Worst-Case Portfolio Optimization in a Market with Bubbles

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Abstract

We investigate a utility maximization problem in the presence of asset price bubbles. At random times, the investor receives warnings that a bubble has formed in the market which may lead to a crash in the risky asset. We propose a regime switching model for the warnings and we make no assumptions about the distribution of the timing and the size of the crashes. Instead, we assume that the investor takes a worst-case perspective towards their impacts, i.e. the investor maximizes her expected utility under the worst-case crash scenario. We characterize the value function by a system of Hamilton-Jacobi-Bellman equations and derive a coupled system of ordinary differential equations for the optimal strategies. Numerical examples are provided.

Keywords: optimal investment, market crashes, worst-case scenario, regime switching, financial bubbles

Mathematics Subject Classification (2010): 91G80, 93E20

JEL Classification: G11

1 Introduction

We study a utility maximization problem in the presence of asset price bubbles which may lead to crashes in the risky asset. The investor is assumed to take a worst-case perspective towards the impact of such crashes. We assume that the investor receives a warning about a potential bubble at random times and the bubble leads to a crash in the risky asset if it bursts. While we assume a probabilistic structure for the arrival times of the warnings, we make no assumption about the distribution of the potential crash. Instead we assume that the investor optimizes her performance functional in the worst-case crash scenario. This leads to a stochastic differential game of the form

$$\sup_{\pi} \inf_{\vartheta} \mathbb{E}[U(X^{\pi,\vartheta}(T))], \tag{1.1}$$

i.e. for each trading strategy π the investor tries to identify the worst-case crash scenario ϑ in the sense that expected utility of terminal wealth X_T is minimized and then maximizes the terminal utility in the worst-case scenario.

Note that the optimization problem is the same as in Korn and Wilmott [22], however, observe that the crucial difference is that we allow for a random number of crashes (where Korn and Wilmott [22] assume to know an upper bound on the total number of crashes). This leads to optimal portfolio strategies where the crash risk has not only a short-term impact (as in Korn and Wilmott [22]) but also a long-term impact (see Belak et al. [4]).

In a previous paper of Belak et al. [4], the warnings occur at the jump times of an independent Poisson process, i.e. the times at which warnings occur are exponentially distributed. In addition, it is assumed that the maximum crash size is fixed a priori and does not change with time. In the present paper we extend the model in Belak et al. [4] to allow for changing maximum crash sizes and changing

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market coefficients by proposing a regime switching model. That is, we assume that the market is in one of $d+1$ states, where in the first state no bubble is present and hence the investor does not have to fear a crash. In the other states, we assume that there is a potential bubble and each state is allowed to have different market coefficients and different maximum crash sizes. The setup of this model also allows to consider more general distributions for the arrival times of the warnings. In particular, it covers phase-type distributions and, hence, we can approximate any arbitrary distribution on $[0, \infty)$.

Our model may also be seen as an extension of the model considered in Bäuerle and Rieder [3] (see also Sotomayor and Cadenillas [30] and Escobar et al. [12] for more general regime-switching models) which allows for crashes in some of the regimes.

In the absence of crash threats, Bäuerle and Rieder [3] show that the optimal wealth fractions in a Markov modulated regime switching portfolio framework are regime-dependent and of the same form as in the classical Merton model, i.e., the parameters (drift and volatility) defining the optimal strategy depend on the regime in place. More precisely, in the case of logarithmic and power utility the optimal strategy is

$$\pi_M^i := \frac{\alpha_i}{(1-p)\sigma_i^2}, \tag{1.2}$$

where α_i and σ_i denote the excess return and volatility of the stock in state $i \in \{0, \dots, d\} =: E$ and where p denotes the risk-aversion coefficient of the investor (see Equation (2.11) below for details). Moreover, in the case of power utility the value function $\mathcal{V}_{RS}$ is given by

$$\mathcal{V}_{RS}(t, x, i) = \frac{1}{p} x^p f_i(t), \tag{1.3}$$

where the family $(f_i)_{i \in E}$ solves the coupled system of ordinary differential equations

$$\frac{\partial}{\partial t} f_i(t) = -\frac{1}{2} \frac{p}{1-p} \frac{\alpha_i^2}{\sigma_i^2} f_i(t) - \sum_{j=0}^{d} q_{i,j} f_j(t), \qquad f_i(T) = 1, \tag{1.4}$$

and where $Q = (q_{i,j})_{i,j \in E}$ denotes the transition rate matrix of the underlying Markov chain. Similarly, in the case of logarithmic utility, the value function can be written as

$$\mathcal{V}_{RS}(t, x, i) = \log(x) + f_i(t), \tag{1.5}$$

where the family $(f_i)_{i \in E}$ solves

$$\frac{\partial}{\partial t} f_i(t) = -\frac{1}{2} \frac{\alpha_i^2}{\sigma_i^2} - \sum_{j=0}^{d} q_{i,j} f_j(t), \qquad f_i(T) = 0. \tag{1.6}$$

A model closely related to our model was recently introduced in Capponi and Figueroa-López [8]. In this article, the authors consider a regime-switching model similar to Bäuerle and Rieder [3] but with an additional defaultable bond. The problem is solved by a stochastic control approach for logarithmic and power utility. Capponi et al. [9] extend the results to unobservable regimes, see also Nagai and Runggaldier [26] for the case without a defaultable bond. While in our paper we will not allow for defaults, we note that by formally setting the crash size to be equal to 100% we are able to model the situation in which the risky asset may default, but is immediately replaced by a new equivalent asset.[1]

The existence, formation, and modeling of financial bubbles has been studied extensively over the last decades. From a microeconomic view, the formation of bubbles has been studied e.g. in Tirole [31], Scheinkman and Xiong [28], and Abreu and Brunnermeier [1]. There is also a growing literature on bubbles from a pricing point of view, see e.g. Loewenstein and Willard [23, 24], Cox and Hobson [10], Jarrow et al. [18, 19], Heston et al. [13], Jarrow and Protter [16, 17], Jarrow et al. [15], and Biagini et al. [6]. In these papers, a price bubble is typically modeled as a local martingale that sits on top of the fundamental value of the asset. In our setting, the main concern for the investor is not the presence of the bubble itself, but the potential burst which may lead to significant losses due to price drops. Nevertheless, we allow the drift and volatility of the risk-bearing asset to differ in the various warning

[1] Alternatively, one can think of a default as the case where the asset drops only by β with $\beta < 1$ and $1-\beta$ being the expected recovery rate. Of course, the asset is delisted but might still be traded over the counter.

states $i = 0, \dots, d$, i.e. the price rally associated with a bubble may be different in each state. The birth and the burst of a bubble is modeled similar to the approach in Jarrow et al. [19] in the sense that the bubble appears and disappears at a random time. However, the focus of our paper is not so much on the formation and detection of a bubble as in the above mentioned papers, but to develop an investment strategy such that this strategy protects the investor if a bubble bursts, that is, a crash occurs.

The worst-case modeling approach towards the impact of market crashes was initiated in Hua and Wilmott [14] in a discrete-time option pricing model. Korn and Wilmott [22] pioneered the worst-case modeling approach in a continuous-time portfolio optimization context in which the investor aims to maximize the expected logarithmic utility of terminal wealth. The results of Korn and Wilmott [22] have been extended in several directions: Korn and Menkens [20] and Korn and Steffensen [21] extend the original model to more general utility functions (by means of an indifference principle and a Hamilton-Jacobi-Bellman (HJB)-system approach, respectively), Seifried [29] considers more general price dynamics and solves the problem by means of a martingale approach and Belak et al. [5] introduce transaction costs. Menkens [25] points out that the worst case approach is an alternative interpretation of Wald's maximin approach and investigates what happens if the probability of a crash is known. Finally, Desmettre et al. [11] consider worst-case optimal consumption over an infinite time horizon. All above-mentioned papers assume that the maximum number of crashes is known to the investor. To be more precise, the investor starts in a situation in which at most n crashes may occur in between the initial time and the investment horizon. Once the first crash has been observed, there are at most $n-1$ crashes left, and so on. In contrast, our model allows for a possibly unbounded number of crashes and the maximum number of crashes is unknown since the crash warnings arrive at random times.

In contrast to the existing worst-case models with a deterministic maximum number of crashes, the optimal strategies in our models exhibit some previously unobserved features. To be more precise, we show the following:

(1) In general, the optimal strategies do not converge to the Merton fraction as the investment horizon tends to infinity. This shows that a random number of total crashes introduces an additional long-term effect on the optimal strategies (see also Belak et al. [4]).

(2) While the investor is always indifferent between an immediate crash and no crash at all in the model considered in Belak et al. [4], this is not necessarily true in our regime-switching model. As is known from Korn and Menkens [20] and Seifried [29], this effect can also occur in the classical worst-case models if the market coefficients change after a crash. In our model however this effect can occur already if the market coefficients are independent of the state as soon as we allow for changing crash sizes.

(3) Finally, the optimal strategies may not necessarily be monotonically decreasing in time. For example, in the numerical investigations at the end of this paper, we construct a (in some sense degenerate) example in which the optimal strategies are oscillating over time.

This paper is organized as follows. We introduce the regime switching bubble model in Section 2, in which the market is divided into a finite number of regimes. Each regime corresponds to a potential bubble, where the bubble can be different for the different regimes. Thus, this may lead to a crash of a different maximum size in the different regimes. In Section 3, we derive a coupled system of Hamilton-Jacobi-Bellman equations and present a verification theorem. We then apply the verification theorem in Section 4 to solve the power utility case and derive the optimal strategies. We conclude in Section 5 with numerical examples.

2 The Market Model

We let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space which supports a standard Brownian motion $W = (W(t))_{t\geq 0}$ and an independent time-homogeneous continuous-time Markov chain $Z = (Z(t))_{t\geq 0}$ with state space $E := \{0, \dots, d\}$ for some $d \geq 1$. We denote by $Q = (q_{i,j})_{0\leq i,j\leq d}$ the transition rate matrix of Z such that

$$q_{i,j} \geq 0 \qquad \text{for all } i, j \in E \ \text{ with } \ i \neq j \tag{2.1}$$

and we set

$$\lambda_i := -q_{i,i} := \sum_{\substack{j=0\\ j\neq i}}^{d} q_{i,j} \qquad \text{for all } i \in E. \tag{2.2}$$

We assume moreover that the state 0 cannot be reached from any other state, i.e. $q_{i,0} = 0$ for all $i \in E$. We denote the augmented filtration generated by W and Z by $\mathbb{F} = (\mathcal{F}(t))_{t\geq 0}$.

Let us now fix some investment horizon $T > 0$ as well as some initial time $t \in [0, T)$. For simplicity, we assume that the bond price $P^0 = (P^0(u))_{u\in[t,T]}$ is given as

$$dP^0(u) = 0, \qquad u \in [t, T], \qquad P^0(t) = 1. \tag{2.3}$$

Note that the assumption of a constant bond price is without loss of generality, since we can always measure all asset prices in the market in units of the bond. We denote by $Z_{t,i} = (Z_{t,i}(u))_{u\geq t}$ the process $(Z(u))_{u\geq t}$ conditioned on $Z(t) = i$. We assume that in the absence of crashes the stock price $P^1 = (P^1(u))_{u\in[t,T]}$ has state-dependent excess return and volatility, i.e.

$$dP^1(u) = \alpha_j P^1(u)du + \sigma_j P^1(u)dW(u), \qquad \text{on } \{Z(u) = j\}, \qquad u \in [t, T], \tag{2.4}$$

where $\alpha_j, \sigma_j > 0$ for all $j \in E$ and where we set $P^1(t) = 1$.

To each state $i \in E$ we associate a maximum crash size $\beta_i \in [0, 1)$. We assume that $\beta_0 = 0$ (i.e. no crash in state 0) and assume without loss of generality that the maximum crash sizes are ordered:

$$0 = \beta_0 \leq \beta_1 \leq \ldots \leq \beta_d > 0. \tag{2.5}$$

Moreover, we set $i_{min} = \min\{i \in E : \beta_i > 0\}$. We denote the jump times of the Markov chain $Z_{t,i}$ by $(T_k)_{k\in\mathbb{N}}$ and set $T_0 = t$. The crash times are now given by a sequence $(\tau_k)_{k\in\mathbb{N}_0}$ of $\mathbb{F}$-stopping times taking values in $[T_k, T] \cup \{\infty\}$ and we assume that a crash occurs only if $Z_{t,i}(\tau_k-) \geq i_{min}$ and

$$T_k \leq \tau_k < T_{k+1}. \tag{2.6}$$

The sequence $(\tau_k)_{k\in\mathbb{N}_0}$ now acts as an impulse control strategy for $Z_{t,i}$ and P^1 as follows: Whenever $\tau_k < T_{k+1}$ and $Z_{t,i}(\tau_k-) \geq i_{min}$, the Markov chain $Z_{t,i}$ is sent to the state 0 at time τ_k and the asset crashes in the following sense:

$$P^1(\tau_k) = (1 - \beta_j)P_1(\tau_k-) \qquad \text{on } \{Z_{t,i}(\tau_k-) = j \geq i_{min}\}. \tag{2.7}$$

We write $\vartheta = (\tau_k)_{k\in\mathbb{N}_0}$ and denote the corresponding controlled Markov chain by $Z^{\vartheta}_{t,i}$. Moreover, we denote by $\mathcal{T}(t, i)$ the set of all sequences of crash times as defined above.

We interpret this market model as follows: Whenever $Z^{\vartheta}_{t,i}$ is in state $j < i_{min}$, then the market is in a safe regime in the sense that no crash can occur. As soon as $Z^{\vartheta}_{t,i}$ jumps into a state $k \geq i_{min}$ a bubble forms in the market which may (or may not) burst at the unknown time τ_k, leading to a crash in the risky asset and bringing the market back into the crash-free state 0. Since we will allow the investor to observe the process $Z^{\vartheta}_{t,i}$, we can interpret the jump times T_k (which are not caused by ϑ) of $Z^{\vartheta}_{t,i}$ as the times at which warnings are issued to the investor that a bubble has formed in the market.

The investor specifies a strategy $\pi = (\pi^0, \ldots, \pi^d) = (\pi^0(u), \ldots, \pi^d(u))_{u\in[t,T]}$ where π^i denotes the fraction of wealth invested in the stock when the market is in state i. We assume that each π^i is adapted, right-continuous, and bounded. Given a crash scenario $\vartheta = (\tau_k)_{k\in\mathbb{N}_0} \in \mathcal{T}(t, i)$ and a trading strategy $\pi = (\pi^0, \ldots, \pi^d)$, the investor's wealth process $X = X^{\pi,\vartheta}_{t,x,i} = (X^{\pi,\vartheta}_{t,x,i}(u))_{u\in[t,T]}$ is given by $X(t) = x$ at initial time,

$$dX(u) = \alpha_j \pi^j(u)X(u)du + \sigma_j \pi^j(u)X(u)dW(u), \qquad u \in [t, T], \tag{2.8}$$

on $\{Z^{\vartheta}_{t,i}(u) = j\} \cap \{u \neq \tau_k\}$,

$$X(\tau_k) = \left\{ \begin{array}{ll} X(\tau_k-) & \text{if } Z^{\vartheta}_{t,i}(\tau_k-) = j < i_{min} \\ (1 - \pi^j(\tau_k)\beta_j)X(\tau_k-) & \text{if } Z^{\vartheta}_{t,i}(\tau_k-) = j \geq i_{min} \end{array} \right\} \tag{2.9}$$

on $\{\tau_k < T_{k+1}\} \cap \{\tau_k \leq T\}$, and $X(\tau_k) = X(\tau_k-)$ on $\{\tau_k \geq T_{k+1}\} \cap \{\tau_k \leq T\}$. We denote by $\mathcal{A}(t, x)$ the set of all trading strategies which lead to a strictly positive wealth process $X^{\pi,\vartheta}_{t,x,i}$ for every $\vartheta \in \mathcal{T}(t, i)$.

The worst-case optimization problem is given by

$$\mathcal{V}(t,x,i) := \sup_{\pi\in\mathcal{A}(t,x)} \inf_{\vartheta\in\mathcal{T}(t,i)} \mathbb{E}\left[U_p\left(X^{\pi,\vartheta}_{t,x,i}(T)\right)\right], \tag{2.10}$$

where the utility function $U_p : (0,\infty) \to \mathbb{R}$ is given by

$$U_p(x) := \left\{ \begin{array}{ll} \frac{1}{p}x^p & \text{if } p<1, p\neq 0 \\ \log(x) & \text{if } p=0 \end{array} \right\}. \tag{2.11}$$

Remark 2.1. The optimization problem (2.10) is to be understood as follows: At any fixed time t, the investor commits to a trading strategy $\pi \in \mathcal{A}(t,x)$ and only after the investor's decision does the market decide on the crash strategy $\vartheta \in \mathcal{T}(t,i)$. This prohibits the investor to set her risky fraction equal to zero at the moment a crash occurs, i.e. she cannot prevent being negatively affected by a crash. In particular, switching the supremum and the infimum in (2.10) leads to a different value.

At first sight it seems contradictory that τ is a stopping time. However, this implies only that the market is going to tell the investor how it is going to act on any portfolio strategy of the investor. With this knowledge, the investor chooses a portfolio strategy, which in turn leads to the reaction of the market already announced to the investor. If τ is the optimal crash scenario of the market, then the investor cannot benefit from knowing the reaction of the market beforehand. $\diamond$

Remark 2.2. Note that the case $d=1$ corresponds to the situation considered in Belak et al. [4] if the market coefficients α_i and σ_i are independent of the state i. One can immediately generalize the situation by considering the case

$$0=\beta_0=\ldots=\beta_{d-1}, \qquad \beta_d=\beta>0, \tag{2.12}$$

and the state d is absorbing for Z. Then the time

$$S=\inf\{t\geq 0 : Z(t)=d\} \tag{2.13}$$

for Z started in 0 is of phase-type, see Asmussen [2, III.4]. Since the distributions of phase-type are dense in all probability distributions on $[0,\infty)$ (with respect to convergence in distribution) we can approximate arbitrary waiting-time distributions between a crash and the next warning. $\diamond$

Remark 2.3. Due to the monotonicity of the utility function U_p, we can assume without loss of generality that $\mathcal{T}(t,i)$ contains only those crash strategies $\vartheta=(\tau_k)_{k\in\mathbb{N}_0}$ for which

$$X(\tau_k) \leq X(\tau_k-) \tag{2.14}$$

for every $k\in\mathbb{N}_0$. $\diamond$

3 The Verification Theorem

In this section, we provide a coupled system of Hamilton-Jacobi-Bellman equations which is inspired by the system of HJBs introduced in Korn and Steffensen [21]. We then present a verification theorem which shows that under some technical assumptions any classical solution of the system of HJBs coincides with the value function. In Section 4, we solve the system of HJBs and derive a coupled system of ordinary differential equations for the optimal strategies.

We fix $K>0$ and let $\mathcal{A}_K(t,x)$ be the subset of all $\pi=(\pi^0,\ldots,\pi^d)\in\mathcal{A}(t,x)$ such that each π^i takes values in $\mathcal{K}:=[-K,K]$. We assume K to be large enough such that $\pi^i_M := \alpha_i/(1-p)\sigma_i^2 < K$ and $K>1/\beta_i$ for all $i\in E$. We denote by

$$\mathcal{V}_K(t,x,i) := \sup_{\pi\in\mathcal{A}_K(t,x)} \inf_{\vartheta\in\mathcal{T}(t,i)} \mathbb{E}\left[U_p\left(X^{\pi,\vartheta}_{t,x,i}(T)\right)\right] \tag{3.1}$$

the value function which is restricted to admissible strategies taking values in $\mathcal{K}$. This restriction of the set of admissible strategies allows us to prove the following growth estimate.

Lemma 3.1. *Let $\pi \in \mathcal{A}_K(t,x)$ and $\vartheta \in \mathcal{T}(t,i)$. Then there exists a constant $C > 0$ such that*

$$\mathbb{E}\Big[\sup_{u\in[t,T]} |X^{\pi,\vartheta}_{t,x,i}(u)|^2\Big] \leq C(1+|x|^2). \tag{3.2}$$

Proof. Denote by $\hat{\vartheta} \in \mathcal{T}(t,i)$ the no-crash scenario. Then the result for $\hat{\vartheta}$ is classical and follows e.g. from Pham [27, Theorem 1.3.15]. For an arbitrary $\vartheta \in \mathcal{T}(t,i)$, we note that

$$\mathbb{E}\Big[\sup_{u\in[t,T]} |X^{\pi,\vartheta}_{t,x,i}(u)|^2\Big] \leq \mathbb{E}\Big[\sup_{u\in[t,T]} |X^{\pi,\hat{\vartheta}}_{t,x,i}(u)|^2\Big] \tag{3.3}$$

since the wealth decreases at the moment a crash happens (see Remark 2.3). □

Assume for now that $\mathcal{V}(\cdot_1,\cdot_2,i) \in C^{1,2}([0,T)\times(0,\infty))$ for all $i \in E$. For each $(t,x) \in [0,T)\times(0,\infty)$, we can define

$$\mathcal{K}'_i(t,x) := \Big\{\pi \in \mathcal{K} : \mathcal{L}^\pi_i \mathcal{V}(t,x,i) + \sum_{j=0}^{d} q_{i,j}\mathcal{V}(t,x,j) \geq 0\Big\}, \tag{3.4}$$

$$\mathcal{K}''_i(t,x) := \Big\{\pi \in \mathcal{K} : \mathcal{V}(t,x,i) \leq \mathcal{V}(t,(1-\pi\beta_i)x,0)\Big\}, \tag{3.5}$$

where the operator $\mathcal{L}^\pi_i$ is given by

$$\mathcal{L}^\pi_i := \frac{\partial}{\partial t} + \alpha_i\pi x\frac{\partial}{\partial x} + \frac{1}{2}\sigma_i^2\pi^2x^2\frac{\partial^2}{\partial x^2} \qquad \text{for } i \in E. \tag{3.6}$$

In any state $i < i_{min}$ the investor does not have to fear the consequences of a possible crash so that she is essentially in the same situation as an investor in a regime switching model without crashes as considered in Bäuerle and Rieder [3]. It is therefore reasonable to expect that the value function in this state solves

$$0 = \sup_{\pi\in\mathcal{K}}\Big\{\mathcal{L}^\pi_i\mathcal{V}(t,x,i) + \sum_{j=0}^{d} q_{i,j}\mathcal{V}(t,x,j)\Big\}. \tag{3.7}$$

On the other hand, if $i \geq i_{\min}$ the investor has to take the possibility of a crash into account and hence (up to the possibility of switching to a different state) we are in a situation similar to Korn and Steffensen [21]. Therefore, we expect that the value function in this state solves

$$0 = \min\Big\{\sup_{\pi\in\mathcal{K}''_i(t,x)}\Big\{\mathcal{L}^\pi_i\mathcal{V}(t,x,i) + \sum_{j=0}^{d} q_{i,j}\mathcal{V}(t,x,j)\Big\}, \sup_{\pi\in\mathcal{K}'_i(t,x)}\Big\{\mathcal{V}(t,(1-\pi\beta_i)x,0) - \mathcal{V}(t,x,i)\Big\}\Big\}. \tag{3.8}$$

This idea is formalized in the following verification theorem. The proof of the theorem can be found in Appendix (A).

Theorem 3.2. *Let $V : [0,T]\times(0,\infty)\times E \to \mathbb{R}$ and assume that we have $V(\cdot_1,\cdot_2,i) \in C^{1,2}([0,T)\times(0,\infty)) \cap C([0,T]\times(0,\infty))$ for each $i \in E$.*

1. *Assume that the function $V(\cdot_1,\cdot_2,i)$ satisfies* (3.7) *with terminal condition $V(T,x,i) = U_p(x)$ for each $i = 0,\dots,i_{min}-1$ and the function $V(\cdot_1,\cdot_2,i)$ satisfies* (3.8) *with terminal condition $V(T,x,i) = U_p(x)$ for each $i = i_{min},\dots,d$.*

2. *Assume that V satisfies a quadratic growth condition in x, i.e. there exists a constant $C > 0$ independent of t and i such that*

$$|V(t,x,i)| \leq C\left(1+|x|^2\right). \tag{3.9}$$

3. *Moreover, suppose that there exists a measurable function $\hat{\pi}^i : [0,T)\times(0,\infty) \to \mathcal{K}$ for each $i = 0,\dots,i_{min}-1$, such that*

$$\hat{\pi}^i(t,x) = \arg\max_{\pi\in\mathcal{K}}\Big\{\mathcal{L}^\pi_i V(t,x,i) + \sum_{j=0}^{d} q_{i,j}V(t,x,j)\Big\} \tag{3.10}$$

and that, for each $i = i_{min}, \dots, d$, there exists a measurable function $\hat{\pi}^i : [0,T) \times (0,\infty) \to \mathcal{K}$ such that

$$\hat{\pi}^i(t,x) = \arg \max_{\pi \in \mathcal{K}_i''(t,x)} \Big\{ \mathcal{L}_i^\pi V(t,x,i) + \sum_{j=0}^d q_{i,j} V(t,x,j) \Big\}. \tag{3.11}$$

Write $\hat{\pi} = (\hat{\pi}^1, \dots, \hat{\pi}^d)$ and suppose additionally that, for each $(t,x,i) \in [0,T) \times (0,\infty) \times E$ and for every $\vartheta \in \mathcal{T}(t,i)$, the SDE (2.8)-(2.9) admits a solution $X^{*,\vartheta} = X^{\pi^*,\vartheta}_{t,x,i}$ under the trading strategy $\pi^* := (\hat{\pi}(u, X^*(u-)))_{u \in [t,T]}$ with $\pi^*(T) = 0$. Finally, assume that $\pi^* \in \mathcal{A}_K(t,x)$.

4. *Given any $(t,x,i) \in [0,T) \times (0,\infty) \times E$ and $\pi \in \mathcal{A}_K(t,x)$, we suppose that we can iteratively define a crash strategy $\vartheta^*(\pi) = (\tau_k^*)_{k \in \mathbb{N}_0} \in \mathcal{T}(t,i)$ through*

$$\tau_k^* := \infty \tag{3.12}$$

on $\{T_k \le T\} \cap \{Z^{\vartheta^*(\pi)}_{t,x,i}(T_k) < i_{min}\}$ and

$$\tau_k^* := \inf \Big\{ u \in [T_k, T_{k+1}) \cap [t,T] : V(u, X(u-), j) \ge V(u, (1 - \pi^j(u)\beta_j) X(u-), 0) \Big\} \tag{3.13}$$

on $\{T_k \le T\} \cap \{Z^{\vartheta^*(\pi)}_{t,x,i}(T_k) = j \ge i_{min}\}$.

Then $V(t,x,i) = \mathcal{V}_K(t,x,i)$, the strategy π^* is optimal, and the corresponding optimal crash strategy is $\vartheta^*(\pi^*)$.

Theorem 3.2 is tailor-made for the case $p \in (0,1)$. For $p \le 0$ the corresponding V does not satisfy the quadratic growth condition used in steps 3 and 5 of the proof. We will return to this problem in Appendix B after solving the system of HJBs since it is easier to verify optimality if we have a specific candidate at hand.

4 Derivation of the Optimal Strategies

Let us now apply Theorem 3.2 to find the value function and determine the optimal strategies. We start with the power utility case $p < 1$, $p \neq 0$.

4.1 Solution of the Coupled System of HJBs for Power Utility

We expect that V takes the form

$$V(t,x,i) = \frac{1}{p} x^p f_i(t), \qquad i \in E. \tag{4.1}$$

Moreover, we assume that f_i is strictly positive on $[0,T]$ for every $i \in \mathbb{E}$. Note that we must have $f_i(T) = 1$ for all $i \in E$. Our first aim is to solve

$$0 = \sup_{\pi \in \mathcal{K}} \Big\{ \mathcal{L}_i^\pi V(t,x,i) + \sum_{j=0}^d q_{i,j} V(t,x,j) \Big\} \tag{4.2}$$

for $i < i_{min}$ in order to find $V(t,x,i)$ and $\pi^{i,*}$. Using (4.1), this equation simplifies to

$$0 = \sup_{\pi \in \mathcal{K}} \left[\frac{1}{p} \frac{\partial}{\partial t} f_i(t) + \Big(\alpha_i \pi - \frac{1}{2}(1-p)\sigma_i^2 \pi^2 \Big) f_i(t) + \frac{1}{p} \sum_{j=0}^d q_{i,j} f_j(t) \right]. \tag{4.3}$$

Formally optimizing with respect to π gives the candidate optimal strategy

$$\pi^{i,*}(t) = \frac{\alpha_i}{(1-p)\sigma_i^2} = \pi_M^i, \tag{4.4}$$

which is indeed the maximum if $f_i(t) > 0$ for all $t \in [0,T]$. Plugging the candidate optimal strategy $\pi^{i,*}$ back into the HJB yields the following ODE for f_i:

$$\frac{\partial}{\partial t} f_i(t) = -\frac{1}{2} p \frac{\alpha_i^2}{(1-p)\sigma_i^2} f_i(t) - \sum_{j=0}^{d} q_{i,j} f_j(t). \tag{4.5}$$

Let us consider the case $i \geq i_{min}$ such that $\beta_i > 0$. We have to solve

$$0 = \min\left\{ \sup_{\pi \in \mathcal{K}_i''(t,x)} \left\{ \mathcal{L}_i^\pi V(t,x,i) + \sum_{j=0}^{d} q_{i,j} V(t,x,j) \right\}, \sup_{\pi \in \mathcal{K}_i'(t,x)} \left\{ V(t,(1-\pi\beta_i)x,0) - V(t,x,i) \right\} \right\}. \tag{4.6}$$

With this and (4.1), we see that the first operator satisfies

$$0 \leq \sup_{\pi \in \tilde{\mathcal{K}}_i''(t)} \left\{ \frac{1}{p}\frac{\partial}{\partial t} f_i(t) + \Big(\alpha_i \pi - \frac{1}{2}(1-p)\sigma_i^2\pi^2\Big) f_i(t) + \frac{1}{p}\sum_{j=0}^{d} q_{i,j} f_j(t) \right\}, \tag{4.7}$$

where

$$\tilde{\mathcal{K}}_i''(t) := \left\{ \pi \in \mathcal{K} : \frac{1}{p}(1-\pi\beta_i)^p f_0(t) - \frac{1}{p} f_i(t) \geq 0 \right\}. \tag{4.8}$$

Similarly, the second operator satisfies

$$0 \leq \sup_{\pi \in \tilde{\mathcal{K}}_i'(t)} \left\{ \frac{1}{p}(1-\pi\beta_i)^p f_0(t) - \frac{1}{p} f_i(t) \right\}, \tag{4.9}$$

where

$$\tilde{\mathcal{K}}_i'(t) := \left\{ \pi \in \mathcal{K} : \frac{1}{p}\frac{\partial}{\partial t} f_i(t) + \Big(\alpha_i \pi - \frac{1}{2}(1-p)\sigma_i^2\pi^2\Big) f_i(t) + \frac{1}{p}\sum_{j=0}^{d} q_{i,j} f_j(t) \geq 0 \right\}. \tag{4.10}$$

Let us first consider (4.9). Since f_0 is assumed to be strictly positive and since $(1-\pi\beta_i)^p/p$ is a decreasing function of π, the supremum in (4.9) is attained for the smallest value of π which satisfies the constraint in $\tilde{\mathcal{K}}_i'(t)$, i.e.

$$\frac{1}{p}\frac{\partial}{\partial t} f_i(t) + \Big(\alpha_i \pi - \frac{1}{2}(1-p)\sigma_i^2\pi^2\Big) f_i(t) + \frac{1}{p}\sum_{j=0}^{d} q_{i,j} f_j(t) \geq 0. \tag{4.11}$$

Notice that the left-hand side of (4.11) is a quadratic and concave function of π tending to $-\infty$ as $|\pi| \to \infty$. We must therefore have that the supremum in (4.9) is attained for the smallest value of π which satisfies the constraint (4.11) with equality. If the right-hand side of (4.9) is equal to zero we therefore have that $\pi^{i,*}$ and f_i are determined by

$$f_i(t) = (1-\pi^{i,*}(t)\beta_i)^p f_0(t), \tag{4.12}$$

$$\frac{\partial}{\partial t} f_i(t) = -p\Big(\alpha_i \pi^{i,*}(t) - \frac{1}{2}(1-p)\sigma_i^2[\pi^{i,*}(t)]^2\Big) f_i(t) - \sum_{j=0}^{d} q_{i,j} f_j(t). \tag{4.13}$$

If the supremum in (4.9) is strictly positive then the complementarity of the two equations in the HJB shows that

$$0 = \sup_{\pi \in \tilde{\mathcal{K}}_i''(t)} \left\{ \frac{1}{p}\frac{\partial}{\partial t} f_i(t) + \Big(\alpha_i \pi - \frac{1}{2}(1-p)\sigma_i^2\pi^2\Big) f_i(t) + \frac{1}{p}\sum_{j=0}^{d} q_{i,j} f_j(t) \right\}. \tag{4.14}$$

Optimizing this equation formally with respect to π yields

$$\pi^{i,*}(t) = \frac{\alpha_i}{(1-p)\sigma_i^2} = \pi_M^i. \tag{4.15}$$

If $\pi_M^i \in \tilde{\mathcal{K}}_i''(t)$ then it is indeed a maximizer of (4.14). Otherwise, we have

$$\frac{1}{p}(1-\pi_M^i\beta_i)^p f_0(t) < \frac{1}{p} f_i(t). \tag{4.16}$$

Since the left-hand side of this equation is decreasing as a function of π and since $\alpha_i\pi - \frac{1}{2}(1-p)\sigma_i^2\pi^2$ is an increasing function of π on $(-\infty, \pi_M^i)$, it follows that if $\pi_M^i \notin \tilde{\mathcal{K}}_i''(t)$ then the supremum in (4.14) is attained for $\pi^{i,*}(t) < \pi_M^i$, which satisfies

$$\frac{1}{p}(1-\pi^{i,*}(t)\beta_i)^p f_0(t) = \frac{1}{p} f_i(t). \tag{4.17}$$

Therefore, we have argued that $[0,T)$ can be decomposed into the set $\mathcal{I}_i$ on which $\pi^{i,*}$ and f_i are determined by

$$f_i(t) = (1-\pi^{i,*}(t)\beta_i)^p f_0(t), \tag{4.18}$$

$$\frac{\partial}{\partial t} f_i(t) = -p\Big(\alpha_i\pi^{i,*}(t) - \frac{1}{2}(1-p)\sigma_i^2[\pi^{i,*}(t)]^2\Big) f_i(t) - \sum_{j=0}^{d} q_{i,j} f_j(t), \tag{4.19}$$

and the set $\mathcal{N}_i$ on which $\pi^{i,*}$ and f_i are determined by

$$\pi^{i,*}(t) = \pi_M^i, \tag{4.20}$$

$$\frac{\partial}{\partial t} f_i(t) = -p\Big(\alpha_i\pi^{i,*}(t) - \frac{1}{2}(1-p)\sigma_i^2[\pi^{i,*}(t)]^2\Big) f_i(t) - \sum_{j=0}^{d} q_{i,j} f_j(t). \tag{4.21}$$

Moreover, note that we have $\pi^{i,*} < \pi_M^i$ on $\mathcal{I}_i$ and by solving

$$f_i(t) = (1-\pi^{i,*}(t)\beta_i)^p f_0(t) \tag{4.22}$$

for $\pi^{i,*}$ we can rewrite the differential equation for f_i as

$$\frac{\partial}{\partial t} f_i(t) = -p\frac{\alpha_i}{\beta_i}\left(1-\left[\frac{f_i(t)}{f_0(t)}\right]^{1/p}\right) f_i(t) + \frac{1}{2}p(1-p)\frac{\sigma_i^2}{\beta_i^2}\left(1-\left[\frac{f_i(t)}{f_0(t)}\right]^{1/p}\right)^2 f_i(t) - \sum_{j=0}^{d} q_{i,j} f_j(t). \tag{4.23}$$

The two differential equations for f_i on $\mathcal{I}_i$ and $\mathcal{N}_i$ can hence by combined to

$$\begin{aligned}\frac{\partial}{\partial t} f_i(t) = &-p\alpha_i \min\left\{\frac{1}{\beta_i}\left(1-\left[\frac{f_i(t)}{f_0(t)}\right]^{1/p}\right), \pi_M^i\right\} f_i(t) \\ &+ \frac{1}{2}p(1-p)\sigma_i^2\left[\min\left\{\frac{1}{\beta_i}\left(1-\left[\frac{f_i(t)}{f_0(t)}\right]^{1/p}\right), \pi_M^i\right\}\right]^2 f_i(t) - \sum_{j=0}^{d} q_{i,j} f_j(t).\end{aligned} \tag{4.24}$$

We are left with showing that we can solve this system of differential equations for $(f_i)_{i\in E}$ and that $f_i(t) > 0$ for all $(t,i) \in [0,T]\times E$. It then follows that V is indeed a solution of the system of HJBs and that $\pi^{i,*}$ is the candidate optimal strategy. For the proof of the following lemma we refer to Appendix A.

Lemma 4.1. *The system of ODEs given by* (4.5) *for* $i = 0,\dots,i_{min}-1$ *and by* (4.24) *for* $i = i_{min},\dots,d$ *with terminal condition* $f_i(T) = 1$ *for all* $i \in E$ *possesses a unique solution on* $[0,T]$*. Moreover, this solution is strictly positive.*

The next step is to check if the candidate optimal strategy $\pi^* = (\pi^{0,*},\dots,\pi^{d,*})$ is admissible. For every $i \in E$ with $i \geq i_{min}$, we can write

$$\pi^{i,*}(t) = \min\{\pi_M^i, \pi^{i,\mathrm{ind}}(t)\}, \tag{4.25}$$

where $\pi^{i,\mathrm{ind}}$ is given by

$$f_i(t) = (1-\pi^{i,\mathrm{ind}}(t)\beta_i)^p f_0(t). \tag{4.26}$$

Taking the logarithm and then the derivative with respect to t, we arrive at the following differential equation for $\pi^{i,\text{ind}}$:

$$\frac{\partial}{\partial t}\pi^{i,\text{ind}}(t) = \frac{1}{\beta_i}(1-\pi^{i,\text{ind}}(t)\beta_i)\left[\frac{1}{pf_0(t)}\frac{\partial}{\partial t}f_0(t) - \frac{1}{pf_i(t)}\frac{\partial}{\partial t}f_i(t)\right], \tag{4.27}$$

$$\pi^{i,\text{ind}}(T) = 0. \tag{4.28}$$

Using the ODE for f_0 in (4.5), the ODE for f_i in $\mathcal{I}_i$ given by (4.19), and using (4.26) shows that

$$\begin{aligned}\frac{\partial}{\partial t}\pi^{i,\text{ind}}(t) = \frac{1}{\beta_i}(1-\pi^{i,\text{ind}}(t)\beta_i)\Bigg[&\Psi_i - \Psi_0 - \frac{1}{2}(1-p)\sigma_i^2\left(\pi^{i,\text{ind}}(t)-\pi_M^i\right)^2 \\ &+ \frac{1}{p}\sum_{j=0}^{d} q_{i,j}\frac{f_j(t)}{f_0(t)}(1-\pi^{i,\text{ind}}(t)\beta_i)^{-p} \\ &- \frac{1}{p}\sum_{\substack{j=0\\ j\neq i}}^{d} q_{0,j}\frac{f_j(t)}{f_0(t)} - q_{0,i}\frac{1}{p}(1-\pi^{i,\text{ind}}(t)\beta_i)^{p}\Bigg],\end{aligned} \tag{4.29}$$

where we denote by

$$\Psi_i := \frac{1}{2}\frac{\alpha_i^2}{(1-p)\sigma_i^2} \tag{4.30}$$

the utility growth potential in regime $i \in E$. We now show that the strategy $\pi^{i,\text{ind}}$ is admissible for each $i \in E$ with $i \geq i_{min}$ and hence so is π^*. The proof of this statement can be found in Appendix A.

Lemma 4.2. *There exists a unique solution of the differential equation*

$$\frac{\partial}{\partial t}y = F(t,y), \qquad y(T)=0, \qquad (t,y)\in[0,T]\times(-\infty,1/\beta_i), \tag{4.31}$$

where

$$\begin{aligned}F(t,y) \;=\; \frac{1}{\beta_i}(1-y\beta_i)\Bigg[&\Psi_i - \Psi_0 - \frac{1}{2}(1-p)\sigma_i^2\left(y-\pi_M^i\right)^2 + \frac{1}{p}\sum_{j=0}^{d} q_{i,j}\frac{f_j(t)}{f_0(t)}(1-y\beta_i)^{-p} \\ &-\frac{1}{p}\sum_{\substack{j=0\\ j\neq i}}^{d} q_{0,j}\frac{f_j(t)}{f_0(t)} - q_{0,i}\frac{1}{p}(1-y\beta_i)^{p}\Bigg].\end{aligned} \tag{4.32}$$

It follows that π^* is an admissible strategy which leads to a strictly positive wealth process in every crash scenario. Moreover, the function $V(t,x,i)$ obviously satisfies a quadratic growth condition in x uniformly in (t,i) as long as $p\in(0,1)$. Finally, given any trading strategy $\pi = (\pi^0,\dots,\pi^d)\in\mathcal{A}_K(t,x)$ the corresponding optimal crash time $\vartheta^*(\pi)$ is obviously well-defined since it is just the first time at which π^i exceeds $\pi^{i,\text{ind}}$. It follows that $V=\mathcal{V}_K$ and that π^* is optimal. Moreover, since the optimal strategy π^* attains its values in the interior of $\mathcal{K}$ it is immediately clear that π^* is also optimal in the class of all bounded trading strategies $\mathcal{A}(t,x)$ and hence $V=\mathcal{V}$.

4.2 Solution of the System of HJBs for Logarithmic Utility

Let us now turn to the case $p=0$. We guess that the value function takes the form

$$V(t,x,i) = \log(x) + f_i(t), \qquad i\in E \tag{4.33}$$

for some functions f_i with $f_i(T)=0$. We can then proceed as in the power utility case to show that the candidate optimal strategy for $i < i_{min}$ is given by

$$\pi^{i,*}(t) = \frac{\alpha_i}{\sigma_i^2} = \pi_M^i \tag{4.34}$$

and that f_i solves

$$\frac{\partial}{\partial t} f_i(t) = -\frac{1}{2}\frac{\alpha_i^2}{\sigma_i^2} - \sum_{j=0}^{d} q_{i,j} f_j(t). \tag{4.35}$$

For $i \geq i_{min}$, the interval $[0,T]$ decomposes into a set $\mathcal{I}_i$ on which $\pi^{i,*}$ and f_i are determined by

$$f_i(t) = \log(1-\pi^{i,*}(t)\beta_i) + f_0(t), \tag{4.36}$$

$$\frac{\partial}{\partial t} f_i(t) = -\alpha_i \pi^{i,*}(t) + \frac{1}{2}\sigma_i^2[\pi^{i,*}(t)]^2 - \sum_{j=0}^{d} q_{i,j} f_j(t), \tag{4.37}$$

and a set $\mathcal{N}_i$ on which $\pi^{i,*}$ and f_i are determined by

$$\pi^{i,*}(t) = \pi_M^i, \tag{4.38}$$

$$\frac{\partial}{\partial t} f_i(t) = -\alpha_i \pi^{i,*}(t) + \frac{1}{2}\sigma_i^2[\pi^{i,*}(t)]^2 - \sum_{j=0}^{d} q_{i,j} f_j(t). \tag{4.39}$$

The existence of $(f_i)_{i\in E}$ can be proved in a very similar fashion to Lemma 4.1. The candidate optimal strategy is given by

$$\pi^{i,*} = \min\{\pi_M^i, \pi^{i,\mathrm{ind}}(t)\}, \tag{4.40}$$

where $\pi^{i,\mathrm{ind}}$ solves

$$\begin{aligned}\frac{\partial}{\partial t}\pi^{i,\mathrm{ind}}(t) \quad = \quad & \frac{1}{\beta_i}(1-\pi^{i,\mathrm{ind}}(t)\beta_i)\Bigg[\Psi_i - \Psi_0 - \frac{1}{2}\sigma_i^2\left(\pi^{i,\mathrm{ind}}(t) - \pi_M^i\right)^2 + \sum_{\substack{j=0\\ j\neq i}}^{d}(q_{i,j}-q_{0,j})f_j(t) \\ & +(q_{i,i}-q_{0,i})\left[f_0(t) + \log(1-\pi^{i,\mathrm{ind}}(t)\beta_i)\right]\Bigg]\end{aligned} \tag{4.41}$$

with terminal condition $\pi^{i,\mathrm{ind}}(T) = 0$. The admissibility of π^* and $\pi^{i,\mathrm{ind}}$ follows by very similar arguments as in Lemma 4.2.

Remark 4.3. Assume that $d = 1$ and that the excess return and the volatility of the stock are state-independent. Then the differential equation for $\pi^{1,\mathrm{ind}}$ simplifies to

$$\frac{\partial}{\partial t}\pi^{1,\mathrm{ind}}(t) = \frac{1}{\beta_1}(1-\pi^{1,\mathrm{ind}}(t)\beta_1)\left[-\frac{1}{2}\sigma_1^2\left(\pi^{1,\mathrm{ind}}(t)-\pi^{0,*}(t)\right)^2 - \lambda_0\log(1-\pi^{1,\mathrm{ind}}(t)\beta_1)\right], \tag{4.42}$$

which is exactly the candidate optimal strategy derived in Belak et al. [4, Equation (3)]. $\diamond$

5 Numerical Results

We conclude this paper with numerical examples. We consider two cases: power utility with five bubble states and phase-type distributed arrival times of warnings.

5.1 Power Utility with Five Bubble States

We assume that

$$\alpha_0 = \ldots = \alpha_d = \alpha = 0.096 \qquad \text{and} \qquad \sigma_0 = \ldots = \sigma_d = \sigma = 0.4, \tag{5.1}$$

and let $T = 25$ and $\lambda = 1/T$. We furthermore choose $d = 5$, $p = 0.1$, and let the generator matrix of Z and the crash sizes β_i be given by

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 \\ 0 & -\lambda & \lambda & 0 & 0 & 0 \\ 0 & 0 & -\lambda & \lambda & 0 & 0 \\ 0 & 0 & 0 & -\lambda & \lambda & 0 \\ 0 & 0 & 0 & 0 & -\lambda & \lambda \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta_4\\ \beta_5\end{pmatrix} = \begin{pmatrix}0.1\\ 0.3\\ 0.5\\ 0.7\\ 0.9\end{pmatrix}. \tag{5.2}$$

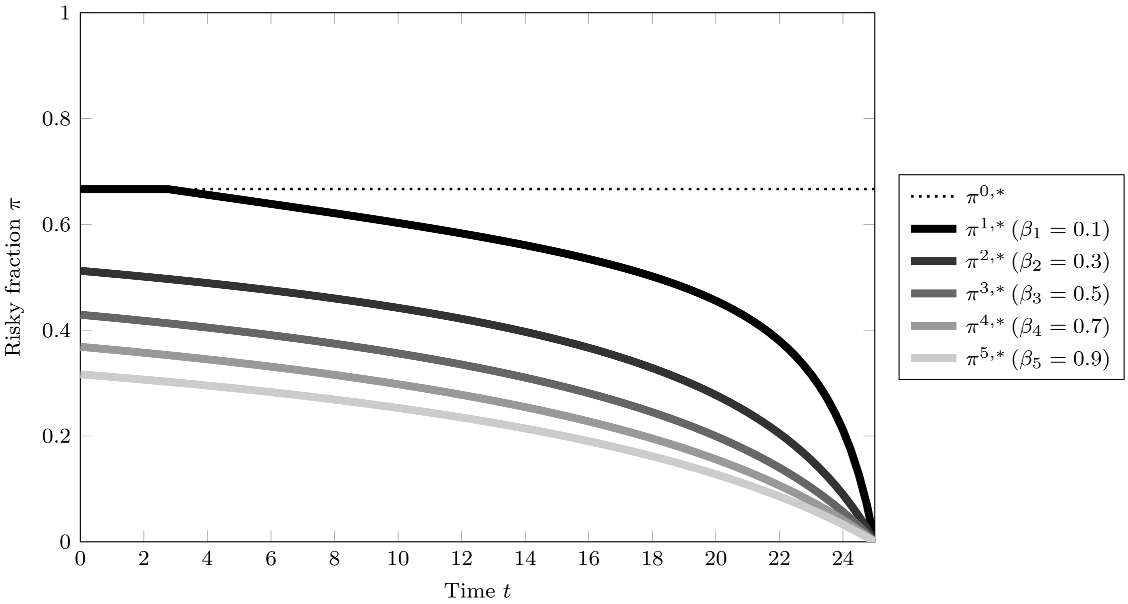

Figure 1: *Optimal strategies with increasing maximum crash sizes.* Given are the optimal portfolio strategies for power utility (with $p = 0.1$) with five different bubble states where the maximum possible crash size *increases* with each state. Observe that the optimal strategy in the first bubble state, that is $\pi^{1,*}$, is initially (that is for the years 0 to 3 or so) capped at the Merton fraction $\pi^{0,*}$ if the investment horizon is very long (in this example 25 years).

So, in particular, the process Z can only jump from state i to state $i+1$ and the last state is absorbing.

The numerical approximations of the optimal strategies can be found in Figure 1. As can be seen, the optimal strategies are decreasing for an increasing maximum crash size but still display similar qualitative features as the optimal strategy obtained in the simplified model considered in Belak et al. [4]. Note however that the optimal strategy in state 1 is equal to the Merton fraction π^i_M for small t (approximately $t < 2.75$), i.e. for small t we are inside the set $\mathcal{N}_1$. In contrast to the simplified model and the models considered in Korn and Menkens [20] and Seifried [29] this is a new feature. As Korn and Menkens [20] show, the optimal strategy in the presence of crashes is always smaller than the Merton fraction and only if one considers changing market coefficients after a crash it may be optimal for the investor to follow the Merton strategy despite the presence of a crash threat. In our model this phenomenon can already occur without considering state-dependent market coefficients. Note that this phenomenon can only be observed if $d > 1$. That is, this cannot be observed in the model of Belak et al. [4].

In the example considered in Figure 1 the market jumps from regimes with lower crash sizes to higher crash sizes from 0.1 to 0.9. Let us now consider the opposite direction, i.e. the market jumps from the safe state 0 to the state with crash size 0.9, from there to the state with crash size 0.7 and so on. This can be modeled by considering the generator matrix

$$Q = \begin{pmatrix} -\lambda & 0 & 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & -\lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & -\lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & -\lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda & -\lambda \end{pmatrix}, \tag{5.3}$$

and keeping the remaining parameters as before.

The resulting optimal strategies can be found in Figure 2. First, notice that by looking at time $t = 0$ the investor is strikingly more conservative since after a crash has occurred the next warning brings the market right back into the most dangerous state 5 with a maximum crash size of 0.9. Also, note that in states 1 and 2, the optimal strategies present a previously unobserved pattern — the strategies are no longer monotone in t but increasing for small values of t and decreasing for larger values of t. The

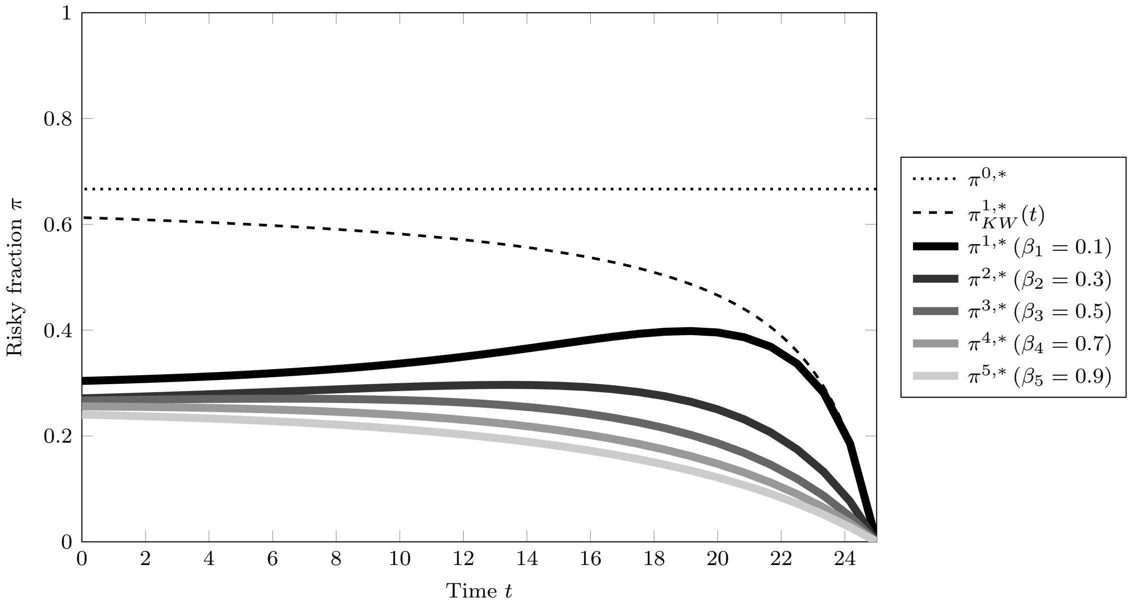

Figure 2: *Optimal strategies with decreasing maximum crash sizes.* Given are the optimal portfolio strategies for power utility (with $p = 0.1$) with five different bubble states where the maximum possible crash size *decreases* with each state. Observe that the optimal strategy in the first bubble state, that is $\pi^{1,*}$, increases for the first 20 years before it decreases in the last 5 years as in the Korn and Wilmott case (compare with the optimal strategy in the Korn and Wilmott case, denote here as $\pi^{1,*}_{KW}(t)$).

rationale behind this observation is as follows: If a crash occurs at time $t << T$ then the probability of another warning coming in before terminal time T is quite high (as compared to a crash close to T). Hence it is quite likely that the market ends up in the dangerous state 5 again. Thus, in order to avoid big losses the investor chooses a small risky fraction. On the other hand, if t gets closer to T and a crash occurs, then the probability of jumping back into state 5 becomes smaller and smaller and hence the investor has to be less and less concerned with this threat as t approaches T. In states 1 and 2, this leads to an increase in the optimal strategy. However as t gets even closer to T the losses due to an immediate crash begin to dominate the threat of jumping back into state 5 and hence the strategies start to decrease again and converge to 0 as $t \to T$. This also explains why in states 3 to 5 the strategies are monotone — because the threat of losing utility due to an immediate crash is bigger than the threat of jumping back into state 5 after this crash.

Observe that the optimal strategy in the case $i = 1$ in Figure 2 verifies the findings in Belak et al. [4]: A fixed number of possible crashes has just a short term (meaning close to the investment horizon) impact while a random (unknown) number of crashes has an additional long term impact. With Figure 2, we can make this more precise. In the short term the imminent threat of a crash is dominating and the investor can almost ignore the long term threat of the unknown number of possible crashes (by investing more in the risky asset in the short term than one would in the long term). On the other hand, the unknown number of potential crashes has only a long term impact. This becomes clear by comparing the optimal strategy in state $i = 1$ with the corresponding optimal strategy $\pi^{1,*}_{KM}$ in Korn and Menkens [20] with exactly one crash (with $\beta = \beta_1$): While the behavior close to maturity of the two strategies is very similar, the long-term difference between the two strategies is significant.

We investigate the feature of non-monotone optimal strategies again from a different point of view in the next example where we replace the exponential arrival time distribution of warnings with various phase-type distributions.

5.2 Phase-Type Distributed Warning Times

We conclude our numerical examples by comparing the optimal strategies which arise for different choices of the distribution of the arrival times of the crash warnings. As pointed out in Remark 2.2, by an appropriate choice of the transition rate matrix Q of the Markov process Z, by making the market coefficients state-independent, and by setting $0 = \beta_0 = \beta_1 = \ldots = \beta_{d-1}$, $\beta_d = \beta = 0.5$, the time S needed to reach state d from state 0 is phase-type distributed. In this section, we consider three different types of phase-type distributions: exponential, Erlang, and Coxian.

Remark 5.1. Note that this setup is degenerate in the sense that the investor receives warnings whenever Z jumps to a state $i \geq 0$, but since the maximum crash size is equal to zero for $i < d$ she does not have to fear a crash as long as she is in one of these states. $\diamond$

For our example, we choose $T = 50$. In order to normalize the different types of distributions and make them comparable we choose the parameters of the distributions so that we always have

$$\mathbb{E}[S] = 25, \tag{5.4}$$

i.e. we expect to see two warnings if we start in state 0 at time $t = 0$. (Actually, the expectation will be to see $2d$ warnings. However, $2d-2$ warnings are artificial/degenerate with no potential interpretation).

To obtain an exponential distribution we need to choose the transition matrix Q_{Exp} of the Markov process Z to be

$$Q_{\text{Exp}} := \begin{pmatrix} q_{0,0} & q_{0,1} \\ q_{1,0} & q_{1,1} \end{pmatrix} = \begin{pmatrix} -2/T & 2/T \\ 0 & 0 \end{pmatrix}. \tag{5.5}$$

We obtain an Erlang distribution by choosing $Q_{\text{Erl}} = (q^{\text{Erl}}_{i,j})_{0 \leq i,j \leq d^{\text{Erl}}}$ such that

$$q^{\text{Erl}}_{i,i} = -\frac{d^{\text{Erl}}}{25}, \qquad q^{\text{Erl}}_{i,i+1} = \frac{d^{\text{Erl}}}{25}, \qquad i = 0, \ldots, d^{\text{Erl}} - 1, \tag{5.6}$$

and $0 = q^{\text{Erl}}_{d^{\text{Erl}},0} = \ldots = q^{\text{Erl}}_{d^{\text{Erl}},d^{\text{Erl}}}$. In our example we consider the three cases $d^{\text{Erl}} = 5$, $d^{\text{Erl}} = 50$, and $d^{\text{Erl}} = 500$.

To obtain the Coxian distribution we need to choose $Q_{\text{Cox}} = (q^{\text{Cox}}_{i,j})_{0 \leq i,j \leq d^{\text{Cox}}}$ such that

$$q^{\text{Cox}}_{i,i} = -\lambda_i, \quad q^{\text{Cox}}_{i,i+1} = p_i \lambda_i, \quad q^{\text{Cox}}_{i,d^{\text{Cox}}} = (1 - p_i)\lambda_i, \quad i = 0, \ldots, d^{\text{Cox}} - 2, \tag{5.7}$$

$q^{\text{Cox}}_{d^{\text{Cox}}-1,d^{\text{Cox}}-1} = -q^{\text{Cox}}_{d^{\text{Cox}}-1,d^{\text{Cox}}} = -\lambda_{d^{\text{Cox}}-1}$ and 0 otherwise. The constants p_i, $i = 0, \ldots, d^{\text{Cox}} - 2$, have to be chosen such that $0 < p_i \leq 1$. For our numerical example, we consider $d^{\text{Cox}} = 2$ and

$$\lambda_0 = 5, \qquad \lambda_1 = \frac{1}{30}, \qquad p_0 = \frac{124}{150}, \tag{5.8}$$

such that

$$Q_{\text{Cox}} = \begin{pmatrix} -5 & \frac{124}{150} \cdot 5 & \frac{26}{150} \cdot 5 \\ 0 & -\frac{1}{30} & \frac{1}{30} \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.9}$$

The resulting probability density functions and cumulative distribution functions are depicted in Figure 3 and Figure 4. As we can see, the Coxian distribution puts a lot of mass on small values of t, i.e. the probability of jumping into the warning state after a short amount of time is quite high compared to the other distributions. The Erlang distribution with $d^{\text{Erl}} = 500$ on the other hand has a significant peak around the mean arrival time $\mathbb{E}[S] = 25$ and puts almost no weight in the tails. Note also that the Erlang distribution converges to the Dirac measure at $\mathbb{E}[S] = 25$ as $d^{\text{Erl}} \to \infty$.

The resulting optimal strategies can be found in Figure 5. The Coxian strategy is the most conservative for $t > 25$ which is due to the high mass on the small time values — since it is more likely to jump back into the crash state shortly after a crash, the investor has to take this into account in order to be indifferent. The Erlang strategy with five phases ($d^{\text{Erl}} = 5$) has a similar behavior as the strategies for state 1 and 2 in the previous example. That is, the strategy is first increasing and then decreasing. In the case of $d^{\text{Erl}} = 50$ and $d^{\text{Erl}} = 500$ phases we can even see an oscillation in the optimal strategies. The reason for this can be found in the density of the Erlang distribution. As $d^{\text{Erl}} \to \infty$ the Erlang distribution converges to the Dirac measure at $t = 25$. That is, the Erlang distribution puts increasingly more mass around the point $t = 25$. This means that after a crash at time t there is a very high probability that the next crash warning will arrive in roughly 25 years and the probability of an

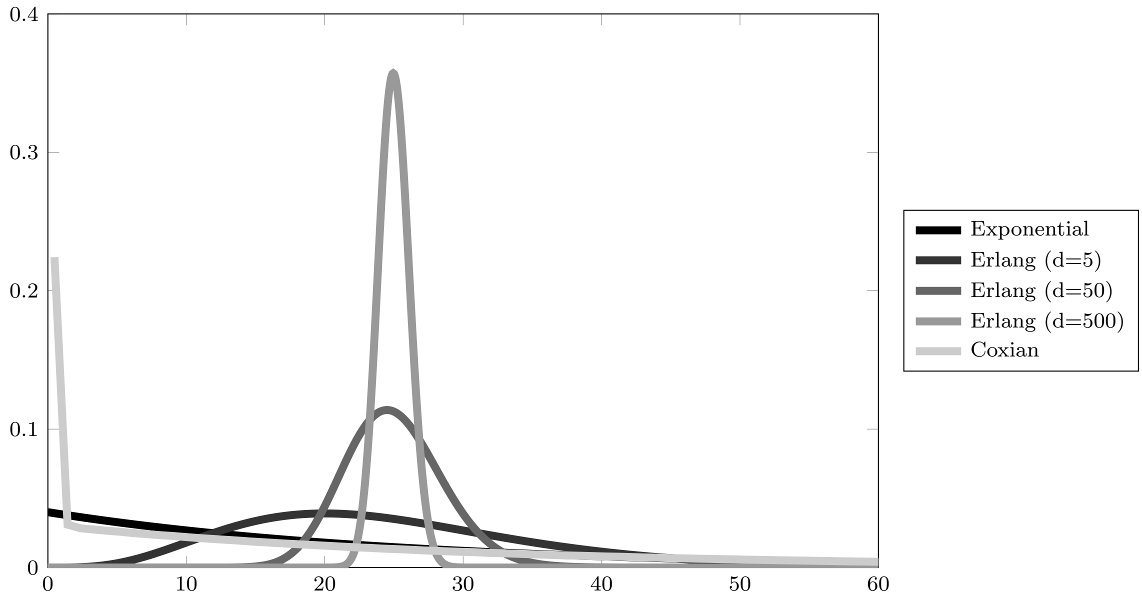

Figure 3: *Probability density functions of various phase-type distributions.* Depicted are the exponential density, the Erlang densities for $d = 5$, 50, and 500, and the Coxian density for $d = 2$.

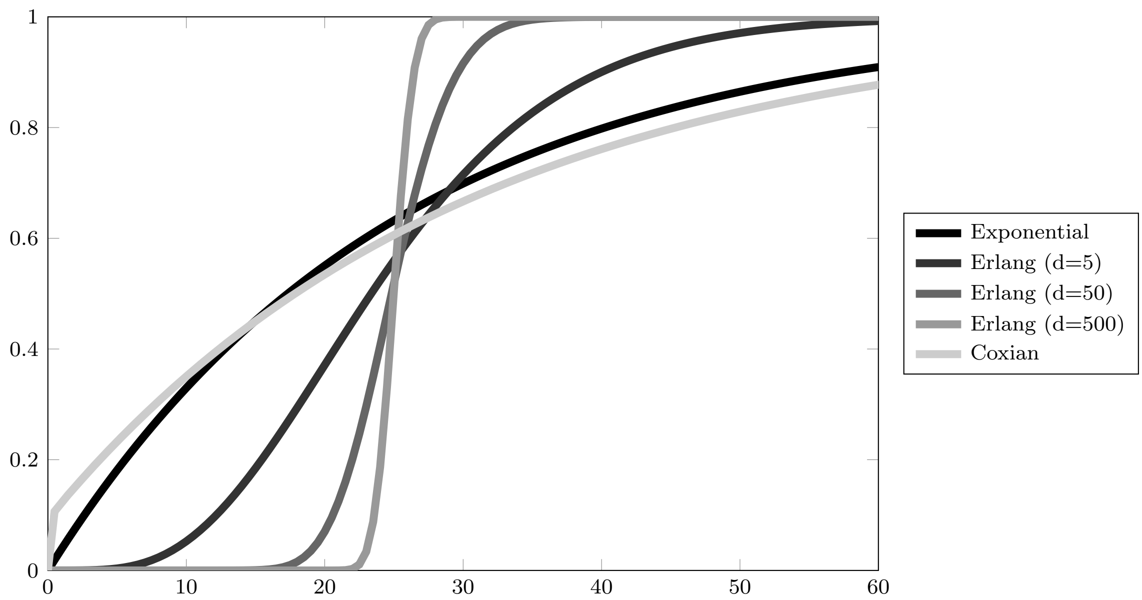

Figure 4: *Cumulative distribution functions of various phase-type distributions.* Depicted are the exponential cumulative distribution function, the Erlang cumulative distribution functions for $d = 5$, 50, and 500, and the Coxian cumulative distribution function for $d = 2$.

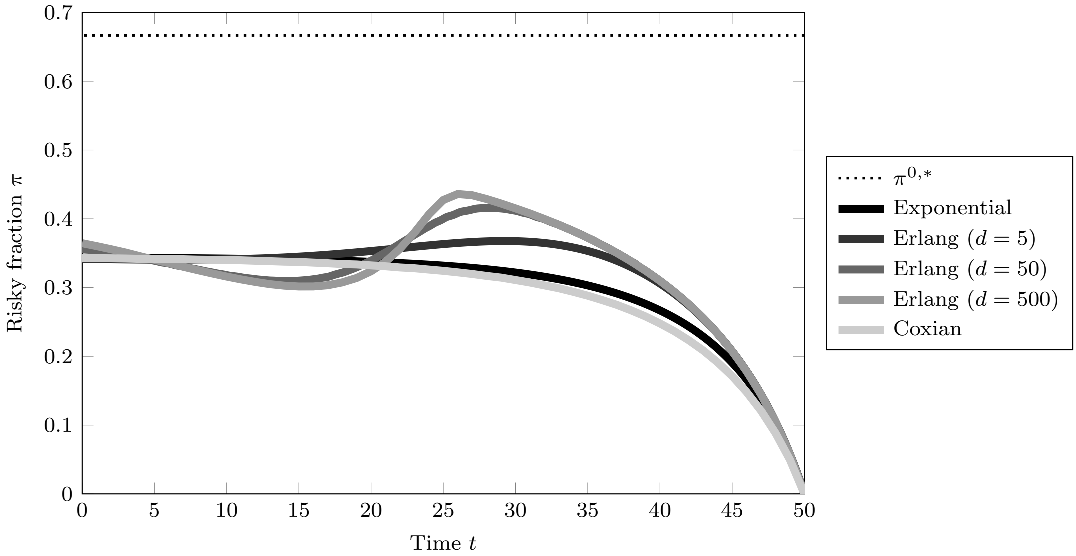

Figure 5: *Optimal strategies for phase-type distributed arrival times of warnings.* Most important are the different Erlang cases. The case $d = 5$ is similar to the optimal strategy in the first state of Figure 2 in that it is increasing for the first 30 years or so and decreasing in the last 20 years or so. However, the Erlang cases with $d = 50$ and 500 are even more extreme — the optimal strategies are oscillating with a period of 25 years.

earlier warning is small. Hence if t is close to T the investor essentially has to prepare for one more crash — since the likelihood of another warning after a crash is small. However, as $T - t$ increases so does the probability of another crash warning occurring after a crash at time t. Hence around $t = 25$ the investor begins to fear that another warning may arrive before the investment horizon — so she has to be afraid of two more crashes. This explains why the strategy in the $d^{\mathrm{Erl}} = 50$ case is increasing for $t \in [16.5, 29.5]$ (approximately). For even smaller values of t, the strategy is again decreasing since the probability of an additional crash warning remains high. The effect becomes more pronounced as d^{Erl} becomes larger due to the convergence property of the Erlang distribution against the Dirac measure — the investor becomes increasingly more certain of how long it will take for another warning to arrive after a crash. Also, note that in the long run all strategies considered in the above example converge to the same level, since the stationary distribution of the Markov chain dominates the investor's decisions for large time horizons.

A Proofs

Proof of Theorem 3.2. Step 1: Fix $(t, x, i) \in [0, T) \times (0, \infty) \times E$, let θ be any $[t, T]$-valued stopping time, fix $\pi = (\pi^1, \dots, \pi^d) \in \mathcal{A}_K(t, x)$ and let $\vartheta = (\tau_k)_{k \in \mathbb{N}_0} \in \mathcal{T}(t, i)$. Write $X = X^{\pi,\vartheta}_{t,x,i}$ and $Z = Z^{\vartheta}_{t,i}$ for short.

Then Itô's formula shows that, for each $k \in \mathbb{N}_0$, we have[2]

$$
\begin{aligned}
V(\theta\wedge\tau_k, & X(\theta\wedge\tau_k), Z(\theta\wedge\tau_k)) \\
&= V(\theta\wedge\tau_{k+1}-, X(\theta\wedge\tau_{k+1}-), Z(\theta\wedge\tau_{k+1}-)) \qquad \text{(A.1)}\\
&\quad - \sum_{j=0}^{d}\int_{\theta\wedge\tau_k}^{\theta\wedge\tau_{k+1}-}\Big[\mathcal{L}_j^{\pi^j(u)}V(u,X(u),j) + \sum_{l=0}^{d} q_{j,l}V(u,X(u),l)\Big]\mathbb{1}_{\{Z(u-)=j\}}\,du\\
&\quad - \sum_{j=0}^{d}\int_{\theta\wedge\tau_k}^{\theta\wedge\tau_{k+1}-}\sigma_j\pi^j(u)X(u)\frac{\partial}{\partial x}V(u,X(u),j)\mathbb{1}_{\{Z(u-)=j\}}\,dW(u)\\
&\quad - \sum_{j=0}^{d}\sum_{l=0}^{d}\int_{\theta\wedge\tau_k}^{\theta\wedge\tau_{k+1}-}\Big[V(u,X(u),l) - V(u,X(u),j)\Big]\mathbb{1}_{\{Z(u-)=j\}}\,\nu_k(du,l),
\end{aligned}
$$

where ν_k denotes the compensated jump measure of the uncontrolled process Z started in state 0 at time τ_k.

Step 2: Consider the strategy $\pi^* \in \mathcal{A}_K(t,x)$ together with an arbitrary $\vartheta \in \mathcal{T}(t,i)$. Since $\hat{\pi}^j \in \mathcal{K}_j''(u,y)$ for each $(u,y,j) \in [t,T)\times(0,\infty)\times E$ with $j \geq i_{min}$, this implies that, for any $k \in \mathbb{N}_0$ and any $j \in E$ with $j \geq i_{min}$, we have on $\{\tau_k \leq \theta\}\cap\{Z(\tau_k-)=j\}$

$$
\begin{aligned}
V(\tau_k, X^{*,\vartheta}(\tau_k), Z(\tau_k)) &= V\big(\tau_k, (1-\pi^{j,*}(\tau_k)\beta_j)X^{*,\vartheta}(\tau_k-), 0\big) &\text{(A.2)}\\
&\geq V\big(\tau_k, X^{*,\vartheta}(\tau_k-), Z(\tau_k-)\big), &\text{(A.3)}
\end{aligned}
$$

where we denote $X^{*,\vartheta} := X^{\pi^*,\vartheta}$. Thus, using this in (A.1) and then iteratively applying (A.1) shows that, for each $N \in \mathbb{N}$, we have

$$
\begin{aligned}
V(t,x,i) \leq {} & V(\theta\wedge\tau_N-, X^{*,\vartheta}(\theta\wedge\tau_N-), Z(\theta\wedge\tau_N-)) \qquad \text{(A.4)}\\
&- \sum_{k=0}^{N}\sum_{j=0}^{d}\int_{\theta\wedge\tau_{k-1}}^{\theta\wedge\tau_k-}\Big[\mathcal{L}_j^{\pi^j(u)}V(u,X^{*,\vartheta}(u),j) + \sum_{l=0}^{d} q_{j,l}V(u,X^{*,\vartheta}(u),l)\Big]\mathbb{1}_{\{Z(u-)=j\}}\,du\\
&- \sum_{k=0}^{N}\sum_{j=0}^{d}\int_{\theta\wedge\tau_{k-1}}^{\theta\wedge\tau_k-}\sigma_j\pi^j(u)X^{*,\vartheta}(u)\frac{\partial}{\partial x}V(u,X^{*,\vartheta}(u),j)\mathbb{1}_{\{Z(u-)=j\}}\,dW(u)\\
&- \sum_{k=0}^{N}\sum_{j=0}^{d}\sum_{l=0}^{d}\int_{\theta\wedge\tau_{k-1}}^{\theta\wedge\tau_k-}\Big[V(u,X^{*,\vartheta}(u),l) - V(u,X^{*,\vartheta}(u),j)\Big]\mathbb{1}_{\{Z(u-)=j\}}\,\nu_k(du,l),
\end{aligned}
$$

where we set $\tau_{-1} := t$. Now, since for $j = 0,\dots,i_{min}-1$ the function $\hat{\pi}^j$ is a pointwise maximizer of

$$
\sup_{\pi\in\mathcal{K}}\Big\{\mathcal{L}_j^{\pi}V(t,x,j) + \sum_{l=0}^{d} q_{j,l}V(t,x,l)\Big\} \geq 0 \qquad \text{(A.5)}
$$

and since for $j = i_{min},\dots,d$ the function $\hat{\pi}^j$ is a pointwise maximizer of

$$
\sup_{\pi\in\mathcal{K}_j''(t,x)}\Big\{\mathcal{L}_j^{\pi}V(t,x,j) + \sum_{l=0}^{d} q_{j,l}V(t,x,l)\Big\} \geq 0, \qquad \text{(A.6)}
$$

we can estimate the first integral in (A.4) to obtain

$$
\begin{aligned}
V(t,x,i) \leq {} & V(\theta\wedge\tau_N-, X^{*,\vartheta}(\theta\wedge\tau_N-), Z(\theta\wedge\tau_N-)) \qquad \text{(A.7)}\\
&- \sum_{k=0}^{N}\sum_{j=0}^{d}\int_{\theta\wedge\tau_{k-1}}^{\theta\wedge\tau_k-}\sigma_j\pi^j(u)X^{*,\vartheta}(u)\frac{\partial}{\partial x}V(u,X^{*,\vartheta}(u),j)\mathbb{1}_{\{Z(u-)=j\}}\,dW(u)\\
&- \sum_{k=0}^{N}\sum_{j=0}^{d}\sum_{l=0}^{d}\int_{\theta\wedge\tau_{k-1}}^{\theta\wedge\tau_k-}\Big[V(u,X^{*,\vartheta}(u),l) - V(u,X^{*,\vartheta}(u),j)\Big]\mathbb{1}_{\{Z(u-)=j\}}\,\nu_k(du,l).
\end{aligned}
$$

[2] To ease the notational burden, we assume that taking the left limit is less binding than taking the minimum of two stopping times. That is, the expression $\theta\wedge\tau_{k+1}-$ is to be read as $(\theta\wedge\tau_{k+1})-$.

Now send $N \to \infty$ to obtain

$$
\begin{aligned}
V(t,x,i) \le\ & V(\theta-, X^{*,\vartheta}(\theta-), Z(\theta-)) && \text{(A.8)}\\
& - \sum_{k=0}^{\infty}\sum_{j=0}^{d}\int_{\theta\wedge\tau_{k-1}}^{\theta\wedge\tau_k-} \sigma_j \pi^j(u) X^{*,\vartheta}(u) \frac{\partial}{\partial x} V(u, X^{*,\vartheta}(u), j) \mathbb{1}_{\{Z(u-)=j\}}\, dW(u) \\
& - \sum_{k=0}^{\infty}\sum_{j=0}^{d}\sum_{l=0}^{d}\int_{\theta\wedge\tau_{k-1}}^{\theta\wedge\tau_k-} \Big[V(u, X^{*,\vartheta}(u), l) - V(u, X^{*,\vartheta}(u), j)\Big] \mathbb{1}_{\{Z(u-)=j\}}\, \nu_k(du, l) \\
=\ & V(\theta-, X^{*,\vartheta}(\theta-), Z(\theta-)) && \text{(A.9)}\\
& - \sum_{j=0}^{d}\int_{t}^{\theta-} \sigma_j \pi^j(u) X^{*,\vartheta}(u) \frac{\partial}{\partial x} V(u, X^{*,\vartheta}(u), j) \mathbb{1}_{\{Z(u-)=j\}}\, dW(u) \\
& - \sum_{j=0}^{d}\sum_{l=0}^{d}\int_{t}^{\theta-} \Big[V(u, X^{*,\vartheta}(u), l) - V(u, X^{*,\vartheta}(u), j)\Big] \mathbb{1}_{\{Z(u-)=j\}}\, \nu_0(du, l),
\end{aligned}
$$

where for the last equation we used that $\nu_k = \nu_l$ for all $k, l \in \mathbb{N}_0$.

Step 3: Note that, for any $\pi \in \mathcal{A}_K(t,x)$ and $\vartheta \in \mathcal{T}(t,i)$, the last integral in (A.1) is a martingale. Indeed, by Brémaud [7, Excercise I.E2], we only need to show that

$$
\mathbb{E}\left[\int_t^T \sum_{j=0}^{d}\sum_{l=0}^{d} \Big| V(u, X(u), l) - V(u, X(u), j)\Big|\, du\right] < +\infty. \tag{A.10}
$$

By the growth condition on V, we have for each $j, l \in E$

$$
\mathbb{E}\left[\int_t^T \Big| V(u, X(u), l) - V(u, X(u), j)\Big|\right] \le 2CT\left(1 + \mathbb{E}\left[\sup_{u\in[t,T]} |X(u)|^2\right]\right), \tag{A.11}
$$

which is finite by Lemma 3.1.

Let $n \in \mathbb{N}$ and define θ_n to be the minimum of T and the infimum over all u with $u > t$ such that

$$
\sum_{j=0}^{d}\int_t^u \left|\sigma_j \pi^j(r) X^{*,\vartheta}(r) \frac{\partial}{\partial x} V(r, X^{*,\vartheta}(r), j) \mathbb{1}_{\{Z(r-)=j\}}\right|^2 dr \tag{A.12}
$$

exceeds n. Now replace θ by θ_n in (A.9) and take expectations to obtain

$$
V(t,x,i) \le \mathbb{E}\left[V(\theta_n-, X^{*,\vartheta}(\theta_n-), Z(\theta_n-))\right]. \tag{A.13}
$$

Note that, for each $n \in \mathbb{N}$, we have

$$
\mathbb{E}\left[|V(\theta_n-, X^{*,\vartheta}(\theta_n-), Z(\theta_n-))|\right] \le C\left(1 + \mathbb{E}\left[\sup_{u\in[t,T]} |X^{*,\vartheta}(u)|^2\right]\right) < +\infty \tag{A.14}
$$

due to the growth condition on V and by Lemma 3.1. Hence, we can send $n \to \infty$ in (A.13) and use dominated convergence to obtain

$$
V(t,x,i) \le \mathbb{E}\left[V(T-, X^{*,\vartheta}(T-), Z(T-))\right]. \tag{A.15}
$$

Since V is continuous, satisfies $V(T, x, \cdot) = U_p(x)$, and since $\pi^*(T) = 0$ it follows that

$$
V(t,x,i) \le \mathbb{E}\left[U_p\left(X^{*,\vartheta}(T)\right)\right]. \tag{A.16}
$$

Since ϑ was chosen arbitrarily this implies that

$$
V(t,x,i) \le \inf_{\vartheta\in\mathcal{T}(t,i)} \mathbb{E}\left[U_p\left(X^{*,\vartheta}(T)\right)\right] \le \mathcal{V}_K(t,x,i). \tag{A.17}
$$

Step 4: Consider an arbitrary $\pi \in \mathcal{A}_K(t,x)$ together with the associated crash scenario $\vartheta^*(\pi) \in \mathcal{T}(t,i)$. We denote $X := X^{\pi,\vartheta^*(\pi)}$. Now, for each $k \in \mathbb{N}_0$ and each $j = 0, \dots, i_{min} - 1$, we have

$$\Big[\mathcal{L}_j^{\pi(u)} V(u, X(u), j) + \sum_{l=0}^{d} q_{j,l} V(u, X(u), l)\Big] \mathbb{1}_{\{Z(u-)=j\}} \le 0 \tag{A.18}$$

for all $u \in [t,T)$. Moreover, for each $k \in \mathbb{N}_0$ and each $j = i_{min}, \dots, d$, we have

$$V(u, X(u-), Z(u-)) < V(u, (1-\pi^j(u)\beta_j) X(u-), 0) \tag{A.19}$$

for each $u \in [\tau_k^*, \tau_{k+1}^*)$ on $\{\tau_{k+1}^* \le T\} \cap \{Z(u-) = j\}$ by the construction of $\vartheta^*(\pi)$. We must now distinguish two situations: Either we are in case (a) in which

$$\Big[\mathcal{L}_j^{\pi^j(u)} V(u, X(u), j) + \sum_{l=0}^{d} q_{j,l} V(u, X(u), l)\Big] \mathbb{1}_{\{Z(u-)=j\}} < 0 \tag{A.20}$$

or we are in case (b) in which

$$\Big[\mathcal{L}_j^{\pi^j(u)} V(u, X(u), j) + \sum_{l=0}^{d} q_{j,l} V(u, X(u), l)\Big] \mathbb{1}_{\{Z(u-)=j\}} \ge 0. \tag{A.21}$$

The latter case implies in particular that $\pi^j(u) \in \mathcal{K}'_j(u, X(u-))$ and hence (A.19) shows that

$$\sup_{\pi \in \mathcal{K}'_j(u, X(u-))} \Big[V(u, (1-\pi\beta_j)X(u-), 0) - V(u, X(u-), j)\Big] > 0. \tag{A.22}$$

Since V solves the system of HJBs and since $j \ge i_{min}$, we therefore have

$$\sup_{\pi \in \mathcal{K}''_j(u, X(u-))} \Big[\mathcal{L}_j^{\pi} V(u, X(u), j) + \sum_{l=0}^{d} q_{j,l} V(u, X(u), l)\Big] = 0. \tag{A.23}$$

Since $\pi^j(u) \in \mathcal{K}''_j(u, X(u-))$ by (A.19), we conclude that

$$\Big[\mathcal{L}_j^{\pi^j(u)} V(u, X(u), j) + \sum_{l=0}^{d} q_{j,l} V(u, X(u), l)\Big] \mathbb{1}_{\{Z(u-)=j\}} = 0 \tag{A.24}$$

in case (b). Therefore, combining case (a) and case (b) implies that

$$\Big[\mathcal{L}_j^{\pi^j(u)} V(u, X(u), j) + \sum_{l=0}^{d} q_{j,l} V(u, X(u), l)\Big] \mathbb{1}_{\{Z(u-)=j\}} \le 0 \tag{A.25}$$

on $\{\tau_{k+1}^* \le T\}$. Thus, using this with (A.18) in (A.1) shows that

$$\begin{aligned} V(\theta \wedge \tau_k^*, &X(\theta \wedge \tau_k^*), Z(\theta \wedge \tau_k^*)) \\ &\ge V(\theta \wedge \tau_{k+1}^* -, X(\theta \wedge \tau_{k+1}^* -), Z(\theta \wedge \tau_{k+1}^* -)) \\ &\qquad - \sum_{j=0}^{d} \int_{\theta \wedge \tau_k^*}^{\theta \wedge \tau_{k+1}^*} \sigma_j \pi^j(u) X(u) \frac{\partial}{\partial x} V(u, X(u), j) \mathbb{1}_{\{Z(u-)=j\}}\, dW(u) \\ &\qquad - \sum_{j=0}^{d} \sum_{l=0}^{d} \int_{\theta \wedge \tau_k^*}^{\theta \wedge \tau_{k+1}^*} \Big[V(u, X(u), l) - V(u, X(u), j)\Big] \mathbb{1}_{\{Z(u-)=j\}}\, \nu_k(du, l) \end{aligned} \tag{A.26}$$

for each $k \in \mathbb{N}_0$. Moreover, by the construction of $\vartheta^*(\pi)$, using that the process Z jumps back to state zero at crash times, that is $Z(\tau_k^*) = 0$ for all $k \in \mathbb{N}_0$, and the right-continuity of π, we have for every $k \in \mathbb{N}_0$ on $\{\tau_k^* \le \theta\} \cap \{Z(\tau_k^* -) = j \ge i_{min}\}$:

$$V(\tau_k^*, X(\tau_k^*), Z(\tau_k^*)) = V\big(\tau_k^*, (1-\pi^j(\tau_k^*)\beta_j) X(\tau_k^* -), 0\big) \le V\big(\tau_k^*, X(\tau_k^* -), Z(\tau_k^* -)\big). \tag{A.27}$$

Using this in (A.26) and applying inductively (A.26) shows therefore that

$$
\begin{aligned}
V(t,x,i) \geq\; & V(\theta-, X(\theta-), Z(\theta-)) && \text{(A.28)}\\
& - \sum_{j=0}^{d} \int_t^{\theta-} \sigma_j \pi^j(u) X(u) \frac{\partial}{\partial x} V(u, X(u), j) \mathbb{1}_{\{Z(u-)=j\}}\, dW(u) \\
& - \sum_{j=0}^{d} \sum_{l=0}^{d} \int_t^{\theta-} \Big[V(u, X(u), l) - V(u, X(u), j) \Big] \mathbb{1}_{\{Z(u-)=j\}}\, \nu_0(du, l).
\end{aligned}
$$

Step 5: Define a sequence of stopping times $(\theta_n)_{n\in\mathbb{N}}$ as in step 3, but with (π^*, ϑ) replaced by $(\pi, \vartheta^*(\pi))$. Thus, taking expectations in (A.28) shows that

$$
V(t,x,i) \geq \mathbb{E}\left[V(\theta_n-, X(\theta_n-), Z(\theta_n-)) \right]. \tag{A.29}
$$

Sending $n \to \infty$, we conclude by dominated convergence that

$$
V(t,x,i) \geq \mathbb{E}\left[V(T-, X(T-), Z(T-)) \right]. \tag{A.30}
$$

Moreover, it follows from the definition of $\vartheta^*(\pi)$ that

$$
\begin{aligned}
\mathbb{E}\left[V(T-, X(T-), Z(T-)) \right] &\geq \mathbb{E}\left[V(T, X(T), Z(T)) \right] = \mathbb{E}\left[U_p\left(X(T) \right) \right] && \text{(A.31)}\\
&\geq \inf_{\vartheta\in\mathcal{T}(t,i)} \mathbb{E}\left[U_p\left(X^{\pi,\vartheta}(T) \right) \right] && \text{(A.32)}
\end{aligned}
$$

and since π was chosen arbitrarily this implies

$$
V(t,x,i) \geq \mathcal{V}_K(t,x,i). \tag{A.33}
$$

Hence, $V(t,x,i) = \mathcal{V}_K(t,x,i)$ by (A.17). Thus, replacing V by $\mathcal{V}_K$ on the right-hand side of (A.17) shows that

$$
\mathcal{V}_K(t,x,i) \leq \inf_{\vartheta\in\mathcal{T}(t,i)} \mathbb{E}\left[U_p\left(X^{*,\vartheta}(T) \right) \right], \tag{A.34}
$$

which proves the optimality of π^*. The optimality of $\vartheta^*(\pi^*)$ follows similarly by using $V(t,x,i) = \mathcal{V}_K(t,x,i)$ and the optimality of π^* together with (A.29) and (A.32). □

Proof of Lemma 4.1. Note that the differential operator (4.5) is globally Lipschitz continuous and the differential operator (4.24) is locally Lipschitz continuous in f_i. Hence, by the theorem of Picard-Lindelöf, there exists a unique local solution of the system of differential equations. In order to show that there exists a strictly positive solution on $[0,T]$ it suffices to show that each f_i is strictly positive on $[0,T]$ and f_i does not explode on $[0,T]$. We only consider the case $p \in (0,1)$, the case $p < 0$ can be handled similarly.

Let therefore $p \in (0,1)$. Define

$$
g_i(y) := \alpha_i y - \frac{1}{2}(1-p)\sigma_i^2 y^2 \tag{A.35}
$$

and note that g_i attains its maximum at π_M^i. We let

$$
M := \max_{i\in E} g_i(\pi_M^i) > 0 \qquad \text{and} \qquad \bar{\lambda} := \max_{i\in E} \lambda_i. \tag{A.36}
$$

Now, for any $i \in E$ and any $t \in [0,T]$ with $f_i(t) > 0$, we have

$$
\begin{aligned}
\frac{\partial}{\partial t} f_i(t) &= -p g_i(\pi^{i,*}(t)) f_i(t) - \sum_{j=0}^{d} q_{i,j} f_j(t) && \text{(A.37)}\\
&\geq -pM f_i(t) - q_{i,i} f_i(t) - \sum_{\substack{j=0\\ j\neq i}}^{d} q_{i,j} f_j(t) && \text{(A.38)}\\
&\geq -pM f_i(t) - \max_{j\in E}\{f_j(t)\} \sum_{\substack{j=0\\ j\neq i}}^{d} q_{i,j} && \text{(A.39)}\\
&\geq -[pM + \bar{\lambda}] \max_{j\in E}\{f_j(t)\}. && \text{(A.40)}
\end{aligned}
$$

Thus, Gronwall's inequality shows that

$$f_i(t) \le f_i(T) + \int_t^T (pM + \bar{\lambda}) \max_{j\in E}\{f_j(u)\}\, du \tag{A.41}$$

for every $i \in E$ and therefore

$$\max_{j\in E}\{f_j(t)\} \le 1 + \int_t^T (pM + \bar{\lambda}) \max_{j\in E}\{f_j(u)\}\, du. \tag{A.42}$$

Applying Gronwall's inequality again shows that

$$\max_{j\in E}\{f_j(t)\} \le e^{(pM+\bar{\lambda})(T-t)}. \tag{A.43}$$

Furthermore, we have

$$\frac{\partial}{\partial t} f_i(t) = -\frac{1}{2} p \frac{\alpha_i}{(1-p)\sigma_i^2} f_i(t) - \sum_{j=0}^{d} q_{i,j} f_j(t) \tag{A.44}$$

$$\le \bar{\lambda} f_i(t) \tag{A.45}$$

for each $i = 0, \dots, i_{min} - 1$ as long as $f_j(t) > 0$ for all $j \in E$.

Let us now assume that there exists $t_0 \in [0, T)$ such that

$$\lim_{t\downarrow t_0} f_k(t) = 0 \tag{A.46}$$

for some $k \in E$ and that $f_j(t) > 0$ for all $t \in (t_0, T]$ and all $j \in E$ (Note that $t_0 < T$ is clear from the terminal condition on f_i). It follows from (A.43) and (A.45) that

$$f_j(t) \le e^{(pM+\bar{\lambda})(T-t)} \qquad \text{for all } (t,j) \in [t_0, T] \times E, \tag{A.47}$$

$$f_j(t) \ge e^{-\bar{\lambda}(T-t)} \qquad \text{for all } (t,j) \in [t_0, T] \times E \text{ with } j < i_{min}. \tag{A.48}$$

This implies in particular that $k \ge i_{min}$. Moreover, we have

$$0 \le \frac{f_k(t)}{f_0(t)} \le e^{(pM+2\bar{\lambda})(T-t)}, \qquad t \in [t_0, T], \tag{A.49}$$

and hence

$$\frac{1}{\beta_k}\left(1 - e^{\frac{1}{p}(pM+2\bar{\lambda})(T-t)}\right) \le \frac{1}{\beta_k}\left(1 - \left[\frac{f_k(t)}{f_0(t)}\right]^{1/p}\right) \le \frac{1}{\beta_k}. \tag{A.50}$$

Define

$$L := \frac{1}{\beta}\left(1 - e^{\frac{1}{p}(pM+2\bar{\lambda})T}\right). \tag{A.51}$$

Since $\pi^{k,*}(t) \le \pi_M^k$ and since g_k is increasing on $(-\infty, \pi_M^k)$ it follows that

$$g_k(L) \le g_k(\pi^{k,*}(t)), \qquad t \in [t_0, T]. \tag{A.52}$$

Therefore,

$$\frac{\partial}{\partial t} f_k(t) = -p g_k(\pi^{k,*}(t)) f_k(t) - \sum_{j=0}^{d} q_{k,j} f_j(t) \le [-p \min\{g_k(L), 0\} + \bar{\lambda}] f_k(t) \tag{A.53}$$

for every $t \in [t_0, T]$ which shows that

$$f_k(t) \ge e^{(-p\min\{g_k(L),0\}+\bar{\lambda})(T-t)}, \qquad t \in [t_0, T], \tag{A.54}$$

in contradiction to

$$\lim_{t\downarrow t_0} f_k(t) = 0. \tag{A.55}$$

Thus, combining this with (A.43) shows that $f_i > 0$ on $[0, T]$ and f_i is non-exploding for each $i \in E$. □

Proof of Lemma 4.2. Since $F(t,y)$ is continuous in t and globally Lipschitz continuous in y on any closed subinterval of $(-\infty, 1/\beta_i)$, it suffices to show that we can find constants $-\infty < a < b < 1/\beta_i$ such that the solution of the differential equation stays inside the interval $[a,b]$. We only consider the case $0 < p < 1$, the case $p \leq 0$ can be proved similarly.

Step 1: We prove the existence of a constant a such that $F(t,y) \leq 0$ whenever $y \leq a$. For this, note that the sign of F only depends on the term

$$\Psi_i - \Psi_0 - \frac{1}{2}(1-p)\sigma_i^2\left(y - \pi_M^i\right)^2 + \frac{1}{p}\sum_{j=0}^{d} q_{i,j}\frac{f_j(t)}{f_0(t)}(1-y\beta_i)^{-p} - \frac{1}{p}\sum_{\substack{j=0\\ j\neq i}}^{d} q_{0,j}\frac{f_j(t)}{f_0(t)} - q_{0,i}\frac{1}{p}(1-y\beta_i)^{p}. \quad (A.56)$$

Furthermore, note that there exist constants $\underline{M}, \overline{M} > 0$ independent of $t \in [0,T]$ and $j \in E$ according to the proof of Lemma 4.1 such that

$$\underline{M} \leq \frac{f_j(t)}{f_0(t)} \leq \overline{M}. \quad (A.57)$$

Then

$$\begin{aligned} &\Psi_i - \Psi_0 - \frac{1}{2}(1-p)\sigma_i^2\left(y - \pi_M^i\right)^2 + \frac{1}{p}\sum_{j=0}^{d} q_{i,j}\frac{f_j(t)}{f_0(t)}(1-y\beta_i)^{-p} - \frac{1}{p}\sum_{\substack{j=0\\ j\neq i}}^{d} q_{0,j}\frac{f_j(t)}{f_0(t)} - q_{0,i}\frac{1}{p}(1-y\beta_i)^{p} \\ &\qquad \leq \Psi_i - \Psi_0 - \frac{1}{2}(1-p)\sigma_i^2\left(y - \pi_M^i\right)^2 + \frac{1}{p}\left(\lambda_0 + \sum_{\substack{j=0\\ j\neq i}}^{d} q_{i,j}\overline{M}(1-y\beta_i)^{-p}\right), \quad (A.58)\end{aligned}$$

which is less or equal to 0 if and only if

$$\frac{1}{p}\left(\lambda_0 + \sum_{\substack{j=0\\ j\neq i}}^{d} q_{i,j}\overline{M}(1-y\beta_i)^{-p}\right) \leq \Psi_0 - \Psi_i + \frac{1}{2}(1-p)\sigma_i^2\left(y - \pi_M^i\right)^2. \quad (A.59)$$

Since $(1-y\beta_i)^{-p} \to 0$ and $(y - \pi_M^i)^2 \to +\infty$ as $y \to -\infty$ we see that there exists a constant a such that $F(t,y) \leq 0$ whenever $y \leq a$.

Step 2: Next we show that there exists a constant $b < 1/\beta_i$ independent of t such that $F(t,y) \geq 0$ whenever $y \geq b$. We have

$$\begin{aligned} &\Psi_i - \Psi_0 - \frac{1}{2}(1-p)\sigma_i^2\left(y - \pi_M^i\right)^2 + \frac{1}{p}\sum_{j=0}^{d} q_{i,j}\frac{f_j(t)}{f_0(t)}(1-y\beta_i)^{-p} - \frac{1}{p}\sum_{\substack{j=0\\ j\neq i}}^{d} q_{0,j}\frac{f_j(t)}{f_0(t)} - q_{0,i}\frac{1}{p}(1-y\beta_i)^{p} \\ \geq\ & \frac{1}{p}\sum_{\substack{j=1\\ j\neq i}}^{d} q_{i,j}\underline{M}(1-y\beta_i)^{-p} - \frac{1}{p}\lambda_i - \frac{1}{p}\sum_{\substack{j=1\\ j\neq i}}^{d} q_{0,j}\overline{M} - q_{0,i}\frac{1}{p}(1-y\beta_i)^{p} + \Psi_i - \Psi_0 \quad (A.60) \\ &-\frac{1}{2}(1-p)\sigma_i^2\left(y - \pi_M^i\right)^2,\end{aligned}$$

which is greater or equal than 0 if and only if

$$\frac{1}{p}\sum_{\substack{j=1\\ j\neq i}}^{d} q_{i,j}\underline{M}(1-y\beta_i)^{-p} - \frac{1}{p}\lambda_i - \frac{1}{p}\sum_{\substack{j=1\\ j\neq i}}^{d} q_{0,j}\overline{M} - q_{0,i}\frac{1}{p}(1-y\beta_i)^{p} \geq \Psi_0 - \Psi_i + \frac{1}{2}(1-p)\sigma_i^2\left(y - \pi_M^i\right)^2. \quad (A.61)$$

Now since $(1-y\beta_i)^{-p}$ approaches $+\infty$ and $(1-y\beta_i)^{p}$ approaches 0 as $y \to 1/\beta_i$ and since $(y - \pi_M^i)^2$ is bounded in y on $[0, 1/\beta_i]$, we see that there exists a constant $b < 1/\beta_i$ such that $F(t,y) \geq 0$ whenever $y \geq b$. $\square$

B Verification for Logarithmic and Negative Power Utility

Let us now verify that the solutions of the coupled system of HJBs constructed in Section 4.1 and Section 4.2 are indeed the value functions. For $p \in (0,1)$, this is clear by Theorem 3.2. For $p \leq 0$, the function V does not satisfy the quadratic growth condition which was used in step 3 and step 5 of the proof of Theorem 3.2. However, the explicit nature of our solutions allows us to verify these two steps also for $p \leq 0$.

Theorem B.1. *Let $p = 0$ and let V be the solution of the system of HJBs given in* (4.33)*. Then $V = \mathcal{V}_K = \mathcal{V}$.*

Proof. Let both $\pi \in \mathcal{A}(t,x)$ and $\vartheta \in \mathcal{T}(t,i)$ be arbitrary. We show that the two stochastic integrals in (A.1) are martingales. With this, we can choose $\theta = T$ and conclude as before. Note that

$$x\frac{\partial}{\partial x}V(t,x,i) = 1 \tag{B.1}$$

for every $i \in E$ and hence the integrand of the Brownian integral is bounded (uniformly in t and i) so that the integral is indeed a martingale. Moreover, for each $i, j \in E$, we have

$$|V(t,x,i) - V(t,x,j)| = |f_i(t) - f_j(t)| \tag{B.2}$$

which is again bounded (uniformly in t, i and j) and hence the integral with respect to the compensated jump measure is a martingale as well. □

For $p < 0$, we need to rule out some admissible trading strategies first. Because the pure bond strategy $\pi \equiv 0$ is admissible (and recalling that the interest rate is equal to zero), we may without loss of generality restrict the set of admissible strategies to those $\pi \in \mathcal{A}_K(t,x)$ which satisfy

$$U_p(x) \leq \inf_{\vartheta \in \mathcal{T}(t,i)} \mathbb{E}\left[U_p\left(X^{\pi,\vartheta}_{t,x,i}(T)\right)\right]. \tag{B.3}$$

Theorem B.2. *Let $p < 0$ and let V be the solution of the system of HJBs given in* (4.1)*. Then $V = \mathcal{V}_K = \mathcal{V}$.*

Proof. We simply prove step 3 and step 5 of the proof of Theorem 3.2 without relying on the quadratic growth condition.

Step 3: Recall that we have for π^* and any arbitrary $\vartheta \in \mathcal{T}(t,i)$ (according to (A.9))

$$\begin{aligned} V(t,x,i) \leq{}& V(\theta-, X^{*,\vartheta}(\theta-), Z(\theta-)) \\ &- \sum_{j=0}^{d} \int_t^{\theta-} \sigma_j \pi^j(u) X^{*,\vartheta}(u) \frac{\partial}{\partial x} V(u, X^{*,\vartheta}(u), j) \mathbb{1}_{\{Z(u-)=j\}}\, dW(u) \\ &- \sum_{j=0}^{d}\sum_{l=0}^{d} \int_t^{\theta-} \Big[V(u, X^{*,\vartheta}(u), l) - V(u, X^{*,\vartheta}(u), j)\Big] \mathbb{1}_{\{Z(u-)=j\}}\, \nu_0(du, l) \end{aligned} \tag{B.4}$$

for any $[t,T]$-valued stopping time θ. We have to show that

$$V(t,x,i) \leq \mathcal{V}_K(t,x,i). \tag{B.5}$$

Using that $V(\theta-, X^{*,\vartheta}(\theta-), Z(\theta-)) \leq 0$ in (B.4) shows that the sum of the two stochastic integrals is a local martingale bounded from above by $-V(t,x,i)$ and thus it is a submartingale. Choosing $\theta = T$ in (B.4) and taking expectations shows furthermore that

$$V(t,x,i) \leq \mathbb{E}\left[V(T-, X^{*,\vartheta}(T-), Z(T-))\right]. \tag{B.6}$$

Since V is continuous, satisfies $V(T,x,\cdot) = U_p(x)$, and since $\pi^*(T) = 0$ it follows that

$$V(t,x,i) \leq \mathbb{E}\left[U_p\left(X^{*,\vartheta}(T)\right)\right]. \tag{B.7}$$

Since ϑ was chosen arbitrarily this implies that

$$V(t,x,i) \leq \inf_{\vartheta \in \mathcal{T}(t,i)} \mathbb{E}\left[U_p\left(X^{*,\vartheta}(T)\right)\right] \leq \mathcal{V}_K(t,x,i). \tag{B.8}$$

Step 5: Let $\pi \in \mathcal{A}_K(t,x)$ and let $\vartheta^*(\pi)$ be the corresponding candidate optimal crash strategy. Recall that by (A.28) we have

$$\begin{aligned} V(t,x,i) \geq\ & V(\theta-, X(\theta-), Z(\theta-)) \\ & - \sum_{j=0}^{d} \int_t^{\theta-} \sigma_j \pi^j(u) X(u) \frac{\partial}{\partial x} V(u, X(u), j) \mathbb{1}_{\{Z(u-)=j\}}\, dW(u) \\ & - \sum_{j=0}^{d} \sum_{l=0}^{d} \int_t^{\theta-} \Big[V(u, X(u), l) - V(u, X(u), j)\Big] \mathbb{1}_{\{Z(u-)=j\}}\, \nu_0(du, l) \end{aligned} \tag{B.9}$$

for any $[t,T]$-valued stopping time θ. We have to show that

$$V(t,x,i) \geq \mathcal{V}_K(t,x,i). \tag{B.10}$$

For every $n \in \mathbb{N}$, we define

$$\theta_n := \inf\{u \geq t : |V(u, X(u), Z(u))| \geq n\} \wedge T. \tag{B.11}$$

Observe that this together with

$$x \frac{\partial}{\partial x} V(t,x,i) = pV(t,x,i) \tag{B.12}$$

implies that the stochastic integrals in (B.9) stopped at θ_n are martingales and, hence, replacing θ by θ_n in (B.9) and taking expectations shows that

$$V(t,x,i) \geq \mathbb{E}\left[V(\theta_n-, X(\theta_n-), Z(\theta_n-))\right]. \tag{B.13}$$

If we can show that

$$\lim_{n\to\infty} \mathbb{E}\left[V(\theta_n-, X(\theta_n-), Z(\theta_n-))\right] = \mathbb{E}\left[V(T-, X(T-), Z(T-))\right] \tag{B.14}$$

then we can conclude as in the proof of Theorem 3.2.

First, let us note that

$$\mathbb{E}\left[V(\theta_n-, X(\theta_n-), Z(\theta_n-))\mathbb{1}_{\{\theta_n=T\}}\right] = \mathbb{E}\left[V(T-, X(T-), Z(T-))\mathbb{1}_{\{\theta_n=T\}}\right] \tag{B.15}$$

and therefore

$$\lim_{n\to\infty} \mathbb{E}\left[V(\theta_n-, X(\theta_n-), Z(\theta_n-))\mathbb{1}_{\{\theta_n=T\}}\right] = \mathbb{E}\left[V(T-, X(T-), Z(T-))\right] \tag{B.16}$$

by monotone convergence. In order to prove (B.14), it is therefore sufficient to show that

$$\lim_{n\to\infty} \mathbb{E}\left[V(\theta_n-, X(\theta_n-), Z(\theta_n-))\mathbb{1}_{\{\theta_n<T\}}\right] = 0. \tag{B.17}$$

Let us note that (by Lemma 4.1) there exist constants $\underline{M}, \overline{M} > 0$ such that

$$\underline{M} U_p(x) \leq V(t,x,i) \leq \overline{M} U_p(x). \tag{B.18}$$

Next, it is clear that there exists a constant $L > 0$ such that

$$U_p(x) \leq \mathcal{V}_K(t,x,i) \leq L U_p(x). \tag{B.19}$$

Indeed, the first inequality follows from (B.3) and the second inequality follows from considering the no-crash strategy in $\mathcal{V}_K(t,x,i)$, which implies that $\mathcal{V}_K \leq \mathcal{V}_{RS}$, where $\mathcal{V}_{RS}$ denotes the value function in the regime switching model without crashes, see (1.3). Therefore, combining (B.18) and (B.19), we can find a constant $C > 0$ (independent of x) such that

$$V(t,x,i) \geq C\mathcal{V}_K(t,x,i). \tag{B.20}$$

Using this, we obtain

$$\begin{aligned}
0 &\geq \lim_{n\to\infty} \mathbb{E}\left[V(\theta_n-, X^{\pi,\vartheta^*}_{t,x,i}(\theta_n-), Z^{\vartheta^*}_{t,i}(\theta_n-))\mathbb{1}_{\{\theta_n<T\}}\right] && \text{(B.21)}\\
&\geq \lim_{n\to\infty} C\mathbb{E}\left[\mathcal{V}_K(\theta_n-, X^{\pi,\vartheta^*}_{t,x,i}(\theta_n-), Z^{\vartheta^*}_{t,i}(\theta_n-))\mathbb{1}_{\{\theta_n<T\}}\right] && \text{(B.22)}\\
&\geq \lim_{n\to\infty} C\mathbb{E}\left[\inf_{\vartheta\in\mathcal{T}(\theta_n-,Z(\theta_n-))} \mathbb{E}\left[U_p\left(X^{\pi,\vartheta}_{\theta_n-,X(\theta_n-),Z(\theta_n-)}(T)\right)\right]\mathbb{1}_{\{\theta_n<T\}}\right] && \text{(B.23)}\\
&\geq \lim_{n\to\infty} C\mathbb{E}\left[\inf_{\vartheta\in\mathcal{T}(t,i)} \mathbb{E}\left[U_p\left(X^{\pi,\vartheta}_{t,x,i}(T)\right)\right]\mathbb{1}_{\{\theta_n<T\}}\right] && \text{(B.24)}\\
&\geq \lim_{n\to\infty} C\mathbb{E}\left[U_p(x)\mathbb{1}_{\{\theta_n<T\}}\right] && \text{(B.25)}\\
&= 0. && \square
\end{aligned}$$

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