Worst-Case Portfolio Optimization with Proportional Transaction Costs

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Abstract

We study optimal asset allocation in a crash-threatened financial market with proportional transaction costs. The market is assumed to be in either a normal state, in which the risky asset follows a geometric Brownian motion, or in a crash state, in which the price of the risky asset can suddenly drop by a certain relative amount. We only assume the maximum number and the maximum relative size of the crashes to be given and do not make any assumptions about their distributions. For every investment strategy, we identify the worst-case scenario in the sense that the expected utility of terminal wealth is minimized. The objective is then to determine the investment strategy which yields the highest expected utility in its worst-case scenario.

We solve the problem for utility functions with constant relative risk aversion using a stochastic control approach. We characterize the value function as the unique viscosity solution of a second-order nonlinear partial differential equation. The optimal strategies are characterized by time-dependent free boundaries which we compute numerically. The numerical examples suggest that it is not optimal to invest any wealth in the risky asset close to the investment horizon, while a long position in the risky asset is optimal if the remaining investment period is sufficiently large.

Keywords: Portfolio optimization, worst-case scenarios, crash modeling, transaction costs, dynamic programming, viscosity solutions

AMS Subject Classification: 91B28, 91A15, 90C39, 35D05

1 Introduction

The continuous-time portfolio optimization problem was introduced by Merton in his seminal articles [25] and [26]. Since then, this classical model has been extended in numerous directions in order to account for a vast variety of different investment objectives, trading constraints and market conditions. In the present paper we make a first step towards unifying two streamlines

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which emerged from Merton's original work: Worst-case portfolio optimization as first introduced by Korn and Wilmott [21] and optimal trading in the presence of proportional transaction costs as first analyzed by Magill and Constantinides [24].

1.1 The worst-case approach to portfolio optimization

In Merton's papers [25] and [26], the price processes of the risky assets have continuous sample paths which do not fully explain extreme price movements (especially sudden downward movements, i.e. *crashes*). Even though Merton's results can be extended to price processes with jumps (see e.g. Merton [27] or Aase [1]), the investor may still suffer substantial losses at the moment a crash occurs, since the optimal strategy hedges crashes only on average over the investment period. Korn and Wilmott [21] proposed a model which does not only include crash scenarios, but also leads to optimal strategies which make the investor indifferent about the occurrence of the worst possible crash scenario. Instead of maximizing the expected utility of terminal wealth, the so-called *worst-case bound* of a trading strategy is used as optimization criterion. That is, for each admissible trading strategy, a worst-case scenario is determined, and the strategy yielding the highest expected utility in its worst-case scenario is considered optimal.

The model of Korn and Wilmott has been extended in several directions. A recent overview can be found in Korn and Seifried [19]. Originally, Korn and Wilmott [21] derived the optimal trading strategy using an indifference argument. Korn and Menkens [18] formalized this balancing argument by a dynamic programming argument and thereby embedded the worst-case approach into the stochastic control framework. Korn and Steffensen [20] derived a verification theorem which proves optimality in a bigger class of admissible strategies. Finally, Seifried [29] solved the worst-case problem for more general price processes by means of martingale optimality arguments, Desmettre et al. [13] extended the results to optimal consumption over an infinite time horizon, and Belak et al. [4] extend the model to allow for an unbounded number of crashes.

Korn and Wilmott [21] showed that the optimal risky fraction (i.e. the optimal fraction of the total wealth invested in the risky asset) under the threat of crashes is time-dependent and can be determined as a solution of an ordinary differential equation. Assuming that the excess return of the risky asset over the risk-free asset is positive, this solution is strictly positive before terminal time and decreases as the investment horizon is approached. At terminal time, the optimal risky fraction is zero.

1.2 Portfolio optimization in the presence of transaction costs

All of the above-mentioned models have one feature in common. In order to apply the optimal strategies in these models, the investor would have to adjust the position in the risky asset at every time instant and the trading volume would be of infinite variation. However, since an investor usually has to pay a fee in order to engage in a transaction, this would lead to immediate bankruptcy. Therefore, efforts have been made to include different types of cost structures in Merton's model. In this paper, we assume that the investor faces transaction costs which are proportional to the volume of the transaction.

Due to the extensive number of research articles on portfolio optimization with proportional transaction costs, we only give a quick overview of the articles closest to ours. The interested reader is referred to e.g. Kallsen and Muhle-Karbe [17], Gerhold et al. [14], Choi et al. [7] and the references therein for a different approach to tackle the problem.

The treatment of the portfolio problem under proportional costs in a continuous-time model was initiated by Magill and Constantinides [24] and insights were gained on the nature of the optimal strategy. A rigorous solution can be found in the seminal article of Davis and Norman [10]. Shreve and Soner [30] obtained similar results under weaker assumptions using a viscosity solution approach. A generalization of this model was studied by Kabanov and Klüppelberg [16] and de Vallière and Kabanov [12]. All five papers considered optimal consumption over an infinite (time) horizon. The optimal terminal wealth problem with a finite horizon was first treated in Akian et al. [2]. Davis et al. [11] considered the same dynamic programming equation in an option pricing setting, showed that the value function is a viscosity solution thereof and proved uniqueness in the case of exponential utility. Dai and Yi [9] derived the existence of a regular solution of the dynamic programming equation and Belak et al. [5] proved uniqueness of the value function for a more general class of utility functions. Kunisch and Sass [22] and Herzog et al. [15] proposed algorithms to approximate the value function and the optimal strategies numerically. Liu and Loewenstein [23] obtained a closed form solution under the assumption that the terminal time is random and Bichuch [6] studied the finite-horizon problem by means of an asymptotic analysis.

In a market with proportional transaction costs, the optimal strategy is to keep the fraction of wealth invested in the risky asset inside a certain no-trading region. Only when the boundary of this region is reached does the investor engage in an infinitesimal transaction in order to keep the risky fraction just inside the region. In the optimal consumption model with infinite horizon, the no-trading region is simply an interval, whereas in the finite-horizon problem of optimizing expected utility of terminal wealth the boundaries of the no-trading region are time-dependent.

1.3 Outline of this paper

In this paper, we combine the two different approaches outlined above. That is, we consider a financial market which is under the threat of a crash in the risky asset and where the investor pays a fee proportional to the size of the transaction.

This leads to a stochastic differential game (to be more precise, our model is an instance of Wald's maximin model, see e.g. Wald [31]) between the investor and some opponent (which in the sequel will be assumed to be the market), acting on a two-dimensional state process. The investor is allowed to choose a singular control as to maximize the expected utility of terminal wealth. On the other hand, the market chooses impulse controls consisting of a finite sequence of stopping times and bounded relative crash sizes with the objective of minimizing terminal wealth. The nature of our problem allows us to immediately deduce the optimal relative crash size and hence reduces the market's controls to finite sequences of stopping times, leading to an iterated optimal stopping problem from the market's point of view. The presence of transaction costs and the presence of crashes lead naturally to a constrained state space which changes with every crash.

In Section 2, we give a precise formulation of our model and state the optimization problem. We gather some basic properties of the value function in Section 3. In particular, we show that the value function is finite, concave, homothetic, and continuous. In Section 4, we give a proof of the dynamic programming principle (Theorem 4.5). Then we show in Section 5 that the value function is a viscosity solution of a certain nonlinear second-order partial differential equation (Theorem 5.3) and address the issue of uniqueness in Corollary 5.8. In Section 6, we conclude this paper with numerical examples.

The numerical results suggest that some of the features of the optimal strategy in the presence of crashes change compared to the crash-free and/or zero-costs strategies. For example, we always observe that the upper boundary of the no-trading region falls below the optimal strategy in the case without transaction costs close to terminal time, which in the absence of crashes can only occur when leverage is optimal. Moreover, not only is the buy boundary zero before the investment horizon T is reached (as in the no-crash case), but the sell boundary is also zero strictly before the investment horizon T (unlike in the no-crash case). With other words, the optimal worst-case strategy in the presence of transaction costs is not to invest in the risky asset close to the investment horizon.

2 Market model and problem formulation

In this section, we specify the market model and formulate the optimization problem.

2.1 The market model

We consider a market consisting of two assets, namely a risk free asset P_0 called *bond* and a risky asset P_1 called *stock*. On the finite time interval $[t,T]$, we assume that in *normal times* (i.e. in crash-free times) the prices of the two assets evolve according to

$$\begin{aligned} dP_0(u) &= rP_0(u-)du, && P_0(t) = p_0, \\ dP_1(u) &= \alpha P_1(u-)du + \sigma P_1(u-)dW(u), && P_1(t) = p_1. \end{aligned}$$

We refer to $r \geq 0$ as the *interest rate*, $\alpha \in \mathbb{R}$ as the *trend* and $\sigma > 0$ as the *volatility* of the stock. We assume that W is a standard Brownian motion defined on the canonical Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = C_0([0,\infty))$ denotes the space of continuous functions $\omega : [0,\infty) \to \mathbb{R}$ satisfying $\omega(0) = 0$ and where $\mathbb{P}$ denotes the Wiener measure. We denote the augmented filtration generated by W by $\mathbb{F} = \{\mathcal{F}(u)\}_{u\geq 0}$, and, similarly, we denote by $\mathbb{F}^t = \{\mathcal{F}^t(u)\}_{u\geq t}$ the augmented filtration generated by $\{W(u) - W(t)\}_{u\geq t}$.

The investor's wealth invested in the bond and the stock at time t are denoted by b and s, respectively. We assume that whenever the investor buys or sells stocks, she has to pay a fee proportional to the size of the transaction. That is, if the investor buys stocks for Δs units of money, she has to pay transaction costs of size $\lambda \Delta s$, where $\lambda \in (0,\infty)$. We assume that the investor pays these costs from the bond account. Therefore, after the transaction, she holds $b - (1+\lambda)\Delta s$ units of money in the bond and $s + \Delta s$ units of money in the stock. Similarly, if the investor sells stocks for Δs units of money, she has to pay transaction costs of size $\mu \Delta s$, where $\mu \in (0,1)$.

In the absence of crashes, we can therefore model the investor's trading strategies as follows. Given an initial time $t \in [0,T)$, let L and M be two $\mathbb{F}^t$-adapted, non-decreasing, càdlàg processes (i.e. right-continuous paths with left limits) defined on $[t,T]$, and set $L(t-) = M(t-) = 0$. If L and M represent the cumulative units of money used for stock purchases and sales, respectively, the investor's wealth invested in bond and stock, denoted by B and S, respectively, follows

$$dB(u) = rB(u-)du - (1+\lambda)dL(u) + (1-\mu)dM(u), \qquad u \in [t,T], \tag{1}$$
$$dS(u) = \alpha S(u-)du + \sigma S(u-)dW(u) + dL(u) - dM(u), \qquad u \in [t,T], \tag{2}$$

where we set the initial values to be $B(t-) = b$ and $S(t-) = s$.

A *crash* is modeled as a pair $(\tau, \beta(\tau))$ consisting of a *crash time* τ and an $\mathcal{F}^t(\tau)$-measurable *crash size* $\beta(\tau) \in [0,\beta]$ for some maximal deterministic crash size $\beta \in (0,1)$. τ is assumed to be a $[t,T] \cup \{\infty\}$-valued $\mathbb{F}^t$-stopping time. On $\{\tau \leq T\}$, we assume that the stock price drops by a fraction of $\beta(\tau)$ at time τ, i.e.

$$P_1(\tau) = \big(1 - \beta(\tau)\big)P_1(\tau-).$$

In particular, given a strategy (L,M), this implies for the investor's stock position that

$$S(\tau) = \big(1 - \beta(\tau)\big)S(\tau-) + \big(L(\tau) - L(\tau-)\big) - \big(M(\tau) - M(\tau-)\big).$$

We interpret the event $\{\tau = \infty\}$ as the crash $(\tau, \beta(\tau))$ not occurring within the investment period $[t, T]$. *We assume for simplicity that the crash size is constant and equal to the maximum crash size β.* In light of Korn and Steffensen [20, Remark 1(a)], this does not pose any restriction on our model, since from a worst-case perspective, the optimal crash scenario is either a crash of maximal size β or no crash at all. We denote the set of all crash times of the above form by $\mathcal{B}(t)$.

Throughout this paper, we assume that within the investment period $[t, T]$, at most one crash can occur. However, our results can be extended to the general case of at most n crashes by an iterative procedure.

A crucial point about the definition of a crash is that we do not assume that it has a specific distribution (neither the crash time, nor the crash size). Instead, a crash is regarded as a control variable which can be chosen as to minimize the expected utility of terminal wealth. This will be made more precise in Section 2.2.

In the presence of crashes, the model for the trading strategies becomes more involved, since we want to allow the investor to observe crashes and to switch possibly to a different strategy afterwards. However, since the investor does not know a priori when a crash occurs, she has to choose a whole family of post-crash strategies $\check{\pi} = \{(\check{L}^\tau, \check{M}^\tau)\}_{\tau\in\mathcal{B}(t)}$ and apply the strategy $\check{\pi}^\tau = (\check{L}^\tau, \check{M}^\tau)$ if a crash is observed at time τ. For each $\tau \in \mathcal{B}(t)$, we assume that the pair $(\check{L}^\tau, \check{M}^\tau)$ is $\mathbb{F}^\tau$-adapted and we set $\check{L}^\tau(\tau-) = \check{M}^\tau(\tau-) = 0$. With this setup, the investor is able to observe crashes and react on the new information made available to her. Note that this approach is the same as in Seifried [29].

In order to simplify notations, we write $\pi = (L, M)$ for pre-crash strategies and $\check{\pi} = \{(\check{L}^\tau, \check{M}^\tau)\}_{\tau\in\mathcal{B}(t)}$ for a family of post-crash strategies. More generally, we make the following convention: if we denote a pre-crash quantity by $\aleph$, we denote the corresponding post-crash quantity by $\check{\aleph}$.

Given a pre-crash trading strategy $\pi = (L, M)$, a family of post-crash strategies $\check{\pi} = \{(\check{L}^\tau, \check{M}^\tau)\}_{\tau\in\mathcal{B}(t)}$ and a crash $\tau \in \mathcal{B}(t)$, the investor's bond position on $\{\tau < \infty\}$ with initial position $B(t-) = b$ is given by

$$dB(u) = rB(u-)du - (1 + \lambda)dL(u) + (1 - \mu)dM(u), \qquad u \in [t, \tau), \tag{3}$$

$$B(\tau) = B(\tau-) - (1 + \lambda)\check{L}^\tau(\tau) + (1 - \mu)\check{M}^\tau(\tau), \tag{4}$$

$$dB(u) = rB(u-)du - (1 + \lambda)d\check{L}^\tau(u) + (1 - \mu)d\check{M}^\tau(u), \qquad u \in [\tau, T]. \tag{5}$$

Similarly, the investor's stock position on $\{\tau < \infty\}$ with initial position $S(t-) = s$ is given by

$$dS(u) = \alpha S(u-)du + \sigma S(u-)dW(u) + dL(u) - dM(u), \qquad u \in [t, \tau), \tag{6}$$

$$S(\tau) = (1 - \beta)S(\tau-) + \check{L}^\tau(\tau) - \check{M}^\tau(\tau), \tag{7}$$

$$dS(u) = \alpha S(u-)du + \sigma S(u-)dW(u) + d\check{L}^\tau(u) - d\check{M}^\tau(u), \qquad u \in (\tau, T]. \tag{8}$$

Remark 2.1. *Observe that Equation* (7) *is set up such that the crash is executed first, since it is applied to $S(\tau-)$. The control of the investor $(\check{L}^\tau, \check{M}^\tau)$ is applied only thereafter. Thus, the investor can only react to a crash, but she cannot prevent being negatively affected by a crash at time τ by selling all risky holdings at time τ, since her transaction is executed after the crash.*

The *net wealth* X of the investor after liquidation of the stock position at time u is given by

$$X(u) := \begin{cases} B(u) + (1 - \mu)S(u), & \text{if } S(u) > 0, \\ B(u) + (1 + \lambda)S(u), & \text{if } S(u) \leq 0. \end{cases}$$

It is sometimes necessary to stress the dependence of B, S, and X on the initial values, the trading strategy and the crash. We denote by

$$X_{t,b,s}^{\pi,\breve{\pi},\tau}(u)$$

the net wealth at time u, if $B(t-) = b$, $S(t-) = s$, the pre-crash trading strategy is π, the family of post-crash strategies is $\breve{\pi}$ and the crash occurs at time τ. We may drop some of the indices if they are clear from the context. Similarly, this applies to B and S in the same way.

2.2 Problem formulation

We require admissible strategies to lead to a non-negative net wealth. Taking into account that in case of $s > 0$ a crash decreases the net wealth and that in case $s < 0$ a crash increases the net wealth, the following open *solvency regions* can be defined:

$$\mathcal{S}^1 := \left\{(b,s) \in \mathbb{R}^2 \,\middle|\, b + (1+\lambda)s > 0, b + (1-\mu)(1-\beta)s > 0\right\},$$
$$\mathcal{S}^0 := \left\{(b,s) \in \mathbb{R}^2 \,\middle|\, b + (1+\lambda)s > 0, b + (1-\mu)s > 0\right\}.$$

So, whenever $(b,s) \in \overline{\mathcal{S}}^1$ (here $\overline{A}$ denotes the closure of a set A), the investor can liquidate the stock holdings and end up with non-negative wealth, even if a crash occurs momentarily. The boundaries of the solvency regions are parametrized as follows:

$$\partial\mathcal{S}^1_- := \partial\mathcal{S}^0_- := \left\{(b,s) \in \mathbb{R}^2 \middle| s \leq 0, b + (1+\lambda)s = 0\right\},$$
$$\partial\mathcal{S}^1_+ := \left\{(b,s) \in \mathbb{R}^2 \middle| s > 0, b + (1-\mu)(1-\beta)s = 0\right\},$$
$$\partial\mathcal{S}^0_+ := \left\{(b,s) \in \mathbb{R}^2 \middle| s > 0, b + (1-\mu)s = 0\right\}.$$

Figure 1 sketches the location of the boundaries of the solvency regions for large transaction costs (to emphasize the qualitative features of $\mathcal{S}^1$ and $\mathcal{S}^0$).

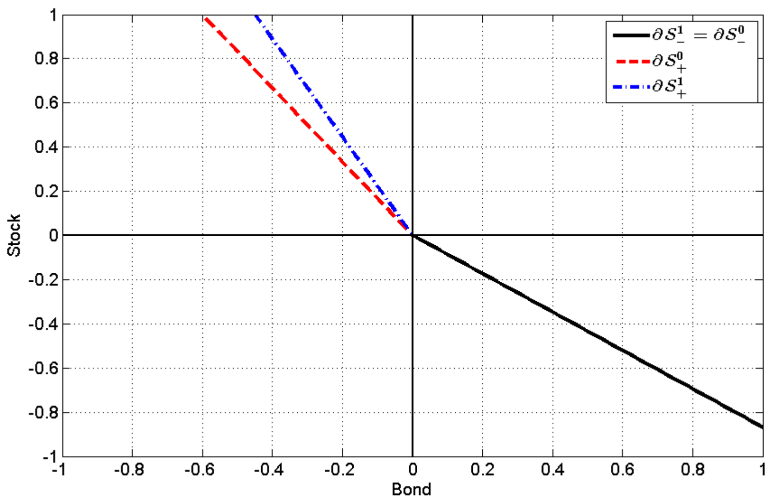

Figure 1: Boundaries of the solvency regions for $\mu = 0.2$, $\lambda = 0.15$ and $\beta = 0.25$.

Let $t \in [0,T]$. A pre-crash trading strategy π is called *admissible* for initial positions $(b,s) \in \overline{\mathcal{S}}^1$, if the pair (B,S) given by Equations (3) and (6) with initial values $B(t-) = b$ and $S(t-) = s$ and for $\tau \equiv \infty$ takes values in $\overline{\mathcal{S}}^1$ for all $u \in [t,T]$. The set of all admissible pre-crash trading strategies of this form is denoted by $\mathcal{A}(t,b,s)$.

A family of post-crash strategies $\check{\pi} = \{(\check{L}^\tau, \check{M}^\tau)\}_{\tau \in \mathcal{B}(t)}$ corresponding to a pre-crash strategy $\pi \in \mathcal{A}(t,b,s)$ is called *admissible*, if for every $\tau \in \mathcal{B}(t)$ and for every $u \in [\tau, T] \cap [t, T]$, the corresponding pair (B, S) given by Equations (3) to (8) takes values in $\overline{\mathcal{S}}^0$. The set of all admissible families of post-crash trading strategies of this form is denoted by $\check{\mathcal{A}}(\pi)$.

With this, the objective of this paper is to maximize the worst-case expected utility of terminal wealth, i.e.

$$\sup_{\substack{\pi \in \mathcal{A}(0,b,s) \\ \check{\pi} \in \check{\mathcal{A}}(\pi)}} \inf_{\tau \in \mathcal{B}(0)} E\left[U_p\left(X_{0,b,s}^{\pi,\check{\pi},\tau}(T)\right)\right] \tag{9}$$

for a utility function $U_p : (0,\infty) \to \mathbb{R}$ of the form

$$U_p(x) := \begin{cases} x^p/p & \text{if } p < 1, p \neq 0, \\ \log(x) & \text{if } p = 0. \end{cases}$$

We extend U_p to $[0,\infty)$ by setting

$$U_p(0) := \begin{cases} 0 & \text{if } 0 < p < 1, \\ -\infty & \text{if } p \leq 0. \end{cases}$$

It is sometimes helpful to interpret the optimization problem as a game between the investor and the market. The investor decides on a trading strategy and aims to maximize expected utility of terminal wealth, whereas the market decides on a crash strategy with the objective of minimizing the investor's expected utility of terminal wealth.

Let $0 \leq t < T$, $(b,s) \in \overline{\mathcal{S}}^1$ and let $\pi \in \mathcal{A}(t,b,s)$, $\check{\pi} \in \check{\mathcal{A}}(\pi)$ and $\tau \in \mathcal{B}(t)$. We define the *performance criterion* of π, $\check{\pi}$ and τ by

$$\mathcal{J}(\pi, \check{\pi}, \tau, t, b, s) := E\left[U_p\left(X_{t,b,s}^{\pi,\check{\pi},\tau}(T)\right)\right].$$

The *worst-case bound* of π and $\check{\pi}$ is defined as

$$\mathcal{W}(\pi, \check{\pi}, t, b, s) := \inf_{\tau \in \mathcal{B}(t)} \mathcal{J}(\pi, \check{\pi}, \tau, t, b, s).$$

Finally, the *value function* in the crash-threatened market is defined by

$$\mathcal{V}(t,b,s) := \sup_{\substack{\pi \in \mathcal{A}(t,b,s) \\ \check{\pi} \in \check{\mathcal{A}}(\pi)}} \mathcal{W}(\pi, \check{\pi}, t, b, s).$$

Let $(\check{L}, \check{M})$ be a single admissible post-crash strategy starting in t for initial values $(b,s) \in \overline{\mathcal{S}}^0$. Abusing notations, we write $\check{\pi} = (\check{L}, \check{M})$ and we denote the set of all admissible post-crash strategies of this form by $\check{\mathcal{A}}(t,b,s)$. Then, we can define the performance criterion of $\check{\pi}$ in the crash-free market by

$$\check{\mathcal{J}}(\check{\pi}, t, b, s) := E\left[U_p\left(X_{t,b,s}^{\check{\pi}}(T)\right)\right].$$

Similarly, the value function in the crash-free market is denoted by

$$\check{\mathcal{V}}(t,b,s) := \sup_{\check{\pi} \in \check{\mathcal{A}}(t,b,s)} \check{\mathcal{J}}(\check{\pi}, t, b, s).$$

3 Basic properties of the value function

In this section we analyze some of the basic properties of the value function $\mathcal{V}$. Note that most of the results in this section are very similar to the results obtained by Shreve and Soner [30]. We therefore keep the exposition to a minimum.

3.1 Some basic properties of the value function

Since the investor can always liquidate the stock position immediately at initial time t and stop trading afterwards, we naturally obtain a lower bound on the value function. Furthermore, this strategy is the only admissible (and hence optimal) strategy on the boundary of the solvency region.

Lemma 3.1. *1. Let $(b,s) \in \partial\mathcal{S}^1$. Then the only admissible strategy is to instantly jump to the position $(0,0)$ and remain there.*

2. For $(b,s) \in \overline{\mathcal{S}}^1$, the trading strategy of instantly closing the stock position and no trading afterwards is an admissible strategy. Furthermore, for every $(b,s) \in \overline{\mathcal{S}}^1$, we have

$$\mathcal{V}(t,b,s) \geq \begin{cases} U_p\left((b+(1+\lambda)s)\, e^{r(T-t)}\right), & \text{if } s \leq 0, \\ U_p\left((b+(1-\mu)(1-\beta)s)\, e^{r(T-t)}\right), & \text{if } s > 0. \end{cases} \tag{10}$$

Proof. 1. The proof is very similar to the proof of [30, Remark 2.1] by Shreve and Soner and will thus not be reproduced here. The only additional difficulty arises due to the presence of crashes. This can be handled as follows:

a) If $(b,s) \in \partial\mathcal{S}^1_-$, then $s \leq 0$. In this case a crash would be beneficial for the investor in the sense that the net wealth increases. Thus, it cannot be optimal from the market's point of view to trigger a crash. At this point the proof follows exactly as in Shreve and Soner [30, Remark 2.1].

b) If $(b,s) \in \partial\mathcal{S}^1_+$, then it must be optimal for the market to crash immediately. To see this, note that in this case the investor's position after the crash at time t is given by $(b,(1-\beta)s) \in \partial\mathcal{S}^0_+$. Since we are in the crash-free market at this point, following Shreve and Soner [30, Remark 2.1], we can conclude that the only admissible strategy is to close the stock position and that the crash is indeed optimal, since it leads to a wealth of $X(T) = 0$.

2. See Shreve and Soner [30, Remark 2.2]. Note that for $s > 0$, the worst-case crash scenario is an immediate crash at time t, since afterwards, once the stock position is closed, crashes do not affect the net wealth. This explains the factor $(1-\beta)$ in the second case of Inequality (10). □

If $(b,s) \in \overline{\mathcal{S}}^1$, Lemma 3.1 allows us to restrict the sets of admissible strategies $\mathcal{A}(t,b,s)$ and $\check{\mathcal{A}}(\pi)$ to those strategies π and $\check{\pi}$, which have a worst-case bound satisfying

$$\mathcal{W}(\pi,\check{\pi},t,b,s) \geq \begin{cases} U_p\big((b+(1+\lambda)s)e^{r(T-t)}\big) & \text{if } s \leq 0, \\ U_p\big((b+(1-\mu)(1-\beta)s)e^{r(T-t)}\big) & \text{if } s > 0. \end{cases}$$

Misusing notations, we denote the sets of such strategies again by $\mathcal{A}(t,b,s)$ and $\check{\mathcal{A}}(\pi)$. We therefore have for all pre-crash strategies $\pi \in \mathcal{A}(t,b,s)$ and post-crash strategies $\check{\pi} \in \check{\mathcal{A}}(\pi)$:

$$\mathcal{W}(\pi,\check{\pi},t,b,s) = -\infty, \qquad \text{if and only if} \qquad (b,s) \in \partial\mathcal{S}^1 \text{ and } p \leq 0.$$

The next proposition gathers some further properties of the value function $\mathcal{V}$.

Lemma 3.2. *1. Let $\gamma \in [1-\mu, 1+\lambda]$ and define*

$$\varphi_\gamma(t,b,s) := \frac{1}{p}(b+\gamma s)^p \exp\left(p\left[r + \frac{(\alpha-r)^2}{2(1-p)\sigma^2}\right](T-t)\right)$$

if $p < 1, p \neq 0$ *and*

$$\varphi_\gamma(t,b,s) := \log(b+\gamma s) + \left[r + \frac{(\alpha - r)^2}{2\sigma^2} \right] (T-t)$$

if $p = 0$. *Then* $\mathcal{V} \leq \check{\mathcal{V}} \leq \varphi_\gamma < +\infty$.

2. *For every* $(t,b,s) \in [0,T] \times \overline{\mathcal{S}}^1$, *we have*

$$\mathcal{V}(t,b,s) \leq \check{\mathcal{V}}(t,b,(1-\beta)s). \tag{11}$$

3. *Let* $t \in [0,T]$. *Then* $\mathcal{V}(t,\cdot,\cdot)$ *is concave on* $\overline{\mathcal{S}}^1$. *In particular,* $\mathcal{V}(t,\cdot,\cdot)$ *is locally Lipschitz-continuous on* $\mathcal{S}^1$.

4. *For every* $\kappa > 0$ *and* $(b,s) \in \overline{\mathcal{S}}^1$, *we have*

$$\mathcal{V}(t,\kappa b,\kappa s) = \begin{cases} \kappa^p \mathcal{V}(t,b,s) & \text{if } p < 1, p \neq 0, \\ \log(\kappa) + \mathcal{V}(t,b,s) & \text{if } p = 0. \end{cases}$$

Proof. 1. The relation $\mathcal{V} \leq \check{\mathcal{V}}$ is obvious. The inequality $\check{\mathcal{V}} \leq \varphi_\gamma$ can be proved by similar arguments as in Shreve and Soner [30, Proposition 5.1], since φ_γ is a classical supersolution of the Dynamic Programming Equation (27).

2. Consider the crash time $\tau^* \equiv t$. Then

$$\begin{aligned} \mathcal{V}(t,b,s) &\leq \sup_{\substack{\pi \in \mathcal{A}(t,b,s) \\ \check{\pi} \in \check{\mathcal{A}}(\pi)}} E\left[U_p\left(X^{\pi,\check{\pi},\tau^*}_{t,b,s}(T) \right) \right] \\ &= \sup_{\check{\pi} \in \check{\mathcal{A}}(t,b,(1-\beta)s)} E\left[U_p\left(X^{\check{\pi}}_{t,b,(1-\beta)s}(T) \right) \right] = \check{\mathcal{V}}\big(t,b,(1-\beta)s\big). \end{aligned}$$

3. The concavity is inherited from the utility function U_p. The details can be found in Shreve and Soner [30, Proposition 3.1]. Note that every concave function is locally Lipschitz-continuous in the interior of its domain.

4. The result follows from the linearity of the bond and stock dynamics and using the homotheticity of U_p, see also Shreve and Soner [30, Proposition 3.3]. □

Remark 3.3. *The interpretation of Inequality* (11) *is not to be confused with a similar result which holds in the absence of costs. In this case, it is known that the optimal trading strategy renders the investor indifferent about the occurrence of a crash. That is, if* π^* *denotes the optimal pre-crash strategy without transaction costs,* v *and* $\check{v}$ *the value functions in the absence of costs, and if* x *denotes the investor's total wealth (that is* $x = b + s$*) at time* t, *then*

$$v(t,x) = \check{v}(t,(1-\beta\pi^*_t)x)$$

holds. For the transaction cost problem, we cannot expect the investor to be indifferent about the occurrence of a crash whenever

$$\mathcal{V}(t,b,s) = \check{\mathcal{V}}(t,b,(1-\beta)s)$$

holds, since the right-hand side of the equation does not explicitly depend on the optimal trading strategy before the crash! However, we can expect that if we have equality, then the investor is *usually exposed to the threat of a crash. More specifically, we expect that indifference holds only on the boundary of the set*

$$\big\{(b,s)\in\mathcal{S}^1 : \mathcal{V}(t,b,s)=\check{\mathcal{V}}(t,b,(1-\beta)s)\big\},$$

and that an optimal crash time from the market's point of view (assuming that the investor also acts optimally) is given by the first hitting time of this set, i.e.

$$\tau^* := \inf\Big\{u\geq t : \mathcal{V}\big(u,B(u),S(u)\big)=\check{\mathcal{V}}\big(u,B(u),(1-\beta)S(u)\big)\Big\}.$$

This is to be expected by the classical optimal stopping theory (see e.g. Peskir and Shiryaev [28]).

3.2 Continuity of the value function

The aim of this section is to prove that $\mathcal{V}$ and $\check{\mathcal{V}}$ are continuous. We prove the continuity of $\mathcal{V}$ in detail, whereas the continuity of $\check{\mathcal{V}}$ is proved in Belak et al. [5], or follows by imitating the arguments leading to the continuity of $\mathcal{V}$. We start by proving a time-shifting property of $\mathcal{V}$, which will allow us to establish the time-continuity by varying the terminal time T instead of the initial time t.

Lemma 3.4. *Denote the value function corresponding to the terminal time T by $\mathcal{V}_T$. Let $t\in[0,T]$ and $h\geq -t$. Then*

$$\mathcal{V}_T(t,b,s)=\mathcal{V}_{T+h}(t+h,b,s),\qquad (b,s)\in\mathcal{S}^1.$$

Proof. We denote by $\mathcal{A}_T(t,b,s)$, $\check{\mathcal{A}}_T(\pi)$, and $\mathcal{B}_T(t)$ the respective sets of admissible strategies corresponding to terminal time T. We will show that there is a one-to-one correspondence between the sets $\mathcal{A}_T(t,b,s)$ and $\mathcal{A}_{T+h}(t+h,b,s)$. By similar arguments, one can then show that there is also a one-to-one correspondence between $\check{\mathcal{A}}_T(\pi)$ and $\check{\mathcal{A}}_{T+h}(\pi)$, and $\mathcal{B}_T(t)$ and $\mathcal{B}_{T+h}(t+h)$, respectively. From this, it easily follows that $\mathcal{V}_T(t,b,s)=\mathcal{V}_{T+h}(t+h,b,s)$.

Let therefore $\pi=(L,M)\in\mathcal{A}_T(t,b,s)$. Since Ω is assumed to be the canonical space and since L and M are $\mathbb{F}^t$-adapted, it follows that we can write

$$L(u,\omega)=L\big(u,\omega([t+\cdot]\wedge u)-\omega(t)\big),\quad M(u,\omega)=M\big(u,\omega([t+\cdot]\wedge u)-\omega(t)\big),$$

where $\omega\in\Omega$ and $u\in[t,T]$. Now define

$$\begin{aligned} L_h(u,\omega) &:= L\big(u-h,\omega([t+h+\cdot]\wedge u)-\omega(t+h)\big), &\qquad \omega\in\Omega,\ u\in[t+h,T+h],\\ M_h(u,\omega) &:= M\big(u-h,\omega([t+h+\cdot]\wedge u)-\omega(t+h)\big), &\qquad \omega\in\Omega,\ u\in[t+h,T+h].\end{aligned}$$

Then (L_h,M_h) is $\mathbb{F}^{t+h}$-adapted and hence $(L_h,M_h)\in\mathcal{A}_{t+h}(t+h,b,s)$. Since we can similarly construct admissible strategies for terminal time T from strategies with terminal time $T+h$, there is a one-to-one correspondence between the two sets $\mathcal{A}_T(t,b,s)$ and $\mathcal{A}_{T+h}(t+h,b,s)$. $\square$

Lemma 3.5. *Let $\pi\in\mathcal{A}(t,b,s)$, $\check{\pi}\in\check{\mathcal{A}}(\pi)$, and $\tau\in\mathcal{B}(t)$ and denote*

$$B(u):=B^{\pi,\check{\pi},\tau}_{t,b,s}(u),\qquad S(u):=S^{\pi,\check{\pi},\tau}_{t,b,s}(u).$$

Assume that τ is such that $S(\tau)\leq S(\tau-)$.

1. *There exists a constant $C_0>0$, such that*

$$E\left[B(T)+S(T)\right]=E\left[|B(T)+S(T)|\right]\leq C_0(b+s).$$

2. There exists a constant $C_1 > 0$, such that

$$E\left[(B(T)+S(T))^2\right] \leq C_1(1+b^2+s^2).$$

Proof. We will frequently make use of the fact that on $\mathcal{S}^1$ and $\mathcal{S}^0$, we have

$$s \leq |s| \leq C(b+s), \qquad b \leq |b| \leq C(b+s)$$

for $C = 1+\max\{1/\mu, 1/\lambda\}$. We note further that by Equations (3)-(8) and by the condition on τ, it can easily be verified that for every stopping time $\theta \leq T$, we have

$$B(\theta)+S(\theta) \leq b+s+\int_t^\theta rB(u)+\alpha S(u)\,du + \int_t^\theta \sigma S(u)\,dW(u).$$

To see this, assume for simplicity that the crash time τ satisfies $\tau \leq \theta$. Using the integral representation for B, we obtain

$$B(\theta) = b+\int_t^\theta rB(u)\,du - (1+\lambda)[L(\tau)+L^\tau(\theta)] + (1-\mu)[M(\tau)+M^\tau(\theta)].$$

For $S(\theta)$, we obtain similarly

$$S(\theta) = S(\tau)+\int_\tau^\theta \alpha S(u)\,du + \int_\tau^\theta \sigma S(u)\,dW(u) + L^\tau(\theta) - M^\tau(\theta). \tag{12}$$

Now, by assumption, we have $S(\tau) \leq S(\tau-)$ and, again by the integral representation of $S(\tau-)$, we have

$$S(\tau) \leq S(\tau-) = s+\int_t^\tau \alpha S(u)\,du + \int_t^\tau \sigma S(u)\,dW(u) + L(\tau) - M(\tau).$$

Using this in (12) yields

$$S(\theta) \leq s+\int_t^\theta \alpha S(u)\,du + \int_t^\theta \sigma S(u)\,dW(u) + L^\tau(\theta) - M^\tau(\theta) + L(\tau) - M(\tau),$$

and upon adding $B(\theta)$ on both sides of the inequality we see that

$$\begin{aligned} B(\theta)+S(\theta) &\leq b+s+\int_t^\theta rB(u)+\alpha S(u)\,du + \int_t^\theta \sigma S(u)\,dW(u) \\ &\qquad - \lambda[L(\tau)+L^\tau(\theta)] - \mu[M(\tau)+M^\tau(\theta)] \\ &\leq b+s+\int_t^\theta rB(u)+\alpha S(u)\,du + \int_t^\theta \sigma S(u)\,dW(u). \end{aligned}$$

1. Let $\tau_n := \inf\{u \geq t : |S(u)| \geq n\} \wedge T$. Setting $K := C(r+\alpha) \wedge 0$, we have

$$\begin{aligned} B(\tau_n)+S(\tau_n) &\leq b+s+\int_t^{\tau_n} rB(u)+\alpha S(u)\,du + \int_t^{\tau_n} \sigma S(u)\,dW(u) \\ &\leq b+s+K\int_t^{\tau_n} B(u)+S(u)\,du + \sigma\int_t^{\tau_n} S(u)\,dW(u). \end{aligned}$$

Taking expectations on both sides implies that

$$E\left[B(\tau_n)+S(\tau_n)\right] \leq b+s+KE\left[\int_t^{\tau_n} B(u)+S(u)\,du\right].$$

Since $B(u) + S(u) \geq 0$, this implies that

$$E\left[B(\tau_n) + S(\tau_n)\right] \leq b + s + KE\left[\int_t^T B(u) + S(u)\,du\right].$$

Taking the limit $n \to \infty$ and using that $\tau_n \to T$, this implies that

$$E\left[B(T) + S(T)\right] \leq b + s + K\int_t^T E\left[B(u) + S(u)\right]\,du,$$

and we conclude by Gronwall's inequality.

2. We have

$$\begin{aligned}(B(T) + S(T))^2 &\leq (1 + B(T) + S(T))^2 \\ &\leq \left(1 + b + s + K\int_t^T B(u) + S(u)\,du + \sigma\int_t^T S(u)\,dW(u)\right)^2.\end{aligned}$$

Using the fact that $(a+b)^2 \leq 2a^2 + 2b^2$, this implies that there exists a constant $L > 0$, such that

$$\begin{aligned}&(B(T) + S(T))^2 \\ &\quad\leq L\left(1 + b^2 + s^2 + KT\int_t^T (B(u) + S(u))^2\,du + \sigma^2\left[\int_t^T S(u)\,dW(u)\right]^2\right). \qquad (13)\end{aligned}$$

Note that

$$E\left[\left(\int_t^T S(u)\,dW(u)\right)^2\right] = E\left[\int_t^T S(u)^2\,du\right] \leq C^2 E\left[\int_t^T (B(u) + S(u))^2\,du\right].$$

Hence, taking expectation in Inequality (13), we see that

$$\begin{aligned}E\left[(B(T) + S(T))^2\right] \leq L\Bigg(1 + b^2 + s^2 + KT\int_t^T E\left[(B(u) + S(u))^2\right]\,du \\ + \sigma^2 C^2\int_t^T E\left[(B(u) + S(u))^2\right]\,du\Bigg),\end{aligned}$$

and we can again conclude by Gronwall's inequality. □

Remark 3.6. *Since we take a worst-case perspective, the condition on τ in Lemma 3.5 poses no restriction to our subsequent analysis, since clearly, an optimal crash should never increase the net wealth. We therefore assume from now on that this condition always holds.*

We can now prove the continuity in time. As a first step, we prove the result in the case $0 < p < 1$ and then extend the result to $p \leq 0$ by means of an approximation procedure.

Proposition 3.7. *Assume that $0 < p < 1$ and let $(b, s) \in \overline{\mathcal{S}}^1$ be fixed. Then $\mathcal{V}(\cdot, b, s)$ is uniformly continuous on $[0, T]$.*

Proof. By Lemma 3.4, we have

$$|\mathcal{V}_T(t, b, s) - \mathcal{V}_T(t + h, b, s)| = |\mathcal{V}_T(t, b, s) - \mathcal{V}_{T-h}(t, b, s)|$$

for every $h \geq -t$, and hence in order to prove continuity in t it suffices to prove continuity in T.

1. We first show that $\mathcal{V}_T$ is increasing in T. Let therefore $T_0 < T_1$ and fix $t \in [0, T_0]$. Let $\pi_0 \in \mathcal{A}_{T_0}(t,b,s)$, $\breve{\pi}_0 \in \breve{\mathcal{A}}(\pi_0)$ and define π_1 and $\breve{\pi}_1$ such that $\pi_1 = \pi_0$ and $\breve{\pi}_1 = \breve{\pi}_0$ (componentwise) on $[t, T_0)$ and such that $S^{\pi_1,\breve{\pi}_1}(u) = 0$ on $[T_0, T_1]$ (i.e. liquidation of the stock position at time T_0 and no trading afterwards). Then $\pi_1 \in \mathcal{A}_{T_1}(t,b,s)$, $\breve{\pi}_1 \in \breve{\mathcal{A}}(\pi_1)$ and, noticing that every crash time for time horizon T_0 is also admissible for horizon T_1, it follows for every $\tau \in \mathcal{B}_{T_0}(t)$ that
$$X^{\pi_1,\breve{\pi}_1,\tau}_{t,b,s}(T_1) = e^{r(T_1-T_0)} X^{\pi_0,\breve{\pi}_0,\tau}_{t,b,s}(T_0) \geq X^{\pi_0,\breve{\pi}_0,\tau}_{t,b,s}(T_0). \tag{14}$$
Thus, since the stock position is closed on $[T_0, T_1]$ and hence the worst-case bound of $(\pi_1, \breve{\pi}_1)$ is not attained for stopping times with values in this interval, we get
$$\mathcal{W}_{T_1}(\pi_1, \breve{\pi}_1, t, b, s) \geq \mathcal{W}_{T_0}(\pi_0, \breve{\pi}_0, t, b, s),$$
and since π_0 and $\breve{\pi}_0$ were chosen arbitrarily it follows that
$$\mathcal{V}_{T_1}(t,b,s) \geq \mathcal{V}_{T_0}(t,b,s).$$

2. Let $\varepsilon > 0$. We are left with showing that
$$\mathcal{V}_{T_1}(t,b,s) - \mathcal{V}_{T_0}(t,b,s) \leq \varepsilon,$$
if $T_1 - T_0$ is sufficiently small. Choose $\pi_1 \in \mathcal{A}_{T_1}(t,b,s)$ and $\breve{\pi}_1 \in \breve{\mathcal{A}}(\pi_1)$ to be ε-optimal, i.e.
$$\mathcal{W}_{T_1}(\pi_1, \breve{\pi}_1, t, b, s) + \varepsilon \geq \mathcal{V}_{T_1}(t,b,s).$$
Denote by π_0 and $\breve{\pi}_0$ the restrictions of the strategies π_1 and $\breve{\pi}_1$ to $[t, T_0]$, then $\pi_0 \in \mathcal{A}_{T_0}(t,b,s)$ and $\breve{\pi}_0 \in \breve{\mathcal{A}}(\pi_0)$. Furthermore, there exists a crash time $\tau^\varepsilon \in \mathcal{B}_{T_0}(t)$ which is ε-optimal in the sense that
$$\mathcal{W}_{T_0}(\pi_0, \breve{\pi}_0, t, b, s) + \varepsilon \geq \mathcal{J}_{T_0}(\pi_0, \breve{\pi}_0, \tau^\varepsilon, t, b, s)$$
and since it is possible to consider $\mathcal{B}_{T_0}(t) \subset \mathcal{B}_{T_1}(t)$, τ^ε also defines an admissible crash time for time horizon T_1. Then we obtain
$$\begin{aligned}
\mathcal{V}_{T_1}(t,b,s) - \mathcal{V}_{T_0}(t,b,s) &\leq \mathcal{W}_{T_1}(\pi_1, \breve{\pi}_1, t, b, s) - \mathcal{W}_{T_0}(\pi_0, \breve{\pi}_0, t, b, s) + \varepsilon \\
&\leq \mathcal{J}_{T_1}(\pi_1, \breve{\pi}_1, \tau^\varepsilon, t, b, s) - \mathcal{J}_{T_0}(\pi_0, \breve{\pi}_0, \tau^\varepsilon, t, b, s) + 2\varepsilon \\
&\leq E\left[U_p\left(X^{\pi_1,\breve{\pi}_1,\tau^\varepsilon}_{t,b,s}(T_1)\right)\mathbf{1}_A\right] - E\left[U_p\left(X^{\pi_0,\breve{\pi}_0,\tau^\varepsilon}_{t,b,s}(T_0)\right)\mathbf{1}_A\right] + 2\varepsilon \\
&= E\left[U_p\left(X^{\pi_1,\breve{\pi}_1,\tau^\varepsilon}_{t,b,s}(T_1)\mathbf{1}_A\right) - U_p\left(X^{\pi_0,\breve{\pi}_0,\tau^\varepsilon}_{t,b,s}(T_0)\mathbf{1}_A\right)\right] + 2\varepsilon \\
&\leq E\left[U_p\left(\left[X^{\pi_1,\breve{\pi}_1,\tau^\varepsilon}_{t,b,s}(T_1) - X^{\pi_0,\breve{\pi}_0,\tau^\varepsilon}_{t,b,s}(T_0)\right]\mathbf{1}_A\right)\right] + 2\varepsilon \\
&\leq U_p\left(E\left[\left(X^{\pi_1,\breve{\pi}_1,\tau^\varepsilon}_{t,b,s}(T_1) - X^{\pi_0,\breve{\pi}_0,\tau^\varepsilon}_{t,b,s}(T_0)\right)\mathbf{1}_A\right]\right) + 2\varepsilon,
\end{aligned} \tag{15}$$
where $A := \{X^{\pi_1,\breve{\pi}_1,\tau^\varepsilon}_{t,b,s}(T_1) - X^{\pi_0,\breve{\pi}_0,\tau^\varepsilon}_{t,b,s}(T_0) > 0\}$. Here, it has been used that $U_p(X(T_1)) - U_p(X(T_0)) \leq 0$ on A^c (to obtain the third line), that $U_p(0) = 0$ (fourth line). Finally, the subadditivity of U_p (fifth line) and Jensen's inequality (last line) have been used.

Next, since $(\pi_1, \breve{\pi}_1)$ and $(\pi_0, \breve{\pi}_0)$ coincide on $[t, T_0]$ and since τ^ε is $[t, T_0] \cup \{+\infty\}$-valued, it is not hard to see that
$$\begin{aligned}
&\left(X^{\pi_1,\breve{\pi}_1,\tau^\varepsilon}_{t,b,s}(T_1) - X^{\pi_0,\breve{\pi}_0,\tau^\varepsilon}_{t,b,s}(T_0)\right)\mathbf{1}_A \\
&\qquad \leq (1+\lambda)\Big|\int_{T_0}^{T_1} rB(u) + \alpha S(u)\,du\Big| + (1+\lambda)\Big|\int_{T_0}^{T_1} \sigma S(u)\,dW(u)\Big|
\end{aligned}$$

with

$$B(u) := B_{t,b,s}^{\pi_1,\check{\pi}_1,\tau^\varepsilon}(u), \qquad S(u) := S_{t,b,s}^{\pi_1,\check{\pi}_1,\tau^\varepsilon}(u).$$

With this, we see that there exists a constant $C > 0$ (which is allowed to change from line to line), such that

$$\begin{aligned}
&E\left[\left(X_{t,b,s}^{\pi_1,\check{\pi}_1,\tau^\varepsilon}(T_1) - X_{t,b,s}^{\pi_0,\check{\pi}_0,\tau^\varepsilon}(T_0)\right)\mathbf{1}_A\right] \\
&\quad \leq (1+\lambda)E\left[\Big|\int_{T_0}^{T_1} rB(u) + \alpha S(u)\,du\Big| + \Big|\int_{T_0}^{T_1} \sigma S(u)\,dW(u)\Big|\right] \\
&\quad \leq CE\left[\int_{T_0}^{T_1} |B(u)+S(u)|\,du\right] + CE\left[\int_{T_0}^{T^1} S(u)^2\,du\right]^{1/2} \\
&\quad \leq C\int_{T_0}^{T_1} E[B(u)+S(u)]\,du + C\left[\int_{T_0}^{T^1} E[(B(u)+S(u))^2]\,du\right]^{1/2}
\end{aligned}$$

By Lemma 3.5, we can hence find a constant $K > 0$ independent of $\pi_1, \check{\pi}_1$ and τ^ε, such that

$$E\left[\left(X_{t,b,s}^{\pi_1,\check{\pi}_1,\tau^\varepsilon}(T_1) - X_{t,b,s}^{\pi_0,\check{\pi}_0,\tau^\varepsilon}(T_0)\right)\mathbf{1}_A\right] \leq K(b+s)(T_1-T_0) + K(1+b^2+s^2)^{1/2}(T_1-T_0)^{1/2}.$$

Combining this with Inequality (15) yields the desired result. □

For $p \leq 0$, we define

$$U_p^j(x) := U_p(x+1/j), \qquad \tilde{U}_p^j(x) = U_p^j(x) - U_p^j(0), \qquad x \in [0,\infty),$$

where $j \in \mathbb{N}$. Note that, with this, $\tilde{U}_p^j(0) = 0$. We denote by $\mathcal{V}^j$ the value function corresponding to $U_p^j(x)$. It can then be verified that $\mathcal{V}^j(\cdot,b,s)$ is also uniformly continuous on $[0,T]$ for all (b,s) fixed. Indeed, in the proof of Proposition 3.7, we only need to replace U_p by $\tilde{U}_p^j$ in Inequality (15) to make the same proof work.

Lemma 3.8. *Let $p \leq 0$ and fix $(b,s) \in \mathcal{S}^1$. Then*

$$\lim_{j\to\infty} \mathcal{V}^j(t,b,s) = \mathcal{V}(t,b,s)$$

uniformly in t.

Proof. First, note that the family

$$\left\{U_p\big(X_{t,b,s}^{\pi,\check{\pi},\tau}(T)\big)\right\}_{t\in[0,T],\pi\in\mathcal{A}(t,b,s),\check{\pi}\in\check{\mathcal{A}}(\pi),\tau\in\mathcal{B}(t)} \tag{16}$$

is uniformly integrable. Indeed, choose $q > 1$ arbitrary. Then

$$E\left[\big|U_p\big(X_{t,b,s}^{\pi,\check{\pi},\tau}(T)\big)\big|^q\right] = \frac{pq}{|p|^q}E\left[U_{pq}\big(X_{t,b,s}^{\pi,\check{\pi},\tau}(T)\big)\right]$$

and since

$$\begin{aligned}
&U_{pq}\big(b + \min\{(1-\mu)(1-\beta)s, (1+\lambda)s\}\big) \\
&\qquad\qquad \leq E\left[U_{pq}\big(X_{t,b,s}^{\pi,\check{\pi},\tau}(T)\big)\right] \leq \frac{1}{pq}(b+s)^{pq}\exp\left(pq\left[r + \frac{(\alpha-r)^2}{2(1-pq)\sigma^2}\right]T\right)
\end{aligned}$$

by Lemma 3.2.1, the uniform integrability follows.

Let us now fix some $j \in \mathbb{N}$, $(t,b,s) \in [0,T] \times \mathcal{S}^1$, $\pi \in \mathcal{A}(t,b,s)$, $\check{\pi} \in \check{\mathcal{A}}(\pi)$, and $\tau \in \mathcal{B}(t)$. Let furthermore $\delta > 0$ be such that $U_p^j(\delta) \leq 0$. We calculate

$$\begin{aligned}
0 &\leq E\left[U_p^j\big(X_{t,b,s}^{\pi,\check{\pi},\tau}(T)\big)\right] - E\left[U_p\big(X_{t,b,s}^{\pi,\check{\pi},\tau}(T)\big)\right] \\
&= E\left[\Big(U_p^j\big(X_{t,b,s}^{\pi,\check{\pi},\tau}(T)\big) - U_p\big(X_{t,b,s}^{\pi,\check{\pi},\tau}(T)\big)\Big)\mathbf{1}_{\{X_{t,b,s}^{\pi,\check{\pi},\tau}(T)>\delta\}}\right] \\
&\qquad\qquad + E\left[\Big(U_p^j\big(X_{t,b,s}^{\pi,\check{\pi},\tau}(T)\big) - U_p\big(X_{t,b,s}^{\pi,\check{\pi},\tau}(T)\big)\Big)\mathbf{1}_{\{X_{t,b,s}^{\pi,\check{\pi},\tau}(T)\leq\delta\}}\right] \\
&\leq U_p^j(\delta) - U_p(\delta) - E\left[U_p\big(X_{t,b,s}^{\pi,\check{\pi},\tau}(T)\big)\mathbf{1}_{\{X_{t,b,s}^{\pi,\check{\pi},\tau}(T)\leq\delta\}}\right],
\end{aligned}$$

where the last inequality follows from the fact that the difference $U_p^j(x) - U_p(x)$ on $[\delta, \infty)$ is maximal at δ, and since $U_p^j(x) \leq 0$ for all $x \leq \delta$. Let now $\varepsilon > 0$. By the uniform integrability of (16), it follows that if δ is small enough, then

$$\left|E\left[U_p\big(X_{t,b,s}^{\pi,\check{\pi},\tau}(T)\big)\mathbf{1}_{\{X_{t,b,s}^{\pi,\check{\pi},\tau}(T)\leq\delta\}}\right]\right| \leq \varepsilon/2,$$

uniformly in $t, \pi, \check{\pi}$ and τ. Next, for this choice of δ, there exists $J \in \mathbb{N}$ large enough, such that

$$U_p^j(\delta) - U_p(\delta) \leq \varepsilon/2$$

for all $j \geq J$. In total, this implies that

$$\sup_{t\in[0,T]} \sup_{\substack{\pi\in\mathcal{A}(t,b,s),\\ \check{\pi}\in\check{\mathcal{A}}(\pi)}} \inf_{\tau\in\mathcal{B}(t)} \left|E\left[U_p^j\big(X_{t,b,s}^{\pi,\check{\pi},\tau}(T)\big)\right] - E\left[U_p\big(X_{t,b,s}^{\pi,\check{\pi},\tau}(T)\big)\right]\right| \leq \varepsilon$$

for all $j \geq J$, from which the result follows. □

Proposition 3.9. *Assume that $p \leq 0$ and let $(b,s) \in \mathcal{S}^1$ be fixed. Then $\mathcal{V}(\cdot,b,s)$ is uniformly continuous on $[0,T]$.*

Proof. Let $\varepsilon > 0$, $t \in [0,T]$, and let t_n, $n \in \mathbb{N}$, be a sequence in $[0,T]$, converging to t. By Lemma 3.8, there exists $j \in \mathbb{N}$, such that

$$\sup_{t\in[0,T]} |\mathcal{V}(t,b,s) - \mathcal{V}^j(t,b,s)| \leq \varepsilon/3$$

and by the continuity of $\mathcal{V}^j$, there exists some $N \in \mathbb{N}$, such that

$$|\mathcal{V}^j(t_n,b,s) - \mathcal{V}^j(t,b,s)| \leq \varepsilon/3$$

for all $n \geq N$. Hence

$$\begin{aligned}
&|\mathcal{V}(t_n,b,s) - \mathcal{V}(t,b,s)| \\
&\qquad \leq |\mathcal{V}(t_n,b,s) - \mathcal{V}^j(t_n,b,s)| + |\mathcal{V}^j(t_n,b,s) - \mathcal{V}^j(t,b,s)| + |\mathcal{V}^j(t,b,s) - \mathcal{V}(t,b,s)| \leq \varepsilon
\end{aligned}$$

for all $n \geq N$. □

Putting the pieces together, we can prove the joint continuity of $\mathcal{V}$ and $\check{\mathcal{V}}$.

Theorem 3.10. *The value function $\mathcal{V}$ is continuous.*

Proof. Since $\mathcal{V}(t,b,s)$ is locally bounded in a small neighborhood of (b,s) uniformly in t, the local Lipschitz continuity (Lemma 3.2.3) of $\mathcal{V}$ holds uniformly in t. With this, it is easy to prove the joint continuity on $[0,T]\times\mathcal{S}^1$. Indeed, let $t\in[0,T]$ and $(b,s)\in\mathcal{S}^1$, and choose a sequence (t_n,b_n,s_n) converging to (t,b,s). Note that, eventually, (b_n,s_n) is contained in a compact subset K of $\mathcal{S}^1$. By the local Lipschitz continuity of $\mathcal{V}$, there exists a constant $L>0$ such that

$$|\mathcal{V}(u,b_n,s_n)-\mathcal{V}(u,b,s)|\le L(|b_n-b|+|s_n-s|)$$

for all $u\in[0,T]$ and n sufficiently large. Hence

$$\begin{aligned}\lim_{n\to\infty}|\mathcal{V}(t_n,b_n,s_n)-\mathcal{V}(t,b,s)| &\le \lim_{n\to\infty}|\mathcal{V}(t_n,b_n,s_n)-\mathcal{V}(t_n,b,s)|+|\mathcal{V}(t_n,b,s)-\mathcal{V}(t,b,s)|\\ &= \lim_{n\to\infty}L(|b_n-b|+|s_n-s|)+|\mathcal{V}(t_n,b,s)-\mathcal{V}(t,b,s)|=0.\end{aligned}$$

In order to show that the continuity of $\mathcal{V}$ extends to the boundary of $\mathcal{S}^1$, we let $(t,b,s)\in[0,T]\times\partial\mathcal{S}^1$ and let (t_n,b_n,s_n), $n\in\mathbb{N}$, be a sequence converging to (t,b,s). If $s\le 0$, we have

$$\lim_{n\to\infty}\mathcal{V}(t_n,b_n,s_n)\le\lim_{n\to\infty}\varphi_{1+\lambda}(t_n,b_n,s_n)=U_p(0)$$

and if $s>0$, we have

$$\lim_{n\to\infty}\mathcal{V}(t_n,b_n,s_n)\le\lim_{n\to\infty}\check{\mathcal{V}}(t_n,b_n,(1-\beta)s_n)\le\lim_{n\to\infty}\varphi_{1-\mu}(t_n,b_n,(1-\beta)s_n)=U_p(0).\qquad\square$$

Corollary 3.11. *The value function $\check{\mathcal{V}}$ is continuous.*

Proof. This follows by imitating the arguments leading to the continuity of $\mathcal{V}$. $\square$

4 The dynamic programming principle

Equipped with the continuity of the value function, we are now in the position to prove the dynamic programming principle. The main problem arising in the proof is the construction of strategies $\pi^\varepsilon\in\mathcal{A}(t,b,s)$ and $\check{\pi}^\varepsilon\in\mathcal{A}(\pi^\varepsilon)$ which are ε-optimal in the sense that

$$\mathcal{V}(t,b,s)\le\mathcal{W}(\pi^\varepsilon,\check{\pi}^\varepsilon,t,b,s)+\varepsilon.$$

The existence of such strategies is clear if the initial time t as well as the initial bond and stock holdings b and s are deterministic. The first aim is the construction of such strategies for random t,b,s.

4.1 Existence of ε-optimal strategies

The problem with the construction of ε-optimal strategies is the following. Denote by τ a random initial time and by (B,S) a random initial position of the investor. Then clearly, for every $\omega\in\Omega$, we can find a strategy $\pi^\varepsilon_\omega\in\mathcal{A}(\tau(\omega),B(\omega),S(\omega))$ which is ε-optimal. However, if we compose such strategies π^ε_ω into a single strategy π^ε, then it is not clear if $\pi^\varepsilon\in\mathcal{A}(\tau,B,S)$ due to measurability issues.

We start by constructing suitable partitions of $(0,T]\times\mathcal{S}^0$ and $(0,T]\times\mathcal{S}^1$, respectively. For this, let $\varepsilon>0$ and let $(t,b,s)\in(0,T]\times\mathcal{S}^1$. By the continuity of $\mathcal{V}$, there exists $r(t,b,s)>0$, such that

$$|\mathcal{V}(\bar{t},\bar{b},\bar{s})-\mathcal{V}(\hat{t},\hat{b},\hat{s})|\le\varepsilon$$

for all $(\bar{t},\bar{b},\bar{s}),(\hat{t},\hat{b},\hat{s}) \in (t-r(t,b,s),t] \times \tilde{K}(b,s;r(t,b,s))$, where $\tilde{K}(b,s;r(t,b,s))$ is the set of all $(\bar{b},\bar{s})$, such that $\|(b,s)-(\bar{b},\bar{s})\|_2 < r(t,b,s)$, and such that there exist $l,m \geq 0$ with

$$b = \bar{b} + (1-\mu)m - (1+\lambda)l, \qquad s = \bar{s} - m + l,$$

i.e. (b,s) can be reached by a transaction (l,m) from $(\bar{b},\bar{s})$. With this, the family $\{(t-r(t,b,s),t] \times \tilde{K}(b,s;r(t,b,s))\}_{(t,b,s)\in(0,T]\times\mathcal{S}^1}$ forms an open covering of $(0,T]\times\mathcal{S}^1$ (in the topology induced by the half open sets of the form $(u,t]\times\tilde{K}(b,s;r))$, and hence there exists a countable sub-covering $(\tilde{t}_i - r(\tilde{t}_i,\tilde{b}_i,\tilde{s}_i),\tilde{t}_i] \times \tilde{K}(\tilde{b}_i,\tilde{s}_i;r(\tilde{t}_i,\tilde{b}_i,\tilde{s}_i))$, $i \in \mathbb{N}$. By the usual method of taking appropriate set differences, it follows that we can construct a sequence $(t_i,b_i,s_i) \in (0,T]\times\mathcal{S}^1$, $i \in \mathbb{N}$, and a corresponding sequence of sets $K_i^1 := K^1(t_i,b_i,s_i)$, $i \in \mathbb{N}$, such that

$$|\mathcal{V}(\bar{t},\bar{b},\bar{s}) - \mathcal{V}(\hat{t},\hat{b},\hat{s})| \leq \varepsilon \tag{17}$$

for all $(\bar{t},\bar{b},\bar{s}),(\hat{t},\hat{b},\hat{s}) \in K_i^1$, such that there exist $l,m \geq 0$ with

$$b_i = b + (1-\mu)m - (1+\lambda)l, \qquad s_i = s - m + l,$$

for all $(t,b,s) \in K_i^1$, and such that the K_i^1 form a partition of $(0,T]\times\mathcal{S}^1$. In the same way, we can construct a sequence $(\check{t}_i,\check{b}_i,\check{s}_i)$, $i \in \mathbb{N}$, and a corresponding partition K_i^0, $i\in\mathbb{N}$, of $(0,T]\times\mathcal{S}^0$ with the same properties, but where the continuity property (17) holds for $\check{\mathcal{V}}$ instead of $\mathcal{V}$.

Lemma 4.1. *Let $\varepsilon > 0$ and let θ be a $[t,T]$-valued stopping time. Fix an arbitrary pre-crash trading strategy $\pi \in \mathcal{A}(t,b,s)$ and a crash time $\tau \in \mathcal{B}(t)$. Then there exists $\pi^\varepsilon \in \mathcal{A}(t,b,s)$ which coincides with π on $[t,\tau\wedge\theta)$ and a post-crash strategy $\check{\pi}^{\varepsilon,\tau}$ corresponding to π^ε and τ such that*

$$\begin{aligned} E\left[U_p\left(X_{t,b,s}^{\pi^\varepsilon,\check{\pi}^{\varepsilon,\tau},\tau}(T)\right)\middle|\mathcal{F}^t\big((\tau\wedge\theta)-\big)\right] + \varepsilon \\ \geq \check{\mathcal{V}}\big(\tau,B(\tau-),(1-\beta)S(\tau-)\big)\mathbf{1}_{\{\tau\leq\theta\}} + \mathcal{V}\big(\theta,B(\theta-),S(\theta-)\big)\mathbf{1}_{\{\tau>\theta\}}. \end{aligned}$$

Proof. Let $\varepsilon > 0$. We let $\{(\check{t}_i,\check{b}_i,\check{s}_i)\}_{i\in\mathbb{N}}$ and $\{K_i^0\}_{i\in\mathbb{N}}$ be the sequences constructed above, such that the K_i^0 form a partition of $(0,T]\times\mathcal{S}^0$. For every $i \in \mathbb{N}$, there exists a strategy $\pi_i \in \check{\mathcal{A}}(\check{t}_i,\check{b}_i,\check{s}_i)$ (in the crash-free world) such that

$$\check{\mathcal{J}}(\pi_i,\check{t}_i,\check{b}_i,\check{s}_i) \geq \check{\mathcal{V}}(\check{t}_i,\check{b}_i,\check{s}_i) - \varepsilon.$$

Let now $h \in [0,\delta)$ such that $(\check{t}_i - h,\check{b}_i,\check{s}_i) \in K_i^0$. As in the proof of Lemma 3.4, we can shift the strategy π_i from $[\check{t}_i,T]$ to $[\check{t}_i-h,T-h]$ and we extend this new strategy, denoted by $\bar{\pi}_i$, in such a way, that the stock position of the investor is closed on $[T-h,T]$. Then $\bar{\pi}_i \in \check{\mathcal{A}}(\check{t}_i-h,\check{b}_i,\check{s}_i)$ and

$$\check{\mathcal{J}}_T(\pi_i,\check{t}_i,\check{b}_i,\check{s}_i) - \check{\mathcal{J}}_T(\bar{\pi}_i,\check{t}_i-h,\check{b}_i,\check{s}_i) \leq \check{\mathcal{J}}_T(\pi_i,\check{t}_i,\check{b}_i,\check{s}_i) - \check{\mathcal{J}}_{T-h}(\bar{\pi}_i,\check{t}_i-h,\check{b}_i,\check{s}_i) = 0. \tag{18}$$

For every $(b,s) \in K_i^0(\check{b}_i,\check{s}_i)$, there exist (by construction) some $m,l \geq 0$, such that

$$\check{b}_i = b - (1+\lambda)l + (1-\mu)m, \qquad \check{s}_i = s + l - m,$$

i.e. $(\check{b}_i,\check{s}_i)$ can be reached by a transaction from (b,s). Define

$$\tilde{\pi}_i = \bar{\pi}_i + (l,m).$$

Then $\tilde{\pi}_i \in \check{\mathcal{A}}(\check{t}_i-h,b,s)$. By the construction of $\tilde{\pi}_i$, we have

$$\check{\mathcal{J}}(\tilde{\pi}_i,\check{t}_i-h,b,s) = \check{\mathcal{J}}(\bar{\pi}_i,\check{t}_i-h,\check{b}_i,\check{s}_i), \tag{19}$$

since the paths of $X^{\tilde{\pi}_i}$ and $X^{\bar{\pi}_i}$ coincide after the initial transaction at time $\check{t}_i - h$. In total, we have therefore

$$\check{\mathcal{J}}(\tilde{\pi}_i, \check{t}_i - h, b, s) = \check{\mathcal{J}}(\bar{\pi}_i, \check{t}_i - h, \check{b}_i, \check{s}_i) \geq \check{\mathcal{J}}(\pi_i, \check{t}_i, \check{b}_i, \check{s}_i) \geq \check{\mathcal{V}}(\check{t}_i, \check{b}_i, \check{s}_i) - \varepsilon \geq \check{\mathcal{V}}(\check{t}_i - h, b, s) - 2\varepsilon,$$

i.e. $\tilde{\pi}_i$ is 2ε-optimal. The first equality is just Equation (19). The following two inequalities follow from Inequality (18) and the ε-optimality of π_i, respectively. The last inequality is a consequence of the continuity of $\check{\mathcal{V}}$ and the construction of K_i^0.

Let now θ be a $[t,T]$-valued stopping time, $\pi \in \mathcal{A}(t,b,s)$ and fix an arbitrary crash time $\tau \in \mathcal{B}(t)$. We can assume without loss of generality that $(\tau, B(\tau-), (1-\beta)S(\tau-)) \in (0,T] \times \mathcal{S}^0$ and $(\theta, B(\theta), S(\theta)) \in (0,T] \times \mathcal{S}^1$, since we know the optimal strategies on $\partial\mathcal{S}^1$ and $\partial\mathcal{S}^0$ (Lemma 3.1.1), and since $\mathcal{F}(0)$ is trivial. We denote by C_i the set $\{\tau \leq \theta\}$ intersected with the event

$$(\tau, B^\pi(\tau-), (1-\beta)S^\pi(\tau-)) \in K_i^0.$$

By the construction of K_i^0, there exist (random) $l_i, m_i \geq 0$ solving

$$\begin{aligned}
\check{b}_i &= B^\pi(\tau-) - (1+\lambda)l_i + (1-\mu)m_i, \\
\check{s}_i &= (1-\beta)S^\pi(\tau-) + l_i - m_i.
\end{aligned}$$

Denote $h_i = \tau - \check{t}_i$. Using l_i, m_i and h_i, we can construct strategies $\tilde{\pi}_i^C$ as above which are 2ε-optimal on C_i. Similarly, if we denote by D_i the set $\{\tau > \theta\}$ intersected with the event

$$(\theta, B^\pi(\theta-), S^\pi(\theta-)) \in K_i^1,$$

we can construct pre-crash strategies $\tilde{\pi}_i^D$ and post-crash strategies $\tilde{\pi}_i^{D,\tau}$ corresponding to $\tilde{\pi}_i^D$ and τ which are 2ε-optimal on D_i. Define

$$\begin{aligned}
\pi^\varepsilon(u) &:= \pi(u)\mathbf{1}_{\{u<\tau\wedge\theta\}} + \mathbf{1}_{\{u\geq\tau\wedge\theta\}} \sum_{i=1}^{\infty} \tilde{\pi}_i^D(u)\mathbf{1}_{D_i}, \\
\check{\pi}^{\varepsilon,\tau}(u) &:= \mathbf{1}_{\{u\geq\tau\wedge\theta\}} \sum_{i=1}^{\infty} \Big[\tilde{\pi}_i^C(u)\mathbf{1}_{C_i} + \tilde{\pi}_i^{D,\tau}(u)\mathbf{1}_{D_i}\Big].
\end{aligned}$$

Then $\pi^\varepsilon \in \mathcal{A}(t,b,s)$ and $\pi = \pi^\varepsilon$ on $[t, \tau\wedge\theta)$. Furthermore, $\check{\pi}^{\varepsilon,\tau}$ is a post-crash strategy corresponding to π^ε and τ. On the set $\{\theta < \tau\}$, by the 2ε-optimality of π^ε and $\check{\pi}^{\varepsilon,\tau}$, we have

$$\begin{aligned}
\mathcal{V}\big(\theta, B(\theta-), S(\theta-)\big) &\leq E\left[U_p\left(X^{\pi^\varepsilon,\check{\pi}^{\varepsilon,\tau},\tau}_{\theta,B(\theta-),S(\theta-)}(T)\right)\right] + 2\varepsilon \\
&= E\left[U_p\left(X^{\pi^\varepsilon,\check{\pi}^{\varepsilon,\tau},\tau}_{t,b,s}(T)\right)\Big|\mathcal{F}^t(\theta-)\right] + 2\varepsilon \\
&= E\left[U_p\left(X^{\pi^\varepsilon,\check{\pi}^{\varepsilon,\tau},\tau}_{t,b,s}(T)\right)\Big|\mathcal{F}^t((\tau\wedge\theta)-)\right] + 2\varepsilon. \qquad (20)
\end{aligned}$$

Similarly, on the set $\{\theta \geq \tau\}$, we have

$$\begin{aligned}
\check{\mathcal{V}}\big(\tau, B(\tau-), (1-\beta)S(\tau-)\big) &\leq \check{\mathcal{J}}\big(\check{\pi}^{\varepsilon,\tau}, \tau, B(\tau-), (1-\beta)S(\tau-)\big) + 2\varepsilon \\
&= E\left[U_p\left(X^{\check{\pi}^{\varepsilon,\tau},\tau}_{\tau,B(\tau-),(1-\beta)S(\tau-)}(T)\right)\right] + 2\varepsilon \\
&= E\left[U_p\left(X^{\pi^\varepsilon,\check{\pi}^{\varepsilon,\tau},\tau}_{t,b,s}(T)\right)\Big|\mathcal{F}^t(\tau-)\right] + 2\varepsilon \\
&= E\left[U_p\left(X^{\pi^\varepsilon,\check{\pi}^{\varepsilon,\tau},\tau}_{t,b,s}(T)\right)\Big|\mathcal{F}^t((\tau\wedge\theta)-)\right] + 2\varepsilon. \qquad (21)
\end{aligned}$$

Combining Inequalities (20) and (21), we see that

$$\begin{aligned}E\left[U_p\left(X_{t,b,s}^{\pi^\varepsilon,\check{\pi}^{\varepsilon,\tau},\tau}(T)\right)\middle|\mathcal{F}^t\big((\tau\wedge\theta)-\big)\right]+2\varepsilon\\ \geq \check{\mathcal{V}}\big(\tau,B(\tau-),(1-\beta)S(\tau-)\big)\mathbf{1}_{\{\tau\leq\theta\}}+\mathcal{V}\big(\theta,B(\theta-),S(\theta-)\big)\mathbf{1}_{\{\tau>\theta\}},\end{aligned}$$

which concludes the proof. □

Remark 4.2. *Note that we can make the strategies $\bar{\pi}_i$ in the previous proof perform worse (by both buying and selling at terminal time). Hence, instead of Inequality* (18)*, we could even obtain*

$$\check{\mathcal{J}}(\pi_i,\check{t}_i,\check{b}_i,\check{s}_i)=\check{\mathcal{J}}(\bar{\pi}_i,\check{t}_i-h,\check{b}_i,\check{s}_i).$$

We will need this observation for the proof of Lemma 4.3.

As a next step, we prove that there exist crash times which are ε-optimal from a random time onwards.

Lemma 4.3. *Let $\varepsilon>0$ and let θ be a $[t,T]$-valued stopping time. Let $\pi\in\mathcal{A}(t,b,s)$ and $\check{\pi}\in\check{\mathcal{A}}(\pi)$ be ε-optimal. Then there exists a crash time $\tau^\varepsilon\in\mathcal{B}(t)$ with $\tau^\varepsilon\geq\theta$ such that*

$$E\left[U_p\left(X_{t,b,s}^{\pi,\check{\pi},\tau^\varepsilon}(T)\right)\middle|\mathcal{F}^t(\theta-)\right]\leq\mathcal{V}\big(\theta,B(\theta-),S(\theta-)\big)+\varepsilon.$$

Proof. We can without loss of generality assume that π is of the same form as constructed in Lemma 4.1[1]. Indeed, if $\pi\in\mathcal{A}(t,b,s)$ and $\check{\pi}\in\check{\mathcal{A}}(\pi)$ are arbitrary ε-optimal strategies and $\pi^\varepsilon\in\mathcal{A}(t,b,s)$ and $\check{\pi}^\varepsilon\in\check{\mathcal{A}}(\pi)$ are as in Lemma 4.1, then π^ε and $\check{\pi}^\varepsilon$ satisfy

$$\mathcal{V}(t,b,s)\leq\mathcal{W}(\pi^\varepsilon,\check{\pi}^\varepsilon,t,b,s)+2\varepsilon,$$

meaning that π^ε and $\check{\pi}^\varepsilon$ are 2ε-optimal. As before, we can furthermore assume $(\theta,B(\theta),S(\theta))\in(0,T]\times\mathcal{S}^1$.

Let therefore $\pi\in\mathcal{A}(t,b,s)$ and $\check{\pi}\in\check{\mathcal{A}}(\pi)$ such that π can be written as

$$\pi(u)=\pi_0(u)\mathbf{1}_{\{u<\theta\}}+\mathbf{1}_{\{u\geq\theta\}}\sum_{i=1}^{\infty}\tilde{\pi}_i\mathbf{1}_{D_i}$$

for some $\pi_0\in\mathcal{A}(t,b,s)$. The sequence $\{D_i\}_{i\in\mathbb{N}}$ consists of the sets

$$D_i=\big\{\omega\in\Omega:\big(\theta,B(\theta-),S(\theta-)\big)\in K_i^1\big\},$$

where $\{K_i^1\}_{i\in\mathbb{N}}$ is the sequence as constructed in the beginning of this section. Also, recalling the corresponding sequence of corner points (t_i,b_i,s_i), the strategies $\{\tilde{\pi}_i\}_{i\in\mathbb{N}}$ can be written as

$$\tilde{\pi}_i=\bar{\pi}_i+(l,m),$$

where l and m are chosen minimally, such that we have on D_i

$$b_i=B(\theta-)-(1+\lambda)l+(1-\mu)m,\qquad s_i=S(\theta-)+l-m.$$

The strategies $\{\bar{\pi}_i\}_{i\in\mathbb{N}}$ are just time-shifted (to the grid points t_i) versions of corresponding ε-optimal strategies $\{\pi_i\}_{i\in\mathbb{N}}$, where $\pi_i\in\mathcal{A}(t_i,b_i,s_i)$. We denote the post-crash strategies corresponding to π_i by $\check{\pi}_i$ and the post-crash strategies corresponding to $\tilde{\pi}_i$ by $\ddot{\pi}_i$. Then we have

$$\mathcal{W}(\pi_i,\check{\pi}_i,t_i,b_i,s_i)\leq\mathcal{W}(\tilde{\pi}_i,\ddot{\pi}_i,u,b,s),\qquad\text{for all }(u,b,s)\in K_i^1,\ i\in\mathbb{N}.$$

[1]For a proof not relying on this assumption see Belak [3].

Note that we even obtain

$$\mathcal{W}(\pi_i, \check{\pi}_i, t_i, b_i, s_i) = \mathcal{W}(\tilde{\pi}_i, \ddot{\pi}_i, u, b, s), \qquad \text{for all } (u, b, s) \in K_i^1,\ i \in \mathbb{N}, \tag{22}$$

if we adjust the strategies $\bar{\pi}_i$ and their corresponding post-crash strategies as discussed in Remark 4.2.

For every $i \in \mathbb{N}$, we can find ε-optimal crash times $\tau_i \in \mathcal{B}(t_i)$ such that

$$\mathcal{W}(\pi_i, \check{\pi}_i, t_i, b_i, s_i) \geq \mathcal{J}(\pi_i, \check{\pi}_i, \tau_i, t_i, b_i, s_i) - \varepsilon. \tag{23}$$

By applying the same time-shift to τ_i as applied to obtain the strategy $\bar{\pi}_i$ from π_i and denoting the resulting crash time by $\bar{\tau}_i$, we see that $\bar{\tau}_i \in \mathcal{B}(u)$ for all u such that $(u, b_i, s_i) \in K_i^1$. By the construction of $\tilde{\pi}_i$, it follows that

$$\mathcal{J}(\pi_i, \check{\pi}_i, \tau_i, t_i, b_i, s_i) \geq \mathcal{J}(\tilde{\pi}_i, \ddot{\pi}_i, \bar{\tau}_i, u, b, s), \qquad \text{for all } (u, b, s) \in K_i^1. \tag{24}$$

So, in total, combining Equations (22) to (24), we have for every $i \in \mathbb{N}$ and every $(u, b, s) \in K_i^1$

$$\mathcal{W}(\tilde{\pi}_i, \ddot{\pi}_i, u, b, s) \geq \mathcal{J}(\tilde{\pi}_i, \ddot{\pi}_i, \bar{\tau}_i, u, b, s) - \varepsilon,$$

meaning that $\bar{\tau}_i$ is ε-optimal on the set D_i. Now, set

$$\tau^\varepsilon = \bar{\tau}_i \qquad \text{on} \qquad D_i$$

to obtain the desired ε-optimal crash time $\tau^\varepsilon \in \mathcal{B}(t)$ satisfying $\tau^\varepsilon \geq \theta$. □

Lemma 4.4. *Let $\varepsilon > 0$ and let θ be a $[t, T]$-valued stopping time. Let $\pi \in \mathcal{A}(t, b, s)$ and $\check{\pi} \in \check{\mathcal{A}}(\pi)$ be ε-optimal and fix $\bar{\tau} \in \mathcal{B}(t)$. Then there exists a crash time $\tau^\varepsilon \in \mathcal{B}(t)$ such that*

$$\begin{aligned} &E\left[U_p\left(X_{t,b,s}^{\pi,\check{\pi},\tau^\varepsilon}(T)\right)\middle|\mathcal{F}^t\big((\bar{\tau} \wedge \theta)-\big)\right] \\ &\qquad\qquad \leq \mathcal{V}\big(\theta, B(\theta-), S(\theta-)\big)\mathbf{1}_{\{\bar{\tau}>\theta\}} + \check{\mathcal{V}}\big(\bar{\tau}, B(\bar{\tau}-), (1-\beta)S(\bar{\tau}-)\big)\mathbf{1}_{\{\bar{\tau}\leq\theta\}} + \varepsilon. \end{aligned}$$

Proof. By the previous Lemma 4.3, there exists a crash time $\bar{\tau}^\varepsilon \in \mathcal{B}(t)$ satisfying $\bar{\tau}^\varepsilon \geq \theta$ such that

$$E\left[U_p\left(X_{t,b,s}^{\pi,\check{\pi},\bar{\tau}^\varepsilon}(T)\right)\middle|\mathcal{F}^t(\theta-)\right] \leq \mathcal{V}\big(\theta, B(\theta-), S(\theta-)\big) + \varepsilon.$$

Set

$$\tau^\varepsilon := \begin{cases} \bar{\tau}^\varepsilon, & \text{on } \{\bar{\tau} > \theta\}, \\ \bar{\tau}, & \text{on } \{\bar{\tau} \leq \theta\}. \end{cases}$$

Then, on $\{\bar{\tau} \geq \theta\}$, we have

$$\begin{aligned} E\left[U_p\left(X_{t,b,s}^{\pi,\check{\pi},\tau^\varepsilon}(T)\right)\middle|\mathcal{F}^t\big((\bar{\tau} \wedge \theta)-\big)\right] &= E\left[U_p\left(X_{t,b,s}^{\pi,\check{\pi},\bar{\tau}^\varepsilon}(T)\right)\middle|\mathcal{F}^t(\theta-)\right] \\ &\leq \mathcal{V}\big(\theta, B(\theta-), S(\theta-)\big) + \varepsilon \\ &\leq \check{\mathcal{V}}\big(\bar{\tau}, B(\bar{\tau}-), (1-\beta)S(\bar{\tau}-)\big)\mathbf{1}_{\{\bar{\tau}=\theta\}} \\ &\qquad + \mathcal{V}\big(\theta, B(\theta-), S(\theta-)\big)\mathbf{1}_{\{\bar{\tau}>\theta\}} + \varepsilon, \end{aligned} \tag{25}$$

where the last inequality follows from Inequality (11). On the set $\{\bar{\tau} \leq \theta\}$, we have

$$\begin{aligned} E\left[U_p\left(X_{t,b,s}^{\pi,\check{\pi},\tau^\varepsilon}(T)\right)\middle|\mathcal{F}^t\big((\bar{\tau} \wedge \theta)-\big)\right] &= E\left[U_p\left(X_{t,b,s}^{\pi,\check{\pi},\bar{\tau}}(T)\right)\middle|\mathcal{F}^t(\bar{\tau}-)\right] \\ &\leq \check{\mathcal{V}}\big(\bar{\tau}, B(\bar{\tau}-), (1-\beta)S(\bar{\tau}-)\big). \end{aligned} \tag{26}$$

Combining Inequalities (25) and (26), we deduce the assertion of the lemma. □

4.2 The dynamic programming principle

With the help of Lemma 4.1 and Lemma 4.4, we can prove the dynamic programming principle.

Theorem 4.5. *Let $(t,b,s) \in [0,T) \times \mathcal{S}^1$ and let θ be a $[t,T]$-valued stopping time. Then*

$$\mathcal{V}(t,b,s) = \sup_{\pi \in \mathcal{A}(t,b,s)} \inf_{\tau \in \mathcal{B}(t)} E\Big[\mathcal{V}\big(\theta, B(\theta-), S(\theta-)\big)\mathbf{1}_{\{\theta<\tau\}} + \check{\mathcal{V}}\big(\tau, B(\tau-), (1-\beta)S(\tau-)\big)\mathbf{1}_{\{\theta\geq\tau\}}\Big].$$

Proof. 1. Let $\varepsilon > 0$ and fix an arbitrary pre-crash strategy $\pi \in \mathcal{A}(t,b,s)$ and a crash time $\tau \in \mathcal{B}(t)$. By Lemma 4.1, we can find $\pi^\varepsilon \in \mathcal{A}(t,b,s)$ which coincides with π on $[t, \tau \wedge \theta)$ and a post-crash strategy $\check{\pi}^\varepsilon \in \check{\mathcal{A}}(\pi)$ corresponding to π^ε and τ such that

$$\begin{aligned} &E\left[U_p\left(X_{t,b,s}^{\pi^\varepsilon,\check{\pi}^\varepsilon,\tau}(T)\right)\Big|\mathcal{F}^t\big((\tau\wedge\theta)-\big)\right] + \varepsilon \\ &\qquad \geq \mathcal{V}\big(\theta, B(\theta-), S(\theta-)\big)\mathbf{1}_{\{\tau>\theta\}} + \check{\mathcal{V}}\big(\tau, B(\tau-), (1-\beta)S(\tau-)\big)\mathbf{1}_{\{\tau\leq\theta\}}. \end{aligned}$$

We have

$$\begin{aligned} \mathcal{V}(t,b,s) &\geq \inf_{\tau\in\mathcal{B}(t)} E\left[U_p\left(X_{t,b,s}^{\pi^\varepsilon,\check{\pi}^\varepsilon,\tau}(T)\right)\right] \\ &= \inf_{\tau\in\mathcal{B}(t)} E\left[E\left[U_p\left(X_{t,b,s}^{\pi^\varepsilon,\check{\pi}^\varepsilon,\tau}(T)\right)\Big|\mathcal{F}^t\big((\tau\wedge\theta)-\big)\right]\right] \\ &\geq \inf_{\tau\in\mathcal{B}(t)} E\Big[\mathcal{V}\big(\theta, B(\theta-), S(\theta-)\big)\mathbf{1}_{\{\tau>\theta\}} \\ &\qquad\qquad + \check{\mathcal{V}}\big(\tau, B(\tau-), (1-\beta)S(\tau-)\big)\mathbf{1}_{\{\tau\leq\theta\}}\Big] - \varepsilon. \end{aligned}$$

Since ε and τ were chosen arbitrarily and since we can find corresponding π^ε and $\check{\pi}^\varepsilon$ for every $\pi \in \mathcal{A}(t,b,s)$, this implies

$$\begin{aligned} \mathcal{V}(t,b,s) \quad \geq \quad & \sup_{\pi\in\mathcal{A}(t,b,s)} \inf_{\tau\in\mathcal{B}(t)} E\Big[\mathcal{V}\big(\theta, B(\theta-), S(\theta-)\big)\mathbf{1}_{\{\theta<\tau\}} \\ &\qquad + \check{\mathcal{V}}\big(\tau, B(\tau-), (1-\beta)S(\tau-)\big)\mathbf{1}_{\{\theta\geq\tau\}}\Big]. \end{aligned}$$

2. Let $\varepsilon > 0$, and let both $\pi \in \mathcal{A}(t,b,s)$ and $\check{\pi} \in \check{\mathcal{A}}(\pi)$ be ε-optimal. By Lemma 4.4, for every $\tau \in \mathcal{B}(t)$, we can find $\tau^\varepsilon \in \mathcal{B}(t)$ such that we have

$$\begin{aligned} &E\left[U_p\left(X_{t,b,s}^{\pi,\check{\pi},\tau^\varepsilon}(T)\right)\Big|\mathcal{F}^t\big((\tau\wedge\theta)-\big)\right] \\ &\qquad \leq \mathcal{V}\big(\theta, B(\theta-), S(\theta-)\big)\mathbf{1}_{\{\tau>\theta\}} + \check{\mathcal{V}}\big(\tau, B(\tau-), (1-\beta)S(\tau-)\big)\mathbf{1}_{\{\tau\leq\theta\}} + \varepsilon. \end{aligned}$$

We therefore have, for every $\tau \in \mathcal{B}(t)$, by the ε-optimality of the trading strategies,

$$\begin{aligned} \mathcal{V}(t,b,s) &\leq \inf_{\tau\in\mathcal{B}(t)} E\left[U_p\left(X_{t,b,s}^{\pi,\check{\pi},\tau}(T)\right)\right] + \varepsilon \\ &\leq E\left[E\left[U_p\left(X_{t,b,s}^{\pi,\check{\pi},\tau^\varepsilon}(T)\right)\Big|\mathcal{F}^t\big((\tau\wedge\theta)-\big)\right]\right] + \varepsilon \\ &\leq E\Big[\mathcal{V}\big(\theta, B(\theta-), S(\theta-)\big)\mathbf{1}_{\{\tau>\theta\}} \\ &\qquad + \check{\mathcal{V}}\big(\tau, B(\tau-), (1-\beta)S(\tau-)\big)\mathbf{1}_{\{\tau\leq\theta\}}\Big] + 2\varepsilon. \end{aligned}$$

Taking the infimum over $\mathcal{B}(t)$ and the supremum over $\mathcal{A}(t,b,s)$, this implies

$$\begin{aligned}\mathcal{V}(t,b,s) \quad &\leq \sup_{\pi\in\mathcal{A}(t,b,s)} \inf_{\tau\in\mathcal{B}(t)} E\Big[\mathcal{V}\big(\theta, B(\theta-), S(\theta-)\big)\mathbf{1}_{\{\theta<\tau\}} \\ &\qquad + \check{\mathcal{V}}\big(\tau, B(\tau-), (1-\beta)S(\tau-)\big)\mathbf{1}_{\{\theta\geq\tau\}}\Big] + 2\varepsilon.\end{aligned}$$

We conclude since ε was chosen arbitrarily.

□

5 The dynamic programming equations and the viscosity property

It is a known result that the value function $\check{\mathcal{V}}$ is the unique viscosity solution of the second-order partial differential equation

$$0 = \min\{\mathcal{L}^{nt}\check{\mathcal{V}}(t,b,s), \mathcal{L}^{buy}\check{\mathcal{V}}(t,b,s), \mathcal{L}^{sell}\check{\mathcal{V}}(t,b,s)\} \tag{27}$$

with suitable boundary conditions. The differential operators $\mathcal{L}^{nt}$, $\mathcal{L}^{buy}$ and $\mathcal{L}^{sell}$ are given by

$$\begin{aligned}\mathcal{L}^{nt} &:= -\frac{\partial}{\partial t} - rb\frac{\partial}{\partial b} - \alpha s\frac{\partial}{\partial s} - \frac{1}{2}\sigma^2 s^2 \frac{\partial^2}{\partial s^2}, \\ \mathcal{L}^{buy} &:= (1+\lambda)\frac{\partial}{\partial b} - \frac{\partial}{\partial s}, \\ \mathcal{L}^{sell} &:= -(1-\mu)\frac{\partial}{\partial b} + \frac{\partial}{\partial s},\end{aligned}$$

respectively. The proof of the viscosity property of $\check{\mathcal{V}}$ can e.g. be found in Davis et al. [11] (in a slightly different context) or by a straightforward adaptation of the methods in Shreve and Soner [30]. The uniqueness is proved in Belak et al. [5].

The aim of this section is to show that $\mathcal{V}$ is the unique viscosity solution of

$$0 = \max\big\{\mathcal{V}(t,b,s) - \check{\mathcal{V}}(t,b,(1-\beta)s), \min\{\mathcal{L}^{nt}\mathcal{V}(t,b,s), \mathcal{L}^{buy}\mathcal{V}(t,b,s), \mathcal{L}^{sell}\mathcal{V}(t,b,s)\}\big\}. \tag{28}$$

We begin with recalling the definition and some properties of viscosity solutions.

5.1 Viscosity solutions

The notion of viscosity solutions of partial differential equations can be defined in many equivalent ways. We recall the definitions which we will use in the sequel. An overview of viscosity solutions and their properties can be found in Crandall et al. [8].

Denote by $\mathbb{S}^2$ the set of symmetric 2×2 matrices with entries in $\mathbb{R}$. Let

$$F : [0,T) \times \mathcal{S}^1 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{S}^2 \to \mathbb{R}$$

be a continuous function satisfying

$$X \leq X' \qquad \Rightarrow \qquad F(t,x,r,q,p,X) \geq F(t,x,r,q,p,X')$$

and

$$q \leq q' \qquad \Rightarrow \qquad F(t,x,r,q,p,X) \geq F(t,x,r,q',p,X),$$

where $t\in[0,T)$, $x\in\mathcal{S}^1$, $r,q,q'\in\mathbb{R}$, $p\in\mathbb{R}^2$, and $X,X'\in\mathbb{S}^2$. By $X\leq X'$ we mean that $X'-X$ is non-negative definite.

Let $w : [0,T] \times \overline{\mathcal{S}}^1 \to \mathbb{R}$. If w is sufficiently smooth, we can consider differential equations of the form

$$F(t, x, w, D_t w, D_x w, D_x^2 w) = 0, \qquad (t,x) \in [0,T) \times \mathcal{S}^1. \tag{29}$$

Here, $D_t w$ denotes the derivative of w with respect to t, $D_x w$ denotes the gradient of w with respect to x and $D_x^2 w$ denotes the matrix of second order partial derivatives of w with respect to the components of x. However, if w is not sufficiently smooth, say only continuous, then we need to introduce a weaker concept of a solution of (29).

Definition 5.1. *1.) Let $w : [0,T] \times \overline{\mathcal{S}}^1 \to \mathbb{R}$ be continuous. Then w is called a* viscosity subsolution *of* (29)*, if for each $(t,x) \in [0,T) \times \mathcal{S}^1$ and all $\varphi \in C^{1,2}([0,T] \times \overline{\mathcal{S}}^1)$ with $\varphi \geq w$ satisfying $\varphi(t,x) = w(t,x)$, we have*

$$F\big(t, x, \varphi(t,x), D_t\varphi(t,x), D_x\varphi(t,x), D_x^2\varphi(t,x)\big) \leq 0.$$

2.) w is called a viscosity supersolution *of* (29)*, if for each $(t,x) \in [0,T) \times \mathcal{S}^1$ and all $\varphi \in C^{1,2}([0,T] \times \overline{\mathcal{S}}^1)$ with $\varphi \leq w$ satisfying $\varphi(t,x) = w(t,x)$, we have*

$$F\big(t, x, \varphi(t,x), D_t\varphi(t,x), D_x\varphi(t,x), D_x^2\varphi(t,x)\big) \geq 0.$$

3.) w is called a viscosity solution *of* (29)*, if it is both a viscosity sub- and supersolution.*

It was shown in Crandall et al. [8], that a sufficiently regular viscosity solution is also a solution in the classical sense and vice versa.

An equivalent definition of viscosity sub- and supersolutions can be given in terms of the sub- and superjets of the function w. We define the *superjet* $J^{2,+}w(t,x)$ of $w(t,x)$ to be the set of all $(q,p,X) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{S}^2$ such that

$$\limsup_{\substack{s\to t,\\ y \to x}} \frac{1}{|t-s| + ||x-y||_2^2} \Big[w(t,x) - w(s,y) - q(t-s) \\ - \langle p, x-y \rangle - \frac{1}{2} \langle X(x-y), x-y \rangle \Big] \leq 0,$$

where we assume that $(s,y) \in [0,T) \times \mathcal{S}^1$ and where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{R}^2$. We define the *subjet* $J^{2,-}w(t,x)$ of $w(t,x)$ by setting

$$J^{2,-}w(t,x) := -J^{2,+}(-w)(t,x).$$

We define the closure $\overline{J}^{2,+}w(t,x)$ of the superjet $J^{2,+}w(t,x)$ to be the set of all $(q,p,X) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{S}^2$, for which we can find a sequence $\{(t_j, x_j, q_j, p_j, X_j)\}_{j\in\mathbb{N}}$ in $[0,T) \times \mathcal{S}^1 \times J^{2,+}w(t_j, x_j)$, such that

$$\lim_{j\to\infty} (t_j, x_j, q_j, p_j, X_j) = (t, x, q, p, X).$$

The closure $\overline{J}^{2,-}w(t,x)$ of the subjet $J^{2,-}w(t,x)$ is defined analogously.

In terms of the sub- and superjets, a viscosity solution can be defined as follows.

Definition 5.2. *1.) Let $w : [0,T] \times \overline{\mathcal{S}}^1 \to \mathbb{R}$ be continuous. Then w is called a* viscosity subsolution *of* (29)*, if for each $(t,x) \in [0,T) \times \mathcal{S}^1$ and all $(q,p,X) \in J^{2,+}w(t,x)$, we have*

$$F\big(t, x, w(t,x), q, p, X\big) \leq 0.$$

2.) w is called a viscosity supersolution *of* (29)*, if for each $(t,x) \in [0,T) \times \mathcal{S}^1$ and all $(q,p,X) \in J^{2,-}w(t,x)$, we have*

$$F\big(t, x, w(t,x), q, p, X\big) \geq 0.$$

3.) w is called a viscosity solution *of* (29)*, if it is both a viscosity sub- and supersolution.*

As mentioned before, Definition 5.1 and Definition 5.2 are equivalent, see Crandall et al. [8]. The first definition is more convenient for proving that $\mathcal{V}$ is a viscosity solution of (28), while the second definition is more convenient when proving uniqueness.

5.2 The viscosity property of the value function

This section addresses the viscosity property of $\mathcal{V}$. We summarize the result in the next theorem.

Theorem 5.3. *$\mathcal{V}$ is a viscosity solution of*

$$0 = \max\big\{\mathcal{V}(t,b,s) - \check{\mathcal{V}}(t,b,(1-\beta)s), \min\{\mathcal{L}^{nt}\mathcal{V}(t,b,s), \mathcal{L}^{buy}\mathcal{V}(t,b,s), \mathcal{L}^{sell}\mathcal{V}(t,b,s)\}\big\}, \quad (30)$$

where $(t,b,s) \in [0,T) \times \mathcal{S}^1$, with boundary condition

$$\mathcal{V}(t,b,s) = \begin{cases} 0, & \text{if } (b,s) \in \partial\mathcal{S}^1 \text{ and } 0 < p < 1, \\ -\infty, & \text{if } (b,s) \in \partial\mathcal{S}^1 \text{ and } p \le 0, \end{cases} \quad (31)$$

and terminal condition

$$\mathcal{V}(T,b,s) = \begin{cases} U_p(b + (1-\mu)(1-\beta)s), & \text{if } s > 0, \\ U_p(b + (1+\lambda)s), & \text{if } s \le 0. \end{cases} \quad (32)$$

Remark 5.4. *To see that $\mathcal{V}$ satisfies the terminal condition* (32)*, note that from the market's point of view, a crash at terminal time must be optimal whenever the stock position is positive.*

We split the proof in two cases, the supersolution and the subsolution property.

Proposition 5.5. *$\mathcal{V}$ is a viscosity supersolution of* (30) *with boundary condition* (31) *and terminal condition* (32)*.*

Proof. The proof works similarly to Shreve and Soner [30, Lemma 7.8]. However, some additional care needs to be taken about the term arising from the crash possibility.

Let $(t,b,s) \in [0,T) \times \mathcal{S}^1$ and let $\varphi \in C^{1,2}([0,T] \times \overline{\mathcal{S}}^1)$ with $\varphi \le \mathcal{V}$ be a function satisfying $\varphi(t,b,s) = \mathcal{V}(t,b,s)$. We want to show that

$$0 \le \max\big\{\varphi(t,b,s) - \check{\mathcal{V}}(t,b,(1-\beta)s), \min\{\mathcal{L}^{nt}\varphi(t,b,s), \mathcal{L}^{buy}\varphi(t,b,s), \mathcal{L}^{sell}\varphi(t,b,s)\}\big\}.$$

By Lemma 3.2.2, we have

$$\varphi(t,b,s) = \mathcal{V}(t,b,s) \le \check{\mathcal{V}}\big(t,b,(1-\beta)s\big).$$

If equality holds we are done. Otherwise, it follows as in Davis et al. [11, Theorem 2, Part (ii)] that

$$\mathcal{L}^{buy}\varphi(t,b,s) \ge 0, \qquad \mathcal{L}^{sell}\varphi(t,b,s) \ge 0.$$

Let $\varepsilon > 0$ be small enough, such that the open ball around (b,s) with radius ε, denoted by $\mathcal{B}_\varepsilon(b,s)$, is contained in $\mathcal{S}^1$ and such that $t+\varepsilon < T$. Consider the strategy $L \equiv M \equiv 0$ on $[t,\tau_\varepsilon]$, where τ_ε is given by

$$\tau_\varepsilon := (t+\varepsilon) \wedge \inf\{u \ge t : (B(u),S(u)) \notin \mathcal{B}_\varepsilon(b,s)\}.$$

Since $\mathcal{V}$ and $\check{\mathcal{V}}$ are continuous and since we have

$$\mathcal{V}(t,b,s) < \check{\mathcal{V}}(t,b,(1-\beta)s),$$

we can choose ε small enough such that

$$\mathcal{V}(\bar{t},\bar{b},\bar{s}) < \check{\mathcal{V}}(\bar{t},\bar{b},(1-\beta)\bar{s}) \qquad \text{for all } (\bar{t},\bar{b},\bar{s}) \in [t,\tau_\varepsilon] \times \overline{\mathcal{B}}_\varepsilon(b,s),$$

which implies that it cannot be optimal to crash on $[t,\tau_\varepsilon]$, since this would contradict the dynamic programming principle. Using $\mathcal{V}(t,b,s) = \varphi(t,b,s)$ together with Itô's formula yields

$$\begin{aligned}\mathcal{V}(t,b,s) &= \varphi(\tau_\varepsilon, B(\tau_\varepsilon), S(\tau_\varepsilon)) + \int_t^{\tau_\varepsilon} \mathcal{L}^{nt}\varphi(u,B(u),S(u))\,du \\ &\quad - \int_t^{\tau_\varepsilon} \sigma S(u-)\varphi_s(u-,B(u-),S(u-))\,dW(u).\end{aligned}$$

Take expectation on both sides to obtain

$$\mathcal{V}(t,b,s) = E\left[\varphi(\tau_\varepsilon, B(\tau_\varepsilon), S(\tau_\varepsilon))\right] + E\left[\int_t^{\tau_\varepsilon} \mathcal{L}^{nt}\varphi(u,B(u),S(u))\,du\right]. \tag{33}$$

By Theorem 4.5, since crashes cannot be optimal on $[t,\tau_\varepsilon]$ and since $\varphi \le \mathcal{V}$, we have

$$\begin{aligned}\mathcal{V}(t,b,s) &\ge E\left[\mathcal{V}\big(\tau_\varepsilon, B(\tau_\varepsilon), S(\tau_\varepsilon)\big)\right] \\ &\ge E\left[\varphi\big(\tau_\varepsilon, B(\tau_\varepsilon), S(\tau_\varepsilon)\big)\right] \\ &= \mathcal{V}(t,b,s) - E\left[\int_t^{\tau_\varepsilon} \mathcal{L}^{nt}\varphi(u,B(u),S(u))\,du\right],\end{aligned}$$

where the last equality follows from Equation (33). Hence,

$$E\left[\int_t^{\tau_\varepsilon} \mathcal{L}^{nt}\varphi(u,B(u),S(u))\,du\right] \ge 0,$$

i.e.

$$\max_{(\bar{t},\bar{b},\bar{s})\in[t,\tau_\varepsilon]\times\overline{\mathcal{B}}_\varepsilon(b,s)} \mathcal{L}^{nt}\varphi(\bar{t},\bar{b},\bar{s}) \ge 0.$$

Let $\varepsilon \downarrow 0$ to obtain

$$\mathcal{L}^{nt}\varphi(t,b,s) \ge 0,$$

which completes the proof. □

Proposition 5.6. $\mathcal{V}$ *is a viscosity subsolution of* (30) *with boundary condition* (31) *and terminal condition* (32).

Proof. Let $(t,b,s) \in [0,T) \times \mathcal{S}^1$ and let $\varphi \in C^{1,2}([0,T]\times\overline{\mathcal{S}}^1)$ with $\varphi \ge \mathcal{V}$ be a function satisfying $\varphi(t,b,s) = \mathcal{V}(t,b,s)$. We want to show that

$$0 \ge \max\big\{\varphi(t,b,s) - \check{\mathcal{V}}(t,b,(1-\beta)s), \min\{\mathcal{L}^{nt}\varphi(t,b,s), \mathcal{L}^{buy}\varphi(t,b,s), \mathcal{L}^{sell}\varphi(t,b,s)\}\big\}.$$

As in the proof of Proposition 5.5 we have $\varphi(t,b,s) - \check{\mathcal{V}}(t,b,(1-\beta)s) \le 0$. That is, we only have to show that

$$\min\{\mathcal{L}^{nt}\varphi(t,b,s), \mathcal{L}^{buy}\varphi(t,b,s), \mathcal{L}^{sell}\varphi(t,b,s)\} \le 0.$$

Assume that, on the contrary, we have

$$\min\{\mathcal{L}^{nt}\varphi(t,b,s), \mathcal{L}^{buy}\varphi(t,b,s), \mathcal{L}^{sell}\varphi(t,b,s)\} > 0.$$

That is, there exist some $\delta, \varepsilon > 0$ such that on $[t, t+\varepsilon] \times \overline{\mathcal{B}}_\varepsilon(b,s) \subset [t,T) \times \mathcal{S}^1$, we have

$$\mathcal{L}^{nt}\varphi(\bar{t},\bar{b},\bar{s}) \geq \delta, \qquad \mathcal{L}^{buy}\varphi(\bar{t},\bar{b},\bar{s}) \geq \delta, \qquad \mathcal{L}^{sell}\varphi(\bar{t},\bar{b},\bar{s}) \geq \delta.$$

Given $\pi \in \mathcal{A}(t,b,s)$ and $h > 0$ small enough such that $t + h \in (t+\varepsilon, T)$, define the stopping times

$$\begin{aligned}
\tau_\varepsilon &:= \inf\left\{u \geq t : \big(B^\pi(u), S^\pi(u)\big) \notin \overline{\mathcal{B}}_\varepsilon(b,s)\right\} \wedge (t+\varepsilon), \\
\tau_h &:= \inf\left\{u \geq t+h : \mathcal{V}\big(u, B^\pi(u), S^\pi(u)\big) = \check{\mathcal{V}}\big(u, B^\pi(u), (1-\beta)S^\pi(u)\big)\right\}.
\end{aligned}$$

Application of Theorem 4.5 yields

$$\begin{aligned}
\mathcal{V}(t,b,s) &= \sup_{\pi\in\mathcal{A}(t,b,s)} \inf_{\tau\in\mathcal{B}(t)} E\Big[\mathcal{V}\big(\tau_\varepsilon, B(\tau_\varepsilon-), S(\tau_\varepsilon-)\big)\mathbf{1}_{\{\tau_\varepsilon<\tau\}} \\
&\qquad\qquad + \check{\mathcal{V}}\big(\tau, B(\tau-), (1-\beta)S(\tau-)\big)\mathbf{1}_{\{\tau_\varepsilon\geq\tau\}}\Big] \\
&\leq \sup_{\pi\in\mathcal{A}(t,b,s)} E\Big[\mathcal{V}\big(\tau_\varepsilon, B(\tau_\varepsilon-), S(\tau_\varepsilon-)\big)\mathbf{1}_{\{\tau_\varepsilon<\tau_h\}} \\
&\qquad\qquad + \check{\mathcal{V}}\big(\tau_h, B(\tau_h-), (1-\beta)S(\tau_h-)\big)\mathbf{1}_{\{\tau_\varepsilon\geq\tau_h\}}\Big] \\
&= \sup_{\pi\in\mathcal{A}(t,b,s)} E\Big[\mathcal{V}\big(\tau_\varepsilon\wedge\tau_h, B((\tau_\varepsilon\wedge\tau_h)-), S((\tau_\varepsilon\wedge\tau_h)-)\big)\Big] \\
&= \sup_{\pi\in\mathcal{A}(t,b,s)} E\Big[\mathcal{V}\big(\tau_\varepsilon, B(\tau_\varepsilon-), S(\tau_\varepsilon-)\big)\Big].
\end{aligned}$$

The second last equality follows from the definition of τ_h and the last equality follows since $h > \varepsilon$. At this point, we can proceed as in the proof of Shreve and Soner [30, Lemma 7.9] after equation (7.9) to obtain a contradiction[2], proving that

$$\min\{\mathcal{L}^{nt}\varphi(t,b,s), \mathcal{L}^{buy}\varphi(t,b,s), \mathcal{L}^{sell}\varphi(t,b,s)\} \leq 0$$

and thus concluding the proof. □

Clearly, combining Proposition 5.5 and Proposition 5.6 yields Theorem 5.3 which finalizes this section.

5.3 Uniqueness of solutions

We now turn our focus on the uniqueness of viscosity solutions of Equations (27) and (28). For convenience, we define

$$\begin{aligned}
F^c(t,b,s,u) &:= u - \check{\mathcal{V}}(t,b,(1-\beta)s), \\
F^{nt}(b,s,q,p,X) &:= -q - rbp_1 - \alpha s p_2 - \frac{1}{2}\sigma^2 s^2 X_{22}, \\
F^{buy}(p) &:= (1+\lambda)p_1 - p_2, \\
F^{sell}(p) &:= -(1-\mu)p_1 + p_2,
\end{aligned}$$

where $(t,b,s) \in [0,T)\times\mathcal{S}^1$, $u, q \in \mathbb{R}$, $p = (p_1,p_2) \in \mathbb{R}^2$, and $X = (X_{ij})_{i,j=1,2} \in \mathbb{S}^2$. Furthermore, set

$$\check{F}(b,s,q,p,X) := \min\Big\{F^{nt}(b,s,q,p,X), F^{buy}(p), F^{sell}(p)\Big\}, \tag{34}$$

$$F(t,b,s,u,q,p,X) := \max\Big\{F^c(t,b,s,u), \check{F}(b,s,q,p,X)\Big\}. \tag{35}$$

[2]See also Belak [3] for full details.

Then (34) corresponds to the Dynamic Programming Equation (27) and (35) corresponds to the Dynamic Programming Equation (28).

Theorem 5.7. *Let $u, v \in C([0,T] \times \overline{\mathcal{S}}^1)$. Assume that u is a viscosity subsolution of* (28) *and v is a viscosity supersolution of* (28) *with $u \leq v$ on $([0,T] \times \partial\mathcal{S}^1) \cup (\{T\} \times \mathcal{S}^1)$. Assume further that*

$$\begin{aligned} &U_p\Big(b + \min\{(1-\mu)(1-\beta)s, (1+\lambda)s\}\Big) \\ &\qquad \leq \quad u(t,b,s), v(t,b,s) \leq U_p\Big((b+s)\Big[2r - \frac{\alpha^2}{(1-p)\sigma^2}\Big](T-t)\Big). \end{aligned} \tag{36}$$

Then $u \leq v$ on $[0,T] \times \overline{\mathcal{S}}^1$.

Proof. u is a viscosity subsolution, i.e. for each $(q,p,X) \in J^{2,+}u(t,b,s)$, we have

$$F(t,b,s,u(t,b,s),q,p,X) \leq 0.$$

In particular,

$$F^c(t,b,s,u(t,b,s)) = u(t,b,s) - \check{\mathcal{V}}(t,b,(1-\beta)s) \leq 0.$$

v is a viscosity supersolution, i.e. for every $(\bar{q},\bar{p},\bar{X}) \in J^{2,-}v(t,b,s)$, we have

$$F(t,b,s,v(t,b,s),\bar{q},\bar{p},\bar{X}) \geq 0.$$

If $F^c(t,b,s,v(t,b,s)) \geq 0$, i.e. $v(t,b,s) \geq \check{\mathcal{V}}(t,b,(1-\beta)s)$, it follows that

$$u(t,b,s) \leq \check{\mathcal{V}}(t,b,(1-\beta)s) \leq v(t,b,s)$$

and we are done. Otherwise, if $\check{F}(b,s,\bar{q},\bar{p},\bar{X}) \geq 0$, we can use the same arguments as in Belak et al. [5] to establish $u(t,b,s) \leq v(t,b,s)$. □

Theorem 5.7 implies that the Dynamic Programming Equation (28) characterizes the value function $\mathcal{V}$ uniquely.

Corollary 5.8. *$\mathcal{V}$ is the unique viscosity solution of* (28) *in the class of continuous functions satisfying the growth condition* (36).

6 Numerical Results

We conclude this paper with some numerical examples. By means of the homotheticity property (see Lemma 3.2.4), we first reduce the dimension of the state space. We then adapt the algorithm introduced by Kunisch and Sass [22] to simulate the value functions and determine the free boundaries. A different algorithm can be found e.g. in Herzog et al. [15], which does not require any structural assumptions on the dynamic programming equations.

Throughout this section, we assume for simplicity that the constants r, α, σ and $p \neq 0$ are such that

$$\frac{1}{1-p}\frac{\alpha - r}{\sigma^2} \in (0,1).$$

6.1 Reduction to risky fractions and the optimal strategies

As a first step, we use the homotheticity property of the value functions to reduce the dimension of the state space. Setting $x := s/(b+s)$ and

$$\mathscr{V}(t,x) := \mathcal{V}(t,1-x,x), \qquad \check{\mathscr{V}}(t,x) := \check{\mathcal{V}}(t,1-x,x),$$

we have, by the homotheticity property (Lemma 3.2.4),

$$\mathcal{V}(t,b,s) = (b+s)^p \mathscr{V}(t,x), \qquad \check{\mathcal{V}}(t,b,s) = (b+s)^p \check{\mathscr{V}}(t,x).$$

By formally expressing the derivatives of $\check{\mathcal{V}}$ in terms of the derivatives of $\check{\mathscr{V}}$, one can show (as in Shreve and Soner [30, Proposition 8.1]) that $\check{\mathscr{V}}$ is the unique viscosity solution of

$$0 = \min\big\{\mathscr{L}^{nt}\check{\mathscr{V}}(t,x), \mathscr{L}^{buy}\check{\mathscr{V}}(t,x), \mathscr{L}^{sell}\check{\mathscr{V}}(t,x)\big\}, \tag{37}$$

on $[0,T) \times \mathscr{S}^0$, with terminal condition

$$\check{\mathscr{V}}(T,x) = \check{\mathcal{V}}(T,1-x,x) = \begin{cases} \frac{1}{p}(1-\mu x)^p, & \text{if } x > 0, \\ \frac{1}{p}(1+\lambda x)^p, & \text{if } x \le 0, \end{cases}$$

where $\mathscr{S}^0 := (-1/\lambda, 1/\mu)$, and where the operators $\mathscr{L}^{nt}$, $\mathscr{L}^{buy}$ and $\mathscr{L}^{sell}$ are given by

$$\begin{aligned} \mathscr{L}^{nt}\mathscr{V} &:= -\mathscr{V}_t - \big(\alpha x + r(1-x) - \frac{1}{2}(1-p)\sigma^2 x^2\big)p\mathscr{V} \\ &\quad - \big((\alpha - r)(1-x) - (1-p)\sigma^2 x(1-x)\big)x\mathscr{V}_x - \frac{1}{2}\sigma^2 x^2(1-x)^2\mathscr{V}_{xx}, \\ \mathscr{L}^{buy}\mathscr{V} &:= p\lambda\mathscr{V} - (1+\lambda x)\mathscr{V}_x, \\ \mathscr{L}^{sell}\mathscr{V} &:= p\mu\mathscr{V} + (1-\mu x)\mathscr{V}_x. \end{aligned}$$

Similarly, $\mathscr{V}$ is the unique viscosity solution of

$$\begin{aligned} 0 \;=\; \max\Big\{&\mathscr{V}(t,x) - (1-\beta x)^p \check{\mathscr{V}}\left(t, \frac{(1-\beta)x}{1-\beta x}\right), \\ &\min\big\{\mathscr{L}^{nt}\mathscr{V}(t,x), \mathscr{L}^{buy}\mathscr{V}(t,x), \mathscr{L}^{sell}\mathscr{V}(t,x)\big\}\Big\} \end{aligned} \tag{38}$$

on $[0,T) \times \mathscr{S}^1$, with terminal condition

$$\mathscr{V}(T,x) = \mathcal{V}(T,1-x,x) = \begin{cases} \frac{1}{p}(1-x+(1-\mu)(1-\beta)x)^p, & \text{if } x > 0, \\ \frac{1}{p}(1+\lambda x)^p, & \text{if } x \le 0, \end{cases}$$

where $\mathscr{S}^1 := (-1/\lambda, 1/[1-(1-\beta)(1-\mu)])$.

As a next step, let us take a look at the optimal strategies. We start with the crash-free case. We define the regions

$$\begin{aligned} \check{\mathscr{R}}^{nt} &:= \left\{(t,x) \in [0,T] \times \overline{\mathscr{S}}^0 : \mathscr{L}^{nt}\check{\mathscr{V}}(t,x) = 0\right\}, \\ \check{\mathscr{R}}^{buy} &:= \left\{(t,x) \in [0,T] \times \overline{\mathscr{S}}^0 : \mathscr{L}^{buy}\check{\mathscr{V}}(t,x) = 0\right\}, \\ \check{\mathscr{R}}^{sell} &:= \left\{(t,x) \in [0,T] \times \overline{\mathscr{S}}^0 : \mathscr{L}^{sell}\check{\mathscr{V}}(t,x) = 0\right\}. \end{aligned}$$

Under the assumption $\alpha > r$, Dai and Yi [9] show that $\check{\mathscr{V}}$ is indeed sufficiently smooth such that we can apply the operators $\mathscr{L}^{nt}$, $\mathscr{L}^{buy}$ and $\mathscr{L}^{sell}$ in the classical sense. Furthermore, the three regions form a partition of $[0,T] \times \overline{\mathscr{S}}^0$ and each of the regions is a connected set. The boundaries between the regions can be described by two continuous functions of time. Additionally, if $(t, x_1) \in \check{\mathscr{R}}^{buy}$, $(t, x_2) \in \check{\mathscr{R}}^{nt}$, $(t, x_3) \in \check{\mathscr{R}}^{sell}$, then $x_1 < x_2 < x_3$. That is, the sell region always lies above the no-trading region, which in turn lies above the buy region. From this, we can conjecture the form of the optimal trading strategy.

1. If $(t, x) \in \check{\mathscr{R}}^{nt}$, then the investor does nothing. We therefore call $\check{\mathscr{R}}^{nt}$ the *no-trading region*.
2. If $(t, x) \in \check{\mathscr{R}}^{buy}$, then the investor buys stock until the new position reaches $\check{\mathscr{R}}^{nt}$. We therefore call $\check{\mathscr{R}}^{buy}$ the *buy region*.
3. If $(t, x) \in \check{\mathscr{R}}^{sell}$, then the investor sells stock until the new position reaches $\check{\mathscr{R}}^{nt}$. We therefore call $\check{\mathscr{R}}^{sell}$ the *sell region*.

With this, the optimal risky fraction process is a diffusion reflected at the boundaries of the no-trading region. Shreve and Soner [30] rigorously prove the existence of the optimal strategy for the infinite-horizon case.

In the sequel, we compute $\check{\mathcal{V}}$ numerically which allows us to determine $\check{\mathscr{R}}^{nt}$, $\check{\mathscr{R}}^{buy}$, and $\check{\mathscr{R}}^{sell}$ which in turn determine the conjectured optimal strategy. However, before we do so, let us consider the crash-threatened case.

In the presence of crashes, we have to adjust the definition of the three regions slightly, since we have an additional operator in the dynamic programming equation. This operator leads to an additional region which we call the *crash region*:

$$\mathscr{R}^{crash} := \left\{ (t,x) \in [0,T] \times \overline{\mathscr{S}}^1 : \mathscr{V}(t,x) = (1-\beta x)^p \check{\mathscr{V}}\left(t, \frac{(1-\beta)x}{1-\beta x}\right) \right\}.$$

Note that $\mathscr{R}^{crash}$ is a closed set. The best crash time from the market's point of view can then be conjectured to be the first hitting time of $\mathscr{R}^{crash}$, see Remark 3.3. Assume now that $\mathscr{V}$ is sufficiently smooth to be a classical solution of (38). Then we define

$$\begin{aligned}
\mathscr{R}^{nt} &:= \left\{ (t,x) \in \left([0,T] \times \overline{\mathscr{S}}^1 \right) \setminus \mathscr{R}^{crash} : \mathscr{L}^{nt}\mathscr{V}(t,x) = 0 \right\}, \\
\mathscr{R}^{buy} &:= \left\{ (t,x) \in \left([0,T] \times \overline{\mathscr{S}}^1 \right) \setminus \mathscr{R}^{crash} : \mathscr{L}^{buy}\mathscr{V}(t,x) = 0 \right\}, \\
\mathscr{R}^{sell} &:= \left\{ (t,x) \in \left([0,T] \times \overline{\mathscr{S}}^1 \right) \setminus \mathscr{R}^{crash} : \mathscr{L}^{sell}\mathscr{V}(t,x) = 0 \right\}.
\end{aligned}$$

We conjecture that these sets form a partition of $([0,T] \times \overline{\mathscr{S}}^1) \setminus \mathscr{R}^{crash}$ and that the optimal trading strategy on these sets is determined in the same way as in the crash-free case. We furthermore conjecture that selling is optimal within the crash region. To see this, we show that if $\mathcal{V}(t,b,s) = \check{\mathcal{V}}(t,b,(1-\beta)s)$ and (t,b,s) is not on the boundary of the crash region, then

$$\mathcal{L}^{sell}\mathcal{V}(t,b,s) < 0,$$

whenever

$$\beta > \frac{\mu + \lambda}{1 + \lambda}.$$

To this end, set $\tilde{s} := (1-\beta)s$ and calculate

$$\begin{aligned}\mathcal{L}^{sell}\mathcal{V}(t,b,s) &= -(1-\mu)\check{\mathcal{V}}_b(t,b,\tilde{s}) + (1-\beta)\check{\mathcal{V}}_{\tilde{s}}(t,b,\tilde{s}) \\ &= -(1-\beta)\Big[(1+\lambda)\check{\mathcal{V}}_b(t,b,\tilde{s}) - \check{\mathcal{V}}_{\tilde{s}}(t,b,\tilde{s})\Big] \\ &\qquad\qquad + \Big[-(1-\mu) + (1-\beta)(1+\lambda)\Big]\check{\mathcal{V}}_b(t,b,\tilde{s}) \\ &= -(1-\beta)\mathcal{L}^{buy}\check{\mathcal{V}}(t,b,\tilde{s}) + \Big[\mu+\lambda-\beta(1+\lambda)\Big]\check{\mathcal{V}}_b(t,b,\tilde{s}).\end{aligned}$$

The claim follows since $\mathcal{L}^{buy}\check{\mathcal{V}} \geq 0$ and $\check{\mathcal{V}}_b > 0$. We therefore have $\mathscr{L}^{sell}\mathscr{V} < 0$ in the interior of $\mathscr{R}^{crash}$ and $\mathscr{L}^{sell}\mathscr{V} \geq 0$ outside of $\mathscr{R}^{crash}$. Since $\mathscr{V}$ is assumed to be of class $C^{1,2}$, this implies $\mathscr{L}^{sell}\mathscr{V} = 0$ on the boundary of $\mathscr{R}^{crash}$. Hence, we expect that the investor should sell whenever she reaches the crash region.

Note that as in the case without costs, the investor switches the strategy after a crash occurs. That is, as long as the investor knows that there is one crash left, she follows the optimal strategy for one crash (determined by $\mathscr{R}^{crash}$, $\mathscr{R}^{nt}$, $\mathscr{R}^{buy}$, and $\mathscr{R}^{sell}$). After observing the crash, the investor switches to the optimal strategy without crashes, i.e. the strategy determined by $\check{\mathscr{R}}^{nt}$, $\check{\mathscr{R}}^{buy}$, and $\check{\mathscr{R}}^{sell}$.

Let us now consider some numerical examples. We start with the crash-free case in Section 6.2 and then compare the results with the crash-threatened case in Section 6.3.

6.2 The crash-free case

Let us first consider the crash-free case, i.e. we are in the same situation as considered in Kunisch and Sass [22]. Therefore, we can apply the algorithm proposed in their paper to determine $\check{\mathcal{V}}$ and the location of the free boundaries between the regions $\check{\mathscr{R}}^{nt}$, $\check{\mathscr{R}}^{buy}$, and $\check{\mathscr{R}}^{sell}$. We consider the following market parameters:

$$\begin{aligned} r &:= 0, & \alpha &:= 0.096, & \sigma &:= 0.4, \\ T &:= 10, & p &:= 0.1, & \mu &:= \lambda := 0.01. \end{aligned}$$

Note that the numerical method applied below also works for negative values of p. The constants α, σ, and p in this example are chosen so that the optimal strategy in the crash-free market without transaction costs is equal to $2/3$. Note that this strategy is not admissible in the presence of transaction costs, since it requires infinite variation trading which would lead to immediate bankruptcy of the investor.

Let us give a short outline of the algorithm, the details of which can be found in Kunisch and Sass [22]. We discretize $[0,T]$ using an equidistant grid with mesh size Δt. Similarly, we discretize $[0,1]$ with an equidistant grid with mesh size Δx. Note that we restrict the approximation of $\check{\mathscr{V}}$ to $[0,1] \subset \mathscr{S}^0$. The derivatives in the operators $\mathscr{L}^{nt}$, $\mathscr{L}^{buy}$, and $\mathscr{L}^{sell}$ are approximated using a central finite-difference scheme. We solve the differential equation (37) backwards in time. In every time step (say, we are at time $t < T$), we make an initial guess $N_0 := [a_0, b_0]$ for the no-trading region. On $[a_0, b_0]$, we solve $\mathscr{L}^{nt}v_1(t,x) = 0$ for v_1 and extend the solution to $[0,1]$ using the explicit solutions of $\mathscr{L}^{buy}v_1(t,x) = 0$ on $[0, a_0)$ and $\mathscr{L}^{sell}v_1(t,x) = 0$ on $(b_0, 1]$ using a smooth pasting condition at a_0 and b_0. For every $x \in [0,1]$, we define

$$\lambda_1^B(t,x) := -\mathscr{L}^{nt}v_1(t,x)\mathbf{1}_{[0,a_0]}, \qquad \lambda_1^S(t,x) := -\mathscr{L}^{nt}v_1(t,x)\mathbf{1}_{[b_0,1]},$$

and we introduce the sets

$$B_1 := \left\{ x \in [0,1] : \lambda_1^B(t,x) + \mathscr{L}^{buy} v_1(t,x) < 0 \right\},$$
$$S_1 := \left\{ x \in [0,1] : \lambda_1^S(t,x) + \mathscr{L}^{sell} v_1(t,x) < 0 \right\}.$$

We set $N_1 = [0,1] \setminus (B_1 \cup S_1)$ to be the new guess for the no-trading region and repeat the procedure until $N_k \approx N_{k-1}$ for some $k > 0$. Once the no-trading region converged, we proceed with the next time step $t - \Delta t$.

Figure 2 depicts the resulting free boundaries. Whenever the investor holds a position which is below the buy boundary, it is optimal to buy stock and if the position is above the sell boundary, it is optimal to sell stock. If the position is in between the buy and sell boundary, the optimal action of the investor is not to trade at all. Nevertheless, the value function of the optimization problem varies in the no–trading area as well. The maximum line shows which risky fraction maximizes the value function.

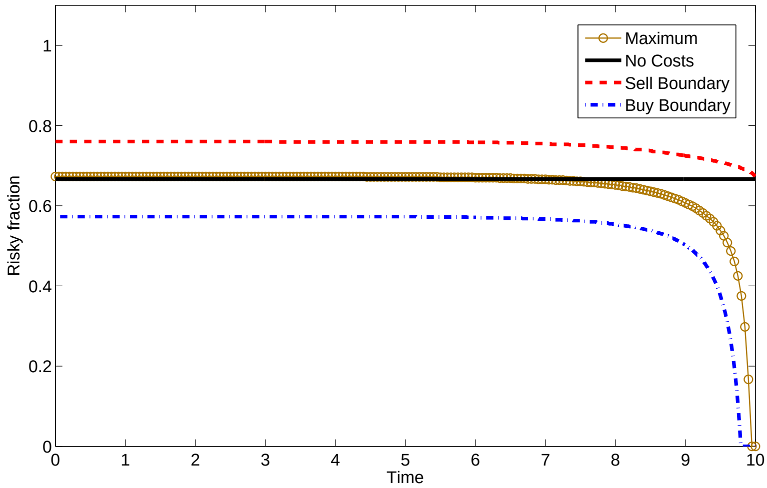

Figure 2: Optimal trading regions without crashes.

For comparison, we also plotted the optimal strategy for the case without transaction costs. It is a well known result, see Merton [26], that in this case the optimal strategy is to keep the risky fraction constant at the level

$$\check{\eta} := \frac{1}{1-p} \frac{\alpha - r}{\sigma^2} = \frac{2}{3}.$$

We call $\check{\eta}$ the Merton optimal portfolio or simply the *Merton fraction*. In Figure 2, note that, if the investment horizon is sufficiently large, the boundaries become stationary. When the investment horizon becomes smaller, we have two effects. First, the sell boundary decreases. This is because we optimize the total wealth *after* liquidation of the stock, i.e. the investor has to close the stock position at terminal time. Since she has to pay transaction costs in the process of liquidation, a lower stock position at terminal time is preferable. On the other hand, the buy boundary also decreases. This shows that the closer the investor comes to maturity, the less she wants to engage in transactions, since there is not enough time left to gain the transaction costs back. Dai and Yi [9, Theorem 4.7] show, that from the point

$$t^* := T - \frac{1}{\alpha - r} \log \left(\frac{1+\lambda}{1-\mu} \right) \tag{39}$$

onwards, buying shares of the stock is never optimal (i.e. t^* is the point at which the buy boundary hits 0). Also, note that the boundaries are only decreasing if the excess return of the stock over the bond is positive, i.e. $\alpha > r$. Otherwise, if $\alpha < r$, the boundaries are increasing (c.f. Herzog et al. [15, Example 3.3]).

Figure 2 also shows the risky fraction which maximizes the value function over time. The maximum is attained in the interior of the no-trading region and has similar qualitative features as the buy boundary. In particular, the maximum is attained at a risky fraction of 0 if the investment period is sufficiently small. This feature has already been observed in Herzog et al. [15]. Also, notice that for long horizons, the maximum can be above the Merton fraction, i.e. the two quantities do not necessarily have to coincide.

6.3 The crash-threatened case

Let us now consider what happens if the market is under the threat of a crash. We assume the crash size to be $\beta = 0.2$ and let the remaining parameters be as in Section 6.2. First, we need to adjust the algorithm in Kunisch and Sass [22] to work with our Dynamic Programming Equation (38). This is done as follows. Assume that we want to approximate the value function at time $t < T$. For the k-th iteration, we first solve as before

$$0 = \mathscr{L}^{nt} v_k(t, x)$$

inside our guess for the no-trading region $[a_{k-1}, b_{k-1}] \subset [0, 1]$ and extend v_k to $[0, 1]$ using the explicit solutions of $\mathscr{L}^{buy} v_k(t, x) = 0$ and $\mathscr{L}^{sell} v_k(t, x) = 0$. Also, we construct the active sets B_k and S_k as before. Then, we check if the crash constraint is satisfied for all $x \in [0, 1]$. For this, define

$$C_k := \left\{ x \in [0, 1] : v_k(t, x) - (1 - \beta x)^p \check{\mathscr{V}} \left(t, \frac{(1 - \beta)x}{1 - \beta x} \right) \leq 0 \right\}.$$

On C_k, we set

$$v_k(t, x) = (1 - \beta x)^p \check{\mathscr{V}} \left(t, \frac{(1 - \beta)x}{1 - \beta x} \right).$$

Now set $N_k = [0, 1] \setminus (B_k \cup S_k \cup C_k)$ and proceed with the next iteration.

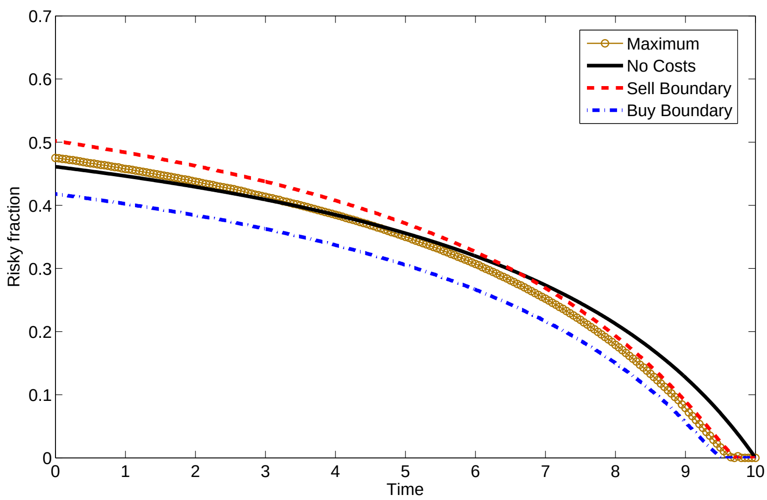

Figure 3: Optimal trading regions for a crash of size $\beta = 0.2$.

The resulting buy and sell boundaries are plotted in Figure 3. Again, for comparison, we plotted the optimal strategy in the case without costs. From Korn and Steffensen [20], the optimal strategy without costs and with one crash is given by the solution $\eta(t)$ of the ordinary differential equation

$$\eta_t(t) = \frac{1}{\beta}(1-\eta(t)\beta)\Big((\alpha-r)(\eta(t)-\check{\eta}) - \frac{1}{2}(1-p)\big((\eta(t))^2-\check{\eta}^2\big)\Big) \tag{40}$$

with terminal condition $\eta(T)=0$. Under the condition

$$0 \leq \eta(t) \leq \check{\eta},$$

the differential equation (40) has a unique solution. Note that the optimal strategy here involves continuous trading which is no longer feasible in the presence of transaction costs.

In Figure 3, we observe a striking feature of the sell boundary. If the time to maturity becomes smaller, the sell boundary crosses the optimal strategy without costs. Even more, the sell boundary hits zero strictly before terminal time T. We would furthermore like to emphasize that the buy and the sell boundary do not reduce to zero at the same time. That is, the buy boundary hits zero at time $t \approx 9.52$, whereas the sell boundary hits zero at time $t \approx 9.74$. Note also that t^*, i.e. the time at which the buy boundary in the crash-free setting hits zero, verifies $t^* \approx 9.79$ (compare this also with Remark 6.1). The maximum is again attained inside the no-trading region, which implies that for small investment horizons it is attained at a risky fraction of 0, since the sell boundary reaches 0 before terminal time. Also, for larger horizons, the maximum is attained above the optimal strategy of the no-costs case and the difference is even more pronounced than in the no-crash case (compare with Figure 2).

We would like to point out that even in the models without crashes one can observe that the sell boundary falls below the optimal strategy without costs (see e.g. Shreve and Soner [30, Equation (11.4)] for the infinite-horizon model and Liu and Loewenstein [23, Equation (22)] for the finite-horizon case). However, in these models, this behavior can only be observed in special cases, e.g. if the Merton fraction is sufficiently high (in particular, $\check{\eta} > 1$). See also Gerhold et al. [14] for a discussion of this effect. In our model, this behavior can be observed as soon as $\beta > 0$, that is, as soon as we allow for crashes.

Remark 6.1. *Let us consider a (heuristic) example to explain why the sell boundary reaches zero strictly before terminal time. Let $r=0$ and assume that $t_0 \in [0,T]$ is such that*

$$T - t_0 < \frac{1}{\alpha}\log\left(\frac{1+\lambda}{1-\mu}\right). \tag{41}$$

In particular, this implies that $t_0 > t^*$. Assume that at time t_0, the investor has a positive stock position $s > 0$ and a positive bond position $b > 0$ sufficiently large, such that $s/(b+s)$ is close to 0. Assume now, that we are in the crash-threatened case and assume that a crash of size β occurs at time t_0, leaving the investor with $(1-\beta)s$ units of money invested in the stock. After the crash we are in the crash-free setting and since t_0 is such that Inequality (41) holds, we see that buying is not optimal (since $t_0 > t^*$). Intuitively, we can conclude that the sell boundary reaches 0 (at time t_0) in the crash case at least as soon as the buy boundary in crash-free case reaches zero, that is $t_0 \leq t^*$. This is, because the investor cannot recoup any losses made in the stock by buying more stock after the crash. Heuristically, the wealth invested in the stock follows approximately a geometric Brownian motion starting in $(1-\beta)s$ (remember that we assume $s/(b+s)$ to be close to 0). The expected terminal stock wealth is then approximately

$$(1-\beta)se^{\alpha(T-t_0)}.$$

In order to benefit from a positive stock position we need

$$(1-\beta)se^{\alpha(T-t_0)} \geq s,$$

which clearly can only hold if $T-t_0$ is large enough. Hence, for small investment periods, it is not optimal to invest any money in the stock at all!

Remark 6.2. *It turns out that the sell region is empty in our numerical example, i.e. $\mathscr{R}^{sell}=\emptyset$, and the crash region $\mathscr{R}^{crash}$ is given as the whole region above and including the sell boundary in Figure 3 (by the discussions at the end of Section 6.1, however, the investor sells whenever the risky fraction touches the crash region). This means that the only reason why the investor sells shares of the stock is to protect herself against the impact of crashes on her portfolio. On the other hand, whenever the risky fraction is below the sell boundary, then the investor would benefit from a crash since in this case $\mathcal{V}(t,b,s)<\check{\mathcal{V}}(t,b,(1-\beta)s)$, and only on the sell boundary is the investor truly indifferent between an immediate crash and no crash at all.*

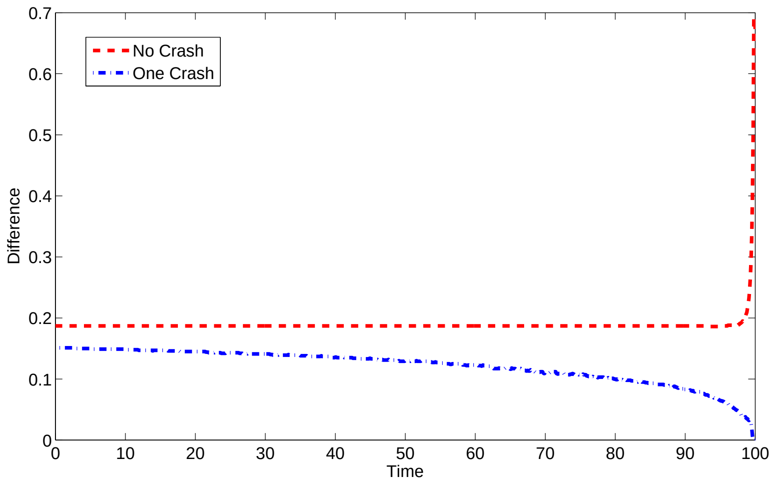

Figure 4: Difference of sell and buy boundaries with and without crashes.

In the case without crashes, the difference between the sell and the buy boundary is known to stabilize quickly as $T-t$ becomes large (see e.g. Gerhold et al. [14]). In the presence of crashes this effect can no longer be observed, since even for long times to maturity, the presence of a crash threat has a significant influence on the optimal trading strategy. This can be seen in Figure 4, where the difference between the buy and the sell boundary is plotted over a time horizon of a hundred years for both the case with and without crashes. Without crashes, the difference appears to be stable for maturities greater than approximately two to three years, meaning that the time-influence of the transaction costs on the optimal strategies is only significant for small investment periods. On the other hand, in the presence of crashes, the difference is increasing with increasing time to maturity even for large time horizons, indicating that the sensitivity of the optimal strategies with respect to time is significantly higher in the presence of crashes. Note, however, that the difference is always smaller for the case with a potential crash than for the case with no crash.

Remark 6.3 (Relative loss of utility)**.** *Clearly, assuming that no crash occurs, an investor following the optimal strategy in the presence of crashes will achieve less expected utility compared*

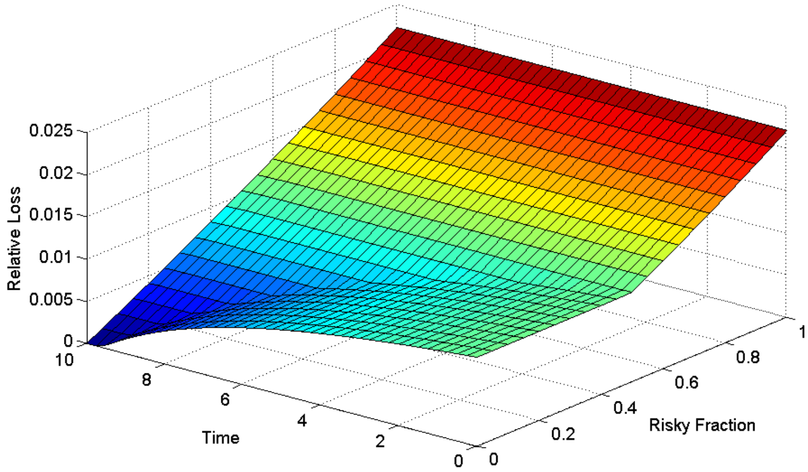

Figure 5: Relative loss of utility.

to the investor who follows the optimal strategy for zero crashes. To estimate the trade-off, we present in Figure 5 the relative loss of utility given by

$$\frac{\check{\mathscr{V}}(t,x) - \mathscr{V}(t,x)}{\check{\mathscr{V}}(t,x)}, \qquad (t,x) \in [0,T] \times [0,1].$$

One can see that the relative loss of utility for protection against a 20% *crash is at most* 2.5% *and, as long as the initial risky fraction is small (meaning that at time* $t = 0$ *it is in the no-trading region), the relative loss is at most around* 1% *even for long investment periods of around* 10 *years.*

7 Conclusions and outlook

We studied the optimal portfolio problem under transaction costs and under the threat of a crash. We characterized the value function as the unique continuous viscosity solution of the Dynamic Programming Equation (28) and analyzed the optimal trading strategies numerically.

Numerical simulations indicate that for the optimal worst-case strategy with transaction costs, the sell boundary will not only be below the optimal no-costs strategy close to the investment horizon as soon as there is a strictly positive crash size, but moreover, the sell boundary will be zero before the investment horizon is reached (see Figure 3 and Remark 6.1). While the first property can be observed also in the case without crashes under suitable conditions, the latter property is unique to the worst-case scenario optimization problem with transaction costs.

It remains for future research to study the regularity of $\mathcal{V}$ in more detail, to prove the existence of optimal controls (both in the case with and without crashes) and to verify the structural assumptions made in Section 6. Furthermore, Figure 4 indicates that the presence of crashes has a significant influence on the spread of the trading boundaries, even for very large time periods, while in the absence of crashes the spread becomes essentially stationary even for short time horizons. It would therefore be interesting to study the sensitivity of the value function with respect to the crash size in more detail and analyze its effects on the liquidity premium.

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