Noncrossing Partitions and Subgroups of Artin Groups of Finite Type

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Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Doctor of Philosophy is entirely my own work, and that I have exercised reasonable care to ensure that the work is original, and does not to the best of my knowledge breach any law of copyright, and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

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Contents

Notation and Abbreviations

Groups		
W	Coxeter group	(p.9)
$B(W)$	Artin group	(p.17)
G'	Commutator subgroup of G	(p.40)
Maps		
AB	Abelianisation homomorphism on $B(W)$	(p.18)
RL	Reflection length homomorphism on $B(W)$	(p.18)
Spaces		
X	Contractible $B(W)$ complex	(p.17)
X_H	Quotient $H\backslash X$	(p.18)
$X_{[0,3]}$	Retract of X in F_4 case	(p.56)
X'	Quotient $X_{[0,3]}/B(F_4)'$	(p.56)
X'_j	Filtration of X'	(p.59)
X_j	Filtration of $X_{[0,3]}$	(p.60)
N	Retract of $X_{\mathrm{ker(RL)}}$	(p.18)
N_i	Filtration of N	(p.22)
N'	Retract of $X_{B(C_3)'}$	(p.44)
$st(v)$	Star of a vertex v	(p.45)
Poset Notation		
$P_{[i,j]}$	Truncation of poset P by its rank function	(p.12)
$\|P\|$	Order complex of poset P	(p.12)
L	The lattice of non-crossing partitions	(p.13)
γ	Coxeter element of L	(p.13)
Homology Notation		
$C_n(Y)$	Free abelian group with basis the n-cells of Y	
$H_n(Y)$	Nth homology group of Y	
$S(v_1 \dots v_{k+1})$	A cycle in $C_{k-1}\|L_{[1,k]}\|$	(p.33)
Miscellaneous		
F_n	Groups with a $K(\Gamma, 1)$ with finite n-skeleton	(p.41)
NCP	Non-crossing partition, an element of L	
Unique rising chain	Factorisation of an NCP in terms of reflections that are all increasing in the total order	(p.17)
p-decreasing	Factorisation of γ which decreases for 1st p reflections and increases for remaining reflections	(p.30)

Abstract

Noncrossing Partitions and Subgroups of Artin Groups of Finite Type

Ben Quigley

In this thesis we use non-crossing partitions(NCP) to examine Artin groups of finite type and their subgroups. We follow the work of Brady-Watt and Bessis to construct a contractible universal cover for the Artin group $B(W)$ using this NCP structure. This space X can be factored out by normal subgroups H of $B(W)$ and the resulting quotient space is a $K(H,1)$. We examine this quotient space to see what information it gives us about the subgroup H.

The main result of this thesis is that $B(F_4)'$, the commutator subgroup of the Artin group $B(F_4)$, is finitely presented. It is already known whether the commutator subgroups of the other irreducible Artin groups are finitely presented. We retract the space X in the F_4 case and filter it appropriately to apply a theorem of Brown. If the filtration is finite mod $B(F_4)'$ and successive stages of the filtration are obtained from the previous stage by the adjunction of 3-cells then $B(F_4)'$ has finiteness type F_2 but not finiteness type F_3.

We also recover the fact that $B(C_3)'$ is finitely generated but not finitely presented. This is done by examining the fundamental group and second homology group of our $K(B(C_3)',1)$.

The other subgroup we are interested in is the kernel of the map which sends the NCP generators of an Artin group to the lengths of the corresponding non-crossing partitions. We define a Morse function on the quotient space in this case to calculate the homology. The Morse function on the quotient space also defines one on truncations of the NCP lattice. The simplification resulting from this Morse function recovers the fact that the homology of these truncations is entirely in the top dimension.

Chapter 1

Introduction

For any Coxeter group W we can define a total length function on the group and use it to construct a partial order on W. Choosing some product of the simple reflections, γ, to be the maximum element of this poset determines the lattice of non-crossing partitions(NCP) for W. These NCPs give an alternate presentation for the associated Artin group, which we denote $B(W)$. We follow [6], [8] and [3] to construct a contractible universal cover X for the Artin group using this presentation. Factoring X out by normal subgroups H of $B(W)$ results in a $K(H,1)$. In this thesis we use these spaces to study the subgroups H. We are generally interested in two particular subgroups; the kernel of the abelianisation map and the kernel of the map RL, which sends the NCP generators of the Artin group to the length of the corresponding NCPs. In the first half of the thesis we recover some results about the RL map, looking at the homology of the space H. In the second half of the thesis we apply our constructions to the abelianisation map, in the C_3 and F_4 cases where it does not coincide with RL. The main result is that $B(F_4)'$, the commutator subgroup of the Artin group of type F_4, is finitely presented.

The layout of the thesis is as follows. In Chapter 2 we review background information regarding Coxeter groups, posets and the lattice of non-crossing partitions. We describe the non-crossing partition approach to Artin groups and how this can be used to produce classifying spaces for subgroups of Artin groups. In Chapter 3 we consider the kernel of the RL map and its classifying space. We define a Morse function on this space and use discrete Morse theory to simplify its homology groups. As a consequence we get a Morse function on truncations of the NCP lattice, which shows that the homology of these truncations is all in the top dimension. In Chapter 4 we detail how we calculate the homology of $\ker(RL)$. The appendix features matlab functions that were used to calculate the homology, following this method, in the A_n case for $n \leq 7$. This confirmed these homology groups match those of Callegaro in [11]. In Chapter 5 we discuss some techniques for analysing finiteness properties of groups utilising their classifying spaces. In Chapter 6 we consider the C_3 case, where the abelianisation and RL maps do not coincide. We study the fundamental group of the $K(B(C_3)', 1)$ to show that $B(C_3)'$ is finitely generated and study the second homology group of the space to show that $B(C_3)'$ is not finitely presented, recovering a result of Squier [22]. Finally in Chapter 7 we use some of the techniques from Chapter 5 to show that the commutator subgroup in the F_4 case is finitely presented but not of finiteness type F_3.

Chapter 2

Coxeter Groups and Non-crossing Partitions

In this chapter we give a brief introduction to Coxeter groups, Artin groups and how the theory of non-crossing partitions can be used to study such groups. For more detail regarding Coxeter groups we recommend Humphreys [17] and Bourbaki [5]. We refer to Armstrong [1] for facts about non-crossing partitions.

2.1 Coxeter groups

Definition 2.1.1. Suppose $S = \{s_1, \ldots, s_n\}$ is a finite set and M is an $n \times n$ matrix with (i, j)th entry $m(s_i, s_j) \in \{1, 2, 3, \ldots, \infty\}$ which also satisfies

$$m(s, s') = m(s', s) \quad \text{and}$$
$$m(s, s') = 1 \Leftrightarrow s = s'.$$

Let W be a group generated by S with one relation of the form $(ss')^{m(s,s')} = 1$ for each pair of generators $(s, s') \in S \times S$. Note that $m(s, s') = \infty$ means that there is no relation between s and s'. Such a group is called a *Coxeter group* and M is called a *Coxeter matrix.*

We call n the *rank* of the group. If there is a partition of the generators $S = S' \cup S''$ such that the elements of S' commute with the elements of S'' then we say W is *reducible.* Note that $W = \langle S' \rangle \times \langle S'' \rangle$ in this case, where both factors are themselves Coxeter groups, otherwise W is *irreducible.*

The numbers $m(s_i, s_j)$ for a particular Coxeter group can also be given by a graph known as a *Coxeter diagram.* This graph has one vertex v_s for each generator $s \in S$. Whenever $m(s_i, s_j) \geq 3$ the vertices v_{s_i} and v_{s_j} are connected by an edge. If $m(s_i, s_j) \geq 4$ then we label this edge by the number $m(s_i, s_j)$.

It can be shown that every Coxeter group W has a geometric representation as a group generated by linear reflections. In the case that W is finite these generating reflections can be chosen to be Euclidean reflections and W is a finite reflection group. The details of this representation can be found in chapters 5 and 6 of [17]. We are mainly interested in the finite case and from now on we will use W to denote a finite irreducible Coxeter group of rank n. The figure below displays the Coxeter diagrams for these groups.

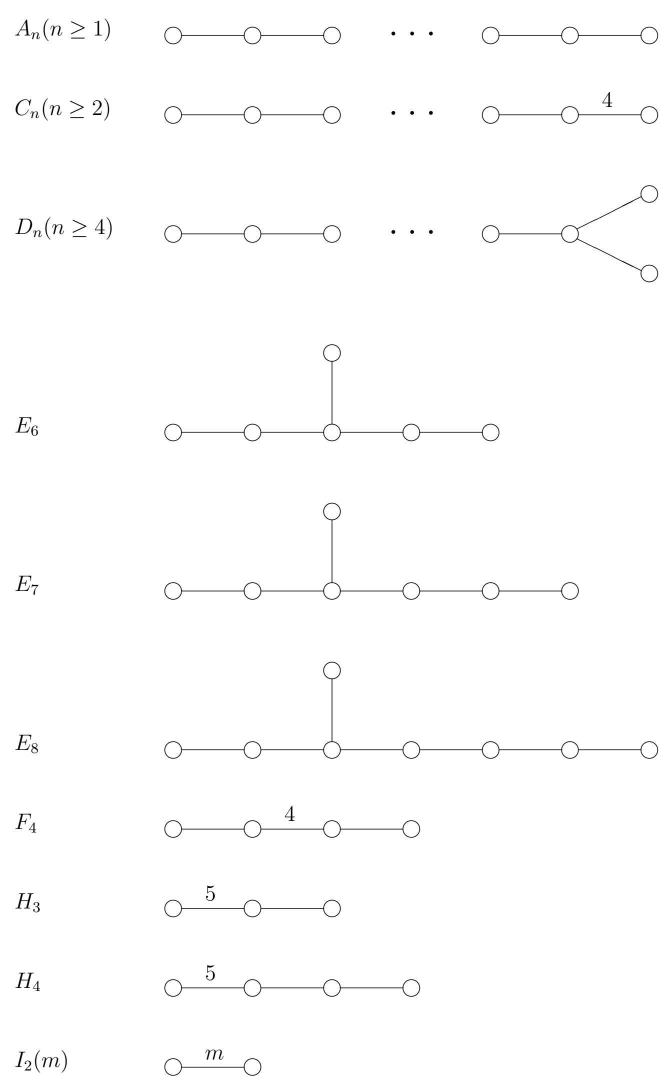


Fig. 2.1.1: Coxeter diagrams for the finite irreducible Coxeter groups

Example 2.1.1. The groups C_n correspond to the symmetry groups of the hypercubes. In the C_2 case $S = \{a, d\}$ and we have two generators, each of order 2, and these satisfy $(ad)^4 = e$. C_2 is isomorphic to the symmetry group of the square. The elements a, d, ada and dad are reflections through the lines in the Figure 2.1.2. The elements $da, dada$ and ad are clockwise rotations through $\pi/2, \pi$ and $3\pi/2$ respectively.

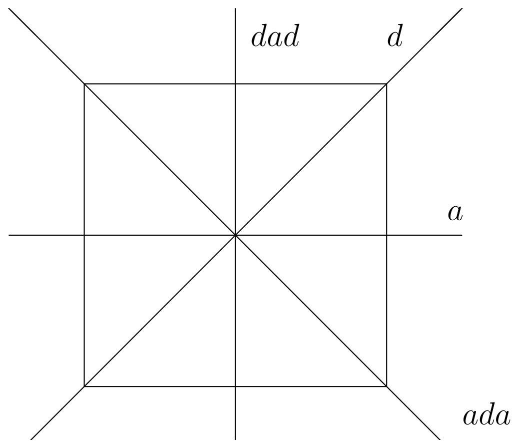


Fig. 2.1.2: Reflections in C_2

This example illustrates the fact that the reflections from the set S are not the only reflections in the geometric representation. We sometimes refer to the elements of S as the set of *simple reflections*. The set of all reflections is used to define the non-crossing partitions for the group W.

Definition 2.1.2. We call

$$T = \{wsw^{-1} | s \in S, w \in W\}$$

the set of *reflections*.

We equip W with a length function, $l : W \to \mathbb{Z}$, with respect to this larger generating set T. That is $l(w)$ is the minimum integer r such that there exists an expression $w = t_1 \dots t_r$ with $t_1, \dots, t_r \in T$. We call such a minimal expression $w = t_1 \dots t_r$ a *reduced word* for w. This function will induce a partial order on W.

2.2 Posets

Definition 2.2.1. A relation $\leq$ on a set P is a *partial order* if it is reflexive, anti-symmetric and transitive. The set P is called a partially ordered set or poset. If for any $x, y \in P$ we have that either $x \leq y$ or $y \leq x$ then we call this a total order.

Let $(P, \leq)$ be a partially ordered set. It is said to be *graded* if there exists a *rank function*, $\rho : P \to \mathbb{N}$, that satisfies

(i) For every $x, y \in P$ such that $x \leq y$ we have that $\rho(x) \leq \rho(y)$.

(ii) If $x \leq y$ and there does not exist any z such that $x \leq z \leq y$ then $\rho(y) = \rho(x) + 1$.

Note we say that y *covers* x in this case.

The poset is said to be bounded if it has a maximum element $\hat{1}$ satisfying $x \leq \hat{1}$ for all $x \in P$ and a minimum element $\hat{0}$ satisfying $\hat{0} \leq x$ for all $x \in P$. The *proper part* of a bounded poset P is $\hat{P} = P \backslash \{\hat{0}, \hat{1}\}$. We will often use the rank function to define other truncations of a poset, let $P_{[i,j]} = \{x \in P \,|\, i \leq \rho(x) \leq j\}$.

Definition 2.2.2. We define the *order complex* $|P|$ of a poset P to be the simplicial complex whose vertex set is elements of P and whose simplices are the non-empty finite chains in P.

Definition 2.2.3. Let $(P, \leq)$ be any partially ordered set.
An element y of P is an upper bound of $x_1, x_2 \in P$ if $x_1 \leq y$ and $x_2 \leq y$. The element y is a least upper bound of x_1, x_2 if it is an upper bound such that $y \leq z$ for any other upper bound z of x_1, x_2.
An element y of P is a lower bound of $x_1, x_2 \in P$ if $y \leq x_1$ and $y \leq x_2$. It is a greatest lower bound if $z \leq y$ for any other lower bound z of x_1, x_2.
P is called a lattice if least upper bounds and greatest lower bounds exist for all pairs of elements in P.

Let $\mathcal{E}(P)$ be the set of covering relations of P, meaning pairs (x, y) such that y covers x and let Λ be a totally ordered set. An *edge labeling* of P with label set Λ is a map $\lambda : \mathcal{E}(P) \to \Lambda$. Let $c = x_0 \leq x_1 \leq \cdots \leq x_r$ be an unrefinable chain of elements in P, that is $(x_i, x_{i+1}) \in \mathcal{E}(P)$ for all $0 \leq i \leq r-1$. We let $\lambda(c) = (\lambda(x_0, x_1), \lambda(x_1, x_2), \ldots, \lambda(x_{r-1}, x_r))$ be the label of c with respect to λ. We say c is *rising* or *falling* with respect to λ if the entries of $\lambda(c)$ are strictly increasing or strictly decreasing, respectively, in the total order of Λ. We say that c is *lexicographically smaller* than an unrefinable chain c' in P of the same length if $\lambda(c)$ precedes $\lambda(c')$ in the lexicographic order induced by the total order on Λ.

Definition 2.2.4. An edge labelling λ of P is called an *EL-labeling* if for every non-singleton interval $[u, v]$ in P

(i) there is a unique rising maximal chain in $[u, v]$ and

(ii) this chain is lexicographically smallest among all maximal chains in $[u, v]$

with respect to λ. The poset P is called EL-shellable if it has an EL-labelling for some label set λ.

2.3 Non-Crossing Partitions

Definition 2.3.1. Define the *absolute order* on W by

$$w_1 \leq w_2 \Leftrightarrow l(w_2) = l(w_1) + l(w_1^{-1}w_2) \quad \forall\, w_1, w_2 \in W.$$

Thus $w_1 \leq w_2$ if and only if there is a shortest expression for w_2 with a prefix which is a shortest expression for w_1.

$(W, \leq)$ is a graded poset with rank function l and the identity $e \in W$ is the unique minimum element. In general though it does not have a unique maximum element. We look at a special class of maximum elements.

Definition 2.3.2. A *standard Coxeter element* is any element of the form

$$\gamma = s_{\sigma(1)}s_{\sigma(2)}...s_{\sigma(n)},$$

where σ is some permutation of the set $\{1, 2, ..., n\}$. A *Coxeter element* is any conjugate of a standard Coxeter element in W.

Lemma 2.3.1.

(1) *Any two standard Coxeter elements are conjugate.*

(2) *If $\gamma \in W$ is a Coxeter element then we have $t \leq \gamma$ for all $t \in T$.*

A proof of this is given in [1], Lemma 2.6.2.

Definition 2.3.3. Fixing a specific Coxeter element γ, we define the poset of *non-crossing partitions* (NCP) to be the set of elements in the interval $[e, \gamma] = \{w \in W | e \leq w \leq \gamma\}$ with the order inherited from $(W, \leq)$.

Note that the isomorphism type of $[e, \gamma]$ is independent of the choice of γ since the Coxeter elements are all conjugate and conjugation by a fixed group element $w \in W$ is an automorphism of the poset.

Theorem 2.3.1. *The poset* $[e, \gamma]$ *is a lattice.*

This property is proved in [9]. We will refer to this non-crossing partition lattice as L.

Example 2.3.1. Returning to our C_2 example, we see $T = \{a, d, b = ada, c = dad\}$ and we could choose $\gamma = ad$ to be our Coxeter element. The non-crossing partition lattice simply consists of $T \cup \{e, \gamma\}$ in this case. The other two elements in C_2 are da and $dada = db$, both of which have reflection length two and do not precede γ. Figure 2.3.1 shows the order complex $|L|$ of this lattice.

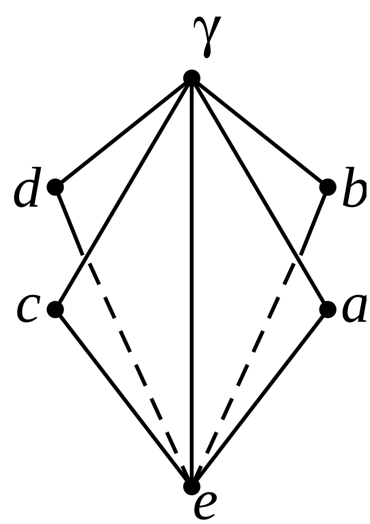


Fig. 2.3.1: Order Complex $|L|$ in the C_2 case

2.4 An Order on Reflections

We will need some of the machinery from [9] in our later calculations. First we review some information regarding root systems for finite reflection groups.

Let V be a real $n-$dimensional Euclidean space with inner product $(\cdot, \cdot)$. A *hyperplane* H in V is defined as $\{x \in V | (x, \alpha) = c\}$, for a non-zero vector $\alpha \in V$. This vector α is called a normal to the hyperplane. In the case $c = 0$ this is called a *linear* hyperplane, otherwise

H is called *affine*. The formula for reflecting a general vector $\beta \in V$ through H is

$$r_\alpha(\beta) = \beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha.$$

A *finite reflection group* is a group generated by such reflections.

Definition 2.4.1. A finite set of non-zero vectors Φ in V is called a *root system* for the finite reflection group W if the following conditions hold:

$$\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\} \text{ for all } \alpha \in \Phi,$$
$$r_\alpha(\Phi) = \Phi \text{ for all } \alpha \in \Phi.$$

A finite *hyperplane arrangement* A is a finite set of hyperplanes in V. The arrangement is called *central* if all hyperplanes pass through the origin, otherwise it is affine. The connected components of $V\backslash A$ are referred to as *regions*. Let A be the central hyperplane arrangement for a finite reflection group W and fix a region C of the arrangement, called the *fundamental chamber*. The inward unit normals of C is the set of *simple roots*, $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. A *positive root* is a linear combination of the elements of Π with non-negative coefficients. Similarly, a *negative root* is a linear combination of these elements with non-positive coefficients. The set of positive roots is denoted by Φ^+ while the set of negative roots is denoted by Φ^-. The set of roots Φ is the disjoint union of positive and negative roots. Note that the reflections in the hyperplanes normal to the simple roots correspond to the simple reflections in the geometric representation of a Coxeter group.

In [23] a total order is put on the set of roots of W. First a specific ordering $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is chosen on the simple roots so that $\{\alpha_1, \ldots, \alpha_s\}$ and $\{\alpha_{s+1}, \ldots, \alpha_n\}$ are orthogonal sets. This can always be done since the Coxeter diagram is a tree. Next a total ordering on the roots is defined by

$$\rho_i = R(\alpha_1)R(\alpha_2)\ldots R(\alpha_{i-1})\alpha_i, \text{ with } \alpha_{i+n} := \alpha_i,$$

where $R(v)$ is the reflection in the linear hyperplane with normal v. Let the Coxeter element be $\gamma = R(\alpha_1)R(\alpha_2)\ldots R(\alpha_n)$. It is shown in [23] this lists the $nh/2$ positive roots first, with $\{\alpha_{s+1}, \ldots, \alpha_n\}$ being the last $n-s$ positive roots, perhaps after some permutation. Here h is the order of γ in W. Also note that this induces a total order on the reflections of W. For $1 \leq i \leq nh/2$, we let R_i be the reflection $R(\rho_i)$.

In [23] another set of auxiliary vectors (they're called Petrie polygon vertices) are defined. These are constructed starting with the dual basis $\{\beta_1, \ldots, \beta_n\}$ to $\{\alpha_1, \ldots, \alpha_n\}$. Define

$$\mu_i = R(\alpha_1)R(\alpha_2)\ldots R(\alpha_{i-1})\beta_i, \text{ with } \alpha_{i+n} := \alpha_i \text{ and } \beta_{i+n} := \beta_i.$$

In our computations of NCP lattices we will make use of Lemma 4.8 of [9], restated for our purposes below. (The requirement on the ordering of the roots in the original Lemma is not necessary.)

Lemma 2.4.1. *For reflections R_i and R_j with corresponding positive roots ρ_i and ρ_j the following are equivalent:*
(a) $l(R_i R_j \gamma) = n - 2$.
(b) $\mu_j \cdot \rho_i = 0$.

Note that if condition (a) is true then $l(R_j R_i) + l(R_i R_j \gamma) = l(\gamma)$ and thus $R_j R_i \in L$.

We also recall Theorem 4.2 of [2], which tells us that each element of L has a unique factorisation as rising chain of reflections.

Theorem 2.4.1. *If T is totally ordered as described above and $\gamma = R(\alpha_1)R(\alpha_2)\dots R(\alpha_n)$ then the natural edge labeling of L with label set T is an EL-Labeling.*

Applying this result to an interval of length 2 we have the following corollary which we use in Chapter 4. Since only one factorisation of the interval can be rising, the rest must be falling.

Corollary 2.4.1. *Let $\sigma \leq \gamma$ have length two and order m in W. Then there are reflections $\tau_1, \tau_2, \dots \tau_m$ in W with $\tau_1 < \tau_2 < \cdots < \tau_m$ in the total order on reflections and*

$$\sigma = \tau_1 \tau_m = \tau_m \tau_{m-1} = \cdots = \tau_2 \tau_1.$$

Example 2.4.1. Again consider the C_2 case. Examining Figure 2.1.2 we choose the fundamental chamber C to be the cone on the points $\{(1,0),(1,1)\}$, thus the simple roots are $\alpha_1 = (0,1)$ and $\alpha_2 = (1,-1)$. Calculating the ρ vectors we find the roots to be

$$\rho_1 = (0,1), \quad \rho_2 = (1,1), \quad \rho_3 = (1,0), \quad \rho_4 = (1,-1),$$
$$\rho_5 = (0,-1), \quad \rho_6 = (-1,-1), \quad \rho_7 = (-1,0), \quad \rho_8 = (-1,1).$$

The first row contains the four positive roots, while the second row contains the four negative roots. The order on the positive roots induces an order on the reflections

$$R(\rho_1) = a, \quad R(\rho_2) = b, \quad R(\rho_3) = c, \quad R(\rho_4) = d.$$

The chain $e < a < \gamma$ is labelled by (a,d) since $\gamma = ad$. Consider this interval $[e, \gamma]$, the only interval of size greater than 1 in this case. It has four possible chains; $(a,d), (b,a), (c,b), (d,c)$. Of these (a,d) is the unique rising chain, the other three are all falling. Note also that (a,d) is the lexicographically smallest chain.

2.5 Artin Groups

Definition 2.5.1. The *Artin group*, or generalised braid group $B(W)$ is the group with generating set S and for each pair of generators (s_i, s_j) with $i \neq j$ in $S \times S$ we have a relation that says the alternating product of s_i and s_j of length $m(s_i, s_j)$, beginning with s_i, is equal to the alternating product of the same two generators, of length $m(s_j, s_i)$ and beginning with s_j. We require $m(s_i, s_j) = m(s_j, s_i) \in \{2, 3, \ldots, \infty\}$ with the convention that there is no relation between s_i and s_j if $m(s_i, s_j) = \infty$.

Example 2.5.1. The most well-known example of an Artin group is the classical braid group $B_n = B(\Sigma_n)$.

Example 2.5.2. Returning to the case $W = C_2$ we find that $B(C_2)$ is the group generated by two generators (which we also denote by a and d) subject to the single relation $adad = dada$.

It is shown in [6], [8] and [3] that $B(W)$ also has a presentation with a generator $[w]$ for each $w \in L \setminus \{e\}$ subject to the relations $[w_1][w_1^{-1}w_2] = [w_2]$ whenever $w_1 \lneqq w_2$.

We also recall from [6], [8] and [3] that the universal cover of the presentation complex of this second presentation is the 2-skeleton of a contractible simplicial complex X of dimension n.

Definition 2.5.2. Let X be the abstract simplicial complex whose $k-$cells are ordered $(k+1)$ tuples from $B(W)$ of the form $\{g_0, g_1, \ldots, g_k\}$ with $g_i = g_0[w_i]$ for $e < w_1 < \cdots < w_k$ a chain in L. In particular, the vertex set of X is $B(W)$.

We use the notation $(g_0, \{e < w_1 < \cdots < w_k\})$ for such a cell. Hence the cells of X are identified with pairs (g, σ) where $g \in B(W)$ and σ is an initialised chain in L, that is a chain beginning with e. The action of $B(W)$ on X is given by

$$g \cdot \{g_0, g_1, \ldots, g_k\} = \{gg_0, gg_1, \ldots gg_k\} \quad \text{or}$$
$$g \cdot (g_0, \sigma) = (gg_0, \sigma).$$

Note that the cell $(g_0, \{e < w_1 < \cdots < w_k\})$ has k faces of the form

$$(g_0, \{e < w_1 < \cdots < \hat{w}_i < \cdots < w_k\}), \quad \text{for} \quad 1 \leq i \leq k,$$

obtained by deleting w_i from σ. The remaining face, which we call the 'top' face, is obtained by deleting e from σ and is given by the set $\{g_0[w_1], g_0[w_2], \ldots, g_0[w_k]\}$ or in pair notation $(g_0[w_1], \{e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k\})$.

Example 2.5.3. Returning to the case $W = C_2$ we see that $B(C_2)$ also has a presentation with generators a, b, c, d, γ and relations $\gamma = ad = dc = cb = ba$. The corresponding 2-complex is a $K(B(C_2), 1)$ since $n = 2$ here and the universal cover coincides with X.

2.6 Subgroups

If $H \trianglelefteq B(W)$ then we can form a CW complex X_H, whose cells are of the form (Hg, σ) where the first component is now a right H coset and σ remains an initialised chain in L. If σ is the chain $e < w_1 < \cdots < w_k$ then this cell has boundary faces of the form

$$(Hg, \{e < w_1 < \cdots < \hat{w}_i < \cdots < w_k\}), \quad \text{for } 1 \leq i \leq k,$$

and 'top' face $(H(g[w_1]), \{e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k)\}$. There is an action of the quotient group $H \backslash B(W)$ on X_H given by

$$(Hg_1)(Hg_2, \sigma) = (H(g_1g_2), \sigma).$$

If H arises as the kernel of a homomorphism $\phi : B(W) \to D$, we can identify the coset Hg with the element $\phi(g) \in D$ through the first isomorphism theorem. Thus the cells of X_H can be described as pairs (d, σ) where $d \in im(\phi)$ and σ is an initialised chain in L.

Note that H acts freely on X and $H \backslash X = X_H$. Since X is contractible this means that X_H is a $K(H, 1)$.

Example 2.6.1. For each W, there is a homomorphism $\mathrm{RL} : B(W) \to \mathbb{Z} : [w] \mapsto l(w)$ which takes each NCP generator $[w]$ to its total reflection length. In [7] it is shown that X_H deformation retracts to a finite $(n-1)$-complex for $H = \ker(\mathrm{RL})$. We will denote this complex by N and now formally define it since we will use it later.

Definition 2.6.1. We let N be the finite subcomplex of $X_{\ker(\mathrm{RL})}$ consisting of the cells of the form

$$(m, e < w_1 < w_2 < \cdots < w_k) \text{ with } m \in \mathbb{Z} \text{ and } 0 \leq m < n - |w_k|.$$

Example 2.6.2. For each W, there is an abelianisation homomorphism $\mathrm{AB} : B(W) \to B(W)/[B(W), B(W)]$. This homomorphism coincides with RL except when W is C_n, F_4 or $I_2(m)$ for m even.

Example 2.6.3. For $W = C_2$, N is the finite 1-complex with two vertices labelled 0 and 1 and four arcs labelled by a, b, c and d each joining 0 to 1. Consequently, $\ker(RL)$ is free of rank 3. By contrast, the cover of N corresponding to $\ker(AB)$, given in Figure 2.6.1, is an infinite graph. The vertices of this cover are labelled $(p, -p)$ or $(p, 1-p)$ where $p \in \mathbb{Z}$. At each $(p, -p)$ there are four edges originating with the edges labelled a and c both terminating at $(p+1, -p)$ while the edges labelled b and d terminate at $(p, 1-p)$. Consequently, $\ker(AB)$ is not finitely generated.

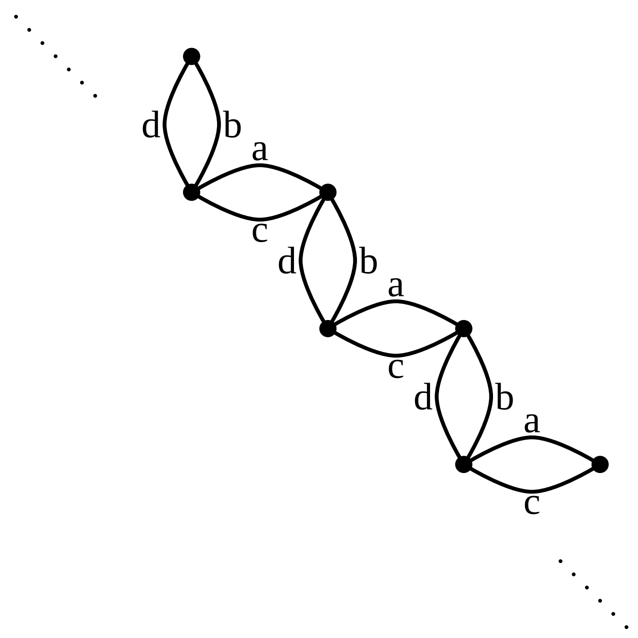


Fig. 2.6.1: Cover of N corresponding to $\ker(AB)$

Chapter 3

A Morse function on N

In this chapter we define a discrete Morse function on the complex N which is a $K(\ker(\mathrm{RL}), 1)$. Because of the structure of N this will be equivalent to a sequence of Morse functions on truncations of the non-crossing partition lattice. We give an introduction to discrete Morse theory and how it can be used to calculate the homology of cell complexes. We explain how to construct a function on N and prove that it is a Morse function. We then use the Morse function to show that the homology of a truncation of the NCP lattice is only in the top dimension. We can also explicitly state the critical cells that give rise to this homology. The important property of the non-crossing partitions that we use is the ordering on the reflections from [23] and that every non-crossing partition has a unique factorisation as a 'rising chain' of reflections with respect to this ordering.

3.1 Introduction to Morse functions

Definition 3.1.1. A Morse function μ on a complex X is an assignment of a distinct real number to each cell. The assignment satisfies the following properties:
1. For any cell σ_1, the number of facets σ_2 of σ_1 which satisfy $\mu(\sigma_2) > \mu(\sigma_1)$ is at most one.
2. For any cell σ_1, the number of cells σ_2 of which σ_1 is a facet which satisfy $\mu(\sigma_2) < \mu(\sigma_1)$ is at most one.

A cell for which both of these numbers is zero is called a critical cell. These critical cells form a Morse complex with the same homology as the original complex X, see [13] or

Chapter 11 of [18]. The boundary map for this Morse complex counts the number of paths from each facet of a critical cell to the critical cells of one lower dimension. By path here we mean a sequence of cells $\{\tau_1, \tau_2, \ldots, \tau_k\}$ where $\mu(\tau_i) > \mu(\tau_{i+1})$ and either τ_{i+1} is a facet of τ_i or vice versa.

Example 3.1.1. Consider the 2-cell shown in Figure 3.1.1. The function μ, defined by Table 3.1.1, is an example of a Morse function on this space.

Note that the only critical cell is the vertex a and this vertex is the Morse complex for this space. We have also included arrows in the diagram connecting pairs of non-critical cells. Each cell σ_1 is paired with its facet σ_2 satisfying $\mu(\sigma_1) > \mu(\sigma_2)$. We can think of deforming the space by 'pushing' in the direction of the arrows to give the Morse complex.

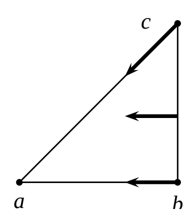


Fig. 3.1.1: A 2-cell collapsing to its Morse complex

Note that when X is finite, the values of μ can be chosen to be positive integers and the Morse function is equivalent to a total order on the cells. Thus we can think of building the cell structure of X by attaching the cells to each other in this order. In the rest of this section we describe a canonical Morse function for N by putting a total order on its cells.

$\mu(\sigma)$	σ
1	a
2	$a < c$
3	c
4	$a < b$
5	b
6	$a < b < c$
7	$b < c$

Table 3.1.1: Morse function on a 2-cell

3.2 Constructing the Morse function

We consider the filtration $N_{n-1} \subset N_{n-2} \subset \cdots \subset N_1 \subset N_0 = N$ of the complex N where N_i is the subcomplex of N given by the union of all cells of the form

$$(i, \{e \lessdot w_1 \lessdot \cdots \lessdot w_{n-i-1}\})$$

and their faces. Here $u \lessdot v$ means that v covers u.

Notation: We order these maximum dimension cells of N_i lexicographically using the total order on reflections. Let $\sigma = \{e \lessdot w_1 \lessdot \cdots \lessdot w_{n-i-1}\}$ and $\overline{\sigma} = \{e \lessdot \overline{w}_1 \lessdot \cdots \lessdot \overline{w}_{n-i-1}\}$. Then we say $(i, \sigma) < (i, \overline{\sigma})$ if $w_{j-1}^{-1} w_j < \overline{w}_{j-1}^{-1} \overline{w}_j$ for some $1 \leq j \leq n-i-1$ and $w_k = \overline{w}_k$ for all $k < j$.

We use the above filtration to build the Morse function starting with N_{n-1}, which consists of the single cell $(n-1, \{e\})$. Thus, $(n-1, \{e\})$ is the first cell in our ordering. Then, assuming that N_{i+1} has been constructed, follow the steps below to attach the other cells of N_i in the appropriate order.

3.2.1 Constructing N_i from N_{i+1}

We will use a recursive algorithm to attach the cells of $N_i \setminus N_{i+1}$. The algorithm will take as input a k-cell, $\tau = (i, \{e < w_1 < \cdots < w_k\})$ in N_i. It will ensure that all the faces of τ are attached. In addition, the algorithm will ensure that either τ itself is attached and 'true' is returned, or it will not attach τ and 'false' will be returned.

Algorithm

1. Check if the cell τ has already been attached. If so, return 'true'. Otherwise move to step 2.
2. Input each of the k facets

$$(i, \{e < w_1 < \cdots < w_{k-1} < \hat{w}_k\}), (i, \{e < w_1 < \cdots < \hat{w}_{k-1} < w_k\}),$$

$$\ldots, (i, \{e < \hat{w}_1 < \cdots < w_{k-1} < w_k\})$$

into the algorithm in this order. If at any stage a facet returns 'false' then attach τ followed by that facet. Proceed then with the remaining facets before returning 'true' for τ. Otherwise do not attach τ and return 'false'.

Note: In step 2 above the face of τ not considered by the algorithm is the top face $(i+|w_i|, \{e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k\})$. However this has already been considered since it belongs to N_{i+1}.

Note: The terms 'true' and 'false' have the following meaning. 'True' is returned for a cell if it has been attached previously or is being attached at this stage. 'False' is returned for a cell if its entire boundary has already been attached when the cell is first considered.

Note: We will see that it is impossible for more than one facet of a cell to return 'false'.

To construct N_i from N_{i+1}, take its first cell of maximum dimension in the lexicographic order. Input this cell into the algorithm. If the cell returns 'false', attach this cell after the algorithm has completed. If the cell returns'true' then it has already been attached. In both cases its entire boundary has been attached. Continue to repeat this process for the next cell of maximum dimension in the lexicographic order until all cells have been attached. Note that once this lexicographic order is chosen there is no choice in the process.

Example 3.2.1. Consider the case $W = C_3$, the classical presentation is

$$C_3 = < a, b, c \quad | \quad (ab)^4 = (ac)^2 = (bc)^3 = a^2 = b^2 = c^2 = 1 > .$$

However C_3 is isomorphic to the set of symmetries of a cube and we wish to use the non-crossing partition presentation in terms of reflections. The following notation is used for the nine reflections in C_3:

$$\begin{aligned}
[1] &: (x,y,z) \mapsto (-x,y,z) \\
[2] &: (x,y,z) \mapsto (x,-y,z) \\
[3] &: (x,y,z) \mapsto (x,y,-z) \\
(1,2) &: (x,y,z) \mapsto (y,x,z) \\
(1,3) &: (x,y,z) \mapsto (z,y,x) \\
(2,3) &: (x,y,z) \mapsto (x,z,y) \\
(1,\bar{2}) &: (x,y,z) \mapsto (-y,-x,z) \\
(1,\bar{3}) &: (x,y,z) \mapsto (-z,y,-x) \\
(2,\bar{3}) &: (x,y,z) \mapsto (x,-z,-y).
\end{aligned}$$

A set of simple roots is $\{(2,3), [1], (1,3)\}$ and we choose $\gamma = (2,3)[1](1,3)$. The nine length

2 elements of L are

$$[1](2,3), \quad [1,2]=[1](1,2), \quad [1,3]=[1](1,3), \quad (1,\bar{2},\bar{3})=(2,3)(1,\bar{3}),$$
$$(1,2,3)=(2,3)(1,3), \quad [2,3]=(2,3)[3], \quad (1,\bar{3})[2], \quad (1,2,\bar{3})=(1,\bar{3})(1,2), \quad (1,2)[3].$$

Here N is 2-dimensional and we follow Chapter 2 to order the roots and the vertices. This ordering is given Table 3.2.1. Note that for the rest of this example we will refer to the reflection R_i simply by the digit i.

number	1	2	3	4	5	6	7	8	9
reflection	$(2,3)$	$[1]$	$(1,\bar{2})$	$(1,\bar{3})$	$[2]$	$(2,\bar{3})$	$(1,2)$	$[3]$	$(1,3)$

Table 3.2.1: Order on the reflections of C_3

Using the geometric model for the NCP lattice from [9], we can visualise the NCP lattice with the following diagram. The vertices of the graph, in black, represent the nine reflections. The midpoint of each edge represents a length two NCP. The different colours denote points which are identified. Note that this graph is homotopy equivalent to the proper part of the lattice and the minimum and maximum elements, e and γ, are not included.

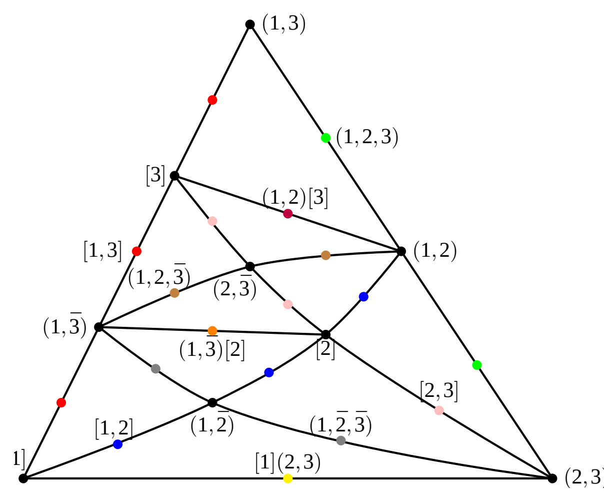


Fig. 3.2.1: Proper part of C_3 lattice

The table on the following page outlines the Morse function on N. Here the critical cells are alone in rows while each non-critical cell in column 1 is paired with its facet in column 2. Cell number 1 is the sole element of N_2. Cells 2 through 11 are in $N_1 \setminus N_2$, the single 0-cell being matched with the lexicographically first 1-cell and the rest of the 1-cells critical. The remaining cells are in $N_0 \setminus N_1$ with, once again, the single 0-cell being matched with the lexicographically first 1-cell but now the remaining 1-cells are matched with eleven 2-cells leaving the remaining ten 2-cells critical. We notice that the critical cells are top-dimensional in the subset of the filtration in which they first appear.

To illustrate how this example is constructed, consider starting to attach $N_0 \setminus N_1$, having already built N_1. We take the lexicographically first 2-cell in the space which is $\sigma = (0, e < (2,3) < (2,3)[1])$ or in the short notation $(0, e < 1 < 12)$. This cell has not been attached so we consider its facets in turn. First $(0, e < 1)$, again this has not been attached and so we look at its facets. The vertex $(0, e)$ returns false at step 2, since it has no smaller facets to input. Hence it is matched with $(0, e < 1)$. We note that the other vertex $(1, e)$ returns true at step 1. We then input the next facet $(0, e < 12)$ of σ into the algorithm. This does not return true at step 1 so again we move to step 2 and examine its facets. This time both vertices return true at step 1, so the cell itself $(0, e < 12)$ returns false at step 2. It is then matched with σ.

Now let $\sigma = (0, e < 3 < 27)$, the first critical cell of $N_0 \setminus N_1$. It does not return true at step 1, so we move to step 2 and look at its facets. The cell $(0, e < 3)$ has already been seen as a facet of $(0, e < 3 < 14)$ and so returns true. The cell $(0, e < 27)$ has already been seen as a facet of the cell $(0, e < 2 < 27)$ and it also returns true at step 1. So σ returns false at step 2. It is a maximum dimension cell, so we attach it now before moving on to the next maximum dimension cell. It has not been matched, so it is critical.

Finally we note that $27 = 32$ and a factorisation of γ is 321, which is falling. In fact all the critical cells of $N_0 \setminus N_1$ correspond to decreasing factorisations of γ. For example 981 is another decreasing factorisation and corresponds to the critical cell $(0, e < 9 < 29)$, note that $29 = 98$. We will see later how we can generalise this property.

Number	Cell	Number	Cell
1	$(2, e)$		
2	$(1, e < 1)$	3	$(1, e)$
4	$(1, e < 2)$		
5	$(1, e < 3)$		
6	$(1, e < 4)$		
7	$(1, e < 5)$		
8	$(1, e < 6)$		
9	$(1, e < 7)$		
10	$(1, e < 8)$		
11	$(1, e < 9)$		
12	$(0, e < 1)$	13	$(0, e)$
14	$(0, e < 1 < 12)$	15	$(0, e < 12)$
16	$(0, e < 1 < 14)$	17	$(0, e < 14)$
18	$(0, e < 1 < 18)$	19	$(0, e < 18)$
20	$(0, e < 1 < 19)$	21	$(0, e < 19)$
22	$(0, e < 2 < 12)$	23	$(0, e < 2)$
24	$(0, e < 2 < 27)$	25	$(0, e < 27)$
26	$(0, e < 2 < 29)$	27	$(0, e < 29)$
28	$(0, e < 3 < 14)$	29	$(0, e < 3)$
30	$(0, e < 3 < 27)$		
31	$(0, e < 4 < 29)$	32	$(0, e < 4)$
33	$(0, e < 4 < 14)$		
34	$(0, e < 4 < 45)$	35	$(0, e < 45)$
36	$(0, e < 4 < 47)$	37	$(0, e < 47)$
38	$(0, e < 5 < 18)$	39	$(0, e < 5)$
40	$(0, e < 5 < 27)$		
41	$(0, e < 5 < 45)$		
42	$(0, e < 6 < 47)$	43	$(0, e < 6)$
44	$(0, e < 6 < 18)$		
45	$(0, e < 7 < 19)$	46	$(0, e < 7)$
47	$(0, e < 7 < 27)$		
48	$(0, e < 7 < 47)$		
49	$(0, e < 7 < 78)$	50	$(0, e < 78)$
51	$(0, e < 8 < 29)$	52	$(0, e < 8)$
53	$(0, e < 8 < 18)$		
54	$(0, e < 8 < 78)$		
55	$(0, e < 9 < 19)$	56	$(0, e < 9)$
57	$(0, e < 9 < 29)$		

3.3 Properties of the Function

In the first step of our algorithm we need to check if a cell has already been attached to the complex. One way to do this is to run through the cells that have been attached and look for it. The lemma below gives a more efficient method of performing this check. It also provides the basis for proving that the algorithm gives a Morse function and for identifying critical cells.

Lemma 3.3.1. *Let σ be a cell first considered by the algorithm as a face of the maximum dimension cell $\theta = (i, \{e \lessdot w_1 \lessdot \cdots \lessdot w_{n-i-1}\})$ and suppose σ is given by deleting from the chain of θ the following (possibly empty) set of entries $\{w_{i_1}, w_{i_2}, \ldots, w_{i_l}\}$. Let R be the first reflection in the total order which precedes $w_{n-i-1}^{-1}\gamma$ and let τ be the facet of σ obtained by deleting the entry w_j. Then the algorithm will return 'true' for τ at step 1 if and only if one of the following properties is satisfied:*
(1): $j > i_1$ or
(2): $w_{j-1}^{-1}w_j > w_j^{-1}w_{j+1}$, where $w_j^{-1}w_{j+1} = R$ in the case $j = n-i-1$.

Proof. The algorithm considers the faces of θ with deleted entry w_{n-i-1} first, followed by w_{n-i-2} and so on. Hence if (1) holds τ would have been previously attached as a face of the cell with deleted entries

$$[\{w_{i_1}, w_{i_2}, \ldots, w_{i_l}\} \setminus \{w_{i_1}\}] \cup \{w_j\}.$$

So 'true' would be returned at step 1 when τ was considered as a face of σ. Note that, for property (1) to hold, we need $\sigma \neq \theta$.

If (2) holds then τ is a face of the maximum dimension cell

$$\widehat{\theta} = (i, \{e \lessdot w_1 \lessdot \cdots \lessdot w_{j-1} \lessdot w_{j-1}(w_j^{-1}w_{j+1}) \lessdot w_{j+1} \lessdot \cdots \lessdot w_{n-i-1}\})$$

which precedes θ in the lexicographical order. In the case $j = n-i-1$, we can set

$$\widehat{\theta} = (i, \{e \lessdot w_1 \lessdot \cdots \lessdot w_{n-i-2} \lessdot w_{n-i-2}R\})$$

and the statement still holds.

For the converse, we note that 'true' would be returned at step 1 of the algorithm if τ had already been attached and there are only two ways this could occur. First, τ could be a facet of more than one face of θ and one of those faces might have previously been considered by the algorithm, so (1) would hold. Second, τ could be a face of a previous maximum dimension cell, that σ was not a face of, in which case (2) holds. $\square$

Note: To elaborate on these properties:

(1) says that the segment of the chain of σ before the element w_j is not saturated, that is, there is a deleted element before w_j.

Every maximum dimension cell gives a factorisation of the last element w_k as a product of reflections

$$w_k = w_1(w_1^{-1}w_2)(w_2^{-1}w_3)\cdots(w_{k-1}^{-1}w_k).$$

Property (2) says that this factorisation is decreasing at w_j. Note that we will often compare $w_{j-1}^{-1}w_j$ with $w_j^{-1}w_{j+1}$ in the propositions that follow. In the case where $j = n-i-1$, we always take $w_j^{-1}w_{j+1} = R$, where R is the first reflection in the total order which precedes $w_{n-i-1}^{-1}\gamma$, as was the case for Lemma 3.3.1.

Theorem 3.3.1. *Let $\theta = (i, \{e \lessdot w_1 \lessdot \cdots \lessdot w_{n-i-1}\})$ be a maximum dimension cell and denote its face σ by the (possibly empty) set of deleted entries $\{w_{i_1}, w_{i_2}, \ldots, w_{i_l}\}$. If $w_{j-1}^{-1}w_j < w_j^{-1}w_{j+1}$, for some $j < i_1$ then exactly one of the facets of σ will return false. Hence σ will return true at step 2.*

In the case that $\sigma = \theta$, and i_1 does not exist, we check this condition for any j.

Proof. We use induction on l, the number of entries w_j which satisfy $w_{j-1}^{-1}w_j < w_j^{-1}w_{j+1}$ and $j < i_1$.

Start with $l = 1$ so that $w_{k-1}^{-1}w_k > w_k^{-1}w_{k+1}$ for all $k < i_1$ except for $k = j$. Then all facets with deleted entry w_k, except for $k = j$ will return true at step 1, since one of the properties of Lemma 3.3.1 is satisfied. Now consider τ with deleted entries $\{w_j, w_{i_1}, \ldots, w_{i_l}\}$. Neither of the two properties of Lemma 3.3.1 are satisfied so the algorithm will not return true at step 1. Instead it will move to step 2 and examine all the facets of τ. Consider the facet with deleted entry w_k. If $k > j$ then this facet satisfies property (1) of Lemma 3.3.1. If $k < j$ we know that $w_{k-1}^{-1}w_k > w_k^{-1}w_{k+1}$, so this facet satisfies property (2) of Lemma 3.3.1. Hence all the facets of τ will return true, which means that τ itself will return false. Now assume that the case $l = p$ holds and consider $l = p + 1$. This means that

$$w_{j_m-1}^{-1}w_{j_m} < w_{j_m}^{-1}w_{j_m+1} \quad \text{for} \quad 1 \le m \le p+1, \quad j_m < j_{m+1}, \quad j_{p+1} < i_1$$
$$\text{and} \quad w_{k-1}^{-1}w_k > w_k^{-1}w_{k+1} \quad \text{for} \quad k < i_1 \quad \text{otherwise}.$$

The facets of σ with deleted entry w_k for $k > i_1$ will all return true by property (1) of Lemma 3.3.1. The facets of σ with deleted entry w_k for $j_{p+1} < k < i_1$ will all return true by property (2) of Lemma 3.3.1. Consider the facet of σ with deleted entry $w_{j_{p+1}}$. The algorithm will not return true at step (1) for this cell. However we note that this cell is a face of θ with deleted entries $\{j_{p+1}, i_1, ..., i_l\}$ and p entries satisfying $w_{k-1}^{-1}w_k < w_k^{-1}w_{k+1}$, $k < j_{p+1}$. By the induction assumption then, it will return true.

Similarly the facets of σ with deleted entries w_k for $1 \leq k < j_{p+1}$ will all return true, either by the induction assumption or property (2) of Lemma 3.3.1, except when $k = j_1$. In this case the algorithm is considering a cell τ which is the face of θ with deleted entries $\{j_1, i_1, ..., i_l\}$ and $w_{k-1}^{-1}w_k > w_k^{-1}w_{k+1}$ for $k < j_1$. It will not return true at step (1) for τ so it will move to step (2). The facets of τ with deleted entry w_k will return true by property (1) of Lemma 3.3.1 for $k > j_1$ and true by property (2) for $k < j_1$. Hence the algorithm will return false for τ and it is the only facet of σ that will return false. $\square$

Corollary 3.3.1. *Let θ and σ be as above but $w_{j-1}^{-1}w_j < w_j^{-1}w_{j+1}$ is not satisfied for any $j < i_1$ (or for any j in the case that $\sigma = \theta$). Then the algorithm will return false for σ.*

Proof. The cell τ at the end of the proof of Theorem 3.3.1 satisfied exactly this condition and we showed that it returned false. $\square$

Corollary 3.3.2. *This algorithm gives a total order on the cells of N_i, we say that $\sigma_1 \leq \sigma_2$ if σ_1 was attached by the algorithm before σ_2. This ordering defines a Morse function μ by assigning the nth cell attached by the algorithm the function value n.*

Proof. A Morse function must satisfy the two properties mentioned in section 3.1. The theorem, together with corollary 3.3.1 imply that at most one facet of a cell σ will return false during step 2 of the algorithm. Only the facet of σ that returns false during step 2 of the algorithm will be after σ in the order and hence have a greater Morse function value. The number of such facets is at most one so the first property holds. For the second property, suppose we are completing the algorithm for a cell σ and look at its facet τ for the first time. This facet will either return true or false, if false then we will have $\mu(\sigma) < \mu(\tau)$. In either case whenever we come across τ again, as a facet of some cell $\hat{\sigma}$, it will return true at step 1 and we have $\mu(\tau) < \mu(\hat{\sigma})$. So this ordering on the cells of N_i satisfies both Morse function properties. $\square$

Corollary 3.3.3. *The only critical cells of N_i that are not in N_{i+1} are of maximum dimension, $n - i - 1$. They are precisely the cells of the form*

$$(i, \{e \lessdot w_1 \lessdot \cdots \lessdot w_{n-i-1}\}) \quad \textit{with} \quad w_{j-1}^{-1}w_j > w_j^{-1}w_{j+1} \quad \textit{for all } j.$$

Proof. Consider the critical cells of N_i. First we have all the critical cells in N_{i+1} and we have new critical cells that are added by the algorithm. A critical cell σ added by the algorithm must have had all its facets return true. Otherwise the facet that returned false would have a greater Morse function value. However this means σ itself will return false. If σ is first considered as a facet of another cell θ then by returning false it will be given a Morse function value greater than θ and it would not be critical. The only cells which are not facets of other cells in N_i are those of maximum dimension. Hence the new critical cells added are all of maximum dimension.

In particular these are maximum dimension cells which have returned false, so that they have a greater Morse function value than all of their facets. By Corollary 3.3.1 this requires that they are of the above form. □

Definition 3.3.1. We will refer to factorisations of γ which are decreasing for the first p steps and increasing afterwards as p-decreasing factorisations. That is $\tau_1 \dots \tau_p \tau_{p+1} \dots \tau_n$ is a factorisation of γ with $l(\tau_i) = 1$ for all i and

$$\begin{aligned} \tau_i > \tau_{i+1} &\quad \text{for } 1 \leq i \leq p, \\ \tau_i < \tau_{i+1} &\quad \text{for } p < i < n. \end{aligned}$$

Theorem 3.3.2. *The critical $n-i-1$ cells in the Morse complex for N are in one to one correspondence with factorisations of γ which are $n-i-1$ decreasing.*

Proof. The critical $n-i-1$ cells in N are precisely those maximum dimension cells in N_i which return false. Hence they must be of the form specified in Corollary 3.3.3.

Now consider factorisations of γ as given in the hypothesis. Each such factorisation determines a maximum dimension cell in X_i of the form:

$$(i, \{e \lessdot \tau_1 \lessdot (\tau_1\tau_2) \lessdot \cdots \lessdot (\tau_1\tau_2 \cdots \tau_{n-i-1})\}).$$

With $w_j = \tau_1\tau_2 \cdots \tau_j$, we have $w_{j-1}^{-1}w_j > w_j^{-1}w_{j+1}$ for $1 \leq j \leq n-i-2$ since the first $n-i-2$ reflections are decreasing. The other requirement for this cell to be critical is for $\tau_{n-i-1} = w_{n-i-2}^{-1}w_{n-i-1} > R$, where R is the first reflection in the unique rising chain for $w_{n-i-1}^{-1}\gamma$. However $w_{n-i-1}^{-1}\gamma = \tau_{n-i} \cdots \tau_n$ and in fact this chain is rising as well since $\tau_j < \tau_{j+1}$ for $n-i \leq j \leq n-1$. Hence $R = \tau_{n-i}$ since the unique rising chain for an interval is also the lexicographically smallest. Since we have $\tau_{n-i-1} > \tau_{n-i}$ this cell is critical by corollary 3.

Similarly every critical cell, $w_1 \lessdot \cdots \lessdot w_{n-i-1}$, in N_i has associated with it such a factorisation of γ. We choose the obvious factorisation for w_{n-i-1} and the unique increasing factorisation for $w_{n-i-1}^{-1}\gamma$. $\square$

We recall from [7] that each N_i has the structure of a mapping cone on a truncation of $|L|$:

Theorem 3.3.3. *(Proposition 6.1 of [7]) N_i is the mapping cone of a map*

$$f : |L_{[1,n-i-1]}| \to N_{i+1} : w_1 < w_2 < \cdots < w_k \mapsto (i + |w_1|, e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k)$$

where $p = n - i - 1$ and $|L_{[1,p]}|$ is the order complex of $L_{[1,p]}$, the truncation of the lattice L to elements w with $1 \leq l(w) \leq p$.

We deduce the following corollary, which can also be derived from Theorem 4.2 of [2] together with [4]

Corollary 3.3.4. *Truncations $|L_{[1,k]}|$ have the homotopy type of a wedge of top dimensional spheres.*

Proof. The Morse function on $N_i \setminus N_{i+1}$ determines a Morse function on $|L_{[1,n-i-1]}|$. Let $\hat{\mu}(R_1) = 0$ and otherwise

$$\hat{\mu}(w_1 \lessdot \cdots \lessdot w_{n-i-1}) = \mu(i, \{e \lessdot w_1 \lessdot \cdots \lessdot w_{n-i-1}\}),$$

where μ is a Morse function determined by the algorithm of section 3.2 with $\mu(\sigma) > 0$ for all $\sigma \in N_0 \setminus N_1$. Note that the cell $(i, e < R_1)$ would be matched with the vertex (i, e) in $N_i \setminus N_{i+1}$. However (i, e) is the cone point and does not have an equivalent in $|L_{[1,p]}|$. Thus R_1 becomes a critical vertex but otherwise $\hat{\mu}$ is the same as μ. This Morse function defines a Morse complex with the same homotopy type as the space. By Corollary 3.3.2 the only critical cells of the Morse function, besides the vertex R_1, are in the top dimension. Hence the Morse complex is a wedge of spheres about R_1. $\square$

Chapter 4

Homology calculations on N

In the appendix we detail matlab functions that were used to calculate the homology of N, which is a $K(\ker(\mathrm{RL},1))$ in the cases of A_2 up to A_7. We confirmed these homology groups match those of Callegaro in [11]. We refer to [7] for more detail on the connection between the different spaces used. In this chapter we detail some of the structure of N which is used to simplify these calculations. The Morse function from Chapter 3 could also be used to calculate the homology of the space but the boundary maps for the Morse complex are complicated. It was easier to use a different geometric basis for $\tilde{H}_{p-1}(|L_{[1,p]}|)$.

4.1 The Boundary Map

In [7] it is shown that each filtration N_i of N has the structure of a mapping cone on truncations of the lattice L. We restated the map that gives rise to this structure in Theorem 3.3.3. This allows the homology of N to be computed as the homology of a chain complex whose groups are homologies of truncations of L:

Theorem 4.1.1. *(Theorem 6.4 of [7]) The homology of N is isomorphic to the homology of the chain complex whose pth group is $\tilde{H}_{p-1}(|L_{[1,p]}|)$ and whose boundary homomorphism is given, at the level of chains, by $\sum a_\sigma \sigma \mapsto \sum a_\sigma \Omega(\sigma)$, where*

$$\Omega(w_1 \lessdot w_2 \lessdot \cdots \lessdot w_p) = (w_1^{-1}w_2 \lessdot w_1^{-1}w_3 \lessdot \cdots \lessdot w_1^{-1}w_p).$$

4.2 Spheres from factorisations.

From the above and the previous chapter we can compute the homology of N by computing the images of the Morse generators for $\tilde{H}_{p-1}(|L_{[1,p]}|)$ under the map induced by Ω. However, since the computation of the Morse generators involves following all alternating paths from the critical cells we will use a different basis. To introduce this basis we first identify a large but finite set of cycles in $C_{p-1}(|L_{[1,p]}|)$.

Let $1 \leq k \leq n-1$, we note that each factorisation of a length $k+1$ NCP determines a cycle in $C_{k-1}(|L_{[1,k]}|)$ which contains $(k+1)!$ simplices of dimension $k-1$, one for each permutation in Σ_{k+1} as follows. Suppose that $\sigma = v_1 \dots v_{k+1}$ is a length $k+1$ NCP where v_i is a reflection in W. For each subset $\{i_1, \dots i_j\}$ of $\{1, 2, \dots, k+1\}$ define the NCP $w(i_1, \dots, i_j)$ to be the product of $v_{i_1}, v_{i_2}, \dots, v_{i_j}$ in increasing order, that is, in the order in which they appear in the given factorisation of σ. We observe that $w(i_1, \dots i_j)$ is the least upper bound of the reflections $v_{i_1}, v_{i_2}, \dots, v_{i_j}$ in the lattice L. Our $(k-1)$-sphere in $|L_{[1,k]}|$ is the signed sum of the simplices corresponding to the chains

$$v_{i_1} = w(i_1) \lessdot w(i_1, i_2) \lessdot \cdots \lessdot w(i_1, \dots, i_k) = \sigma$$

where $(i_1, i_2, \dots, i_{k+1})$ ranges over the permutations of $(1, \dots, k+1)$ and the sign is the usual sign of the permutation given by $(i_1, i_2, \dots, i_{k+1})$.

Notation: We will denote the $(k-1)$-sphere determined by $v_1 \dots v_{k+1}$ by $S(v_1 \dots v_{k+1})$.

Example 4.2.1. Suppose $\sigma = abcd$ is a length four NCP in W. The corresponding two sphere in $|L_{[1,3]}|$ is shown in Figure 4.2.1 stereographically projected from the d vertex. Note that $S(abcd)$ consists of twenty four 2-cells. These can be split into four groups of six, each group being the barycentric subdivision of a larger 2-cell whose vertex set is three of $\{a, b, c, d\}$. For example consider the 2-cell on $\{a, b, c\}$, this consists of the following six 2-cells in $|L_{[1,3]}|$:

$$a < ab < abc, \quad b < ab < abc,$$
$$b < bc < abc, \quad c < bc < abc,$$
$$c < ac < abc, \quad a < ac < abc.$$

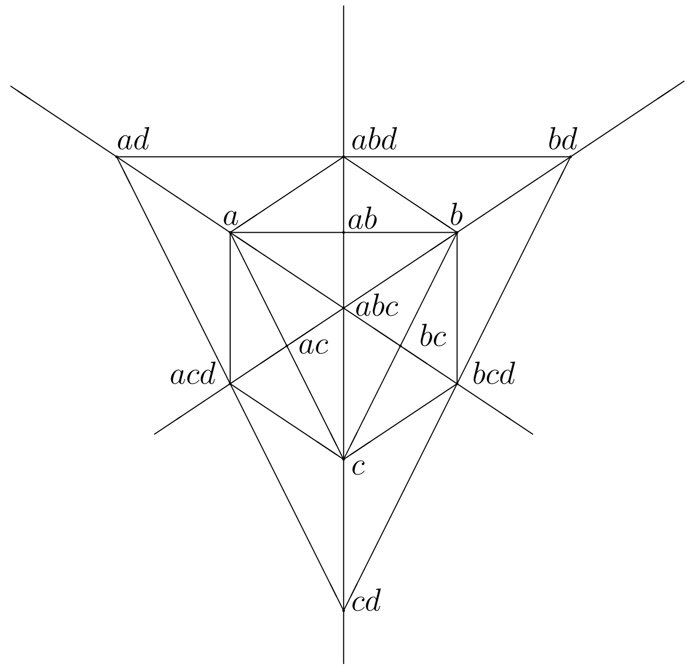


Fig. 4.2.1: $S(abcd)$ in $L_{[1,3]}$

Theorem 4.2.1.

$$\Omega(S(v_1 \dots v_{k+1})) = \sum_j (-1)^{j+1} S(v_1^{v_j} v_2^{v_j} \dots v_{j-1}^{v_j} v_{j+1} \dots v_{k+1}).$$

Proof. We observe that for each j with $2 \leq j \leq k+1$

$$(*) \qquad \sigma = v_1 \dots v_{j-1} v_j v_{j+1} \dots v_{k+1} = v_j v_1^{v_j} v_2^{v_j} \dots v_{j-1}^{v_j} v_{j+1} \dots v_{k+1}$$

and we compute

$$\begin{aligned}
\Omega[S(v_1 \dots v_{k+1})] &= \Omega \left[\sum_{\tau \in \Sigma_{k+1}} \operatorname{sign}(\tau) \left(v_{\tau(1)} < w(\tau(1), \tau(2)) < \dots < w(\tau(1), \dots, \tau(k)) \right) \right] \\
&= \sum_{\tau \in \Sigma_{k+1}} \operatorname{sign}(\tau) \Omega[\left(v_{\tau(1)} < w(\tau(1), \tau(2)) < \dots < w(\tau(1), \dots, \tau(k)) \right)] \\
&= \sum_{\tau \in \Sigma_{k+1}} \operatorname{sign}(\tau) \left(v_{\tau(1)}^{-1} w(\tau(1), \tau(2)) < \dots < v_{\tau(1)}^{-1} w(\tau(1), \dots, \tau(k)) \right).
\end{aligned}$$

Grouping the terms according to the first reflection in the product and applying equation $(*)$ gives the result. $\square$

Example 4.2.2. Consider $\sigma = abcd$, a length four NCP in W as in Example 4.2.1. The six 2-cells around the vertex b are

$$\begin{aligned} b < ab < abd, &\quad - b < ab < abc, \\ b < bd < bcd, &\quad - b < bd < abd, \\ b < bc < abc, &\quad - b < bc < bcd. \end{aligned}$$

The image of each of these cells respectively under Ω is

$$\begin{aligned} a^b < a^b d, &\quad - a^b < a^b c, \\ d < cd, &\quad - d < a^b d, \\ c < a^b c, &\quad - c < cd. \end{aligned}$$

This is $-S(a^b cd)$, a copy of S^1. Grouping the six cells around each of the other three vertices, the image under Ω is $S(bcd), S(a^c b^c d), -S(a^d b^d c^d)$.

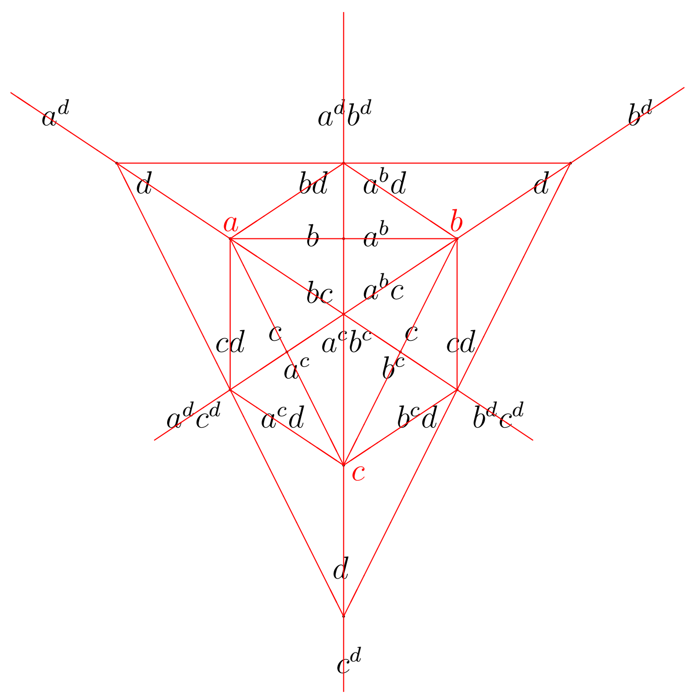


Fig. 4.2.2: Boundary of $S(abcd)$

4.3 A geometric basis for $\tilde{H}_{p-1}(|L_{[1,p]}|)$

Definition 4.3.1. Let σ be the unique rising chain for $(v_1 \dots v_{p+1})^{-1}\gamma$. Then for each p with $1 \leq p \leq n-1$ we define the geometric basis for $\tilde{H}_{p-1}(|L_{[1,p]}|)$ to be the set $\{S(v_1 \dots v_{p+1})\}$ where each of the factorisations $v_1 \dots v_{p+1}\sigma$ is p-decreasing.

Theorem 4.3.1. *The geometric basis is a basis for* $\tilde{H}_{p-1}(|L_{[1,p]}|)$.

Proof. Both the Morse and geometric generators are in one-to-one correspondence with p-decreasing factorisations. Thus we order both sets of generators lexicographically using the total order on reflections. For a specific such factorisation f denote by $M(f)$ and $G(f)$ respectively the corresponding Morse and geometric cycles.

We express each Morse generator $M(f)$ in terms of the geometric basis elements. By Theorem 3.3.2, the facet of $M(f)$ corresponding to f is the only facet whose factorisation is p-decreasing. $G(f)$ also contains this facet but some of its other facets could correspond to other p-decreasing factorisations f'. However, these other p-decreasing factorisations must be lexicographically earlier than f since f is p-decreasing. Repeating this process we find that $M(f) - G(f)$ can be expressed as an integral linear combination of $G(f')$ where f' is lexicographically earlier than f. This means the matrix expressing the ordered set of $M(f)$'s in terms of the ordered set of $G(f)$'s is upper triangular with 1's on the diagonal. This gives $\{G(f) \mid f \text{ } p\text{-decreasing } \}$ a basis for $\tilde{H}_{p-1}(|L_{[1,p]}|)$. $\square$

Example 4.3.1. Recall Example 3.2.1. where we put a Morse function on N in the C_3 case. By Corollary 3.3.4 this also induces a Morse function on $L_{[1,2]}$. The ordering on the reflections was as follows.

number	1	2	3	4	5	6	7	8	9
reflection	$(2,3)$	$[1]$	$(1,\bar{2})$	$(1,\bar{3})$	$[2]$	$(2,\bar{3})$	$(1,2)$	$[3]$	$(1,3)$

Consider the factorisation $f = R_5R_3R_1$. The Morse cycle $M(f)$ consists of the cell $R_5 < R_5R_3 = R_2R_7$ and a complex of non-critical cells that could contract to the critical cell R_1. In this case $M(f) =$

$$R_5 < R_2R_7 \quad -R_5 < R_1R_8 \quad +R_1 < R_1R_8 \quad -R_2 < R_2R_7 \quad +R_2 < R_1R_2 \quad -R_1 < R_1R_2.$$

The cycle $G(f)$ is precisely $S(R_5R_3R_1) =$

$$R_5 < R_2R_7 \quad -R_5 < R_1R_8 \quad +R_3 < R_1R_4 \quad -R_3 < R_2R_7 \quad +R_1 < R_1R_8 \quad -R_1 < R_1R_4.$$

Of these cells, only $R_5 < R_2R_7$ and $R_3 < R_2R_7$ correspond to 2-decreasing factorisations; $R_5R_3R_1$ and $R_3R_2R_1$ respectively. We note that $S(R_3R_2R_1) =$

$$R_3 < R_2R_7 \quad -R_3 < R_1R_4 \quad -R_2 < R_2R_7 \quad +R_2 < R_1R_2 \quad -R_1 < R_1R_2 \quad +R_1 < R_1R_4.$$

None of these other five cells correspond to 2-decreasing factorisations, so the process stops here. We see that the cells in $M(f) - G(f)$ are precisely those in $S(R_3R_2R_1)$.

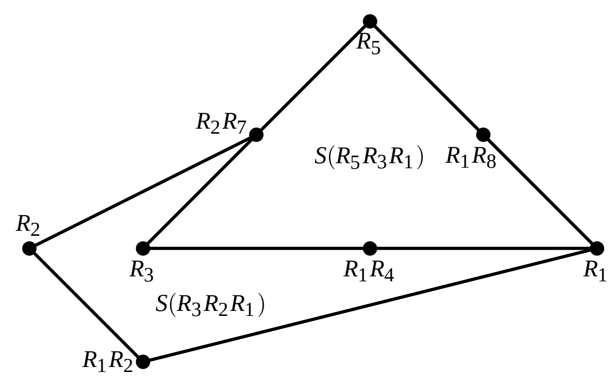


Fig. 4.3.1: Cells in $M(R_5R_3R_1)$

4.4 Syzygies in $C_{p-1}(|L_{[1,p]}|)$.

Recall Corollary 2.4.1, if $\sigma \leq \gamma$ has length two and order m in W, then there are reflections $\tau_1, \tau_2, \ldots, \tau_m$ in W with $\tau_1 < \tau_2 < \cdots < \tau_m$ in the total order on reflections and

$$\sigma = \tau_1\tau_m = \tau_m\tau_{m-1} = \cdots = \tau_2\tau_1. \tag{4.4.1}$$

To compute the homology of the chain complex

$$\tilde{H}_{n-2}(|L_{[1,n-1]}|) \to \tilde{H}_{n-3}(|L_{[1,n-2]}|) \to \ldots \to \tilde{H}_0(|L_{[1,1]}|) \to H_0(X_1)$$

we need to express the image of a geometric generator in $H_{k-1}(|L_{[1,k]}|)$ in terms of geometric generators in $H_{k-2}(|L_{[1,k-1]}|)$. By Theorem 4.3.1, it is sufficient to express an arbitrary $S(v_1 \ldots v_{k+1})$ in terms of those $S(v_1 \ldots v_{k+1})$ corresponding to $k+1$-decreasing sequences. Our approach is to replace each $S(v_1 \ldots v_{k+1})$ with an increasing segment $\sigma = v_jv_{j+1}$ by a combination of $S(v_1 \ldots v_{k+1})$'s where the corresponding σ segments are decreasing. Since the corresponding factorisations are lexicographically later by equation 4.4.1, the process is finite.

Lemma 4.4.1. *Let $S(v_1 \dots v_{k+1})$ be a cycle in $C_{k-1}(|L_{[1,k]}|)$ as above and suppose the product $v_1 \dots v_{k+1}$ factors as $\alpha\sigma\beta$ where σ itself factors as in equation 4.4.1. Then, in $C_{k-1}(|L_{[1,k]}|)$, we have*

$$S(\alpha(\tau_1\tau_m)\beta) + S(\alpha(\tau_m\tau_{m-1})\beta) + \cdots + S(\alpha(\tau_2\tau_1)\beta) = 0.$$

Proof. Consider $S(\alpha(\tau_1\tau_m)\beta)$ first and recall that $S(\alpha(\tau_1\tau_m)\beta)$ is a sum over permutations θ in Σ_{k+1} of the k-cell determined by the permuted reflections with coefficient given by the sign of θ. Fix a specific permutation θ for which τ_m is the ith place and τ_1 is the jth place with $1 \leq i < j \leq k+1$ and consider the corresponding term in the expansion of $S(\alpha(\tau_m\tau_{m-1})\beta)$. Precisely the same k-cell arises in the expansion of $S(\alpha(\tau_m\tau_{m-1})\beta)$ but this time with coefficient $\text{sign}((i,j)\theta) = -\text{sign}(\theta)$ and the corresponding terms cancel.

Similarly, each term of $S(\alpha(\tau_m\tau_{m-1})\beta)$ in which τ_{m-1} appears before τ_m cancels with a corresponding term of $S(\alpha(\tau_{m-1}\tau_{m-2})\beta)$ in which τ_{m-1} appears before τ_{m-2} until eventually each term of $S(\alpha(\tau_2\tau_1)\beta)$ in which τ_1 appears before τ_2 cancels with a corresponding term of $S(\alpha(\tau_1\tau_m)\beta)$ in which τ_1 appears before τ_m. $\square$

Example 4.4.1. Consider the sphere $S(R_2R_7R_1)$ in the C_3 case. Note that $R_2R_7 = R_7R_5 = R_5R_3 = R_3R_2$. Hence by the Lemma we have

$$S(R_2R_7R_1) + S(R_7R_5R_1) + S(R_5R_3R_1) + S(R_3R_2R_1) = 0. \tag{4.4.2}$$

Writing the spheres on the left hand side of equation 4.4.2 in terms of their edges in $|L_{[1,2]}|$ we get:

$$\begin{array}{ccccccc}
+R_2 < R_2R_7 & + & R_7 < R_7R_5 & + & R_5 < R_5R_3 & + & R_3 < R_3R_2 \\
-R_2 < R_2R_1 & - & R_7 < R_7R_1 & - & R_5 < R_5R_1 & - & R_3 < R_3R_1 \\
+R_7 < R_7R_1 & + & R_5 < R_5R_1 & + & R_3 < R_3R_1 & + & R_2 < R_2R_1 \\
-R_7 < R_7R_5 & - & R_5 < R_5R_3 & - & R_3 < R_3R_2 & - & R_2 < R_2R_7 \\
+R_1 < R_1R_2 & + & R_1 < R_1R_9 & + & R_1 < R_1R_8 & + & R_1 < R_1R_4 \\
-R_1 < R_1R_9 & - & R_1 < R_1R_8 & - & R_1 < R_1R_4 & - & R_1 < R_1R_2.
\end{array}$$

The first column are the edges in $S(R_2R_7R_1)$ and three of them cancel with edges in the second column, elements of $S(R_7R_5R_1)$. The remaining three edges in the second column cancel with three in the third column, which are elements of $S(R_5R_3R_1)$. Similarly the remaining three edges in the third column cancel with three edges in $S(R_3R_2R_1)$ and the final three edges of this sphere cancel with the edges remaining in $S(R_2R_7R_1)$. So equation 4.4.2 holds and we can write $S(R_2R_7R_1)$ in terms of spheres with decreasing factorisations.

Note: In the case of $W = A_n$, we use Lemma 4.4.1 in the case $m = 2$ or $m = 3$.

We need one final syzygy in order to write any sphere as a linear combination of basis elements.

Lemma 4.4.2. *Let $v_1 \dots v_{k+2}$ be a factorisation of a length $k+2$ element of L. Then*

$$\sum_{j=1}^{k+2}(-1)^j S(v_1 \dots \hat{v}_j \dots v_{k+2}) = 0. \tag{4.4.3}$$

Proof. Consider the sphere $S(v_1 \dots v_{k+2})$. The cells in this sphere can be split into $k+2$ groups, each of which is the barycentric subdivision of a larger k-cell whose vertex set is $\{v_1, \dots \hat{v}_j, \dots v_{k+2}\}$. Note that the boundary of such a k-cell is simply $S(v_1 \dots \hat{v}_j \dots v_{k+2})$. Since these large k-cells fit together to form a k-sphere, the sum of their boundaries must be 0, that is equation 4.4.3 holds. □

Given a sphere, Lemma 4.4.1 allows us to write it as a combination of spheres with decreasing factorisations. Let $S(v_1 \dots v_{k+1})$ be a sphere with $v_i > v_{i+1}$ for $1 \leq i \leq k$ and $\sigma = v_{k+2} \dots v_n$ be the unique rising chain for $(v_1 \dots v_{k+1})^{-1}\gamma$. If $v_{k+1} < v_{k+2}$ then the factorisation $v_1 \dots v_{k+1}\sigma$ is k-decreasing and $S(v_1 \dots v_{k+1})$ is a member of the geometric basis for $\tilde{H}_{k-1}(|L_{[1,k]}|)$. However if $v_{k+1} > v_{k+2}$ this is not the case, in fact the factorisation is $(k+1)$-decreasing. We apply equation 4.4.3 to write $S(v_1 \dots v_{k+1})$ as a combination other spheres. Note in this case that all the other factorisations are lexicographically earlier, so again this process is finite.

The matlab programmes in the appendix work as follows. First they calculate all factorisations of γ. They use these factorisations and Theorem 4.3.1 to identify a basis for each $\tilde{H}_p(L_{[1,p+1]})$. Theorem 4.2.1 is used to calculate Ω of these basis elements in terms of spheres of a lower dimension. Lemmas 4.4.1 and 4.4.2 are then used to write this as a combination of basis elements of $\tilde{H}_{p-1}(L_{[1,p]})$. We put these results into matrices and find their Smith normal forms to calculate the homology groups.

Chapter 5

Finiteness Properties of groups

The construction from [7]/Chapter 2 gives classifying spaces for subgroups of Artin groups. It is natural to try use these classifying spaces to infer information about the subgroups. For the remainder of this thesis we will concentrate on the case where this subgroup is the commutator subgroup. We will denote the commutator subgroup of G by G'.

Gorin and Lin computed finite presentations for the commutator subgroups of the braid groups, which correspond to the Artin groups $B(A_n)$, in [15]. In [24], Zinde used similar computations to find presentations for the commutator subgroups of the other Artin groups, not all of which are finite presentations. Orevkov adds a finite presentation for the H_3 case in [21] and corrects the claim from [24] that $B(C_3)'$ is a free group on four generators. He notes that $B(C_3)'$ and $B(F_4)'$ are finitely generated but that it is still open whether they are finitely presented. He summarises the presentations of the remaining commutator subgroups as follows.

Theorem 5.0.1. *(Orevkov) The groups $B(I_2(2k))', k \geq 2$ are free groups on countable sets of generators. The commutator subgroups of the remaining irreducible Artin groups, besides the C_3 and F_4 cases, are finitely presented.*

These results essentially use the Reidemeister-Schreier method to compute the presentations. More details of these calculations are given in [19]. (Although they also say $B(C_3)'$ is a free group on 4 generators.) Regarding the C_3 case, Squier shows in [22] that the commutator subgroup is not finitely presented.

We are mainly interested in using our classifying space to show that $B(F_4)'$ is finitely presented; this is done in Chapter 7. In addition, Chapter 6 uses this construction to recover the fact that $B(C_3)'$ is finitely generated but not finitely presented. This current chapter will review some tools we need for these tasks, methods to infer information regarding the presentation of a group from its classifying space.

5.1 Finiteness Properties

For a group Γ, we know that a $K(\Gamma,1)$ exists. See Example 1B.7. of [16], for example. We further classify groups by the existence of $K(\Gamma,1)$'s with particular properties.

Definition 5.1.1. A group Γ has type F_n if there is a $K(\Gamma,1)$ whose n-skeleton is finite.

Γ has type F_1 if and only if Γ is finitely generated. Similarly Γ has type F_2 if and only if Γ is finitely presented. In general, Γ has type F_n if and only if we can find a finite n-dimensional CW complex X with $\pi_1(X)$ isomorphic to Γ and $\pi_i(X)=0$, for $2 \le i \le n-1$.

Example 5.1.1. For any finite W, the groups $B(W)$ are of type F_n for any n by [3] and [8], while the groups $\ker(RL)$ are of type F_n for any n by [7]. On the other hand, we can draw no such immediate conclusions for $B(C_3)'$ and $B(F_4)'$, since, as we will see in Chapters 6 and 7, our classifying space for $\ker(AB)$ is not finite for these two cases. The rest of this thesis will be occupied with finiteness properties of these two groups.

5.2 Hopf's Formula

One way to establish that a group Γ is not finitely presented is to prove that $H_2(\Gamma)$ is not finitely generated. This is the approach we use in Chapter 6 for $\Gamma = B(C_3)'$ and the conclusion is based on the following formula of Hopf.

Theorem 5.2.1. *Suppose a group Γ has a finite presentation F/R, where F is free and R is a normal subgroup of F generated by relation words, then*

$$H_2(\Gamma) = \frac{R \cap [F,F]}{[R,F]}.$$

Our approach will be to compute $H_2(\Gamma)$ as $H_2(X)$ where X is our classifying space for $\ker(AB)$ and $\Gamma = B(C_3)'$. We will also show that Γ is finitely generated by directly finding generators for $\pi_1(X)$.

5.3 Brown's Theorem

Applying the Hopf approach to $B(F_4)'$ did not establish that the group is not finitely presented so we began to wonder if $B(F_4)'$ was, in fact, finitely presented. One approach to showing this is to apply Brown's theorem from [10], or more specifically, Corollary 3.3(b) of that paper. The approach of that paper is to deduce finiteness properties of a group Γ from a filtration of a nice Γ-complex. Essentially, we try to show that the passage from one stage of the filtration to the next is eventually equivalent, up to homotopy, to the adjunction of cells of a fixed dimension.

The finiteness property considered in [10] is defined using projective resolutions, a notion which we will not use in this thesis. We will instead follow section 7.4 of [14] where Brown's Theorem is stated as

Theorem 5.3.1. *(Brown) Let the $(n-1)$-connected free Γ-CW complex Y admit a Γ-filtration $\{K_i\}$ where each $\Gamma\backslash K_i$ has finite n-skeleton. Then Γ has type F_n if and only if $\{K_i\}$ is essentially $(n-1)$-connected, i.e., for each k with $0 \leq k \leq n-1$, the sequence of maps*

$$\{\pi_k(K_0) \to \pi_k(K_1) \to \cdots \to \pi_k(K_i) \to \pi_k(K_{i+1}) \to \dots\}$$

induced by inclusions becomes a sequence of trivial homomorphisms for i large enough.

This is the approach we use in Chapter 7, where Y is the universal cover of our classifying space X for $B(F_4)'$.

Chapter 6

$B(C_3)'$ is not finitely presented.

We now turn our attention to the commutator subgroup of the Artin group of type C_3. We show that the group is finitely generated but not finitely presented.

We recall the notation used in Example 3.1.1 for the elements of L in the C_3 case. We note that the classical presentation of the Artin Group $B(C_3)$ is

$$B(C_3) =< a, b, c \quad | \quad abab = baba, ac = ca, bcb = cbc > .$$

As explained in Chapter 2, the Artin Group has another presentation with a generator $[w]$ for each $w \in L \backslash \{e\}$ and relations $[w_1][w_1^{-1}w_2] = [w_2]$ whenever $w_1 \lneqq w_2$. Note that we will refer to the generator $[w]$ by $\overline{w}$ for the rest of this chapter to avoid awkward notation such as [[1]]. The generators of the classical presentation correspond to the simple reflection generators in the non-crossing partition presentation, that is

$$a = \overline{[1]}, \quad b = \overline{(1,3)}, \quad c = \overline{(2,3)}.$$

We will denote the commutator subgroup of this Artin group by $B(C_3)'$. It is the subgroup generated by all the commutators in $B(C_3)$, that is elements of the form

$$[g, h] = g^{-1}h^{-1}gh, \quad g, h \in B(C_3).$$

Now we follow Section 2.6 to construct $X_{B(C_3)'}$. Since $B(C_3)' \trianglelefteq B(C_3)$ we can form the complex $X_{B(C_3)'} = B(C_3)' \backslash X$ whose cells are of the form $(B(C_3)'g, \sigma)$. Now $B(C_3)'$ is the kernel of the abelianisation homomorphism:

$$\mathrm{AB} : B(C_3) \to B(C_3)/B(C_3)' : \begin{cases} a & \mapsto e_1 \\ b & \mapsto e_2 \\ c & \mapsto e_2 \end{cases}.$$

Note that when we abelianise the Artin group the relation $bcb = cbc$ becomes $b = c$ so the image is $\mathbb{Z} \times \mathbb{Z}$. We write the cells of $X_{B(C_3)'}$ as $((x, y), \sigma)$ where $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ and σ is an initialised chain in L.

We want to study the homotopy type of this space. This would be easier if we could retract it onto a space that is easier to understand. Fortunately we already have an obvious choice for such a retraction. We note that

$$q : X_{B(C_3)'} \to X_{\ker(\mathrm{RL})} : \{(x, y), \sigma\} \mapsto \{x + y, \sigma\}$$

is a covering space map. Further we know that $X_{\ker(\mathrm{RL})}$ retracts onto the subspace N so we can use homotopy lifting to lift this retraction to a subspace of $X_{B(C_3)'}$. Consider the following commutative diagram

$$\begin{tikzcd}
X_{B(C_3)'} \times I \arrow[r, dotted, "h"] \arrow[d, "{(q,id)}"'] \arrow[dr, "g"] & X_{B(C_3)'} \arrow[d, "q"] & q^{-1}(N) \\
X_{ker(p)} \times I \arrow[r, "f"] & X_{ker(p)} \arrow[r, shift left, "r"] & N \arrow[l, shift left, "i"]
\end{tikzcd}$$

The maps i and r are inclusion and retraction maps respectively. The map f is the homotopy with $f(x, 0) = x$, $f(x, 1) = ir(x)$ and $f(n, t) = n$ for all $x \in X_{\ker(\mathrm{RL})}, n \in N$ and $t \in I$. The composition $f \cdot (q, id)$ is denoted by g. Now the identity map on $X_{B(C_3)'}$ is a lift of $g(\cdot, 0)$, that is $q \cdot id = g(\cdot, 0)$. Hence by the homotopy lifting property there exists a unique homotopy $h : X_{B(C_3)'} \times I \to X_{B(C_3)'}$ that lifts g and satisfies $h(\cdot, 0) = id$. So for all $y \in X_{B(C_3)'}$ we have $q \cdot h(y, 1) = g(y, 1) = f(q(y), 1) \in N$ which implies $h(y, 1) \in q^{-1}(N)$. Thus $X_{B(C_3)'}$ and $q^{-1}(N)$ have the same homotopy type.

6.1 A generating set for $B(C_3)'$

Now let us consider this space $N' = q^{-1}(N)$ whose fundamental group is isomorphic to $B(C_3)'$. First we will recover the fact that this group is finitely generated. N' is the union of all cells of the form

$$\{(x,y), e \lessdot w_1 \lessdot w_2\}, \quad x+y=0$$

and their faces. From now on we will refer to the vertices of this complex just by the pair (x,y), rather than the full name $((x,y),\{e\})$. We let the *star* of a vertex v, denoted $st(v)$, consist of every cell that contains v as a vertex and every cell which is a facet of such a cell. Consider the star of the vertex $(0,0)$ illustrated in Figure 6.1.1 .

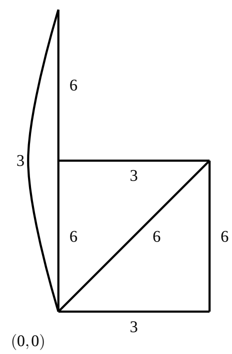


Fig. 6.1.1: $st(0,0)$ in N'

Of the nine reflection generators in $B(C_3)$ the map AB sends three of them, $\overline{[1]}, \overline{[2]}$ and $\overline{[3]}$, to e_1 and the other six to e_2. Of the nine generators corresponding to length 2 non-crossing partitions, three of them, $\overline{(1,2,3)}, \overline{(1,2,\bar{3})}$ and $\overline{(1,\bar{2},\bar{3})}$, are sent to $2e_2$ by AB and the remaining six sent to e_1+e_2. Thus the labels in Figure 6.1.1 refer to the number of edges $((x,y)\{e<w\})$ joining each pair of vertices. We can see there are three types of 2-cells in this subcomplex, the first joining the vertices $(0,0),(1,0)$ and $(1,1)$, the second joining the vertices $(0,0),(0,1)$ and $(1,1)$ and the third joining the vertices $(0,0),(0,1)$ and $(0,2)$. There are nine 2 cells of each type.

To build up N' we sew in another copy of this subcomplex for each simplex (x,y) with $x+y=0$. We will examine the fundamental group of N' in the same way by calculating the fundamental group of $st(0,0)$ and then adding in subsequent copies of this cell structure. The space N' is illustrated in Figure 6.1.2, note that the blue lines here emphasize the six edges where each pair of adjacent stars intersect.

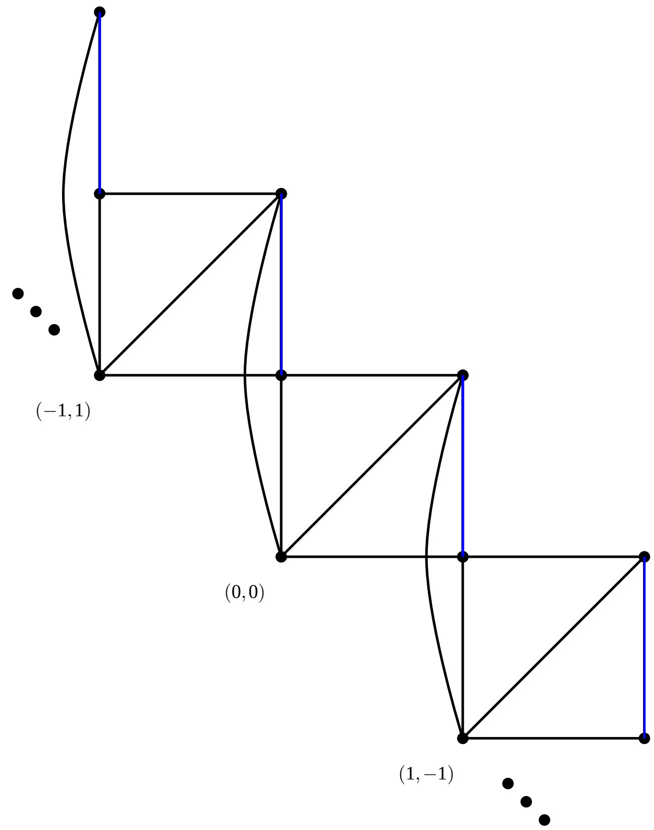


Fig. 6.1.2: Illustration of N'

6.1.1 The fundamental group of the star of $(0,0)$.

We use the following method to compute the fundamental group of this subcomplex. First we choose a maximal tree of its 1-skeleton. Thus each additional edge defines a loop in the subcomplex and a generator in its π_1. We then look at the 2-cells. The boundary of each 2-cell defines a loop that is now homotopic to the trivial loop. So each 2-cell will define a relation on the generators. Many of these relations will end up identifying generators. Eventually we will show that

Theorem 6.1.1. *The group $\pi_1(st(0,0))$ has a presentation with four generators and two relations.*

Proof. First we will change our notation. Each edge in this subcomplex is of the form

$$((x,y),\{e<s\})$$

where (x,y) is the starting vertex and s is an element of L of length one or two. We will instead refer to each set of edges joining a pair of vertices by a letter in the set

$\{T, U, V, W, X, Y, Z\}$. Figure 6.1.3 indicates which letters are used for each vertex set. A specific edge in one of these sets will be denoted by the subscript s, the non-crossing partition that labels the edge.

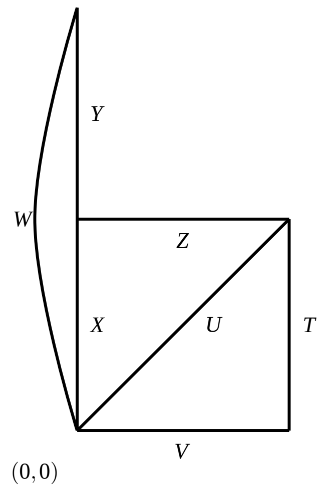


Fig. 6.1.3: Edge labelling in $st(0,0)$

For example we will now refer to the edge $((0,0), e < [2])$ by $V_{[2]}$.

We choose our maximal tree to consist of the edges $X_{(1,2)}, Y_{(1,2)}, V_{[1]}$ and $Z_{[1]}$. The remaining twenty seven edges then determine loops. Table 6.1.1 gives names to each of these generators and says precisely which set of edges determine each loop. We have chosen the vertex $(0,1)$ as the basepoint.

Name	Loop	Possible values of s
$\tilde{T}_s$	$-X_{(1,2)} + V_{[1]} + T_s - Z_{[1]}$	$(1,\bar{3}), (1,3), (1,2), (2,3), (1,\bar{2}), (2,\bar{3})$
$\tilde{U}_s$	$-X_{(1,2)} + U_s - Z_{[1]}$	$[1,3], [1](2,3), [2,3], [1,2], [2](1,\bar{3}), (1,2)[3]$
$\tilde{V}_s$	$-X_{(1,2)} + V_{[1]} - V_s + X_{(1,2)}$	$[2], [3]$
$\tilde{W}_s$	$Y_{(1,2)} - W_s + X_{(1,2)}$	$(1,2,3), (1,2,\bar{3}), (1,\bar{2},\bar{3})$
$\tilde{X}_s$	$-X_{(1,2)} + X_s$	$(1,\bar{3}), (1,3), (2,3), (1,\bar{2}), (2,\bar{3})$
$\tilde{Y}_s$	$Y_{(1,2)} - Y_s$	$(1,\bar{3}), (1,3), (2,3), (1,\bar{2}), (2,\bar{3})$
$\tilde{Z}_s$	$Z_{[1]} - Z_s$	$[2], [3]$

Table 6.1.1: Loops in $st(0,0)$

Table 6.1.2 will deal with the 2-cells. All of the cells start at the vertex $(0,0)$ so we omit this and simply refer to each by its chain in L. We state the boundary of each 2-cell and the relation that occurs when this boundary loop is made trivial.

	2-cell	Boundary	Relation
1	$e < [1] < [1,3]$	$T_{(1,3)} - U_{[1,3]} + V_{[1]}$	$\tilde{T}_{(1,3)} = \tilde{U}_{[1,3]}$
2	$e < [1] < [1,2]$	$T_{(1,2)} - U_{[1,2]} + V_{[1]}$	$\tilde{T}_{(1,2)} = \tilde{U}_{[1,2]}$
3	$e < [1] < [1](2,3)$	$T_{(2,3)} - U_{[1](2,3)} + V_{[1]}$	$\tilde{T}_{(2,3)} = \tilde{U}_{[1](2,3)}$
4	$e < [2] < [1,2]$	$T_{(1,\bar{2})} - U_{[1,2]} + V_{[2]}$	$\tilde{T}_{(1,\bar{2})} = \tilde{V}_{[2]} + \tilde{U}_{[1,2]}$
5	$e < [2] < [2,3]$	$T_{(2,3)} - U_{[2,3]} + V_{[2]}$	$\tilde{V}_{[2]} = \tilde{T}_{(2,3)} - \tilde{U}_{[2,3]} = \tilde{U}_{[1](2,3)} - \tilde{U}_{[2,3]}$
6	$e < [2] < [2](1,\bar{3})$	$T_{(1,\bar{3})} - U_{[2](1,\bar{3})} + V_{[2]}$	$\tilde{T}_{(1,\bar{3})} = \tilde{V}_{[2]} + \tilde{U}_{[2](1,\bar{3})}$
7	$e < [3] < [1,3]$	$T_{(1,\bar{3})} - U_{[1,3]} + V_{[3]}$	$\tilde{T}_{(1,\bar{3})} = \tilde{V}_{[3]} + \tilde{U}_{[1,3]}$
8	$e < [3] < (1,2)[3]$	$T_{(1,2)} - U_{(1,2)[3]} + V_{[3]}$	$\tilde{V}_{[3]} = \tilde{T}_{(1,2)} - \tilde{U}_{(1,2)[3]} = \tilde{U}_{[1,2]} - \tilde{U}_{(1,2)[3]}$
9	$e < [3] < [2,3]$	$T_{(2,\bar{3})} - U_{[2,3]} + V_{[3]}$	$\tilde{T}_{(2,\bar{3})} = \tilde{V}_{[3]} + \tilde{U}_{[2,3]}$
10	$e < (2,3) < (2,3)[1]$	$-X_{(2,3)} + U_{(2,3)[1]} - Z_{[1]}$	$\tilde{U}_{(2,3)[1]} = \tilde{X}_{(2,3)}$
11	$e < (2,3) < [2,3]$	$-X_{(2,3)} + U_{[2,3]} - Z_{[3]}$	$\tilde{U}_{[2,3]} = \tilde{X}_{(2,3)} - \tilde{Z}_{[3]}$
12	$e < (1,3) < [1,3]$	$-X_{(1,3)} + U_{[1,3]} - Z_{[3]}$	$\tilde{Z}_{[3]} = -\tilde{U}_{[1,3]} + \tilde{X}_{(1,3)} = -\tilde{X}_{(1,\bar{3})} + \tilde{X}_{(1,3)}$
13	$e < (1,\bar{3}) < [1,3]$	$-X_{(1,\bar{3})} + U_{[1,3]} - Z_{[1]}$	$\tilde{U}_{[1,3]} = \tilde{X}_{(1,\bar{3})}$
14	$e < (1,\bar{3}) < [2](1,\bar{3})$	$-X_{(1,\bar{3})} + U_{[2](1,\bar{3})} - Z_{[2]}$	$\tilde{U}_{[2](1,\bar{3})} = \tilde{X}_{(1,\bar{3})} - \tilde{Z}_{[2]}$
15	$e < (1,\bar{2}) < [1,2]$	$-X_{(1,\bar{2})} + U_{[1,2]} - Z_{[1]}$	$\tilde{U}_{[1,2]} = \tilde{X}_{(1,\bar{2})}$
16	$e < (1,2) < [1,2]$	$-X_{(1,2)} + U_{[1,2]} - Z_{[2]}$	$\tilde{Z}_{[2]} = -\tilde{U}_{[1,2]} = -\tilde{X}_{(1,\bar{2})}$
17	$e < (1,2) < (1,2)[3]$	$-X_{(1,2)} + U_{(1,2)[3]} - Z_{[3]}$	$\tilde{U}_{(1,2)[3]} = -\tilde{Z}_{[3]}$
18	$e < (2,\bar{3}) < [2,3]$	$-X_{(2,\bar{3})} + U_{[2,3]} - Z_{[2]}$	$\tilde{U}_{[2,3]} = \tilde{X}_{(2,\bar{3})} - \tilde{Z}_{[2]}$
19	$e < (2,3) < (1,\bar{2},\bar{3})$	$Y_{(1,\bar{3})} - W_{(1,\bar{2},\bar{3})} + X_{(2,3)}$	$\tilde{W}_{(1,\bar{2},\bar{3})} = \tilde{Y}_{(1,\bar{3})} - \tilde{X}_{(2,3)}$
20	$e < (2,3) < (1,2,3)$	$Y_{(1,3)} - W_{(1,2,3)} + X_{(2,3)}$	$\tilde{W}_{(1,2,3)} = \tilde{Y}_{(1,3)} - \tilde{X}_{(2,3)}$
21	$e < (1,\bar{3}) < (1,\bar{2},\bar{3})$	$Y_{(1,\bar{2})} - W_{(1,\bar{2},\bar{3})} + X_{(1,\bar{3})}$	$\tilde{Y}_{(1,\bar{2})} = \tilde{W}_{(1,\bar{2},\bar{3})} + \tilde{X}_{(1,\bar{3})}$
22	$e < (1,\bar{3}) < (1,2,\bar{3})$	$Y_{(1,2)} - W_{(1,2,\bar{3})} + X_{(1,\bar{3})}$	$\tilde{W}_{(1,2,\bar{3})} = -\tilde{X}_{(1,\bar{3})}$
23	$e < (1,2) < (1,2,3)$	$Y_{(2,3)} - W_{(1,2,3)} + X_{(1,2)}$	$\tilde{Y}_{(2,3)} = \tilde{W}_{(1,2,3)} = \tilde{Y}_{(1,3)} - \tilde{X}_{(2,3)}$
24	$e < (1,2) < (1,2,\bar{3})$	$Y_{(2,\bar{3})} - W_{(1,2,\bar{3})} + X_{(1,2)}$	$\tilde{Y}_{(2,\bar{3})} = \tilde{W}_{(1,2,\bar{3})}$
25	$e < (1,3) < (1,2,3)$	$Y_{(1,2)} - W_{(1,2,3)} + X_{(1,3)}$	$\tilde{W}_{(1,2,3)} = -\tilde{X}_{(1,3)}$
26	$e < (1,\bar{2}) < (1,\bar{2},\bar{3})$	$Y_{(2,3)} - W_{(1,\bar{2},\bar{3})} + X_{(1,\bar{2})}$	$\tilde{W}_{(1,\bar{2},\bar{3})} = \tilde{Y}_{(2,3)} - \tilde{X}_{(1,\bar{2})}$
27	$e < (2,\bar{3}) < (1,2,\bar{3})$	$Y_{(1,\bar{3})} - W_{(1,2,\bar{3})} + X_{(2,\bar{3})}$	$\tilde{Y}_{(1,\bar{3})} = \tilde{W}_{(1,2,\bar{3})} + \tilde{X}_{(2,\bar{3})}$

Table 6.1.2: 2-cells in $st(0,0)$

We see that relations $1-4$, 7 and 9 express the six $\tilde{T}$ generators in terms of the $\tilde{U}$ and $\tilde{V}$. Relations 5 and 8 then write the $\tilde{V}$ generators in terms of $\tilde{U}$. Next relations 10, 11, 13 - 15 and 17 define these $\tilde{U}$ loops in terms of the $\tilde{X}$ and $\tilde{Z}$. Relations 12 and 16 express the two $\tilde{Z}$ generators in terms of the $\tilde{X}$. The relations 19, 20 and 22 write the $\tilde{W}$ generators as a combination of the $\tilde{X}$ and $\tilde{Y}$. After a bit of simplification, relations 21, 23 - 25 and 27 give the following formulas for the $\tilde{Y}$ generators in terms of the $\tilde{X}$.

$$\begin{array}{ll}
(24) & \tilde{Y}_{(2,\bar{3})} = -\tilde{X}_{(1,\bar{3})} \\
(25) & \tilde{Y}_{(1,3)} = -\tilde{X}_{(1,3)} + \tilde{X}_{(2,3)} \\
(23) & \tilde{Y}_{(2,3)} = -\tilde{X}_{(1,3)} \\
(27) & \tilde{Y}_{(1,\bar{3})} = -\tilde{X}_{(1,\bar{3})} + \tilde{X}_{(2,\bar{3})} \\
(21) & \tilde{Y}_{(1,\bar{2})} = -\tilde{X}_{(1,\bar{3})} + \tilde{X}_{(2,\bar{3})} - \tilde{X}_{(2,3)} + \tilde{X}_{(1,\bar{3})}
\end{array}$$

So, we are left with the five $\tilde{X}$ generators and the three relations 6, 18 and 26. Writing every generator in relation 26 in terms of the $\tilde{X}$, we have

$$\tilde{X}_{(1,\bar{2})} = \tilde{X}_{(2,3)} - \tilde{X}_{(2,\bar{3})} + \tilde{X}_{(1,\bar{3})} - \tilde{X}_{(1,3)}.$$

Similarly, relation 18 gives

$$\tilde{X}_{(1,\bar{2})} = -\tilde{X}_{(2,\bar{3})} + \tilde{X}_{(2,3)} - \tilde{X}_{(1,3)} + \tilde{X}_{(1,\bar{3})}.$$

Setting these two expressions equal to one another gives one relation between the four generators. Finally, using these expressions for $\tilde{X}_{(1,\bar{2})}$, relation 6 simplifies to

$$-\tilde{X}_{(2,3)} + \tilde{X}_{(2,\bar{3})} + \tilde{X}_{(2,3)} - \tilde{X}_{(1,\bar{3})} + \tilde{X}_{(1,3)} - \tilde{X}_{(2,3)} + \tilde{X}_{(1,\bar{3})} - \tilde{X}_{(2,\bar{3})} + \tilde{X}_{(2,3)} - \tilde{X}_{(1,3)} = 0.$$

Hence $\pi_1(st(0,0))$ has four generators and two relations. □

6.1.2 The fundamental group of N'

We are now in a position to show that $B(C_3)'$ is finitely generated.

Theorem 6.1.2. *The fundamental group of N' is generated by the set of four generators for $\pi_1(st(0,0))$.*

Proof. Let $\hat{A} = st(0,0)$, $\hat{B} = st(1,-1)$ and P be an open neighbourhood of $A \cap B$ that retracts onto $A \cap B$. Then $A = \hat{A} \cup P$ and $B = \hat{B} \cup P$ are open sets in $A \cup B$ that retract onto $\hat{A}$ and $\hat{B}$ respectively. We use van Kampen's theorem to calculate the fundamental group of $A \cup B$ from $\pi_1(A)$ and $\pi_1(B)$. Choose $(1,0)$ as the basepoint since it is common to both A and B. Let $i_{AB} : \pi_1(A \cap B) \to \pi_1(A)$ be the homomorphism induced by the inclusion $A \cap B \to A$ and i_{BA} be defined similarly. Then van Kampen's theorem says that

$$\pi_1(A \cup B) \cong \frac{\pi_1(A) * \pi_1(B)}{G},$$

where G is the subgroup generated by all elements of the form $i_{AB}(\omega) i_{BA}(\omega)^{-1}$ for $\omega \in \pi_1(A \cap B)$. We will show that these relations allow us to write the generators of $\pi_1(A)$ in terms of the generators of $\pi_1(B)$ and vice versa.

We will use the same notation as before to refer to edges and loops and a superscript to denote whether we are talking about the subcomplex A or B. Note that the edges that make up a loop in A are those given in Table 6.1.1 conjugated by $-V_{[1]} + X_{(1,2)}$, to deal with the change in basepoint. We consider the four generators of B. Note that we can invert the $\tilde{Y}$ equations (24), (25), (23), (27) and (21) above to get

$$\begin{aligned}
\tilde{X}_{(1,\bar{3})} &= -\tilde{Y}_{(2,\bar{3})}, \\
\tilde{X}_{(1,3)} &= -\tilde{Y}_{(2,3)}, \\
\tilde{X}_{(2,3)} &= -\tilde{Y}_{(2,3)} + \tilde{Y}_{(1,3)}, \\
\tilde{X}_{(2,\bar{3})} &= -\tilde{Y}_{(2,\bar{3})} + \tilde{Y}_{(1,\bar{3})}.
\end{aligned}$$

So we can use these four $\tilde{Y}^B$ loops as the generators of $\pi_1(B)$ instead of the $\tilde{X}^B$.

Choose $T^A_{(1,2)} = Y^B_{(1,2)}$ to be the maximal tree in $A \cap B$. Then the generators of $\pi_1(A \cap B)$ are

$$T^A_{(1,2)} - T^A_s = Y^B_{(1,2)} - Y^B_s = \tilde{Y}^B_s$$

where $s \in \{(1,\bar{2}), (1,3)(1,\bar{3}), (2,3), (2,\bar{3})\}$. The van Kampen relations identify the left and right hand sides of this equation. The left hand side can be written as

$$\begin{aligned}
T^A_{(1,2)} - T^A_s &= (T^A_{(1,2)} - Z^A_{[1]} - X^A_{(1,2)} + V^A_{[1]}) - (V^A_{[1]} + X^A_{(1,2)} + Z^A_{[1]} - T^A_s) \\
&= \tilde{T}^A_{(1,2)} - \tilde{T}^A_s.
\end{aligned}$$

We have already seen that the $\tilde{T}$ loops are homotopic to some combination of the $\tilde{X}$, hence these $\tilde{Y}^B$ generators can also be written in terms of the $\tilde{X}^A$.

To prove the other direction we note that the $\tilde{X}$ can be written in terms of the $\tilde{T}$. The cells 10 - 18 in Table 6.1.2 write the four $\tilde{X}$ and two $\tilde{Z}$ in terms of the $\tilde{U}$. They also write $\tilde{U}_{(2,3)[1]}$, $\tilde{U}_{[2](1,\bar{3})}$ in terms of the other four $\tilde{U}$ and put one relation between these four $\tilde{U}$. Then cells 1 - 9 express these four $\tilde{U}$ along with the two $\tilde{V}$ in terms of the $\tilde{T}$. They also give

$$\tilde{T}_{(1,\bar{2})} = \tilde{T}_{(1,\bar{3})} - \tilde{T}_{(1,3)}, \qquad \tilde{T}_{(2,\bar{3})} = \tilde{T}_{(1,\bar{2})} + \tilde{T}_{(2,3)} - \tilde{T}_{(1,\bar{2})} + \tilde{T}_{(1,2)}$$

and put one more relation between the remaining $\tilde{T}$. We show that the van Kampen relations can be used to write the $\tilde{T}^A$ in terms of the $\tilde{Y}^B$.

Start with the van Kampen relation

$$\begin{aligned}
\tilde{T}^A_{(1,2)} &= \tilde{Y}^B_{(2,\bar{3})} + \tilde{T}^A_{(2,\bar{3})} \\
&= \tilde{Y}^B_{(2,\bar{3})} + \tilde{T}^A_{(1,\bar{2})} + \tilde{T}^A_{(2,3)} - \tilde{T}^A_{(1,\bar{2})} + \tilde{T}^A_{(1,2)} \\
\Rightarrow \quad 0 &= \tilde{Y}^B_{(2,\bar{3})} + \tilde{T}^A_{(1,\bar{2})} + \tilde{T}^A_{(2,3)} - \tilde{T}^A_{(1,\bar{2})} \\
\Rightarrow \tilde{T}^A_{(2,3)} &= -\tilde{T}^A_{(1,\bar{2})} - \tilde{Y}^B_{(2,\bar{3})} + \tilde{T}^A_{(1,\bar{2})}.
\end{aligned}$$

Next we note that $\tilde{T}^A_{(1,2)} = \tilde{Y}^B_{(1,\bar{2})} + \tilde{T}^A_{(1,\bar{2})}$ and thus

$$\begin{aligned}
\tilde{T}^A_{(1,2)} &= \tilde{Y}^B_{(2,3)} + \tilde{T}^A_{(2,3)} \\
\Rightarrow \tilde{Y}^B_{(1,\bar{2})} + \tilde{T}^A_{(1,\bar{2})} &= \tilde{Y}^B_{(2,3)} - \tilde{T}^A_{(1,\bar{2})} - \tilde{Y}^B_{(2,\bar{3})} + \tilde{T}^A_{(1,\bar{2})} \\
\Rightarrow \tilde{T}^A_{(1,\bar{2})} &= -\tilde{Y}^B_{(2,\bar{3})} - \tilde{Y}^B_{(1,\bar{2})} + \tilde{Y}^B_{(2,3)}.
\end{aligned}$$

This gives allows us to express $\tilde{T}^A_{(1,2)}$ in terms of the $\tilde{Y}^B$ and once we have that the other van Kampen relations easily allow us to write the other $\tilde{T}^A$, and hence the $\tilde{X}^A$, as a combination of the $\tilde{Y}^B$.

Note that there are five relations between these generators of $\pi_1(A \cup B)$: two from $\pi_1(A)$, two from $\pi_1(B)$ and one from the fifth van Kampen relation.

Now consider the generalised van Kampen theorem from [12]. Define

$$\hat{E}_n = \bigcup_{k=-n}^{n} st(k,-k),$$

P_1 as an open neighbourhood of $E_n \cap st(n+1,-n-1)$ that retracts onto this subspace, P_2 as an open neighbourhood of $E_n \cap st(-n-1,n+1)$ that retracts onto this subspace and $E_n = \hat{E}_n \cup P_1 \cup P_2$. Then the spaces E_n are open subsets of N', all of which include the basepoint $(0,0)$, such that $N' = \bigcup E_n$. By the generalised van Kampen theorem $\pi_1(N')$ is the direct limit of the system $(\pi_1(E_n), \theta_{ij})$, where the homomorphisms $\theta_{ij} : \pi_1(E_i) \to \pi_1(E_j)$ are induced by inclusion. By repeated application of the argument above for a pair of stars, these homomorphisms are onto and it follows that $\pi_1(N')$ is generated by the four generators of $\pi_1(st(0,0))$.

□

6.2 $B(C_3)'$ is not finitely presented.

As explained in Chapter 5, we can show that $B(C_3)'$ is not finitely presented by examining the second homology group of N'. We now calculate $H_2(N')$ and show it is not finitely generated. First we will look at the homology of $st(0,0)$, as we did for the fundamental group, and then we will use Mayer-Vietoris sequences to add on additional copies of this subcomplex. We used Matlab to calculate $H_2(st(0,0))$ and show it was equal to $\mathbb{Z}^2$. The table of 2-cells in the previous section gives the boundary of each 2-cell. We put these boundaries into a matrix and computed its Smith Normal Form. This showed that there were two generators for the kernel of the map $\partial : C_2 \to C_1$. Since there are no 3-cells, there is no torsion and $H_2(st(0,0)) = \mathbb{Z}^2$.

Theorem 6.2.1. *Fix $n \geq 0$, let $D_n = \bigcup_{k=0}^{n} st(k,-k)$ be the union of $n+1$ stars. Then $H_2(D_n) = \mathbb{Z}^{3n+2}$.*

Proof. We use induction and note that the case $n = 0$ holds since we have seen that $H_2(st(0,0)) = \mathbb{Z}^2$. For the general step define D_{n+1} as the union of

$$A = \bigcup_{k=0}^{n} st(k,-k) \quad \text{and} \quad B = st(n+1,-n-1).$$

We recall the Mayer-Vietoris sequence for A and B.

$$\cdots \to H_2(A \cap B) \to H_2(A) \oplus H_2(B) \to H_2(A \cup B)$$

$$\to H_1(A \cap B) \to H_1(A) \oplus H_1(B) \to H_1(A \cup B) \to \dots$$

The maps in this sequence are defined as

$$\phi_n : H_n(A \cap B) \to H_n(A) \oplus H_n(B) : x \mapsto (x, -x)$$
$$\psi_n : H_n(A) \oplus H_n(B) \to H_n(A \cup B) : (a, b) \mapsto a + b$$
$$\delta_n : H_n(A \cup B) \to H_{n-1}(A \cap B) : x \mapsto \partial a$$

where we write x, a cycle in $A \cup B$, to be a sum $a + b$ of chains in A and B and we have $\partial a = -\partial b$ since $\partial(a + b) = 0$.

There are no 2-cells in $A \cap B$ so

$$0 \to H_2(A) \oplus H_2(B) \xrightarrow{\psi_2} H_2(A \cup B) \to \dots$$

is the first part of this sequence. We know that $H_2(A) = \mathbb{Z}^{3n+2}$ by the induction assumption and $H_2(B) = \mathbb{Z}^2$. Since the sequence is exact we have $\ker(\psi_2) = 0$ and these $3n+4$ homology generators are embedded in $H_2(A \cup B)$.

There is one more homology generator coming from the intersection between A and B. Let C_i be the 2-cell described in row i of Table 6.1.2. In $st(n, -n)$ we have that

$$\partial(C_1 - C_2 + C_4 - C_5 - C_7 + C_9) = T_{(1,3)} - T_{(1,2)} + T_{(1,\bar{2})} - T_{(2,3)} - T_{(1,\bar{3})} + T_{(2,\bar{3})}.$$

In B we have

$$\partial(C_{20} - C_{22} + C_{21} - C_{23} - C_{19} + C_{24}) = Y_{(1,3)} - Y_{(1,2)} + Y_{(1,\bar{2})} - Y_{(2,3)} - Y_{(1,\bar{3})} + Y_{(2,\bar{3})}.$$

Subtracting the cells in $st(-n, n)$ from those in B we have a cycle C. We know that this cycle is not in $\mathrm{Im}(\psi_2)$ since it is not in the kernel of δ_2. Hence this is a new homology class in $H_2(A \cup B)$.

Finally we note that there are no homology generators missing from this description. The next part of the Mayer-Vietoris sequence is

$$\cdots \to H_2(A \cup B) \xrightarrow{\delta_2} H_1(A \cap B) \xrightarrow{\phi_1} H_1(A) \oplus H_1(B) \to \ldots$$

where we know that $H_1(A \cap B) = \mathbb{Z}^5$. We also know that $H_1(B) = \mathbb{Z}^4$ as it is the abelianisation of its fundamental group. These four generators for $H_1(B)$ are clearly in the image of ϕ_1. Suppose there was an additional generator $\hat{C}$ of $H_2(A \cup B)$ that is not a linear combination of the previous generators. Then $\hat{C} \notin \mathrm{Im}(\psi_2)$ which means that $\delta_2(\hat{C}) \neq 0$. Both $\delta_2(C)$ and $\delta_2(\hat{C})$ are in $\ker(\phi_1)$ and they are linear independent. This would contradict the rank-nullity theorem for $H_1(A \cap B)$. Thus

$$H_2(A \cup B) = \mathbb{Z}^{3n+2} \oplus \mathbb{Z}^2 \oplus \mathbb{Z} = \mathbb{Z}^{3n+5}.$$

□

Corollary 6.2.1. *$B(C_3)'$ is not finitely presented*

Proof. Consider the directed system $(H_2(\hat{E}_n), \theta_{ij})$ where

$$\hat{E}_n = \bigcup_{k=-n}^{n} st(k, -k)$$

as in Corollary 6.1.2 and $\theta_{ij} : H_2(\hat{E}_i) \to H_2(\hat{E}_j)$ is the homomorphism induced by the inclusion $\hat{E}_i \to \hat{E}_j$. Note that the space $\hat{E}_n$ is isomorphic to D_{2n} and so it is clear from

Theorem 6.2.1 that the θ_{ij} are embeddings. Hence the direct limit of the system is not finitely generated. We note that N' is the direct limit of the spaces $\hat{E}_n$ and that each compact set of N' is contained in some $\hat{E}_n$. By Proposition 3.33 of [16] we have that the natural map

$$\lim_{\rightarrow} H_2(\hat{E}_n) \to H_2(N')$$

is an isomorphism. It follows that $H_2(N')$ is also not finitely generated and then by Hopf's formula, $\pi_1(N')$ is not finitely presented. $\square$

Chapter 7

$B(F_4)'$ is finitely presented.

7.1 Introduction

In this chapter we show that the commutator subgroup of the Artin group $B(F_4)$ is finitely presented but not of type F_3. First we recall how the non-crossing partition lattice is used to construct the space X from chapter 2, which is a $B(F_4)'$ complex. We also recall how it is retracted in [7] to the space $X_{[0,3]}$. We then examine the quotient of this space by $B(F_4)'$ and get an understanding how cells of this quotient relate to the lattice L. We use the quotient to define an appropriate filtration $\{X_j\}$ of $X_{[0,3]}$ that satisfies the hypothesis of Brown's theorem from Chapter 5. At this stage we need to show that we can obtain X_{j+1} from X_j by the adjunction of 3-cells, up to homotopy. Brown's theorem would then tell us that $B(F_4)'$ is of type F_2 but not of type F_3. In order to do this we split the vertices that are added onto X_{j+1} from X_j into four different types. We add these vertices, and the cells that are incident on them, to X_j one type at a time, and inside these types one vertex at a time. It turns out that each vertex cones off a subcomplex in the space. In a series of propositions we examine these four different subcomplexes and show they are homotopy equivalent either to a single 2-sphere or a wedge of four 2-spheres. Coning off these 2-spheres then is equivalent to adjoining 3-cells to the complex, as required.

7.2 The Space

We recall from Chapter 2 the use of the non-crossing partition lattice of a finite Coxeter group to construct a classifying space for the corresponding Artin group. The universal cover X of this classifying space is the abstract simplicial complex with a vertex for each element g of $B(F_4)$. There is a $k-$simplex $\{g_0, ..., g_k\}$ on these vertices if for each i we have $g_i = g_0[w_i]$ where $e < w_1 < ... < w_k$ is a chain in L. We write such a simplex in the form $\{g_0, e < w_1 < ... < w_k\}$. In [7] it is shown that X deformation retracts, equivariantly with respect to the ker(RL) action, to the three dimensional subspace consisting of all the cells of the form

$$(g, e < w_1 < w_2 < w_3), \quad RL(g) = 0, \quad w_3 \neq \gamma$$

and their faces. We will denote this subspace by $X_{[0,3]}$. Since $\ker(\mathrm{AB}) \leq \ker(\mathrm{RL})$, $X_{[0,3]}$ is the universal cover of a classifying space for $\ker(\mathrm{AB}) = B(F_4)'$. We wish to apply Brown's theorem from Chapter 5 to the group $\Gamma = B(F_4)'$ and the space $Y = X_{[0,3]}$. It remains to choose an appropriate filtration of the space that satisfies the hypotheses of this theorem.

7.3 The Quotient

We first examine the quotient space $X_{[0,3]}/B(F_4)'$, which we will denote X'. We will require the images of the subsets in the filtration to all be finite in this quotient.

For each coset $gB(F_4)'$ with $RL(g) \in [0, 3]$ we have a vertex in X'. In the appendix we show that $B(F_4)/B(F_4)' \cong \mathbb{Z} \times \mathbb{Z}$ as in the C_3 case. We labelled the 24 reflections of F_4 by r_i, $1 \leq i \leq 24$, and we defined the abelianisation map, AB on the generating set $\{[r_i]\}$ by

$$\mathrm{AB} : B(F_4) \to B(F_4)/B(F_4)' : [r_i] \mapsto \begin{cases} (0,1) & \text{if } i \text{ is odd} \\ (1,0) & \text{if } i \text{ is even.} \end{cases}$$

So the vertices of X' are the elements (x, y) of $\mathbb{Z} \times \mathbb{Z}$ with $x, y \geq 0$ and $0 \leq x + y \leq 3$. The last condition follows from the fact that $RL(g) = x + y$ for $\mathrm{AB}(g) = (x, y)$.

Note that we will sometimes refer to the non-crossing partitions $\{r_i |\, i \text{ even}\}$ as even reflections and $\{r_i |\, i \text{ odd}\}$ as odd reflections.

We use the function AB to define the quotient map

$$\mathrm{AB}^* : X_{[0,3]} \to X' : \{g_0, \ldots, g_k\} \mapsto \{\mathrm{AB}(g_0), \ldots, \mathrm{AB}(g_k)\}.$$

This allows us to examine the higher dimensional cells of X'. The space has edges of the form $\mathrm{AB}^*(\{g, e < w_1\})$ joining $\mathrm{AB}(g)$ to $\mathrm{AB}(g[w_1])$, labelled by the element $w_1 \neq \gamma$. Note that $\mathrm{AB}(g[w_1]) = \mathrm{AB}(g) + \mathrm{AB}([w_1])$. The appendix describes the different possible values

$\mathrm{AB}([w_i])$	Number of NCPs w_i
(1,0)	12
(0,1)	12
(2,0)	8
(0,2)	8
(1,1)	39
(2,1)	12
(1,2)	12

Table 7.3.1: Abelianisation of the generators $[w_i]$

for $\mathrm{AB}([w_1])$ and the number of non-crossing partition generators that map to each value. This is summarised in Table 7.3.1.

Thus, for example, at every vertex (x, y) in X' with $x + 1 + y \leq 3$, there are 12 edges joining (x, y) to $(x + 1, y)$.

The space has 2-cells of the form $\mathrm{AB}^*(\{g, e < w_1 < w_2\})$. Such a 2-cell has vertices $\{\mathrm{AB}(g), \mathrm{AB}(g)+\mathrm{AB}([w_1], \mathrm{AB}(g)+\mathrm{AB}([w_2])\}$ and its edges are labelled by the appropriate NCPs: w_1, w_2 and $w_1^{-1}w_2$. Table 7.3.1 shows that there are 55 choices for w_2 where $l(w_2) = 2$ and 24 choices for w_2 where $l(w_2) = 3$. The number of different options for w_1 depends on the factorisations of the element w_2. The appendix lists all factorisations as a product of reflections of an element t_i where $l(t_i) = 2$. Table 7.3.2 lists the number of factorisations for these elements.

If $\mathrm{AB}([t_i]) = (2, 0)$ then t_i must be the product of two even reflections and if $\mathrm{AB}([t_i]) = (0, 2)$ then t_i must be the product of two odd reflections. If $\mathrm{AB}[t_i] = (1, 1)$ then half of the factorisations of t_i are of the form $r_{\mathrm{even}}r_{\mathrm{odd}}$ and the other half are of the form $r_{\mathrm{odd}}r_{\mathrm{even}}$. Hence, for example, at every vertex (x, y) in X' with $x + y + 2 \leq 3$, there are 54 2-cells

$\mathrm{AB}([t_i])$	Number of factorisations of t_i	Parabolic subgroup	Number of NCPs t_i
$(2, 0)$	3	A_2	8
$(0, 2)$	3	A_2	8
$(1, 1)$	2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	24
$(1, 1)$	4	C_2	15

Table 7.3.2: The parabolic subgroups of length 2 NCPs

connecting $(x, y), (x+1, y)$ and $(x+1, y+1)$. Of these 24 come from the NCPs in the third row of Table 7.3.2 and 30 of these come from the NCPs in the last row.

The appendix also lists factorisations of elements τ_i, where $l(\tau_i) = 3$. It lists those factorisations of the form $t_j r_k$, where $l(t_j) = 2$ and $l(r_k) = 1$. For each such r_k there is another factorisation of τ_i beginning with r_k and for each t_j there is another factorisation of τ_i that ends in t_j. The number of factorisations is summarised in Table 7.3.3.

In order to illustrate where these numbers come from, we will consider one example from the third row. The NCP τ_2 satisfies $\mathrm{AB}([\tau_2]) = (1, 2)$. It has four factorisations listed in the appendix:

$$t_8 r_{24} = t_9 r_{23} = t_{54} r_1 = t_{55} r_{21}.$$

It also has factorisations

$$r_{24} t_8 = r_{21} t_9 = r_{23} t_{54} = r_1 t_{55}.$$

Note that of the four r_i preceding τ_2, three have i odd and the fourth must have i even. The subgroup generated by $\{r_1, r_{23}, r_{24}\}$ is a parabolic subgroup of F_4 with parabolic Coxeter element τ_2. The corresponding subgroup is isomorphic to $\mathbb{Z}_2 \times A_2$. In the other case, where τ_i has nine reflections listed, τ_i is the parabolic Coxeter element for a subgroup that is isomorphic to C_3.

Table 7.3.3 can also tell us about the 3-cells in X': $\mathrm{AB}^*(\{e < w_1 < w_2 < w_3\})$. Consider the number of 3-cells joining the vertices $(x, y), (x+1, y), (x+2, y), (x+2, y+1)$, where $x + y = 0$. There are twelve choices for w_3. Six of these are the Coxeter elements of parabolic subgroups isomorphic to $\mathbb{Z}_2 \times A_2$. For each of these there is only one choice for w_2 with $\mathrm{AB}([w_2]) = (2, 0)$. For each such w_2 there are three elements w_1 with $w_1 < w_2$. This gives $6 \times 1 \times 3 = 18$ 3-cells. The other six possible w_3 are the Coxeter elements of parabolic subgroups isomorphic to C_3. Each of these has three elements w_2 with $w_2 < w_3$ and $\mathrm{AB}([w_2]) = (2, 0)$. Again for each such w_2 there are three choices for w_1. This makes

$\mathrm{AB}([\tau_i])$	No. of factorisations of τ_i	No. of NCPs τ_i	No. of r_j with $r_j < \tau_i$, j even
$(2, 1)$	4	6	3
$(2, 1)$	9	6	6
$(1, 2)$	4	6	1
$(1, 2)$	9	6	3

Table 7.3.3: Factorisations of length 3 NCPs

a further $6 \times 3 \times 3 = 54$ 3-cells. In total there are 72 different 3-cells joining these vertices and by symmetry there is also 72 3-cells joining the vertices $(x, y), (x, y+1), (x, y+2)$ and $(x+1, y+2)$.

Now consider the 3-cells joining the vertices $(x, y), (x+1, y), (x+1, y+1), (x+2, y+1)$. Again the possibilities for w_3 are split into two groups of six. One group has six w_2 with $w_2 < w_3$ and $\mathrm{AB}([w_2]) = (x+1, y+1)$. Of these w_2, three have parabolic subgroup C_2 and hence two choices of even reflections w_1 that precede w_2. The other three have parabolic subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ and thus only one such w_1. This makes a total of $6 \times (3 \times 2 + 3 \times 1) = 54$ 3-cells. The other group of six w_3 are the parabolic Coxeter elements of copies of $\mathbb{Z}_2 \times A_2$. Each has three elements w_2 which precede it and satisfy $\mathrm{AB}([w_2]) = (1, 1)$. All of these w_2's are products of a pair of commuting reflections so for each there is only one w_1 with $w_1 < w_2$ and $\mathrm{AB}([w_1]) = (1, 0)$. This gives us another $6 \times 3 \times 1 = 18$ 3-cells and together we get another 72 3-cells on these vertices.

Note that for every $w_1 < w_2$ with $\mathrm{AB}([w_1]) = (1, 0)$ and $\mathrm{AB}([w_2]) = (1, 1)$ there will be a $\hat{w}_1 = w_1^{-1} w_2$ such that $\hat{w}_1 < w_2$ and $\mathrm{AB}([\hat{w}_1]) = (0, 1)$. So for every 3-cell described above on the vertices $(x, y), (x+1, y), (x+1, y+1), (x+2, y+1)$ there will also be a 3-cells on the vertices $(x, y), (x, y+1), (x+1, y+1), (x+2, y+1)$. Thus there are a total of 72 3-cells on these vertices too. By symmetry there are also the same number of 3-cells on the vertices $(x, y), (x, y+1), (x+1, y+1), (x+1, y+2)$ and on the vertices $(x, y), (x+1, y), (x+1, y+1), (x+1, y+2)$. To summarise, for any legal choice of vertices there are 72 different 3-cells joining those vertices and a total of 432 3-cells incident on every vertex (x, y) with $x + y = 0$.

7.4 The Filtration

We define the following filtration on the quotient X' corresponding to the filtration of $\mathbb{Z}$ given by the intervals $[-n, n]$.

$$
\begin{aligned}
X'_0 &= st(0,0) \\
X'_1 &= st(-1,1) \cup st(0,0) \cup st(1,-1) \\
&\vdots \\
X'_j &= \bigcup_{x=-j}^{j} st(x,-x) \\
&\vdots
\end{aligned}
$$

Clearly each X'_j is finite since there are a finite number of cells in the star of each vertex, as described in the previous section. We choose our filtration on $X_{[0,3]}$ to be the pre-image of this filtration under the map $AB^* : X_{[0,3]} \to X'$. Thus

$$\begin{aligned}
X_0 &= \bigcup_{AB(g)=(0,0)} st(g), \\
X_1 &= \left(\bigcup_{AB(g)=(-1,1)} st(g) \right) \cup \left(\bigcup_{AB(g)=(0,0)} st(g) \right) \cup \left(\bigcup_{AB(g)=(1,-1)} st(g) \right), \\
&\ \vdots \\
X_j &= \bigcup_{x=-j}^{j} \left(\bigcup_{AB(g)=(x,-x)} st(g) \right), \\
&\ \vdots
\end{aligned}$$

Note that for each vertex $(-x, x)$ in X', there are an infinite number of vertices g in $X_{[0,3]}$ with $\mathrm{AB}(g) = (-x, x)$, one for each element of $B(F_4)'$.

We need to show that X_{j+1} is obtained from X_j by the adjunction of 3-cells up to homotopy. Moving from X_j to X_{j+1} we add the stars of the vertices g with $\mathrm{AB}(g) = (j+1, -j-1)$ or $\mathrm{AB}(g) = (-j-1, j+1)$. We will show that adding $st(g)$ for all g with $\mathrm{AB}(g) = (j+1, -j-1)$ is equivalent to the adjunction of 3-cells. The other case is similar. We do this adjoining in four steps.

There are 8 vertices in the star of a vertex $(j+1, -j-1)$ in X'. These are

$$(j+1, -j-1), (j+2, -j-1), (j+3, -j-1), (j+3, -j),$$

$$(j+1, -j), (j+2, -j), (j+1, -j+1), (j+2, -j+1).$$

Of these the last four belong to X'_j while the first 4 are new in X'_{j+1}. Thus X_{j+1} is the subcomplex of $X_{[0,3]}$ consisting of cells whose vertices g satisfy

$$AB(g) \in X'_j \cup \{(j+1, -j-1), (j+2, -j-1), (j+3, -j), (j+3, -j-1)\}.$$

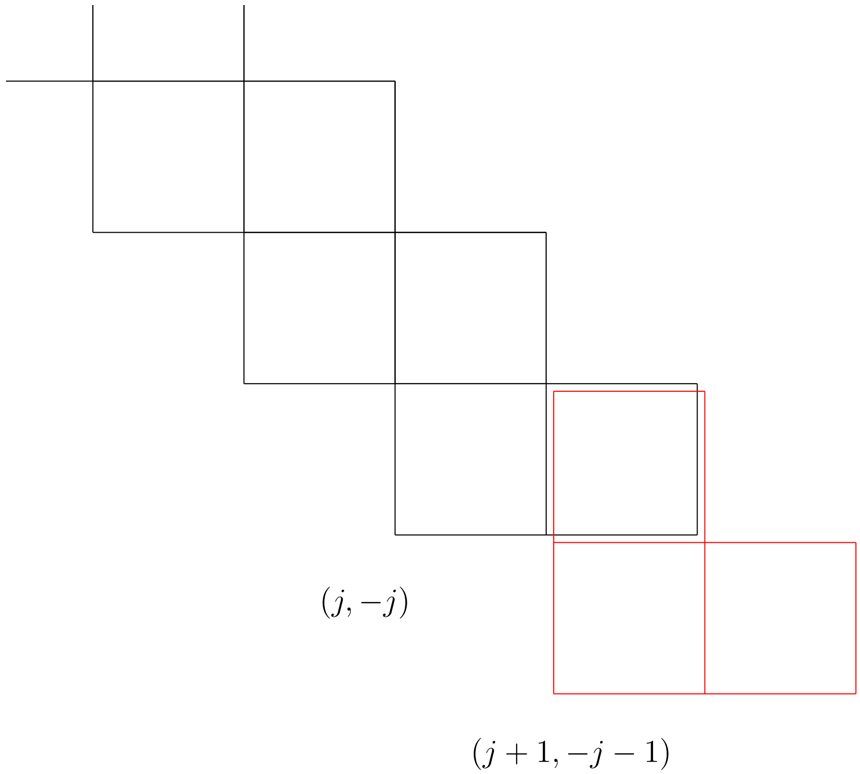


Fig. 7.4.1: Illustration of X'_j and $st(j+1, -j-1)$

Definition 7.4.1. We define the following subcomplexes of X_{j+1}:
$X_{j+1/4}$ is the subcomplex consisting of the cells of X_{j+1} whose vertices g satisfy $AB(g) \in X'_j \cup \{(j+1, -j-1)\}$.
$X_{j+1/2}$ is the subcomplex consisting of the cells of X_{j+1} whose vertices g satisfy $AB(g) \in X'_j \cup \{(j+1, -j-1), (j+2, -j-1)\}$.
$X_{j+3/4}$ is the subcomplex consisting of the cells of X_{j+1} whose vertices g satisfy $AB(g) \in X'_j \cup \{(j+1, -j-1), (j+2, -j-1), (j+3, -j)\}$.

Proposition 7.4.1. *$X_{j+1/4}$ is obtained from X_j by the adjunction of* 3*-cells, up to homotopy.*

Proof. We order the infinitely many vertices g of $X_{[0,3]}$ satisfying $\mathrm{AB}(g) = (j+1, -j-1)$ using some arbitrary total order. We construct $X_{j+1/4}$ from X_j by adjoining each of these vertices g in turn together with the finitely many cells whose vertex set contains g and is contained in $X_j \cup \{g\}$. For such a vertex g, we wish to add in all the cells that start at the vertex g and have the rest of their vertices in X_j. The top facets of these cells are already in X_j, where the top facet of $\sigma = (g, e < w_1 < \cdots < w_k)$ is $(g[w_1], e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k)$. Adding on the cell σ is essentially coning off this top facet. Adding the vertex g and the cells described above is equivalent to coning off the subcomplex $X_j \cap st(g)$, up to homotopy. We will show that this subcomplex is homotopy equivalent to the wedge of four 2-spheres. Coning this off is, up to homotopy, equivalent to adjoining four 3-cells to X_j, as required.

The complex $X_j \cap st(g)$, illustrated in Figure 7.4.2, is 2-dimensional. There are 12 vertices $g[w_1]$ that map to $(j+1, -j)$ under the abelianisation, 12 that map to $(j+2, -j+1)$, 8 that map $(j+1, -j+1)$ and 39 that map to $(j+2, -j)$.

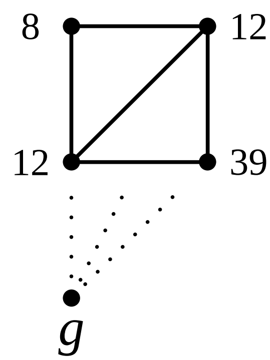


Fig. 7.4.2: The star of g

We will begin our analysis of this intersection by looking at the 1-dimensional graph between the $(j+1, -j)$ vertices and the $(j+2, -j+1)$ ones. We will then add the cells that include the other two types of vertices and examine how this affects the homotopy type. This graph is shown in Figure 7.4.3. Each vertex $g[w_i]$ is simply labelled w_i for brevity. We can see there are 24 vertices in the graph and 54 edges. The blue edges form a maximal tree and each red edge creates a loop, a generator of the fundamental group of the graph, in combination with edges in the maximal tree. There are 31 such loops.

Now consider the effect of adding on the 39 vertices that map to $(j+2, -j)$ and the cells that include these vertices. These vertices are of the form $g[t_i]$ where $\mathrm{AB}([t_i]) = (1, 1)$. Twelve of the corresponding non-crossing partitions; $t_1, t_{13}, t_{21}, t_{27}, t_{32}, t_{36}, t_{40}, t_{44}, t_{47}, t_{50}, t_{53}, t_{55}$, have only one r_{odd} below them and one τ_{even} above them in the lattice L. So each contributes one cell of the form $\{g[r_{odd}] < g[t_i] < g[\tau_{even}]\}$ to the space. Adding these cells does not effect the homotopy type of the graph, as the 2-cell can be collapsed to the edge$\{g[r_{odd}] < g[\tau_{even}]\}$. For example t_1, has the reflection r_1 below it, the length 3 element τ_4 above it and the cell $\{g[r_1] < g[t_1] < g[\tau_4]\}$ collapses to the edge $\{g[r_1] < g[\tau_4]\}$ as show in Figure 7.4.4.

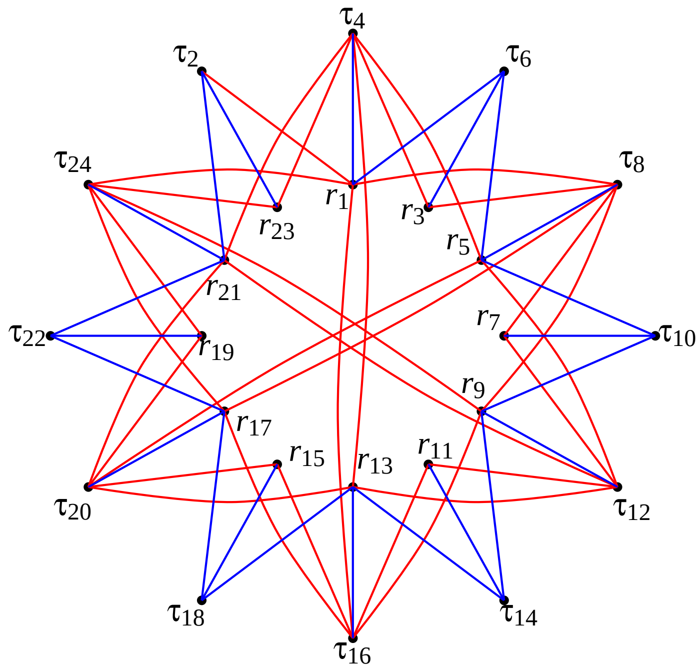


Fig. 7.4.3: Graph of the vertices mapping to $(j+1,-j)$ and $(j+2,-j+1)$

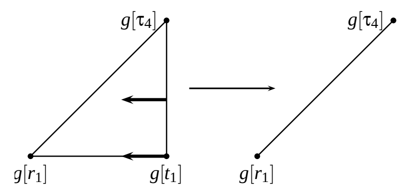


Fig. 7.4.4: Cell $\{g[r_1] < g[t_1] < g[\tau_4]\}$

A further twelve of these non-crossing partitions; $t_3, t_9, t_{15}, t_{23}, t_{28}, t_{34}, t_{37}, t_{42}, t_{45}, t_{49}, t_{51}, t_{54}$, have only one r_{odd} below them and two τ_{even} above them in L. Adding these cells also has no effect on the homotopy type of the graph. For example the element t_3 has the reflection r_1 below it and the length 3 elements τ_{24} and τ_6 above it, contributing the cell $\{g[r_1] < g[t_3] < g[\tau_{24}]\}$, which collapses to the edge $\{g[r_1] < g[\tau_{24}]\}$ and the cell $\{g[r_1] < g[t_3] < g[\tau_6]\}$, which collapses to the edge $\{g[r_1] < g[\tau_6]\}$. This is shown in Figure 7.4.5.

Twelve more of these elements; $t_4, t_7, t_{10}, t_{12}, t_{14}, t_{17}, t_{20}, t_{24}, t_{30}, t_{35}, t_{39}, t_{43}$, have two r_{odd} below them in L and one τ_{even} above them. Following a similar collapse as the previous case, these cells also do not effect the homotopy type of the graph.

That leaves just three of these vertices that do affect the homotopy type of the graph, those being $g[t_5], g[t_{18}]$ and $g[t_{25}]$. Consider t_5, which has r_1 and r_{13} preceding it, while τ_4

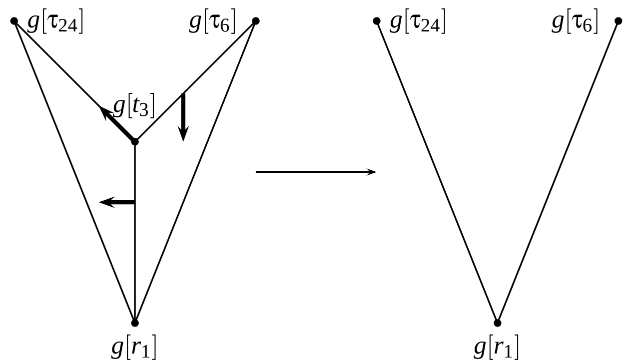


Fig. 7.4.5: Collapse of the cells containing the vertex $g[t_3]$

and τ_{16} come after it in L. These four vertices already form a loop in our graph, shown in Figure 7.4.6. Including the cells $\{g[r_1] < g[t_5] < g[\tau_{16}]\}, \{g[r_1] < g[t_5] < g[\tau_4]\}, \{g[r_{13}] < g[t_5] < g[\tau_{16}]\}$ and $\{g[r_{13}] < g[t_5] < g[\tau_4]\}$ simply cones off this loop and kills it.

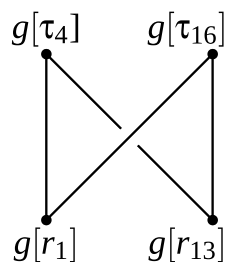


Fig. 7.4.6: Vertices incident on $g[t_5]$

Similarly the 2-cells incident on $g[t_{18}]$ fill in the loop joining $g[r_9], g[r_{21}], g[\tau_{12}]$ and $g[\tau_{24}]$, while the 2-cells incident on $g[t_{25}]$ kill the loop between $g[r_5], g[r_{17}], g[\tau_{20}]$ and $g[\tau_8]$.

We now consider the 8 vertices $g[w_1]$ where $\mathrm{AB}(w_1) = (0, 2)$. Each of these has three r_{odd} below them in L and three τ_{even} above them. This is summarised in Table 7.4.1.

Figure 7.4.7 shows the vertices incident on the vertex $g[t_2]$ and the edges joining them. Recall that the blue edges were part of our maximal tree, while the red edges defined generators in the fundamental group of the graph. By adjoining the star of $g[t_2]$, we cone off this graph, killing these four loops. This pattern repeats for $t_{22}, t_{33}, t_{41}, t_{48}$, we move clockwise around the circle in Figure 7.4.3, killing off four π_1 generators each time.

Figure 7.4.8 shows the graph coned off by the vertex $g[t_8]$. The edge $\{g[r_1] < g[\tau_{24}]\}$ is

t_2	r_1	r_3	r_5	τ_4	τ_6	τ_8
t_{22}	r_5	r_7	r_9	τ_8	τ_{10}	τ_{12}
t_{33}	r_9	r_{11}	r_{13}	τ_{12}	τ_{14}	τ_{16}
t_{41}	r_{13}	r_{15}	r_{17}	τ_{16}	τ_{18}	τ_{20}
t_{48}	r_{17}	r_{19}	r_{21}	τ_{20}	τ_{22}	τ_{24}
t_8	r_{21}	r_{23}	r_1	τ_{24}	τ_2	τ_4
t_6	r_1	r_9	r_{17}	τ_8	τ_{16}	τ_{24}
t_{26}	r_5	r_{13}	r_{21}	τ_4	τ_{12}	τ_{20}

Table 7.4.1: Elements above and below t_i in L, where $\mathrm{AB}([t_i]) = (0, 2)$

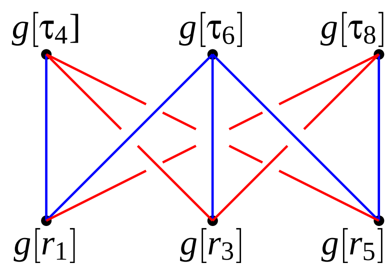


Fig. 7.4.7: Vertices incident on $g[t_2]$

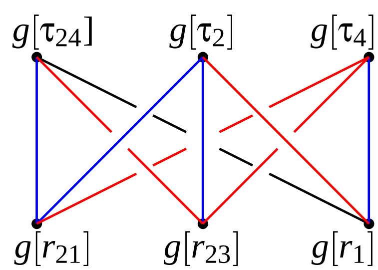


Fig. 7.4.8: Vertices incident on $g[t_8]$

not part of the maximal tree for the larger graph, nor is the loop it creates killed off here.

This leaves seven π_1 generators in the graph, the one contributed by this edge $\{g[r_1] < g[\tau_{24}]\}$ and the six from the edges going from one side of the circle directly to the other in Figure 7.4.3. Figure 7.4.9 shows the graph coned off by the vertex $g[t_6]$. The blue edges were not part of the maximal tree for the graph but a disc has now been glued on along each edge and a portion of this tree. Hence each is homotopic to a path in the maximal tree. Adding these new 2-cells fills in the loops contributed by the four red edges.

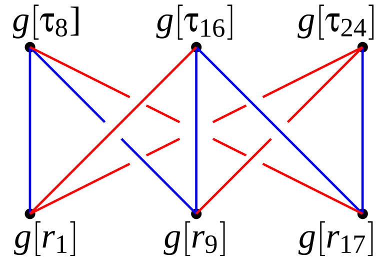


Fig. 7.4.9: Vertices incident on $g[t_6]$

We now have three π_1 generators remaining, contributed by the edges $\{g[r_{13}] < g[\tau_4]\}$, $\{g[r_{21}] < g[\tau_{12}]\}$ and $\{g[r_5] < g_[\tau_{20}]\}$. These correspond to the three loops killed off by the vertices $g[t_5], g[t_{18}]$ and $g[t_{25}]$ that we previously mentioned. The graph now has the homotopy type of a point and the cells attached to $g[t_{26}]$ still have to be added. Now that there are no loops to be killed off, adding these cells gives our space the homotopy type of a wedge of four 2-spheres.

□

Proposition 7.4.2. *$X_{j+1/2}$ is obtained from $X_{j+1/4}$ by the adjunction of 3-cells, up to homotopy.*

Proof. We order the infinitely many vertices g of $X_{[0,3]}$ satisfying $AB(g) = (j+2, -j-1)$ using some arbitrary order. We construct $X_{j+1/2}$ from $X_{j+1/4}$ by adjoining each of these vertices g in turn together with the finitely many cells whose vertex set contains g and is contained in $X_{j+1/4} \cup \{g\}$. The 3-dimensional cells that satisfy this criteria have the form $\{g[r_{even}]^{-1} < g < g[r_{odd}] < g[t_i]\}$ and the facets $\{g[r_{even}]^{-1} < g[r_{odd}] < g[t_i]\}$ are already in $X_{j+1/4}$. We will show that the space $X_{j+1/4} \cap st(g)$ has the homotopy type of a single sphere. Adding the vertex g and these cells is equivalent to coning off this sphere, adjoining a 3-cell to $X_{j+1/4}$.

The complex $X_j \cap st(g)$ is illustrated in Figure 7.4.10. There are twelve vertices $g[r_{even}]^{-1}$, twelve of the form $g[r_{odd}]$ and eight of the form $g[t_i]$ which map to $(j+2, -j+1)$ under the abelianisation.

This space has a very similar structure to the one we studied in the first proof. The 1-dimensional graph between the vertices $g[r_{even}]^{-1}$ and $g[r_{odd}]$ has the same homotopy type as Figure 7.4.3. We repeat this graph below with the appropriate labels for the vertices. In this case the vertex $g[r_{even}]^{-1}$ is simply labelled r_{even}, while the vertex $g[r_{odd}]$ is labelled by

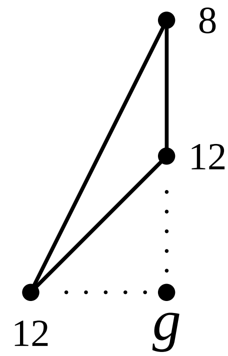


Fig. 7.4.10: The star of g

r_{odd}. Again the blue edges form a maximal tree and each red edge contributes a generator to the fundamental group of the graph.

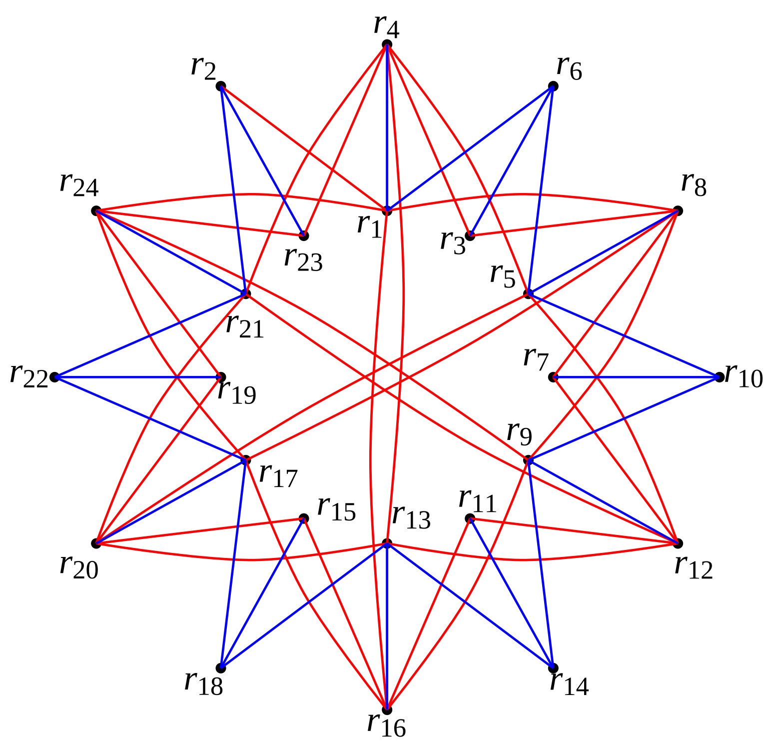


Fig. 7.4.11: Graph of the vertices mapping to $(j+1, -j-1)$ and $(j+2, -j)$

Now we adjoin the eight vertices $g[t_i]$ and the cells incident on them to this graph. Note that these are precisely the same eight t_i with $AB([t_i]) = (0, 2)$ that we encountered at the end of the previous proof. We saw each had three elements r_{odd} preceding them in L

t_2	r_1	r_3	r_5	r_4	r_6	r_8
t_{22}	r_5	r_7	r_9	r_8	r_{10}	r_{12}
t_{33}	r_9	r_{11}	r_{13}	r_{12}	r_{14}	r_{16}
t_{41}	r_{13}	r_{15}	r_{17}	r_{16}	r_{18}	r_{20}
t_{48}	r_{17}	r_{19}	r_{21}	r_{20}	r_{22}	r_{24}
t_8	r_{21}	r_{23}	r_1	r_{24}	r_2	r_4
t_6	r_1	r_9	r_{17}	r_8	r_{16}	r_{24}
t_{26}	r_5	r_{13}	r_{21}	r_4	r_{12}	r_{20}

Table 7.4.2: Labels for the elements in the graph above and below $g[t_i]$

and three elements τ_j above them. Each such τ_j satisfied $AB(\tau_j) = (1, 2)$, so for each we have that $\tau_j = r_k t_i$ for some k even. Hence $g[r_k]^{-1} < g[t_i]$ is an edge in Figure 7.4.11. We summarise the elements $g[r_{odd}]$ and $g[r_{even}]^{-1}$ which are joined to $g[t_i]$ in Table 7.4.2.

Once again adjoining the star of a vertex $g[t_i]$ is equivalent to coning off the graph on the vertices incident with it. This is killing four π_1 generators in the graph. The six subgraphs described by Table 7.4.2 are isomorphic those described at the end of the previous proof. We follow the method there for the first five elements, $t_2, t_{22}, t_{33}, t_{41}$ and t_{48} which collapses all the loops in the graph besides the three contributed by the edges: $\{g[r_4]^{-1} < g[r_{13}]\}$, $\{g[r_{12}]^{-1} < g[r_{21}]\}$ and $\{g[r_{20}]^{-1} < g[r_5]\}$. Figure 7.4.12 shows the graph coned off by the final vertex $g[t_{26}]$. The blue edges were not originally part of the maximal tree for the larger graph, but they are now homotopic to some segment of it. These last three π_1 generators are killed off and in addition one final disc is adjoined to the space, giving it the homotopy type of a 2-sphere.

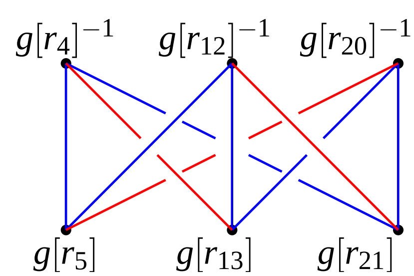


Fig. 7.4.12: Vertices incident on $g[t_{26}]$

□

Proposition 7.4.3. *$X_{j+3/4}$ is obtained from $X_{j+1/2}$ by the adjunction of 3-cells, up to homotopy.*

Proof. We order the vertices g of $X_{[0,3]}$ that satisfy $AB(g) = (j+3, -j)$ in some arbitrary order. We construct $X_{j+3/4}$ from $X_{j+1/2}$ by adjoining each of these vertices g in turn together with the cells whose vertex set contains g and is contained in $X_{j+1/2} \cup \{g\}$. The space $X_{j+1/2} \cap st(g)$ is illustrated in Figure 7.4.13. It is homotopic to the space $X_j \cap st(\hat{g})$, for $\hat{g}$ satisfying $AB(\hat{g}) = (j+1, -j-1)$, that was discussed in the proof of proposition 7.4.1. It was shown that space is homotopic to the wedge of four 2-spheres. Adding each element g cones off these spheres, which is equivalent to adjoining 3-cells to $X_{j+1/2}$.

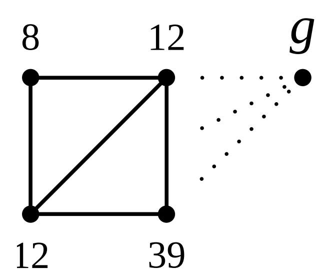


Fig. 7.4.13: Star of g

□

Proposition 7.4.4. *X_{j+1} is obtained from $X_{j+3/4}$ by the adjunction of 3-cells, up to homotopy.*

Proof. We order the vertices g of $X_{[0,3]}$ that satisfy $AB(g) = (j+3, -j-1)$ in some arbitrary order. We construct X_{j+1} from $X_{j+3/4}$ by adjoining each of these vertices g in turn together with the cells whose vertex set contains g and is contained in $X_{j+3/4} \cup \{g\}$. The space $X_{j+3/4} \cap st(g)$ is illustrated in Figure 7.4.14. It is homotopic to the space $X_j \cap st(\hat{g})$, for $\hat{g}$ satisfying $AB(\hat{g}) = (j+2, -j-1)$, that was discussed in the proof of proposition 7.4.2. It was shown that space is homotopic to a single 2-sphere. Adding each element g cones off this sphere, which is equivalent to adjoining a 3-cells to $X_{j+3/4}$.

□

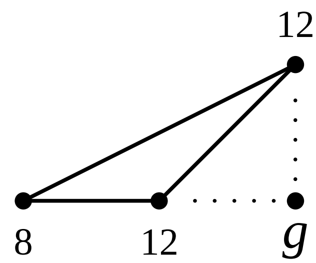


Fig. 7.4.14: Star of g

Theorem 7.4.1. *The group $B(F_4))'$ satisfies the finiteness property F_2, but not the finiteness property F_3.*

Proof. Propositions 7.4.1, 7.4.2, 7.4.3 and 7.4.4 combined with Theorem 5.3.1 give this result. $\square$

Note: The Propositions 7.4.1, 7.4.2, 7.4.3 and 7.4.4 refer to the filtration of $X_{[0,3]}$. They show that the next stage of the filtration can be built from the previous stage by the adjunction of 3-cells. Following a similar method does not give the same result for the quotient X'. If it did we would have a $K(B(F_4)', 1)$ with a finite 2-skeleton, up to homotopy, and would not need Brown's theorem to deduce that $B(F_4)'$ is finitely presented.

In particular, recall the step in $X_{[0,3]}$ when adding the cells of $X_{j+1/4}$ whose vertex set contains g and is contained in $X_j \cup \{g\}$ to X_j. We saw that these cells are homotopy equivalent to coning off the subcomplex $X_j \cap st(g)$. This is not true if we try to do a similar step in X'; adding the cells whose vertex set contains $(j+1, -j-1)$ and is contained in $X'_j \cup \{(j+1, -j-1)\}$ to X'_j. For example consider the cell $\{(j+1, -j), e < r_1 < r_1r_2\}$ in $X'_j \cap st(j+1, -j-1)$. This is coned off not once but twice to the vertex $(j+1, -j-1)$, by the cells $\{(j+1, -j-1), e < r_3 < r_3r_1 < r_3r_1r_2\}$ and $\{(j+1, -j-1), e < r_4 < r_4r_1 < r_4r_1r_2\}$. The corresponding 3-cells in the $X_{[0,3]}$ case have two different top faces they are coning off, those being $\{g[r_3] < g[r_3r_1] < g[r_3r_2r_1]\}$ and $\{g[r_4] < g[r_4r_1] < g[r_4r_1r_2]\}$.

Appendix A

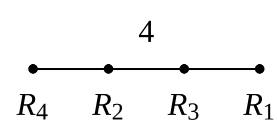
F_4 calculations

A.1 The Non-Crossing Partition Lattice for F_4

This appendix details the non-crossing partition lattice for the Coxeter group F_4. We follow Section 2.4 or [23] to number the reflections in this group. We choose

$$\begin{aligned}
\alpha_1 &= (0.5, -0.5, -0.5, -0.5),\\
\alpha_2 &= (0, -1, 1, 0),\\
\alpha_3 &= (0, 1, 0, 0),\\
\alpha_4 &= (0, 0, -1, 1)
\end{aligned}$$

to be our simple roots. Let R_1, R_2, R_3, R_4 be the corresponding reflections with fixed subspace $\alpha_i^{\perp}$, represented by matrices in $\mathbb{R}^4$. Note that the roots have been ordered such that $\{\alpha_1, \alpha_2\}$ and $\{\alpha_3, \alpha_4\}$ are orthogonal sets. We label the vertices in the Coxeter diagram below to show the relationships between the reflections.



Let $\gamma = R_1R_2R_3R_4$ be the Coxeter element and note that $\gamma^{12} = I$. We calculate the 24 positive roots by $\rho_i = R_1R_2...R_{i-1}\alpha_i$ where the α's and the R's are indexed cyclically modulo 4. We denote the reflection with fixed subspace $\rho_i^{\perp}$ by r_i. So there are 24 reflections in F_4. As in Chapter 2 we put a partial order on the group given by $w_1 < w_2$ when

$l(w_1) + l(w_1^{-1}w_2) = l(w_2)$ and the non-crossing partition lattice L consists of all elements in the interval $[e, \gamma]$ of this partial order.

We now examine the non-crossing partitions whose image under the length function is 2. Not all products of a pair of reflections precede γ, and thus are in L. To determine those that do, we look at the dual basis to $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$:

$$\begin{aligned}
\beta_1 &= (2, 0, 0, 0), \\
\beta_2 &= (2, 0, 1, 1), \\
\beta_3 &= (3, 1, 1, 1), \\
\beta_4 &= (1, 0, 0, 1).
\end{aligned}$$

These vectors satisfy $\beta_i \cdot \alpha_j = \delta_{i,j}$ for $1 \leq i, j \leq n$, where $\delta_{i,j}$ is the Kronecker delta. We define the vectors μ_i by $\mu_i = R_1R_2...R_{i-1}\beta_i$, where the R's and β's are again indexed cyclically modulo 4. By Lemma 2.4.1, the product of two reflections r_jr_i is an element of L if $\mu_j \cdot p_i = 0$. We check all possible products and obtain 55 matrices representing length 2 elements in L. We denote these elements by $t_i, 1 \leq i \leq 55$. Each element has between two and four factorisations, listed in Table A.1.1.

Regarding the non-crossing partitions w with $l(w) = 3$, we note that for each reflection r_i:

$$\begin{aligned}
& l(r_i) + l(r_i^{-1}\gamma) = l(\gamma) \\
\iff & l(\gamma^{-1}r_i\gamma) + l(r_i^{-1}\gamma) = l(\gamma) \\
\iff & r_i^{-1}\gamma < \gamma.
\end{aligned}$$

We denote these 24 elements by $\tau_i = r_i\gamma$, for $1 \leq i \leq 24$, and clearly there can be no additional length 3 elements of L. Each length 3 element has either 4 or 9 factorisations as a product of a length 2 element by a reflection, listed in Table A.1.2.

t_1	r_2r_1	r_1r_2			t_2	r_3r_1	r_5r_3	r_1r_5	
t_3	r_4r_1	r_1r_4			t_4	r_6r_1	r_7r_6	$r_{12}r_7$	r_1r_{12}
t_5	r_8r_1	$r_{13}r_8$	$r_{20}r_{13}$	r_1r_{20}	t_6	r_9r_1	$r_{17}r_9$	r_1r_{17}	
t_7	$r_{16}r_1$	$r_{19}r_{16}$	$r_{22}r_{19}$	r_1r_{22}	t_8	$r_{21}r_1$	$r_{23}r_{21}$	r_1r_{23}	
t_9	$r_{24}r_1$	r_1r_{24}			t_{10}	r_3r_2	r_8r_3	$r_{21}r_8$	r_2r_{21}
t_{11}	r_4r_2	$r_{24}r_4$	r_2r_{24}		t_{12}	r_5r_2	$r_{20}r_5$	$r_{23}r_{20}$	r_2r_{23}
t_{13}	r_4r_3	r_3r_4			t_{14}	r_6r_3	r_9r_6	$r_{24}r_9$	r_3r_{24}
t_{15}	r_5r_4	r_4r_5			t_{16}	r_6r_4	r_8r_6	r_4r_8	
t_{17}	r_7r_4	$r_{10}r_7$	$r_{13}r_{10}$	r_4r_{13}	t_{18}	r_9r_4	$r_{16}r_9$	$r_{21}r_{16}$	r_4r_{21}
t_{19}	$r_{12}r_4$	$r_{20}r_{12}$	r_4r_{20}		t_{20}	$r_{17}r_4$	$r_{22}r_{17}$	$r_{23}r_{22}$	r_4r_{23}
t_{21}	r_6r_5	r_5r_6			t_{22}	r_7r_5	r_9r_7	r_5r_9	
t_{23}	r_8r_5	r_5r_8			t_{24}	$r_{10}r_5$	$r_{11}r_{10}$	$r_{16}r_{11}$	r_5r_{16}
t_{25}	$r_{12}r_5$	$r_{17}r_{12}$	$r_{24}r_{17}$	r_5r_{24}	t_{26}	$r_{13}r_5$	$r_{21}r_{13}$	r_5r_{21}	
t_{27}	r_8r_7	r_7r_8			t_{28}	r_9r_8	r_8r_9		
t_{29}	$r_{10}r_8$	$r_{12}r_{10}$	r_8r_{12}		t_{30}	$r_{11}r_8$	$r_{14}r_{11}$	$r_{17}r_{14}$	r_8r_{17}
t_{31}	$r_{16}r_8$	$r_{24}r_{16}$	r_8r_{24}		t_{32}	$r_{10}r_9$	r_9r_{10}		
t_{33}	$r_{11}r_9$	$r_{13}r_{11}$	r_9r_{13}		t_{34}	$r_{12}r_9$	r_9r_{12}		
t_{35}	$r_{14}r_9$	$r_{15}r_{14}$	$r_{20}r_{15}$	r_9r_{20}	t_{36}	$r_{12}r_{11}$	$r_{11}r_{12}$		
t_{37}	$r_{13}r_{12}$	$r_{12}r_{13}$			t_{38}	$r_{14}r_{12}$	$r_{16}r_{14}$	$r_{12}r_{16}$	
t_{39}	$r_{15}r_{12}$	$r_{18}r_{15}$	$r_{21}r_{18}$	$r_{12}r_{21}$	t_{40}	$r_{14}r_{13}$	$r_{13}r_{14}$		
t_{41}	$r_{15}r_{13}$	$r_{17}r_{15}$	$r_{13}r_{17}$		t_{42}	$r_{16}r_{13}$	$r_{13}r_{16}$		
t_{43}	$r_{18}r_{13}$	$r_{19}r_{18}$	$r_{24}r_{19}$	$r_{13}r_{24}$	t_{44}	$r_{16}r_{15}$	$r_{15}r_{16}$		
t_{45}	$r_{17}r_{16}$	$r_{16}r_{17}$			t_{46}	$r_{18}r_{16}$	$r_{20}r_{18}$	$r_{16}r_{20}$	
t_{47}	$r_{18}r_{17}$	$r_{17}r_{18}$			t_{48}	$r_{19}r_{17}$	$r_{21}r_{19}$	$r_{17}r_{21}$	
t_{49}	$r_{20}r_{17}$	$r_{17}r_{20}$			t_{50}	$r_{20}r_{19}$	$r_{19}r_{20}$		
t_{51}	$r_{21}r_{20}$	$r_{20}r_{21}$			t_{52}	$r_{22}r_{20}$	$r_{24}r_{22}$	$r_{20}r_{24}$	
t_{53}	$r_{22}r_{21}$	$r_{21}r_{22}$			t_{54}	$r_{24}r_{21}$	$r_{21}r_{24}$		
t_{55}	$r_{24}r_{23}$	$r_{23}r_{24}$							

Table A.1.1: Factorisations of length 2 NCPs in F_4

τ_1	τ_4	τ_5	τ_8	τ_9	τ_{12}	τ_{13}	τ_{16}	τ_{17}	τ_{20}	τ_{21}	τ_{24}
$t_{11}r_{23}$	t_1r_{23}	$t_{10}r_{24}$	t_2r_{24}	t_3r_{20}	$t_{15}r_{21}$	$t_{23}r_{24}$	t_5r_{17}	$t_{18}r_{20}$	$t_{25}r_{21}$	t_5r_{24}	t_3r_{23}
$t_{12}r_{24}$	t_2r_2	$t_{11}r_{21}$	t_4r_5	t_4r_4	$t_{17}r_5$	$t_{24}r_8$	t_6r_{20}	$t_{19}r_{21}$	$t_{26}r_{24}$	t_7r_{20}	t_6r_4
$t_{15}r_2$	t_5r_5	$t_{13}r_2$	t_6r_{12}	t_5r_{12}	$t_{18}r_{13}$	$t_{25}r_{16}$	$t_{28}r_1$	$t_{34}r_4$	$t_{37}r_5$	t_9r_{22}	t_7r_{17}
$t_{19}r_5$	t_8r_{20}	$t_{14}r_4$	t_9r_{17}	$t_{16}r_1$	$t_{22}r_4$	$t_{29}r_5$	$t_{30}r_9$	$t_{35}r_{12}$	$t_{39}r_{13}$	$t_{31}r_1$	t_8r_{22}
$t_{20}r_{20}$	$t_{10}r_1$	$t_{16}r_3$	$t_{14}r_1$	$t_{17}r_8$	$t_{24}r_9$	$t_{30}r_{12}$	$t_{33}r_8$	$t_{38}r_9$	$t_{41}r_{12}$	$t_{42}r_8$	$t_{18}r_1$
$t_{25}r_4$	$t_{12}r_{21}$	$t_{18}r_8$	$t_{21}r_3$	$t_{19}r_{13}$	$t_{26}r_{16}$	$t_{31}r_{17}$	$t_{35}r_{13}$	$t_{39}r_{16}$	$t_{43}r_{17}$	$t_{43}r_{16}$	$t_{20}r_{21}$
$t_{49}r_{12}$	$t_{23}r_3$	$t_{28}r_6$	$t_{22}r_6$	$t_{27}r_6$	$t_{32}r_7$	$t_{36}r_{10}$	$t_{40}r_{11}$	$t_{44}r_{14}$	$t_{47}r_{15}$	$t_{46}r_{13}$	$t_{45}r_9$
$t_{52}r_{17}$	$t_{26}r_8$	$t_{31}r_9$	$t_{25}r_9$	$t_{29}r_7$	$t_{33}r_{10}$	$t_{38}r_{11}$	$t_{41}r_{14}$	$t_{46}r_{15}$	$t_{48}r_{18}$	$t_{50}r_{18}$	$t_{48}r_{16}$
$t_{55}r_{22}$	$t_{51}r_{13}$	$t_{54}r_{16}$	$t_{34}r_7$	$t_{37}r_{10}$	$t_{42}r_{11}$	$t_{45}r_{14}$	$t_{49}r_{15}$	$t_{51}r_{18}$	$t_{54}r_{19}$	$t_{52}r_{19}$	$t_{53}r_{19}$

τ_2	τ_3	τ_6	τ_7	τ_{10}	τ_{11}	τ_{14}	τ_{15}	τ_{18}	τ_{19}	τ_{22}	τ_{23}
t_8r_{24}	t_1r_{24}	t_2r_4	$t_{15}r_8$	$t_{22}r_8$	$t_{28}r_{12}$	$t_{33}r_{12}$	$t_{37}r_{16}$	$t_{41}r_{16}$	$t_{45}r_{20}$	$t_{48}r_{20}$	$t_{51}r_{24}$
t_9r_{23}	t_3r_2	t_3r_5	$t_{16}r_5$	$t_{23}r_9$	$t_{29}r_9$	$t_{34}r_{13}$	$t_{38}r_{13}$	$t_{42}r_{17}$	$t_{46}r_{17}$	$t_{49}r_{21}$	$t_{52}r_{21}$
$t_{54}r_1$	t_9r_4	$t_{13}r_1$	$t_{21}r_4$	$t_{27}r_5$	$t_{32}r_8$	$t_{36}r_9$	$t_{40}r_{12}$	$t_{44}r_{13}$	$t_{47}r_{16}$	$t_{50}r_{17}$	$t_{53}r_{20}$
$t_{55}r_{21}$	$t_{11}r_1$	$t_{15}r_3$	$t_{23}r_6$	$t_{28}r_7$	$t_{34}r_{10}$	$t_{37}r_{11}$	$t_{42}r_{14}$	$t_{45}r_{15}$	$t_{49}r_{18}$	$t_{51}r_{19}$	$t_{54}r_{22}$

Table A.1.2: Factorisations of length 3 NCPs in F_4

A.2 Abelianisation of the Artin Group

We are interested in the abelianisation of the associated Artin group $B(F_4)$. The presentation of $B(F_4)$ can be read from the Coxeter diagram, the generators are $[R_1], [R_2], [R_3], [R_4]$ with relations

$$\begin{aligned}
&[R_4][R_2][R_4] = [R_2][R_4][R_2],\\
&[R_4][R_3] = [R_3][R_4],\\
&[R_4][R_1] = [R_1][R_4],\\
&[R_2][R_3][R_2][R_3] = [R_3][R_2][R_3][R_2],\\
&[R_2][R_1] = [R_1][R_2],\\
&[R_3][R_1][R_3] = [R_1][R_3][R_1].
\end{aligned}$$

There is a natural set inclusion map $i : F_4 \to B(F_4) : R_i \mapsto [R_i]$ that extends to the other elements of L. When we abelianise, the generators $[R_4]$ and $[R_2]$ get identified and so do generators $[R_3]$ and $[R_1]$. Any element g of $B(F_4)$ can be written as a word in these four generators. The abelianisation map is

$$AB : B(F_4) \to \mathbb{Z} \times \mathbb{Z} : g \mapsto (x, y)$$

where x is the sum of the exponents of $[R_4]$ and $[R_2]$ and y is the sum of the exponents of $[R_3]$ and $[R_1]$ in such a word for g.

We note that it follows from the total order on reflections that

$$r_i = \gamma^{k+1} R_j \gamma^{-k-1}, \quad j > 2,$$
$$\text{where } i = j + 4k, \quad 1 \leq j \leq 4.$$

Since $r_1 = R_1$, $r_2 = R_2$, $r_{23} = R_3$ and $r_{24} = R_4$ we have

$$AB([r_i]) = \begin{cases} (0,1) & \text{if } i \text{ is odd} \\ (1,0) & \text{if } i \text{ is even.} \end{cases}$$

Note that there are 8 elements g with $AB([g]) = (0,2)$, each of which has three factorisations as a product of reflections. Similarly there are 8 elements g with $AB([g]) = (2,0)$, each of which has three factorisations. Finally there are 39 elements g with $AB([g]) = (1,1)$. Of these, 24 have two factorisations as a product of a pair of commuting reflections, while 15 of them have four different factorisations.

Note that there are 12 elements g with $AB([g]) = (1,2)$ and 12 with $AB([g]) = (2,1)$. In each case 6 of these elements have 4 factorisations and 6 have 9 factorisations as a product of a length 2 element by a reflection. Note also that for any length 3 element g of NCP, we can easily read off the reflection set that precedes it and the set of length 2 elements that precede it from this list of factorisations.

Appendix B

Matlab Functions for Calculating Homology of Artin Groups A_n

```
%R is the set of reflections in the symmetric group $S_n$

%This function will take in the set R and create the two
%relationship sets, C the set of commutative elements and D for sets of
%dihedral groups with 3 elements.

function[C,D]=Relationships(R)

    [n,useless]=size(R);
    k=1;
    l=1;

    for i=1:n

        for j=i+1:n

            %Check if R(i,:) and R(j,:) commute
            if [R(i,1),R(i,2),R(i,2),R(i,1)]~=[R(j,1),R(j,2),R(j,1),R(j,2)]
                if R(j,1)>R(i,2) || R(i,1)>R(j,2) || (R(i,1)<R(j,1) && R(i,2)>R(j,2)) || (R(j,1)<R(i,1) && R(j,2)>R(i,2))
                    C(k,:)=[i,j];
                    k=k+1;
                end
            end

            %Check if R(i,:) and R(j,:) are part of a dihedral set
            if R(i,2)==R(j,1)
                %Run thro. R again and find the 3rd element of the set
                m=0;
                found='no';
                while strcmp(found,'no')
                    m=m+1;
                    if R(m,:)==[R(i,1),R(j,2)]

                        found='yes';

                    end

                end

                D(l,:)=[i,j,m];
                l=l+1;
```

```
            elseif R(i,1)==R(j,2)
                m=0;
                found='no';
                while strcmp(found,'no')
                    m=m+1;
                    if R(m,:)==[R(j,1),R(i,2)]

                        found='yes';

                    end

                end

                D(l,:)=[i,j,m];
                l=l+1;
            end

        end

    end

end

%This function takes an array of 2 characters and the relationship sets C
%and D. It uses these to move the second character to the first position
%and replace the appropriate character in the 2nd position.

function[A]=Replace(S,C,D)

    i=1;
    arg='no';
    [n,useless]=size(C);
    [m,useless]=size(D);

    while (strcmp(arg,'no') && i<=n)

        if (S(1)==C(i,1) && S(2)==C(i,2)) || (S(1)==C(i,2) && S(2)==C(i,1))
            arg='yes';
        end

        i=i+1;

    end

    if strcmp(arg,'yes')

        A=[S(2),S(1)];

    else

        i=1;
        while (strcmp(arg,'no') && i<=m)

        if (S(1)==D(i,1) && S(2)==D(i,2))
            A=[S(2),D(i,3)];
            arg='yes';

        elseif (D(i,1)==S(1) && D(i,3)==S(2))
            A=[S(2),D(i,2)];
            arg='yes';

        elseif (D(i,2)==S(1) && D(i,1)==S(2))
            A=[S(2),D(i,3)];
            arg='yes';

        elseif (D(i,2)==S(1) && D(i,3)==S(2))
            A=[S(2),D(i,1)];
            arg='yes';

        elseif (D(i,3)==S(1) && D(i,1)==S(2))
            A=[S(2),D(i,2)];
```

```
            arg='yes';

        elseif (D(i,3)==S(1) && D(i,2)==S(2))
            A=[S(2),D(i,1)];
            arg='yes';

        end

        i=i+1;

        end

    end

    if strcmp(arg,'no')
        S
    end

end

%This function will that in an n character array S, a number i and the
%relationship matrix M. It will use these relationships to move the ith
%character to the front of the array changing the rest of the characters
%appropriately.

function[S]=Move(S,i,M)

    while i>1

                temp=S(i-1);
        S(i-1)=S(i);
        if M(temp,S(i))==-1
                        S(i)=temp;
                else
                        S(i)=M(temp,S(i));
                end

        i=i-1;

    end

end

%This function takes an array of 2 characters and the relationship sets C
%and D. It will return a third character  if these three
%form a dihedral set or -1 if the original two commute.
%If they cross, it returns 0.

function[a]=Ascending(S,C,D)

    [n,useless]=size(C);
    [m,useless]=size(D);
    i=1;
    arg='no';

    while (strcmp(arg,'no') && i<=n)

        if (S(1)==C(i,1) && S(2)==C(i,2)) || (S(1)==C(i,2) && S(2)==C(i,1))
            arg='yes';
        end

        i=i+1;

    end

    if strcmp(arg,'yes')

        a=-1;

    else
```

```
        i=1;
        while (strcmp(arg,'no') && i<=m)

        if (S(1)==D(i,1) && S(2)==D(i,2))
            a=D(i,3);
            arg='yes';

        elseif (D(i,1)==S(1) && D(i,3)==S(2))
            a=D(i,2);
            arg='yes';

        elseif (D(i,2)==S(1) && D(i,1)==S(2))
            a=D(i,3);
            arg='yes';

        elseif (D(i,2)==S(1) && D(i,3)==S(2))
            a=D(i,1);
            arg='yes';

        elseif (D(i,3)==S(1) && D(i,1)==S(2))
            a=D(i,2);
            arg='yes';

        elseif (D(i,3)==S(1) && D(i,2)==S(2))
            a=D(i,1);
            arg='yes';

        end

        i=i+1;

        end

    end

        if strcmp(arg,'no')
                a=0;
        end

end

%This function will take a set of n rows representing a loop in the
%complex. The first entry gives the sign of the loop and the number of times that generator is included. This
%function will write that loop in terms of the generating loops.
%The array G will consist of the generators of the n facets giving a
%straightening algorithm for n-1 dim elements.

function[A]=Generators(A,M,G)

    [n, m]=size(A);

    alldescending='no';

    %Continue running through the for loop below until it doesn't need to
    %change anything, which means each element must be descending.
    while strcmp(alldescending,'no')

    alldescending='yes';

    %This loop checks if any pair of each element of A is ascending and if
    %so changes that element. Sometimes this involves adding extra elements
    %to the end of A.
    for i=1:n

        for j=2:m-1

        S=A(i,:);
        if S(j)<S(j+1)

            alldescending='no';
```

```
        if M(S(j),S(j+1))==-1 %They commute
            A(i,1)=S(1)*(-1);
            A(i,j)=S(j+1);
            A(i,j+1)=S(j);

        % elseif M(S(j),S(j+1))==0 %they cross
                    % 'error in Generators: elements cross'

                    else %they're part of a dihedral set
            A(i,1)=S(1)*(-1);
            A(i,j)=S(j+1);
            A(i,j+1)=M(S(j),S(j+1));

            n=n+1;
            A(n,:)=A(i,:);
            %A(n,1)=S(1)*(-1);
            A(n,j)=M(S(j),S(j+1));
            A(n,j+1)=S(j);

        end
    end

    end
end

end

%Once we have written this as a set of descending loops we use the
%straightening algorithm to write it in terms of the generators.

[p,q]=size(G);

% if q ~= m
    % 'error'
% else
    i=1;
    while i<=n

        j=1;
        while j<=p

            if A(i,2:m)==G(j,1:q-1)

                                temp=A(i,1);
                                A(i,:)=[];
                                n=n-1;

                for k=1:q-1

                    S=G(j,:);
                    S(q-k)=[];
                    A(n+k,:)=[temp*(-1)^(k-1),S];

                end

                                n=n+q-1;
                i=i-1;
                j=p; %Break this j loop,move to the next i.

            end
            j=j+1;

        end

        i=i+1;
    end

%end


%Finally, we add up the generators
```

```
    i=1;
    while i<n

       j=i+1;
       while j<=n

           if A(j,2:m)==A(i,2:m)
               A(i,1)=A(i,1)+A(j,1);
               A(j,:)=[];
               j=j-1;
               n=n-1;
           end

           j=j+1;

       end

       %If the final answer is 0, then just delete the row
       if A(i,1)==0
           A(i,:)=[];
           i=i-1;
           n=n-1;
       end

       i=i+1;

    end

end

%This function will take in the relationship matrix M and 2 sets of Generators, 1 for dim n
% and 1 for dim n-1. It will use the other functions defined to calculate a matrix H which
%satisfies Hc=0 if f(c.x)=0

function[H]=Matrix(M,A,G)

[m,n]=size(A);
[p,q]=size(G);

if (q~=n-1)
    -1
end

H=zeros(p,m);

%Run thro. each element of A, calculate its boundary, perform the
%straightening algorithm on the answer
for i=1:m
    i

    B=Boundary(A(i,:),M);
    B=Generators(B,M,A);

    [l,useless]=size(B);

    %Run thro. each element of B and enter it in the appropriate place in
    %the matrix
    for j=1:l

        foundalready='no';
        k=1;
        while (k<=p) && strcmp(foundalready,'no')

            if B(j,2:n)==G(k,:)

                foundalready='yes';
                H(k,i)=B(j,1);

            end

            k=k+1;
```

```
        end

    end

end

end

%This function will take in an array of permutations expressing an element
%of length n. It will return the set of these chains which are descending
%for the first l terms and then ascending for the remainder.

function[A]=Sort(Chains,l)

    [n,m]=size(Chains);
    A=zeros(n,m);
    k=1;

    for i=1:n

        j=1;
        correct='yes';
        while(strcmp(correct,'yes') && j<l)
            if Chains(i,j)<Chains(i,j+1)
                correct='no';
            end
            j=j+1;
        end

        while(strcmp(correct,'yes') && j<m)
            if Chains(i,j)>Chains(i,j+1)
                correct='no';
            end
            j=j+1;
        end

        if strcmp(correct,'yes')
            A(k,:)=Chains(i,:);
            k=k+1;
        end

    end

    A(k:n,:)=[];

end

%The function will take in the relationship matrix M and an n character
%array S, and will return its boundary. The first entry in each row of the new array will
%be either +1 or -1, representing the sign of the face.

function[A]=Boundary(S,M)

    n=length(S);
    A=zeros(n,n);

    for i=1:n

        temp=Move(S,i,M);
        A(i,1)=(-1)^(i-1);
        A(i,2:n)=temp(2:n);

    end

end


%Given a length n element S and the relationship sets C and D, this
%function will return all possible expressions for that element.
```

```
function[A]=Expressions(S,C,D)

    n=length(S);
    if n==2

        temp=Replace(S,C,D);
        A=[S;temp];
        if temp(2)~=S(1)
            A(3,:)=[temp(2),S(1)];
        end

    else

        %Initialise

        Done=[S(1)];
        B=Expressions(S(2:n),C,D);
        [m,useless]=size(B);

        for i=1:m

            A(i,:)=[S(1),B(i,:)];

        end

        %Run through A, if we see an entry that hasn't been moved to the
        %front(isn't in Done) then add the expressions with it at the front
        %and add it to Done.

        i=1;
        p=1; %this is the original length of Done
        while(i<=m)

            for j=1:n

                %Check if A(i,j)is in Done
                in='no';
                for k=1:p
                    if A(i,j)==Done(k)

                        in='yes';

                    end
                end

                %If it isn't move it to the front and get the expressions
                %for the remaining terms
                if strcmp(in,'no')

                    R=A(i,:);
                    R=Move(R,j,C,D);
                    B=Expressions(R(2:n),C,D);
                    [q,useless]=size(B);
                    for l=1:q
                        A(m+l,:)=[A(i,j),B(l,:)];
                    end

                    %Record the increased length of A
                    m=m+q;

                    %Add A(i,j) to Done
                    Done(p+1)=A(i,j);
                    p=p+1;

                end

            end

            i=i+1;
        end
```

```
    end
end
```

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