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Designing, Implementing and Assessing Guided-Inquiry based Tutorials in Introductory Physics

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Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Doctor of Philosophy is entirely my own work, that I have exercised reasonable care to ensure that the work is original, and does not to the best of my knowledge breach any law of copyright, and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

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Contents

Abstract

This work concerns the development of instruction to address identified difficulties students have in introductory physics courses. Tests were designed to investigate students conceptual understanding and use of mathematical tools in an electromagnetism context, and students understanding of the relationship between position, velocity, and acceleration, and the graphing of these quantities with time in the context of simple harmonic motion. The resulting instruction specifically tackles difficulties discovered after initial testing, and takes the form of structured worksheets which students complete in small groups, with tutors acting as facilitators. Through the comparison of students answers to pretest and post test questions the effectiveness of the developed instruction has been assessed.

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- Doughty, L., McLoughlin, E. van Kampen, P. *How students think about and use integration in an electromagnetism context*, ICPE-EPEC, Prague, August 2013

- Doughty, L., van Kampen, P. McLoughlin, E. *Teaching kinematics concepts and graphing through the context of simple harmonic motion*, ICPE-EPEC, Prague, August 2013

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Paper:

- Doughty, L., McLoughlin, E. van Kampen, P. *What integration cues, and what cues integration in electromagnetism.* Am. J. Phys. 2013 (Under Review)

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Chapter 1

Introduction and Background

This work involves the identification of student difficulties with different concepts in introductory and intermediate physics courses and the development of instruction to address these difficulties. Students' responses to conceptual and calculation questions which are unseen and cannot be answered from memory are analysed to determine the level of their understanding of a topic and to find any misconceptions and difficulties they have relating to the topic. This information enables us to tailor and design instruction to address the main areas of difficulty.

In my view, physics instruction should not advocate the rote memorization of facts that will not benefit students. Instead it should aim to improve students' ability to use basic knowledge to predict and explain more complex phenomena, a skill that will help them succeed in tasks they will encounter in other contexts[1]. McDermott[2] states that in order for instruction to be effective it must address the needs and abilities of the students and she warns that students' performance on quantitative problems is not an accurate measure of students' understanding of concepts, as students can often successfully apply memorized formulae without having a functional understanding of the physics involved. We have attempted to foster these ideas in the designing of our instruction: carefully constructing tests that probe for conceptual understanding; and matching the content of the instruction to students' starting knowledge, while tackling specific misconceptions and difficulties and developing reasoning skills.

The development of instruction is an iterative process consisting of multiple steps. The approach that we have taken is similar to that used by the Physics Education Group at the University of Washington[3]. They describe this process in three parts: the conducting of systematic investigations of student understanding; the development of instructional strategies to specifically address the identified difficulties; and the continuous cycle of testing, modifying, and revising the materials. In our study, we have employed a pre/post test strategy to both identify students' difficulties and examine the effectiveness of the resulting instruction. We have also made use of teaching-learning interviews with a smaller number of students to gain deeper understanding of students' difficulties and to pilot new strategies for addressing these difficulties, before modified materials are administered to a whole class group. The pre/post method and the teaching-learning interviews are outlined in detail in Chapter 2, along with a description of how data from both were analysed.

This chapter will discuss the style of the instruction we developed and will provide a motivation for why this style was chosen. It will also give a background to the two courses that this research was centred on, and the specific topics and difficulties that I have chosen to present in this thesis will be introduced.

1.1 Instruction: structure and style

Although we do not have an explicit theoretical framework in this study, we share many of the views outlined by McDermott[2] and Redish[4] about the factors influencing students' learning of physics concepts. McDermott[2] states that in order for students to have the ability to apply a concept in a variety of contexts, they must be able to define that concept and also know how that concept relates to connected concepts. To develop this ability an emphasis must be placed on concept development and model building. Redish[4] recognises though that students will have pre-existing, often incorrect or incomplete, models upon entry to physics courses, and that each student's model will be unique due to their own prior experience with the physical world. These established models are often difficult to change and McDermott[2] suggests that to overcome students' misconceptions they must be presented with conceptual conflicts in a number of contexts. Both authors also state that a necessary skill students should obtain from physics instruction is qualitative reasoning. In many cases, students' lack of reasoning ability hinders them in coming to an understanding of physics concepts. Specific attention should be given to improving students' reasoning skills, by providing them with practice at solving qualitative problems and explaining their thought processes.

The constructing of complete functional models and good reasoning skills are not usually a result of traditional instruction (lectures and tutorials where the tutor illustrates solutions to numerical problems) mainly because meaningful learning is not passive, students must be actively engaged in the learning process to gain an operational understanding[2;4;5]. As a result, lecture instruction was used in this study only to introduce key terms and present the basic ideas, so that the concepts were discussed before they were elaborated in more qualitative detail in the tutorials. The lecture notes were also intended to be a reference for students.

The tutorial instruction in this study is based around guided-inquiry worksheets[6] which are completed by students in groups of four to five. Tutorial instruction in both physics courses in this study include worksheets adapted from Tutorials in Introductory Physics[7] and the newly designed worksheets follow the same format as these materials. The worksheets are structured in a way that guides students through a set of prescribed tasks towards the construction of their own definition of a concept, identification of the relationship between concepts, or the solving of a complex problem. Each task requires an explanation of reasoning and the emphasis is on qualitative understanding[3]. During tutorials the tutors act as facilitators, promoting and monitoring discussions, and supporting students to reach their own answers.

Hmelo-Silver et al[8] state that the scaffold that is provided in a guided-inquiry approach gradually trains students into becoming better problem-solvers. They also mention how breaking a complex problem into manageable tasks allows the instructor to highlight the important aspects of the solution and forces students to engage with the key strategies. Shaffer and McDermott[3] used a control study to compare the effectiveness of guided-inquiry tutorials they developed with traditional instruction for teaching electric circuits. They found that students in the tutorial group performed significantly better in questions requiring conceptual understanding, and slightly better on quantitative problems despite spending less time of these type problems than the traditional group.

The tutorials are completed by students in groups so that they are engaged in discussion and can share their conceptual and procedural knowledge. In this type of cooperative learning, students can request explanations and justifications from one another. There is evidence that indicates that students who take part in cooperative learning achieve higher levels of thought, experience higher academic gains [9], and retain information longer [10] than students who work individually. Collaborative learning also encourages students to take responsibility for their own learning and enhances their critical thinking skills [10].

Heller et al [11] investigated group versus individual problem solving in an introductory university physics course. They analysed students' solutions to a problem by comparing it to an expert solution in terms of conceptual understanding, planning, descriptions, logical progression, and use of appropriate mathematics. They found that solutions produced by a group were always significantly better than even those produced by the best individual problem solver in each group, especially on qualitative aspects. More importantly, each individual student's problem solving performance improved at the same rate over time, regardless of their ability.

Four to five students is the recommended group size for the most effective collaborative learning [12], and hence is the number that was used for tutorial completion in this study. Heller et al [13] observed that students working in pairs did not have sufficient combined conceptual and procedural knowledge to solve context-risk problems and tended to go off track or get stuck on a single approach. However, when the group size exceeds four or five students, at least one student will become a passive participant, not contributing to group discussions in a meaningful way [12], and hence not reaping the benefits of the collaborative environment. The groups for the tutorials in this study were self-selected by the students, and hence sometimes changed from tutorial-to-tutorial.

1.2 Electromagnetism

In recent years, a significant number of publications have appeared on electromagnetism beyond the introductory level in US universities[14–24]. Early work by Manogue et al[14] considered problems involving Ampere's law. They observed that in moving beyond the introductory level, students are required for the first time to connect various pieces of mathematics and physics knowledge they already have into a coherent problem-solving strategy, and that there is generally a mismatch between the aims of calculus courses and physics courses.

Work carried out at the University of Colorado[16–21] has charted the territory in terms of developing a junior-level approach to electromagnetism. They have identified persistent conceptual and mathematical difficulties, and developed their course around three central ideas of developing mathematical sophistication, problem-solving expertise, and developing the students as physicists[20].

We have investigated students' conceptual and mathematical difficulties with the topics involved in an intermediate electromagnetism course. When students are presented with a problem involving the calculation of a physics quantity, they need to: interpret the physical context in which the problem is set; draw on their conceptual understanding to identify factors influencing that quantity; and use mathematical tools to create and compute expressions. For example, if asked to calculate the electric flux through a sheet due to a point charge located directly below its centre, students would have to: recognise that field lines from the test charge cut the surface of the sheet (although they don't need to use the idea of field lines, we do recommend its use because in our experience it is the only way students can make some sense of the concept); recognise that the electric field will have a different magnitude and direction at all points on the sheet, that only the electric field that is in the same direction as the normal of the sheet contributes to the flux through the sheet, and as a result the flux through small segments of the sheet where the flux can be assumed to be constant must be found and added together; calculate the dot product and use integration to sum infinitesimal 'bits' of flux over the sheet. Even when students are aware of the mathematical tools they are required to implement there are still many technical aspects they have overcome in order to use them successfully.

In preliminary investigations where students were given questions similar to the type just described, students often did not recognise when particular mathematical tools were required and struggled to use them correctly. It some cases it seemed that students' difficulties in using the mathematical tools actually hindered their conceptual reasoning, while in others it was difficult to determine if it was students' lack of conceptual understanding of the quantity involved that prevented students applying the mathematical tools. Because they are widely relied upon for both the calculation of many electromagnetism quantities (as demonstrated above) and the conceptual understanding of most of these quantities and laws, we focused mainly on examining students' ability to recognise when they need to use integration and vector operations, and the difficulties they have in applying integration, vector addition, and dot products in a number of different contexts.

1.2.1 Integration

Detailed research aimed at students' use of integration within an electromagnetism course was carried out Meredith and Marrongelle[15] and Rebello and co-workers at Kansas State University[24]. Their papers have used student interviews as their primary research instrument, and focussed in the main on two important issues: (i) What cues integration? and (ii) What difficulties do students encounter with setting up and evaluating an integral?

This thesis describes work that adds to these studies in a number of ways. We investigate these questions on what cues students to use integration in electromagnetism in a semi-quantitative manner. We have also returned to the earlier work by Manogue et al[14] and investigated to what extent the calculus courses our students take provide adequate preparation for an electromagnetism course. In particular, we address the question: (iii) What does integration cue? In other words, what do students think when they see an integral? We cast this part of the research in the framework of concept image that has gained considerable traction in the mathematics education literature. We have also examined (iv) how students interpret integrals in basic physics contexts, i.e. do students know the physical meaning of simple integrals? These questions, the results we obtained and the consequent instruction are described in detail in Chapter 3.

1.2.2 Vector operations

Previous studies examining students' use of vectors in physics have found that after traditional introductory physics courses many students cannot add vectors and do not treat physical quantities such as force as vector quantities. We investigated our students' ability to (i) add vectors without a context and (ii) calculate vector components, before instruction in the course. We also examined (iii) students' conceptual reasoning about the net electric force due to multiple sources, and (iv) students' ability to combine conceptual and mathematical knowledge in numerical field calculations. Tests designed to investigate these aspects of vector addition and analysis of students' answers to these test questions are described in Chapter 4, along with a discussion of the effectiveness of original and tailored instruction in improving students' conceptual and mathematical abilities relating to vector addition.

Students' inability to recognise a quantity as a vector and to calculate vector components would almost certainly affect their understanding of dot products which are required for their understanding of Gauss' Law and Ampere's Law, and their ability to calculate line integrals and quantities such as electric flux. We developed a tutorial aimed at improving students' conceptual understanding of the dot product using the context of work (a context students have previous experience with). Details of the development of this tutorial and students' answers to pretest and post test work questions are described in Chapter 5. A comparison of students' answers to two questions involving the calculation of electric flux (requiring the use of dot products and integration) in years before and after specific instruction on integration and dot products was implemented, is also shown.

1.2.3 Course content and delivery

The research described in this part of the thesis pertains to an intermediate electromagnetism course which overlaps both with a calculus-based first-year electromagnetism course and the first 5 chapters of Griffiths' textbook on Electrodynamics[25]. The course begins with electric charge, force, and field, and works through potential and flux, to Gauss' Law. When magnetic fields are introduced, Ampères Law and Faraday's Law are covered. The course concludes with Maxwell's equations.

This content is delivered via one lecture and two tutorials per week during a twelve week semester. Most lectures set out to reacquaint students with materials they had already seen in a different context (for example within mathematics courses, within non-calculus contexts, etc) and to provide some historical background. Reif[1] recommends that instruction should ensure that students can adequately identify any concept before being asked to use in a problem-solving task. Hence, in many cases the tutorials were paired with a conceptual physics tutorial preceding a conceptual mathematical tutorial. Most conceptual tutorials were adapted from Tutorials in Introductory Physics[7] and all conceptual mathematical tutorials were patterned after these. The tutorials are facilitated by one of my supervisors (the lecturer), myself, and two other post-graduate students, so that there is generally one tutor for every three groups. Prior to the tutorial session all tutors meet to discuss the worksheet content, potential areas of student difficulty, and questioning strategies to help students overcome these difficulties. After the tutorial, all tutors reconvene to review the session. These discussions usually do not result in major structural changes to the worksheets but they give an indication of students' progress during the tutorial and sometimes result in the re-wording of questions that some students found difficult to interpret (usually with worksheets being administered to the whole class group for the first time).

1.2.4 Students

The course is taken by a diverse cohort of students: second-year engineering and physics students, as well as fourth year pre-service science and mathematics teachers. Typically, between 50 and 70 students enrol in the course annually. Students' attainment is also quite diverse. All students have successfully completed mathematics courses that include a more or less standard treatment of introductory level integration; some students have also completed a vector calculus course, while others take such a course concurrent with the electromagnetism course. Most students have taken a calculus-based introductory physics course; some have taken an algebra-based physics course.

1.3 Simple Harmonic Motion

The second part of this thesis pertains to an introductory Waves and Optics course, which aims to introduce students to the fundamentals of wave propagation, simple acoustics and optical phenomena. The course is delivered via two hours of whole group lectures per week which have both knowledge-based lectures to introduce concepts and whole group problem-solving, facilitated by the lecturer. In addition, there is one small group tutorial session per week, where students complete selected conceptual tutorials from Tutorials in Introductory Physics[7]. The facilitators for these tutorials are one of my supervisors (the lecturer) and myself. For the most part, the concepts covered and the ordering of the these tutorials match that of the lectures. When this study began however, there was no tutorial available for the topic of simple harmonic motion.

Simple harmonic motion is a context that brings together many concepts previously introduced in an introductory mechanics course. Some of the aspects to be considered include: the relationship between the restoring force and position; the varying position, velocity, and acceleration with time and the sinusoidal nature of their graphs; and the energy of the simple harmonic system. Simple harmonic motion is the first of many contexts where these aspects will be combined and so we felt it was important that students were given the opportunity to develop their qualitative reasoning surrounding these concepts using this initial context.

1.3.1 Tutorial content

Initial drafts of the simple harmonic motion tutorial incorporated most of the mentioned aspects of simple harmonic motion. However, due to time constraints within the course and tests showing some students' persistent difficulties relating the concepts of position, velocity, and acceleration, and the graphing of these concepts with respect to time (similar to difficulties previously identified in other studies[26–30]), the focus shifted solely to addressing these aspects. In this way, the purpose of the instruction changed from teaching students all aspects of simple harmonic motion to using the context of a simple harmonic oscillator to re-address and cement students' understanding of these three motion concepts. Chapter 7 describes three different approaches taken by the worksheets in three different years and a comparison of the effectiveness of each approach is discussed using analysis of post-test results from each of the years.

1.3.2 Students

The Waves and Optics course is taken in the second-semester by all first-year physics students, some students from a Common Entry into Science programme (meaning that they have yet to decide their subject major), and some first-year pre-service science teachers. Typically, 25 to 40 students enrol in this course per annum. All these students have completed a calculus-based introductory mechanics course where they were required to qualitatively describe motion in one and two dimensions.

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Chapter 2

Research Methodology

This chapter begins by summarising the research questions described in Chapter 1. Section 2.2 describes the instruments we implemented to answer these research questions and Section 2.3 describes how the data we obtained from these instruments were analysed and how this analysis is presented in the results chapters.

2.1 Research Questions

As stated in the first chapter, the aims of this study were: to develop instruction to improve students' conceptual understanding of the mathematical tools that are required in electromagnetism contexts; and to develop students' understanding of the relationships between position, velocity and acceleration, and the graphing of these quantities with time, using the context of a simple harmonic oscillator.

Preliminary investigations highlighted that the mathematical tools required most often in electromagnetism were integration and vector operations, and that students struggled to employ them. In order to design useful instruction we needed to determine what difficulties our students have in applying integration and vector operations. This gave rise to a number of research questions:

- Do students know when to apply the tool?

- How do students think about the tool (i.e. what is their conceptual understanding of the tool)?
- How do students use the tool (i.e. do technical difficulties with algebra, geometry, trigonometry prevent students from applying the tool correctly)?

A number of research questions materialised when the design of instruction to develop students' understanding of the relationships between position, velocity and acceleration, and the graphing of these quantities with time began:

- Can students interpret varying position, velocity and acceleration versus time graphs?
- Can students draw varying position, velocity and acceleration versus time graphs?
- Can students reason conceptually about the relationships between position, velocity, and acceleration generally?
- Can students identify how both the magnitude and sign of position, velocity, and acceleration changes during one period of a simple harmonic oscillation?

When specific difficulties were identified, in order to continually improve instruction, two more research questions which apply to both parts of the research emerged:

- What is the best approach to take to tackle these difficulties and improve students' conceptual understanding of the mathematical tools?
- How effective was the developed instruction at improving students' understanding and use of the concepts involved?

2.2 Research design and instruments

This study makes use of questionnaires and teaching-learning student interviews as part of the process of instruction design and development. These research methods and the analysis of data obtained from their implementation are elements of a qualitative research design[1]. In order to design effective instruction, it was important to first obtain a detailed picture of students' difficulties which a qualitative approach allowed us to do. However, because the instruction is for an entire class group and the goal is to develop instruction that is widely applicable, it is also essential to gain a representative account of the most frequent difficulties. Therefore, following qualitative analysis where categories are defined, the numbers of students fitting those categories are determined.

2.2.1 Pre/post tests

Pretests, post-tests and their comparison are an integral part of the instructional design process. Students' answers to carefully structured questions can give useful insights into the level of their understanding of a topic and can pinpoint their main difficulties. As mentioned in the introduction, instruction is most effective when it meets students' current abilities and specifically tackles their difficulties and misconceptions. Pretests provide us with the information that allows us to do that. Post tests inform us of the extent to which students have reached the expected learning outcomes of the instruction, and when pretest and post-tests address the same aspects of a concept, an indication of the level of students' conceptual change/gain can be obtained through comparison of their performance on pre- and post-tests. Because we have seen that student cohorts entering the course typically have the same starting ability, comparing students' answers to similar post-test questions in different years has assisted us in comparing the relative effectiveness of the instruction in those years.

Pretest questions typically use basic contexts. In each question we try to only address one aspect of a concept. This reduces the number of difficulties that could influence students' answers, so that we can determine specific difficulties. Also we have found that if the question is too complex many students will not attempt the question, and this means we have no way of determining where their main difficulties lie.

By contrast, the post-test questions typically use unseen contexts that combine various aspects of a concept. The questions are structured similarly to tutorials but many of the steps that were present in the tutorials to guide students' reasoning are removed. The questions still generally consist of multiple parts enabling us to identify students' persistent difficulties.

In the Electromagnetism course, pretest questions were administered online using the Moodle quiz platform. There was one pretest per week, which opened directly after the lecture and students had until just before the first tutorial of that week to complete it. This way students will have at least some relevant knowledge of the topics being tested. Students were given credit for completing the pretest regardless of the correctness of their answers. They had approximately 25 minutes to complete the entire Moodle quiz which consisted of opinion questions on the previous weeks' tutorials and typically three to four pretest questions. Most questions were multiple-choice questions followed by open questions asking for explanations of reasoning behind the multiple-choice question they had just answered. Because the online pretests were unsupervised, students could consult their lecture notes, text-books, the web, etc. However, the time constraint seemed to prevent this occurring often, and generally when there was evidence that students tried to use the lecture notes, apart from identifying that the question was testing a particular concept, they could not relate the examples from the lecture to the pretest context.

In the Waves and Optics course, pretests took the form of written questionnaires which were administered at the start of the relevant tutorial session. Students were given approximately 10 minutes to answer and they completed the questions on their own.

Post-tests in both courses were questions asked on either the mid-term continuous assessment exam or the end-of-semester exam.

All students were given an opt-out clause which was approved by the research ethics committee in Dublin City University (Appendix A). They still had to do the assessment but would not be quoted. Typically between 0 and 3 students did not consent to be quoted.

2.2.2 Teaching-learning interviews

Teaching-learning interviews serve a dual purpose: they can help in the gaining of further insight into students' difficulties with a particular topic, and they can be used for piloting strategies for tackling those difficulties. A standard clinical interview has a semi-structured format with pre-planned content, tasks and questions and the goal of such an interview is solely to identify students' current reasoning and understanding without trying to change it[2]. Although a teaching-learning interview is structured similarly to a clinical interview, the aim is to determine how students learn about a certain topic, and so the interviewer provides scaffolding and hints where required.[3] The interviewer observes what approaches were effective in changing students' reasoning so that a picture of how learning of the topic can be best facilitated is obtained.

The teaching-learning interview appears to be adapted from the 'Teaching Experiment' which consists of three components: modelling, teaching episodes, and individual or group interviews[2]. Teaching episodes are conducted with a small group of students, and the interview aspects are the recorded dialogue from during the teaching episode. The modelling refers to the combining of students' responses so that students' progress over time can be seen.

Engelhardt et al[2] state that the advantages of the teaching experiment is that you can test new techniques and the analysis will show what techniques helped develop the most conceptual growth in students. However, Chini et al[4] warn that the interview setting is different from the classroom setting. Because students have to explain their ideas to the interviewer they may pay more attention to their thinking process than they normally would, so there is no guarantee that what works in the environment of a teaching-learning interview will work the same way in a whole class setting.

In this study, I conducted 10 individual teaching-learning interviews to pilot possible approaches I designed to develop students' conceptual understanding of the dot-product, and to develop students' understanding of the relationships between position, velocity, and acceleration. All students gave their permission for the interviews to be audio-taped. The approach for each topic was piloted during the same interview, and approximately 20 minutes was spent on each topic. The students partaking in the interviews were all in first-semester of first year. The students were either taking an algebra-based introductory physics course or a calculus-based introductory physics course, and all topics involved in the interview had been recently taught in both courses. All students volunteered for the interview.

2.3 Data Analysis

2.3.1 Pre/post test data

Analysis of students' answers to the pretest and post-test questions was a multiple step process. Initially, students' answers were scanned to formulate a draft set of categories. The next step involved considering students' answers in more depth to begin populating the categories. At this point, after a greater insight into students' reasoning had been obtained, the categories were re-examined, in some cases leading to the merging of some categories and creation of new ones. Lastly, the categories and any unclear quotes were discussed with my supervisors. In some cases, we would independently categorize students' answers.

For the most part, data presentation in the results chapters for all pretest and post-test questions follows the same format. Tables containing the categories and the quantity of students whose answers fit that category will be shown first. This is succeeded by a discussion of the individual categories, where quotes are used to illustrate the types of answer in each category. In some of the quotes presented, students use language that we would not consider to be technically correct and may even hint at other misconceptions that students have about a concept or context being used. However, the student quotes are being used to demonstrate a particular aspect of the question and so these other incorrect aspects of their answers may not be commented on during the discussion.

2.3.2 Teaching-learning interview data

This type of systematic analysis was not conducted on the data from the teaching-learning interviews. The main goal of these interviews was to trial a theoretical approach to indicate if it could be a sensible approach to take when designing the instruction. All interviews were transcribed and coded to investigate if the transcripts provided support for the general design of the instruction.

In the results chapters, when the development of the tutorial is being described excerpts from the transcripts are shown, to demonstrate the questioning strategy applied in the interviews and how that sequence was adapted for tutorial worksheets. In some of the quotes presented, students use language that we would not consider to be technically correct and may even hint at other misconceptions that students have about a concept or context being used. However, the student quotes are being used to demonstrate a particular aspect of the interview and so these other incorrect aspects of their answers may not be commented on during the discussion.

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Chapter 3

How Do Students Think About and Use Integration in an Electromagnetism Context?

3.1 Introduction

Integration is a mathematical tool that is widely utilized throughout various physics topics. In an intermediate electromagnetism course, it is employed for many basic operations involving the calculation of, e.g., the total charge on an object of varying charge density ($\int\lambda(x)\mathrm{d}x$,$\int\sigma(x,y)\mathrm{d}x\mathrm{d}y$); the electric field at a certain location due to a charged object; the electric flux through a surface due to a non-uniform field ($\int\int(\vec{E}\cdot\widehat{n})\mathrm{d}A$); the magnetic field at a certain location due to a current carrying wire; the circulation of a magnetic field around a loop ($\int\vec{B}\cdot\mathrm{d}\vec{l}$). Although these concepts and the contexts in which they are being calculated may vary in complexity and difficulty, the application of the integration process remains the same.

In order to successfully complete these calculations students must recognise the need for integration, be capable of setting up the definite integral, and be able to evaluate the integral. Integration makes it possible to sum infinitely many infinitesimally small bits of a quantity to give the exact value of that quantity. Therefore, for students to recognise that integration must be applied to a problem, students must both be aware of this function of integration and be capable of identifying when terms involved in calculating a quantity are varying.

Setting up the definite integral is the step that varies the most in difficulty, but even in the simplest cases this may cause many students problems. The integrand and the infinitesimal term are determined in the expression for an infinitely small bit of the quantity being calculated. For example, when calculating the total charge on a non-uniformly charged rod, the infinitesimal corresponds to an infinitesimal length over which the linear charge density can be considered constant (to first order). The integrand corresponds to the linear charge density, and the exact algebraic form of the integrand depends on where the origin is chosen. The choice of origin also affects the values of the upper and lower limits. For example, the total charge on a rod of length L with a linear charge density that varies linearly between λ_0 and λ_1 could be written as

$$Q = \int_0^L \left[\lambda_0 + \frac{\lambda_1 - \lambda_0}{L} x\right] \mathrm{d}x,$$

or as

$$Q = \int_{-L/2}^{L/2} \left[\frac{\lambda_1 + \lambda_0}{2} + \frac{\lambda_1 - \lambda_0}{L} x\right] \mathrm{d}x.$$

While such expressions are simple for experts, we have found that many of our students struggle with setting up and interpreting even these integrals.

Writing an integral expression for electric field is more complex as it involves taking possible variations in charge density, distance, and direction into consideration. As in the example for charge, an origin must be defined, the exact location of which will determine the exact form of the integrand and the limits. In these type of problems it becomes more important that students learn what location for the origin is convenient. Again, the concept and context will determine the magnitude of the challenge involved. For example, when finding the electric field due to a uniformly charged rod at a certain location, not all of the rod is the same distance from that location, and hence the contributions to the electric field from different sections of the rod will different in magnitude and direction at that location. Students must see that Coulomb's law cannot be applied to the rod as a whole because of the varying distance and direction. However, when a segment of the rod with infinitesimal length is considered, all of the charge on that segment is an equal distance from the location in question and the electric field due to that segment is in one direction. This segment can be treated like a point charge and so Coulomb's law can be applied to this particular segment.

The rules of vector addition suggest that an efficient way of adding the contributions from each infinitesimal segment is through integrating expressions for the x and y components of the electric fields between the two ends of the rod. The form of the integrand and the limits will depend on the location you are considering. Students must learn that e.g. when asked to calculate the electric field some distance above the midpoint of the rod, it is advantageous to choose the origin to be at the centre of the rod; in that case, the limits will be -$\frac{1}{2}L$ and $\frac{1}{2}L$ and symmetry arguments can be used to reduce the amount of calculational effort. Students must also have considerable proficiency with geometry and algebra to set up the integral correctly. Complexity increases again for electric flux as the integral expression includes the dot product of two vectors, and a two-dimensional integration.

When the integral is eventually set up, it needs to be evaluated. This last step can be carried out with only a procedural knowledge of integration, but the first two steps require a proficient conceptual understanding of the process. Mathematics educators have recognised that after instruction students have often gained an effective procedural knowledge of calculus without acquiring an adequate conceptual understanding behind the processes involved.[1–3]

The aim of this part of the study was to investigate how students think about integration and how they use integration in a variety of electromagnetic contexts. In other words, to examine the conceptual and procedural difficulties students experience when integration is required. These identified difficulties were then used to inform changes to instruction.

Section 3.2 outlines students' "concept image" of integration before they start the course. Section 3.3 shows what cues the use of integration for students in an electromagnetism context. Technical difficulties students have when using integration are identified in Section 3.4. Section 3.5 presents results from our investigation into how students interpret integrals in a physical setting. The tutorials designed to specifically address students' conceptual understanding of integration and the other tutorials that require the use of integration are described in Section 3.6. Section 3.7 shows how students answered similar questions after the new instruction and Section 3.8 discusses further aspects that need to be examined.

3.2 Students' views of integration

3.2.1 Integration concept image

In a situation where a concept needs to be recalled and used, many mental processes are activated that will affect the meaning and usage of that concept. Tall and Vinner[4] define the mental pictures of properties and processes relating to a concept that are built from experience as the *concept image*. A concept image is unique to each individual. In the case of integration, a person's concept image may contain generic ideas like antiderivative, area under the curve, Riemann sum, but also specific instances such as "the indefinite integral of $2x$ is x^2." The concept image thus may, and generally does, contrast with the *concept definition*, which uses words to specify a concept and is generally accepted by the wider community. Despite there being a functional difference between informal statements that describe a mathematical concept and a formal definition that prescribes an instance of that concept, instructors move readily between the two, without necessarily making the distinction clear to students. As a result the students do not always focus on formal definitions, nor do they use them in their reasoning about a concept; rather, they rely on the ideas they form from their experience with a concept.[4]

Alcock and Simpson[5] found that even in situations where students are introduced to the formal mathematical definition of a concept before being given examples, they may still base their learning mainly on the latter. In many calculus courses, including those taken by our students, the concept definition and the idea of a Riemann sum are introduced first, but the conceptual emphasis is placed on the definite integral as a way of finding the area under a curve. This may lead to students developing a limited concept image, only knowing how to use the concept in a small number of contexts.[4] According to Thompson and Silverman,[6] it is vital that students regard the integral not only as area under the curve, but also as the sum of infinitely small bits. Moreover, the practical emphasis in mathematics courses is overwhelmingly on mastering a plethora of techniques such as partial integration and substitution to evaluate an integral. If Alcock and Simpson are right, then there is a serious risk that the knowledge of integration students bring to an electromagnetism class is of little use to them.

3.2.2 Our students' views of integration

We investigated what integration cued in our students by administering a pretest before students received any instruction in the Electromagnetism module, in the form of a short written questionnaire handed out at the start of the first lecture (see Fig.3.1). We feel this ensured we obtained as true a reflection as possible of students' concept images of integration.

We presented our students with two integrals, $\int_a^b \mathrm{d}x$ and $\int_a^b n(x)\mathrm{d}x$, and asked them

Interpret (i.e. write down everything you think of when you see) the following integrals.

$$\int_a^b \mathbf{d}\boldsymbol{x}$$

$$\int_a^b \boldsymbol{n}(\mathbf{x})\mathbf{d}\boldsymbol{x}$$

Figure 3.1: Pretest question probing students' concept image of integration.

to interpret these integrals. By asking students to write down everything they think of when they see these integrals, all pieces of a student's concept image should have emerged, exposing how complete their concept image of integration is. On the conceptual side, definite integral, anti-derivative, and sum of an infinitesimal quantity would all be valid aspects of a student's concept image of both integrals. By contrast, we expected area under the curve to be a more natural interpretation of $\int_a^b n(x)\mathrm{d}x$ than (for the curve $y = 1$) of $\int_a^b \mathrm{d}x$; and evaluation as a possible correct response for $\int_a^b \mathrm{d}x$, but not for $\int_a^b n(x)\mathrm{d}x$. Furthermore, having prior knowledge of students' views of a general integral like $\int_a^b n(x)\mathrm{d}x$ would help us develop teaching strategies for specific integrals such as $\int_a^b \lambda(x)\mathrm{d}x$.

Students' responses to these questions are shown in Table 3.1. Some students gave more than one interpretation for both integrals and therefore the percentages sum to more than 100%.

Table 3.1: Categorization of students' interpretations of two integrals.

Category	$\int_a^b \mathrm{d}x$ %(N=50)	$\int_a^b n(x)\mathrm{d}x$ %(N=50)
sum	6 (3)	2 (1)
area under a curve	20 (10)	16 (8)
definite integral	16 (8)	12 (6)
anti-derivative	6 (3)	2 (1)
evaluation	76 (38)	60 (30)
verbalized integral	50 (25)	44 (22)
other	6 (3)	2 (1)
no answer	2 (1)	6 (3)

Few students made mention of any aspect of integration we would deem conceptual. Fewer than 10% of our students mentioned summation, and at most 20% viewed integration as the area under the curve of a function. Three students described the integral $\int_a^b \mathrm{d}x$ as a sum, with varying levels of correctness and completeness:

> *"Another word for summation"*

> *"sum of x from a to b"*

> *"this integral describes the sum of parts between a and b of infinitesimal length $\mathrm{d}x$"*

Only one student mentioned summation in the description of $\int_a^b n(x)\mathrm{d}x$:

> *"sum of parts between a and b of infinitesimal length $\mathrm{d}x$ for the function $n(x)$"*

Ten students appeared to interpret the first integral as the area under a curve. Five students explicitly mentioned a curve in their answer: two students simply stated "area under the curve" and did not elaborate on this; two students used $\mathrm{d}x$ as the function and believed that the integral would give them "the area under the curve $\mathrm{d}x$"; while one of the five stated that there is no curve in this case and concluded that this means that there is nothing to be evaluated.

The remaining five students wrote that the integral will yield an area. Although these students did not mention a curve, their explanations strongly suggest that they meant area under a curve:

"area from b to a closed by the x-axis"

"used to find area's from a point a on the x-axis to b"

For the second integral, seven students mentioned area under the curve, all answering similarly to the following:

"area under the curve $n(x)$ between a and b on the x-axis"

One student mentioned that the integral would give an area without elaborating on this.

All other responses concerned what might be termed technical aspects of integration. Eight students identified the first integral as a definite integral; six students did so for the second integral. Similarly, three students mentioned that the first integral gives the anti-derivative or that integration is the "reverse process of differentiation" without elaboration; one student mentioned that the second integral gives an anti-derivative.

The evaluation category was the largest by far, with over 75% of students either evaluating $\int_a^b \mathrm{d}x$ or describing the evaluation process. Typical examples are:

"$[x]_a^b \Rightarrow b - a$"

"Increase the power of x by 1 and divide by the value of the new power. Insert the limits (upper limit - lower limit)"

For just over half of all students this was the only part of their answer that could be considered an aspect of their evoked concept image for integration, which suggests that despite knowing the process of integration, they have little or no understanding of it. A further strong indication for this conclusion comes from the finding that 60%

of students evaluated $\int_a^b n(x)\mathrm{d}x$, despite it not being possible to do so. Twenty-one of the 30 students that did so treated n as a constant, getting an answer $\frac{nb^2}{2} - \frac{na^2}{2}$, for a variety of reasons. Six students mentioned in their answer that n is a constant. It is hard to know whether they really thought so or simply reshaped the question so that they could do the only thing integration triggered with them: evaluation. Moreover, it cannot be denied that there is an ambiguity in the notation: $n(x)$ *could* mean: multiply n times x. It is also instructive to note that four students identified that $n(x)$ is a function, but still ended up with the same evaluation.

Finally, for both integrals approximately half of the students simply restated in words what the symbols mean at a very basic level. Such a verbalization should not really form part of a concept image, but it is probably a very strong indication of how these students approach any symbolic expression. Typical answers in this category were:

> *"Integral of* $1\mathrm{d}x$ *from* a *to* b*"*
>
> *"An integral with limits* b *and* a *integrated with respect to* x*"*
>
> *"Integration of a function "*$n(x)$*" with respect to a variable "x" between parameters* a *and* b*"*

For five students this type answer was their only response for both integrals, indicating that they have neither a conceptual nor procedural understanding of integration. A further three students that had evaluated and verbalized the integral $\int_a^b \mathrm{d}x$ but did not attempt to evaluate $\int_a^b n(x)\mathrm{d}x$. This implies that although these students have a poor conceptual understanding of integration, they know that this second integral cannot be evaluated.

3.2.3 Implications for integration in physics

Our results support and quantify the ideas on students' concept image of integration expressed in the aforementioned literature.[4;5] Just over 5% of our students spontaneously mentioned summation when asked what the two definite integrals of Fig.3.1 mean to them, and at most 20% linked the integrals to an area under the curve. Overwhelmingly students wished to evaluate the integral, whether this is possible or not, and verbalized the symbols.

Based on these findings, it is clear that teaching and learning of electromagnetism is a much more formidable task than one might anticipate. Calculus-based introductory texts and junior level texts such as Griffiths[7] routinely include calculations that involve integration e.g. the calculation of the electric field above the mid-point of a uniformly charged rod. Once the techniques have been mastered, it is easy to overlook how many pieces of mathematics knowledge must be called upon: not only integration, but also superposition, vector addition, and understanding of what a function means. However, an additional problem now emerges: the knowledge of integration our students have obtained is not the kind that is likely to be useful in a physics context.

Some researchers have suggested that the reason students perform poorly when required to use mathematics in physics problem solving is an inability to transfer knowledge from their calculus courses to physics[8] or an inability to apply the mathematical skills they have in a physics context.[9] All eight students interviewed in a study by Cui *et al*[8] were confident of their calculus knowledge and almost all felt that their knowledge would be adequate for use in physics. However, it was found that these students were unclear as to when calculus should be utilized in physics problems. The authors also found that, while seven from a different set of eight students were able to use calculus in solving physics problems, only three could provide an adequate explanation for why integration was required.

Our results suggest that students' difficulties with integration in physics are unlikely to be a transfer issue; the problem is that the kind of knowledge of integration our students acquire from their mathematics courses is inadequate. Our findings are in line with those of Grundmeier et al[1] and Petterson and Scheja.[10]. This problem appears not to be a local issue: Orton[3] and Grundmeier et al[1] found that upon completion of calculus courses students often have little conceptual knowledge of integration despite being proficient at computing integrals.

3.3 Cueing integration in a physics context

3.3.1 Previous findings

If students primarily associate integration with evaluating a symbolic expression, it is not unreasonable to wonder what would cue students that integration is required to solve a particular physics problem. Meredith and Marrongelle[11] identified three important types of cue that may lead students to recognize when integration is called for:

1. *Recall cue:* Students may remember having used integration in similar contexts, without understanding why it is used.
2. *Dependence cue:* Recognition that when a quantity varies (for example with position or with time), integration may be needed.
3. *Parts-of-a-whole cue:* Recognition that a quantity (the whole) consists of many parts, which may be evaluated individually.

The types of cue are listed in increasing order of usefulness and desirability. Students relying on the recall cue are unlikely to recognize the need for integration in problems that involve contexts or situations that are unseen. Although, recognizing that some quantity is varying will trigger the use of integration for students using the dependence cue, it may cause students to set up the integral incorrectly in situations where the infinitesimal term is not obtained from the varying quantity (discussed in more detail with technical difficulties, 3.4.1). Students relying on a parts-of-a-whole are likely to have a more complete picture of what is being asked. It is also clear that the parts-of-a-whole cue is a natural fit with the summation/accumulation view of integration. Meredith and Marrongelle[11] suggested that a parts-of-a-whole cue is conditional on identifying a dependence.

In their study, Meredith and Marrongelle[11] interviewed students working on four electrostatics problems that required the use of integration. Five of the nine students who participated in the study did not use integration in at least one of the problems. The authors found evidence that students did not understand the physics involved in the problem and that this hindered them identifying the need for integration in these cases. They concluded that a correct interpretation of the physical situation presented in a problem is essential for integration to be cued.

For the problems where students did integrate, the dependence cue was employed most often with eight students using it for at least one question. The recall cue was only used in problems involving concepts where students had repeatedly experienced integrals being applied. Five students used the parts-of-a-whole cue at some point, and mostly in conjunction with the dependence cue. Its use became less frequent with increasing complexity of the problem. This does suggest again that a lack of understanding of the physics may interfere with the cueing of integration, especially with students' ability to reason that the integral is as a sum of the quantity in question.

A similar study by Nguyen and Rebello[12] found that students did not have difficulty in recognizing the need for an integral, using the recall cue in problems that were familiar to them and the dependence cue in those that were not. Several of their students mentioned the sum of infinitesimally small elements at some point indicating that they had an idea of the parts-of-a-whole resource. However, these students still set up incorrect expressions for the infinitesimal quantity or did not attend to how it should be added. The authors stated that although parts-of-a-whole is the most useful cue, employing it did not ensure that the students would reach the correct expression.

3.3.2 Treatment of integration in the course

In the Electromagnetism course the students took, integration was treated as follows. The first time students encountered integration, they were reminded in lecture that integration of any function $f(x)$ allows determination of the area under a curve, with an emphasis on the addition of the area of infinitely many rectangles. Later in the same lecture, integration was used in an example, involving the calculation of the electric field above the midpoint of a uniformly charged rod. In later lectures, it was demonstrated to students how integrals are used in a number of more or less standard sample problems; in each case, the idea of summing infinitely many contributions from infinitely many parts would be emphasized. In tutorial, students would again mix the graphical representation of an integral with the idea of a Riemann sum in determining the total charge on a non-uniformly charged rod, and in five more physics problems.

3.3.3 Post-test

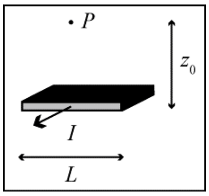
The post-instruction exam presented students with a question where integration was required to find the magnetic field due to a current-carrying thin ribbon (the full question is shown in Fig. 3.2). The question was quite unlike any integration questions the students had seen in tutorials: they had integrated to find electric field, electric flux, potential, and magnetic force, but not magnetic field (except in the derivation of Ampère's Law from the Biot-Savart Law), and had not seen linear current density in any form. A recall cue as defined by Meredith and Marrongelle could therefore be eliminated as a likely trigger for integration. The setting is conceptually quite simple, so that confusion about what is being asked is unlikely to be a major factor in students' responses.

While the question relies on Ampère's law, in solving the problem students do not encounter any of the difficulties associated with *using* it.[13–15] The problem is the magnetostatic equivalent of finding the electric field due to a uniformly charged rod:

it concerns vector addition of infinitesimal magnetic fields and the recognition of symmetry in doing so, but not the problem of finding clever loops that allow calculation of the magnetic field.

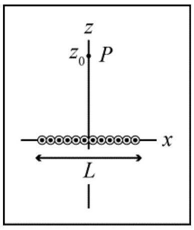
Part (a) of the question was designed to check whether students saw a dependence on distance and a cancellation of vertical components of the magnetic field. Part (b) allowed us to verify that students understood the unseen definition of K as a linear current density. Providing students with the expression for the infinitesimal magnetic field due to the small segment, in part (c), was an attempt at separating problems with the physics from problems with integration. It is the latter that we are interested in for the purposes of this research. This separation also allowed us to check to what extent Meredith and Marrongelle's assertion holds true that understanding of the physics is an important part of integration. The indefinite integral was provided on the exam paper, so students only needed to evaluate the integral at the correct limits.

A infinitely long, infinitely thin conducting sheet of width L carries a constant uniform current I as shown. In this question, you will consider the magnetic field at a point P a distance z_0 above the central axis of the sheet.



(a) Explain that the expression $B = \mu_0 I / 2\pi z_0$ for the magnitude of the magnetic field at P is not exact but only approximate.

To find an exact expression for the magnetic field at P, we can define a linear current density K, defined as the current per unit length in the sideways direction. Also, choose the origin of a Cartesian coordinate system to coincide with the centre of the sheet directly below P.



(b) Explain that the following equation holds: $I = KL$.

(c) Show that the magnitude of the magnetic field at P, $\mathrm{d}B$, due to a segment of the sheet located at x of width $\mathrm{d}x$ is given by

$$\mathrm{d}B = \frac{\mu_0 K \mathrm{d}x}{2\pi\sqrt{z_0^2 + x^2}}$$

(d) In which direction does the net magnetic field point at P? Explain.

(e) Derive an expression for the net magnetic field $\vec{B}$ at P due to the entire sheet.

(f) Calculate the line integral $\int \vec{B} \cdot \mathrm{d}\vec{l}$ going from point P to point R. (The points are shown in the diagram at right.) Explain briefly.

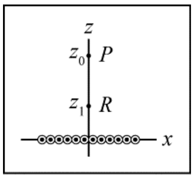


Figure 3.2: Post-test question on the calculation of magnetic field

3.3.4 Results

Forty-five students (N=45) took the exam. When asked to find the exact magnetic field at point P due to the entire sheet, for half the students (49%) integration was not cued, with twenty students (44%) not responding to this part ((e)) of the question and two students (4%) giving an answer that did not involve integration. We found a similar pattern in student answers in a similar integration question that asked students to evaluate the total electric flux Φ_E from an infinitesimal flux $\mathrm{d}\Phi_E$. This is strong evidence that understanding the physics of the problem does indeed play an important role, not only in *how*, but even in *whether* they attempt to carry out an integration. We note that for almost half the students, being given the expression for an infinitesimal quantity before being asked to evaluate the whole quantity did not trigger integration.

Table 3.2: Dependence cue investigation, correlating whether students' recognition that the magnetic field due to different parts of the current-carrying ribbon was consistent with them using integration ($N = 45$).

	non-constancy recognized	non-constancy not recognized
integration cued	42% (19)	9% (4)
integration not cued	18% (8)	31% (14)

Kanim[16] found similar percentages when he investigated students' approaches to calculating the net electric field at a given point due to a charged rod. In his case, 40% of the students showed that they knew integration was required, and only 10% correctly calculated the field.

To investigate how frequently students who knew to use integration were triggered by a dependence cue, and how likely it was that students who recognized that the quantity was non-constant would use integration, the responses to part (a) were correlated with the response for parts (d) and (e). This data is shown in Table 3.2. In nineteen of the twenty-three cases where integration was cued, students had identified at least one reason for why the magnetic field due to the sheet could not be expressed using the given expression. Most frequently, this was the varying distance from the point to different parts of the sheet. On the other hand, fourteen of the twenty-two students for whom integration was not cued had either not answered the part (a) or had not correctly identified why the given expression was only approximate. These two links show how strongly reliant students are on the dependence cue. However, eight students who responded correctly in the first part did not use integration in part (e), suggesting that the dependency cue was not sufficient, and seven students used integration despite not recognizing why the expression in part (a) was not exact, indicating that the dependence cue is not necessary.

There was evidence that eight of the students who used integration used a parts-of-a-whole cue, as they included a description of integration as a sum or an explanation that a sum was required in some part of the question. Although none of the following typical answers were completely correct, they reveal an understanding of integration as a process of accumulation:

> *"K divides the sheet with a current I into many small sheets with current ΔI... by adding all the ΔI we get the original total current"*
>
> *"Looking at the sheet as a group of... n wires, each with current I/n, the magnetic field would be a simple sum of each wire"*
>
> *"$\overrightarrow{B}$ = sum of all small segments"*

It is interesting to note that all students who used the parts-of-a-whole cue also used the dependence cue. Five of the eight, went on to give an essentially correct answer when finding the expression for the exact magnetic field due to the entire sheet. Our findings support those by Meredith and Marrongelle[11] that use of the parts-of-a-whole cue occurs in conjunction with the dependency cue, and that it is the most potent cue of the three.

3.3.5 Implications for teaching

Thus far, our investigation has shown that prior to the electromagnetism course between 60% and 75% of our students saw integration as an evaluation, while only 5% identified integration as a process of accumulation.

It is clear that this only exacerbates the difficulties students have with solving problems that require integration. If students had internalized that an integral is a summation before starting the course, a teacher could concentrate on helping students learn to identify when a continuously varying quantity is being added. In the present situation, a teacher needs to convince students that integration is required because it is a summation – a tautology to an expert, but a source of wonder for a student who sees an integral mostly as an evaluation tool.

We have found that less than half of our students recognized the need for integration on completion of a course that comprised five tutorials that addressed the idea of integration as summation in an electromagnetism context, in a manner similar to the one required in the post-test of Fig. 3.2.

This suggests to us that for instruction in integration within an electromagnetism course to be successful, students must repeatedly and exclusively be confronted with the idea that integration is an infinite summation. Far from "wasting time" on re-teaching mathematics within a physics course, this augments and enhances what students have learned in a mathematics context. To this end, changes were made to the instruction, including the omitting of any reference to integration as finding the area under the curve, both in lecture and in tutorial, and the replacement of these with an introductory lecture (described in Section 3.5.1) and an introductory tutorial on integration (described in Section 3.6) that expose students to the infinite summation concept.

3.4 Technical difficulties with integration

3.4.1 Known difficulties

The studies by Meredith and Marrongelle[11] and Nguyen and Rebello[12] also discussed the difficulties students have in applying integration (i.e. setting up and evaluating integrals) in physics problems. When setting up an integral, understanding the origin of the infinitesimal term was found to be the main difficulty. Although the infinitesimal term has its own physical meaning and writing an integral without it is physically meaningless, Nguyen and Rebello[12] found that many students either neglected to include it in the set up of an integral or placed it after the integrand without realizing how it changed the quantity being summed. Meredith and Marrongelle[11] also identified the latter difficulty and described it as a failure of the dependence cue when the non-constant quantity is not a density or rate of change. For example when asked to find the electric field due to a uniformly charged rod, students using the dependence cue identified that the varying quantity was distance and appended $\mathrm{d}r$ next to the formula for a point charge.

A failure to pay attention to how infinitesimal terms should be added is another identified difficulty. Approximately half the students involved in the Nguyen and Rebello [12] study did not account for the vector nature of electric field and integrated the whole of $\mathrm{d}\vec{E}$. The authors suggested that this was due to students' lack of visualization of the physical situation presented. This study also found that when evaluating the integral students also had difficulties understanding the physical meaning of symbols in the integral, recalling basic mathematical equations needed to write all variables in terms of the infinitesimal term, determining the limits for integration, and computing the integrals.

3.4.2 Our results

The twenty-three students who used integration in the post-test question described in Section 3.3.3 did experience technical difficulties in setting up the integral (even though the corresponding infinitesimal quantity was given), and in evaluating it. Nineteen students obtained the given expression for the magnetic field due to a small segment of the sheet. All of these showed that the distance to each infinitesimally thin wire was $\sqrt{z_0^2 + x^2}$, but ten students substituted $K\mathrm{d}x$ for current without explanation. While it is impossible to tell whether they are merely inferring that $\mathrm{d}I = K\mathrm{d}x$ from what is given without understanding what it means, all nine students who explained why they replaced I with $K\mathrm{d}x$ did so correctly. A typical explanation was:

> *"$K = I/L$ and $\mathrm{d}x$ is a very small length, thus multiplying K by $\mathrm{d}x$ should give a small current."*

The four students who answered incorrectly gave very varied responses; two of these students added the infinitesimal term to their answer, or without explaining where it comes from. For example, one student started from Ampère's Law,

$$\mathrm{d}B = \frac{\mu_0 I \hat{\phi}}{2\pi z_0} \tag{3.1}$$

but multiplied by a factor $\cos\theta$ which allowed them to obtain an expression that looked correct apart from notationally equating a scalar to a vector:

$$\mathrm{d}B = \frac{\mu_0 I}{2\pi\sqrt{z_0^2 + x^2}} \cdot \hat{\phi} \tag{3.2}$$

This student then stated:

> *"however now we are not assuming I to act at one point so we must replace I with a charge density K."*

In the final answer, a term $\mathrm{d}x$ appeared in the numerator, accompanied by the text "width sheet." This example is not unlike some responses quoted by Nguyen and Rebello[12] to illustrate that their students often append an infinitesimal term without appearing to understand its origin. It is also clear that this student has only a tenuous grasp of the physics involved. However, our findings generally neither confirm nor contradict those from Meredith and Marrongelle[11] and Nguyen and Rebello[12]: our question was not designed to elicit difficulties with the infinitesimal, and allowed such problems to remain hidden.

Only 1 student evaluated the integral completely correctly. Table 3.3 shows the types of integration errors made by the other 22 students. Some students made more than one of these errors.

Table 3.3: Errors made during integration

Error	**Magnetic Field %(N=23)**
Incorrect accumulation	83(19)
No limits	22(5)
Incorrect limits	43(10)
No process attempted	22(5)
Mistake in process	17(4)

Nineteen students integrated the expression for $\mathrm{d}B$ as if the magnetic field were a scalar. A number of factors could be at play here: perhaps they did not think of the infinitesimal magnetic field as a vector, they may not have known how to add vectors, or they may not see integration as an accumulation process, or a combination of these reason. Failure to consider how quantities should be added up has previously been identified as a difficulty.[12]

Determining the limits for integration was also a problem for students, with five students leaving them out and ten students using incorrect limits. Four students used 0 as the lower limit and L as the upper limit, and this was their only mistake during the integration process. This suggests that students struggle to determine the limits when the lower limit is non-zero.

Although five students had recognized the need for integration they did not attempt to compute the integral at all. Four more students made different mistakes during the evaluation process.

Our findings support the conclusions by Meredith and Marrongelle[11] and Nguyen and Rebello[12] that students recognizing the need for integration is necessary but not sufficient to come to a correct answer. As pointed out by Manogue et al[13], students struggle because so many aspects of physics and mathematics knowledge need to be cued and applied correctly to come to an answer.

3.5 Students' interpretation of one-dimensional integrals

While we feel it is imperative that students see integration as an accumulation, it is also important, in a physics context, that students understand what quantity they are accumulating and why.[17] In Section 3.3.4 students' strong reliance on the dependence cue when recognising the need for integration and the correlation between the dependence and parts-of-a-whole cues were shown. If students can interpret the physical representation in a problem they would be more likely to recognise a varying quantity, hence increasing the probability of integration being cued. Sealey[18] found that until students could make sense of the concepts involved in a problem, they did not recognise the need for approximations when the quantities were varying. Also, Nguyen and Rebello[12] reported that their students struggled when asked to interpret infinitesimal quantities, for example confusing $\mathrm{d}A$ as a changing area rather than as a small element of area. Such an interpretation mirrors closely what they would have seen in a mathematics contexts, where x is the parameter that changes in the expression $(x)\mathrm{d}x$. In setting up integrals, their students often did not include the infinitesimal term or appended it to the integrand without realising how it changed the physical meaning of the integral.

To investigate the extent that these difficulties were present in our students, a pretest asking students to interpret two basic integrals (representing length and charge) was administered after a lecture had introduced integration, the concepts of charge and charge density, and the use of integration in the calculation of quantities such as charge and electric field.

3.5.1 Integration lecture

The aim of this lecture is to introduce integration as a process of accumulation. Von Korff and Rebello[17] describe a framework for understanding integration as a summation. Their multi-route network of objects and processes is shown in Figure 3.3. The authors state that successful instruction does not have to include all possible routes.

The top line shows the macroscopic route towards building the product for summation, while the bottom line follows the same principle but at an infinitesimal level. The authors suggest that you can move from the top to the bottom line at any point. They suggest that the starting point is the macroscopic or infinitesimal quantity, which will provide the variable of integration. The next step is the building of the macroscopic or infinitesimal object by writing the product of the constant function and the macroscopic or infinitesimal quantity. Next this product must be written as a sum

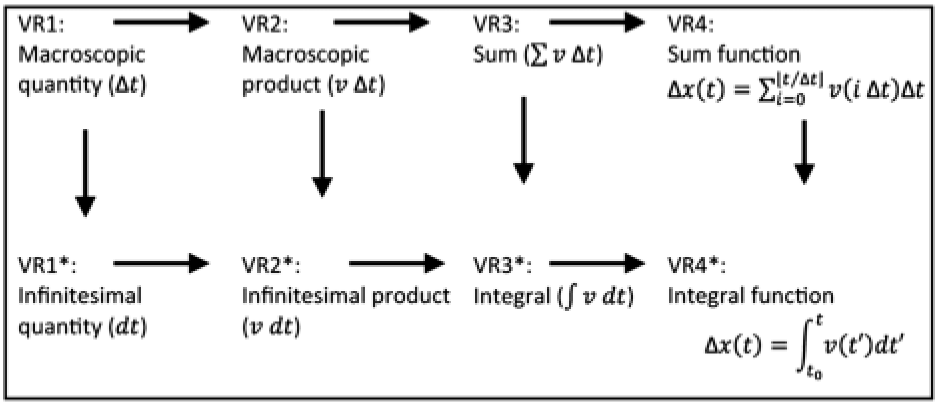


Figure 3.3: Network of routes towards understanding integration as a summation

(or an integral) and lastly the macroscopic quantity is made to approach 0 so that the function becomes continuous and an exact value of the product between two limits can be found.

The lecture takes the macroscopic route to the integral (VR1$\longrightarrow$ VR2$\longrightarrow$ VR3$\longrightarrow$ VR4$\longrightarrow$ VR4*). Integration is introduced as a means of calculating the volume of a pyramid. This example was chosen because it shows the concepts behind scalar integration in a context that is easily understood. First the pyramid is sliced very thinly so that each slice resembles a rectangular block with a height Δz_i (VR1). The area of the slice will depend on what height it is at so the volume of each slice is $\Delta V_i = A(z_i)\Delta z_i$ (VR2). The volumes of all slices can then be added together to get the total volume $V = \sum \Delta V_i$. Before this is developed further students are reminded of the idea of a Riemann sum in general terms: $\sum_{i=1}^{N} f(x_i)\Delta x_i$ (VR3). When N is large this expression is unworkable in that the quantity in question would need to be calculated individually for each segment Δx before being added together. However, if you make Δx infinitesimally small ($\Delta x \to 0$), so that the number of segments tends to infinity ($N \to \infty$), then the Riemann sum expression becomes a Riemann integral (VR4$\longrightarrow$ VR4*). In this way the concept definition for an integral is introduced, and the point is made that the sum becomes workable because the integral is equal to the anti-derivative. The lecture then returns to finding the volume of the pyramid using the idea of a Riemann integral.

Later, the lecture shows how integration can be used to find the charge on a rod with linear charge density that changes linearly along the rod. The term $\lambda(x)$ is explained and expression for the charge on the rod is written as $Q = \int_0^L \mathrm{d}Q = \int_0^L \lambda(x)\mathrm{d}x$.

3.5.2 Pretest Question

To see whether students could relate integrals to physical situations in electromagneism, they were given a question about a rod with non-uniform linear charge density $\lambda(x)$ and ends located at $x = a$ and $x = 5a$ as shown in Figure 3.4. Students were asked to give the physical quantity given by the integrals $\int_a^{5a}\mathrm{d}x$ and $\int_{2a}^{4a}\lambda(x)\mathrm{d}x$ and to explain their answer briefly. They were told that they did not need to calculate the integral.

Figure 3.4: Non-uniformly charged rod

The physical quantity represented by $\int_a^{5a}\mathrm{d}x$ is the length of the rod. A fully correct interpretation involves students reasoning that $\mathrm{d}x$ is the length of a small piece of the rod, the integral sums all the small pieces of length $\mathrm{d}x$ between the given limits, and because the limits are either end of the rod you get the total length of the rod.

$\int \mathrm{d}x$ is the simplest possible integral but by asking students to interpret it the question investigated if students could give a physical meaning to an infinitesimal term. Through analysis of their explanations it was also possible to determine if students were able to reason how the process of integration uses this infinitesimal term to find the total quantity.

The charge on the rod between $2a$ and $4a$ is given by $\int_{2a}^{4a} \lambda(x)\mathrm{d}x$. A full explanation here would have involved students reasoning that although the charge density is non-uniform, the charge on a infinitesimally small segment is constant. $\lambda(x)\mathrm{d}x$ gives the charge on a infinitesimally small segment of length $\mathrm{d}x$ and so the integral sums the charge from all the infinitesimally small segments along the rod between the limits $2a$ and $4a$.

$\int \lambda(x)\mathrm{d}x$ is slightly more complex in that it contains a function as the integrand, but it is still a basic integral in an electrostatic context. This question allowed us to examine if students gave a physical meaning to the infinitesimal term when there was an integrand to consider. Also, students could have evaluated the integral $\int_a^{5a} \mathrm{d}x$, found that it yielded $4a$ and then connected this with being the length of the rod. As $\int_a^{5a} \lambda(x)\mathrm{d}x$ could not be evaluated, it was more likely that students would need to access the Riemann sum idea from their concept image in order to answer correctly.

The first section considers students physical interpretation of the two integrals. Then students responses and reasoning are placed in the concept image framework, using a categorisation similar to that used in Section 3.2. This allowed us to investigate how students apply their concept image when the infinitesimal term and the function represent a physical quantity, hence converting from the more general to a specific case. If the same categories were present in each case, this would confirm and strengthen our interpretation of students beginning concept image.

This pretest question was given to two different student cohorts after essentially the same lecture instruction. In Year 1 48 students answered and in Year 2 49 students answered, giving us a total of 97 student answers for this question. In Table 3.4 and Table 3.5 the numbers for each category of answer are given for the total and for each year separately. As a χ^2 test showed that there was no statistical difference between the two years, only the totals are used in the description.

3.5.3 Students' physical interpretation of the integrals $\int \mathrm{d}x$ and $\int \lambda(x)dx$

3.5.3.1 Length Integral

Students' answers for $\int_a^{5a} \mathrm{d}x$ is shown in Table 3.4.

Table 3.4: Students' answers for the length integral

Category	Year 1 %(N=48)	Year 2%(N=49)	Total %(N=97)
Length	31(15)	43(21)	37(36)
Charge or charge density	19(9)	18(9)	19(18)
Area/Volume	8(4)	10(5)	9(9)
No physical quantity	8(4)	2(1)	5(5)
No interpretation	23(11)	20(10)	22(21)
Other	8(4)	8(4)	8(8)

68% of students linked $\int_a^{5a} \mathrm{d}x$ to a physical quantity. Thirty-six students (37%) correctly said that the integral gave the length of the rod. Types of reasoning used by students are quoted in Section 3.5.4.

Nine students (9%) related the integral to charge and another nine students (9%) related the integral to charge density:

> *"This integral describes the charge with respect to the position on the rod"*
>
> *"length x is integrated from a to $5a$, so the integration gives the overall charge on it"*
>
> *"...as it is a wire it gives linear charge density"*

It is most likely that these students are trying to make a connection between this integral and the examples shown in lecture.

Six students (6%) interpreted the integral as an area and three students (3%) stated that it would yield volume. From their explanations, it seems that most of these students are also trying to relate this integral to those seen during the lecture:

> *"Area, logic relating to the pyramid example"*

> *"Volume is the integral of area as a function of height, the height of the above* [the rod shown in the question]*varies from a to $5a$ in this specific case"*

However, it does seem that at least three students simply mis-interpreted $\mathrm{d}x$ as an infinitesimal area instead of an infinitesimal length.

> *"sums all small pieces of area from a to $5a$"*

Five students (5%) stated that the integral had no physical meaning and explained:

> *"no function has been integrated just a constant "1""*

> *"There is no f(x) in this equation leaving the answer always as $4a$"*

> *"Integrating dx gives us x but this doesn't represent any physical quantity for this particular question."*

These students claim that there was nothing to integrate,indicating that they are not aware of the meaning of an infinitesimal term or its function in the integration process.

The eight (8%) responses in the Other category gave different incorrect interpretations of the integral:

> *"Defines the rod between a and $5a$"*

> *"Velocity, first derivative of position"*

22% of the students did not give a physical interpretation but instead evaluated the integral, getting an answer "$4a$", despite the question explicitly stating that there was no need to do so. For these students, it is likely that the integral sign simply triggers evaluation, completion of a process without consideration of what the integral means or how it relates to the situation in question. This confirms how prominent evaluation is in students' concept image of an integral. For many students evaluation is the only response even when the integral is based on a physical setting and they were told that they did not need evaluate.

3.5.3.2 Charge Integral

Students' answers for $\int_{2a}^{4a} \lambda(x)\mathrm{d}x$ are shown in Table 3.5.

Table 3.5: Students' answers for the charge integral

Category	Year 1 %(N=48)	Year 2 %(N=49)	Total %(N=97)
Charge	25(12)	39(19)	32(31)
Charge density	23(11)	27(13)	25(24)
Relating charge density and length	8(4)	4(2)	6(6)
No interpretation	35(17)	22(11)	29(28)
Other	8(4)	8(4)	8(8)

71% of students linked $\int_{2a}^{4a} \lambda(x)\mathrm{d}x$ to a physical quantity. Thirty-one students (32%) correctly said that the integral gave the charge on the rod between $2a$ and $4a$. Examples of the types of answers students gave will be shown in Section 3.5.4.

25% of our students appear to look at the integrand only (neglecting the effect of the infinitesimal term on the integral), and infer that the result of the integration must be a charge density:

> *"We are integrating the charge density and then subbing in our limits of 4a and 2a indicating that we are just getting the charge density between these two points"*

"$\lambda(x)$ represents linear charge density"

Interestingly, six of these students (6% of the total) had identified that $\int_a^{5a} \mathrm{d}x$ gave the length of the rod, but yet did not consider how the infinitesimal term would affect the meaning of this second integral. This is consistent with findings from other studies [11;12] and could be due to the treatment of integration in maths where students evaluate integrals without the need to consider the infinitesimal term.

Six students (6%) linked charge density and length showing that they gave attention to the infinitesimal term. However, their explanations do not state that the integral gives charge and some were difficult to interpret:

"Length or distance in relation to linear charge density. The same as before except we now have $\lambda(x)$ which is linear charge density."

"Finding the anti-derivative of λ which describes linear charge density, we see that the function is dependent on length also"

The remaining answers or explanations were unclear, or were too difficult to be categorized.

29% of the students did not give a physical interpretation of the integral. Twenty-one students (22%) calculated the integral (even though the linear charge density had not been defined). Six students treated $\lambda(x)$ as a constant λ to be multiplied by x, carried out the integration correctly, and then omitted the constant λ from their answer, $6a^2$. At least six students treated $\lambda(x)$ as a constant, integrated $\lambda \mathrm{d}x$, and obtained either $2a$, $2a\lambda$, or $2x\lambda$. Seven students recognised that they could not evaluate the integral but did not attempt to give a physical meaning to the integral:

"Without knowing the function in question I cannot integrate it with the techniques I know"

Table 3.6: Students' explanations about the length and charge integrals compared with explanations from the concept image pretest described in Section3.2.2

Category	$\int_a^b \mathrm{d}x$ %(N=50)	$\int_a^b n(x)\mathrm{d}x$ %(N=50)	$\int_a^{5a} \mathrm{d}x$ %(N=97)	$\int_{2a}^{4a} \lambda(x)\mathrm{d}x$ %(N=97)
sum	6(3)	2(1)	8(8)	13(13)
area under a curve	20(10)	16(8)	3(3)	0(0)
definite integral	16(8)	12(6)	1(1)	1(1)
anti-derivative	6(3)	2(1)	2(2)	2(2)
evaluation	76(38)	60(3)	38(37)	34(33)
verbalized integral	50(25)	44(22)	13(13)	14(14)
explaining terms	-	-	19(18)	30(29)
other	6(3)	2(1)	13(13)	2(2)
no explanation	2(1)	6(3)	10(10)	10(10)

"Can't integrate λ with respect to x"

This supports the results from the concept image investigation and further shows these students' strong inclination to evaluate when presented with an integral.

3.5.4 Concept image categorisation of students' reasoning

To determine if there was a consistency in students' concept image of integration when presented with essentially the same integrals in and out of a physical context, we placed students' explanations and reasoning into the categorisation developed for the concept image pretest.

The categorization of students' responses is shown in Table 3.6. The categories are the same as those in Table 3.1 for the concept image pretest, with the addition of an *explaining terms* category where students interpreted the infinitesimal, the integrand, or both but did not explain the integration. As some students gave more than one type of reasoning for both integrals the percentages sum to more than 100%.

Few students describe what we would consider to be conceptual aspects of integration, i.e. the idea of summation or area under the curve. Just over 10% mentioned summation with varying completeness and not all with the correct physical interpretation of the integral. For the length integral, students explanations it this category were similar to:

> *"sum of all infinitesimal* $\mathrm{d}x$*'s along the rod"*

> *"the sum of charges from* a *to* $5a$*"*

and for the charge integral were similar to:

> *"the charge is got at infinite slices and added together"*

> *"charge density is being summed over infinitely small points between* $2a$ *and* $4a$*"*

There is a slight increase in the number of students using sum reasoning compared to the concept image pretest, though many of the students using it did not give a correct physical meaning to the integrals. It is likely that the increase is largely due to students trying to relate the integrals to those they had seen in the lecture.

Only three students mentioned area under the curve for the length integral and none mentioned it for the charge integral:

> *"the integral relates usually to the area under a curve so I think as the length of the rod is integrated it must relate to the density"*

> *"usually integral will determine area under the curve, but since straight line here, it just determines length"*

Considering that not many students could correctly interpret the integrals and certainly not many see integration as a summation, it is not surprising that few students mentioned area under the curve. Previous studies have shown that students do not spontaneously invoke the area under the curve concept when solving physics problems [19], and that it is only useful when students already see integration as a process of accumulation [6] and understand the structure of the definite integral (i.e. see the integral as having two components, an integrand and an infinitesimal term)[19;20]

Like the concept image pretest, the technical aspects of integration were more frequently seen in students' reasoning than conceptual aspects. Definite integral and anti-derivative would not be very useful aspects of students' concept image when giving a physical interpretation of the given integrals and this was reflected in the small numbers mentioning these aspects with only 1% and 2% mentioning either, respectively.

As mentioned in the interpretation section (Section 3.5.3) for these integrals, evaluation was still a common response. Even some students who gave a physical interpretation for the integrals still used an evaluation or a description of an evaluation in their explanation:

> *"Integral of* $\mathrm{d}x$ *is* x*"*

> *"the integral is of a constant, so it will be one point to another point giving a length"*

> *"We are integrating the charge density and then subbing in our limits of* $4a$ *and* $2a$ *indicating that we are just getting the charge density between these two points."*

Another popular response was to explain the terms in the expression without attempting to explain the process of integration. For example four students who said that the $\int_{2a}^{4a} \lambda(x)\mathrm{d}x$ gives charge density explained this by stating:

> *"*$\lambda(x)$ *is linear charge density"*

In many cases, this was intertwined with the integral being verbalized:

> *"Integral of linear charge density varying with length"*

> *"Integral of linear charge density function gives the charge on the rod"*

10% of students did not provide an explanation for their interpretation and the students in the Other category gave explanations that were difficult to interpret or that did not fit into the main categories. For example, for the length integral:

> *"it is a rod"*

or for the charge integral:

> *"logic relating to the pyramid example"*

3.5.5 Conclusions and implications for teaching

The analysis of this pretest shows that students have difficulty in relating integrals to a physical situation, with only 37% identifying $\int_a^{5a} \mathrm{d}x$ as the length of the rod and only 31% interpreting $\int_{2a}^{4a} \lambda(x)\mathrm{d}x$ as the charge on the rod between $2a$ and $4a$. Perhaps this is unsurprising for the charge integral, considering that students have only just been introduced to the relationship between charge and charge density, but for $\int_a^{5a} \mathrm{d}x$ the same argument cannot be made. It confirms the difficulty students have giving a physical meaning to the infinitesimal term, previously identified by Nguyen and Rebello[12] and Meredith and Marongelle[11], and highlights how strongly seeing an integral triggers evaluation for some students.

In terms of students concept image, the re-occurrence of the same categories in students reasoning suggests that they may be generalisable, and shows that students concept image of integration does not change when given a physical context. Most students who gave a physical interpretation for the integrals either did not attempt or struggled to explain how the integral gave that physical quantity.

This confirms the need for further instruction on integration focusing of the accumulative nature of integration, but also indicates that attention must be paid to developing students' understanding of the infinitesimal term and the setting up of integrals in various physical contexts.

3.6 Tutorial instruction

In this section, we describe the tutorials developed by the lecturer to improve students' understanding of charge density and charge (the physics quantity being used for the integration tutorial) and to describe integration as a process of accumulation.

3.6.1 *A Van de Graaff Generator* tutorial

This tutorial followed the lecture described in Section 3.5.1. As mentioned in the previous section (Section 3.5) it is important that students understand the concepts involved in the physical setting where integration is required. Although the main aim in this instance was to teach integration as a process of summation, it was necessary that students had an understanding of the physical context in which the integration instruction would be set. As the physical context being used for integration would be the calculation of the total charge on a rod with a continuously varying charge distribution, there was first a tutorial aimed an improving students knowledge of the concept of charge density and the idea of how charge density is constant when considering a small enough segment.(For full tutorial refer to Appendix B)

Students examine six small pieces of the rod (each about 14 cm) where the linear charge density seems to be constant, but by looking at a 2 m stretch they see that constancy does not hold. In doing so, students get familiar with the concept of linear charge density and its relation with charge.

3.6.2 *Charged Objects* tutorial

This tutorial followed the *A Van de Graaff Generator* tutorial. The aim of this tutorial is to guide students through the building of an integral, using it as a summation. Using the Von Korff and Rebello framework of integration[17] described in Section 3.5.1 the route taken in the *Charged Objects* tutorial to reach this aim is VR2$\longrightarrow$ VR3$\longrightarrow$ VR4$\longrightarrow$ VR4*. The full tutorial is in Appendix C.

Students start by considering the charge on a rod with constant linear charge density. Students write expressions for the charge on rods consisting of two parts of different lengths and different linear charge densities, and a different rod consisting of N uniformly charged parts. Expressions like $Q = \lambda_1 L_1 + \lambda_2 L_2 + ... + \lambda_N L_N$ would be acceptable at this stage and aim to make students more at ease with the more complex notation they are to develop: $Q = \Sigma_{i=1}^{N} \lambda(x_i^*)_i$.

Students are then asked to consider a rod with a continuously varying charge distribution. Students are then asked to explain why $Q = \lambda(x_i^*)L$ is not a correct value for the charge on the rod in this case but that $Q = \Sigma_{i=1}^{5} \lambda(x_i^*)_i$ is a better approximation. Students are then asked to explain how they could improve on the approximation, and what is meant by the expression:

$$\lim_{\substack{N\to\infty \\ \Delta x_i \to 0}} \sum_{i=1}^{N} \lambda(x_i^*)\Delta x_i \tag{3.3}$$

Here, we wanted students to see that the smaller Δx_i becomes the closer the charge density for that segment gets to being constant, hence making the charge calculation more accurate, and as Δx_i approaches 0 you get the exact charge. Students are then given the definition that equation 3.3 is equal to $\int \lambda(x)\mathrm{d}x$, and are asked to explain why this integral give the exact charge on the rod when evaluated between the limits 0 and L. Finally, they are asked to calculate this integral for $\lambda(x) = \lambda_0 + \frac{\lambda_1 - \lambda_0}{l}x$, and to summarise in their own words how the technique of integration allows them to find the total charge for any physically sensible charge distribution.

3.6.3 Other instruction involving integration

The approach to integration taken in all other tutorials in which it is used is VR1* $\longrightarrow$ VR2* $\longrightarrow$ VR3* $\longrightarrow$ VR4* (see Von Korff and Rebello framework of integration[17] in Figure 3.3). This is mainly because now that integration has been introduced as a summation, this is now the more straight-forward path to take. In all instances students were given the required definite integral.

In the *Electric Field of a Charged Rod* tutorial students use integration to add the vector components of the electric field due to segments of a uniformly-charged rod, of infinitesimal length $\mathrm{d}x$, to find the electric field at a point a distance above the left end of the rod. In the *Potential and Electric Field* tutorial potential difference is described as $\Delta V = -\int \vec{E} \cdot \mathrm{d}\vec{l}$ and the ordering of limits is investigated. Students use double integration to calculate the total charge on a non-uniformly charged square sheet and a circular disk with a varying surface charge density in the *Two-dimensional charge distributions* tutorial. Integration is also used in the *Calculating electric flux* tutorial to calculate the electric flux through a square sheet due to a charge that is a distance below one of the corners of the sheet, and in the *Circulation* tutorial to calculate the circulation of the magnetic field around a rectangular loop. These tutorials are Appendices D through to H.

3.7 Post Tests

This section will describe two post-tests administered to students surrounding integration. The first post test is structured similar to the pretest question described in Section 3.5.2. It asks students about the physical meaning of integrals, thus focusing primarily on how students think about integration. It should be noted here that there was no specific instruction on the interpretation of integrals. A second post test involved the calculation of a physical quantity, mainly assessing if and how integration was cued. As these post-test questions are counterparts of the previously described questions, they could be directly compared to determine if the instruction influenced how students think about and use integration.

3.7.1 Students' interpretation of one-dimensional integrals

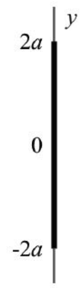
3.7.1.1 Question

There was a number of things that we wanted to investigate in this post-test question: Are students still experiencing similar difficulties after instruction?; Are students still as keen to evaluate when presented with an integral?; Are students more inclined to use sum reasoning in their explanations, demonstrating that this idea is now part of their concept image of integration?

Figure 3.5 shows the post test question asking students to give the physical meaning of two different integrals for a situation involving a non-uniformly charged rod lying vertically along the y-axis. The integral in the first part represents the length of the rod and the second integral gives the charge on the rod between the upper and lower limit.

Question 2 (30 marks)

A rod has a non-uniform linear charge density $\lambda(y)$. Its ends are located at $y = -2a$ and $y = 2a$ as shown. First consider the general case.



a) **[2 marks]** What physical quantity is given by $\int_{-2a}^{2a} \mathrm{d}y$? Explain briefly.

b) **[3 marks]** What physical quantity is given by $\int_{a}^{2a} \lambda(y)\mathrm{d}y$? Explain briefly.

Figure 3.5: Post-test question involving the interpretation of integrals

The only difference between the situation presented here and the one given in the pretest question is that the length of an infinitesimally small segment of the rod is now $\mathrm{d}y$ instead of $\mathrm{d}x$. This means that the two integrals students must interpret here are essentially the same as the integrals in the pretest question, allowing for a direct comparison, although a test effect cannot be ruled out.

3.7.1.2 Students' physical interpretation of the integrals, $\int_{-2a}^{2a} \mathrm{d}y$ and $\int_{a}^{2a} \lambda(y)\mathrm{d}y$

Length Integral:

Students' answers for $\int_{-2a}^{2a} \mathrm{d}y$ is shown in Table 3.7.

Table 3.7: Students' answers for the length integral

Category	Total %(N=48)
Length	73(35)
Charge or charge density	17(8)
Area/Volume	4(2)
Other	6(3)

All students linked $\int_{-2a}^{2a} \mathrm{d}y$ to a physical quantity. Thirty-five students (73%) correctly stated that the integral gave the length of the rod. This is almost double the percentage of students who interpreted the corresponding integral in the pretest.

Seven students (15%) related the integral to charge and one student (2%) related the integral to charge density:

"Charge on the length $\mathrm{d}y$ *of the rod"*

"integral to find the charge distribution along the rod"

The same fraction of students misinterpret the corresponding integrals in this way in the pre and post tests. However, unlike the pretest where students were not providing explanations and appeared to be attempting to link the integral to what they had seen in lecture, there was evidence here that at least three students in this category knew that for this integral to be charge, that charge density needed to be also considered. These students either said that the charge density was equal to 1 here or that the charge density was constant and so could be taken outside the integral:

"$Q = \lambda(x)\mathrm{d}x$, when $\lambda(x)$ is constant it will not be integrated"

Only two students (4%) interpreted the integral as an area:

"area of the charged rod enclosed"

"how many small area's $\mathrm{d}y$*"*

No student stated that the integral did not represent a physical quantity or claimed that there was nothing to integrate. This suggests that the students who stated the opposite for the corresponding pretest integral now know that the infinitesimal term has a physical meaning.

The three responses (6%) in the Other category gave different incorrect interpretations of the integral, but were much more relevant to the situation they were presented with than the responses from the Other category in the pretest:

"Position of y*"*

"average change in length over the rod"

Unlike the pretest no student evaluated the integral without giving a physical interpretation, though some students still evaluated as an explanation or as part of their explanation (see Section 3.7.1.3).

Charge Integral:

Students' answers for $\int_a^{2a} \lambda(y)\mathrm{d}y$ are shown in Table 3.8.

Table 3.8: Student answers' for the charge integral

Category	Total %(N=48)
Charge	65(31)
Charge density	33(16)
No interpretation	2(1)

98% of students linked $\int_a^{2a} \lambda(y)\mathrm{d}y$ to a physical quantity. Only one student attempted to evaluate the integral, not giving a physical interpretation. Thirty-one students

(65%) correctly said that the integral gave the charge on the rod between a and $2a$. This is double the percentage of correct responses compared to the corresponding charge integral in the pretest.

Sixteen students (33%) appear to look at the integrand only (neglecting the effect of the infinitesimal term on the integral), and infer that the result of the integration must be a charge density. Seven of these students (15% of the total) had identified that $\int_{-2a}^{2a} \mathrm{d}y$ gave the length of the rod, but yet did not consider how the infinitesimal term would affect the meaning of this second integral.

Conclusions:

Overall, there was an obvious improvement in students' answers to this post-test question compared to the corresponding pretest question, despite there being no instruction specifically relating to interpreting integrals.

Almost all students gave a physical interpretation for the integrals in the post-test, an increase of 30% from the pretest where many students just evaluated the integrals. This shows that these students at least now see integration as more than just a "machine" and the product of integral as having a physical meaning.

It is unlikely that this is just the result of students being "trained" by the pretest, since the percentage of students correctly interpreting the integrals doubled. This shows that more students do understand that the infinitesimal term is a physical quantity and how it affects the meaning of the integral. However, there are still a large number of students who ignore the infinitesimal term when the integrand is a function. Although no attention was given solely to the infinitesimal term during instruction, students had used them in explaining the set-up of integrals in many context, as described in Section 3.6. Students' continued poor understanding of the infinitesimal term suggests that instruction should explicitly address this aspect of the integral.

3.7.1.3 Concept image categorisation of students' reasoning

This part of the analysis was concerned with placing students explanations and reasoning into the concept image categorisation that we had developed for the concept image pretest.

Students' responses are shown in Table 3.9. The responses were analysed under the same categories used in Table 3.6. No students mentioned area under the curve, definite integral, or anti-derivative in their explanations and so these categories were removed. Unlike in the other concept image analyses all students only used one type of reasoning for both integrals.

Table 3.9: Categorization of students' interpretations of two integrals.

Category	$\int_{-2a}^{2a} \mathrm{d}y$ (N=48)	$\int_{a}^{2a} \lambda(y)\mathrm{d}y$ (N=48)
sum	32 (15)	35 (17)
evaluation	21 (10)	2 (1)
verbalized integral	4 (2)	2 (1)
explaining terms	6 (3)	25 (12)
other	6 (3)	2 (1)
no explanation	31 (15)	33 (16)

Summation was the only conceptual aspect of integration mentioned in students explanations (i.e. no students mentioned area under the curve unlike the corresponding pretest). Approximately 35% of the students (a 20% increase from the pretest) described integration as a sum although again not all with the correct physical interpretation of the integral. For the length integral, students explanations it this category were similar to:

> *"sum of all length segments* $\mathrm{d}y$*"*

> *"rod is divided into small pieces* $\mathrm{d}y$ *and you are summing up all these small pieces"*

and for the charge integral were similar to:

"Charge is linear charge density times the length, here you are taking the charge on a small piece of the rod $\mathrm{d}y$ *and summing all the individual charges on the* $\mathrm{d}y$ *pieces"*

"integral finds the linear charge density over many different points along the length and adds them together"

The technical aspects of integration were not mentioned regularly in students reasoning for either integral. Only 10% evaluated the length integral and only 1% evaluated the charge integral. This shows that integration does not trigger evaluation as strongly any more.

Explaining the terms in the integral, without attempting to describe the process of integration was again seen in students' responses, especially for the charge integral:

"Charge is equal to linear charge density by length"

Over 30% of students did not provide an explanation for their interpretation.

Overall, significant differences can be seen in students concept image of integration between the pretest and post-test. Firstly, there is an increase in the percentage of students describing integration as a sum, with approximately one-third of students describing it as such post-instruction.

There is also a large decrease in the amount of students evaluating the integrals, down to 10% for the length integral and 1% for charge integral. All students who evaluated the length integral, also correctly interpreted the integral as giving the length of the rod. For the charge integral, the decrease may be partially be explained by students now understanding the meaning of $\lambda(x)$ and hence that $\int \lambda(x)\mathrm{d}x$ cannot be integrated.

All other students do not describe the process of integration in any way, with over 30% not attempting to explain their interpretation at all. This suggests that although students no longer evaluate when they see an integral, not many students can use conceptual aspects to describe the process of integration in a meaningful way.

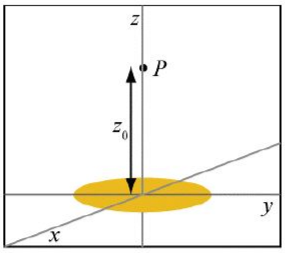
3.7.2 Cueing integration

3.7.2.1 Question

In a previous post-test question described in Section 3.3, we saw that integration was not cued for half the students when asked to calculate the magnetic field at a point due to a current-carrying ribbon. We also found that the dependency cue seemed to be the cue most frequently used by students and we hypothesised that the low number of students integrating could be a result of students not understanding integration as a process of summation.

In this post-test question, now that instruction had focused on integration as a process of accumulation, we wanted to investigate if integration would be cued more frequently when students were asked to calculate physical quantities. The question is shown in Figure 3.6. Integration is required to calculate three quantities asked for in the question: the total charge on the disk, the potential at the centre of the disk and the magnitude of the electric field at a point a distance directly above the centre of the disk. By asking for the three different physical quantities it allowed us to determine if context affected whether or not integration was cued for students. It should be noted here that students had completed a similar charge calculation in tutorial (the only difference being the charge distribution) and an electric field due to a uniformly charged rod calculation in tutorial. Past examination papers contain electric field calculations in various settings, while charge calculation requiring integration had not been asked previously. Potential calculations like this had not been covered in tutorial but similar questions had appeared on past exams.

A circular disk of radius R and negligible thickness has a surface charge density given, in polar coordinates, by $\sigma = \alpha r^2$. The disk lies in the x,y-plane with its centre at $z = 0$ as shown at right.



a) **[6 marks]** Calculate the total charge on the disk in terms of the given quantities. Explain and show your work.

b) **[8 marks]** Calculate the potential at the centre of the disk. Explain and show your work.

Consider a point P, located on the z-axis at $z = z_0$.

c) **[8 marks]** Show that the z-component of the electric field at point P due to a small segment of length dr and angular width dϕ, located on the disk at (r,ϕ), is given by
$$\frac{1}{4\pi\varepsilon_0} \cdot \frac{\alpha z_0 r^3 \mathrm{d}r\mathrm{d}\phi}{\left(r^2 + z_0^2\right)^{3/2}}.$$

d) **[8 marks]** Calculate the magnitude of the electric field at point P due to the entire disk. Explain and show your work.

Figure 3.6: Post-test question on the calculation of charge, potential and electric field

3.7.2.2 Results

The number of students who used integration for the calculation of charge and electric field is shown in Table 3.10. Students' answers to the potential calculation turned out to be difficult to analyse in this way. Although 16% answered using the correct method, 18% used the definition for potential difference, $\int \vec{E} \cdot \mathrm{d}\vec{l}$, and attempted to evaluate this integral. Another 16% incorrectly stated that the potential at the centre of the disk would be zero because the electric field is zero there, and so would be unlikely to use integration.

Table 3.10: Integration cued for the calculation of charge, potential, and electric field

Question	Integration cued %(N=56)
Charge	38(21)
Electric field	77(43)

Table 3.10 shows that a low number of students integrate when calculating charge, yet over three-quarter of the students integrate to find the electric field. For the electric field calculation the question provided students with the expression for the electric field due to a small segment of the disk (dE), whereas dQ is not given for the charge

calculation. It is possible that the infinitesimal terms triggered integration or that seeing $\mathrm{d}E$ indicated that the segment is part-of-a-whole and hinted towards the use of integration to sums these parts to get the whole. We acknowledge that in the magnetic field calculation question described in Section 3.3 the infinitesimal quantity was also provided and in that case only half the students integrated. The increase here could be explained by more students understanding integration as a summation, which combined with seeing $\mathrm{d}E$ may have cued integration. As students typically did not provide explanations with their answers we cannot say this for certain.

Another possible reason for the difference is that students may not have understood $\sigma = \alpha r^2$ as being a varying charge density. For students relying on the dependence cue this would explain integration not being cued. Two students that calculated the charge on the disk by multiplying the charge density by the area of the entire disk even stated that the charge density is uniform having used $\sigma = \alpha r^2$:

> *"since it is not stated otherwise, I am assuming the disk has a uniform charge density"*

> *"$Q = \sigma A$ for a uniform disk"*

We also cannot rule out the possibility that students practised past examination papers and that students who used integration for electric field and not for charge were simply using the recall cue.

3.7.3 Technical Difficulties

The students who used integration still experience technical difficulties in setting up the integrals (even when the corresponding infinitesimal quantity was given for the electric field), and in evaluating them. Half the students (11/21) who integrated for the charge calculation had the correct expression for $\mathrm{d}Q$. The most common incorrect expression was $\mathrm{d}Q = \alpha r^2 \mathrm{d}r$. This could be due to students' familiarity only considering one dimension when calculating charge through their regular use of $Q = \lambda L$.

Again only half the students (23/46) who integrated for the electric field calculation could explain how to obtain the expression for the z-component of the electric field due to a small segment of the disk. Seven students (15%) started with an incorrect formula, either EA or $\vec{E} \cdot \mathrm{d}\vec{l}$. For six students (13%), it was the finding of the z-component that was the difficulty, while for four students (8%) this was the only aspect of the expression that they explained. Three students (7%) added the infinitesimal term to their answer without explaining where it comes from.

66% and 33% of students evaluated the charge and electric field correctly, respectively. For charge, this percentage includes those who evaluated their incorrect expression for $\mathrm{d}Q$ correctly. This increase compared to the magnetic field question is most likely due to students not having to consider how to accumulate the quantity in either case, with the z-component of electric field already having been taken into consideration for $\mathrm{d}E$.

Table 3.11 shows the types of integration errors made by the other students.

Table 3.11: Errors made during integration

Error	**Charge %(N=21)**	**Electric Field % (N=43)**
No limits	5(1)	9(4)
Incorrect limits	5(1)	4(2)
No process attempted	5(1)	7(3)
Mistake in process	20(4)	47(22)

Compared to the magnetic field question, limits were not near as big a problem for students, most likely because the lower limit is 0 in this case. A similar percentage of students make a mistake during the evaluation process again compared to the magnetic field question. For the electric field question there was a large percentage of students that when substituting in the lower limit, 0, let that whole piece of the expression equal to 0 even though it was not.

3.8 Conclusions

In this study we have investigated students' concept image of integration, if and how integration is cued in a physics context, how students interpret integrals in a physical setting, and the technical difficulties students have in using integration. We have also developed instruction that describes integration as a process of summation.

Prior to specific instruction on integration in this course, students' concept image of integration consisted mainly of technical aspects of integration with evaluation being the most common and most applied aspect in and out of a physics context. Very few students mentioned integration as a summation. In the post-test, evaluation was no longer a prominent piece of students' concept image and the percentage of students describing integration as a process of summation increased. This has helped in students ability to interpret of integrals, where in pretest when asked to do so many students evaluated instead of providing a physical meaning for the integral.

Before students took tutorials on integration in the course, it was found that when calculating physical quantities integration was cued in only half the students. In a more recent post-test there was an increase to 75% of students cueing integration for an electric field question, although the exact cause of this increase is unclear.

The meaning of the infinitesimal term and how it changes an integral, the setting up of integrals, and the consideration of how a quantity should be accumulated before an integral is evaluated were other aspects of integration that our students were found to have difficulty with.

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Chapter 4

Vector Addition

4.1 Introduction

Vector addition is a mathematical process regularly required in electromagnetism problem solving. Problems involving electric and magnetic forces and fields frequently involve contributions from more than one source and so students will need to know how to add these.

An example of this type of problem is the calculation of the magnetic field at a point above a current-carrying thin ribbon, like the post-test question described in Chapter 3. In that case, we found that over 80% of students who integrated the given expression for the magnetic field due to a small segment of the ribbon ($\mathrm{d}B$) failed to consider how the infinitesimal contributions due to each segment of length $\mathrm{d}x$ along the ribbon should be added. There are a number of possible explanations for why this was the case: because students may not see integration as a process of summation, when integrating they were not thinking about adding the $\mathrm{d}B$ vectors contributions along the ribbon; students may not realise that the magnetic field is a vector quantity; they may not know how vectors should be added; they may be unable to calculate vector components due to difficulties with geometry or algebra.

If not understanding integration as a process of accumulation was the root cause of students not considering how the magnetic fields should be added, then this problem may already have been addressed by the instruction described in Chapter 3. However, if it is as a result of any or a combination of the other three possible explanations then further action is required, as perceiving and treating electric and magnetic fields as vector quantities is vital for the conceptual understanding and calculation of almost all other electromagnetic quantities. This chapter reports on an investigation of students' understanding of vector addition.

Before students enter this intermediate electromagnetism courses they will have encountered and added multiple vector quantities, e.g. displacement, velocity, acceleration in mechanics and even electric and magnetic fields in either an introductory electricity and magnetism course or as part of their introductory general physics course. Vector components and vector addition would also have been covered in their mathematics course. In this study, pretest questions were designed to examine students' ability to (i) add two vectors without a context and (ii) calculate vector components before instruction. Students' conceptual understanding of vector addition within the context of electric force was tested both before and after instruction. A numerical question that combines this conceptual understanding and component calculation was also asked as a post-test. Analysis of students' answers to these questions highlighted the main mathematical and conceptual difficulties students have adding vector quantities and comparison of pretests and post-tests indicated aspects of the instruction that were beneficial and those that appeared not to be effective.

Relevant previous findings on students' difficulties in treating physical quantities as vectors and the addition of vector quantities are discussed in Section 4.2. Section 4.3 describes the instruction students received on vector addition in and out of the electric force context before this investigation began and the resulting changes to instruction. Section 4.4 shows our students' ability to add vectors, Section 4.5 discusses students' conceptual understanding of vector addition, and Section 4.6 looks at how students answer numerical problems that involve vector addition. The main conclusions and implications of our findings are outlined in Section 4.7.

4.2 Previous findings

Nguyen and Rebello[1] report that in a number of electricity problems involving integration, almost all students in their study started to integrate the infinitesimal quantity without considering how the quantity should be added up. When calculating the electric field due to a charged arch, half of their students did not appear to notice the vector nature of $\mathrm{d}E$, integrating the whole of $\mathrm{d}E$. The authors do not provide a real insight into what they think is at the root of this problem, but it seems that they consider it to be a result of students' lack of understanding of the integration process.

In a study by Kanim[2] investigating students' understanding of vectors in electrostatics, the author found that students had difficulty reasoning about net electric forces and fields from a number of point sources and that this difficulty was increased when trying to reason about field and forces from continuous charge distributions. Flores et al[3] report that in interviews using the context of electrostatics many students struggled to reason qualitatively about vector addition and about the relationship between vectors and their components. Like Kanim[2] their students were found to make mistakes when reasoning how the magnitude of a resultant depended on the directions of the individual forces.

Flores et al[3] also investigated students use of vectors in a mechanics context. They found that after traditional instruction, when asked to find a net force or an average acceleration, students did not consider the vector nature of these quantities at all. When students were asked similar questions after specifically modified instruction, most students seemed to recognize that they were adding vectors but used various incorrect methods and reasoning about vector addition.

Some studies have examined students' skills in manipulating vectors without using a physics context both before and after instruction in calculus-based introductory physics courses.[3–5] Knight[5] administered his *Vector Knowledge Test* to determine how sophisticated his students' starting knowledge of vector mathematics was. He found that, when asked to define a vector in their own words, only 60% of the student cohort had some idea of what a vector was, despite the majority of students stating that they had studied vectors previously. He also found that only half his students could add vectors and that only 40% could recognise and use vector components. The studies by Nguyen and Meltzer[4] and Flores et al[3] showed that students' ability to carry out two-dimensional vector addition does not improve much after traditional instruction. Nguyen and Meltzer[4] found that students were often able to find the direction of the resultant vector using the head-to-tail or the parallelogram rules without being able to determine the correct magnitude. Flores et al[3] saw that students frequently and persistently applied the Pythagorean theorem inappropriately. The authors also state that considering the students' lack of skills in adding vectors, skills that are often required when tackling questions involving vector quantities, it is unsurprising that students do not consider the vector nature of these quantities.

4.3 Instruction

This section describes the two different sets of instruction on vectors, electric force, and electric field given to students throughout the four years of this study. In years one and two students received the same lecture instruction and completed the same tutorials. As a result of findings from the analysis of students' answers to the pretest and post-test questions described in Sections 4.4, 4.5 and 4.6, changes were made to the instruction in years three and four.

4.3.1 Year one and year two

Relevant instruction in these years consisted of a lecture, where multiple electric force and electric field calculations were shown, and two tutorials, the first focusing only on the addition of vectors through the head-to-tail and components methods, and the second being a conceptual look at the addition of forces due to multiple charge contributions. The first tutorial, *Vectors* was developed by the lecturer and the second tutorial, *Charges*, was adapted from Tutorials in Introductory Physics.[6]

4.3.1.1 Lecture

Electric force was the first vector quantity introduced. Students were reminded that the net electric force on a charge is the vector sum of all individual forces: $\vec{F}_{on1} = \vec{F}_{12} + \vec{F}_{13} + ...$ An example of a force calculation was then shown. The question involved the calculation of the resultant force on one charge due to two other charges when the three charges were arranged as shown in Figure 4.1 (all charges had a different magnitude). The direction of each force was indicated first and then Coulomb's law was applied to find the magnitude of each force. The vectors were added component-wise, but the calculation of the each component was not gone through stepwise. Instead the value of each component was given and used to find the resultant. A similar process was followed to find the electric field at point X due to the three point charges.

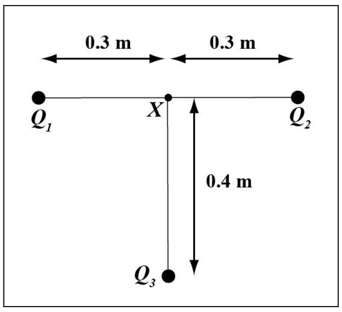


Figure 4.1: Arrangement of three charges for electric force and field calculations

4.3.1.2 *Vectors* tutorial

This tutorial followed the lecture described above. The full tutorial is given in Appendix I. The tutorial aimed to remind students of and improve their understanding of the head-to-tail and component methods of vector addition. Students used a number of unit vectors to describe the position of one point in relation to a reference point, first in one dimension (where the points lie along the same line as the unit vector they are using to describe its position) and then in two dimensions. In the two dimensional case the point is moved to a location that can be reached using the original unit vector and its perpendicular unit vector. Students are asked to draw the unit vectors head-to-tail from the reference point to reach the point the new location in two different ways and then to write an expression for the position vector in terms of the unit vectors. Students are asked to describe what they think is meant by the term vector addition.

Next students described a vector, $\vec{t}$ in terms of a horizontal unit vector, $\widehat{x}$ and a vertical unit vector, $\widehat{y}$. By doing this students build on what they had used in the first part of the tutorial to move towards the more conventional way of describing vectors using horizontal and vertical components. Students then wrote the x- and y-components of $\vec{t}$ in terms of the unit vectors and then used $\vec{t_x}$ and $\vec{t_y}$ when writing the expression for the length of the vector in terms of its components and the length of $\vec{t_x}$ in terms to t and the angle between $\vec{t}$ and $\vec{t_x}$.

In the last part of the tutorial, students added two vectors on a grid using the head-to-tail method. They were then asked to draw the x-components of each individual vector and the resultant vector and to write an equation that showed the relationship between these three vectors. They repeated the same for the y-components and were asked to explain that the method of adding components is an alternative way of adding vectors.

4.3.1.3 *Charges* tutorial

This tutorial followed the *Vectors* tutorial and is shown in full in Appendix J. In this tutorial students are introduced to the superposition of forces due to multiple point sources firstly, and then due to a continuous charge distribution. Students apply Coulomb's law initially to a pair of charges q and Q separated by a distance s, and are then asked to consider student statements comparing the force on q in that initial situation to when there are two more Q charges placed the same distance from q, shown in Figure 4.2. One student states that says that the force will be three times as large because there are now three charges exerting equal forces, while the other student states that the net force will remain the same because the force due to the two outside charges will cancel. Students should find here that neither student is correct and that only the horizontal component of the outside charges will cancel, leaving the vertical components which when added will be larger that the original force but will not be three times as large.

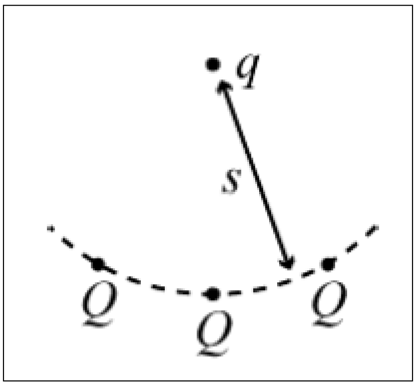


Figure 4.2: Three Q charges placed an equal distance from q

This idea is reinforced in the next part where students rank the electric force in three different situations shown in Figure 4.3. It also combines the three aspects students need to consider when calculating force, i.e. magnitude of the charges, distance, and direction.

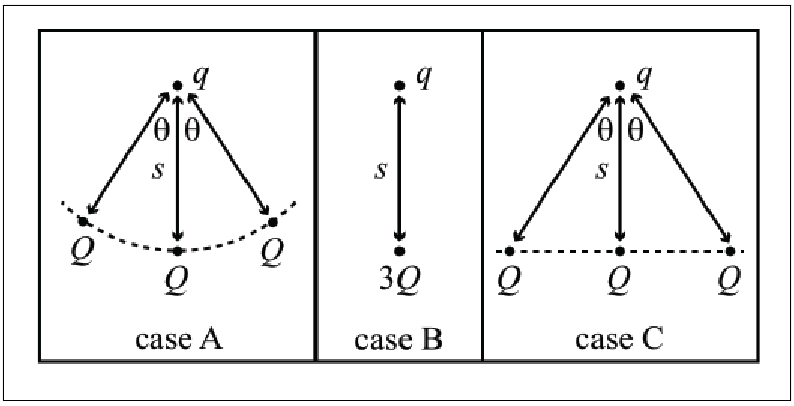


Figure 4.3: Three different arrangements of charges for force comparison

Students then compare the force in three more situations: one where there is a Q directly below q; another where Q is replaced with three $Q/3$ charges, one the same distance directly below q and the other two are placed the same distance either side of that $Q/3$; and another where there are five $Q/5$ charges arranged in the same fashion. This builds up the reasoning required for a continuous charge distribution which students then consider when they compare the force on the q charge when it is directly above a point charge of charge Q and when it is directly above the centre of a uniformly charged rod of total charge Q.

4.3.2 Year three and year four

As mentioned before, changes to instruction were made based on analysis of students' answers to pretest and post-test questions. The changes to the lecture and *Charges* tutorial are described in this section. The *Vectors* tutorial was omitted after post test results, which are presented later in this chapter, suggested that it had not been effective in improving students' use of vectors when reasoning about or calculating electric forces and fields.

4.3.2.1 Changed lecture

Electric forces were introduced in the same way as the original lecture. When electric force is first described in terms of a vector sum, there was an additional slide showing the addition and subtraction of two vectors using the head-to-tail method. There was an example electric force question, and only one electric field calculation similar to the question in the original lecture. However, the solution was carried out in a step-wise fashion showing all the pieces necessary for finding the components of the field: the use of trigonometry to relate the field vector and its components using an angle in the right-handed triangle they make; identifying a 'usable' angle using similar triangles; and expressing the cosine and sine in terms of the known lengths. The electric field at a point above the centre of a uniformly charged rod is also considered conceptually in a similar way to the old *Charges* tutorial, and then the calculation is carried out.

4.3.2.2 Changes to the *Charges* tutorial

This tutorial now followed the lecture. It is shown in Appendix K. This first part remained unchanged. After students rank the electric force on q in the three cases in Figure 4.3, students now evaluate the net force in each case. They write the expression for the force due to $3Q$ themselves and are then asked to derive the expressions $\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{s^2}(2\cos\theta + 1)\widehat{y}$ for case A and $\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{s^2}(2\cos^3\theta + 1)\widehat{y}$ for case C. This gives students practice at component calculation within the context, directly following a conceptual treatment of the situation. Students are asked to show that their derived expressions for the net force in each case are consistent with the ranking they gave previously. This gives students the opportunity to interpret the expressions, seeing that the larger the angle the smaller the net force becomes, and to get more at ease with the geometry and algebraic manipulations.

The addition of the evaluation section meant that there was no longer time for the force due to a continuous charge distribution section. Instead, a similar approach was taken to this in lecture (mentioned above).

4.4 Testing students' mathematical ability to manipulate vectors

Pretest questions were designed to help distinguish a general lack of skills in manipulating vectors from difficulties that are more specific to the context of electric force and field. Previous studies[3;4] have shown that many students do not have the skills required to perform basic vector operations after completion of standard introductory courses similar to those our students have taken. We wanted to find out if this was true for our students.

Component calculation in an electrostatics context is arguably more difficult than in mechanics. Due to the way in which electric force and field vectors are drawn, students need to use similar triangles in order to determine a 'usable' angle for calculating the magnitude of the components. So we also wanted to examine students' ability to complete the process of component calculation using the context of electric force.

4.4.1 Vector addition

In all four years of this study students were administered the same pretest question testing their understanding of vector addition. The question is shown in Figure 4.4. Students were asked to give an expression that linked the three vectors, $\vec{a}$, $\vec{b}$, and $\vec{c}$. Students were expected to see that when the vectors $\vec{a}$ and $\vec{b}$ are added, either using the head-to-tail method or component-wise, the resultant vector is $\vec{c}$. Therefore the correct relationship is some form of $\vec{c} = \vec{b} + \vec{a}$.

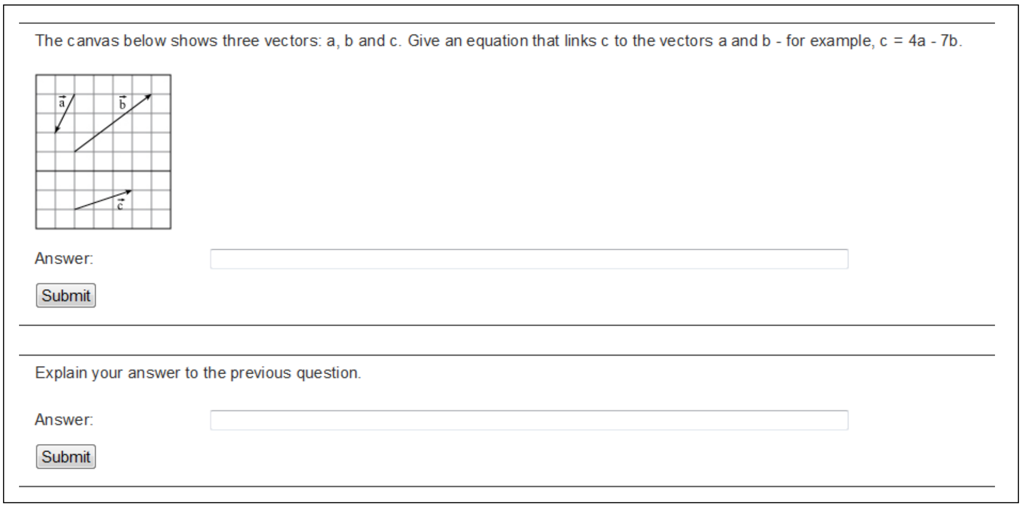


Figure 4.4: Pretest testing students' ability to add two vectors

Over the four years a total of 186 students answered this pretest question. Table 4.4 shows the types of relationships between the three vectors that students identified. Students' responses from years 1 and 2, and years 3 and 4, are grouped together because in each of these year sets the lecture students received prior to answering the pretest was the same and a χ^2 test showed that there was no statistical difference between the years.

Table 4.1: Analysis of student answers for the vector addition pretest question

Category	**Years 1 and 2 %(N=87)**	**Years 3 and 4 %(N=99)**
$\vec{c} = \vec{b} + \vec{a}$	48(42)	44(44)
$\vec{c} = \vec{b} - \vec{a}$	15(13)	20(20)
Other forms of addition and subtraction	24(21)	28(28)
Other/No answer	13(11)	7(7)

4.4.1.1 $\vec{c} = \vec{b} + \vec{a}$

In both sets of years, less than half the students identified the correct relationship between the three vectors. This indicates that even this basic general form of vector addition is a difficulty for students. The reminder example of vector addition and subtraction shown in the lecture (described in Section 4.3.2.1) did not increase the number of correct responses. This suggests that students' difficulty with adding vectors is not that easily fixed. Even after instruction where an emphasis was placed on graphical vector manipulation Flores et al[3] found that only about two-thirds of their students gave correct answers for both magnitude and direction when adding two vectors.

The head-to-tail method of adding vectors was a popular method used by students in both sets of years, with answers either stating that this was the method used or attempting to explain this method:

> *"When vectors $\vec{a}$ and $\vec{b}$ are added nose to tail the answer is $\vec{c}$"*

> *"When you add $\vec{a}$ and $\vec{b}$ head to tail the displacement is the same as $\vec{c}$"*

> *"If one joins $\vec{a}$ to $\vec{b}$ and joins the two other ends you get $\vec{c}$"*

Students using component type reasoning, either gave explanations that explicitly mentioned the vectors' x and y components, or talked about how many units to the left or right, up or down, the vectors were:

> *"$\vec{a} = -x - 2y$, $\vec{b} = 4x + 3y$, $\vec{c} = 3x + y$. I decomposed the vectors into orthogonal components and solved"*

> *"Sum of individual vertical and horizontal components"*

> *"The effect of $\vec{b}$ is to move up 3 and right 4, $\vec{a}$ moves down 2 and left one, combined this gives up 1 and right 3 which is the same as $\vec{c}$"*

Some students either just stated that the relationship was the result of *"vector addition"* or repeated the relationship in words:

> *"Vector $\vec{a}$ added to vector $\vec{b}$ gives vector $\vec{c}$"*

Other students gave explanations that are mostly correct, but suggest that they are not using a particular method:

> *"Vectors include direction so they end up subtracting when directions are included"*

> *"As the vectors are going in opposite directions by adding them you will get a shorter vector with a net direction in the direction of $\vec{b}$"*

> *"Direction and length of $\vec{a}$ make $\vec{b}$ less steep"*

The frequency of each type of reasoning in both sets of years is shown in Table 4.2.

Table 4.2: Students' reasoning for why $\vec{c} = \vec{b} + \vec{a}$

Reasoning	**Years 1 and 2 %(N=42)**	**Years 3 and 4 %(N=44)**
Head-to-tail	36(15)	52(23)
Components	30(13)	25(11)
Other	33(14)	23(10)

In the second two years there was an increase in the number of students using head-to-tail reasoning. This is possibly due to the example showing vector addition using this method given in the changed lecture.

4.4.1.2 $\vec{c} = \vec{b} - \vec{a}$

Between 15% and 20% of students linked the three vectors in this way. For half of these students, their explanations suggest that this answer is a result of incorrectly using one of the correct methods for vector addition. For those attempting to use the head-to-tail method here, it is likely that they use $-\vec{a}$ because $\vec{a}$ is in a 'negative' direction:

> *"Moving from the origin to $\vec{b}$ and the following the negative path of $\vec{a}$ leaves you at $\vec{c}$"*

"If you walk vector $\vec{b}$ and then $\vec{a}$ you end up at $\vec{c}$'s arrow"

For those students, using component reasoning, it seems that they neglected to consider $\vec{a}$ as being negative:

"$b = (4,3)$, $a = (1,2)$, so $b - a = (3,1)$ which is c"

It seems that some students subtracted the vectors because $\vec{c}$ is smaller than $\vec{b}$:

"Because vector $\vec{c}$ is smaller than $\vec{b}$ we know that there is subtraction used"

"$\vec{c}$ is greater in magnitude than $\vec{a}$ but less than $\vec{b}$"

Again, some students write the relationship in words as an explanation or explain it as a result of the *"laws of vectors"*.

4.4.1.3 Other forms of vector addition and subtraction

Answers in this category were of the form of the example given to students in the question, where there was a numerator before both vectors $\vec{b}$ and $\vec{a}$. Most relationships given only occurred once. Most students did not provide an explanation for their answers here and when they did their reasoning was difficult to interpret. For example one student said that the relationship between the three vectors was $\vec{b} - 3\vec{a}$ because it was

"the sum of multiple vectors".

For a few students, there was evidence that they were attempting to use correct vector addition methods:

"$1/3\vec{a} + 1/3\vec{b}$, eventually they will collide with each other and form a triangle"

"$7\vec{b} - 3\vec{a}$, the x and y components of vector $\vec{a}$ are subtracted from the x and y components of vector $\vec{b}$"

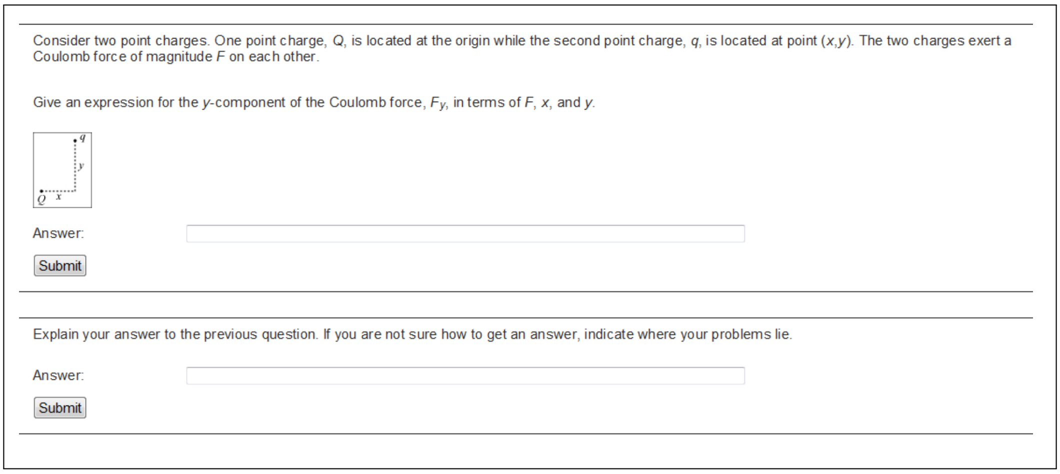
4.4.1.4 Conclusions

We have found that vector addition was a difficulty for the majority for our students at this stage, with only 45% correctly identifying the relationship between the three vectors and only 35% reasoning correctly about the head-to-tail or component methods for adding vectors. Approximately 20% used vector addition methods incorrectly, demonstrating that they had some starting knowledge of the process, but were unable to execute it. That leaves a large portion of students who were unable to answer sensibly; and for many their answers seemed to be guesswork.

4.4.2 Component Calculation

4.4.2.1 Question

The pretest designed to investigate students' ability to calculate vector components in an electrostatics context is shown in Figure 4.5. The question was asked in Year 3 only. Students needed to complete multiple steps to correctly answer this question: recognise the direction of the force vector; draw the x and y components of the force vector, completing a right-angled triangle; choose an angle and identify the corresponding angle in the original triangle (similar triangles); write the sine or cosine of that angle (depending on their chosen angle) in terms of the known quantities. Students should have reached the expression $F_y = F.\frac{y}{\sqrt{x^2+y^2}}$. They were asked to indicate what their difficulties were if they were unsure how to answer.



Consider two point charges. One point charge, Q, is located at the origin while the second point charge, q, is located at point (x,y). The two charges exert a Coulomb force of magnitude F on each other.

Give an expression for the y-component of the Coulomb force, F_y, in terms of F, x, and y.

Answer:

Submit

Explain your answer to the previous question. If you are not sure how to get an answer, indicate where your problems lie.

Answer:

Submit

Figure 4.5: Pretest testing students' ability to calculate vector components

4.4.2.2 Results

Forty-eight students answered this pretest question. Only four of these (8%) gave the required expression. Their explanations were not detailed, only stating that

> *"the vertical component is equal to $F.\sin\theta$"*

Five students (10%) knew that they need to use an angle to write F_y in the required terms, but they multiplied by the the cosine of the wrong angle. Three of these then substituted the cosine correctly ($\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$) while one substituted incorrectly, $\cos\theta = \frac{x+y}{x}$, and one did not substitute at all.

Fifteen students (31%) used expressions from the lecture notes. The most popular of these was the y component of the electric field expression from the example with a similar structural setting to the pretest question. Most students admitted to taking the expression from the notes. Those who tried to explain it, correctly explained the distance part (Pythagoras) but stated that they did not know where the rest came from. One student even stated:

> *"I'm not sure if this is just for the y component, if it is then why is there*

x's in it?"

Another student who gives the expression $\frac{1}{4\pi\varepsilon_0}\frac{Q}{y^2}$ says

> *"we are looking at the y component so we sub in y into Coulomb's equation"*

Even though these students consulted their lecture notes where there was a component calculation detailed in the solution, they still could not explain where the terms from the component calculation from.

Two students (4%) identified a correct relationship between the the force vector and its constituent vectors, $F_y^2 = F^2 - F_x^2$. Although this is not the required expression, it shows that these students at least knew the relationship between a vector and its components.

Another nine students (18%) gave various other expressions like

> *"$F_y = 1.5x$"*
>
> *"$F_y = y - x$"*
>
> *"$F_y\widehat{y}$"*

Finally, thirteen students (27%) said that they did not know. Five students did not give any indication as to what their problem with the question was, three students said that they did not understand the question, and five students identified that the difficulty in the question was in calculating the y component for various reasons:

> *"$F = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$ is the possible equation. Not sure how to get it in terms of y"*
>
> *"Don't understand the concept of the x and y axis with charge"*

4.4.2.3 Conclusions

Although the way in which the question was asked could have allowed for some interference from difficulties with the physics involved, the low percentage of students who answered correctly and the wide range of other answers given indicates that component calculation is a mathematics skill that many students are struggling with. All of these students had encountered electric force calculations prior to this that would have involved the calculation of components. Admittedly before now students were more than likely given a value for the required angle and possibly would have applied what they perceived to be a general rule that when looking for the x-component you multiply by the cosine of that angle and for the y-component you multiply by sine of the angle. When asked to derive this like they were in this question, the majority of students appear to be lost.

4.5 Conceptual tests

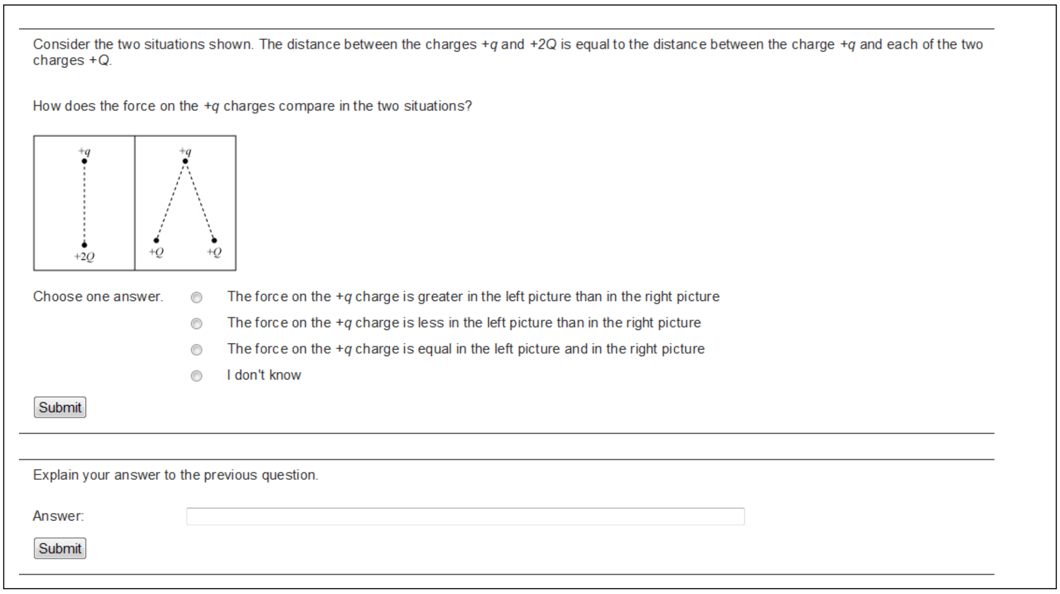
As mentioned previously, one of our main motivations for investigating if students understand and correctly use vectors in an electromagnetism context was the large number of students who did not consider the vector nature of the magnetic field when integrating $\mathrm{d}B$. We wanted to ask conceptual questions that did not involve integration to determine to what extent this was caused by students' understanding of the quantity as a vector.

Reasoning about net electric forces due to a collection of point charges and continuous charge distributions has been previously identified as a source of difficulty for students[2;3]. We wanted to investigate if this posed a problem for our students and if so how effective was the current instruction at improving their reasoning about net forces.

4.5.1 Pretest

4.5.1.1 Question

In years one and two students were given the conceptual pretest question shown in Figure 4.6. The question asked students to compare the force on a test charge in two situations: one where a $+2Q$ charge was placed directly below the test charge; and another where two test charges, each of magnitude $+Q$, were placed the same distance from the test charge but were separated by an angle. The total charge and the distance are the same in both situations. The only way students can rank the magnitude of the net force is by considering the direction of each individual force in the second situation and recognising that the horizontal components of the force vectors will cancel when added, and the vertical components add to a smaller vertical force.


Consider the two situations shown. The distance between the charges $+q$ and $+2Q$ is equal to the distance between the charge $+q$ and each of the two charges $+Q$.

How does the force on the $+q$ charges compare in the two situations?

Choose one answer.

- The force on the $+q$ charge is greater in the left picture than in the right picture
- The force on the $+q$ charge is less in the left picture than in the right picture
- The force on the $+q$ charge is equal in the left picture and in the right picture
- I don't know

Submit

Explain your answer to the previous question.

Answer:

Submit

Figure 4.6: Pretest testing students' conceptual knowledge of vector addition in a physics context

4.5.1.2 Results

The type of reasoning students used in their explanations is shown in Table 4.3. A χ^2 test showed that there was no statistical difference between the two years, and so only the totals are used in the description.

Table 4.3: Types of reasoning used by students for the conceptual pretest question

Types of reasoning	Year 1 %(N=43)	Years 2%(N=44)	Total %(N=87)
Components (correct)	5(2)	11(5)	8(7)
Consider direction (correct but incomplete)	14(6)	11(5)	13(11)
Adding force as a vector (incorrect)	9(4)	14(6)	12(10)
Adding force (not as a vector)	14(6)	14(6)	14(12)
Considering elements for force calculation	37(16)	23(10)	30(26)
Other	19(9)	25(11)	22(19)
Don't know	2(1)	2(1)	2(2)

It appears that only one third of students recognised that force is a vector quantity, and only 21% knew how to treat the vectors to correctly determine that force in the first situation was greater that the net force in the second situation. 8% of all students explained that this was due to the cancellation of the horizontal components in the second situation:

> *"Each charging Q is exerting a force on the charge q, but in the second picture the x-components cancel each other out, and only leave the y-components. In the first picture only the y-components are relevant as all charges lie in the same line along the y-axis. So the force is equal to $F = (1/4\pi\varepsilon_0)2Qq/r^2$ in the first picture, but is less in the second".*

> *"The $2Q$ force is perpendicular so all its force is in the vertical component, each of the Q charges lose force to horizontal components which cancel each other out."*

A further 13% of students used reasoning that is incomplete, but know that the direction the forces act in is important and that the net force due to the two $+Q$ charges will be less than the force due to the single $+2Q$ charge:

"Angle reduces force in the vertical direction"

"In the left picture both charges act in the same direction so all of the force acts to repel q. In the right some of the forces are acting in different directions, these forces act to cancel each other out and so the net force is less."

11% of students mentioned that force is a vector and in some cases talked about the addition of vector components, yet concluded that the net force in both situations will be equal:

"Because both Q charges on the right picture are repelling q, the net horizontal charge cancels out, leaving only the vertical component, like the left picture."

"The addition of the component vectors for the 2 $+Q$ charges in the right picture will equal to the vector force acting on the $+q$ charge in the left sided picture"

"The vectors added will be the same."

Another 14% of students thought the combined force of the two separate $+Q$ charges will be the same as the single $+2Q$ charge, treating the forces as scalar quantities:

"The same force is being applied in both situations just in different format."

"If you work out the individual force for each of the $+Q$ charges and add them it is the same for the diagram on the left."

The largest category contained students who were looking at the contributing factors in a force calculation. Most students in this category stated here that because the total charge or distance from the charges was the same in both situations, the force must be the same. Sometimes it was difficult to determine whether these students were reasoning as in the previous category, or whether they were adding the charges first and then used Coulomb's law:

> *"Both $+q$ charges have $2Q$ charge acting on them from the same distance away"*
>
> *"The $+Q$ charges are summed to give $2Q$."*
>
> *'Because the distance is equal and the charge is the same, the force is the same."*

The remaining 20% either did not provide an explanation or gave explanations that were incorrect and difficult to categorize, for example:

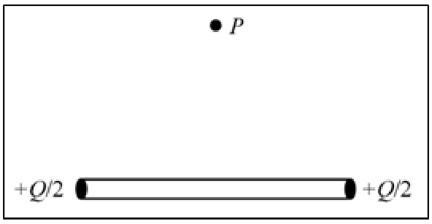
> *"Force between two charges not affected by outside charges"*
>
> *"if the charge was greater in the left hand picture it would create a greater distance as the identical forces would repel further with greater magnitudes of charge"*

4.5.2 Post test

4.5.2.1 Question

Figure 4.7 shows the conceptual post test question given to students mid-semester in Year 1. The question involves a continuous charge distribution which Kanim[2] found added extra difficulties when students were trying to reason about force vectors. The question asks students to compare the direction and magnitude of the electric field at a point, P, a distance above the centre of a rod with total charge $+Q$, when the rod is uniformly charged and when $+\frac{Q}{2}$ is located at each end of the rod. When considering the direction a complete answer would have involved reasoning that in both cases the horizontal components of the electric field due to charge distributions on either side of the centre of the rod will cancel and the vertical components are added to give a net electric field that points straight up. For the magnitude, students should have reasoned that it would decrease when the charge is moved to either end of the rod because now all the charge is a greater distance from point P, and the angle between the two electric fields is greater. Hence the magnitude of the electric force decreases and the vertical component is smaller.

A test charge is placed at rest at point P directly above the centre of a uniformly charged rod of charge $+Q$. Suppose the charge distribution on the rod is changed so that each end of the rod has a charge $+Q/2$, as indicated by the shaded regions in the diagram at right.



a) Would the direction of the electric force exerted on the original test charge at point P change? If so, how? Explain.

b) Would the magnitude of the electric force exerted on the original test charge at point P change? If so, how? Explain.

Figure 4.7: Post-test testing students' conceptual knowledge of vector addition in a physics context

4.5.2.2 Results

Direction:

82% of students correctly stated that the direction of the electric force would not change. Two-thirds of our students treated the electric force as a vector. Only 7% merely stated that this was the result of *"vector addition"*, and 59% used component reasoning to explain that the direction of the net force did not change when the charge distribution changed:

> *In both cases there is always a cancellation of the components in the x-direction thus leaving only the vertical component*

Another 7% of students correctly stated that the direction will not change and made correct inferences about the situation. However, they did not give evidence of using vector reasoning:

> *charge distribution of each end of the rod is the same*

> *Each end acts similar to a point charge*

9% did not give an explanation for their answer.

14% of our students stated that the direction of the force would change. Their reasoning suggests that either they do not understand the situation presented in the question, or other aspects of the concept of force.

Magnitude:

Only 36% of students consider the vector nature of the electric force when considering if its magnitude would change. 25% correctly use components to reason that the magnitude of the electric force would be less when the charge is moved to either end of the rod:

> *"It would be less, as on the uniformly charged rod there are more charges toward the center giving a larger y-component of the electric force'"*

> *"The force vectors in the second scenario will have a maximum x-component... the electric force is "wasted" in the x-direction as they cancel, therefore the magnitude of the y-component of force is affected by this and is less"*

34% of students only considered distance to explain the decrease in the magnitude of the net force:

> *"The magnitude would decrease because although we still have the total $+Q$ charge it is now coming from two points that are further away so as distance increases, force decreases"*

The decrease in students using component reasoning for the magnitude could be due to students not deeming it necessary to provide a second reason once they had identified that the net force was less because of the increased distance. This argument is supported by the fact that seven of the eleven students who used correct component reasoning did not mention the distance as part of their explanation. Another possibility is that some students who had used component reasoning for the distance may not have felt that is was necessary to do so again.

It is possible that although students could reason through components for the direction, reasoning about the decreasing magnitude of the vertical component due to the increasing angle was too large a step for students to make, causing them to ignore the vector nature of the force for this case. However, this is unlikely because almost all students that used component reasoning for the direction reasoned correctly (if incompletely) about the decrease in magnitude of the net force.

25% of students gave various incorrect answers that did not include looking at either the distance or the addition of vectors. In some cases not considering the force as a vector quantity appeared to main difficulty:

> *"The magnitude is the same, because the charge has remained the same although it's now coming from different directions its magnitude hasn't been increased or decreased"*

However, it seemed from some students' explanations that they had misinterpreted the physical situation, reasoning that the net force had decreased because *"at the ends the charge is less"*, or that the net force increased because *"the charge increased"*.

4.5.3 Conclusions and implications

Our findings from the conceptual pretest question confirm that our students experience difficulties reasoning about net electric forces and fields, similar to those reported by Kanim[2] and Flores et al[3]. Two-thirds of our students did not consider the vector nature of the electric force and only 8% correctly explained their answer using components. For 11% of students recognizing that the force was a vector was not enough to stop them adding the individual forces like scalars.

When comparing the direction of the net electric force in the post-test question, two-thirds of students identified or treated the electric force as a vector quantity, with 59% of students using component reasoning. Compared to the pretest, that is double the percentage of students who considered the vector nature of the electric force, and a 50% increase in the use of vector components.

The percentage of students using component reasoning when considering the change in the magnitude of the electric force decreased to 36%, meaning that although 70% of students stated correctly that the magnitude of the electric force would be less for the two $+Q/2$ charges, less than 10% gave a complete explanation using both distance and component reasoning. Reasoning based on distance alone was the more common response. This shows that although many students now treat electric force as a vector and more students can talk about vector components correctly for direction, students are still not inclined to use components when considering magnitude.

As stated previously this conceptual question was made difficult because of the continuous charge distribution, and overall there was a significant improvement in reasoning about the addition of electric forces. This indicates that the *Charges* tutorial had a positive impact on students' understanding of the nature of electric forces.

4.6 Numerical tests

A numerical post-test question, that requires the combined conceptual knowledge of vector addition and the ability to calculate vector components, was asked on the midterm exam in the first three years of this study. This question allowed us to determine the overall effectiveness of the instruction because it tests both the conceptual and technical aspects of vector addition in an electric field context.

4.6.1 Question

The question that was given in Year one is shown in Figure 4.8. Students were asked to calculate the magnitude of the net electric field at a point P due to a $+3pC$ point charge and a $-2pC$ point charge. There are multiple steps students needed to take here: they needed to apply Coulomb's law separately for each point charge; recognise that because the electric field is a vector quantity, the electric field due to each point charge must be added component-wise; to find the components of each vector students needed to use trigonometry and similar triangles; lastly the corresponding components of each vector needed to be summed and the resultant found through the use of Pythagoras' theorem.

Two charges Q_1 and Q_2 are placed on a grid as shown. Q_1 has a positive charge of +3 pC and Q_2 has a negative charge of –2 pC. (1 pC = 10^{-12} C.) Each square on the grid is 1 cm^2.

a) Calculate the magnitude of the net electric field at point P. (Values of physical constants are given at the end of the paper.)

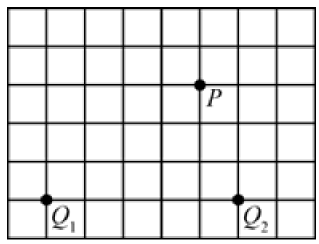


Figure 4.8: Question on the numerical calculation of the electric field due to two point charges

In Years two and three the question was changed so that the $-2pC$ charge was placed directly below the point P. This was done mainly to reduce the amount of time that would be needed to complete the question, without simplifying the question. Past midterm exam papers are not made available to students and they will not have seen this exact question previously. However, many text books show very similar worked examples.

4.6.2 Year one and Year two results

A combined total of 90 students answered this question in Years one and two. A categorization of their answers are shown in Table 4.4. A χ^2 test showed that their was no statistical difference between the two years. Calculation mistakes such as not converting the distance from cm to m or incorrect conversions when the distance had been squared were not taken into consideration here.

Table 4.4: Categorization of students' answers to the numerical question

Category	**%(N=90)**
Fully correct method	17(15)
Correct component calculation	10(9)
Incorrect component calculation	9(8)
Scalar addition	40(36)
Other/No answer	24(22)

Only 17% of students used a fully correct method taking all the steps described above. A further 10% correctly calculated components but left their answer in the form $a\widehat{x} + b\widehat{y}$, not finding the magnitude of the resultant force. It is possible that these students did not see the need to complete this final step. Another 9% recognised the need for component calculation but attempted various incorrect methods to do this. For most students this involved multiplying the expression for the electric field by an incorrect trigonometric function, in some cases even $\tan\theta$. However, some students did not use trigonometry at all, instead using the 'horizontal distance' from the point charge to point P in the electric field formula when calculating the x-component, and the 'vertical distance' from the point charge to point P in the when calculating the y-component. These students for the x and y components of the electric field of Q_1 then looked like this: $E_x = \frac{1}{4\pi\varepsilon_0}\frac{3pC}{(0.4)^2}$ and $E_y = \frac{1}{4\pi\varepsilon_0}\frac{3pC}{(0.2)^2}$.

40% of our students calculated the electric field at point P due to each point charge and then added the magnitudes as scalars. For some students there was evidence that they had considered the vector nature of the field, either drawing the field and its components, or writing $E_x = E\cos\theta$, $E_y = E\sin\theta$. This suggests that for some students adding the fields as scalars was caused by a lack of understanding of how to approach a component calculation. It is also interesting to note here that in Year one when the conceptual question described in Section 4.5.2 was asked in the same exam, 25% of students treated the field as a scalar in the numerical question, yet reasoned about the direction of the net electric force using vector components in the conceptual question.

The Other category includes students who calculated the electric field due to one or both of the charges only. Most of the students in this category, however, did not calculate the electric field due to either point charge correctly. Some students used Coulomb's law, substituting in the magnitude of both point charges and using the distance that one of the point charges was from point P. A small number of students added the charges and used this value and the distance that one of the point charges was from point P for their calculation of the electric field. It seems that most of the student in the Other category do not understand the concept of electric field, are unaware of how to use Coulomb's law, or both.

Overall, only 27% of students correctly calculated the vector components of the electric field. While 40% appeared not to consider the vector nature of the electric field when adding the two fields, it is possible that for some of these students this is caused by their lack of knowledge of how to approach a component calculation. 10% attempted the component calculation process unsuccessfully. All this suggests that the instruction had been unproductive in improving students' ability to add vector quantities. Perhaps this is to be expected, considering that many students cannot successfully complete component calculations at the start of the course (indicated by their answers to the vector component pretest question described in Section 4.4.2), and how component calculation was addressed in the instruction. Despite the *Vectors* tutorial looking at the use of trigonometry to relate a vector to its components, it did not carry the level of difficulty that is involved in doing this in the context of electric force and field. During instruction students had not been given much opportunity calculate net forces and fields.

We decided that more emphasis needed to be placed on component calculation in the context in which it needed to be used. The lecture was changed to include a step-by-step solution to a field calculation. The *Vectors* tutorial was omitted and an extra section was added to the *Charges* tutorial where students got to practice the technical aspects of a force calculation directly following a conceptual introduction (changes discussed in more detail in Section 4.3.2.

4.6.3 Year three results

Table 4.4 was extended to include students' answers to the same question from Year three. Students' answers mostly fit the same categories, though there was a change in the distribution of these answers among the categories. One extra category where students only calculated one component and appeared to neglect the other was added to Table 4.5. It should be noted for the comparison that in Year three students had performed the same in the vector addition pretest.

Table 4.5: Categorization of students' answers to the numerical question

Category	**Years one and two %(N=90)**	**Year three %(N=61)**
Fully correct method	17(15)	23(14)
Correct component calculation	10(9)	11(7)
Only one component calculated	-	10(6)
Incorrect component calculation	9(8)	25(15)
Scalar addition	40(36)	11(7)
Other/No answer	24(22)	19(12)

Although there was almost no change in percentage of students completing a fully correct calculation, there was a significant increase in the percentage of students who attempted to calculate components. There was a rise in the number of students who calculated the components by changing the distance in the electric field as described above. We are unable to explain why students believe this to be a method for component calculation. A similar percentage of students appear not to understand the concept of electric field, and/or are unaware of how to use Coulomb's law.

Changes to instruction seem to have been moderately successful. When students could use Coulomb's law the majority treated the electric field as a vector. Component calculation, although no longer hindering students viewing the electric field as a vector, is still a difficulty for students.

4.7 Conclusions

The results of investigations we have carried out in this part of the study indicate that students neglecting to consider how infinitesimal contributions to a net magnetic field should be added is definitely caused by more than students' lack of understanding of integration as a summation. In more basic contexts, even after instruction, students often do not treat quantities as vectors and they struggle to reason about and calculate vector components.

Like previous studies[3–5], we found that even after completion of an introductory physics course, many of our students struggled with vector addition at the start of the course, with just over 40% correctly adding two vectors not placed in a physical context. Component calculation involving the use of trigonometry and similar triangles proved to be even more difficult for students.

In a conceptual electric force pretest question, only one-third of students treated the force as a vector quantity and only 8% reasoned completely using components. For 14% of students recognizing that force is a vector quantity did not stop them treating it as a scalar when adding. After instruction where students used the head-to-tail and components methods of adding vectors and reasoned conceptually about the addition of forces for multiple point charges and continuous charge distributions, two-thirds of students treated the electric force as a vector quantity using component reasoning to determine the direction of the net force in a conceptual force question.

However, when asked about the magnitude of the net force the number of students using the vector nature of the force in their reasoning decreased, and in a numerical question 40% of students added the electric field due to two point charges as if they were scalars. Changing the instruction so that students got the chance to practise the calculation of net forces along with using conceptual reasoning, did increase the percentage of students treating the electric force as a vector quantity in a numerical question, but the calculation of components is still a difficulty for students.

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Chapter 5

Dot Products and Integration

5.1 Introduction

A complete understanding of Gauss' Law and Ampère's Law are two of the ultimate learning outcomes of any electromagnetism course. Central to these two laws are the mathematical tools of integration and dot products. In Gauss' Law the dot product is used to find the magnitude of the electric field perpendicular to the surface and integration is used to sum infinitesimal products $(\vec{E} \cdot \widehat{n})\mathrm{d}A$ when the electric field through that surface is varying in magnitude or direction. Similarly, in Ampère's Law both tools are required to calculate the line integral of a magnetic field.

Students face many complexities when building an understanding of Gauss' Law and Ampère's Law, and considering situations that require the application of these laws[1–4]. Struggling to interpret the meaning of $\vec{E} \cdot \widehat{n}$ and $\vec{B} \cdot \mathrm{d}\vec{l}$ would hinder them further in this process. Chapter 4 identified that students often fail to recognise quantities as vectors and have difficulty reasoning about vector components and calculating them, and this suggests that the idea of a dot product would also cause problems for students. When this is combined with the conceptual and procedural difficulties students have with integration (discussed in Chapter 3), it is likely that students would struggle to both reason conceptually about flux and line integrals, and to calculate these quantities. Section 5.2 outlines previous findings on students' difficulties with Gauss' Law and Ampère's Law relevant to the ideas discussed in this chapter; how students reason about dot products in the context of work; and difficulties students have with interpreting and evaluating graphical representations of the dot product.

Section 5.3 shows results from a post-test question involving the calculation of simple line integrals of a magnetic field which confirms that the combined requirement of dot products and integration proved indeed to be problematic for students. Since Chapter 3 already focussed on integration, this chapter will examine students' ability to use dot products in an electromagnetism context. Sections 5.4 and 5.6 present students' answers to both conceptual and technical dot product questions before and after newly designed instruction. This new tutorial used the context of work to develop students' conceptual understanding of dot products. A description of the tutorial and how it was developed is given in Section 5.5. Lastly, Section 5.7 compares students' answers to two similar calculating electric flux post-test questions from before and after the new dot product instruction. These questions also are a pre/post for the integration instruction and so students use of both tools will be discussed for both questions.

5.2 Previous findings

Studies have shown the many pitfalls that exist when students have to apply Gauss' Law and Ampère's Law in physics problems. Guisasola et al[1] describe a question where students were asked if they agree that Gauss' Law can be simplified from $\oint\oint \vec{E} \cdot \mathrm{d}\vec{S} = \frac{q}{\varepsilon_0}$ to $E = \frac{q}{S\varepsilon_0}$ to give the electric field on a Gaussian surface that surrounds a charge q. They found that the majority of students agree, not taking into account the pattern of field lines. In a corresponding Ampère's Law question, they found the same and they suppose that students consider the B and $\mathrm{d}l$ vectors are parallel and that the magnetic field is constant along the path.

Lindsey et al[5] and Loverude et al[6] have investigated students' understanding of work, their ability to apply it to systems, and the ability to relate it to energy and the ideal gas law. Both of these studies identified that when students were asked about the sign of the work students often only considered the direction of the force relative to the co-ordinate system. Also, some students associated work done only with forces perceived to be active. Lindsey et al[5] explain most of students' incorrect reasoning about work as a result of a lack of an operational definition for work. The operational definition the authors seem to advocate for is in terms of energy, rather than the dot product of force and displacement.

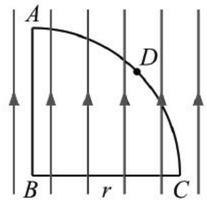
In a study by Christensen et al,[7] students' understanding of the graphical representation of the dot product was tested after traditional instruction. Students were given three pairs of vectors each with different angles between them and were asked to rank the dot product of each pair. The authors found that when the angle between the vectors was 90° or less approximately 75% of their students could rank the dot products correctly, however this decreased to 60% when the angle between one pair of vectors was made greater than 90°. Their students generally answered correctly when they used a component method but students frequently used an equation representation, similar to $\vec{a} \cdot \vec{b} = |veca|.|vecb|. \cos\theta$, with varying degrees of success. Students using this equation usually failed to reach a negative dot product reasoning that the equation takes the absolute value of the vector and the cosine of the angle is not negative.

5.3 Line integrals of the magnetic field

5.3.1 Loop in Uniform Magnetic Field Question

Figure 5.1 shows the entire Loop in Uniform Magnetic Field Question. A loop consisting of two perpendicular line segments of length r and a quarter circle of radius r is placed in a uniform magnetic field of magnitude B_0. Students were asked ques-

A flat conducting loop $ABCA$ consists of a quarter circle of radius r and two straight line segments of length r. It is placed in a uniform magnetic field $\vec{B}$ of magnitude B_0 in the orientation shown.

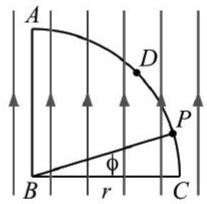


(a) **[4 marks]** What is the value of $\oint \mathrm{d}l$, where the integration is to be taken around the entire loop? Explain briefly.

(b) **[4 marks]** Evaluate the integral $\int_A^B \vec{B}\cdot \mathrm{d}\vec{l}$.

(c) **[4 marks]** Evaluate the integral $\int_B^C \vec{B}\cdot \mathrm{d}\vec{l}$.

Point D is located midway along the arc CA.



(d) **[4 marks]** Is the integral $\int_C^D \vec{B}\cdot \mathrm{d}\vec{l}$ *greater than, less than,* or *equal to* $\int_D^A \vec{B}\cdot \mathrm{d}\vec{l}$? Explain.

(e) **[5 marks]** To evaluate $\int_C^D \vec{B}\cdot \mathrm{d}\vec{l}$, first consider the point P along the arc CD. What angle does the vector $\mathrm{d}\vec{l}$ make with the magnetic field at point P? Use a diagram to obtain your answer.

(f) **[6 marks]** Evaluate $\int_C^D \vec{B}\cdot \mathrm{d}\vec{l}$. Show your work.

Figure 5.1: Loop in Uniform Magnetic Field Problem

tions on the circulation of $\mathrm{d}l$ and the line integral of the magnetic field along each of the three segments. Because the magnetic field is uniform this question does not require students to use much physics knowledge.

5.3.2 Circulation of $\mathrm{d}l$

Question (a) asks students to calculate the circulation of $\mathrm{d}l$. The question was designed to test students' understanding of integration and circulation. To answer correctly, students must realize that the integral is independent of the magnetic field and asks them to calculate the perimeter of the loop.

Table 5.1 shows a categorization of students' answers.

Table 5.1: Students' calculation of the circulation of $\mathrm{d}l$

Category	**%(N=44)**
Correct length of loop	27 (12)
Incorrect length of loop	11 (5)
Length of loop (no value)	5 (2)
0	23 (10)
Other	11 (5)
No answer	23 (10)

43% identified that the circulation of $\mathrm{d}l$ gave the length of the loop, but only 27% gave a correct value for the length, $2r + \frac{\pi}{2}r$. Incorrect expressions for the length included 2, $\frac{1}{4}$, and r^2. Most students did not provide an explanation but those who did used sum reasoning:

"summation of segments of length that make up the loop"

"integral adds up all the tiny lengths $\mathrm{d}l$ *for the entire loop"*

23% of students stated that the circulation of $\mathrm{d}l$ is zero. 5% appear to have interpreted $\mathrm{d}l$ as a displacement $\mathrm{d}\vec{l}$:

"loop is complete so $\mathrm{d}l$ *is zero, position has not changed"*

"ends up at the same point, $\int_A^B \mathrm{d}l + \int_B^C \mathrm{d}l + \int_C^D \mathrm{d}l + \int_D^A \mathrm{d}l$*"*

The other 18% do not understand the physical meaning of $\mathrm{d}l$ and have reasoned about the integral in terms of magnetic flux, Ampère's Law, work, or potential:

"any magnetic field lines that enter the loop also exit the loop"

"no enclosed current"

"work done is zero, it ends back where it started in a uniform field"

From the Other category, 5% treated the circulation of $\mathrm{d}l$ as a line integral reasoning about the relative directions of the magnetic field and the displacement:

"From A to B, direction of motion is opposite to the magnetic field lines, therefore $\vec{B} \cdot \mathrm{d}\vec{l}$ is negative..."

Hence less than half of our students could correctly interpret the circulation of $\mathrm{d}l$, which is a difficulty we have seen previously (discussed in Chapter 3). Even when they understood the physical meaning of the integral, some struggled to evaluate it in this context.

5.3.3 Uniform magnetic field antiparallel to direction of integration

Question (b) asks students to evaluate the integral $\int_A^B \vec{B} \cdot \mathrm{d}\vec{l}$. To answer this question correctly, students must recognize that, in general, evaluating the line integral requires them to calculate the projection of the magnetic field on an infinitesimal displacement $\mathrm{d}\vec{l}$, calculate the product of this projection and the length of $\mathrm{d}\vec{l}$, and add these products for all infinitesimal segments between the starting point A and end point B. In other words, students must realize that they have to calculate $\int |\vec{B}||\mathrm{d}\vec{l}| \cos\theta$, and that θ is the angle between $\vec{B}$ and $\mathrm{d}\vec{l}$. In this case, $\vec{B}$ and $\mathrm{d}\vec{l}$ are constant along the path and antiparallel, so the integration simplifies to a multiplication of the magnitude of the magnetic field, the segment length, and a factor -1. Therefore the line integral has a value $-B_0 r$. However, our classroom experience suggests to us that beyond these cognition-related considerations, students overcoming their fear of integrals and dot products is an equally important requirement.

Table 5.2 summarises students' responses to the question.

Table 5.2: Students' evaluation of the integral $\int_A^B \vec{B} \cdot \mathrm{d}\vec{l}$

Category	**%(N=44)**
Negative, Correct expression	5 (2)
Negative, Incorrect expression	18 (8)
Positive, Correct expression	5 (2)
Positive, Incorrect expression	27 (12)
0	20 (9)
Other	2 (1)
No answer	23 (10)

We first consider issues related to the sign. About 20% of our students (10/44) responded that the line integral was negative. However, only two of these obtained the correct expression $-B_0r$; and only one of those gave an explanation that shows an understanding of the relevance of the direction of the vectors in calculating a dot product:

> *"motion in opposite direction to field lines"*

The other seven students all reasoned that the relative direction of the two vectors was opposite or considered that the cosine angle between the two vectors equals -1, but failed to obtain the correct answer:

> "$-B_0\frac{1}{2}(B^2 - A^2)$, $\cos 180° = -1$"

> "-1, *dot product of parallel vectors=1, and they are in opposite directions*"

This suggests that although these students can reason about the relative direction aspect of the dot product, possible problems with the physics concepts involved, the interpreting of the physical setting, and integration prevent students from obtaining the correct expression.

About 30% of our students (14/44) gave positive expressions for the line integral; in two cases, this expression was B_0r. Two of these students considered the relative direction of the two vectors, but stated that the two were in the same direction. This suggests that these students interpreted the limits or the "direction" of the integral incorrectly. The other 12 students did not provide explanations but it is possible that they just did not consider the relative directions of the two vectors when evaluating the dot product.

Another 20% (9/44) stated that the value of the line integral was zero. Five of these students considered the direction of the two vectors. However, their explanations suggest that they interpreted the line integral as flux:

"BA is parallel to the magnetic fields, so no intersection"

"A to B is parallel to the field lines"

About 25% of the students (10/44) did not provide an answer at all.

Many students encountered difficulties with the integration aspect of the question. A quarter of the students (10/44) carried out the integration and substituted the limits B and A. These students appear not to have realized that A and B are points rather than coordinates, as they obtained expressions such as $B_0(B - A)$.

Students also encountered difficulties with the physics. About 15% (7/44) assumed a non-uniform magnetic field and attempted to calculate it using the Biot-Savart law. These students clearly do not understand the situation presented in the question.

5.3.4 Uniform magnetic field perpendicular to direction of integration

By and large, the same considerations apply to question (c). To evaluate this line integral correctly, students had to recognize that $\vec{B}$ and $\mathrm{d}\vec{l}$ are perpendicular to each other at all points along the path, and so the dot product of the two vectors, and hence the value of the line integral, is zero.

About 40% of the students (17/44) correctly stated that the line integral was zero. About 25% (11/44) explained that this was the case because $\vec{B}$ and $\mathrm{d}\vec{l}$ are perpendicular to each other or that $\cos 90° = 0$. Three students gave no explanation, while the other three gave incorrect explanations that seem to relate the line integral to potential difference and work:

"As no work is done and both points have the same potential"

"When we go from B to C we end up at a point with the same value magnitude and when taken away from each other we get zero"

"$Br - Br$'

About 20% of the students (8/44) gave the same expression as they had given for the first line integral, with students who had the length $(B - A)$ for the fist line integral changing it to $(C - B)$.

9% (4/44) gave similar incorrect expressions, but had stated that the first line integral was 0. It is possible that these students have confused the angle for which the cosine gives 0.

Fifteen students (34%) did not answer this part of the question.

5.3.5 Uniform magnetic field with varying orientation with respect to the direction of integration

5.3.5.1 Comparison of $\int_C^D \vec{B} \cdot \mathrm{d}\vec{l}$ and $\int_D^A \vec{B} \cdot \mathrm{d}\vec{l}$

In question (d), students are asked to compare $\int_C^D \vec{B} \cdot \mathrm{d}\vec{l}$ and $\int_D^A \vec{B} \cdot \mathrm{d}\vec{l}$. To obtain a complete and correct answer, students needed to recognize that the angle between $\vec{B}$ and $\mathrm{d}\vec{l}$ varies, and that the angle increases continuously along the path from C to A. Hence the component of the magnetic field in the direction of the infinitesimal displacement $\mathrm{d}\vec{l}$ is greater anywhere along the path from C and D than along the path from D to A, and therefore $\int_C^D \vec{B} \cdot \mathrm{d}\vec{l}$ is greater than $\int_D^A \vec{B} \cdot \mathrm{d}\vec{l}$. Equivalently, students may reason that the cosine of the angle between $\vec{B}$ and $\mathrm{d}\vec{l}$ decreases continuously between C and A, with the same conclusion.

Table 5.3: Comparison of $\int_C^D \vec{B} \cdot \mathrm{d}\vec{l}$ and $\int_D^A \vec{B} \cdot \mathrm{d}\vec{l}$

Category	**%(N=44)**
Greater than	27(12)
Less than	18(8)
Equal to	25(11)
Other	2 (1)
No answer	27 (12)

27% of the students (12/44) correctly stated that $\int_C^D \vec{B} \cdot \mathrm{d}\vec{l}$ is greater than $\int_D^A \vec{B} \cdot \mathrm{d}\vec{l}$. Almost all of these students (11/44) gave an explanation that shows that they used correct reasoning:

> *"From C to D the displacement* $\mathrm{d}l$ *is much more in the direction of B"*

> *"The vertical component of* $\mathrm{d}l$ *for* $\mathrm{d}l$ *from C to D is bigger than from D to A"*

> *"The angle that B makes with* $\mathrm{d}l$ *along the loop between C and D is smaller than the angle B makes with* $\mathrm{d}l$ *between D and A"*

Only one of these 12 students used incorrect reasoning:

> *"D to A is the sum of negative integrals, C to D is the sum of positive integrals"*

Another 25% of our students (11/44) stated that the line integrals were equal. About 20% (8/44) appeared not to take into account the relative direction of the vectors, mentioned these in an unspecified way, or stated they were the same. In each case, they argued that the field and the length are the same for the two integrals:

> *"B is constant and the length of the segments are equal therefore their products are equal"*

> *"as both here are the same distance and are in the same direction"*

> *"area and angle are the same"*

These responses suggest that when the students see a dot product they realize that the (relative) directions of the vectors are important, but that they cannot use this knowledge meaningfully because they cannot interpret the physical situation.

A further 20% of students (8/44) stated that $\int_C^D \vec{B}\cdot \mathrm{d}\vec{l}$ is greater than $\int_D^A \vec{B}\cdot \mathrm{d}\vec{l}$. Some 10% (5/44) interpreted the integral as flux, and based their responses on the number of field lines "passing through" each path:

> *"there are more field lines through the section of the loop AD"*

"AD has a greater horizontal component for field lines to pass through"

A quarter of the students (12/44) did not answer this question and one student gave an answer that was difficult to interpret.

5.3.5.2 Calculation of $\int_C^D \vec{B} \cdot \mathrm{d}\vec{l}$

Questions (e) and (f) assess whether students can calculate the line integral of the magnetic field along part of a circular arc. In question (e) they needed to consider the angle between $\vec{B}$ and $\mathrm{d}\vec{l}$ at any point along the path in terms of an auxiliary variable, the angle with the horizontal axis ϕ. This question requires students to have mastered basic geometry, and be able to apply it to this situation. Question (f) assesses students' ability to carry out the integration for a varying angle between magnetic field and integration direction. To answer this question correctly, students needed to identify this variability. They were expected to write each $\mathrm{d}\vec{l}$ in terms of the corresponding infinitesimal angular change $\mathrm{d}\phi$, i.e., to use the equality $\mathrm{d}l = r\mathrm{d}\phi$, write the dot product for an infinitesimal length as $B_0.r\mathrm{d}\phi.\cos\phi$, and then integrate this expression between the limits 0 and $\frac{\pi}{4}$. 70% did not give a response to question (f). One student obtained the correct expression for the line integral using the method described above. 9% set up the integral correctly but did not include the limits, nor did they attempt to evaluate the integral. The remaining 19% did not consider the angle in between the two vectors. 7% simply multiplied the magnitude of the magnetic field by the length of the piece of the loop ($B_0\frac{\pi}{4}r$). The remainder gave various incorrect answers that were difficult to interpret.

Most of the students who did not answer question (f) had not answered question (e). For question (e) students may not have been able to picture what was being asked of them, and this may have then prevented them attempting question (f).

5.3.6 Conclusions

Even in the simplest cases presented (where the angle between the magnetic field and the path was constant along that path) approximately one quarter of the students were able to correctly determine the sign of the dot product. The same percentage of students were able to reason about how the varying angle the path made with the magnetic field would affect the magnitude of the line integral along the curved part of the loop. This suggests that there are serious problems with students' understanding of the dot product.

In terms of integration, only 44% correctly interpreted the circulation of $\mathrm{d}l$ as the length of the loop, and only 27% were able to correctly determine the value for the length. This is in line with the findings of Chapter 3. When required to integrate a dot product, only 9% correctly evaluated the magnitude of the first line integral and only one student could do the same for the curved part of the loop. Despite the situation being undemanding in terms of the physics, students still seemed to have difficulty interpreting the situation, with many using a Biot-Savart type expression for the magnetic field. It is also worth noting that students who used non-constant expressions for the magnetic field did not attempt to integrate.

Anecdotally we know that students find it difficult to work with the line integral of the magnetic field, because to them it has no direct physical interpretation (unlike the line integral of the electric field, which they can link to the potential difference between the end points of the integration). We have also, in previous chapters, identified difficulties with integration. To investigate the extent to which students' inability to calculate these type line integrals was caused by difficulties with the dot product, we administered conceptual and technical pretest questions that used a more familiar context and did not involve integration.

5.4 Dot product pretests

This section presents students' answers to pretest questions designed to test their understanding of the dot product. Before entry to the course students would have experienced the use of dot products in mechanics for work calculations and in their mathematics courses. In lecture, the dot product had been introduced generally, as the projection on one vector along the other and the expression $\vec{a} \cdot \vec{b} = |\vec{a}|.|\vec{b}|. \cos\theta$ was explained. In the same lecture, students were re-introduced to the term, *normal*, and the dot product was used in the calculation of the flux through a sphere due to an enclosed charge, as a means of introducing Gauss' Law. In a another lecture given before the conceptual question work was described as a line integral.

5.4.1 Technical pretest

5.4.1.1 Question

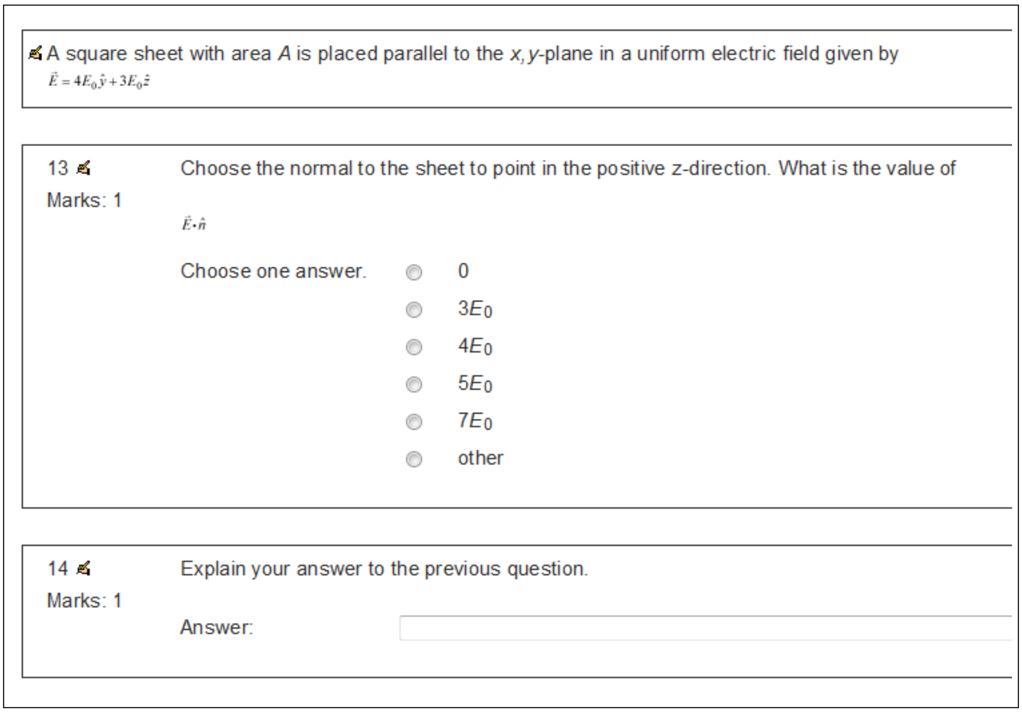
This pretest question, shown in Figure 5.2, examined students' basic technical knowledge of dot products. Students were given the expression for a uniform electric field in terms of its y- and z-components, $4E_0\widehat{y} + 3E_0\widehat{z}$, and were asked for the value of $\vec{E} \cdot \widehat{n}$ when a sheet whose normal is in the positive z-direction is placed in the electric field. The question was multiple choice so students were given the options 0,$3E_0$, $4E_0$, $5E_0$ and $7E_0$.

To answer this question students would not need to understand the meaning of the two vectors. Students only needed to know that the dot product is the projection of one vector along the other, or in other words that it is the component of one vector that is in the same direction as the other vector. The components of each vector were given, and so students with the required knowledge of the dot product should have seen that as the y-component and the normal were perpendicular their dot product was 0 and because the other component was in the same direction the dot product was $3E_0$ in this case. The distracters were chosen based on the different ways could

Table 5.4: Students' evaluation of $\vec{E} \cdot \widehat{n}$

Category	**%(N=31)**
0	10(3)
$3E_0$	45(14)
$4E_0$	13(4)
$5E_0$	10 (3)
$7E_0$	13 (4)
Other	6(2)
No answer	3(1)

possible manipulate the electric field expression or incorrectly apply the dot product. Students were also asked to explain their answer.


A square sheet with area A is placed parallel to the x,y-plane in a uniform electric field given by $\vec{E} = 4E_0\hat{y} + 3E_0\hat{z}$

13
Marks: 1

Choose the normal to the sheet to point in the positive z-direction. What is the value of

$\vec{E} \cdot \hat{n}$

Choose one answer.

- 0
- $3E_0$
- $4E_0$
- $5E_0$
- $7E_0$
- other

14
Marks: 1

Explain your answer to the previous question.

Answer:

Figure 5.2: Technical dot product pretest question

5.4.1.2 Results

Thirty-one students responded to this pretest. Table 5.4 shows their choices for the value of $\vec{E} \cdot \widehat{n}$.

Only 45% of students chose the correct value for the dot product of the two vectors.

Two students (6%) gave a complete explanation, using technical reasoning:

> *"The dot product of $\vec{E}$ with $\widehat{n}$ is got by multiplying the two x-components, then the two y-components and finally the two z-components. As $\vec{E}$ has no x-component we get 0 for multiplying the x's, we get the same result for the y's as $\widehat{n}$ has no y-component but for the z we get $(1)(3) = 3$"*

26% stated that $3E_0$ is the component that is in the z-direction but did not explain why this is the component that contributes to the value of the dot product and why the y-component does not contribute in this case:

> *"The normal is in the z-direction and has a value of 1 so multiplied with the $3E_0$ in the z-direction is $3E_0$"*

The other 13% either did not explain their choice, or gave explanations that suggest that their choice was a guess:

> *"because the $3E_0$ is in the equation"*

The 10% who stated that $\vec{E} \cdot \widehat{n}$ was 0 seemed not to consider the electric field vector:

> *"the normal is perpendicular to the area"*

13% of students chose $4E_0$, with only one these providing an explanation. This student stated that because both vectors contained a z-component that these terms would somehow cancel each other leaving only the y term:

> *" because the equation has z in it and $\widehat{n}$ has z in it so it takes the value of $\widehat{z}$ away so we are left with $4E_0\widehat{y}$"*

10% of students do not consider $\widehat{n}$ at all, instead they calculate the resultant electric field using Pythagoras' theorem ($5E_0$). Another 13% simply add the magnitudes of the two components of the electric field to get $7E_0$. Most of these students do not really explain their answer but just verbalize what they have done. However one student has attempted to find the dot product:

“ Inner product is the sum of products $\widehat{n}.4E_0\widehat{y}$ *and* $\widehat{n}.3E_0\widehat{z} = 7E_0$*”*

One student only considered the normal vector stating that as it is a unit vector it will have a magnitude of 1, and another student stated that we need more information.

5.4.2 Conceptual pretest

The chosen context for the dot product instruction was the concept of work (reasons for choice will be described in Section 5.5). Although the tutorial used a mechanics context, we wanted to investigate if students would consider the relative directions of force and displacement when considering the work done by an electric force, and if they would be able to reason correctly about how the relative directions influenced the work done by the force.

5.4.2.1 Question

In this question students considered the work done by an electric field when a test charge is moved along a path perpendicular to that electric field. This path was chosen to make it easier for us to determine if students were reasoning using relative directions. Some students may determine the sign of the work by considering only the direction of the force or the displacement, students only looking at the force might reason that the work is positive because the net force is pointing upwards, or those looking at displacement may state that it is negative because the charge is moving to the left. Also because the magnitude and direction of the electric field does not change along the path, it eliminates the difficulty of considering an integral.

Students were presented with the situation shown in Figure 5.3 and were asked if the work done by the electric field due to the two point charges is positive, negative, or zero when a positive test charge moves from B to D. To answer the question correctly, students must see that at all points along the path from B to D the electric field is perpendicular to the integration path, and hence the work done by the electric field is zero.

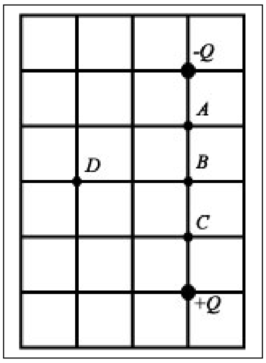


Figure 5.3: Pretest question on the work done by an electric field perpendicular to the path

5.4.2.2 Results

Forty-one students responded to this pretest. Their answers are shown in Table 5.5.

Table 5.5: Students' answers for the work along a path perpendicular to the electric field

Category	**%(N=41)**
Zero	46(19)
Positive	17(7)
Negative	24(10)
No answer/Don't know	12 (5)

46% of the students (19/41) stated that the work done by the electric field in moving the point charge from B to D was zero. Six of these students correctly reason this by considering the relative directions of the electric field and the displacement:

> *because the distance moved by the test charge is perpendicular to the electric field*

One student used reasoning based on electric potential:

> *The test charge stays at the same electric potential, and thus no work is done on it*

Three students reasoned that if the electric field were doing work then the test charge would not move horizontally from B to D but upwards from B towards $-Q$. This may be taken as a (logically) equivalent statement, but previous research has shown that students often think that a force only does work if it "causes" the motion.[6;5]

If the electric field was at work here the charge would move from $+Q$ to $-Q$ not horizontally towards D

The electric field will only move charges in the y-direction as the x's cancel

Another three students think that the electric field/force is zero along the path from B to D:

the electric field at each point is zero as the charges are equidistant from each respectively

The other six students either did not provide an explanation or gave an explanation that was difficult to interpret. Seven students (20%) stated that the work done by the electric field would be positive. Two of these students stated that:

work is the product of force and displacement in the direction of the force

These students either did not understand what this statement means, did not know how to find it, or had incorrectly determined the direction of the force along the path. The other explanations did not consider work as a dot product.

Ten students (25%) stated that the work done by the electric field would be negative. Three of these students gave explanations that suggest that they understand that the relative directions of the force and displacement vectors are important but are unable to determine the dot product correctly possibly because they did not know the correct direction of the electric field along the path:

the y-components in the electric field are moving against the direction of the moving charge

The charge moves in the opposite direction as the electric field

It is possible that this last student incorrectly determined the direction of the force along the path. Three students determined the sign of work by looking only at the direction of the displacement:

There is no displacement in the y-axis and negative displacement in the x direction

The other explanations did not consider work as a dot product. Five students (10%) either did not answer this question or stated that they didn't know.

5.4.3 Conclusions

The simple cases presented in the pretests revealed that many students have problems with the dot product, both technically and conceptually. When considering the work done by the electric field, 27% of students tried to reason through the relative directions of force and displacement. Due to difficulties in determining the correct direction of the electric field along the path some of these students did not find that the work was zero along the path. Before answering this question students had only experienced work in an electrostatics context during the lecture, and this possibly explains in part students' poor performance on this question.

Only 45% of students were able to find the dot product of two vectors in a situation where students would have been relatively familiar with the little physics that was required to answer the question. This suggests that unless students' understanding of the dot product is addressed their understanding of more complex physics concepts may be impeded.

5.5 Dot product instruction

5.5.1 Development of tutorial on Work

The context of work was chosen for two main reasons. The first encounter students have with the dot product within the electromagnetism course is when considering the work done by an electric field/force. Work is a concept that students will have been previously introduced in mechanics (although most likely without any emphasis being placed on its dot product nature). The aim was to use students' familiarity with the concept of work as a basis for developing their understanding of dot products.

Interviews were carried out with 10 students with the aim of gaining a deeper understanding of what students know about dot products in a work context and what type of reasoning helped them in developing their understanding. The participants were first year physics students taking a calculus-based mechanics course and first year non-physics science students taking an algebra-based mechanics course. The concept of work had been covered recently in both courses.

The protocol for the interviews was quite open, but there was a loose structure, in that there were a number of particular aspects that we wanted to investigate.

5.5.1.1 Positive, negative, and zero work

The first of these was to see if students knew under what conditions work is positive, negative, and zero. To do this the interview began by pushing a pen across the table and asking students if the work done by the hand on the pen was positive, negative or zero. Three students say that it depends on what direction you decide is positive and discuss conventions for choosing a sensible positive direction:

> **Student 1:** *it really kinda depends what you want to take as positive and negative...like if you say going in that direction its positive, then backwards its negative*

Student 2: *Positive but maybe depends on what direction you define positive to be*

Interviewer: *So if I define this direction* (pointing in the direction the pen was moved)*as being a positive displacement and a positive force...*

Student 2: *Then moving in this direction is positive work and moving in the opposite direction is negative work*

Student 3: *Positive because on the x and y axis the y goes up the way*

Three students used energy reasoning to explain why the work done by the hand was positive:

Student 4: *When the object is by itself it has potential energy and when you push it you give it kinetic energy so its positive*

Student 5: *The work done is positive because it's performing an action, it's not taking energy from it, it's giving energy to a system, yeah?*

One student (Student 6) says the work done in this case is positive because the *"it makes a pen move a distance"* but could not elaborate on this, while the remaining three students, although stating that the work was positive could not provide an explanation.

At this point the interviewer asked students (those who had not already done so) to first consider the kinetic energy before and after the hand applied a force and then comment on the relative direction of the force and the displacement. This approach was taken because relating a change in kinetic energy due to a force to work done by that force was judged to be a more concrete explanation for when work can be considered to be positive, negative or zero, and comparing the relative directions of force and displacement vectors was considered to be more abstract. Also, the only students who were able to correctly explain the sign of the work in this case without being prompted had used energy reasoning. So, in this case students first see that the work done by the hand is positive because it caused an increase in the pen's kinetic energy. Then, students can relate the two vectors being in the same direction to this positive work.

After this, the interviewer pushed the pen, released it, and when it came to a stop asked students if the work done by friction was positive, negative, or zero. Here, eight students identified that the work done by friction was negative. Six of these students used kinetic energy type reasoning, explaining that it was negative because it brings the pen to a stop, or that it is taking energy away:

> **Student 1:** *it has to be negative for the pencil to come to a stop*

> **Student 8:** *You're giving it energy but the friction is taking it away*

One student used the relative directions of force and displacement to explain that the work done was negative:

> **Student 9:** *Because its going against the direction of the pen*

Student 6 stated that the work is negative because:

> *stops a positive work*

Student 3 thought that the work was zero because:

> *it is just something acting on it*

Student 10 said the work is positive despite recognizing that the friction slowed the pen down:

> *Positive, it's making the pen slow down...its working like opposite*

As no student used both types of reasoning the interviewer asked students to comment on the alternate reason, or in the case of the three students that used neither, went through both again.

Then, the interviewer asked students whether the work done by either gravity or the normal force due to the table on the pen was positive, negative or zero, while pushing the pen across the table. It was expected at this point that most students would use kinetic energy reasoning at this point, due to its prevalence in the previous two scenarios and also because the force and displacement being perpendicular to each other yielding zero work might not be a logical step for students to take without knowing the cosine dependency (which no student had mentioned before this point). Nine students correctly stated that the work done by both these forces in this case is zero. Five of these students seem to compare the directions of the force and the displacement:

> **Student 2:** *None, because it is in an irrelevant direction to the displacement*
>
> **Student 8:** *Zero, because its not going up or down*
>
> **Interviewer:** *The pen?*
>
> **Student 8:** *Yeah, the pen*

Two students used kinetic energy type reasoning:

> **Student 5:** *Because there's no increase or decrease in energy given to the system by the table*

Student 1, uses a combination of both types of reasoning, although is not explicit about either:

> *No because you're not pushing the pen down, the normal doesn't have to do anything*

Student 7 could not explain why the work done was zero without a lot of prompting from the interviewer.

Student 6 said initially that the work was negative and again required a lot of guidance from the interview to reach the correct expression.

As can be seen from students' reasoning, by the end of this sequence six of the students were able to use force and displacement reasoning for the most difficult of the cases, and two more used kinetic energy correctly. This suggested that my initial thoughts were justified, and that developing a tutorial on work starting from kinetic energy and then moving to the relative direction of force and displacement would work for this particular cohort of students. As a result the first section of the tutorial (Appendix L) follows a similar structure to this interview sequence. This first section is shown in Figure 5.4.

WORK EM 21

I. Positive, negative, and zero work

The *work-energy theorem* states that the net work done on an object is equal to the change in the object's total energy. In cases where the potential energy is constant, the net work done is equal to the change in kinetic energy.

A. Place your pen at rest on the table. Use your hand to give the pen some speed, then push the pen at constant speed, then let go of the pen.

1. Why does the pen come to a stop after you release it?

   Is the change in the pen's kinetic energy *positive, negative,* or *zero*?

   Is the work done on the pen *positive, negative,* or *zero*? Explain.

2. While the pen was moving at constant speed, which forces were acting on it?

   Is the net work done *positive, negative,* or *zero*? Explain.

   Is the work done by each force *positive, negative,* or *zero*? Explain.

3. While the pen was speeding up, was the change in the pen's kinetic energy *positive, negative,* or *zero*? Explain.

   During that time interval, was the work done by your hand *positive, negative,* or *zero*? Explain.

4. While the pen was moving, was gravity changing the kinetic energy of the pen?

   Is the work done by gravity on the pen *positive, negative,* or *zero*? Explain.

B. Draw arrows to indicate the directions of each force and displacement.

hand	friction	gravity

Comment on the relative directions of force and displacement when the work done is positive, negative and zero.

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Physics Education Group, Dublin City University

Figure 5.4: Work Tutorial: Positive, negative and zero work section

The tutorial begins by reminding students of the work-energy theorem. Students are then asked to consider the forces acting on the pen, the change in the pen's kinetic energy, and the work done on the pen when it comes to a stop, when it was moving at constant speed, and when it was speeding up. They are also asked about the work done by gravity, first considering if it changed the kinetic energy of the pen.

Then students are asked to draw arrows to represent the force and displacement when the hand, friction and gravity were applying forces on the pen, and were asked to comment on the relative directions of the vectors when the work done was positive, negative and zero.

5.5.1.2 Calculating work

This sequence consisted of two work calculations, one was a basic one-dimensional scenario and one was a slightly more complex two-dimensional problem. The first involved a person pushing a box horizontally along a flat surface. Students were asked what information they would need and how they would use it to calculate the work done by the person on the box. The second was a person using a rope at an angle to pull a box horizontally across a flat surface. Students were asked if they could use the same calculation as they had done for the first question (force times displacement) for finding the work done on the box by the rope. A simple diagram was drawn for the students by the interviewer in each case showing the force being applied on the box and an arrow for the displacement, and students were given the same magnitudes for force and displacement in both questions.

The idea here was to first test if students could recall how to calculate work using force and displacement when the two vectors are in the same direction (a basic calculation students would have encountered in their Junior Cert science course). In the next scenario, the force and displacement vectors made a non-zero angle with each other. This allowed us to prove whether our students know that they cannot apply the exact same method, see how far they can go in such a calculation by themselves and what parts of the dot product calculation process they found most difficult.

The teaching sequence that was decided upon for guiding students through this second calculation was decided upon in advance of the interviews. The method tried involved first getting students to draw the force vector and then resolve this vector into horizontal and vertical components. Students were then prompted to reflect upon the previous questioning sequence to determine that the vertical component of the force does not do work on the box and that the horizontal component does positive work. Next students needed to calculate the horizontal component using trigonometry (multiplying the force vector by the cosine of the angle in between the force vector and its horizontal component). Lastly, students had to multiply the horizontal component of the force by the displacement.

For the first question, seven of the students either said that they would need to know the force and displacement (or distance), or information that that they could use to calculate the force (mass and acceleration). Three students said that they would need mass and velocity to calculate the kinetic energy. When asked what the value of the work would be if the force the person applied to the box was 10 N and the box was displaced 2 m all students gave the correct response of 20 J.

When the second question was described and the students were asked if they could apply the same calculation here and get an answer of 20 J nine of the students said no explaining that this was due to the angle. Five students were able to proceed through the calculation with minimal probing or guiding from the interviewer. The following is an example of this:

> **Interviewer:** *This time can I do the same calculation that we did up here?*
>
> **Student 1:** *No, to resolve that you need to know the angle the rope is at from the centre of the box to the man's hand. You resolve it, then you get the force that's going in the same direction as where you want the box to travel.*

Interviewer draws the force vector at this point.

> **Student 1:** *So, you resolve your parallel and perpendicular..*
>
> **Interviewer:** *Can you show me?*

Student 1 drew the horizontal and vertical components completing a right-angled triangle and labelled the angle between the force vector and the horizontal component α. Student 1 then labelled the horizontal component $\cos\alpha 10$ and the vertical component $\sin\alpha 10$.

> **Student 1:** *We want this.* (pointing to the horizontal component) *It is the direction the box is going*
>
> **Interviewer:** *OK. So what do I still need to multiply by to get work?*
>
> **Student 1:** *By distance*

Only one student did not recognise initially that the force being applied at an angle to the displacement would change how the work was calculated. The sequence of questioning used by the interviewer to guide this student through the calculation was typical of the remaining five students:

> **Interviewer:** *Can I just calculate the work the same way as before?*
>
> **Student 8:** *I guess so*
>
> **Interviewer:** *So I can just multiply the two directly?*
>
> **Student 1:** .....

At this point the interviewer drew the force and displacement vectors and asked the question again.

> **Student 1:** *Would you have to find their vector components?*

The interviewer split the force vector into its horizontal and vertical components.

> **Interviewer:** *Does the y-component matter in this case?*
>
> **Student 1:** *No... it's moving on a horizontal*

Interviewer: *Does the x-component matter here?*

Student 1: *Yeah...*

Interviewer: *Because it is in the same direction as the...*

Student 1: *displacement*

Interviewer: *So how do I find F_x? What would I need to know?*

Student 1: *The angle*

Interviewer: *OK, so if I knew the angle between the force and the displacement I can calculate the force in the direction of the displacement. What trig function should I use?*

Student 1: cos, *force time* $\cos\theta$ *would give you* F_x

Interviewer: *So now I have F_x what do I need to multiply by to get work?*

Student 1: *The displacement*

As mentioned, this was the typical teaching sequence for the work calculation, however, some students needed extra guidance reasoning why the y-component of the force did not contribute to the work done by the force. In those cases, the interviewer referred back to the normal or gravity force questions from the previous part of the interview.

This teaching sequence was replicated for the second section of the work tutorial (shown in Figure 5.5).

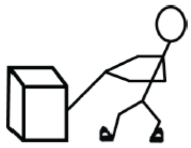
II. Calculating work

A. A person pushes a box with a constant force of magnitude *F* across a frictionless surface over a distance *d*.

What is the sign of the work done by the person on the box? Explain.

What is the magnitude of the work done by the person on the box?

B. This time the person decides to use a rope to pull the box across the surface, as shown. The rope makes an angle θ with the horizontal. The tension force on the box has magnitude *F*, and the box still moves a distance *d*.



1. Explain why the expression you derived in part II-A for the work done cannot be used in this case.

2. Resolve the force into two components, in such a way that one component does zero work. Sketch the components in the box at right. Explain.

   Is the work done by the other component *positive, negative,* or *zero*? Explain.

3. Write an expression for the magnitude of each component in terms of *F* and θ. Explain briefly.

Figure 5.5: Work Tutorial: Calculating work

The last part of the tutorial asks students to compare the work done by gravity in moving a ball down two different segments of a ramp (situation shown in Figure 5.6). First, students considered a student statement reasoning that the work done by gravity is negative because the force due to gravity acts in a negative direction, which readdresses this common misconception held by students. Students were then asked to draw the force and displacement vectors for each part of the ramp. Students saw that when they found $F.\cos\theta$ for each part of the ramp they could not compare the work done along each path. However, the comparison is easy when $d.\cos\theta$ is considered for each path. This also demonstrated that it is irrelevant whether the force vector is projected onto the displacement vector or the displacement vector is projected onto the force vector.

III. Work done by gravity

A ball is rolling down a ramp, shown at right. The ramp consists of two sections, 1 and 2, of equal height. The length of section 2 is greater than the length of 1, and makes a smaller angle with the horizontal.

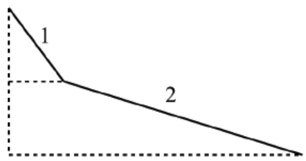


Figure 5.6: Work Tutorial: Work done by gravity

5.5.2 Other relevant instruction

Following the *Work* tutorial students complete the *Electric Potential Difference* tutorial (Appendix M). Here, students consider the work done by an electric field along multiple paths, before examining the relationship between work and electric potential difference. This tutorial was used in all years of the study, and based on classroom observations students seemed to reason more completely following the introduction of the new work tutorial.

Dot products were used in potential, flux, and circulation calculations in the remainder of the course.

5.6 Conceptual dot product post-test

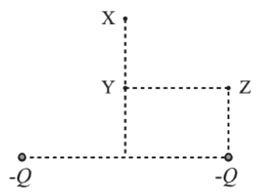
Again, the context of work done by an electric force was used to examine students' reasoning about the relative directions of force and displacement, to examine whether the work tutorial had an impact on students' reasoning or not.

5.6.1 Question

This post-test question presented students (N=61) with the situation shown in Figure 5.7. Students were first asked if the work done by the electric force is positive, negative, or zero when a positive test charge is moved from X to Y. To answer this question correctly, students should have reasoned that the force on the test charge due to the two point charges would act directly downwards. Hence, the force and displacement are in the same direction resulting in the force doing positive work.

The next question asked students if the work done by the electric force is positive, negative, or zero, when a positive test charge is moved from Y to Z. Here, students were expected to reason that at most points along the path there was a component of the electric force in the same direction as the movement along the path, even though a completely correct answer involves slightly more subtle reasoning.

Two negative point charges -Q are held in place. A positive test charge q_0 is moved in the vicinity of the two point charges. Points X and Y are equidistant from the two point charges, and the line through Y and Z is parallel to the line joining the point charges.



(a) **[5 marks]** Is the work done by the electric force on the test charge *positive, negative,* or *zero* when the test charge is moved from X to Y? Explain.

(b) **[5 marks]** Is the work done by the electric force on the test charge *positive, negative,* or *zero* when the test charge is moved from Y to Z? Explain.

Figure 5.7: Work post-test question

5.6.2 Work done by force parallel to the path

Table 5.6 shows the students' answers for the work done by the electric field when the test charge is moved from X to Y.

Table 5.6: Students' answers for the work done by force parallel to the path

Category	**%(N=61)**
Positive	80 (49)
Negative	11 (7)
Zero	8 (5)

Some 80% of students stated that the work done by the electric force is positive in this case, and 66% considered the relative direction of force and displacement:

"The x-components of the force due to each cancel, so the force on the test charge is downwards. This is the same direction as the displacement and so the work is positive"

The remaining 14% reasoned that the work was positive because the test charge is moving closer to the point charges or just because it is attracted to the point charges:

"Test charge is attracted to the point charges, therefore there is work done by the system in bringing q_0 from X to Y"

'moving in a direction closer to the negative charges"

11% stated that the work would be negative. 3% only considered either the direction of the force or the direction of the displacement, and the other students' reasoning was difficult to interpret.

5% out of the 8% of students who stated that the work was zero, used correct reasoning based on their incorrect interpretations of the situation. Two students had one positive and one negative point charge, instead of the two negative and so the net electric force would be perpendicular to the path in that case. One student interpreted point X as being located behind point Y rather than above it.

5.6.3 Work done by force at an acute angle to the path

Table 5.7 shows the students' answers for the work done by the electric field when the test charge is moved from Y to Z.

Table 5.7: Students' answers for the work done by force at an acute angle to the path

Category	**%(N=61)**
Positive	39 (24)
Negative	20 (12)
Zero	41 (25)

Only 39% of students stated that work done by the electric force would be positive from Y to Z, with 26% stating that the electric force has a component in the direction of the displacement at most points along the path:

> *"The horizontal force due to the charge on the right is greater, therefore the force and displacement are in the same direction along the path"*

> *'The force due to the charge on the right is stronger, therefore the force in the most part is in the direction of movement"*

41% stated that the work was zero in this case; two-thirds of these students (28% of all students) only considered the relative direction of force and displacement at point Y. Thus, a total of 67% of students in this case could determine the sign of the dot product correctly. 10% of students know they must consider the relative directions, but incorrect reasoning about the force along the path appeared to prevent them from determining the work correctly:

> *'Horizontal components of Q_1 and Q_2 cancel. No work is done on the test charge, vertical forces don't affect path"*

> *'work done by the charge on the left is negative, work done by the charge on the right is positive, therefore total work is zero"*

The same 5% who misinterpreted the situation but reasoned correctly for the first question did they the same here, finding that the work done was negative.

5.6.4 Conclusions

In the context of work done by an electric force, about three quarters of our students appear to be able to use the dot product correctly, or at least use the relative orientation of the electric field and displacement vectors to determine the sign of the work done. Only one quarter of students were able to do this in the Loop in Uniform Magnetic Field post-test question. This suggests that, while difficulties with the dot product are an obstacle for a minority of students after completion of the new instruction, familiarity with or confidence in being able to interpret the physical situation is also an important factor.

5.7 Calculating electric flux

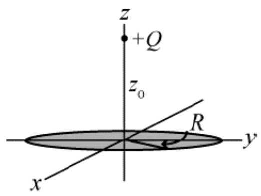
The post-test questions described in this section involve the calculation of the electric flux through a surface due to a point charge. Because the electric field varies in both magnitude and direction over the surface, students were required to use both the dot product and integration. The first question described was asked before instruction and had been designed specifically for these mathematical tools and the second question was administered to a different cohort in the year where both the integration and dot product instruction had been implemented.

5.7.1 Post-test after old instruction

5.7.1.1 Post-test question

The students were told to consider the electric flux through a circular disk due to a point charge a distance above the centre of the disk, shown in Figure 5.8.

A point charge $+Q$ is placed a distance z_0 above the centre of a hypothetical circular disk with radius R located in the x,y-plane. In this question, you will consider the electric flux through the disk, assuming its normal to point downwards.



(a) Explain that for very small values of R, the electric flux through the disk is approximately given by $\Phi_E = \frac{QR^2}{4\varepsilon_0 {z_0}^2}$.

(b) Explain that for very large values of R, the electric flux through the disk is approximately given by $\Phi_E = \frac{Q}{2\varepsilon_0}$.

(c) In the general case, show that the electric flux $\mathrm{d}\Phi_E$ through a small segment of length $\mathrm{d}r$ and angular width $\mathrm{d}\phi$ located at (r,ϕ) is given by $\mathrm{d}\Phi_E = \frac{1}{4\pi\varepsilon_0}\frac{Qrz_0}{\left(r^2+{z_0}^2\right)^{3/2}}\mathrm{d}r\mathrm{d}\phi$.

(d) Hence, show that in the general case the flux through the entire disk is given by $\Phi_E = \frac{Q}{2\varepsilon_0}\left[1-\frac{1}{\sqrt{1+R^2/z_0^2}}\right]$.

(e) Show that, for large R, this expression approaches the expression in part b).

(f) Show that, for small R, this expression approaches the expression in part a). (*Hint:* Use the approximation $(1+x)^n \approx 1+nx$.)

Figure 5.8: Calculation of the electic flux through a circular disk

5.7.1.2 Issues with dot product

Part (c) asked students to show that the electric flux $\mathrm{d}\Phi_E$ through a small segment of length $\mathrm{d}r$ and angular width $\mathrm{d}\phi$ located at (r, ϕ) is given by

$$\mathrm{d}\Phi_E = \frac{1}{4\pi\varepsilon_0}\frac{Qrz_0}{(r^2+z_0^2)^{\frac{3}{2}}}\mathrm{d}r\mathrm{d}\phi. \tag{5.1}$$

To obtain equation 5.1 students would have needed to take a large number of steps. One successful line of reasoning would be:

1. Start with the general expression for infinitesimal flux: $\mathrm{d}\Phi = (\overrightarrow{E}.\widehat{n})\mathrm{d}A$
2. Choose a small segment with area, $\mathrm{d}A = r\mathrm{d}r\mathrm{d}\phi$
3. Find the distance from the charge Q to the segment, in terms of the given values, using Pythagoras' Theorem

4. Write an expression for the magnitude of the electric field at the location of the segment
5. Find the magnitude of the electric field in the direction of the normal of the segment ($\overrightarrow{E}.\widehat{n}$) by multiplying the electric field by the cosine of the angle between the electric field and normal vectors. Students would need to use similar triangles to write the cosine of this angle in terms of the given values.
6. Multiply the expression for $\overrightarrow{E}.\widehat{n}$ by the expression for $\mathrm{d}A$

We focus on those elements of students' answers that relate to the dot product. In total, 31% of the students derived the expression correctly. The majority of these students started from a correct formula for electric flux (($\overrightarrow{E}.\widehat{n}$)$\mathrm{d}A$, $E.A.\cos\theta$, or similar). The others either started by writing $\Phi = EA$ or did not start with a formula, but calculated all the elements and multiplied them together correctly.

Almost one-fifth of students did not answer the question at all, and half of the students either did not obtain the given expression or did so by incorrect manipulations. 7% started from $\overrightarrow{E}.\widehat{n}\mathrm{d}A$, but did not incorporate the dot product in their answer, suggesting that they did not know what the expression means. 9% of students started with $\Phi = EA$ and did not include $\cos\theta$ in their answer. These students either did not think that a dot product applied in the situation that is presented here, or they did not know how to find it.

5.7.1.3 Issues with integration

In part (c), 16% of students neglected to include the area in their answer, and 13% used an incorrect expression (either $1/2r^2\phi$ or $\mathrm{d}r\mathrm{d}\phi$). 11% of students just appended the infinitesimal terms at the end of their derivation, seemingly without knowing where they came from or how they affect the expression.

To check students' ability to use the information on an infinitesimal flux to obtain an expression for the total flux, part (d) asked students to show that the latter is given by

$$\Phi_E = \frac{Q}{2\varepsilon_0}\left[1 - \frac{1}{\sqrt{1 + R^2/z_0^2}}\right]. \tag{5.2}$$

Because the electric flux varies across the the disk (the magnitude and direction of the electric field changes as you move out from the centre of the disk along the radius), integration was required to sum all the small segments of different constant electric flux values. Thus integrating, the given expression for the electric flux through a small segment with the correct limits ($\int_0^R \int_0^{2\pi}$) will give the electric flux through the entire disk. Students were not required to compute the integral themselves as they were provided with the anti-derivative.

Table 5.8 shows the number of students who answered correctly, incorrectly or did not answer at all for the electric flux through the entire disk. As the indefinite integral was provided, students did not need to compute the integral just fill the limits in and simplify. The number of no answers is high with approximately 45% of the respondents not answering this part of the question. This strongly suggests that they do not see integration as a process of adding infinitesimals. This finding is in line with that of Chapter 2, where a similar percentage of students were unable to use the given expression for $\mathrm{d}B$ to B .

Table 5.8: Answers for electric flux through the entire disk

Category	**Electric Flux %(N=45)**
Correct	27(12)
Incorrect(integration used)	22(10)
Incorrect(no integration used)	4(2)
No answer	47(21)

5.7.2 Post-test after new instruction

5.7.2.1 Question

Figure 5.9 shows the full post-test question. This question is similar to the earlier post-test question in Section 5.7.1: it concerns flux due to a point charge, and asks students to derive an expression for $\mathrm{d}\Phi_E$ before evaluating Φ_E from that expression.

A point charge $+Q$ is located at the bottom of a hypothetical open cylinder of radius R and height h. In this question, consider the electric flux through the <u>curved</u> part of the cylinder. Choose the z-axis to coincide with the cylinder axis, and the origin to coincide with the point charge $+Q$.

(a) **[15 marks]** Show that the electric flux $\mathrm{d}\Phi_E$ through a small segment of height $\mathrm{d}z$ and angular width $\mathrm{d}\phi$ located at (R,ϕ,z) is given by $\mathrm{d}\Phi_E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{QR^2}{\left(R^2+z^2\right)^{3/2}}\mathrm{d}\phi\mathrm{d}z$.

(b) **[9 marks]** Hence, find an expression for the electric flux through the entire <u>curved</u> surface of the cylinder.

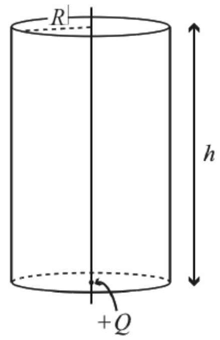


Figure 5.9: Post-test question on the calculation of electric flux through the curved surface of a cylinder

5.7.2.2 Issues with dot product

Sixty-three students (N=63) answered this post-test question. As before, we focus on those elements of students' answers that relate to the dot product when answering part (a). 40% of all students derived the expression correctly, with the majority starting from a correct formula for electric flux. The others started by writing $\Phi = EA$, but calculated all the elements and multiplied them together correctly. However, a further 24% did start from $\overrightarrow{E}.\widehat{n}\mathrm{d}A$ or $\Phi = EA$ and did include $\cos\theta$ in their answer, but failed to obtain the correct expression due to other errors. Thus, almost two-thirds of the students showed that they identified the dot product as an essential step in the solution, compared to just under one-third after the old instruction.

Only 5% of students did not answer the question - again a marked improvement from the 18% who did not answer the question after the old instruction. 17% brought area into the expression by substituting $Q = Q\mathrm{d}\phi\mathrm{d}z$ or $Q = \sigma\mathrm{d}\phi\mathrm{d}z$ into the expression for electric field, even though some of these students had started with a correct formula for electric flux. These 11 students probably do not know what electric flux is, and may be recalling processes from questions on electric field. It is impossible to judge whether these students had issues with dot product or not.

5.7.2.3 Issues with integration

With regard to the area of a small segment, 38% of students either neglected this in their answer or used an incorrect expression. 14% did not obtain the given expression, 14% just added the infinitesimal terms onto their expressions (apparently without knowing where they came from or how they affect the expression) the remaining 10% completed incorrect manipulations to obtain it.

To obtain the electric flux through the entire curved surface students had to integrate the given expression for the electric flux through a small segment of the surface with the limits $0 \to h$ for $\mathrm{d}z$ and $0 \to 2\pi$ for $\mathrm{d}\phi$. As the indefinite integral was provided students only needed to fill in the limits and simplify the expression obtained.

In this case, 83% of students now performed an integration, with 40% doing so correctly. As with the dot product, we see a doubling of the fraction of students attempting the integration, without this resulting in a corresponding increase in correct responses. 10% of students did not use integration in their answer, and 8% gave no answer at all (compared to 45% after the old instruction).

5.8 Conclusion

We have identified a number of difficulties students have with the dot product. In the line integral of a magnetic field post-test question, students are specifically asked for a dot product, most students did not consider the relative directions of force and displacement. Difficulties students have considering the dot product may be context dependent, as the same percentage of students reason about the relative directions of force and displacement in a pretest questions when asked about the work done by an electric force (dot product not mentioned explicitly).

Technical issues have also emerged. Less than half of our students could give the value of the dot product $\vec{E} \cdot \widehat{n}$, when given the components of the electric field and the direction of the normal to eliminate difficulties with setting interpretation and component calculation.

On changing the instruction to try and address some of the issues identified, the fraction of students obtaining correct answers did not change significantly. However, many more students attempted the question and gave evidence of having taken the first steps towards understanding dot product issues.

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Chapter 6

Developing students' understanding of the relationships between position, velocity and acceleration in the context of Simple Harmonic Motion

6.1 Introduction

This chapter describes the development of a curriculum for combining the conceptual understanding and the mathematical and graphical aspects of displacement, velocity, and acceleration using the context of a simple harmonic oscillator. These kinematics concepts and graphing skills are regularly intertwined and students' ability to apply this combination is called upon frequently in many physics contexts. Describing the motion of a simple harmonic oscillator is one such context and for our students Simple Harmonic Motion (SHM) is the first topic where they will encounter these concepts since their introductory calculus based mechanics course, taken in the previous semester.

Previous studies have identified difficulties students have with these concepts, and with interpreting and drawing the corresponding graphs [1–5]. Section 6.2 describes the relevant findings from these studies. We have investigated the extent to which our students experience these difficulties and have found other difficulties from the analysis of students' responses to pretest and post-test questions. Pretest and student interview results establishing students' ability to interpret and draw graphs representing simple harmonic motion are presented in Section 6.3.

Three different approaches to tackling difficulties determined in pretests and post-tests have been implemented. Sections 6.4, 6.5, and 6.6 describe each of these approaches individually along with results from the post-test questions administered each year. Finally, Section 6.7 discusses the efficacy of these three approaches and the implications for teaching these concepts.

6.2 Previous findings

6.2.1 Velocity and acceleration

Trowbridge and McDermott [2;3] have investigated students' understanding of the concepts of velocity and acceleration in one dimension. In their first paper focusing on the concept of velocity, they begin by highlighting the importance of these concepts, stating that a comprehension of them is vital when studying many other areas of physics. They argue that it is probable that students in introductory physics courses will have incomplete and analogous ideas about motion as a result of their own observations and interpretation of prior instruction. They report on a number of misconceptions of velocity that they determined by means of interviews where students were asked questions about different demonstrations involving balls and tracks. One such misconception reported is thinking that two objects would have the same speed when they are at the same position or at the instant of passing each other. Halloun and Hestenes[1] also found that students believed that when two particles have the same speed when they simultaneously occupy the same position, even if the two particles were moving with different constant speeds.

In a second paper focusing on the concept of acceleration, Trowbridge and McDermott's[3] main objective was to evaluate students' understanding of acceleration as the ratio of $\Delta v/\Delta t$. Students were asked to consider the motion of two balls, ball A travelling at a constant velocity while ball B is released at a later time with a greater initial velocity but slowing down as it travels up an incline. Approximately half the students they interviewed believed that the acceleration of the balls is the same when they have the same velocity, erroneously reasoning that at this point they have the same Δv and Δt. The authors state that students are unable to make the necessary distinction between the concepts of velocity and change in velocity. They also found that some students believed that catching up implies having a greater acceleration.

6.2.2 Graphing

Research investigating students' abilities in drawing and interpreting kinematics graphs has been carried out[1;4;5]. McDermott et al[4] state that undergraduate students in introductory physics courses seem to lack the ability to use graphs either for imparting or extracting information. They found that students have more difficulty interpreting curved graphs than straight line graphs. They suggest that this could be because on curved graphs the slope changes as well as the height. They gave their students a curved graph of position versus time that also continued below the x-axis. Some students believed that the slowest motion was where the curve meets the x-axis (where height is zero). A typical error made by students is stating that the object is speeding up at a point on the graph where the height is increasing but the slope is decreasing. Some students believed that at a point below the x-axis where the slope is increasing represented an object slowing down because the slope was negative. Many students select the point where the curve cuts the x-axis as a turning point because the position is going from plus to minus.

Both Beichner[5] and McDermott et al[4] state that students have difficulties translating between position, velocity and acceleration versus time graphs. Beichner[5] found that approximately 25% of the students he surveyed believed that switching between kinematics concepts would not change the shape of the graph. Similarly, McDermott et al[4] found that in the laboratory students regularly draw graphs that have the same shape for position, velocity and acceleration versus time even when they make measurements that do not support this. They also found that students often do not realize that they should use the slope of the position versus time graph as the height of the velocity versus time graph.

Another area that students have difficulty with is representing information extracted from a narrative passage onto a graph. Students often cannot show a negative velocity on a graph. They also find drawing an acceleration graph for an object that slows down, turns around and speeds up in the opposite direction difficult to represent

If these difficulties are experienced by our students when entering this course, it would be unlikely that they would have the ability to reason completely about or draw the motion of a simple harmonic oscillator. Therefore, it was important to use the pretests to examine students starting knowledge of the motion concepts.

6.3 Initial student understanding

6.3.1 Lecture instruction

The lecture introduces many aspects of the topic of SHM: the relationship between the position, restoring force, and acceleration; the total energy of the harmonic oscillator; and the expression for the wave function representing a simple harmonic oscillation. The lecture uses the example of a mass attached horizontally on a spring to consider these aspects. However, how the position, velocity and acceleration changes throughout an oscillation is not discussed qualitatively in the lecture, and the graphs of these three quantities versus time are not presented.

6.3.2 Pretest questions

6.3.2.1 Year one

Students were asked to consider a compression spring mounted horizontally, attached to a block of mass m positioned on a frictionless surface. The change in position of the block after it is compressed to a position A and then released, is described in the question. The question also explains that the equilibrium position is the 'zero' position and that position B (the corresponding maximum position) is a positive position. Students were asked to consider whether the velocity of the block is increasing, decreasing, or is at a constant rate, and whether it is positive, negative, or zero during the intervals, A, E, and B of the block's motion (shown in Figure 6.1). They were also asked if the acceleration was positive, negative, or zero during the same intervals. The full question is presented in Appendix N.

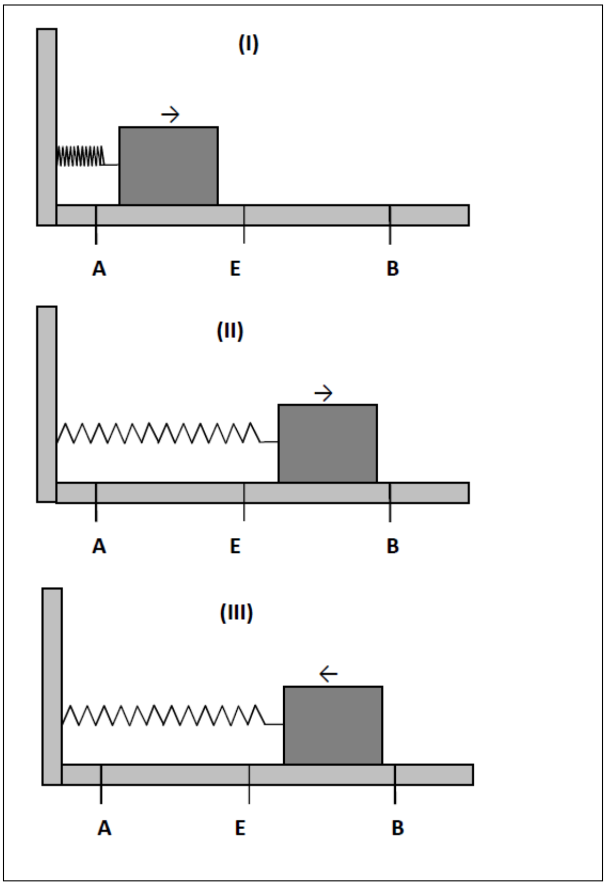


Figure 6.1: Year one pretest: three intervals of the motion

This question allowed us to investigate if students could reason about how the velocity of the block was changing throughout the motion. Students were not expected to recognise exactly how the velocity would change between points as this would have required them to reason about the relationship between position and restoring force, and then force and acceleration. Rather they were expected to determine at which points the velocity would have the largest and smallest values, and state generally whether the block's velocity was increasing or decreasing between the two points in each of the intervals.

By asking students for the sign of the acceleration in each interval, we could investigate how students related it to a change in velocity.

6.3.2.2 Year two

Figure 6.2 shows the pretest administered in year two and the full pretest question is included in Appendix O. The pretest uses the same situation as the pretest in year one. The question provides students with the position versus time graph for one period of the block's motion and asks them to label positions A, B, and E on the graph. From this, we could determine if students could map the physical situation presented onto a graph. Students would have to realise that the equilibrium position, E, was the 'zero' position, that A was the maximum negative position, and that B was the maximum positive position. They would then need to know how each of these are represented on a graph.

A block of mass m is placed on a frictionless surface and is attached to a compression spring with spring constant k as shown in *Figure 1*. The block is compressed to position A and is then released. *Figure 2* shows one full period of the blocks position versus time graph.

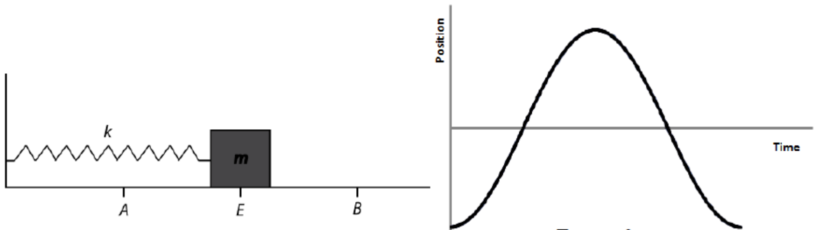


Figure 1 *Figure 2*

(a) Label clearly where positions A, B and E occur on the graph.

(b) Draw the corresponding graphs of velocity versus time and acceleration versus time for the same situation. Explain why you drew the graphs this way.

Figure 6.2: Year two pretest

The next part of the question asked students to draw the corresponding graphs of velocity and acceleration versus time and explain why they drew them this way. Students could have approached this question either by reasoning about the physical situation, determining how the force and hence acceleration changed throughout the motion, or they could have used the slope of the position versus time graph to draw the velocity versus time, and then the slope of the velocity versus time graph to draw acceleration versus time. Students would not have needed to understand the motion of the block in any way to draw the velocity and acceleration versus time graphs.

Even if students did not provide explanations for their graphs, this question would at least allow us to see if students recognized that the shape of the graphs for each of the kinematics quantities should be different from one another.

6.3.2.3 Year three

Two pretest questions were posed in this year, one in the context of SHM and one in the context of the motion of a car.

SHM context question: Appendix P shows this full pretest question. The question utilizes the same situation as the previous two pretests. Students were presented with the information shown in Figure 6.3. Students were given a velocity versus time graph and were asked to label where the positions A, B and E occurred on this graph. Students were expected to identify whether the velocity was a maximum or minimum at the three positions and when the velocity would be positive or negative, and to use this information to interpret the graph in terms of the three positions.

The next part of the question focussed on students understanding of acceleration. Students were asked if the acceleration of the block was increasing or decreasing in magnitude and whether it was positive or negative during each of the four intervals, $E - B$, $B - E$, E_A, and $A - E$. Students were then asked to plot this information on an acceleration versus time graph.

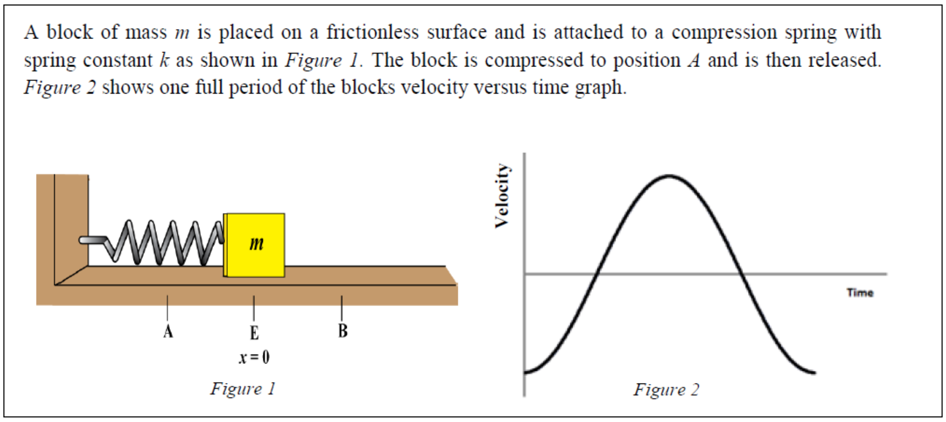


Figure 6.3: Year three SHM context pretest

Non-SHM context question: (See Appendix Q) This question presented students with the position versus time graph for two cars, as shown in Figure 6.4. Students were asked at what time the velocity was zero and at what time it was greatest for each car. They were also asked at what time the acceleration was zero in the case of the straight-line graph.

The question probes if students can use the slope of the position versus time graph to reason about the velocity of the two cars. It also allowed us to determine if there is a difference in how students understand the motion described by a straight-line graph and a curved graph.

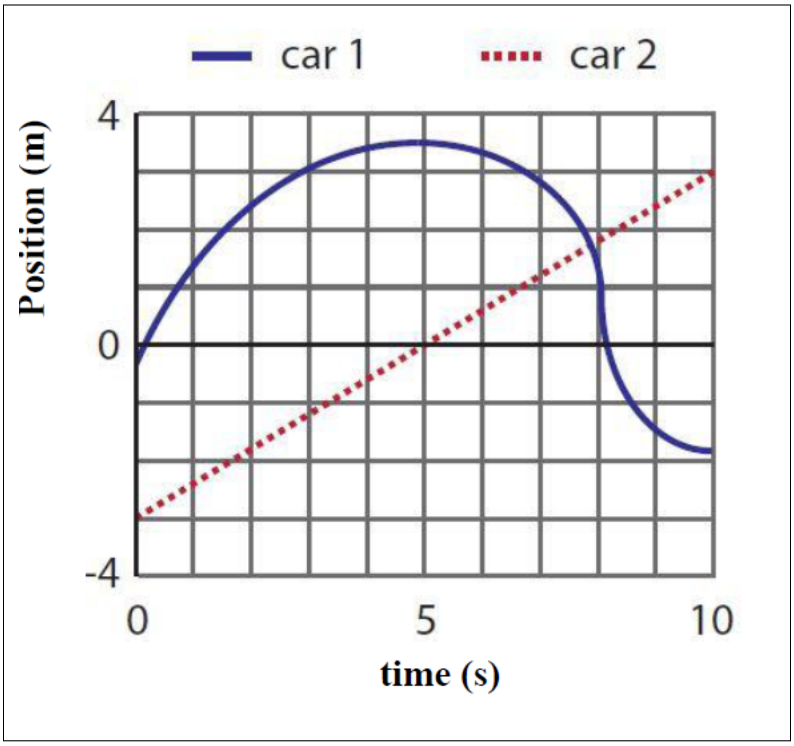


Figure 6.4: Year three non-SHM context pretest

6.3.3 Pretest results

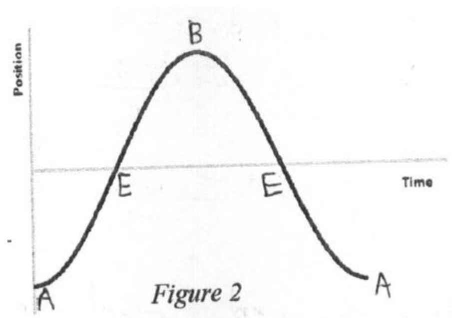
The concepts of position, velocity, and acceleration are each discussed in the following section, drawing evidence from the parts of the pretests that relate to these concepts.

6.3.3.1 Position

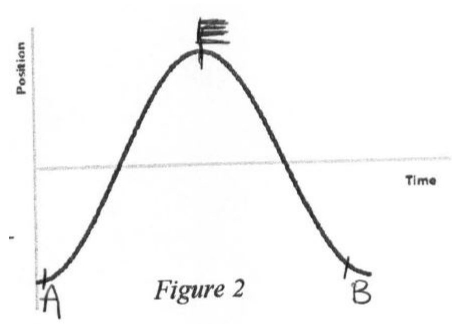
Table 6.1 shows a categorization of students' labelling of the position versus time graph from the year two pretest question. Figure 6.5 shows examples of labelling from each category, in terms of the position of equilibrium, E.

Table 6.1: Students' labelling of the position versus time graph

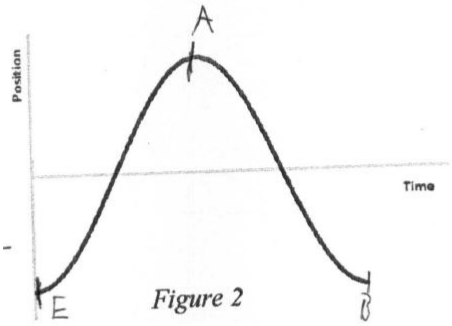
Category	**%(N=17)**
Correct	47(8)
E in the middle	24(4)
E at the start	12(2)
E at the end	18(3)



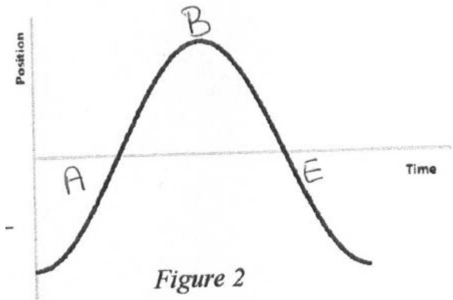
(a) Correct labels



(b) E in the middle



(c) E at the start



(d) E at the end

Figure 6.5: Students labelling of the position versus time graph

Less than half of the students in this cohort correctly related the physical situation to the given position versus time graph. The students who labelled E in the middle possibly see the graph as a picture and have simply mapped the situation where the motion is symmetric about E. It is difficult to interpret how the students who represented E at the start or end are considering the different aspects of the graph represented. Perhaps those who start with E are mapping the part described in the question where the block starts at E, is compressed to A, and when it is released moves to B. It is unlikely that students would have trouble explaining the order of positions the block would move through during the motion, hence this type labelling suggests that more than half of our students have difficulties' interpreting the various aspects of a graph, such as slope, height, and positive and negative axes.

6.3.3.2 Velocity

Interpreting velocity from a non-SHM position versus time graph: Students' answers for what times Car 1 and Car 2 (from Figure 6.4, year three) have zero velocity are shown in Table 6.2.

Table 6.2: Students' answers for where the velocity of Car 1 and Car 2 is zero

Answer	Car 1 %(N=36)	Car 2 %(N=36)
5 s	**47(17)**	6(2)
Never	25(9)	**75(27)**
0 s	22(8)	11(4)
Other/No answer	6(2)	8(3)

Less than half the student cohort (47%) identified the correct time at which Car 1 had zero velocity. The type of reasoning used by these students was split fairly evenly between: position does not change at that time; the car is changing direction at that time; it is the turning point of the graph; and slope is zero at that time. Only a small number of students did not provide an explanation for their answer.

One quarter of students stated that the velocity of Car 1 was never zero because the position was constantly changing. Despite knowing the condition for which the velocity is zero these students were still unable to identify the feature of the graph that showed where the position was not changing.

A greater number of students (75%) answered correctly for Car 2, stating that the velocity was never zero because the car had a constant velocity or that the graph had a straight line. Interestingly, all students who reasoned that the velocity of Car 2 was zero because it was a straight line graph had not answered correctly for Car 1. Most students would have experienced straight line graphs much more frequently than curved graphs, and so it is likely that students do better for the straight line graph more because of familiarity than understanding.

For both graphs some students stated that the velocity was zero at 0 s because the car is "starting off".

Table 6.3 shows students' answers for the time where the velocity of each car is greatest. Again, more students answered correctly for the straight line graph. For Car 1, the steepest slope gave a negative velocity and so it would have been correct for students to reason that this is in the time where Car 1 has the lowest velocity, however there was no evidence that any students considered the sign of the slope. Hence, we considered any answer that reasoned about the magnitude of the velocity at this point to be correct.

Table 6.3: Students' answers for where the velocities of Car 1 and Car 2 are greatest

Answer	**Car 1 %(N=36)**	**Car 2 %(N=36)**
Correct	**47(17)**	**83(30)**
Max position time	28(10)	8(3)
Start of the graph	14(5)	-
Other/No answer	11(4)	8(3)

The students who correctly identified the time when Car 1 had zero velocity, also correctly identified the time when Car 1 had greatest velocity (at $t = 8$ s), using similar type reasoning: greatest distance covered in a short time interval, or steepest slope. Some students stated that the velocity was greatest at the start of Car 1's motion (0, 1 s, 2 s) because it is accelerating the most or because it has the steepest slope. The reasoning used by these students is correct and perhaps they did consider the acceleration at $t = 8$ s to be negative and hence the time of the lowest acceleration of the car.

28% stated that the Car 1's velocity was greatest at 5 s, which is the greatest value of the position of Car 1. Some students stated that they chose this time because it was the highest point or because it was the maximum displacement versus time. These students have difficulty distinguishing position from a change in position.

Again, students answered significantly better for Car 2 with 83% correctly stating that the velocity of Car 2 is constant. The remaining students used the same incorrect reasoning as they had for Car 1. The difference in performance between the straight-line graph and the curved graph shows that although students recognise that a constant slope for position versus time means that the velocity is constant, many students cannot apply the same slope reasoning to a curve graph and revert to previously held misconceptions such as relating the greatest velocity to the greatest position reached rather than the greatest change in position.

Interpreting how the velocity changes during SHM: This section relates to the pretest question administered in year one, Section 6.3.2.1(N=11). Tables 6.4, 6.5, and 6.6 show how students interpreted how the velocity changed in terms of magnitude and direction for three intervals of the blocks' motion. The question asked students whether the velocity was increasing, decreasing, or constant for the three intervals. For the first two intervals, students should find the same result regardless of whether they consider the velocity or the speed. However, in the third interval the velocity becomes more negative; and so velocity is decreasing but speed is increasing. We found that students appeared to look only at speed for each of the intervals, possibly because they were also asked for the sign of the velocity. As tutorials require students to consider how the magnitude of the velocity changes, students who only considered speed were deemed to be correct here.

Table 6.4: Students' answers for velocity A to E interval

Answer	**%(N=11)**
Increasing, positive	**73(8)**
Increasing, negative	9(1)
Constant rate, negative	9(1)
-, negative	9(1)

Table 6.5: Students' answers for velocity E to B interval

Answer	%(N=11)
Decreasing, positive	**73(8)**
Increasing, positive	18(2)
-, positive	9(1)

Table 6.6: Students' answers for velocity B to E interval

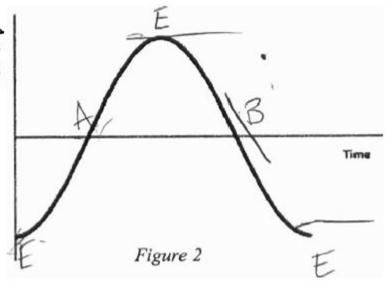
Answer	%(N=11)
Increasing, negative	**91(10)**
Decreasing, negative	9(1)

Eight students reasoned consistently about how the magnitude and the sign of the velocity changed throughout the motion. This suggests that most students know when the velocity is increasing and decreasing throughout the motion. In later pretests, we see that this does not mean that they understand how it increases and decreases.

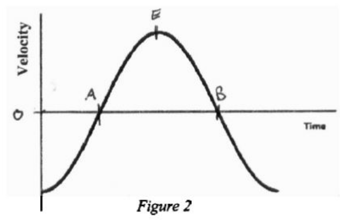
Labelling a SHM velocity versus time graph: Table 6.7 shows a categorization of students' labelling of the velocity versus time graph given in the year three pretest and Figure 6.6 shows an example of a labelling from the four main categories.

Table 6.7: Students' labelling of the velocity versus time graph

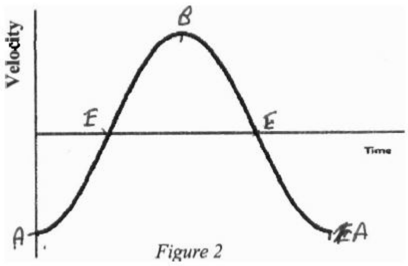
Category	%(N=27)
Fully correct	7(2)
Correct points (E only labelled once)	22(6)
Labelled like position versus time	22(6)
A and B negative peaks, E positive peak	33(9)
Other	15(4)



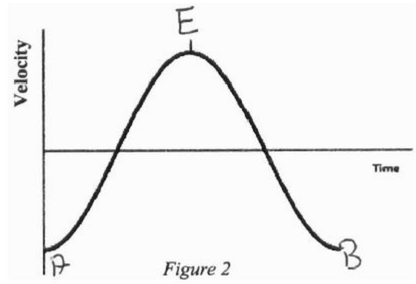
(a) Correct labels



(b) E only labelled once



(c) Labelled like position versus time



(d) A and B negative peaks, E positive peak

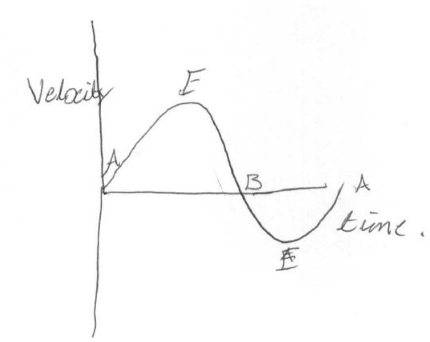
Figure 6.6: Students labelling of the velocity versus time graph

29% of students correctly identified A, B, and E on the velocity versus time graph. Considering that the majority of students reasoned correctly about the sign and the increasing and decreasing velocity, students labelling this graph incorrectly is most likely caused by an inability to interpret the graph. Also there is the added difficulty for students that they are labelling positions on the velocity versus time graph. We have seen previously that students have a difficulty with the relationship between position and velocity and this could have been the reason for why so many label the graph in a way that would be correct for position versus time.

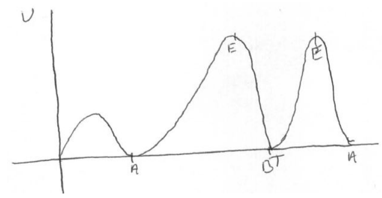
Drawing a SHM velocity versus time graph when given the position versus time graph: Table 6.8 shows a categorization of students' drawing of the velocity versus time graph when given the position versus time graph from the pretest in year two and Figure 6.7 shows an example of a drawn graph from the four main categories.

Table 6.8: Students' drawing of velocity versus time graph when given position versus time graph

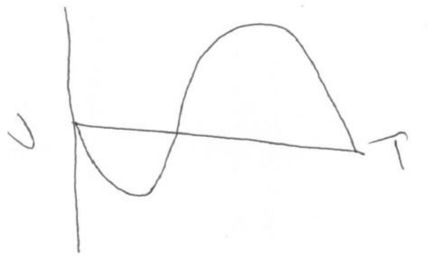
Category	**%(N=17)**
Correct	41(7)
Correct max and min velocities, no negative	12(2)
Correct max and min velocities, incorrect sign	18(3)
Half period	18(3)
Other	12(2)



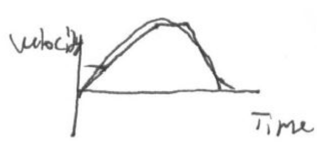
(a) Correct graph



(b) Correct max and min velocities, no negative



(c) Correct max and min velocities, incorrect sign



(d) Half period

Figure 6.7: Students drawing of velocity versus time graph given position versus time graph

41% of students were able to draw the corresponding graph of velocity versus time in the pretest. Although it may be argued that these students may simply recall a memorized graph, most correctly labelled the positions A, B and E on both the given position versus time graph and the graph they drew for velocity versus time.

6.3.3.3 Acceleration

Interpreting acceleration from a non-SHM constant position versus time graph: This section refers to the question involving the position versus time graph for two cars. Students were asked at what time the acceleration of Car 2 (straight-line graph) was zero. Their answers are summarized in Table 6.9.

Table 6.9: Categorization of students' answers for where the acceleration of Car 2 is zero

Answer	**%(N=36)**
Correct (always)	78(28)
0 s	6(2)
Constant acceleration	11(4)
Other/No answer	6(2)

78% correct stated that the acceleration of Car 2 was zero at all times because its velocity was constant, but 11% appear to confuse constant acceleration with constant speed.

Interpreting how the sign of acceleration changes during SHM: This section relates to the pretest from year one. Table 6.10 shows students' responses to whether the acceleration of the block was positive, negative, or zero during three intervals of the motion of the block.

Table 6.10: Students' answers for acceleration for the three intervals

Answer	A **%(N=11)**	E **%(N=11)**	B **%(N=11)**
Positive	**82(9)**	36(4)	45(5)
Negative	9(1)	**45(5)**	**45(5)**
Zero	9(1)	18(2)	9(1)

Nine students correctly stated that the acceleration would be positive as the block moves from A to E. However, students' answers for the next interval indicate that only five students related the acceleration to the change in velocity over the interval. The other four students appeared not to distinguish between acceleration and velocity, as they give acceleration the same sign as velocity.

Interpreting how the acceleration changes during SHM given velocity versus time graph: In the first part of the year three pretest students were asked to label a given velocity versus time graph. As mentioned in Section 6.3.3.2 only eight out of 27 students labelled the graph correctly. In this part of the question students were asked to consider whether the magnitude of the acceleration was increasing or decreasing and whether it was positive or negative during each of the intervals.

No student did this correctly for all intervals. Over 60% of students described the acceleration for two or more intervals the same way. For half of our students their misinterpretation may be because they only labelled the positions A, B, and E at the peaks of the velocity versus time graph and so it would not have been possible for these students to reason about how the acceleration was changing from the velocity versus time graph. In some cases, their graphs indicated that the acceleration was, for example, decreasing and negative, and increasing and positive, in the same interval.

Drawing acceleration versus time given the velocity versus time graph: Students were asked to draw the acceleration versus time graph using the qualitative information about how the acceleration was changing during the motion, which they obtained from the given velocity versus time graph, described in the previous section.

22% of students correctly drew and labelled a graph for acceleration versus time. Most of these students had not correctly labelled the velocity versus time graph. Overall, the majority of students did not provide an explanation for their graph, and it appeared that students had essentially guessed the shape of their graph. This is supported by the fact that only one student drew a graph that represented how they had interpreted how the acceleration was changing from the velocity versus time graph. Less than 20% could have drawn a complete sine or cosine shaped graph from how the had interpreted acceleration.

6.3.3.4 Conclusions and implications for teaching

Our students were found to experience many of the same difficulties with relating position, velocity, and acceleration, and the graphing of these quantities, reported from previous studies. When qualitatively interpreting the motion of a block moving with SHM, most students could correctly identify how the magnitude and the sign of the velocity of the block would change throughout the motion. However, when considering the acceleration some students treated it the same as the velocity, a misconception previously discovered by Trowbridge and McDermott[3].

An inability to extract and present information in the form of a graph discussed by McDermott et al[4] was illustrated by our students, when despite being capable of reasoning about the velocity throughout SHM, only between 30% and 40% could draw and label a velocity versus time graph for the motion.

We also saw that students have greater difficulty interpreting non-linear graphs than linear graphs[4], which affects how they use the slope of a SHM graph to reason about the corresponding concept.

The year three pretest highlighted how a combination of these difficulties hindered all students in reasoning correctly and in drawing a correct graph for the acceleration of the block despite being provided with the velocity versus time graph for the motion.

Graphs are a widely used mode of presenting information in physics and so reading and drawing graphs are essential skills that students will require frequently in many courses. It is clear from these pretest results that our students struggle with this form of representation upon entering the Waves and Optics course. Hence, a concious effort must be made to address the identified difficulties students have with graphing. Students prior experience with graphs at this point would have been plotting quantitative data in their mechanics labs and calculating a numerical value for a slope. To improve students' conceptual understanding of graphs, the developed SHM instruction focuses on qualitative reasoning about the various aspects of a graph.

Students incorrectly reasoning about acceleration as velocity over time rather than a change in velocity over time is another difficulty that must be addressed. Simple harmonic motion is an ideal context to both investigate and develop students' reasoning about the motion concepts and the graphing of these quantities with time, because there is a changing magnitude and sign for position, velocity, and acceleration. The curved but repetitive line shape of the graph presents an ideal opportunity to develop students ability to interpret slope and even use this interpretation to relate the motion concepts.

6.3.4 Student interviews

Before changes were made to instruction in year three, individual teaching-learning interviews were carried out in order to examine students' ability to interpret the slope of velocity versus time graphs and so investigate if using the slope of velocity versus time would be a sensible approach to developing students' understanding of acceleration in SHM.

Interviews were carried out with 10 students with the aim of gaining a deeper understanding of the difficulties students have with interpreting and drawing graphs, and understanding the motion of a simple harmonic oscillator. In particular, we wanted to investigate if students' ability to interpret the slope of a velocity versus time graph could be used to help them reason about acceleration. Because of the limited time for each interview, sometimes students had to be reminded about the relationship between velocity and acceleration so that determining if students could use the slope could be focused on.

The participants were first year physics students taking a calculus-based mechanics course (5) and first year non-physics science students taking an algebra-based mechanics course (5). Position, velocity, and acceleration had been covered in lectures earlier in the semester in both courses and students had graphed how these quantities changed with time in different contexts during laboratories. No differences were found between students from both courses during the interviews and so they are not distinguished between during the discussion of the interviews.

The interviews entailed students considering situations similar to the pre-test questions described in Section 6.3.2.3, which involved answering questions on the velocity of two cars from their position versus time graph, and Section 6.3.2.2, where students had to draw the velocity and acceleration versus time graphs for a one period of SHM given the position versus time graph.

In the excerpts from the interviews, "..." represents a prolonged pause in students' answers.

6.3.4.1 Acceleration from the slope of velocity versus time

The graph of velocity versus time presented to students is shown in Figure 6.8. Asking students where the acceleration of the cars was zero and where it was greatest allowed us to determine (i) if students appreciated that acceleration is a change in velocity over a change in time, (ii) recognized how the slope of a velocity versus time represents this, and (iii) if students could use the slope of this graph to interpret changes in velocity at a particular time.

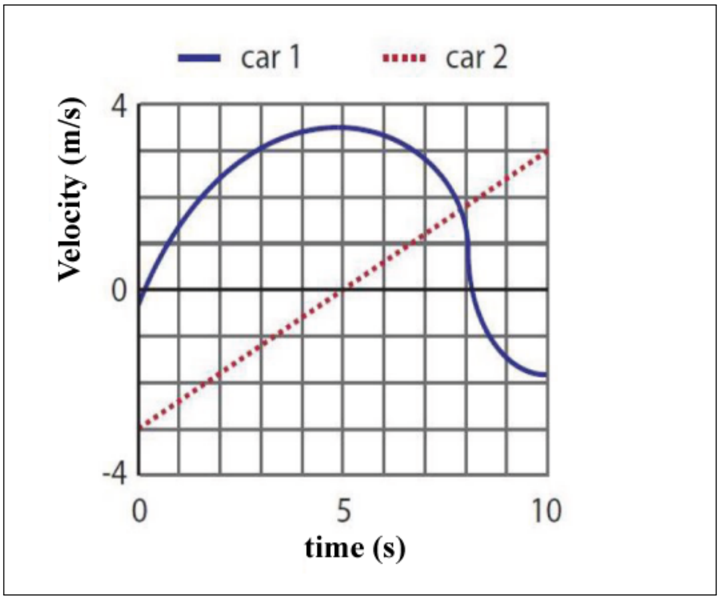


Figure 6.8: Velocity graph for two cars

When students were asked at what time Car 1 had zero acceleration only four students answered correctly. The reasoning used by these four students included that it was “the turning point of the curve”, that it was “a horizontal line”, or that there was “no change in velocity at that time”.

For the remaining six students, it was their inability to relate acceleration to velocity that appeared to be the main difficulty. The following excerpt is taken from an interview with student 5, after they had been asked to indicate where the acceleration was zero on the velocity versus time graph and the student indicated $t = 0$ s. This student is struggling to relate acceleration to $\Delta v/\Delta t$ and initially links acceleration to force.

Interviewer: *What is acceleration?*

Student 5: *The initial force...*

Interviewer: *So in terms of speed?*

Student 5: *The speed is increasing, acceleration, I can’t remember the formula. Speed over time or something*

Interviewer: *Ok so change in speed over time, so if I had no acceleration would I have a change in speed?*

Student 5: *No*

Interviewer: *Ok so I need to find a place on the graph where there is no change in speed*

At this point the student indicated the correct time ($t = 5$ s).

The other five students initially treated acceleration as velocity versus time but when reminded that it was a change in velocity versus time, had no difficulty indicated where on the graph the car had zero acceleration.

After students recognised where Car 1 had zero acceleration, all 10 students correctly identified the time where Car 1 had greatest acceleration, and stated that the acceleration of Car 2 was constant. Students' answers to this part of the interview indicate that once students can relate velocity and acceleration correctly they have no difficulty identifying how the slope represents this relationship.

6.3.4.2 Labelling a SHM position versus time graph

In the next part of the interview, the interviewer used Figure 6.9 to explain to students the full motion of a block on a spring after the spring was compressed to point A, and released. The interviewer also stated that the equilibrium position was the $x = 0$ position, that any position to the right of equilibrium was to be considered a positive position, and any position to the left of equilibrium as a negative position. Students were initially asked to label the equilibrium position (E), the maximum positive position (B), and the maximum negative position (A) on the given position versus time graph (Figure 6.9).

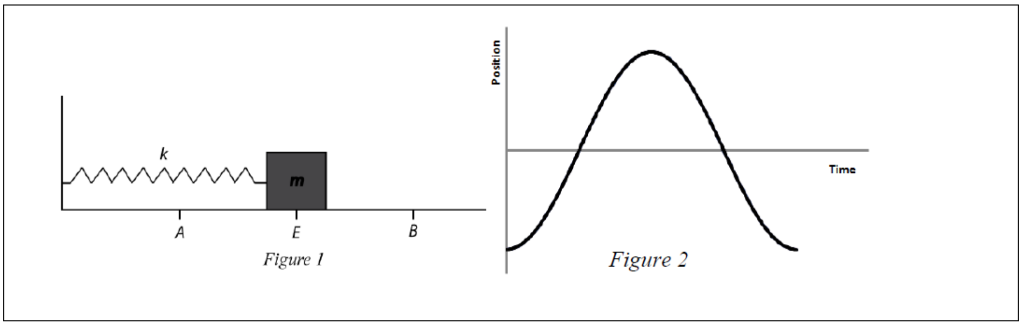


Figure 6.9: Simple harmonic oscillator and a graph of position versus time for one period of its motion

Eight students labelled the graph correctly without prompting. The other two students initially labelled E at the maximum positive position and A and B in the maximum negative positions, as we had seen some students do in the pretest question in year two. However, these students only needed slight hints to correct their labelling:

Interviewer: *Why is* E *at the max point on the graph?*

Student 7: *It stops*

Interviewer: *If* A *is our max negative, where is the max positive?*

Student 7: [points to the maximum positive position] *So* E *is there.* [correct position ($x = 0$)]

6.3.4.3 Qualitative reasoning about the velocity and the graphing of velocity versus time

After successfully labelling the position versus time graph (Figure 6.9) students were asked to draw the corresponding graph for velocity versus time. No student could begin to draw the velocity versus time graph on their own. The approach taken by the interviewer was to get students to determine first at which positions the magnitude of the velocity was zero and greatest, then whether the magnitude of the velocity was increasing or decreasing between each of the positions, and what the sign of the velocity was in each interval. The interviewer then asked students to plot the velocity versus time graph.

Seven of the students were able to reason how the magnitude of the velocity changed throughout the motion, stating that the velocity of the block had to be zero at A and B as it turned around at these points. However, the interviewer had to explain to the other three students how the restoring force of the spring affected the motion before they could reason about how the velocity changes.

Most students determined the sign of velocity for each interval without further help from the interviewer. In some cases, students needed to be reminded of the convention of choosing a direction towards the right as the positive direction.

Once this information had been determined students had few difficulties representing it on the graph. Two of the students drew constant slopes for each interval, but when asked about what this indicated about the velocity, they seemed to recognise that the graph should be curved, although confusing what the slope of the velocity graph represented:

> **Interviewer:** *Do you think that the velocity increases at a constant rate?*
>
> **Student 9:** *I seem to consistently make that mistake, not knowing the difference between a kind of rigid graph versus a curved one. But obviously it's not going to go at* 10 *m/s from* E *to* B *and* 10 *m/s back all the time*
>
> **Interviewer:** *Well you have drawn a changing velocity but a constant acceleration, which would mean that you would have a horizontal slope for the acceleration versus time...*
>
> **Student 9:** *I don't think that's right, I think it should be curved.*

6.3.4.4 Qualitative reasoning about the acceleration and the graphing of acceleration versus time

When asking the students to draw the corresponding graph for acceleration versus time, the interviewer took the same approach as with velocity, first getting students to reason qualitatively about the changing acceleration and then getting them to plot it. When asked to compare the magnitude of the acceleration at points A and E seven students stated that the block had a greater acceleration at A, but none of these students could explain why this was the case. When questioned further two of these students thought that the block would have greater acceleration at E than B, stating that A was greatest because when you release the block at A it goes from zero velocity to moving rapidly. Three students stated initially that E would have the greatest acceleration, again relating acceleration and velocity incorrectly. With all students the interviewer, drew tangents on the curve of their velocity versus time graph and asked them to explain acceleration from the change in velocity:

Interviewer: *From A to E the acceleration is...?*

Student 1: *Positive and increasing*

Interviewer: *So..*

Student 1: *There's more acceleration at E*

Interviewer: *So what is acceleration, again?*

Student 1: *A change in velocity...*

Interviewer: *So if we look back to our velocity graph and look at the points close to A and E there's...*[drawing tangents]

Student 1: *More change in velocity at A than E*

Interviewer: *So where is the acceleration greater?*

Student 1: *At A*

With regard to drawing this graph, all students needed to be prompted to consider again where the magnitude of the acceleration was greatest and needed reassurance about how this would be represented on the graph. Once they drew the first interval all students were able to complete the graph by themselves.

6.3.4.5 Conclusions and implications for teaching

In terms of reasoning about the motion in SHM, students experienced little difficulty when determining the changing magnitude and sign of the position and velocity. However, we observed the same difficulty students have relating acceleration to velocity as we had in pretests.

Students had no trouble determining how the slope of the velocity versus time graph represents no change in velocity, a large change in velocity, and constant velocity.

It was interesting to see the difference in students confidence when drawing a graph that started at the origin and increased initially compared to when the graph started on the positive y-axis and decreased initially. Once students had drawn the first interval, most of them completed the other interval without consulting the qualitative information about the motion they had determined previously.

Using the slope of the velocity versus time graph to determine the positions where acceleration was greatest and where it was zero and to determine the sign of the acceleration seemed to be successful during the interviews and as a result this approach was adapted for tutorials in year three.

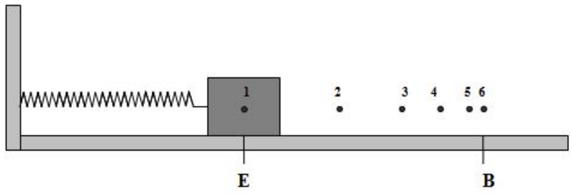
6.4 Approach one

This section describes the tutorial instruction and the results of the post-test question administered to students in year one. At this point in the study the aim of the instruction was primarily on developing students' understanding of SHM. The tutorials were developed prior to the difficulties our students experienced with the motion concepts and their graphing being determined.

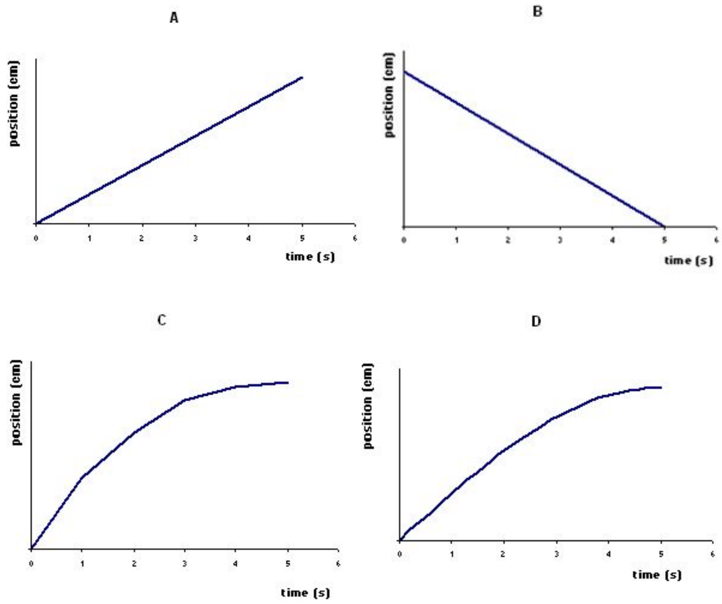
6.4.1 Instruction

Appendix R shows the tutorial and homework administered in year one. The context of a compression spring mounted horizontally attached to a block of mass m positioned on a frictionless surface was used. Instruction followed the sequence was: position versus time; velocity versus time; acceleration versus time; and the relationship between position and restoring force. The tasks surrounding each motion concept involved students first reasoning qualitatively about the concept for one interval (E) of the motion before plotting that concept with time for the particular interval. Students then repeated this for the other three intervals.

The tutorial began by reminding students where the block would move when it is compressed and then released. Students were presented with Figure 6.10(a) which shows the position of the block at one second intervals as it moves from E to B. Four different line shapes for position versus time (Figure 6.10(b)) were shown (a constant increasing slope, a constant decreasing slope, a line consisting of different constant increasing slopes, a continuously varying slope) and students chose which one best represents how the position of the block changes with time. Students were expected to reason that the slope could not be constant as the change in position each second is not the same, and that it also unlikely that the block travels at a constant rate for a second and then changes to a slower constant rate for the next second, and so on. In this way students saw that the curved graph is the only sensible line shape for this motion. Students used this line as a reference point for how the shape of the line for the remaining intervals should be drawn when they determined the sign and change in magnitude of the position for those intervals.



(a) position changing with time from E to B



(b) line shapes for position versus time

Figure 6.10: Year one instruction

The velocity versus time section of the tutorial was structured similarly to the position versus time section. Students drew velocity vectors at each of the positions in Figure 6.10(a) and determined whether the block was speeding up or slowing down. Again four line shapes were presented: a constant increasing slope, a constant decreasing slope, a continuously varying negative slope, and a continuously varying positive slope. Students considered the sign of the velocity and whether the block was speeding up or slowing down in each interval, before plotting this information on a graph.

Students used the change in velocity to reason about the sign and changing magnitude of the acceleration and the same line shape procedures applied.

Lastly, students considered the relationship between position and restoring force (a section that is adapted from Tutorials in Introductory Physics[6]). Students drew vectors to represent the position of the block, and the force on the block by the spring when the block was to the left, and right of the equilibrium position. Based on these vectors students reasoned about the relationship between position and restoring force. Using Newton's Second Law students linked the position and the acceleration of the block.

6.4.2 Post-test

This section describes the post-test question administered in the end-of-semester exam in year one and the responses students gave.

6.4.2.1 Post-test question

Figure 6.11 shows the post-test administered in year one. This question is similar to the pretest question described earlier for year two. The question provides students with the position versus time graph for one period of the blocks motion and asks them to label positions A, B, and E on the graph. From this, we could determine if students could map the physical situation presented onto a graph. Students would have to realise that the equilibrium position, E, was the 'zero' position, that A was the maximum negative position, and that B was the maximum positive position. They would then need to know how each of these are represented on a graph.

The next part of the question asked students to draw the corresponding graph of acceleration versus time and explain why they drew them this way. Students could have approached this question either by reasoning about the physical situation, determining how the force and hence acceleration changed throughout the motion, or they could have used the slope of the position versus time graph to draw a velocity versus time, and then the slope of this velocity versus time graph to draw acceleration versus time.

A block of mass m is placed on a frictionless surface and is attached to a compression spring with spring constant k as shown in *Figure 1*. The block is compressed to position A and is then released. *Figure 2* shows one full period of the blocks position versus time graph.

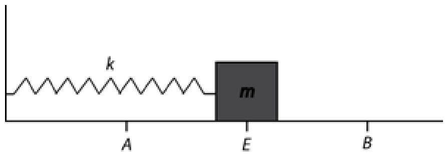


Figure 1

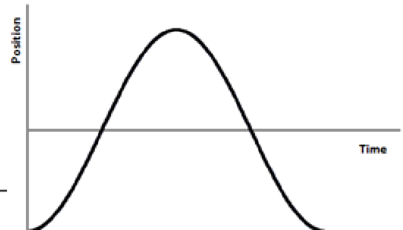


Figure 2

(a) Copy *Figure 2* into your answer booklets and label clearly where positions A, B and E occur on the graph.

(b) Draw the corresponding graph of acceleration versus time for the same situation. Explain why you drew the graph this way.

Figure 6.11: Year one post-test

6.4.2.2 Position versus time

Table 6.11 shows a categorization of students' labelling of the position versus time graph. Students' answers fit the same categories as the same pretest (described in Section 6.3.3.1). 85% of students correctly related the physical situation to the given position versus time graph.

Table 6.11: Students' labelling of the position versus time graph

Category	**%(N=20)**
Correct	85(17)
E in the middle	5(1)
E at the start	5(1)
E at the end	5(1)

6.4.2.3 Acceleration versus time

45% drew the correct graph for acceleration versus time. 25% did not use physics reasoning but stated the positions where acceleration was a maximum and where it was zero, and the sign of the acceleration for the different intervals as an explanation for drawing the graph this way. The other 20% explained why acceleration was a maximum and zero at the corresponding positions, and why the acceleration was positive and negative in certain intervals, all using restoring force reasoning:

> *"$F = -kx$, $\vec{a} = -\frac{k}{m}x$, $\vec{a} \propto -x$. At E, x is 0 so a is 0. At A, x is at its most negative so a is at its most positive"*

> *"At A the maximum force is exerted by the spring as it is most compressed. This force results in an acceleration in the $+x$- direction until the block reaches equilibrium position, E. Here there is no force acting i.e. no acceleration, but the block overshoots this position. The restoring force now acts in the $-x$-direction causing an increasing negative acceleration which is at a maximum at B."*

30% of students drew a graph that would be correct for velocity versus time with some students providing an explanation that suggests that they relate acceleration directly to velocity and not a change in velocity:

> *"at point B for the block to completely change direction it must come to a complete stop, i.e. acceleration $= 0$"*

The remaining 25% drew incorrect acceleration versus time graphs either reasoning incorrectly about how the acceleration changed during the motion, or incorrectly plotting their correct statements about the acceleration, for example one student thought that a downward slope on the acceleration versus time graph represented negative acceleration and visa versa.

6.4.3 Conclusions and implications for teaching

In the post-test most students could identify the maximum positive and negative positions on the position versus time graph but less than half could draw the correct corresponding acceleration versus time graph.

The 30% of students who continued to treat acceleration the same as velocity rather than as a change in velocity confirm that this is a persistent difficulty. As a result, acceleration became a more central aspect of the SHM tutorial instruction in year two. Also, considering how students who could reason completely about the changing acceleration all did so through the relationship between position and restoring force, we decided that this was the approach that we should promote among our students.

6.5 Approach two

6.5.1 Instruction

Based on the results from the post-test in year one we decided to explore a different approach. While most of the tasks were the same, the sequence used in year one was reversed. Instead, students began by considering the relationship between restoring force and position, and using this relationship and Newton's Second Law to reason qualitatively about the acceleration throughout the motion. The velocity tasks remained the same as in year one. The instruction is shown in full in Appendix S.

6.5.2 Post-test

This section describes the post-test question administered in the end-of-semester exam in year two and the responses given by students.

6.5.2.1 Post-test question

Figure 6.12 shows the post-test administered in year two ($N = 17$). The parts of this question were structured in the same sequence that the tutorial instruction had followed. The same simple harmonic oscillator situation was presented and students were shown a labelled position versus time graph.

Students were asked to draw restoring force vectors for each of the intervals. We assumed that this would be a task that would present little difficulty for students but we intended the task to prompt students to consider the restoring force when considering the acceleration in the next part of the question.

This next part required students to identify the correct acceleration versus time graph from a choice of four graphs. Graph 3 was the correct graph for acceleration versus time. We expected that: students who correctly identified the positions where the magnitude of the acceleration was greatest and and zero but neglected to consider the sign might choose Graph 4; students who treated acceleration as velocity over time might choose Graph 1; and students who treated acceleration as velocity over time but neglected to consider the sign might choose Graph 2.

The last part of the question asked students to draw the corresponding graph of velocity versus time.

A block of mass *m* is placed on a frictionless surface and is attached to an extension spring with spring constant *k* as shown in *Figure 1*. The block is stretched to position *B* and is then released. *Figure 2* shows one full period of the blocks position versus time graph.

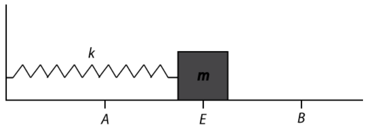


Figure 1

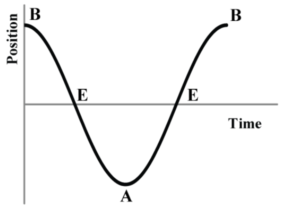
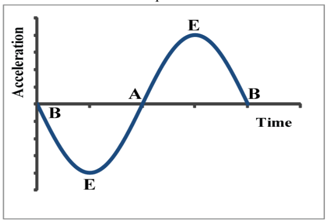


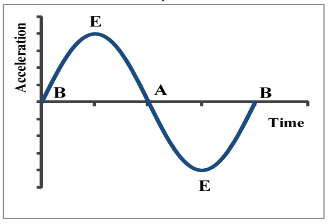
Figure 2

(a) Draw the position of the block for each interval (B-E, E-A etc.) like in *Figure 1*.
(b) On the same diagrams you drew in (a), indicate the direction and the sign of the restoring force for each interval. Explain.
(c) Which of the following acceleration versus time graphs correctly represents the motion described? Explain your reason for choosing this graph.

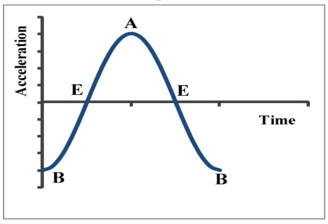
Graph 1



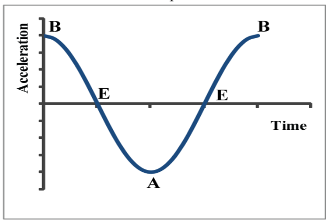
Graph 2



Graph 3



Graph 4



(d) Draw the corresponding graph of velocity versus time for the same situation. Explain why you drew the graph this way.

Figure 6.12: Year two post-test

6.5.2.2 Restoring force vectors

Only 59% of our students (10/17) were able to correctly draw the restoring force vectors for each of the intervals. The other 41% drew vectors that showed the displacement of the block during the intervals. The force of the spring acting towards the equilibrium position at all times is the basis of SHM and had been described to students during lecture. Also, students had drawn restoring force vectors in tutorial when the block was to the left and right of the equilibrium position.

40% of students either cannot reason why the block moves between the positions A and B or they are unable to draw vectors that represent how the force acts on the block. So a task intended to guide students in their reasoning about the acceleration throughout the motion, actually indicates that for many students thinking about the motion in this way would not be useful in answering this question.

6.5.2.3 Acceleration versus time graph

The number of students who chose each of the graphs is shown in Table 6.12.

Table 6.12: Students' choice of acceleration versus time graph

Graph	**%(N=17)**
1	12(2)
2	35(6)
3	**29(5)**
4	24(4)

Only 29% of students identified the correct acceleration versus time graph. Only one student correctly used the restoring force reasoning:

> *"The acceleration is at its greatest when the spring is at A and B... I also know that the force from B to E is a negative vector, which would give negative acceleration"*

The other explanations were similar to the following:

> *"When it is released from point B it begins to accelerate. It reaches point E and continues to A even though it is slowing down it is still accelerating because its velocity is changing."*

The explanations from the 24% of students who chose Graph 4 state correctly where the magnitude of the acceleration is greatest and zero but most did not mention sign. One student stated that the acceleration from B to E was negative and that the downward slope in Graph 4 was a result of that.

The majority of the 47% of students who chose Graphs 1 or 2, incorrectly relate the acceleration to the velocity:

> *"Graph 2 correctly represents the acceleration as it starts at rest, then accelerates to E, decelerates to A and stops again"*

> *"Graph 1 describes the motion... it accelerates in the negative direction towards E and decelerates to A where it stops, and accelerates in the positive direction toward E..."*

6.5.2.4 Velocity versus time graph

Table 6.13 shows a categorization of students' drawing of the velocity versus time graph.

Table 6.13: Students' drawing of velocity versus time graph

Category	**%(N=17)**
Correct	29(5)
Correct max and min velocities, incorrect sign	41(7)
Same as position versus time graph	18(3)
No answer	12(2)

71% of students correctly determined the positions where the velocity of the block was greatest and zero, correctly representing these instances on a graph. However, only 29% considered the sign of the velocity.

18% of students drew the same graph as the given position versus time graph. These students had also chosen the same graph for acceleration versus time. In each case they reason that the maximum positions are the locations of the greatest acceleration and velocity.

6.5.3 Conclusions and implications for teaching

The change in approach in year two was not successful in improving students reasoning about the acceleration during SHM. Some students could not identify the direction of the restoring force during the intervals of the motion, and most of those who did did not apply restoring force reasoning when considering acceleration. Students continued to reason about acceleration using velocity, and students' difficulties relating these two concepts were evident again.

These results prompted us to return to the tutorial sequence from year one. However, a new approach to tackling students' difficulties in understanding the relationship between velocity and acceleration would be required.

6.6 Approach three

This section will describe the changed instruction resulting from the student interviews described in Section 6.3.4. Results from students' answers to the post-test from year three are also described.

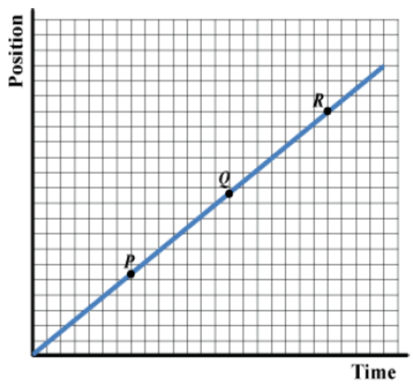
6.6.1 Instruction

Appendix T shows the tutorial instruction used in year three.

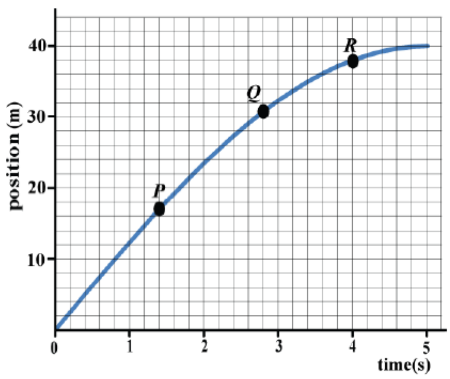
6.6.1.1 *Position, Velocity, and Acceleration* tutorial

A pre-SHM tutorial was introduced in year three to remind students how to use the slope of uniform and non-uniform position versus time graphs to draw velocity versus time graphs, and then in the case of the non-uniform motion graph how to use the slope of the velocity versus time graph to draw acceleration versus time graphs.

The graphs students were asked to consider are shown in Figure 6.13. Both graphs were placed on a grid.



(a) uniform motion graph



(b) non-uniform motion graph

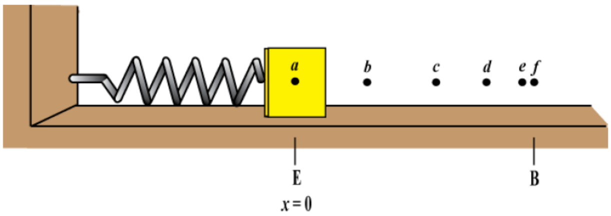
Figure 6.13: Pre-SHM tutorial position versus time graphs

For the uniform motion graph, students first considered the change in position at different points on the graph and explained what this indicates about the velocity. Students then plotted the velocity versus time on a given axis.

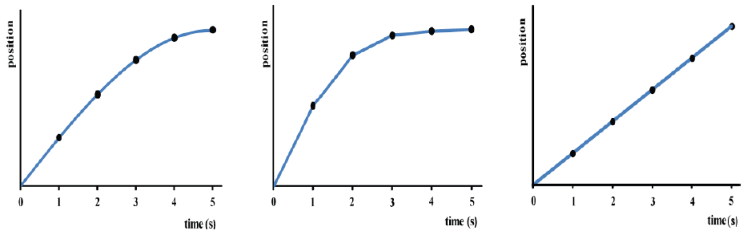
A similar approach is taken for the non-uniform motion graph. In order to determine the velocity at the different points on the graph, students drew tangents to the curve at those points and calculated the value of the slope of the tangent using the grid. The non-uniform motion graph is a quarter-sine curve and so the process of calculating the slope of the tangents of the curve, demonstrates to students in advance the general line shape of the position, velocity, and acceleration versus time curves for SHM.

6.6.1.2 *Position in Simple Harmonic Motion* tutorial

This tutorial begins the same way as in the previous two years. The context of a compression spring mounted horizontally attached to a block of mass m positioned on a frictionless surface was used. Students were presented with Figure 6.14(a) which shows the position of the block at one second intervals as it moves from E to B. Three different line shapes for position versus time (Figure 6.14(b)) were shown(a constant increasing slope, a line consisting of different constant increasing slopes, a continuously varying slope) and students chose which one best represents how the position of the block changes with time. Students were expected to reason that the slope could not be constant as the change in position each second is not the same, and that it also unlikely that the block travels at a constant rate for a second and then changes to a slower constant rate for the next second, and so on. In this way students saw that the curved graph is the only sensible line shape for this motion. Students used this line as a reference point for how the shape of the line for the remaining intervals should be drawn when they determined the sign and change in magnitude of the position for those intervals.



(a) position changing with time from E to B



(b) line shapes for position versus time

Figure 6.14: Year one instruction

Students drew the position of the block at one second intervals as it moves back from B to E, and reason about the difference in position versus time in the two intervals. Students complete this process for the other two interval, summarize the position versus time for all intervals in terms of increasing and decreasing magnitude and sign, and then drawn the position versus time graph for the full motion.

6.6.1.3 *Velocity in Simple Harmonic Motion* and *Acceleration in Simple Harmonic Motion* tutorials

In the velocity tutorial students are initially provided with a graph of position versus time for the interval, E to B, and in the acceleration tutorial students are initially provided with a graph of velocity versus time for the same interval. Students answer questions similar to those in the pre-SHM tutorial, described above. This process is repeated for each interval of the motion.

6.6.2 Post-test

This section describes the post-test question administered in the mid-semester continuous assessment exam in year three and the responses given by students.

6.6.2.1 Post-test question

Figure 6.15 shows the post-test administered in year three ($N = 40$). The SHM context used is different than the context used in tutorials, and other pretest and post-test questions. Students are asked to consider the motion of a boy on a fairground ride that moves vertically between ground level and a height of 20 m above the ground. They are told that the motion is simple harmonic.

The first part asked students for the centre of motion, which students were expected to identify as the half-way point between the ground and the maximum height, 10 m. Students were then told to set the centre of motion as the origin and that upwards was a positive direction. Students were asked to draw the position versus time graph, beginning when the boy is at ground level. Students should have determined that the center of motion is the equilibrium position ($x = 0$) in this case, that the ground is the maximum negative position, and that 20 m is the maximum positive position. As the motion they asked to consider starts at ground level, their graph of position versus time should have started on the negative y-axis and moved to zero, and so on.

The next part of the question asked students to draw the corresponding velocity versus time graph. From the information given in the question students should have recognized that when the boy is moving from ground to 20 m that the velocity is positive and when he is moving back down the velocity is negative. Students also had to realise that the velocity of the boy would be zero at the turning points, ground level and at a height of 20 m), and that his greatest velocity would be at the centre of motion.

Lastly, students were asked to use their velocity versus time graph to identify where the boy's acceleration is greatest. Students were expected to reason that because the slope of the velocity versus time graph was steepest near the maximum positive negative positions that these were the locations was where the rate of change of the velocity was greatest during the motion, and hence the acceleration is greatest at ground level and the maximum height of 20 m.

A boy of mass 40kg sits on a seat in a fairground ride. The seat travels with simple harmonic motion vertically between a point at ground level and a point that is at a height of 20m above the ground.

(a) At what height is the centre of motion?

Set the origin of the coordinate system at the centre of motion and choose upward as the positive direction.

(b) Beginning at the instant the boy is at ground level, sketch the position versus time graph. Explain why you drew the graph the way you did.

(c) Sketch the velocity versus time graph again beginning at the instant the boy is at ground level. Explain why you drew the graph the way you did.

(d) At which point(s) does the boy have the greatest acceleration? Explain with reference to your velocity versus time graph.

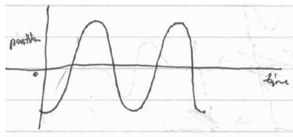
Figure 6.15: Year three post-test

6.6.2.2 Position versus time graph

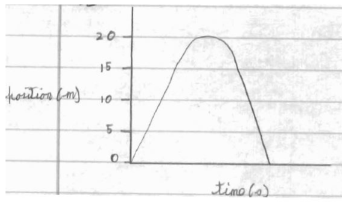
Table 6.14 shows a categorization of the position versus time graphs drawn by the students. Figure 6.16 shows examples of a graph from each category.

Table 6.14: Students' position versus time graphs

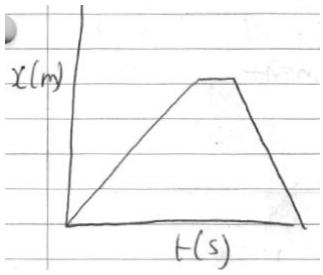
Category	**%(N=40)**
Correct	75(30)
Correct based on incorrect center of motion	13(5)
Other	10(4)
No answer	3(1)



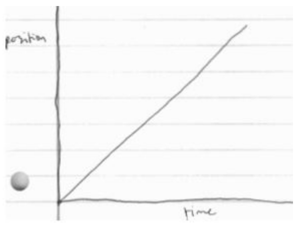
(a) Correct graph



(b) Correct based on incorrect centre of motion



(c) Incorrect graph a



(d) Incorrect graph b

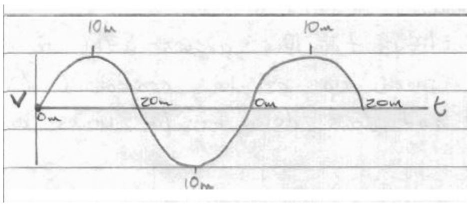
Figure 6.16: Students' position versus time graphs

6.6.2.3 Velocity versus time graph

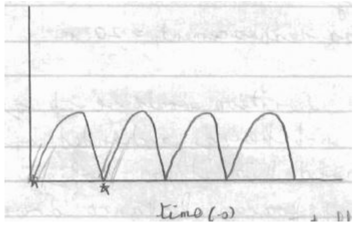
Table 6.15 shows a categorization of the velocity versus time graphs drawn by the students. Figure 6.17 shows examples of a graph from each category.

Table 6.15: Students' velocity versus time graphs

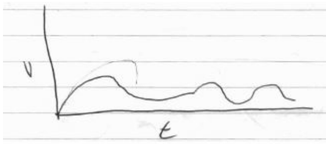
Category	**%(N=40)**
Correct	40(16)
Correct magnitude, no negative	20(8)
Same as position versus time	8(3)
Other	8(3)
No graph, correct velocity reasoning	5(2)
No answer	20(8)



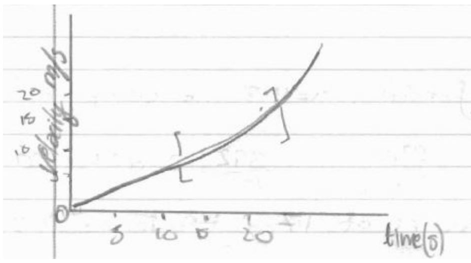
(a) Correct graph



(b) Correct magnitude, no negative



(c) Incorrect graph a



(d) Incorrect graph b

Figure 6.17: Students' velocity versus time graphs

6.6.2.4 Greatest acceleration

Table 6.16 shows a categorization of the answers students gave for where the acceleration was greatest.

Table 6.16: Students' answers for the position(s) where acceleration is greatest

Category	%(N=40)
Correct (used graph in reasoning)	28(11)
Correct (did not use graph)	33(13)
Equilibrium (velocity versus time graph correct)	8(3)
Other	8(3)
No answer	25(10)

61% of our students correctly identified that the acceleration would be greatest at ground level and the maximum height (20 m). 28% used the slope of their velocity versus time graph to reason why this was the case (8% had an incorrect velocity versus time graph but determined the correct position of greatest acceleration based on this):

> *"You can tell this from the velocity versus time graph as these are the*

points where the tangent to the curve will have the greatest slope"

The other 33% did not appear to use their graph, with the majority of these students just stating that the acceleration is greatest when the velocity is zero. It is difficult to tell if these students know why this is the case or if they are simply recalling this fact from memory.

8% of the students stated that the acceleration would be greatest at a height of 10 m, despite drawing a correct velocity versus time graph, and 25% did not answer this question (most of these students had not drawn the velocity versus time graph).

6.6.3 Conclusions

The post-test question was more difficult than the post-test questions from the other two years, in that the context was different to that used by the tutorials and it did not show students a graph of the position versus time for the motion. Yet students answered considerably better than the other years.

75% of the students drew the correct graph for the position versus time. Although it is true that incorrectly defining ground level as the centre of motion made the velocity versus time graph, a further 13% of students drew the correct position versus time graph based on this mistake.

Although the percentage of students correctly drawing the velocity versus time only increased slightly compared to the post-test in year two, there was a significant in percentage of students who could relate the acceleration to velocity correctly.

6.7 Conclusions

Pretest questions examined multiple aspects involved in relating position, velocity, and acceleration, and the interpretation and drawing of these graphs. Most students can interpret how the slope of a graph represents the rate of change of the quantity.

However, students experience difficulties interpreting velocity as $\Delta x/\Delta t$ and acceleration as $\Delta v/\Delta t$.

We have implemented three different instructional approaches to developing students reasoning about and graphing of position, velocity, and acceleration in a SHM context. Students answers to post-test questions in each of these years indicated some improvement in their understanding of these concepts. In year one, the most common difficult we found was students' inability to distinguish between velocity and acceleration. Students who answered correctly, all reasoned about the acceleration in terms of the restoring force, and so it seemed that a sensible way to develop students reasoning of acceleration was to promote this approach. However, in year two we saw students persistence in reasoning about acceleration in terms of velocity, and again most students used an incorrect relationship between the two.

After student interviews confirmed that a sensible approach might be to get students to reason about acceleration using the slope of the velocity versus time graph, this was the approach adopted in year three. This approach was the most successful of the three in terms of students reasoning about acceleration in SHM.

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Chapter 7

Conclusions

This chapter will summarise the main findings from both parts of this study, answering the research questions outlined in Chapter 2.

7.1 Mathematical tools in electromagnetism

For many electromagnetic calculations a combination of the mathematical tools discussed in this thesis are required. This means that when students perform poorly on test questions asking them to calculate quantities such as the line integral of an electric or magnetic field, it is difficult to determine where exactly students' difficulties lie. One of the aims of this thesis was to investigate students' understanding and use of integration, vector addition, and dot products separately, so that instruction could be designed to develop students' conceptual understanding of these tools and to tackle specific difficulties students have in using them. The following paragraphs will answer the research questions outlined in Chapter 2 for each tool individually.

We have found that it cannot be assumed that students enter the course with a working understanding of integration. We found that less than half our students knew to use integration in post-test questions. In cases where integration was cued, students often experienced difficulties determining limits and evaluating the integral. Analysis of students' answers to pretest questions showed that after completing an introductory calculus course few students had any conceptual aspects of integration as part of their concept image and evaluation was the most prominent aspect of their concept image. Students also struggled to interpret basic integrals in a simple setting, with many students not recognising the meaning of the infinitesimal term especially when an integrand was present. Instruction was developed where students considered integration as a process of accumulation using the calculation of total charge on a non-uniformly charged rod as a context. Although students still experienced technical difficulties when applying integration, the number of cases where integration is cued increased and students' reasoning about integral improved.

With regard to vector addition, pretests showed that even after completion of an introductory physics course, many of our students neglected to consider the vector nature of quantities such as force and field. In some cases recognizing that force is a vector quantity did not stop students treating it as a scalar when adding. The technical aspect of vector addition, where basic trigonometry is required to calculate vector components, was even more problematic for students at this point. A tutorial aimed solely at developing students' understanding of the head-to-tail and component-wise addition of vectors did not appear to have a significant impact on students reasoning in post-tests. However, a tutorial where students complete field component calculations following a conceptual introduction appeared to improve the approach students took in a net electric field post-test calculation.

The technical aspects involved with dot products appeared to be the most problematic for students. Many students could not evaluate the dot product $\vec{E} \cdot \widehat{n}$ in the simplest possible setting. Developing students' understanding of the process involved in a dot product calculation was achieved in the context of work, which students were familiar with and which many students seemed to be able to reason about already. Comparison of students' calculations of electric flux in years before and after this instruction showed a significant increase in the number of students who consider the cosine dependence.

7.2 Students' understanding of the relationships between position, velocity and acceleration

Many physics contexts require an understanding of the relationships between position, velocity, and acceleration, and the ability to interpret and draw graphs of these quantities with time. This part of the study aimed to determine the extent to which these difficulties were present for our students after completing introductory mechanics courses. This section will address the research questions outlined in Chapter 2.

Can students interpret varying position, velocity, and acceleration versus time graphs? We found that most students could interpret how the slope of a graph represents the rate of change of the quantity in a non-SHM context. However, when asked to label given position and velocity versus time graphs for SHM less than half our students could label the maximum positive and maximum negative positions of these graphs, despite being able to reason quantitatively about these quantities.

Can students draw varying position, velocity, and acceleration versus time graphs? In this study, we only asked students to draw these graphs in the context of SHM. For velocity versus time, less than half our students could draw the correct graph. Incorrect graphs were mostly a result of students neglecting to consider the sign of the velocity or representing a negative velocity incorrectly on the graph. Fewer students can draw acceleration versus time correctly. However, this appeared to be a consequence of their misinterpretation of how the acceleration changed throughout the motion. If students had based their graph on their quantitative description of the acceleration, the vast majority of students could not have drawn a continuous graph for the motion.

Can students reason conceptually about the relationships between position, velocity, and acceleration generally and in the context of SHM? We found that many students persistently treated velocity as position over time, and acceleration as velocity versus time.

What was the best approach to take to tackle these difficulties and improve students' understanding of the relationship between position, velocity, and acceleration? Students answers to post-test questions in each of three years suggested some improvement in their understanding of these concepts and their ability to graph them with time. Using the restoring force to reason about the acceleration during SHM, did not appear to be effective in changing students reasoning of acceleration, with many students still incorrectly relating the acceleration to velocity. The approach taken in year three, reasoning about acceleration using the slope of the velocity versus time graph, was the most successful of the three in terms of students reasoning about acceleration in SHM.

7.3 Overall Conclusions

The student difficulties presented in this study highlight the significant role that mathematics and formal representations play in physics. If our students are representative of the general population then many students enter introductory and intermediate physics courses without the mathematics knowledge required to understand physics concepts and solve problems, or the ability to interpret and use common representations of physics phenomena.

We have identified specific difficulties that students have with the mathematical tools of integration and vector operations, and with interpreting and drawing graphs. When these specific difficulties were addressed in tutorials, there was evident gains in students' conceptual understanding of the tools and graphs, and in students' use of the tool or graph. In some cases, it was clear that this increased understanding of the tool or representation improved students' reasoning about the physics concepts involved. This indicates that awareness of students' beginning knowledge and difficulties can lead to more effective instruction.

Appendix A

Opt-Out Clause

This Appendix contains the permission sheet administered to students where they could choose to consent or not consent for their answers to test questions to be quoted anonymously.

Name: ______________________________ Class: __________

We do research on the efficacy of this module. Very occasionally, we may want to quote somebody's answer anonymously and non-attributably - *i.e.,* in a way that it can't be related back to the student. Typical examples would be: "30% of students gave an answer like X", or "Three students commented that the charge was negative."

Clearly, we hope that all of you will consent to being quoted (however unlikely this is), so that we have all your answers available. However, it is entirely up to you whether you give your consent to be quoted anonymously and non-attributably or not - there are no repercussions if you opt out, and we will not ask you to explain why.

☐ I consent to be quoted anonymously and non-attributably

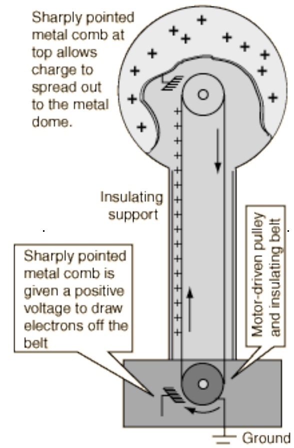
☐ I do not consent to be quoted anonymously and non-attributably

Appendix B

A Van De Graff Generator Tutorial

This Appendix contains the tutorial worksheet which introduces the concept of charge density.

I. An experiment with a Van de Graaff generator



A Van de Graaff generator uses a moving belt to separate charges. A pulley drives the insulating belt by a sharply pointed metal comb which has been given a positive charge by a power supply and removes electrons from the belt, leaving it positively charged. A similar comb at the top then gives electrons to the belt, leaving the sphere positively charged.

It is possible to measure the total positive charge on the sphere, which is equal to the total positive charge given to the belt. In an experiment, Student 1 measures the total charge on the sphere every time the belt moves 2 cm (starting after the belt has been moving for some time). Her results are given in the table at right. The first row should be read: after 118 cm of belt had passed under the upper comb, a charge of 111 nC has accumulated on the sphere.

position (cm)	charge (nC)
118	111
120	115
122	119
124	123
126	127
128	131
130	135
132	139

A. Describe the charge distribution on the part of the belt investigated by the student. Explain briefly.

B. Based on your answer in part A, predict how much charge there is on any 1 m of the belt. (Disregard anything that may have happened before the student made her measurements.) Explain.

Also predict how much charge there is on 1 cm of the belt. Explain.

C. In your own words, describe a procedure you could use to calculate the linear charge density on the belt. Do *not* use a formula.

D. Determine the linear charge density on the belt.

Give an *interpretation* of the linear charge density of the belt, *i.e.*, explain the meaning of the number you just calculated. (*Hint:* Which of the charges that you calculated in part B is numerically equal to the linear charge density?)

E. A belt that generates equal amounts of charge on the sphere when equal lengths of the belt pass the comb is said to be *uniformly charged*. What assumption is being made when this phrase is applied? (*Note:* You are *not* asked to describe the procedure you used in part D.)

Discuss with your partners whether the belt was uniformly charged, and write down your conclusions.

II. Further experiments

A. Four other students take data from the Van de Graaff generator, starting at different instants. Their data are shown in the table below.

Student 2		Student 3		Student 4		Student 5	
position (cm)	charge (nC)	position (cm)	charge (nC)	position (cm)	charge (nC)	position (cm)	charge (nC)
56	25	180	259	150	180	86	59
58	27	182	265	152	185	88	62
60	29	184	271	154	190	90	65
62	31	186	277	156	195	92	68
64	33	188	283	158	200	94	71
66	35	190	289	160	205	96	74
68	37	192	295	162	210	98	77
70	39	194	301	164	215	100	80

Does each of the students find a constant linear charge density? Explain briefly.

B. Student 6 makes similar measurements, but at 30 cm intervals. Is belt used by Student 6 uniformly charged? Explain briefly.

position (cm)	charge (nC)
0	0
30	7
60	29
90	65
120	115
150	180
180	259
210	353

C. Compare the measurements of Student 6 with those of the first 5 students. Explain that it is likely that all students made their measurements using the *same* belt.

Is the examination of a small part of the belt sufficient to reveal whether the entire belt has constant or varying linear charge density?

How can you modify the interpretation of the linear charge density so that it applies even to belts with varying linear charge density?

D. Consider the following statement:

> "*The linear charge density on the belt is equal to the total charge on the belt divided by its length, so it is equal to 353 nanocoulombs over 2.10 metres, which is equal to 1.68 nC/m.*"

Do you agree with this statement? If not, does the calculation give any useful information about the rod? Explain.

Note: Under normal circumstances, the belt on a Van de Graaff generator is uniformly charged, and provides a constant current to the sphere.

Appendix C

Charged Objects Tutorial

This Appendix contains the tutorial worksheet which aims to develop students' understanding of integration as a process of summation using the context of a charged rod.

CHARGED OBJECTS

I. Charge on a Uniformly Charged Rod

A. A uniformly charged rod has total charge Q and length L.

Describe in your own words what is meant by "uniformly charged". Avoid technical terms.

B. Imagine the rod is cut in half.

1. What is the charge on each half rod? Explain.

2. What is the ratio of the charge and the length of each half rod?

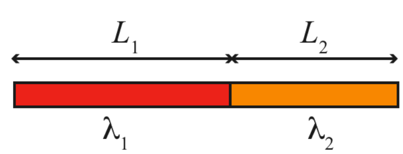
The ratio of charge and length is called the *linear charge density*, symbol λ.

C. What is the unit of linear charge density?

Describe again what is meant by "uniformly charged rod", this time using the term "linear charge density."

D. One part of a charged rod has a constant linear charge density λ_1 and length L_1. What is the total charge on this part of the rod?

The remainder of this rod has a constant linear charge density λ_2 and length L_2. What is the total charge on the rod?



E. A different rod consists of N uniformly charged parts. Each part has linear charge density $\lambda_1, \lambda_2, \ldots, \lambda_N$ and length $L_1, L_2, \ldots, L_N$. What is the total charge on this rod?

To make it easier to tackle continuously varying linear charge distributions, it is helpful to define a variable, x, that indicates the position on the rod between 0 and the total length, L. We define a set of constants x_i such that x_i is the position of the end of the i-th rod, another set of constants Δx_i such that Δx_i is the length of the i-th rod. This allows you to think of the linear charge density as a function of x, $\lambda(x)$.

F. Explain in some detail that in this notation, the charge on the second part of the rod in the example of part E is given by $\lambda(x_2^*)\Delta x_2$, where x_2^* is any value between x_1 and x_2.

Write an expression for the total charge on the rod in the notation described above.

II. Rods with continuously varying charge distributions

Consider a different non-uniformly charged rod of length L. The position on the rod is indicated by a variable x that runs from 0 to L. The linear charge density at $x = 0$ is λ_0 and the linear charge density at $x = L$ is λ_1.

A. Explain why calculating $\lambda(x^*)L$, where x^* is any value between 0 and L, does not generally give the correct value for the total charge on this rod.

B. Explain how the expression $Q = \sum_{i=1}^{5} \lambda(x_i^*) \cdot \Delta x_i$ approximates the charge on this rod.

C. Describe how you could improve on the approximation.

D. Explain what is meant by the expression $\lim\limits_{\substack{N\to\infty \\ \Delta x_i \to 0}} \sum_{i=1}^{N} \lambda\left(x_i^*\right)\Delta x_i$.

E. By definition, $\int \lambda\left(x\right)\mathrm{d}x \equiv \lim\limits_{\substack{N\to\infty \\ \Delta x_i \to 0}} \sum_{i=1}^{N} \lambda\left(x_i\right)\Delta x_i$. Explain why $Q = \int_0^L \lambda\left(x\right)\mathrm{d}x$ gives the *exact* charge on the rod.

F. For a particular rod of length L, the charge distribution is described by $\lambda(x) = \lambda_0 + \dfrac{\lambda_1 - \lambda_0}{L}x$.

1. Calculate the linear charge density at the centre of the rod.

2. Calculate the total charge on this rod.

G. In your own words, explain how the technique of integration allows you to find the total charge for any physically sensible charge distribution.

Appendix D

Electric Field of a Charged Rod Tutorial

This Appendix contains the tutorial worksheet where students calculate the electric field due to a uniform charge distribution.

ELECTRIC FIELD OF A CHARGED ROD

I. Introduction

A. A rod has a uniform linear charge density λ_0 and length L. Consider the following statement:

"The charge on the rod is equal to $\lambda_0 L$, so the electric field is given by $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{\lambda_0 L}{r^2}$*."*

Explain why this statement is generally incorrect.

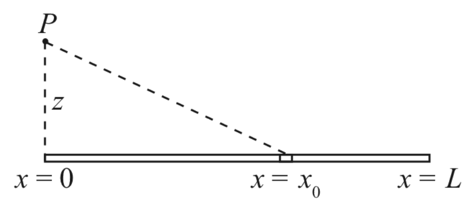
B. For which values of r is this a good approximation? Explain briefly.

Describe a procedure that would allow you to calculate the electric field due to the rod at any point outside the rod.

II. Electric field

To calculate the electric field at point P a distance z above the left end of the rod, it is useful to choose a coordinate system where the rod lies on the x-axis with the left end at $x = 0$, as shown.

Also shown is a small segment of the rod, of length dx and located at x_0.



A. Explain that the electric field at P, dE, due to the small segment is given by $\mathrm{d}E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{\lambda_0 \mathrm{d}x}{x_0^2 + z^2}$.

B. A student writes for the x-component of this electric field: $\mathrm{d}E_x = -\dfrac{1}{4\pi\varepsilon_0}\cdot\dfrac{\lambda_0 \mathrm{d}x}{r^2}\cdot\sin\theta$. Indicate the variables r and θ in the diagram.

Explain why the expression contains a minus sign.

C. In the next step, the student makes the substitutions $\sin\theta = x_0 / r$ and $r = \left(x_0^2 + z^2\right)^{1/2}$. Explain why such a substitution is useful.

Carry out the substitution, and simply the expression you obtain as much as possible.

The x-component of the electric field due to the entire rod is given by $E_x = -\frac{\lambda_0}{4\pi\varepsilon_0}\int_0^L \frac{x\mathrm{d}x}{(x^2+z^2)^{3/2}}$. Explain.

Use the indefinite integral given at the end of the tutorial to show that $E_x = \frac{\lambda_0}{4\pi\varepsilon_0}\left[\frac{1}{(L^2+z^2)^{1/2}} - \frac{1}{z}\right]$.

D. Show that the electric field at P is given by $\vec{E} = \dfrac{\lambda_0}{4\pi\varepsilon_0}\left[\left(\dfrac{1}{(L^2+z^2)^{1/2}} - \dfrac{1}{z}\right)\hat{x} + \dfrac{L}{z(L^2+z^2)^{1/2}}\hat{z}\right]$.

Useful integrals:

$$\int \frac{\mathrm{d}x}{\left(a^2+x^2\right)^{3/2}} = \frac{a^2}{x\left(a^2+x^2\right)^{1/2}} + C$$

$$\int \frac{x\mathrm{d}x}{\left(a^2+x^2\right)^{3/2}} = -\frac{1}{\left(a^2+x^2\right)^{1/2}} + C$$

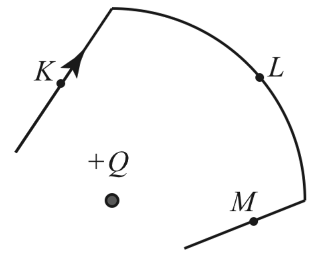
Appendix E

Potential and Electric Field Tutorial

This Appendix contains the tutorial worksheet where students consider the electric potential in a uniform electric field and electric potential due to a point charge.

I. Equipotential lines

A test charge q_0 is moved along the path shown in the vicinity of a positive charge $+Q$ as shown.

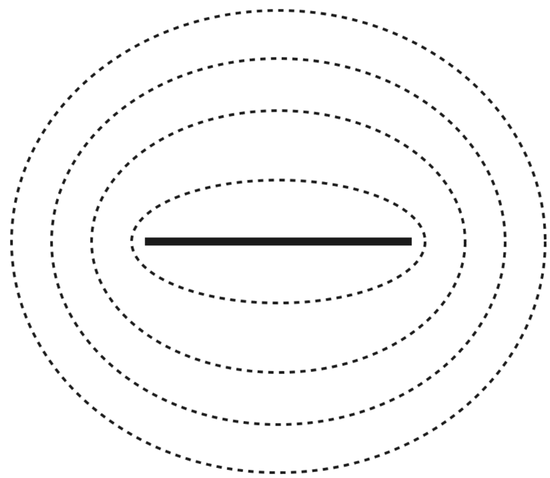


A. Near each of the following points, is $\vec{E} \cdot d\vec{l}$ *positive, negative,* or *zero*? Explain.

- Point K:
- Point L:
- Point M:

B. Shown at right are a negatively charged rod and a number of paths. The work done along any part of each path is zero.

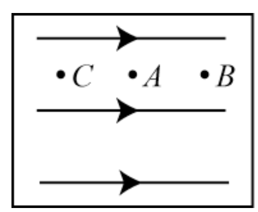
Use this information to draw field lines. Explain.



Paths along which the work is always zero are called equipotential lines. Explain why this is a reasonable name.

II. Electric potential in a uniform electric field

A test charge q_0 is brought from point A to point B in a uniform electric field of magnitude E_0.



A. Is the work done by the electric field *positive, negative,* or *zero*? Explain.

Is the potential difference $V_B - V_A$ *positive, negative,* or *zero*? Explain.

B. Suppose the test charge were moved from A to C instead. Is the work done by the electric field *positive, negative,* or *zero*? Explain.

Is the potential difference $V_C - V_A$ *positive, negative,* or *zero*? Explain.

A student calculates the potential difference $V_B - V_A$. She chooses a coordinate system whereby the electric field points in the positive x-direction.

C. Starting from the definition $V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{l}$, show that $V_B - V_A = -\int_{x_A}^{x_B} E_0 dx$.

Evaluate the integral.

Is your answer consistent with your answer to part A? If not, resolve the inconsistency.

D. A student now considers the potential difference between points A and C, and makes the following statement:

> *"When I go from A to C, the electric field points in the opposite direction to* $d\vec{l}$ *so the dot product gives me a minus sign. Therefore,*
>
> $V_C - V_A = E_0(x_C - x_A)$.

Do you agree with this statement? Explain briefly.

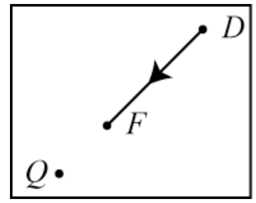
Is your answer consistent with your answer to part B? If not, part E may help you resolve the inconsistency.

The integral of a function $f(x)$ is equal to $\int_a^b f(x)dx = F(b) - F(a)$, where $F(x)$ is the antiderivative of $f(x)$, only when $a < b$. If $a > b$, then $\int_a^b f(x)dx = -[F(b) - F(a)]$.

E. Use this information to find a correct expression for $V_C - V_A$ in terms of E_0, x_A and x_C, if you have not already done so.

III. The potential of a point charge

A test charge q is brought from point D to point F in the electric field of a positive point charge Q as shown.



A. Examine the following calculation:

$$V_F - V_D = -\int_D^F \vec{E}\cdot \mathrm{d}\vec{l} = \int_{r_D}^{r_F} \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\mathrm{d}r = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}\bigg|_{r_D}^{r_F} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_F} - \frac{1}{r_D}\right).$$

Explain why a minus sign appears in the first step.

Explain why a minus sign disappears in the second step.

Explain why no minus sign appears in the third step, even though the antiderivative of r^{-2} is equal to $-r^{-1}$.

B. Explain that the potential at F is given by $V_F = \dfrac{Q}{4\pi\varepsilon_0}\dfrac{1}{r_F}$.

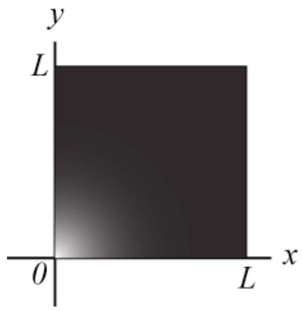
Appendix F

Two-Dimensional Charge Distributions Tutorial

This Appendix contains the tutorial worksheet where students calculate the total charge on a square sheet and a circular disk both with varying charge density.

I. A square charged sheet

A coordinate system is chosen so that the x and y axes coincide with two sides of a non-uniformly charged square sheet as shown at right. The sheet has sides of length of L, negligible thickness, and a surface charge density given by $\sigma = Cx^2y^3$.



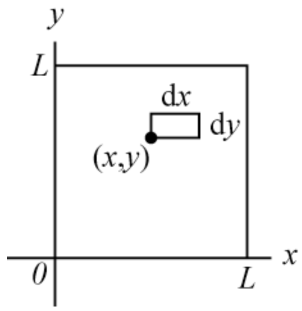
A. Consider the following conversation:

Student 1: *"To get the charge on the sheet, set x and y equal to L. Then the surface charge density is CL^5, the area of the sheet is L^2, so the total charge is CL^7."*

Student 2: *"I think you need to get the values of x and y in the middle of the sheet, because you want to get the average charge density. I worked out that the total charge is then $\frac{1}{32}CL^7$."*

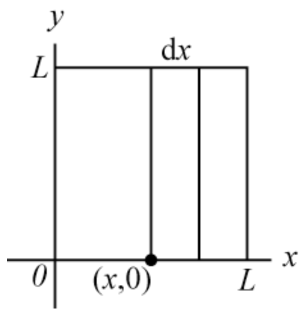
Both statements are <u>incorrect</u>. Indicate what is wrong with each statement.

B. Consider a small rectangular segment of the sheet. Its sides are parallel to the x and y axes, and have length dx and dy, and the left bottom corner of the segment is located at (x,y) as shown.



1. Explain why the charge dQ on this segment can be approximated by $\mathrm{d}Q = Cx^2y^3\mathrm{d}x\mathrm{d}y$.

2. How can you use your answer under 1 to evaluate the charge on the strip of width dx and length L shown at right?

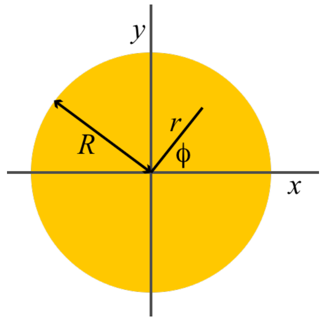


3. Show that the charge on the strip is given by $Q_{\text{strip}} = \frac{1}{4}CL^4x^2\mathrm{d}x$.

4. Show that the charge on the sheet is given by $Q_{\text{sheet}} = \frac{1}{12}CL^7$.

II. Calculating the charge on a non-uniformly charged disk using polar coordinates

A circular disk of radius R and negligible thickness has a surface charge density given by $\sigma = \alpha r^2 \cos^2 \phi$. The disk lies in the x,y-plane, and the centre of the circle is located at the origin as shown at right.



In this section, you will calculate the net charge on this disk using polar coordinates.

A. Explain why the total charge Q on the disk <u>cannot</u> be found by calculating $Q = \sigma \pi R^2$.

B. First consider a small segment of the disk of radial length dr and angular width dϕ. By analogy with the rectangular segment of part I, the corners of the segment located at (r,ϕ) are at (r,ϕ), $(r,\phi+\mathrm{d}\phi)$, $(r+\mathrm{d}r,\phi)$, and $(r+\mathrm{d}r,\phi+\mathrm{d}\phi)$.

1. Sketch this segment in the diagram at right. What shape does the segment have?

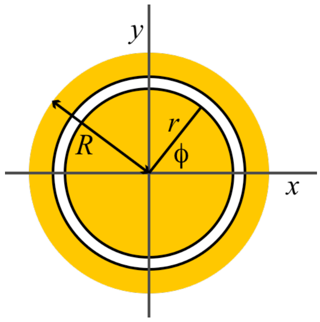
2. What is the arc length of the small segment given that its angular width is dϕ, and all angles are given in radians? Explain briefly.

3. Explain that, when dr and dϕ become smaller, the area dA of the segment will get closer to $\mathrm{d}A = r\mathrm{d}r\mathrm{d}\phi$. (*Hint:* A sketch may help.)

4. Write an expression for the charge dQ of this segment in terms of the given quantities.

5. Identify three segments in the diagram that contain the same amount of charge. Explain.

C. Now consider a ring with radius r and width dr as shown at right.



1. Show that the total charge on the ring depicted in the diagram is given by

$$Q_{\text{ring}} = \int_{\text{ring}} \mathrm{d}Q = \int_0^{2\pi} \alpha r^3 \cos^2\phi \,\mathrm{d}r\,\mathrm{d}\phi$$

2. Explain that an alternative expression for the total charge on the ring is:

$$Q_{\text{ring}} = 4\alpha r^3 \mathrm{d}r \int_0^{\pi/2} \cos^2\phi \,\mathrm{d}\phi$$

3. Evaluate the charge on the ring.

D. Show that the total charge on the disk is given by $Q_{\text{disk}} = \frac{1}{4}\pi\alpha R^4$.

Appendix: Useful integral

$$\int \cos^2 x \mathrm{d}x = \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x + C$$

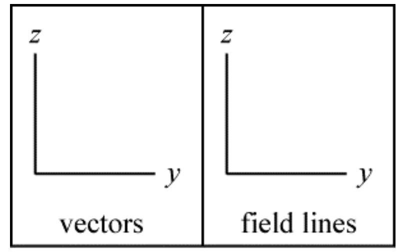
Appendix G

Calculating Electric Flux Tutorial

This Appendix contains the tutorial worksheet where students calculate the electric flux through a square sheet due to a point charge located below one of the corners of the sheet.

I. Flux through a square sheet in a uniform field

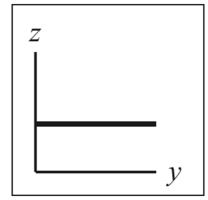
A. An electric field is given by $\vec{E} = 2E_0\hat{y} + 3E_0\hat{z}$.



Is the electric field uniform? Explain how you can tell.

Draw a number of electric field vectors in the left box, and a number of electric field lines in the right box.

B. Consider a hypothetical square sheet with sides of length a parallel to the x,y-plane as shown in the side view diagram at right.



Do the magnitude and direction of the electric field vary across the sheet?

Generally, the electric flux through a surface can be calculated from $\Phi_E = \iint \left(\vec{E} \cdot \hat{n}\right) \mathrm{d}A$.
Explain why in this case the expression simplifies to $\Phi_E = \left(\vec{E} \cdot \hat{n}\right) \iint \mathrm{d}A$.

C. Choose the normal $\hat{n}$ to point in the positive z direction. Draw $\hat{n}$, and show that $\vec{E} \cdot \hat{n} = 3E_0$.

Given an expression for $\iint \mathrm{d}A$ in terms of given quantities. Explain briefly. (*Hint:* you do not need to do integrate.

Write down an expression for Φ_E in terms of E_0 and a.

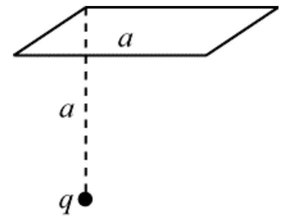
D. Calculate the magnitude E of the electric field.

Calculate the angle ϕ the electric field makes with the normal.

Write down an expression for Φ_E in terms of E, ϕ, and a.

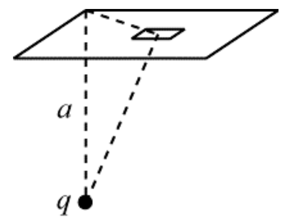
II. Flux through a square sheet in a non-uniform field

The original electric field is removed, and a small object with positive charge q is now fixed at a distance a below one of the corners of the sheet.



A. Are the magnitude and direction of the electric field approximately constant across the sheet? Explain.

B. Consider the small segment of length dx and width dy on the sheet, centred at a point (x,y), shown at right.



Is the electric field approximately constant across this segment?

Justify the expression for the electric flux $d\Phi_E$ through the segment, $d\Phi_E = \left(\vec{E}\cdot\hat{n}\right)dx dy$.

C. Show that for this segment, $d\Phi_E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{x^2+y^2+a^2}\dfrac{a}{\left(x^2+y^2+a^2\right)^{1/2}}dx dy$.

D. Explain that the flux through the sheet is given by $\Phi_E = \dfrac{qa}{4\pi\varepsilon_0}\displaystyle\int_0^a\int_0^a \frac{1}{\left(x^2+y^2+a^2\right)^{3/2}}dx dy$

Evaluate integral in x to find $\Phi_E = \dfrac{qa^2}{4\pi\varepsilon_0}\displaystyle\int_0^a \frac{1}{\left(y^2+a^2\right)\left(y^2+2a^2\right)^{1/2}}dy$

Show that $\Phi_E = \dfrac{q}{24\varepsilon_0}$, and evaluate this expression for q = 1 nC.

E. Can you make sense of the fact that the flux through the sheet does not depend on a?

III. Appendix: Physical constants and table of integrals

$$\varepsilon_0 = 8.85\times10^{-12}\,\mathrm{C^2N^{-1}m^{-2}}$$

$$\int \frac{1}{\left(x^2+y^2+a^2\right)^{3/2}}\,\mathrm{d}x = \frac{x}{\left(y^2+a^2\right)\left(x^2+y^2+a^2\right)^{1/2}} + C$$

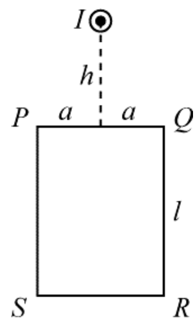
$$\int \frac{1}{\left(x^2+a^2\right)\left(x^2+2a^2\right)^{1/2}}\,\mathrm{d}x = \frac{1}{a^2}\arctan\left(\frac{x}{\left(x^2+2a^2\right)^{1/2}}\right) + C$$

Appendix H

Circulation Tutorial

This Appendix contains the tutorial worksheet where students calculate the line integral of the magnetic field due to a line current around a rectangular loop.

In this tutorial, you will calculate the circulation of the magnetic field around a rectangular loop *PQRS*. The loop has sides of length $2a$ and l, and is placed outside a wire carrying a current I as shown at right. The loop lies in the plane of the paper, and the top of the loop is a perpendicular distance h from the wire. You may take it as given that the magnitude B of the magnetic field produced by a current I at a distance r is given by $B=\mu_0 I/2\pi r$.



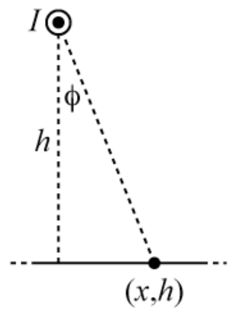
A. Sketch some of the magnetic field lines in the region of loop *PQRS*.

Sketch the magnetic field vector at point P.

B. Show that the magnetic field at P is given by $\vec{B} = \dfrac{\mu_0 I}{2\pi\left(a^2+h^2\right)^{1/2}}\hat{\phi}$.

For convenience, choose the origin to be the point where the wire intersects the plane of the paper. The positive x-axis points to the right, and the positive z-axis points down towards the bottom of the page.

C. Consider a small displacement $\mathrm{d}\vec{l}$ from a point (x,h) along the top section of the loop in the direction from P to Q as shown at right. Draw the magnetic field vector and the displacement vector, and show that the angle between the two vectors is equal to the angle ϕ shown.



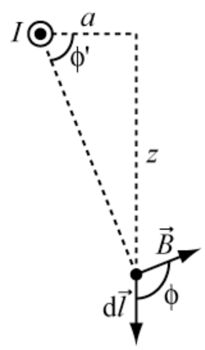
D. Show that for a small displacement $\mathrm{d}\vec{l}$ at point (x,h) along the top section of the loop in the direction from P to Q,

$$\vec{B}\cdot\mathrm{d}\vec{l} = \frac{\mu_0 I}{2\pi}\frac{h}{x^2+h^2}\mathrm{d}x.$$

E. Show that the line integral of the magnetic field along the top section of the loop is given by

$$\int_P^Q \vec{B} \cdot \mathrm{d}\vec{l} = \frac{\mu_0 I}{\pi} \arctan\left(\frac{a}{h}\right).$$

F. Show that, in the diagram at right, $\phi' = \pi - \phi$.



Hence, show that for a small displacement $\mathrm{d}\vec{l}$ along the right section of the loop in the direction from Q to R,

$$\vec{B} \cdot \mathrm{d}\vec{l} = -\frac{\mu_0 I}{2\pi} \frac{a}{a^2 + z^2} \mathrm{d}z.$$

G. Show that the line integral of the magnetic field along the right section of the loop is given by

$$\int_Q^R \vec{B}\cdot \mathrm{d}\vec{l} = \frac{\mu_0 I}{2\pi}\left[\arctan\left(\frac{h}{a}\right) - \arctan\left(\frac{h+l}{a}\right)\right].$$

H. Write down expressions for the line integrals $\int_R^S \vec{B}\cdot \mathrm{d}\vec{l}$ and $\int_S^P \vec{B}\cdot \mathrm{d}\vec{l}$. You should justify, but not derive, your answers.

⇒ Check your answers with an instructor.

I. Use your previous answers, and the trigonometric identity at the bottom of the page, to show that the circulation of the magnetic field around the loop is zero, as required by Ampère's Law.

Useful formulae:

$$\int \frac{a}{a^2+x^2}\,\mathrm{d}x = \arctan\left(\frac{x}{a}\right) + C$$

$$\arctan(-x) = -\arctan x$$

$$\arctan x + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$$

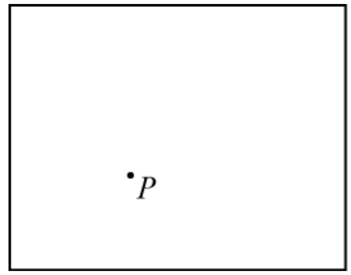
Appendix I

Vectors Tutorial

This Appendix contains the tutorial worksheet that introduces vector addition using the head-to-tail method and the component method (implemented in year one and year two).

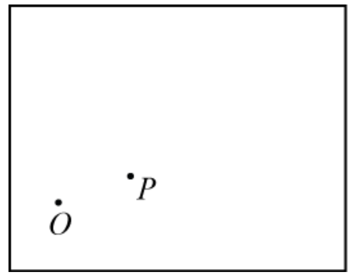
VECTORS

I. Position vectors in one dimension



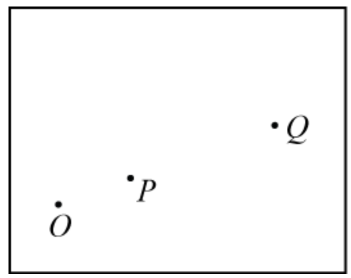
Shown in the box at right is a point P. You wish to describe its position.

A. Describe how you would do this.



A student decides to describe the position of point P by choosing a reference point O somewhere in the box, as shown at right.

B. Explain how you could use this point O to describe the position of point P.



C. How can you use the position of P to describe the position of Q?

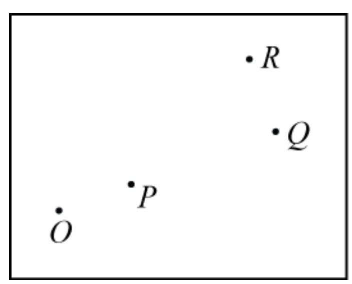
Do you think it is possible to describe the position of any point in this way? Explain briefly.

We can choose OP as a *unit vector*. We will write this unit vector as $\hat{e}_1$.

D. Draw the vector $\hat{e}_1$ in the box at right above. This vector is the *position vector* for point P.

Write the position vector for point Q in terms of the unit vector $\hat{e}_1$. Show your work.

What is the position vector for a point midway between O and P?



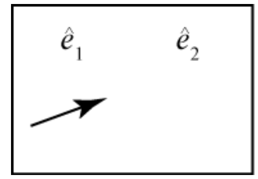
E. Can you describe the position of point R in terms of the unit vector $\hat{e}_1$ alone? Explain briefly.

If so, give an expression for the position vector for point R. If not, give an expression for the position vector of the point that is closest to R that you can describe in terms of $\hat{e}_1$ alone.

II. Position vectors in two dimensions

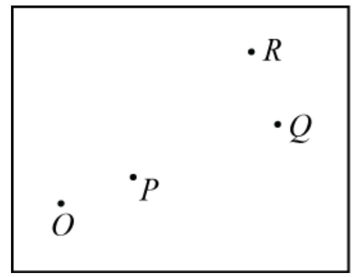
To describe the position of a point in a plane it is often useful to define two unit vectors that are perpendicular to each other. By convention, the second unit vector $\hat{e}_2$ can be obtained by rotating $\hat{e}_1$ 90 degrees anti-clockwise.

A. In the box at right, a unit vector vector $\hat{e}_1$ is drawn in. Draw the unit vector $\hat{e}_2$ in the same box.



B. Draw a number of unit vectors head-to-tail in such a way that the head of the last vector you draw coincides with point *R*.

Is there more than one way of doing this? Explain briefly.



C. Consider the following conversation:

Student 1: *"One possible way of getting from O to R is by adding 3 unit vectors* $\hat{e}_1$ *head-to-tail and then one unit vector* $\hat{e}_2$. *A short way of writing this is* $3\hat{e}_1 + \hat{e}_2$*."*

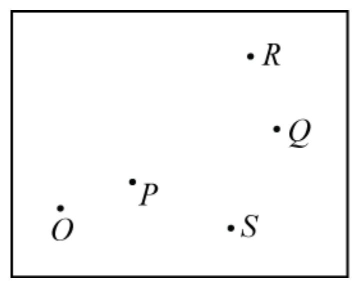
Student 2: *"You go from O to R by taking 4 steps of unit length. A short way of writing this is* $4\hat{e}_1$*."*

Briefly discuss both students' answers.

D. Can you draw a number of unit vectors head-to-tail in such a way that the head of the last vector you draw coincides with point *S*?

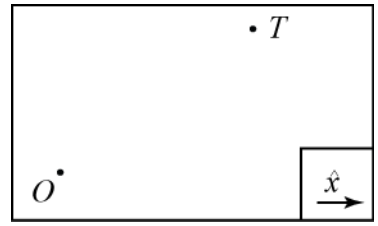
Write down an expression for the position vector for point *S* in terms of the unit vectors.

In your own words, describe what you think is meant by the terms *vector addition* and *vector subtraction*.



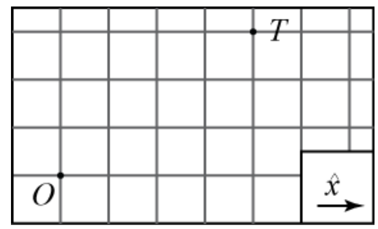
III. Cartesian coordinates.

To describe the position of a point T, a student picks a reference point O and calls her first unit vector $\hat{x}$, as shown at right.



A. She calls the second unit vector $\hat{y}$. How long is this unit vector, and in which direction does it point? Explain briefly.

It is not easy to see how many unit vectors point T is removed from point O. To make it easier to see this, you can construct a grid. You can think of the grid as tracing out all possible unit vector steps. The figure at right shows points O and T with a grid superimposed.



B. Write the position vector for T, $\vec{t}$, in terms of the chosen unit vectors.

Show that it is possible to write $\vec{t} = t_x\hat{x} + t_y\hat{y}$, where t_x and t_y are scalars. What are the values of t_x and t_y?

The vectors $\vec{t}_x \equiv t_x\hat{x}$ and $\vec{t}_y \equiv t_y\hat{y}$ are called the x- and y-components of $\vec{t}$.

C. Draw the vectors $\vec{t}$, $\vec{t}_x$, and $\vec{t}_y$ in a single diagram.

Write expressions for the following quantities, showing your work in each case:

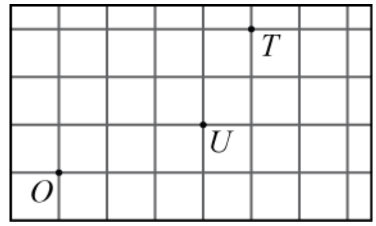
- $\vec{t}$ in terms of $\vec{t}_x$ and $\vec{t}_y$;

- the length of $\vec{t}$, t, in terms of t_x and t_y;

- t_x in terms of t and the angle ϕ the vector $\vec{t}$ makes with $\vec{t}_x$.

IV. Vector addition and subtraction.

An object is displaced from point T by a vector $\vec{v}$. It ends up at point U.



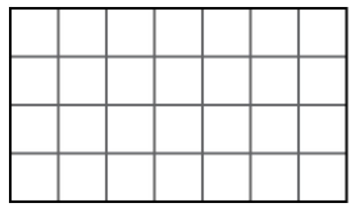
A. *Construct* the position vector for U, $\vec{u}$, using the head-to-tail method.

B. Consider the following statement:

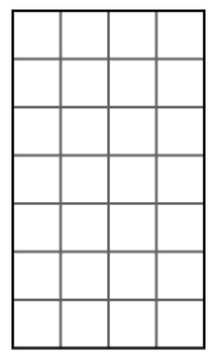
> *"When I combine the vectors $\vec{t}$ and $\vec{v}$ to get $\vec{u}$, I end up closer to the origin. From that I can see that $\vec{u} = \vec{t} - \vec{v}$."*

Do you agree? Explain your reasoning.

C. Draw the component vectors $\vec{t}_x$, $\vec{u}_x$ and $\vec{v}_x$ in the grid at right. Write an equation that links these three vectors.



Draw the component vectors $\vec{t}_y$, $\vec{u}_y$ and $\vec{v}_y$ in the grid at right, and write an equation that links these three vectors.



In your own words, explain that the method of *adding components* is an alternative way of adding vectors.

Appendix J

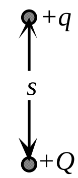
Charges Tutorial

This Appendix contains the tutorial worksheet where students consider the net electric force due to multiple point sources (implemented in year one and year two).

I. Superposition

Coulomb's law states that the electric force between two *point charges* acts along the line connecting the two points. (A *point charge* is a charged object that is sufficiently small that the charge can be treated as if it were all located at a single point.) The magnitude of the force on either of the charges is proportional to the product of the charges and is inversely proportional to the square of the distance between the charges.

A. Two positive point charges $+q$ and $+Q$ (with $|Q| > |q|$) are held in place a distance s apart.



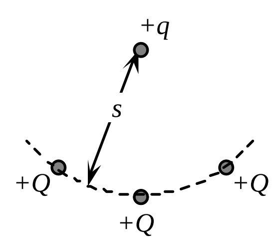
1. Indicate the direction of the electric force exerted on each charge by the other.

2. Is the force on the $+q$ charge by the $+Q$ charge *greater than, less than,* or *equal to* the force on the $+Q$ charge by the $+q$ charge? Explain.

3. By what factor would the magnitude of the electric force on the $+q$ charge change if the charges were instead separated by a distance $2s$?

4. By what factor would the magnitude of the electric force on the $+q$ charge change if the $+Q$ charge were replaced by a charge $-4Q$?

B. Two more $+Q$ charges are held in place the same distance s away from the $+q$ charge as shown. Consider the following student dialogue concerning the net force on the $+q$ charge:



Student 1: *"The net electric force on the +q charge is now three times as large as before, since there are now three positive charges exerting forces on it."*

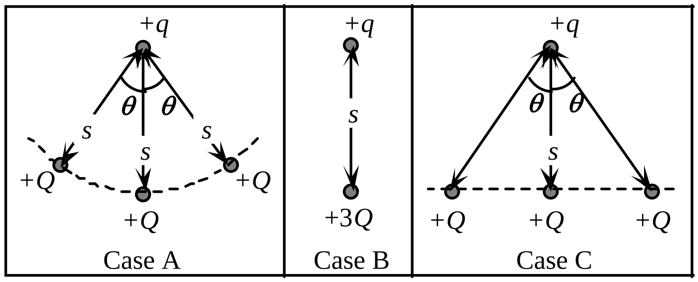
Student 2: *"I don't think so. The force from the +Q charge on the left will cancel the force from the +Q charge on the right. The net electric force will be the same as in part A."*

1. Do you agree with either student? Explain.

2. Indicate the direction of the net electric force on the $+q$ charge. Explain.

3. What, if anything, can be said about how the magnitude of the net electric force on the $+q$ charge changes when the two $+Q$ charges are added? Explain.

C. Rank the four cases below according to the magnitude of the net electric force on the $+q$ charge. Explain how you determined your ranking.



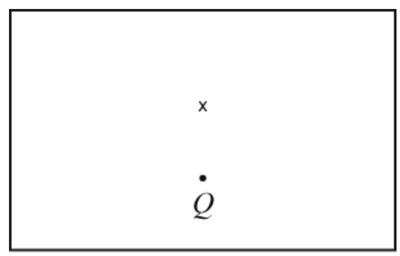
✓ Check your ranking with a tutorial instructor before continuing.

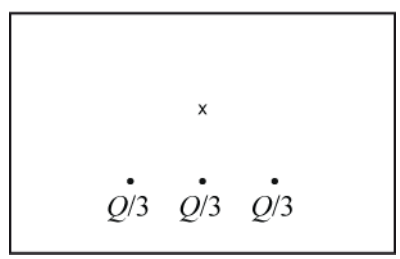
II. Linear charge distributions

In this section you will investigate qualitatively what happens to the direction of the electric force when charge is distributed along a line. In a homework assignment, you will investigate what happens to the magnitude and direction of the forces quantitatively.

A. We start by comparing the two situations shown at right. In the first case, there is a single charge Q; in the second case, the charge Q has been replaced by a charge $Q/3$, and two more charges $Q/3$ have been placed on a line, equidistant on either side of it.

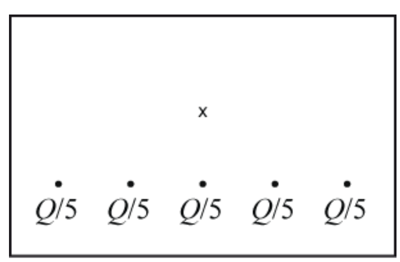
Draw qualitatively correct vectors representing the net electric force on a point charge placed at point "x". Explain briefly.



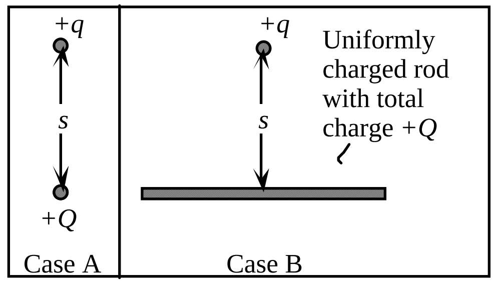


B. Suppose you were to replace the $Q/3$ charges with $Q/5$ charges, and add two more $Q/5$ charges at either end of the line of charges as shown at right.

How would the magnitude and direction of the net electric force change compared to part A? Explain.



C. In case A at right, a point charge $+q$ is a distance s from the center of a small ball with charge $+Q$.



In case B the $+q$ charge is a distance s from the center of an acrylic rod with a total charge $+Q$.

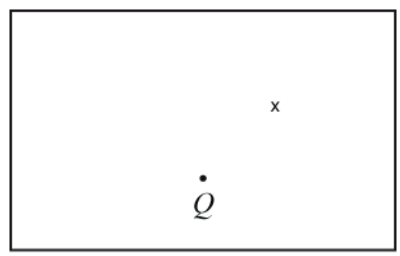
Consider the following student dialogue:

Student 1: *"The charged rod and the charged ball have the same charge, +Q, and are the same distance from the point charge, +q. So the force on +q will be the same in both cases."*

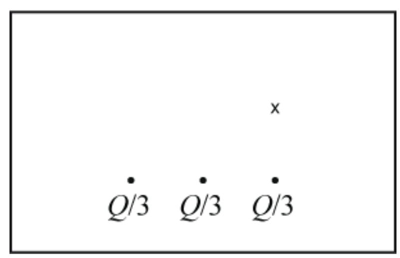
Student 2: *"No, in case B there are charges spread all over the rod. The charge directly below the point charge will exert the same force on +q as the ball in case A. The rest of the charge on the rod will make the force in case B bigger."*

Neither student is correct. Discuss with your partners the errors made by each student. Write a correct description of how the forces compare in the space below. Explain.

D. Now consider the same process for a point that is not located above the centre of the line as shown at right.



How does the *direction* of the electric force exerted on a point charge fixed at the location marked "x" due to *middle* charge $Q/3$ compare to that exerted by the Q charge? Explain.

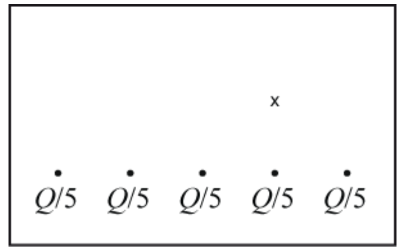


Compare the electric forces exerted by the left and right $Q/3$ charges in terms of magnitude and direction.

Which of these would affect the point charge at "x" more? Explain.

Indicate the direction of the net electric force exerted by the three $Q/3$ charges.

E. The three $Q/3$ charges are replaced by five $Q/5$ charges as shown at right.

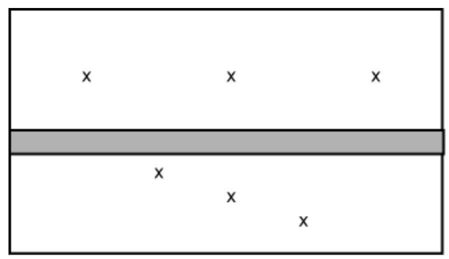


Indicate the direction of the net electric force due to the *three* middle charges. Explain briefly.

Describe how the electric forces exerted by the left and right $Q/5$ charges would affect the direction of the net electric force.

F. Imagine this process to continue for a long time, so that there are N Q/N charges spaced equidistantly along a line, where N is a large integer. Describe the direction of the net electric field for each of the two positions discussed in this part of the tutorial – the one directly above the centre of the charge distribution, and the one just to the right of it.

G. Sketch vectors to represent the net electric force exerted by an infinitely long uniform linear charge distribution on a test charge at each of the points marked by an "×". Explain.



Appendix K

Revised Charges Tutorial

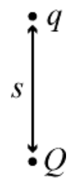
This Appendix contains the tutorial worksheet where students consider the net electric force due to multiple point sources conceptually and mathematically (implemented in year three and year four).

CHARGES

I. Superposition

Coulomb's law states that the electric force between two *point charges* acts along the line connecting the two points. (A *point charge* is a charged object that is sufficiently small that the charge can be treated as if it were all located at a single point.) The magnitude of the force on either of the charges is proportional to the product of the charges and is inversely proportional to the square of the distance between the charges. In this section, you will investigate how addition of electric force vectors can be used to compare the net force in different cases qualitatively – *i.e.*, without doing numerical calculations.

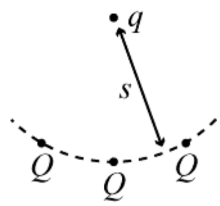
A. Two positive point charges q and Q (with $|Q| > |q|$) are held in place a distance s apart. Indicate the direction of the electric force exerted on each charge by the other.



1. Is the force on the q charge by the Q charge *greater than, less than,* or *equal to* the force on the Q charge by the q charge? Explain.

2. By what factor would the magnitude of the electric force on the q charge change if the charges were instead separated by a distance $2s$?

B. Two more identical Q charges are held in place the same distance s away from the q charge as shown. Consider the following dialogue concerning the net force on the q charge:



Student 1: *"The net electric force on the q charge is now three times as large as before, since there are now three positive charges exerting forces on it."*

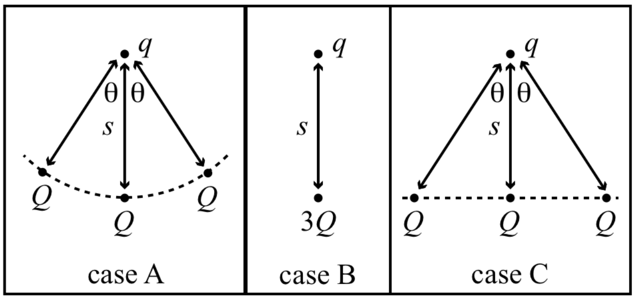
Student 2: *"I don't think so. The force from the Q charge on the left will cancel the force from the Q charge on the right. The net electric force will be the same as in part A."*

1. Do you agree with either student? Explain.

2. Indicate the direction of the net electric force on the $+q$ charge. Explain.

3. What, if anything, can be said about how the magnitude of the net electric force on the $+q$ charge changes when the two Q charges are added? Explain.

C. Rank the three cases below according to the magnitude of the net electric force on the q charge. Explain how you determined your ranking.



✓ Check your ranking with a tutorial instructor before continuing.

II. Evaluating the net force on a point charge

In this section, you will investigate the forces in cases A–C above quantitatively. Choose the positive x-direction to be to the right, and the positive y-direction to be upwards.

A. Write an expression for the electric force on charge q in case B. (*Note:* A force is a vector!)

B. The net force on charge q in case A is given by $\vec{F}_{q,A} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{s^2}\left(2\cos\theta + 1\right)\hat{y}$. Derive this expression.

C. The net force on charge q in case C is given by $\vec{F}_{q,C} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Qq}{s^2}\left(2\cos^3\theta + 1\right)\hat{y}$. Derive this expression in two steps:

1. Show that the distance r from the right and left charges Q to the charge q is given by $r = s/\cos\theta$.

2. Derive the expression.

D. Show that your answers in this section are consistent with your ranking in section I.

Section I of this tutorial was adapted from *Tutorials in Introductory Physics* by McDermott, Shaffer, & P.E.G., U. Wash. First Edition, 2002.

Appendix L

Work Tutorial

This Appendix contains the tutorial worksheet which addresses conceptual aspects of the dot product using the context of mechanical work.

WORK

I. Positive, negative, and zero work

The *work-energy theorem* states that the net work done on an object is equal to the change in the object's total energy. In cases where the potential energy is constant, the net work done is equal to the change in kinetic energy.

A. Place your pen at rest on the table. Use your hand to give the pen some speed, then push the pen at constant speed, then let go of the pen.

1. Why does the pen come to a stop after you release it?

   Is the change in the pen's kinetic energy *positive, negative,* or *zero*?

   Is the work done on the pen *positive, negative,* or *zero*? Explain.

2. While the pen was moving at constant speed, which forces were acting on it?

   Is the net work done *positive, negative,* or *zero*? Explain.

   Is the work done by each force *positive, negative,* or *zero*? Explain.

3. While the pen was speeding up, was the change in the pen's kinetic energy *positive, negative,* or *zero*? Explain.

   During that time interval, was the work done by your hand *positive, negative,* or *zero*? Explain.

4. While the pen was moving, was gravity changing the kinetic energy of the pen?

   Is the work done by gravity on the pen *positive, negative,* or *zero*? Explain.

B. Draw arrows to indicate the directions of each force and displacement.

hand	friction	gravity

Comment on the relative directions of force and displacement when the work done is positive, negative and zero.

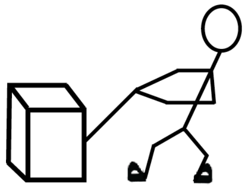
II. Calculating work

A. A person pushes a box with a constant force of magnitude F across a frictionless surface over a distance d.

What is the sign of the work done by the person on the box? Explain.

What is the magnitude of the work done by the person on the box?

B. This time the person decides to use a rope to pull the box across the surface, as shown. The rope makes an angle θ with the horizontal. The tension force on the box has magnitude F, and the box still moves a distance d.



1. Explain why the expression you derived in part II-A for the work done cannot be used in this case.

2. Resolve the force into two components, in such a way that one component does zero work. Sketch the components in the box at right. Explain.

   Is the work done by the other component *positive, negative,* or *zero*? Explain.

3. Write an expression for the magnitude of each component in terms of F and θ. Explain briefly.

   Write an expression for the net work done in terms of F, d, and θ. Explain briefly.

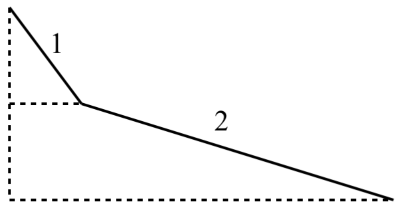
The process you completed to find the work done by the rope on the box is called calculating the dot product. The dot product of two vectors is generally written $\vec{a}\cdot\vec{b}$; it is equal to the product of the magnitudes of the vectors and the cosine of the angle between them. The product is a scalar quantity.

C. In section I, you used the change in kinetic energy to determine whether the work done by each force was positive, negative or zero. Now use the dot product of force and displacement to do this.

Are your answers in parts I and II consistent with each other?

III. Work done by gravity

A ball is rolling down a ramp, shown at right. The ramp consists of two sections, 1 and 2, of equal height. The length of section 2 is greater than the length of 1, and makes a smaller angle with the horizontal.



A. Consider the following student statement regarding the sign of the work done by gravity on the ball.

> *"The work done by gravity is negative because the force is acting downwards, which is in the negative direction."*

Do you agree with this statement? Explain.

B. Draw the force vectors, $\vec{F}_1$ and $\vec{F}_2$, and displacement vectors, $\vec{d}_1$ and $\vec{d}_2$, for the two sections of the ramp, and indicate the angle between them (θ_1 and θ_2).

How do the force vectors compare? Explain.

How do the displacement vectors compare? Explain.

To compare the work done by gravity on the two different sections of the ramp, it makes sense to try a similar approach to the one used in part II.

C. What is the physical meaning of $F_1 \cos\theta_1$ in this case?

Is the magnitude of $F_1 \cos\theta_1$ *greater than, less than,* or *equal to* the magnitude of $F_2 \cos\theta_2$?

Based on your answers, is the magnitude of the work done by gravity in section 1 *greater than, less than,* or *equal to* the work done by gravity in section 2? If you can't tell, state so. Explain.

D. What is the physical meaning of $d_1 \cos\theta_1$ in this case?

Is the magnitude of $d_1 \cos\theta_1$ *greater than, less than,* or *equal to* the magnitude of $d_2 \cos\theta_2$?

Based on your answers, is the magnitude of the work done by gravity in section 1 *greater than, less than,* or *equal to* the work done by gravity in section 2? If you can't tell, state so. Explain.

E. Comment on the two different ways of using the dot product to calculate work.

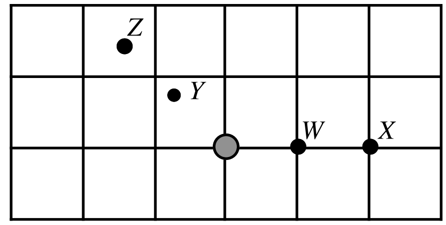
Appendix M

Electric Potential Difference Tutorial

This Appendix contains the tutorial worksheet which introduces students to the relationship between electric potential difference and the work done by the field.

I. Work and electric fields

The diagram at right shows a top view of a positively charged rod. Points *W, X, Y,* and *Z* lie in a plane near the center of the rod. Points *W* and *Y* are equidistant from the rod, as are points *X* and *Z*.



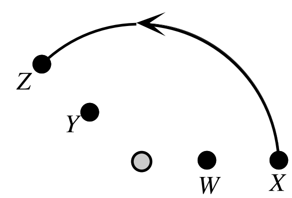
A. Draw electric field vectors at points *W, X, Y,* and *Z*.

B. A particle with charge $+q_o$ travels along a straight line path from point *W* to point *X*.

Is the work done *by the electric field* on the particle *positive, negative,* or *zero?* Explain using a sketch that shows the electric force on the particle and the displacement of the particle.

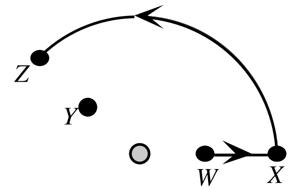
Compare the work done by the electric field when the particle travels from point *W* to point *X* to that done when the particle travels from point *X* to point *W*.

C. The particle travels from point *X* to point *Z* along the circular arc shown.

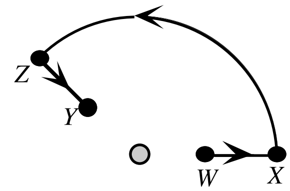


1. Is the work done *by the electric field* on the particle *positive, negative,* or *zero?* Explain. (*Hint:* Sketch the direction of the force on the particle and the direction of the displacement for several short intervals during the motion.)

D. Suppose the particle travels from point W to point Y along the path $WXZY$ as shown.

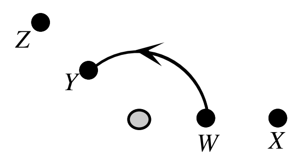


1. Compare the work done by the electric field when the particle travels from point W to point X to that done when the particle travels from point Z to point Y. Explain.

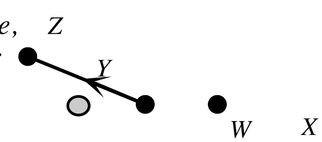


   What is the total work done on the particle by the electric field as it moves along the path *WXZY?*

2. Suppose the particle travels from W to Y along the arc shown. Is the work done on the particle by the electric field *positive, negative,* or *zero?* Explain using force and displacement vectors.



3. Suppose the particle travels along the straight path WY. Is the work done on the particle by the electric field *positive, negative,* or *zero?* Explain using force and displacement vectors. (*Hint:* Compare the work done along the first half of the path to the work done along the second half.)



E. Compare the work done as the particle travels from point *W* to point *Y* along the three different paths in part D.

It is often said that the work done by a static electric field is *path independent*. Explain how your results in part D are consistent with this statement.

II. Electric potential difference

A. Suppose the charge of the particle in section II is increased from $+q_o$ to $+1.7q_o$.

1. Is the work done by the electric field as the particle travels from *W* to *X* *greater than, less than,* or *equal to* the work done by the electric field on the original particle? Explain.

2. How is the quantity *the work divided by the charge* affected by this change?

The *electric potential difference* ΔV_{WX} between two points *W* and *X* is defined to be:

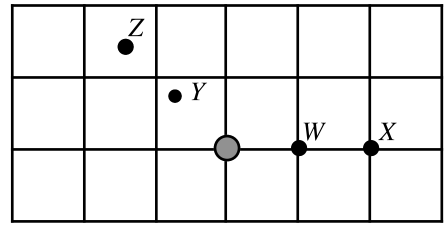
$$\Delta V_{WX} = -\frac{W_{\text{elec}}}{q}$$

where W_{elec} is the work done by the field as a charge q travels from point *W* to point *X*.

3. Does this quantity depend on the *magnitude* of the charge of the particle that is used to measure it? Explain.

4. Does this quantity depend on the *sign* of the charge of the particle that is used to measure it? Explain.

B. Shown at right are four points near a positively charged rod. Points *W* and *Y* are equidistant from the rod, as are points *X* and *Z*. A charged particle with mass $m_o = 3 \times 10^{-8}$ kg is released from rest at point *W* and later is observed to pass point *X*.



1. Is the particle positively or negatively charged? Explain.

2. Suppose that the magnitude of the charge on the particle is 2×10^{-6} C and that the speed of the particle is 40 m/s as it passes point *X*.

   a. Find the change in kinetic energy of the particle as it travels from point *W* to point *X*.

   b. Find the work done on the particle by the electric field between point *W* and point *X*. (*Hint:* See part D of section I.)

   c. Find the electric potential difference between point *W* and point *X*.

   d. If the same particle were released from point *Y,* would its speed as it passes point *Z* be *greater than, less than,* or *equal to* 40 m/s? Explain.

3. Suppose that a second particle with the same mass as the first but nine times the charge (*i.e.,* 18×10^{-6} C) were released from rest at point W.

   a. Would the electric potential difference between points *W* and *X* change? If so, how, if not, why not?

   b. Would the speed of the second particle as it passes point *X* be *greater than, less than,* or *equal to* the speed of the first particle as it passed point *X?* Explain.

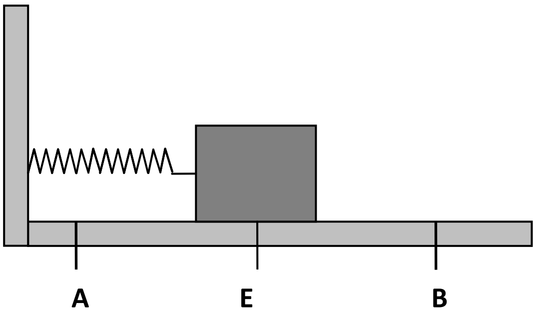
Appendix N

Simple Harmonic Motion Pretest

Year One

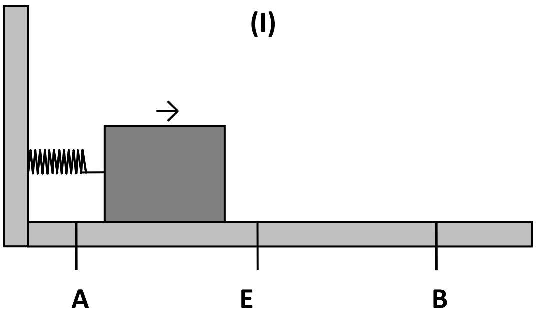
This Appendix contains the pretest administered in year one testing students qualitative understanding of the changing position, velocity, and acceleration during simple harmonic motion.

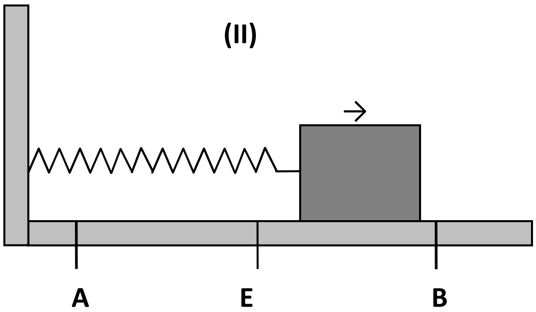
Consider a compression spring mounted horizontally attached to a block of mass *m* positioned on a frictionless surface, as shown at right. The spring is at its natural length and the block is said to be at its "equilibrium position" (E). At the equilibrium position, the position of the block *x* is equal to zero and at position B *x* is positive.

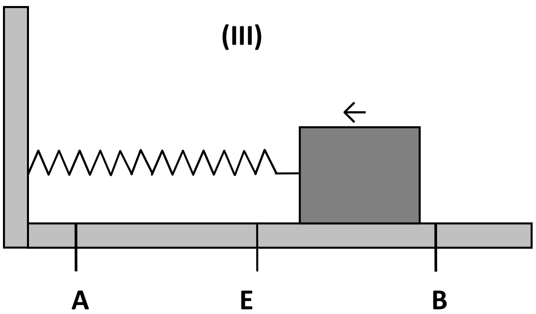


Positions A and B are both the same distance from the equilibrium position. When the spring is compressed to position A and then released, the block will go back through the equilibrium position to position B. The block then returns through the equilibrium position to position A. This motion will continue indefinitely.

A. For the three instances shown below state:
   (i) whether the velocity of the block is increasing, decreasing or at a constant rate and also whether the velocity is positive or negative
   (ii) whether the acceleration of the block is positive, negative or zero. Explain.







Appendix O

Simple Harmonic Motion Pretest

Year Two

This Appendix contains the pretest administered in year two testing students ability to graph the changing position, velocity, and acceleration with time during simple harmonic motion.

Simple Harmonic Motion Pretest

A block of mass *m* is placed on a frictionless surface and is attached to a compression spring with spring constant *k* as shown in *Figure 1*. The block is compressed to position *A* and is then released. *Figure 2* shows one full period of the blocks position versus time graph.

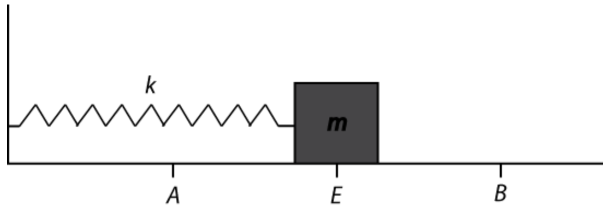


Figure 1

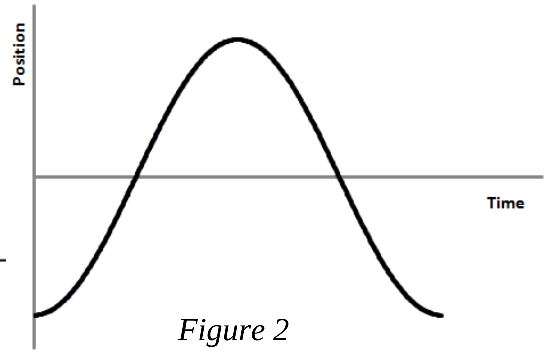


Figure 2

(a) Label clearly where positions A, B and E occur on the graph.

(b) Draw the corresponding graphs of velocity versus time and acceleration versus time for the same situation. Explain why you drew the graphs this way.

Appendix P

Simple Harmonic Motion Pretest

Year Three

This Appendix contains the pretest administered in year three testing students ability to label, interpret and draw graphs of the changing velocity and acceleration with time during simple harmonic motion.

<u>Simple Harmonic Motion Pretest</u>

A block of mass *m* is placed on a frictionless surface and is attached to a compression spring with spring constant *k* as shown in *Figure 1*. The block is compressed to position *A* and is then released. *Figure 2* shows one full period of the blocks velocity versus time graph.

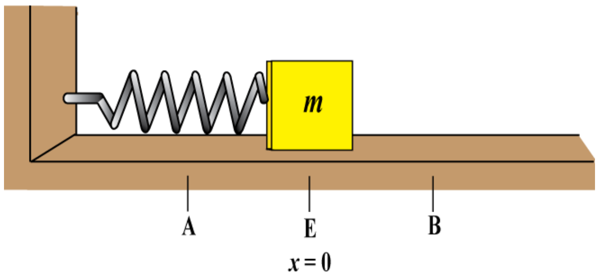


Figure 1

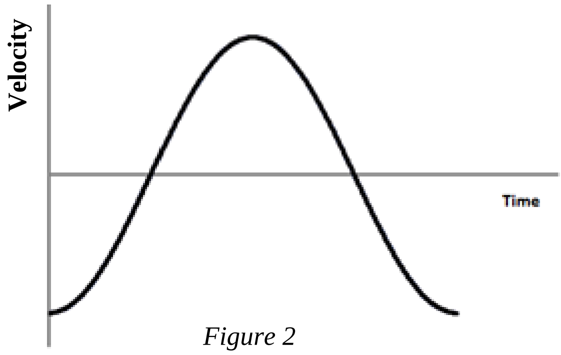


Figure 2

(a) Label clearly where positions A, B and E occur on the velocity versus time graph. Explain.

(b) In the table below, state the sign for *<u>acceleration</u>* and whether its magnitude is increasing or decreasing for each interval e.g. as the block moves from E to B.

Interval	**Sign**	**Magnitude**
E-B		
B-E		
E-A		
A-E		

(c) Sketch the information from your table on the axis below. Explain why you drew it the way you did.

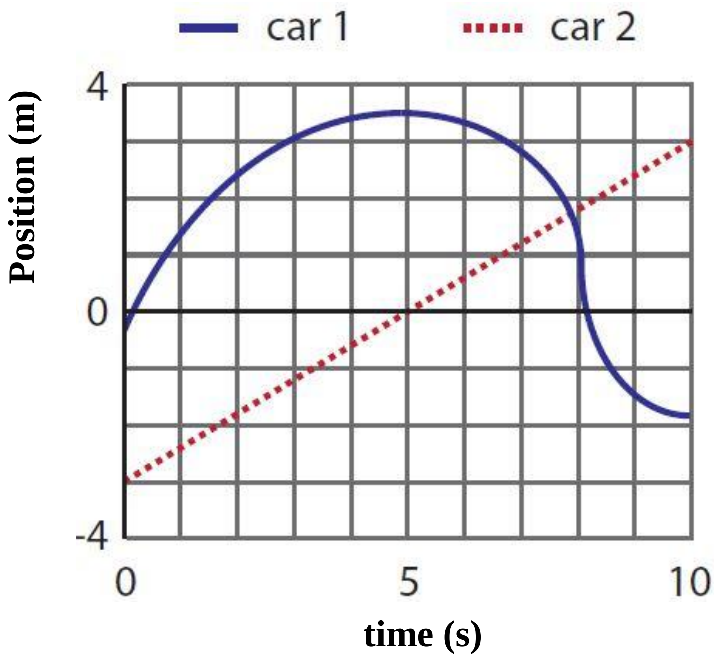
Appendix Q

Non-SHM Context Pretest

Year Three

This Appendix contains the pretest administered in year three testing students ability to interpret velocity and acceleration from a position versus time graph.

Position, Velocity and Acceleration Pretest



1. At what time(s) does car 1 have zero velocity? If it does not have zero velocity at any time state so explicitly. Explain.

2. At what time(s) does car 1 have greatest velocity? Explain.

3. At what time(s) does car 2 have zero velocity? If it does not have zero velocity at any time state so explicitly. Explain.

4. At what time(s) does car 2 have greatest velocity? Explain.

5. At what time(s) does car 2 have zero acceleration? If it does not have zero acceleration at any time state so explicitly. Explain.

4. By Newton's Second Law, if mass is constant what is the relationship between acceleration and force? Hence, or otherwise, state the relationship between force and position.

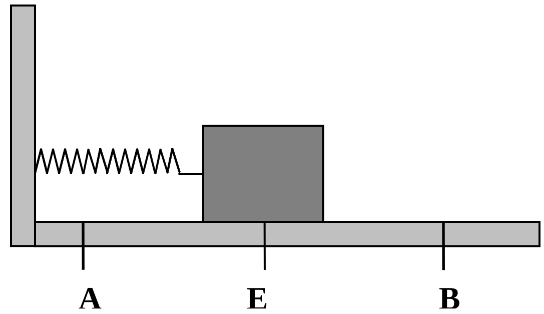
C. Consider the following statements made by two students regarding the relationship, $F \alpha -x$.

Student 1: "*I don't think that this relationship holds throughout. When the object is moving away from equilibrium, position and force are in opposite directions, so* $F \alpha -x$ *is true here. But when the object is moving towards the equilibrium position, force and displacement are in the same direction so* $F \alpha -x$ *does not hold."*

Student 2: *"No. The relationship* $F \alpha - x$ *will always hold. The x is just position, not a change in position or displacement. Anywhere to the right of equilibrium x is greater than zero and force is less than zero. Anywhere to the left of equilibrium x is less than zero and force is greater than zero. Therefore, the relationship will always hold."*

When we know the spring constant k, we can say that F = -kx.

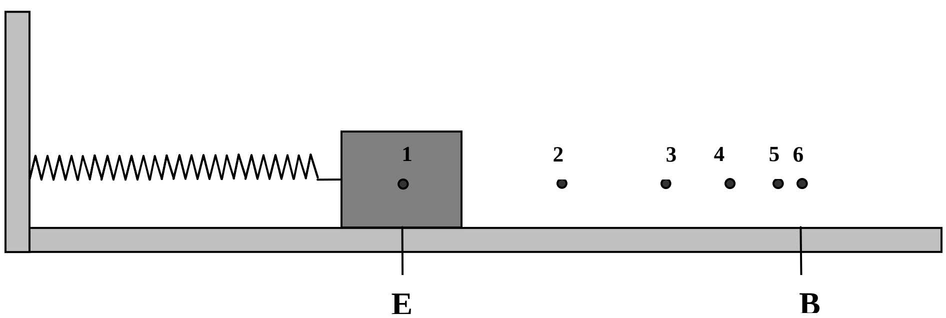
Simple Harmonic Motion Tutorial



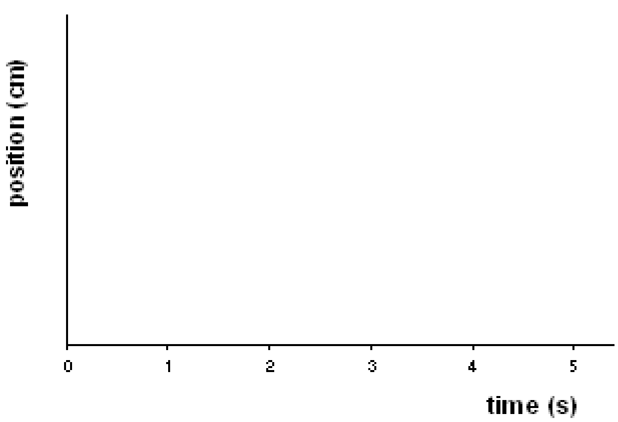
A. Consider a compression spring mounted horizontally attached to a block of mass *m* positioned on a frictionless surface, as shown at right. The spring is at its natural length and the block is said to be at its "equilibrium position" (E).

Positions A and B are both the same distance from the equilibrium position. When the spring is compressed to position A and then released, the block will go back through the equilibrium position to position B. The block then returns through the equilibrium position to position A. This motion will continue indefinitely.

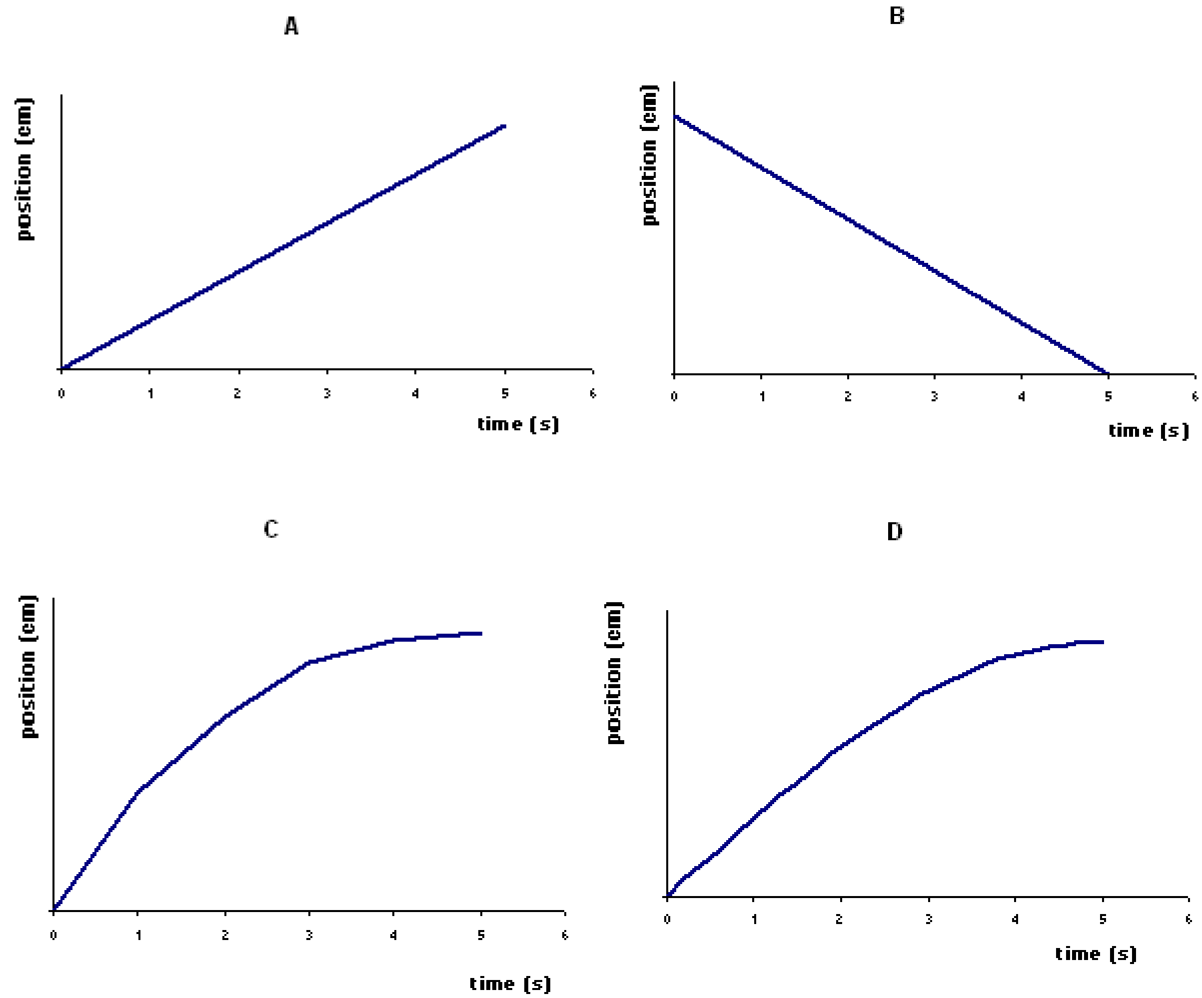
Now, we look more closely at the motion of the block from the equilibrium position to position B. The diagram below shows the position of the block after it passes the equilibrium position at instants separated by equal time intervals. Point 1 is at the equilibrium position and point 6 is at position B.



1. On the axis given below, plot position (distance from equilibrium position) versus time for the points 1 to 6 (Points 1, 2, 3, 4, 5, and 6 are taken at 0s, 1s, 2s, 3s, 4s and 5s respectively).



2. Which of the following line shapes correctly fits the data points you have drawn for position vs time? If none of the four options suit your plotted points, chose the graph you think correctly represents the position versus time, and examine where you went wrong with the plotting of the points.



Next we will look at the motion of the block from position B to the equilibrium position.

3. If we say that at equilibrium x = 0, will x be less than, greater than or equal to zero, to the right of the equilibrium position? What sign can we denote to any position on the right?

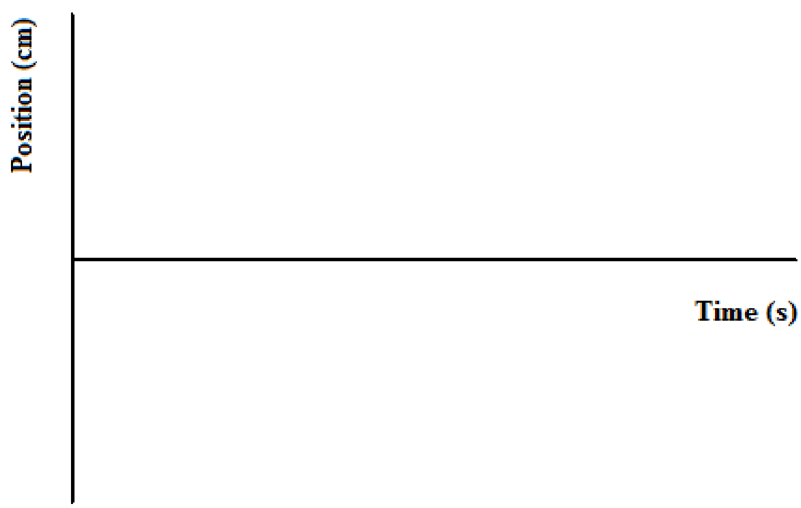
So is position versus time for the block going from position B to equilibrium going to be positive or negative?

Is displacement in the same direction as when the block moved from equilibrium to B?

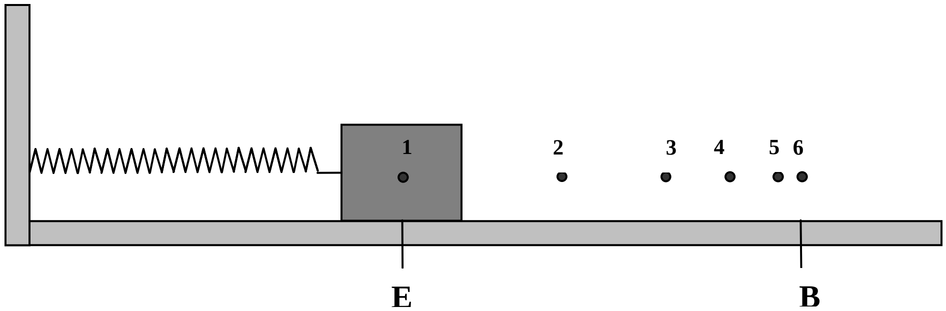
What similarity and difference exists between position versus time for the block on moving from equilibrium to B and on moving from B to equilibrium.

Redraw the line shape for position versus time on moving from equilibrium to position B on the axis provided below and also use the information from part 3 to add the motion from position B to equilibrium to the graph.

4. Repeat step 3 in order to add the motion of the block from equilibrium to position A and from position A to equilibrium to your graph.



B. 1.



Looking again at this diagram, draw vectors on the diagram that represents the instantaneous velocity of the block at each of the points 1 to 6. If the velocity is zero at any point, state so explicitly.

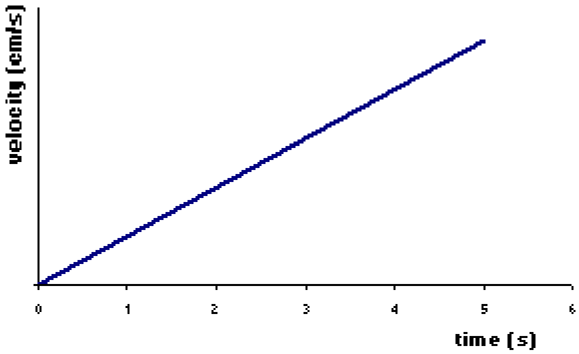
2. At which of the points will the instantaneous velocity of the block be greatest? Explain

At which of the points will the instantaneous velocity of the block be least? Explain.

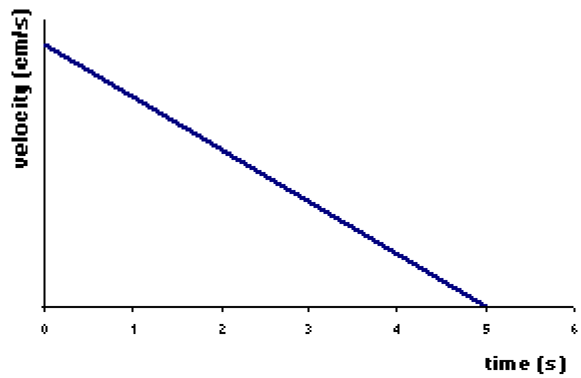
Is the block speeding up, slowing down, or moving at a constant rate as it moves from point 1 to 6? Explain.

Use this information to help you decide which of the following graphs correctly represents velocity versus time for the block moving from equilibrium to position B. Explain why you chose the graph you do.

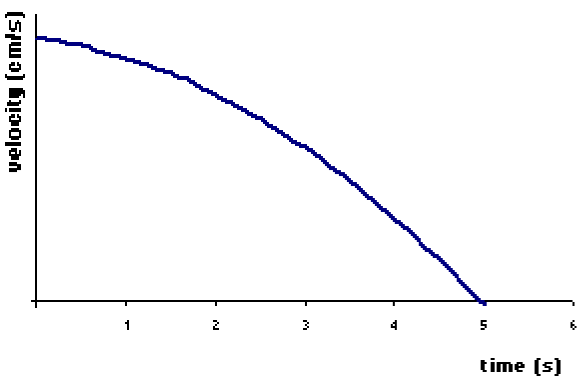
A



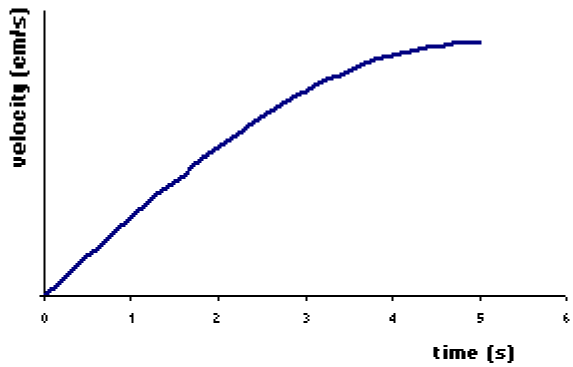
B



C



D



Next we will look at the velocity of the block from position B to the equilibrium position.

3. If we say that on moving from equilibrium to B velocity is positive, what will it be when moving from B to equilibrium? How can you represent this on the velocity versus time graph?

Where will the magnitude of instantaneous velocity be a maximum? Where will it be a minimum? Hence, is the block speeding up or slowing down as it moves from position B to equilibrium?

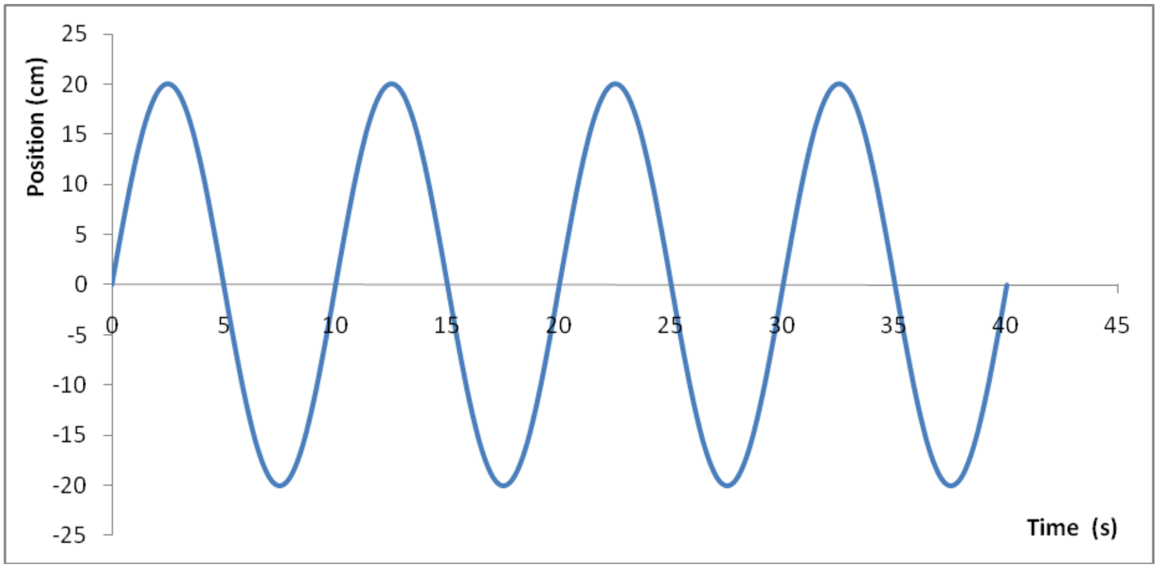
Redraw the line shape for velocity versus time on moving from equilibrium to position B on the axis provided below and also use the information from part 3 to add the motion from position B to equilibrium to the graph.

4. Repeat step 3 in order to add the motion of the block from equilibrium to position A and from position A to equilibrium to your graph.


Velocity (cm/s)
Time (s)

E. Because the surface that the block is situated on is frictionless, the block will continue to repeat this motion until an external force acts on it. Hence its motion is said to be periodic.

The graph below is an extended position vs. time graph for the block. This graph includes values for time at various points and gives the amplitude (distance from equilibrium to position A or B). You will notice that the graph has a sinusoidal line shape and so we can say that it is harmonic.



1. If we say that one cycle involves the movement of the block from equilibrium to position B, from B back through equilibrium to position A, and then from A back to equilibrium, how long did it take for the block to complete one full cycle?

The time it takes to complete one full cycle is called the period and is given the symbol T. Frequency (f) is defined as the number of cycles per second.

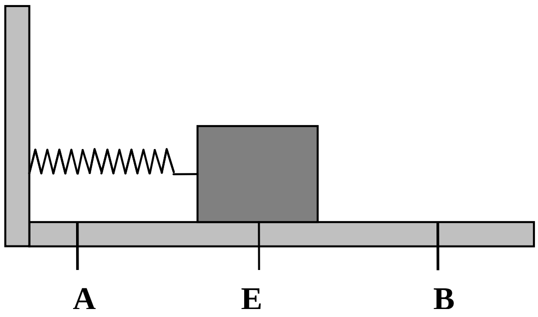
Write the frequency f in terms of the period T. What is the frequency of the block?

However, when dealing with sine waves, physicists prefer to work in radians as it leaves differentiation easier (for more information on this see http://mathforum.org/library/drmath/view/54181.html). As a result we use angular frequency more often than frequency. Angular frequency is given by $2\pi f$ and is given the symbol ω.

Simple Harmonic Motion Homework

I. Acceleration in Simple Harmonic Motion

Consider a compression spring mounted horizontally attached to a block of mass *m* positioned on a frictionless surface, as shown at right. The spring is at its natural length and the block is said to be at its "equilibrium position" (E).



Positions A and B are both the same distance from the equilibrium position. When the spring is compressed to position A and then released, the block will go back through the equilibrium position to position B. The block then returns through the equilibrium position to position A. This motion will continue indefinitely.

In the Simple Harmonic Motion Tutorial we looked at how position and velocity changed with time as the block moved from equilibrium to position B, from position B to equilibrium, from equilibrium to position A, and from position A to equilibrium. Here we will look at how acceleration changes with time for the same motion sequence.

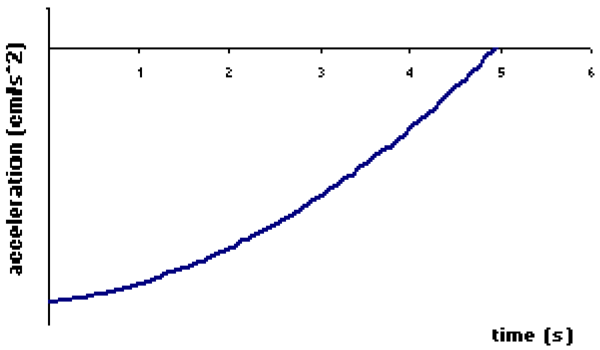
We will start with the motion of the block from equilibrium to position B.

1.Will acceleration and velocity be in the same direction? So if velocity is in the positive direction, what direction will acceleration be in?

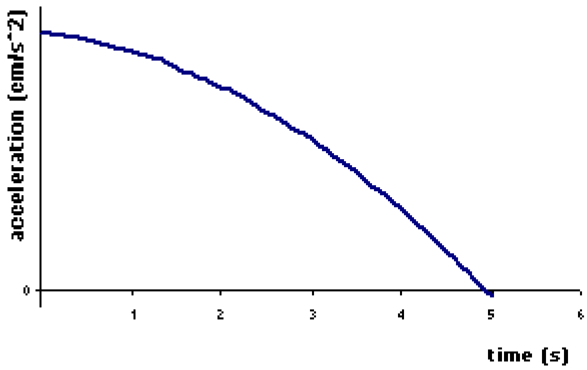
At which of the points will the magnitude of acceleration be a maximum? At which of the points will the magnitude of acceleration be a minimum?

Taking this into consideration, which of the following graphs correctly represents acceleration versus time for the block, on moving from equilibrium to position B? Explain.

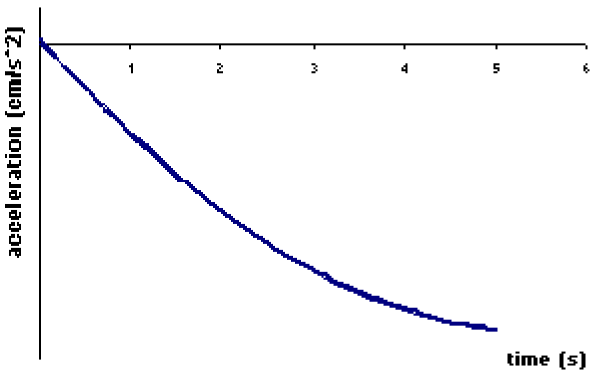
A



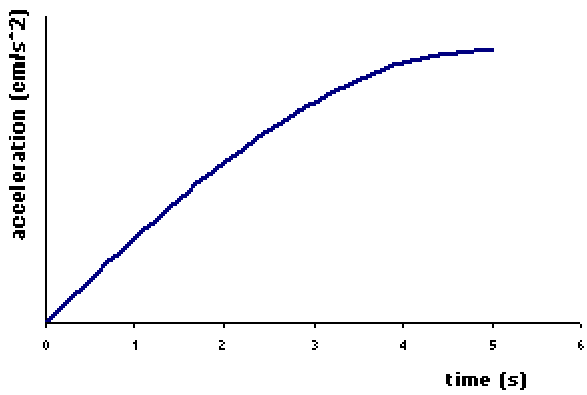
B



C



D



Next we will look at the motion of the block from position B to the equilibrium position.

2. As you deducted from looking at velocity versus time the block is speeding up from B to E, so is acceleration and velocity in the same or opposite direction?

Where will the magnitude of acceleration be greatest? Where will it be least?

Redraw the line shape for acceleration versus time on moving from equilibrium to position B on the axis provided below and also use the information from part 2 to add the motion from position B to equilibrium to the graph.

3.Repeat step 2 in order to add the motion of the block from equilibrium to position A and from position A to equilibrium to your graph.


Acceleration (cm/s^2)
Time (s)

II. Relationship between position and restoring force

Consider a compression spring mounted horizontally attached to a block of mass *m* positioned on a frictionless surface. When the spring is at its natural length, the block is said to be at its equilibrium position (E) as shown in figure 1.

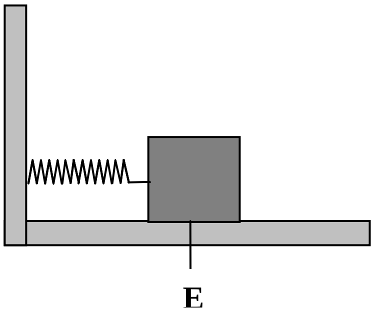


Figure 1

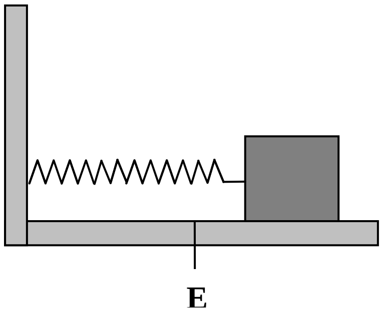


Figure 2

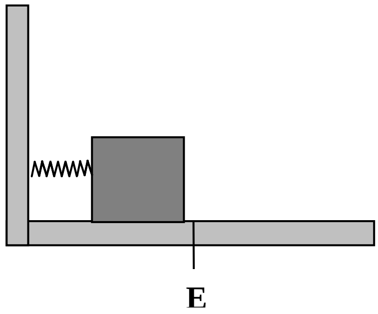


Figure 3

A.
In the boxes at right, draw arrows to represent the directions of:

- the position of the block, x, taking x = 0 when the block is at its equilibrium position and
- the force on the block by the spring, F.

Arrows for figure 2	Arrows for figure 3
Position of block	Position of block
Force on block by spring	Force on block by spring

Use the arrows you drew to explain why the minus sign is necessary in the expression F α –x.

B.
1. When the object moves to the right is x less than, greater than or equal to zero? So what sign can we denote to any position on the right?

2. When the object moves to the left is x less than, greater than or equal to zero? So what sign can we denote to any position on the left?

3. Looking back to our acceleration graph in section I of the Simple Harmonic Motion tutorial, is acceleration proportional to +x or –x? Explain.

4. By Newton's Second Law, if mass is constant what is the relationship between acceleration and force? Hence, or otherwise, state the relationship between force and position.

C. Consider the following statements made by two students regarding the relationship, $F \alpha - x$.

Student 1: "*I don't think that this relationship holds throughout. When the object is moving away from equilibrium, position and force are in opposite directions, so* $F \alpha - x$ *is true here. But when the object is moving towards the equilibrium position, force and displacement are in the same direction so* $F \alpha - x$ *does not hold."*

Student 2: *"No. The relationship* $F \alpha - x$ *will always hold. The x is just position, not a change in position or displacement. Anywhere to the right of equilibrium x is greater than zero and force is less than zero. Anywhere to the left of equilibrium x is less than zero and force is greater than zero. Therefore, the relationship will always hold."*

When we know the spring constant k, we can say that F = -kx.

Appendix S

Simple Harmonic Motion Instruction

Year Two

This Appendix contains the simple harmonic motion tutorial and homework instruction administered in year two.

Simple Harmonic Motion Tutorial

I. Relationship between position and restoring force

Consider a compression spring mounted horizontally attached to a block of mass *m* positioned on a frictionless surface. When the spring is at its natural length, the block is said to be at its equilibrium position (E) as shown in figure 1.

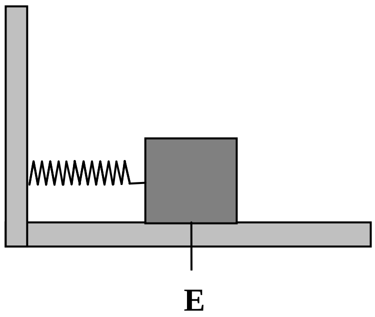


Figure 1

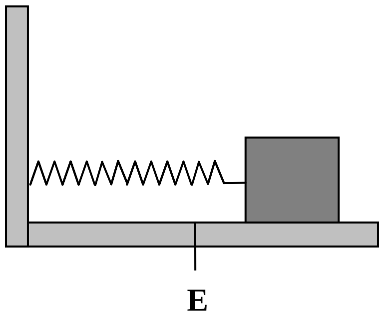


Figure 2

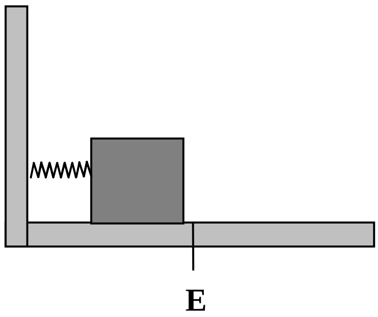


Figure 3

A.
In the boxes at right, draw arrows to represent the directions of:

- the position of the block, x, taking x = 0 when the block is at its equilibrium position and
- The restoring force (F) on the block by the spring.

Arrows for figure 2	Arrows for figure 3
Position of block	Position of block
Restoring force on block	Restoring force on block

From the arrows, explain the relationship between restoring force and position.

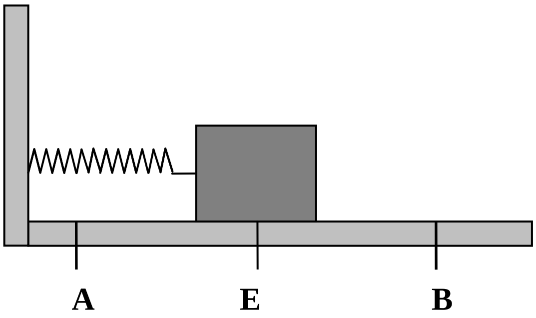
Consider the following statements made by two students regarding the relationship, $F \propto -x$. Which student do you agree with? Explain.

Student 1: "*I don't think that this relationship holds throughout. When the object is moving away from equilibrium, displacement and force are in opposite directions, so $F \propto -x$ is true here. But when the object is moving towards the equilibrium position, force and displacement are in the same direction so $F \propto -x$ does not hold.*"

Student 2: *"No. The relationship $F \propto -x$ will always hold. The x is just position, not a change in position or displacement. Anywhere to the right of equilibrium x is greater than zero and* *force is less than zero. Anywhere to the left of equilibrium x is less than zero and force is greater than zero. Therefore, the relationship will always hold.*

B.

Positions A and B are both the same distance from the equilibrium position. When the spring is compressed to position A and then released, the block will go back through the equilibrium position to position B. The block then returns through the equilibrium position to position A. This motion will continue indefinitely on a frictionless surface.



Where will the magnitude of the restoring force be a maximum? Where will the magnitude of the restoring force be a minimum? Explain.

II. Acceleration in simple harmonic motion

A.

What can we say about the magnitude of the acceleration at points A, B and E? Explain. (Hint: Newton's second law)

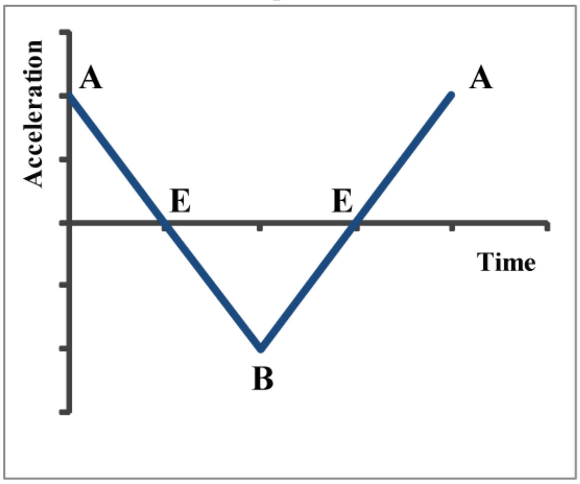
We define the direction to the right as positive and the direction to the left as negative. When the restoring force is acting towards the right is it positive or negative? When the restoring force is acting towards the left is it positive or negative?

In the following table state whether the acceleration is positive or negative and whether it is increasing or decreasing for each interval. Explain.

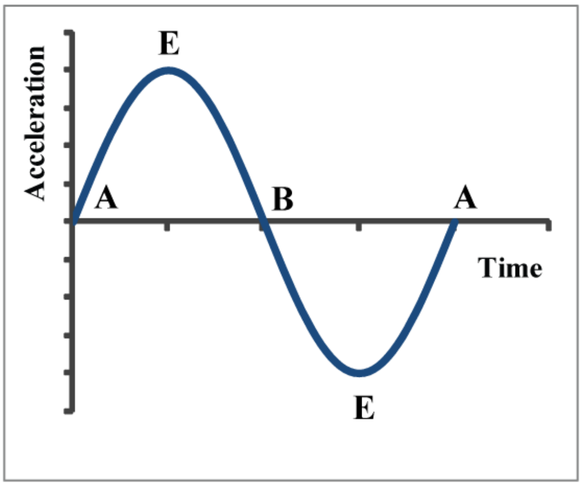
Interval	**Sign of acceleration**	**Change in acceleration**
A-E		
E-B		
B-E		
E-A		

B.
Which of the following graphs best represents the acceleration versus time for the above situation? Explain.

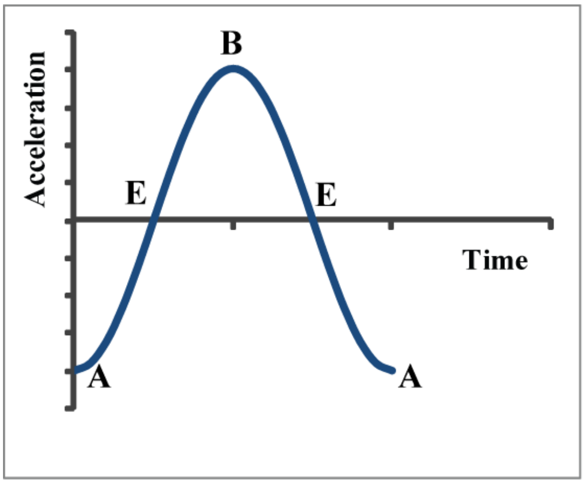
Graph 1



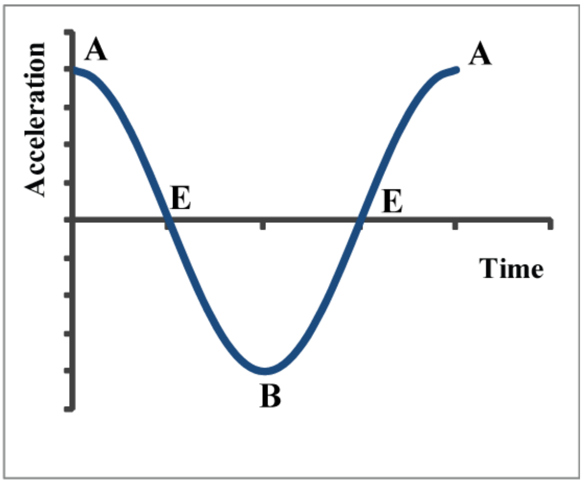
Graph 2



Graph 3



Graph 4



Describe what the other graphs show about acceleration versus time.

III. Velocity in simple harmonic motion

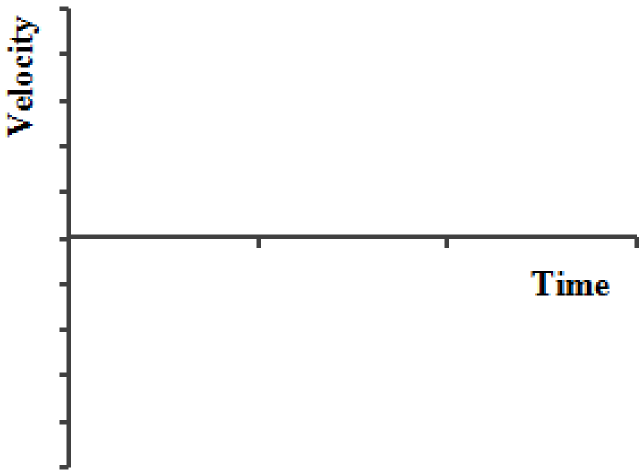
A.

What can we say about the magnitude of the velocity at points A, B and E? Explain.

In the following table state whether the velocity is positive or negative and whether it is increasing or decreasing for each interval. Explain.

Interval	Sign of velocity	Change in velocity
A-E		
E-B		
B-E		
E-A		

Represent the velocity versus time for the four intervals, starting at A, on the axis given.



B. Consider the following student statements. Which student do you agree with? Explain.

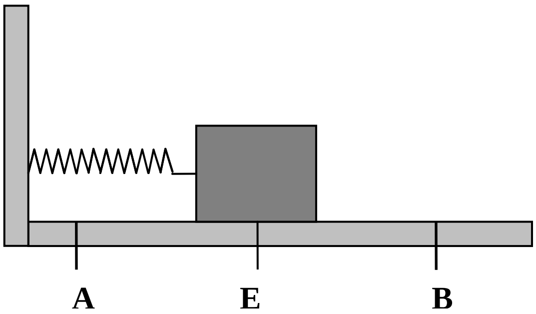
Student 1: *At points A and B the velocity is zero. This should mean that acceleration is zero at these points also. Our graph for acceleration versus time doesn't show this – it must be incorrect.*

Student 2: *I disagree. At A and B the restoring force is greatest. We know that restoring force is directly proportional to acceleration, therefore our graph for acceleration versus time is correct.*

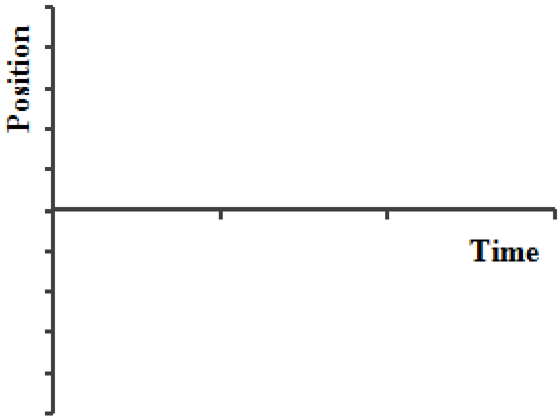
Simple Harmonic Motion Homework

A.

Positions A and B are both the same distance from the equilibrium position. When the spring is compressed to position A and then released, the block will go back through the equilibrium position to position B. The block then returns through the equilibrium position to position A. This motion will continue indefinitely on a frictionless surface.



Draw a graph to represent position versus time for one full cycle of the motion. Start your graph from point E and make sure to label all points.

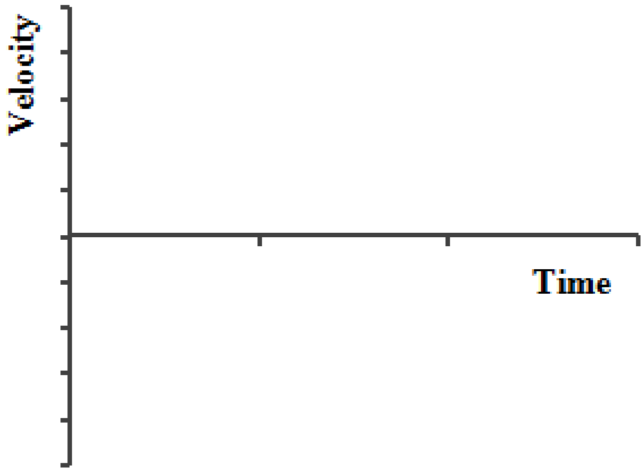


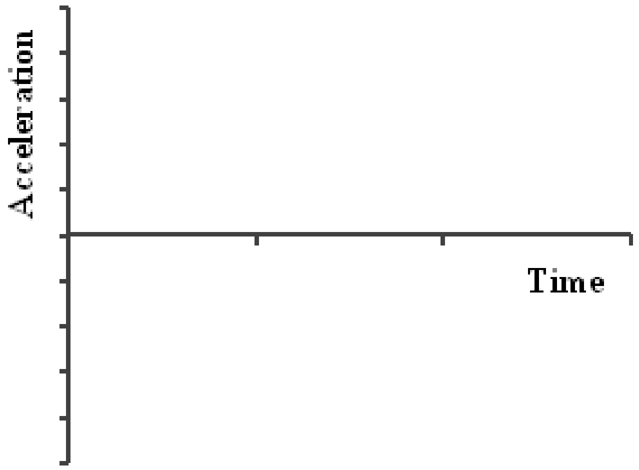
Explain why you drew the graph the way you did.

B. In the following table, state the sign of each and whether it is increasing or decreasing.

Interval	Position	Velocity	Acceleration
A-E	-, *decreasing*		
E-B			
B-E			
E-A			

C. Draw velocity and acceleration versus time graphs for one full cycle of the motion both starting at point E.



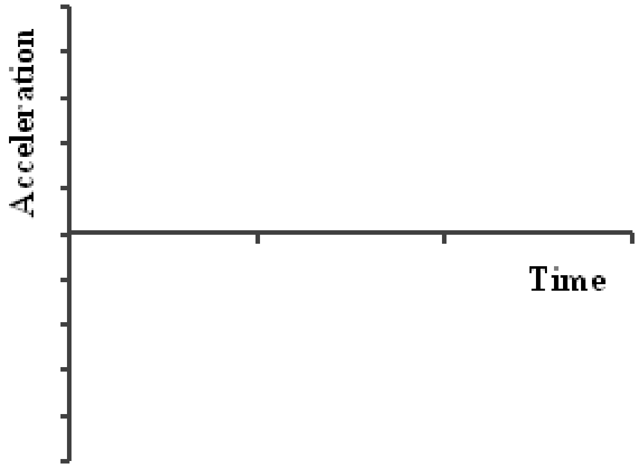


D. Suppose the mass of the block is doubled.

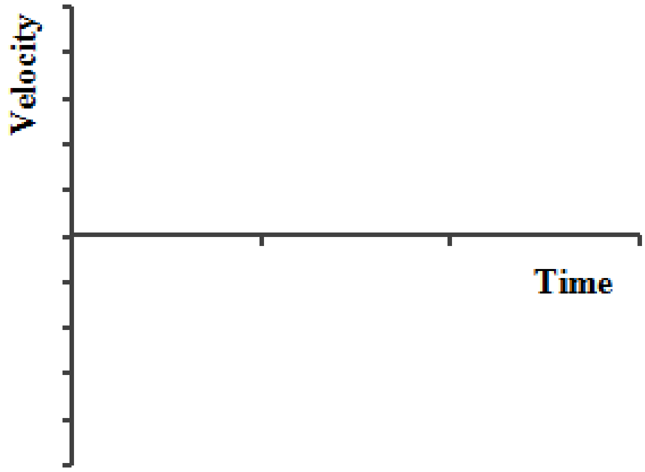
1. Will the springs restoring force be affected by the change in mass? Explain.

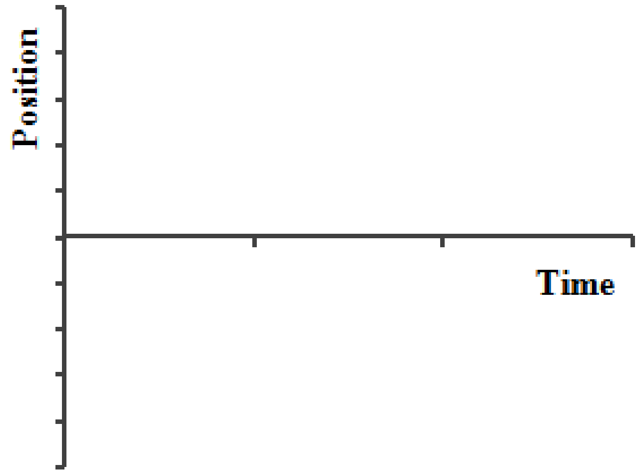
   Will the acceleration of the block be affected by the change in mass? If so, how? Explain.

2. Describe the changes in the acceleration versus time graph that result from the mass being doubled and show them on the axis given.



3. Describe the changes in the velocity and position versus time graphs due to the change in acceleration and show them on the axis given.





Appendix T

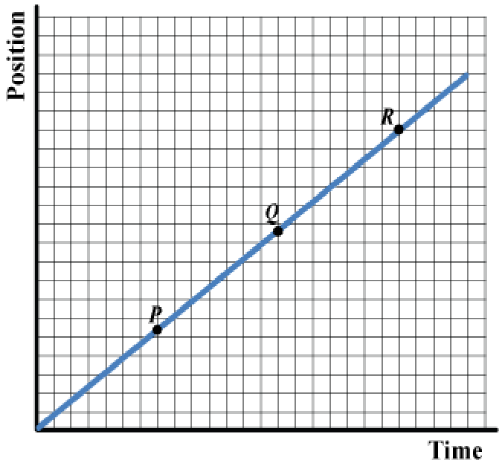
Simple Harmonic Motion Instruction

Year Three

This Appendix contains the simple harmonic motion tutorial and homework instruction administered in year three.

I. Uniform motion

A. An object travels from one position to another. The graph at right shows how its position changes with time.



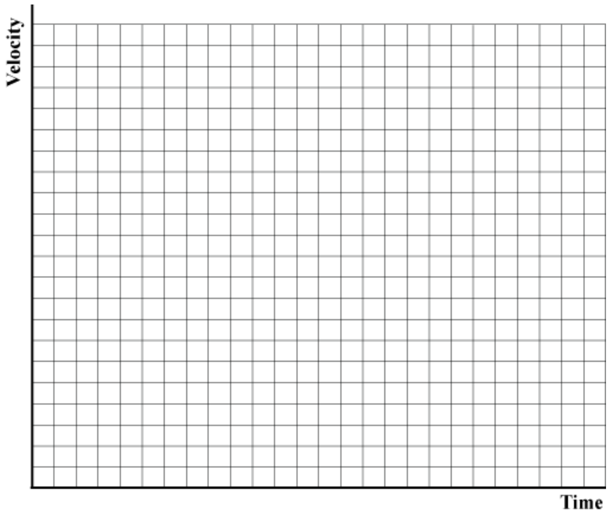
Compare the position of the object at points *P*, *Q* and *R*.

Compare the change in position between points *P* and *Q*, and *Q* and *R*.

B. What can you say about the velocity of the object? Explain.

How can we find the velocity of the object at points *P*, *Q* and *R*? What property of the position versus time graph does velocity represent?

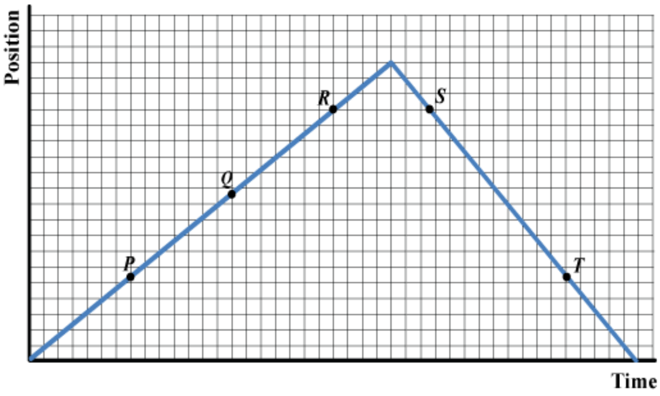
Plot the velocity of the object at points *P*, *Q* and *R* on the axis below and draw the corresponding line shape.



C. Is there a change in the velocity of the object? Hence, what can we say about the acceleration of the object?

D. The object now travels back to its original position. Again the graph shows how its position changes with time.

Compare the change in position between points *P* and *R*, and *S* and *T* (consider both magnitude and sign)



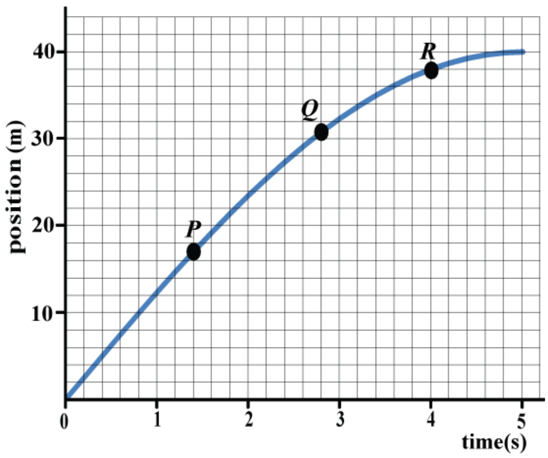
On the grid provided draw the graph for velocity versus time for the object as it returns to its original position.

What can we say about the acceleration of the object as it returns to its original position?

II. Non-uniform motion

A. The position versus time graph for another moving object is shown at right. The object travels 40m in 5s.

Compare the change in position between points *P* and *Q*, and *Q* and *R*.

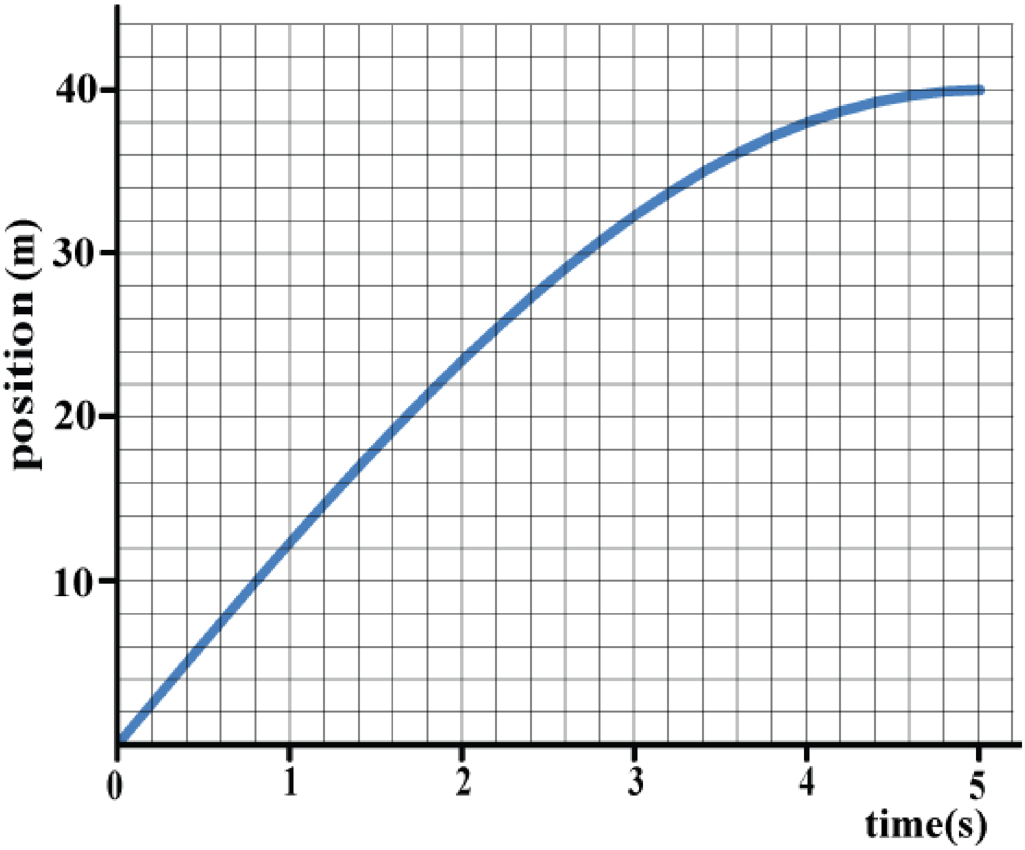


Is the velocity of the object increasing or decreasing? Explain.

Is the velocity positive or negative? Explain.

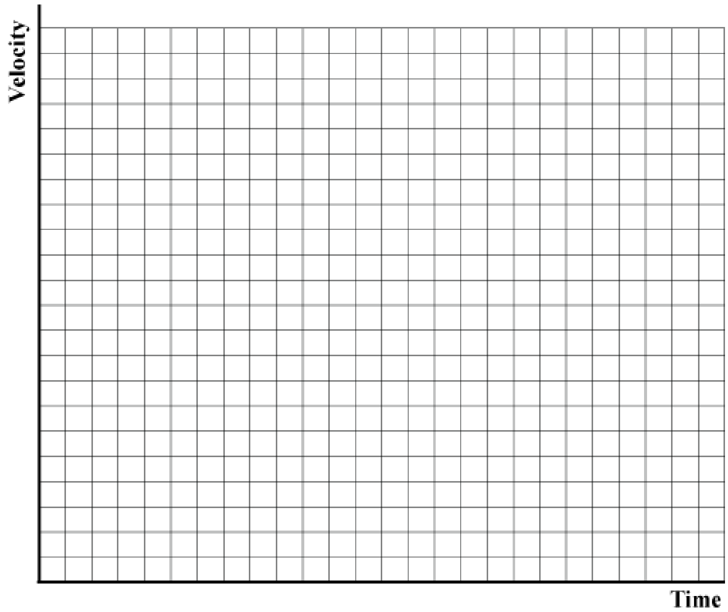
B. On the position versus time graph below draw tangents to the curve at each second interval starting at 0s.

What will the slope of these tangents tell us? Explain.



Calculate the slope of each of the tangents.

Plot the velocity points that you have calculated on the axis below and draw an appropriate line shape.



C. At what times will the acceleration be greatest and least? Explain.

Will the acceleration be positive or negative?

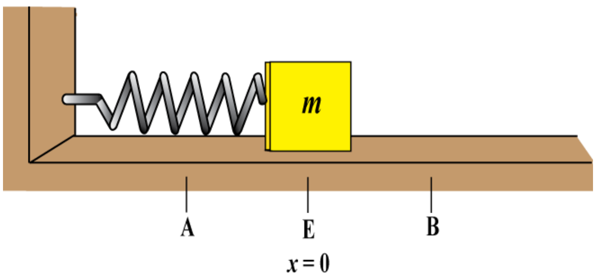
How can we use the velocity versus time graph that you have drawn to find the acceleration versus time graph?

Draw the graph for acceleration versus time on the grid below.

D. Compare the position, velocity and acceleration for the object.

POSITION IN SIMPLE HARMONIC MOTION

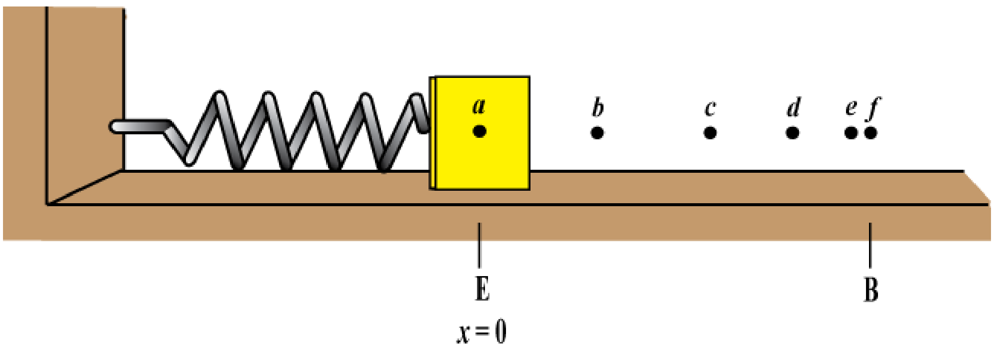
I. Position versus time from Position B to Equilibrium



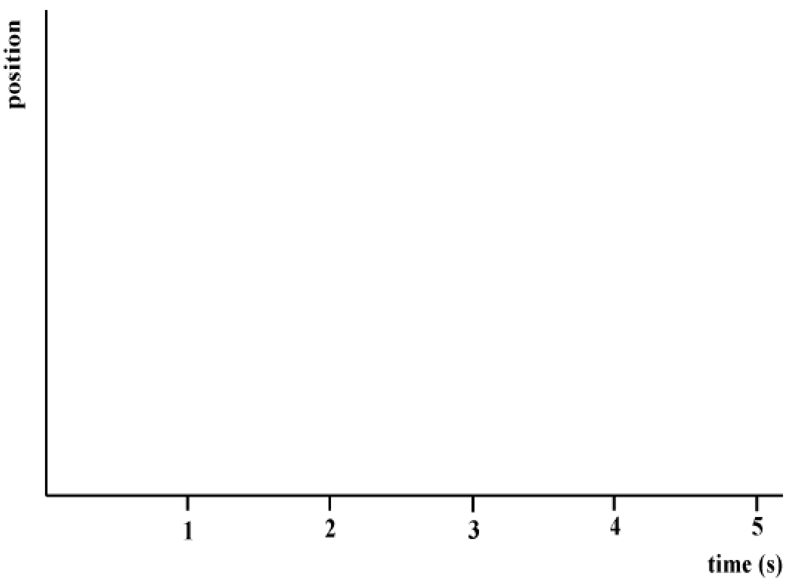
Consider a compression spring mounted horizontally attached to a block of mass *m* positioned on a frictionless surface. When the spring is at its natural length, the block is said to be at its "equilibrium position", E.

Positions A and B are both the same distance from the equilibrium position. When the spring is compressed to position A, and then released, the block will go back through the equilibrium position to position B. The block then returns through the equilibrium position to position A. This motion will continue indefinitely.

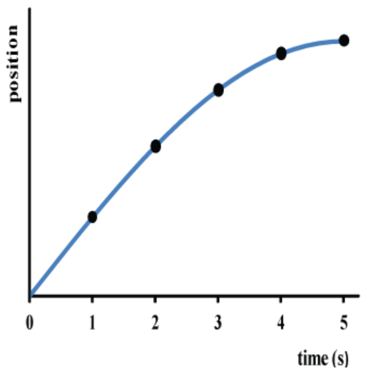
A. The diagram below shows the position of the block after it leaves position B returning to the equilibrium position at instants separated by equal time intervals. Point *a* is at the position B and point *f* is at the equilibrium position ($x = 0$).

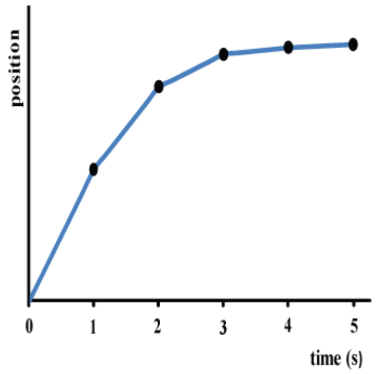


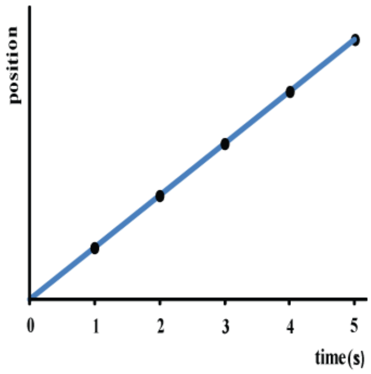
On the axis given below, plot position (distance from equilibrium position) versus time for the points *a* to *f* (Points *a, b, c, d, e,* and *f* are taken at 0s, 1s, 2s, 3s, 4s, and 5s respectively).



B. Describe what the following three line shapes represent about the motion of the block.







Which of the line shapes do you think correctly fits the data points you have drawn for position versus time? Explain.

II. Position versus time for all intervals

A. Consider the following statements made by two students regarding the sign of the position when the block is moving from position A to equilibrium.

Student 1: *When the block was moving from equilibrium to position B the position was positive. When it moves from B to equilibrium it is moving in the opposite direction, therefore position must be negative.*

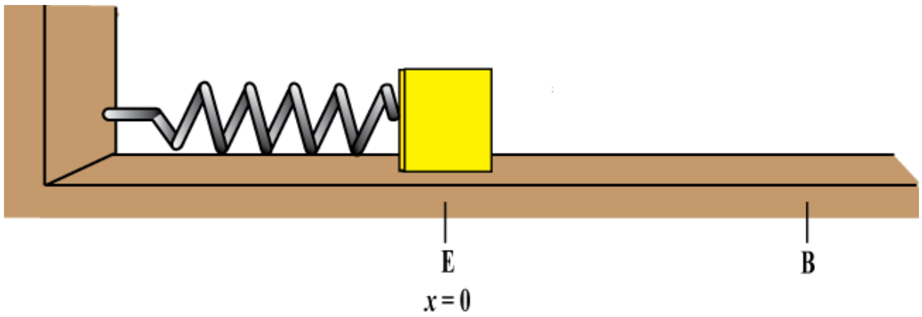
Student 2: *We are just looking at position, not a change in position or displacement. At equilibrium the position is zero. I think that when the block is at a position anywhere to the right of equilibrium it is a positive position.*

Which of the students do you agree with?

What will the sign for position be when the block is moving from equilibrium to position A, and when it is moving from position A to equilibrium? Explain brielfly.

B. Again consider the block moving from position B to equilibrium. Is the magnitude of the position increasing or decreasing? Explain.

On the diagram below indicate the position of the block at equal time intervals as it moves from position B to equilibrium (label the points *g-l*).

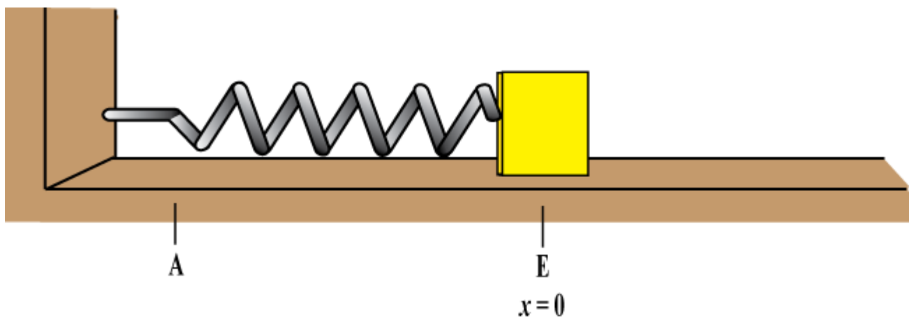


Will the graph of position versus time for B to equilibrium start at the origin? Explain.

Draw an axis and sketch the line shape for position versus time from B to equilibrium. Explain briefly.

C. Now consider the block moving from equilibrium to position A. Is the magnitude of the position increasing or decreasing? Explain briefly.

On the diagram below indicate the position of the block at equal time intervals as it moves from equilibrium to position A (label the points *m-r*).



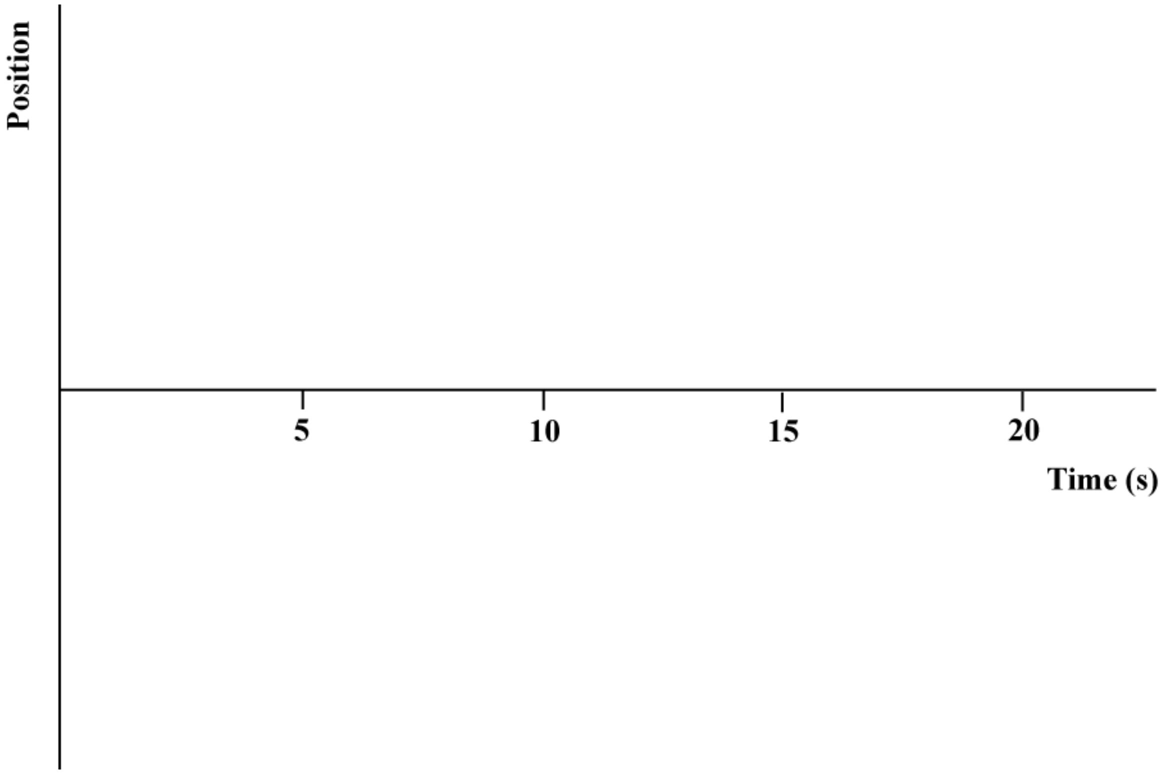
Compare the positions at points a-f when the block moves from equilibrium to position B to the positions at points *m-r* when the block moves from equilibrium to position A.

Draw an axis and sketch the line shape for position versus time from equilibrium to position A. Explain briefly.

D. In the table below, state the sign for position and whether its magnitude is increasing or decreasing for each interval.

Interval	**Sign**	**Magnitude**
E-B	Positive	Increasing
B-E		
E-A		
A-E		

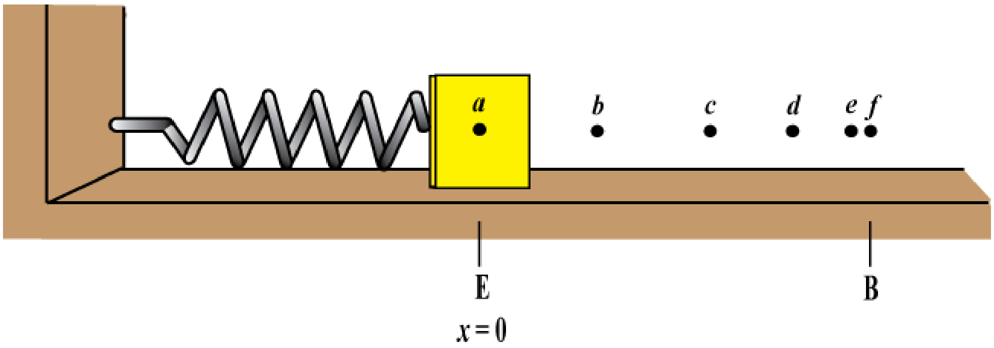
E. Use the information from the table to draw the graph of position versus time for one full cycle of motion starting from position A.



VELOCITY IN SIMPLE HARMONIC MOTION

I. Velocity versus time from equilibrium to position B

A. The diagram below shows the position of the block after it passes the equilibrium position going to position B at instants separated by equal time intervals. Position B is 40mm from the equilibrium position.



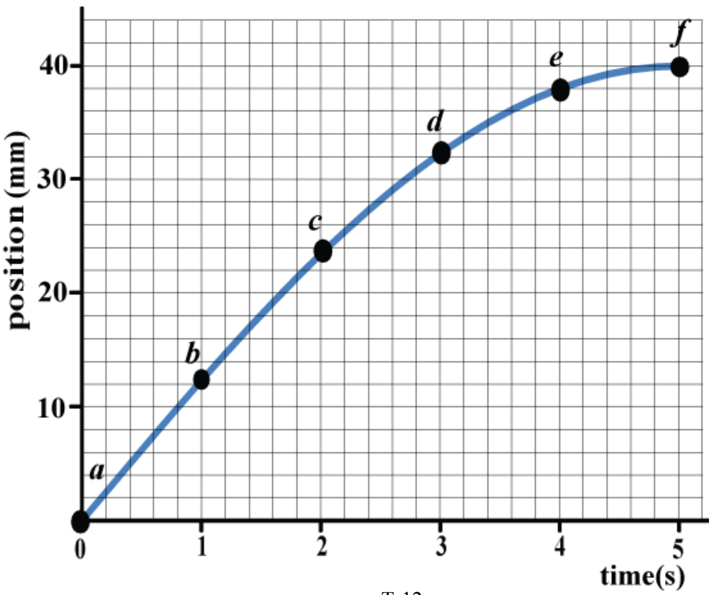
Between which consecutive points *a-b, b-c*, etc. is the change in position greatest and least?

Between which consecutive points *a-b, b-c*, etc. will the velocity be greatest? Where will it be least? Explain briefly.

Is the magnitude of the block's velocity increasing or decreasing as it moves from equilibrium to position B?

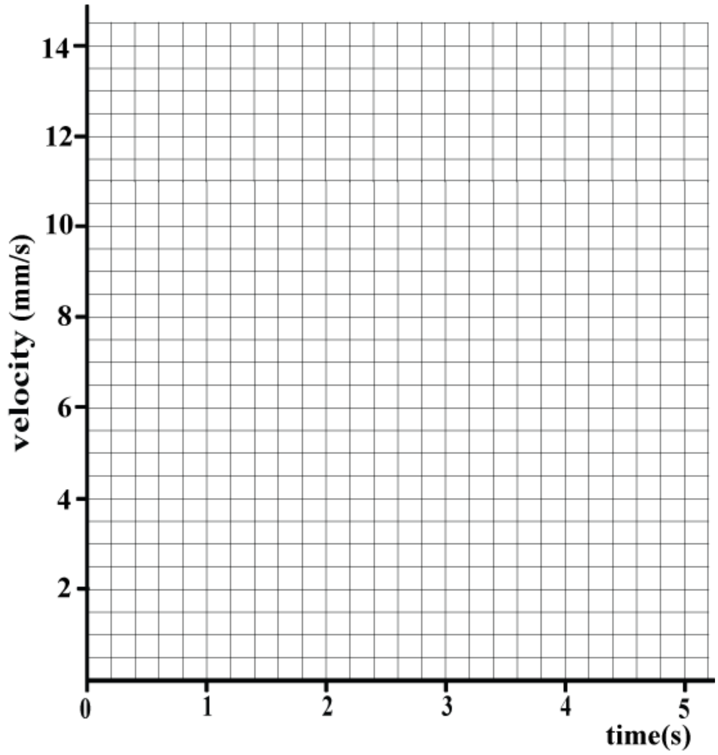
Draw vectors on the diagram that represent the instantaneous velocity of the block at each of the points *a-f*. If the velocity is zero at any point, state so explicitly.

B. In the *Position in Simple Harmonic Motion Tutorial* you obtained the following graph for position versus time for the block when it moves from equilibrium to position B.

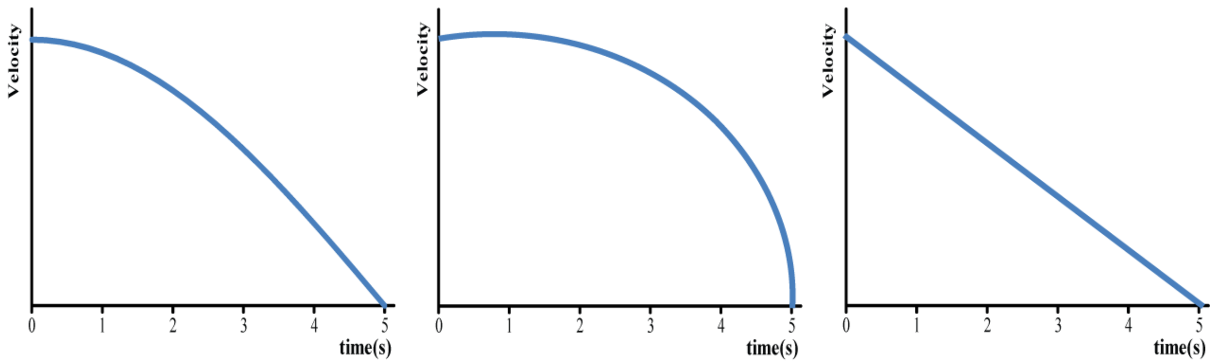


VELOCITY IN SIMPLE HARMONIC MOTION

Determine the velocity of the block at points *a, c* and *f.* Hence plot these points on the velocity versus time graph below.

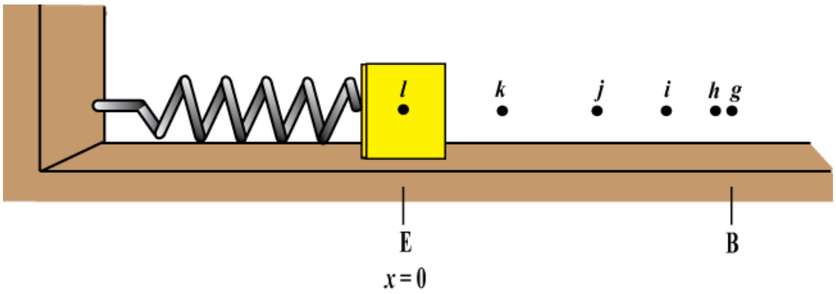


Explain what the following line shapes suggest about velocity versus time from equilibrium to position B. Which one is correct?



II. Velocity versus time for all intervals

A. The diagram below shows the position of the block as it moves from position B to equilibrium.



Is the magnitude of the block's velocity increasing or decreasing as it moves from position B to equilibrium?

Draw vectors on the diagram that represent the instantaneous velocity of the block at each of the points *g*-*l*. If the velocity is zero at any point, state so explicitly.

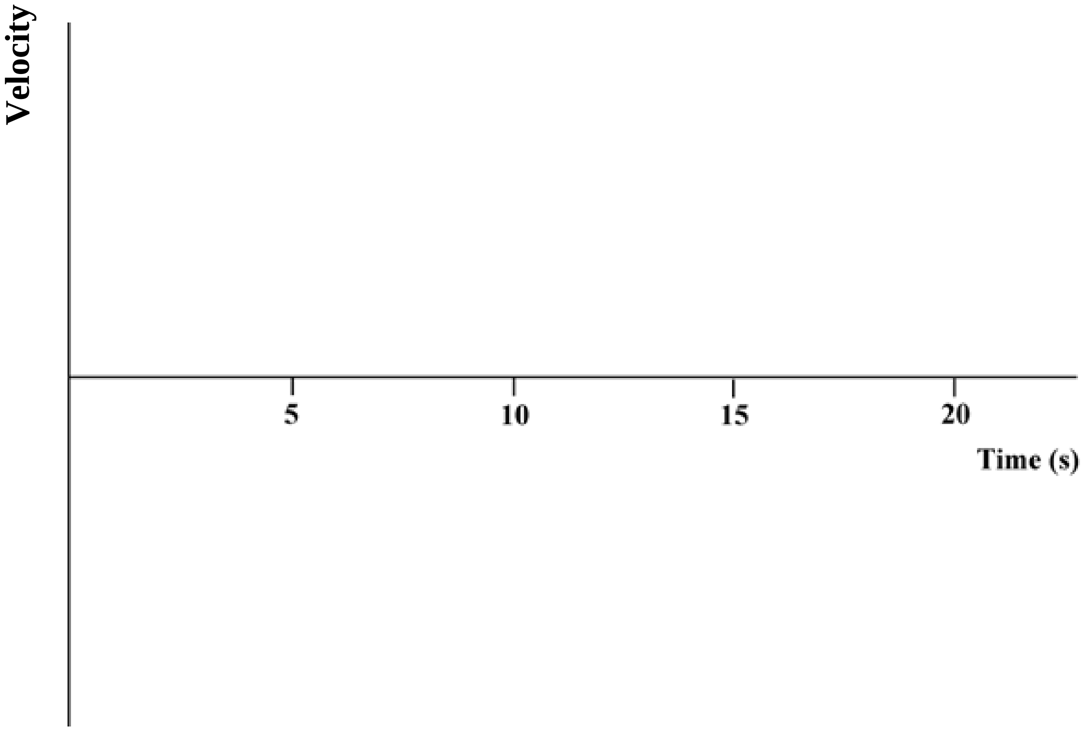
Based on the change of position is the velocity of the block positive or negative as it moves from position B to equilibrium?

Draw an axis and sketch the line shape for velocity versus time from position B to equilibrium. Explain briefly.

B. In the table below, state the sign for velocity and whether its magnitude is increasing or decreasing for each interval.

Interval	Sign	Magnitude
E-B	Positive	Decreasing
B-E		
E-A		
A-E		

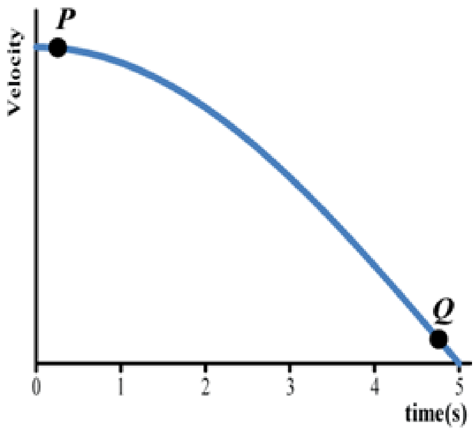
C. Use the information from the table to draw the graph of velocity versus time for one full cycle of motion starting from position A.



D. Compare position and velocity versus time for one full cycle of the blocks motion.

ACCELERATION IN SIMPLE HARMONIC MOTION

I. Acceleration versus time from equilibrium to position B



A. The graph at right shows velocity versus time for the block as it moves from equilibrium to position B. Point *P* is close to equilibrium and *Q* is close to position B.

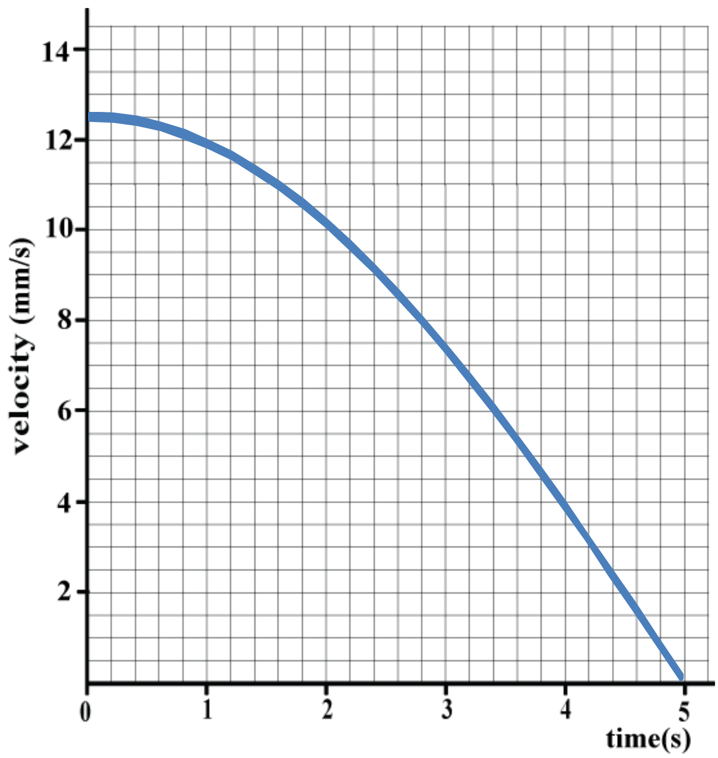
Is the change in velocity at *P* greater than or less than the change in velocity at *Q*? How can you tell from the graph?

Is the magnitude of the acceleration at *P* greater than or less than the magnitude of the acceleration at *Q*? Explain briefly.

Is the magnitude of the acceleration increasing or decreasing as the block moves from equilibrium to position B?

Is the change in velocity between point *P* and *Q* positive or negative? Hence is the acceleration positive or negative as it moves from equilibrium to position B?

B. In the *Velocity in Simple Harmonic Motion Tutorial* you obtained the following graph for velocity versus time for the block when it moves from equilibrium to position B.



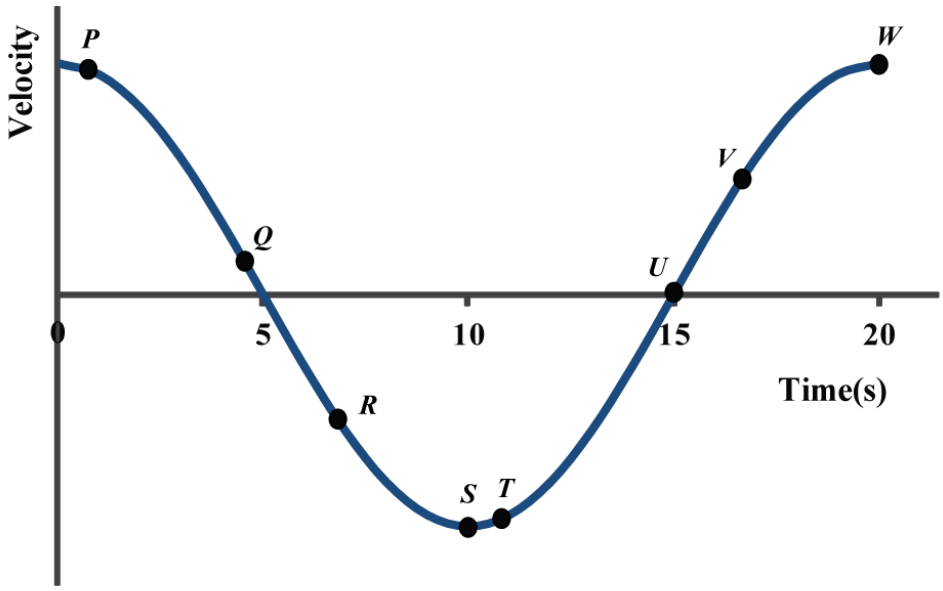
ACCELERATION IN SIMPLE HARMONIC MOTION

Determine the acceleration of the block at various points that will allow you determine the line shape of the acceleration versus time graph. Create an axis on the grid provided below and draw the graph of acceleration versus time for the block as it moves from equilibrium to position B.

C. Compare the magnitude and the sign of the blocks velocity and acceleration as it moves from equilibrium to position B.

II. Acceleration versus time for all intervals

A. The following graph shows velocity versus time for one full cycle of the blocks motion from equilibrium to position B, back through equilibrium to position A and again back to equilibrium.



Rank the magnitude of the acceleration at points *P*-*W* from greatest to least. Explain briefly.

At which points is the acceleration positive? At which points is the acceleration negative? If it is zero at any of the points state so explicitly. Explain.

B. In the table below, state the sign for acceleration and whether its magnitude is increasing or decreasing for each interval.

Interval	Sign	Magnitude
E-B	Negative	Increasing
B-E		
E-A		
A-E		

C. Use the information from the table to draw the graph of acceleration versus time for one full cycle of motion starting from position A.