(t)) &= \gamma_{\underline{v}}(t)X(t)[g(t,\underline{\pi}_t) - \tilde{g}(t,\underline{v}_t) + \underline{\pi}^\top(t)\underline{v}(t)]dt \\ &\quad + \gamma_{\underline{v}}(t)X(t)\underline{\pi}^\top\sigma(t)dW_{\underline{v}}(t) - \gamma_{\underline{v}}(t)dc(t). \end{aligned} \tag{B.2}$$

We wish to solve this for the wealth process $X(t)$ by firstly solving the *homogenous* part given by

$$\begin{aligned} d(\gamma_{\underline{v}}(t)X(t)) &= \gamma_{\underline{v}}(t)X(t)[g(t,\underline{\pi}_t) - \tilde{g}(t,\underline{v}_t) + \underline{\pi}^\top(t)\underline{v}(t)]dt \\ &\quad + \gamma_{\underline{v}}(t)X(t)\underline{\pi}^\top\sigma(t)dW_{\underline{v}}(t), \end{aligned}$$

and then *guessing* a solution to the *inhomogenous* equation and solving this by variation of parameters. By Ito's lemma

$$
\begin{aligned}
d[\ln(\gamma_{\underline{v}}(t)X(t))] &= \frac{1}{\gamma_{\underline{v}}(t)X(t)}d(\gamma_{\underline{v}}(t)X(t)) - \frac{1}{(\gamma_{\underline{v}}(t)X(t))^2}[d(\gamma_{\underline{v}}(t)X(t))]^2 \\
&= [g(t,\underline{\pi}_t) - \tilde{g}(t,\underline{v}_t) + \underline{\pi}^{\top}(t)\underline{v}_t]dt + \underline{\pi}^{\top}(t)\sigma(t)dW_{\underline{v}}(t) \\
&\quad -\frac{1}{2}\|\underline{\pi}^{\top}(t)\sigma(t)\|^2 dt.
\end{aligned}
$$

The solution to the homogenous equation is then given by

$$
\begin{aligned}
\gamma_{\underline{v}}(t)X(t) &= x\exp\left\{\int_0^t g(s,\underline{\pi}_s) - \tilde{g}(s,\underline{v}_s) + \underline{\pi}^{\top}(s)\underline{v}_s ds\right\} \\
&\quad \cdot\exp\left\{\int_0^t \underline{\pi}^{\top}(s)\sigma(s)dW_{\underline{v}}(s) - \frac{1}{2}\int_0^t \|\underline{\pi}^{\top}(s)\sigma(s)\|^2 ds\right\}.
\end{aligned}
$$

Therefore, we guess the solution to the inhomogenous equation is in the form

$$
\gamma_{\underline{v}}(t)X(t) = f(t)h_{\underline{v}}(t),
$$

with $\gamma_{\underline{v}}(0)X(0) = x = f(0)$ and $h_{\underline{v}}(t)$ given above. Also note that $h_{\underline{v}}(t)$ satisfies the homogenous equation, i.e.,

$$
\begin{aligned}
dh_{\underline{v}}(t) &= h_{\underline{v}}(t)\Big\{[g(t,\underline{\pi}_t) - \tilde{g}(t,\underline{v}_t) + \underline{\pi}^{\top}(t)\underline{v}_t]dt \\
&\quad +\underline{\pi}^{\top}(t)\sigma(t)dW_{\underline{v}}(t)\Big\}.
\end{aligned}
$$

Now, assuming $f(t)$ is deterministic we apply Ito's Rule to obtain

$$
\begin{aligned}
d[\gamma_{\underline{v}}(t)X(t)] &= df(t)h_{\underline{v}}(t) + dh_{\underline{v}}(t)f(t) \\
&= df(t)h_{\underline{v}}(t) + \gamma_{\underline{v}}(t)X(t)\Big\{[g(t,\underline{\pi}_t) - \tilde{g}(t,\underline{v}_t) + \underline{\pi}(t)\underline{v}_t]dt + \underline{\pi}(t)\sigma(t)dW_{\underline{v}}(t)\Big\},
\end{aligned}
$$

which from the inhomogenous equation of (B.2) must equal

$$
-\gamma_{\underline{v}}(t)dc(t) + \gamma_{\underline{v}}(t)X(t)\Big\{[g(t,\underline{\pi}_t) - \tilde{g}(t,\underline{v}_t) + \underline{\pi}(t)\underline{v}_t]dt + \underline{\pi}(t)\sigma(t)dW_{\underline{v}}(t)\Big\},
$$

giving the separable equation

$$
\begin{aligned}
df(t)h_{\underline{v}}(t) &= -\gamma_{\underline{v}}(t)dc(t), \\
\Rightarrow df(t) &= -\frac{\gamma_{\underline{v}}(t)}{h_{\underline{v}}(t)}dc(t), \\
\Rightarrow f(t) &= x - \int_0^t \gamma_{\underline{v}}(s)h_{\underline{v}}^{-1}(s)dc(s),
\end{aligned}
$$

from the initial condition for f. Therefore the solution to the inhomogenous equation is given by :

$$
\begin{aligned}
\gamma_{\underline{v}}(t)X(t) &= \left\{x - \int_0^t \gamma_{\underline{v}}(t)h_{\underline{v}}(t)dc(s)\right\}h_{\underline{v}}^{-1}(t), \\
\Rightarrow X(t) &= \left\{x - \int_0^t \gamma_{\underline{v}}(s)h_{\underline{v}}^{-1}(s)dc(s)\right\}h_{\underline{v}}(t)/\gamma_{\underline{v}}(t) \\
&= \left\{x - \int_0^t \exp\left[-\int_0^s g(u,\underline{\pi}_u) + \underline{\pi}^{\top}(u)\underline{v}_u du\right]\right. \\
&\qquad \left.\exp\left[-\int_0^s \underline{\pi}^{\top}(u)\sigma(u)dW_{\underline{v}}(u) + \frac{1}{2}\int_0^s \|\underline{\pi}^{\top}(u)\sigma(u)\|^2 du\right]dc(s)\right\} \\
&\quad \cdot \exp\left\{\int_0^t g(s,\underline{\pi}_s) + \underline{\pi}^{\top}(s)\underline{v}_s ds\right. \\
&\qquad \left.+ \int_0^t \underline{\pi}^{\top}(s)\sigma(s)dW_{\underline{v}}(s) - \frac{1}{2}\int_0^t \|\underline{\pi}^{\top}(s)\sigma(s)\|^2 ds\right\}.
\end{aligned}
$$

◇