Gravitational Collapse of a Self-Similar Cylindrical Scalar Field with Non-Minimal Coupling

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Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Doctor of Philosophy is entirely my own work, that I have exercised reasonable care to ensure the work is original, and does not to the best of my knowledge breach any law of copyright, and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

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Coordinates

The coordinates used for the vast majority of the thesis are $\{u, v, \theta, z\}$. The similarity variable is defined by $\eta = v/u$. We use the rescaling $\tau = -\log\eta$ in Chapters 2,3 and 4. We use the rescaling $t = \log(-\eta)$ in Chapters 5 and 6. $\mathcal{N}_-$ and $\mathcal{N}_+$ are used to denote the past and future null cones of the scaling origin $\mathcal{O}$. They are the hypersurfaces defined by $v = 0$ and $u = 0$, respectively.

Metric functions

The metric functions for the non-self-similar metric are $\bar{\gamma}(u,v), \bar{\phi}(u,v)$ and $r(u,v)$. The self-similar metric functions are $\gamma(\eta), \phi(\eta)$ and $S(\eta)$. They are related by

$$\bar{\gamma} = \gamma, \qquad \bar{\phi} = \phi - \frac{1}{2}\log|u|, \qquad r = |u|S.$$

Parameters

The system of field equations has two parameters, k and V_0. We also make common use of the translation $\lambda = 1 - k^2/2$.

Matter functions

The scalar field is denoted by $\psi(u,v)$ and the potential by $V(\psi)$. The self-similar form of the scalar field is then given by $\psi = F(\eta) + \log|u|^{k/2}$. The translation $l = 2F/k - \log|\eta|^{1/2}$ is used much more widely in the analysis. The potential is shown to satisfy $V = V_0 e^{\lambda l}$.

Other variables

We introduce the variable $R = e^{\tau/2}S$ in Chapter 2. The analysis in Chapter 4 is carried out using the system of variables

$$u_1(\tau) = \frac{\dot{R}}{R}, \qquad u_2(\tau) = V_0 e\lambda l, \qquad u_3(\tau) = \dot{l},$$

where the overdot denotes differentiation with respect to τ.

In Chapters 5 and 6 we use $S(\eta) = \sigma(t)$ and the analysis is carried out using the system of variables

$$x_1 = \frac{\sigma'(t)}{\sigma}, \qquad x_2 = V_0 e^{\lambda l}, \qquad x_3 = \frac{dl}{dt} + \frac{1}{2}.$$

Abstract

The aim of this thesis is to give a rigorous analysis of the Einstein field equations which arise when modelling the collapse of a scalar field in self-similar, whole-cylinder symmetry. The principal motivation is to discover whether, and under what conditions, this class of spacetimes admit the existence of a naked singularity. Imposing self-similarity on the spacetime gives rise to a set of single variable functions describing the metric. Furthermore, it is shown that the scalar field is dependent on a single unknown function of the same variable and that the scalar field potential has exponential form. The Einstein equations then take the form of a set of ODEs, with two degrees of freedom and a free initial datum, where initial data is given on the regular axis. Self-similarity also gives rise to a scalar curvature singularity at the scaling origin, to the future of the regular axis. The field equations have singular points along the axis and along the past and future null cones of the singularity, labelled $\mathcal{N}_-$ and $\mathcal{N}_+$, respectively. We label the region between the axis and $\mathcal{N}_-$ as region **I** and the region bounded by $\mathcal{N}_-$ and $\mathcal{N}_+$ as region **II**. The problem naturally divides into two stages, that is, solving the equations in these two separate regions. The independent variable may be rescaled in each separate region to obtain an autonomous system of field equations and a dynamical systems approach is used to obtain qualitative solutions. It is shown that some solutions have a maximal interval of existence ending either on or before $\mathcal{N}_-$, where the termination of the solution corresponds to either a spacetime singularity or future null infinity, and that some solutions may be extended into region **II**. All of these solutions are then shown to terminate in a spacelike singularity before reaching $\mathcal{N}_+$. This supports the Cosmic Censorship Hypothesis.

Chapter 1

Introduction

Gravitational collapse is one of the most intriguing processes in nature, leading to such exotic phenomena as supernovae, white dwarfs, neutron stars and black holes. It is one the most fruitful topics in the theory of General Relativity (GR). In the words of John Wheeler "..one feels that he has at last in gravitational collapse a phenomenon where general relativity comes into its own.." [41] A black hole is formed when the collapse of a sufficiently massive body is unimpeded by atomic pressures and continues to contract to a vanishingly small radius, forming a singularity which is hidden behind an event horizon. The event horizon prevents the escape of matter and radiation from the region within a certain radius of the singularity, thus shielding the external universe from it. The existence of black holes was predicted by the famous Schwarzschild solution and there is now strong observational evidence that they do indeed exist. Indeed, the singularity theorems of Hawking and Penrose prove the existence of singularities in the theory of GR. A related phenomenon which arises in some theoretical models of collapse is the naked singularity, which is when the singularity is formed before the event horizon, thus rendering it visible to outside world. This is an undesirable effect since the predictability of classical physics is compromised; infinite amounts of energy may be emitted from the singularity and, therefore, one cannot predict its causal future. In response to this, Penrose conjectured that the collapse of physically realistic matter could only result in the creation of singularities which are censored by an event horizon. Known as the Cosmic Censorship Hypothesis (CCH), it has yet to be proven and its veracity is one of the principal outstanding questions in

GR. However, the models which do exhibit naked singularity formation are highly idealised and are generally thought to be unrealistic for one reason or another and a plausible counter-example to the CCH has yet to be found. The purpose of this work is to add to the literature on this subject and to determine whether or not the class of self-similar cylindrically symmetric scalar field spacetimes obeys the CCH.

In the following section we give a brief account of the notation used throughout the thesis. In Section 1.2, we describe the mathematical concepts of GR necessary to formulate the problem at hand. Section 1.3 gives a review of some of the existing literature on cylindrical symmetry and Section 1.4 is a brief on note the role of self-similarity in GR. We summarise the findings and layout of the thesis in Section 1.5.

1.1 Notation

We use the notation $\mathcal{M}$ for a manifold and $\boldsymbol{g}$ for the metric tensor. A bold symbol is used for index-free representations of all tensors, including vectors and one-forms. We use the signature (-,+,+,+) for the metric. Latin indices are used for all abstract indices for tensors and these always run over the values (0,1,2,3). Components in a particular coordinate basis are labelled using the coordinates, for example, the g_{22} component of the metric in the coordinate basis $\{u, v, \theta, z\}$ would be labelled $g_{\theta\theta}$. We use the Einstein summation convention so that repeated indices are summed over all values, for example,

$$V^a w_a = V^0 w_0 + V^1 w_1 + V^2 w_2 + V^3 w_3. \qquad (1.1.1)$$

Partial derivatives are denoted by $\partial f/\partial x = \partial_x f = f_{,x}$. The Lie derivative with respect to $\boldsymbol{\xi}$ is denoted by $\mathcal{L}_{\boldsymbol{\xi}}$ and the covariant derivative by ∇_a. The covariant derivative along a vector field $\boldsymbol{X}$ is given by $\nabla_{\boldsymbol{X}}$. An affine parameter along a geodesic is always given by μ and an overdot represents the derivative with respect to the affine parameter. We also use an overdot to denote a derivative with respect to the independent variable τ in Chapter 4. We refer to ordinary and partial differential equations as ODEs and PDEs, respectively.

1.2 General Relativity

In this section, we outline some of the fundamentals of GR which lead us to the field equations of the theory, which relate the matter to the geometry of the spacetime. Central to the theory of GR is the geometry of curved spaces, as it is the curvature of the four dimensional spacetime manifold in which the gravitational field manifests itself. Hence, an understanding of differential geometry is essential and we outline some of the fundamentals below.

1.2.1 Manifolds and tensor fields

An n-dimensional manifold $\mathcal{M}$ is roughly defined as a space which, at every point, looks locally like Euclidean space of the corresponding dimension, $\mathbb{R}^n$, but which may have a much different global structure. One can set up a non-degenerate coordinate system which parameterises $\mathcal{M}$ locally, although the system may not extend to the whole manifold. An atlas for $\mathcal{M}$ is a set of local coordinate systems such that their union covers $\mathcal{M}$ entirely. The definition of curves and surfaces in $\mathcal{M}$ as constraints on, or parameterisations of, the coordinates follows naturally. $\mathcal{M}$ is said to be differentiable (smooth) if the transformations between any two coordinate systems are differentiable (smooth) on subsets of $\mathcal{M}$ on which they overlap. All manifolds are assumed to be of this type in GR. We can define vectors on a smooth manifold as tangents to curves in the manifold and we denote by $\mathcal{T}_p(\mathcal{M})$ the space of all tangent vectors at a point p in $\mathcal{M}$. It may be shown that $\mathcal{T}_p(\mathcal{M})$ is a vector space of the same dimension as $\mathcal{M}$. Given a basis for the tangent space $\{e_a\} = \{\partial/\partial x^a\}$ then $\boldsymbol{V} \in \mathcal{T}_p(\mathcal{M})$ can be written as

$$\boldsymbol{V} = V^a \frac{\partial}{\partial x^a} = V^a \partial_a, \tag{1.2.1}$$

where V^a are the components of $\boldsymbol{V}$ in the basis $\{e_a\}$. A dual vector, or one-form, acts on vectors to produce a real number. The space of all one forms at a point p is denoted $\mathcal{T}_p^*(\mathcal{M})$ and is dual to $\mathcal{T}_p(\mathcal{M})$. Introducing a basis $\{e^a\} = \{dx^a\}$ on $\mathcal{T}_p^*(\mathcal{M})$ such that $e^a e_b = \delta^a_b$, then we can write $\boldsymbol{w} \in \mathcal{T}_p^*(\mathcal{M})$ as

$$\boldsymbol{w} = w_a dx^a, \tag{1.2.2}$$

where w_a are the components of $\boldsymbol{w}$ in the basis $\{e_a\}$ and δ^a_b is the Kronecker delta defined by

$$\delta^a_b = \begin{cases} 1, & \text{if} \quad a = b \\ 0, & \text{if} \quad a \neq b \end{cases} \tag{1.2.3}$$

Then $\boldsymbol{w}(\boldsymbol{V})$ produces the scalar $V^a w_a$. A tensor is a geometrical object which maps any product space of $\boldsymbol{V} \in \mathcal{T}_p(\mathcal{M})$ and $\boldsymbol{w} \in \mathcal{T}^*_p(\mathcal{M})$ to the real numbers, is linear in its arguments and is invariant under coordinate transformations. The rank of a tensor corresponds to the space on which it acts. Vectors and one-forms are tensors of rank (1,0) and (0,1), respectively. When changing from one coordinate system to another, the components of vectors and one-forms change according to

$$V^a = X^a_{b'} V^{b'}, \qquad w_a = X^{b'}_a w^{b'}, \tag{1.2.4}$$

where $x^{b'}$ are the new coordinates and

$$X^a_{b'} = \frac{\partial x^a}{\partial x^{b'}}, \qquad X^{b'}_a = \frac{\partial x^{b'}}{\partial x^a}, \tag{1.2.5}$$

are the transformation matrices or Jacobians of the transformation. A tensor $\boldsymbol{T}$ of rank (1,1) maps the product space $\mathcal{T}_p(\mathcal{M}) \times \mathcal{T}^*_p(\mathcal{M})$ into the reals and its components transform according to

$$T^a_b = X^{d'}_b X^a_{c'} T^{c'}_{d'}. \tag{1.2.6}$$

Tensors of higher rank transform in an analagous way.

1.2.2 Derivatives on a manifold

It is not difficult to show from (1.2.4) that the partial derivative of a tensor is generally not a tensor, that is, it is not invariant under coordinate transformations. We introduce two types of differentiation on a manifold which are both tensorial; the covariant derivative and the Lie derivative.

Consider the vector field in a particular basis V^a evaluated at points p and q. The covariant derivative of a vector field measures the difference between $V^a(p)$ and the vector parallel to $V^a(q)$ at p in the limit as $p \to q$. It may be shown that this leads to

$$\nabla_b V^a = \partial_b V^a + \Gamma^a_{cb} V^c, \tag{1.2.7}$$

where $\nabla_b V^a$ is the covariant derivative of V^a. The term Γ^a_{cb}, called an affine connection, is a set of functions which transform in such a way that the combination of the terms on the righthand side above are invariant under coordinate transformations, i.e. they are tensorial. Hence, $\nabla_b V^a$ is a tensor of rank (1,1). The covariant derivative of tensors of general rank have a similar definition. Note that $\nabla_a \psi = \partial_a \psi$ for a scalar ψ.
The covariant derivative of a vector $\boldsymbol{V}$ along a vector $\boldsymbol{X}$ is defined as

$$\nabla_{\boldsymbol{X}} V^b = X^a \nabla_a V^b, \tag{1.2.8}$$

which is the analogue of the directional derivative in Euclidean space. $\boldsymbol{V}$ is said to be parallel-transported along $\boldsymbol{X}$ if $\nabla_{\boldsymbol{X}} V^a = 0$. The Lie derivative of a tensor field with respect to a vector field $\boldsymbol{\xi}$ measures the change in that tensor field transported along the integral curves or the flow of $\boldsymbol{\xi}$. The Lie derivative of a vector field V^a with respect to $\boldsymbol{\xi}$ is given by

$$\mathcal{L}_{\boldsymbol{\xi}} V^a = \xi^b \partial_b V^a - V^b \partial_b \xi^a, \tag{1.2.9}$$

with similar definitions for tensors of other rank. The Lie derivative preserves the rank of a tensor. As we shall see, the Lie derivative is useful in defining the symmetries of a manifold.

1.2.3 The metric tensor

The metric tensor $\boldsymbol{g}$ on a manifold is a privileged tensor which takes as its input two vectors and returns a scalar. It is symmetric, of rank (0,2) and completely determines the curvature of the manifold. It allows us to measure the length of intervals between points in the manifold through the relation

$$ds^2 = g_{ab} dx^a dx^b, \tag{1.2.10}$$

which is known as the line element. We define the inverse metric, g^{ab}, by

$$g^{ab} g_{bc} = \delta^a_c \tag{1.2.11}$$

where δ^a_c is the Kronecker delta. It may be used to raise and lower indices so that for a vector V^a we have $g_{ab}V^a = V_b$, which shows that when acting on a vector it produces a one-form. An affine connection may be defined in terms of the metric tensor by

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}). \tag{1.2.12}$$

This connection is known as the metric connection and the connection will always be defined in this way in GR. It is easy to see that Γ^a_{bc} is symmetric in its lower indices and one can check that $\nabla_c g_{ab} = 0$ follows immediately from this definition.

The isometries of the spacetime are defined in terms of the metric tensor by

$$\mathcal{L}_{\boldsymbol{\xi}} g_{ab} = 0, \tag{1.2.13}$$

where $\boldsymbol{\xi}$ is a Killing vector field of the manifold. This equation demonstrates the fact that the metric is unchanged as we follow the flow of $\boldsymbol{\xi}$ which implies the spacetimes is symmetric along these orbits.

1.2.4 Geodesics on a manifold

Geodesics are a special class of curves whose tangent vectors $\boldsymbol{X}$ satisfy $\nabla_{\boldsymbol{X}} X^a = 0$. Given any two points on a manifold p and q which are sufficiently close together, there is a unique geodesic joining p and q which extremises the invariant spacetime interval between them. Geodesics are of central importance to GR as they are the paths followed by unaccelerated or free-falling particles and radiation in a gravitational field. Of particular interest to us are the null geodesics of the spacetime, the paths along which radiation travels, which form the boundaries of the causal past and future of points in the spacetime. The null geodesics may be parameterised such that the parametric equations are given by

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0, \tag{1.2.14}$$

where the overdot denotes differentiation with respect to an affine parameter, which we always denote by μ. They may be derived from the Euler-Lagrange equation

$$\frac{d}{d\mu}\frac{\partial \mathscr{L}}{\partial \dot{x}^a} - \frac{\partial \mathscr{L}}{\partial x^a} = 0, \tag{1.2.15}$$

where

$$\mathscr{L} = g_{ab}\dot{x}^a\dot{x}^b, \tag{1.2.16}$$

is the Langrangian, and we make use of both (1.2.14) and (1.2.15) in what follows. For the majority of the thesis we use double null coordinates u and v which pick out the outgoing and ingoing null geodesics, respectively.

1.2.5 The Riemann curvature tensor and the Einstein equation

An important measure of the curvature of a manifold is the Riemann tensor, which captures the failure of a vector to return to itself after being parallel-transported along a closed curve. It is a tensor of rank (1,3) with components

$$R^a{}_{bcd} = \partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed}. \tag{1.2.17}$$

For a vector field X^a we have $\nabla_c\nabla_d X^a - \nabla_d\nabla_c X^a = R^a{}_{bcd}X^b$, which can be thought of as the change in X^a having been transported around an infinitesimal closed loop. We have seen that the connection Γ^a_{bc} depends on first derivatives of the metric g_{ab} and so the Riemann tensor is of second order in derivatives of g_{ab}. Contracting the Riemann tensor on its first and third indices we have $R^c{}_{acb} = R_{ab}$, which is defined as the Ricci tensor. The trace $g^{ab}R_{ab} = \mathcal{R}$ is then defined as the Ricci scalar. To write down the Einstein equation, we must define the energy-momentum tensor, which measures the distribution of matter and energy throughout the spacetime. It is a symmetric tensor of rank (0,2), commonly denoted T_{ab}. For a general scalar field ψ it is given by

$$T_{ab} = \nabla_a\psi\nabla_b\psi - \frac{1}{2}g_{ab}\nabla^c\psi\nabla_c\psi - g_{ab}V(\psi), \tag{1.2.18}$$

where $V(\psi)$ is the scalar field potential. This is the form of T_{ab} which we use throughout the thesis. Other forms arise for electromagnetic fields, fluids, vacua etc. The cases $V \equiv 0$ and $V \neq 0$ are referred to as the minimally and non-minimally coupled cases, respectively. We deal with both, however, the vast majority of the work is on the non-minimally coupled case. We are now in a position to state the Einstein equation:

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}\mathcal{R} = \frac{8\pi G}{c^4}T_{ab}, \tag{1.2.19}$$

where G is Newton's gravitational constant, G_{ab} is the Einstein tensor and c is the speed of light. In the most general set up, this is a set of ten independent, non-linear, second order PDEs, the solutions of which give the metric of the spacetime manifold. They may be greatly simplified by assuming certain symmetries in the spacetime geometry and, as we shall see, in the case of self-similar cylindrically symmetric solutions they reduce to ODEs. We also make use of the definition $E_{ab} = G_{ab} - 8\pi G/c^4 T_{ab}$ and use units such that $8\pi G = c = 1$ throughout.

1.3 A review of research on cylindrical symmetry

Cylindrically symmetric spacetimes have been of interest since first studied as sources for gravitational waves (GW) by Beck [3] in 1925 and later by Einstein and Rosen [12] in 1937 who, as stated in [18] "gave the first clear demonstration the theory of general relativity predicts the existence of exact gravitational waves". The existence of GW is a fundamental question in GR as it is one of the principal analogies to the other classical field theories. Although controversial at the time, it is now widely accepted that GW exist and they are expected to be observed experimentally within a decade. Kennefick [24] gives an interesting account of the controversies surrounding GW. During the 1960s, huge advances were made in the understanding of gravitational radiation and gravitational collapse. In the context of cylindrical symmetry, the work of Thorne [40] was important in both areas. He proposed an energy-like measure for cylindrical systems, called 'C-energy'. As stated in [20], "It has several interesting and useful features: It is covariant and is associated with a conserved flux vector; it has the correct Newtonian limit, the mass per specific length of the cylinder; it is propagated by Einstein-Rosen waves." It has proven useful in studying the dynamics of several models, such as cylindrical electromagnetic universes and the collapse of a cylindrical shell of counter-rotating dust particles. However, it was shown in [20] that C-energy may be non-zero in the absence of a gravitational field, which casts doubt on its suitability as an energy measure. Thorne also proved the important result that there cannot exist horizons in the collapse of an infinite cylinder with a vacuum exterior [40]. He also showed that naked singularities are formed in the collapse of a thin cylindrical dust shell. The important work of Shapiro and Teukolsky [36] in non-spherical collapse gave strong numerical evidence that naked singularities can form in the collapse of dust spheroids, of which infinite cylinders are an approximation. Apostolatos and Thorne (AT) [1] used C-energy arguments to demonstrate that an infinitesimal amount of rotation halts the collapse of a cylindrical shell of counter-rotating dust particles and that the shell bounces and oscillates before settling down to a static equilibrium. However, they make some strong assumptions such as the total angular momentum being zero and the existence of fixed times such that the shell is momentarily static and radiation free. Piran [33] showed that, in some numerical examples, realistic pressure can also prevent collapse to a singularity. Echeverria [11] expands on the non-rotating dust case by giving more detail about the nature of the singularity and describing a sharp burst of gravitational waves just before the singularity forms. Letelier and Wang [26] subsequently showed that the collision of ingoing and outgoing cylindrical null fluids, which may model the interaction of gravitational waves with incoming radiation during the cylindrical collapse, results in the formation of a naked singularity. Sheel, Shapiro and Teukolsky [34] used C-energy to prove the stability of relativistic cylindrical polytropes and highlight some of the short-comings in approximating finite, non-spherical collapse using infinite cylindrical models.

Chiba [7] studied the full dust collapse scenario using analytical and numerical methods and found that no gravitational waves are emitted during the free fall time. Stachel [38] examined the behaviour of Einstein-Rosen waves at null infinity in directions orthogonal to the axis of symmetry and found that they are not asymptotically flat. Ashtekar, Bičák and Schmidt generalised this study by examining these waves in all directions and they found that, in generic directions, the curvature 'peels off' much better than in directions orthogonal to the axis of symmetry. Pereira and Wang (PW) [32] later generalised the work of AT to include the emission of gravitational waves during the collapse. They found that, in some special cases, the rotation isn't strong enough to halt the collapse to a singularity. Although there were some errors in their calculations, these were corrected by Gleiser [14] who found that this model does indeed admit singularity solutions. Soon after, detail was added to this study by Gonçalves and Jhingan [17] who claimed to establish that the previous results of PW hold for generic initial data and that there are no trapped surfaces in the spacetime. Nolan [30] followed this up soon after with a proof that these spacetimes admitted global naked singularity solutions. However, Seriu's [35] analysis contradicts the results of PW. His imposition of the weak energy condition outside the collapsing shell causes it to bounce rather than collapse to a singularity in all cases.

Hamity, Cécere and Barraco [19] performed a numerical analysis of this model and found both stable and unstable equilibrium solutions. They also studied a two-shell model. Two recent papers by Gleiser and Ramirez [15],[16] revisit this problem using linearised approximations to the dynamical equations governing the motion of the shell. The work of Nakao, Ida and Kurita [28],[29] claims to have disproven some of the AT results, specifically, that the collapsing shell of counter-rotating dust particles does not always settle to a static equilibrium final state.

Investigation of the collapse of a self-similar massless scalar field in cylindrical symmetry was performed by Wang [42]. He found exact solutions, some of which correspond to collapse to a black hole and others which correspond to a censored point-like singularity. However, it appears that the singularity existed in the initial configuration and so it wasn't a realistic model of collapse. We show below that the massless model is mathematically equivalent to the vacuum case which, given regular initial data, yields global, singularity-free solutions. He performed linear perturbations of the self-similar solutions, and found that some of the black hole solutions are stable under these perturbations. This study is closest to our own and the subject of this thesis is a generalisation of this problem to the massive scalar field case. One cannot generally obtain exact solutions to the field equations in this case, making the analysis much more complicated. Note that we have not carried out any perturbation analysis here.

Nakao and Morisawa [27] used a linear perturbation of the cylindrical dust spacetime to model the high-speed collapse of a perfect fluid. This was later generalised to the collapse of two perfect fluids by Sharif and Ahmad, who used the same high-speed approximation scheme. Nolan and Nolan [31] considered the cylindrical analogue of the Oppenheimer-Snyder model, that is, the matching of an isotropic dust interior to a vacuum exterior and showed that it is impossible to perform such a matching. Harada, Nolan and Nakao [20] gave a complete description of self-similar, cylindrical vacuum spacetimes and it was here that they indentified a problem with the definition and interpretation of C-energy.

1.4 Self-similarity

Self-similarity plays an important role in many theories of classical physics. In GR, Carr's self-similarity hypothesis asserts that under certain physical conditions, solutions naturally evolve to a self-similar form [4], thus self-similar solutions are highly relevant to the study of gravitational collapse. Another motivation for the assumption of self-similarity is that it brings about a significant simplification, reducing the field equations to ODEs, and self-similar cylindrical spacetimes are now well understood (see, for example, [8],[6],[5]). Indeed, many of the spacetimes violating the CCH are self-similar, although some of the more realistic models in which naked singularities arise have been shown to be unstable under perturbations [9],[13]. We note that this work is the cylindrical analogue of the work carried out in [8].

There are two types of self-similarity; continuous or homothetic self-similarity and discrete self-similarity, also known as the first and second kinds, respectively. We only consider self-similarity of the first kind here and we simply refer to it a self-similarity henceforth. Spacetimes which are self-similar admit a homothetic Killing vector field $\boldsymbol{\xi}$ such that

$$\mathcal{L}_{\boldsymbol{\xi}} g_{ab} = 2g_{ab}. \tag{1.4.1}$$

This captures the notion of invariance under scale transformations [4]. The coefficient of g_{ab} on the right may be rescaled and is chosen as 2 for convenience.

1.5 Summary of results

This work gives a rigorous, analytical study of the class of self-similar, cylindricallly symmetric spacetimes coupled to a non-linear scalar field, where the Killing vectors are assumed to be hypersurface orthogonal and where the homothetic vector field is assumed to be orthogonal to the cylinders of symmetry. These spacetimes contain a singularity at a point along the axis and so to model a collapse scenario we assume a regular axis to the past of this singularity. The field equations for this class reduce to a system of ODEs for three metric functions and the scalar field. The system has two degrees of freedom and a free initial datum. Solutions for the entire parameter space are examined and, apart from a few critical values of the parameters where exact solutions may be obtained, qualitative solutions are given. The global structure of these solutions is presented in all cases and it is shown that the singularity of the spacetime is censored in all cases, thereby upholding the cosmic censorship hypothesis for these spacetimes.

In Chapter 2, we give a description of the spacetimes in question which leads to the formulation of the initial value problem with the regular axis as the initial data point. In Chapter 3, we prove existence and uniqueness of solutions emanating from the regular axis and Chapter 4 gives all solutions to the causal past of the singularity. These solutions may be divided broadly into two classes; those that terminate on or before the boundary of the causal past of the singularity and those which are regular there. In Chapter 5, the solutions which fall into the latter class are extended into the region beyond this surface, which leads up to the boundary of the causal future of the singularity. They are shown to terminate in a spacelike singularity in all cases. Chapter 6 is devoted to an alternative method for finding these solutions which may be used to obtain more detail about their behaviour in some cases and which is of interest in its own right.

Chapter 2

Self-similar cylindrically symmetric scalar field spacetimes

The purpose of this chapter is to give a description of the self-similar, cylindrically symmetric spacetimes coupled to a scalar field, which ultimately leads to the form of the Einstein equations and initial data which we work with for the remainder of the thesis. We start by giving the line element for spacetimes with whole-cylinder symmetry in a double null coordinate system, that is, a coordinate system where the two null coordinates pick out the paths of ingoing and outgoing radial null rays. We give the energy-momentum tensor for a general scalar field and the corresponding field equations. Before imposing self-similarity, we deal with the special case of non-minimal coupling, i.e. with vanishing potential, in the more general setting. We show that it is, essentially, mathematically equivalent to the vacuum case and that the field equations are practically identical to those studied in [2]. We then specialise to the self-similar setting, showing that the self-similar line element and scalar field essentially depend on a single variable and that the scalar field potential has exponential form. To ensure that the collapse ensues from an initially regular configuration we impose regular axis conditions [22] to the past of the scaling origin $\mathcal{O}$, $(u, v) = (0, 0)$. This is defined as the point at which the homothetic Killing vector field is identically zero. This gives a restricted initial data set from which to evolve the equations. We finish with a proof of the existence of a curvature singularity at $\mathcal{O}$.

2.1 Cylindrically symmetric spacetimes

Cylindrically symmetric spacetimes are invariant under translations along and rotations about an axis of symmetry. They admit a pair of commuting, spacelike Killing vectors, which may be written as

$$\boldsymbol{\xi}_{(\theta)} = \partial_\theta, \qquad \boldsymbol{\xi}_{(z)} = \partial_z, \tag{2.1.1}$$

where the Killing coordinates are θ and z and θ is identified at 0 and 2π. We make the further assumption that these are hypersurface orthogonal, which is known as whole-cylinder symmetry [40]. We choose our other two coordinates u and v such that lines of constant u and v represent the paths of outgoing and ingoing radial null rays, respectively. By outgoing we mean travelling from the axis of symmetry to future null infinity (if it exists) and vice versa. Given these symmetries, the line element may generally be written as

$$ds^2 = -2e^{2\bar{\gamma}+2\bar{\phi}}dudv + e^{2\bar{\phi}}r^2 d\theta^2 + e^{-2\bar{\phi}}dz^2, \tag{2.1.2}$$

where the functions $\bar{\gamma}, \bar{\phi}$ and r depend only on the null coordinates u and v. The function r gives the radius of the cylinders of symmetry. This form of the metric is invariant under the following coordinate transformations.

$$u \to \bar{u}(u), \quad v \to \bar{v}(v), \quad z \to \lambda z, \tag{2.1.3}$$

for constant λ. Note that $\theta \in [0, 2\pi)$ where 0 and 2π are identified and so transformations of the kind $\theta \to \lambda\theta$ are not allowed in general.

2.2 Field equations for a cylindrically symmetric scalar field

As mentioned in Section 1.2 we take our matter source to be a cylindrically symmetric, self-interacting scalar field $\psi(u, v)$ with stress-energy tensor given by (1.2.18). Coupling this to the line element (2.1.2) produces the

Einstein field equations, which reduce to

$$2r_{,u}\bar{\gamma}_{,u} - r_{,uu} - 2r\bar{\phi}_{,u}^{2} = r\psi_{,u}^{2}, \tag{2.2.1a}$$

$$r_{,uv} = re^{2\bar{\gamma}+2\bar{\phi}}V(\psi), \tag{2.2.1b}$$

$$2r_{,v}\bar{\gamma}_{,v} - r_{,vv} - 2r\bar{\phi}_{,v}^{2} = r\psi_{,v}^{2}, \tag{2.2.1c}$$

$$2(\bar{\phi}_{,u}\bar{\phi}_{,v} + \bar{\gamma}_{,uv}) = -\psi_{,u}\psi_{,v} + e^{2\bar{\gamma}+2\bar{\phi}}V(\psi), \tag{2.2.1d}$$

$$2r\bar{\phi}_{,uv} + r_{,u}\bar{\phi}_{,v} + r_{,v}\bar{\phi}_{,u} + r_{,uv} = 0. \tag{2.2.1e}$$

The last equation here corresponds to $E_{zz} - r \cdot E_{\theta\theta} = 0$. Our scalar field ψ satisfies the Klein-Gordon equation

$$\nabla^a\nabla_a\psi - V'(\psi) = 0, \tag{2.2.2}$$

which implies $\nabla^a T_{ab} = 0$. Letting $g = -e^{4\bar{\phi}+4\bar{\gamma}}r^2$ denote the metric determinant we have

$$\begin{aligned}\nabla_a\nabla^a\psi &= \frac{1}{(-g)^{1/2}}\partial_a\left[(-g)^{1/2}g^{ab}\partial_b\psi\right]\\ &= \frac{1}{(-g)^{1/2}}\left[\partial_u((-g)^{1/2}g^{uv}\partial_v\psi) + \partial_v((-g)^{1/2}g^{vu}\partial_u\psi)\right]\\ &= -\frac{1}{e^{2\bar{\gamma}+2\bar{\phi}}r}\left[\partial_u(r\psi_{,v}) + \partial_v(r\psi_{,u})\right].\end{aligned} \tag{2.2.3}$$

Combining this with (2.2.2) and simplifying, we arrive at

$$2r\psi_{,uv} + r_{,v}\psi_{,u} + r_{,u}\psi_{,v} + re^{2\bar{\gamma}+2\bar{\phi}}V'(\psi) = 0, \tag{2.2.4}$$

which is the wave equation for the scalar field ψ. Note that (2.2.4) can be derived from (2.2.1a)-(2.2.1e) and that from this point onwards we make use of (2.2.1a)-(2.2.1c),(2.2.4) and (2.2.1e) in our analysis.

2.3 The minimally coupled case

Before imposing self-similarity we deal with the case where the scalar field potential V is equal to zero, i.e. the minimally coupled case, which has already been studied in depth in both the general and self-similar cases.

With $V = 0$, equation (2.2.1b) gives

$$r_{,uv} = 0, \qquad r = f(u) + g(v). \tag{2.3.1}$$

We require the absence of trapped cylinders in the initial configuration so the gradient of r must be spacelike [41]. This reduces to the condition

$$f'(u)g'(v) < 0. \tag{2.3.2}$$

Using the coordinate freedom (2.1.3), we then set

$$r = \frac{v - u}{\sqrt{2}}. \tag{2.3.3}$$

To demonstrate equivalence to the vacuum case, we follow the example of [20] and introduce time and radial coordinates

$$T = \frac{v + u}{\sqrt{2}}, \qquad X = \frac{v - u}{\sqrt{2}}. \tag{2.3.4}$$

The line element is given by

$$ds^2 = e^{2\bar{\gamma}+2\bar{\phi}}(dX^2 - dT^2) + X^2 e^{2\bar{\phi}} d\theta^2 + e^{-2\bar{\phi}} dz^2, \tag{2.3.5}$$

the remaining field equations then reduce to

$$\bar{\gamma}_{,X} = X\left(\bar{\phi}_{,T}^2 + \bar{\phi}_{,X}^2 + \frac{\psi_{,T}^2}{2} + \frac{\psi_{,X}^2}{2}\right), \tag{2.3.6a}$$

$$\bar{\gamma}_{,T} = X(2\bar{\phi}_{,T}\,\bar{\phi}_{,X} + \psi_{,X}\,\psi_{,T}), \tag{2.3.6b}$$

$$\psi_{,TT} - \psi_{,XX} - \frac{\psi_{,X}}{X} = 0, \tag{2.3.6c}$$

$$\bar{\phi}_{,TT} - \bar{\phi}_{,XX} - \frac{\bar{\phi}_{,X}}{X} = 0. \tag{2.3.6d}$$

Given regular initial data, the linear wave equations for ψ and $\bar{\phi}$ yield unique, globally hyperbolic, singularity free spacetimes. Solutions of $\bar{\gamma}$ may be then be obtained from (2.3.6a) and (2.3.6b). We note that this is, essentially, mathematically equivalent to the vacuum case, which results from setting $\psi = 0$, and refer the reader to [2] and [20] (for the self-similar case) for a full treatment of the problem.

2.4 The self-similar field equations

We assume self-similarity of the first kind which is equivalent to the existence of a homothetic Killing vector field $\boldsymbol{\xi}$ satisfying (1.4.1). We also assume that $\boldsymbol{\xi}$ has the form

$$\boldsymbol{\xi} = \alpha(u,v)\frac{\partial}{\partial u} + \beta(u,v)\frac{\partial}{\partial v}, \tag{2.4.1}$$

so that $\boldsymbol{\xi}$ is orthogonal to the the cylinders of symmetry. Thus $\boldsymbol{\xi}$ is cylindrically symmetric. Equation (1.4.1) is equivalent to

$$\nabla_a \xi_b + \nabla_b \xi_a = 2g_{ab}. \tag{2.4.2}$$

The equations given by $(a,b) = (0,0)$ and $(a,b) = (1,1)$ simplify to

$$\beta_{,u} = 0, \qquad \alpha_{,v} = 0, \tag{2.4.3}$$

so we have $\alpha = \alpha(u)$ and $\beta = \beta(v)$. Given the coordinate freedom (2.1.3) we may choose u and v such that $\alpha = 2u$ and $\beta = 2v$. The remaining equations then simplify to

$$u\gamma_{,u} + v\gamma_{,v} = 0, \tag{2.4.4a}$$

$$vr_{,v} + ur_{,u} = r, \tag{2.4.4b}$$

$$2v\phi_{,v} + 2u\phi_{,u} = -1. \tag{2.4.4c}$$

We define

$$\eta = \frac{v}{u}, \tag{2.4.5}$$

which we label the similarity variable. Using the method of characteristics we find that (2.4.4a)-(2.4.4c) have solutions

$$\bar{\gamma} = \gamma(\eta), \qquad \bar{\phi} = \phi(\eta) - \log|u|^{1/2}, \qquad r = |u|S(\eta). \tag{2.4.6}$$

These solutions give rise to the self-similar cylindrical metric:

$$ds^2 = -2|u|^{-1}e^{2\gamma(\eta)+2\phi(\eta)}dudv + |u|e^{2\phi(\eta)}S^2(\eta)d\theta + |u|e^{-2\phi(\eta)}dz^2. \tag{2.4.7}$$

The coordinate transformations that preserve this form of the metric are

$$u \to \lambda u, \quad v \to \mu v, \quad z \to \sigma z, \tag{2.4.8}$$

for constants λ, μ, σ. We now show that, in self-similar spacetimes, the scalar field reduces to a function of essentially one variable and that the potential has exponential form. Note that a proof of the following result appears in [43] and is quoted in [25]. Note also that it doesn't depend on the form of the homothetic Killing vector $\boldsymbol{\xi}$ and holds under either of the separate assumptions $\mathcal{L}_{\boldsymbol{\xi}} g_{ab} = 2g_{ab}$ and $\mathcal{L}_{\boldsymbol{\xi}} T_{ab} = 0$. Note that we use the notation $\nabla_a \psi = \psi_a$ below.

Proposition 2.4.1. *For a self-similar scalar field ψ with energy-momentum tensor (1.2.18) and $V(\psi) \neq 0$, admitting a homothetic Killing vector $\boldsymbol{\xi}$ such that (1.4.1) holds, the potential $V(\psi)$ has the exponential form*

$$V(\psi) = \bar{V}_0 e^{-2\psi/k}, \tag{2.4.9}$$

where $\bar{V}_0 \neq 0, k \neq 0$ are constants.

Proof. It can be shown that (1.4.1) leads to $\mathcal{L}_{\boldsymbol{\xi}} T_{ab} = 0$ via the Einstein equations, which is known as a matter collineation [4]. For T_{ab} given by (1.2.18) we have

$$\psi_a \mathcal{L}_{\boldsymbol{\xi}} \psi_b + \psi_b \mathcal{L}_{\boldsymbol{\xi}} \psi_a - g_{ab} \Big(\psi^c \psi_c + \frac{1}{2} \psi^c \mathcal{L}_{\boldsymbol{\xi}} \psi_c + \frac{1}{2} \psi_c \mathcal{L}_{\boldsymbol{\xi}} \psi^c + 2V + V'(\psi) \mathcal{L}_{\boldsymbol{\xi}} \psi \Big) = 0. \tag{2.4.10}$$

Now $\mathcal{L}_{\boldsymbol{\xi}} \psi_c = \mathcal{L}_{\boldsymbol{\xi}} g_{bc} \psi^b = 2\psi_c + g_{bc} \mathcal{L}_{\boldsymbol{\xi}} \psi^b$, and so

$$\psi^c \mathcal{L}_{\boldsymbol{\xi}} \psi_c = 2\psi^c \psi_c + \psi_c \mathcal{L}_{\boldsymbol{\xi}} \psi^c. \tag{2.4.11}$$

Combining this with (2.4.10) and taking the trace then yields

$$-\psi^c \mathcal{L}_{\boldsymbol{\xi}} \psi_c = 4V + 2V'(\psi) \mathcal{L}_{\boldsymbol{\xi}} \psi. \tag{2.4.12}$$

Using (2.4.12) to eliminate $2V + V'(\psi)\mathcal{L}_{\boldsymbol{\xi}} \psi$ from (2.4.10) and simplifying produces

$$\psi_a \mathcal{L}_{\boldsymbol{\xi}} \psi_b + \psi_b \mathcal{L}_{\boldsymbol{\xi}} \psi_a - \frac{1}{2} g_{ab} \psi^c \mathcal{L}_{\boldsymbol{\xi}} \psi_c = 0. \tag{2.4.13}$$

Contracting with ψ^a gives

$$\psi^c \psi_c \mathcal{L}_{\boldsymbol{\xi}} \psi_b + \frac{1}{2} \psi_b \psi^c \mathcal{L}_{\boldsymbol{\xi}} \psi_c = 0, \tag{2.4.14}$$

and contracting with ψ^b gives

$$\frac{3}{2}\psi^b\psi_b(\psi^c\mathcal{L}_{\boldsymbol{\xi}}\psi_c) = 0. \tag{2.4.15}$$

In the case $\psi^c\psi_c = 0$ we have $\psi^c\mathcal{L}_{\boldsymbol{\xi}}\psi_c = 0$, from (2.4.14), since $\psi^c\mathcal{L}_{\boldsymbol{\xi}}\psi_c = 0$ is given by $\psi_b = 0$. It then follows from (2.4.11) that $\psi_c\mathcal{L}_{\boldsymbol{\xi}}\psi^c = 0$. Contracting (2.4.13) with $\mathcal{L}_{\boldsymbol{\xi}}\psi^b$ produces

$$\psi_a\mathcal{L}_{\boldsymbol{\xi}}\psi^b\mathcal{L}_{\boldsymbol{\xi}}\psi_b + \psi_b\mathcal{L}_{\boldsymbol{\xi}}\psi^b\mathcal{L}_{\boldsymbol{\xi}}\psi_a = \psi_a\mathcal{L}_{\boldsymbol{\xi}}\psi^b\mathcal{L}_{\boldsymbol{\xi}}\psi_b = 0, \tag{2.4.16}$$

and we see that $\mathcal{L}_{\boldsymbol{\xi}}\psi_b$ is null, since it is automatically null if $\psi_a = 0$. Given that it is also orthogonal to ψ_b, it must be parallel to it, i.e. $\mathcal{L}_{\boldsymbol{\xi}}\psi_b = \Lambda\psi_b$ for some quantity Λ. Putting this into (2.4.13) gives $2\Lambda\psi_a\psi_b = 0$, which reveals that Λ must be zero, i.e. $\mathcal{L}_{\boldsymbol{\xi}}\psi_b = 0$.
In the case $\psi^c\psi_c \neq 0$, we also have $\psi^c\mathcal{L}_{\boldsymbol{\xi}}\psi_c = 0$, by (2.4.15). It follows immediately from (2.4.14) that $\mathcal{L}_{\boldsymbol{\xi}}\psi_b = 0$ in this case also. It is straightforward to show that $\partial_b\mathcal{L}_{\boldsymbol{\xi}}\psi = \mathcal{L}_{\boldsymbol{\xi}}\psi_b$, so we have $\partial_b\mathcal{L}_{\boldsymbol{\xi}}\psi = 0$, and thus $\mathcal{L}_{\boldsymbol{\xi}}\psi = k$, for some constant k. Equation (2.4.12) then simplifies to $2V + kV' = 0$, which yields (2.4.9) for $k \neq 0$. Note that $k = 0$ gives $V = 0$, which is dealt with in section 2.3. □

Corollary 2.4.1. *If $\boldsymbol{\xi}$ has the form (2.4.1) with $\alpha = 2u$ and $\beta = 2v$, then ψ and $V(\psi)$ may be written as*

$$\psi = F(\eta) + \frac{k}{2}\log|u|, \qquad V(\psi) = \frac{\bar{V}_0 e^{-\frac{2}{k}F(\eta)}}{|u|}. \tag{2.4.17}$$

Proof. In this case, $\mathcal{L}_{\boldsymbol{\xi}}\psi = k$ reduces to

$$\mathcal{L}_{\boldsymbol{\xi}}\psi = 2u\psi,_u + 2v\psi,_v = k, \tag{2.4.18}$$

from which $\psi = F(\eta) + \log|u|^{k/2}$ follows. Proposition 2.4.1 then gives the potential V. □

We are now in a position to formulate the field equations as a set of ODEs. In terms of γ, ϕ, S and F, (2.2.1a)-(2.2.1c),(2.2.4) and (2.2.1e) are given by

$$2\eta\gamma'(S-\eta S') + \eta^2 S'' + 2S\left(\eta\phi' + \frac{1}{2}\right)^2 = -S\left(\eta F' - \frac{k}{2}\right)^2, \quad (2.4.19a)$$

$$\eta S'' = -\bar{V}_0 S e^{2\gamma+2\phi-2F/k}, \quad (2.4.19b)$$

$$2S'\gamma' - S'' - 2S\phi'^2 = SF'^2, \quad (2.4.19c)$$

$$2\eta S'' + 4\eta S\phi'' + 4\eta S'\phi' + 2S\phi' + S' = 0, \quad (2.4.19d)$$

$$2\eta SF'' + 2\eta S'F' + SF' - \frac{kS'}{2} + \frac{2\bar{V}_0}{k} S e^{2\gamma+2\phi-2F/k} = 0. \quad (2.4.19e)$$

Now, (2.4.19a)+η^2 (2.4.19c) simplifies to

$$\frac{1}{2} + 2\eta\gamma' + 2\eta\phi' = k\eta F' - \frac{k^2}{4}. \quad (2.4.20)$$

Dividing by η and integrating gives

$$2\gamma + 2\phi = kF - \left(\frac{1}{2} + \frac{k^2}{4}\right)\log|\eta| + c_1, \quad (2.4.21)$$

for some constant c_1. Equation (2.4.19b) then reduces to

$$\eta S'' = V_0 e^{(k-2/k)F}|\eta|^{-(1/2+k^2/4)}S, \quad (2.4.22)$$

where $V_0 = \bar{V}_0 e^{c_1}$ and we have used (2.4.21) to replace $e^{2\gamma+2\phi}$. We define

$$l = \frac{2F}{k} - \log|\eta|^{1/2}, \qquad \lambda = \frac{k^2}{2} - 1, \quad (2.4.23)$$

which gives

$$\eta S'' = -V_0|\eta|^{-1}e^{\lambda l}S. \quad (2.4.24)$$

Equation (2.4.19d) is exact and may be integrated to give

$$2S\phi' + S' = c_2|\eta|^{-1/2}, \quad (2.4.25)$$

for some constant c_2. Written in terms of l and S, (2.4.19e) becomes

$$\eta S l'' + \eta S' l' + \frac{Sl'}{2} - \frac{S}{4\eta} + \frac{2V_0}{k^2|\eta|} S e^{\lambda l} = 0. \quad (2.4.26)$$

2.5 The regular axis conditions

To ensure that the collapse ensues from an initially regular configuration we impose regular axis conditions [22] to the past of the scaling origin $(u, v) = (0, 0)$. The circumferential radius ρ and the specific length L of the cylinders are given by the norms of the Killing vectors

$$\rho = \sqrt{\xi^a_{(\theta)} \cdot \xi_{(\theta)a}} = |u|^{\frac{1}{2}} e^{\phi} S, \qquad L = \sqrt{\xi^a_{(z)} \cdot \xi_{(z)a}} = |u|^{\frac{1}{2}} e^{-\phi}. \tag{2.5.1}$$

The axis is defined by $\rho = 0$. We rule out the case $u = 0$ as this is a null hypersurface and we require the axis to be timelike. For a regular axis, the specific length L must be non-zero and finite, and so ϕ must also be finite. A regular axis must therefore correspond to $S(\eta) = 0$. Hence, η must be constant along the axis and a rescaling of u and v using the coordinate freedom (2.4.8) places the axis at $\eta = 1$.
Note that the past null cone of the origin $\mathcal{N}_-$ corresponds to $\eta = 0$ and the interval $\eta \in (0, 1]$ constitutes region **I**. Note, however, that all of region **I** may not exists as part of the spacetime, as shown in Chapter 4. Further conditions for a regular axis are as follows [22]:

$$\nabla^a \rho \nabla_a \rho = 1 + O(\rho^2), \quad \nabla^a \rho \nabla_a L = O(\rho), \quad \nabla^a L \nabla_a L = O(1), \tag{2.5.2}$$

where the big-oh relations hold in the limit $\rho \to 0$. The first condition ensures the standard 2π-periodicity of the azimuthal coordinate near the axis, while the remaining conditions ensure the absence of any curvature singularities at the axis.

Proposition 2.5.1. *The regular axis conditions reduce to the following set of data:*

$$S(1) = 0, \quad S'(1) = -1, \quad \phi'(1) = -\frac{1}{4}, \quad \gamma'(1) = 0, \quad l'(1) = 0. \tag{2.5.3}$$

This sets $c_2 = -1$ *in (2.4.25).*

Proof. Note that $u < 0$ to the past of the origin (0,0) and, therefore, $u < 0$

on the axis. The equations (2.5.2) then give

$$\lim_{\eta\to 1} 2e^{-2\gamma}(S' + S\phi')^2 = 1, \tag{2.5.4a}$$

$$\lim_{\eta\to 1} e^{-2\gamma}\left(S' + S\phi'\right)\left(\frac{1}{2} + 2\phi'\right) = 0, \tag{2.5.4b}$$

$$\lim_{\eta\to 1} 2e^{-2\gamma}\left(\frac{1}{2} + \phi'\right)\phi' = L_0, \tag{2.5.4c}$$

for some $L_0 \in \mathbb{R}$. Equation (2.5.4a) gives

$$\lim_{\eta\to 1} e^{-\gamma}(S' + S\phi') = \pm\frac{1}{\sqrt{2}}, \tag{2.5.4d}$$

which may be used to simplify (2.5.4b) to

$$\lim_{\eta\to 1} \frac{e^{-\gamma}}{\sqrt{2}}\left(\frac{1}{2} + 2\phi'\right) = 0. \tag{2.5.4e}$$

So we either have $\lim_{\eta\to 1}\phi' = -1/4$ or $\lim_{\eta\to 1} e^{-\gamma} = 0$. In the latter case, it follows from (2.5.4e) that $\lim_{\eta\to 1} e^{-\gamma}\phi' = 0$ also. Now (2.4.25) may be used to replace $S' + S\phi'$ with $c_2 - S\phi'$ in (2.5.4d), and so

$$\lim_{\eta\to 1} e^{-\gamma}(c_2 - S\phi') = \pm\frac{1}{\sqrt{2}}. \tag{2.5.4f}$$

This is inconsistent with $\lim_{\eta\to 1} e^{-\gamma} = \lim_{\eta\to 1} e^{-\gamma}\phi' = 0$ since $\lim_{\eta\to 1} S = 0$, and so we must have $\phi'(1) = -1/4, S'(1) = c_2$ and $e^{-\gamma(1)} = \pm\sqrt{2}c_2 \neq 0$.
The areal radius ρ, and therefore S, must increase away from the axis. Recall that $\eta \in [0,1]$ in region **I**, so that η is decreasing away from the axis at $\eta = 1$. We must then have $S'(1) < 0$. Note that the field equations (2.4.19a)-(2.4.19e) are invariant under the transformation $\bar{S} \to S/(-S'(1))$, so we may set $S'(1) = -1$.
It follows from finiteness of $\gamma(1), \phi(1)$ and equations (2.4.21),(2.4.23) that $F(1)$ and $l(1)$ must also be finite. Equation (2.4.24) then gives $S''(1) = 0$ and, using this fact, (2.4.19c) gives $\gamma'(1) = 0$. Inserting $\phi'(1) = -1/4$ and $\gamma'(1) = 0$ into (2.4.20), we arrive at $F'(1) = k/4$, which is equivalent to $l'(1) = 0$. □

The following result deals with the subcase $\psi^c\psi_c = 0$ which is one of two which arise from (2.4.15).

Proposition 2.5.2. *In the case $\psi^c\psi_c = 0$, solutions to the Einstein equation with line element and energy-momentum tensor given by (2.4.7) and (1.2.18), respectively, admit a regular axis if and only if $k = 0$.*

Proof. First note that $\psi^c\psi_c = 2g^{01}\psi_{,u}\,\psi_{,v} = 0$ leads to either $\psi_{,u} = 0$ or $\psi_{,v} = 0$. Equation (2.4.17) then gives

$$\psi_{,u} = -\frac{vF'}{u^2} + \frac{k}{2u} = 0, \qquad \psi_{,v} = \frac{F'}{u} = 0, \tag{2.5.5a}$$

$$\Rightarrow F' = \frac{k}{2\eta}, \qquad \Rightarrow F' = 0. \tag{2.5.5b}$$

In both cases, $F'(1) = k/4$, which was established in the previous proof, holds if and only if $k = 0$, i.e. if and only if $V \equiv 0$. □

2.6 Singular nature of the scaling origin

As an immediate consequence of the assumption of self-similarity, in the non-minimally coupled case, there exists a spacetime singularity where the homothetic Killing vector is identically zero, i.e., at $(u, v) = (0, 0)$.

Proposition 2.6.1. *Let $\mathcal{T}$ denote the scalar invariant $T^{ab}T_{ab}$. Then*

$$\lim_{u\to 0}\left(\mathcal{T}|_{\eta=1}\right) = \infty. \tag{2.6.1}$$

Proof. It is straightforward to show that

$$\begin{aligned}\mathcal{T} &= 2(g^{uv})^2\left(T_{uu}T_{vv} + T_{uv}^2\right) + \left(g^{\theta\theta}T_{\theta\theta}\right)^2 + (g^{zz}T_{zz})^2 \\ &\geq 2(g^{uv})^2 T_{uu}T_{vv}.\end{aligned} \tag{2.6.2}$$

Now,

$$(g^{uv})^2 T_{uu}T_{vv} = \frac{e^{-4\gamma-4\phi}}{u^2}\psi_{,u}^2\,\psi_{,v}^2 = e^{-4\gamma-4\phi}\left(\eta F' - \frac{k}{2}\right)^2 \frac{F'^2}{u^2}. \tag{2.6.3}$$

Using $F'(1) = k/4$ we have

$$\mathcal{T}(1) \geq 2e^{-4\gamma(1)-4\phi(1)}\left(\frac{k^2}{16u}\right)^2. \tag{2.6.4}$$

Since $k \neq 0$, taking the limit $u \to 0$ completes the proof. □

Chapter 3

Solutions emanating from the regular axis

This main purpose of this chapter is to prove existence and uniqueness of solutions to the self-similar field equations (2.4.19a)-(2.4.19e) in a neighbourhood of the regular axis. Note that (2.4.19e) is singular at the axis $S = 0$ and so this is a non-trivial problem. We also cast the field equations as an autonomous set using a rescaling of η which is valid throughout region **I**.

3.1 The field equations in region I

Here we derive the form of the Einstein equations with which we will work for the remainder of the chapter. The following choice of dependent and independent variables gives an autonomous system.

Proposition 3.1.1 (Autonomous field equations)**.** *Let*

$$\tau = -\log \eta, \qquad R = e^{\tau/2} S. \tag{3.1.1}$$

*Then the interval $\eta \in [1, 0)$, i.e. region **I** of the spacetime, corresponds to*

$\tau \in [0,\infty)$ and the field equations are equivalent to

$$2\gamma + 2\phi = \frac{k^2 l}{2} + \frac{\tau}{2} + c_1, \tag{3.1.2a}$$

$$\ddot{R} = \left(\frac{1}{4} - V_0 e^{\lambda l}\right) R, \tag{3.1.2b}$$

$$\dot{R} + \left(2\dot{\phi} - \frac{1}{2}\right) R = 1, \tag{3.1.2c}$$

$$R\ddot{l} + \dot{R}\dot{l} = \left(\frac{1}{4} - \frac{2}{k^2} V_0 e^{\lambda l}\right) R, \tag{3.1.2d}$$

$$\frac{\dot{R}^2 - 1}{R^2} + \frac{k^2 \dot{R}\dot{l}}{R} + 2V_0 e^{\lambda l} - \frac{2+k^2}{8} - \frac{k^2 \dot{l}^2}{2} = 0, \tag{3.1.2e}$$

$$R(0) = 0, \quad \dot{R}(0) = 1, \quad l(0) = l_0, \quad \dot{l}(0) = 0, \tag{3.1.2f}$$

where the overdot denotes a derivative with respect to τ. The regular axis conditions correspond to the initial conditions (3.1.2f).

Proof. For a general function f we have $\eta f' = -\dot{f}$ and $\eta^2 f'' = \ddot{f} + \dot{f}$. Then for $\eta > 0$, multiplying equations (2.4.24),(2.4.25),(2.4.26) by η and changing variables gives

$$\ddot{S} + \dot{S} = -V_0 e^{\lambda l} S, \tag{3.1.3a}$$

$$2S\dot{\phi} + \dot{S} = -c_2 e^{-\tau/2} = e^{-\tau/2}, \tag{3.1.3b}$$

$$S\ddot{l} + \dot{S}\dot{l} + \frac{S\dot{l}}{2} = S\left(\frac{1}{4} - \frac{2V_0 e^{\lambda l}}{k^2}\right). \tag{3.1.3c}$$

Writing these in terms of R gives (3.1.2b)-(3.1.2d). Equation (3.1.2a) comes directly from (2.4.21),(2.4.23) and the definition of τ. Multiplying (2.4.19c) by η^2 and changing variables yields

$$2\left(\dot{R} - \frac{R}{2}\right)\dot{\gamma} + V_0 e^{\lambda l} R - 2R\dot{\phi}^2 = \frac{k^2 R}{4}\left(\dot{l} - \frac{1}{2}\right)^2, \tag{3.1.4}$$

where we have used (2.4.23) and (2.4.24) to replace $\eta F'$ and $\eta^2 S''$, respectively. Using (3.1.2c) and the derivative of (3.1.2a) to replace $\dot{\gamma}$ and $\dot{\phi}$ with expressions in R and l then produces

$$\left(\dot{R}-\frac{R}{2}\right)\left(\frac{k^2\dot{l}}{2}-\frac{1}{R}+\frac{\dot{R}}{R}\right)+V_0e^{\lambda l}R$$
$$-\frac{R}{2}\left(\frac{1}{R}-\frac{\dot{R}}{R}+\frac{1}{2}\right)^2=\frac{k^2R}{4}\left(\dot{l}-\frac{1}{2}\right)^2. \qquad (3.1.5)$$

Multiplying by 2/R and simplifying, we arrive at (3.1.2e). Finally, the axis is at $\tau=0$, so $R(0)=0, l(0)=l_0$. Furthermore, $\dot{l}(\tau=0)=-l'(\eta=1)=0$ and $\dot{S}(\tau=0)=-S'(\eta=1)=\dot{R}(0)-R(0)/2$ which gives $\dot{R}(0)=1$. □

Equations (3.1.2) are equivalent to the Einstein field equations with line element (2.4.7), energy-momentum tensor (1.2.18) and a regular axis. Henceforth, we study equations (3.1.2b),(3.1.2d) and (3.1.2e) to determine solutions for R and l. Solutions for ϕ are then obtained by integrating (3.1.2c) and, once this is found, γ is given by (3.1.2a). Note that $\tau\in[0,\infty)$ in region **I** and $\tau\to\infty$ at $\mathcal{N}_-$, if the solution extends this far.

3.2 Existence and uniqueness of solutions with a regular axis

In this section we present a series of results which culminate in the proof of existence and uniqueness of solutions emanating from the regular axis. The axis is a singular point of (3.1.2d) and so existence and uniqueness of a solution is not guaranteed. However, using a fixed point argument, we can prove that for fixed values of k^2, V_0 and l_0, a unique solution to (3.1.2) exists on an interval $[0,\tau_*]$, for some $\tau_*>0$.
The argument involves and application of the following theorem which may found in [10].

Theorem 3.2.1 (Banach's Fixed Point Theorem)**.** *Let χ be a closed subset of a Banach space E and let T be a contraction mapping from χ into χ. Then there exists a unique x in χ such that $T(x)=x$.*

□

Note that on the axis, R is zero to first order only, that is, $\dot{R}(0)=dR/d\tau|_{\tau=0}\neq 0$. It is convenient to work with a new variable $x=R/\tau$, which is non-zero on the axis, and to look for solutions which are C^2. We use a first-order reduction and write the system as a set of integral equations according to the following results:

Lemma 3.2.1. *Let $x = R/\tau$. Then initial data for $x, \dot{x}$ corresponding to a regular axis are given by*

$$x(0) = 1, \quad \dot{x}(0) = 0. \tag{3.2.1}$$

Proof. Using Taylor's theorem about $\tau = 0$, with the assumption that $R \in C^2[0, \tau_*)$ for some $\tau_* > 0$, we write R and thus x as

$$R(\tau) = \tau + \ddot{R}(\hat{\tau}(\tau))\frac{\tau^2}{2}, \tag{3.2.2a}$$
$$x(\tau) = 1 + \ddot{R}(\hat{\tau}(\tau))\frac{\tau}{2}, \tag{3.2.2b}$$

for some $\hat{\tau} \in [0, \tau]$ and $0 < \tau < \tau_*$. Evaluating $\dot{x}(0)$ from first principles, we find

$$\dot{x}(0) = \lim_{\tau \to 0} \frac{x(\tau) - x(0)}{\tau} = \lim_{\tau \to 0} \frac{\ddot{R}(\hat{\tau})\tau}{2\tau} = \lim_{\tau \to 0} \frac{\ddot{R}(\hat{\tau}(\tau))}{2}. \tag{3.2.3}$$

Now, $\hat{\tau} \in [0, \tau]$ goes to zero in the limit $\tau \to 0$ and so $\dot{x}(0) = \ddot{R}(0)/2 = 0$, from (3.1.2b). □

Lemma 3.2.2. *Let $x_1 = x, x_2 = \dot{x}, x_3 = l, x_4 = \dot{l}$. Then (3.1.2b),(3.1.2d) and (3.1.2e) are equivalent to the following set of integral equations:*

$$x_1(\tau) = 1 + \int_0^\tau x_2(t)dt, \tag{3.2.4a}$$
$$x_2(\tau) = \int_0^\tau \frac{t^2}{\tau^2} x_1(t)\alpha(x_3(t))dt, \tag{3.2.4b}$$
$$x_3(\tau) = l_0 + \int_0^\tau x_4(t)dt, \tag{3.2.4c}$$
$$x_4(\tau) = \int_0^\tau \frac{t x_1(t)}{\tau x_1(\tau)}\beta(x_3(t))dt, \tag{3.2.4d}$$

where

$$\alpha(x_3(t)) = \frac{1}{4} - V_0 e^{\lambda x_3(t)}, \quad \beta(x_3(t)) = \frac{1}{4} - \frac{2}{k^2} V_0 e^{\lambda x_3(t)}. \tag{3.2.4e}$$

Proof. From (3.1.2b) and $R = \tau x$, we have

$$\tau \ddot{R} = \tau^2 \ddot{x} + 2\tau \dot{x} = \tau^2 x \left(\frac{1}{4} - V_0 e^{\lambda l} \right) \tag{3.2.5}$$

which can be integrated to give

$$\dot{x} = \frac{1}{\tau^2} \int_0^\tau t^2 x(t) \left(\frac{1}{4} - V_0 e^{\lambda l(t)} \right) dt.$$

Equation (3.1.2d) may also be written in the integral form

$$\begin{aligned} \dot{l} &= \frac{1}{R} \int_0^\tau R \left(\frac{1}{4} - \frac{2}{k^2} V_0 e^{\lambda l(t)} \right) dt, \qquad (3.2.6) \\ &= \int_0^\tau \frac{t x_1(t)}{\tau x_1(\tau)} \beta(x_3(t)) dt. \end{aligned}$$

Equations (5.3.10) and (5.3.9) follow immediately from the definitions. □

A solution of (3.2.4) corresponds to a fixed point of the the mapping $T : \boldsymbol{x} \to T(\boldsymbol{x}) = \boldsymbol{y}$ where

$$y_1(\tau) = 1 + \int_0^\tau x_2(t) dt, \tag{3.2.7a}$$

$$y_2(\tau) = \int_0^\tau \frac{t^2}{\tau^2} x_1(t) \alpha(x_3(t)) dt, \tag{3.2.7b}$$

$$y_3(\tau) = l_0 + \int_0^\tau x_4(t) dt, \tag{3.2.7c}$$

$$y_4(\tau) = \int_0^\tau \frac{t x_1(t)}{\tau x_1(\tau)} \beta(x_3(t)) dt. \tag{3.2.7d}$$

We aim to use Banach's fixed point theorem (the contraction mapping principle)[10] to show that T has a unique fixed point. We begin by defining the space χ in which $\boldsymbol{x}$ lies, which we require to be a closed subset of a Banach space. Let $E = C^0([0, \tau_*], \mathbb{R}^4)$, with the norm of a vector $\boldsymbol{x}$ given by

$$||\boldsymbol{x}||_E = \sup_{\tau \in [0, \tau_*]} ||\boldsymbol{x}(\tau)|| = \sup_{\tau \in [0, \tau_*]} \max_{1 \le i \le 4} |x_i(\tau)|. \tag{3.2.8}$$

E is therefore a Banach space [10]. Let

$$\chi(\tau_*, b, B) = \{\boldsymbol{x} \in C^0([0,\tau_*], \mathbb{R}^4) : \boldsymbol{x}(0) = \boldsymbol{x}_0, \sup_{\tau\in[0,\tau_*]} ||\boldsymbol{x} - \boldsymbol{x}_0|| \le B, \inf_{\tau\in[0,\tau_*]} x_1(\tau) \ge b > 0\}, \tag{3.2.9}$$

where

$$\boldsymbol{x}_0 = (1, 0, l_0, 0)^T, \qquad \text{and} \qquad b < 1. \tag{3.2.10}$$

Then χ is a closed subset of E, and is therefore also a Banach space. We wish to show that it is possible to choose τ_*, b and B such that T is a contraction mapping on χ, i.e. T maps χ into itself and that there is a number $0 < \kappa < 1$ such that for any vectors $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)} \in \chi$,

$$||\boldsymbol{y}^{(1)} - \boldsymbol{y}^{(2)}|| \le \kappa ||\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(2)}||. \tag{3.2.11}$$

The following four results verify that $T\boldsymbol{x} = \boldsymbol{y} \in \chi$.

Lemma 3.2.3. *The image $T\boldsymbol{x} = \boldsymbol{y}$ of $\boldsymbol{x} \in \chi$, has the same initial data as $\boldsymbol{x}$. That is, $\boldsymbol{y}(0) = \boldsymbol{x}_0$.*

Proof. It is straightforward to show that the integrals in (3.2.7) equal zero at $\tau = 0$ by using the weighted mean value theorem for integrals. □

Lemma 3.2.4. *Let*

$$M_1 = \max\left\{B, (B+1)\left(\frac{1}{4} + \delta\right), \frac{(B+1)}{b}\left(\frac{1}{4} + \frac{2\delta}{k^2}\right)\right\}, \tag{3.2.12}$$

where $\delta = |V_0|e^{|\lambda|(B+|l_0|)}$. If $\tau_* \le B/M_1$, then for $\boldsymbol{x} \in \chi(\tau_*, b, B)$

$$\sup_{\tau\in[0,\tau_*]} ||\boldsymbol{x} - \boldsymbol{x}_0|| \le B, \quad \Rightarrow \quad \sup_{\tau\in[0,\tau_*]} ||\boldsymbol{y} - \boldsymbol{y}_0|| \le B. \tag{3.2.13}$$

Proof. We first note the following inequalities, which hold on $[0, \tau_*]$ for $\boldsymbol{x} \in \chi(\tau_*, b, B)$.

$$b \le x_1 \le B + 1, \quad x_2 \le B, \quad x_3 \le B + |l_0|, \quad x_4 \le B, \tag{3.2.14a}$$

$$|\alpha(x_3)| = \left|\frac{1}{4} - V_0 e^{\lambda x_3}\right| \le \frac{1}{4} + \delta, \tag{3.2.14b}$$

$$|\beta(x_3)| = \left|\frac{1}{4} - \frac{2}{k^2} V_0 e^{\lambda x_3}\right| \le \frac{1}{4} + \frac{2\delta}{k^2}, \tag{3.2.14c}$$

Now,

$$\sup_{\tau\in[0,\tau_*]} ||\boldsymbol{y}-\boldsymbol{y}_0|| = \sup_{\tau\in[0,\tau_*]} \max(A), \tag{3.2.15}$$

where

$$\begin{aligned} A = \Bigg\{ &\left|\int_0^\tau x_2(t)dt\right|, \left|\int_0^\tau \frac{t^2}{\tau^2} x_1(t)\alpha(x_3(t))dt\right|, \\ &\left|\int_0^\tau x_4(t)dt\right|, \left|\int_0^\tau \frac{tx_1(t)}{\tau x_1(\tau)}\beta(x_3(t))dt\right| \Bigg\}. \end{aligned} \tag{3.2.16}$$

We derive a bound for each element of A as follows:

$$\left|\int_0^\tau x_2(t)dt\right| \leq \int_0^\tau |x_2(t)|dt \leq \int_0^\tau Bdt = B\tau. \tag{3.2.17}$$

$$\begin{aligned} \left|\int_0^\tau \frac{t^2}{\tau^2} x_1(t)\alpha(x_3(t))dt\right| &\leq \int_0^\tau |x_1(t)\alpha(x_3(t))|dt, \\ &\leq \int_0^\tau (B+1)\left(\frac{1}{4}+\delta\right)dt, \\ &= (B+1)\left(\frac{1}{4}+\delta\right)\tau. \end{aligned} \tag{3.2.18}$$

$$\left|\int_0^\tau x_4(t)dt\right| \leq \int_0^\tau |x_4(t)|dt \leq \int_0^\tau Bdt = B\tau. \tag{3.2.19}$$

$$\begin{aligned} \left|\int_0^\tau \frac{tx_1(t)}{\tau x_1(\tau)}\beta(x_3(t))dt\right| &\leq \int_0^\tau \left|\frac{x_1(t)}{x_1(\tau)}\beta(x_3(t))\right| dt, \\ &\leq \int_0^\tau \frac{(B+1)}{b}\left(\frac{1}{4}+\frac{2\delta}{k^2}\right)dt \\ &= \frac{B+1}{b}\left(\frac{1}{4}+\frac{2\delta}{k^2}\right)\tau. \end{aligned} \tag{3.2.20}$$

We then have $\max(A) \leq M_1\tau$ and so $\sup_{\tau\in[0,\tau_*]}\max(A) \leq M_1\tau_*$. We choose $\tau_* \leq B/M_1$ and (3.2.13) is satisfied. □

Lemma 3.2.5. *If $\tau_* \leq (1-b)/B$, then*

$$\inf_{\tau\in[0,\tau_*]} x_1 \geq b > 0 \quad \Rightarrow \quad \inf_{\tau\in[0,\tau_*]} y_1 \geq b > 0. \tag{3.2.21}$$

Proof. It follows from (3.2.17) that that

$$\inf_{\tau\in[0,\tau_*]}\int_0^{\tau} x_2(t)dt \geq -B\tau_*. \tag{3.2.22}$$

Hence,

$$\inf_{\tau\in[0,\tau_*]} y_1 = 1 + \inf_{\tau\in[0,\tau_*]}\int_0^{\tau} x_2(t)dt \geq 1 - B\tau_*. \tag{3.2.23}$$

We then choose $\tau_* \leq (1-b)/B$ so that (3.2.21) is satisfied. Note that $x_1(0) = 1 > b$ and so the upper bound on τ_* is strictly positive. □

Proposition 3.2.1. *For a given $\boldsymbol{x} \in \chi(\tau_*, B, b)$, with $\tau_* \leq \min\{b/M_1, (1-b)/B\}$, we have $T\boldsymbol{x} = \boldsymbol{y} \in \chi$, where T is defined by (37).*

Proof. The proof follows immediately from the three preceding lemmas. □

Proposition 3.2.2. *Let*

$$\tilde{M} = \max\left\{\frac{1}{4} + \delta, \lambda(B+1)\delta\right\} \tag{3.2.24a}$$

$$\bar{M} = \max\left\{\frac{1}{4} + \frac{2\delta}{k^2}, \frac{2\lambda}{k^2}(B+1)\delta\right\}, \tag{3.2.24b}$$

$$M_2 = \max\left\{1, \tilde{M}, (B+1)\left(\frac{1}{4} + \frac{2\delta}{k^2}\right)\frac{1}{b^2} + \frac{1}{b}\bar{M}\right\}. \tag{3.2.24c}$$

The mapping

$$T : \chi \to \chi; \mapsto T\boldsymbol{x} = \boldsymbol{y}, \tag{3.2.25}$$

with χ defined by (3.2.9), with $\tau_* < \min\{b/M_1, (1-b)/B, 1/M_2\}$, is a contraction mapping, i.e., there exists a number κ satisfying $0 < \kappa < 1$ such that

$$||T\boldsymbol{x}^{(1)} - T\boldsymbol{x}^{(2)}||_{\chi} = ||\boldsymbol{y}^{(1)} - \boldsymbol{y}^{(2)}||_{\chi} \leq \kappa||\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(2)}||_{\chi}, \tag{3.2.26}$$

for any $\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}$ in χ.

Proof. First note that the hypothesis of the previous result is satisfied in this case and so T maps χ into itself. Recall

$$||\boldsymbol{y}^{(1)} - \boldsymbol{y}^{(2)}||_{\chi} = \sup_{\tau\in[0,\tau^*]}\max_{1\leq i\leq 4}|y_i^{(1)} - y_i^{(2)}|. \tag{3.2.27}$$

We show that for each i, $1 \leq i \leq 4$, we have $|y_i^{(1)} - y_i^{(2)}| \leq a\sigma\tau^*$ where a is some constant and $\sigma = ||\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(2)}||_{\mathcal{X}}$. Then by choosing an appropriate value for τ_*, we show that T is a contraction on the interval $[0, \tau_*]$. We have

$$|y_1^{(1)} - y_1^{(2)}| = \left| \int_0^\tau x_2^{(1)}(t) - x_2^{(2)}(t) dt \right| \leq \int_0^\tau \left| x_2^{(1)}(t) - x_2^{(2)}(t) \right| dt, \\ \leq \int_0^\tau \sigma dt = \sigma\tau. \tag{3.2.28}$$

$$\begin{aligned} |y_2^{(1)} - y_2^{(2)}| = & \left| \int_0^\tau (\alpha^{(1)}(t) x_1^{(1)}(t) - \alpha^{(2)}(t) x_1^{(2)}(t)) \frac{t^2}{\tau^2} dt \right|, \\ \leq & \int_0^\tau \left| (\alpha^{(1)}(t) x_1^{(1)}(t) - \alpha^{(2)}(t) x_1^{(2)}(t)) \right| dt, \end{aligned} \tag{3.2.29}$$

where $\alpha^{(j)} = \alpha(x_3^{(j)})$. Let $\boldsymbol{p}^{(i)} = (x_1^{(i)}, x_3^{(i)})^T$, $f(\boldsymbol{p}^{(i)}) = \alpha(x_3^{(i)}) x_1^{(i)}$ for $i = 1, 2$. Then, by the mean value theorem, there exists some point $\hat{\boldsymbol{p}} = (\hat{x}_1, \hat{x}_3)$ on the line segment joining $\boldsymbol{p}^{(1)}$ to $\boldsymbol{p}^{(2)}$ such that

$$\alpha^{(1)} x_1^{(1)} - \alpha^{(2)} x_1^{(2)} = f(\boldsymbol{p}^{(1)}) - f(\boldsymbol{p}^{(2)}) = \vec{\nabla} f(\hat{\boldsymbol{p}}) \cdot (\boldsymbol{p}^{(1)} - \boldsymbol{p}^{(2)}), \tag{3.2.30}$$

where the $\cdot$ denotes the Euclidean inner product. Then, by the Cauchy-Schwarz inequality, we have

$$||\alpha^{(1)} x_1^{(1)} - \alpha^{(2)} x_1^{(2)}|| \leq \left(||\vec{\nabla} f(\hat{\boldsymbol{p}})|| \right) ||\boldsymbol{p}^{(1)} - \boldsymbol{p}^{(2)}||, \tag{3.2.31}$$

where the norm here is Euclidean. Note that f is differentiable everywhere and thus satisfies the hypotheses of the mean value theorem. We have

$$\vec{\nabla} f(\hat{\boldsymbol{p}}) = \left(\frac{1}{4} - V_0 e^{\lambda \hat{x}_3}, -\lambda V_0 e^{\lambda \hat{x}_3} \hat{x}_1 \right)^T. \tag{3.2.32}$$

Using the inequalities $\hat{x}_1 \leq B + 1$, $\hat{x}_3 \leq B + |l_0|$, we find

$$||\vec{\nabla} f(\hat{\boldsymbol{p}})|| \leq \sqrt{\left(\frac{1}{4} + \delta \right)^2 + (\lambda(B+1)\delta)^2} = \tilde{M}. \tag{3.2.33}$$

Using (3.2.31) and (3.2.33) we find $|\alpha^{(1)} x_1^{(1)} - \alpha^{(2)} x_1^{(2)}|| \leq \tilde{M} ||\boldsymbol{p}^{(1)} - \boldsymbol{p}^{(2)}||$.

It is easy to show that $||\boldsymbol{p}^{(1)} - \boldsymbol{p}^{(2)}|| \leq \sqrt{2}\sigma = \sqrt{2}||\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(2)}||_{\chi}$, and so

$$\begin{aligned}\int_0^\tau \left|y_2^{(1)} - y_2^{(2)}\right| dt &= \int_0^\tau \left|(\alpha^{(1)}(t)x_1^{(1)}(t) - \alpha^{(2)}(t)x_1^{(2)}(t))\right| dt \\ &\leq \int_0^\tau \sqrt{2}\sigma\tilde{M}dt = \sqrt{2}\sigma\tilde{M}\tau. \qquad (3.2.34)\end{aligned}$$

Similarly, we have

$$\begin{aligned}\left|y_3^{(1)} - y_3^{(2)}\right| = \left|\int_0^\tau x_4^{(1)}(t) - x_4^{(2)}(t)dt\right| &\leq \int_0^\tau \left|x_4^{(1)}(t) - x_4^{(2)}(t)\right| dt \qquad (3.2.35) \\ &\leq \int_0^\tau \sigma dt = \sigma\tau.\end{aligned}$$

Finally,

$$\begin{aligned}\left|y_4^{(1)} - y_4^{(2)}\right| &= \left|\int_0^\tau \left(\frac{\beta^{(1)}x_1^{(1)}(t)}{x_1^{(1)}(\tau)} - \frac{\beta^{(2)}x_1^{(2)}(t)}{x_1^{(2)}(\tau)}\right)\frac{t}{\tau}dt\right| \qquad (3.2.36) \\ &\leq \int_0^\tau \left|\frac{\beta^{(1)}x_1^{(1)}(t)}{x_1^{(1)}(\tau)} - \frac{\beta^{(2)}x_1^{(2)}(t)}{x_1^{(2)}(\tau)}\right| dt \\ &= \int_0^\tau \left|\beta^{(1)}x_1^{(1)}(t)\left(\frac{x_1^{(2)}(\tau) - x_1^{(1)}(\tau)}{x_1^{(1)}(\tau)x_1^{(2)}(\tau)}\right) + \frac{\beta^{(1)}x_1^{(1)}(t) - \beta^{(2)}x_1^{(2)}(t)}{x_1^{(2)}(\tau)}\right| dt \\ &\leq \int_0^\tau \left|\beta^{(1)}x_1^{(1)}(t)\right|\frac{\sigma}{b^2}dt + \int_0^\tau \frac{1}{b}\left|\beta^{(1)}x_1^{(1)}(t) - \beta^{(2)}x_1^{(2)}\right| dt = I_1 + I_2,\end{aligned}$$

using $1/b \geq 1/x_1$ and $|x_1^{(2)} - x_1^{(1)}| \leq \sigma$. Using the mean value theorem and the Cauchy-Schwarz inequality again here we find $I_2 \leq \bar{M}\sigma\tau/b$, where

$$\bar{M} = \max\left\{\frac{1}{4} + \frac{2\delta}{k^2}, \frac{2\lambda}{k^2}(B+1)\delta\right\}. \qquad (3.2.37)$$

Using the bounds defined by (3.2.7) we find

$$I_1 \leq (B+1)\left(\frac{1}{4} + \frac{2\delta}{k^2}\right)\frac{\sigma}{b^2}\tau. \qquad (3.2.38)$$

So we have

$$|y_4^{(1)} - y_4^{(2)}| \leq \left[(B+1)\left(\frac{1}{4} + \frac{2\delta}{k^2}\right)\frac{1}{b^2} + \frac{\bar{M}}{b}\right]\sigma\tau \qquad (3.2.39)$$

Gathering these bounds we find that $\sup_{\tau\in[0,\tau_*]}\max_{1\leq i\leq 4}|y_i^1 - y_i^2| \leq M_2\tau_*\sigma$, where

$$M_2 = \max\left\{1, \tilde{M}, (B+1)\left(\frac{1}{4} + \frac{2\delta}{k^2}\right)\frac{1}{b^2} + \frac{\bar{M}}{b}\right\}. \tag{3.2.40}$$

Let $\kappa = M_2\tau_*$. Then if $\tau_* < 1/M_2$, we have $||\boldsymbol{y}^{(1)} - \boldsymbol{y}^{(2)}||_\chi \leq \kappa||\boldsymbol{x}^{(1)} - \boldsymbol{x}^{(2)}||_\chi$ where $0 < \kappa < 1$ and so T is a contraction mapping. □

Proposition 3.2.3. *For τ_* sufficiently small, the mapping T defined above has a unique fixed point on $[0, \tau_*]$.*

Proof. Given any constants B and b, let $m = \min\{B/M_1, (1-b)/B, 1/M_2\}$. For $\tau_* < m$, then, Propositions 3.2.1 and 3.2.2 hold, so T is a contractive mapping from a closed subset χ of a Banach space E, into itself. Using Banach's fixed point theorem completes the proof. [10] □

In light of this result, we know that there is a unique $\boldsymbol{x}$ on $[0, \tau_*]$, such that $T\boldsymbol{x} = \boldsymbol{x}$. Combining this with (3.2.7) shows that there is a unique $\boldsymbol{x}$ such that (3.2.4) holds, hence (3.1.2) has a unique solution on some interval $[0, \tau_*]$.

Proposition 3.2.4. *(3.1.2b) and (3.1.2d) subject to (3.1.2f) have a unique solution on $[0, \tau_*]$, for some $\tau_* > 0$.*

Proof. Proposition 3.2.3 shows that we have a unique solution for l and $x = R\tau$ on $[0, \tau_*]$, so we have a unique solution for R and l. □

Theorem 3.2.2 (Existence and Uniqueness)**.** *Let $k, V_0 \in \mathbb{R}$. For each $l_0, \phi_0 \in \mathbb{R}$ there exists a unique solution of (3.1.2) on an interval $[0, \tau_*]$ corresponding to a spacetime with line element (2.4.7) and energy-momentum tensor (1.2.18). The spacetime admits a regular axis for $u + v < 0$.*

Proof. Using Proposition 3.2.4 we have a unique solution for R and l on some interval $[0, \tau_*]$. Integrating (3.1.2c) we have

$$\phi = \phi_0 + \frac{\tau}{4} + \int_0^\tau \frac{1 - \dot{R}}{R} dt,$$

which gives a unique solution for ϕ. Equation (3.1.2a) then gives a unique solution for γ. □

Chapter 4

From the axis to $\mathcal{N}_-$

In this chapter we determine the global structure of all solutions in the region bounded by the axis and $\mathcal{N}_-$, region **I**. Section 4.1 deals with the most general cases and Section 4.2 deals with a number of subcases where exact solutions may be found, which correspond to specific values of the parameters. We finish the chapter with a theorem which summarises the various global structures that obtain throughout the parameter space in Section 4.3. Of particular interest are the solutions with $k^2 < 2, V_0 < 0$ and $k^2 < 2, V_0 > 0, V_0 e^{\lambda l_0} > k^2/8$, as these have regular behaviour throughout region **I**, including on its future boundary, $\mathcal{N}_-$. These solutions are extended into region **II** in the following chapter.

4.1 Qualitative solutions

We may reduce (3.1.2b),(3.1.2d) and (3.1.2f) to a set of three first order equations by the following change of variables.

$$u_1 = \frac{\dot{R}}{R}, \qquad u_2 = |V_0| e^{\lambda l}, \qquad u_3 = \dot{l}. \tag{4.1.1}$$

They satsify

$$\dot{u}_1 = \frac{1}{4} - \epsilon u_2 - u_1^2, \tag{4.1.2a}$$

$$\dot{u}_2 = \lambda u_2 u_3, \tag{4.1.2b}$$

$$\dot{u}_3 = \frac{1}{4} - \epsilon \frac{2u_2}{k^2} - u_1 u_3, \tag{4.1.2c}$$

$$u_2(0) = |V_0| e^{\lambda l_0} > 0, \qquad u_3(0) = 0, \tag{4.1.2d}$$

where

$$\epsilon = \operatorname{sgn}(V_0), \qquad \lambda = \frac{k^2}{2} - 1. \tag{4.1.3}$$

Note that u_1 is not defined on $\tau \leq 0$ and that $\lim_{\tau \to 0^+} u_1 = \infty$. Note also that $u_2 > 0$ by definition.

Using the results from Chapter 3, there exists τ_M such that $\boldsymbol{u} = (u_1, u_2, u_3)$ has a unique solution on $(0, \tau_M)$. The following standard result proves useful in determining the maximal interval of existence in each case(see, for example, [39]).

Theorem 4.1.1. *Let $\Psi_{\boldsymbol{a}}(t)$ be the unique solution of the DE $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$, where $\boldsymbol{f} \in C^1(\mathbb{R}^n)$, which satisfies $\boldsymbol{x}(0) = \boldsymbol{a}$, and let $(t_{\min}, t_{\max})$ be the maximal interval of existence on which $\Psi_{\boldsymbol{a}}(t)$ is defined. If $t_{\max}$ is finite, then*

$$\lim_{t \to t_{\max}^-} ||\Psi_{\boldsymbol{a}}(t)|| = +\infty, \tag{4.1.4}$$

where $||.||$ denotes the standard norm on $\mathbb{R}^n$.

This result may be adapted to our system by defining $(0, \tau_M)$ as the maximal interval of existence for the unique solution $\boldsymbol{u}(\tau)$ of (4.1.2). It follows from Theorem 4.1.1 that if the components of the solution u_1, u_2 and u_3 satisfy finite lower and upper bounds for all $\tau \in (0, \tau_M)$, then we have $\tau_M = \infty$. Furthermore, if τ_M is finite then we have $\lim_{\tau \to \tau_M^-} |u_i| = +\infty$ for at least one $i \in \{1, 2, 3\}$.

The system has three parameters $\{k^2, V_0, l_0\}$. The qualitative picture of solutions depends primarily on the signs of V_0 and $\lambda = k^2/2 - 1$ and so we devote a subsection to each of the four permutations.

4.1.1 $V_0 < 0, \lambda > 0$.

In this case, we find that τ_M is finite and the hypersurface $\tau = \tau_M$ corresponds to radial null infinity . Note that $\epsilon = -1$ and $k^2 > 2$ here.

Lemma 4.1.1. *If $V_0 < 0$, $\lambda > 0$ then $u_3 > 0, \dot{u}_3 > 0$ and $u_1 > u_3$ for all $\tau \in (0, \tau_M)$.*

Proof. First note that since $\lim_{\eta\to 0^+} u_1 = +\infty$, then $u_1 > u_3$ holds initially. Now consider

$$\dot{u}_1 - \dot{u}_3 = -\epsilon\frac{2\lambda}{k^2}u_2 - u_1(u_1 - u_3) = \frac{2\lambda}{k^2}u_2 - u_1(u_1 - u_3). \tag{4.1.5}$$

for $\epsilon = -1$. Since $u_2 > 0, \lambda > 0$, it is clear that $u_1 - u_3$ cannot cross zero from above, so $u_1 > u_3$ for $\tau \in (0, \tau_M)$. At $u_1 = 1/2$ we have $\dot{u}_1 = -\epsilon u_2 > 0$ if $\epsilon = -1$, so $u_1 = \dot{R}/R > 1/2$ for all $\tau \in (0, \tau_M)$. Hence, $R > 0$ for all $\tau \in (0, \tau_M)$. Equation (3.1.2d) may be integrated to give

$$\dot{l} = u_3 = \frac{1}{R}\int_0^\tau \left(\frac{1}{4} - \epsilon\frac{2u_2}{k^2}\right) R\, d\tau', \tag{4.1.6}$$

which is clearly positive for $R > 0, \epsilon = -1$. Since $u_3(0) = 0$ we must have $\dot{u}_3 > 0$ initially. It is straightforward to check that, at $\dot{u}_3 = 0$, we have $\ddot{u}_3 = (u_1 - u_3)u_1 u_3$ which is positive for $u_1 > u_3 > 0$. □

Lemma 4.1.2. *Let $\hat{\lambda} = (\lambda + \sqrt{\lambda^2 + 16})/4$. If $V_0 < 0$ and $\lambda > 0$, then $u_1 > u_2^{1/2}/\hat{\lambda}$ for all $\tau \in (0, \tau_M)$ and there exists $\tau_1 \in (0, \tau_M)$ such that u_1 is monotonically increasing and bounded above by $\sqrt{1/4 + u_2}$ for all $\tau \in (\tau_1, \tau_M)$.*

Proof. First note that $u_1 > (1/4 + u_2)^{1/2} > u_2^{1/2}/\hat{\lambda}$ on some initial interval, where the second inequality holds due to $\hat{\lambda} > 1$. The preceding lemma tells us that $u_3 > 0, \dot{u}_3 > 0$, from which it follows that $\dot{u}_2 > 0, \ddot{u}_2 > 0$, for all $\tau \in (0, \tau_M)$. Since $\dot{u}_2/\lambda u_2 = u_3 < u_1$, the u_i are bounded, and solutions therefore exist, while u_1 is decreasing. By inspection of (4.1.2a) with $\epsilon = -1$, u_1 is decreasing for $u_1 > (1/4 + u_2)^{1/2}$. Since $\ddot{u}_2 > 0$, there must exist some $\tau_1 \in (0, \tau_M)$ such that $u_1(\tau_1) = (1/4 + u_2(\tau_1))^{1/2}$. Note that at $u_1 = (1/4 + u_2)^{1/2}$ we have $\dot{u}_1 = 0$ and

$$\frac{d}{d\tau}\left[u_1 - \left(\frac{1}{4} + u_2\right)^{1/2}\right] = -\frac{\dot{u}_2}{2(1/4 + u_2)^{1/2}} < 0. \tag{4.1.7}$$

u_1 is therefore increasing and bounded above by $(1/4+u_2)^{1/2}$ for all $\tau \in (\tau_1, \tau_M)$. Now consider

$$\frac{d}{d\tau}\left(\frac{u_2^{1/2}}{\hat{\lambda}}\right) = \frac{\lambda u_2^{1/2} u_3}{2\hat{\lambda}} < \frac{\lambda u_2^{1/2} u_1}{2\hat{\lambda}}. \tag{4.1.8}$$

Suppose there exists τ_* such that $u_1(\tau_*) = u_2^{1/2}(\tau_*)/\hat{\lambda}$. Then, using (4.1.2a) and (4.1.8)

$$\begin{aligned}\dot{u}_1(\tau_*) - \frac{d}{d\tau}\left(\frac{u_2^{1/2}}{\hat{\lambda}}\right)\bigg|_{\tau=\tau_*} &> \frac{1}{4} + u_2(\tau_*) - \frac{u_2(\tau_*)}{\hat{\lambda}^2} - \frac{\lambda u_2(\tau_*)}{2\hat{\lambda}^2} \\ &= \frac{1}{4} + \left(1 - \frac{1}{\hat{\lambda}^2} - \frac{\lambda}{2\hat{\lambda}}\right) u_2(\tau_*) = \frac{1}{4} > 0,\end{aligned} \tag{4.1.9}$$

so u_1 cannot cross $u_2^{1/2}/\hat{\lambda}$ from above. □

Lemma 4.1.3. *If $V_0 < 0, \lambda > 0$, then there exists $\tau_2 \in (0, \tau_M)$ such that $u_1(\tau_2) = k^2 u_3(\tau_2)$ and $u_1 < k^2 u_3$ for all $\tau \in (\tau_2, \tau_M)$.*

Proof. Consider

$$\dot{u}_1 - k^2 \dot{u}_3 = -\frac{k^2-1}{4} - u_2 - u_1\left(u_1 - k^2 u_3\right) \leq -\frac{1}{4} - u_2, \tag{4.1.10}$$

provided $u_1 > k^2 u_3$, which holds initially. This also shows that $u_1 - k^2 u_3$ can only cross zero from above. If $\tau_M = \infty$, then the result must follow from (4.1.10). If not then $\lim_{\tau \to \tau_M^-} |u_i| = \infty$ for at least one u_i, by Theorem 4.1.1. Since $u_2^{1/2}/\hat{\lambda} < u_1 < (1/4+u_2)^{1/2}$ and $0 < u_3 < u_1$ for $\tau \in (\tau_1, \tau_M)$, we must then have $\lim_{\tau \to \tau_M^-} u_2 = \lim_{\tau \to \tau_M^-} u_1 = \infty$. We know from Lemma 4.1.2 that $u_1 > u_2^{1/2}/\hat{\lambda}$ for all $0 < \tau < \tau_M$, which gives $\dot{u}_1 < 1/4 + \lambda u_2/2$, since $1 - 1/\hat{\lambda}^2 = \lambda/2$. We then have

$$\lim_{\tau \to \tau_M} \int_{\tau_1}^{\tau} \frac{\lambda u_2}{2}\, d\tau' > \lim_{\tau \to \tau_M}\left(u_1 - u_1(\tau_1) - \frac{\tau - \tau_1}{4}\right) = \infty. \tag{4.1.11}$$

Suppose then that $u_1 > k^2 u_3$ for all $\tau \in (0, \tau_M)$. Integrating (4.1.10) and taking the limit gives $\lim_{\tau \to \tau_M^-}(u_1 - k^2 u_3) = -\infty$, using (4.1.11), so we have a contradiction. □

Lemma 4.1.4. *If $V_0 < 0$ and $\lambda > 0$, then $\tau_M < \infty$ and for $i = \{1, 2, 3\}$,*

$$\lim_{\tau \to \tau_M} u_i = \infty. \tag{4.1.12}$$

Proof. Using Lemmas 4.1.2 and 4.1.3 we have

$$\dot{u}_2 > \frac{\lambda u_1 u_2}{k^2} > \frac{\lambda u_2^{3/2}}{k^2 \hat{\lambda}}, \tag{4.1.13}$$

for $\tau \in (\tau_2, \tau_M)$. Integrating over $[\tau_2, \tau]$ and rearranging we find

$$u_2 > \left(\frac{1}{u^{1/2}(\tau_2)} - \frac{\lambda(\tau - \tau_2)}{2k^2 \hat{\lambda}} \right)^{-2}, \tag{4.1.14}$$

so we have $\tau_M \leq k^2 \hat{\lambda} u_2^{-1/2}(\tau_2)/\lambda + \tau_2$ and $\lim_{\tau \to \tau_M^-} u_2 = \infty$. It follows directly from Lemma 4.1.2 that $\lim_{\tau \to \tau_M^-} u_1 = \infty$. Lemmas 4.1.1 and 4.1.3 tell us that $u_1/k^2 < u_3 < u_1$ approaching τ_M and so the the proof is complete. □

Proposition 4.1.1. *For $V_0 < 0$, $\lambda > 0$, the surface corresponding to $\tau = \tau_M$ represents future null infinity and the Ricci scalar decays to zero there.*

Proof. We aim to show that along outgoing radial null geodesics, an infinite amount of affine parameter time is required to reach the surface $\tau = \tau_M$. These geodesics correspond to lines of constant u, θ, z. The geodesic equation (1.2.14) then reduces to

$$\ddot{v} + (2\bar{\gamma}_{,v} + 2\bar{\phi}_{,v})\dot{v}^2 = 0, \tag{4.1.15}$$

where here the dot denotes differentiation with respect to an affine parameter μ, which is chosen such that $\dot{v} > 0$ and $\mu(\tau = 0) = 0$. Dividing by $\dot{v}$ and integrating, we find

$$e^{2\bar{\gamma} + 2\bar{\phi}} \dot{v} = \frac{1}{|u_0|} e^{2\gamma + 2\phi} \dot{v} = C, \tag{4.1.16}$$

for constants $u_0 < 0, C > 0$, recalling (2.4.6). Substituting $2\gamma + 2\phi$ using (3.1.2a) gives

$$\frac{1}{|u_0|} e^{(k^2 l + \tau)/2 + c_1} \dot{v} = C. \tag{4.1.17}$$

Along the geodesics, we also have $v = u_0\eta = u_0 e^{-\tau}$, and thus $dv = -u_0 e^{-\tau} d\tau$. Integrating (4.1.17) then leads to

$$\frac{1}{|u_0|}\int_{v_0}^{v} e^{(k^2 l+\tau')/2+c_1} dv' = \int_0^{\tau} e^{(k^2 l-\tau')/2+c_1} d\tau' = C\mu. \tag{4.1.18}$$

Clearly $e^{k^2 l/2} = |V_0|^{-1} e^l u_2 > u_2$ holds for τ sufficiently close to τ_M, since $e^l \to \infty$. Using (4.1.11) and the fact that τ_M is finite then gives $\lim_{\tau\to\tau_M}\mu = \infty$. This confirms that the surface $\tau = \tau_M$ corresponds to radial null infinity. To demonstrate the decay of the Ricci scalar, which we label $\mathcal{R}$, it is convenient to consider the trace of the energy-momentum tensor:

$$\begin{aligned}
g^{ab}T_{ab} &= -2g^{01}\psi_{,u}\,\psi_{,v} - 4V \\
&= 2|u|e^{-2\gamma-2\phi}\left(-\frac{\eta F'}{u} + \frac{k}{2u}\right)\frac{F'}{u} - \frac{4\bar{V}_0 e^{-2F/k}}{|u|} \qquad (4.1.19)\\
&= \frac{2e^{-2\gamma-2\phi}}{|u|\eta}\left[\left(-\frac{k\eta l'}{2} + \frac{k}{4}\right)\left(\frac{k\eta l'}{2} + \frac{k}{4}\right) - 2V_0 e^{\lambda l}\right] \\
&= \frac{e^{-k^2 l/2+\tau/2-c_1}}{|u|}\left(\frac{k^2}{2}\left(\frac{1}{4} - \dot{l}^2\right) - 4V_0 e^{\lambda l}\right) \\
&= \frac{e^{\tau/2-c_1}}{|u|}\left(\frac{k^2}{2}\left(\frac{1}{4} - \dot{l}^2\right)e^{-k^2 l/2} - 4V_0 e^{-l}\right) = -\mathcal{R}. \qquad (4.1.20)
\end{aligned}$$

Recall that $1/2 < \dot{l} = u_3 < u_1 < (1/4 + u_2)^{1/2}$ approaching τ_M. It follows that $0 > e^{-k^2 l/2}(1/4 - \dot{l}^2) > e^{-k^2 l/2} u_2 = |V_0| e^{-l}$. Since $e^{-l} \to 0$, both terms in the bracket have limit zero as $\tau \to \tau_M$. Since $\tau_M < \infty$, it follows that $\lim_{\tau\to\tau_M}\mathcal{R} = 0$. □

4.1.2 $V_0 < 0, \lambda < 0$

Here we find that the surface $\mathcal{N}_-$ corresponds to a fixed point of the u-system, is regular and is reached by radial null rays in finite affine time. These are some of the solutions which may be extended into region **II**. Note that $\epsilon = -1, k^2 < 2$ here.

Lemma 4.1.5. *If $V_0 < 0$ and $\lambda < 0$, then $\tau_M = +\infty$ and*

$$\lim_{\tau\to\infty}(u_1, u_2, u_3) = \left(\frac{1}{2}, 0, \frac{1}{2}\right). \tag{4.1.21}$$

Proof. We have seen in the proof of Lemma 4.1.1 that if $V_0 < 0$, then $u_3 > 0$

for all $\tau \in (0, \tau_M)$. Equation (4.1.2b) with $\lambda < 0$ then tells us that u_2 is monotonically decreasing on $(0, \tau_M)$. Equation (4.1.7) then tells us that u_1 cannot cross $(1/4 + u_2)^{1/2}$ from above. Since $\dot{u}_1 < 0$ if $u_1 > (1/4 + u_2)^{1/2}$, u_1 is decreasing and bounded below by this term for all $\tau \in (0, \tau_M)$. For any $\tau_* \in (0, \tau_M)$ we then have $u_2 < u_2(\tau_*)$ and $1/2 < u_1 < u_1(\tau_*)$ for all $\tau \in (\tau_*, \tau_M)$. Hence, for all $\tau \in (\tau_*, \tau_M)$ we have

$$\frac{1}{4} - u_1(\tau_*)u_3 < \dot{u}_3 < \frac{1}{4} + \frac{2u_2(\tau_*)}{k^2} - \frac{u_3}{2}, \tag{4.1.22}$$

from which it follows that

$$\min\{1/4u_1(\tau_*), u_3(\tau_*)\} < u_3 < \max\{1/2 + 4u_2(\tau_*)/k^2, u_3(\tau_*)\}. \tag{4.1.23}$$

We have thus far proven that each of the u_i are bounded above and below for all $\tau \in (\tau_*, \tau_M)$ and so $\tau_M = +\infty$ by Theorem 4.1.1. Now, it follows from (4.1.5) that $u_1 - u_3$ can only change sign once. Recalling $\ddot{u}_3 = (u_1 - u_3)u_1u_3$ at $\dot{u}_3 = 0$, $\dot{u}_3$ can only change sign a finite number of times and so u_3 must be monotone as $\tau \to \infty$. Hence, each of the u_i are bounded and monotone in the limit as $\tau \to \infty$ and so the system must evolve to a fixed point. It is easily checked that the only fixed point of the system (4.1.2a)-(4.1.2c) consistent with the given analysis is $(1/2, 0, 1/2)$. □

We now prove that the metric is regular in the limit as $\tau \to \infty$ in this case. The following theorem, which may be found in Chapter 9 of [21], proves useful.

Theorem 4.1.2. *In the differential equation*

$$\boldsymbol{x}'(t) = E\boldsymbol{x} + F(\boldsymbol{x}), \tag{4.1.24}$$

let $F(\boldsymbol{x})$ be of class C^1 with $F(0) = 0, \partial_{\boldsymbol{x}} F(0) = 0$. Let the constant matrix E possess $d > 0$ eigenvalues having negative real parts, say, d_i eigenvalues with real parts equal to α_i, where $\alpha_1 < \ldots < \alpha_r < 0$ and $d_1 + \ldots + d_r = d$, whereas the other eigenvalues, if any, have non-positive real parts. If $\alpha_r \omega < 0$, then (4.1.24) has solutions $\boldsymbol{x} = \boldsymbol{x}(t) \neq 0$, satisfying

$$||\boldsymbol{x}(t)|| e^{\omega t} = 0, \qquad \text{as} \qquad t \to +\infty, \tag{4.1.25}$$

where $||\boldsymbol{x}(t)||$ *denotes the Euclidean norm, and any such solution satisfies*

$$\lim_{t\to+\infty} t^{-1}\log||\boldsymbol{x}(t)|| = \alpha_i, \qquad \textit{for some } i. \tag{4.1.26}$$

□

Proposition 4.1.2. *If* $V_0 < 0$ *and* $\lambda < 0$*, then the metric is regular in the limit as* $\tau \to \infty$*, i.e. on* $\mathcal{N}_-$*, and outgoing radial null rays reach* $\mathcal{N}_-$ *in finite parameter time.*

Proof. We define a new system of variables via

$$\hat{\boldsymbol{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3), \qquad \begin{aligned} \hat{u}_1 &= u_1 - \tfrac{1}{2} \\ \hat{u}_2 &= u_2 \\ \hat{u}_3 &= u_3 - \tfrac{1}{2} \end{aligned} \tag{4.1.27}$$

Then it is easy to check that the $\hat{\boldsymbol{u}}$-system of equations is of the form (4.1.24), satisfying $F(0) = 0, \partial_{\boldsymbol{x}} F(0) = 0$, where the matrix

$$E = \begin{pmatrix} -1 & -\epsilon & 0 \\ 0 & \lambda/2 & 0 \\ -1/2 & -2\epsilon/k^2 & -1/2 \end{pmatrix} \tag{4.1.28}$$

has 3 negative eigenvalues, $\lambda/2 -, 1/2$ and -1, of which $\lambda/2$ is the greatest. Using (4.1.26) and $\alpha_i \leq \lambda/2$, for any $\varepsilon > 0$ there exists $T(\varepsilon)$ such that $|\hat{u}_1| \leq ||\hat{\boldsymbol{u}}|| < e^{(\lambda/2+\varepsilon)\tau}$. Note that $\hat{u}_1 = \dot{S}/S$, and so

$$-e^{(\lambda/2+\varepsilon)\tau} < \frac{\dot{S}}{S} < e^{(\lambda/2+\varepsilon)\tau} \tag{4.1.29}$$

for $\tau > T(\varepsilon)$. Integrating and taking the limit $\tau \to \infty$ then shows that $0 < \lim_{\tau\to\infty} S < +\infty$. Rearranging (3.1.3b) we have

$$2\dot{\phi} = \frac{e^{-\tau/2}}{S} - \frac{\dot{S}}{S}, \tag{4.1.30}$$

which may be integrated using (4.1.29) to show $\lim_{\tau\to\infty} |\phi| < +\infty$. Hence the metric components $g_{\theta\theta} = |u|e^{2\phi}S^2$ and $g_{zz} = |u|e^{-2\phi}$ are regular on $\mathcal{N}_-$. Notice, however, that $2\gamma \sim (k^2/4 + 1/2)\tau$ as $\tau \to \infty$, by (3.1.2a), and so the component $g_{uv} = |u|^{-1} e^{2\gamma+2\phi}$ blows up at $\mathcal{N}_-$. This turns out to be a coordinate singularity and may be avoided by making the transformation $|v| \to \bar{v} = |v|^{-\lambda/2}$. It is straightforward to check that the metric component is then given by $g_{u\bar{v}} = |u|^{-1}|v|^{1+\lambda/2}e^{2\gamma+2\phi}$. Note that $v = u\eta = ue^{-\tau}$ and $1+\lambda/2 = k^2/4+1/2$, so the metric is well behaved in this coordinate system. We also have $e^{(k^2 l-\tau)/2} \sim e^{\lambda\tau/2}$ as $\tau \to +\infty$ and so it follows from (4.1.18) that $\lim_{\tau\to\infty} \mu < +\infty$. □

4.1.3 $V_0 > 0, \lambda < 0$

There are two subcases here, distinguished the sign of $u_2(0) - k^2/8$. When negative, the solutions have a similar structure to those outlined in the previous section. In the positive case, we have a finite interval of existence and a singularity at τ_M. We deal with the case $u_2(0) = k^2/8$ in Section 4.2.2. Note that $\epsilon = 1, k^2 < 2$ here.

Lemma 4.1.6. *If $V_0 > 0, \lambda < 0$ and $u_2(0) < k^2/8$, then $\tau_M = \infty$ and*

$$\lim_{\tau\to\infty} (u_1, u_2, u_3) = \left(\frac{1}{2}, 0, \frac{1}{2}\right). \tag{4.1.31}$$

Proof. Using equation (4.1.6), with $\epsilon = 1$, and $u_2(0) < k^2/8$, we must have u_3 initially positive, since R is initially positive.
Since u_3 cannot cross zero from above while $u_2 < k^2/8$ and $u_3 > 0, \lambda < 0$ give $\dot{u}_2 < 0$, we have $\dot{u}_2 < 0, u_2 < k^2/8$ and $u_3 > 0$ for all $\tau \in (0, \tau_M)$. At $\dot{u}_1 = 0$ we have $\ddot{u}_1 = -\dot{u}_2 > 0$. Given that $\dot{u}_1 < 0$ initially, it must then hold for all $\tau \in (0, \tau_M)$. Note also that $\dot{u}_1 > -\lambda/4 - u_1^2$ and so $u_1 > \sqrt{|\lambda|}/2$ for all $\tau \in (0, \tau_M)$. It then follows that $\dot{u}_3 < 1/4 - \sqrt{|\lambda|}u_3/2$, from which it follows that $u_3 < 1/2\sqrt{|\lambda|}$, for all $\tau \in (0, \tau_M)$. Hence, all the u_i are bounded and so $\tau_M = \infty$. The remainder of the proof is analogous to that of Lemma 4.1.5. □

Proposition 4.1.3. *If $V_0 < 0$ and $\lambda < 0$, then the metric is regular in the limit as $\tau \to \infty$, i.e. on $\mathcal{N}_-$, and outgoing radial null rays reach $\mathcal{N}_-$ in finite parameter time.*

Proof. Note that the sign of V_0 does not affect the arguments in Proposition 4.1.2 and so the proof is identical. □

Lemma 4.1.7. *For $V_0 > 0$, suppose there exists $\tau_0 \in (0, \tau_M)$ such that $u_1(\tau_0) \leq -1/2$. Then $\tau_M < \infty$ and $\lim_{\tau\to\tau_M^-} u_1 = -\infty$.*

Proof. We define a new variable $\bar{u}_1 = u_1 + 1/2$, which satisfies

$$\dot{\bar{u}}_1 = \bar{u}_1 - u_2 - \bar{u}_1^2 < \bar{u}_1 - \bar{u}_1^2. \tag{4.1.32}$$

If $\bar{u}_1 < 0$ then $\dot{\bar{u}}_1 < 0$ so we have $\bar{u}_1 < 0$ for all $\tau \in (\tau_0, \tau_M)$. Integrating (4.1.32) then shows that there exists some $\tau_1 > \tau_0$ such that $\lim_{\tau\to\tau_1^-} \bar{u}_1 = -\infty$. Then we must have $\tau_M \leq \tau_1$ and, using Theorem 4.1.1, $\lim_{\tau\to\tau_M^-} |u_i| = +\infty$ for some i. Suppose that $\lim_{\tau\to\tau_M^-} u_1 > -\infty$. It is clear from (4.1.2b) that u_2 is finite provided u_3 and τ are finite and so we must have $\lim_{\tau\to\tau_M^-} |u_3| = \infty$. If $\lim_{\tau\to\tau_M^-} u_3 = +\infty$, it follows from (4.1.6) and the fact that $u_2 > 0$ and $0 < R < R(\tau_0)$ that $\lim_{\tau\to\tau_M^-} R = 0$. Note that $R < R(\tau_0)$ follows from $u_1 < 0$ here. Note also that

$$R = R(\tau_0) \exp\left(\int_{\tau_0}^{\tau} u_1 \, d\tau'\right), \tag{4.1.33}$$

and so $\lim_{\tau\to\tau_M^-} R = 0$ implies that

$$\lim_{\tau\to\tau_M^-} \int_{\tau_0}^{\tau} u_1 \, d\tau' = -\infty, \tag{4.1.34}$$

from which it must follow that $\lim_{\tau\to\tau_M^-} u_1 = -\infty$. If $\lim_{\tau\to\tau_M^-} u_3 = -\infty$ then we have either $\lim_{\tau\to\tau_M^-} R = 0$ or

$$\lim_{\tau\to\tau_M^-} \int_{0}^{\tau} u_2 \, d\tau' = +\infty, \tag{4.1.35}$$

where we have used (4.1.6) and the fact that R is bounded above again. It follows immediately from (4.1.2a) and (4.1.35) that $\lim_{\tau\to\tau_M^-} u_1 = -\infty$ in this case also. Hence $\tau_1 = \tau_M$ and the proof is complete. □

Lemma 4.1.8. *If $V_0 > 0, \lambda < 0$ and $u_2(0) > k^2/8$, then $\tau_M < \infty$ and*

$$\lim_{\tau\to\tau_M^-} u_1 = -\infty, \quad \lim_{\tau\to\tau_M^-} u_3 = -\infty, \quad \lim_{\tau\to\tau_M^-} u_2 = +\infty. \tag{4.1.36}$$

Proof. In this case we have $u_2 > k^2/8, u_3 < 0$ on some initial interval, using equation (4.1.6). Moreover, u_3 cannot cross zero from below while $u_2 > k^2/8$ and since u_2 is increasing for $u_3 < 0$, these conditions hold for all $\tau \in (0, \tau_M)$. We then have

$$\dot{u}_1 < \frac{|\lambda|}{4} - u_1^2, \tag{4.1.37}$$

for all $\tau \in (0, \tau_M)$. For $u_1 > \sqrt{|\lambda|/2}$, this may be integrated over $(0, \tau)$ to give

$$u_1 < m \coth(m\tau), \tag{4.1.38}$$

where $m = \sqrt{|\lambda|}/2$ and we have used $\lim_{\tau \to 0^+} u_1 = \infty$. Note that (4.1.38) automatically holds for $u_1 < m$ also, since $\coth(m\tau) > 1$ for all τ.

Combining (4.1.38) with (4.1.2c) and $u_2 < u_2(0)$ gives

$$\dot{u}_3 < -b - (m \coth(m\tau))u_3, \tag{4.1.39}$$

where $b = 2u_2(0)/k^2 - 1/4 > 0$. This may be integrated to give

$$u_3 < -\frac{b(\cosh(m\tau) - 1)}{m \sinh(m\tau)}, \tag{4.1.40}$$

$$\lambda \int_0^\tau u_3 \, d\tau' > -\frac{2\lambda b}{m^2} \log\left(\cosh\left(\frac{m\tau}{2}\right)\right) = 8b \log\left(\cosh\left(\frac{m\tau}{2}\right)\right). \tag{4.1.41}$$

We then arrive at

$$u_2 = u_2(0) \exp\left[\int_0^\tau u_3 \, d\tau'\right] > \cosh^{8b}\left(\frac{m\tau}{2}\right). \tag{4.1.42}$$

Suppose that τ_M is infinite. Then we would have $\tau_* \in (0, \infty)$ such that $u_2 > 1/2$, which gives $\dot{u}_1 < -1/4 - u_1^2$, for all $\tau \in (\tau_*, \infty)$, using (4.1.2a) with $\epsilon = 1$. Integrating this inequality shows that $\lim_{\tau \to \tau_M^-} u_1 = -\infty$ for τ_M finite, which is a contradiciton. So we must have $\lim_{\tau \to \tau_M^-} |u_i| = \infty$ for at least one of the u_i, by Theorem 4.1.1. Now consider $X = u_1 - k^2 u_3/2$, which satisfies

$$\dot{X} = -\frac{\lambda}{4} - u_1 X > -u_1 X. \tag{4.1.43}$$

Note that X is initially positive and at $X = 0$ we have $\dot{X} > 0$, so $X > 0$ for $\tau \in (0, \tau_M)$. Suppose that $u_1 \geq -1/2$, which gives $\dot{X} < X/2$, for all $\tau \in (0, \tau_M)$. Then X is bounded above, which in turn gives a lower bound for u_3. This gives an upper bound on u_2, which contradicts $\lim_{\tau \to \tau_M^-} ||\boldsymbol{u}(\tau)|| = \infty$. Hence, u_1 must cross $-1/2$ in finite τ and thus $\lim_{\tau \to \tau_M^-} u_1 = -\infty$, using the

preceding lemma. It follows immediately from $X > 0$ that $\lim_{\tau\to\tau_M^-} u_3 = -\infty$ also.
To complete the proof, we assume $\lim_{\tau\to\tau_M^-} u_2 = B + 1/4 < \infty$ for some constant B, and then arrive at a contradiction. Note that u_2 is monotone and so the limit must exist. The assumption gives $u_2 < B + 1/4$, and thus $\dot{u}_1 > -B - u_1^2$, for all $\tau \in (0, \tau_M)$. Dividing by u_1 we have $\dot{u}_1/u_1 < -B/u_1 - u_1$ for $u_1 < 0$. Choosing τ_0 such that $u_1(\tau_0) < 0$ and integrating over $[\tau_0, \tau]$ gives

$$u_1 > u_1(\tau_0) \exp\left[\int_{\tau_0}^{\tau} \frac{-B}{u_1} - u_1 \, d\tau'\right]. \tag{4.1.44}$$

Since $\lim_{\tau\to\tau_M^-} u_1 = -\infty$, it must follow that

$$\lim_{\tau\to\tau_M^-} \int_{\tau_0}^{\tau} u_1 \, d\tau' = -\infty. \tag{4.1.45}$$

Now, $u_1 > k^2 u_3/2$ gives $\dot{u}_2 > 2\lambda u_2 u_1/k^2$. Dividing by u_2, integrating and using the above, we find $\lim_{\tau\to\tau_M^-} u_2 = \infty$, which is our contradiction. □

Proposition 4.1.4. *If $V_0 > 0$, $\lambda < 0$, and $u_2(0) > k^2/8$ then $\tau_M < \infty$ and there is a singularity at τ_M.*

Proof. In terms of u_2, u_3, with $V_0 > 0$, the Ricci scalar is given by

$$\mathcal{R} = \frac{e^{\tau/2 - c_1}}{|u|} \left(\frac{k^2}{2} \left(u_3^2 - \frac{1}{4} \right) u_2^{-k^2/2\lambda} + 4 u_2^{-1/\lambda} \right). \tag{4.1.46}$$

Using the previous lemma we have $\tau_M < \infty$, $\lim_{\tau\to\tau_M^-} u_2 = \infty$ and $\lim_{\tau\to\tau_M^-} u_3 = -\infty$. Since $\lambda < 0$, it is clear that $\lim_{\tau\to\tau_M^-} \mathcal{R} = \infty$. □

4.1.4 $V_0 > 0, \lambda > 0$.

Similarly to the previous section, we have two different pictures depending on the sign of $u_2(0) - k^2/8$. When positive, u_3 is initially negative, and vice-versa. Hence, u_2 either starts above $k^2/8$ and is decreasing, or vice-versa. We show that the maximal interval of existence of solutions is finite and give the limiting behaviour at τ_M in each of the two subcases. We then prove the existence of a singularity at τ_M in each case. The case $u_2(0) = k^2/8$ is dealt with in Section 4.2.2. Note that $\epsilon = 1, k^2 > 2$ here.

$u_2(0) > k^2/8$

Lemma 4.1.9. *If $V_0 > 0, \lambda > 0$ and $u_2(0) > k^2/8$, then there exists $\tau_0 \in (0, \tau_M)$ such that $u_1(\tau_0) = 0$ and $u_2 > k^2/8$ for all $\tau \in [0, \tau_0]$.*

Proof. Recall that $u_2(0) > k^2/8$ gives $u_3 < 0, \dot{u}_3 < 0$ on some initial interval. Differentiating (4.1.2c) gives

$$\ddot{u}_3 = \left(\frac{2u_2}{k^2} - \frac{1}{4} + u_1^2 \right) u_3 - u_1 \dot{u}_3. \qquad (4.1.47)$$

At $\dot{u}_3 = 0$ we have $\ddot{u}_3 = (u_1 - u_3)u_1 u_3$, which is negative for $u_1 > 0 > u_3$, and so $\dot{u}_3 < 0$ holds while $u_1 > 0$. We then have $0 < u_2 < u_2(0)$ and $\dot{u}_3 > 1/4 - 2u_2(0)/k^2$ for $u_1 > 0$. Hence, the u_i are all bounded above and below for $u_1 > 0$, and so either $\tau_M = \infty$, or there exists τ_0 such that $u_1(\tau_0) = 0$. Consider (4.1.43) with $u_1 > 0$ and $u_3 < 0$, which give $\dot{X} < -\lambda/4 - u_1^2$. It is obvious that either X or u_1 must cross zero in finite τ. However, $u_1 < X$ if $u_3 < 0$ and so there must exist τ_0 such that $u_1(\tau_0) = 0$. We then have $\dot{u}_3(\tau_0) = 1/4 - 2u_2(\tau_0)/k^2 < 0$, from which $u_2(\tau_0) > k^2/8$ immediately follows. The fact that $\dot{u}_2 = \lambda u_2 u_3 < 0$ for $\tau \in (0, \tau_0]$ completes the proof. □

Lemma 4.1.10. *If $V_0 > 0, \lambda > 0, u_2(0) > k^2/8$, then $u_1 < 0, u_3 < 0, \dot{u}_3 < 0, \ddot{u}_3 < 0$ for all $\tau \in (\tau_0, \tau_M)$, where $u_1(\tau_0) = 0$.*

Proof. If $u_2 \geq k^2/8$ and $\epsilon = 1$ then $\dot{u}_1 \leq -\lambda/4 - u_1^2 < 0$, which preserves $u_1 < 0$. If $u_2 < k^2/8$ and $\dot{u}_3 < 0$, then by (4.1.2c) we have $u_1 u_3 > 0$. For $\tau \in (\tau_0, \tau_M)$ then, $u_1 < 0$ holds while $u_3 < 0, \dot{u}_3 < 0$ hold. Using the preceding lemma and (3.1.2b), we have $\ddot{R} < -\lambda R/4$ for all $\tau \in [0, \tau_0]$. This may be integrated to give $R < m^{-1} \sin(m\tau) \leq m^{-1}$, which gives $R^{-2} > m^2 = \lambda/4$, for $\tau \in [0, \tau_0]$. Now, (3.1.2e) may be rearranged to give

$$\frac{1}{R^2} - \frac{\lambda}{4} + \lambda u_1^2 + \frac{k^2}{2} (u_1 - u_3)^2 = k^2 \left(\frac{2u_2}{k^2} - \frac{1}{4} + u_1^2 \right). \qquad (4.1.48)$$

Then, for $R^{-2} > \lambda/4 > 0$, which follows from $u_1 < 0$, we must have

$$\frac{2u_2}{k^2} - \frac{1}{4} + u_1^2 > 0. \qquad (4.1.49)$$

We see from equation (4.1.47) that this inequality, along with $u_1 < 0, u_3 < 0$

and $\dot{u}_3 < 0$, gives $\ddot{u}_3 < 0$, which preserves $\dot{u}_3 < 0$. Since $R^{-2} > \lambda/4$ holds for $u_1 = \dot{R}/R < 0$, we must have $u_1, u_3, \dot{u}_3, \ddot{u}_3$ negative for $\tau \in [\tau_0, \tau_M)$. □

Proposition 4.1.5. *If $V_0 > 0, \lambda > 0$ and $u_2(0) > k^2/8$, then $\tau_M < \infty$ and $\lim_{\tau\to\tau_M^-} u_1 = \lim_{\tau\to\tau_M^-} u_3 = -\infty$.*

Proof. We know that there exists τ_0 such that $\dot{u}_3 < 0, \ddot{u}_3 < 0$ for all $\tau \in (\tau_0, \tau_M)$. Supposing that $\tau_M = +\infty$, then we must have $\lim_{\tau\to\tau_M^-} u_3 = -\infty$ and thus u_2 finite for all $\tau \in (0, \infty)$. We must also have $u_1 > -1/2$ for all $\tau \in (0, \tau_M)$, by Lemma 4.1.7. We also know from the preceding proof that $R^{-2} > \lambda/4$ for all $\tau \in (0, \tau_M)$. Equation (4.1.48) then gives

$$\frac{k^2}{2}(u_1 - u_3)^2 < k^2\left(\frac{2u_2}{k^2} - \frac{1}{4} + u_1^2\right). \tag{4.1.50}$$

If $u_1 > -1/2$, then the lefthand side blows up at τ_M which, given that u_2 is finite, is a clear contradiction. Hence, u_1 crosses $-1/2$ at some finite τ and $\lim_{\tau\to\tau_M^-} u_1 = -\infty$ for τ_M finite. Dividing (4.1.2a) by u_1, integrating and taking the limit $\tau \to \tau_M$ we find

$$\lim_{\tau\to\tau_M^-} \int_{\tau_*}^{\tau} \frac{1 - 4u_2}{4u_1} - u_1\, d\tau' = \lim_{\tau\to\tau_M^-} \log \frac{u_1}{u_1(\tau_*)} = \infty, \tag{4.1.51}$$

where τ_* is chosen such that $u_1 < 0$ for $\tau \in [\tau_*, \tau_M)$. Since $\lim_{\tau\to\tau_M^-}(1 - 4u_2)/4u_1 = 0$, it follows that

$$\lim_{\tau\to\tau_M^-} \int_{\tau_*}^{\tau} u_1\, d\tau' = -\infty. \tag{4.1.52}$$

Integrating $\dot{u}_3/u_3$ and taking the limit we find

$$\lim_{\tau\to\tau_M^-} \log \frac{u_3}{u_3(\tau_*)} = \lim_{\tau\to\tau_M^-} \int_{\tau_*}^{\tau} \frac{k^2 - 8u_2}{4k^2 u_3} - u_1\, d\tau' = +\infty, \tag{4.1.53}$$

using the fact that $(k^2 - 8u_2)/4k^2 u_3$ is bounded for $\tau \in (0, \tau_M)$. The result immediately follows. □

$u_2(0) < k^2/8$

Lemma 4.1.11. *If $V_0 > 0, \lambda > 0$ and $u_2(0) < k^2/8$, then $\tau_M > \pi/2m$ and $u_1(\pi/2m) > 0, u_2(\pi/2m) < k^2/8, u_3(\pi/2m) > 0, R(\pi/2m) > m^{-1}$ and*

$k^2u_3/2 > u_1$ for all $\tau \in [\pi/2m, \tau_M)$.

Proof. For $u_2 < k^2/8$ we have $\dot{u}_1 > -\lambda/4 - u_1^2$. Integrating over $(0, \tau)$ gives

$$u_1 > m\cot(m\tau), \tag{4.1.54}$$

where we have used $\lim_{\tau\to 0^+} u_1 = \infty$. While $u_3 > 0$, which holds initially, we have $u_2 > u_2(0)$. Combined with the above, this gives

$$\dot{u}_3 < \frac{1}{4} - \frac{2u_2(0)}{k^2} - (m\cot(m\tau))u_3. \tag{4.1.55}$$

Integrating over $(0, \tau)$ gives

$$u_3 < \frac{\tilde{b}(1-\cos(m\tau))}{m\sin(m\tau)} = \frac{\tilde{b}\sin(m\tau/2)}{m\cos(m\tau/2)}, \tag{4.1.56}$$

where $\tilde{b} = 1/4 - 2u_2(0)/k^2$. Integrating again we find

$$\lambda\int_0^\tau u_3\, d\tau' < -\frac{2\lambda\tilde{b}}{m^2}\log\left[\cos\left(\frac{m\tau}{2}\right)\right] = -8\tilde{b}\log\left[\cos\left(\frac{m\tau}{2}\right)\right], \tag{4.1.57}$$

and so using $\dot{u}_2 = \lambda u_2 u_3$,

$$u_2 < u_2(0)\cos^{-8\tilde{b}}\left(\frac{m\tau}{2}\right). \tag{4.1.58}$$

Note that u_3 cannot cross zero from above if $u_2 < k^2/8$. The bounds $u_3 > 0$, (4.1.54),(4.1.56) and (4.1.58) therefore hold, and solutions exist, as long as $u_2 < k^2/8$ holds. Assuming $\tau_M > \pi/2m$ we have $u_2(\pi/2m) < 2^{4\tilde{b}}u_2(0)$. Letting $z = 8u_2(0)/k^2 < 1$, and using $4\tilde{b} = 1 - 8u_2(0)/k^2 = 1 - z$, we have

$$\frac{8u_2(\pi/2m)}{k^2} < 2^{1-z}z \leq 1, \tag{4.1.59}$$

for all $z \leq 1$, which is equivalent to $u_2(\pi/2m) < k^2/8$. Our assumption is then validated and $u_3(\pi/2m) > 0$. We also have $u_1(\pi/2m) > 0$ from (4.1.54), and it is straightforward to show by integrating (4.1.54) that $R > m^{-1}\sin(m\tau)$ on $[0, \pi/2m]$, which gives $R(\pi/2m) > m^{-1}$.
Recall $X = u_1 - k^2u_3/2$, which satisfies

$$\dot{X} < -\frac{\lambda}{4} - X^2, \tag{4.1.60}$$

provided $u_3 > 0, X \geq 0$, using (4.1.43). Integrating over $(0, \tau)$ we find $X < m\cot(m\tau)$. Since $\cot(m\tau) = 0$ at $\tau = \pi/2m$ and $u_3 > 0$ for $\tau \in (0, \pi/2m)$, there must exist $\tau_* \in (0, \pi/2m)$ such that $X(\tau_*) = 0$. Note also that X cannot cross zero from below if $\lambda > 0$ and so $X < 0$ for $\tau \in (\tau_*, \tau_M)$. □

Lemma 4.1.12. *If $V_0 > 0, \lambda > 0$ and $u_2(0) < k^2/8$, then there exists $\tau_0 \in (0, \tau_M)$ such that $u_1(\tau_0) = 0, u_3(\tau_0) > 0$ and $R(\tau_0) > m^{-1}$.*

Proof. Suppose $u_1 > 0$ for all $\tau \in (0, \tau_M)$. Then $u_3 > 0$ for all $\tau \in (0, \tau_M)$, since $u_3(\pi/2m) > 0$ and $u_3 > 2u_1/k^2$ for all $\tau \in (\pi/2m, \tau_M)$ by the previous lemma. We then have $\dot{u}_3 < 1/4$ for all $\tau \in (0, \tau_M)$, which gives a finite upper bound on u_3, and thus u_2 and u_1, for finite τ. Hence, $\tau_M = +\infty$. If $u_2 \leq 1/4$ we have

$$\dot{u}_3 \geq \frac{\lambda}{2k^2} - u_1 u_3 > \frac{\lambda}{2k^2} - \frac{k^2 u_3^2}{2}, \tag{4.1.61}$$

using $u_1 < k^2 u_3/2$. It follows that $u_3 > u_m = \min\{u_3(\pi/2m), \sqrt{\lambda}/k^2\}$ for $u_2 \leq 1/4$ and $\tau \in (\pi/2m, \tau_M)$. This gives $\dot{u}_2 > \lambda u_m u_2$, and so there must exist $\tau_* \in (\pi/2m, \tau_M)$ such that $u_2 > 1/4$ for all $\tau \in (\tau_*, \tau_M)$. By inspection of (4.1.2a), there must then exist $\tau_0 \in (\tau_*, \tau_M)$ such that $u_1(\tau_0) = 0$. Using Lemma 4.1.11, we have $\tau_0 > \pi/2m$ and thus $u_3(\tau_0) > 0$ and $R(\tau_0) > R(\pi/2m) > m^{-1}$. □

Lemma 4.1.13. *For $V_0 > 0, \lambda > 0$, suppose there exists $\tau_0 \in (0, \tau_M)$ such that $u_1(\tau_0) = 0$ and $u_2(\tau_0) > k^2/8$. Then $\tau_M < +\infty$ and $\lim_{\tau \to \tau_M^-} u_1 = -\infty, \lim_{\tau \to \tau_M^-} u_3 = +\infty$.*

Proof. Equation (4.1.48) may also be written as

$$u_1^2 - \frac{1}{R^2} + \frac{\lambda}{4} - \frac{k^2 u_3^2}{2} = k^2 \left(\frac{1}{4} - \frac{2u_2}{k^2} - u_1 u_3 \right) = k^2 \dot{u}_3. \tag{4.1.62}$$

Now define

$$\Gamma = u_1^2 - \frac{1}{R^2} + \frac{\lambda}{4}, \tag{4.1.63}$$

which satisfies

$$\begin{aligned} \dot{\Gamma} = 2u_1\dot{u}_1 + \frac{2\dot{R}}{R^3} &= 2u_1 \left(\frac{1}{4} - u_2 - u_1^2 + \frac{1}{R^2} \right) \\ &= 2u_1 \left(\frac{k^2}{8} - u_2 - \Gamma \right). \end{aligned} \tag{4.1.64}$$

If $\Gamma > 0, u_2 > k^2/8$ and $u_1 < 0$, then $\dot{\Gamma} > 0$. From the hypothesis we have $\Gamma(\tau_0) > 0$ and as long as $u_2 > k^2/8$ holds we have $\dot{u}_1 < -\lambda/4 - u_1^2$. If $\Gamma > 0$, equation (4.1.62) tells us that $\dot{u}_3 > 0$ if $u_3 < \sqrt{2\Gamma/k^2}$ and so we have $u_3 > 0$, which gives $u_2 > k^2/8$, while $\Gamma > 0$. Hence, $\Gamma > 0, u_2 > k^2/8, u_1 < 0$ and $u_3 > 0$ hold for all $\tau \in (\tau_0, \tau_M)$. We then have $\dot{u}_1 < -\lambda/4 - u_1^2$ for $\tau \in (\tau_0, \tau_M)$, which may be integrated to show that $\tau_M < +\infty$ and $\lim_{\tau\to\tau_M^-} u_1 = \lim_{\tau\to\tau_M^-} X = -\infty$, since $X < u_1$. Integrating $\dot{X}/X = -\lambda/4X - u_1$, then shows that

$$\lim_{\tau\to\tau_M^-} \int_{\tau_*}^{\tau} u_1 = -\infty, \tag{4.1.65}$$

where τ_* is chosen such that $u_1(\tau_*) < 0$. We can use this to show $\lim_{\tau\to\tau_M^-} \Gamma = +\infty$ by integrating (4.1.64). Since $\dot{u}_3 < 0$ for $u_3 > \sqrt{2\Gamma/k^2}$, we must have $u_3 < \sqrt{2\Gamma/k^2}$ and $\dot{u}_3 > 0$, for τ sufficiently close to τ_M and so $\lim_{\tau\to\tau_M^-} u_3$ must exist. Now suppose $\lim_{\tau\to\tau_M^-} u_3 < +\infty$. We then have $\lim_{\tau\to\tau_M^-} u_2 < +\infty$ and

$$\lim_{\tau\to\tau_M^-} \log\left(\frac{u_3}{u_3(\tau_*)}\right) = \lim_{\tau\to\tau_M^-} \int_{\tau_*}^{\tau} \left(\frac{1}{4u_3} - \frac{2u_2}{k^2 u_3} - u_1\right) d\tau' = +\infty, \tag{4.1.66}$$

since $1/4u_3 - 2u_2/k^2u_3$ is bounded above and below under the assumption. Hence, we have $\lim_{\tau\to\tau_M^-} u_3 = +\infty$, by contradiction. □

Lemma 4.1.14. *For $V_0 > 0, \lambda > 0$, suppose there exists $\tau_0 \in (0, \tau_M)$ such that $u_1(\tau_0) = 0, u_2(\tau_0) < k^2/8$ and $u_1 > 0$ for $\tau \in (0, \tau_0)$. Then $\tau_M < +\infty$ and $\lim_{\tau\to\tau_M^-} u_1 = -\infty$, $\lim_{\tau\to\tau_M^-} u_3 = +\infty$.*

Proof. Since $\dot{u}_1(\tau_0) = 1/4 - u_2(\tau_0)$ and u_1 is approaching zero from above we must have $u_2(\tau_0) \geq 1/4$. Lemma 4.1.12 tells us that $u_3(\tau_0) > 0$. It is clear from (4.1.2c) that if $u_1 < 0, u_2 < k^2/8$ and $u_3 > 0$, then $\dot{u}_3 > 0$. Suppose there exists $\tau_* \in (\tau_0, \tau_M)$ such that $u_2(\tau_*) = k^2/8$. Then $u_1 < 0$, since $\dot{u}_1 < 0$, on (τ_0, τ_*). These combine to give $\dot{u}_3 > 0$ on (τ_0, τ_*). It follows from (4.1.47) that $\dot{u}_3$ cannot cross zero from above if $u_2 \geq k^2/8$ and $u_3 > 0$. Hence, $\dot{u}_3 > 0$ for all $\tau \in (\tau_0, \tau_M)$. We then also have $u_3 > 0, \dot{u}_2 > 0, \ddot{u}_2 >$ on (τ_0, τ_M). Suppose $\tau_M = \infty$. Then u_2 must exceed $k^2/8$ in finite τ which gives $\dot{u}_1 < -\lambda/4 - u_1^2$. We have seen that this implies $\lim_{\tau\to\tau_M^-} u_1 = -\infty$ for $\tau_M < +\infty$, so we have a contradiction. A similar argument to one given in the preceding lemma gives $\lim_{\tau\to\tau_M^-} u_3 = +\infty$. □

Lemma 4.1.15. *For $V_0 > 0, \lambda > 0$, suppose there exists $\tau_0 \in (0, \tau_M)$ such that $u_1(\tau_0) = 0, u_2(\tau_0) = k^2/8$. Then $\tau_M < +\infty$ and $\lim_{\tau\to\tau_M^-} u_1 = -\infty$, $\lim_{\tau\to\tau_M^-} u_3 = +\infty$.*

Proof. At τ_0 we have $\dot{u}_3 = \ddot{u}_3 = 0$, and it is not hard to check that $\dddot{u}_3(\tau_0) = -\dot{u}_1(\tau_0)u_3(\tau_0)^2 > 0$. A similar argument to the one given above then shows that $u_2 > k^2/8, \dot{u}_3 > 0$ obtain for $\tau \in (\tau_0, \tau_M)$ and the rest follows in a similar fashion. □

Proposition 4.1.6. *If $V_0 > 0, \lambda > 0$ and $u_2(0) > k^2/8$ then τ_M is finite and $\lim_{\tau\to\tau_M^-} u_1 = -\infty, \lim_{\tau\to\tau_M^-} u_3 = +\infty$*

Proof. Lemma 4.1.12 shows that there exists τ_0 such that $u_1(\tau_0) = 0$ which we can assume, without loss of generality, that it is the first such τ_0. Depending on the sign of $u_2 - k^2/8$, Lemma 4.1.13, 4.1.14 or 4.1.15 completes the proof. □

Lemma 4.1.16. *For $V_0 > 0$, $\lambda > 0$, suppose that $\tau_M < \infty$ and*

$$\lim_{\tau\to\tau_M^-} u_1 = -\infty, \qquad \lim_{\tau\to\tau_M^-} u_3 = -\infty. \tag{4.1.67}$$

Then $\lim_{\tau\to\tau_M^-} \mathcal{R} = \infty$ and $\lim_{\tau\to\tau_M^-} \mu < +\infty$, i.e. there exists a singularity at $\tau = \tau_M$ which is reached by outgoing null rays in finite affine time.

Proof. If $\lim_{\tau\to\tau_M^-} u_3 = -\infty$ then $\lim_{\tau\to\tau_M^-} u_2 < \infty$ and, since $\lambda > 0$, $\lim_{\tau\to\tau_M^-} u_2^{-k^2/2\lambda} \neq 0$. Using (4.1.46) then shows $\lim_{\tau\to\tau_M^-} \mathcal{R} = \infty$. The solution to the geodesic equation (4.1.18) and the finiteness of l reveal $\lim_{\tau\to\tau_M^-} \mu < \infty$. □

Lemma 4.1.17. *For $V_0 > 0$, $\lambda > 0$, suppose that $\tau_M < \infty$ and $\lim_{\tau\to\tau_M^-} u_1 = -\infty, \lim_{\tau\to\tau_M^-} u_3 = +\infty$. Then for any $\tau_* < \tau_M$ such that $u_3 > 0$ for all $\tau \in (\tau_*, \tau_M)$ we have*

$$\lim_{\tau\to\tau_M^-} \int_{\tau_*}^{\tau} u_1 d\tau = -\infty. \tag{4.1.68}$$

Proof. Note that for $u_3 > 0$ and $\epsilon = 1$ we have $\dot{u}_3/u_3 < 1/4u_3 - u_1$. Integrating then gives

$$u_3 < u_3(\tau_*) \exp\left(\int_{\tau_*}^{\tau} \frac{1}{4u_3} - u_1 d\tau'\right). \tag{4.1.69}$$

Rearranging and taking the limit we have

$$\lim_{\tau\to\tau_M^-} \exp\left(-\int_{\tau_*}^{\tau} u_1 d\tau'\right) > \lim_{\tau\to\tau_M^-} \frac{u_3}{u_3(\tau_*)} \exp\left(-\int_{\tau_*}^{\tau} \frac{1}{4u_3} d\tau'\right) = +\infty, \tag{4.1.70}$$

since $\lim_{\tau\to\tau_M^-} 1/4u_3 = 0$. The result immediately follows. □

Lemma 4.1.18. *For $V_0 > 0$, $\lambda > 0$, suppose that $\tau_M < \infty$ and $\lim_{\tau\to\tau_M^-} u_1 = -\infty$, $\lim_{\tau\to\tau_M^-} u_3 = +\infty$. Then $\lim_{\tau\to\tau_M^-} x_1/x_3 = \ell$ exists and*

$$\lim_{\tau\to\tau_M^-} u_2 = +\infty, \qquad -\infty < \ell < -k^2 - \frac{1}{4}. \tag{4.1.71}$$

Proof. First we define $q_1 = u_1/u_3$, which satisfies

$$\dot{q}_1 = \frac{1}{u_3}\left(\frac{1}{4} - u_2\right) - \frac{q_1}{u_3}\left(\frac{1}{4} - \frac{2u_2}{k^2}\right). \tag{4.1.72}$$

Now, suppose u_2 is bounded above for all $\tau \in (0, \tau_M)$. Then it straightforward to show that q_1 must be bounded below, since the first term and the coefficient of q_1 in (4.1.72) are then bounded below and above, respectively. So there exists some $\ell_* < 0$ such that $u_1/u_3 > \ell_*$, for $\tau \in (0, \tau_M)$. We then have $\dot{u}_2/u_2 > \lambda u_1/\ell_*$ for $\tau \in (0, \tau_M)$. Integrating and using Lemma 4.1.17 then shows that $\lim_{\tau\to\tau_M^-} u_2 = +\infty$, and so u_2 is unbounded, by contradiction. Given that u_2 is monotone increasing for $u_3 > 0$, which obtains for τ sufficiently close to τ_M, we must have $\lim_{\tau\to\tau_M^-} u_2 = +\infty$.
It is clear from (4.1.72) that $\dot{q}_1$ is negative if $u_2 > k^2/8 > 1/4$, $q_1 < 0$ and $u_3 > 0$, which all hold for τ sufficiently close to τ_M. Thus, q_1 is monotone decreasing for τ sufficiently close to τ_M and the limit $\lim_{\tau\to\infty} q_1 = \ell < 0$ exists. We now prove by contradiction that ℓ is finite. Assuming $\lim_{\tau\to\tau_M^-} q_1 = -\infty$, there must exist some $\tau_* \in (0, \tau_M)$ such that $q_1 < -2\lambda$, which gives $\lambda u_3 < -u_1/2$, for $\tau \in (\tau_*, \tau_M)$. Now define $q_2 = u_2/u_1$, which satisfies

$$\frac{\dot{q}_2}{q_2} = \lambda u_3 - \frac{1}{4u_1} + \frac{u_2}{u_1} + u_1 < -\frac{1}{4u_1} + \frac{u_1}{2}, \tag{4.1.73}$$

for $u_1 < 0, u_2 > 1/4, q_1 < -2\lambda$. Integrating and taking the limit $\tau \to \tau_*$ we have

$$\lim_{\tau\to\tau_M^-} q_2 \geq q_2(\tau_*) \lim_{\tau\to\tau_M^-} \exp\left(\frac{1}{2}\int_{\tau_*}^{\tau} u_1 - \frac{1}{2u_1} d\tau'\right) = 0, \tag{4.1.74}$$

where we've used Lemma 4.1.17, $\lim_{\tau\to\tau_M^-} 1/u_1 = 0$ and $q_2 < 0$. It follows that $\lim_{\tau\to\tau_M^-} q_2 = 0$. Defining $q_3 = u_2/u_3$ we have

$$\frac{\dot{q}_3}{q_3} = \lambda u_3 - \frac{1}{4u_3} + \frac{2u_2}{k^2 u_3} + u_1. \tag{4.1.75}$$

Since $\lim_{\tau\to\tau_M^-} q_2 = 0$ we can choose τ_* such that we also have

$$\frac{2u_2}{k^2 u_3} = \frac{2q_2 u_1}{k^2 u_3} < -\frac{u_1}{4}, \tag{4.1.76}$$

and thus

$$\frac{\dot{q}_3}{q_3} < \frac{u_1}{4}, \tag{4.1.77}$$

for $\tau \in (\tau_*, \tau_M)$, where we have used $\lambda u_3 < -u_1/2$ and (4.1.75). Integrating and using Lemma 4.1.17 then shows that $\lim_{\tau\to\tau_M^-} q_3 = 0$.
However, it follows from $u_1 < 0, u_3 > 0$ and (4.1.72) that

$$\dot{q}_1 > -q_3 + \frac{2q_3}{k^2} q_1. \tag{4.1.78}$$

It is clear that if $\lim_{\tau\to\tau_M^-} q_3 = 0$, then $\lim_{\tau\to\tau_M^-} q_1$ is finite and so we have a contradiction. Therefore, ℓ must be finite.

To estimate ℓ, we divide (3.1.2e) across by u_3^2 which, in the case $\epsilon = 1$, gives

$$\left(1 - \frac{1}{\dot{R}^2}\right) q_1^2 + \frac{2u_2}{u_3^2} + k^2 q_1 + \frac{2+k^2}{8u_3^2} - \frac{k^2}{2} = 0, \tag{4.1.79}$$

in terms of R, u_1, u_2, u_3. Let $q_4 = u_2/u_3^2$. To determine the limiting behaviour of q_4, we consider its derivative, which may be written as

$$\dot{q}_4 = \frac{u_2}{u_3}\left(\lambda + \frac{1}{2u_3^2} + \frac{4q_4}{k^2} + 2q_1\right) = q_2 Y, \tag{4.1.80}$$

where $Y = \lambda + 1/2u_3^2 + 4q_4/k^2 + 2q_1$. Recall that q_1 is monotone decreasing and $u_3 > 0, \dot{u}_3 > 0$ sufficiently close to τ_M, say, on an interval (τ_0, τ_M). Suppose there exists $\tau_1 \in (\tau_0, \tau_M)$ such that $\dot{q}_4(\tau_1) = 0$. Then we must have $Y(\tau_1) = 0$, since $q_2(\tau_1) > 0$. It is easily shown that this gives $\ddot{q}_4(\tau_1) = q_2(\tau_1)\dot{Y}(\tau_1)$. Moreover, since $\dot{Y} = -\dot{u}_3/u_3^3 + 4\dot{q}_4/k^2 + 2\dot{q}_1$, we have $\dot{Y}(\tau_1) = -2\dot{u}_3(\tau_1)/u_3^3(\tau_1) + 2\dot{q}_1(\tau_1) < 0$, and thus $\ddot{q}_4(\tau_1) < 0$. So $\dot{q}_4$ can only cross zero in (τ_0, τ_M) with negative slope, i.e. it may only change sign once on (τ_0, τ_M). Therefore, q_4 must be monotone close to τ_M and, therefore, $\lim_{\tau\to\tau_M^-} q_4$ exists. Suppose $\lim_{\tau\to\tau_M^-} q_4 \neq 0$. Then, since q_4 is positive, there must exist some $\varepsilon > 0, \delta > 0$ such that $u_2 > \varepsilon u_3^2 > 1/4$ for $\tau_M - \tau < \delta$. Using this, $u_1/u_3 > \ell$ and (4.1.72) produces

$$\dot{q}_1 < -\frac{\ell}{4u_3} + \frac{2\varepsilon u_1}{k^2}, \tag{4.1.81}$$

for $\tau \in (\tau_M - \delta, \tau_M)$. It follows that

$$\lim_{\tau\to\tau_M^-} q_1 < \lim_{\tau\to\tau_M^-} \left(q_1(\tau_M - \delta) + \int_{\tau_M-\delta}^{\tau} \left(\frac{2\varepsilon u_1}{k^2} - \frac{\ell}{4u_3} \right) d\tau' \right) = -\infty, \tag{4.1.82}$$

using Lemma 4.1.17 and $\lim_{\tau\to\tau_M^-} \ell/u_3 = 0$. This contradicts the fact the ℓ is finite and so we have $\lim_{\tau\to\tau_M^-} q_4 = 0$. Taking the limit of (4.1.79) then yields

$$\omega\ell^2 + k^2\ell - \frac{k^2}{2} = 0, \tag{4.1.83}$$

where $\omega = \lim_{\tau\to\tau_M^-}(1 - \dot{R}^{-2})$. Note that $\omega \leq 0$ gives $\ell \geq 1/2$ which contradicts $\ell < 0$. We then have

$$\ell = -\frac{k^2}{2\omega} - \sqrt{\frac{k^4}{4\omega^2} + \frac{k^2}{2\omega}}, \tag{4.1.84}$$

since the upper root of (4.1.84) is positive and, therefore, not allowed. Clearly $\omega < 1$, so $\ell < -k^2/2 - \sqrt{k^4/4 + k^2/2} < -k^2 - 1/4$, if $k^2 > 1/4$. □

Lemma 4.1.19. *For $V_0 > 0$, $\lambda > 0$, suppose that $\tau_M < \infty$ and*

$$\lim_{\tau\to\tau_M^-} u_1 = -\infty, \qquad \lim_{\tau\to\tau_M^-} u_3 = +\infty. \tag{4.1.85}$$

Then $\lim_{\tau\to\tau_M^-} \mathcal{R} = \infty$ and $\lim_{\tau\to\tau_M^-} \mu < +\infty$, i.e. there exists a singularity at $\tau = \tau_M$ which is reached by outgoing null rays in finite affine time.

Proof. Lemma 4.1.18 and $\lambda > 0$ give $\lim_{\tau\to\tau_M^-} u_2 = \lim_{\tau\to\tau_M^-} l = \infty$. Using equation (4.1.20) we have

$$\lim_{\tau\to\tau_M^-}\mathcal{R}=\frac{k^2e^{\tau_M/2-c_1}}{2|u|}\lim_{\tau\to\tau_M^-}\left(e^{-k^2l}u_3\right)^{1/2}u_3^{3/2}. \tag{4.1.86}$$

Define $Z=e^{-k^2l}u_1$, which satisfies

$$\dot{Z}=e^{-k^2l}\left(\frac{1}{4}-u_2-u_1^2-k^2u_1u_3\right)<\left(-u_1-k^2u_3\right)Z, \tag{4.1.87}$$

for $u_2>1/4$. Using Lemma 4.1.18, we may choose some $\tau_*\in(0,\tau_M)$ such that $u_1/u_3<-k^2-1/8$ and $Z<0$ for all $\tau\in(\tau_*,\tau_M)$. We then have $\dot{Z}/Z>u_3/8>-u_1/8\ell$ for all $\tau\in(\tau_*,\tau_M)$. Integrating and using Lemma 4.1.17 then proves $\lim_{\tau\to\tau_M^-}Z=-\infty$. Hence, $\lim_{\tau\to\tau_M^-}e^{-k^2l}u_3=\ell^{-1}\lim_{\tau\to\tau_M^-}Z=+\infty$, which gives $\lim_{\tau\to\tau_M^-}\mathcal{R}=+\infty$, by (4.2.13).
Recall that μ parameterises the outgoing radial null geodesics. To show that $\lim_{\tau\to\tau_M^-}\mu<+\infty$, note that $e^{-k^2l}u_1=Z<Z(\tau_*)$ for some $\tau_*\in(0,\tau_M)$. This gives $e^{k^2l}<u_1/Z(\tau_*)$, which in turn gives $e^{k^2l/2}<(u_1/Z(\tau_*))^{1/2}$. Now let $p=(-u_1)^{1/2}$ and consider

$$\dot{p}=\frac{1}{2p}\left(-\frac{1}{4}+u_2+u_1^2\right)>\frac{p^3}{2}, \tag{4.1.88}$$

for $u_2>1/4$. Dividing by p^2 and integrating we have

$$\int_{\tau_*}^{\tau}\frac{\dot{p}}{p^2}\,d\tau'=\frac{1}{p(\tau_*)}-\frac{1}{p}>\int_{\tau_*}^{\tau}\frac{p}{2}\,d\tau'. \tag{4.1.89}$$

Using equation (4.1.18) we have

$$\begin{aligned} C\mu=\int_0^{\tau}e^{k^2l/2-\tau'/2}d\tau'&<\int_0^{\tau_*}e^{k^2l/2}d\tau'+\int_{\tau_*}^{\tau}\left(\frac{u_1}{Z(\tau_*)}\right)^{1/2}d\tau' \\ &=\int_0^{\tau_*}e^{k^2l/2}d\tau'+\int_{\tau_*}^{\tau}\frac{p}{(-Z(\tau_*))^{1/2}}d\tau'. \end{aligned} \tag{4.1.90}$$

Taking the limit and using (4.1.89) we find that $\lim_{\tau\to\tau_M^-}\mu<+\infty$. □

Proposition 4.1.7. *For $V_0>0,\lambda>0,u_2(0)\neq k^2/8$ we have $\tau_M<\infty$ and there is a curvature singularity at τ_M, which is reached by outgoing null rays in finite affine time.*

Proof. In the subcase $u_2(0)>k^2/8$, Proposition 4.1.5 and Lemma 4.1.16

give the result. If $u_2(0) < k^2/8$ Proposition 4.1.6 and Lemma 4.1.19 give the result. $\square$

4.2 Exact solutions

4.2.1 $k^2 = 2$

In this case we have $\lambda = 0$ which gives us constant potential $V = V_0$. We then have

$$\ddot{R} = \frac{d}{d\tau} R\dot{l} = \left(\frac{1}{4} - V_0\right) R. \tag{4.2.1}$$

Proposition 4.2.1. *If $k^2 = 2$ and $0 < V_0 \leq 1/4$, then there is a curvature singularity along $\mathcal{N}_-$ which is reached by outgoing radial null rays in finite affine time.*

Proof. In the case $V_0 < 1/4$, solutions of (4.2.1) in terms of S are given by

$$S = \upsilon^{-1} e^{-\tau/2} \sinh(\upsilon\tau), \qquad l = l_0 + \log\left[\frac{1}{2}(1 + \cosh(\upsilon\tau))\right], \tag{4.2.2}$$

where $\upsilon = \sqrt{1/4 - V_0}$. Clearly $\tau_M = +\infty$. We also have

$$\lim_{\tau\to\infty} \dot{l} = \lim_{\tau\to\infty} \frac{\upsilon \sinh(\upsilon\tau)}{1 + \cosh(\upsilon\tau)} = \upsilon. \tag{4.2.3}$$

For $k^2 = 2$ we then have

$$\lim_{\tau\to\infty} \mathcal{R} = \lim_{\tau\to\infty} \frac{e^{\tau/2 - l}}{|u|} \left(\frac{1}{4} - \dot{l}^2 + 4V_0\right) = \lim_{\tau\to\infty} \frac{10 V_0 e^{\tau/2 - l_0}}{|u|(1 + \cosh(\upsilon\tau))}. \tag{4.2.4}$$

The solution to the geodesic equation (4.1.18) reduces to

$$\frac{1}{2}\int_0^\tau e^{l_0 - \tau'/2}(1 + \cosh(\upsilon\tau)')d\tau' = C\mu. \tag{4.2.5}$$

Note that $V_0 > 0$ gives $\upsilon < 1/2$ for which

$$\lim_{\tau\to\infty} S = 0, \qquad \lim_{\tau\to\infty} \mathcal{R} = \infty, \qquad \lim_{\tau\to\infty} \mu < \infty. \tag{4.2.6}$$

For $V_0 = 1/4$ we have

$$S = \tau e^{-\tau/2}, \qquad l = l_0, \qquad \mathcal{R} = \frac{5 e^{\tau/2 - l_0}}{4|u|}, \tag{4.2.7}$$

which give

$$\lim_{\tau\to\infty} S = 0, \qquad \lim_{\tau\to\infty} \mathcal{R} = \infty, \qquad \lim_{\tau\to\infty} \mu < \infty. \tag{4.2.8}$$

□

Proposition 4.2.2. *If $k^2 = 2$, $V_0 < 0$, then $\mathcal{N}_-$ corresponds to radial null infinity and the Ricci scalar decays to zero there.*

Proof. In the case $\upsilon > 1/2\,(V_0 < 0)$, (4.2.2),(4.2.4) and (4.2.5) tell us that

$$\lim_{\tau\to\infty} S = \infty, \qquad \lim_{\tau\to\infty} \mathcal{R} = 0, \qquad \lim_{\tau\to\infty} \mu = \infty. \tag{4.2.9}$$

We remind the reader that we are not considering the case $V_0 = 0(\upsilon = 1/2)$. □

Proposition 4.2.3. *If $k^2 = 2$, $V_0 > 1/4$, there exists a curvature singluarity along $\tau = \pi/\bar{\upsilon}$ where $\bar{\upsilon} = \sqrt{V_0 - 1/4}$.*

Proof. In the case $V_0 > 1/4$, solutions to (4.2.1) are given by

$$S = \bar{\upsilon}^{-1} e^{-\tau/2} \sin(\bar{\upsilon}\tau), \qquad l = l_0 + \log\left[\frac{1}{2}(1 + \cos(\bar{\upsilon}\tau))\right]. \tag{4.2.10}$$

At $\bar{\upsilon}\tau = \pi$ we have $S = 0$ and

$$\lim_{\tau\to\pi/\bar{\upsilon}} l = -\infty, \qquad \lim_{\tau\to\pi/\bar{\upsilon}} \dot{l} = -\lim_{\tau\to\pi/\bar{\upsilon}} \bar{\upsilon}\tan\left(\frac{\bar{\upsilon}\tau}{2}\right) = -\infty, \tag{4.2.11}$$

which give $\lim_{\tau\to\pi/\bar{\upsilon}} \mathcal{R} = \infty$. □

4.2.2 $V_0 e^{\lambda l_0} = k^2/8$

Lemma 4.2.1. *u_3 is monotone in a neighbourhood of the axis.*

Proof. Note that there exists $\tau_1 \in (0, \tau_M)$ such that $u_1 > u_3$ and $u_1 > 0$ hold for $\tau \in (0, \tau_1)$. Suppose there exists $\tau_0 \in (0, \tau_1)$ with $\dot{u}_3(\tau_0) = 0$. We then have $\ddot{u}_3(\tau_0) = (u_1(\tau_0) - u_3(\tau_0))u_1(\tau_0)u_3(\tau_0)$, which has the same sign as $u_3(\tau_0)$, since $u_1(\tau_0) > u_3(\tau_0), u_1(\tau_0) > 0$. So either $u_3(\tau_0) < 0$ and is a local max, or $u_3(\tau_0) > 0$ and is a local min. Since $u_3(0) = 0$, in the former case we must then have $\tau_* \in (0, \tau_0)$ such that $u_3(\tau_*) < 0$ is a local min, which is contradiction. Similarly for the latter case. Hence, u_3 is monotone on $(0, \tau_1)$. □

Lemma 4.2.2. *If $u_2(0) = k^2/8$, then $u_2 = k^2/8$ and $u_3 = 0$ for all $\tau \in [0, \tau_M)$.*

Proof. First note that $u_2 = k^2/8, u_3 = 0$ is an invariant manifold of the system (4.1.2a)-(4.1.2c) with $\epsilon = -1$. The system (4.1.2a)-(4.1.2c) is not defined at $\tau = 0$ and so we must show that there exists $\tau_0 > 0$ such that $u_2(\tau_0) = k^2/8, u_3(\tau_0) = 0$. Using the preceding result, u_3 is monotone and, since $u_3(0) = 0$, has the same sign while $u_1 > u_3$ and $u_1 > 0$ hold. There must therefore exist τ_1 such that u_2 is monotone on $[0, \tau_1]$. It follows that $u_2 - k^2/8$ has the same sign on $(0, \tau_1)$. Suppose that $u_2 - k^2/8 > 0$ on $(0, \tau_1)$. We can choose τ_1 such that $R > 0$ on $(0, \tau_1)$. Then, using (4.1.6) and $R > 0$, $\dot{l} = u_3$ must be negative on $(0, \tau_1)$, which is a contradiction. A similar argument rules out $u_2 - k^2/8 < 0$ on $(0, \tau_1)$, so we have $u_2 - k^2/8 = 0$ for all $\tau \in (0, \tau_1)$. If u_2 is constant on $(0, \tau_1)$ then $u_3 = 0$ must also hold there. □

Proposition 4.2.4. *Recall $m = \sqrt{\lambda}/2$. If $V_0 e^{\lambda l_0} = k^2/8$ and $\lambda < 0$ then there is a singularity at $\tau = \infty$, which is reached by radial null rays in finite affine time. If $V_0 e^{\lambda l_0} = k^2/8$ and $\lambda > 0$ then there is a singularity at $\tau = \pi/m$, which is reached by radial null rays in finite affine time.*

Proof. Using the preceding result, we have $u_2 = k^2/8, u_3 = 0$, and thus $\ddot{R} = -\lambda R/4$, for all $\tau \in (0, \tau_M)$. The solutions in terms of S are

$$S = \begin{cases} m^{-1} e^{-\tau/2} \sin(m\tau), & \text{if} \quad \lambda > 0. \\ m^{-1} e^{-\tau/2} \sinh(m\tau), & \text{if} \quad \lambda < 0. \end{cases} \tag{4.2.12}$$

Note that the case $\lambda = 0, u_2(0) = k^2/8$ is precisely the case $k^2 = 2, V_0 = 1/4$ covered in Proposition 4.2.2. If $\lambda < 0$ then we clearly have $\tau_M = +\infty$. In this case we also have $m = \sqrt{1/4 - k^2/8} < 1/2$ and so $\lim_{\tau \to \infty} S = 0$. Using $\dot{l} = 0$,$V_0 e^{\lambda l} = k^2/8$ and (4.1.20) the Ricci scalar reduces to

$$\mathcal{R} = \frac{3k^2 e^{-k^2 l_0/2 + \tau/2 - c_1}}{8|u|}, \tag{4.2.13}$$

and it immediately apparent that $\lim_{\tau \to \infty} \mathcal{R} = +\infty$.
In the case $\lambda > 0$ we have $S(\pi/m) = 0$. In the cases studied thus far, surfaces characterised by $S = 0$, other than the regular axis, have been singular, which was demonstrated by an infinite curvature invariant. In this

case, however, it is clear from (4.2.13) above that $\mathcal{R}$ is finite if τ is finite, and one can check that this is the case for other invariants such as $\mathcal{T} = T^{ab}T_{ab}$ and the Kretschmann scalar $R^{abcd}R_{abcd}$. However, the specific length of the cylinders L limits to zero as $\tau \to \pi/m$, which violates the regular axis conditions. This may be seen solving (3.1.2c) for ϕ given the solutions for $R = e^{\tau/2}S$ given above, which yields

$$e^{\phi} = \frac{e^{\phi_0+\tau/4}}{\cos(m\tau/2)}. \tag{4.2.14}$$

Recalling $L = |u|e^{-\phi}$, we have $\lim_{\tau\to\pi/m} L = 0$. We speculate that we have a non-scalar curvature spacetime singularity at $\tau = \pi/m$ in this case. The solution to the geodesic equation (4.1.18) with $l = l_0$ shows that μ is finite for all $\tau > 0$ in both cases. □

4.3 Global structure of solutions in region I

In this section we gather the results from the two previous sections which give the global structure of solutions in region **I** for the entire parameter space. They are summarised by the following theorem and the corresponding spacetime diagrams are depicted in Fig. 4.1.

Theorem 4.3.1 (Global structure of solutions in the causal past of the scaling origin $\mathcal{O}$)**.** *For each $k, V_0, l_0 \in \mathbb{R}$, let $[0, \tau_M)$ be the maximal interval of existence for the unique solution of (4.1.2). The global structure of the solution is given by one of the following cases:*

Case 1. If $k^2 > 2, V_0 < 0$, then $\tau_M < \infty$ and the hypersurface $\tau = \tau_M$ corresponds to radial null infinity, with the Ricci scalar decaying to zero there.

Case 2. If $k^2 = 2$ and $V_0 < 0$, then $\tau_M = +\infty$ and $\mathcal{N}_-$ corresponds to radial null infinity, with the Ricci scalar decaying to zero there.

Case 3. If

$$(i) \quad k^2 < 2 \quad and \quad V_0 < 0,$$
$$(ii) \quad k^2 < 2, V_0 > 0 \quad and \quad V_0 e^{\lambda l_0} < k^2/8,$$

then $\tau_M = \infty$, the metric is regular on $\mathcal{N}_-$, which is reached in finite affine time along radial null rays. Hence, $\mathcal{N}_-$ exists as part of the spacetime.

Case 4. If

$$(i) \quad k^2 < 2, V_0 > 0 \quad and \quad V_0 e^{\lambda l_0} > k^2/8,$$
$$(ii) \quad k^2 > 2 \quad and \quad V_0 > 0,$$
$$(iii) \quad k^2 = 2 \quad and \quad V_0 > 1/4,$$

then $\tau_M < \infty$ and there is a singularity at $\tau = \tau_M$, with the radius of the cylinders of symmetry equal to zero there and which is reached by outgoing radial null rays in finite affine time.

Case 5. If

$$(i) \quad k^2 = 2 \quad and \quad 0 < V_0 \leq 1/4,$$
$$(ii) \quad k^2 < 2 \quad and \quad V_0 e^{\lambda l_0} = k^2/8,$$

then $\tau_M = +\infty$ and there is a curvature singularity on $\mathcal{N}_-$, with the radius of the cylinders of symmetry equal to zero there and which is reached by outgoing radial null rays in finite affine time.

Proof. The proof of cases 1 and 2 are given by Propositions 4.1.1 and 4.2.2, respectively. For case 3, part (i) is given by Proposition 4.1.2 and part (ii) is given by Proposition 4.1.3. Case 4 part (i) is given by Propositions 4.1.4, part (ii) is given by Propositions 4.1.5 and 4.2.4, and part (iii) by Proposition 4.2.3. Case 5 is proven by Propositions 4.2.1 and 4.2.4. □

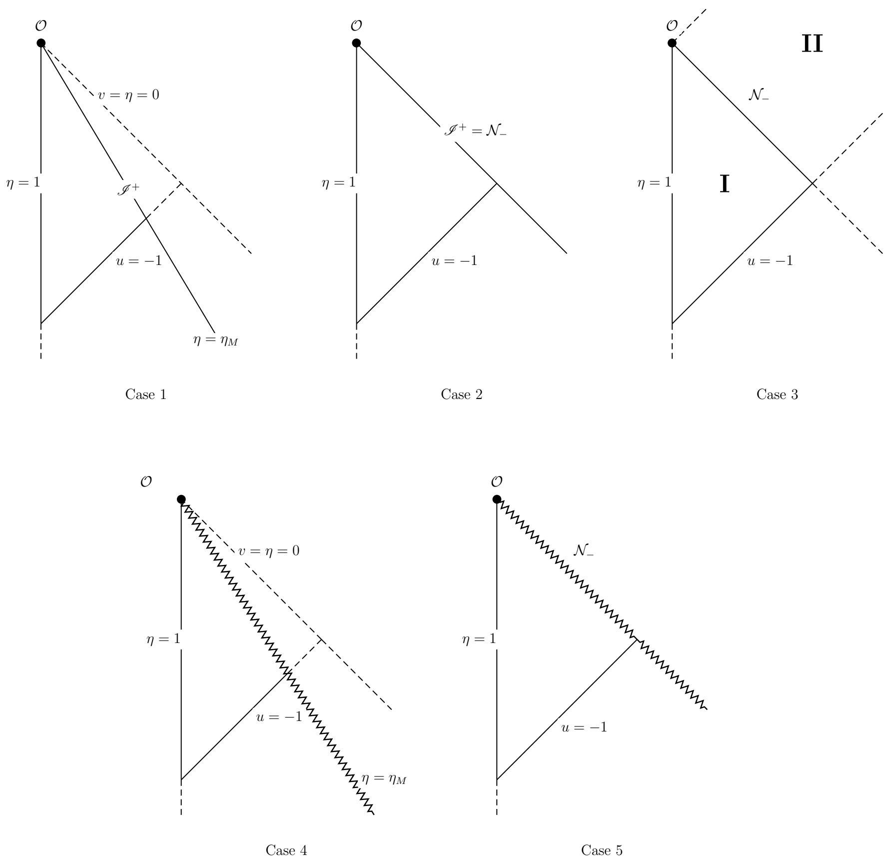

Figure 4.1: Global structure of the spacetime in region **I** for each subcase.

Chapter 5

The future of $\mathcal{N}_-$

In this chapter we investigate those solutions which are regular up to $\mathcal{N}_-$ and which may be extended to its future, that is, the solutions corresponding to values of $k^2 < 2, V_0 < 0$ and $k^2 < 2, V_0 > 0, V_0 e^{\lambda l_0} > k^2/8$. As in region **I**, we work with a rescaling of the independent variable, $t = \log(-\eta)$, which gives an autonomous set of field equations. $\mathcal{N}_-$ and $\mathcal{N}_+$ are singular points of the field equations and correspond to the limits $t \to -\infty$ and $t \to +\infty$, respectively. We first determine the asymptotic behaviour of solutions emanating from $\mathcal{N}_-$ and then given an analysis of the fixed points of the autonomous system. These fixed points are possible end states for solutions and give the behaviour of solutions on the surface $\mathcal{N}_+$, if it exists as part of the spacetime. It is shown that, for one of three available fixed points, if a solution evolves to this point then the metric is regular on $\mathcal{N}_+$, the future null cone of $\mathcal{O}$. A spacetime corresponding to that solutions would, therefore, admit a naked singularity.
However, we then examine the evolution of solutions emanating from $\mathcal{N}_-$ and show that in all cases the maximal interval of existence has a finite upper limit t_M and that there exists a spacelike curvature singularity along the surface $t = t_M$. Thus, no solution to the field equations evolves to any of the fixed points mentioned above, and the surface $\mathcal{N}_+$ does not exist as part of the spacetime, ruling out the possibility of a naked singularity solution.

5.1 Asymptotic behaviour of solutions at $\mathcal{N}_{-}$

Proposition 5.1.1. *Let* $t = \log(-\eta), \epsilon = \mathrm{sgn}(\mathrm{V_0}), \sigma(\mathrm{t}) = \mathrm{S}(\eta)$ *and*

$$\begin{aligned} x_0(t) &= \frac{e^{t/2}}{\sigma(t)}, & x_1(t) &= \frac{\eta S'(\eta)}{S} = \frac{\sigma'(t)}{\sigma(t)}, \\ x_2(t) &= |V_0| e^{\lambda l(\eta)}, & x_3(t) &= \eta l'(\eta) + \frac{1}{2} = \frac{dl}{dt} + \frac{1}{2}. \end{aligned} \tag{5.1.1}$$

Then x_0, x_1, x_2, x_3 *satisfy*

$$x_1'(t) = x_1 + \epsilon x_2 - x_1^2, \tag{5.1.2a}$$

$$x_2'(t) = |\lambda| \left(\frac{1}{2} - x_3 \right) x_2, \tag{5.1.2b}$$

$$x_3'(t) = \frac{x_3}{2} + \frac{x_1}{2} + \epsilon \frac{2x_2}{k^2} - x_1 x_3, \tag{5.1.2c}$$

$$x_1^2 - x_0^2 - \left(\frac{k^2}{2} + 1 \right) x_1 - \frac{k^2 x_3^2}{2} + k^2 x_1 x_3 - 2\epsilon x_2 = 0, \tag{5.1.2d}$$

$$\lim_{t \to -\infty} (x_0, x_1, x_2, x_3) = (0, 0, 0, 0). \tag{5.1.2e}$$

Proof. First note that (5.1.2b) comes directly from the definitions of x_2 and x_3. Given $f(\eta)$, defining $F(t) = f(\eta)$ yields $\eta f'(\eta) = F'(t)$ and $\eta^2 f''(\eta) = F''(t) - F'(t)$. Equations (5.1.2a) and (5.1.2c) follow directly from (2.4.24) and (2.4.26). Equation (2.4.25) is equivalent to

$$\frac{d\phi}{dt} = \frac{x_0 - x_1}{2}. \tag{5.1.3}$$

Differentiating (3.1.2a) with respect to t gives

$$2\frac{d\gamma}{dt} = -2\frac{d\phi}{dt} + \frac{k^2}{2}\frac{dl}{dt} - \frac{1}{2} = x_1 - x_0 + \frac{k^2 x_3}{2} - \frac{k^2}{4} - \frac{1}{2}. \tag{5.1.4}$$

Dividing (3.1.2c) by S, changing variables and replacing $d\gamma/dt$ and $d\phi/dt$ using (5.1.3) and (5.1.4) produces

$$x_1 \left(x_1 - x_0 + \frac{k^2 x_3}{2} - \frac{k^2}{4} - \frac{1}{2} \right) - \epsilon x_2 - \frac{1}{2}(x_0 - x_1)^2 = \frac{k^2 x_3^2}{4}. \tag{5.1.5}$$

Multiplying by 2 and simplifying gives (5.1.2d). It was shown in Chapter 4 that if $k^2 < 2$ then

$$\lim_{\eta\to 0+}\left(\eta l'(\eta), |V_0|e^{\lambda l}, \frac{\eta S'(\eta)}{S}\right) = \left(-\frac{1}{2}, 0, 0\right), \tag{5.1.6}$$

and that S is non-zero and finite at $\eta = 0$. The condition (5.1.2e) follows immediately. □

We note that the equations (5.1.2a)-(5.1.2c) subject to (5.1.2e) define a dynamical system and may be studied independently of (5.1.2d).

Proposition 5.1.2. *Let*

$$\mu_1 = x_1 + Ax_2, \qquad \mu_3 = x_3 + Bx_2, \tag{5.1.7a}$$

$$A = \epsilon\frac{4}{2+k^2}, \qquad B = \epsilon\frac{16}{k^2(2+k^2)}. \tag{5.1.7b}$$

Then μ_1, μ_3 satisfy

$$\mu_1' = \mu_1 - x_1^2 - A\lambda x_2 x_3, \tag{5.1.7c}$$

$$\mu_3' = \frac{\mu_3}{2} + \frac{\mu_1}{2} - x_1 x_3 - B\lambda x_2 x_3. \tag{5.1.7d}$$

Proof. It is straightforward to check that (5.1.7c),(5.1.7d) follow directly from (5.1.7a)(5.1.7b) and (5.1.2a)-(5.1.2c). □

Note that Theorem 4.1.2 may also be used in analysing limits as $t \to -\infty$ in an analagous way. We define the vector $\boldsymbol{x}$ by

$$\boldsymbol{x} = (x_1, x_2, x_3). \tag{5.1.8}$$

The system defined by (5.1.2a)-(5.1.2c) and (5.1.2e) satisfies the hypothesis of Theorem 4.1.2, which grants local existence of solutions near the origin of the $\boldsymbol{x}$-system, which is at $t = -\infty$. We denote by $(-\infty, t_M)$ the maximal interval of existence for a given solution.

Lemma 5.1.1. *For any $\varepsilon > 0$, there exists $T(\varepsilon) \in (-\infty, t_M)$ such that*

$$|x_i| < e^{(|\lambda|/2-\varepsilon)t}, \tag{5.1.9}$$

for $t < T(\varepsilon)$ and each $i \in \{1, 2, 3\}$.

Proof. The system defined by (5.1.2a)-(5.1.2c) is of the form (4.1.24), satisfying $F(0) = 0, \partial_{\boldsymbol{x}} F(0) = 0$, where the matrix

$$E = \begin{pmatrix} 1 & \epsilon & 0 \\ 0 & |\lambda|/2 & 0 \\ 1/2 & 2\epsilon/k^2 & 1/2 \end{pmatrix} \tag{5.1.10}$$

has 3 positive eigenvalues, $|\lambda|/2, 1/2$ and 1, of which $|\lambda|/2$ is the smallest. Solutions to the system (5.1.2a)-(5.1.2c) therefore exist, which satisfy (4.1.25) and (4.1.26). Using (4.1.26), for any $\varepsilon > 0$, there exists $T(\varepsilon) < 0$ such that

$$\log ||\boldsymbol{x}(t)|| < (|\lambda|/2 - \varepsilon)t, \tag{5.1.11}$$

for all $t < T(\varepsilon)$. Since $|x_i| < ||\boldsymbol{x}||$ for each $i \in \{1, 2, 3\}$, the result follows. □

Lemma 5.1.2. *For $\epsilon = 1\,(\epsilon = -1)$, there exists $T \in (-\infty, t_M)$ such that $x_1 < 0, x_3 < 0\,(x_1 > 0, x_3 > 0)$ for $t \in (-\infty, T)$.*

Proof. Using Lemma 5.1.1 we have $|x_i| = O(e^{(|\lambda|/2-\varepsilon)t})$ in the limit $t \to -\infty$, for any $\varepsilon > 0$. From (5.1.7c) we then have

$$\frac{d}{dt}\left(e^{-t}\mu_1\right) = -e^{-t}(x_1^2 + A\lambda x_2 x_3) = O(e^{(|\lambda|-2\varepsilon-1)t}) \quad \text{as} \quad t \to -\infty, \tag{5.1.12}$$

which may be integrated to give

$$\mu_1 = c_3 e^t + O(e^{(|\lambda|-2\varepsilon)t}) = O(e^{(|\lambda|-2\varepsilon)t}), \tag{5.1.13}$$

and so

$$x_1 = -Ax_2 + O(e^{(|\lambda|-2\varepsilon)t}) \quad \text{as} \quad t \to -\infty, \tag{5.1.14}$$

for some constant c_3, since $|\lambda| - 2\varepsilon < 1$. A similar process using (5.1.7d) yields $\mu_3 = O(e^{\tilde{m}t})$, which gives

$$x_3 = -Bx_2 + O(e^{\tilde{m}t}) \quad \text{as} \quad t \to -\infty, \tag{5.1.15}$$

where $\tilde{m} = \min\{1/2, |\lambda| - 2\varepsilon\}$. Since $\lim_{t\to-\infty} x_3 = 0$, we may choose $T(\varepsilon)$ such that $|\lambda x_3| < \varepsilon$ for $t < T(\varepsilon)$. We then have

$$\frac{x_2'}{x_2} < \frac{|\lambda|}{2} + \varepsilon, \qquad \text{for } t \in (-\infty, T(\varepsilon)).$$

Integrating over $[t, T]$ shows that $x_2(t) > x_2(T)e^{(|\lambda|/2+\varepsilon)(t-T)}$ on the same interval. Choosing ε such that $|\lambda|/2+\varepsilon < \min\{1/2, |\lambda|-2\varepsilon\}$ shows that the x_2 terms in equations (5.1.14) and (5.1.15) are dominant for t sufficiently close to $-\infty$. T may be then chosen, without loss of generality, such that x_1 and x_3 have the same sign as $-Ax_2$ and $-Bx_2$ on $(-\infty, T)$, respectively. Note from (5.1.7b) that A and B have the same sign as ϵ. □

Proposition 5.1.3. *There exists* $c_4 > 0$ *such that*

$$\lim_{t\to-\infty} e^{-|\lambda|t/2}\boldsymbol{x} = c_4\,(A, 1, B)\,. \tag{5.1.16}$$

Proof. Integrating (5.1.2b) over $[t, T]$ we have

$$e^{-|\lambda|t/2}x_2(t) = e^{-|\lambda|T/2}x_2(T) + \int_t^T e^{-|\lambda|t'/2}|\lambda|x_2x_3\,dt'. \tag{5.1.17}$$

Consider the case $\epsilon = -1$. By Lemma 5.1.2 we have $x_3 > 0$ on $t \in (-\infty, T)$, and by choosing T sufficiently small such that the bounds of Lemma 5.1.1 hold, we have

$$e^{-|\lambda|T/2}x_2(T) < e^{-|\lambda|t/2}x_2 < e^{-|\lambda|T/2}x_2(T) + \int_t^T |\lambda|e^{(|\lambda|/2-2\varepsilon)t'}\,dt'. \tag{5.1.18}$$

The integral here is finite in the limit as $t \to -\infty$ for $\varepsilon < |\lambda|/4$ and so $e^{-|\lambda|t/2}x_2$ has positive and finite upper and lower bounds in the limit as $t \to -\infty$. It also monotone for $t < T$ since $x_3 > 0$ there, and so we have $\lim_{t\to-\infty} e^{-|\lambda|t/2}x_2 = c_4$, for some $c_4 > 0$. A similar argument gives this result in the case $\epsilon = 1$. Multiplying (5.1.14) and (5.1.15) by $e^{-|\lambda|t/2}$ and taking the limit $t \to -\infty$ gives $\lim_{t\to-\infty} \boldsymbol{x} = c_3(A, 1, B)$. □

Remark For convenience, we define t_* by $c_4 = e^{-|\lambda|t_*/2}$. Notice then that the result of Proposition 5.1.3 may be written as $\lim_{\bar{t}\to-\infty} e^{-|\lambda|\bar{t}/2}\boldsymbol{x} = (A, 1, B)$. Noting that (5.1.2a)-(5.1.2d) is invariant under translations of the independent variable we drop the bar and let $\bar{t} = t$. Hence

$$\lim_{t\to-\infty} e^{-|\lambda|t/2}\boldsymbol{x} = (A, 1, B)\,. \tag{5.1.19}$$

This describes the asymptotic behaviour of solutions to the future of $\mathcal{N}_-$, as they emerge from $\mathcal{N}_-$.

5.2 Analysis of fixed points

Proposition 5.2.1. *Let*

$$P_1 = (1, 0, 1), \quad P_2 = \left(\alpha_-, \frac{k^2}{8}, \frac{1}{2}\right), \quad P_3 = \left(\alpha_+, \frac{k^2}{8}, \frac{1}{2}\right), \tag{5.2.1}$$

where

$$\alpha_\pm = \frac{1 \pm \sqrt{|\lambda|}}{2}. \tag{5.2.2}$$

If $\epsilon = 1$ then the equilibria of the system (5.1.2a)-(5.1.2c) are P_1 and the origin. If $\epsilon = -1$ then the equilibria are P_1, P_2, P_3 and the origin.

Proof. This is straightforward to check. □

Proposition 5.2.2. *Define $F(\boldsymbol{x})$ by setting $\boldsymbol{x}'(t) = F(\boldsymbol{x})$, where the component equations are given by (5.1.2a)-(5.1.2c). Let*

$$\boldsymbol{y}(s) = (y_1, y_2, y_3), \qquad \begin{aligned} y_1(s) &= 1 - x_1(t) \\ y_2(s) &= x_2(t) \\ y_3(s) &= 1 - x_3(t) \end{aligned} \qquad s = -t. \tag{5.2.3}$$

Then

$$\boldsymbol{y}'(s) = F(\boldsymbol{y}). \tag{5.2.4}$$

Proof. This is straightforward to check using (5.1.2a)-(5.1.2c). □

Proposition 5.2.3. *Suppose $\lim_{t\to\infty} \boldsymbol{x} = P_1$. Then*

$$\lim_{t\to\infty} e^{|\lambda|t/2}(1 - x_1, x_2, 1 - x_3) = c_5(A, 1, B), \tag{5.2.5}$$

for some constant $c_5 > 0$.

Proof. First note that $\lim_{t\to\infty} \boldsymbol{x} = P_1$ is equivalent to $\lim_{s\to-\infty} \boldsymbol{y} = (0, 0, 0)$. Since $\boldsymbol{y}'(s) = F(\boldsymbol{y})$, solutions emanating from the origin of the y-system satisfy the exact conditions satisfied by solutions emanating from the origin of the x-system used in the proofs of section 5.1. We may, therefore, carry out an identical analysis to find

$$\lim_{s\to-\infty} e^{-|\lambda|s/2}\boldsymbol{y} = c_5(A, 1, B), \tag{5.2.6}$$

which is our result. □

Proposition 5.2.4. *Let $k^2 < 2$ and suppose that $\lim_{t\to\infty} \boldsymbol{x} = P_1$. Then the metric is regular in the limit approaching $\mathcal{N}_+$ along lines of constant v. Furthermore, if $k^2 \neq \sqrt{3} - 1$ then $\lim_{t\to\infty} \mathcal{R} \neq 0$.*

Proof. Using Proposition 5.2.3, for any $\varepsilon > 0$ there exists $T(\varepsilon)$ such that

$$1 - (Ac_5 + \varepsilon)e^{-|\lambda|t/2} < x_1 < 1 - (Ac_5 - \varepsilon)e^{-|\lambda|t/2}, \tag{5.2.7}$$

for $t > T(\varepsilon)$. Recalling $x_1 = \sigma'/\sigma$, it is straightforward to show that this leads to

$$c_6 < e^{-t}\sigma < c_7, \qquad t > T(\varepsilon), \tag{5.2.8}$$

for positive constants c_6, c_7. We also have

$$\frac{d}{dt}(e^{-t}\sigma) = e^{-t}\sigma\,(x_1 - 1)\,. \tag{5.2.9}$$

Using the asymptotic behaviour of solutions at P_1 derived in Proposition 5.2.3 shows that $e^{-t}\sigma$ is monotone near P_1. Hence, $e^{-t}\sigma$ has a finite, positive limit as $t \to \infty$. In region **II** of the spacetime we have $v > 0, u < 0$ and thus $|u| = -v/\eta = ve^{-t}$, and so $r = |u|\sigma = ve^{-t}\sigma$ has a positive finite limit approaching $\mathcal{N}_+$ $(u = 0)$ along lines of constant v. Combining (5.1.3),(5.2.7) and (5.2.8) we have

$$\frac{1}{c_6 e^{t/2}} + (Ac_5 - \varepsilon)e^{-|\lambda|t/2} < 2\frac{d\phi}{dt} + 1 < \frac{1}{c_7 e^{t/2}} + (Ac_5 + \varepsilon)e^{-|\lambda|t/2}, \tag{5.2.10}$$

for $t > T(\varepsilon)$. We see that if $\varepsilon \leq Ac_5$ then $2\phi + t$ is monotone in t. Integrating and taking exponentials then shows that $\lim_{t\to\infty} e^{2\phi+t}$ exists and is non-zero and finite. Hence, $\lim_{t\to\infty} |u|e^{-2\phi} = \lim_{t\to\infty} ve^{-2\phi-t}$ is non-zero and finite. So far we have shown that $g_{\theta\theta} = |u|e^{2\phi}r^2$ and $g_{zz} = |u|e^{2\phi}$ have non-zero, finite limits as $t \to +\infty$. Using similar arguments, it may shown that the metric component $|u|^{-1}e^{2\gamma+2\phi}$ behaves like $e^{(1-|\lambda|/2)t}$ in the limit as $t \to +\infty$ and, therefore, has limit $+\infty$. However, by making the coordinate transformation $\bar{u} = -2|u|^{|\lambda|/2}/|\lambda|$ we avoid this problem. The corresponding metric component in this coordinate system is $2|\bar{u}|^{-1}e^{2\gamma+2\phi}/|\lambda|$ and it may be shown in a similar fashion that this has a non-zero, finite limit as $t \to +\infty$. Following (4.1.20), the Ricci scalar may be written as

$$\mathcal{R} = \frac{e^{-k^2 l/2 + t/2 - c_1}}{v}\left(\frac{k^2}{2}(1 - x_3)\,x_3 - 4\epsilon x_2\right). \tag{5.2.11}$$

It may be shown, using (5.2.5) in a similar way, that for $t > T(\epsilon)$ we have

$$c_8 e^{|\lambda| t/2} < e^{-k^2 l/2 + t/2 - c_1} < c_9 e^{|\lambda| t/2}, \tag{5.2.12}$$

for positive constants c_8, c_9. It also follows from (5.2.5) that

$$\begin{aligned} \lim_{t\to\infty} e^{|\lambda| t/2} \left(\frac{k^2}{2} (1 - x_3) x_3 - 4\varepsilon x_2 \right) &= \frac{k^2 B c_5}{2} - 4\epsilon c_5 \\ &= \left(\frac{8}{k^2(2+k^2)} - 4 \right) \epsilon c_5 \neq 0, \end{aligned} \tag{5.2.13}$$

for $k^2 \neq \sqrt{3} - 1$. Combining this with (5.2.11) and (5.2.12) shows that $\lim_{t\to\infty} \mathcal{R}$ is non-zero and finite for all $k^2 < 2$ with $k^2 \neq \sqrt{3} - 1$. □

This result shows that in spacetimes where the solutions to the field equations satisfy $\lim_{t\to\infty} \boldsymbol{x} = P_1$, the future null cone of the singularity $\mathcal{N}_+$ is regular and exists are part of the spacetime, thus rendering the singularity at the origin naked. However, it is shown in later sections that none of the solutions actually do evolve to P_1.

Proposition 5.2.5. *If* $\lim_{t\to\infty} \boldsymbol{x} = P_2$ *or* $\lim_{t\to\infty} \boldsymbol{x} = P_3$*, then* $\lim_{t\to\infty} r = 0$ *and* $\lim_{t\to\infty} \mathcal{R} = +\infty$*, where* r *is the radius of the cylinders and* $\mathcal{R}$ *is the Ricci scalar.*

Proof. If $\lim_{t\to\infty} x_1 = \alpha_\pm$ then for any $\varepsilon > 0$ there exists $T(\varepsilon)$ such that $x_1 < \alpha_+ + \varepsilon$ for $t > T(\varepsilon)$, since $\alpha_- < \alpha_+$. Note that $\sqrt{|\lambda|} = \sqrt{1 - k^2/2} < 1 - k^2/4$, which gives $\alpha_+ < 1 - k^2/8$. This leads to $\sigma < \sigma(T) e^{(1-k^2/8+\varepsilon)(t-T)}$ for $t > T$. It follows that $r = |u|\sigma < v\sigma(T) e^{(-k^2/8+\varepsilon)(t-T)}$ for $t > T$. Choosing $\varepsilon < k^2/8$ shows that $\lim_{t\to\infty} r = 0$, for $v \in (0, \infty)$. It is straightforward to show that $\lim_{t\to\infty} e^{-k^2 l/2 + t/2} = +\infty$ follows from $\lim_{t\to\infty} l'(t) = 0$, which is equivalent to $\lim_{t\to\infty} x_3 = 1/2$. Then using $\lim_{t\to\infty} x_2 = k^2/8$ and $\epsilon = -1$ we find that $\lim_{t\to\infty} \mathcal{R} = +\infty$. □

Proposition 5.2.6. *Let* $\epsilon = -1$*. Then there is no solution of (5.1.2a)-(5.1.2d) which satisfies* $\lim_{t\to\infty} \boldsymbol{x} = P_2$*.*

Proof. First note that x_0 satisfies

$$x_0' = x_0 \left(\frac{1}{2} - x_1 \right). \tag{5.2.14}$$

If $\lim_{t\to\infty} x_1 = 1/2 - \sqrt{|\lambda|}/2$ then $x_0' \sim \sqrt{|\lambda|}/2$ as $t \to +\infty$. Since $x_0 > 0$ we must have $\lim_{t\to\infty} x_0 = +\infty$. This contradicts $\lim_{t\to\infty} \boldsymbol{x} = P_2$ and (5.1.2d). $\square$

Comment 5.2.1. *We note that* $\lim_{t\to t_M^-} \boldsymbol{x} = P_1$ *or* P_3 *are consistent with (5.1.2d).*

5.3 Evolution of solutions

Lemma 5.3.1. *For* $V_0 < 0$*, suppose there exists* $t_0 \in (-\infty, t_M)$ *such that* $x_3(t_0) = 1/2$*. Then* $x_2(t_0) < k^2/8$ *and* $x_3 > 1/2$*,* $x_2 \leq k^2/8$ *hold for all* $t \in (t_0, t_M)$*.*

Proof. First note that $x_3 = 1/2, x_2 = k^2/8$ defines an invariant manifold of the system, so if $x_3(t_0) = 1/2$, $x_2(t_0) = k^2/8$, then we would have $x_3 = 1/2, x_2 = k^2/8$ for all $t \in (-\infty, t_M)$, which is clearly not the case. Moreover, at $x_3 = 1/2$ we have

$$x_3' = \frac{1}{4} - \frac{2x_2}{k^2}, \tag{5.3.1}$$

and so x_3 cannot reach 1/2 from below if $x_2 > k^2/8$. Hence, we must have $x_2(t_0) < k^2/8$ and $x_3'(t_0) > 0$. Equation (5.3.1) also shows that x_3 cannot cross 1/2 from above if $x_2 < k^2/8$. Given that x_2 is decreasing if $x_3 > 1/2$, we must have $x_3 > 1/2, x_2 < k^2/8$, and $\dot{x}_2 < 0$ for all $t \in (t_0, t_M)$. $\square$

This lemma highlights the importance of $x_3(t_0) = 1/2$ for some t_0. If such a t_0 exists then $x_3 > 1/2$ is preserved for all $t \in (t_0, t_M)$ and x_2 is monotonically decreasing there.

Lemma 5.3.2. *Let* $i \in \{1, 2, 3\}$*. Then each* x_i *is monotone in the limit as* $t \to t_M^-$*. Hence, either* $\lim_{t\to t_M^-} x_i$ *exists or* $\lim_{t\to t_M^-} x_i = \pm\infty$*.*

Proof. Lemma 5.3.1 tells us that if $\epsilon = -1$, then $x_3 - 1/2$ can only change sign once. If $\epsilon = 1$, then at $x_3 = 1/2$ we have $x_3' = 1/4 + 2x_2/k^2 > 0$, so $x_3 - 1/2$, and thus x_2', can only change sign once in this case also. At $x_1' = 0$ we have $x_1'' = \epsilon x_2'$, which means that x_1' can only change sign twice. At $x_1 - x_3 = 0$ we have $x_1' - x_3' = \epsilon(1 - 2/k^2)x_2$ which always has the same sign, specifically, the opposite sign to ϵ. Hence, $x_1 - x_3$ can only change sign once also. Now, at $x_3' = 0$ we have

$$x_3'' = \left(\frac{1}{2} - x_3\right)\left(\frac{1}{2} - x_1\right)(x_1 - x_3)\,. \tag{5.3.2}$$

The righthand side here may only change sign a finite number of times. Hence, x_3' eventually becomes fixed in sign and x_3 becomes monotone. □

Lemma 5.3.3. *Let $t_0 \in (-\infty, t_M)$ be such that $\sigma(t_0) \neq 0$. Then*

$$\lim_{t\to t_M^-} \sigma = 0 \quad \Leftrightarrow \quad \lim_{t\to t_M^-} \int_{t_0}^{t} x_1\, dt' = -\infty. \tag{5.3.3}$$

Proof. By the definition of x_1 we have

$$\sigma = \sigma(t_0) \exp\left(\int_{t_0}^{t} x_1\, dt'\right). \tag{5.3.4}$$

The result immediately follows. □

Lemma 5.3.4. *If $V_0 < 0\,(V_0 > 0)$ then $x_1 < 1\,(x_1 < 0)$ for all $t \in (-\infty, t_M)$.*

Proof. If $\epsilon = -1$ then it follows directly from (5.1.2a) that x_1 cannot cross 1 from below. If $\epsilon = 1$ then by Lemma 5.1.2 we have $x_1 < 0$ on an initial interval, say $(-\infty, t_0)$. Now suppose that $x_1(t_0) = 0$. It is clear that $x_0' > 0$ on $(-\infty, t_0)$ and so $x_0(t_0) > 0$. At t_0, (5.1.2d) with $\epsilon = 1$ then reduces to

$$-x_0^2(t_0) - \frac{k^2 x_3^2(t_0)}{2} - 2x_2(t_0) = 0, \tag{5.3.5}$$

which clearly contradicts $x_0(t_0) > 0$. Hence, no such t_0 exists. □

Lemma 5.3.5. *If t_M is finite and $\lim_{t\to t_M^-} x_1 = -\infty$ then $\lim_{t\to t_M^-} \sigma = 0$.*

Proof. If $\lim_{t\to t_M^-} x_1 = -\infty$ then either $\lim_{t\to t_M^-} \sigma' = -\infty$ or $\lim_{t\to t_M^-} \sigma = 0$. Writing (2.4.24) in terms of t, x_2 and σ gives

$$\sigma'' = \sigma' + \epsilon x_2 \sigma. \tag{5.3.6}$$

If $\epsilon = 1$ then we have $\sigma'' > \sigma'$, which rules out $\lim_{t\to t_M^-} \sigma = -\infty$, since t_M is finite. Suppose that $\epsilon = -1$, x_2 is bounded above by a constant b for all $t \in (-\infty, t_M]$ and let σ_M be the maximum of σ. Then we have

$\sigma'' > \sigma' - b\sigma_M$, which also rules out $\lim_{t\to t_M^-} \sigma - \infty$.

Now let $\tilde{X} = x_1 - k^2 x_3/2$. If $\tilde{X} > 0$ then

$$\tilde{X}' = \frac{|\lambda| x_1}{2} - \left(\frac{1}{2} - x_1\right)\tilde{X} < \frac{|\lambda|}{2} + \frac{\tilde{X}}{2}, \tag{5.3.7}$$

using $x_1 < 1$, which holds by Lemma 5.3.4. So on intervals where $\tilde{X} > 0$ it is sub-exponential. For finite t then, it is bounded above by a constant $\hat{b}$. This gives

$$\frac{x_2'}{x_2} < |\lambda| \left(\frac{1}{2} + \frac{2\hat{b}}{k^2} - \frac{2x_1}{k^2}\right). \tag{5.3.8}$$

Integrating shows that if x_2 is unbounded above then the integral of x_1 is unbounded below, which gives $\lim_{t\to t_M^-} \sigma = 0$ by Lemma 5.3.3. □

We note that these last two results tell us that $\lim_{t\to t_M^-} \sigma = 0$ if and only if $\lim_{t\to t_M^-} x_1 = -\infty$.

Lemma 5.3.6. *If t_M is finite and $\lim_{t\to t_M^-} x_1 = -\infty$, then $\lim_{t\to t_M^-} x_3 = \pm\infty$.*

Proof. Note that if $|x_3|$ is bounded on $(-\infty, t_M)$, then x_2 is also bounded. Integrating (5.1.2c) we have

$$x_3 = e^{(t-t_0)/2} x_3(t_0) + \int_{t_0}^{t} e^{(t-t')/2} \left(\left(\frac{1}{2} - x_3\right) x_1 + \epsilon \frac{2x_2}{k^2}\right) dt', \tag{5.3.9}$$

for any $t_0 \in (-\infty, t_M)$. Given that $\lim_{t\to t_M^-} x_1 = -\infty$, we have

$$\lim_{t\to t_M^-} \int_{t_0}^{t} x_1 \, dt' = -\infty, \tag{5.3.10}$$

by Lemmas 5.3.3 and 5.3.4. By Lemma 5.3.2 we either have $\lim_{t\to t_M^-} x_3 = \pm\infty$ or $\lim_{t\to t_M^-} x_3 = h \in \mathbb{R}$. Suppose that $\lim_{t\to t_M^-} x_3 = h \neq 1/2$. Then $\lim_{t\to t_M^-} x_2 < \infty$ and by inspection of (5.3.9) with (5.3.10) we have $\lim_{t\to t_M^-} x_3 = \pm\infty$, which is a contradiction. We may rule out the case $\lim_{t\to t_M^-} x_3 = 1/2$ as follows. Consider

$$x_1 + x_0 = \frac{\sigma' + e^{t/2}}{\sigma}. \tag{5.3.11}$$

Since $\lim_{t\to t_M^-} \sigma = 0$, we either have $\lim_{t\to t_M^-} \sigma' + e^{t/2} = 0$ or $\lim_{t\to t_M^-} x_1 + x_0 = \pm\infty$. In the latter case, if $\lim_{t\to t_M^-} x_3 = 1/2$ then (5.1.2d) could not be satisfied, since $x_1^2 - x_0^2$ would then dominate and the left hand side would become infinite in the limit as $t \to t_M$. In the former case, using l'Hôpital's rule, we have

$$\lim_{t \to t_M^-} (x_1 + x_0) = \lim_{t \to t_M^-} \frac{2\sigma'' + e^{t/2}}{2\sigma'} = \lim_{t \to t_M^-} \frac{2\sigma' - 2x_2\sigma + e^{t/2}}{2\sigma'} = -\frac{1}{2}, \quad (5.3.12)$$

where we have used $\lim_{t \to t_M^-} \sigma' + e^{t/2} = 0, \lim_{t \to t_M^-} \sigma = 0$ and $\lim_{t \to t_M^-} x_2 < +\infty$. Taking the limit of (5.1.2d) then gives

$$-2 \lim_{t \to t_M^-} x_1 - \frac{1}{4} - \frac{k^2}{8} - 2\epsilon \lim_{t \to t_M^-} x_2 = 0, \quad (5.3.13)$$

where we have used $\lim_{t \to t_M^-} x_3 = 1/2$. This is a clear contradiction of $\lim_{t \to t_M^-} x_1 = -\infty$, $\lim_{t \to t_M^-} x_2 < +\infty$. □

Lemma 5.3.7. *If t_M is finite and $\lim_{t \to t_M^-} x_3 = \pm\infty$, then $\lim_{t \to t_M^-} x_1 = -\infty$ and $\lim_{t \to t_M^-} \sigma = 0$.*

Proof. If $\lim_{t \to t_M^-} x_3 = +\infty$ then $\lim_{t \to t_M^-} x_2 < \infty$, by (5.1.2b). By inspection of (5.1.2c), we see that x_1 must be unbounded below approaching t_M. Using Lemma 5.3.2, we then have $\lim_{t \to t_M^-} x_1 = -\infty$. Recall that $\tilde{X} = x_1 - k^2 x_3/2 < \hat{b}$ for all $t \in (-\infty, t_M)$. It follows immediately that if $\lim_{t \to t_M^-} x_3 = -\infty$, then $\lim_{t \to t_M^-} x_1 = -\infty$. □

Lemma 5.3.8. *If t_M is finite then*

$$\lim_{t \to t_M^-} x_1 = -\infty, \qquad \lim_{t \to t_M^-} x_3 = \pm\infty, \qquad \lim_{t \to t_M^-} \sigma = 0. \quad (5.3.14)$$

Proof. Using Theorem 4.1.1 and Lemma 5.3.2 we must have $\lim_{t \to t_M^-} |x_i| = +\infty$ for some $i \in \{1, 2, 3\}$. Given that if x_3 is bounded and t is finite then x_2 is bounded, and $x_1 < 1$ for all $t \in (-\infty, t_M)$, we must have either $\lim_{t \to t_M^-} x_1 = -\infty$ or $\lim_{t \to t_M^-} x_3 = \pm\infty$. Lemmas 5.3.5, 5.3.6 and 5.3.7 complete the proof. □

Lemma 5.3.9. *For $V_0 < 0$, suppose there exists $t_0 \in (-\infty, t_M)$ such that $x_1(t_0) < 0$. Then t_M is finite.*

Proof. If $x_1(t_0) < 0$ and $\epsilon = -1$ then (5.1.2a) yields $x_1'(t_0) < -x_1(t_0)^2$ and so $x_1 < 0$ persists. That is, $x_1 < 0$ and $x_1' < -x_1^2$ for all $t \in (t_0, t_M)$.

Integrating shows x_1 diverges to $-\infty$ in finite time and so t_M must be finite. □

Lemma 5.3.10. *For $V_0 < 0$, suppose there exists $t_0 \in (-\infty, t_M)$ such that $x_2(t_0) = k^2/8$. Then $x_3(t_0) < 1/2$ and $x_3 < 1/2$, $x_2 > k^2/8$ hold for all $t \in (t_0, t_M)$.*

Proof. Using Lemma 5.3.1, we must have $x_3 < 1/2$ for all $t \in (-\infty, t_0]$. We also have $x_2' > 0$ while $x_3 < 1/2$ and since x_3 cannot cross $1/2$ from below while $x_2 \geq k^2/8$ then we must have $x_2 > k^2/8, \dot{x}_2 > 0$ and $x_3 < 1/2$ for all $t \in (t_0, t_M)$. □

Lemma 5.3.11. *For $V_0 < 0$, suppose there exists $t_0 \in (-\infty, t_M)$ such that $x_1(t_0) = 1/2$. Then t_M is finite.*

Proof. At $x_1 = 1/2$, equation (5.1.2d) with $\epsilon = -1$ simplifies to

$$\frac{1}{4} + x_0^2 + \frac{k^2}{4} + \frac{k^2}{2}\left(x_3^2 - x_3\right) = 2x_2. \tag{5.3.15}$$

Using the fact the $x_3^2 - x_3 \geq -1/4$ we then have

$$x_2(t_0) > \frac{1}{8} + \frac{k^2}{16} + \frac{x_0^2}{2} > \frac{k^2}{8}. \tag{5.3.16}$$

There must then exist $t_* \in (-\infty, t_0)$ such that $x_2(t_*) = k^2/8$. Using Lemma 5.3.10 we have $x_3 < 1/2$, and thus $x_2' > 0$, for all $t \in (t_*, t_M)$. Using (5.3.16) we have $x_2 > 1/8 + k^2/16$, from which it follows that

$$x_3' < \frac{\lambda}{4k^2} - \left(\frac{1}{2} - x_1\right)\left(\frac{1}{2} - x_3\right) < \frac{\lambda}{4k^2} + \frac{1}{2}\left(\frac{1}{2} - x_3\right) \tag{5.3.17}$$

for all $t \in (t_0, t_M)$, where we have used $x_1 < 1$. This shows that $x_3' < 0$ if $x_3 > 1/2 + \lambda/2k^2$. It follows that $x_3 - 1/2 < \hat{m} = \max\{x_3(t_0) - 1/2, \lambda/2k^2\} < 0$, which gives $x_2' > \lambda\hat{m}x_2 > 0$, for all $t \in (t_0, t_M)$. Supposing that $t_M = +\infty$, there must exist $t_1 \in (t_0, t_M)$ such that $x_2 > 1/2$, and thus $x_1' < -1/4$, for $t \in (t_1, t_M)$. Then we must have $t_2 \in (t_1, t_M)$ such that $x_1(t_2) < 0$, which contradicts $t_M = +\infty$, by Lemma 5.3.9. □

Proposition 5.3.1. *If $V_0 < 0$, then t_M is finite.*

Proof. The preceding lemma rules out the possibility that $\boldsymbol{x}$ limits to P_1 or P_3 as $t \to \infty$, since the x_1 components of P_1 and P_3 are greater than one half. Proposition 5.2.6 rules out the possibility that $\boldsymbol{x}$ limits to P_2. Taking note of Lemma 5.3.2 which rules out limit cycles and other behaviours, we see that we must either have $\lim_{t\to\infty} ||\boldsymbol{x}|| = +\infty$ or t_M finite with $\lim_{t\to t_M^-} ||\boldsymbol{x}|| = \infty$. We may rule out the former case as follows. Assuming $t_M = \infty$, Lemma 5.3.2 tells us that each $\lim_{t\to\infty} x_i$ exists. Now, we can't have $\lim_{t\to\infty} x_1 = -\infty$, because in that case there would exists $t_0 < +\infty$ such that $x_1(t_0) < 0$ and thus t_M would be finite by Lemma 5.3.9. Nor can we have $\lim_{t\to\infty} x_2 = +\infty$ since this would cause x_1 to become negative in finite t, via (5.1.2a), so we would have t_M finite here also. This also rules out $\lim_{t\to\infty} x_3 = -\infty$ since this would give $\lim_{t\to\infty} x_2 = +\infty$, by (5.1.2b). Given that $x_2 > 0$ by definition and $x_1 < 1$ by Lemma 5.3.4, this leaves the possibility that $\lim_{t\to\infty} x_3 = +\infty$. However, it is easy to see that if $\lim_{t\to\infty} x_3 = +\infty$ and $\lim_{t\to\infty} |x_1| > -\infty$, then (5.1.2d) is not satisfied, since in that case we have $\lim_{t\to\infty} x_2 < \infty$ and the left hand side has limit $-\infty$. We must, therefore, have t_M finite. □

Proposition 5.3.2. *If $V_0 > 0$ then t_M is finite.*

Proof. It is easily checked that (5.1.2d) with $\epsilon = 1$ may be written as

$$\left(1 + \frac{k^2}{2}\right)\left(x_1 - x_1^2\right) = -2x_2 - x_0^2 - \frac{k^2}{2}\left(x_1 - x_3\right)^2 < -x_0^2, \tag{5.3.18}$$

from which it follows that

$$-\left(x_1 - \frac{1}{2}\right)^2 < -\left(\frac{x_0}{\kappa}\right)^2, \tag{5.3.19}$$

$$x_1 - \frac{1}{2} < -\frac{x_0}{\kappa}, \tag{5.3.20}$$

where $\kappa^2 = 1 + k^2/2$ and we have used $x_0 > 0$ and $x_1 < 0$, which is given by Lemma 5.3.4, and $x_0 > 0$. This is equivalent to

$$x_0' > \frac{x_0^2}{\kappa}. \tag{5.3.21}$$

Integrating shows that $x_0 = e^{t/2}/\sigma$ blows up in finite time, and so t_M is finite with $\lim_{t\to t_M^-} \sigma = 0$. □

Proposition 5.3.3. *For all solutions to (5.1.2d)-(5.1.2c) with $k^2 < 2$ we have t_M finite and* $\lim_{t\to t_M^-} \mathcal{T} = +\infty$.

Proof. Propositions 5.3.1 and 5.3.2 tell us that t_M is finite in all cases and Lemma 5.3.8 tells us that (5.3.14) holds in each case. In Proposition 2.6.1 it was shown that

$$\mathcal{T} \geq \frac{k^4 e^{-4\gamma-4\phi}}{16v^2}\left(\eta^2 l'(\eta)^2 - \frac{1}{4}\right)^2 = \frac{k^4 e^{-k^2 l - t - 2c_1}}{16v^2} x_3^2 (1-x_3)^2. \quad (5.3.22)$$

If $\lim_{t\to t_M^-} x_3 = -\infty$, then clearly $\lim_{t\to t_M^-} e^{-k^2 l} > 0$, so we have $\lim_{t\to t_M^-} \mathcal{T} = +\infty$. We now consider the case $\lim_{t\to t_M^-} x_3 = +\infty$, where it is possible that $\lim_{t\to t_M^-} e^{-k^2 l} = 0$. Given that $\lim_{t\to t_M^-} x_2 < +\infty$ in this case, dividing (5.1.2d) by x_1^2 and taking the limit $t \to t_M$ yields

$$1 - \lim_{t\to t_M^-}\left[\frac{e^t}{\sigma'^2} - \frac{k^2}{2}\left(\frac{x_3}{x_1}\right)^2 - k^2\left(\frac{x_3}{x_1}\right)\right] = 0. \quad (5.3.23)$$

Now, given that $\lim_{t\to t_M^-} \sigma = 0$ and $\lim_{t\to t_M^-} x_2 < \infty$, it is clear that $\sigma'' \sim \sigma' < 0$ as $t \to t_M$. Hence, σ' is monotone near t_M and has a finite limit. It follows that $\lim_{t\to t_M^-} e^t/\sigma'^2$ exists and, in light of (5.3.23), $\lim_{t\to t_M^-} x_3/x_1$ also exists. Letting $\lim_{t\to t_M^-} e^t/\sigma'^2 = \omega > 0$ and $\lim_{t\to t_M^-} x_3/x_1 = \ell$ we have

$$\ell^2 - 2\ell = \frac{2}{k^2}(1-\omega) < \frac{2}{k^2}, \quad (5.3.24)$$

from which it follows that

$$\ell > 1 - \sqrt{1 + \frac{2}{k^2}} > -\frac{1}{k^2}. \quad (5.3.25)$$

Now consider $Y = e^{-k^2 l/4} x_3$, which obeys

$$\frac{Y'}{Y} = \frac{1}{2} + \frac{k^2}{8} + \epsilon\frac{2x_2}{k^2 x_3} - x_1\left(1 - \frac{1}{2x_3}\right) - \frac{k^2 x_3}{4}. \quad (5.3.26)$$

Using $\lim_{t\to t_M^-} x_2/x_3 = 0$ and $\lim_{t\to t_M^-} x_3/x_1 = \ell > -1/k^2$, there must exist t_0 sufficiently close to t_M such that $x_3 < -2x_1/k^2$ and

$$\frac{Y'}{Y} > -x_1\left(\frac{1}{2} - \frac{1}{2x_3}\right), \quad (5.3.27)$$

on (t_0, t_M). We may choose t_0, without loss of generality, such that (5.3.3) holds and so integrating the inequality above shows that $\lim_{t\to t_M^-} Y = +\infty$. We observe that the lower bound for $\mathcal{T}$ seen in (5.3.22) behaves like Y^4 as $t \to t_M$, so the proof is complete. □

Theorem 5.3.1. *For all solutions to the system (11), t_M is finite and there is a spacelike curvature singularity along $t = t_M$.*

Proof. Note that surfaces of constant $t \in (-\infty, +\infty)$ are spacelike. Proposition 5.3.3 completes the proof. □

Proposition 5.3.4. *The surface corresponding to $t = t_M$ is reached by outgoing radial null rays in finite parameter time.*

Proof. Recall Propositions 4.1.2 and 4.1.3 which proved that, if $k^2 < 2$, then outgoing radial null geodesics reach $\mathcal{N}_-$ in finite paramater time. For convenience, we choose a new parameter μ such that $\dot{v} > 0$ and $\mu(\eta = 0) = 0$. Then a similar calculation to the one in Proposition 4.1.1 reveals that

$$\tilde{C}\mu = \int_{-\infty}^{t} e^{k^2 l/2 + t/2 - c_1} dt', \tag{5.3.28}$$

for constant $\tilde{C} > 0$. Given that $\lim_{t\to-\infty} l = -1/2$ we have $e^{k^2 l/2 + t/2} \sim e^{|\lambda| t/2}$ as $t \to -\infty$ and it is straightforward to show that there exists $t_0 \in (-\infty, t_M)$ such that the portion of the integral in (5.3.28) over the interval $(-\infty, t_0)$ is finite. In fact, this is true of any t_0 which is bounded away from t_M. In the case $\lim_{t\to t_M^-} x_3 = -\infty$ we have $\lim_{t\to t_M^-} e^{k^2 l/2} = 0$ and, given that t_M is finite, taking the limit $t \to t_M$ of (5.3.28) shows that $\lim_{t\to t_M^-} \mu < +\infty$. We now turn to the case $\lim_{t\to t_M^-} x_3 = +\infty$.

Let $\tilde{Y} = e^{-2k^2 l/3} x_1$. It may be shown in an analogous way to the proof of Proposition 5.3.3, using $\lim_{t\to t_M^-} x_3/x_1 > -1/k^2$, that $\lim_{t\to t_M^-} \tilde{Y} = -\infty$. We can then choose $t_0 \in (-\infty, t_M)$ such that $\tilde{Y} < -1$, which gives $e^{k^2 l/2} < (-x_1)^{3/4}$, for $t \in (t_0, t_M)$. Letting $\tilde{p} = (-x_1)^{3/4}$ we have

$$\tilde{p}' = \frac{3(-x_1)^{-1/4}}{4} \left(-x_1 - \epsilon x_2 + x_1^2\right). \tag{5.3.29}$$

We can choose t_0 such that $-x_1 - \epsilon x_2 > 0$ on $t \in (t_0, t_M)$, since $\lim_{t\to t_M^-} x_2 < +\infty$ in this case. We then have $3\tilde{p}^{7/3}/4 < \dot{\tilde{p}}$, which we divide by $3\tilde{p}^{4/3}/4$

and integrate to obtain

$$\int_{t_0}^{t} e^{k^2 l/2}\, dt' < \int_{t_0}^{t} \tilde{p}\, dt' < \frac{4}{\tilde{p}_0^{1/3}} - \frac{4}{\tilde{p}^{1/3}}. \tag{5.3.30}$$

Taking the limit $t \to t_M$ shows that

$$\lim_{t \to t_M^-} \int_{t_0}^{t} e^{k^2 l/2}\, dt' < \infty. \tag{5.3.31}$$

Since t_M is finite, the result follows from (5.3.28). □

Proposition 5.3.5. *Ingoing radial null geodesics have infinite affine length.*

Proof. For ingoing radial null geodesics we have $v = v_0, \dot{\theta} = \dot{z} = 0$ and $u = v/\eta = -v_0 e^{-t}, du = -u dt$. Solutions to the geodesic equation are then given by

$$e^{2\gamma+2\phi} \frac{dt}{d\mu} = e^{k^2 l/2 - t/2 + c_1} \frac{dt}{d\mu} = \hat{C}, \tag{5.3.32}$$

for constant $\hat{C} > 0$. To determine whether the spacetime has a past null infinity we look for $\lim_{u \to -\infty} \mu$, which is given by $\lim_{t \to -\infty} \mu$, since $t \to -\infty$ as $u \to -\infty$ along lines of constant v. Integrating over (t, t_0) and taking this limit we find

$$\tilde{C} \lim_{t \to -\infty} (\mu_0 - \mu) = \lim_{t \to -\infty} \int_{t}^{t_0} e^{k^2 l/2 - t/2 + c_1} dt'. \tag{5.3.33}$$

Given that $\lim_{t \to -\infty} l = -1/2$, we clearly have $\lim_{t \to -\infty} \mu = -\infty$.
For completeness we now examine the behaviour of the null geodesic along $\mathcal{N}_-$. Our current coordinate system is not suited to the task since the metric is constant on this surface and we have a coordinate singularity there. Specifically, we have seen that $e^{2\gamma+2\phi} \sim e^{-(k^2/4+1/2)t} = (-\eta)^{-(k^2/4+1/2)}$ in the limit as $t \to -\infty, \eta \to 0^-$. We define a new coordinate and constant by

$$\xi(\eta) = \int_0^{\eta} e^{2\gamma(\eta')+2\phi(\eta')} d\eta', \qquad \beta = \frac{k^2}{4} + \frac{1}{2}. \tag{5.3.34}$$

Since $e^{2\gamma+2\phi} \sim (-\eta)^{-\beta}$ and $0 < \beta < 1$ for $k^2 < 2$, it is straightforward to show that $\xi \sim \eta^{1-\beta}$ at $\eta = 0$ and that $\xi(0) = 0$. Writing the line element in terms of ξ we have

$$\begin{aligned} ds^2 &= 2e^{2\gamma+2\phi}(dud\eta + u^{-1}\eta\, du^2) + e^{2\phi}r^2d\theta^2 + e^{-2\phi}dz^2 \\ &= 2dud\xi + 2e^{2\gamma(\eta(\xi))+2\phi(\eta(\xi))}u^{-1}\eta(\xi)\, du^2 \\ &\qquad\qquad + e^{2\phi}r^2d\theta^2 + e^{-2\phi}dz^2. \end{aligned} \tag{5.3.35}$$

To derive the geodesic equation we consider the Lagrangian $\mathscr{L}$ which simplifies to

$$\mathscr{L} = 2\dot{u}\dot{\xi} + 2u^{-1}\eta e^{2\gamma+2\phi}\dot{u}^2, \tag{5.3.36}$$

along ingoing radial null geodesics. We then have

$$\frac{d}{d\mu}\frac{\partial\mathscr{L}}{\partial\dot{\xi}} - \frac{\partial\mathscr{L}}{\partial\xi} = 2\ddot{u} - 2u^{-1}\frac{d}{d\xi}\left(\eta e^{2\gamma+2\phi}\right)\dot{u}^2 = 0. \tag{5.3.37}$$

Using $d\xi = e^{2\gamma+2\phi}d\eta$ and the derivative of (2.4.21) we have

$$\frac{d}{d\xi}\left(\eta e^{2\gamma+2\phi}\right) = 1 + \eta(2\gamma'(\eta) + 2\phi'(\eta)) = \frac{k^2\eta l'}{2} + \frac{1}{2}. \tag{5.3.38}$$

Given that $\eta l' = -1/2$ everwhere on $\mathcal{N}_-$, the geodesic equation reduces to

$$\frac{\ddot{u}}{\dot{u}} - \frac{|\lambda|\dot{u}}{2u} = 0, \tag{5.3.39}$$

which may be integrated to give

$$\frac{\dot{u}}{\dot{u}_0} = \left(\frac{u}{u_0}\right)^{|\lambda|/2}. \tag{5.3.40}$$

Integrating again shows that $\mu \sim |u|^{1-|\lambda|/2}$ and so $\lim_{u\to-\infty}\mu = +\infty$. $\square$

These three final results determine the global structure of the spacetime for all solutions corresponding to $k^2 < 2$. It is depicted in Fig 5.1 below.

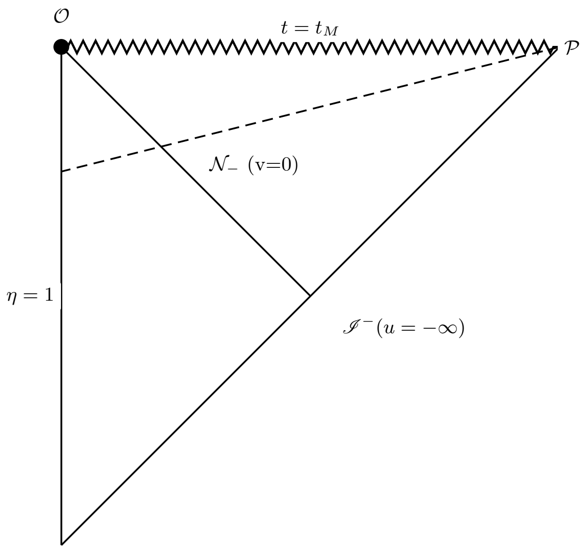

Figure 5.1: Global structure of the spacetimes with $k^2 < 2$.

Chapter 6

Analytic series solutions in region II

In this chapter we present an alternative method for solving the dynamical system (5.1.2a)-(5.1.2c). It involves using analytic series solutions to these equations using power series in $e^{|\lambda|t/2}$. By writing the solutions in this way, we obtain recurrence relations for the coefficients of these series. We then show that, for almost all values $k^2 < 2$, the coefficients of the series satisfy a geometric bound and that the series themselves are, therefore, uniformly convergent for small enough t, i.e. in a neighbourhood of the origin of the system, $\mathcal{N}_-$. We derive an analytical bound for the truncation error of the series for a given fixed number of terms, which enables us to approximate the solutions and their derivatives, to any required degree of accuracy, within the interval on which the series converge. In many cases, this may be used to assert that a particular solution satisfies the hypothesis of one or more results which then give the global structure of the solution. Although we have given an adequate description of the possible spacetimes for all parameter values in chapters 3,4 and 5 to rule out the existence of naked singularities in each case, the present approach allows us to add more detail in certain cases. For example, for $k^2 < 2$ in region **II** we have shown that $\lim_{t \to t_M^-} x_3 = \pm\infty$ in all cases but have not given any indication which of those two situations prevails. Using this method, we show that in fact both may arise depending on the values of k^2 and V_0. Although superfluous in terms of the question fo cosmic censorship, this information might be useful for other physical interpretations. Furthermore, the method could easily be generalised to other dynamical systems of a similar type, i.e. those with quadratic non-linearities and a similar eigen-structure, and is therefore of interest in its own right.

6.1 Analytic series solutions

Let $\zeta = e^{|\lambda|t/2}$. We begin by writing formal series solutions to (5.1.2a)-(5.1.2c) as follows:

$$x_1 = \sum_{n=1}^{\infty} a_n \zeta^n, \quad x_2 = \sum_{n=1}^{\infty} b_n \zeta^n, \quad x_3 = \sum_{n=1}^{\infty} c_n \zeta^n. \tag{6.1.1}$$

Substituting these expressions into in the system yields

$$\sum_{n=1}^{\infty} \frac{n|\lambda|}{2} a_n \zeta^n = \sum_{n=1}^{\infty} (a_n + \epsilon b_n)\zeta^n - \sum_{n=1}^{\infty} \left(\sum_{j=1}^{n-1} a_j a_{n-j} \right) \zeta^n, \tag{6.1.2a}$$

$$\sum_{n=1}^{\infty} \frac{n|\lambda|}{2} b_n \zeta^n = \sum_{n=1}^{\infty} \frac{|\lambda|}{2} b_n \zeta^n - |\lambda| \sum_{n=1}^{\infty} \left(\sum_{j=1}^{n-1} b_j c_{n-j} \right) \zeta^n, \tag{6.1.2b}$$

$$\sum_{n=1}^{\infty} \frac{n|\lambda|}{2} c_n \zeta^n = \sum_{n=1}^{\infty} \left(\frac{c_n}{2} + \frac{a_n}{2} + \epsilon \frac{2b_n}{k^2} \right) \zeta^n - \sum_{n=1}^{\infty} \left(\sum_{j=1}^{n-1} a_j c_{n-j} \right) \zeta^n. \tag{6.1.2c}$$

The coefficients of the series then satisfy the following recurrence relations, obtained by equating powers of ζ:

$$a_n = \left(\frac{n|\lambda|}{2} - 1 \right)^{-1} \left(\epsilon b_n - \sum_{j=1}^{n-1} a_j a_{n-j} \right), \tag{6.1.3a}$$

$$b_n = -\frac{2}{n-1} \sum_{j=1}^{n-1} b_j c_{n-j}, \tag{6.1.3b}$$

$$c_n = \left(\frac{n|\lambda|}{2} - \frac{1}{2} \right)^{-1} \left(\frac{a_n}{2} + \epsilon \frac{2b_n}{k^2} - \sum_{j=1}^{n-1} a_j c_{n-j} \right). \tag{6.1.3c}$$

Recall the asymptotic behaviour given in Proposition 5.1.3. This gives us the first coefficient in each series by

$$a_1 = A = \epsilon \frac{4}{2+k^2}, \qquad b_1 = 1, \qquad c_1 = B = \epsilon \frac{16}{k^4(2+k^2)}. \tag{6.1.4}$$

The coefficients for $n > 1$ are then given by the recurrence relations. Note that b_n depends only on coefficients with index $j < n$. Once b_n is found, a_n followed by c_n may be computed. Notice that these relations are problematic at $n = 2/|\lambda|, n = 1/|\lambda|$. However, these only arise when k^2 is such that $2/|\lambda| \in \mathbb{N}$. We define the set $K = \{2 - 4/n; n \in \mathbb{N}/\{0, 1\}\}$ and note that if $k^2 < 2$ and $k^2 \notin K$ then $2/|\lambda| \notin \mathbb{N}$. This is a countably infinite set with an accumulation at $k^2 = 2$. As a subset of the reals it is a set of measure zero and so we do not see this as a serious limitation of this method. As an example, in the case $\epsilon = 1$, we have

$$a_4 = \frac{1024(64 + 112k^2 - 480k^4 + 340k^6 + 96k^8 - 161k^{10} + 38k^{12})}{3k^{12}(-1 + k^2)^2(2 + k^2)^4(-4 + 3k^2)}. \quad (6.1.5)$$

We see this quantity is not defined at $k^2 = 1, k^2 = 0$ and $k^2 = 4/3$, which are all in K, but is otherwise well-defined. The coefficients become very complex polynomials in k^2 as n gets large but we may still prove the following useful results.

Proposition 6.1.1. *Let $k^2 \notin K, \alpha > 0, \delta > 0$ and $N \in \mathbb{N}$ be such that $|a_j|, |b_j|, |c_j| \leq \alpha\delta^{j-1}/j$ for all $j < N$, and that*

$$N > \frac{2}{|\lambda|}\left(1 + \frac{2}{k^2} + \frac{2\alpha}{\delta}H_{N-1}\right), \quad (6.1.6)$$

where H_n is the n-th harmonic number defined by

$$H_n = \sum_{j=1}^{n} \frac{1}{j}. \quad (6.1.7)$$

Then $|a_N|, |b_N|, |c_N| \leq \alpha\delta^{N-1}/N$.

Proof. It is easy to see that if $k^2 \notin K$ and N is finite then a_j, b_j, c_j with $j < N$ are also finite. Then we can always choose α and δ large enough such that $|a_j|, |b_j|, |c_j| \leq \alpha\delta^{j-1}/j$ for all $j < N$. From (6.1.3b) we have

$$|b_N| \leq \frac{2}{N-1}\sum_{j=1}^{N-1}|b_j||c_{N-j}| \leq \frac{2}{N-1}\sum_{j=1}^{N-1}\frac{\alpha^2\delta^{N-2}}{j(N-j)} \quad (6.1.8)$$
$$= \frac{2\alpha^2\delta^{N-2}}{N-1}\sum_{j=1}^{N-1}\frac{1}{j(N-j)}.$$

Now,

$$\sum_{j=1}^{N-1} \frac{1}{j(N-j)} = \sum_{j=1}^{N-1} \frac{1}{N}\left(\frac{1}{j} + \frac{1}{N-j}\right) = \frac{2}{N} H_{N-1}, \tag{6.1.9}$$

and so $|b_N| \leq \alpha\delta^{N-1}/N$ is satisfied if

$$N \geq \frac{4\alpha H_{N-1}}{\delta} + 1 \tag{6.1.10}$$

Note that $\lim_{n\to\infty} H_n/n = 0$ and so for any choice of α, δ, this is satisfied for large enough N. It follows from (6.1.6) that $N|\lambda|/2 - 1 > 0$, and so

$$\begin{aligned} |a_N| &\leq \left(\frac{N|\lambda|}{2} - 1\right)^{-1} \left(|b_N| + \sum_{j=1}^{N-1} |a_j||a_{N-j}|\right) \\ &\leq \left(\frac{N|\lambda|}{2} - 1\right)^{-1} \left(\frac{\alpha\delta^{N-1}}{N} + \sum_{j=1}^{N-1} \frac{\alpha^2\delta^{N-2}}{j(N-j)}\right) \\ &= \left(\frac{N|\lambda|}{2} - 1\right)^{-1} \left(\frac{\alpha\delta^{N-1}}{N} + \frac{2\alpha^2\delta^{N-2}}{N} H_{N-1}\right). \end{aligned} \tag{6.1.11}$$

$|a_N| \leq \alpha\delta^{N-1}/N$ is then satisfied if

$$\left(\frac{N|\lambda|}{2} - 1\right)^{-1} \left(\frac{\alpha\delta^{N-1}}{N} + \frac{2\alpha^2\delta^{N-2}}{N} H_{N-1}\right) \leq \frac{\alpha\delta^{N-1}}{N}, \tag{6.1.12}$$

which simplifies to

$$N \geq \frac{4}{|\lambda|}\left(1 + \frac{\alpha H_{N-1}}{\delta}\right). \tag{6.1.13}$$

Similarly

$$\begin{aligned} |c_N| &\leq \left(\frac{N|\lambda| - 1}{2}\right)^{-1} \left(\frac{|a_N|}{2} + \frac{2|b_N|}{k^2} + \sum_{j=1}^{N-1} |a_j||c_{N-j}|\right) \\ &\leq \left(\frac{N|\lambda| - 1}{2}\right)^{-1} \left(\left(\frac{1}{2} + \frac{2}{k^2}\right)\frac{\alpha\delta^{N-1}}{N} + \sum_{j=1}^{N-1} \frac{\alpha^2\delta^{N-2}}{j(N-j)}\right), \end{aligned} \tag{6.1.14}$$

which ensures $|c_N| \leq \alpha\delta^{N-1}/N$ is satisfied if (6.1.6) holds. It is straightforward to show that (6.1.6) implies (6.1.10) and (6.1.13). □

Corollary 6.1.1. *If the hypotheses of Proposition 6.1.1 hold for some* N, α, δ, *then we have* $|a_n|, |b_n|, |c_n| \leq \alpha\delta^{n-1}/n$ *for all* $n \geq 1$.

Proof. Since Proposition 6.1.1 holds for N, it follows that $|a_j|, |b_j|, |c_j| \leq \alpha\delta^{j-1}/j$, for all $j < N+1$. Using (6.1.6) and $H_{N-1} = H_N - 1/N$ we have

$$N + 1 > \frac{2}{|\lambda|}\left(1 + \frac{2}{k^2} + \frac{2\alpha}{\delta}H_N\right) - \frac{2\alpha}{\delta|\lambda|N} + 1. \tag{6.1.15}$$

It is straightforward to show that (6.1.6) gives $1 > 2\alpha/\delta|\lambda|N$ and so we have

$$N + 1 > \frac{2}{|\lambda|}\left(1 + \frac{2}{k^2} + \frac{2\alpha}{\delta}H_N\right). \tag{6.1.16}$$

The hypothesis of Proposition 6.1.1 then holds for $N+1$ and, by induction, for all $n > N$. □

Proposition 6.1.2. *Suppose that* $|a_n|, |b_n|, |c_n| \leq \alpha\delta^{n-1}/n$ *for all* $n \geq 1$. *Then the series (6.1.1) converge uniformly for* $\zeta \in (0, 1/\delta)$.

Proof. Define

$$M_n = \frac{\alpha\delta^{n-1}\zeta^n}{n} = \frac{\alpha(\delta\zeta)^n}{\delta n}. \tag{6.1.17}$$

Then $|a_n\zeta^n|, |b_n\zeta^n|, |c_n\zeta^n| \leq M_n$ for all $n \geq 1$, and the series $\sum_{n=1}^{\infty} M_n$, converges for $\delta\zeta < 1$. The result is then given by the Weierstrass M-test. □

Proposition 6.1.3. *Let* $\alpha > 0, \delta > 0$ *and* $N \in \mathbb{N}$ *be such that the hypotheses of Proposition 6.1.1 are satisfied. Let* $\bar{\zeta} = \delta\zeta < 1$ *and* $\bar{\boldsymbol{x}}_N$ *be the approximation to* $\boldsymbol{x}$ *given by the first* $N-1$ *terms of the series (6.1.1). Then the truncation error* $e_N = ||\bar{\boldsymbol{x}}_N - \boldsymbol{x}||$ *satisfies*

$$e_N \leq \frac{\alpha\bar{\zeta}^N}{\delta}\Phi(\bar{\zeta}, 1, N), \tag{6.1.18}$$

and $\lim_{N\to\infty} e_N = 0$, *where* Φ *is the Lerch transcendant.*

Proof. Given that $|a_n|, |b_n|, |c_n| \leq \alpha\delta^{n-1}/n$ for all $n \in \mathbb{N}$ and the series (6.1.1) are uniformly convergent we have

$$\begin{aligned} e_N = ||\bar{\boldsymbol{x}}_N - \boldsymbol{x}|| &\leq \sum_{j=N}^{\infty} \frac{\alpha\delta^{j-1}\zeta^j}{j} \\ &= \frac{\alpha\bar{\zeta}^N}{\delta}\sum_{i=0}^{\infty}\frac{\bar{\zeta}^i}{N+i} = \frac{\alpha\bar{\zeta}^N}{\delta}\Phi(\bar{\zeta}, 1, N) \end{aligned} \tag{6.1.19}$$

Note that $\Phi(\bar{\zeta}, 1, N) < 1/N - \log(1-\bar{\zeta})$, and so if $\bar{\zeta} < 1$ then $\lim_{N\to\infty} e_N = 0$. □

In the following sections, we also make use of approximations to the first derivatives of the x_i and the so the following result is necessary to bound the truncation errors for these approximations.

Corollary 6.1.2. *Let*

$$\bar{x}_1' = \bar{x}_1 + \epsilon\bar{x}_2 - \bar{x}_1^2, \qquad \bar{x}_3' = \frac{\bar{x}_3}{2} + \frac{\bar{x}_1}{2} + \epsilon\frac{2\bar{x}_2}{k^2} - \bar{x}_1\bar{x}_3, \tag{6.1.20}$$

be approximations to x_1' and x_3', respectively, where $\bar{x}_i$ are the components of $\bar{\boldsymbol{x}}_N$ mentioned above. Then

$$\begin{aligned} |x_1' - \bar{x}_1'| &\le (2 + 2|\bar{x}_1| + e_N)e_N, \\ |x_3' - \bar{x}_3'| &\le \left(1 + \frac{2}{k^2} + 2|\bar{x}_3| + e_N\right) e_N. \end{aligned} \tag{6.1.21}$$

Proof.

$$\begin{aligned} |x_1' - \bar{x}_1'| &= |x_1 - \bar{x}_1 + \epsilon(x_2 - \bar{x}_2) - (x_1 + \bar{x}_1)(x_1 - \bar{x}_1)| \\ &\le (1 + |x_1 + \bar{x}_1|)|x_1 - \bar{x}_1| + |x_2 - \bar{x}_2| \\ &\le (2 + 2|\bar{x}_1| + e_N)e_N, \end{aligned} \tag{6.1.22}$$

where we have used $|\bar{x}_1| - e_N \le |x_1| \le |\bar{x}_1| + e_N$ to obtain the last inequality. The remainder of the proof is similar. □

So we have shown that for $k^2 \notin K$ the series solutions (6.1.1) are uniformly convergent in a neighbourhood of the origin of the system and that the solutions may be approximated to any required degree of accuracy within that neighbourhood by calculating a sufficient number of terms of the series. In practice, we take a fixed value of k^2 and compute the first N coefficients for a chosen value of N, using a suitable computer programme such as Mathematica. We then choose α and δ such that the hypotheses of Proposition 6.1.1 are satisfied and plot the approximate solution over the interval of validity, looking for information which allows us to determine the future evolution. The first 100 coefficients for each of the series may be computed in a few seconds using a standard laptop computer and this is more than adequate for the majority of values of k^2. In the following section we give some examples of this process and some analytical results which are adapted to the information acquired.

6.2 Some analytical results adapted to the method

Here we present some results which prove useful in determining the future evolution of solutions which pass through certain points in the phase space. These build on and add to the results of Chapter 5. In the following section, we prove that the hypotheses of these lemmas are satisfied in certain cases. This is established using the results of the previous section.

Lemma 6.2.1. *Suppose that for* $V_0 < 0$*, there exists* t_0 *such that* $x_1(t_0) < 0$ *and* $x_3(t_0) > 1/2$*. Then* t_M *is finite,* $\lim_{t \to t_M^-} x_1 = -\infty$ *and* $\lim_{t \to t_M^-} x_3 = +\infty$*.*

Proof. Given that $x_1(t_0) < 0$, Lemmas 5.3.9 and 5.3.8 tell us that t_M is finite and $\lim_{t \to t_M^-} x_1 = -\infty, \lim_{t \to t_M^-} x_3 = \pm\infty$. Lemma 5.3.1 tells us that $x_3 > 1/2$ holds for $t \in (t_0, t_M)$, which completes the proof. □

Lemma 6.2.2. *For* $V_0 < 0$*, suppose there exists* t_0 *such that* $x_1(t_0) < 0$ *and* $x_3(t_0) < 0$*. Then* t_M *is finite and* $\lim_{t \to t_M^-} x_1 = -\infty, \lim_{t \to t_M^-} x_3 = -\infty$*.*

Proof. Note that $\epsilon = -1, x_1 < 0, x_3 < 0$ give $x_1' < 0$ and $x_3' < 0$ and so we must have $x_1 < 0, x_3 < 0$ for all $t \in (t_0, \zeta_M)$. Lemmas 5.3.9 and 5.3.8 complete the proof. □

Lemma 6.2.3. *Let* $V_0 > 0$*. Suppose there exists* t_0 *such that*

$$(1 - \Lambda)x_3(t_0) + \Lambda - \Lambda^{1/2} > 0, \quad \textit{where} \quad \Lambda = \frac{x_1(t_0)}{x_1(t_0) - 1}. \tag{6.2.1}$$

Then $\lim_{t \to t_M^-} x_1 = -\infty, \lim_{t \to t_M^-} x_3 = +\infty$*.*

Proof. If $\epsilon = 1$ then $x_1' > x_1 - x_1^2$, which may be integrated over $[t_0, t]$ to give $x_1 > -\Lambda e^{\bar{t}}/(1 + \Lambda e^{\bar{t}})$, where $\bar{t} = t - t_0$. It then follows from (5.1.2c) that if $x_3 \le 1/2$ we have

$$x_3' > -\frac{\Lambda e^{\bar{t}}}{1 + \Lambda e^{\bar{t}}} + \left(\frac{1}{2} + \frac{\Lambda e^{\bar{t}}}{1 + \Lambda e^{\bar{t}}}\right) x_3, \tag{6.2.2}$$

which may be integrated to give

$$x_3 > \frac{(1-\Lambda)x_3(t_0) - \Lambda(e^{\bar{t}/2} - 1)}{e^{-\bar{t}/2}(1 - \Lambda e^{\bar{t}})}. \tag{6.2.3}$$

Using Lemma 5.3.8, solutions exists while x_1 is bounded below, which is true for $e^{\bar{t}} > -\Lambda^{-1}$. Using $(1-\Lambda)x_3(t_0) + \Lambda - \Lambda^{1/2} > 0$, the righthand side of (6.2.3) has limit $+\infty$ as $e^{\bar{t}} \to \Lambda^{-1}$. Given that (6.2.3) is valid for all $x_3 \le 1/2$ and $t \in (t_0, t_M)$ there must exist $t_1 \in (t_0, t_M)$ such that $x_3(t_1) \ge 1/2$, which must then hold for all $t \in (t_1, t_M)$, as we have seen that x_3 can only cross $1/2$ once. We know by Proposition 5.3.2 that t_M is finite, so Lemma 5.3.8 and $x_3 \ge 1/2$ completes the proof. □

Lemma 6.2.4. *Let $V_0 > 0$, $z = x_3/x_1$ and $w = -x_2/x_1$. Suppose there exists t_0 such that*

$$-\frac{1}{\Delta k^2} < z(t_0) < \frac{1}{|\lambda|}, \quad z'(t_0) < 0, \quad w(t_0) < A^{-1}, \tag{6.2.4}$$

where

$$\Delta = \frac{x_1(t_0)}{|\lambda| - 2x_1(t_0)} < 0. \tag{6.2.5}$$

Then $\lim_{t\to t_M^-} x_1 = -\infty, \lim_{t\to t_M^-} x_3 = -\infty.$

Proof. Note that z and w satisfy

$$z' = \left(w - \frac{1}{2}\right) z + \frac{1}{2} - \frac{2w}{k^2}, \tag{6.2.6}$$

$$w' = w\left(w - A^{-1} + x_1(1 - |\lambda| z)\right). \tag{6.2.7}$$

Recall from Lemma 5.3.4 that if $\epsilon = 1$ then $x_1 < 0$, and thus $w > 0$, for all $t \in (-\infty, t_M)$. It then follows from (6.2.7) that if $w < A^{-1}$ and $z < 1/|\lambda|$ then $w' < 0$. It is straightforward to show that if $z' = 0$ then $z'' = (z - 2/k^2)w'$ which is negative if $z > 2/k^2$ and $w' < 0$. Then while $z > 2/k^2$ holds we have $z' < 0$ and $w' < 0$, which give $z < z_0, w < A^{-1}$ and $x_1 < -Ax_2$, where $z_0 = z(t_0)$. These combine to give

$$x_1' < |\lambda| x_2 - x_1^2, \qquad \frac{w'}{w} < x_1(1 - |\lambda| z_0), \tag{6.2.8}$$

using (5.1.2a) and (6.2.7). The first inequality may be integrated over (t_0, t)

to give

$$x_1 < \frac{|\lambda|\Delta}{1 + 2\Delta e^{|\lambda|\bar{t}/2}}. \tag{6.2.9}$$

We may rearrange (6.2.6) as

$$z' = (z - 2/k^2)w + (1 - z)/2. \tag{6.2.10}$$

It becomes clear that if $z > 2/k^2$ and $w > k^2/4$ then $z' > -|\lambda|z/2$, which may be integrated to give $z > z_0 e^{|\lambda|\bar{t}/2}$. Since $-2\Delta z_0 > 2/k^2$ from the hypothesis, $z > 2/k^2$ holds provided $e^{|\lambda|\bar{t}/2} \leq -2\Delta$ and $w > k^2/4$. Hence, (6.2.8) and (6.2.9) hold while $e^{|\lambda|\bar{t}/2} \leq -2\Delta$ and $w > k^2/4$. Suppose $w > k^2/4$ holds for all $t \in (-\infty, t_M)$. Then (6.2.9) holds for all $e^{|\lambda|\bar{t}/2} \leq -2\Delta$. But then t_M must be such that $e^{|\lambda|\bar{t}_M/2} \leq -2\Delta$ since the right-hand side blows up at $e^{|\lambda|\bar{t}_M/2} = 2\Delta$. This means that $w > k^2/4$ and (6.2.8) hold for all $t \in (t_0, t_M)$. However, integrating the second inequality in (6.2.8) using 5.3.10 shows that this cannot happen, so we have a contradiction. Hence, there must exist $t_1 \in (t_0, t_M)$ such that $w(t_1) < k^2/4$ and $2/k^2 < z(t_1) < 1/|\lambda|$. Note that 5.3.10 holds for all cases, as proven by Lemmas 5.3.4, 5.3.5. and 5.3.8

Now, if $w < k^2/4$ then either $z < 2/k^2$ or $z > 2/k^2$ and $z' < -|\lambda|z/2$, using (6.2.10). Since $2/k^2 < z(t_1) < 1/|\lambda|$ we must have $z < 1/|\lambda|$ provided $w < k^2/4$. It is clear from (6.2.7) that if $z < 1/|\lambda|$ and $w < k^2/4$ then $w' < 0$. Hence, $w < k^2/4$ and $z < 1/|\lambda|$ hold for all $t \in (t_1, t_M)$. Finally, at $z = 0$ we have $z' = 1/2 - 2w/k^2 > 0$ and so $z > 0$, which gives $x_3 < 0$, for all $t \in (t_0, t_M)$. Proposition 5.3.2 and Lemma 5.3.8 complete the proof. □

6.3 Computation of some approximate solutions

Here we give information regarding the computation of the series (6.1.1) and show that this can be used in conjuction with the results of the previous section to prove certain properties of the global solutions. We present a small sample to give the reader a flavour of the application of the method which is by no means exhaustive. In fact, the global structure of solutions may be obtained for the majority of values of k^2 with both $\epsilon = \pm 1$ using this method and a range of other analytical results which we have omitted for the sake of brevity. We define

$$m = \max_{1\leq j\leq N}\left\{\frac{j|a_j|}{\alpha\delta^{j-1}}, \frac{j|b_j|}{\alpha\delta^{j-1}}, \frac{j|c_j|}{\alpha\delta^{j-1}}\right\}, \quad \Omega = \frac{2}{|\lambda|}\left(1+\frac{2}{k^2}+\frac{2\alpha}{\delta}H_{N-1}\right).$$

Then, for a given value of k^2, if α, δ and N are chosen such that $m < 1$ and $N > \Omega$ then Proposition 6.1.1 is satisfied.
Let $\bar{x}_1$ and $\bar{x}_3$ be the approximations to x_1 and x_3 using the first N terms in the series (6.1.1) for a given value of N. Table 6.1 gives values for $m, \Omega, \bar{x}_1(\zeta_0), \bar{x}_3(\zeta_0)$ and an upper bound on $e_N(t_0)$ corresponding to $V_0 < 0$ and chosen values of k^2, δ, α and ζ_0.
Note that $N = 100$ and all values are rounded to three significant figures in all tables presented in this section.
We emphasise that these tables represent (rounded) values of exact calculations, rather than numerical approximations.

k^2	δ	α	m	Ω	ζ_0	$\bar{x}_1(\zeta_0)$	$\bar{x}_3(\zeta_0)$	$e_N(\zeta_0) <$
0.57	18	50	.878	93.1	.03	-.160	.784	10^{-28}
0.60	18	50	.631	94.6	.03	-.122	.687	10^{-28}
0.63	18	50	.462	96.2	.03	-.0939	.610	10^{-28}
0.66	18	50	.345	97.9	.03	-.0723	.546	10^{-28}

Table 6.1: Information for solutions with $k^2 \in \{0.57, 0.60, 0.63, 0.66\}$ and $V_0 < 0$ showing that Lemma 6.2.1 holds in each case.

It shows that if $V_0 < 0, k^2 \in \{0.57, 0.60, 0.63, 0.66\}, \delta = 18, \alpha = 50$ and $N = 100$, then $m < 1$ and $N > \Omega$, which proves $|a_n|, |b_n|, |c_n| \leq \alpha\delta^{n-1}/n$ for all $n \in \mathbb{N}$. Given the values of the approximations and the bound on the truncation error it also shows that the actual solutions satisfy $x_1(\zeta_0 = 0.3) < 0$ and $x_3(\zeta_0 = 0.3) > 1/2$ and, therefore, the hypothesis of Lemma 6.2.1. This proves that $\lim_{t\to t_M^-} x_1 = -\infty$ and $\lim_{t\to t_M^-} x_3 = +\infty$ in these cases. Figures 6.1 and 6.2 plot the approximations to x_1 and x_3 for $V_0 < 0$ and the relevant values of k^2. They give a very strong indication that the solutions corresponding to the values between these discrete values have similar behaviours.

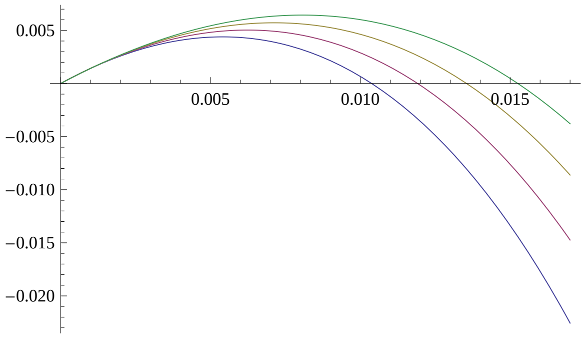

Figure 6.1: Plot of solutions for x_1 with $k^2 = 0.57, 0.6, 0.63, 0.66$ and $V_0 < 0$ over the interval $\zeta \in [0, 0.17]$. In each case x_1 crosses 0 in finite ζ.

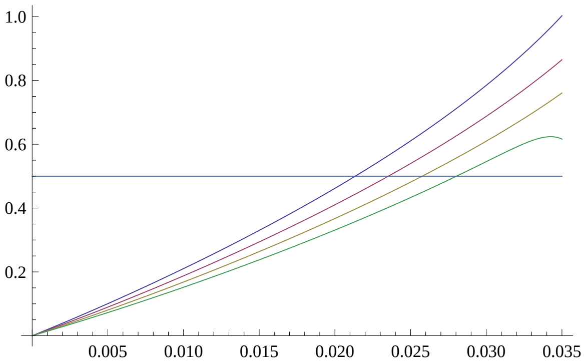

Figure 6.2: Plot of solutions for x_3 with $k^2 = 0.57, 0.6, 0.63, 0.66$ and $V_0 < 0$ over the interval $\zeta \in [0, 0.35]$. In each case x_3 crosses $1/2$ in finite ζ. Note also the curve for $k^2 = 0.66$ starts to turn down toward the end of the interval where the series is no longer convergent.

Table 6.2 shows that solutions with $\epsilon = -1, k^2 = 0.99$ also satisfy Lemma 6.2.1 and those with $\epsilon = -1, k^2 \in \{1.01, 1.02\}$ have both x_1 and x_3 negative at some finite value of ζ and, therefore, satisfy Lemma 6.2.2. This switching of behaviour of the solutions as the value of k^2 crosses 1 is interesting and is demonstrated in Fig. 6.3 below. Recall that $k^2 = 1$ is an element of the exceptional set K. This switching of behaviour occurs at other elements of K also, which suggests that these values have some significance relative to the dynamical system.

k^2	δ	α	m	Ω	ζ_0	$\bar{x}_1(\zeta_0)$	$\bar{x}_3(\zeta_0)$	$e_N(\zeta_0) <$
0.98	25	50	.919	93.1	.028	-.114	.550	10^{-16}
0.99	34	70	.930	96.4	.022	-.145	.593	10^{-12}
1.01	33	68	.917	98.3	.025	-.05	-.602	10^{-6}
1.02	25	40	.992	79.7	.036	-.0151	-.495	10^{-6}

Table 6.2: Information for solutions with $k^2 \in \{0.98, 0.99, 1.01, 1.02\}$ and $V_0 < 0$ showing that Lemma 6.2.1 holds for these values which are less than one and that Lemma 6.2.2 holds for these values that are greater than one.

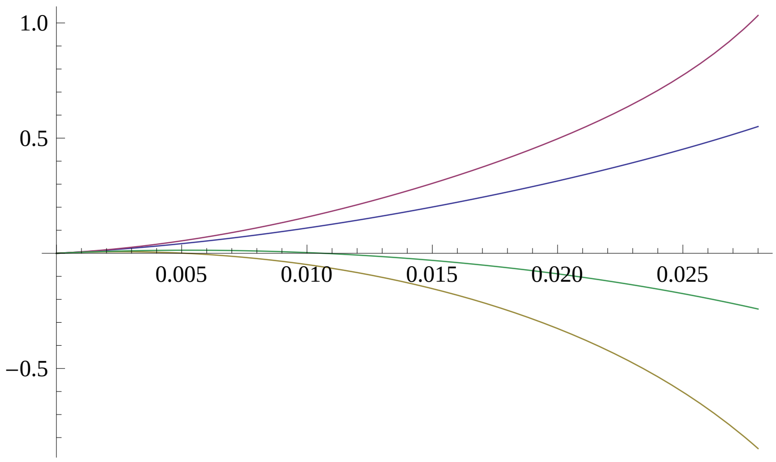

Figure 6.3: Plot of approximation to solutions for x_3 with $k^2 \in \{0.98, 0.99, 1.01, 1.02\}$ and $V_0 < 0$ over the interval $\zeta \in [0, 0.16]$. The behaviour of x_3 is markedly different for values of k^2 either side of 1.

We complete the section with two short tables showing that Lemmas 6.2.3 and 6.2.4 are satisfied for some values of k^2. In table 6.3, $\hat{\Lambda} = (1 - \bar{\Lambda})\bar{x}_3(t_0) + \bar{\Lambda} - \bar{\Lambda}^{1/2}$ where $\bar{\Lambda} = \bar{x}_1/(\bar{x}_1 - 1)$. We note that it is trivial to derive a bound on the truncation error for w and z which is of the order of e_N.

k^2	δ	α	m	Ω	ζ_0	$\hat{\Lambda}$	$e_N(\zeta_0) <$
0.84	12.5	30	.904	97.4	.072	.0116	10^{-5}
0.87	13	30	.867	96.3	.068	.0239	10^{-6}
0.90	13.5	30	.947	95.4	.067	.0123	10^{-5}

Table 6.3: Information for solutions with $k^2 \in \{0.84, 0.87, .90\}$ and $V_0 > 0$ showing that Lemma 6.2.3 applies in each case.

k^2	δ	α	m	Ω	ζ_0	$\bar{w} - A^{-1}$	$-(\bar{\Delta}k^2)^{-1}$	$\bar{z}_0$	$\lvert\lambda\rvert^{-1}$	$\bar{z}'_0$	$e_N(\zeta_0) <$
1.30	7.7	11	.982	99.0	.012	-.132	1.91	2.76	2.86	-.0328	10^{-4}
1.31	8.2	11	.948	95.2	.11	-.159	1.99	2.84	2.90	-.0424	10^{-5}
1.32	9	12.5	.976	99.4	.105	-.281	1.87	2.92	2.94	-.189	10^{-5}

Table 6.4: Information for solutions with $k^2 \in \{1.30, 1.31, 1.32\}$ and $V_0 > 0$ showing that Lemma 6.2.4 applies in each case.

Conclusions and further work

As stated in the introduction, the main result of this thesis is that the singularity resulting in the collapse of self-similar, cylindrically symmetric scalar field is censored for all values of the parameters and initial data of the Einstein equations relevant to these spacetimes. This is interesting as it differs in this way to many other models of collapse in both spherical and cylindrical self-similar spacetimes.

In the future we would like to attempt to analyse the more general analogue of this spacetime in spherical symmetry, that is, the collapse of a non-minimally coupled scalar field in self-similar, spherical symmetry. This would build on the work of Christodoulou who considered the minimally coupled case in [8]. We would also like to develop the method presented in Chapter 6 by generalising it to systems of the form

$$\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{x}^T B\boldsymbol{x}, \tag{6.3.1}$$

with constant matrices A and B.

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