Highly Nonlinear Stochastic and Deterministic Differential Equations with Time-Varying Shocks: Asymptotic Behaviour and Numerical Analysis

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Declaration

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Doctor of Philosophy is entirely my own work, and that I have exercised reasonable care to ensure that the work is original, and does not to the best of my knowledge breach any law of copyright, and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work.

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To my parents

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This thesis concerns the asymptotic behaviour for nonlinear differential equations, and also considers how this behaviour can be recovered by appropriate numerical schemes. In particular, perturbed equations are studied, where the equation without perturbations has known asymptotic behaviour. The restoring force is generally not of linear order close to the equilibrium, and the perturbation, which is time–varying forcing function, may be very irregular.

The thesis addresses three questions: first, what conditions on the forcing function characterize the case when the rate of decay of the solution of the unperturbed equation is preserved, and what is the decay rate for more slowly decaying forcing functions? Equations for which there is faster than power decay in the solution of the unperturbed equation are considered. This analysis involves generalising the class of regularly varying functions, as well generalising the notion of the Liapunov exponent to equations without leading order linear terms. Perturbation theorems, for which the decay rates of the unperturbed solutions are directly recovered, are also given.

Second, we prove that continuous time behaviour can be reproduced numerically. This is done when faster–than–power law, but slower than exponential, decay occurs. A semi–implicit method is used to cope with strong global nonlinearities. If the nonlinearity is smaller than linear order close to equilibrium, a fixed step–size scheme recovers the asymptotic behaviour.

Thirdly, it can be shown that the results can be applied to stochastically forced equations if the shocks have state–independent intensity. Numerical results are also presented, and the method in the deterministic case can be adapted to deal with the asymptotic behaviour of the perturbation, as well as the nonlinearity.

Chapter 1

Introduction

1.1 Summary of Main Results

This thesis is concerned with the asymptotic behaviour of perturbations of the scalar autonomous ordinary differential equation

$$y'(t) = -f(y(t)), \quad t > 0. \tag{1.1.1}$$

This equation is assumed to have a unique, globally asymptotically stable equilibrium at zero, which is assumed not to attract solutions in finite–time. Natural questions in any such investigation are: when does the perturbed equation still behave in the same way as the unperturbed equation, and when does it differ? Indeed, it would be better yet if it can be shown that certain types of behaviour in the solution of the perturbed equation can happen if and only if the perturbations have certain properties. It is a theme of the thesis that we try in many cases to determine such necessary and sufficient conditions.

Mainly, two classes of perturbation are considered: additive deterministic and stochastic perturbations, where the intensity of the forcing term is state–independent. The perturbed equations therefore are

$$x'(t) = -f(x(t)) + g(t), \quad t > 0 \tag{1.1.2}$$

and

$$dX(t) = -f(X(t))\,dt + \sigma(t)\,dB(t), \quad t \geq 0, \tag{1.1.3}$$

where g and σ are continuous scalar deterministic functions, and B is a one–dimensional Brownian motion. Since we generally assume

$$f \in C(\mathbb{R};\mathbb{R}), \quad f(0) = 0, \quad xf(x) > 0 \text{ for } x \neq 0,$$

continuous solutions will exist on $[0, \infty)$, due to the dissipative condition on f. We do not generally concern ourselves with questions of uniqueness of solutions, as it happens that all solutions to any of the equations will generally possess the same asymptotic properties. However, uniqueness can be achieved by the harmless addition of Lipschitz or monotonicity conditions on f, if desired. Such assumptions would not limit the main scope of our results.

The thesis has two main themes. First, we wish to determine conditions (usually necessary and sufficient) on the problem data for which the *rate of convergence of the solutions of the perturbed differential equations is the same as that of the unperturbed equation*. This also leads to an investigation as to what happens if the perturbations are above a critical size. The second main theme of the thesis is to investigate whether this *precise asymptotic behaviour can be recovered in simulations*. We have addressed this second question in the framework of theoretical numerical analysis, and give proofs that the discrete dynamical system that describes the numerical method does indeed possess the right properties.

The results we prove are for differential equations that we call "highly" nonlinear in the thesis title. This means that we largely consider equations for which either

$$\lim_{x \to 0} \frac{f(x)}{x} = 0 \text{ or } \lim_{x \to 0} \frac{f(x)}{x} = +\infty$$

so that results from the linear theory cannot hope to be sharp, especially if precise asymptotic behaviour is required. Some results apply to linear (or linearisable) equations also, so it can be seen that some convergence results in the linear case are specialisations or corollaries of the more general nonlinear theory.

The rough outline of the thesis is as follows. In Chapter 2, we consider deterministic perturbations of (1.1.1), including (1.1.2), but also study state–dependent or non–autonomous and multiplicative perturbations such as

$$x'(t) = -f(x(t)) + \delta(x(t)), \quad t > 0,$$

or

$$x'(t) = -f(x(t))(1 + \eta(t)), \quad t > 0$$

where δ and η are sufficiently regular. The main theme in Chapter 2 is to ask, if g, δ and η possess properties which ensure that the perturbed equations have positive solutions, can we establish nonlinear analogues of Hartman–Wintner type theorems that have been developed over many years (and for many types of differential system) in the linear case? This leads to the introduction of two measures for ascertaining the preservation of asymptotic decay properties. One is suitable for direct comparison of the asymptotic behaviour of perturbed and unperturbed equations, leading to what we call a *Hartman–Wintner*–type result, which will be roughly of the form

$$\lambda_{HW} := \lim_{t\to\infty} \frac{x(t)}{y(t)} \in (0,\infty)$$

(and typically $\lambda_{HW} = 1$). The other we call a *Hartman–Grobman*–type result, and uses an analogue of the Liapunov exponent in the nonlinear case, which will be of the form

$$\lambda_{HG} := \lim_{t\to\infty} \frac{F(x(t))}{t}$$

(and typically $\lambda_{HG} = 1$), where F is the well–defined function

$$F(x) = \int_x^1 \frac{1}{f(u)}\,du$$

which is of key importance in solving the unperturbed problem (1.1.1):

$$F(y(t)) = F(y(0)) + t, \quad t \geq 0.$$

The goal for all the perturbed equations is to give necessary and sufficient conditions on the data for which $\lambda_{HW} = 1$, and to give sufficient conditions under which λ_{HW} is either zero or infinite, so that the perturbed equations do not inherit the asymptotic behaviour of the unperturbed equations.

Another important development in Chapter 2 is the introduction of a subclass of nonlinear functions we term asymptotic preserving. It transpires that this class of nonlinear functions is also of great utility in studying Hartman–Grobman type results (which is the main focus of Chapter 3) and which give sufficient local control to enable precise asymptotic results to be proven for numerical methods (in Chapters 5 and 6). Most of the interesting properties of these asymptotic preserving functions are established in Chapter 3, and sufficient conditions for such functions to possess other desirable properties are also carefully investigated.

Chapter 3 is the heart of the thesis; it is the longest chapter, establishes results for both deterministic and stochastic equations, and identifies the class of equations and conditions that we wish to consider when discretising in Chapters 4, 5 and 6.

It is well–known in the linear case that the Hartman–Grobman type of result is more robust to perturbations than Hartman–Wintner type results. Therefore, although Hartman–Grobman type results are less refined than their Hartman–Wintner analogues, they can possibly be established under weaker restrictions on the problem data in general, and the perturbation behaviour in particular. This enables us to make some useful relaxations to the perturbation in Chapter 3. First, we can study stochastic as well as deterministic equations, in contrast to Chapter 2. Second, the sign of the perturbation is not restricted (and indeed it could oscillate rapidly). Third, superlinear equations (with $f(x)/x \to \infty$ as $x \to 0$) can be considered in a Hartman–Grobman framework. Such equations do not readily admit Hartman–Wintner type theorems, due to the sensitivity of the nonlinearity f.

The analysis in Chapter 3 deals with equations in which the underlying unperturbed equation has a rapidly varying solution, so that the solution of (1.1.1) obeys

$$\lim_{t\to\infty} \frac{y(\lambda t)}{y(t)} = 0, \quad \text{for all } \lambda > 1.$$

This implies that the rate of decay in the unperturbed solution is faster than any negative power of t, as $t \to \infty$. Indeed it can happen that the rate of decay of y can be slower than, or faster than, a negative exponential function. Exponential decay is also permitted, since functions f obeying

$$\lim_{x\to 0} \frac{f(x)}{x} = \alpha \in (0, \infty)$$

have all the required properties. Therefore, our results also apply to the linear case.

The deterministic results in Chapter 3 have the following character (which is largely shared by stochastic results). Decay rates of the solution of (1.1.2) are determined if g has the property that

$$\int_0^t g(s)\, ds \text{ tends to a finite limit as } t \to \infty, \tag{1.1.4}$$

and define

$$\Gamma(t) = -\int_t^\infty g(s)\, ds, \quad t \geq 0. \tag{1.1.5}$$

If this holds, the limit

$$\lambda_\Gamma := \liminf_{t\to\infty} \frac{F(|\Gamma(t)|)}{t}$$

is well defined, and moreover $x(t) \to 0$ as $t \to \infty$, so the limit

$$\lambda_x := \liminf_{t\to\infty} \frac{F(|x(t)|)}{t}$$

is also well–defined. In the unperturbed case, $\lambda_x = 1$. Essentially, we show that $\lambda_x \geq 1$ if $\lambda \geq 1$, so small perturbations do not retard the decay rate (at least in a sense that can be detected by the nonlinear analogue of the Liapunov exponent λ_x). If the decay in the perturbations can be compared with the decay of the unperturbed equation, but the perturbations decay more slowly, in the sense that $\lambda \in (0, 1)$, then $\lambda_x = \lambda$. In other words, the slow decay of the perturbation is inherited quite strongly by the solution. Also, very slowly decaying perturbations relative to the unperturbed equation lead likewise to very slow decay in x: we have that $\lambda = 0$ implies $\lambda_x = 0$. In the case of positive perturbations and solutions, one can show

$$\lambda \geq 1 \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{F(x(t))}{t} = 1.$$

However, examples show that we cannot expect to get limits in general, granted only the existence of limits inferior in the perturbation, so in a certain sense our results are hard to improve without restricting the class of perturbations. Interestingly, even very rapidly decaying perturbations do not guarantee that the rate of decay of x and y are the same: it is possible to achieve very fast decay in x relative to y using a very rapidly decaying (and very carefully chosen) g.

Chapter 4 considers for the first time results about discretisations. The results show that a type of implicit method, called the split step backward Euler (SSBE) method, built for dealing with stochastically perturbed equations, inherits the convergence to zero of the perturbed continuous equation under a discrete analogue of the condition (1.1.4). We mainly use a constant step size $h > 0$, and in that situation, the SSBE method has the form

$$x_h(0) = \zeta; \tag{1.1.6a}$$

$$x_h^*(n) = x_h(n) - hf(x_h^*(n)), \quad n \geq 0; \tag{1.1.6b}$$

$$x_h(n+1) = x_h^*(n) + \gamma_h(n+1), \quad n \geq 0, \tag{1.1.6c}$$

where $x_h(n)$ approximates the solution of (1.1.2) at time $t = nh$ i.e., $x(nh)$ and $\gamma_h(n+1)$ approximates $\int_{nh}^{(n+1)h} g(s)\,ds$. It turns out that there is an advantage in studying the SSBE method in this form, as convergence results can also be proven for the stochastic equation (1.1.3) without much extra effort in the case when $\sigma \in L^2(0,\infty)$.

Chapter 4 also develops an analogue of a result in Chapter 3 which is used to determine the asymptotic behaviour of the SDE (1.1.3). In the case of sequences $\sigma_n \in \ell^2(\mathbb{N})$ it concerns the rate at which the discrete–time Gaussian martingale

$$M(n) := \sum_{j=0}^{n-1} \sigma_j \xi(j+1)$$

approaches its almost sure finite limit. The noise sequence ξ is a sequence of independent and identically distributed normal random variables, so with appropriate choices of σ_n, $M(n)$ can be used to approximate the Itô integral

$$I(t) := \int_0^t \sigma(s)\,dB(s)$$

where σ is a deterministic function. In rough terms, we show that the convergence of $M(n)$ to $M(\infty)$ as $n \to \infty$ takes place in a similar manner to the way that $I(t)$ tends to $I(\infty)$ as $t \to \infty$, and that this is controlled by an iterated logarithm law. This is an important ingredient in ensuring that the asymptotic behaviour of the SSBE of (1.1.3) mimicks that of the solution of (1.1.3) itself.

Some flexibility is built into the proofs in Chapter 4 that the work in this thesis does not fully exploit. This flexibility concerns the step size. We are able to show that results for the SSBE scheme and for martingale convergence still apply if time steps vary, as long as the sum of the steps diverges. This flexibility, we believe, is of great importance for studying superlinear equations (in which $f(x)/x \to \infty$ as $x \to \infty$) or for perturbations which may change values rapidly.

Chapters 5 and 6 apply the SSBE method to determine analogues of the Liapunov exponent results in Chapter 3, using the basic convergence results proven in Chapter 4. Chapter 5 applies the SSBE method to the deterministic differential equation (1.1.2), while Chapter 6 studies the discretisation of the stochastic differential equation (1.1.3). There is one important restriction imposed on the nonlinearity in Chapters 5 and 6, however, that is not present in Chapter 3. In Chapters 5 and 6 we assume that f is sublinear (roughly in the sense that $f'(x) \to 0$ as $x \to 0$). This restriction means that the SSBE with a constant step size can recover the asymptotic behaviour of the perturbed differential equations; without this restriction, we show that the rate cannot be recovered with a constant step size.

Nevertheless, our discretisation results are very successful. We give some notation and a specimen result here in the deterministic case. For results on rates in (1.1.2) to be similar to those of (1.1.1), we have seen above that (1.1.4) must hold. The discrete analogue of this condition is

$$\sum_{j=0}^{n-1} \gamma_h(j+1) \text{ has a finite limit as } n \to \infty,$$

and this gives rise to a null sequence which is the analogue of the function Γ defined in (1.1.5) above:

$$\Gamma_h(n) = -\sum_{j=n}^{\infty} \gamma_h(j+1), \quad n \geq 0.$$

This gives rise to a discrete analogue of λ, namely

$$\lambda_h = \liminf_{n\to\infty} \frac{F(|\Gamma_h(n)|)}{nh}.$$

Since $x_h(n) \to 0$ as $n \to \infty$, we have an analogue of λ_x, denoted by

$$\lambda_x(h) = \liminf_{n\to\infty} \frac{F(|x_h(n)|)}{nh}.$$

Our results now are exactly analogous to the continuous time results. We show that $\lambda_h \geq 1$ implies $\lambda_x(h) \in [1, \lambda_h]$ and that $\lambda_h \in [0, 1]$ implies $\lambda_x(h) = \lambda_h$.

These, in a certain sense, are results for certain discrete dynamical systems. However, since these discrete equations are suppose to model solutions of differential equations, we would like to know under which circumstances the discrete problem recovers the behaviour of the continuous one. We do this by considering some nice classes of perturbation g. We show, for example, if g positive decreasing and integrable, that the discrete approximation λ_h of λ in fact obeys $\lambda_h = \lambda$ for any choice of h. Therefore, we can prove a result showing that x_h and x enjoy exactly the same asymptotic behaviour. Indeed, if $\lambda \geq 1$ then both λ_x and $\lambda_x(h)$ are both in $[1, \lambda]$ and if $\lambda \in [0, 1]$ then $\lambda_x = \lambda_x(h) = \lambda$.

Very similar results regarding the stochastic equation (1.1.3) are given in Chapter 6.

The final chapter of the thesis, Chapter 7, indicates areas where the results in the thesis can be extended. We particular highlight the potential to applying the methods in this work to the split step method for superlinear equations. We also indicate how bad perturbations might be dealt with numerically, and how to extend the results with non–positive perturbations in both continuous and discrete time to the situation when the solution of the underlying unperturbed equation (1.1.1) has asymptotic preserving solutions i.e.,

$$\lim_{\epsilon \to 0^+} \liminf_{t \to \infty} \frac{y((1+\epsilon)t)}{y(t)} = 1.$$

1.2 Notation and Regular Variation

We employ the notational convention that $\mathbb{R}^+ = [0, \infty)$. First, we define a useful equivalence relation on the space of positive continuous functions.

Definition 1.2.1. *Suppose $b, c \in C(\mathbb{R}^+; (0, \infty))$. b and c are asymptotically equivalent if $\lim_{t\to\infty} b(t)/c(t) = 1$; written $b(t) \sim c(t)$ as $t \to \infty$, or $b \sim c$ for extra brevity.*

The same notation is used if limits are taken to 0. Occasionally, we employ the standard Landau notation. If c is as above and $b \in C(\mathbb{R}^+; \mathbb{R})$, we write $b(t) = O(c(t))$ if $|b(t)| \leq Kc(t)$ for some $K \in (0, \infty)$ and all t sufficiently large, and $b(t) = o(c(t))$ if $b(t)/c(t) \to 0$ as $t \to \infty$. Again, we employ the same notation if the limits are taken for $t > 0$ sufficiently small.

Since we refer often to the class of regularly varying functions, we pause here to remind the reader of the definition (see Karamata [41], or Bingham, Goldie and Teugels [24] for a more modern account). They are a natural enlargement of the class of power functions which arise in many applications. A measurable function $f : (0, \infty) \to (0, \infty)$ with $f(x) > 0$ for $x > 0$ is said to be regularly varying at 0 with index $\beta \in \mathbb{R}$ if

$$\lim_{x \to 0^+} \frac{f(\lambda x)}{f(x)} = \lambda^\beta \quad \text{for all } \lambda > 0. \tag{1.2.1}$$

We use the notation $f \in \mathrm{RV}_0(\beta)$. If $\beta = 0$, f is said to be slowly varying at zero and we denote this by $f \in \mathrm{RV}_0(0)$ or $f \in \mathrm{SV}_0(0)$. Regular variation at infinity arises when the limit in (1.2.1) is taken as $x \to \infty$, and we write $f \in \mathrm{RV}_\infty(\beta)$ in this instance.

The convergence in (1.2.1) is uniform in λ; this result is called the uniform convergence theorem for regularly varying functions. Integrals of regularly varying functions are regularly varying, and the asymptotic behaviour is captured by a result often referred to as *Karamata's theorem*. Regular variation is an important quantitative property because we can quantify the change in value of the function when the argument is scaled by a factor of λ. Important results about regular variation that we used in this work can be found in the monograph [24].

The sign and size of β give information about the qualitative behaviour of f. The larger the value of β then the quicker the rate-of-increase as $x \to \infty$, but the slower the rate of increase as we increase away from 0. The functions

$$x^\beta, \quad x^\beta \log(1/x), \quad (x \log(1/x))^\beta,$$

are regularly varying at zero with index β. Typical examples of slowly varying functions are positive constants, functions converging to a positive constant, logarithms and iterated logarithms.

1.3 Relevant Literature

The thesis is concerned with perturbed scalar ordinary differential equations, scalar stochastic differential equations with additive noise, and numerical methods to recover their long–time behaviour. All these topics have attracted great attention in the literature. We give some general references concerning these topics here, and pause in more depth to review works which have had a greater influence on the work in this thesis.

First, we think about the asymptotic theory of perturbed ordinary differential equations. We note that the autonomous differential equation (1.1.1) is the unique positive limiting equation of the differential equation (1.1.2) if either $g(t) \to 0$ as $t \to \infty$ or if $g \in L^1(0,\infty)$. Therefore the problem studied here is connected strongly with work which relates the asymptotic behaviour of original non–autonomous equations to their limiting equations. Especially interesting work in this direction is due to Artstein in a series of papers [21, 22, 23]. Among the major conclusions of his work show that in some sense asymptotic stability and attracting regions of the limiting equation are synonomous with the asymptotic stability and attracting regions of the original nonautonomous equation. However, these results do not apply directly to the problems considered here, because the non–autonomous differential equation (1.1.2) does not have zero as a solution. Moreover, equation (1.1.2) does not exhibit the property that its limiting equation is not an ordinary differential equation, so the extension of the limiting equation theory given in in e.g., [21] is not needed to explain the difference in the asymptotic behaviour between the original equation and its limiting equation. Other interesting works on asymptotically autonomous equations in this direction include Strauss and Yorke [77, 78] and D'Anna, Maio and Moauro [30].

A very significant topic in the theory of perturbed differential systems is the degree to which the underlying unperturbed equation can be perturbed, and yet the solution of the perturbed equation inherits its asymptotic behaviour. If we narrow the focus of the investigation further, to consider the *exact rates* at which perturbed solutions decay or grow, we are lead into the field of asymptotic integration and what may be broadly called Hartman–Wintner theorems. There is a very comprehensive theory available in the case of linear equations. For ordinary differential equations, this was systematically instigated by Hartman and Wintner in [36], with an excellent synopsis of important results appearing in Hartman's monograph [35] (see in particular [35, Cor X.16.4]). The situation for perturbations of linear functional differential equations was settled comprehensively in Pituk [66]. Important works in the linear theory of asymptotic integration include those of Arino and Győri [19, 20], Castillo and Pinto [25], Haddock and Sacker [34], and Pinto [65]. Works on the nonlinear (especially sublinear theory) include those of Appleby and Patterson, Graef, and Kusano and Onose [12, 14, 15, 33, 45]. An excellent overview of some of the important literature on asymptotic integration may be found in the introductions to [25] and Nesterov [64].

Another question concerns the rates at which solutions of perturbed nonlinear ordinary differential equations tend to equilibrium. Here we have taken inspiration from work of Appleby and his co–authors, especially recent work with Patterson, in which deterministic and stochastic equations are essentially considered at the same time. The first paper [8] considers SDEs of the form (1.1.3) in which $f(x) \sim a\text{sgn}(x)|x|\beta$ as $x \to 0^+$ for some $\beta > 1$ and $a > 0$. In that work, sufficient conditions are established under which the solution of (1.1.2) inherits the asymptotic behaviour of (1.1.1). This is generalised to deal with the case of f in $\text{RV}_0(\beta)$ for $\beta > 1$ in [11] to deal with both equations (1.1.2) and (1.1.3). For (1.1.3), sign conditions on g were not needed, and necessary and sufficient conditions on g for the decay rates of (1.1.1) and (1.1.2) to be equal were determined. Sharp sufficient conditions on σ were also established under which (1.1.1) and (1.1.3) have the same rate of decay. In [13] on the other hand, with positivity conditions imposed on g, exact rates of decay for the solutions of (1.1.2), including the case of slowly decaying perturbations, were established for equations in which f is in $\text{RV}_0(\beta)$ or is rapidly varying at zero. Our methods in this thesis often make use the "constructive comparison proofs" that occur often in these papers. The scheme of such proofs involves constructing functions which satisfy upper or lower differential inequalities, thereby trapping the solution of the target differential equation above or below the constructed bounds.

In this thesis, the novelty of our work stems from removing regular variation hypotheses on the nonlinearity, as well as allowing faster than power law decay in the solutions to be observed. Also, we have adapted the methods of proof in [13] to allow for the analysis of large perturbations. Naturally, this faster than power–law decay can include superexponential decay in the solutions, even for solutions of stochastic differential equations. Another paper which considers such rapid decay in solutions, arising from autonomous SDEs with superlinear drift is Appleby et al [10]. However, in that work, the fast rate of decay relies on the dominant nonlinearities being regularly varying with unit index.

Nevertheless, our asymptotic preservation condition on the nonlinearity, together with rapid variation condition on the solution of (1.1.1) (which forces the function F to be in the class of slowly varying functions), means that our results are related, at least in spirit, with the work using the theory of regular variation in differential equations. There is a burgeoning literature regarding the application of the theory of regular variation to the asymptotic behaviour of ordinary and functional differential equations (see for example the monographs of Marić [59], and Řehák [72] and recent representative papers such as those of Evtukhov and Samoilenko [31], Chatzarakis et al, [28], Kozḿa [46], Matucci and Řehák [60, 61], and Kusano and Manojlović [47]). Besides, we consider illustrative examples throughout the thesis in which a regular variation assumption is imposed on f.

An important part of the work in this thesis involves the asymptotic behaviour of stochastic differential equations. A number of important textbooks and monographs have been written on the subject. Classical work on the asymptotic behaviour, especially asymptotic stability of stochastic differential equations, was undertaken in Gikhman and Skorohod [32] and in Khas'minski [43]. The work of Skorohod emphasised linear stochastic equations [75]. Mao has made a number of important contributions, particularly with regard to the exponential stability of solutions in [53], with further developments, including extensions to functional and neutral equations appearing in Mao [54]. A very comprehensive monograph on stochastic functional differential equations is Kolmanovskii and Myshkis [44], which devotes a lot of space to different modes of convergence, especially in p–th mean. Further results on the asymptotic behaviour and stability of stochastic partial differential equations and stochastic delay partial equations are in the book of Liu [49].

A number of our results concern subexponential decay in solutions of SDEs. Precise asymptotic results on the nonexponential growth or decay in solutions of autonomous SDEs were pioneered by Gikhman and Skorohod [32], with follow–up work by Zhang and Tsoi [79, 80]. The paper of Appleby, Rodkina and Schurz [17] relaxed significantly the requirements on the size of the diffusion term so that exact rates of decay could be recovered, but with additional regular variation hypotheses being needed on the nonlinearities.

Largely, however, these works are concerned largely with SDEs where the diffusion coefficient depends upon the state, and a better comparison can be made with stochastic differential equations with state–independent noise. Such equations have attracted a lot of attention. Liapunov function techniques have been applied to study their asymptotic stability in Khas'minski [43], with a lot of emphasis given to equations with perturbations σ being in $L^2(0,\infty)$. This is the hypothesis we impose on σ in this work, as it is within this class of perturbations that there is the possibility that the solution of (1.1.3) inherits the asymptotic behaviour of (1.1.1). Indeed, it can be shown that when $\sigma \notin L^2(0,\infty)$, the solution of (1.1.3) is oscillatory a.s. and therefore cannot inherit the positive asymptotic behaviour of the solution of (1.1.1).

However, in a pair of papers in 1989, Chan and Williams [27] and Chan[26] showed that the stability of global equilibria in these systems could be preserved with a much slower rate of decay in σ. These results required strong assumptions on the strength of the nonlinear feedback. Rajeev [67] later demonstrated that these results could be generalised to equations with some non–autonomous features, and some results on bounded solutions were obtained. In parallel, Mao demonstrated in [51, 52] that a polynomial rate of decay of solutions was possible if the perturbation intensity decayed at a polynomial rate. These results were extended to neutral functional differential equations by Mao and Liao in [48], with exponential decaying upper bounds on the intensity giving rise to an exponential convergence rate in the solution. Another work of Mao, together with Liu, [50] exploits non–autonomous features to give rise to power–law decay in the solutions.

In Appleby, Gleeson and Rodkina [7], the topic of stability in [27] was revisited, with the monotonicity assumptions on f and σ being relaxed. Indeed, necessary and sufficient conditions on stability, boundedness and unboundedness in the linear case was obtained in Appleby, Cheng and Rodkina [4], and in the nonlinear case in [5]. The papers [7, 5] are relevant to this discussion because in contrast to the other works on state–independent perturbations, it exploits the dynamics of an internally perturbed ordinary differential equation with smooth sample paths to study the perturbed stochastic differential equation. This is the approach that we exploit to study the SDE in this thesis.

Lastly, we indicate how our work fits into a broader theme within numerical analysis, especially as it relates to the asymptotic behaviour. Works which study the qualitative behaviour numerical simulations of solutions of differential equations, and which seek conditions under which this behaviour is preserved by discretisation, have become more commonplace in recent years, but the monographs of Stuart and Humphries [76] and Mickens [62] are among the first comprehensive treatments.

For stochastic equations, when this programme of research started, the major emphasis was on the mean square asymptotic behaviour of linear SDEs. Among the early and important contributions the highlight work of Saito and Mitsui [68], Schurz [69] and [70] and Higham [37]. The papers of Schurz and Higham also demonstrate the usefulness of implicit methods for dealing with problems in which the continuous solutions converge to the equilibrium state. Necessary and sufficient conditions for exponential stability in the solution of SDEs and the corresponding discretisation were given in Higham, Mao, Stuart [39]. The success of the balanced–implicit method was highlighted in Milstein, Platen and Schurz [63].

More recently, attention has focussed on preserving the non–exponential pathwise stability and decay rates of solutions of stochastic differential equations which arise due to nonlinear drift and diffusion coefficients. Examples of papers in this direction include Appleby, Rodkina, and Berkolaiko [16] and Appleby, Rodkina and Mackey [9]. The latter paper is interesting in the context of this thesis, as it concerns equations with state–independent perturbations.

Recently, the limitations of using explicit Euler methods for simulating stochastic differential equations have been explored. For the equation studied in this work, the paper of Appleby, Rodkina, Berkolaiko [2] demonstrates that if f does not obey a global linear bound (in the sense that $\lim_{|x|\to\infty} |f(x)|/|x| = +\infty$), then for sufficiently large initial conditions, the solution will oscillate unboundedly with probability arbitrarily close to unity, even though all solutions of the corresponding continuous equation tend to zero with probability one. However, local stability is preserved, in the sense that if the noise intensity remains arbitrarily small and the initial condition is sufficiently small, then solutions of the explicit scheme will converge with probability arbitrarily close to unity. Examples which demonstrate that explicit Euler methods will suffer from these problems when it is desired to preserve stationarity in SDEs, are presented in Mattingly, Stuart, and Higham [58].

Given, therefore, that we desire to preserve the asymptotic behaviour for general nonlinear f (which need not obey global linear bounds), it becomes necessary to use a method other than explicit Euler–Maruyama. Implicit methods have been recognised as performing well in these circumstances, as evidence by work of Schurz [71] and Rodkina and Schurz [73]. Among possible implicit methods, a good candidate would appear to be the split step backward Euler method (SSBE) developed by Higham, Mao, and Stuart in [38] and in [58], as it has been shown to ensure convergence of numerical approximations of solutions of SDEs on compact intervals, and preserves a.s. exponential stability in SDEs (see e.g., Higham, Mao, and Yuan [40]) and in hybrid SDEs (see e.g., Mao, Shen, Gray [55]). General stability under weaker assumptions on the drift and diffusion coefficient is in Mao and Szpruch [56, 57] in which a dissipative condition on f is used. However, in contrast to the equations studied here, the diffusion term depends only on the state, and equilibria are preserved by the stochastic perturbation.

In each case, the algorithms perform well with weak or no restrictions on the uniform step size, in contrast to explicit Euler methods. In this thesis, we show that making a very modest strengthening on the drift coefficient f at 0 that the SSBE method preserves all possible types of asymptotic behaviour, without restriction being made on the step size $h > 0$.

Chapter 2

Hartman–Wintner theorems for ordinary differential equations

2.1 Introduction

One of the main themes in this thesis is to explore conditions under which perturbed differential equations inherit the asymptotic behaviour of an underlying unperturbed equation. In particular, we wish to understand when the rate of decay is preserved by the perturbation.

In this section, we try to explain why we have chosen in this chapter to prove certain results, but not others, and that this puts relatively natural limits on the scope of our investigation. In simple terms, we eliminate from consideration in this chapter very strong nonlinearities because these do not allow us to study the *explicit asymptotic behaviour* of perturbed solutions. We readmit a large class of these nonlinearities in Chapter 3, because there is a good implicit measure of asymptotic behaviour available which is based on a reasonable generalisation of the Liapunov exponent. We also eliminate very weak nonlinearities, because in this case the explicit asymptotic behaviour provides inferior asymptotic information to the implicit behaviour.

This is not to say the "strong" and "weak" cases are not interesting, but merely that the asymptotic behaviour appears to be of a very different character to that seen in the linear case. In a certain sense our analysis expands from linear functions the class of nonlinearities for which a statement about explicit asymptotic behaviour is more powerful than an implicit statement which is based on a Liapunov exponent. In the strong case, an implicit result may be all that can be proven; in the weak case the explicit result appears weaker.

In this chapter, we therefore study "moderate" nonlinearities which have a nice asymptotic preserving property and are sublinear. This covers a very wide range of decay types in the unperturbed equation, covering all decay types between exponential and of power law type. Decay which is exponential, or faster than exponential, is studied in Chapter 3. Thus, in the end, it is only the case where the unperturbed equation has super–slow decay (i.e., slower than any negative power of t) that we eliminate.

Let us now try to make the discussion precise. Naturally, it would be useful to give a formulation of what "preserving the decay rate" means. One obvious way to proceed is to try to specify when

$$\lim_{t\to\infty} \frac{x(t)}{y(t)} = 1 \tag{2.1.1}$$

where x is the solution of the perturbed equation, and y the solution of the unperturbed equation. We will call a result of this type (when the limit on the right–hand side is finite and nontrivial) a Hartman–Wintner theorem. In our case, the unperturbed equation will be the autonomous scalar ordinary differential equation

$$y'(t) = -f\left(y(t)\right), \quad t > 0; \quad y(0) = \xi > 0. \tag{2.1.2}$$

In order that the zero solution is attracting, is a unique equilibrium solution, and that there is a well–defined (and positive) continuous solution of (2.1.2), we ask that

$$f(0) = 0, \quad xf(x) > 0 \quad x > 0, \quad f \in C([0,\infty);[0,\infty)). \tag{2.1.3}$$

In later chapters, we will be interested in the case in which solutions of perturbed equations can change in sign, and in this case we will extend these properties appropriately from $[0,\infty)$ to $\mathbb{R}$.

Before proceeding further, we eliminate the case in which the solution of the unperturbed equation tends to zero in finite time. In fact we have that

$$\text{There exists } T > 0 \text{ such that } \lim_{t\to T^-} y(t) = 0,\ y(t) > 0 \text{ for all } t \in [0,T) \iff \int_{0^+}^{1} \frac{1}{f(u)}\,du < +\infty. \tag{2.1.4}$$

Therefore, from now on, we assume that F defined by

$$F(x) := \int_x^1 \frac{1}{f(u)}\,du, \tag{2.1.5}$$

obeys

$$\lim_{x\to 0^+} F(x) = \infty. \tag{2.1.6}$$

This means that solutions of (2.1.2) obey $y(t) > 0$ for all $t \geq 0$ and $y(t) \to 0$ as $t \to \infty$.

We eliminate the finite–time stable equations from our study in this chapter (and in the rest of the thesis too) because their asymptotic behaviour is very sensitive to even small perturbations. For example, suppose that (2.1.4) holds, and consider the equation

$$x'(t) = -f\left(x(t)\right) + g(t), \quad t \geq 0; \quad x(0) = \zeta > 0 \tag{2.1.7}$$

where g is continuous and $g(t) > 0$ for all $t \geq 0$. Then $x(t) > 0$ for all $t \geq 0$, in contrast to the solution when $g(t) \equiv 0$. Of course, even if $x(t) \to 0$ as $t \to \infty$, the rate at which this occurs is completely different from the asymptotic behaviour of $y(t)$ as $t \to T^-$.

Thus, we start by examining, under (2.1.6), when (2.1.1) happens. This condition can be achieved in certain cases, but is unsuitable for some class of differential equation even when $F(x) \to \infty$ as $x \to \infty$. The next part of our discussion rules out the class of equations for which

$$\lim_{x \to 0^+} \frac{f(x)}{x} = +\infty$$

for the purposes of this chapter, at least.

The asymptotic relation $x(t)/y(t) \to 1$ as $t \to \infty$ suggests that the unperturbed equation should behave according to

$$\lim_{t \to \infty} \frac{y(t;\xi_1)}{y(t;\xi_2)} = 1 \quad \forall \xi_1, \xi_2, \tag{2.1.8}$$

where we use the notation $y(t;\xi)$ to denote the solution of (2.1.2) with $y(0) = \xi$. This seems reasonable to request , as we are perturbing the initial condition only, and have not considered yet the effect of perturbing by means of a forcing function. However, even this invariance of decay rate with respect to the initial condition (as measured by (2.1.8)) is not achieved in the case when $f(x)/x \to \infty$ as $x \to 0^+$. Let now y be the solution of the unperturbed ODE with $y(0) = 1$. Then

$$\lim_{t \to \infty} \frac{y'(t)}{y(t)} = \lim_{t \to \infty} -\frac{f(y(t))}{y(t)} = -\infty,$$

so by taking the integral over $[t-c, t]$ for any $c > 0$, and letting $t \to \infty$, we get

$$\lim_{t \to \infty} \frac{y(t-c)}{y(t)} = +\infty. \tag{2.1.9}$$

Now, the solution to the problem with initial condition ξ, $y(t;\xi)$, is $y(t;\xi) = F^{-1}(F(\xi) + t) = y(F(\xi) + t)$. Hence for $\xi_1 > \xi_2$, we have $F(\xi_1) < F(\xi_2)$, so

$$\begin{aligned} \lim_{t \to \infty} \frac{y(t;\xi_1)}{y(t;\xi_2)} &= \lim_{t \to \infty} \frac{y(F(\xi_1) + t)}{y(F(\xi_2) + t)} \\ &= \lim_{\tau \to \infty} \frac{y(F(\xi_1) - F(\xi_2) + \tau)}{y(\tau} \\ &= \lim_{\tau \to \infty} \frac{y(\tau - (F(\xi_2 - F(\xi_1)))}{y(\tau)} = +\infty, \end{aligned}$$

using (2.1.9) with $c =: F(\xi_2) - F(\xi_1) > 0$. Therefore, for *superlinear* equations i.e., those for which $f(x)/x \to \infty$ as $x \to 0^+$, it is unrealistic to expect

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} \in (0,\infty)$$

as even arbitrary small changes in the initial condition of the unperturbed equation give rise to completely different rates.

This result shows the need for another way of measuring the rate of decay for superlinear unperturbed equations which does not display such extreme sensitivity to the initial condition. We explore in depth in later chapters a measure based on the Liapunov–exponent which is successful for dealing with small perturbations of linear differential equations, but make some simple remarks here. The alternative measure is

$$\lim_{t\to\infty} \frac{F\left(y(t)\right)}{t}.$$

This measure of the rate of decay *is not sensitive to the initial data*, because $y(t;\xi) = F^{-1}\left(F(\xi)+t\right)$ for all $t \geq 0$, implies $F(y(t;\xi)) = t + F(\xi)$, for $t \geq 0$ so we have

$$\lim_{t\to\infty} \frac{F\left(y(t;\xi)\right)}{t} = 1 \quad \text{for all } \xi > 0. \tag{2.1.10}$$

Moreover, the measure is also *robust to state–dependent perturbations* which are also smaller than $f(x)$ to leading order as $x \to 0^+$. To see this, consider the perturbed equation

$$x'(t) = -f\left(x(t)\right) + \delta\left(x(t)\right), \quad t > 0; \quad x(0) = \zeta > 0. \tag{2.1.11}$$

We assume that δ is continuous, $\delta(x) = o\left(f(x)\right)$ as $x \to 0^+$, $\delta(0) = 0$. This automatically implies that for all $\zeta > 0$ sufficiently small we have $x(t) > 0$ for all $t \geq 0$ and $x(t) \to 0$ as $t \to \infty$. Moreover, as $\delta(x)/f(x) \to 0$ as $x \to 0^+$ we have

$$\lim_{t\to\infty} \frac{x'(t)}{f(x(t))} = -\lim_{t\to\infty} \left\{-1 + \frac{\delta(x(t))}{f(x(t))}\right\} = -1.$$

Thus, for all $\epsilon \in (0,1)$ there is a $T(\epsilon) > 0$ such that for $t \geq T(\epsilon)$ we have

$$-1-\epsilon < \frac{x'(t)}{f(x(t))} < -1+\epsilon.$$

Integration on both sides of this inequality for $t \geq T(\epsilon)$ leads to

$$-(1+\epsilon)(t-T) \leq \int_{x(T)}^{x(t)} \frac{1}{f(u)}\,du < -(1-\epsilon)(t-T).$$

Dividing by t, letting $t \to \infty$, and then letting $\epsilon \to 0^+$, we get

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 1. \tag{2.1.12}$$

Therefore, in this case, perturbations in the initial condition and perturbations to the nonlinear term do not change the measure used to estimate the rate of decay, provided that the nonlinear perturbation is of lower order than the nonlinear term in (2.1.2). We note that (2.1.10) and (2.1.12) are properties shared by the Liapunov exponent in the linear case when $f(x) = ax$, $a > 0$.

One of the questions we examine in this thesis to what extent a limiting result like (2.1.12) can be achieved in the *superlinear* case that $f(x)/x \to \infty$ as $x \to 0^+$ and the perturbed differential equation is of the form (2.1.7). This will form a good part of the analysis in Chapter 3, for example. In fact, we will show in Chapter 3 that even in the case that $f(x)/x \to \infty$ as $x \to 0^+$, the nonlinear analogue of the Liapunov exponent for equation (2.1.7) still enjoys the property (2.1.12) when the perturbation g decays to zero sufficiently quickly. In this thesis, results which focus on the limiting behaviour of $F(x(t))/t$ will be called *Hartman–Grobman–type* results.

However it may be unnecessary to seek as result as conservative as this when $f(x)/x \to 0$ as $x \to 0$ even for the solution of (2.1.7). To motivate this statement start by observing that the solution of the unperturbed equation obeys

$$\lim_{t\to\infty} \frac{y(t;\xi)}{F^{-1}(t)} = 1 \quad \forall \xi > 0 \tag{2.1.13}$$

when $f(x)/x \to 0$ as $x \to 0^+$. This is because

$$\lim_{t\to\infty} \frac{y'(t)}{y(t)} = \lim_{t\to\infty} \frac{-f(y(t))}{y(t)} = 0$$

and so by integrating over $[t, t+c]$ and letting $t \to \infty$, we see that

$$\lim_{t\to\infty} \frac{y(t+c)}{y(t)} = 1 \quad \forall c \in \mathbb{R}$$

from which (2.1.13) follows.

Therefore, in the sublinear case (i.e, when $f(x)/x \to 0$ as $x \to 0^+$), it first becomes important to understand the connection between the asymptotic relations

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = 1 \quad \text{and} \quad \lim_{t\to\infty} \frac{F(x(t))}{t} = 1. \tag{2.1.14}$$

We do this for a class of nonlinear f in the thesis which has the property of *preserving* *asymptotic behaviour at* 0. For simplicity, let f be increasing, and suppose it obeys

$$\lim_{\epsilon \to 0^+} \liminf_{x \to 0^+} \frac{f((1-\epsilon)x)}{f(x)} = 1. \tag{2.1.15}$$

We call such functions f *uniformly asymptotic preserving at* 0 (or UAP at 0, for short), and this condition will be imposed on the function f in (2.1.2) or (2.1.7) directly, or on another function which is asymptotic to f at zero.

The reason for the terminology "asymptotic preserving" is that condition (2.1.15) implies that f preserves asymptotic rates of convergence to zero in the sense that

$$b(t) > 0 \text{ and } a(t) \sim b(t) \text{ as } t \to \infty \text{ implies } f(a(t)) \sim f(b(t)) \text{ as } t \to \infty.$$

We make use of (2.1.15) rather than the last limit partly for technical reasons, as it allows us to construct estimates in this chapter and subsequently. But (2.1.15) is also convenient as it frees us from the requirement to check that $f \circ a \sim f \circ b$ for all choices of function a and b for which $a \sim b$.

The class of UAP functions includes many functions which obey $f(x)/x \to \infty$ as $x \to 0^+$, but it excludes functions which are excessively flat at 0. For example

$$f(x) = x^\beta, \quad \beta > 0$$

is UAP at 0, but

$$f(x) = \begin{cases} e^{\frac{-1}{x}}, & x > 0 \\ 0, & x = 0 \end{cases}$$

is not UAP at zero. The class of asymptotic preserving functions is closely related to, but not the same as, the class of regularly varying functions. If f is regularly varying at 0, it obeys (2.1.15); however, there exist increasing functions f which obey (2.1.15), but are not regularly varying. An example of such a function is

$$f(x) = x \exp\left(-\frac{1}{2}\left\{1 - \cos(\log(1/x))\right\}\right), \quad x \in (0, \tfrac{1}{2})$$

with $f(0) = 0$ and f defined on $[1/2, \infty)$ so that it is positive, continuous and increasing on $[0, \infty)$. Other functions which are UAP but not RV can be constructed by appealing to the representation theorem for regularly varying functions (see [24, Thm 1.3.1]).

We will show that the condition (2.1.15) forces a similar asymptotic preserving property on F:

$$\lim_{\epsilon \to 0^+} \limsup_{x \to 0^+} \frac{F((1-\epsilon)x)}{F(x)} = 1. \tag{2.1.16}$$

This means that if f is UAP at zero, and $a(t) \sim b(t)$ as $t \to \infty$ with $b(t) > 0$, then

$F(a(t)) \sim F(b(t))$ as $t \to \infty$. Therefore, if f obeys (2.1.15), we have

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = 1 \Rightarrow \lim_{t\to\infty} \frac{F(x(t))}{t} = 1. \tag{2.1.17}$$

We will also show later that if f obeys (2.1.15), then slower than power-law decay in y is impossible; indeed, we have that (2.1.15) implies that

$$\limsup_{t\to\infty} \frac{\log y(t)}{\log t} =: -\beta_0 \in [-\infty, 0).$$

Thus, in very rough terms, if we exclude very flat functions which do not obey (2.1.15), then the direct asymptotic decay rate $x(t)/F^{-1}(t) \to 1$ as $t \to \infty$ for a perturbed differential equation is a stronger result than the Liapunov exponent limit captured by $F(x(t))/t \to 1$ as $t \to \infty$.

In this chapter, we examine to what extent a result like $x(t)/F^{-1}(t) \to 1$ as $t \to \infty$ can be proven for perturbed differential equations. From our experience above, we confine attention to the case when $f(x)/x \to 0$ as $x \to 0^+$. Since, for the unperturbed equation, it is always possible to obtain a Hartman–Wintner type relation of the form (2.1.13), regardless of the initial condition, it is not unreasonable to wonder whether it can always be shown that $x(t)/F^{-1}(t) \to 1$ as $t \to \infty$ whenever perturbations are sufficiently small that $F(x(t))/t \to 1$ as $t \to \infty$.

In fact, there is a large class of uniform asymptotic persevering and sublinear functions for which this is true. For instance, let $f \in RV_0(\beta)$ for $\beta > 1$. Then f is sublinear and UAP, and $F(x) \to \infty$ as $x \to 0^+$. Moreover, we have that $F \in RV_0(1-\beta)$ by Karamata's theorem (see e.g, Theorem 1.5.11 in [24]), and thus $F^{-1} \in RV_\infty(-1/\beta - 1)$. Hence F^{-1} is asymptotic preserving (and indeed UAP) at $+\infty$. Thus, if $F(x(t))/t \to 1$ as $t \to \infty$, we conclude that $x(t)/F^{-1}(t) \to 1$ as $t \to \infty$. Therefore, in the case that $f \in \text{RV}_0(\beta)$ for $\beta > 1$, Hartman–Wintner and Hartman–Grobman results are equivalent.

However, it transpires that we cannot in general make such conclusion for sublinear f. In fact, the above argument suggests that problems may arise if $f \in RV_0(1)$ because even though $F \in RV_0(0)$ and it is thus UAP, F^{-1} is nevertheless rapidly varying, and thus is not asymptotic preserving. Thus we can not conclude in general, when $f \in RV_0(1)$, that $F(x(t))/t \to 1$ as $t \to \infty$ implies $x(t)/F^{-1}(t) \to 1$ as $t \to \infty$.

Of course, this is well–known in the linear case, which is a special case of f being in $\text{RV}_0(1)$. Here is a simple illustration of the phenomenon for (2.1.7). Consider $f(x) = ax$ for $a > 0$ and let $g(t) = e^{-at}$ for $t \geq 0$. Let $\xi > 0$ and x be the solution of (2.1.7). Then

$$x(t) = e^{-at}\xi + e^{-at} \int_0^t e^{as} \cdot e^{-as}\, ds = e^{-at}(\xi + t).$$

Since $F(x) = -\frac{1}{a}\log x$, we have $F^{-1}(t) = e^{-at}$. Therefore we have

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = +\infty$$

even though, as $t \to \infty$, we have

$$\frac{F(x(t))}{t} = -\frac{1}{at}\log(e^{-at}(\xi + t)) = 1 - \frac{\log(\xi + t)}{at} \to 1.$$

Finally, we remark that the elimination from our study of very flat f's that are not UAP is not a purely technical matter, but is essential if we are to consider cases in which Hartman–Wintner results are considered stronger than Hartman–Grobman ones. We mentioned earlier that $f(x) = e^{-1/x}$ for $x > 0$ is not UAP. In fact, it has the property

$$\lim_{x\to 0^+} \frac{f(\lambda x)}{f(x)} = \begin{cases} 0 & 0 < \lambda < 1 \\ 1 & \lambda = 1 \\ +\infty & \lambda > 1, \end{cases}$$

which forces the decreasing function F to have the rapid variation property

$$\lim_{x\to 0^+} \frac{F(\lambda x)}{F(x)} = \begin{cases} 0 & \lambda > 1 \\ 1 & \lambda = 1 \\ +\infty & 0 < \lambda < 1. \end{cases}$$

This in turn forces F^{-1} to be slowly varying and hence UAP. Therefore, we always have the implication

$$\lim_{t\to\infty} \frac{F(a(t))}{t} = 1 \quad \Rightarrow \lim_{t\to\infty} \frac{a(t)}{F^{-1}(t)} = 1$$

but it can be that the function a obeys $a(t) \sim F^{-1}(t)$ as $t \to \infty$, but that $F(a(t))$ is not asymptotic to t as $t \to \infty$, because F^{-1} is rapidly varying. Therefore the "usual" relative strength of Hartman–Wintner and Hartman–Grobman results are reversed in this situation, suggesting that the direct asymptotic measure $\lim_{t\to\infty} x(t)/F^{-1}(t)$ is more robust to larger perturbations than the Liapunov–exponent $\lim_{t\to\infty} F(x(t))/t$.

In this chapter, we show that when f is UAP, increasing and sublinear, in general more restrictive conditions are needed on the size of perturbations (for some classes of perturbations) if a Hartman–Wintner–type result is to established than when a Hartman–Grobman–type result is to be proven. At this moment, we will be able to make such a comparison if the perturbation is "state-dependent" (as in the differential equation (2.1.11)) or "multiplicative", so that perturbed differential equation has the form

$$x'(t) = -f\left(x(t)\right)\left(1 + \eta(t)\right), \quad t > 0, \tag{2.1.18}$$

where η is a continuous function on $[0,\infty)$ such that $t+\int_0^t \eta(s)\,ds \to \infty$ as $t\to\infty$.

Later in the thesis, when a Hartman–Grobman–type result is available for an "additively" perturbed equation of the form (2.1.7), we will see that the Hartman–Wintner result of this chapter requires stronger conditions on g. This chapter will contain Hartman–Wintner type results for equations of the form (2.1.7), (2.1.11) and (2.1.18). We will also show that the conditions on the perturbations cannot be relaxed significantly without destroying the asymptotic equivalence of the perturbed and unperturbed solutions, so that the results are in some sense sharp.

2.2 Absence of Hartman–Wintner numerical results

Before proving these continuous–time Hartman–Wintner results, we address now what may at this instant appear an omission in this thesis. Our results on the numerical behaviour of differential equations in Chapters 6 and 7 do not attempt (beyond the case of $f\in RV_0(\beta)$ for $\beta>1$ which is covered in Chapter 5) to establish Hartman–Wintner type results, even though, in this chapter, we show that they are available in continuous time. One reason, beyond restriction on space and time, is that we can not be confident that behaviour would be easily recovered by seemingly reasonable numerical methods.

To explain this remark, we consider the unperturbed differential equation

$$y'(t) = -f\left(y(t)\right), \quad t>0; \quad y(0)=\xi>0$$

where $f(x)/x\to 0$ as $x\to 0^+$. As already remarked this implies

$$\lim_{t\to\infty}\frac{y(t;\xi)}{F^{-1}(t)}=1, \quad \lim_{t\to\infty}\frac{F\left(y(t)\right)}{t}=1. \tag{2.2.1}$$

Since f is sublinear, it seems reasonable, and is in the spirit of numerical work in this thesis, to use a scheme with a constant step size, $\Delta>0$, to simulate y. Suppose we use a one–step implicit method to do this, which is precisely the split–step method outlined later in this work when applied to the perturbed equation (2.1.7) when $g(t)\equiv 0$.

We suppose as before that f obeys (2.1.3). Then, for any $\Delta>0$, there is (at least one) positive sequence $(y_n)_{n\geq 0}$ which satisfies

$$y_{n+1}=y_n-\Delta f(y_{n+1}), \quad n\geq 0; \quad y_0=\xi>0. \tag{2.2.2}$$

The idea of course, is that y_n approximates $y(n\Delta)$. All solutions of (2.2.2) tend monotonically to zero as $n\to\infty$. A unique solution can be guaranteed if we ask that f is increasing, and for simplicity, we do so now. Moreover, if we strengthen the assumption $f(x)/x\to 0$ as $x\to 0^+$ to f being in C^1 in an open interval $(0,\delta)$ for some $\delta>0$ and also ask that $f'(x) \to 0$ as $x \to 0^+$, then it can be shown that

$$\lim_{n\to\infty} \frac{F(y_n)}{n\Delta} = 1. \tag{2.2.3}$$

This is the correct analogue of $F(y(t))/t \to 1$ as $t \to \infty$. Note that this limit is of Hartman–Grobman, rather than Hartman–Wintner, type.

We now prove (2.2.3). It can be deduced easily by rewriting (2.2.2) in the form $y_n = y_{n+1} + \Delta f(y_{n+1})$, and Taylor expending to first order to get

$$F(y_{n+1}) - F(y_n) = \Delta \frac{f(y_{n+1})}{f(\tilde{y}_n)}, \tag{2.2.4}$$

where $\tilde{y}_n \in [y_n, y_{n+1}]$. A further Taylor expansion to first order in f show that there exists $y_n^* \in [\tilde{y}_n, y_{n+1}]$ and $\theta_n \in [0,1]$ such that

$$\frac{f(\tilde{y}_n)}{f(y_{n+1})} = 1 + f'(y_n^*)\theta_n\Delta \to 1, \quad n \to \infty.$$

Using this limit, summing across (2.2.4), dividing by n and taking limits, gives (2.2.3).

Now the reasonable question to ask is whether the asymptotic behaviour in (2.2.1) is also preserved by the implicit scheme. The answer, for general f, is "no" provided f grows faster than a critical *sublinear* rate. In fact, the following result has been recently proven by Appleby [1].

Theorem 2.2.1. *Suppose that f obeys (2.1.3), is in C^1, increasing and obeys $f'(x) \to 0$ as $x \to 0^+$. Suppose further that f' is UAP at 0 and $(y_n)_{n\geq 0}$ solves (2.2.2). Then the following are equivalent:*

(i)

$$\lim_{x\to 0^+} \frac{f(x)}{x/\log(\frac{1}{x})} = 0$$

(ii)

$$\lim_{n\to\infty} \frac{y_n}{F^{-1}(n\Delta)} = 1.$$

Moreover, if

$$\lim_{x\to 0^+} \frac{f(x)}{x/\log(\frac{1}{x})} = +\infty,$$

then

$$\lim_{n\to\infty} \frac{y_n}{F^{-1}(n\Delta)} = +\infty.$$

This result shows that we should be very careful when trying to infer information above the rate of decay of certain sublinear differential equations from numerical simulation. In particular, given that the rate of decay is not persevered for unperturbed equation in certain situations, it cannot be expected that the situation is better for perturbed equations. Therefore we prefer at present the robust information given in the limit (2.1.9), which does not depend on the strength of the sublinear function f. This means our numerical results always are written with a view to preserving Hartman–Grobman type results, rather than Hartman–Wintner–type results.

2.3 Hartman-Wintner Theorems

Let f obey

$$f \in C([0,\infty);[0,\infty)),\ f(x)>0 \text{ for all } x>0,\ f(0)=0 \text{ and } f \text{ is increasing.} \tag{2.3.1}$$

We also assume that f is sublinear at 0 i.e.,

$$\lim_{x\to 0^+} \frac{f(x)}{x} = 0. \tag{2.3.2}$$

This forces F defined by (2.1.5) to obey (2.1.6).

Finally, it is assumed that f is uniformly asymptotic preserving so that

$$\lim_{\epsilon\to 0^+} \limsup_{x\to 0^+} \frac{f\left((1+\epsilon)x\right)}{f(x)} = 1. \tag{2.3.3}$$

We now consider three perturbations of the autonomous initial value problem

$$y'(t) = -f\left(y(t)\right), \quad t\ge 0, \quad y(0)>0.$$

(I) *Non–autonomous, multiplicative perturbation*

$$x'(t) = -f\left(x(t)\right)\left(1+\eta(t)\right), \quad t\ge 0; \quad x(0)=\xi>0 \tag{2.3.4}$$

where $\eta \in C([0,\infty);\mathbb{R})$.

(II) *Non–autonomous, additive perturbation*

$$x'(t) = -f\left(x(t)\right) + g(t), \quad t\ge 0; \quad x(0)=\xi>0 \tag{2.3.5}$$

where $g \in C([0,\infty);(0,\infty))$.

(III) *Autonomous, additive perturbation*

$$x'(t) = -f\left(x(t)\right) + \delta\left(x(t)\right), \quad t\ge 0; \quad x(0)=\xi>0, \tag{2.3.6}$$

where $\delta(x) = o\left(f(x)\right)$ as $x\to 0^+$ and δ is continuous. In particular, we have $|\delta(x)| <$

$f(x)$ for all $x \in (0, \xi]$ provided that $\xi > 0$ is sufficiently small.

The first set of theorems give necessary and sufficient conditions for $x(t)/y(t) \to 1$ as $t \to \infty$ for each of the perturbed equations (2.3.4), (2.3.5), (2.3.6).

Theorem 2.3.1. *Suppose f obeys (2.3.1), (2.3.2), (2.3.3). Let F be given by (2.1.5). Let η be continuous on $[0, \infty)$ with $t + \int_0^t \eta(s)\, ds \to \infty$ as $t \to \infty$. Then the solution of (2.3.4) obeys $x(t) \to 0$ as $t \to \infty$ and the following are equivalent:*

(i)
$$\lim_{t\to\infty} \frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)} \int_0^t \eta(s)\, ds = 0.$$

(ii)
$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = 1.$$

Theorem 2.3.2. *Suppose f obeys (2.3.1), (2.3.2), (2.3.3). Let F be given by (2.1.5). Let $g \in C([0, \infty); (0, \infty))$. Then the following are equivalent for the solution of (2.3.5)*

(i)
$$\lim_{t\to\infty} \frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)} \int_0^t \frac{g(s)}{f\left(F^{-1}(s)\right)}\, ds = 0.$$

(ii)
$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = 1.$$

Theorem 2.3.3. *Suppose f obeys (2.3.1), (2.3.2), (2.3.3). Let F be given by (2.1.5). Let δ be continuous on $[0, \infty)$, $\delta(0) = 0$ and $|\delta(x)| < f(x)$ for all $x \in (0, \xi]$. Then the following are equivalent for the solution of (2.3.6):*

(i)
$$\lim_{x\to 0^+} \frac{f(x)}{x} \int_x^\xi \frac{\delta(u)}{\left(f(u) - \delta(u)\right) f(u)}\, du = 0.$$

(ii)
$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = 1$$

The proofs of these results are deferred to later in the chapter.

Scrutiny of the proofs of these results show that, to a certain extent, Theorem 2.3.1 is the most fundamental. This may be surprising, since the solution of the equation with multiplicative perturbations can be obtained explicitly, whereas the solution of the equation with an additive perturbation cannot. Nevertheless, Theorem 2.3.2 proves largely to be a corollary of Theorem 2.3.1, while the converse half of Theorem 2.3.3 is a corollary of the converse in Theorem 2.3.1. Later results concerning the non–preservation of the decay rate of the unperturbed equation follow a similar pattern:

results for the additive and state–dependent perturbations either follow the pattern of the proof for the multiplicative perturbation, or they are direct corollaries of the multiplicative results.

Notice that one way of stating Theorem 2.3.1 which abstracts the result from the theory of ordinary differential equations to the theory of functions is

$$\lim_{t\to\infty} \frac{f(F^{-1}(t))}{F^{-1}(t)} N(t) = 0 \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{F^{-1}(t+N(t))}{F^{-1}(t)} = 1,$$

where N is a continuous function such that $t+N(t)\to\infty$ as $t\to\infty$, and f enjoys the properties in Theorem 2.3.1.

We show now in case (III) that terms that are small enough to be neglected when we seek to preserve the nonlinear analogue of the Liapunov exponent cannot be neglected if we wish to prove a Hartman–Wintner type of result.

Proposition 2.3.1. *Suppose f obeys (2.3.1), (2.3.2). Moreover let f be in C^1 and obey $xf'(x)/f(x)\to 1$ as $x\to 0^+$.*

(i) If $\delta(x) = o\left(xf'(x) - f(x)\right)$ as $x\to 0^+$, and $x\mapsto f(x)/x$ is increasing on $(0,\delta')$ for some $\delta'>0$, then

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = 1.$$

(ii) If $\delta \sim K\left(xf'(x) - f(x)\right)$ as $x\to 0^+$ for some $K\neq 0$, then $\delta(x) = o\left(f(x)\right)$ as $x\to 0^+$, and the solution x of (2.3.6) obeys

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 1,$$

but $x(t)$ is not asymptotic to $F^{-1}(t)$ as $t\to\infty$.

Thus in the case (III), we can not neglect a $\delta = o(f)$ term if we want asymptotic behaviour preservied in a Hartman-Wintner sense.

Proof. We establish claim (ii) first. Here $K\neq 0$, and we have

$$\frac{d}{dx}\left(\frac{x}{f(x)}\right) = \frac{f(x) - xf'(x)}{f^2(x)} \sim \frac{-1/K\delta(x)}{f^2(x)}, \quad x\to 0^+.$$

Thus, as $x/f(x)\to\infty$ as $x\to 0^+$

$$\frac{-x}{f(x)} \sim \frac{1}{f(1)} - \frac{x}{f(x)} \sim \int_x^1 \frac{-1/K\delta(u)}{f^2(u)}\,du, \quad x\to 0^+.$$

Therefore $|\int_x^1 \delta(u)/f^2(u)\,du| \to\infty$ as $x\to 0^+$. Hence

$$\frac{1}{K}\int_x^1 \frac{\delta(u)}{f^2(u)}\,du \sim \frac{x}{f(x)}, \quad x\to 0^+,$$

or

$$\lim_{x\to 0} \frac{f(x)}{x} \int_x^1 \frac{\delta(u)}{f^2(u)}\,du = K \neq 0,$$

so by Theorem 2.3.3, we cannot have $x(t) \sim F^{-1}(t)$, $t \to \infty$. Modifying the proof of Theorem 2.3.3 slightly, it can be shown that $x(t) \sim c(K)F^{-1}(t)$, $t \to \infty$ for $c = c(K) > 0$. Since $xf'(x) \sim f(x)$ as $x \to 0^+$, it is clear that $\delta(x) = o\left(f(x)\right)$ as $x \to 0^+$. Hence, by applying the argument following (2.1.11), we see that the solution of (2.3.6) obeys

$$\lim_{t\to\infty} \frac{F\left(x(t)\right)}{t} = 1.$$

To prove part (i), write

$$\frac{1}{f(1)} - \frac{x}{f(x)} = \int_x^1 \frac{f(u) - uf'(u)}{f^2(u)}\,du.$$

Thus

$$\frac{x}{f(x)} \sim \int_x^1 \frac{f(u) - uf'(u)}{f^2(u)}\,du \to \infty, \quad x \to 0^+.$$

Since $x \mapsto f(x)/x$ is increasing on $(0, \delta')$ for some $\delta' > 0$, $xf'(x) - f(x) > 0$ for all $x < \delta'$. Now $\delta(x) = o\left(xf'(x) - f(x)\right)$ as $x \to 0^+$. Hence, for every $\epsilon > 0$ there is $x(\epsilon) > 0$ such that

$$|\delta(x)| < \epsilon\left(xf'(x) - f(x)\right), \quad x < x(\epsilon).$$

Now

$$\frac{f(x)}{x}\left|\int_x^{x(\epsilon)} \frac{\delta(u)}{f^2(u)}\,du\right| \leq \epsilon \frac{f(x)}{x}\left|\int_x^{x(\epsilon)} \frac{uf'(u) - f(u)}{f^2(u)}\,du\right| \to \epsilon, \quad x \to 0^+.$$

Since $\epsilon > 0$ is arbitrary we can infer

$$\lim_{x\to 0^+} \frac{f(x)}{x} \int_x^1 \frac{\delta(u)}{f^2(u)}\,du = 0.$$

Theorem 2.3.1 now yields $x(t) \sim F^{-1}(t)$ as $t \to \infty$. □

A similar argument can be developed for the equation (2.3.5).

Proposition 2.3.2. *Suppose f obeys (2.3.1), (2.3.2). Moreover let f be in C^1 and obey $xf'(x)/f(x) \to 1$ as $x \to 0^+$. Suppose also that there is $\delta' > 0$ such that $x \mapsto f(x)/x$ is increasing on $(0, \delta')$. Let $g \in C([0,\infty);(0,\infty))$ and suppose there is $c > 0$ such that*

$$g(t) \sim c\left\{F^{-1}(t)f'\left(F^{-1}(t)\right) - f\left(F^{-1}(t)\right)\right\}, \quad t \to \infty. \tag{2.3.7}$$

Then

(i)

$$\lim_{t\to\infty} \frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)} \int_0^t \frac{g(s)}{f\left(F^{-1}(t)\right)}\,ds = c.$$

(ii)

$$\liminf_{t\to\infty} \frac{F\left(\int_t^\infty g(s)\,ds\right)}{t} \geq 1.$$

Therefore, the solution x of (2.3.7) *obeys*

$$\lim_{t\to\infty} \frac{F\left(x(t)\right)}{t} = 1,$$

but $x(t)$ is not asymptotic to $F^{-1}(t)$ as $t\to\infty$.

Proof. Notice that

$$\begin{aligned}\frac{d}{dt}\left(\frac{F^{-1}(t)}{f\left(F^{-1}(t)\right)}\right) &= \frac{-f\left(F^{-1}(t)\right) + f'\left(F^{-1}(t)\right)F^{-1}(t)}{f\left(F^{-1}(t)\right)} \\ &\sim \frac{c^{-1}g(t)}{f\left(F^{-1}(t)\right)}, \quad t\to\infty.\end{aligned}$$

Hence, if $D(t) := F^{-1}(t)/f\left(F^{-1}(t)\right)$, then $D(t)\to\infty$, $t\to\infty$ and

$$D'(t) \sim \frac{c^{-1}g(t)}{f\left(F^{-1}(t)\right)}, \quad t\to\infty. \tag{2.3.8}$$

Since $g(t)/f\left(F^{-1}(t)\right) > 0$, we have either that $\int_0^t g(s)/f(F^{-1}(s))\,ds$ tends to a finite limit, or to $+\infty$ as $t\to\infty$. But if the former is true, then $D(t)$ should tend to a finite limit by (2.3.8), and this is a contradiction. Hence

$$\lim_{t\to\infty} \int_0^t \frac{g(s)}{f\left(F^{-1}(s)\right)}\,ds = +\infty.$$

Now, as $D(t)\to\infty$ as $t\to\infty$, we may apply L'Hôpital's rule to get

$$\begin{aligned}\lim_{t\to\infty} \frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)} \int_0^t \frac{g(s)}{f\left(F^{-1}(s)\right)}\,ds &= \lim_{t\to\infty} \frac{\frac{g(t)}{f(F^{-1}(t))}}{D'(t)} \\ &= \lim_{t\to\infty} \frac{g(t)/f\left(F^{-1}(t)\right)}{c^{-1}g(t)/f\left(F^{-1}(t)\right)} = c,\end{aligned}$$

proving (i). This also shows that $x(t)$ cannot be asymptotic to $F^{-1}(t)$ as $t\to\infty$, by Theorem 2.3.2.

We now prove (ii) as follows: using the asymptotic estimate on g and integration by substitution yields

$$\int_t^\infty g(s)\,ds \sim c\int_t^\infty \left\{F^{-1}(s)f'\left(F^{-1}(s)\right) - f\left(F^{-1}(s)\right)\right\}\,ds$$
$$= c\int_0^{F^{-1}(t)} \left(\frac{uf'(u)}{f(u)} - 1\right)\,du.$$

Since f is in $\text{RV}_0(1)$, by Karamata's theorem (see e.g, Theorem 1.5.11 in [24]) F is slowly varying and hence is asymptotic preserving. Therefore with $\lambda(x) := xf'(x)/f(x) - 1$, we have

$$\frac{F\left(\int_0^\infty g(s)\,ds\right)}{t} \sim \frac{F\left(\int_0^{F^{-1}(t)}\lambda(u)\,du\right)}{t}, \quad t \to \infty.$$

Taking the limit inferior, we get

$$\liminf_{t\to\infty} \frac{F\left(\int_0^\infty g(s)\,ds\right)}{t} = \liminf_{x\to 0^+} \frac{F\left(\int_0^x \lambda(u)\,du\right)}{F(x)}.$$

To show the last limit is greater than or equal to unity, note first that $\lambda(x) \to 0$ as $x \to 0^+$. Hence $\int_0^x \lambda(u)\,du = o(x)$ as $x \to 0^+$, and so there is an $x^* > 0$ such that $\int_0^x \lambda(u)\,du < x$ for all $x < x^*$. Thus $F\left(\int_0^x \lambda(u)\,du\right) > F(x)$ for all $x < x^*$, and so

$$\liminf_{x\to 0^+} \frac{F\left(\int_0^x \lambda(u)\,du\right)}{F(x)} \geq 1,$$

as required. □

The above calculation reveals a sufficient condition on g which guarantees asymptotic preservation in Theorem 2.3.2.

Proposition 2.3.3. *If f is in C^1 and*

$$g(t) = o\left(F^{-1}(t)f'(F^{-1}(t)) - f(F^{-1}(t))\right), \quad t \to \infty,$$

Then

$$\lim_{t\to\infty} \frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)} \int_0^t \frac{g(s)}{f\left(F^{-1}(s)\right)}\,ds = 0.$$

Therefore, if f obeys (2.3.1), (2.3.2), then the solution x of (2.3.7) obeys $x(t) \sim F^{-1}(t)$ as $t \to \infty$.

Proof. Write $\phi(t) := F^{-1}(t)f'\left(F^{-1}(t)\right) - f\left(F^{-1}(t)\right)$ and as above $D(t) := F^{-1}(t)/f\left(F^{-1}(t)\right)$. Note that $\lim_{t\to\infty} g(t)/\phi(t) \to 0$ as $t \to \infty$, $D(t) \to \infty$ as $t \to \infty$ and $D'(t) = \phi(t)/f\left(F^{-1}(t)\right)$. Since $g(t) > 0$, there are two cases to consider:

$$\text{(I)} \quad \lim_{t\to\infty} \int_0^t \frac{g(s)}{f\left(F^{-1}(s)\right)}\,ds = \infty \quad \text{or} \quad \text{(II)} \lim_{t\to\infty} \int_0^t \frac{g(s)}{f\left(F^{-1}(s)\right)}\,ds < +\infty.$$

In the second case (II), we automatically have

$$\lim_{t\to\infty} \frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)} \int_0^t \frac{g(s)}{f\left(F^{-1}(s)\right)}\,ds = 0,$$

as needed. In case (I), by L'Hôpital's rule we get

$$\begin{aligned}\lim_{\to\infty} \frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)} \int_0^t \frac{g(s)}{f\left(F^{-1}(s)\right)}\,ds &= \lim_{t\to\infty} \frac{\int_0^t g(s)/f\left(F^{-1}(s)\right)\,ds}{D(t)} \\ &= \lim_{t\to\infty} \frac{g(t)/f\left(F^{-1}(t)\right)}{\phi(t)/f\left(F^{-1}(t)\right)} = \lim_{t\to\infty} \frac{g(t)}{\phi(t)} = 0,\end{aligned}$$

as claimed □

Our preliminary discussion showed for the solution x of (2.3.6) that $\delta(x) = o(f(x))$ as $x \to 0^+$ ensured $F(x(t))/t \to 1$ as $t \to \infty$. In the case that $f \in \mathrm{RV}_0(\beta)$ for $\beta > 1$, we have that F^{-1} is asymptotic preserving, so we have $x(t) \sim F^{-1}(t)$ as $t \to \infty$. The next result checks this conclusion independently using Theorem 2.3.3 and shows that the condition $\delta(x) = o(f(x))$ as $x \to 0^+$ is sharp.

Proposition 2.3.4. *If $f \in RV_0(\beta)$ for $\beta > 1$ and $\delta(x) = o\left(f(x)\right)$, then*

$$\lim_{x\to 0^+} \frac{f(x)}{x} \int_x^\xi \frac{\delta(u)}{\left(f(u) - \delta(u)\right) f(u)}\,du = 0,$$

and so the solution x of (2.3.6) obeys

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = 1.$$

Proof. Since $\delta(x) = o\left(f(x)\right)$ as $x \to 0^+$, we have that

$$\int_x^\xi \frac{\delta(u)}{\left(f(u) - \delta(u)\right) f(u)}\,du \sim \int_x^\xi \frac{\delta(u)}{f^2(u)}\,du, \quad x \to 0^+$$

in the case that the integrals diverge. If they do not, then clearly

$$\lim_{x\to 0^+} f(x)/x \int_x^\xi \delta(u)/\left(f(u) - \delta(u)\right) f(u)\,du = 0.$$

Also, in the divergent case, it follows from the fact that $\delta(x) = o(f(x))$ as $x \to 0^+$ that

$$\int_x^\xi \frac{\delta(u)}{\left(f(u) - \delta(u)\right) f(u)}\,du = o\left(\int_x^\xi \frac{1}{f(u)}\,du\right) = o\left(F(x)\right), \quad x \to 0^+.$$

Hence

$$\frac{f(x)}{x} \int_x^\xi \frac{\delta(u)}{\left(f(u) - \delta(u)\right) f(u)}\,du = o\left(F(x)\frac{f(x)}{x}\right), \quad x \to 0^+.$$

By Karamata's Theorem (see e.g, Theorem 1.5.11 in [24]), since $\beta > 1$, we get $F(x) \sim (\beta-1)^{-1}x/f(x)$ as $x \to 0^+$. Therefore

$$\lim_{x\to 0^+} \frac{f(x)}{x}\int_x^\xi \frac{\delta(u)}{(f(u)-\delta(u))\, f(u)}\,du = 0,$$

as needed. □

We now state a second set of results which show that if the limits in Theorem 2.3.1–2.3.3 on the data change from being zero to being infinite, then $x(t)/F^{-1}(t)$ tends to either zero or infinity. In other words, for each of the perturbed differential equations, the asymptotic behaviour of $x(t)$ is no longer described precisely by $F^{-1}(t)$, and the direct asymptotic behaviour of the unperturbed equation is not preserved.

Theorem 2.3.4. *Suppose f obeys (2.3.1), (2.3.2), (2.3.3). Let F be given by (2.1.5). Let $\eta \in C([0,\infty);\mathbb{R})$ and $t + \int_0^t \eta(s)\,ds \to \infty$ as $t \to \infty$. Suppose further that*

$$\lim_{t\to\infty} \frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)}\int_0^t \eta(s)\,ds =: L,$$

and let x be the solution of (2.3.4).

(i) If $L = +\infty$, then $x(t)/F^{-1}(t) \to 0$ as $t \to \infty$.

(ii) If $L = -\infty$, then $x(t)/F^{-1}(t) \to \infty$ as $t \to \infty$

Theorem 2.3.5. *Suppose f obeys (2.3.1), (2.3.2), (2.3.3). Let F be given by (2.1.5). Let $g \in C([0,\infty);(0,\infty))$, and x be the solution of (2.3.5). Suppose further that*

$$\lim_{t\to\infty} \frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)}\int_0^t \frac{g(s)}{f\left(F^{-1}(t)\right)}\,ds = +\infty.$$

Then

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = +\infty.$$

Theorem 2.3.6. *Suppose f obeys (2.3.1), (2.3.2), (2.3.3). Let F be given by (2.1.5). Let δ be continuous on $[0,\infty)$, $\delta(0) = 0$ and $|\delta(x)| < f(x)$ for all $x \in (0,\xi]$. Suppose further that*

$$\lim_{t\to\infty} \frac{f(x)}{x}\int_x^\xi \frac{\delta(u)}{f(u)\,(f(u)-\delta(u))}\,du =: L,$$

and let x be the solution of (2.3.6).

(i) If $L = +\infty$, then $x(t)/F^{-1}(t) \to +\infty$ as $t \to \infty$.

(ii) If $L = -\infty$, then $x(t)/F^{-1}(t) \to 0$ as $t \to \infty$.

2.4 Proofs

2.4.1 Proof of Theorem 2.3.1

Let $N(t) = \int_0^t \eta(s)\,ds$. Then

$$x(t) = F^{-1}\left(\xi + t + N(t)\right), \quad t \geq 0.$$

Since $f(x)/x \to 0$ as $x \to 0^+$, we have $x(t) \sim F^{-1}\left(t + N(t)\right)$ as $t \to \infty$. Therefore, we need to show that $F^{-1}\left(t + N(t)\right) \sim F^{-1}(t)$ as $t \to \infty$. Fix $t > 0$ and define

$$\delta_t(x) := t + N(t) - F(x).$$

Put $x = F^{-1}\left(t + N(t)\right)$. Then $\delta_t\left(F^{-1}(t + N(t)\right) = 0$. Because $\delta_t'(x) = -F'(x) = 1/f(x) > 0$, we see $x = F^{-1}\left(t + N(t)\right)$ is the unique zero of δ_t. By hypothesis $N(t)f\left(F^{-1}(t)\right)/F^{-1}(t) \to 0$ as $t \to \infty$. Let $k > 0$ and set $x := F^{-1}(t) \cdot k$ and so $F(x/k) = t$. Then

$$\delta_t(x) = \delta_t(kF^{-1}(t)) = F(x/k) - F(x) + N\left(F(x/k)\right).$$

Define $N(t)f\left(F^{-1}(t)\right)/F^{-1}(t) =: a(t) \to 0$ as $t \to \infty$. Hence

$$a(F(x/k)) = N\left(F(x/k)\right)\frac{f(x/k)}{x/k}.$$

Therefore

$$\delta_t(x) = \int_{x/k}^{x} \frac{1}{f(u)}\,du + a\left(F(x/k)\right)\frac{x/k}{f(x/k)}.$$

This yields

$$\frac{f(x/k)}{x/k}\delta_t(x) = k\int_{\alpha=1/k}^{1} \frac{f(x/k)}{f(\alpha x)}\,d\alpha + a\left(F\left(\frac{x}{k}\right)\right). \tag{2.4.1}$$

Put $k = 1 + \epsilon$ for $\epsilon \in (0, 1)$. Then, as f is increasing, and $x = (1+\epsilon)F^{-1}(t)$, we have

$$\begin{aligned}
&\frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)}\delta_t\Big((1+\epsilon)F^{-1}(t)\Big) \\
&= (1+\epsilon)\int_{\alpha=1/(1+\epsilon)}^{1} \frac{f(x/(1+\epsilon))}{f(\alpha x)}\,d\alpha + a\left(F(x/(1+\epsilon))\right) \\
&\geq (1+\epsilon)\left(1 - \frac{1}{1+\epsilon}\right)\frac{f(x/(1+\epsilon))}{f(x)} + a\left(F(x/(1+\epsilon))\right) \\
&= \epsilon\frac{f(x/(1+\epsilon))}{f(x)} + a\left(F(x/(1+\epsilon))\right).
\end{aligned}$$

Since

$$\limsup_{x\to 0^+} \frac{f((1+\epsilon)x)}{f(x)} =: \Phi_\epsilon,$$

we have that

$$\limsup_{y\to 0^+} \frac{f(y)}{f(y/(1+\epsilon))} = \Phi_\epsilon,$$

and therefore

$$\liminf_{y\to 0^+} \frac{f(y/(1+\epsilon))}{f(y)} = \frac{1}{\Phi_\epsilon}.$$

Thus, for every $\epsilon > 0$ there exists $x_1(\epsilon) > 0$ such that $x < x_1(\epsilon)$ implies

$$\frac{f(x/(1+\epsilon))}{f(x)} > \frac{1}{\Phi_\epsilon}\frac{1}{1+\epsilon}.$$

Since $F(x) \to \infty$ as $x \to 0^+$ and $a(t) \to 0$ as $t \to \infty$, for every $\epsilon > 0$ there is $x_2(\epsilon) > 0$ such that

$$\left|a\left(F\left(\frac{x}{1+\epsilon}\right)\right)\right| < \frac{\epsilon}{16}\frac{1}{\Phi_\epsilon}, \quad x < x_2(\epsilon).$$

Let $x(\epsilon) = \min(x_1(\epsilon), x_2(\epsilon))$ and so $(1+\epsilon)F^{-1}(t) < x(\epsilon)$ provided that $t \geq F(x(\epsilon)/(1+\epsilon)) =: T(\epsilon)$. Then, for $t \geq T(\epsilon)$ we have $x := (1+\epsilon)F^{-1}(t) < x(\epsilon)$ and so

$$\begin{aligned}\frac{f(F^{-1}(t))}{F^{-1}(t)}\delta_t\left((1+\epsilon)F^{-1}(t)\right) \geq & \epsilon\frac{f(x/1+\epsilon)}{f(x)} + a(F(x/1+\epsilon)) \\ & > \frac{\epsilon}{2}\frac{1}{\Phi_\epsilon} - \frac{\epsilon}{16}\frac{1}{\Phi_\epsilon} = \frac{7}{16}\epsilon\frac{1}{\Phi_\epsilon} > 0.\end{aligned}$$

Therefore $\delta_t((1+\epsilon)F^{-1}(t)) > 0$ for all $t \geq T(\epsilon)$. By the monotonicity of δ_t and the fact that $\delta_t(y) = 0$ if and only if $y = F^{-1}(t + N(t))$, we have

$$F^{-1}(t + N(t)) < (1+\epsilon)F^{-1}(t), \quad t \geq T(\epsilon).$$

We now get a corresponding lower bound. Take $k = 1/(1+\epsilon)$ in (2.4.1) for $\epsilon \in (0,1)$; then with $x := 1/(1+\epsilon)F^{-1}(t)$, we have

$$\frac{f(F^{-1}(t))}{F^{-1}(t)}\delta_t\left(\frac{F^{-1}(t)}{1+\epsilon}\right) = -\frac{1}{1+\epsilon}\int_{\alpha=1}^{1+\epsilon}\frac{f((1+\epsilon)x)}{f(\alpha x)}\,d\alpha + a(F(x(1+\epsilon))).$$

Since f is increasing we have

$$\frac{1}{1+\epsilon}\int_1^{1+\epsilon}\frac{f((1+\epsilon)x)}{f(\alpha x)}\,d\alpha \geq \frac{\epsilon}{1+\epsilon} > \frac{\epsilon}{2}.$$

Hence

$$\frac{f(F^{-1}(t))}{F^{-1}(t)}\delta_t\left(\frac{1}{1+\epsilon}F^{-1}(t)\right) \leq \frac{-\epsilon}{2} + a(F(x(1+\epsilon))).$$

Now, since $F(x) \to \infty$ as $x \to 0^+$, and $a(t) \to 0$ as $t \to \infty$, there exists $x_3(\epsilon) > 0$

such that $|a\left(F(x(1+\epsilon))\right)| < \frac{\epsilon}{4}$ for $x \le x_3(\epsilon)$. Let $T'(\epsilon) = F\left(x_3(\epsilon)(1+\epsilon)\right)$. Then, for $t \ge T'(\epsilon)$ we have $x = 1/(1+\epsilon)\cdot F^{-1}(t) \le x_3(\epsilon)$ and so

$$\frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)}\delta_t\left(\frac{1}{1+\epsilon}F^{-1}(t)\right) \le \frac{-\epsilon}{2} + \frac{\epsilon}{4} = \frac{-\epsilon}{4} < 0.$$

Hence

$$\delta_t\left(\frac{1}{1+\epsilon}F^{-1}(t)\right) < 0, \quad t \ge T'(\epsilon),$$

and so

$$F^{-1}\left(t+N(t)\right) > \frac{1}{1+\epsilon}F^{-1}(t), \quad t \ge T'(\epsilon).$$

Let $T''(\epsilon) = \max\left(T(\epsilon), T'(\epsilon)\right)$. Then

$$\frac{1}{1+\epsilon}F^{-1}(t) < F^{-1}\left(t+N(t)\right) < (1+\epsilon)F^{-1}(t), \quad t \ge T''(\epsilon).$$

Since $\epsilon > 0$ is arbitrary, we have

$$\lim_{t\to\infty}\frac{F^{-1}\left(t+N(t)\right)}{F^{-1}(t)} = 1,$$

so $x(t) \sim F^{-1}(t)$ as $t \to \infty$, as required.

To prove the converse, define $y(t) = F^{-1}(t)$, which solves $y'(t) = -f\left(y(t)\right)$ with $y(0) = 1$. Since $f(x) = o(x)$ as $x \to 0^+$ and $N(t) := \int_0^t \eta(s)\,ds$, we have

$$x(t) = F^{-1}\left(\zeta + t + N(t)\right) \sim F^{-1}\left(t+N(t)\right) = y(t+N(t)), \quad t \to \infty.$$

By hypothesis, $x(t) \sim F^{-1}(t) = y(t)$ as $t \to \infty$. Therefore, we have $y\left(t+N(t)\right) \sim y(t)$, $t \to \infty$. Now, write

$$\begin{aligned} y\left(t+N(t)\right) - y(t) &= -\int_t^{t+N(t)} f\left(y(s)\right)\,ds \\ &= f\left(y(t)\right)\cdot -N(t)\int_0^1 \frac{f\left(y\left(t+\alpha N(t)\right)\right)}{f\left(y(t)\right)}\,d\alpha \\ &=: f\left(y(t)\right)\cdot -N(t)I(t). \end{aligned}$$

Since $I(t) > 0$, and $y(t) = F^{-1}(t)$, this yields

$$\frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)}\cdot N(t) = \frac{1}{I(t)}\left(1 - \frac{y\left(t+N(t)\right)}{y(t)}\right). \tag{2.4.2}$$

Now, take the case $N(t) \ge 0$. Since y is decreasing, $y(t) \ge y\left(t+\alpha N(t)\right) \ge y\left(t+N(t)\right)$ for $\alpha \in [0,1]$. Hence, as f is increasing

$$\frac{f\left(y(t+N(t)\right)}{f\left(y(t)\right)} \le I(t) \le 1.$$

which implies

$$|I(t)-1| \leq \left|\frac{f\left(y(t+N(t))\right)}{f\left(y(t)\right)}-1\right|, \quad N(t) \geq 0. \tag{2.4.3}$$

Now consider the case when $N(t)<0$. First $y(t) \leq y\left(t+\alpha N(t)\right) \leq y\left(t+N(t)\right)$ for $\alpha \in [0,1]$, and therefore we get $f\left(y(t)\right) \leq f(y\left(t+\alpha N(t)\right)) \leq f(y\left(t+N(t)\right))$. Hence

$$1 \leq I(t) \leq \frac{f(y\left(t+N(t)\right))}{f\left(y(t)\right)}$$

which implies

$$|I(t)-1| \leq \left|\frac{f(y\left(t+N(t)\right))}{f\left(y(t)\right)}-1\right|, \quad N(t) < 0. \tag{2.4.4}$$

By (2.4.3) and (2.4.4), we have the consolidated estimate

$$|I(t)-1| \leq \left|\frac{f(y\left(t+N(t)\right))}{f\left(y(t)\right)}-1\right|, \quad t \geq 0.$$

Since $y\left(t+N(t)\right) \sim y(t)$ as $t \to \infty$, and f is asymptotic preserving, $I(t) \to 1$ as $t \to \infty$. Now, by (2.4.2) and $y\left(t+N(t)\right) \sim y(t)$ as $t \to \infty$, we have

$$\lim_{t\to\infty} \int_0^t \eta(s)\, ds \frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)} = 0,$$

as required.

2.4.2 Proof of Theorem 2.3.2

We start by observing that, as $g(t)>0$, $x(t)>z(t)$ for all $t \geq 0$ where $z(t)$ solves $z'(t) = -f\left(z(t)\right)$ for all $t \geq 0$ with $z(0) = x(0)/2$, and thus we have $x(t) \geq F^{-1}\left(x(0)/2+t\right)$ for all $t \geq 0$. Therefore

$$f\left(x(t)\right) \geq f\left(F^{-1}(x(0)/2+t)\right) =: \alpha(t) f\left(F^{-1}(t)\right)$$

and we notice that $\alpha(t) \to 1$ as $t \to \infty$, because the fact that $f(x)/x \to 0$ as $x \to 0$ implies $F^{-1}\left(x(0)/2+t\right) \sim F^{-1}(t)$ as $t \to \infty$ and as f is asymptotic preserving $f\left(F^{-1}(x(0)/2+t)\right) \sim f\left(F^{-1}(t)\right)$ as $t \to \infty$. Therefore, as $g(t)>0$ for all $t \geq 0$

$$0 < \frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)} \int_0^t \frac{g(s)}{f\left(x(s)\right)}\, ds \leq \frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)} \int_0^t \frac{g(s)}{\alpha(s) f\left(F^{-1}(s)\right)}\, ds. \tag{2.4.5}$$

Since $\alpha(t)>0$ and $\alpha(t) \to 1$ as $t \to \infty$, if

$$\lim_{t\to\infty} \int_0^t \frac{g(s)}{f\left(F^{-1}(s)\right)}\, ds = \infty, \tag{2.4.6}$$

we have

$$\int_0^t \frac{g(s)}{\alpha(s) f\left(F^{-1}(s)\right)}\, ds \sim \int_0^t \frac{g(s)}{f\left(F^{-1}(s)\right)}\, ds, \quad t \to \infty$$

in which case, as $t \to \infty$, we get

$$\frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)} \int_0^t \frac{g(s)}{\alpha(s) f\left(F^{-1}(s)\right)}\, ds \sim \frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)} \int_0^t \frac{g(s)}{f\left(F^{-1}(s)\right)}\, ds \to 0,$$

the last limit holding by assumption. This implies by (2.4.5) that

$$\lim_{t\to\infty} \frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)} \int_0^t \frac{g(s)}{f\left(x(s)\right)}\, ds = 0, \tag{2.4.7}$$

in the case that (2.4.6) holds. The only other alternative to (2.4.6) is

$$\lim_{t\to\infty} \int_0^t \frac{g(s)}{f\left(F^{-1}(s)\right)}\, ds =: K < +\infty. \tag{2.4.8}$$

Hence, as $\alpha(t) \to 1$ as $t \to \infty$, we have that (2.4.8) implies

$$\lim_{t\to\infty} \int_0^t \frac{g(s)}{\alpha(s) f\left(F^{-1}(s)\right)}\, ds = K' < +\infty.$$

Due to the limits $F^{-1}(t) \to 0$ as $t \to \infty$, $f(x)/x \to 0$ as $x \to 0^+$ and $f\left(F^{-1}(t)\right)/F^{-1}(t) \to 0$ as $t \to \infty$, we have

$$\lim_{t\to\infty} \frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)} \int_0^t \frac{g(s)}{\alpha(s) f\left(F^{-1}(s)\right)}\, ds = 0.$$

Hence by (2.4.5), (2.4.7) still prevails. Therefore (2.4.7) is a consequence of the hypotheses. Next, as $g(t) > 0$ for all $t \geq 0$ implies $x(t) > 0$ for all $t \geq 0$, we may introduce

$$\eta(t) := \frac{-g(t)}{f\left(x(t)\right)}, \quad t \geq 0.$$

Then the differential equation can be rewritten as

$$x'(t) = -f\left(x(t)\right)\left(1 + \eta(t)\right), \quad t \geq 0 \tag{2.4.9}$$

and (2.4.7) becomes

$$\lim_{t\to\infty} \frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)} \int_0^t \eta(s)\, ds = 0. \tag{2.4.10}$$

Now, apply Theorem 2.3.1 to the differential equation (2.4.9) under the condition (2.4.10) to get $x(t) \sim F^{-1}(t)$ as $t \to \infty$. This proves the first half of the equivalence.

To prove the converse, we have by assumption that $x(t) \sim F^{-1}(t)$ as $t \to \infty$. Write as above $\eta(t) = -g(t)/f\left(x(t)\right)$ for all $t \geq 0$, and let the differential equation be in the form of (2.4.9). Then, the converse in Theorem 2.3.1 states that $x(t) \sim F^{-1}(t)$ as

$t \to \infty$ implies

$$\lim_{t\to\infty} \frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)} \int_0^t \eta(s)\, ds = 0.$$

Hence, by the definition of η, we have

$$\lim_{t\to\infty} \frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)} \int_0^t \frac{g(s)}{f\left(x(s)\right)}\, ds = 0.$$

Now, as f is asymptotic preserving and $x(t) \sim F^{-1}(t)$ as $t \to \infty$, we have $f\left(x(t)\right) \sim f\left(F^{-1}(t)\right)$ as $t \to \infty$. Therefore, with $f\left(x(t)\right) =: f\left(F^{-1}(t)\right)\beta(t)$, we have $\beta(t) \to 1$ as $t \to \infty$ and

$$\lim_{t\to\infty} \frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)} \int_0^t \frac{g(s)}{f\left(F^{-1}(s)\right)} \cdot \frac{1}{\beta(s)}\, ds = 0.$$

Arguing as above, we see that this implies

$$\lim_{t\to\infty} \frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)} \int_0^t \frac{g(s)}{f\left(F^{-1}(s)\right)}\, ds = 0,$$

proving the desired converse.

2.4.3 Proof of Theorem 2.3.3

Define

$$\Phi(x) = \int_x^\xi \frac{1}{f(u) - \delta(u)}\, du.$$

Then $x(t) = \Phi^{-1}(t)$ for all $t \geq 0$ and $y(t) = F^{-1}(t)$ for all $t \geq 0$. We want to show that $\Phi^{-1}(t) \sim F^{-1}(t)$ as $t \to \infty$. Note that

$$\begin{aligned} F(x) - \Phi(x) &= \int_x^\xi \frac{1}{f(u)}\, du - \int_x^\xi \frac{1}{f(u) - \delta(u)}\, du + \int_\xi^1 \frac{1}{f(u)}\, du \\ &= F(\xi) - \int_x^\xi \frac{\delta(u)}{f(u)\left(f(u) - \delta(u)\right)}\, du =: \Psi(x). \end{aligned}$$

By hypothesis, we have that $\lim_{x\to 0^+} f(x)\Psi(x)/x = 0$. Now, for fixed x, write $\delta_x(z) := \Phi(z) - x$ then $\delta_x\left(\Phi^{-1}(x)\right) = 0$ and $z = \Phi^{-1}(x)$ is the unique solution of $\delta_x(z) = 0$. Note also that $F(x) - \Phi(x) = \Psi(x)$, and so $\delta_x(z) = F(z) - \Psi(z) - x$. Let $K > 0$ and $z := KF^{-1}(x)$ and write $\bar{\epsilon}(x) := \Psi(x)f(x)/x$. Then $\bar{\epsilon}(x) \to 0$ as $x \to 0$ and we have that $\Psi(x) = \bar{\epsilon}(x)x/f(x)$. Thus $x = F(z/K)$ and

$$\delta_x(z) = \frac{z}{f(z)}\left(-\bar{\epsilon}(z) + \frac{f(z)}{z}\int_z^{z/K} \frac{1}{f(u)}\, du\right),$$

so with $z := KF^{-1}(x)$, we have

$$\frac{f(z)}{z}\delta_x(z) = -\bar{\epsilon}(z) + \int_1^{1/K} \frac{f(z)}{f(\alpha z)}\, d\alpha. \tag{2.4.11}$$

Let $K = 1/(1+\epsilon)$ for $\epsilon \in (0,1)$. Then

$$\frac{f(z)}{z}\delta_x(z) = \int_1^{1+\epsilon} \frac{f(z)}{f(\alpha z)}\, dz - \bar{\epsilon}(z).$$

Since f is increasing, we have

$$\int_1^{1+\epsilon} \frac{f(z)}{f(\alpha z)}\, dz \geq \int_1^{1+\epsilon} \frac{f(z)}{f\left((1+\epsilon)z\right)}\, dz = \epsilon \frac{f(z)}{f\left((1+\epsilon)z\right)}.$$

Now, since f is increasing and UAP, there is $\tilde{z}_1(\eta,\epsilon) > 0$ such that $z < \tilde{z}_1(\eta,\epsilon)$ implies

$$\frac{f\left((1+\epsilon)z\right)}{f(z)} < \Phi_\epsilon(1+\eta).$$

Now, let $\eta = \epsilon$ and so $z_1(\epsilon) = \tilde{z}_1(\epsilon,\epsilon)$. Then

$$\frac{f\left((1+\epsilon)z\right)}{f(z)} < \Phi_\epsilon(1+\epsilon), \quad z < z_1(\epsilon).$$

Therefore, for $z < z_1(\epsilon)$

$$\int_1^{1+\epsilon} \frac{f(z)}{f(\alpha z)}\, d\alpha > \epsilon \frac{1}{\Phi_\epsilon(1+\epsilon)} > \epsilon \frac{1}{2\Phi_\epsilon}.$$

Since $\bar{\epsilon}(z) \to 0$ as $z \to 0^+$, there is $z_2(\epsilon) > 0$ such that

$$|\bar{\epsilon}(z)| < \frac{\epsilon}{4\Phi_\epsilon} \wedge \frac{\epsilon}{4}, \quad z < z_2(\epsilon),$$

because $\Phi_\epsilon \to 1$ as $\epsilon \to 0^+$. Therefore, for $z < \min\left(z_1(\epsilon), z_2(\epsilon)\right) =: z(\epsilon)$

$$\int_1^{1+\epsilon} \frac{f(z)}{f(\alpha z)}\, d\alpha - \bar{\epsilon}(z) > \frac{\epsilon}{4\Phi_\epsilon} > 0.$$

Thus, as $z = 1/(1+\epsilon)\cdot F^{-1}(x)$, we have for $1/(1+\epsilon)\cdot F^{-1}(x) < z(\epsilon)$ that $z < z(\epsilon)$, so

$$\frac{f\left(1/(1+\epsilon)\cdot F^{-1}(x)\right)}{1/(1+\epsilon)\cdot F^{-1}(x)}\delta_x\left(1/(1+\epsilon)\cdot F^{-1}(x)\right) > \frac{\epsilon}{4\Phi_\epsilon} > 0.$$

Thus $x > F\left((1+\epsilon)z(\epsilon)\right) =: x_1(\epsilon)$ implies $\delta_x\left(1/(1+\epsilon)\cdot F^{-1}(x)\right) > 0$.

Next, suppose that $K = 1+\epsilon$. Then

$$\frac{f(z)}{z}\delta_x(z) = -\int_{1/(1+\epsilon)}^{1} \frac{f(z)}{f(\alpha z)}\, d\alpha - \bar{\epsilon}(z).$$

Since f is increasing, we get

$$\int_{1/(1+\epsilon)}^{1} \frac{f(z)}{f(\alpha z)}\, d\alpha \geq 1 - \frac{1}{1+\epsilon} = \frac{\epsilon}{1+\epsilon} > \frac{\epsilon}{2}.$$

Note that $|\bar{\epsilon}(z)| < \epsilon/4$ for $z < z(\epsilon)$, so for $z < z(\epsilon)$ we have

$$\frac{f(z)}{z}\delta_x(z) = -\int_{1/1+\epsilon}^{1} \frac{f(z)}{f(\alpha z)}\, d\alpha - \bar{\epsilon}(z) < \frac{-\epsilon}{2} + |\bar{\epsilon}(z)| < \frac{-\epsilon}{4} < 0.$$

Thus as $z = (1+\epsilon)F^{-1}(x)$, we have for $(1+\epsilon)F^{-1}(x) < z(\epsilon)$ that $z < z(\epsilon)$, so

$$\frac{f\left((1+\epsilon)F^{-1}(x)\right)}{(1+\epsilon)F^{-1}(x)}\delta_x\left((1+\epsilon)F^{-1}(x)\right) = -\bar{\epsilon}(z) + \int_{1}^{1/1+\epsilon} \frac{f(z)}{f(\alpha z)}\, d\alpha < \frac{-\epsilon}{4} < 0.$$

Therefore, for $x > F\left(1/(1+\epsilon)\cdot z(\epsilon)\right) =: x_2(\epsilon)$, $\delta_x\left((1+\epsilon)F^{-1}(x)\right) < 0$. Let $x(\epsilon) = \max\left(x_1(\epsilon), x_2(\epsilon)\right)$. Then for $x > x(\epsilon)$, $\delta_x\left((1+\epsilon)F^{-1}(x)\right) < 0$ and $\delta_x\left((1/1+\epsilon)F^{-1}(x)\right) > 0$. Hence, as $\delta_x\left(\Phi^{-1}(x)\right) = 0$ and $z = \Phi^{-1}(x)$ is the unique solution of $\delta_x(z) = 0$, we have

$$\frac{1}{1+\epsilon}F^{-1}(x) < \Phi^{-1}(x) < (1+\epsilon)F^{-1}(x), \quad x > x(\epsilon).$$

Therefore, as $\epsilon \in (0,1)$ is arbitrary, we can let $\epsilon \to 0$ to get

$$\lim_{t\to\infty} \frac{\Phi^{-1}(t)}{F^{-1}(t)} = 1,$$

and thus we have shown the first part of the equivalence.

To prove the converse, define

$$\eta(t) := \frac{-\delta(x(t))}{f(x(t))}, \quad t \geq 0.$$

Then η is well-defined, as $x(t) > 0$ for all $t \geq 0$, and we have

$$x'(t) = -f(x(t))\left(1 + \eta(t)\right), \quad t \geq 0.$$

Define also

$$\begin{aligned} N(t) := \int_0^t \eta(s)\, ds &= \int_0^t -\frac{\delta\left(x(s)\right)}{f\left(x(s)\right)}\, ds \\ &= \int_0^t -\frac{\delta\left(x(s)\right)}{f\left(x(s)\right)} \cdot \frac{x'(s)}{-f\left(x(s)\right) + \delta\left(x(s)\right)}\, du \\ &= \int_{x(t)}^{\xi} \frac{-\delta(u)}{f(u)} \frac{1}{f(u) - \delta(u)}\, du. \end{aligned}$$

Since $x(t) \sim F^{-1}(t)$ as $t \to \infty$, so as f is UAP, we have $f\left(F^{-1}(t)\right) \sim f\left(x(t)\right)$ as $t \to \infty$.

Hence, as $t \to \infty$, we have

$$\begin{aligned}\frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)}N(t) &= -\frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)}\int_{x(t)}^{\xi}\frac{\delta(u)}{f(u)\left(f(u)-\delta(u)\right)}\,du \\ &\sim -\frac{f(x(t))}{x(t)}\int_{x(t)}^{\xi}\frac{\delta(u)}{f(u)\left(f(u)-\delta(u)\right)}\,du.\end{aligned}$$

By the converse part of Theorem 2.3.1, we have that the left hand side of the last asymptotic relation tends to zero as $t \to \infty$. Hence

$$\lim_{t\to\infty}\frac{f\left(x(t)\right)}{x(t)}\int_{x(t)}^{\xi}\frac{\delta(u)}{f(u)\left(f(u)-\delta(u)\right)}\,du = 0.$$

Since $t \mapsto x(t)$ is continuous and $x(t) \to 0$ as $t \to \infty$, this last limit yields

$$\lim_{x\to 0^+}\frac{f(x)}{x}\int_{x(t)}^{\xi}\frac{\delta(u)}{f(u)\left(f(u)-\delta(u)\right)}\,du = 0,$$

as required.

2.4.4 Proof of Theorem 2.3.4

Since f is sublinear, if we write $N(t) := \int_0^t \eta(s)\,ds$, the usual considerations yield $x(t) \sim F^{-1}\left(t + N(t)\right)$, $t \to \infty$. Define

$$L(t) := \frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)}\int_0^t \eta(s)\,ds = \frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)}N(t).$$

Suppose $L(t) \to \infty$ as $t \to \infty$; we hypothesise that $x(t) = o\left(F^{-1}(t)\right)$ as $t \to \infty$, which is true if $F^{-1}\left(t + N(t)\right)/F^{-1}(t) = o(1)$ as $t \to \infty$. Define for a fixed $t > 0$

$$\delta_t(x) = t + N(t) - F(x).$$

Then $\delta_t(x) = 0$ if and only if $x = F^{-1}\left(t + N(t)\right)$. Also, $\delta_t'(x) = -F'(x) = 1/f(x) > 0$ and thus $\delta_t(x) > 0$ if and only if $x > F^{-1}\left(t + N(t)\right)$. Let $K \in (0,\infty)$ and consider $\delta_t\left(KF^{-1}(t)\right)$. Then

$$\begin{aligned}\delta_t\left(KF^{-1}(t)\right) &= t + N(t) - F\left(KF^{-1}(t)\right) \\ &= \int_{F^{-1}(t)}^{1}\frac{1}{f(u)}\,du - \int_{KF^{-1}(t)}^{1}\frac{1}{f(u)}\,du + \frac{F^{-1}(t)}{f\left(F^{-1}(t)\right)}L(t) \\ &= \frac{F^{-1}(t)}{f\left(F^{-1}(t)\right)}L(t) - \int_{KF^{-1}(t)}^{F^{-1}(t)}\frac{1}{f(u)}\,du.\end{aligned}$$

Hence

$$\frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)}\delta_t\left(KF^{-1}(t)\right) = L\left(F(F^{-1}(t))\right) - \frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)}\int_{KF^{-1}(t)}^{F^{-1}(t)}\frac{1}{f(u)}\,du$$
$$=: \mu_K\left(F^{-1}(t)\right),$$

where

$$\mu_K(x) = L\left(F(x)\right) - \int_K^1 \frac{f(x)}{f(\lambda x)}\,d\lambda.$$

Let $\mu_0 > 1$ be arbitrary. Let $N = N(K) \in \mathbb{N}$ such that $\mu_0^{-N} \geq K$, $\mu_0^{-(N+1)} < K$. Then

$$\int_K^1 \frac{f(x)}{f(\lambda x)}\,d\lambda = \sum_{j=0}^{N-1}\int_{\mu_0^{-(j+1)}}^{\mu_0^{-j}}\frac{f(x)}{f(\lambda x)}\,d\lambda + \int_K^{\mu_0^{-N}}\frac{f(x)}{f(\lambda x)}\,d\lambda.$$

Since f is increasing, we have

$$\frac{1}{f(\mu_0^{-(j+1)}x)} > \frac{1}{f(\lambda x)}, \quad \text{for} \quad \lambda \in [\mu_0^{-(j+1)}, \mu_0^{-j}].$$

Hence

$$\int_K^1 \frac{f(x)}{f(\lambda x)}\,d\lambda \leq \sum_{j=0}^{N-1}\left[\mu_0^{-j} - \mu_0^{-(j+1)}\right]\frac{f(x)}{f\left(\mu_0^{-(j+1)}x\right)} + \left(\mu_0^{-N} - K\right)\frac{f(x)}{f(Kx)}$$
$$= \left(1 - \mu_0^{-1}\right)\sum_{j=0}^{N-1}\mu_0^{-j}\frac{f(x)}{f\left(\mu_0^{-(j+1)}x\right)} + \left(\mu_0^{-N} - K\right)\frac{f(x)}{f(Kx)}.$$

Now, since f is UAP, it follows that there exists $\Phi_{\mu_0} \in (1, \infty)$ such that

$$\limsup_{x\to 0^+}\frac{f(\mu_0 x)}{f(x)} =: \Phi_{\mu_0}.$$

Hence

$$\limsup_{x\to 0^+}\frac{f(x)}{f\left(\mu_0^{-(j+1)}x\right)} = \limsup_{y\to 0^+}\frac{f(\mu_0^{j+1}y)}{f(y)} \leq \Phi_{\mu_0}^{j+1}.$$

Also, as $K > \mu_0^{-(j+1)}$, we have

$$\frac{1}{f(Kx)} < \frac{1}{f\left(\mu_0^{-(N+1)}x\right)}.$$

Therefore

$$\limsup_{x\to 0^+}\int_K^1 \frac{f(x)}{f(\lambda x)}\,d\lambda \leq \left(1 - \mu_0^{-1}\right)\sum_{j=0}^{N-1}\mu^{-j}\Phi_\mu^{j+1} + \left(\mu^{-N} - K\right)\Phi_{\mu_0}^{N+1}$$
$$= \left(1 - \mu^{-1}\right)\Phi_{\mu_0}\sum_{j=0}^{N-1}\left(\frac{\Phi_{\mu_0}}{\mu_0}\right)^j + \left(\mu_0^{-N} - K\right)\Phi_{\mu_0}^{N+1}.$$

Since Φ_{μ_0}, μ_0 are finite and K-independent, and $N = N(K)$, we see that the lim sup is finite, and bounded by a K-dependent quantity i.e.,

$$\limsup_{x\to 0^+} \int_K^1 \frac{f(x)}{f(\lambda x)}\, d\lambda =: A(K) \in (0,\infty),$$

for any choice of $K \in (0,1)$. Therefore, for each $K \in (0,1)$, there is $x_1(K) > 0$ such that $x < x_1(K)$ implies

$$\int_K^1 \frac{f(x)}{f(\lambda x)}\, d\lambda \le 2A(K).$$

Since $L(t) \to \infty$ as $t \to \infty$, and $F(x) \to \infty$ as $x \to 0^+$, $L(F(x)) \to \infty$ as $x \to 0^+$. Thus, there is $\tilde{x}_2(K) > 0$ such that $x < \tilde{x}_2(K)$ implies

$$L(F(x)) > 4A(K) + 1, \quad x < \tilde{x}_2(K).$$

Let $x_3(K) = \min(x_1(K), \tilde{x}_2(K))$. Then for $x < x_3(K)$

$$\mu_K(x) = L(F(x)) - \int_{\lambda=K}^1 \frac{f(x)}{f(\lambda x)}\, d\lambda > 2A(K) + 1 > 0.$$

Now, let $t \ge T(K) = F(x_3(K))$. Then $F^{-1}(t) < x_3(K)$ and $\mu_K(F^{-1}(t)) > 2A(K) + 1$. Thus

$$\frac{f(F^{-1}(t))}{F^{-1}(t)} \delta_t\left(KF^{-1}(t)\right) = \mu_K\left(F^{-1}(t)\right) > 2A(K) + 1 > 0$$

and so $\delta_t(KF^{-1}(t)) > 0$ for $t \ge T(K)$. Therefore, $KF^{-1}(t) > F^{-1}(t + N(t))$ for all $t \ge T(K)$. Hence

$$\limsup_{t\to\infty} \frac{F^{-1}(t + N(t))}{F^{-1}(t)} \le K.$$

Now, as $K \in (0,1)$ is arbitrary, we may let $K \to 0^+$ to get

$$\lim_{t\to\infty} \frac{F^{-1}(t + N(t))}{F^{-1}(t)} = 0,$$

so $x(t) = o(F^{-1}(t))$ as $t \to \infty$, as required.

To prove (ii), take as before $\delta_t(x) = t + N(t) - F(x)$, so that $\delta_t(x) < 0$ if and only if $x < F^{-1}(t + N(t))$. Let $K > 1$ and consider $\delta_t(KF^{-1}(t))$; if $\delta_t(KF^{-1}(t)) > 0$ for all $t \ge T(K)$ then $KF^{-1}(t) < F^{-1}(t + N(t))$ for all $t \ge T(K)$ and from this we get $x(t)/F^{-1}(t) \to \infty$ as $t \to \infty$.

As before, with $L_-(t) := -L(t) \to \infty$ as $t \to \infty$,

$$\begin{aligned}\frac{f(F^{-1}(t))}{F^{-1}(t)} \delta_t\left(KF^{-1}(t)\right) &= \frac{f(F^{-1}(t))}{F^{-1}(t)} \int_{F^{-1}(t)}^{KF^{-1}(t)} \frac{1}{f(u)}\, du - L_-(t) \\ &=: \mu_K^-\left(F^{-1}(t)\right),\end{aligned}$$

where
$$\mu_K^-(x) = \int_{\lambda=1}^{K} \frac{f(x)}{f(\lambda x)}\, d\lambda - L_-\left(F(x)\right).$$
Now, as f is increasing, $f(\lambda x) \geq f(x)$ for $\lambda \geq 1$, $\mu_K^-(x) \leq K - 1 - L_-\left(F(x)\right)$. Since $F(x) \to \infty$ as $x \to 0+$ and $L_-(t) \to \infty$ as $t \to \infty$, for all $x \leq x_1(K)$, $L_-\left(F(x)\right) > K+1$. Therefore, for $x \leq x_1(K)$, $\mu_K^-(x) < -2 < 0$. Now, let $T(K) = F\left(x_1(K)\right)$; for $t \geq T(K)$, $F^{-1}(t) \leq x_1(K)$. Hence $\mu_K^-\left(F^{-1}(t)\right) < -2 < 0$. Therefore, for $t \geq T(K)$
$$\frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)} \delta_t\left(KF^{-1}(t)\right) = \mu_K^-\left(F^{-1}(t)\right) < 0,$$
so $\delta_t\left(KF^{-1}(t)\right) < 0$ for $t \geq T(K)$. Hence $KF^{-1}(t) < F^{-1}\left(t + N(t)\right)$ for $t \geq T(K)$. Therefore, for any $K > 1$,
$$\liminf_{t\to\infty} \frac{F^{-1}\left(t+N(t)\right)}{F^{-1}(t)} \geq K.$$
Since $K > 1$ is arbitrary, we may let $K \to \infty$ to get
$$\liminf_{t\to\infty} \frac{F^{-1}\left(t+N(t)\right)}{F^{-1}(t)} = +\infty.$$
Hence
$$\lim_{t\to\infty} \frac{F^{-1}\left(t+N(t)\right)}{F^{-1}(t)} = +\infty$$
and therefore
$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = +\infty,$$
as required.

2.4.5 Proof of Theorem 2.3.5

Define $\eta(t) = -g(t)/\left(f(x)(t)\right)$; this is well-defined as $x(t) > 0$ for all $t \geq 0$. Then $g(t) = -f\left(x(t)\right)\eta(t)$, so
$$x'(t) = -f(x(t))\left(1 + \eta(t)\right). \tag{2.4.12}$$
Now, as $g(t) > 0$, $x'(t) > -f\left(x(t)\right)$ for all $t \geq 0$ with $x(0) = \xi$. Therefore $x(t) > F^{-1}(t)/\left(F(\xi) + t\right)$ for all $t \geq 0$. Hence $f\left(x(t)\right) > \left(f \circ F^{-1}\right)\left(F(\xi) + t\right)$ for all $t \geq 0$. Since $F^{-1}(t)/\left(F(\xi) + t\right) \sim F^{-1}(t)$ as $t \to \infty$ and f is asymptotic preserving, we have that $\liminf_{t\to\infty} f(x(t))/f\left(F^{-1}(t)\right) \geq 1$. Suppose
$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t)} < +\infty. \tag{2.4.13}$$
Then $x(t) \leq C_1 F^{-1}(t)$ for all $t \geq T$ and $f\left(x(t)\right) \leq f\left(C_1 F^{-1}(t)\right)$ for all $t \geq T$. Since f is UAP, $f\left(x(t)\right) \leq C_2 f\left(F^{-1}(t)\right)$ for all $t \geq T$ for some $C_2 > 0$. From

$\liminf_{t\to\infty} f(x(t))/f\left(F^{-1}(t)\right) \geq 1$, there is $C_0 > 0$ such that $C_0 f\left(F^{-1}(t)\right) \leq f\left(x(t)\right)$ for $t \geq T$. Thus

$$\frac{1}{C_2}\frac{g(t)}{f\left(F^{-1}(t)\right)} \leq -\eta(t) \leq \frac{1}{C_0}\frac{g(t)}{f\left(F^{-1}(t)\right)}, \quad t \geq T.$$

This yields

$$\frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)}\frac{1}{C_2}\int_T^t \frac{g(s)}{f\left(F^{-1}(s)\right)}\,ds \leq \frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)}\int_T^t -\eta(s)\,ds$$

Thus

$$\lim_{t\to\infty}\frac{f\left(F^{-1}(t)\right)}{F^{-1}(t)}\int_T^t \eta(s)\,ds = -\infty,$$

which implies, by Theorem 2.3.4, that $x(t)/F^{-1}(t) \to \infty$ as $t \to \infty$, which is a contradiction to the supposition in (2.4.13). Hence we must have that

$$\limsup_{t\to\infty}\frac{x(t)}{F^{-1}(t)} = +\infty,$$

as required.

2.4.6 Proof of Theorem 2.3.6

We have that $f(x) > \delta(x)$ for all $x \in (0,\delta') \supseteq (0,\xi)$. Therefore $x(t) \to 0$ as $t \to \infty$, $x(t) > 0$ for all $t \geq 0$. Define

$$\Phi)(x) := \int_x^\xi \frac{1}{f(u)-\delta(u)}\,du.$$

Then $x(t) = \Phi^{-1}(t)$ for all $t \geq 0$. Let $F(x) = \int_x^1 1/f(u)\,du$ for $0 < x \leq 1$. Note

$$\Phi(x) - F(x) = \int_x^\xi \frac{\delta(u)}{f(u)\left(f(u)-\delta(u)\right)}\,du - F(\xi).$$

Since $f(x)/x \to 0$ as $x \to 0^+$

$$L(x) := \frac{f(x)}{x}\left(\Phi(x) - F(x)\right) \to L, \quad x \to 0^+.$$

Next, define for fixed $x > 0$

$$\delta_x(y) := \Phi(y) - x = L(y)\frac{y}{f(y)} + F(y) - x.$$

Clearly $\delta_x'(y) < 0$ for all $y < \xi$ and $y = \Phi^{-1}(x)$ is the unique solution of $\delta_x(y) = 0$. Let $K > 0$ and consider $\delta_x\left(KF^{-1}(x)\right)$. Then

$$\frac{f\left(KF^{-1}(x)\right)}{KF^{-1}(x)}\delta_x\left(KF^{-1}(x)\right) = L\left(KF^{-1}(x)\right) - \frac{f\left(KF^{-1}(x)\right)}{KF^{-1}(x)}\int_{F^{-1}(x)}^{KF^{-1}(x)}\frac{1}{f(u)}\,du. \quad (2.4.14)$$

We first consider the case $L(y) \to \infty$ as $y \to 0^+$. Let $K > 1$ be arbitrary, then

$$\frac{f\left(KF^{-1}(x)\right)}{KF^{-1}(x)}\delta_x\left(KF^{-1}(x)\right) = \mu_K\left(F^{-1}(x)\right),$$

where

$$\mu_K(y) := L(Ky) - \frac{1}{K}\int_1^K \frac{f(Ky)}{f(\lambda y)}\,d\lambda.$$

We now estimate the integral term. Let $\mu_0 > 1$ be such that

$$\limsup_{x\to 0^+}\frac{f(\mu_0 x)}{f(x)} =: \Phi_{\mu_0} < +\infty,$$

the finiteness arising from the fact that f is UAP. Since $\mu_0 > 1$, there is $N = N(K)$ such that $\mu_0^N \le K$, and $\mu_0^{N+1} > K$. Write

$$\frac{1}{K}\int_1^K \frac{f(Ky)}{f(\lambda y)}\,d\lambda = \frac{1}{K}\sum_{j=0}^{N}\int_{\mu_0^{j-1}}^{\mu_0^j}\frac{f(Ky)}{f(\lambda y)}\,d\lambda + \frac{1}{K}\int_{\mu_0^N}^{K}\frac{f(Ky)}{f(\lambda y)}\,d\lambda.$$

By the monotonicity of f, $1/f(\lambda y) \le 1/f(\mu_0^{j-1}y)$ for $\lambda \in [\mu_0^{j-1}, \mu_0^j]$, and $1/f(\lambda y) \le 1/f(\mu_0^N y)$ for $\lambda \in [\mu_0^N, K]$. Hence

$$\frac{1}{K}\int_1^K \frac{f(Ky)}{f(\lambda y)}\,d\lambda \le \frac{1}{K}\sum_{j=1}^{N}\left(\mu_0^j - \mu_0^{j-1}\right)\frac{f(Ky)}{f(\mu_0^{j-1}y)} + \frac{K-\mu_0^N}{K}\frac{f(Ky)}{f(\mu_0^N y)}.$$

Since $K < \mu_0^{N+1}$, we get

$$\frac{1}{K}\int_1^K \frac{f(Ky)}{f(\lambda y)}\,d\lambda \le \frac{1}{K}(\mu_0 - 1)\sum_{j=1}^{N}\mu_0^{j-1}\frac{f(\mu_0^{N+1}y)}{f(\mu_0^{j-1}y)} + \frac{K-\mu_0^N}{K}\frac{f(\mu_0^{N+1}y)}{f(\mu_0^N y)}. \quad (2.4.15)$$

Next, we have

$$\limsup_{y\to 0^+}\frac{f(\mu_0^{N+1}y)}{f(\mu_0^{j-1}y)} \le \Phi_{\mu_0}^{(N+1)-(j-1)} = \Phi_{\mu_0}^{N-j+2}.$$

Therefore, as the sums in (2.4.15) are finite

$$\limsup_{y\to 0^+}\frac{1}{K}\int_1^K \frac{f(Ky)}{\lambda y}\,d\lambda \le \frac{1}{K}(\mu_0 - 1)\sum_{j=1}^{N}\mu_0^{j-1}\Phi_{\mu_0}^{N-j+2} + \frac{K-\mu_0^N}{K}\Phi_{\mu_0}.$$

Since $N = N(K)$, the right hand side is K-dependent (μ_0 having been chosen independently of K). Hence

$$\limsup_{y\to 0^+} \frac{1}{K}\int_1^K \frac{f(Ky)}{f(\lambda y)}\,d\lambda =: A(K) \in (0,\infty),$$

Therefore, for every $K > 1$ and $\eta \in (0,1)$ there exists $\tilde{y}_1(K,\eta) > 0$ such that $y \le \tilde{y}_1(K,\eta)$ implies

$$\frac{1}{K}\int_1^K \frac{f(Ky)}{f\lambda y)}\,d\lambda \le A(K) + \eta.$$

Pick $\eta = 1/2$ and $y_1(K) = \tilde{y}_1(K, 1/2)$. Then $y \le y_1(K)$ implies

$$\frac{1}{K}\int_1^K \frac{f(Ky)}{f\lambda y)}\,d\lambda \le A(K) + \frac{1}{2}.$$

Next $L(y) \to \infty$ as $y \to 0^+$; in particular, for each $K > 1$ there exists $\tilde{y}_2(K) > 0$ such that

$$L(y) > 2A(K) + 1, \quad y \le \tilde{y}_2(K).$$

Take $y_2(K) = \tilde{y}_2(K)/K$; then $y \le y_2(K)$ implies $Ky < \tilde{y}_2(K)$. Hence for $y \le y_2(K)$ we have that $L(Ky) > 2A(K) + 1$. Now, let $y^*(K) = \min(y_1(K), y_2(K))$. Then for $y \le y^*(K)$

$$\mu_K(y) > 2A(K) + 1 - \left(A(K) + \frac{1}{2}\right) = A(K) + \frac{1}{2} > 0.$$

Next, let $x \ge x^*(K) := F(y^*(K))$. Then $F^{-1}(x) \le y^*(K)$, and therefore $\mu_K(F^{-1}(x)) > A(K) + 1/2 > 0$. Hence for $x \ge x^*(K)$

$$\frac{f(KF^{-1}(x))}{KF^{-1}(x)}\delta_x\left(KF^{-1}(x)\right) = \mu_K\left(F^{-1}(x)\right) > 0,$$

Thus $\delta_x(KF^{-1}(x)) > 0$. Thus, as $y \mapsto \delta_x(y)$ is decreasing and $\delta_x(\Phi^{-1}(x)) = 0$, $KF^{-1} < \Phi^{-1}(x)$ for $x \ge x^*(K)$. Therefore for each $K > 1$, there exists $x^*(K) > 0$ such that

$$\frac{\Phi^{-1}(x)}{F^{-1}(x)} > K, \quad x \ge x^*(K).$$

This implies $\lim_{x\to\infty} \Phi^{-1}(x)/F^{-1}(x) = +\infty$, or $\lim_{t\to\infty} x(t)/F^{-1}(t) = +\infty$, as required.

We now consider the case (ii) when $L(y) \to -\infty$ as $y \to 0^+$. Let $K \in (0,1)$ be arbitrary. Then by (2.4.14)

$$\frac{f(KF^{-1}(x))}{KF^{-1}(x)}\delta_x\left(KF^{-1}(x)\right) = L\left(KF^{-1}(x)\right) + \frac{1}{K}\int_K^1 \frac{f(KF^{-1}(x))}{f(\lambda F^{-1}(x))}\,d\lambda$$
$$=: \mu_K^-\left(F^{-1}(x)\right)$$

where
$$\mu_K^-(y) := L(Ky) + \frac{1}{K}\int_K^1 \frac{f(Ky)}{f(\lambda y)}\,d\lambda.$$
Since f is increasing,
$$\frac{1}{K}\int_K^1 \frac{f(Ky)}{f(\lambda y)}\,d\lambda \le \frac{1-K}{K}.$$
Hence we have that
$$\mu_K^-(y) \le L(Ky) + \frac{1-K}{K}.$$
Since $L(y) \to -\infty$ as $y \to 0^+$, there exists $\tilde{y}_1(K) > 0$ such that $y \le \tilde{y}_1(K)$ implies $-L(y) > 1 + (1-K)/K$. Next, let $y_1(K) = \tilde{y}_1(K)/K$. Then, for $y \le y_1(K)$ we have $Ky \le Ky_1(K) = \tilde{y}_1(K)$, and so $-L(Ky) > 1 + (1-K)/K$. Hence for $y \le y_1(K)$ we have that $L(Ky) < -1 - (1-K)/K$ and therefore

$$\mu_K^-(y) \le L(Ky) + \frac{1-K}{K} < -1 - \frac{(1-K)}{K} + \frac{1-K}{K} = -1 < 0.$$

Next, let $x \ge x_1(K) := F(y_1(K))$. Then $F^{-1}(x) \le y_1(K)$, so $\mu_K^-(F^{-1}(x)) < 0$. Thus $x \ge x_1(K)$ implies

$$\frac{f(KF^{-1}(x))}{KF^{-1}(x)}\delta_x\left(KF^{-1}(x)\right) = \mu_K^-\left(F^{-1}(x)\right) < 0,$$

so $\delta_x(KF^{-1}(x)) < 0$ for all $x \ge x_1(K)$. Hence, as δ_x is increasing and $\delta_x(\Phi^{-1}(x)) = 0$, we have that
$$\Phi^{-1}(x) < KF^{-1}(x), \quad x \ge x_1(K).$$
Therefore, we have shown that for every $K \in (0,1)$, there exists $x_1(K) > 0$ such that

$$0 < \frac{\Phi^{-1}(x)}{F^{-1}(x)} < K \quad x \ge x_1(K).$$

Hence
$$\lim_{x\to\infty} \frac{\Phi^{-1}(x)}{F^{-1}(x)} = 0,$$
which implies that
$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = 0,$$
completing the proof.

Chapter 3

Asymptotic Behaviour of Continuous-time Equations with Rapidly Decaying Solutions

3.1 Introduction

In the last chapter, we considered what we call Hartman–Wintner–type results, where the solution of the perturbed ODE

$$x'(t) = -f(x(t)) + g(t), \quad t > 0; \quad x(0) = \zeta. \tag{3.1.1}$$

is compared to that of the unperturbed ODE

$$y'(t) = -f(y(t)), \quad t > 0. \tag{3.1.2}$$

The type of result desired in that case is that

$$\lim_{t\to\infty} \frac{x(t)}{y(t)} = 1.$$

or

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = 1$$

where

$$F(x) := \int_x^1 \frac{1}{f(u)}\,du \tag{3.1.3}$$

obeys $F(x) \to \infty$ as $x \to 0^+$.

However, we also saw that if f obeys $f(x)/x \to \infty$ as $x \to 0^+$ (or more generally when f is such that the function F^{-1} is not asymptotic preserving) we may not be able to achieve the direct (or explicit) asymptotic behaviour of $x(t)$ as $t \to \infty$. Instead, we saw that the Liapunov–exponent measure

$$\lim_{t\to\infty} \frac{F(x(t))}{t}$$

seems to be more robust to changes in the initial conditions or the righthand side of the differential equation. Furthermore, focussing on (3.1.1) in particular, our results in Chapter 2 do not give results when the sign of g changes. This is a limitation in itself, and suggests that more work is needed, especially if we wish to prove results for stochastic differential equations, in which control of the sign of perturbations is likely to be very difficult.

For this reason, we attempt in this chapter to emulate the results obtained in Appleby and Patterson [11] concerning the equation (3.1.1) as well as the related stochastic differential equation

$$dX(t) = -f(X(t))\,dt + \sigma(t)\,dB(t), \quad t \geq 0. \tag{3.1.4}$$

where σ is a continuous and deterministic function. In that paper, it was assumed that $f \in \text{RV}_0(\beta)$ for $\beta > 1$. One implication of this hypothesis is that the solution of the unperturbed equation (3.1.2) exhibits power–law decay, in the sense that

$$\lim_{t\to\infty} \frac{\log y(t)}{\log t} = -c,$$

for some $c > 0$ depending on β. Moreover, F^{-1} preserves asymptotic behaviour (so that Hartman–Wintner and Hartman–Grobman results are equivalent).

We wish to extend these results to deal with faster than power decay in the solution of (3.1.2), in the sense that

$$\lim_{t\to\infty} \frac{\log y(t)}{\log t} = -\infty, \tag{3.1.5}$$

Moreover, the rate of decay to 0 can be subexponential, exponential or superexponential according as to whether

$$\alpha := \lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} f'(0), \tag{3.1.6}$$

is zero, finite but nonzero or infinite. We may define a function y to be *subexponential* if

$$\lim_{t\to\infty} \sup_{0\leq s\leq T} \left| \frac{y(t-s)}{y(t)} - 1 \right| = 0, \quad \text{for all } T > 0, \tag{3.1.7}$$

and (decaying) *superexponential* if

$$\lim_{t\to\infty} \frac{y(t-T)}{y(t)} = +\infty, \quad \text{for all } T > 0. \tag{3.1.8}$$

We note that (3.1.7) implies
$$\lim_{t\to\infty} \frac{1}{t}\log y(t) = 0,$$
and (3.1.8) implies
$$\lim_{t\to\infty} \frac{1}{t}\log y(t) = -\infty.$$
In fact using (3.1.6), we have
$$\lim_{t\to\infty} \frac{\log y(t)}{t} = -\alpha.$$

The results in Appleby and Patterson [11] concern rapidly decaying perturbations, are of a Hartman–Wintner type, and assume regular variation in f. Here, in contrast, we study both slowly and rapidly decaying perturbations, drop the regular variation assumption on f in favour or the asymptotic preserving property introduced in the last chapter, and concentrate on Hartman–Grobman type results. The last choice arises because, when solutions of (3.1.2) decay more rapidly than a power, they are likely to be rapidly varying in the sense that

$$\lim_{t\to\infty} \frac{y(\lambda t)}{y(t)} = \begin{cases} 0, & \lambda > 1; \\ 1, & \lambda = 1; \\ \infty, & \lambda < 1. \end{cases}$$

and this is equivalent to asking that F^{-1} is a rapidly varying function. Hence F^{-1} will not be asymptotic preserving. From the experience of the previous chapter, we know that this makes the proof of Hartman–Wintner type results very difficult or even impossible. A sufficient condition to ensure the asymptotic preserving properties of f and the rapid variation of F^{-1} is to assume that f is regularly varying at zero with index 1, but we prefer to rely less on such a *quantitative regular variation hypothesis* in favour of the more *qualitative asymptotic preserving* assumption. It is to be hoped, incidentally, that the earlier work of Appleby and Patterson [11] can be generalised in this spirit by replacing regular variation hypotheses with asymptotic preserving ones, but this does not form part of this thesis.

Despite these differences, the paper [11] provides much inspiration as to how to proceed in our case. A typical result, which is of Hartman–Wintner type, is that the solution of (3.1.1) obeys
$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t)} = \lambda \in \{-1, 0, 1\}$$
if and only if
$$\lim_{t\to\infty} \int_0^t g(s)\,ds \text{ is finite}, \qquad \lim_{t\to\infty} \frac{\int_t^\infty g(s)\,ds}{F^{-1}(t)} = 0.$$
The sufficiency part of this result requires some symmetry in f at zero, and is proven by

considering the asymptotic behaviour of the internally perturbed differential equation

$$z'(t) = -f(z(t) + \Gamma(t)), \quad t \geq 0,$$

where $\Gamma(t) = -\int_t^\infty g(s)\,ds$ and $x(t) = z(t) + \Gamma(t)$ for $t \geq 0$. This asymptotic behaviour is established by careful comparison arguments, the most important of which rely on the construction of explicit upper and lower bounding functions. We adopt this general strategy here too; a comparable "small" perturbation result in this chapter would take a Hartman–Grobman form, and has the general conclusion that

$$\lambda_x := \liminf_{t\to\infty} \frac{F(|x(t)|)}{t} \geq 1$$

if and only if

$$\lim_{t\to\infty} \int_0^t g(s)\,ds \text{ is finite}, \quad \lambda_\Gamma := \liminf_{t\to\infty} \frac{F(|\Gamma(t)|)}{t} \geq 1,$$

where

$$\Gamma(t) = -\int_t^\infty g(s)\,ds, \quad t \geq 0$$

For large perturbations, our results show that $\lambda_\Gamma \in [0,1]$ implies $\lambda_x = \lambda_\Gamma$.

Besides these general results for deterministic equations, we establish related results for the SDE (3.1.4). The method of proof is strikingly similar to the deterministic case, and indeed is an easy corollary of results for internally perturbed equations with $\int_t^\infty \sigma(s)\,dB(s)$ (which is well–defined for $\sigma \in L^2(0,\infty)$ standing in place of Γ. Fortunately, the asymptotic behaviour of this family of random variables is known, and given by

$$\limsup_{t\to\infty} \frac{|\int_t^\infty \sigma(s)\,dB(s)|}{\Sigma(t)} = 1, \quad \text{a.s.}$$

where

$$\Sigma(t) = \sqrt{2\int_t^\infty \sigma^2(s)\,ds \log\log\left(\frac{1}{\int_t^\infty \sigma^2(s)\,ds}\right)}.$$

Therefore, to a certain extent the deterministic function Σ can play the role of Γ in (3.1.1), and if this is done sharp results relying on properties of Σ can be formulated. Indeed, with

$$\lambda_\Sigma = \lim_{t\to\infty} \frac{F(\Sigma(t))}{t},$$

and

$$\lambda_X = \liminf_{t\to\infty} \frac{F(|X(t)|)}{t}$$

the analogue of the "small perturbation" result in this framework is that $\lambda_\Sigma > 1$ implies $\lambda_X \geq 1$ a.s. and for "large" perturbations the analogous result is that $\lambda_\Sigma \in [0,1]$ implies $\lambda_X = \lambda_\Sigma$ a.s.

3.2 Introduction of technical hypotheses

As in Chapter 2, we assume that the unperturbed equation (3.1.2) has a unique and globally stable equilibrium at 0 by imposing on f the conditions

$$f \in C(\mathbb{R};\mathbb{R}) \tag{3.2.1}$$

and

$$xf(x) > 0 \quad \text{for} \quad x \neq 0, \quad f(0) = 0. \tag{3.2.2}$$

We have extended the continuity and attraction properties to $\mathbb{R}$ from $[0,\infty)$ because solutions are no longer guaranteed to be positive.

We make again the assumption that f preserves asymptotic behaviour at 0 through a function φ with some enhanced monotonicity, symmetry and regularity properties. Assume that φ is asymptotic preserving at zero, in the sense that

$$\lim_{\epsilon\to 0^+} \liminf_{x\to 0^+} \frac{\varphi((1-\epsilon)x)}{\varphi(x)} = 1, \tag{3.2.3}$$

where

$$\varphi \text{ is an increasing, odd and continuous function such that } \lim_{x\to 0} \frac{f(x)}{\varphi(x)} = 1. \tag{3.2.4}$$

The asymptotic oddness of f is assumed so as to ensure that convergence rates from both sides of the equilibrium are the same.

As noted in Chapter 2, (3.2.3) does not force y to obey (3.1.5), but it does imply that solutions must decay at least as fast as a negative power of t i.e., there exists a $\gamma > 0$ such that

$$\limsup_{t\to\infty} \frac{\log y(t)}{\log(t)} \leq -\gamma.$$

We prove this assertion in this chapter. However, as suggested above, we will assume directly that the solution (3.1.2) decays faster than any power, in that sense that, if $y(0) > 0$, then

$$\lim_{t\to\infty} \frac{y(\lambda t)}{y(t)} = 0, \quad \text{for each } \lambda > 1. \tag{3.2.5}$$

Since y is determined entirely by f, this can also be thought of as a condition on f. (3.2.5) is implied by $f \in RV_0(1)$, but if (3.2.5) holds, it does not necessarily mean that mean that f must be in $f \in RV_0(1)$. However, (3.2.5) does imply that

$$\int_0^1 \frac{1}{f(u)}\, du = +\infty,$$

that F defined by (3.1.3) is slowly varying (i.e., $F \in RV_0(0)$) and that F^{-1} is rapidly varying.

We remark that (3.2.3) and (3.2.5) are compatible. (3.2.3) does however ensure that y must decay at least as fast as some negative power of t. (3.2.3) rules out functions like $f(x) = e^{-1/x}$ which give decay slower than any power of t in y; (3.2.5) rules out functions like $f(x) = x^\beta$ for $\beta > 1$ which give power–like decay in t.

We term the property (3.2.3) "asymptotic preserving at 0", since one important implication that can be deduced from it is that if $a(t) \sim b(t) \to 0$ as $t \to \infty$, and $b(t) > 0$, then $\varphi(a(t)) \sim \varphi(b(t))$ as $t \to \infty$.

Lemma 3.2.1. *Suppose $z(t) > 0$ and obeys $z(t) \to 0$ as $t \to \infty$. Suppose further that $\mu(t) \to 1$ as $t \to \infty$. If φ obeys (3.2.4) and*

$$\lim_{\epsilon\to 0^+} \liminf_{x\to 0^+} \frac{\varphi((1-\epsilon)x)}{\varphi(x)} = 1. \tag{3.2.6}$$

Then

$$\lim_{t\to\infty} \frac{\varphi\left(\mu(t)z(t)\right)}{\varphi(z(t))} = 1.$$

Proof. For every $\epsilon \in (0,1)$ there is $T_1(\epsilon)$ such that $1-\epsilon < \mu(t) < 1/1-\epsilon$ for all $t \ge T_1(\epsilon)$ and so

$$(1-\epsilon)z(t) < \mu(t)z(t) < \frac{1}{1-\epsilon}z(t),$$

which implies

$$\frac{\varphi\left((1-\epsilon)z(t)\right)}{\varphi\left(z(t)\right)} < \frac{\varphi\left(\mu(t)z(t)\right)}{\varphi\left(z(t)\right)} < \frac{\varphi\left(\frac{1}{1-\epsilon}z(t)\right)}{\varphi\left(z(t)\right)}.$$

Therefore

$$\liminf_{t\to\infty} \frac{\varphi\left(\mu(t)z(t)\right)}{\varphi\left(z(t)\right)} \ge \liminf_{t\to\infty} \frac{\varphi\left((1-\epsilon)z(t)\right)}{\varphi\left(z(t)\right)} = \liminf_{x\to 0^+} \frac{\varphi\left((1-\epsilon)x\right)}{\varphi\left(x\right)}.$$

Take $\epsilon \to 0^+$ on both sides and by (3.2.6), we get

$$\liminf_{t\to\infty} \frac{\varphi\left(\mu(t)z(t)\right)}{\varphi\left(z(t)\right)} \ge 1. \tag{3.2.7}$$

Now take the limit superior to get

$$\begin{aligned}\limsup_{t\to\infty} \frac{\varphi\left(\mu(t)z(t)\right)}{\varphi\left(z(t)\right)} &\le \limsup_{t\to\infty} \frac{\varphi\left(\frac{1}{1-\epsilon}z(t)\right)}{\varphi\left(z(t)\right)} = \limsup_{x\to 0^+} \frac{\varphi\left(\frac{1}{1-\epsilon}x\right)}{\varphi\left(x\right)} \\ &= \limsup_{y\to 0^+} \frac{\varphi(y)}{\varphi\left((1-\epsilon)y\right)} = \frac{1}{\limsup_{y\to 0^+} \frac{\varphi((1-\epsilon)y)}{\varphi(y)}}.\end{aligned}$$

Let $\epsilon \to 0^+$ on both sides yields

$$\limsup_{t\to\infty} \frac{\varphi\left(\mu(t)z(t)\right)}{\varphi\left(z(t)\right)} \le \frac{1}{\lim_{\epsilon\to 0^+} \limsup_{y\to 0^+} \frac{\varphi((1-\epsilon)y)}{\varphi(y)}} = 1. \tag{3.2.8}$$

Combining (3.2.7) and (3.2.8) gives

$$\lim_{t\to\infty} \frac{\varphi\left(\mu(t)z(t)\right)}{\varphi\left(z(t)\right)} = 1,$$

as claimed. □

We now turn to properties of g. It is reasonable to suppose that when g decays to zero sufficiently rapidly, then the asymptotic behaviour of (3.1.2) is likely to be preserved in (3.1.1), in an appropriate sense. On the other hand, if g decays to zero more slowly, it may be that the rate of convergence of $x(t) \to 0$ as $t \to \infty$ can be slower than that of $y(t) \to 0$ as $t \to \infty$.

We now assume g is continuous with the following properties, the second of which will be relaxed at the end to deal with some slowly decaying cases:

$$g \in C([0,\infty);\mathbb{R}); \tag{3.2.9}$$

$$\lim_{t\to\infty} \int_0^t g(s)ds \quad \text{exists and finite.} \tag{3.2.10}$$

With f being continuous, it follows that the unperturbed equation (3.1.2) has a unique continuous solution. With g being continuous, it follows that there is at least one continuous solution of the perturbed ODE (3.1.1). Lipschitz conditions on f can be imposed to guarantee that there is a unique solution. However, since f is often an increasing function in our analysis, uniqueness can be obtained for the continuous solution of the perturbed ODE (3.1.1) under this condition.

The proof is quite standard, but we give it for completeness. Suppose x_1, x_2 are distinct continuous solutions of (3.1.1). Let

$$\Delta(t) := (x_1(t) - x_2(t))^2, \quad t \geq 0.$$

Since f is continuous, x_1 and x_2 are also C^1, and so therefore is Δ. Since f is increasing, we get

$$(x-y)\left(f(x) - f(y)\right) \geq 0, \quad x \neq y.$$

Now $\Delta(0) = 0$, $\Delta(t) \geq 0$ for $t \geq 0$ and for $t > 0$, we have

$$\begin{aligned}\Delta'(t) &= 2\left(x_1(t) - x_2(t)\right)\left(x_1'(t) - x_2'(t)\right) \\ &= -2\left(x_1(t) - x_2(t)\right)\left(f(x_1(t) - f(x_2(t))\right) \leq 0.\end{aligned}$$

Hence $\Delta(t) = 0$ for all t, and therefore (3.1.1) has a unique solution.

It is also possible to impose monotonicity on f close to zero, and Lipschitz behaviour elsewhere, and still guarantee uniqueness. The above proof also can be used to get the uniqueness of continuous solutions to the stochastic differential equation studied in this work.

3.3 Preliminary Results

In this section, we explore some of the implications of the hypotheses on f (and φ) introduced above. This is achieved by a series of results (eight lemmas and one proposition), whose results we quickly list now.

The first two lemmas show that assuming F is slowly varying and that F^{-1} is rapidly varying are equivalent. We have preferred to impose the second of these assumptions, as it makes precise the idea that the rate of decay of the unperturbed equation is fast. The condition of asymptotic preservation in f is novel, and we give some other conditions under which it holds. The third and fourth result state that f is asymptotic preserving, and F^{-1} rapidly varying under the condition that $x \mapsto f(x)/x$ is decreasing. Therefore, the results in this chapter apply very directly to the situation when $f(x)/x \to \infty$ as $x \to 0^+$ especially when that divergence is monotone; in this situation, superexponential decay in y is seen. On the other hand, the next two results show that that if $f(x)/x \to 0$ monotonically, but that $f(x)/x^{1+\epsilon}$ is decreasing, then once again f is asymptotic preserving, and F^{-1} rapidly varying. This result deals with the subexponential (but faster than power) case.

It has been claimed that the asymptotic preserving property of f extends to F, and that this forces decay in y at least as fast as a negative power of t. The next three results map this out precisely, including making a distinction between power law and faster than power law decay. The last result in the section shows certain types of rapid decay in the solution of the unperturbed equation actually imply that f is regularly varying with index 1 at zero, which gives the required asymptotic preserving and rapid variation properties of f and F^{-1} that are needed for our general results.

Lemma 3.3.1. *Suppose $f(x) > 0$ for all $x > 0$ and $f \in C([0,\infty);[0,\infty))$. Let F be given by (3.1.3) with $F(x) \to \infty$ as $x \to 0^+$. If*

$$\lim_{x\to 0^+} \frac{F(\lambda x)}{F(x)} = 1, \qquad \textit{for all } \lambda > 0,$$

then for every $\eta > 0$

$$\lim_{y\to\infty} \frac{F^{-1}\left((1+\eta)y\right)}{F^{-1}(y)} = 0.$$

Proof. Let $\epsilon \in (0,1)$. By the hypothesis, we have $F(\epsilon x)/F(x) \to 1$ as $x \to 0^+$. For every $\eta > 0$ there is $\tilde{x}(\eta,\epsilon)$ such that $x < \tilde{x}(\eta,\epsilon)$ implies $F(x) < F(\epsilon x) < (1+\eta)F(x)$. Now, let $\tilde{y}(\eta,\epsilon) = F\left(\tilde{x}(\eta,\epsilon)\right)$, and let $y > \tilde{y}(\eta,\epsilon)$. Then $F^{-1}(y) < F^{-1}\left(\tilde{y}(\eta,\epsilon)\right) = \tilde{x}(\eta,\epsilon)$. Hence $y < F\left(\epsilon F^{-1}(y)\right) < (1+\eta)y$ or $F^{-1}(y) > \epsilon F^{-1}(y) > F^{-1}\left((1+\eta)y\right)$. Therefore, for every $\eta > 0$ and $\epsilon \in (0,1)$ there is $\tilde{y}(\eta,\epsilon) > 0$ such that

$$0 < \frac{F^{-1}\left((1+\eta)y\right)}{F^{-1}(y)} < \epsilon, \quad y > \tilde{y}(\eta,\epsilon).$$

Hence

$$\lim_{y\to\infty} \frac{F^{-1}\left((1+\eta)y\right)}{F^{-1}(y)} = 0, \quad \text{for all } \eta > 0,$$

as required. □

Lemma 3.3.2. *Suppose $f(x) > 0$ for all $x > 0$ and $f \in C([0,\infty);[0,\infty))$. Let F be given by (3.1.3). If for every $\eta > 0$*

$$\lim_{y\to\infty} \frac{F^{-1}\left((1+\eta)y\right)}{F^{-1}(y)} = 0,$$

and $F^{-1}(y) \to 0$ as $y \to \infty$, then F is in $RV_0(0)$.

Proof. Let $\lambda > 1$ be arbitrary, but fixed. Also, let $\epsilon \in (0,1)$. Then there is $T(\epsilon,\lambda) > 0$ such that $F^{-1}(\lambda t)/F^{-1}(t) < \epsilon$ for all $t \geq T(\epsilon,\lambda)$. Hence $F^{-1}(\lambda t) < \epsilon F^{-1}(t)$ for all $t \geq T(\epsilon,\lambda)$ or, as F is decreasing, we have $\lambda t > F\left(\epsilon F^{-1}(t)\right)$ for all $t \geq T(\epsilon,\lambda)$. Write $x = F^{-1}(t)$; then for all $x < x(\epsilon,\lambda) := F^{-1}\left(T(\epsilon,\lambda)\right)$ we have $\lambda F(x) > F(\epsilon x)$. Therefore $F(\epsilon x)/F(x) < \lambda$ for all $x < x(\epsilon,\lambda)$. Since $\epsilon < 1$ and F is decreasing $F(\epsilon x) > F(x)$. Therefore, we have $\lambda > F(\epsilon x)/F(x) > 1$ for all $x < x(\epsilon,\lambda)$, which implies

$$\lambda \geq \limsup_{x\to 0^+} \frac{F(\epsilon x)}{F(x)} \geq \liminf_{x\to 0^+} \frac{F(\epsilon x)}{F(x)} \geq 1.$$

Since λ is arbitrary, letting $\lambda \to 1$ yields

$$\lim_{x\to 0^+} \frac{F(\epsilon x)}{F(x)} = 1, \quad \epsilon \in (0,1).$$

Now, let $\epsilon > 1$. Since $\beta := 1/\epsilon \in (0,1)$, we have

$$\lim_{x\to 0^+} \frac{F(\epsilon x)}{F(x)} = \lim_{x\to 0^+} \frac{F(\frac{1}{\beta}x)}{F(x)} = \lim_{z\to 0^+} \frac{F(z)}{F(\beta z)} = 1.$$

Hence

$$\lim_{x\to 0^+} \frac{F(\epsilon x)}{F(x)} = 1, \quad \epsilon \in (0,\infty),$$

which implies that $F \in RV_0(0)$, as required. □

Lemma 3.3.3. *Suppose $f(0) = 0$, $xf(x) > 0$ for all $x \neq 0$ and $f \in C(\mathbb{R};\mathbb{R})$. Suppose that*

$$\lim_{x\to 0^+} \frac{f(x)/x}{\psi(x)} = 1$$

where ψ is decreasing function, and that

$$\lim_{x\to 0}\frac{f(x)}{\varphi(x)}=1$$

where φ is an increasing and odd function. Then for every $\epsilon\in(0,1)$ there is $x(\epsilon)>0$ such that

$$\frac{\varphi\left((1-\epsilon)x\right)}{\varphi(x)}>\frac{1-\epsilon}{1+\epsilon},\quad 0<x<x(\epsilon).$$

Proof. For every $\epsilon\in(0,1)$, there is $x_1(\epsilon)>0$ such that

$$(1+\epsilon)^{-1/4}\psi(x)<\frac{f(x)}{x}<(1+\epsilon)^{\frac{1}{4}}\psi(x),\quad 0<x<x_1(\epsilon).$$

Also, for every $\epsilon\in(0,1)$, there is $x_2(\epsilon)>0$ such that

$$(1+\epsilon)^{-\frac{1}{4}}<\frac{f(x)}{\varphi(x)}<(1+\epsilon)^{\frac{1}{4}},\quad 0<|x|<x_2(\epsilon),$$

where φ is increasing and odd. Let $0<x<x_1(\eta)$. Then

$$\frac{f\left((1-\epsilon)x\right)}{(1-\epsilon)x}>(1+\eta)^{-1/4}\psi\left((1-\epsilon)x\right)>(1+\eta)^{-1/4}\psi(x)>(1+\eta)^{-2/4}\frac{f(x)}{x}.$$

Thus, for $x\in(0,x_1(\eta))$ we have that

$$\frac{f\left((1-\epsilon)x\right)}{f(x)}>(1+\eta)^{-2/4}(1-\epsilon).$$

Also, for every $x\in(0,x_2(\eta))$ we get

$$(1+\eta)^{-1/4}<\frac{f\left((1-\epsilon)x\right)}{\varphi\left((1-\epsilon)x\right)}<(1+\eta)^{1/4},\quad (1+\eta)^{-1/4}<\frac{f(x)}{\varphi(x)}<(1+\eta)^{1/4}.$$

Now let $x_3(\eta)=\min(x_1(\eta),x_2(\eta))$. Then for $x\in(0,x_3(\eta))$

$$\begin{aligned}\frac{\varphi\left((1-\epsilon)x\right)}{\varphi(x)}&>\frac{f\left((1-\epsilon)x\right)(1+\eta)^{-1/4}}{f(x)(1+\eta)^{1/4}}=\frac{f\left((1-\epsilon)x\right)}{f(x)}(1+\eta)^{-2/4}\\&>(1+\eta)^{-1}(1-\epsilon).\end{aligned}$$

Let $\eta=\epsilon$. Then for $x\in(0,x_3(\epsilon))$

$$\frac{\varphi\left((1-\epsilon)x\right)}{\varphi(x)}>\frac{1-\epsilon}{1+\epsilon},$$

as claimed. □

Lemma 3.3.4. *Suppose $f(0)=0$, $xf(x)>0$ for all $x>0$ and $f\in C([0,\infty);[0,\infty))$.*

Suppose that

$$\lim_{x\to 0^+} \frac{f(x)/x}{\psi(x)} = 1$$

where ψ is decreasing function, and that F defined by (3.1.3) *obeys $F(x) \to \infty$ as $x \to \infty$. Then $F \in RV_0(0)$.*

Proof. Define

$$\Psi(x) = \int_x^1 \frac{1}{u\psi(u)}\,du.$$

Then $F(x) \sim \Psi(x)$ as $x \to 0^+$. Next, let $\alpha \in (0,1), 0 < x < 1$. Then

$$\Psi(\alpha x) = \Psi(x) + \int_{\alpha x}^{x} \frac{1}{u\psi(u)}\,du = \Psi(x) + \int_{\alpha}^{1} \frac{1}{v\psi(vx)}\,dv.$$

Since ψ is decreasing and $x < 1$ for $v \in (0,1)$, we have $0 < vx \le v$ and hence $1/\psi(vx) \le 1/\psi(v)$. Thus for all $0 < \alpha < 1$, $x \in (0,1)$

$$\Psi(x) \le \Psi(\alpha x) \le \Psi(x) + \int_{\alpha}^{1} \frac{1}{v\psi(v)}\,dv = \Psi(x) + \Psi(\alpha).$$

Thus as $\Psi(x) \to \infty$ as $x \to 0^+$ we obtain

$$\lim_{x\to 0^+} \frac{\Psi(\alpha x)}{\Psi(x)} = 1, \text{ for all } \alpha \in (0,1).$$

Now, let $\alpha > 1$. Since $\beta := 1/\alpha \in (0,1)$, we have

$$\lim_{x\to 0^+} \frac{\Psi(\alpha x)}{\Psi(x)} = \lim_{x\to 0^+} \frac{\Psi(\frac{1}{\beta}x)}{\Psi(x)} = \lim_{z\to 0^+} \frac{\Psi(z)}{\Psi(\beta z)} = 1.$$

Therefore

$$\lim_{x\to 0^+} \frac{\Psi(\alpha x)}{\Psi(x)} = 1, \quad \text{for all } \alpha > 0.$$

Since $F(x) \sim \Psi(x)$ as $x \to 0^+$, we have $\lim_{x\to 0^+} F(\alpha x)/F(x) = 1$ for all $\alpha > 0$, which implies that $F \in RV_0(0)$. □

Lemma 3.3.5. *Suppose $f(0) = 0$, $xf(x) > 0$ for all $x > 0$ and $f \in C([0,\infty);[0,\infty))$. Suppose that $x \mapsto f(x)/x$ be increasing and $x \mapsto f(x)/x^{1+\epsilon}$ is decreasing for all $\epsilon > 0$. Let F defined by* (3.1.3)*. Then*

(i) $F \in RV_0(0)$;

(ii) f obeys

$$\lim_{\epsilon\to 0^+} \liminf_{x\to 0^+} \frac{f((1-\epsilon)x)}{f(x)} = 1.$$

Proof. We start by proving (ii). Let $0 < u < x$. Then $f(u)/u < f(x)/x$ and

$f(u)/u^{1+\epsilon} > f(x)/x^{1+\epsilon}$. Therefore

$$\frac{x}{u} < \frac{f(x)}{f(u)} < \left(\frac{x}{u}\right)^{1+\epsilon}. \tag{3.3.1}$$

Putting $u = (1-\eta)x$ (which forces $0 < \eta < 1$), we get

$$1-\eta > \frac{f((1-\eta)x)}{f(x)} > (1-\eta)^{1+\epsilon}.$$

Now taking the limit as $x \to 0^+$ and then the limit as $\eta \to 0^+$, we see that

$$\lim_{\eta\to 0^+} \liminf_{x\to 0^+} \frac{f((1-\eta)x)}{f(x)} = 1,$$

as required.

To prove (i), let $0 < \lambda < 1$. Then

$$F(\lambda x) = F(x) + \int_{\lambda x}^{x} \frac{1}{f(u)}\, du. \tag{3.3.2}$$

By (3.3.1) we obtain

$$\int_{\lambda x}^{x} \frac{x}{u}\frac{1}{f(x)}\, du \le \int_{\lambda x}^{x} \frac{1}{f(u)}\, du \le \int_{\lambda x}^{x} \left(\frac{x}{u}\right)^{1+\epsilon} \frac{1}{f(x)}\, du.$$

Evaluating the left and right hand sides for $\lambda \in (0,1)$ we have

$$\frac{x}{f(x)} \log\left(\frac{1}{\lambda}\right) \le \int_{\lambda x}^{x} \frac{1}{f(u)}\, du \le \frac{1}{\epsilon}(\lambda^{-\epsilon}-1)\frac{x}{f(x)}. \tag{3.3.3}$$

Let $\mu > 1$ be arbitrary, $x \in (0, 1/\mu)$. Then there is $N = N(x) \in \mathbb{N}$ such that $x\mu^N \le 1$, $x\mu^{N+1} > 1$. Thus for $x < 1/\mu$,

$$\begin{aligned} F(x) &= \sum_{j=1}^{N} \int_{x\mu^{j-1}}^{x\mu^j} \frac{1}{f(u)}\, du + \int_{\mu^N x}^{1} \frac{1}{f(u)}\, du \\ &\ge \sum_{j=1}^{N} \int_{x\mu^{j-1}}^{x\mu^j} \frac{1}{f(u)}\, du. \end{aligned} \tag{3.3.4}$$

Put $y := x\mu^j$, and $\lambda := 1/\mu$. Then by (3.3.3)

$$\int_{x\mu^{j-1}}^{x\mu^j} \frac{1}{f(u)}\, du = \int_{\lambda y}^{y} \frac{1}{f(u)}\, du \ge \frac{y}{f(y)} \log\left(\frac{1}{\lambda}\right) = \frac{x\mu^j}{f(x\mu^j)} \log\mu.$$

We estimate this lower bound below. Since $x\mu^j \ge x$, $f(x\mu^j)/(x\mu^j)^{1+\epsilon} < f(x)/x^{1+\epsilon}$, which implies that

$$\frac{1}{f(x\mu^j)} > \frac{1}{\mu^{j(1+\epsilon)}}\frac{1}{f(x)}.$$

Therefore

$$\int_{x\mu^{j-1}}^{x\mu^j} \frac{1}{f(u)}\,du \geq (\mu^{-\epsilon})^j \frac{x}{f(x)} \log \mu. \tag{3.3.5}$$

Thus by (3.3.4) and (3.3.5)

$$F(x) \geq \sum_{j=1}^{N} (\mu^{-\epsilon})^j \log \mu \frac{x}{f(x)} = \frac{x}{f(x)} \log \mu \cdot \frac{\mu^{-\epsilon}}{1-\mu^{-\epsilon}} \left(1 - (\mu^N)^{-\epsilon}\right). \tag{3.3.6}$$

Now $1/(x\mu) \leq \mu^N \leq 1/x$, and thus $x^\epsilon \mu^\epsilon \geq (\mu^N)^{-\epsilon} \geq x^\epsilon$. Hence by (3.3.6), we obtain

$$\frac{F(x)}{x/f(x)} \geq \log \mu \cdot \frac{\mu^{-\epsilon}}{1-\mu^{-\epsilon}} (1 - x^\epsilon \mu^\epsilon).$$

Therefore, as $\epsilon > 0$ we obtain

$$\liminf_{x \to 0^+} \frac{F(x)}{x/f(x)} \geq \frac{\log \mu}{\mu^\epsilon - 1}.$$

Thus as μ is arbitrary, we may let $\mu \to 1$ to get

$$\liminf_{x \to 0^+} \frac{F(x)}{x/f(x)} \geq \frac{1}{\epsilon}.$$

Letting $\epsilon \to 0^+$, gives

$$\lim_{x \to 0^+} \frac{F(x)}{x/f(x)} = +\infty. \tag{3.3.7}$$

For $\lambda \in (0,1)$, by (3.3.2) and (3.3.3) we have

$$1 < \frac{F(\lambda x)}{F(x)} \leq 1 + \frac{1}{\epsilon}(\lambda^{-\epsilon} - 1)\frac{x/f(x)}{F(x)}.$$

Thus by (3.3.7)

$$1 \leq \liminf_{x \to 0^+} \frac{F(\lambda x)}{F(x)} \leq \limsup_{x \to 0^+} \frac{F(\lambda x)}{F(x)} \leq 1 + \frac{\lambda^{-\epsilon} - 1}{\epsilon} \cdot 0 = 1.$$

Hence for all $\lambda \in (0,1)$, $\lim_{x \to 0^+} F(\lambda x)/F(x) = 1$. As in other proofs, this can be extended for all $\lambda > 1$ and so $F \in \mathrm{RV}_0(0)$ as required. □

Before proceeding further, we summarise our progress. We have shown that the rapid variation of F^{-1} and slow variation of F are equivalent. It is appealing from the point of view of applications to ask for a certain type of decay phenomenon on the unperturbed equation. Our analysis shows that this implies $F \in RV_0(0)$, which will be very useful in proofs.

Moreover, it is usually easier to work with F than with F^{-1}; if one establish that F is slowly varying, one can know that F^{-1} is rapidly varying without needing to compute it explicitly. Moreover, sufficient conditions on f may force F to be slowly varying, which can be even more readily checked.

In this direction, we have shown

(i) If $x \mapsto f(x)/x$ is decreasing, f is odd and increasing then $F \in RV_0(0)$ and f is asymptotic preserving.

(ii) If f is odd $x \mapsto f(x)/x$ is increasing, and $x \mapsto f(x)/x^{1+\eta}$ is decreasing for all $\eta > 0$ sufficiently small then $F \in RV_0(0)$ and f is asymptotic preserving.

The condition in (i) and in (ii) can be checked directly on f. We note that with a little extra effort, the oddness and strict monotonicity in these conditions can be replaced by asymptotic oddness or asymptotic monotonicity in the sense that its is necessary only that f can be asymptotic at 0 to an odd function, or that $x \mapsto f(x)$, or $x \mapsto f(x)/x$, or $x \mapsto f(x)/x^{1+\eta}$ is asymptotic to a strictly monotone function.

We close this section with an investigation of how asymptotic preservation in f forces the solution of the unperturbed equation to decay at least at a power–law rate.

Lemma 3.3.6. *Suppose φ is increasing, positive, continuous on $[0,\infty)$ and obeys*

$$\lim_{\epsilon\to 0^+} \liminf_{x\to 0^+} \frac{\varphi\left((1-\epsilon)x\right)}{\varphi(x)} = 1.$$

Define

$$\Phi(x) = \int_x^1 \frac{1}{\varphi(x)}\, du, \quad x > 0,$$

and suppose $\Phi(x) \to \infty$ as $x \to \infty$. Then Φ obeys

$$\lim_{\epsilon\to 0^+} \limsup_{x\to 0^+} \frac{\Phi((1-\epsilon)x)}{\Phi(x)} = 1, \tag{3.3.8}$$

$$\limsup_{t\to\infty} \frac{\Phi^{-1}(Kt)}{\Phi^{-1}(t)} := \alpha_K < 1, \quad \textit{for each } K > 1.$$

Furthermore, if the positive continuous function f is such that $f(x)/\varphi(x) \to 1$ as $x \to 0^+$, and F is given by (3.1.3), then

$$\lim_{\epsilon\to 0^+} \limsup_{x\to 0^+} \frac{F((1-\epsilon)x)}{F(x)} = 1,$$

$$\limsup_{t\to\infty} \frac{F^{-1}(Kt)}{F^{-1}(t)} := a_K < 1, \quad \textit{for each } K > 1.$$

Proof. Let

$$\liminf_{x\to 0^+} \frac{\varphi((1-\epsilon)x)}{\varphi(x)} =: \Phi_\epsilon.$$

Since φ is increasing, $\Phi_\epsilon \le 1$ and $\epsilon \mapsto \Phi_\epsilon$ is monotone with $\Phi_\epsilon \to 1$ as $\epsilon \to 0^+$. Thus there is $\epsilon_1 \in (0,1)$ such that $\Phi_\epsilon > 1/2$ for all $\epsilon < \epsilon_1$. Therefore, for $\epsilon < \epsilon_1$

$$\limsup_{x\to 0^+} \frac{\varphi(x)}{\varphi((1-\epsilon))} = \frac{1}{\Phi_\epsilon} \in [1,\infty).$$

Hence, for each fixed $\epsilon \in (0,\epsilon_1)$ and for all $\eta > 0$, there is $x_1(\eta,\epsilon) > 0$ such that $x < x_1(\eta,\epsilon)$ implies $\varphi(x)/\varphi((1-\epsilon)x) < \eta + 1/\Phi_\epsilon$. Write

$$\Phi((1-\epsilon)x) = \int_x^{\frac{1}{1-\epsilon}} \frac{1-\epsilon}{\varphi((1-\epsilon)v)}\, dv.$$

Now, let $x < x_1(\eta,\epsilon) \wedge 1/(1-\epsilon) \wedge 1 =: x(\eta,\epsilon)$

$$\begin{aligned}\Phi((1-\epsilon)x) &= (1-\epsilon)\int_x^{x(\eta,\epsilon)} \frac{1}{\varphi((1-\epsilon)v)}\, dv + (1-\epsilon)\int_{x(\eta,\epsilon)}^{\frac{1}{1-\epsilon}} \frac{1}{\varphi((1-\epsilon)v)}\, dv \\ &=: (1-\epsilon)\int_x^{x(\eta,\epsilon)} \frac{\varphi(v)}{\varphi((1-\epsilon)v)}\frac{1}{\varphi(v)}\, dv + \Phi_1(\eta,\epsilon) \\ &\le (1-\epsilon)\left(\frac{1}{\Phi_\epsilon}+\eta\right)\int_x^{x(\eta,\epsilon)} \frac{1}{\varphi(v)}\, dv + \Phi_1(\eta,\epsilon) \\ &\le (1-\epsilon)\left(\frac{1}{\Phi_\epsilon}+\eta\right)\int_x^{1} \frac{1}{\varphi(v)}\, dv + \Phi_1(\eta,\epsilon).\end{aligned}$$

Hence

$$\Phi((1-\epsilon)x) \le (1-\epsilon)\left(\frac{1}{\Phi_\epsilon}+\eta\right)\Phi(x) + \Phi_1(\eta,\epsilon), \quad x < x(\eta,\epsilon).$$

Therefore, as $\Phi(x) \to \infty$ as $x \to 0^+$

$$\limsup_{x\to 0^+} \frac{\Phi((1-\epsilon)x)}{\Phi(x)} \le (1-\epsilon)\left(\frac{1}{\Phi_\epsilon}+\eta\right).$$

Since $\eta > 0$ is arbitrary, and Φ is decreasing, we have for $\epsilon < \epsilon_1$

$$1 \le L_\epsilon := \limsup_{x\to 0^+} \frac{\Phi((1-\epsilon)x)}{\Phi(x)} \le \frac{1-\epsilon}{\Phi_\epsilon} =: \Lambda_\epsilon.$$

Now as $\Phi_\epsilon \to 1$ as $\epsilon \to 0^+$, $\Lambda_\epsilon \to 1$ as $\epsilon \to 0^+$. Also since $1 \le L_\epsilon \le \Lambda_\epsilon$ and $\Lambda_\epsilon \to 1$ as $\epsilon \to 0^+$, we have that $L_\epsilon \to 1$ as $\epsilon \to 0^+$. Hence (3.3.8) holds as claimed. Note that as $F(x) \sim \Phi(x)$ as $x \to 0^+$, this gives the corresponding property of F also, and moreover the limit superior L_ϵ is the same for F as it is for Φ.

To find the asymptotic behaviour of Φ^{-1}, by the definition of L_ϵ we have that for every $\gamma > 0$, there is a $\bar{x}(\gamma,\epsilon) > 0$ such that $x < \bar{x}(\gamma,\epsilon)$ implies

$$\frac{\Phi((1-\epsilon)x)}{\Phi(x)} < L_\epsilon + \gamma. \tag{3.3.9}$$

Let $t > T_1(\gamma,\epsilon) := \Phi(\bar{x}(\gamma,\epsilon))$. Then $\Phi^{-1}(t) < \bar{x}(\gamma,\epsilon)$, and thus $\Phi((1-\epsilon)\Phi^{-1}(t)) <$

$(L_\epsilon + \gamma)t$. Therefore for $\epsilon < \epsilon_1$, we have

$$\frac{\Phi^{-1}\left((L_\epsilon+\gamma)t\right)}{\Phi^{-1}(t)} < 1-\epsilon, \quad t > T_1(\gamma,\epsilon).$$

Now, let $K > 1$ be arbitrary and fixed. Since $L_\epsilon \to 1$ as $\epsilon \to 0^+$, there is $\epsilon_2 > 0$ such that $L_\epsilon < K$ for all $\epsilon < \epsilon_2(K)$. Pick $\epsilon_0(K) = \epsilon_2(K)/2 \wedge \epsilon_1/2$, and define $\gamma = \gamma(K) = K - L_{\epsilon_0(K)}$. Then $\epsilon_0 < \epsilon_1$, $\epsilon_0 < \epsilon_2$ and $L_{\epsilon_0} < K$, and thus $\gamma > 0$. Therefore, for all $t > T_1(\gamma(K), \epsilon_0(K)) = T_1\left((K - L_{\epsilon_0(K)}), \epsilon_0(K)\right) =: T(K)$ we have

$$\frac{\Phi^{-1}(Kt)}{\Phi^{-1}(t)} < 1-\epsilon_0(K), \quad t > T(K).$$

Finally, define $\tilde{\alpha}(K) := 1 - \epsilon_0(K) < 1$. Then, for every $K > 1$, there is $T(K) > 0$ such that $\Phi^{-1}(Kt)/\Phi^{-1}(t) < \tilde{\alpha}(K)$ for $t \geq T(K)$. Therefore for each $K > 1$

$$\alpha_k := \limsup_{t\to\infty} \frac{\Phi^{-1}(Kt)}{\Phi^{-1}(t)} \leq \tilde{\alpha}(K) = 1-\epsilon_0(K) < 1,$$

as claimed.

The argument used to obtain the corresponding result for F^{-1} is identical to that used above for Φ^{-1}, with F^{-1} replacing Φ^{-1} throughout. The estimate (3.3.9) holds, except perhaps for a different interval of x–values. At the end, the exact limsup α_K may differ between Φ^{-1} and F^{-1}, even though the estimates $1-\epsilon_0(K)$ will be the same. □

We now show that φ being asymptotic preserving implies that $F^{-1}(t)$ must tend to zero at least as fast as a negative power of t, as $t \to \infty$. This means that the solution of the unperturbed differential equation

$$y'(t) = -f(y(t)), \quad t > 0$$

must have this property also. Therefore, the hypothesis that φ be asymptotic preserving prevents slower than power decay in y. We also notice that requesting that φ be asymptotic preserving does not contradict our other hypothesis in this chapter that

$$\lim_{t\to\infty} \frac{y(Kt)}{y(t)} = 0, \qquad \text{for each } K > 1$$

because this merely requires a_K to be zero for all $K > 1$.

Lemma 3.3.7. *Let φ and f obey the hypotheses of Lemma 3.3.6. Then*

$$\limsup_{t\to\infty} \frac{\log F^{-1}(t)}{\log t} =: -\beta < 0.$$

Proof. By Lemma 3.3.6, for each $K > 1$, that there is $\alpha_K < 1$ such that

$$\limsup_{t\to\infty} \frac{F^{-1}(Kt)}{F^{-1}(t)} =: a_K < 1.$$

Fix $K_0 > 1$. Then, for every $\epsilon > 0$ there is $T(\epsilon, K_0) > 0$ such that $F^{-1}(K_0 t)/F^{-1}(t) < \alpha_{K_0} + \epsilon$ for $t \geq T(\epsilon, K_0)$. Take $\epsilon = \frac{1-\alpha_{K_0}}{2}$. Then $\alpha_{K_0} + \epsilon = \frac{1+\alpha_{K_0}}{2} =: \bar{\alpha} < 1$. Thus for all $t \geq T'(K_0) = T(\frac{1-\alpha_{K_0}}{2}, K_0)$, we obtain $F^{-1}(K_0 t)/F^{-1}(t) < \bar{\alpha}$. Next, as $K_0 > 1$, there is $N(K_0) > 1$ such that $K_0^n > T(K_0)$ for all $n \geq N$. Let $t_n := K_0^n$ for all $n \geq N(K_0)$; then $F^{-1}(K_0^{n+1})/F^{-1}(K_0^n) < \bar{\alpha}$. Next, let $y_n := \log F^{-1}(K_0^n)$. Then $y_{n+1} - y_n < \log \bar{\alpha} =: -\bar{\beta} < 0$ for all $n \geq N(K_0)$. Hence

$$\limsup_{n\to\infty} \frac{y_n}{n} \leq -\bar{\beta},$$

which implies that

$$\limsup_{n\to\infty} \frac{\log F^{-1}(K_0^n)}{n} \leq -\bar{\beta}.$$

Now, let $t > K_0$. Then, there is $n = n(t) \in \mathbb{N}$ such that $K_0^n \leq t < K_0^{n+1}$. Hence $n \log K_0 \leq \log t < (n+1)\log K_0$, and so because $n(t) \to \infty$ as $t \to \infty$, we get

$$\lim_{t\to\infty} \frac{\log t}{n(t)} = \log K_0 > 0.$$

Since $F^{-1}(K_0^{n(t)}) \geq F^{-1}(t)$, we obtain

$$\frac{\log F^{-1}(t)}{\log t} \leq \frac{\log F^{-1}(K_0^{n(t)})}{n(t)} \cdot \frac{n(t)}{\log t}.$$

Therefore

$$\limsup_{t\to\infty} \frac{\log F^{-1}(t)}{\log t} \leq -\bar{\beta} \frac{1}{\log K_0} =: -\beta < 0,$$

as required. □

If the limit a_K is zero, we show next that the rate of decay of $F^{-1}(t) \to 0$ as $t \to \infty$ is faster than any negative power of t. For the analysis in this chapter, we desire that a_K should always be zero. Therefore, the following lemma shows that the hypothesis

$$\lim_{t\to\infty} \frac{y(\lambda t)}{y(t)} = 0, \quad \text{for each } \lambda > 1$$

forces faster than power decay in the solution y of the unperturbed differential equation.

Lemma 3.3.8. *If*

$$\limsup_{t\to\infty} \frac{F^{-1}(Kt)}{F^{-1}(t)} = 0, \quad \textit{for each } K > 1,$$

then

$$\lim_{t\to\infty} \frac{\log F^{-1}(t)}{\log t} = -\infty.$$

Proof. Let $K > 1$ be arbirary. Consider the sequence $t_n = K^n \to \infty$, and write $y_n := \log F^{-1}(K^n)$. Then $\lim_{n\to\infty} F^{-1}(K^{n+1})/F^{-1}(K^n) = 0$, from which we deduce $\lim_{n\to\infty} y_{n+1} - y_n = -\infty$. Therefore $y_n/n \to -\infty$ as $n \to \infty$. For each $t \geq 1$, there is $n = n(t)$ such that $K^n \leq t < K^{n+1}$. Thus $\log K \cdot n(t) \leq \log t < (n(t) + 1) \log K$, and

$$\lim_{t\to\infty} \frac{n(t)}{\log t} = \frac{1}{\log K}.$$

Moreover, $F^{-1}(K^{n(t)}) \geq F^{-1}(t)$ and we have

$$\limsup_{t\to\infty} \frac{\log F^{-1}(t)}{t} \leq \limsup_{t\to\infty} \frac{\log F^{-1}(K^{n(t)})}{n(t)} \cdot \frac{n(t)}{\log t} = -\infty,$$

as required. □

We saw above that asymptotic preserving properties of f force decay at least as fast as a negative power of t in $y(t)$, but we can certainly have f being asymptotic preserving without y having the property

$$\lim_{t\to\infty} \frac{y(\lambda t)}{y(t)} = 0 \quad \text{for } \lambda > 1.$$

An instance of this is when $f(x) = x^\beta$ for $x > 0$ and $\beta > 1$. However, we now show that it appears when the solution y decays faster than any power of t (as characterised by the above limit), then, to a certain degree f should be asymptotic preserving.

Proposition 3.3.1. *Consider the class of decreasing positive and C^2 functions y on $[0, \infty)$ such that*

$$\lim_{t\to\infty} \frac{y(t)y''(t)}{y'(t)^2} = 1, \tag{3.3.10}$$

and let y be the solution of $y'(t) = -f(y(t))$, $t > 0$, $y(0) = 1$. Then f is asymptotic preserving and y obeys

$$\lim_{t\to\infty} \frac{y(\lambda t)}{y(t)} = 0, \quad \textit{for each } \lambda > 1.$$

If the limit (α, say) in (3.3.10) is greater than unity, it can be shown that f must be regularly varying (and hence asymptotic preserving). However, y is also regularly varying, and hence y cannot be rapidly varying.

Proof. We start by showing that this implies that y is rapidly varying at 0. In fact, we will prove that if the unperturbed differential equation has a solution with this property, then f must also be asymptotic preserving, and moreover, be regularly varying at zero with index 1.

This follows because the class of increasing positive functions z in C^2 on $[0,\infty)$ for which

$$\lim_{t\to\infty}\frac{z(t)z''(t)}{z'(t)^2}=1 \tag{3.3.11}$$

is rapidly varying at infinity, in the usual sense that

$$\lim_{t\to\infty}\frac{z(\lambda t)}{z(t)}=+\infty, \quad \text{for each } \lambda>1. \tag{3.3.12}$$

The class of functions obeying (3.3.11) are intimately related to the class Γ (see e.g., Section 3.10 in [24]) . We give a definition and some properties of this class now.

Definition 3.3.1. *The class* Γ *consists of those functions* $\phi:\mathbb{R}\to(0,\infty)$ *non-decreasing and right-continuous for which there exists a measurable function* $g:\mathbb{R}\to(0,\infty)$ *the auxiliary function of* ϕ *such that*

$$\frac{\phi(t+ug(t))}{\phi(t)}\to e^u, \quad t\to\infty \quad \textit{for all } u\in\mathbb{R}.$$

We record some important facts about Γ. First if g is an auxiliary function of $\phi\in\Gamma$ we must have that

$$g(t)\sim\frac{\int_0^t\phi(u)\,du}{\phi(u)}, \quad \text{as } t\to\infty.$$

Moreover, if $\phi\in\Gamma$ then so is $x\mapsto\int_0^t\phi(u)\,du=:\phi_1(t)$ and ϕ_1 has the same auxiliary function as ϕ. Therefore

$$\lim_{t\to\infty}\frac{\phi(t)\int_0^t\int_0^y\phi(z)\,dzdy}{\left(\int_0^t\phi(y)\,dy\right)^2}=1. \tag{3.3.13}$$

This limit also characterises Γ (i.e., if ϕ satisfies this limit, then $\phi\in\Gamma$). Moreover, it is relatively easy to check (3.3.13) in contrast to using the definition directly, where one also needs to construct the auxiliary function ϕ. Lastly, if $\phi\in\Gamma$, then ϕ is rapidly varying at infinity, in the usual sense that

$$\lim_{t\to\infty}\frac{\phi(\lambda t)}{\phi(t)}=+\infty, \quad \text{for all } \lambda>1.$$

It is now evident what is the connection between z in (3.3.11) and ϕ in (3.3.13). Suppose z is a C^2 function such that $z''\in\Gamma$. Then, according to the above discussion, so are z' and z. Also, since z'' is also rapidly varying at infinity, we must have $z''(t)\to\infty$ as $t\to\infty$. Hence $z'(t)\to\infty$ and $z(t)\to\infty$ as $t\to\infty$. Therefore, putting $\phi(t):=z''(t)$, we see that $\phi \in \Gamma$. Hence it must obey (3.3.13), so

$$\lim_{t\to\infty} \frac{z''(t)\int_0^t [z'(y)-z'(0)]\,dy}{(z'(t)-z'(0))^2} = 1.$$

The denominator is asymptotic to $z'(t)^2$ as $t \to \infty$. The integral in the numerator is $z(t) - tz'(0) - z(0)$. Since z is rapidly varying, we have that $z(t)/t \to 0$ as $t \to \infty$. Hence the integral in the numerator is asymptotic to $z(t)$ as $t \to \infty$. The displayed limit and these observations imply that z obeys (3.3.11). Of course, conversely if z obeys (3.3.11), it is also true that z'' is in Γ. Hence assuming (3.3.11) forces us to consider the class of functions Γ.

Thus, to show that y is rapidly varying at zero in the manner required by our main results, take $z(t) = 1/y(t)$. It is now easy to show that if y obeys (3.3.10), then z obeys (3.3.11), and hence z is rapidly varying. Therefore for $\lambda > 1$, by putting $z(t) = 1/y(t)$ in (3.3.12), we get

$$\lim_{t\to\infty} \frac{y(\lambda t)}{y(t)} = 0, \quad \text{for all } \lambda > 1.$$

This is the claimed rapid variation.

To show that y gives rise to an asymptotic preserving f, note that y is the solution of the differential equation $y'(t) = -f(y(t))$, $t \geq 0$, $y(0) = 1$. Then, since y is decreasing, f must be given by $f(x) = -y'(y^{-1}(x))$, $x > 0$. We now prove that (3.3.10) implies that $f \in RV_0(1)$, and so f must also be asymptotic preserving. Since y is in C^2, we may compute f'. Since $(y^{-1})'(x) = 1/y'(y^{-1}(x))$, we get

$$\frac{xf'(x)}{f(x)} = \frac{xy''(y^{-1}(x))}{y'(y^{-1}(x))^2}.$$

Thus

$$\lim_{x\to 0^+} \frac{xf'(x)}{f(x)} = \lim_{x\to 0^+} \frac{xy''(y^{-1}(x))}{y'(y^{-1}(x))^2} = \lim_{t\to\infty} \frac{y(t)y''(t)}{y'(t)^2} = 1.$$

We now show that this implies the regular variation of f. The last limit implies that for every $\epsilon \in (0,1)$ there is $x(\epsilon) > 0$ such that

$$(1-\epsilon)\frac{1}{x} < \frac{f'(x)}{f(x)} < (1+\epsilon)\frac{1}{x}, \quad x < x(\epsilon).$$

Let $0 < \lambda < 1$, and $x < x(\epsilon)$. Then, $\lambda x \leq x < x(\epsilon)$ and thus

$$(1-\epsilon)\int_{\lambda x}^{x} \frac{1}{u}\,du \leq \int_{\lambda x}^{x} \frac{f'(u)}{f(u)}\,du \leq (1+\epsilon)\int_{\lambda x}^{x} \frac{1}{u}\,du.$$

Integrating each of the terms and simplifying leads to

$$\log\left(\frac{1}{\lambda^{1-\epsilon}}\right) \leq \log\left(\frac{f(x)}{f(\lambda x)}\right) \leq \log\left(\frac{1}{\lambda^{1+\epsilon}}\right), \quad x < x(\epsilon).$$

Therefore for every $\lambda \in (0,1)$ and every $\epsilon \in (0,1)$ there is $x(\epsilon) > 0$ such that

$$\lambda^{1-\epsilon} \geq \frac{f(\lambda x)}{f(x)} \geq \lambda^{1+\epsilon}, \quad x < x(\epsilon).$$

Rearranging and taking limits leads to

$$\lambda^{1-\epsilon} \geq \limsup_{x\to 0^+} \frac{f(\lambda x)}{f(x)} \geq \liminf_{x\to 0^+} \frac{f(\lambda x)}{f(x)} \geq \lambda^{1+\epsilon}.$$

Now, letting $\epsilon \to 0^+$, we see that for each $\lambda \in (0,1)$ it follows that $\lim_{x\to 0^+} f(\lambda x)/f(x) = \lambda$. By the usual considerations, this implies

$$\lim_{x\to 0^+} \frac{f(\lambda x)}{f(x)} = \lambda \quad \text{for all } \lambda \in (0,\infty),$$

and therefore f is regularly varying at zero with index 1, as needed. □

As an example of this result, suppose that $y(t) = \exp(-e^t)$ is the solution of the unperturbed equation $y'(t) = -f(y(t))$ with $y(0) = e$. It is an easy computation to show that $y'(t) = -y(t)e^t$ and $y''(t) = y(t)(e^{2t} - e^t)$. Hence

$$\lim_{t\to\infty} \frac{y(t)y''(t)}{y'(t)^2} = 1$$

Thus by the last result y is rapidly varying, and f is asymptotic preserving

3.4 Internally Perturbed ODEs

As suggested in the introduction, the asymptotic analysis of the "externally" perturbed differential equation

$$x'(t) = -f(x(t)) + g(t), \quad t > 0; \quad x(0) = \zeta,$$

is facilitated by considering the related "internally" perturbed ordinary differential equation

$$z'(t) = -f(z(t) + \Gamma(t)), \quad t > 0; \quad z(0) = \xi. \tag{3.4.1}$$

This section is devoted to the analysis of (3.4.1). The connection between the equations is the following: if (3.2.10) holds i.e.,

$$\lim_{t\to\infty} \int_0^t g(s)\, ds \text{ exists and is finite}$$

then the function

$$\Gamma(t) := -\int_t^\infty g(s)\, ds, \quad t \geq 0 \tag{3.4.2}$$

is well–defined, as is the function $z(t) = x(t) - \Gamma(t)$, $t \geq 0$ where x is the solution of the externally perturbed ODE. It is immediate that z is a solution of (3.4.1).

One potential advantage in studying (3.4.1) rather than directly attacking the original ODE is that pointwise conditions on g may no longer be needed to get decay properties, contingent on Γ being well–defined.

We demonstrate for (3.4.1) that when the "internal" perturbation Γ decays to zero so rapidly that

$$\liminf_{t\to\infty} \frac{F(|\Gamma(t)|)}{t} \geq 1, \tag{3.4.3}$$

and the solution of (3.4.1) tends to zero as $t \to \infty$, the asymptotic behaviour of the unperturbed equation is preserved in the following sense.

Theorem 3.4.1. *Suppose that f obeys (3.2.1), (3.2.2), (3.2.4) and (3.2.3). Suppose F defined by (3.1.3) obeys $F(x) \to \infty$ as $x \to \infty$. Let Γ be continuous and z be the continuous solution of (3.4.1). If Γ obeys (3.4.3), then*

$$\liminf_{t\to\infty} \frac{F(|z(t)|)}{t} \geq 1.$$

We now explore how Theorem 3.4.1 can be applied to determine sufficient conditions for certain asymptotic decay in (3.1.1).

3.4.1 Application of Theorem 3.4.1 to (3.1.1)

Consider the solution x of (3.1.1) which we suppose obeys $x(t) \to 0$ as $t \to \infty$. Introduce the function $u(t) = \int_0^t g(s)\,ds$ and assume that it tends to a finite limit as $t \to \infty$, which we call $u(\infty)$. We are therefore free to define $\Gamma(t) = u(t) - u(\infty)$ for $t \geq 0$. Clearly, Γ is continuous and obeys $\Gamma(t) \to 0$ as $t \to \infty$. Of course, $u'(t) = g(t)$. Consider now $z(t) = x(t) - u(t) + u(\infty) = x(t) - \Gamma(t)$ for $t \geq 0$. Then z is in $C^1((0,\infty);\mathbb{R})$ and we have that $z(t) \to 0$ as $t \to \infty$. Then $z(0) = \xi + \int_0^\infty g(s)\,ds =: \xi'$ and

$$z'(t) = x'(t) - u'(t) = -f(x(t)) = -f(z(t) + \Gamma(t)), \quad t \geq 0.$$

Therefore, we see that if $\Gamma(t) = \int_t^\infty g(s)\,ds$ obeys (3.4.3), we can apply Theorem 3.4.1 to z to obtain

$$\liminf_{t\to\infty} \frac{F(|z(t)|)}{t} \geq 1.$$

Then, as Γ obeys (3.4.3), we have that for every $\epsilon \in (0,1)$ there exists $T(\epsilon) > 0$ such that for $t \geq T(\epsilon)$ we have $F(|z(t)|) > (1-\epsilon)t$ and $F(|\Gamma(t)|) > (1-\epsilon)t$. Therefore for $t \geq T(\epsilon)$ we have $|z(t)| < F^{-1}((1-\epsilon)t)$ and $|\Gamma(t)| < F^{-1}((1-\epsilon)t)$. Hence $x(t) = z(t) + \Gamma(t)$ obeys $|x(t)| \leq 2F^{-1}((1-\epsilon)t)$ for $t \geq T(\epsilon)$. Since F is decreasing, this yields

$F(|x(t)|/2) \geq (1-\epsilon)t$ for $t \geq T(\epsilon)$. Therefore

$$\liminf_{t\to\infty} \frac{F(|x(t)|/2)}{t} \geq 1-\epsilon.$$

Letting $\epsilon \to 0^+$ yields $\liminf_{t\to\infty} F(|x(t)|/2)/t \geq 1$. Finally, since $F \in \mathrm{RV}_0(0)$ and $x(t) \to 0$ as $t \to \infty$, we have that

$$\liminf_{t\to\infty} \frac{F(|x(t)|)}{t} = \liminf_{t\to\infty} \frac{F(|x(t)|)}{F(|x(t)|/2)} \cdot \frac{F(|x(t)|/2)}{t} = \liminf_{t\to\infty} \frac{F(|x(t)|/2)}{t} \geq 1.$$

Therefore, we have established the following result, having introduced the notation

$$\lambda_\Gamma := \liminf_{t\to\infty} \frac{F(|\Gamma(t)|)}{t}, \tag{3.4.4}$$

which is well–defined for any function Γ such that $\Gamma(t) \to 0$ as $t \to \infty$.

Theorem 3.4.2. *Suppose that f obeys (3.2.1), (3.2.2), (3.2.4) and (3.2.3). Suppose F defined by (3.1.3) obeys $F(x) \to \infty$ as $x \to \infty$. Let g be a continuous function such that (3.2.10) holds and let Γ be defined by (3.4.2). If λ_Γ is defined by (3.4.4) and*

$$\lambda_\Gamma \geq 1, \tag{3.4.5}$$

then the continuous solution of (3.1.1) obeys

$$\lambda_x := \liminf_{t\to\infty} \frac{F(|x(t)|)}{t} \geq 1. \tag{3.4.6}$$

3.4.2 Existence and Stability of Solutions

It is important to note the standing assumptions (3.2.1), (3.2.2), (3.2.9) and (3.2.10) in this section. Now, considering equation (3.4.1), the continuity of f allows us to apply Peano's Theorem to guarantee a solution on some interval $[0, T)$. This can then be extended to a global solution using the hypothesis $\lim_{t\to\infty} \Gamma(t) = 0$. The details of this, and the existence of a global solution of (3.1.1) are given below.

Theorem 3.4.3. *Let f obey (3.2.1), (3.2.2). Suppose Γ is a continuous function obeying $\lim_{t\to\infty} \Gamma(t) = 0$. Then there exists a continuous solution z of (3.4.1) on $[0, \infty)$. Moreover, any continuous solution is uniformly bounded on $[0, \infty)$*

Proof. Since $\Gamma(t) \to 0$ as $t \to \infty$ and Γ is continuous, it follows that there is a $K > 0$ such that $-K \leq \Gamma(t) \leq K$ for all $t \geq 0$. Let $z(0) = \xi$. By Peano's theorem, there is a continuous solution of (3.4.1) on $[0, T)$, where we either have $T = +\infty$, or $T < +\infty$ and $\limsup_{t\to T^-} |z(t)| = +\infty$. Let us now assume that we are in the second case, and obtain contradictions. We consider the cases

$$\text{(I) } |\xi| \leq 2K, \quad \text{(II) } |\xi| > 2K.$$

In case (II), we either have (IIa) $\xi > 2K$ or (IIb) $\xi < -2K$. Consider (IIa). Let $z_+(t) = \xi + 1$ for all $t \geq 0$. We have that $z_+(0) > z(0)$ and since $z_+(t) + \Gamma(t) = \xi + 1 + \Gamma(t) \geq \xi + 1 - K > K + 1 > 0$, we have

$$z_+'(t) = 0 > -f(z_+(t) + \Gamma(t)), \quad t \geq 0.$$

From this it follows that $z(t) < z_+(t) = \xi + 1$ for all $t \in [0, T)$. On the other hand, let $z_-(t) = -(\xi + 1)$ for $t \geq 0$. Then $z_-(0) < z(0)$ and $z_-(t) + \Gamma(t) = -\xi - 1 + \Gamma(t) \leq -\xi - 1 + K < -2K - 1 + K = -(1 + K) < 0$. Thus

$$z_-'(t) = 0 < -f(z_-(t) + \Gamma(t)), \quad t \geq 0.$$

From this it follows that $z(t) > z_-(t) = -(\xi + 1)$ for all $t \in [0, T)$. Therefore we have that $|z(t)| < \xi + 1$ for all $t \in [0, T)$. Therefore $\limsup_{t \to T^-} |z(t)| \leq \xi + 1$, contradicting the hypothesis that $\limsup_{t \to T^-} |z(t)| = +\infty$. The argument in case (IIb) is similar. Therefore, we have $T = +\infty$ in the case (II) when $|\xi| > 2K$ and indeed we have that the solution obeys $|z(t)| < \xi + 1$ for all $t \geq 0$ by recapitulating the above argument.

In the case (I), we may proceed in an almost identical manner. Take the case (Ia) where $0 \leq \xi \leq 2K$. Define $z_+(t) = 2K + 1$ for all $t \geq 0$ and $z_-(t) = -(2K + 1)$ for all $t \geq 0$. Proceeding as above, we have that $z_+(0) > z(0) > z_-(0)$ and that z_+ and z_- satisfy the differential inequalities

$$z_+'(t) > -f(z_+(t) + \Gamma(t)), \quad t \geq 0; \quad z_-'(t) < -f(z_-(t) + \Gamma(t)), \quad t \geq 0,$$

from which we conclude that $2K + 1 = z_+(t) > z(t) > z_-(t) = -(2K + 1)$ for all $t \in [0, T)$. Therefore $|z(t)| < 2K + 1$ for all $t \in [0, T)$, and the contradiction $+\infty = \limsup_{t \to T^-} |z(t)| \leq 2K + 1 < +\infty$ results. Thus $T = +\infty$ in the case (I) when $|\xi| \leq 2K$ and indeed we have that the solution obeys $|z(t)| < 2K + 1$ for all $t \geq 0$ by recapitulating the above argument. □

Theorem 3.4.4. *Let f obey (3.2.1), (3.2.2). Suppose g obeys (3.2.9) and (3.2.10). Then there exists a continuous solution x to (3.1.1) on $[0, \infty)$.*

Proof. The continuity of f and g guarantee that (3.1.1) has a continuous solution x on a maximal interval of existence $[0, T)$, where we either have $T = +\infty$ (and the solution is globally defined) or $T < +\infty$ and $\limsup_{t \to T^-} |x(t)| = +\infty$. Now consider $w(t) := x(t)^2$ for $t \in [0, T)$. We wish to show that $T < +\infty$ is impossible. Suppose to the contrary. Then $\limsup_{t \to T^-} w(t) = +\infty$. On the other hand, since f obeys (3.2.2), and x is C^1

on $(0,T)$, we have that w obeys the differential inequality

$$w'(t) = 2x(t)x'(t) = -2x(t)f(x(t)) + 2x(t)g(t) \leq 2x(t)g(t) \leq w(t) + g^2(t)$$

for $t \in [0,T)$, where we used the inequality $2ab \leq a^2 + b^2$ for arbitrary $a, b \in \mathbb{R}$ at the last step. From the differential inequality, we get for $t \in [0,T)$ that

$$w(t) \leq \zeta^2 e^t + e^t \int_0^t e^{-s} g^2(s)\, ds =: \bar{w}(t)$$

noting that $w(0) = x(0)^2 = \zeta^2$. Since g is continuous on $[0,\infty)$, we see that on the right hand side $\limsup_{t\to T^-} \bar{w}(t) = \lim_{t\to T^-} \bar{w}(t) = \bar{w}(T)$ is finite. But on the left hand side $\limsup_{t\to T^-} w(t) = +\infty$, a contradiction. Hence x is well–defined on $[0,\infty)$, as claimed. □

In Theorem 3.4.1 we note that the hypothesis $z(t) \to 0$ as $t \to \infty$ can be deduced from the hypotheses which we make to ensure the preservation of the rate of convergence. We can now state this formally in terms of the internally perturbed equation: a convergence result for (3.1.1) is a simple corollary of this.

Theorem 3.4.5. *Let f obey (3.2.1), (3.2.2). Suppose Γ is a continuous function obeying $\lim_{t\to\infty} \Gamma(t) = 0$. Let z be a continuous solution of (3.4.1). Then $z(t) \to 0$ as $t \to \infty$.*

Corollary 3.4.1. *Let f obey (3.2.1), (3.2.2). Suppose g obeys (3.2.9) and (3.2.10). Then there exists a continuous solution x to (3.1.1) on $[0,\infty)$ and it obeys $x(t) \to 0$ as $t \to \infty$.*

3.5 Externally Perturbed ODEs

The last result shows that when g obeys (3.2.10), then $x(t) \to 0$ as $t \to \infty$. In this case, it is legitimate to define

$$\lambda_x = \liminf_{t\to\infty} \frac{F(|x(t)|)}{t}. \tag{3.5.1}$$

This notation streamlines the statement of the results on decay rates for solutions of (3.1.1) that now follow

We start by stating results which show that the sufficient conditions under which the perturbed equations inherit the asymptotic behaviour of the unperturbed, are also necessary.

A converse of Theorem 3.4.2 requires that (3.4.6) implies (3.4.5) and (3.2.10). We prove first that (3.4.6) implies (3.2.10).

Theorem 3.5.1. *Let f obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3). Let g be continuous. If any continuous solution x of (3.1.1) obeys (3.4.6), then g obeys (3.2.10).*

Theorem 3.5.2. *Let f obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3). Let g be continuous. If any continuous solution x of (3.1.1) obeys (3.4.6), then g obeys (3.4.5).*

Consolidating these results and Theorem 3.4.2, we can characterise the situation in which $\lambda_x \geq 1$.

Theorem 3.5.3. *Let f obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3). Let g be continuous. Then the following are equivalent:*

(A) g obeys (3.2.10) and Γ defined by (3.4.2) obeys $\lambda_\Gamma \geq 1$ where λ_Γ is given by (3.4.4)

(B) λ_x given by (3.5.1) is well–defined and obeys $\lambda_x \geq 1$.

Knowing that the solution of (3.1.1) decays sufficiently quickly that $\lambda_x > 0$ gives information about the existence and value of λ_Γ.

Theorem 3.5.4. *Let f obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3). Let g be continuous. If the the continuous solution of (3.1.1) obeys*

$$\lambda_x > 0,$$

where λ_x is given by (3.5.1), then g obeys (3.2.10) and Γ defined by (3.4.2) obeys

$$\lambda_\Gamma \geq \lambda_x,$$

where λ_Γ is given by (3.4.4).

We next consider the case when the perturbations decay moderately fast, but slower than the solution of the unperturbed equation, in the following theorem:

Theorem 3.5.5. *Let f obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3). Let g be continuous and obey (3.2.10), and for Γ defined by (3.4.2), let λ_Γ given by (3.4.4) obey $\lambda_\Gamma \in (0, 1]$. Let x be continuous solution of (3.1.1), and λ_x be given by (3.5.1).*

(i) If $\lambda_\Gamma \in (0, 1)$, then $\lambda_x = \lambda_\Gamma$.

(ii) If $\lambda_\Gamma = 1$, then $\lambda_x = 1$.

This covers perturbations which have the same of type decay, as measured by generalised Liapunov exponent λ_Γ but which nevertheless decay more slowly to zero than y; indeed for $\lambda_\Gamma \in (0, 1)$, we have

$$\limsup_{t\to\infty} \frac{|\Gamma(t)|}{y(t)} = +\infty.$$

If g continuous and (3.4.5) holds, but $\lambda_\Gamma = 0$, then the decay in Γ is much slower than that of y, measured by the generalised Liapunov exponent and x inherits this slow decay, as demonstrated in the following theorem.

Theorem 3.5.6. *Let f obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3). Let g be continuous and obey (3.2.10), and for Γ defined by (3.4.2), let λ_Γ given by (3.4.4) obey $\lambda_\Gamma = 0$. Let x be continuous solution of (3.1.1) and λ_x be given by (3.5.1). Then*

$$\lambda_x = 0.$$

We can consolidate our results into as single statement as follows.

Theorem 3.5.7. *Let f obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3). Let g be continuous and obey (3.2.10), and for Γ defined by (3.4.2), let λ_Γ given by (3.4.4) obey $\lambda_\Gamma \in [0,\infty]$. Let x be continuous solution of (3.1.1) and λ_x be given by (3.5.1). Then*

(i) $\lambda_\Gamma \in [1,\infty]$ implies $\lambda_x \in [1,\lambda_\Gamma]$.

(ii) $\lambda_\Gamma \in [0,1]$ implies $\lambda_x = \lambda_\Gamma$.

Next, we notice in the case when g is positive and the initial condition is positive (so that $x(t) > 0$ for all $t \geq 0$), the limit inferior in (3.4.6) is unity, and in fact the upper bound on the solution implied by it is sharp.

Theorem 3.5.8. *Let f obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3). Let g be continuous and positive. If x is any continuous solution (3.1.1) with $x(0) = \xi > 0$, then the following are equivalent:*

(A) g obeys (3.2.10) and Γ defined by (3.4.2) obeys $\lambda_\Gamma \geq 1$ where λ_Γ is given by (3.4.4);

(B)

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 1.$$

We now show that if condition (3.2.10) does not hold, then either $x(t)$ does not tend to zero or if it does, the rate of decay is very slowl as in Theorem 3.5.5.

Theorem 3.5.9. *Let f obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3). Suppose f is continuous, and obeys (3.2.4) and (3.2.3). Suppose that g be a continuous function and (3.2.10) does not hold. Let x be a continuous solution of (3.1.1). Then exactly one of the following statements hold*

(i) $x(t)$ does not tend to zero as $t \to \infty$;

(ii) $x(t) \to 0$ as $t \to \infty$ and λ_x given by (3.5.1) obeys $\lambda_x = 0$.

3.6 Stochastic Equation

We now turn our attention to the stochastic differential equation (3.1.4).

3.6.1 Existence and stability

We rely primarily on existing work to deal with both existence and stability. Define the auxiliary linear SDE

$$dY(t) = -Y(t)dt + \sigma(t)dB(t), \quad t \geq 0; \quad Y(0) = \xi. \tag{3.6.1}$$

From [54] we know that this auxiliary equation has a unique, continuous solution and by Theorem 3.1 in [4] we know that it is globally stable when $\sigma \in L^2(0,\infty)$ is continuous. We then form the following ODE for each ω in the a.s. event Ω_Y on which (3.6.1) has a unique, continuous solution

$$Z'(t,\omega) = -f(Z(t,\omega) + Y(t,\omega)) + Y(t,\omega), \quad t \in [0, T(\omega)), \quad Z(0,\omega) = 0. \tag{3.6.2}$$

We once more appeal to Peano to give us the existence of a solution $Z(\omega)$ to this equation on some interval $[0, T(\omega))$. We now let $X(t,\omega) = Y(t,\omega) + Z(t,\omega)$ for $t \in [0, T(\omega))$. Thus X obeys (3.1.4) on $[0, T(\omega))$. Then by Proposition 3.1 of [7] we have that $T(\omega) = +\infty$ for a.e. ω in some a.s. event Ω_X.

For the stability of solutions of (3.1.4) we note that $\sigma \in L^2(0,\infty)$ allows us to directly apply Theorem 1 in [5] to obtain $\lim_{t\to\infty} X(t) = 0$ a.s.

In summary, we may conclude from this section that we need not concern ourselves with questions of existence or stability since our characterisation conditions are always sufficient to guarantee global solutions with the desired properties.

3.6.2 Convergence rates

We apply Theorem 3.4.1 to determine the rate of convergence of solutions of (3.1.4) where $\sigma \in L^2(0,\infty)$. Let Ω_0 be almost sure event on which X is well-defined. Introduce the function $U(t) := \int_0^t \sigma(s)\,dB(s)$, $t \geq 0$. Since $\sigma \in (0,\infty)$, we note that $U(t) \to U(t) \in (-\infty,\infty)$ as $t \to \infty$ a.s. on Ω_1, also we note that $X(t) \to 0$ as $t \to \infty$ a.s. on Ω_2. Let $\Omega_3 = \Omega_0 \cap \Omega_1 \cap \Omega_2$ and define $Y(t) = X(t) - U(t)$ for all $t \geq 0$. Clearly, for $\omega \in \Omega_3$ $Y(t,\omega) \to -U(\infty,\omega)$ as $t \to \infty$. Also, define $Z(t,\omega) := Y(t,\omega) + U(\infty,\omega)$ for all $t \geq 0$ and $\Gamma(t,\omega) = -U(\infty,\omega) + U(t,\omega)$ for all $t \geq 0$. We note that Y is an adapted process which obeys $Y(t) = X(t) - U(t) = \zeta - \int_0^t f(X(s))\,ds$ for all $t \geq 0$. Since f and the paths of X are continuous on Ω_3, we have that $t \mapsto Y(t,\omega)$ is in $C^1(0,\infty)$ for $\omega \in \Omega_3$, and moreover

$$Y'(t,\omega) = -f\left(X(t,\omega)\right) = -f\left(Y(t,\omega) + U(t,\omega)\right).$$

Thus $t \mapsto Z(t,\omega)$ is in $C^1(0,\infty)$ for $\omega \in \Omega_3$ and

$$\begin{aligned} Z'(t,\omega) =& Y'(t,\omega) = -f\left(Y(t,\omega)+U(t,\omega)\right) \\ &= -f\left(Z(t,\omega)-U(\infty,\omega)+U(t,\omega)\right) \\ &= -f\left(Z(t,\omega)+\Gamma(t,\omega)\right). \end{aligned}$$

Hence

$$Z'(t,\omega) = -f\left(Z(t,\omega)+\Gamma(t,\omega)\right), \quad t>0 \tag{3.6.3}$$

Observe moreover for $\omega \in \Omega_2$ that $Z(t,\omega) \to 0$ as $t \to \infty$. Notice that Γ is given by

$$\Gamma(t,\omega) := -\left(\int_t^\infty \sigma(s)\,dB(s)\right)(\omega). \tag{3.6.4}$$

Therefore, we have that Z obeys an internally perturbed differential equation, the internal perturbation $\Gamma(t) \to 0$ as $t \to \infty$ is independent of X, and $X = Z + \Gamma$.

It is now possible to prove theorems based on the pathwise behaviour of Γ on the event on which $\Gamma(t) \to 0$ as $t \to \infty$. On such an event, we can define

$$\lambda_\Gamma(\omega) := \liminf_{t\to\infty} \frac{F(|\Gamma(t,\omega)|)}{t}, \tag{3.6.5}$$

and on the event on which $X(t) \to 0$ as $t \to \infty$ we can likewise define

$$\lambda_X(\omega) := \liminf_{t\to\infty} \frac{F(|\Gamma(t,\omega)|)}{t}. \tag{3.6.6}$$

The following theorem can now be proven by appealing to the internally perturbed differential equation (3.6.3) path by path.

Theorem 3.6.1. *Let f obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3). Let σ be continuous and $\sigma \in L^2(0,\infty)$ and for Γ defined by (3.6.4), let λ_Γ given by (3.6.5). Let X be a continuous solution of (3.1.1) and λ_X be given by (3.6.6). Then λ_Γ and λ_X are defined a.s. Moreover for each ω in an a.s. event, we have*

(i) $\lambda_\Gamma(\omega) \in [1,\infty]$ *implies* $\lambda_X(\omega) \in [1,\infty]$.

(ii) $\lambda_\Gamma(\omega) \in [0,1]$ *implies* $\lambda_X(\omega) = \lambda_\Gamma(\omega)$.

Converse theorems can be established on a pathwise basis as well.

It is our view that while of theoretical interest, a result of the type Theorem 3.6.1 is not very easy to apply in practice, since the asymptotic behaviour of Γ is not observable, and may in any event vary from path to path. Therefore, what is needed is a result connecting explicitly the asymptotic behaviour of σ with that of Γ, preferably on non–trivial events. Such a result is known, and the proof of the following lemma can be found in [6].

Lemma 3.6.1. *Suppose $\sigma \in C([0,\infty);\mathbb{R})$ is such that $\sigma \in L^2([0,\infty);\mathbb{R})$ and*

$$\int_t^\infty \sigma^2(s)\,ds > 0 \text{ for all } t \geq 0, \tag{3.6.7}$$

and define

$$\Sigma(t) := \sqrt{2\int_t^\infty \sigma^2(s)\,ds \log_2 \frac{1}{\int_t^\infty \sigma^2(s)\,ds}} \tag{3.6.8}$$

Then

$$\limsup_{t\to\infty} \frac{\int_t^\infty \sigma(s)\,dB(s)}{\Sigma(t)} = 1, \quad a.s.$$

and

$$\liminf_{t\to\infty} \frac{\int_t^\infty \sigma(s)\,dB(s)}{\Sigma(t)} = -1, \quad a.s.$$

The condition (3.6.7) is harmless, and indeed covers the more interesting cases; if it does not hold, then there must exist (a minimal) deterministic $T \geq 0$ such that $\int_T^\infty \sigma^2(s)\,ds = 0$. But then $\sigma(t) = 0$ a.e. on $[T,\infty)$ and the stochastic differential equation collapses to the unperturbed differential equation

$$X'(t) = -f(X(t)), \quad t \geq T$$

with the value of $X(T)$ will be random, as X obeys the authentic stochastic differential equation

$$dX(t) = -f(X(t))\,dt + \sigma(t)\,dB(t), \quad t \in [0,T)$$

with σ not identically zero on $[0,T)$. In this case, unless $X(T) = 0$, we will have that $F(X(t))/t \to 1$ as $t \to \infty$, while if $X(T) = 0$, we have $X(t) = 0$ for all $t \geq T$.

We will now use Lemma 3.6.1 to obtain sharp sufficient conditions under which the rate of decay of X can be established. Assume further that the asymptotic behaviour of Σ is known relative to the solution of the unperturbed differential equation, in the sense that

$$\lim_{t\to\infty} \frac{F(\Sigma(t))}{t} =: L_\Sigma \text{ exists.} \tag{3.6.9}$$

Notice that this is not a limit inferior, but a limit. This is partly for technical convenience. However, Σ is quite a regular function, being positive, in C^1 and decreasing. In particular, the monotonicity and positivity enjoyed by Σ is not always present in $\Gamma(t) := -\int_t^\infty g(s)\,ds$ in the deterministic case, which in part justifies strengthening the hypothesis from a limit inferior to a limit.

We are now in a position to prove our main results. Assume now that f obeys (3.2.1), (3.2.2), (3.2.4) and (3.2.3). Let σ be continuous and $\sigma \in L^2(0,\infty)$ and suppose that $\Sigma(t) > 0$ for all $t \geq 0$. By Lemma 3.6.1, we have

$$\limsup_{t\to\infty} |U(\infty) - U(t)| / \Sigma(t) = 1, \qquad \text{a.s. on } \Omega_4.$$

Let $\Omega_5 = \Omega_3 \cap \Omega_4$ and thus for $\omega \in \Omega_5$ we have $\limsup_{t\to\infty} |\Gamma(t,\omega)| / \Sigma(t) = 1$. Therefore, we have for every $\epsilon > 0$ that there is $T(\epsilon,\omega) > 0$ such that $|\Gamma(t,\omega)| < (1+\epsilon)\Sigma(t)$ for all $t \geq T(\epsilon,\omega)$. Hence

$$\frac{F(|\Gamma(t,\omega)|)}{F(\Sigma(t))} > \frac{F((1+\epsilon)\Sigma(t))}{F(\Sigma(t))}, \quad t \geq T(\epsilon,\omega).$$

Taking the limit inferior as $t \to \infty$ and using the fact that $F \in RV_0(0)$, we have

$$\liminf_{t\to\infty} \frac{F(|\Gamma(t,\omega)|)}{F(\Sigma(t))} \geq 1, \quad \omega \in \Omega_5. \tag{3.6.10}$$

On the other hand, for each $\omega \in \Omega_5$ and $\epsilon \in (0,1)$, there is a sequence $(t_n(\omega))_{n\geq 1} \nearrow \infty$ such that $|\Gamma(t_n(\omega),\omega)| > (1-\epsilon)\Sigma(t_m(\omega))$ for all $n \geq N(\epsilon,\omega)$. Thus

$$\frac{F(|\Gamma(t_n,\omega)|)}{F(\Sigma(t_n(\omega)))} < \frac{F((1-\epsilon)\Sigma(t_n(\omega)))}{F(\Sigma(t_n(\omega)))}.$$

Since $t_n \to \infty$ as $n \to \infty$ and F is in $RV_0(0)$,

$$\limsup_{n\to\infty} \frac{F(|\Gamma(t_n(\omega))|)}{F(\Sigma(t_n(\omega)))} \leq 1, \quad \omega \in \Omega_5.$$

Since (t_n) is a sequence, we have

$$\liminf_{t\to\infty} \frac{F(|\Gamma(t,\omega)|)}{F(\Sigma(t))} \leq \limsup_{n\to\infty} \frac{F(|\Gamma(t_n(\omega),\omega)|)}{F(\Sigma(t_n(\omega)))} \leq 1. \tag{3.6.11}$$

Combining (3.6.10) and (3.6.11), we get

$$\liminf_{t\to\infty} \frac{F(|\Gamma(t,\omega)|)}{F(\Sigma(t))} = 1, \quad \omega \in \Omega_5.$$

Hence for $\omega \in \Omega_5$ we have

$$\lambda_\Gamma(\omega) = \liminf_{t\to\infty} \frac{F(|\Gamma(t,\omega)|)}{t} = \liminf_{t\to\infty} \frac{F(|\Gamma(t,\omega)|)}{F(\Sigma(t))} \frac{F(\Sigma(t))}{t} = L_\Sigma. \tag{3.6.12}$$

Now consider the case when $L_\Sigma \geq 1$. Then $\lambda_\Gamma(\omega) \geq 1$ for all $\omega \in \Omega_5$. Hence by Theorem 3.4.1, for each $\omega \in \Omega_5$,

$$\lambda_Z = \liminf_{t\to\infty} \frac{F(|Z(t,\omega)|)}{t} \geq 1.$$

Since $X(t,\omega) = Z(t,\omega) + \Gamma(t,\omega)$ for $t \geq 0$, $\lambda_\Gamma \geq 1$ and $\lambda_Z \geq 1$ on Ω_5 we have that

$$\lambda_X(\omega) := \liminf_{t\to\infty} \frac{F\left(|X(t,\omega)|\right)}{t}$$

obeys $\lambda_X(\omega) \geq 1$ for all $\omega \in \Omega_5$. Therefore we have the following result.

Theorem 3.6.2. *Let f obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3). Let σ be continuous and $\sigma \in L^2(0,\infty)$ and suppose that $\Sigma(t) > 0$ for all $t \geq 0$ where Σ is given by (3.6.8). Also, let L_Σ defined by (3.6.9) obey $L_\Sigma \geq 1$. Then a continuous solution X of (3.1.1) obeys $X(t) \to 0$ as $t \to \infty$ a.s. and λ_X defined by (3.6.6) obeys*

$$\lambda_X \geq 1 \quad a.s.$$

Now we deal with the case of slowly decaying stochastic perturbations. Recall that if the perturbation Γ in (3.4.1) is large, in the sense that $\lambda_\Gamma \in (0,1)$ in (3.4.3), we have shown in Lemma 3.8.5 that

$$\lambda_z = \liminf_{t\to\infty} \frac{F\left(|z(t)|\right)}{t}$$

obeys $\lambda_z \geq \lambda_\Gamma$. Suppose now that $L_\Sigma \in (0,1)$. Then once again we get $\lambda_\Gamma = L_\Sigma \in (0,1)$. Hence applying Lemma 3.8.5 to $Z(\omega)$, we have $\lambda_Z(\omega) \geq L_\Sigma$ for all $\omega \in \Omega_5$. Since $\lambda_\Gamma(\omega) = \lambda_\Sigma$ and $X(t,\omega) = Z(t,\omega) + \Gamma(t,\omega)$, we have that

$$\lambda_X(\omega) \geq L_\Sigma, \quad \omega \in \Omega_5. \tag{3.6.13}$$

To get the corresponding lower bound on λ_X, we prove next the following result.

Theorem 3.6.3. *Let f obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3). Let σ be continuous. Let X be a continuous solution of (3.1.1) such that λ_X defined by (3.6.6) obeys*

$$\lambda_X > 0, \quad \text{on } \Omega^*,$$

where Ω^* is an event of positive probability. Then $\sigma \in L^2(0,\infty)$, Γ given by (3.6.4) is well–defined a.s. and λ_Γ given by (3.6.5) obeys

$$\lambda_\Gamma \geq \lambda_X \quad a.s. \text{ on } \Omega^*.$$

Proof. By hypothesis, we have $\lambda_X(\omega) > 0$ for all $\omega \in \Omega^*$ where Ω^* is an event of positive probability. Hence

$$0 < \lambda_X(\omega) = \liminf_{t\to\infty} \frac{F\left(|X(t,\omega)|\right)}{t}, \quad \omega \in \Omega^*.$$

We first assume $\lambda_X < +\infty$, with the proof in the second case when $\lambda_X = +\infty$ being similar. Since $F(x) \sim \Phi(x)$ as $x \to 0^+$, by definition, for every $\omega \in \Omega^*$ and $\epsilon \in (0,1)$, there is $T_1(\epsilon,\omega) > 0$ such that $\Phi\left(|X(t,\omega)|\right)/t > \lambda_X(\omega)(1-\epsilon)$ for all $t \geq T_1(\epsilon,\omega)$. Hence, dropping the ω–dependence, we have that $|X(t)| < \Phi^{-1}\left(\lambda_X(1-\epsilon)t\right)$ for all $t \geq T_1(\epsilon)$. Since $f(x) \sim \varphi(x)$ as $x \to 0$, there is $\delta > 0$ such that $|f(x)| \leq 2\varphi(|x|)$ for all $x < \delta$. Since $X(t) \to 0$ as $t \to \infty$, it follows that $|X(t)| < \delta$ for all $t \geq T_2$. Let $T_3(\epsilon) = \max\left(T_1(\epsilon), T_2\right)$. Since φ is increasing, then for $t \geq T_3(\epsilon)$

$$|f(X(t))| \leq 2\varphi\left(|X(t)|\right) \leq 2\left(\varphi \circ \Phi^{-1}\right)\left(\lambda_X(1-\epsilon)t\right).$$

Therefore, for $t \geq T_3(\epsilon)$, we have

$$\begin{aligned}\int_{T_3}^t |f\left(X(s)\right)|\,ds &\leq 2\int_{T_3}^t \varphi\left(\Phi^{-1}(\lambda_X(1-\epsilon)s)\right)\,ds\\ &= \frac{2}{\lambda_X(1-\epsilon)}\left[\Phi^{-1}\left(\lambda_X(1-\epsilon)T_3(\epsilon)\right) - \Phi^{-1}\left(\lambda_X(1-\epsilon)t\right)\right]\\ &\leq \frac{2}{\lambda_X(1-\epsilon)}\Phi^{-1}\left(\lambda_X(1-\epsilon)T_3(\epsilon)\right) < +\infty.\end{aligned}$$

Therefore for $\omega \in \Omega^*$, we have that

$$\int_0^\infty |f(X(s))|\,ds \leq \bar{X}(\omega) < +\infty. \tag{3.6.14}$$

We now show that $\sigma \in L^2(0,\infty)$, so that Γ given by (3.6.4) is well–defined. Start by writing

$$\int_0^t \sigma(s)\,dB(s) = X(t) - \zeta + \int_0^t f\left(X(s)\right)\,ds, \quad t \geq 0. \tag{3.6.15}$$

By (3.6.14), the last term in (3.6.15) tends to a finite limit on Ω^*. Since $\lambda_X(\omega) > 0$ for all $\omega \in \Omega^*$, $X(t) \to 0$ as $t \to \infty$ on Ω^*, so the right hand side in (3.6.15) tends to a finite limit on Ω^*. Therefore

$$\lim_{t\to\infty}\int_0^t \sigma(s)\,dB(s) \quad \text{is finite on } \Omega^*.$$

This implies that $\sigma \in L^2(0,\infty)$, for if σ were not in $L^2(0,\infty)$ it would follow that

$$\limsup_{t\to\infty}\left|\int_0^t \sigma(s)\,dB(s)\right| = +\infty, \quad \text{a.s.}$$

which is incompatible with the fact that Ω^* is an event of positive probability.

$$\int_t^\infty \sigma(s)\,dB(s) := \left(\lim_{t\to\infty}\int_0^t \sigma(s)\,dB(s)\right) - \int_0^t \sigma(s)\,dB(s)$$

is well-defined on a.s. (and hence on Ω^*), and so

$$\Gamma(t) = -\int_t^\infty \sigma(s)\,dB(s), \quad t \geq 0$$

is well-defined on Ω^*, a.s. Moreover, since $\sigma \in L^2(0,\infty)$, we may now conclude $X(t) \to 0$ as $t \to \infty$ a.s. so that λ_X is well–defined a.s.

Next we may take (on Ω^*) the limit as $t \to \infty$ on both sides of (3.6.15) to get

$$\int_0^\infty \sigma(s)\,dB(s) = -\zeta + \int_0^\infty f\left(X(s)\right)\,ds. \tag{3.6.16}$$

Therefore on Ω^*, we get from (3.6.15) and (3.6.16) that

$$-\Gamma(t) = -X(t) + \int_t^\infty f\left(X(s)\right)\,ds.$$

Thus for $t \geq T_3(\epsilon)$, we have

$$\begin{aligned}
|\Gamma(t,\omega)| &\leq |X(t,\omega)| + \int_t^\infty |f\left(X(s,\omega)\right)|\,ds \\
&\leq \Phi^{-1}\left(\lambda_X(1-\epsilon)t\right) + \int_t^\infty 2\varphi\left(\Phi^{-1}(\lambda_X(1-\epsilon)s)\right)\,ds \\
&= \left[1 + \frac{2}{\lambda_X(\omega)(1-\epsilon)}\right]\Phi^{-1}\left(\lambda_X(\omega)(1-\epsilon)t\right).
\end{aligned}$$

Therefore, with $m_\epsilon(\omega) := 1 + \frac{2}{\lambda_X(\omega)(1-\epsilon)}$, we have

$$\Phi\left(\frac{1}{m_\epsilon(\omega)}|\Gamma(t,\omega)|\right) \geq \lambda_X(\omega)(1-\epsilon)t, \quad t \geq T_3(\epsilon,\omega).$$

Since Φ is slowly varying, we get $\lim_{x\to 0^+}\Phi(\frac{1}{m}x)/\Phi(x) = 1$. Thus, for every $\eta \in (0,1)$, there is $\tilde{x}_1(\eta,m) > 0$ such that $0 < x < \tilde{x}_1(\eta,m)$ implies $1+\eta > \Phi(\frac{1}{m}x)/\Phi(x) > 1$, since Φ is decreasing. Let $x_1(\epsilon,\omega) = \tilde{x}_1(\epsilon, m_\epsilon(\omega))$. Then for $0 < x < x_1(\epsilon,\omega)$, we have $1+\epsilon > \Phi(\frac{1}{m_\epsilon(\omega)}x)/\Phi(x) > 1$. Since $\Gamma(t,\omega) \to 0$ as $t \to \infty$, there is $T_4(\epsilon,\omega) > 0$ such that $t \geq T_5(\epsilon,\omega) := \max\left(T_3(\epsilon,\omega), T_4(\epsilon,\omega)\right)$ implies

$$|\Gamma(t,\omega)| < x_1(\epsilon,\omega), \quad \Phi\left(\frac{1}{m_\epsilon(\omega)}|\Gamma(t,\omega)|\right) \geq \lambda_X(\omega)(1-\epsilon)t, \quad t \geq T_5(\epsilon,\omega).$$

If $t \geq T_5(\epsilon,\omega)$ and $|\Gamma(t,\omega)| > 0$, then

$$(1+\epsilon)\Phi\left(|\Gamma(t,\omega)|\right) > \Phi\left(\frac{1}{m_\epsilon(\omega)}|\Gamma(t,\omega)|\right) \geq \lambda_X(\omega)(1-\epsilon)t.$$

Thus

$$\frac{\Phi\left(|\Gamma(t,\omega)|\right)}{t} \geq \lambda_X(\omega)\frac{1-\epsilon}{1+\epsilon}, \quad t \geq T_5(\epsilon,\omega); \quad \Gamma(t,\omega) \neq 0. \tag{3.6.17}$$

If $t \geq T_5(\epsilon,\omega)$ and $|\Gamma(t,\omega)| = 0$, inequality (3.6.17) holds trivially, because $\Phi\left(|\Gamma(t,\omega)|\right) = +\infty$. Hence for each $\omega \in \Omega^*$ and every $\epsilon \in (0,1)$, there exists $T_5(\epsilon,\omega) > 0$ such that

$$\frac{\Phi\left(|\Gamma(t,\omega)|\right)}{t} \geq \lambda_X(\omega)\frac{1-\epsilon}{1+\epsilon}, \quad t \geq T_5(\epsilon,\omega).$$

Since $F(x) \sim \Phi(x)$ as $x \to 0^+$, we can argue that there is $T_6(\epsilon,\omega) \geq T_5(\epsilon,\omega)$ such that

$$\frac{F\left(|\Gamma(t,\omega)|\right)}{t} \geq \lambda_X(\omega)\frac{1-\epsilon}{(1+\epsilon)^2}, \quad t \geq T_6(\epsilon,\omega).$$

Letting $t \to \infty$ on both sides, we see that $\lambda_\Gamma(\omega) \geq \lambda_X(\omega)$ for each $\omega \in \Omega^*$, as required. □

It is now straightforward to prove the following result.

Theorem 3.6.4. *Let f obeys (3.2.1), (3.2.2), (3.2.4) and (3.2.3). Let σ be continuous and $\sigma \in L^2(0,\infty)$ and suppose that $\Sigma(t) > 0$ for all $t \geq 0$ where Σ is given by (3.6.8). Also, let L_Σ defined by (3.6.9) obey $L_\Sigma \in (0,1)$. Then a continuous solution of (3.1.1) obeys $X(t) \to 0$ as $t \to \infty$ a.s. and λ_X defined by (3.6.6) obeys*

$$\lambda_X = L_\Sigma \quad a.s..$$

Proof. We have already shown (in (3.6.13)) that there is an a.s. event Ω_5 such that

$$\lambda_X(\omega) \geq L_\Sigma, \quad \omega \in \Omega_5.$$

Since $L_\Sigma > 0$, it follows that $\lambda_X(\omega) > 0$ for all $\omega \in \Omega_5$. Set $\Omega^* := \Omega_5$ in Theorem 3.6.3. Then, by Theorem 3.6.3 we have that λ_Γ given by (3.6.5) obeys

$$\lambda_\Gamma \geq \lambda_X \quad \text{a.s. on } \Omega_5.$$

Finally, from (3.6.12) we have $\lambda_\Gamma(\omega) = L_\Sigma$ for $\omega \in \Omega_5$, so combining this equality with the displayed inequalities, we have that $\lambda_X = L_\Sigma$ a.s., as claimed. □

It remains to deal with the cases when $L_\Sigma = 0$ and $\sigma \in L^2(0,\infty)$, and when $\sigma \notin L^2(0,\infty)$.

Theorem 3.6.5. *Let f obeys (3.2.1), (3.2.2), (3.2.4) and (3.2.3). Let σ be continuous and $\sigma \in L^2(0,\infty)$ and suppose that $\Sigma(t) > 0$ for all $t \geq 0$ where Σ is given by (3.6.8). Also, let L_Σ defined by (3.6.9) obey $L_\Sigma = 0$. Then a continuous solution X of (3.1.1) obeys $X(t) \to 0$ as $t \to \infty$ a.s. and λ_X defined by (3.6.6) obeys*

$$\lambda_X = 0 \quad a.s..$$

Proof. Suppose, contrary to the conclusion, that there is an event A of positive probability on which $\lambda_X(\omega) > 0$ for all $\omega \in A$. Now, by Theorem 3.6.3 (with $\Omega^* := A$), we have that $\sigma \in L^2(0,\infty)$ and $\lambda_\Gamma(\omega) \geq \lambda_X(\omega)$ for all $\omega \in A$. Hence $\lambda_\Gamma(\omega) > 0$ for all $\omega \in A$. Since $\sigma \in L^2(0,\infty)$, we have that $\limsup_{t\to\infty} |\Gamma(t)|/\Sigma(t) = 1$, a.s. and so by the slow variation of F we have

$$\liminf_{t\to\infty} \frac{F(|\Gamma(t)|)}{F(\Sigma(t))} = 1, \quad \text{a.s.}$$

Therefore

$$\lambda_\Gamma = \liminf_{t\to\infty} \frac{F(|\Gamma(t)|)}{t} = \liminf_{t\to\infty} \frac{F(|\Gamma(t)|)}{F(\Sigma(t))} \cdot \frac{F(\Sigma(t))}{t} = 0, \quad \text{a.s.}$$

Hence $\lambda_\Gamma(\omega) = 0$ for all $\omega \in \Omega^*$, where Ω^* is an a.s. event. But $\lambda_\Gamma(\omega) > 0$ for all $\omega \in A$, where $\mathbb{P}[A] > 0$. This yields a contradiction. Hence $\lambda_X = 0$, a.s. □

Lastly, if $\sigma \notin L^2(0,\infty)$, it is impossible to define Γ and also L_Σ. In this case, we must have $\lambda_X = 0$ on any event on which $X(t) \to 0$ as $t \to \infty$. More precisely we prove the following result

Theorem 3.6.6. *Let f obeys (3.2.1), (3.2.2), (3.2.4) and (3.2.3). Let σ be continuous and $\sigma \notin L^2(0,\infty)$. Let X be a continuous solution of (3.1.1), and define $A := \{\omega : X(t,\omega) \to 0$ as $t \to \infty\}$. Then exactly one of the following statements holds*

(A) $\mathbb{P}[A] = 0$;

(B) $\mathbb{P}[A] > 0$ *and* λ_X *defined by* (3.6.6) *obeys* $\lambda_X = 0$ *a.s. on* A.

Proof. Either $\mathbb{P}[A] = 0$ or $\mathbb{P}[A] > 0$. Let the latter hold. Then on A we may define λ_X. Suppose, contrary to statement (B) that there is an event $A' \subseteq A$ with $\mathbb{P}[A'] > 0$ such that $\lambda_X(\omega) > 0$ for all $\omega \in A'$. But then we can apply Theorem 3.6.3, with A' in the role of Ω^*. This forces $\sigma \in L^2(0,\infty)$, which contradicts one of our hypotheses. Hence $\lambda_X = 0$ a.s. on A in case (B), as claimed. □

3.7 Examples

In this section we present examples to illustrate the sharpnesss of the key results of the chapter relating to solutions of (3.1.1). We defer the sometimes untidy calculations to the end of the chapter. We firstly investigate the possibility of limits for the quantities denoted by λ_Γ and λ_x in (3.4.4) and (3.5.1). We also introduce the notation

$$\Lambda_\Gamma := \limsup_{t\to\infty} \frac{F(|\Gamma(t)|)}{t},$$
$$\Lambda_x := \limsup_{t\to\infty} \frac{F(|x(t)|)}{t}.$$

Example 3.7.1. *For every continuous function f which obeys (3.2.5), there exists a continuous function g such that*

$$\lambda_\Gamma = +\infty \text{ implies } \lambda_x = +\infty,$$

where x is a continuous solution of (3.1.1).

Example 3.7.2. *For every continuous function f which obeys (3.2.5) and every $K \in (0,\infty) \setminus \{1\}$, there exists a continuous function g such that*

$$\lambda_\Gamma = \Lambda_\Gamma = K \text{ implies } \lambda_x = \Lambda_x = K,$$

where x is a continuous solution of (3.1.1).

These first two examples demonstrate that when we have a limit for Γ, it is possible to get a limit for x. This naturally leads us to ask if our results which ask for liminfs and guarantee liminfs can be improved to give limits. The following examples show that if we ask only for liminfs for Γ, then we can only expect liminfs for x.

Example 3.7.3. *Suppose that we have f continuous and in $RV_0(1)$. Then for each $K > 1$ there exists a continuous function g such that*

$$\lambda_\Gamma = K, \quad \Lambda_\Gamma = +\infty, \quad \text{and} \quad \lambda_x = \lambda_\Gamma, \quad \Lambda_x = \Lambda_\Gamma,$$

where x is a continuous solution of (3.1.1).

Example 3.7.4. *Suppose that $f \in C^1(0,\infty) \cap C^1(-\infty,0)$ is odd, increasing and in $RV_0(1)$, and obeys*

$$\lim_{x\to 0} \frac{f(x)}{x} = +\infty.$$

Then there exists $g \in C([0,\infty);\mathbb{R})$ such that

$$\lambda_\Gamma = 1, \quad \Lambda_\Gamma = +\infty, \quad \text{and} \quad \lambda_x = 1, \quad \Lambda_x = +\infty,$$

where x is a continuous solution of (3.1.1).

Our final example of the absence of limits shows that we can have distinct *finite* liminfs and limsups for both Γ and x.

Example 3.7.5. *Suppose that f is in $RV_0(1)$, f is increasing with $f \in C^1(0,\infty)$. Then for every $K \in (1,\infty)$ and (finite) $\bar{K} > K$ there exists $g \in C([0,\infty);\mathbb{R})$ such that*

$$\lambda_\Gamma = K, \quad \Lambda_\Gamma = \bar{K} \quad \text{and} \quad \lambda_x = K, \quad \Lambda_x = \bar{K},$$

where x is a continuous solution of (3.1.1).

It is notable that in our examples thus far (and in all examples to follow) we do not find any instances in which $\lambda_x \in (1, \lambda_\Gamma)$. We in fact make the conjecture that this is not possible. We now present an example in the case of a linear f which supports our conjecture that rather than having $\lambda_x \in [1, \lambda_\Gamma]$, as given by Theorem 3.6.3, it is fact the case that $\lambda_x \in \{1, \lambda_\Gamma\}$.

Example 3.7.6. *Let $x(0)$ be known and consider the equation*

$$x'(t) = -x(t) + g(t), \ \ t > 0,$$

where g is continuous.

(i) If $\lambda_\Gamma \in (1, \infty]$, then $\lambda_x = \Lambda_x = 1$ or $\lambda_x = \lambda_\Gamma$.

(ii) If $\lambda_\Gamma = 1$, then

$$\lambda_x = 1. \tag{3.7.1}$$

In light of a limit emerging in this example when $K > 1$ it is natural to ask if (3.7.1) can be improved to give a limit and the next example shows that such an improvement is not possible.

Example 3.7.7. *Once more, we consider the equation $x'(t) = -x(t) + g(t)$, $t \geq 0$. With $g(t) = e^{-t}\cos(t)$ we have $\lambda_\Gamma = 1$ and the solution is $x(t) = e^{-t}\sin(t)$. This gives us*

$$\lambda_x = 1, \quad \Lambda_x = +\infty.$$

3.8 Proofs of Main Results

3.8.1 Proof of Theorem 3.4.2

Lemma 3.8.1. *Suppose that f obeys (3.2.1), (3.2.2), (3.2.4) and (3.2.3). Suppose F defined by (3.1.3) obeys $F(x) \to \infty$ as $x \to \infty$. Let Γ be a continuous function such that $\Gamma(t) \to 0$ as $t \to \infty$. If λ_Γ is defined by (3.4.4) and $\lambda_\Gamma \geq 1$, then the continuous solution z of (3.4.1) obeys*

$$\Lambda_z := \limsup_{t\to\infty} \frac{F(|z(t)|)}{t} \geq 1.$$

Proof. Suppose to the contrary that $\Lambda_z < 1$. We consider separately that cases when $\lambda_\Gamma = +\infty$ and $\lambda_\Gamma < +\infty$.

If $\lambda_\Gamma < +\infty$, for every $\epsilon \in (0, \lambda_\Gamma \wedge 1)$, there is $T_1(\epsilon) > 0$ such that $F(|\Gamma(t)|) > (\lambda_\Gamma - \epsilon)t$ for all $t \geq T_1(\epsilon)$. Let $\epsilon > 0$ be so small that $\Lambda_z + \epsilon < \lambda_\Gamma - \epsilon$. Then as F^{-1} is rapidly varying, $F^{-1}((\lambda_\Gamma - \epsilon)t)/F^{-1}((\Lambda_z + \epsilon)t) \to 0$ as $t \to \infty$. Moreover, there is $T_2(\epsilon) > 0$ such that $F(|z(t)|) < (\Lambda_z + \epsilon)t$ for all $t \geq T_2(\epsilon)$. Let $T_3(\epsilon) = \max(T_1(\epsilon), T_2(\epsilon))$; then for $t \geq T_3(\epsilon)$, we have

$$|z(t)| > F^{-1}((\Lambda_z + \epsilon)t), \quad |\Gamma(t)| < F^{-1}((\lambda_\Gamma - \epsilon)t).$$

Therefore
$$\limsup_{t\to\infty}\left|\frac{\Gamma(t)}{z(t)}\right| \le \limsup_{t\to\infty}\frac{F^{-1}((\lambda_\Gamma-\epsilon)t)}{F^{-1}((\Lambda_z+\epsilon)t)}=0$$
which implies that $\Gamma(t)/z(t)\to 0$ as $t\to\infty$. By construction of $T_2(\epsilon)$, we have either $z(t)>0$ for all $t\ge T_2(\epsilon)$ or $z(t)<0$ for all $t\ge T_2(\epsilon)$. Consider first the case when $z(t)>0$ for all $t\ge T_2(\epsilon)$. Define

$$\mu(t)=1+\frac{\Gamma(t)}{z(t)},\quad t\ge T_2$$

which implies that $\mu(t)\to 1$ as $t\to\infty$. Write

$$\frac{z'(t)}{f(z(t))}=-\frac{f(z(t)\mu(t))}{\varphi(z(t)\mu(t))}\cdot\frac{\varphi(z(t)\mu(t))}{\varphi(z(t))}\cdot\frac{\varphi(z(t))}{f(z(t)},$$

so that as $f(x)\sim\varphi(x)$ as $x\to 0^+$, $z(t)\to 0$ as $t\to\infty$, the first and last factors have a unit limit as $t\to\infty$. Since $\mu(t)\to 1$ as $t\to\infty$ and φ is asymptotic preserving, by Lemma 3.2.1 we have that the second term has a unit limit. Therefore $\lim_{t\to\infty} z'(t)/f(z(t))=-1$, and integrating gives

$$\lim_{t\to\infty}\frac{F(|z(t)|)}{t}=\lim_{t\to\infty}\frac{F(z(t))}{t}=1.$$

Since the limit exists, we have $\Lambda_z=1$ which contradicts the supposition when $z(t)>0$ for $t\ge T_2(\epsilon)$.

Next, consider the other case when $z(t)<0$ for all $t\ge T_2(\epsilon)$. Set $z_-(t):=-z(t)$ for $t\ge T_2(\epsilon)$. Then

$$z_-'(t)=-\,z'(t)=f(z(t)+\Gamma(t))=f(-z_-(t)+\Gamma(t))=f(-z_-(t)\mu(t)),$$

where $\mu(t)=1+\Gamma(t)/z(t)=1-\Gamma(t)/z_-(t)$ and thus $\mu(t)\to 1$ as $t\to\infty$. Hence for $t\ge T_2$ we have

$$\frac{z_-'(t)}{\varphi(z_-(t))}=-\frac{f(-z_-(t)\mu(t))}{\varphi(-z_-(t)\mu(t))}\cdot\frac{-\varphi(-z_-(t)\mu(t))}{\varphi(z_-(t)\mu(t))}\cdot\frac{\varphi(z_-(t)\mu(t))}{\varphi(z_-(t))}.$$

The first factor has unit limit since $f(x)\sim\varphi(x)$ as $x\to 0$. The second factor is identically one, since φ is odd. The third factor tends to unity by by Lemma 3.2.1. Therefore
$$\lim_{t\to\infty}\frac{z_-'(t)}{\varphi(z_-(t))}=-1.$$
Since f is asymptotic to φ, we have

$$\lim_{t\to\infty}\frac{z_-'(t)}{f(z_-(t))}=-1.$$

Integrating and recalling that $|z(t)| = z_-(t)$ for $t \geq T_2$, we have

$$\lim_{t\to\infty} \frac{F\left(|z(t)|\right)}{t} = 1,$$

which yields $\Lambda_z = 1$, a contradiction. Therefore, if $\lambda_\Gamma \geq 1$, we must have $\Lambda_z \geq 1$.

Now consider the case when $\lambda_\Gamma = +\infty$. Then there is $T_1(\epsilon) > 0$ such that $F(|\Gamma(t)|) > 2t$ for all $t \geq T_1(\epsilon)$. Let $\epsilon \in (0,1)$ so small that $\Lambda_z + \epsilon < 2$. Also, there is $T_2(\epsilon) > 0$ such that $F(|z(t)|) < (\Lambda_z + \epsilon)t$ for all $t \geq T_2(\epsilon)$. Let $T_3(\epsilon) = \max(T_1(\epsilon), T_2(\epsilon))$; then we can show again that $\Gamma(t)/z(t) \to 0$ as $t \to \infty$, and that there are only two possibilities for the sign of z, namely that $z(t) > 0$ for all $t \geq T_2(\epsilon)$ or that $z(t) < 0$ for all $t \geq T_2(\epsilon)$. By the same arguments as before when $\lambda_\Gamma < +\infty$, we have once again that $\Lambda_z < 1$ is impossible. Hence, if $\lambda_\Gamma \geq 1$ we must have $\Lambda_z \geq 1$. $\square$

Lemma 3.8.2. *Suppose that f obeys (3.2.1), (3.2.2), (3.2.4) and (3.2.3). Suppose F defined by (3.1.3) obeys $F(x) \to \infty$ as $x \to \infty$. Let Γ be a continuous function such that $\Gamma(t) \to 0$ as $t \to \infty$. If λ_Γ is defined by (3.4.4) and $\lambda_\Gamma \geq 1$, then the continuous solution z of (3.4.1) obeys*

$$\lambda_z := \liminf_{t\to\infty} \frac{F(|z(t)|)}{t} \geq 1.$$

Proof. We consider separately the cases when (I) $\lambda_\Gamma < +\infty$ and (II) $\lambda_\Gamma = +\infty$.

Since φ is asymptotic preserving, there is an $\epsilon_0 \in (0,1)$ such that

$$\Phi_\epsilon := \liminf_{x\to 0^+} \frac{\varphi\left((1-\epsilon)x\right)}{\varphi(x)}$$

is well-defined with $\Phi_\epsilon \in (0,1]$ for all $\epsilon < \epsilon_0$. Φ_ϵ is non-increasing with $\Phi_\epsilon \to 1$ as $\epsilon \to 0^+$, so for every $\epsilon \in (0, 1/2 \wedge \epsilon_0)$ there is $x_2(\epsilon) > 0$ such that

$$\frac{\varphi\left((1-\epsilon)x\right)}{\varphi(x)} > \Phi_\epsilon(1-\epsilon), \quad |x| \leq x_2(\epsilon).$$

Now, consider the case (I) $1 \leq \lambda_\Gamma < +\infty$. Since $\Phi_\epsilon \leq 1$ for all $\epsilon \in (0, \epsilon_0)$, there is $\epsilon_1 \leq \epsilon_0 \wedge 1/2 \wedge 1/3$ so small that

$$(1-\epsilon)^3\Phi_\epsilon + \epsilon < \lambda_\Gamma - \epsilon, \quad \epsilon \in (0, \epsilon_1). \tag{3.8.1}$$

This is clearly true for $\lambda_\Gamma > 1$, but for $\lambda_\Gamma = 1$, (3.8.1) is still valid because

$$(1-\epsilon)^3\Phi_\epsilon + 2\epsilon \leq (1-\epsilon)^3 + 2\epsilon = 1 - \epsilon(1-3\epsilon) - \epsilon^3 \leq 1 - \epsilon(1-3\epsilon) < \lambda_\Gamma.$$

Define $K(\epsilon) = \lambda_\Gamma - \epsilon$, and

$$\lambda_\epsilon := \min\left((1-\epsilon)^3\Phi_\epsilon, K(\epsilon) - \epsilon, 1 - 2\epsilon\right), \quad \epsilon \leq \epsilon_1. \tag{3.8.2}$$

Note that $\lambda_\epsilon \to 1$ as $\epsilon \to 0^+$, and if $\lambda_\Gamma = 1$ then $\lambda_\epsilon = \min\left((1-\epsilon)^3\Phi_\epsilon, 1-2\epsilon\right)$. If $(1-\epsilon)^3\Phi_\epsilon \le 1-2\epsilon$, then $\lambda_\epsilon = (1-\epsilon)^3\Phi_\epsilon \le 1-2\epsilon < 1-\epsilon$. On the other hand, if $(1-\epsilon)^3\Phi_\epsilon > 1-2\epsilon$ then $\lambda_\epsilon = 1-2\epsilon < 1-\epsilon$. Therefore, $\lambda_\Gamma = 1$ implies $\lambda_\epsilon < 1-\epsilon$. If $\lambda_\Gamma > 1$, the same argument prevails. Since $\lambda_\Gamma < +\infty$, there is $T_1(\epsilon) > 0$ such that

$$|\Gamma(t)| < F^{-1}\left(K(\epsilon)t\right), \quad t \ge T_1(\epsilon). \tag{3.8.3}$$

Also, $\lambda_\epsilon \le K(\epsilon) - \epsilon$ and thus $K(\epsilon) \ge \lambda_\epsilon + \epsilon > \lambda_\epsilon$. Therefore, by the properties of f, φ and F^{-1}, we have $1-\epsilon < f(x)/\varphi(x) < \frac{1}{1-\epsilon}$ for all $|x| \le x_1(\epsilon)$,

$$2F^{-1}(\lambda_\epsilon t) < x_1 \wedge x_2, \quad t \ge T_2(\epsilon), \tag{3.8.4}$$

and $F^{-1}(K(\epsilon)t)/F^{-1}(\lambda_\epsilon t) < \epsilon$ for all $t \ge T_3(\epsilon)$. Therefore, if $t \ge T_1 \vee T_2 \vee T_3$, we have $|\Gamma(t)| < F^{-1}(K(\epsilon)t) < \epsilon F^{-1}(\lambda_\epsilon t)$, and thus $F^{-1}(\lambda_\epsilon t) + \Gamma(t) > (1-\epsilon)F^{-1}(\lambda_\epsilon t) > 0$. Also, $F^{-1}(\lambda_\epsilon t) + \Gamma(t) \le 2F^{-1}(\lambda_\epsilon t) < x_1(\epsilon)$ for all $t \ge T_1 \vee T_2 \vee T_3$. We now prepare an upper estimate when the solution of (3.4.1) is positive. By the above constructions, we get

$$\begin{aligned}
f\left(F^{-1}(\lambda_\epsilon t) + \Gamma(t)\right) &> (1-\epsilon)\varphi\left(F^{-1}(\lambda_\epsilon t) + \Gamma(t)\right) \\
&> (1-\epsilon)\varphi\left((1-\epsilon)F^{-1}(\lambda_\epsilon t)\right) \\
&> (1-\epsilon)\Phi_\epsilon(1-\epsilon)\varphi\left(F^{-1}(\lambda_\epsilon t)\right) \\
&> (1-\epsilon)^2\Phi_\epsilon(1-\epsilon) f\left(F^{-1}(\lambda_\epsilon t)\right) \\
&= (1-\epsilon)^3\Phi_\epsilon f\left(F^{-1}(\lambda_\epsilon)\right) \\
&\ge \lambda_\epsilon f\left(F^{-1}(\lambda_\epsilon t\right),
\end{aligned}$$

where $\lambda_\epsilon \le (1-\epsilon)^3\Phi_\epsilon$. Thus

$$f\left(F^{-1}(\lambda_\epsilon t) + \Gamma(t)\right) > \lambda_\epsilon f\left(F^{-1}(\lambda_\epsilon t\right), \quad t \ge T_1 \vee T_2 \vee T_3. \tag{3.8.5}$$

The next part of the construction of the upper solution requires the choice of an advantageous starting point. To do this, we start by noting from Lemma 3.8.1 that

$$\Lambda_z := \limsup_{t\to\infty} \frac{F(|z(t)|)}{t} \ge 1.$$

We now consider the subcases:

$$\text{(A)} \quad \Lambda_z < +\infty \quad \text{(B)} \quad \Lambda_z = +\infty$$

First, in the case (A), for every $\epsilon \in (0,1)$ there is a sequence $t_j(\epsilon) \nearrow \infty$ such that

$$\frac{F(|z(t_j)|)}{t_j} > \Lambda_z - \epsilon,$$

or

$$|z(t_j)| < F^{-1}\left((\Lambda_z - \epsilon)t_j\right).$$

If $\Lambda_z = 1$, then $\Lambda_z - \epsilon = 1 - \epsilon > \lambda_\epsilon$. Also, if $\Lambda_z > 1$, then $\Lambda_z - \epsilon > 1 - \epsilon > \lambda_\epsilon$. Hence $\Lambda_z - \epsilon > \lambda_\epsilon$, and thus $(\Lambda_z - \epsilon)t_j > \lambda_\epsilon t_j$. Therefore

$$|z(t_j)| < F^{-1}\left((\Lambda_z - \epsilon)t_j\right) < F^{-1}(\lambda_\epsilon t_j).$$

In the case (B), there exist $t_j \nearrow \infty$ such that $F(|z(t_j)|)/t_j > 2$. But $\lambda_\epsilon < 1 < 2$ and thus we have once again

$$|z(t_j)| < F^{-1}(\lambda_\epsilon t_j).$$

Therefore, irrespective of the level of Λ_z, for ever $\epsilon > 0$ there is a sequence $t_j(\epsilon) \nearrow \infty$ such that

$$|z(t_j(\epsilon))| < F^{-1}\left(\lambda_\epsilon t_j(\epsilon)\right). \tag{3.8.6}$$

The next part of the construction involves finding an estimate which will be used in getting a lower bound for the solution of (3.4.1) in the case the solution becomes negative. It is the analogue of (3.8.5) above. We start by noting that $\lambda_\epsilon \le K(\epsilon) - \epsilon < K(\epsilon)$, and thus for $t \ge T_1 \vee T_2 \vee T_3$, we have $-F^{-1}(\lambda_\epsilon t) + \Gamma(t) < -F^{-1}(\lambda_\epsilon t) + F^{-1}(K(\epsilon)t) < 0$. Therefore, for $t \ge T_1 \vee T_2 \vee T_3$

$$0 < F^{-1}(\lambda_\epsilon t) - \Gamma(t) = |\Gamma(t) - F^{-1}(\lambda_\epsilon t)| < 2F^{-1}(\lambda_\epsilon t) < x_1 \wedge x_2.$$

Then, for $t \ge T_1 \vee T_2 \vee T_3$ we have

$$0 > f\left(-F^{-1}(\lambda_\epsilon t) + \Gamma(t)\right), \quad 1 - \epsilon < \frac{f\left(-F^{-1}(\lambda_\epsilon t) + \Gamma(t)\right)}{\varphi\left(-F^{-1}(\lambda_\epsilon t) + \Gamma(t)\right)} < \frac{1}{1-\epsilon}.$$

Since $\varphi\left(-F^{-1}(\lambda_\epsilon t) + \Gamma(t)\right) < 0$, we get

$$(1-\epsilon)\varphi\left(-F^{-1}(\lambda_\epsilon t) + \Gamma(t)\right) > f\left(-F^{-1}(\lambda_\epsilon t) + \Gamma(t)\right).$$

Therefore for $t \geq T_1 \vee T_2 \vee T_3$, since φ is odd

$$
\begin{aligned}
-f\left(-F^{-1}(\lambda_\epsilon t)+\Gamma(t)\right) &> -(1-\epsilon)\varphi\left(-F^{-1}(\lambda_\epsilon t)+\Gamma(t)\right) \\
&= (1-\epsilon)\varphi\left(F^{-1}(\lambda_\epsilon t)-\Gamma(t)\right) \\
&> (1-\epsilon)\varphi\left(F^{-1}(\lambda_\epsilon t)-F^{-1}(K(\epsilon)t)\right) \\
&> (1-\epsilon)\varphi\left(F^{-1}(\lambda_\epsilon t)(1-\epsilon)\right) \\
&> (1-\epsilon)^2\Phi_\epsilon\varphi\left(F^{-1}(\lambda_\epsilon t)\right) \\
&> (1-\epsilon)^3\Phi_\epsilon f\left(F^{-1}(\lambda_\epsilon t)\right) \\
&\geq \lambda_\epsilon f\left(F^{-1}(\lambda_\epsilon t)\right).
\end{aligned}
$$

Therefore

$$
-f\left(-F^{-1}(\lambda_\epsilon t)+\Gamma(t)\right) > \lambda_\epsilon f\left(F^{-1}(\lambda_\epsilon t)\right), \quad t \geq T_1 \vee T_2 \vee T_3. \tag{3.8.7}
$$

We now have all the ingredients for the construction of our bounding solutions. Recalling that $(t_j(\epsilon))$ is the sequence from (3.8.6), define $T^*(\epsilon) = \min\{t_j(\epsilon) : t_j(\epsilon) > T_1 \vee T_2 \vee T_3\}$. Define $z_u(t) := F^{-1}(\lambda_\epsilon t)$ for all $t \geq T^*(\epsilon)$. Then by (3.8.6), and the fact that $T^* = t_j$ for some j, we have

$$
z_u(T^*) = F^{-1}(\lambda_\epsilon T^*) = F^{-1}(\lambda_\epsilon t_j) > |z(t_j)| = |z(T^*)|.
$$

Suppose there is a minimal $t_0 > T^*$ such that $|z_u(t_0)| = |z(t_0)|$. Our aim is to show that this is impossible, proving that $|z(t)| < z_u(t)$ for all $t \geq T^*$. Note that $z_u'(t) = -\lambda_\epsilon f(z_u(t))$ for $t \geq T^*$ by construction.

Now, consider first the case that $z(t_0) > 0$. Then $z(t_0) = z_u(t_0) = F^{-1}(\lambda_\epsilon t_0)$ and $z'(t_0) \geq z_u'(t_0)$ by minimality. Hence

$$
\begin{aligned}
-\lambda_\epsilon f\left(F^{-1}(\lambda_\epsilon t_0)\right) &= -\lambda_\epsilon f(z_u(t_0)) = z_u'(t_0) \leq z'(t_0) \\
&= -f(z(t_0)+\Gamma(t_0)) = -f\left(F^{-1}(\lambda_\epsilon t_0)+\Gamma(t_0)\right).
\end{aligned}
$$

Hence

$$
f\left(F^{-1}(\lambda_\epsilon t_0)+\Gamma(t_0)\right) \leq \lambda_\epsilon f\left(F^{-1}(\lambda_\epsilon t_0)\right). \tag{3.8.8}
$$

But since $t_0 > T_1 \vee T_2 \vee T_3$, by (3.8.5), we have

$$
f\left(F^{-1}(\lambda_\epsilon t_0)+\Gamma(t_0)\right) > \lambda_\epsilon f\left(F^{-1}(\lambda_\epsilon t_0)\right),
$$

a contradiction to (3.8.8). Hence we cannot have $z(t_0) > 0$.

Now, consider the case $z(t_0) < 0$. Then $z(t_0) = -z_u(t_0) = -F^{-1}(\lambda_\epsilon t_0)$ and $z'(t_0) \leq$

$-z_u'(t_0)$ by minimality. Therefore

$$\begin{aligned}-\lambda_\epsilon f\left(F^{-1}(\lambda_\epsilon t_0)\right) &= -\lambda_\epsilon f\left(z_u(t_0)\right) = z_u'(t_0) \leq -z'(t_0)\\ &= f\left(z(t_0)+\Gamma(t_0)\right) = f\left(-F^{-1}(\lambda_\epsilon t_0)+\Gamma(t_0)\right).\end{aligned}$$

Hence

$$-f\left(-F^{-1}(\lambda_\epsilon t_0)+\Gamma(t_0)\right) \leq \lambda_\epsilon f\left(F^{-1}(\lambda_\epsilon t_0)\right). \tag{3.8.9}$$

But since $t_0 > T^* \geq T_1 \vee T_2 \vee T_3$, by (3.8.7) we have

$$-f\left(-F^{-1}(\lambda_\epsilon t_0)+\Gamma(t_0)\right) > \lambda_\epsilon f\left(F^{-1}(\lambda_\epsilon t_0)\right),$$

a contradiction to (3.8.9). Hence $z(t_0) < 0$ is impossible.

Finally, note that we need not consider a case when $z(t_0) = 0$, since $z_u(t) > 0$ for all $t \geq T^*$.

Therefore, there cannot exist a $t_0 > T^*(\epsilon)$ such that $|z(t_0)| = z_u(t_0)$. Hence $|z(t)| < z_u(t) = F^{-1}(\lambda_\epsilon t)$ for all $t \geq T^*(\epsilon)$. Hence $F(|z(t)|)/t > \lambda_\epsilon$ for all $t \geq T^*(\epsilon)$. Therefore

$$\lambda_z = \liminf_{t\to\infty} \frac{F(|z(t)|)}{t} \geq \lambda_\epsilon.$$

Since $\epsilon > 0$ can be taken arbitrarily small, and $\lambda_\epsilon \to 1$ as $\epsilon \to 0^+$, we have

$$\lambda_z = \liminf_{t\to\infty} \frac{F(|z(t)|)}{t} \geq 1,$$

as claimed. This completes the proof in the case (I) that $\lambda_\Gamma < +\infty$.

Now, we consider briefly the case (II) when $\lambda_\Gamma = +\infty$. The proof now follows in exactly the same manner as the case (I) when $\lambda_\Gamma < +\infty$, if we let $K(\epsilon) = 2$ above. This is legitimate, because $\lambda_\Gamma = +\infty$ implies that there exists a $T_1 > 0$ such that

$$\frac{F\left(|\Gamma(t)|\right)}{t} > 2, \quad t \geq T_1$$

which yields $|\Gamma(t)| < F^{-1}\left(K(\epsilon)t\right)$ for all $t \geq T_1$. This is precisely the estimate in (3.8.3). Since no other construction is affected, the rest of the argument in case (I) remains valid. Hence the proof in case (II) is complete. □

Lemma 3.8.3. *Suppose that f obeys (3.2.1), (3.2.2), (3.2.4) and (3.2.3). Suppose F defined by (3.1.3) obeys $F(x) \to \infty$ as $x \to \infty$. Let g be continuous and obey (3.2.10). Let Γ be the function defined by (3.4.2), and suppose that λ_Γ defined by (3.4.4) obeys $\lambda_\Gamma \geq 1$. Then any continuous solution x of (3.1.1) obeys*

$$\lambda_x := \liminf_{t\to\infty} \frac{F(|x(t)|)}{t} \geq 1.$$

Proof. By (3.2.10), Γ is well–defined and $\Gamma(t) \to 0$ as $t \to \infty$. We have that $x(t) \to 0$ as $t \to \infty$. Let $z(t) := x(t) - \Gamma(t)$ for $t \geq 0$. Then $z(t) \to 0$ as $t \to \infty$. Moreover, z is a continuous solution of (3.4.1) by construction. Since $\lambda_\Gamma \geq 1$, it follows from Lemma 3.8.2 that

$$\lambda_z := \liminf_{t\to\infty} \frac{F(|z(t)|)}{t} \geq 1.$$

To prove $\lambda_x \geq 1$, we work through the following cases:

$$\text{(I) } \lambda_\Gamma,\ \lambda_z < +\infty; \quad \text{(II) } \lambda_\Gamma < +\infty,\ \lambda_z = +\infty;$$
$$\text{(III) } \lambda_\Gamma = +\infty,\ \lambda_z < +\infty; \quad \text{(IV) } \lambda_\Gamma = \lambda_z = +\infty.$$

First, in the case (I), for every $\epsilon \in (0,1)$ there exist $T_1(\epsilon) > 0$, $T_2(\epsilon) > 0$ such that $F(|\Gamma(t)|) > (\lambda_\Gamma - \epsilon)t$ for all $t \geq T_1(\epsilon)$ and $F(|z(t)|) > (\lambda_z - \epsilon)t$ for all $t \geq T_2(\epsilon)$. Let $T_3(\epsilon) = \max(T_1(\epsilon), T_2(\epsilon))$. Define $\lambda := \min(\lambda_\Gamma, \lambda_z) \geq 1$, then we have that $F(|\Gamma(t)|) > (\lambda - \epsilon)t$ and $F(|z(t)|) > (\lambda - \epsilon)t$ for all $t \geq T_3(\epsilon)$. Hence

$$|x(t)| \leq |z(t)| + |\Gamma(t)| < 2F^{-1}((\lambda - \epsilon)t), \quad t \geq T_3(\epsilon), \tag{3.8.10}$$

which gives $F(\frac{1}{2}|x(t)|) > (\lambda - \epsilon)t$ for $t \geq T_3(\epsilon)$. Since F is slowly varying, by taking the limit as $t \to \infty$, and then as $\epsilon \to 0^+$, we get

$$\lambda_x := \liminf_{t\to\infty} \frac{F\left(|x(t)|\right)}{t} \geq \lambda \geq 1,$$

as required. In case (II), we may bound z according to $F(|z(t)|) > \lambda_\Gamma t$ for all $t \geq T_2(\epsilon)$. Then, for $T_3(\epsilon) = \max(T_1(\epsilon), T_2(\epsilon))$, we have $F(|z(t)|) > (\lambda_\Gamma - \epsilon)t$, $F(|\Gamma(t)|) > (\lambda_\Gamma - \epsilon)t$ for all $t \geq T_3(\epsilon)$. The argument now concludes as in case (I). In case (III), take $T_1(\epsilon) > 0$ to be the number such that $F(|\Gamma(t)|) > \lambda_z t$ for all $t \geq T_1(\epsilon)$, and $T_2(\epsilon) > 0$ such that $F(|\Gamma(t)|) > (\lambda_z - \epsilon)t$ for all $t \geq T_2(\epsilon)$. Then, for $T_3(\epsilon) = \max(T_1(\epsilon), T_2(\epsilon))$, we have that $F(|\Gamma(t)|) > (\lambda_z - \epsilon)t$, $F(|z(t)|) > (\lambda_z - \epsilon)t$ for all $t \geq T_3(\epsilon)$. Putting $\lambda := \lambda_z$, the estimate in (3.8.10) again holds. In case (IV), take $T_3(\epsilon) > 0$ to be that number such that $F(|z(t)|) > 2t$, $F(|\Gamma(t)|) > 2t$ for all $t \geq T_3(\epsilon)$. Then, we have $F(\frac{1}{2}|x(t)|) > 2t$ for $t \geq T_3(\epsilon)$ which gives

$$\lambda_x := \liminf_{t\to\infty} \frac{F\left(|x(t)|\right)}{t} \geq 1$$

as required. □

3.8.2 Proof of Theorem 3.4.5

Fix $\epsilon > 0$ arbitrarily. By hypothesis, we have that there exists $T_1(\epsilon)$ such that $|\Gamma(t)| < \frac{\epsilon}{2}$ for all $t \geq T_1(\epsilon)$. We now split the problem into cases as follows. Either

$$\text{(I) } |z(T_1)| > \epsilon \quad \text{or} \quad \text{(II) } |z(T_1)| \leq \epsilon.$$

In case (I), we may have either that (Ia) $z(T_1) > 0$ or (Ib) $z(T_1) < 0$. We start by examining (Ia).

In case (Ia), $z(T_1) + \Gamma(T_1) > \epsilon/2 > 0$. Thus $z'(T_1) = -f(z(T_1) + \Gamma(T_1)) < 0$. We now claim that there exists $T_2 > T_1$ such that $z(T_2) = \epsilon$. Suppose to the contrary that this is not the case. Then $z(t) > \epsilon$ for all $t \geq T_1$. Therefore $z(t) + \Gamma(t) > \epsilon/2$ for all $t \geq T_1$. Thus $z'(t) = -f(z(t) + \Gamma(t)) < 0$ for all $t \geq T_1$. Thus as z is decreasing on $[T_1, \infty)$ and $|\Gamma(t)| < \epsilon/2$ for $t \geq T_1$, we have $\frac{\epsilon}{2} + z(T_1) > z(t) + \Gamma(t) > \frac{\epsilon}{2}$ for $t \geq T_1$. Therefore

$$f(z(t) + \Gamma(t)) \geq \min_{x \in [\epsilon/2, \epsilon/2 + z(T_1)]} f(x) := \alpha_\epsilon > 0.$$

Hence $z'(t) < -\alpha_\epsilon$ for all $t \geq T_1$. But this implies that $z(t) \to -\infty$ as $t \to \infty$, which contradicts the supposition that $z(t) > \epsilon$ for all $t \geq T_1$. We have therefore shown

$$z(T_1) > \epsilon \text{ implies that there is } T_2 > T_1 \text{ such that } z(T_2) = \epsilon.$$

We next show that $z(t) < \frac{3\epsilon}{2}$ for all $t \geq T_2$. Suppose to the contrary. Then there exists a minimal $T_3 > T_2$ such that $z(T_3) = \frac{3\epsilon}{2}$. Then $z'(T_3) \geq 0$ by minimality of T_3. In addition, as $z(T_3) + \Gamma(T_3) = \frac{3\epsilon}{2} + \Gamma(T_3) > \epsilon > 0$, we have $f(z(T_3) + \Gamma(T_3)) > 0$. But $0 \leq z'(T_3) = -f(z(T_3) + \Gamma(T_3)) < 0$, a contradiction. Therefore, it must follow that for all $t \geq T_2(\epsilon)$ we have $z(t) < \frac{3\epsilon}{2}$. Hence we have shown that

$$z(T_1) > \epsilon \text{ implies there is } T_2 > T_1 \text{ such that } z(t) < \tfrac{3\epsilon}{2} \text{ for all } t \geq T_2(\epsilon).$$

Next, recall that $z(T_2) = \epsilon$. We show next that $z(t) > -\frac{3\epsilon}{2}$ for all $t \geq T_2$. Suppose to the contrary. Then there exists a minimal $T_4 > T_2$ such that $z(T_4) = -\frac{3\epsilon}{2}$. Then $z'(T_4) \leq 0$ by minimality of T_4. In addition, as $z(T_4) + \Gamma(T_4) = -\frac{3\epsilon}{2} + \Gamma(T_4) < -\epsilon < 0$, we have $f(z(T_4) + \Gamma(T_4)) < 0$. But $0 \geq z'(T_4) = -f(z(T_4) + \Gamma(T_4)) > 0$, a contradiction. Therefore, it must follow that for all $t \geq T_2(\epsilon)$ we have $z(t) > -\frac{3\epsilon}{2}$. Hence we have shown that

$$z(T_1) > \epsilon \text{ implies there is } T_2 > T_1 \text{ such that } z(t) > -\tfrac{3\epsilon}{2} \text{ for all } t \geq T_2(\epsilon).$$

Combining the two displayed statements, we see that

$$z(T_1) > \epsilon \text{ implies there is } T_2 > T_1 \text{ such that } |z(t)| < \tfrac{3\epsilon}{2} \text{ for all } t \geq T_2(\epsilon). \quad (3.8.11)$$

The argument in case (Ib) is symmetric, and leads to the conclusion that

$$z(T_1) < -\epsilon \text{ implies there is } T_2 > T_1 \text{ such that } |z(t)| < \tfrac{3\epsilon}{2} \text{ for all } t \geq T_2(\epsilon). \quad (3.8.12)$$

Therefore, combining case (Ia) and case (Ib), we see that

$$\text{If } |z(T_1)| > \epsilon, \text{ then there exists } T_2(\epsilon) > T_1(\epsilon) \text{ such that } |z(t)| < \tfrac{3\epsilon}{2} \text{ for all } t \geq T_2(\epsilon). \quad (3.8.13)$$

We now consider case (II) where $|z(T_1)| \leq \epsilon$. We wish to show that $|z(t)| < \frac{3\epsilon}{2}$ for all $t \geq T_1(\epsilon)$. Suppose to the contrary. Then there is a minimal $T_3 > T_1$ such that $|z(T_3)| = 3\epsilon/2$. In the case that $z(T_3) = \frac{3\epsilon}{2}$, the minimality implies that $z'(T_3) \geq 0$. Since $T_3 > T_1$, $\Gamma(T_3) > -\frac{\epsilon}{2}$, so $f(z(T_3) + \Gamma(T_3)) > 0$. But $0 \leq z'(T_3) = -f(z(T_3) + \Gamma(T_3)) < 0$, a contradiction. In the case that $z(T_3) = -\frac{3\epsilon}{2}$, the minimality implies that $z'(T_3) \leq 0$. Since $T_3 > T_1$, $\Gamma(T_3) < \frac{\epsilon}{2}$, so $f(z(T_3) + \Gamma(T_3)) < 0$. But $0 \geq z'(T_3) = -f(z(T_3) + \Gamma(T_3)) > 0$, a contradiction. Hence we have shown that

$$\text{If } |z(T_1)| \leq \epsilon, \text{ then } |z(t)| < \tfrac{3\epsilon}{2} \text{ for all } t \geq T_1(\epsilon). \quad (3.8.14)$$

Therefore, combining (3.8.13) and (3.8.14), we see that for every $\epsilon > 0$ there exists $T(\epsilon) > 0$ such that for all $t \geq T(\epsilon)$ we have $|z(t)| < 3\epsilon/2$. Thus $\lim_{t\to\infty} z(t) = 0$, as required.

3.8.3 Proof of Theorem 3.5.1

We start by making uniform asymptotic estimates of the terms involving x in the integrated form of (3.1.1), namely

$$x(t) = x(0) + \int_0^t f(x(s))\,ds + \int_0^t g(s)\,ds. \quad (3.8.15)$$

This entails making a pointwise estimate of $f(x(t))$. If it can be shown that the function $t \mapsto \int_0^t f(x(s))\,ds$ tends to a finite limit as $t \to \infty$, the result is secured, because the hypothesis (3.4.6) implies that $x(t) \to 0$ as $t \to \infty$, and therefore that g obeys (3.2.10). By hypothesis, there is a function φ such that $|f(x)| \leq 2|\varphi(x)|$ for $|x| \leq x_1$ for some $x_1 > 0$, where φ is increasing, odd and φ obeys (3.2.3). Since φ is odd, we have that $|\varphi(x)| = \varphi(|x|)$. Hence $|f(x)| \leq 2\varphi(|x|)$ for $|x| \leq x_1$. Since $x(t) \to 0$ as $t \to \infty$, $|x(t)| < x_1$ for all $t \geq T_1$. Thus $|f(x(t))| \leq 2\varphi(|x(t)|)$ for $t \geq T_1$. Since $F(x) \sim \Phi(x)$ as $x \to 0$ and x obeys (3.4.6), it follows that for every $\epsilon \in (0,1)$ there exists $T_2(\epsilon) > 0$ such that $\Phi(|x(t)|) > (1-\epsilon)t$ for all $t \geq T_2(\epsilon)$. Let $T(\epsilon) = 1 + \max(T_1, T_2(\epsilon))$. Since φ is increasing, for $t \geq T(\epsilon)$ we have $|f(x(t))| \leq 2\varphi(|x(t)|) \leq 2(\varphi \circ \Phi^{-1})((1-\epsilon)t)$. Now we estimate the integral involving $f(x(t))$. For $t \geq T(\epsilon)$, we use the fact that φ is continuous and $\Phi(x) = \int_x^1 du/\varphi(u)$ to get

$$\left|\int_T^t f(x(s))\,ds\right| \leq \int_T^t 2\varphi(\Phi^{-1}((1-\epsilon)s))\,ds = \frac{2}{1-\epsilon}\left(\Phi^{-1}((1-\epsilon)T) - \Phi^{-1}((1-\epsilon)t)\right). \tag{3.8.16}$$

Since T is finite, $\lim_{t\to\infty}\int_0^t f(x(s))\,ds$ is finite, and so g obeys (3.2.10), as required.

3.8.4 Proof of Theorem 3.5.2

By Theorem 3.5.1, we have that $\lim_{t\to\infty}\int_0^t g(s)\,ds$ exists and is finite. By (3.4.6), it follows that $\lim_{t\to\infty} x(t) = 0$. Also, by Theorem 3.5.1 it follows that

$$\lim_{t\to\infty}\int_0^t f(x(s))\,ds$$

is finite, so $\int_t^\infty f(x(s))\,ds$ is well defined for all $t \geq 0$. Hence

$$\int_t^\infty g(s)\,ds = \int_t^\infty f(x(s))\,ds - x(t), \quad t \geq 0. \tag{3.8.17}$$

We now analyse the asymptotic behaviour of the right–hand side of (3.8.17) to prove (3.4.5). Using the argument which was applied to deduce (3.8.16), we see that for $t \geq T(\epsilon)$ defined in the proof of Theorem 3.5.1 that

$$\left|\int_t^\infty f(x(s))\,ds\right| \leq \int_t^\infty 2\varphi(\Phi^{-1}((1-\epsilon)s))\,ds = \frac{2}{1-\epsilon}\Phi^{-1}((1-\epsilon)t). \tag{3.8.18}$$

Furthermore, for $t \geq T(\epsilon)$ we also have $|x(t)| \leq \Phi^{-1}((1-\epsilon)t)$. Thus, taking absolute values on each side of (3.8.17), we obtain for $t \geq T(\epsilon)$

$$|\Gamma(t)| \leq \left|\int_t^\infty f(x(s))\,ds\right| + |x(t)| \leq \left(1 + \frac{2}{1-\epsilon}\right)\Phi^{-1}((1-\epsilon)t).$$

Call the ϵ–dependent prefactor on the right hand side $m(\epsilon) > 3$. Then for $t \geq T(\epsilon)$

$$\Phi\left(\frac{1}{m(\epsilon)}|\Gamma(t)|\right) \geq (1-\epsilon)t.$$

Hence

$$\liminf_{t\to\infty}\frac{\Phi\left(\frac{1}{m(\epsilon)}|\Gamma(t)|\right)}{t} \geq 1-\epsilon.$$

Since $\Phi \in \text{RV}_0(0)$, this implies $\liminf_{t\to\infty}\Phi(|\Gamma(t)|)/t \geq 1-\epsilon$, and letting $\epsilon \to 0^+$ yields $\liminf_{t\to\infty}\Phi(|\Gamma(t)|)/t \geq 1$. Recalling that $F(x)/\Phi(x) \to 1$ as $x \to 0$, we obtain (3.4.6), as required.

3.8.5 Proof of Theorem 3.5.4

Since F is asymptotic to Φ, by hypothesis it follows for every $\epsilon \in (0, \lambda_x/2)$ there exists $T_1(\epsilon)$ such that $\Phi(|x(t)|) > (\lambda_x - \epsilon)t$ for all $t \geq T_1(\epsilon)$. Hence it follows that $|x(t)| < \Phi^{-1}((\lambda_x - \epsilon)t)$ for all $t \geq T_1(\epsilon)$. Then, as f is asymptotic to φ, we have for every $\epsilon \in (0, 1)$ there exists $x_1(\epsilon)$ and $T_2(\epsilon) > 0$ such that $1 - \epsilon < f(x)/\varphi(x) < 1 + \epsilon$ for $|x| \leq x_1(\epsilon)$, and thus $\Phi^{-1}((\lambda_x - \epsilon)t) < x_1(\epsilon)$ for all $t \geq T_2(\epsilon)$. Now, let $T_3(\epsilon) = \max(T_1(\epsilon), T_2(\epsilon))$. For $t \geq T_3(\epsilon)$, using the fact that φ is odd and increasing, we have

$$|f(x(t))| < (1 + \epsilon)\varphi(|x(t)|) < (1 + \epsilon)(\varphi \circ \Phi^{-1})((\lambda_x - \epsilon)t), \quad t \geq T_3(\epsilon).$$

Thus for any $T > T_3$ and $t \in [T_3, T]$, we have

$$\left| \int_t^T f(x(s))\, ds \right| \leq \int_t^T |f(x(s))|\, ds \leq \int_t^T (1 + \epsilon)(\varphi \circ \Phi^{-1})((\lambda_x - \epsilon)s)\, ds.$$

We now estimate the integral as follows:

$$\begin{aligned} \int_t^T |f(x(s))|\, ds &\leq (1 + \epsilon) \int_t^T \varphi(\Phi^{-1}((\lambda_x - \epsilon)s))\, ds \\ &= \frac{1 + \epsilon}{\lambda_x - \epsilon} \left[\Phi^{-1}((\lambda_x - \epsilon)t) - \Phi^{-1}((\lambda_x - \epsilon)T) \right]. \end{aligned}$$

Therefore, for all $T > t > T_3$, we have

$$\int_t^T |f(x(s)| ds \leq \frac{1 + \epsilon}{\lambda_x - \epsilon} \Phi^{-1}((\lambda_x - \epsilon)t)$$

and thus $\int_0^\infty |f(x(s)|\, ds < +\infty$. Letting $T \to \infty$, we get

$$\int_t^\infty |f(x(s))|\, ds \leq \frac{1 + \epsilon}{\lambda_x - \epsilon} \Phi^{-1}((\lambda_x - \epsilon)t), \quad t \geq T_3(\epsilon).$$

Since $x(t) \to 0$ as $t \to \infty$, we have that $\int_0^t g(s)\, ds$ tends to a finite limit, from which we get

$$\Gamma(t) = x(t) - \int_t^\infty f(x(s)\, ds, \quad t \geq 0.$$

Since $|\Gamma(t)| \leq |x(t)| + \int_t^\infty |f(x(s)|\, ds$, for $t \geq T_3(\epsilon)$, we have

$$\begin{aligned} |\Gamma(t)| &\leq \Phi^{-1}((\lambda_x - \epsilon)t) + \frac{1 + \epsilon}{\lambda_x - \epsilon} \Phi^{-1}((\lambda_x - \epsilon)t) \\ &= \left(1 + \frac{1 + \epsilon}{\lambda_x - \epsilon}\right) \Phi^{-1}((\lambda_x - \epsilon)t). \end{aligned}$$

Since $\epsilon < \lambda_x/2$, with $A =: 2 + 2/\lambda_x$, we have $|\Gamma(t)| \leq A\Phi^{-1}((\lambda_x - \epsilon)t)$ for $t \geq T_3(\epsilon)$. Therefore $\Phi(\frac{1}{A}(|\Gamma(t)|) \geq (\lambda_x - \epsilon)t$ for all $t \geq T_3(\epsilon)$ which implies that

$$\liminf_{t\to\infty} \frac{\Phi(\frac{1}{A}|\Gamma(t)|)}{t} \geq \lambda_x.$$

Since $\Phi \in RV_0(0)$ and Φ is asymptotic to F we have that

$$\lambda_\Gamma = \liminf_{t\to\infty} \frac{F(|\Gamma(t)|)}{t} \geq \lambda_x,$$

as required.

3.8.6 Proof of Theorem 3.5.5

Lemma 3.8.4. *Let f obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3). Let Γ be a continuous function with $\Gamma(t) \to 0$ as $t \to \infty$ and let λ_Γ given by (3.4.4) obey $\lambda_\Gamma \in (0, 1)$. Let z be any continuous solution of (3.4.1). Then*

$$\limsup_{t\to\infty} \frac{F(|z(t)|)}{t} \geq \lambda_\Gamma.$$

Proof. Define

$$M = \limsup_{t\to\infty} \frac{F(|z(t)|)}{t}.$$

and suppose that $M < \lambda_\Gamma$. Let $\epsilon > 0$ be so small that $M + \epsilon < \lambda_\Gamma - \epsilon$. For every $\epsilon \in (0, \lambda_\Gamma)$ there is $T_1(\epsilon) > 0$ such that $F(|\Gamma(t)|)/t > (\lambda_\Gamma - \epsilon)t$ for all $t \geq T_1(\epsilon)$. Therefore $|\Gamma(t)| \leq F^{-1}((\lambda - \epsilon)t)$ for all $T_1(\epsilon) > 0$. Also, there exists $T_2(\epsilon) > 0$ such that $F^{-1}(|z(t)|) < (M + \epsilon)t$ for all $t \geq T_2(\epsilon)$, so $|z(t)| > F^{-1}((M + \epsilon)t)$ for all $t \geq T_2(\epsilon)$. Let $T_3(\epsilon) = \max(T_1(\epsilon), T_2(\epsilon))$. Then for $t \geq T_3(\epsilon)$ we have

$$\left|\frac{\Gamma(t)}{z(t)}\right| < \frac{F^{-1}((\lambda - \epsilon)t)}{F^{-1}((M + \epsilon)t)}.$$

The slow variation of F entails the rapid variation of F^{-1}, so as $M + \epsilon < \lambda - \epsilon$, we have $F^{-1}((\lambda - \epsilon)t)/F^{-1}((M + \epsilon)t) \to 0$ as $t \to \infty$. Now, consider two cases $z(t) > 0$ for all $t \geq T_2(\epsilon)$ or $z(t) < 0$ for all $t \geq T_2(\epsilon)$. Suppose $z(t) > 0$ for all $t \geq T_2(\epsilon)$, and let $\mu(t) := 1 + \Gamma(t)/z(t) \to 1$ as $t \to \infty$. Then

$$\frac{z'(t)}{f(z(t))} = -\frac{f(z(t)\mu(t))}{\varphi(z(t)\mu(t))} \cdot \frac{\varphi(z(t)\mu(t))}{\varphi(z(t))} \cdot \frac{\varphi(z(t))}{f(z(t))},$$

the asymptotic preserving property of φ, the fact that $\mu(t) \to 1$ as $t \to \infty$, $f(x) \sim \varphi(x)$ as $x \to 0^+$, and $z(t) \to 0$ as $t \to \infty$, mean that each quotient on the right hand side tends to 1 as $t \to \infty$. Hence

$$\lim_{t\to\infty} \frac{z'(t)}{f(z(t))} = -1,$$

and, as $|z(t)| = z(t)$ for all t sufficiently large, we have $\lim_{t\to\infty} F(|z(t)|)/t = 1$. Hence $M = 1$. But we supposed that $M < \lambda_\Gamma < 1$, a contradiction.

Now consider the case when $z(t) < 0$ for all $t \geq T_2(\epsilon)$. Let $z_-(t) = -z(t)$. Then $z_-'(t) = -z'(t) = f(z(t) + \Gamma(t)) = f(-z_-(t) + \Gamma(t)) = f(-z_-(t)\mu(t))$, where $\mu(t) = 1 + \Gamma(t)/-z_-(t)$ and $\mu(t) \to 1$ as $t \to \infty$, and $\mu(t) > 0$ for all $t \geq T_2'(\epsilon)$. Write

$$\frac{z_-'(t)}{\varphi(z_-(t))} = -\frac{-f(-z_-(t)\mu(t))}{-\varphi(-z_-(t)\mu(t))} \cdot \frac{-\varphi(-z_-(t)\mu(t))}{\varphi(z_-(t)\mu(t))} \cdot \frac{\varphi(z_-(t)\mu(t))}{\varphi(z_-(t))}.$$

Since φ is odd, the second factor is unity. The asymptotic preserving property of φ, along with $\mu(t) \to 1$ as $t \to \infty$ and $z_-(t) \to 0$ as $t \to \infty$ means the last term tends to 1 as $t \to \infty$. The first factor tends to 1 as $t \to \infty$ because the f is asymptotic to φ. Hence $\lim_{t\to\infty} z_-'(t)/\varphi(z_-(t)) = -1$ and integrating yields $\lim_{t\to\infty} \Phi(z_-(t))/t = 1$. Since $|z(t)| = z_-(t)$ for all t sufficiently large we have $\lim_{t\to\infty} \Phi(|z(t)|)/t = 1$, which implies $\lim_{t\to\infty} F(|z(t)|)/t = 1$. Thus $M = 1$. But $1 = M < \lambda_\Gamma < 1$, a contradiction. Therefore, we must have

$$\limsup_{t\to\infty} \frac{F(|z(t)|)}{t} \geq \lambda_\Gamma,$$

as required. □

Lemma 3.8.5. *Suppose that f obeys (3.2.1), (3.2.2), (3.2.4) and (3.2.3). Suppose F defined by (3.1.3) obeys $F(x) \to \infty$ as $x \to \infty$. Let Γ be a continuous function such that $\Gamma(t) \to 0$ as $t \to \infty$. If λ_Γ is defined by (3.4.4) and $\lambda_\Gamma \in (0, 1)$, then the continuous solution z of (3.4.1) obeys*

$$\lambda_z := \liminf_{t\to\infty} \frac{F(|z(t)|)}{t} \geq \lambda_\Gamma.$$

Proof. By hypothesis, there exists $T_1(\epsilon)$ such that $|\Gamma(t)| < F^{-1}((\lambda_\Gamma - \epsilon)t)$ for all $t \geq T_1(\epsilon)$. Define

$$\Phi_\epsilon = \liminf_{x\to 0^+} \frac{\varphi\left((1-\epsilon)x\right)}{\varphi(x)}.$$

Since $\lambda_\Gamma < 1$ and $\Phi_\epsilon \to 1$ as $\epsilon \to 0^+$, and $\Phi_\epsilon > 0$ for all $\epsilon > 0$ sufficiently small, there is $\epsilon_0 > 0$ such that $\lambda_\Gamma < 2\epsilon + (1-\epsilon)^3\Phi_\epsilon$ for all $\epsilon < \epsilon_0$. Take $\epsilon < \lambda_\Gamma/2 \wedge 1/2 \wedge \epsilon_0$. Since φ is increasing we have $\Phi_\epsilon \leq 1$ for all $\epsilon \in (0, 1/2)$. Also, for every $\epsilon \in (0, 1/2)$ there is $x_2(\epsilon) > 0$ such that

$$\frac{\varphi\left((1-\epsilon)x\right)}{\varphi(x)} > \Phi_\epsilon(1-\epsilon), \quad 0 < x \leq x_2(\epsilon).$$

Define $\lambda_\epsilon = \lambda_\Gamma - 2\epsilon > 0$. Then $\lambda_\epsilon < (1-\epsilon)^3\Phi_\epsilon$.

By hypothesis, there exists $T_1(\epsilon)$ such that $|\Gamma(t)| < F^{-1}((\lambda_\Gamma - \epsilon)t)$ for all $t \geq T_1(\epsilon)$.

Thus

$$F^{-1}(\lambda_\epsilon t) + \Gamma(t) > F^{-1}((\lambda_\Gamma - 2\epsilon)t) - F^{-1}((\lambda_\Gamma - \epsilon)t) > 0.$$

Thus $0 < F^{-1}(\lambda_\epsilon t) + \Gamma(t) < F^{-1}(\lambda_\epsilon t) + F^{-1}((\lambda_\Gamma - \epsilon)t) < 2F^{-1}((\lambda_\Gamma - 2\epsilon)t)$. Since f is asymptotic to φ and F^{-1} tends to zero and is rapidly varying, for every $\epsilon \in (0,1)$ there exists $x_1(\epsilon)$, $T_2(\epsilon)$ and $T_3(\epsilon)$ such that

$$\begin{aligned} 1-\epsilon < \frac{f(x)}{\varphi(x)} < \frac{1}{1-\epsilon}, \quad |x| < x_1(\epsilon), \\ 2F^{-1}((\lambda_\Gamma - \epsilon)t) < x_1(\epsilon) \wedge x_2(\epsilon), \quad t \geq T_2(\epsilon), \\ \frac{F^{-1}((\lambda_\Gamma - \epsilon)t)}{F^{-1}((\lambda_\Gamma - 2\epsilon)t)} < \epsilon, \quad t \geq T_3(\epsilon). \end{aligned}$$

Now, let $t \geq T_1(\epsilon) \vee T_2(\epsilon) \vee T_3(\epsilon)$. Thus we have

$$\begin{aligned} f(F^{-1}(\lambda_\epsilon t) + \Gamma(t)) &> (1-\epsilon)\varphi\left(F^{-1}(\lambda_\epsilon t) + \Gamma(t)\right) \\ &> (1-\epsilon)\varphi\left((1-\epsilon)F^{-1}(\lambda_\epsilon t)\right). \end{aligned}$$

Now, as $F^{-1}(\lambda_\epsilon t) < x_2(\epsilon)$ for $t \geq T_1 \vee T_2 \vee T_3$, we have $\varphi((1-\epsilon)F^{-1}(\lambda_\epsilon t)) > \Phi_\epsilon(1-\epsilon)\varphi(F^{-1}(\lambda_\epsilon t))$. Also, since $\varphi(x) > (1-\epsilon)f(x)$ for $x < x_1(\epsilon)$, we have

$$\varphi\left((1-\epsilon)F^{-1}(\lambda_\epsilon t)\right) > \Phi_\epsilon(1-\epsilon)\varphi\left(F^{-1}(\lambda_\epsilon t)\right) > \Phi_\epsilon(1-\epsilon)^2 f\left(F^{-1}(\lambda_\epsilon t)\right). \tag{3.8.19}$$

Therefore for $t \geq T_1 \vee T_2 \vee T_3$, $f(F^{-1}(\lambda_\epsilon t) + \Gamma(t)) > \Phi_\epsilon(1-\epsilon)^3 f(F^{-1}(\lambda_\epsilon t))$. Next, since $(1-\epsilon)^3\Phi_\epsilon > \lambda_\epsilon$, we have

$$f\left(F^{-1}(\lambda_\epsilon t) + \Gamma(t)\right) > \lambda_\epsilon f\left(F^{-1}(\lambda_\epsilon t)\right), \quad t \geq T_1 \vee T_2 \vee T_3. \tag{3.8.20}$$

We get a corresponding estimate for a lower bound. Let $t \geq T_1$. Then $|\Gamma(t)| < F^{-1}((\lambda_\Gamma - \epsilon)t)$, and so $\Gamma(t) - F^{-1}(\lambda_\epsilon t) < F^{-1}((\lambda_\Gamma - \epsilon)t) - F^{-1}((\lambda_\Gamma - 2\epsilon)t) < 0$. Also $|\Gamma(t) - F^{-1}(\lambda_\epsilon t)| = F^{-1}(\lambda_\epsilon t) - \Gamma(t) < 2F^{-1}((\lambda_\Gamma - 2\epsilon)t)$. Let $t \geq T_1 \vee T_2 \vee T_3$. Then $1 - \epsilon < f(-F^{-1}(\lambda_\epsilon t) + \Gamma(t))/\varphi(-F^{-1}(\lambda_\epsilon t) + \Gamma(t)) < 1/(1-\epsilon)$ and $f(-F^{-1}(\lambda_\epsilon t) + \Gamma(t)) < 0$. Since $\varphi(-F^{-1}(\lambda_\epsilon t) + \Gamma(t)) < 0$,

$$\begin{aligned} (1-\epsilon)\varphi\left(-F^{-1}(\lambda_\epsilon t) + \Gamma(t)\right) &> f\left(-F^{-1}(\lambda_\epsilon t) + \Gamma(t)\right) \\ &> \frac{1}{1-\epsilon}\varphi\left(-F^{-1}(\lambda_\epsilon t) + \Gamma(t)\right), \end{aligned}$$

and thus as φ is odd, we get for $t \geq T_1 \vee T_2 \vee T_3$,

$$\begin{aligned} -f\left(-F^{-1}(\lambda_\epsilon t) + \Gamma(t)\right) &> -(1-\epsilon)\varphi\left(-F^{-1}(\lambda_\epsilon t) + \Gamma(t)\right) \\ &= (1-\epsilon)\varphi\left(F^{-1}(\lambda_\epsilon t) - \Gamma(t)\right) \\ &> (1-\epsilon)\varphi\left(F^{-1}(\lambda_\epsilon t) - F^{-1}((\lambda-\epsilon)t)\right) \\ &> (1-\epsilon)\varphi\left(F^{-1}(\lambda_\epsilon t)(1-\epsilon)\right). \end{aligned}$$

Applying (3.8.19) to this estimate, we obtain for $t \geq \vee T_1 \vee T_2 \vee T_3$

$$-f\left(-F^{-1}(\lambda_\epsilon t) + \Gamma(t)\right) > \Phi_\epsilon(1-\epsilon)^3 f\left(F^{-1}(\lambda_\epsilon t)\right) > \lambda_\epsilon f\left(F^{-1}(\lambda_\epsilon t)\right), \tag{3.8.21}$$

since $\lambda_\epsilon < \Phi_\epsilon(1-\epsilon)^3$.

Lastly, we need an estimate on the starting value. By Lemma 3.8.1 $\limsup_{t\to\infty} F(|z(t)|)/t \geq \lambda_\Gamma$, so there is a sequence $t_j(\epsilon) \nearrow \infty$ such that $F(|z(t_j)|)/t_j > \lambda_\Gamma - 2\epsilon$, which implies that $|z(t_j)| < F^{-1}\left((\lambda_\Gamma - 2\epsilon)t_j\right) = F^{-1}(\lambda_\epsilon t_j)$. Let $T^*(\epsilon) = \min\{t_j(\epsilon) : t_j(\epsilon) > T_1 \vee T_2 \vee T_3\}$. Define $z_u(t) = F^{-1}(\lambda_\epsilon t)$ for all $t \geq T^*(\epsilon)$. Then we have that $z_u(T^*) = F^{-1}(\lambda_\epsilon T^*) = F^{-1}(\lambda_\epsilon t_j) > |z(t_j)| = |z(T^*)|$. Note that $z_u'(t) = -\lambda_\epsilon f(z_u(t))$ for $t \geq T^*(\epsilon)$.

Suppose there is a minimal $t_0 > T^*$ such that $z_u(t_0) = |z(t_0)|$. Suppose $z(t_0) > 0$. Then $z_u(t_0) = z(t_0)$ and $z_u'(t_0) \leq z'(t_0)$. Then

$$\begin{aligned} -\lambda_\epsilon f\left(F^{-1}(\lambda_\epsilon t_0)\right) &= -\lambda_\epsilon f\left(z_u(t_0)\right) = z_u'(t_0) \leq z'(t_0) \\ &= -f\left(z(t_0) + \Gamma(t_0)\right) = -f\left(F^{-1}(\lambda_\epsilon t_0) + \Gamma(t_0)\right), \end{aligned}$$

or $f\left(F^{-1}(\lambda_\epsilon t_0) + \Gamma(t_0)\right) \leq \lambda_\epsilon f\left(F^{-1}(\lambda_\epsilon t_0)\right)$. Since $t_0 > T^* > T_1 \vee T_2 \vee T_3$

$$\lambda_\epsilon f\left(F^{-1}(\lambda_\epsilon t_0)\right) \geq f\left(F^{-1}(\lambda_\epsilon t_0) + \Gamma(t_0)\right) > \lambda_\epsilon f\left(F^{-1}(\lambda_\epsilon t_0)\right),$$

the last inequality owing to (3.8.20), and giving a contradiction. Hence we cannot have $z(t_0) > 0$. Next $z(t_0) = 0$ is clearly impossible, because $z_u(t_0) > 0 = |z(t_0)|$. Therefore, it remains to eliminate the possibility that $z(t_0) < 0$. In that case, $z_u(t_0) = -z(t_0)$ and by minimality $z'(t_0) \geq -z_u'(t_0)$. Hence

$$\begin{aligned} -\lambda_\epsilon f\left(F^{-1}(\lambda_\epsilon t_0)\right) &= -\lambda_\epsilon f\left(z_u(t_0)\right) = z_u'(t_0) \leq -z'(t_0) \\ &= f\left(f(z(t_0) + \Gamma(t_0)\right) = f\left(-F^{-1}(\lambda_\epsilon t_0) + \Gamma(t_0)\right), \end{aligned}$$

or $-f\left(-F^{-1}(\lambda_\epsilon t_0) + \Gamma(t_0)\right) \leq \lambda_\epsilon f\left(F^{-1}(\lambda_\epsilon)\right)$. Since $t_0 > T^* > T_1 \vee T_2 \vee T_3$, by (3.8.21) we have $-f\left(-F^{-1}(\lambda_\epsilon t_0) + \Gamma(t_0)\right) > \lambda_\epsilon f\left(F^{-1}(\lambda_\epsilon t_0)\right)$, and the last two inequalities are contradictory. Therefore $z(t_0) < 0$ is impossible and thus $|z(t)| < z_u(t)$ for all $t \geq T^*(\epsilon)$. Hence $|z(t)| < F^{-1}\left((\lambda_\Gamma - 2\epsilon)t\right)$ for all $t \geq T^*(\epsilon)$, which implies that $F(|z(t)|) > (\lambda_\Gamma -$

$2\epsilon)t$ for all $t \geq T^*(\epsilon)$. Hence

$$\liminf_{t\to\infty} \frac{F(|z(t)|)}{t} \geq \lambda_\Gamma,$$

as claimed. □

3.8.7 Proof of Theorem 3.5.5

Let x be any continuous solution of (3.1.1). Since Γ is defined by (3.4.2), we may define the function $z(t) := x(t) - \Gamma(t)$ for $t \geq 0$. Then z is a continuous solution of (3.4.1).

Consider first the case that $\lambda_\Gamma \in (0,1)$. Since λ_Γ defined by (3.4.4) obeys $\lambda_\Gamma \in (0,1)$, by Lemma 3.8.5 we have that $\lambda_z \geq \lambda_\Gamma$. Therefore using this fact, and the fact that λ_Γ is finite and positive, for every $\epsilon \in (0,1)$ there exists $T(\epsilon) > 0$ such that

$$\frac{F(|\Gamma(t)|)}{t} > \lambda_\Gamma(1-\epsilon), \quad \frac{F(|z(t)|)}{t} > \lambda_\Gamma(1-\epsilon), \quad t \geq T(\epsilon).$$

(The situation here is simpler than the related Lemma 3.8.3, where we needed to consider cases because certain limits inferior may be infinite.) Therefore $|x(t)| \leq |z(t)| + |\Gamma(t)| \leq 2F^{-1}(\lambda_\Gamma(1-\epsilon)t)$ for $t \geq T(\epsilon)$. By the usual considerations this leads to

$$\lambda_x = \liminf_{t\to\infty} \frac{F(|x(t)|)}{t} \geq \lambda_\Gamma. \tag{3.8.22}$$

Therefore, as $\lambda_\Gamma > 0$, we have $\lambda_x > 0$. Hence, by Theorem 3.5.4, we have that $\lambda_x \leq \lambda_\Gamma$. Combining this with (3.8.22) gives $\lambda_x = \lambda_\Gamma$, as claimed.

Now consider the case when $\lambda_\Gamma = 1$. By Theorem 3.4.2, we have that $\lambda_x \geq 1$. Thus $\lambda_x > 0$. Therefore, we may use Theorem 3.5.4 to conclude that $\lambda_\Gamma \geq \lambda_x$, and since $\lambda_\Gamma = 1$, we have $\lambda_x \leq 1$. Combining with $\lambda_x \geq 1$ gives $\lambda_x = 1$, as required.

3.8.8 Proof of Theorem 3.5.6

By hypothesis, we have $\lambda_\Gamma = 0$. Since $F(x) \to \infty$ as $x \to 0^+$ and $x(t) \to 0$ as $t \to \infty$ we must have $\lambda_x \geq 0$. Suppose now $\lambda_x > 0$. Then by Theorem 3.5.4 we have that $\lambda_\Gamma \geq \lambda_x > 0$. But $\lambda_\Gamma = 0$, so we have a contradiction. Hence we must have

$$\lambda_x = \liminf_{t\to\infty} \frac{F(|x(t)|)}{t} = 0,$$

as claimed.

3.8.9 Proof of Theorem 3.5.9

Since $x(t) > 0$ for all $t \geq 0$, we do not need to assume that f is asymptotically odd, as in other results in this section. If (A) holds, then as g is positive, we have that

$g \in L^1([0,\infty);(0,\infty))$. Therefore, we have that $x(t) \to 0$ as $t \to \infty$. Therefore, by Theorem 3.4.2 and the fact that x is positive, we have that

$$\liminf_{t\to\infty} \frac{F(x(t))}{t} \geq 1.$$

On the other hand, define $z'(t) = -f(z(t))$ for $t \geq 0$ and $z(0) = x(0)/2 > 0$. Then $x(t) > z(t)$ for all $t \geq 0$. Integration yields that $F(z(t))/t \to 1$ as $t \to \infty$. Since F is decreasing, $F(x(t)) < F(z(t))$ for all $t \geq 0$, so $\limsup_{t\to\infty} F(x(t))/t \leq 1$. Combining this with the limit inferior implies that x obeys (B). Conversely, if (B) holds, we have that $x(t) \to 0$ as $t \to \infty$, and since all other hypotheses of Theorem 3.5.2 are true, we have that (A) holds.

3.9 Justification of Examples

3.9.1 Justification of Example 3.7.1

Let α be a differentiable function such that $\alpha(0) = 0$ and $\alpha'(t) \to \infty$ as $t \to \infty$. Now define $g(t) := (1-\alpha'(t))(f \circ F^{-1})(\alpha(t))$ for $t \geq 0$. Define $x(t) = F^{-1}(\alpha(t))$, $t \geq 0$. Then

$$\begin{aligned} x'(t) &= -\alpha'(t)f(x(t)) = -\alpha'(t)f(F^{-1}(\alpha(t))) \\ &= -f(F^{-1}(\alpha(t))) + (1-\alpha'(t))f(F^{-1}(\alpha(t))) = -f(x(t)) + g(t). \end{aligned}$$

By construction it is clear that $\lim_{t\to\infty} F(|x(t)|)/t = +\infty$. Also,

$$\left|\int_t^\infty g(s)ds\right| = \left|-\int_t^\infty g(s)ds\right| = \int_t^\infty (\alpha'(s)-1)(f\circ F^{-1})(\alpha(s))ds, \; t > T'.$$

We now note that

$$\lim_{t\to\infty} \frac{\int_t^\infty (\alpha'(s)-1)(f\circ F^{-1})(\alpha(s))ds}{\int_t^\infty (\alpha'(s))(f\circ F^{-1})(\alpha(s))ds} = \lim_{t\to\infty} \frac{\alpha'(t)-1}{\alpha'(t)} = 1.$$

Hence

$$\lim_{t\to\infty} \frac{|\int_t^\infty g(s)ds|}{\int_t^\infty (\alpha'(s))(f\circ F^{-1})(\alpha(s))ds} = 1.$$

Using the substitution $u = \alpha(s)$ and the fact that $\alpha(t) \to \infty$ as $t \to \infty$ we obtain

$$\int_t^\infty (\alpha'(s))(f\circ F^{-1})(\alpha(s))ds = F^{-1}(\alpha(t)).$$

Therefore we have

$$\lim_{t\to\infty} \frac{|\int_t^\infty g(s)ds|}{F^{-1}(\alpha(t))} = 1.$$

Since $F \in RV_0(0)$ we obtain

$$\lim_{t\to\infty} \frac{F(|\int_t^\infty g(s)ds|)}{\alpha(t)} = 1,$$

and hence

$$\lim_{t\to\infty} \frac{F(|\int_t^\infty g(s)ds|)}{t} = \lim_{t\to\infty} \frac{F(|\int_t^\infty g(s)ds|)}{\alpha(t)} \frac{\alpha(t)}{t} = +\infty.$$

3.9.2 Justification of Example 3.7.2

Define $g(t) := (1-K)(f \circ F^{-1})(Kt),\ t \geq 0$ and $x(t) = F^{-1}(Kt),\ t \geq 0$. Then

$$x'(t) = -Kf(x(t)) = -f(x(t)) + (1-K)f(F^{-1}(Kt)) = -f(x(t)) + g(t).$$

Once more, it is clear that we have $\lim_{t\to\infty} F(|x(t)|)/t = K$. Also,

$$\left| \int_t^\infty g(s)ds \right| = |K-1| \int_t^\infty (f \circ F^{-1})(Ks)ds = \frac{|K-1|}{K} F^{-1}(Kt).$$

Since $F \in RV_0(0)$ we have

$$\lim_{t\to\infty} \frac{F(|\int_t^\infty g(s)ds|)}{t} = \left\{ \lim_{t\to\infty} \frac{F(\frac{|K-1|}{K} F^{-1}(Kt))}{Kt} \right\} K = K.$$

3.9.3 Justification of Example 3.7.3

Let $A \in (0,1)$, $K > 1$ and
$x(t) = F^{-1}(Kt)(\sin(t) + A)$, $t \geq 0$. Then $|x(t)| \leq (1+A)F^{-1}(Kt)$, so

$$\liminf_{t\to\infty} \frac{F(|x(t)|)}{t} \geq K.$$

Now we consider the sequence $t_n = n\pi$, $n \geq 0$, so that $x(t_n) = AF^{-1}(Kt_n)$. Thus, since $F \in RV_0(0)$, we have

$$\lim_{n\to\infty} \frac{F\left(\frac{|x(t_n)|}{A}\right)}{F(|x(t_n)|)} = \lim_{n\to\infty} \frac{Kt_n}{F(|x(t_n)|)} = 1.$$

Hence

$$\liminf_{t\to\infty} \frac{F(|x(t)|)}{t} \leq \lim_{n\to\infty} \frac{F(|x(t_n)|)}{t_n} = K.$$

Along the sequence $\tau_n = 2n\pi - \pi/2,\ n \geq 1$, we have $x(\tau_n) < 0$, and we obtain

$$\limsup_{n\to\infty} \frac{F(|x(\tau_n)|)}{\tau_n} = +\infty.$$

It remains to demonstrate that g also satisfies the required conditions. We have $g(t) = x'(t) + f(x(t))$, $t \geq 0$ and since $x(t) \to 0$ as $t \to \infty$ and $(f \circ x)(t) \in L^1(0,\infty)$,

$$\int_t^\infty g(s)ds = \int_t^\infty x'(s)ds + \int_t^\infty f(x(s))ds = -x(t) + \int_t^\infty f(x(s))ds.$$

Hence

$$\begin{aligned}\int_t^\infty g(s)ds &= -F^{-1}(Kt)(A+\sin(t)) + \int_t^\infty f(F^{-1}(Ks)(A+\sin(s)))ds \\ &= -F^{-1}(Kt)(A+\sin(t)) + \frac{1}{K}\int_0^{F^{-1}(Kt)} \frac{f(u\mu(u))}{f(u)}du,\end{aligned}$$

where $\mu(u) = A + \sin(\frac{F(u)}{K}) \in (A-1, A+1)$. Since $f \sim \phi$ we have

$$\lim_{x\to 0} \frac{\int_0^x \frac{f(u\mu(u))}{f(u)}du}{\int_0^x \frac{\phi(u\mu(u))}{\phi(u)}du} = \lim_{x\to 0} \frac{\frac{f(x\mu(x))}{f(x)}}{\frac{\phi(x\mu(x))}{\phi(x)}} = \lim_{x\to 0} \frac{f(x\mu(x))}{\phi(x\mu(x))} = 1.$$

The final equality above follows from the fact that $x\mu(x) \to 0$ as $x \to 0$. Thus

$$\lim_{t\to\infty} \frac{\int_0^{F^{-1}(Kt)} \frac{f(u\mu(u))}{f(u)}du}{\int_0^{F^{-1}(Kt)} \frac{\phi(u\mu(u))}{\phi(u)}du} = 1.$$

For $u \in (0,1)$ we have

$$-(1-A)u \leq u\mu(u) \leq (A+1)u,$$

and since ϕ is increasing and odd we obtain

$$\frac{-\phi\left((1-A)u\right)}{\phi(u)} \leq \frac{\phi\left(u\mu(u)\right)}{\phi(u)} \leq \frac{\phi\left((A+1)u\right)}{\phi(u)}.$$

Therefore

$$\int_0^x \frac{-\phi\left((1-A)u\right)}{\phi(u)}du \leq \int_0^x \frac{\phi\left(u\mu(u)du\right)}{\phi(u)} \leq \int_0^x \frac{\phi\left((A+1)u\right)}{\phi(u)}du.$$

Since $\phi \in RV_0(1)$

$$-(1-A) \leq \liminf_{x\to 0^+} \frac{1}{x}\int_0^x \frac{\phi\left(u\mu(u)\right)}{\phi(u)}du;\ \limsup_{x\to 0^+} \frac{1}{x}\int_0^x \frac{\phi\left(u\mu(u)\right)}{\phi(u)}du \leq 1+A.$$

It follows that we have

$$-(1-A) \le \liminf_{t\to\infty} \frac{1}{F^{-1}(Kt)} \int_0^{F^{-1}(Kt)} \frac{\phi\left(u\mu(u)\right)}{\phi(u)} du,$$

and

$$\limsup_{t\to\infty} \frac{1}{F^{-1}(Kt)} \int_0^{F^{-1}(Kt)} \frac{\phi\left(u\mu(u)\right)}{\phi(u)} du \le 1 + A.$$

Define

$$J(t) := \int_0^{F^{-1}(Kt)} \frac{f\left(u\mu(u)\right)}{f(u)} du$$

and therefore with $\tilde{J}(t) := J(t)/F^{-1}(Kt)$ we have

$$\liminf_{t\to\infty} \tilde{J}(t) \ge -(1-A); \quad \limsup_{t\to\infty} \tilde{J}(t) \le 1 + A.$$

We also have

$$\frac{\int_t^\infty g(s)ds}{F^{-1}(Kt)} = -\left(A + \sin(t)\right) + \frac{\tilde{J}(t)}{K}.$$

Now we consider the sequence $\tau_n = 2n\pi - \pi/2,\ n \ge 1$, on which $\sin(\tau_n) = -1$. Then for every $\epsilon \in (0,1)$ there exists $N(\epsilon)$ such that $n \ge N(\epsilon)$ implies $\tilde{J}(\tau_n) \ge -(1-A) - \epsilon$. Hence, for $n \ge N(\epsilon)$,

$$\frac{\int_{\tau_n}^\infty g(s)ds}{F^{-1}(K\tau_n)} = 1 - A + \frac{\tilde{J}(\tau_n)}{K} \ge (1-A)\{1 - \frac{1}{K}\} - \frac{\epsilon}{K} > 0.$$

Next we take the sequence $t_n = 2m\pi + \pi/2,\ n \ge 1$, on which $\sin(t_n) = 1$. Then for every $\epsilon \in (0,1)$ there is an $N(\epsilon)$ such that $n \ge N(\epsilon)$ implies that $\tilde{J}(t_n) \le 1 + A + \epsilon^2$, with $\epsilon < \frac{1}{2}K(1+A)(1-\frac{1}{K})$. Hence

$$\frac{\int_{t_n}^\infty g(s)ds}{F^{-1}(Kt_n)} = -A - 1 + \frac{\tilde{J}(t_n)}{K} \le -(1+A)\{1 - \frac{1}{K}\} + \frac{\epsilon}{K} < 0.$$

Thus $t \mapsto \int_t^\infty g(s)ds$ is zero on a sequence of times σ_n such that $\sigma_n \to \infty$ as $n \to \infty$, and so

$$\limsup_{t\to\infty} \frac{F(|\int_t^\infty g(s)ds|)}{t} = +\infty.$$

Clearly we have $\liminf_{t\to\infty} F(|\int_t^\infty g(s)ds|)/t \ge K$. Also,

$$\liminf_{n\to\infty} \frac{\int_{t_n}^\infty g(s)ds}{F^{-1}(Kt_n)} \ge (1-A)\left(1 - \frac{1}{K}\right).$$

Therefore

$$\liminf_{t\to\infty} \frac{F(|\int_t^\infty g(s)ds|)}{Kt} \leq \liminf_{n\to\infty} \frac{F(|\int_{t_n}^\infty g(s)ds|)}{Kt_n} \leq 1.$$

3.9.4 Justification of Example 3.7.4

Define

$$\alpha(t) = \log\left(\frac{1}{F^{-1}(t)}\right), \quad t \geq 0, \quad z(t) = F^{-1}(t)\sin(\alpha(t)), \quad t \geq 0.$$

Both functions are well–defined. Then α is in C^1 and $\alpha(t) \to \infty$ as $t \to \infty$. Note that

$$\alpha'(t) = \frac{f(F^{-1}(t))}{F^{-1}(t)}.$$

Since f is in $C^1(0,\infty)$, it is moreover true that $\alpha' \in C^1(0,\infty)$. Furthermore, $z \in C^1$. Now define

$$x(t) = f^{-1}(-z'(t)), \quad t \geq 0, \quad \gamma(t) = x(t) - z(t), \quad t \geq 0.$$

Since z is C^1 and f is increasing, both functions are well–defined and continuous. We can calculate

$$\begin{aligned} z'(t) &= -f(F^{-1}(t))\sin(\alpha(t)) + F^{-1}(t)\alpha'(t)\cos(\alpha(t)) \\ &= f(F^{-1}(t))(-\sin(\alpha(t)) + \cos(\alpha(t))). \end{aligned}$$

From this formula and the fact that $f \in C^1(0,\infty)$, it can be seen that $z' \in C^1$. Therefore x is automatically differentiable at times t for which $z'(t) \neq 0$. Therefore, to show that x is differentiable, we need only focus on points t' at which $\tan(\alpha(t')) = 1$. Now

$$\lim_{t\to t'} \frac{x(t) - x(t')}{t - t'} = \lim_{t\to t'} \frac{f^{-1}(f(F^{-1}(t))\cos(\alpha(t))(\tan(\alpha(t)) - \tan(\alpha(t'))))}{t - t'}.$$

Now, since $f(F^{-1}(t') \neq 0$, $\alpha'(t') > 0$, and $\cos(\alpha(t')) = \pm 1/\sqrt{2}$ (which implies $\sec^2(\alpha(t')) = 2$), as $t \to t'$ we have

$$\begin{aligned} &f(F^{-1}(t))\cos(\alpha(t))(\tan(\alpha(t)) - \tan(\alpha(t'))) \\ &\qquad\qquad \sim f(F^{-1}(t'))\cos(\alpha(t'))\sec^2(\alpha(t'))\alpha'(t')(t - t'). \end{aligned}$$

Therefore, as f^{-1} is in $\text{RV}_0(1)$, we have

$$\begin{aligned}
&\lim_{t\to t'} \frac{f^{-1}(f(F^{-1}(t))\cos(\alpha(t))(\tan(\alpha(t))-\tan(\alpha(t'))))}{t-t'}\\
&=\lim_{t\to t'} \frac{f^{-1}(f(F^{-1}(t'))\cos(\alpha(t'))2\alpha'(t')(t-t'))}{t-t'}\\
&=c\lim_{x\to 0}\frac{f^{-1}(cx)}{cx}=c\lim_{y\to 0}\frac{f^{-1}(y)}{y},
\end{aligned}$$

where $c=2f(F^{-1}(t'))\cos(\alpha(t'))\alpha'(t')>0$. Since $f(x)/x\to\infty$ as $x\to 0$, it follows that the last limit is zero. Hence

$$\lim_{t\to t'}\frac{x(t)-x(t')}{t-t'}=0,$$

so x is differentiable at t' with $x'(t')=0$. Hence x is differentiable on $[0,\infty)$ (note that $\tan(\alpha(0))=\tan(0)=0$, so x has a one–sided derivative at $t=0$). To show that x is in fact C^1, we simply need to show that

$$\lim_{t\to t'} x'(t)=0.$$

For $t\neq t'$, we have $x'(t)=-(f^{-1})'(-z'(t))z''(t)$. Because $(f^{-1})'(x)=1/f'(f^{-1}(x))$ so as $z'(t)\to 0$ as $t\to t'$ and $f'(x)\to\infty$ as $x\to 0$, we have that $(f^{-1})'(-z'(t))\to 0$ as $t\to t'$. Since $z\in C^2$, we have that $z''(t)\to z''(t')$, which is finite. Thus $x'(t)\to 0$ as $t\to t'$, as required. Hence $x\in C^1(0,\infty)$. Therefore, we have that $\gamma\in C^1(0,\infty)$. Define $g(t)=-\gamma'(t)$ for $t\geq 0$. Then g is continuous on $[0,\infty)$. Moreover for any $0\leq t<T$ we have

$$\int_t^T g(s)\,ds=\int_t^T -\gamma'(s)\,ds=\gamma(t)-\gamma(T).$$

Notice that $x(t)\to 0$ as $t\to\infty$ and $z(t)\to 0$ as $t\to\infty$, so $\gamma(t)\to 0$ as $t\to\infty$. Therefore $\int_t^\infty g(s)\,ds$ is well–defined for each $t\geq 0$ and

$$\int_t^\infty g(s)\,ds=\gamma(t).$$

We now show that $x'(t)=-f(x(t))+g(t)$ for all $t\geq 0$. Since $x(t)=f^{-1}(-z'(t))$, we have $z'(t)=-f(x(t))$. But as x is C^1, and $\gamma=x-z$, we have that γ is in C^1 also. Moreover $\gamma'(t)=g(t)$. Hence

$$x'(t)=z'(t)+\gamma'(t)=z'(t)+g(t)=-f(x(t))+g(t),$$

as required. We now obtain asymptotic estimates for z, γ and x. Recall that

$$x(t)=f^{-1}(f(F^{-1}(t))(\sin(\alpha(t))-\cos(\alpha(t)))).$$

Since f is odd and increasing, we have

$$\begin{aligned}|x(t)| &= |f^{-1}(f(F^{-1}(t))(\sin(\alpha(t)) - \cos(\alpha(t))))| \\ &= f^{-1}(|f(F^{-1}(t))(\sin(\alpha(t)) - \cos(\alpha(t)))|) \leq f^{-1}(2f(F^{-1}(t))).\end{aligned}$$

Since $f \in \mathrm{RV}_0(1)$, we have

$$\limsup_{t\to\infty} \frac{|x(t)|}{F^{-1}(t)} \leq \limsup_{t\to\infty} \frac{f^{-1}(2f(F^{-1}(t))}{f^{-1}(f(F^{-1}(t)))} = 2.$$

Since $F \in \mathrm{RV}_0(0)$, this implies

$$\liminf_{t\to\infty} \frac{F(|x(t)|)}{t} \geq 1.$$

Consider the sequence of times t_n such that $\sin(\alpha(t_n)) = 1$. Then $x(t_n) = F^{-1}(t_n)$. Thus $F(|x(t_n)|) = t_n$. Therefore

$$\liminf_{t\to\infty} \frac{F(|x(t)|)}{t} \leq \liminf_{n\to\infty} \frac{F(|x(t_n)|)}{t_n} = 1,$$

and so

$$\liminf_{t\to\infty} \frac{F(|x(t)|)}{t} = 1,$$

as claimed. Consider the sequence of times τ_n such that

$$\sin(\alpha(\tau_n)) = \cos(\alpha(\tau_n)) = 1/\sqrt{2}.$$

Then

$$x(\tau_n) = f^{-1}(f(F^{-1}(\tau_n))(\sin(\alpha(\tau_n)) - \cos(\alpha(\tau_n)))) = f^{-1}(0) = 0.$$

Therefore $\limsup_{t\to\infty} F(|x(t)|)/t \geq \limsup_{n\to\infty} F(|x(\tau_n)|)/\tau_n = +\infty$, so

$$\limsup_{t\to\infty} \frac{F(|x(t)|)}{t} = +\infty,$$

as claimed.

Now we determine similar estimates for γ. Since $\gamma = x - z$, we have $|\gamma(t)| \leq |x(t)| + |z(t)| \leq |x(t)| + F^{-1}(t)$. Therefore

$$\limsup_{t\to\infty} \frac{|\gamma(t)|}{F^{-1}(t)} \leq 3,$$

so as F is in $\mathrm{RV}_0(0)$, we have

$$\liminf_{t\to\infty} \frac{F\left(|\int_t^\infty g(s)\,ds|\right)}{t} = \liminf_{t\to\infty} \frac{F(|\gamma(t)|)}{t} \geq 1.$$

On the other hand

$$\gamma(t) = x(t) - z(t) = f^{-1}(f(F^{-1}(t))(\sin(\alpha(t)) - \cos(\alpha(t)))) - F^{-1}(t)\sin(\alpha(t)),$$

so along the sequence of times t_n such that $\sin(\alpha(t_n)) = 1$ we have

$$\gamma(t_n) = f^{-1}(f(F^{-1}(t_n))) - F^{-1}(t_n) = 0.$$

Therefore, we have $\limsup_{t\to\infty} F(|\gamma(t)|)/t \geq \limsup_{n\to\infty} F(|\gamma(t_n)|)/t_n = +\infty$. Hence

$$\limsup_{t\to\infty} \frac{F\left(\left|\int_t^\infty g(s)\,ds\right|\right)}{t} = \limsup_{t\to\infty} \frac{F(|\gamma(t)|)}{t} = +\infty,$$

as claimed. Finally, consider the sequence of times θ_n for which $\sin(\alpha(\theta_n)) = 0$, $\cos(\alpha(\theta_n)) = -1$. Then

$$\gamma(\theta_n) = f^{-1}(f(F^{-1}(\theta_n))) = F^{-1}(\theta_n).$$

Hence $F(|\gamma(\theta_n)|) = \theta_n$. Therefore

$$\liminf_{t\to\infty} \frac{F\left(\left|\int_t^\infty g(s)\,ds\right|\right)}{t} = \liminf_{t\to\infty} \frac{F(|\gamma(t)|)}{t} \leq \liminf_{n\to\infty} \frac{F(|\gamma(\theta_n)|)}{\theta_n} = 1.$$

Hence

$$\liminf_{t\to\infty} \frac{F\left(\left|\int_t^\infty g(s)\,ds\right|\right)}{t} = 1,$$

as required.

3.9.5 Justification of Example 3.7.5

Define

$$A = \frac{K + \bar{K}}{2}, \quad B = \frac{\bar{K} - K}{2} > 0, \quad C = \exp\left(\frac{\bar{K} - K}{K}\right).$$

Notice that $A - B = K > 1$, $A + B = \bar{K} > K$ and $C > 1$. Define

$$\alpha(t) = \log\log(t + C), \quad t \geq 0, \quad \lambda(t) = t(A + B\sin(\alpha(t))), \quad t \geq 0.$$

We see that α and λ are well–defined. Moreover α is in C^2 with

$$\alpha'(t) = \frac{1}{(t + C)\log(t + C)} > 0.$$

Thus $t\alpha'(t) \to 0$ as $t \to \infty$. Furthermore

$$0 \leq t\alpha'(t) < \frac{1}{\log(t + C)} \leq \frac{1}{\log(C)} = \frac{K}{\bar{K} - K} < \frac{2K}{\bar{K} - K} = \frac{A - B}{B}.$$

Thus $Bt\alpha'(t) < A - B$. Note next that $\lambda \in C^2$, and that

$$\lambda'(t) = Bt\alpha'(t)\cos(\alpha(t)) + A + B\sin(\alpha(t)).$$

Hence $\lambda'(t) > B - A + A - B = 0$. Furthermore, as $t\alpha'(t) \to 0$ as $t \to \infty$, we see that

$$\liminf_{t\to\infty} \frac{\lambda(t)}{t} = A - B = K, \quad \limsup_{t\to\infty} \frac{\lambda(t)}{t} = A + B = \bar{K},$$
$$\liminf_{t\to\infty} \lambda'(t) = A - B = K, \quad \limsup_{t\to\infty} \lambda'(t) = A + B = \bar{K}.$$

Next, define

$$z(t) = F^{-1}(\lambda(t)), \quad t \geq 0.$$

Then, as $\lambda(0) = 0$, we have that $\lambda(t) > 0$ for all $t \geq 0$, and hence z is well–defined. Moreover, as λ is in C^2, we have that $z'(t) = -(f \circ F^{-1})(\lambda(t))\lambda'(t)$ for $t \geq 0$, so z' is in C^1 and $z'(t) < 0$. Since f is increasing on $(0, \infty)$, we have that

$$x(t) = f^{-1}(-z'(t)), \quad t \geq 0$$

is well–defined. Since $f \in C^1$ on $(0, \infty)$, for every $x > 0$ we have that $(f^{-1})'(x) = 1/f'(f^{-1}(x))$. Therefore, as $-z'(t) > 0$ and z'' is continuous (this follows from the fact that f is in $C^1(0, \infty)$ and $\lambda \in C^2(0, \infty)$), we have that $x'(t)$ is well–defined for all $t \geq 0$ and moreover $x \in C^1(0, \infty)$. Finally, as x and z are both C^1, we may define $\gamma \in C^1(0, \infty)$ by

$$\gamma(t) = x(t) - z(t), \quad t \geq 0.$$

Notice also that the right derivative of γ is defined at 0 and $\gamma'(0+) = x'(0) - z'(0) = \lim_{t\to 0^+} \gamma'(t)$. Define finally $g(t) = \gamma'(t)$ for $t \geq 0$. Then g is continuous on $[0, \infty)$. Moreover, as $x(t) \to 0$ as $t \to \infty$ and $z(t) \to 0$ as $t \to \infty$, we have $\gamma(t) \to 0$ as $t \to \infty$. Hence for any $0 \leq t < T$

$$\int_t^T g(s)\,ds = \int_t^T \gamma'(s)\,ds = \gamma(T) - \gamma(t),$$

so

$$\int_t^\infty g(s)\,ds = -\gamma(t), \quad t \geq 0.$$

Finally, we notice for $t \geq 0$ that $x'(t) = z'(t) + \gamma'(t) = -f(x(t)) + g(t)$. Now

$$x(t) = f^{-1}((f \circ F^{-1})(\lambda(t))\lambda'(t)), \quad t \geq 0, \quad \gamma(t) = x(t) - F^{-1}(\lambda(t)), \quad t \geq 0.$$

Let $\epsilon < 1 < K$ be arbitrary. Then there exists $T_1(\epsilon) > 0$ such that

$$\lambda'(t) > K - \epsilon, \quad \lambda'(t) < \bar{K} + \epsilon, \quad t \geq T_1(\epsilon).$$

Since f is increasing, we have

$$f^{-1}((K-\epsilon)(f\circ F^{-1})(\lambda(t))) < x(t) < f^{-1}((\bar{K}+\epsilon)(f\circ F^{-1})(\lambda(t))).$$

Let $\mu(t) = (f\circ F^{-1})(\lambda(t))$. Note that $\mu(t)\to 0$ as $t\to\infty$, and that

$$\frac{f^{-1}((K-\epsilon)\mu(t))}{f^{-1}(\mu(t))} < \frac{x(t)}{F^{-1}(\lambda(t))} < \frac{f^{-1}((\bar{K}+\epsilon)\mu(t))}{f^{-1}(\mu(t))}$$

for $t\geq T_1(\epsilon)$. Since $f^{-1}\in \mathrm{RV}_0(1)$, we have

$$\limsup_{t\to\infty}\frac{x(t)}{F^{-1}(\lambda(t))}\leq \bar{K}+\epsilon, \quad \limsup_{t\to\infty}\frac{x(t)}{F^{-1}(\lambda(t))}\leq \bar{K}.$$

Similarly, we can deduce that

$$\liminf_{t\to\infty}\frac{x(t)}{F^{-1}(\lambda(t))}\geq K.$$

Therefore, for every $\epsilon\in(0,1)<K$, there is $T_2(\epsilon)>0$ such that for $t\geq T_2(\epsilon)$ we have

$$x(t) < (\bar{K}+\epsilon)F^{-1}(\lambda(t)), \quad x(t) > (K-\epsilon)F^{-1}(\lambda(t)).$$

Therefore, with $\eta(t) := F^{-1}(\lambda(t))\to 0$ as $t\to\infty$, for $t\geq T_2(\epsilon)$, we have

$$\frac{F((\bar{K}+\epsilon)\eta(t))}{F(\eta(t))}\cdot\frac{\lambda(t)}{t} < \frac{F(x(t))}{t} < \frac{F((K-\epsilon)\eta(t))}{F(\eta(t))}\cdot\frac{\lambda(t)}{t}.$$

Since $F\in\mathrm{RV}_0(0)$ and $\eta(t)\to 0$, the limit as $t\to\infty$ of the first factor on each right hand side is unity. Therefore applying the limit superior to both inequalities yields

$$\limsup_{t\to\infty}\frac{F(x(t))}{t}\geq \limsup_{t\to\infty}\frac{\lambda(t)}{t}=\bar{K}, \quad \limsup_{t\to\infty}\frac{F(x(t))}{t}\leq \limsup_{t\to\infty}\frac{\lambda(t)}{t}=\bar{K}.$$

Hence

$$\limsup_{t\to\infty}\frac{F(x(t))}{t}=\bar{K}.$$

Similarly, applying the limit inferior to these inequalities yields

$$\liminf_{t\to\infty}\frac{F(x(t))}{t}=K.$$

Next, as $\gamma(t)=x(t)-F^{-1}(\lambda(t))$, we may exploit the pointwise bounds on $x(t)/F^{-1}(\lambda(t))$ to get

$$\limsup_{t\to\infty}\frac{\gamma(t)}{F^{-1}(\lambda(t))}\leq \bar{K}+1; \quad \liminf_{t\to\infty}\frac{\gamma(t)}{F^{-1}(\lambda(t))}\geq K-1>0.$$

One consequence of this is that $\gamma(t) > 0$ for all t sufficiently large, and so we may replace $\gamma(t)$ by $|\gamma(t)|$ in these estimates. Note also that

$$\left|\int_t^\infty g(s)\,ds\right| = |-\gamma(t)| = |\gamma(t)|.$$

By applying the same line of argument used to estimate $F(x(t))/t$ from estimates of $x(t)/F^{-1}(\lambda(t))$ we obtain

$$\limsup_{t\to\infty} \frac{F(|\gamma(t)|)}{t} = \bar{K}, \quad \liminf_{t\to\infty} \frac{F(|\gamma(t)|)}{t} = K,$$

as required.

3.9.6 Justification of Example 3.7.6

If $K = 1$, then it follows that we must have

$$\liminf_{t\to\infty} \frac{-\log|x(t)|}{t} = K = 1.$$

by Theorem 3.5.5. If $K \in (1,\infty)$ then we proceed using the formula for variation of constants, which gives

$$x(t) = x(0)e^{-t} + \int_0^t e^{-(t-s)} g(s)ds.$$

Hence, integration by parts yields

$$x(t) = e^{-t}\left\{x(0) + \int_0^\infty g(s)ds + \int_0^t \gamma(s)e^s ds\right\} - \gamma(t), \tag{3.9.1}$$

where we define $\gamma(t) := \int_t^\infty g(s)ds$. Since $K > 1$, the quantity

$$x(0) + \int_0^\infty g(s)ds + \int_0^\infty \gamma(s)e^s ds$$

is well defined. Therefore we are justified in taking cases as follows.

Case 1: $x(0) + \int_0^\infty g(s)ds + \int_0^\infty \gamma(s)e^s ds \neq 0$.
From (3.9.1) we obtain

$$\lim_{t\to\infty} e^t x(t) = x(0) + \int_0^\infty g(s)ds + \int_0^\infty \gamma(s)e^s ds \neq 0.$$

It follows readily that we have

$$\lim_{t\to\infty} \frac{-\log|x(t)|}{t} = 1.$$

Case 2: $x(0) + \int_0^\infty g(s)ds + \int_0^\infty \gamma(s)e^s ds = 0$.
Using this supposition and (3.9.1) we have

$$\begin{aligned} x(t) &= e^{-t}\left\{x(0) + \int_0^\infty g(s)ds + \int_0^t \gamma(s)e^s ds\right\} - \gamma(t) \\ &= -e^{-t}\int_0^\infty \gamma(s)e^s ds + e^{-t}\int_0^t \gamma(s)e^s ds - \gamma(t) \\ &= -e^{-t}\int_t^\infty \gamma(s)e^s ds - \gamma(t). \end{aligned}$$

Hence we obtain

$$|x(t)| = \left|e^{-t}\int_t^\infty \gamma(s)e^s ds + \gamma(t)\right|. \tag{3.9.2}$$

By hypothesis, for all $t \geq T(\epsilon)$, we have

$$|\gamma(t)| < e^{-(K-\epsilon)t}, \ \epsilon \in (0, K-1).$$

Taking $t \geq T(\epsilon)$ in (3.9.2) we have

$$|x(t)| \leq e^{-t}\int_t^\infty e^{-(K-\epsilon-1)s}ds + e^{-(K-\epsilon)t} = \left(1 + \frac{1}{K-1-\epsilon}\right)e^{-(K-\epsilon)t}.$$

Hence we obtain

$$\frac{-\log|x(t)|}{t} \geq -\frac{1}{t}\log\left(1 + \frac{1}{K-1-\epsilon}\right) + K - \epsilon, \ t \geq T(\epsilon).$$

Taking liminfs and letting $\epsilon \to 0$ then gives us

$$\liminf_{t\to\infty} \frac{-\log|x(t)|}{t} \geq K.$$

But our more general theorem tells us that we must have

$$\liminf_{t\to\infty} \frac{-\log|x(t)|}{t} \leq K.$$

In the case when $K = +\infty$ a calculation exactly analogous to the one outlined above prevails.

Chapter 4

Basic Discretisation Results for Equations and Martingales

4.1 Introduction

After this chapter, most of the rest of the thesis is devoted to showing that *the rates of convergence of discretised versions* of the perturbed equations

$$x'(t) = -f(x(t)) + g(t), \quad t > 0; \quad x(0) = \zeta \tag{4.1.1}$$

and

$$dX(t) = -f(X(t))\,dt + \sigma(t)\,dB(t), \quad t \geq 0; \quad X(0) = \zeta \tag{4.1.2}$$

to the equilibrium (at zero) of the unperturbed equation

$$y'(t) = -f(y(t))$$

are *the same as observed in the corresponding continuous equation.* In this chapter we deal with *whether convergence in the discrete equations takes place*, and also study the *asymptotic behaviour of discrete analogues of the "tail martingales"*

$$\int_t^\infty \sigma(s)\,dB(s)$$

defined for continuous and square integrable σ, that proved so important in understanding the asymptotic rates of convergence of (4.1.2) in the previous chapter.

We have noticed in studying the continuous deterministic case the importance of the hypothesis

$$\lim_{t\to\infty} \int_0^t g(s)\,ds \text{ exists and is finite} \tag{4.1.3}$$

and the hypothesis

$$\sigma \in L^2(0,\infty)$$

in the continuous stochastic case, which implies

$$\lim_{t\to\infty}\int_0^t \sigma(s)\,dB(s) \text{ exists and is finite, a.s.} \tag{4.1.4}$$

These hypotheses force $x(t)\to 0$ as $t\to\infty$ and $X(t)\to 0$ as $t\to\infty$ a.s. assuming f only obeys

$$f\in C(\mathbb{R};\mathbb{R}),\quad f(0)=0,\quad xf(x)>0\quad \text{for all } x\neq 0. \tag{4.1.5}$$

Writing the differential equations in integral form, namely,

$$x(t)=\zeta-\int_0^t f(x(s))\,ds+\int_0^t g(s)\,ds,\quad t\geq 0,$$
$$X(t)=\zeta-\int_0^t f(X(s))\,ds+\int_0^t \sigma(s)\,dB(s),\quad t\geq 0,$$

we see that the solution tends to zero if the last term on the righthand side in each case tends to a finite limit. Therefore, we would wish our discretisation, in both the deterministic and stochastic cases, to share this property without making too many extra hypotheses on f, g or σ.

For the discrete arguments in this thesis, we generally assume that f will be well–behaved close to the equilibrium: in particular, we have not included in this work an analysis of what happens in the numerical schemes when

$$\lim_{x\to 0}\frac{f(x)}{x}=+\infty.$$

This is largely due to length restrictions, as it is well–understood that equations with this type of nonlinearity often require variable time–stepping to obtain good results on rates (in the final chapter of the thesis we give a brief summary of our findings). However, we do not restrict f to be e.g., globally Lipschitz continuous or globally linearly bounded: we would wish our convergence results to still work satisfactorily even if f were to obey e.g.,

$$\lim_{|x|\to\infty}\frac{|f(x)|}{|x|}=+\infty$$

or $\lim_{|x|\to\infty}|f'(x)|=+\infty$.

If such f's are to be admitted, we should not expect an explicit discretisation of the differential equations to be successful. This is well-understood, but for motivation, consider the unperturbed differential equation

$$y'(t)=-y(t)^3,\quad t\geq 0;\quad y(0)=\xi.$$

For definiteness let $\xi>0$. Then $y(t)>0$ for all $t\geq 0$ and $y(t)$ tends monotonically to zero as $t\to\infty$.

Discretise this explicitly using a one–step Euler method with a constant step size

$h > 0$, and let $y(nh)$ be approximated by y_n. Then $(y_n)_{n\geq 0}$ solves the difference equation

$$y_{n+1} = y_n - hy_n^3, \quad n \geq 0, \quad y_0 = \xi.$$

Now suppose that $\xi^2 > 2/h$. Then $y_1 < 0$ and $|y_1| > |y_0|$, so the positivity and monotonicity of the solution are destroyed after one time step. In fact, it can be shown that (y_n) alternates in sign and $|y_{n+1}| > |y_n|$ for $n \geq 0$ with $|y_n| \to \infty$ as $n \to \infty$. Thus, even if h is chosen arbitrarily small, the discrete approximation behaves entirely differently to the continuous equation if the initial condition is too large.

One way to resolve this instability is to use an implicit method: for the one–step Euler scheme this is

$$y_{n+1} = y_n - hy_{n+1}^3, \quad n \geq 0, \quad y_0 = \xi$$

Then y_n preserves positivity, monotonicity and convergence.

For perturbed differential equations (both deterministic and stochastic) a popular numerical method which enjoys these stability properties is the so–called split–step backward Euler method (or SSBE method, for short). It is a semi–implicit method, which allows for perturbations.

More precisely, let $h > 0$ be a fixed step size and $\zeta \in \mathbb{R}$ be deterministic. Consider the system of deterministic difference equations described by

$$x_h(0) = \zeta; \tag{4.1.6a}$$
$$x_h^*(n) = x_h(n) - hf(x_h^*(n)), \quad n \geq 0; \tag{4.1.6b}$$
$$x_h(n+1) = x_h^*(n) + hg(nh); \quad n \geq 0. \tag{4.1.6c}$$

(4.1.6) is the so-called split-step method for discretising the deterministic differential equation (4.1.1). For the stochastic equation (4.1.2), the scheme would be

$$X_h(0) = \zeta; \tag{4.1.7a}$$
$$X_h^*(n) = X_h(n) - hf(X_h^*(n)), \quad n \geq 0; \tag{4.1.7b}$$
$$X_h(n+1) = X_h^*(n) + \sqrt{h}\sigma(nh)\xi_{n+1}; \quad n \geq 0, \tag{4.1.7c}$$

where $(\xi_n)_{n\geq 0}$ are a sequence of independently and identically distributed normal random variables with mean zero and variance 1.

4.2 The Discretisation of Continuous Equation

In this chapter, we do not study *both* of these SSBE schemes. It turns out that it suffices to study a single scheme with the appropriate properties on the perturbing term. Notice the common form of the equations (4.1.6) and (4.1.7) : it is

$$x_h(0) = \zeta; \tag{4.2.1a}$$

$$x_h^*(n) = x_h(n) - hf(x_h^*(n)), \quad n \geq 0; \tag{4.2.1b}$$

$$x_h(n+1) = x_h^*(n) + \gamma_h(n+1), \quad n \geq 0, \tag{4.2.1c}$$

where $\gamma_h(n+1)$ is the approximation to $\int_{nh}^{(n+1)h} g(s)\,ds$ for equation (4.1.1) and $\gamma_h(n+1)$ is the approximation to $\int_{nh}^{(n+1)h} \sigma(s)\,dB(s)$ for equation (4.1.2). Observe that the hypothesis (4.1.3) implies

$$\int_0^{nh} g(s)\,ds = \sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} g(s)\,ds \text{ tends to a finite limit as } n \to \infty$$

and that the hypothesis (4.1.4) implies

$$\int_0^{nh} \sigma(s)\,dB(s) = \sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} \sigma(s)\,dB(s)$$

tends to a finite limit as $n \to \infty$, a.s. This suggests that we should ask the approximation to the perturbation to obey

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} \gamma_h(j+1) \text{ exists and is finite.} \tag{4.2.2}$$

In fact this is exactly what we do: we show below that if f obeys (4.1.5) and γ_h obeys (4.2.2), then $x_h(n) \to 0$ as $n \to \infty$. It is then a separate matter to check, under the appropriate regularity conditions on g and σ, that the conditions (4.1.3) or (4.1.4) ensure that the approximation γ_h obeys (4.2.2). We do not expend a great deal of effort on this, but note that it will hold if g is of one sign and monotone (in the case of (4.1.3)) or if σ^2 is monotone (in the case of (4.1.4)). Another situation where we can proceed without knowing more about regularity on g and σ is when the integrals

$$\int_0^t g(s)\,ds \text{ or } \int_0^t \sigma^2(s)\,ds \text{ can be computed in closed form for any } t \geq 0.$$

In this case, we can take

$$\gamma_h(n+1) = \int_{nh}^{(n+1)h} g(s)\,ds, \quad \gamma_h(n+1) = \left(\int_{nh}^{(n+1)h} \sigma^2(s)\,ds\right)^{1/2} \xi_{n+1},$$

so that (4.2.2) is true if and only if (4.1.3) is true in the deterministic case, and (4.2.2) is true if and only if (4.1.4) is true in the stochastic case.

Essentially, therefore, our discretisation results say that if we can recover the limiting behaviour of the perturbed term by some quadrature method, then the solution of the

SSBE scheme recovers the convergence of the underlying continuous equation. It is an interesting question as to whether a quadrature method can be found which deals with badly–behaving g or σ (which nevertheless obey (4.1.3) or (4.1.4)) which can be incorporated into the split step scheme. For this reason, we give a generalisation of the SSBE scheme which involves variable time steps and which reaches the same conclusion: provided the mesh does not accumulate, and the discrete perturbation term is summable, then the solution of the SSBE scheme tends to zero as $n \to \infty$. There will be further discussion on this point in the final chapter of the thesis.

We close this short section by showing that there is indeed a solution to the SSBE scheme (4.2.1) assuming only that f obeys (4.1.5). Two other facts about the solution are also recorded: first, any $x_h^*(n)$ solving the first equation in (4.2.1) is closer to 0 than $x_h(n)$ itself; and second, uniqueness of the solution of (4.2.1) is ensured for any choice of $h > 0$ if, in addition, f is increasing. We note that these are the type of conditions imposed on f to guarantee existence and uniqueness in the continuous–time equations.

Theorem 4.2.1. *Suppose that f obeys (4.1.5).*

(i) (4.2.1) *has at least one solution (x_h, x_h^*);*

(ii) If $x_h(n) > 0$, then $0 < x_h^*(n) < x_h(n)$; $x_h(n) < 0$ implies $x_h(n) < x_h^*(n) < 0$; and $x_h(n) = 0$ implies $x_h^*(n) = 0$.

(iii) If moreover, f is increasing, then (4.2.1) *has a unique solution (x_h, x_h^*).*

Proof. We start by proving part (i). Let $h > 0$ and for each $x > 0$, define

$$\Phi_x(y) = y - x + hf(y), \quad y \in \mathbb{R}.$$

Then $y \mapsto \Phi_x(y)$ is continuous and if $x > 0$ we have that

$$\Phi_x(0) = -x < 0, \quad \Phi_x(x) = hf(x) > 0.$$

Therefore, there exists $y = y(x) \in (0, x)$ such that $\Phi_x(y(x)) = 0$. Also, if $x < 0$, we have that

$$\Phi_x(0) = -x > 0, \quad \Phi_x(x) = hf(x) < 0.$$

Therefore, there exists $y(x) \in (x, 0)$ such that $\Phi_x(y(x)) = 0$. Finally, if $x = 0$, we have that

$$\Phi_x(0) = 0, \quad \Phi_x(y) = y + hf(y).$$

Notice that $y\Phi_x(y) = y^2 + hyf(y) > 0$ for all $y \neq 0$, so $\Phi_x(y) = 0$ has the unique solution $y = 0$.

By these observations it follows that if for a given $n \in \mathbb{N}$, $x_h(n) \in \mathbb{R}$ is well–defined, then there exists $x_h^*(n)$ such that $\Phi_{x_h(n)}(x_h^*(n)) = 0$, so $x_h^*(n)$ obeys the first equation in (4.2.1). This means that $x_h(n+1)$ is well–defined by the second equation in (4.2.1). Hence, since $x_h(0) = \zeta$, by induction $x_h(n)$ is well–defined for all $n \in \mathbb{N}$. Hence there is a solution to (4.2.1).

To prove (ii), we show that $x_h^*(n)$ and $x_h(n)$ have the same sign and that $|x_h^*(n)| < |x_h(n)|$ if $x_h(n) \neq 0$. If $x_h(n) = 0$, we have already seen that the unique solution to $\Phi_{x_h(n)}(y) = 0$ is $y = 0$, so we have that $x_h^*(n) = 0$ in this case.

Let $x > 0$. Let $y > x$ and then, as $f(y) > 0$, we have

$$\Phi_x(y) = y - x + hf(y) > 0,$$

so there is no solution to $\Phi_x(y) = 0$ in (x, ∞). Also, if we let $y < 0$, then, as $y < 0$, $-x < 0$ and $f(y) < 0$, we have that $\Phi_x(y) < 0$. Hence there is no solution to $\Phi_x(y) = 0$ in $(-\infty, 0)$. Therefore, if $x > 0$ and $\Phi_x(y) = 0$, we must have $y \in (0, x)$. Thus, if $x_h(n) > 0$, it follows that $x_h^*(n) \in (0, x_h(n))$.

The calculation in the case when $x_h(n) < 0$ is identical, and we conclude that $x_h^*(n) \in (x_h(n), 0)$.

To prove part (iii), for fixed $x \neq 0$, consider $\Phi_x(y_2) - \Phi_x(y_1)$ where $y_2 > y_1$. Then

$$\begin{aligned}\Phi_x(y_2) - \Phi_x(y_1) &= y_2 - x + hf(y_2) - [y_1 - x + hf(y_1)] \\ &= y_2 - y_1 + h\,[f(y_2) - f(y_1)] > 0.\end{aligned}$$

Since f is increasing, $y \mapsto \Phi_x(y)$ is also increasing. Therefore, as $\Phi_x(x)\Phi_x(0) < 0$, and $y \mapsto \Phi_x(y)$ is continuous, there is a unique $y = y(x)$ such that $\Phi_x(y(x)) = 0$ and moreover $y(x) \in (\min(x, 0), \max(x, 0))$.

If $x = 0$, we have already seen that $\Phi_x(y) = 0$ has the unique solution $y = 0$. Therefore, with the above observation, it means that given $x_h(n), x_h^*(n)$ is uniquely defined and so $x_h(n+1)$ is uniquely defined. Since $x_h(0) = \zeta$ is uniquely determined, it follows that $x_h(n)$ is uniquely determined for each $n \in \mathbb{N}$. Hence the SSBE method (4.2.1) has a unique solution. □

4.3 Auxiliary Difference Equation and Convergence

At different points in our analysis, it is useful to be able to represent the dynamics of the discrete problem (4.2.1) in different forms. We are assuming from now on that γ_h obeys (4.2.2). Therefore, the sequence $(\Gamma_n(n))_{n\geq 0}$

$$\Gamma_h(n) = -\sum_{j=n}^{\infty} \gamma_h(j+1), \quad n \geq 0 \tag{4.3.1}$$

is well–defined and obeys $\Gamma_h(n) \to 0$ as $n \to \infty$.

The next result collects together some of the more useful ways in which the solution can be represented.

Proposition 4.3.1. *Suppose* (x, x^*) *is a solution of* (4.2.1). *Suppose further that* z_h *is defined by*

$$z_h(n) = x_h(n) - \Gamma_h(n), \quad n \geq 0, \tag{4.3.2}$$

where Γ_h *is defined by* (4.3.1). *Then the sequence* $(z_h(n))_{n\geq 0}$ *can be written in each of the following forms:*

(i)

$$z_h(n+1) = z_h(n) - h_n f(z_h(n) + \Gamma_h(n)), \quad n \geq 0, \tag{4.3.3}$$

where

$$h_n = \begin{cases} h\frac{f(x_h^*(n))}{f(x_h(n))}, & \text{if } x_h(n) \neq 0, \\ h, & \text{if } x_h(n) = 0. \end{cases}$$

(ii)

$$z_h(n+1) = z_h(n) - hf\left(x_h^*(n)\right), \quad n \geq 0.$$

(iii)

$$z_h(n+1) = z_h(n) - hf\left(z_h(n+1) + \Gamma_h(n)\right), \quad n \geq 0.$$

Proof. Let $(\Gamma_h(n))_{n\geq 0}$ be the sequence defined in (4.3.1). Clearly, $\Gamma_h(n) \to 0$ as $n \to \infty$. For $n \geq 0$ from (4.3.2) we get that

$$z_h(n) = x_h(n) + \sum_{j=n}^{\infty} \gamma_h(j+1), \quad n \geq 0. \tag{4.3.4}$$

Then, by (4.2.1) we have that

$$x_h(n+1) = x_h(n) - hf(x_h^*(n)) + \gamma_h(n+1) \quad \text{for} \quad n \geq 0.$$

Now divide and multiply by $f(x_h(n)) \neq 0$ on the second term on the right hand side to get

$$x_h(n+1) = x_h(n) - hf(x_h(n))\frac{f(x_h^*(n))}{f(x_h(n))} + \gamma_h(n+1), \quad n \geq 0.$$

Letting

$$h_n = \begin{cases} h\frac{f(x_h^*(n))}{f(x_h(n))}, & \text{if } x_h(n) \neq 0, \\ h, & \text{if } x_h(n) = 0, \end{cases}$$

we have that

$$x_h(n+1) = x_h(n) - h_n f(x_h(n)) + \gamma_h(n+1), \quad n \geq 0, \tag{4.3.5}$$

which is valid even when $x_h(n) = 0$.

The next step is to get a difference equation for z_h which involves $\Gamma_h(n)$. Now add $\sum_{j=n+1}^{\infty} \gamma_h(j+1)$ to both sides of (4.3.5). Hence for $n \geq 0$ we have

$$x_h(n+1) + \sum_{j=n+1}^{\infty} \gamma_h(j+1) = x_h(n) + \sum_{j=n+1}^{\infty} \gamma_h(j+1) \\ - h_n f(x_h(n)) + \gamma_h(n+1).$$

Then from (4.3.2) and (4.3.1) we get

$$z_h(n+1) = z_h(n) - h_n f(x_h(n)), \quad n \geq 0. \tag{4.3.6}$$

Since $x_h(n) = z_h(n) + \Gamma_h(n)$, hence we have the desired formula of z_h

$$z_h(n+1) = z_h(n) - h_n f(z_h(n) + \Gamma_h(n)), \quad n \geq 0. \tag{4.3.7}$$

To prove the first equation in part (ii), we consider separately for $n \geq 0$ the cases when $x_h(n)$ is zero or non–zero. First, if $n \geq 0$ and $x_h(n) \neq 0$ then from (4.3.6) we have that

$$\begin{aligned} z_h(n+1) =& z_h(n) - h_n f\left(x_h(n)\right) \\ &= z_h(n) - h \frac{f\left(x_h^*(n)\right)}{f\left(x_h(n)\right)} f\left(x_h(n)\right). \end{aligned}$$

Hence

$$z_h(n+1) = z_h(n) - hf\left(x_h^*(n)\right),$$

as required. In the case $n \geq 0$ and $x_h(n) = 0$, we have $z_h(n+1) = z_h(n)$ from (4.3.6). Notice that $x_h(n) = 0$ implies $x_h^*(n) = 0$, so as $f(0) = 0$ we still have

$$z_h(n+1) = z_h(n) - hf\left(x_h^*(n)\right),$$

as required.

To get part (iii), note that for $n \geq 0$ we have

$$\begin{aligned} x_h^*(n) &= x_h(n+1) - \gamma_h(n+1) = z_h(n+1) + \Gamma_h(n+1) - \gamma_h(n+1) \\ &= z_h(n+1) - \sum_{j=n+1}^{\infty} \gamma_h(j+1) - \gamma_h(n+1) \\ &= z_h(n+1) - \sum_{j=n}^{\infty} \gamma_h(j+1) = z_h(n+1) + \Gamma_h(n). \end{aligned}$$

Hence for $n \geq 0$ we may use part (ii) to get

$$z_h(n+1) = z_h(n) - hf(x_h^*(n)) = z_h(n) - hf(z_h(n+1) + \Gamma_h(n)),$$

as required. □

Let f obey (4.1.5). In the next result, we show that the solution x_h of (4.2.1) obeys $x_h(n) \to 0$ as $n \to \infty$ under the condition (4.2.2).

Theorem 4.3.1. *Suppose that f is continuous and obeys (4.1.5). Suppose that γ obeys (4.2.2). Then we have that all solutions of (4.2.1) obeys $\lim_{x\to\infty} x_h(n) = 0$. Moreover, $\lim_{n\to\infty} z_h(n) = 0$ and $\lim_{n\to\infty} x_h^*(n) = 0$.*

Proof. From the part (ii) of the last proposition we have

$$z_h(n+1) - z_h(n) = -hf(x_h^*(n)), \quad n \geq 0. \tag{4.3.8}$$

By (4.2.2), we obtain

$$\lim_{n\to\infty} \gamma_h(n+1) = \lim_{n\to\infty} \sum_{j=0}^{n} \gamma_h(j+1) - \lim_{n\to\infty} \sum_{j=0}^{n-1} \gamma_h(j) = 0.$$

and thus $\gamma_h(n+1) \to 0$ as $n \to \infty$. Note also that $\Gamma_h(n)$ defined by (4.3.1) obeys $\Gamma_h(n) \to 0$ as $n \to \infty$. Therefore, for every $\epsilon > 0$ there is $N_1(\epsilon) > 0$ such that $|\Gamma_h(n)| < \frac{\epsilon}{4}$, and $|\gamma_h(n+1)| < \frac{\epsilon}{4}$ for all $n \geq N_1(\epsilon)$.

Fix $n \geq N_1(\epsilon)$. We proceed by considering cases. Suppose first that $z_h(n) \geq 0$. There are two cases: either $z_h(n) > \epsilon$ or $0 \leq z_h(n) \leq \epsilon$.

Case one. If $z_h(n) > \epsilon$, then by definition of $z_h(n)$ we have

$$x_h(n) = z_h(n) + \Gamma_h(n) \geq \frac{\epsilon}{4}.$$

From (4.2.1), we notice that $0 < x_h^*(n) < x_h(n)$, and so we have $f(x_h^*(n)) > 0$. On the other hand, we also notice from (4.3.8) that $z_h(n+1) < z_h(n)$. Since $x_h(n+1) = x_h^*(n) + \gamma_h(n+1) > -\epsilon/4$, and $z_h(n+1) = x_h(n+1) - \Gamma_h(n+1)$, we get $z_h(n+1) > -\frac{\epsilon}{2}$. Thus we conclude the following:

$$\text{If } n \geq N_1(\epsilon) \text{ and } z_h(n) > \epsilon, \text{ then } -\frac{\epsilon}{2} < z_h(n+1) < z_h(n). \tag{4.3.9}$$

Case two. If $0 \leq z_h(n) \leq \epsilon$, then by definition of $z_h(n)$ we have $x_h(n) = z_h(n) + \Gamma_h(n) < \frac{5\epsilon}{4}$ and $x_h(n) = z_h(n) + \Gamma_h(n) > -\frac{\epsilon}{4}$. Now we study these two subcases:

(i). If $0 \leq x_h(n) < \frac{5\epsilon}{4}$, then $0 \leq x_h^*(n) \leq x(n) < \frac{5\epsilon}{4}$. Hence, $z_h(n+1) \leq z_h(n)$ as above. Also, we have $z_h(n+1) = x_h(n+1) - \Gamma_h(n+1) = x_h^*(n) + \gamma_h(n+1) - \Gamma_h(n+1) > -\frac{\epsilon}{2}$ and thus $-\frac{\epsilon}{2} < z_h(n+1) \leq z_h(n)$.

(ii). If $-\frac{\epsilon}{4} < x_h(n) < 0$ then we have $x_h^*(n) < 0$ and $f(x_h^*(n)) < 0$. Hence from (4.3.8), we notice that $z_h(n+1) > z_h(n)$. Also, $z_h(n+1) = x_h^*(n) + \gamma_h(n+1) - \Gamma_h(n+1) < \frac{\epsilon}{2}$. Thus we get $\frac{\epsilon}{2} > z_h(n+1) > z_h(n) \geq 0$.

We conclude from (i) and (ii) that if

$$\text{If } n \geq N_1(\epsilon) \text{ and } 0 \leq z_h(n) \leq \epsilon, \text{ then } |z_h(n+1)| \leq \epsilon. \tag{4.3.10}$$

We now suppose that for a given $n \geq N_1(\epsilon)$ that $z_h(n) < 0$. We have two cases: either $z_h(n) < -\epsilon$ or $-\epsilon \leq z_h(n) < 0$.

Case one. If $z_h(n) < -\epsilon$, then by the same argument we get $x_h(n) = z_h(n) + \Gamma_h(n) < -\frac{3\epsilon}{4} < 0$. Thus $x_h(n) < x_h^*(n) < 0$ so that $f(x_h^*(n)) < 0$. Also, by (4.3.8) we get $z_h(n+1) > z_h(n)$. However, by definition of $z_h(n)$ we have $z_h(n+1) = x_h^*(n) + \gamma_h(n+1) - \Gamma_h(n+1) < \frac{\epsilon}{2}$. Hence

$$\text{If } n \geq N_1(\epsilon) \text{ and } z_h(n) < -\epsilon \text{ then } z_h(n) < z_h(n+1) < \frac{\epsilon}{2}. \tag{4.3.11}$$

Case two. If $-\epsilon \leq z_h(n) < 0$, then $x_h(n) = z_h(n) - \Gamma_h(n) < \frac{\epsilon}{4}$ and $x_h(n) = z_h(n) - \Gamma_h(n) > -\frac{5\epsilon}{4}$. We now need to analyse all possibles cases:
(i). If $-\frac{5\epsilon}{4} < x_h(n) \leq 0$, then $x_h^*(n) \leq 0$, $f(x_h^*(n)) \leq 0$ and thus $z_h(n+1) \geq z_h(n)$. Also, $z_h(n+1) = x_h^*(n) + \gamma_h(n+1) - \Gamma_h(n+1) < \frac{\epsilon}{2}$. Therefore $\frac{\epsilon}{2} > z_h(n+1) \geq z_h(n)$. Hence, if $z_h(n+1) \geq 0$, then $|z_h(n+1)| \leq \frac{\epsilon}{2}$ and if $z_h(n+1) < 0$, then we have $0 > z_h(n+1) > z(n) \geq -\epsilon$. Thus $|z_h(n+1)| \leq \epsilon$.
(ii). If $0 < x_h(n) \leq \frac{\epsilon}{4}$, we have $x_h^*(n) > 0$, so $f(x_h^*(n)) > 0$ and $z_h(n+1) < z_h(n)$. Also, $z_h(n+1) = x_h^*(n) + \gamma_h(n+1) - \Gamma_h(n+1) > -\frac{\epsilon}{2}$. Therefore, $-\frac{\epsilon}{2} < z_h(n+1) < z_h(n) < 0$ and thus we get that $|z_h(n+1)| < \frac{\epsilon}{2}$. Hence, from (i) and (ii)

$$-\epsilon \leq z_h(n) < 0 \quad \text{implies} \quad |z_h(n+1)| \leq \epsilon, \qquad \text{if } n \geq N_1(\epsilon). \tag{4.3.12}$$

Combining (4.3.10) and (4.3.12) gives

$$\text{If } n \geq N_1(\epsilon) \text{ and } |z_h(n)| \leq \epsilon, \text{ then } \quad |z_h(n+1)| \leq \epsilon. \tag{4.3.13}$$

Now consider (4.3.9) and (4.3.11). We notice from (4.3.9) that $z_h(n) > \epsilon$ gives $0 \leq z_h(n+1) < z_h(n)$ or $-\frac{\epsilon}{2} < z_h(n+1) < 0$. In the first case $|z_h(n+1)| < |z_h(n)|$, and in the second case this is also true. From (4.3.11), $z_h(n) < -\epsilon$ gives either $z_h(n) < z_h(n+1) \leq 0$ or $0 < z_h(n+1) < \frac{\epsilon}{2}$. Once again we have $|z_h(n+1)| < |z_h(n)|$ in each case. Therefore, combining (4.3.9) and (4.3.11) we have that

$$\text{If } n \geq N_1(\epsilon) \text{ and } |z_h(n)| > \epsilon, \text{ then } |z_h(n+1)| < |z_h(n)|. \tag{4.3.14}$$

Finally, let $\epsilon > 0$. If there is $N_4(\epsilon) \geq N_1(\epsilon)$ such that $|z_h(N_4(\epsilon))| \leq \epsilon$, then by (4.3.13), we have $|z_h(n)| \leq \epsilon$ for all $n \geq N_4(\epsilon)$.

Suppose to the contrary, that $|z_h(n)| > \epsilon$ for all $n \geq N_1(\epsilon)$. By (4.3.14), it follows $\epsilon < |z_h(n+1)| < |z_h(n)|$ for all $n \geq N_1(\epsilon)$. Hence $n \mapsto |z_h(n)|$ is a monotone sequence, bounded below by $\epsilon > 0$, and therefore

$$\lim_{n\to\infty} |z_h(n)| = L_1 \geq \epsilon. \tag{4.3.15}$$

Therefore, for every $\epsilon > 0$ there is $N_2(\epsilon)$ such that $||z_h(n)| - L_1| < \frac{\epsilon}{16}$ for all $n \geq N_2(\epsilon)$. Take $N_3(\epsilon) = \max(N_1(\epsilon), N_2(\epsilon))$.

There are three possibilities: (I) $z_h(n) \to L_1 \geq \epsilon$ as $n \to \infty$, (II) $z_h(n) \to -L_1 \leq -\epsilon$ as $n \to \infty$ or that $(z_h(n))$ alternates in sign infinitely often, so that for every $\epsilon > 0$ there exists a sequence $n_j(\epsilon) \geq N_3(\epsilon)$ such that either (IIIa)

$$|z_h(n_j) - L_1| < \frac{\epsilon}{16}, \quad |z_h(n_j+1) + L_1| < \frac{\epsilon}{16},$$

or (IIIb)

$$|z_h(n_j) + L_1| < \frac{\epsilon}{16}, \quad |z_h(n_j+1) - L_1| < \frac{\epsilon}{16}.$$

We show that (I) rapidly leads to absurdities; the proof for (II) is essentially identical. For (I), since $\Gamma_h(n) \to 0$ as $n \to \infty$, we have $x_h(n) \to L_1$ as $n \to \infty$. Therefore, as $\gamma_h(n+1) \to 0$ as $n \to \infty$ too, we have $x_h^*(n) = x_h(n+1) - \gamma_h(n+1) \to L_1$ as $n \to \infty$. Hence

$$L_1 = \lim_{n\to\infty} x_h^*(n) = \lim_{n\to\infty} \{x_h(n) - hf(x_h^*(n))\} = L_1 - hf(L_1).$$

Therefore $f(L_1) = 0$, so $L_1 = 0$. But $L_1 \geq \epsilon > 0$, a contradiction.

Now consider (IIIa) (the case (IIIb) being similar). We have

$$|x_h(n_j) - L_1| = |z_h(n_j) + \Gamma_h(n_j) - L_1| < \frac{\epsilon}{16} + \frac{\epsilon}{4} = \frac{5\epsilon}{16},$$
$$|x_h(n_j+1) + L_1| = |z_h(n_j+1) + \Gamma_h(n_j+1) + L_1| < \frac{\epsilon}{16} + \frac{\epsilon}{4} = \frac{5\epsilon}{16}.$$

Therefore $x_h(n_j) > 0$, so we have $x_h^*(n_j) > 0$. Now $|\gamma_h(n_j+1)| < \frac{\epsilon}{4}$. Therefore as $L_1 \geq \epsilon$, we have

$$0 < x_h^*(n_j) = x_h(n_j+1) - \gamma_h(n_j+1) < -L_1 + \frac{5\epsilon}{16} + \frac{\epsilon}{4} = -L_1 + \frac{9\epsilon}{16} \leq -\frac{7\epsilon}{16} < 0,$$

a contradiction.

Hence all the possible conclusions that result from the assumption that $|z_h(n)| > \epsilon$ for all $n \geq N_1(\epsilon)$ lead to contradictions. Therefore it follows that for every $\epsilon > 0$ that there exists $N_4(\epsilon) > 0$ such that for all $n \geq N_4(\epsilon)$ we have $|z_h(n)| \leq \epsilon$. This implies that $z_h(n) \to 0$ as $n \to \infty$. Since $\Gamma_h(n) \to 0$ as $n \to \infty$, it follows that $x_h(n) \to 0$ as $n \to \infty$ and moreover that $x_h^*(n) \to 0$ as $n \to \infty$, because $\gamma_h(n+1) \to 0$ as $n \to \infty$.

□

If $f(x)/x \to \infty$ as $x \to 0^+$, it will prove that taking a constant step size is no longer sufficient to recover rates of decay. Instead, suppose that we take a variable step size

h_n such that

$$h_n > 0, \quad \lim_{n\to\infty} \sum_{n=0}^{\infty} h_n = +\infty \tag{4.3.16}$$

and consider the spilt–step scheme

$$x_h^*(n) = x_h(n) - h_n f(x_h^*(n)), \quad n \geq 0 \tag{4.3.17a}$$

$$x_h(n+1) = x_h^*(n) + \gamma_h(n+1), \quad n \geq 0, \tag{4.3.17b}$$

where now $x_h(n)$ approximate the solution of the differential equation at $t_n = \sum_{j=0}^{n-1} h_j$. In deterministic case, we should take $\gamma_h(n+1) = h_n g(t_n)$ or even $\gamma_h(n+1) = \int_{t_n}^{t_n+h_n} g(s)\,ds$ in the case that $\int_a^b g(s)\,ds$ can be computed in closed form.

Theorem 4.3.2. *Suppose that f obeys (4.1.5). Suppose h_n obeys (4.3.16) and x_h solves (4.3.17). Suppose that γ obeys (4.2.2). Then all solutions of (4.3.17) obeys $\lim_{x\to\infty} x_h(n) = 0$. Moreover, $\lim_{n\to\infty} z_h(n) = 0$ and $\lim_{n\to\infty} x_h^*(n) = 0$.*

Proof. We can follow the proof of the previous Theorem 4.3.1 exactly up to (4.3.15). Therefore, we have either that there is $N_4(\epsilon) \geq N_1(\epsilon)$ such that $|z_n(n)| \leq \epsilon$ for all $n \geq N_4(\epsilon)$, or that there exists $L_1 \geq \epsilon$ such that

$$\lim_{n\to\infty} |z_h(n)| = L_1 \geq \epsilon.$$

We can emulate the proof that this rules out case (III) in the proof of Theorem 4.3.1 above, but the proof given above which eliminates the possibilities (I) and (II) (namely $z_h(n) \to L_1 \geq \epsilon$ or $z_h(n) \to -L_1 \leq -\epsilon$ as $n \to \infty$) cannot be reused. These possibilities imply that $z_h(n) \to L_2 \neq 0$ as $n \to \infty$. Since $\Gamma_h(n) \to 0$ as $n \to \infty$ and $\gamma_n(n+1) \to 0$ as $n \to \infty$, it follows that $x_h(n)$ and $x_h^*(n) \to L_2 \neq 0$ as $n \to \infty$. Since for any $j \geq 0$ we have

$$x_h(j+1) = x_h(j) - h_j f\left(x_h^*(j)\right) + \gamma_h(j+1),$$

by summing on both sides we get

$$x_h(n+1) = x(0) - \sum_{j=0}^{n} h_j f\left(x_h^*(j)\right) + \sum_{j=0}^{n} \gamma_h(j+1).$$

Since $x_h(n) \to L_2$ as $n \to \infty$, and (4.2.2) holds, we have that

$$\lim_{n\to\infty} \sum_{j=0}^{n} h_j f\left(x_h^*(j)\right) \quad \text{exists and is finite.}$$

Now $f\left(x_h^*(n)\right) \to f(L_2) \neq 0$ as $n \to \infty$. Since $\sum_{j=0}^{n} h_j \to \infty$ as $n \to \infty$, it must follows that

$$\lim_{n\to\infty} \sum_{j=0}^{n} f\left(x_h^*(j)\right) = \pm\infty,$$

according to the sign of L_2, which yields a contradiction. Therefore, the supposition that $|z_h(n)| > \epsilon$ for all $n \geq N_1(\epsilon)$ is false. Thus, we can rejoin the previous proof and by the same argument we arrive at the desired result. □

4.4 Asymptotic Behaviour of Discrete Gaussian Martingales

The next result in this chapter concerns solely stochastic equations. We consider the split step method applied to the SDE (4.1.2). Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and ξ is a stochastic sequence in $\mathbb{R}$ with the property $\xi = \{\xi(n); n \geq 1\}$ is a sequence of independent identically distributed normal variables. With time–varying *deterministic* step sizes obeying (4.3.16), the split step method is

$$X_h(0) = \zeta; \tag{4.4.1a}$$
$$X_h^*(n) = X_h(n) - h_n f(X_h^*(n)), \quad n \geq 0; \tag{4.4.1b}$$
$$X_h(n+1) = X_h^*(n) + \sqrt{h_n}\sigma_h(n)\xi(n+1); \quad n \geq 0. \tag{4.4.1c}$$

Once again, we view $X_h(n)$ as the approximation to $X(t_n)$ where $t_n = \sum_{j=0}^{n-1} h_j$ for $n \geq 1$ and $t_0 = 0$. Note that the times t_n are deterministic.

Notice that $\sigma_h(n)$ here is an approximation for

$$\left(\frac{1}{h_n}\int_{t_n}^{t_n+h_n} \sigma^2(s)\,ds\right)^{1/2}.$$

This means that the random term in the second equation in (4.4.1) has the same distribution as the stochastic integral it is approximating, namely

$$\int_{t_n}^{t_{n+1}} \sigma(s)\,dB(s).$$

In the case when we can compute $\int_a^b \sigma^2(s)\,ds$ in closed form, we may use the above formula for $\sigma_h(n)$; if the integral is not known in closed form, we can simply take $\sigma_h(n) = \sigma(t_n)$.

The next two results concern the asymptotic behaviour of the sum

$$\sum_{j=0}^{n-1} \sigma_j \xi(j+1)$$

as $n \to \infty$ in the case that (σ_n) is in $\ell^2(\mathbb{N})$. Under this summability condition, the sum tends to a finite limit a.s. by the martingale convergence theorem. The lemmas give iterated logarithm bounds on the speed of convergence of the sum to its limit. We state the results in a general form, but aim to apply them in the case that

$$\sigma_n = \sqrt{h_n}\sigma_h(n).$$

Both proofs are inspired by the proof of the classical Law of the Iterated Logarithm for standard Brownian motion (see Karatzas and Shreve [42, Thm 2.9.23]). The result states that if B is a standard Brownian motion, then

$$\limsup_{t\to 0^+} \frac{B(t)}{\sqrt{2t\log\log(1/t)}} = -\liminf_{t\to 0^+} \frac{B(t)}{\sqrt{2t\log\log(1/t)}} = 1, \quad \text{a.s.}$$

with related results being available for the limits as $t\to\infty$.

Lemma 4.4.1. *Suppose σ_j obeys that $\sigma_j \in \ell^2((N))$ and*

$$\sum_{j=n}^{\infty} \sigma_j^2 > 0 \quad \textit{for all} \quad n \geq 0, \tag{4.4.2}$$

then

$$\limsup_{n\to\infty} \frac{\sum_{j=n}^{\infty} \sigma_j \xi(j+1)}{\sqrt{2\sum_{j=n}^{\infty}\sigma_j^2 \log\log\frac{1}{\sum_{j=n}^{\infty}\sigma_j^2}}} \leq 1, \quad a.s,$$

and

$$\liminf_{n\to\infty} \frac{\sum_{j=n}^{\infty} \sigma_j \xi(j+1)}{\sqrt{2\sum_{j=n}^{\infty}\sigma_j^2 \log\log\frac{1}{\sum_{j=n}^{\infty}\sigma_j^2}}} \geq -1, \quad a.s.$$

Proof. Let $\xi = \{\xi(n); n\geq 1\}$ be a sequence of independent and identically standard normal random variables and $\mathcal{F}_n = \sigma(\{\xi(j); 1\leq j\leq n\})$ for $n\geq 1$, so that $(\mathcal{F}_n)_{n\geq 1}$ is the natural filtration of ξ. Also, let $B = \{B(t) : t\geq 0\}$ be a standard Brownian Motion on the probability space $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\right)$ with its natural filtration $\left(\tilde{\mathcal{F}}(t)\right)_{t\geq 0}$. Hence $\{B(n)-B(n-1); n\geq 1\}$ and $\{\xi(n); n\geq 1\}$ have the same distributions. Let $\bar{\sigma} : [0,\infty) \to \mathbb{R}$ defined by $\bar{\sigma}(t) = \sigma_j, j\leq t\leq (j+1)$. Define next

$$X(t) = \int_0^t \bar{\sigma}(s)\,dB(s), \quad t\geq 0$$

and

$$Y(n) = \sum_{j=0}^{n-1} \sigma_j \xi(j+1), \quad n\geq 1.$$

We notice that $Y(n)$ tends to a finite limit as $n\to\infty$ which we call it $Y(\infty) = \sum_{j=0}^{\infty}\sigma_j\xi(j+1)$. Also, $\bar{\sigma}\in L^2(0,\infty)$. Therefore X is a continuous $\tilde{\mathcal{F}}(t)$–martingale with quadratic variation $\langle X\rangle(t) = \int_0^t \bar{\sigma}^2(s)\,ds$ and $t\mapsto \langle X\rangle(t)$ is also piecewise differentiable.

Moreover, $\bar{X}(t) \to \int_0^\infty \bar{\sigma}(s)\,dB(s)$ as $t \to \infty$ a.s. Hence

$$\tilde{\mathbb{P}}\left[\limsup_{t\to\infty} \frac{\int_t^\infty \bar{\sigma}(s)\,dB(s)}{\sqrt{2\int_t^\infty \bar{\sigma}^2(s)\,ds \log\log \frac{1}{\int_t^\infty \bar{\sigma}^2(s)\,ds}}} = 1\right]$$
$$= \tilde{\mathbb{P}}\left[\limsup_{t\to\infty} \frac{X(\infty) - X(t)}{\sqrt{2\left(\langle X\rangle(\infty) - \langle X\rangle(t)\right)\log_2 \frac{1}{\langle X\rangle(\infty) - \langle X\rangle(t)}}} = 1\right].$$

Note by the martingale time change theorem (see [42, Thm. 3.4.6]) that there is a standard Brownian motion B^* such that $B^*(\langle X\rangle(t)) = X(t)$. Hence

$$\tilde{\mathbb{P}}\left[\limsup_{t\to\infty} \frac{X(\infty) - X(t)}{\sqrt{2\left(\langle X\rangle(\infty) - \langle X\rangle(t)\right)\log_2 \frac{1}{\langle X\rangle(\infty) - \langle X\rangle(t)}}} = 1\right]$$
$$= \tilde{\mathbb{P}}\left[\limsup_{\tau\uparrow T} \frac{B^*(T) - B^*(\tau)}{\sqrt{2(T-\tau)\log_2\left(\frac{1}{T-\tau}\right)}} = 1\right]$$
$$= \tilde{\mathbb{P}}\left[\limsup_{t\downarrow 0} \frac{B^*(T) - B^*(T-t)}{\sqrt{2t\log\log\frac{1}{t}}} = 1\right] = 1,$$

because we may apply at the last step the Law of the iterated logarithm to the standard Brownian motion $B^{**}(t) := B^*(T) - B^*(T-t)$, where we have defined $T := \int_0^\infty \bar{\sigma}^2(s)\,ds$. Thus we have

$$\tilde{\mathbb{P}}\left[\limsup_{t\to\infty} \frac{X(\infty) - X(t)}{\sqrt{2\left(\langle X\rangle(\infty) - \langle X\rangle(t)\right)\log_2 \frac{1}{\langle X\rangle(\infty) - \langle X\rangle(t)}}} = 1\right] = 1$$

or

$$\tilde{\mathbb{P}}\left[\limsup_{t\to\infty} \frac{\int_t^\infty \bar{\sigma}(s)\,dB(s)}{\sqrt{2\int_t^\infty \bar{\sigma}^2(s)\,ds \log_2 \frac{1}{\int_t^\infty \bar{\sigma}^2(s)\,ds}}} = 1\right] = 1.$$

A corresponding limit inferior is similarly deduced:

$$\tilde{\mathbb{P}}\left[\liminf_{t\to\infty} \frac{\int_t^\infty \bar{\sigma}(s)\,dB(s)}{\sqrt{2\int_t^\infty \bar{\sigma}^2(s)\,ds \log_2 \frac{1}{\int_t^\infty \bar{\sigma}^2(s)\,ds}}} = -1\right] = 1.$$

Using the limsup and liminf and making the evident observation that taking a limit superior through the integers will give a smaller limit than the corresponding limit taken through the positive reals, we get a.s. that

$$\limsup_{n\to\infty} \frac{|\int_n^\infty \bar{\sigma}(s)\,dB(s)|}{\sqrt{2\int_n^\infty \bar{\sigma}^2(s)\,ds \log\log \frac{1}{\int_n^\infty \bar{\sigma}^2(s)\,ds}}} \leq \limsup_{t\to\infty} \frac{|\int_t^\infty \bar{\sigma}(s)\,dB(s)|}{\sqrt{2\int_t^\infty \bar{\sigma}^2(s)\,ds \log_2 \frac{1}{\int_t^\infty \bar{\sigma}^2(s)\,ds}}} = 1.$$

Since for $n \geq 1$ we have

$$\int_{n-1}^{n} \bar{\sigma}(s)\,dB(s) = \bar{\sigma}(n-1)(B(n)-B(n-1)) = \sigma_{n-1}(B(n)-B(n-1)),$$

we get that

$$\tilde{\mathbb{P}}\left[\limsup_{n\to\infty} \frac{\left|\sum_{j=n}^\infty \sigma_j(B(j+1)-B(j))\right|}{\sqrt{2\sum_{j=n}^\infty \sigma_j^2 \log_2 \frac{1}{\sum_{j=n}^\infty \sigma_j^2}}} \leq 1\right] = 1.$$

Also, since $\{B(j+1)-B(j), j \geq 0\}$ and $\{\xi(j+1), j \geq 0\}$ have the same probability distributions we have that

$$\mathbb{P}\left[\limsup_{n\to\infty} \frac{\left|\sum_{j=n}^\infty \sigma_j \xi(j+1)\right|}{\sqrt{2\sum_{j=n}^\infty \sigma_j^2 \log_2 \left(\frac{1}{\sum_{j=n}^\infty \sigma_j^2}\right)}} \leq 1\right] = \tilde{\mathbb{P}}\left[\limsup_{t\to\infty} \frac{\left|\sum_{j=n}^\infty \sigma_j(B(j+1)-B(j))\right|}{\sqrt{2\sum_{j=n}^\infty \sigma_j^2 \log_2 \frac{1}{\sum_{j=n}^\infty \sigma_j^2}}} \leq 1\right] = 1,$$

as claimed. □

The condition (4.4.2) is important to ensure that the dynamics of the discretisation of SDE would not collapse to those of unperturbed ODE.

We now impose further assumption on σ_j^2 to ensure the limit superior in Lemma 4.4.1 has a unit limit.

Theorem 4.4.1. *Suppose (σ_n) obeys $(\sigma_n) \in \ell^2(\mathbb{N})$ and (4.4.2). If*

$$\lim_{n\to\infty} \frac{\sigma_n^2}{\sum_{j=n}^\infty \sigma_j^2} = 0 \tag{4.4.3}$$

then

$$\limsup_{n\to\infty} \frac{\left|\sum_{j=n}^\infty \sigma_j \xi(j+1)\right|}{\sqrt{2\sum_{j=n}^\infty \sigma_j^2 \log\log \frac{1}{\sum_{j=n}^\infty \sigma_j^2}}} = 1, \quad a.s.$$

Proof. We have the sequence of identities

$$\begin{aligned}
&\mathbb{P}\left[\limsup_{n\to\infty}\frac{\sum_{j=n}^{\infty}\sigma_j\xi(j+1)}{\sqrt{2\sum_{j=n}^{\infty}\sigma_j^2\log_2\frac{1}{\sum_{j=n}^{\infty}\sigma_j^2}}}\geq 1\right]\\
&=\tilde{\mathbb{P}}\left[\limsup_{n\to\infty}\frac{\int_n^{\infty}\bar{\sigma}(s)\,dB(s)}{\sqrt{2\int_n^{\infty}\bar{\sigma}^2(s)\,ds\log_2\left(\frac{1}{\int_n^{\infty}\bar{\sigma}^2(s)\,ds}\right)}}\geq 1\right]\\
&=\tilde{\mathbb{P}}\left[\limsup_{n\to\infty}\frac{B^*\left(\int_0^{\infty}\bar{\sigma}^2(s)\,ds\right)-B^*\left(\int_0^{n}\bar{\sigma}^2(s)\,ds\right)}{\sqrt{2\left(\int_0^{\infty}\bar{\sigma}^2(s)\,ds-\int_0^{n}\bar{\sigma}^2(s)\,ds\right)\log_2\frac{1}{\int_0^{\infty}\bar{\sigma}^2(s)\,ds-\int_0^{n}\bar{\sigma}^2(s)\,ds}}}\geq 1\right]\\
&=\tilde{\mathbb{P}}\left[\limsup_{n\to\infty}\frac{B^{**}\left(\int_n^{\infty}\bar{\sigma}^2(s)\,ds\right)}{\sqrt{2\int_n^{\infty}\bar{\sigma}^2(s)\,ds\log\log\frac{1}{\int_n^{\infty}\bar{\sigma}^2(s)\,ds}}}\geq 1\right]\\
&=\tilde{\mathbb{P}}\left[\limsup_{n\to\infty}\frac{B^{**}\left(\sum_{j=n}^{\infty}\sigma_j^2\right)}{\sqrt{2\sum_{j=n}^{\infty}\sigma_j^2\log_2\frac{1}{\sum_{j=n}^{\infty}\sigma_j^2}}}\geq 1\right],
\end{aligned}$$

where $B^{**}(t) := B^*(T) - B^*(T-t)$ is another standard Brownian motion. Fix $\theta \in (0, 1/20)$ and define $a_n = \sup\left\{m \in \mathbb{N} : \sum_{j=m}^{\infty}\sigma_j^2 > \theta^n\right\}$. Then $a_n \to \infty$ as $n \to \infty$ and a_n is non-decreasing. Now, define $\tau_n = \sum_{j=a_n}^{\infty}\sigma_j^2$ and thus $\tau_n > \theta^n$. Also, $\sum_{j=a_n+1}^{\infty}\sigma_j^2 \leq \theta^n$ and so $\tau_n \leq \theta^n + \sigma_{a_n}^2$. Since $\theta^n/\tau_n < 1$ and $\theta^n/\tau_n \geq 1 - \sigma_{a_n}^2/\sum_{j=a_n}^{\infty}\sigma_j^2 \to 1$ as $n \to \infty$ by (4.4.3), we have $\tau_n/\theta^n \to 1$ as $n \to \infty$. Define next $X_n = B^{**}(\tau_n) - B^{**}(\tau_{n+1})$ for $n \geq 1$. Then (X_n) is a sequence of independent (normal) random variables. Also, define $h(t) = \sqrt{2t\log\log\frac{1}{t}}$ and $A_n := \left\{\frac{B^{**}(\tau_n)-B^{**}(\tau_{n+1})}{h(\tau_n)} \geq x\right\}$ for fixed $x > 0$. Then the A_n's are independent events. Letting Z below stand for a standard normal random variable, and recalling the bound

$$\Pr[Z > x] \geq \frac{1}{\sqrt{2\pi}}\frac{x}{1+x^2}e^{-x^2/2}, \quad x > 0,$$

we obtain

$$\begin{aligned}
\tilde{\mathbb{P}}[A_n] &= \tilde{\mathbb{P}}\left[\frac{B^{**}(\tau_n)-B^{**}(\tau_{n+1})}{\sqrt{\tau_n-\tau_{n+1}}} \geq \frac{xh(\tau_n)}{\sqrt{\tau_n-\tau_{n+1}}}\right]\\
&= \Pr\left[Z \geq \frac{xh(\tau_n)}{\sqrt{\tau_n-\tau_{n+1}}}\right]\\
&\geq \frac{1}{\sqrt{2\pi}}\frac{xh(\tau_n)/\sqrt{\tau_n-\tau_{n+1}}}{1+x^2h^2(\tau_n)/\sqrt{\tau_n-\tau_{n+1}}}\cdot\exp\left(-\frac{1}{2}\frac{x^2h^2(\tau_n)}{\tau_n-\tau_{n+1}}\right) =: \eta_n.
\end{aligned}$$

Next as $n \to \infty$, we have

$$\frac{1}{\log n}\frac{1}{2}\frac{h^2(\tau_n)}{\tau_n - \tau_{n+1}} = \frac{\log\log\frac{1}{\tau_n}}{1-\frac{\tau_{n+1}}{\tau_n}}\frac{1}{\log n} \sim \frac{\log\log\frac{1}{\tau_n}}{\log n}\frac{1}{1-\theta} \to \frac{1}{1-\theta}.$$

Thus

$$\lim_{n\to\infty}\frac{1}{\sqrt{2\log n}}\frac{h(\tau_n)}{\sqrt{\tau_n - \tau_{n+1}}} = 1/\sqrt{1-\theta}$$

or $h(\tau_n)/\sqrt{\tau_n - \tau_{n+1}} \sim \sqrt{2\log n}/\sqrt{1-\theta}$ as $n \to \infty$. The first factor in η_n is asymptotic to $\frac{1}{xh(\tau_n)/\sqrt{\tau_n-\tau_{n+1}}}$ which is itself asymptotic to $\frac{1}{x\cdot\sqrt{2\log n}/\sqrt{1-\theta}}$ as $n \to \infty$. Now let $x = \sqrt{1-2\theta}$, $\theta < 1/20$. Thus $0 < x^2/(1-\theta) = (1-2\theta)/(1-\theta) < 1$. Hence we have

$$\lim_{n\to\infty}\frac{-\frac{x^2}{2}\frac{h^2(\tau_n)}{\tau_n-\tau_{n+1}}}{\log n} = \frac{-x^2}{1-\theta} \in (-1,0).$$

This means η_n is not summable. Thus $\sum_{n=1}^\infty \tilde{\mathbb{P}}[A_n] = +\infty$. By the second Borel–Cantelli lemma, there exists for an event Ω_θ of probability one such that for every $\omega \in \Omega_\theta$ and every $l \geq 1$, there is an integer $m = M(l,\omega) \geq l$ such that

$$B^{**}(\tau_m)(\omega) - B^{**}(\tau_{m+1})(\omega) \geq \sqrt{1-2\theta}h(\tau_m).$$

On the other hand, applying the (continuous) Law of the Iterated Logarithm to the Brownian motion $-B^{**}$, we have there is an a.s. event Ω_θ^* and a random integer $N_\theta = N^*(\omega) \in \mathbb{N}$ such that for $n \geq N^*(\omega)$ we have

$$-B^{**}(\tau_{n+1})(\omega) < 2h(\tau_{n+1}) = h(\tau_n)\cdot 2\frac{h(\tau_{n+1})}{h(\tau_n)} < 4\sqrt{\theta}h(\tau_n).$$

for all $\omega \in \Omega_\theta^*$, the last inequality coming from

$$\lim_{n\to\infty}\frac{h(\tau_{n+1})}{h(\tau_n)} = \lim_{n\to\infty}\sqrt{\frac{\tau_{n+1}}{\tau_n}\frac{\log\log\frac{1}{\tau_{n+1}}}{\log\log\frac{1}{\tau_n}}} = \sqrt{\theta} < \sqrt{2}\sqrt{\theta}.$$

Thus for every $\omega \in \Omega_\theta \cap \Omega_\theta^*$ and every integer $l \geq 1$, there is an $m \geq M(l,\omega) \vee N^*(\omega)$ such that

$$B^{**}(\tau_m)(\omega) - B^{**}(\tau_{m+1})(\omega) \geq \sqrt{1-2\theta}h(\tau_m),$$

and $-B^{**}(\tau_{m+1})(\omega) < 4\sqrt{\theta}h(\tau_m)$. Hence

$$B^{**}(\tau_m)(\omega) \geq \left(\sqrt{1-2\theta} - 4\sqrt{\theta}\right)h(\tau_m).$$

Therefore, $\tilde{\Omega}_\theta = \Omega_\theta \cap \Omega_\theta^*$ defined by

$$\tilde{\Omega}_\theta = \left\{\omega : \limsup_{m\to\infty}\frac{B^{**}(\tau_m,\omega)}{h(\tau_m)} \geq \sqrt{1-2\theta} - 4\sqrt{\theta}\right\}$$

is a $\tilde{\mathbb{P}}$-almost sure event for $\theta < 1/20$. By the definition of τ_n we have

$$\tilde{\Omega}_\theta = \left\{ \omega : \limsup_{n\to\infty} \frac{B^{**}(\sum_{j=a_n}^\infty \sigma_j^2)}{\sqrt{2\sum_{j=a_n}^\infty \sigma_j^2 \log\log \frac{1}{\sum_{j=a_n}^\infty \sigma_j^2}}} \geq \sqrt{1-2\theta} - 4\sqrt{\theta} \right\}.$$

Since $a_n \to \infty$ as $n \to \infty$ and $a_n \in \mathbb{N}$ we have that

$$\Omega_\theta^{**} = \left\{ \omega : \limsup_{n\to\infty} \frac{B^{**}(\sum_{j=n}^\infty \sigma_j^2)}{\sqrt{2\sum_{j=n}^\infty \sigma_j^2 \log\log \frac{1}{\sum_{j=n}^\infty \sigma_j^2}}} \geq \sqrt{1-2\theta} - 4\sqrt{\theta} \right\}$$

is also a $\tilde{\mathbb{P}}$-a.s. event. Now consider $\Omega^{**} = \bigcap_{\theta\in(0,1/20)\cap\mathbb{Q}} \Omega_\theta^{**}$. Then Ω^{**} is a.s. where

$$\Omega^{**} = \left\{ \omega : \limsup_{n\to\infty} \frac{B^{**}(\sum_{j=n}^\infty \sigma_j^2)}{\sqrt{2\sum_{j=n}^\infty \sigma_j^2 \log\log \frac{1}{\sum_{j=n}^\infty \sigma_j^2}}} \geq 1 \right\}.$$

Therefore, from the initial sequence of identities

$$\mathbb{P}\left[\limsup_{n\to\infty} \frac{\sum_{j=n}^\infty \sigma_j \xi(j+1)}{\sqrt{2\sum_{j=n}^\infty \sigma_j^2 \log_2 \frac{1}{\sum_{j=n}^\infty \sigma_j^2}}} \geq 1 \right] = 1.$$

We have already shown in Lemma 4.4.1 that

$$\mathbb{P}\left[\limsup_{n\to\infty} \frac{\sum_{j=n}^\infty \sigma_j \xi(j+1)}{\sqrt{2\sum_{j=n}^\infty \sigma_j^2 \log_2 \frac{1}{\sum_{j=n}^\infty \sigma_j^2}}} \leq 1 \right] = 1,$$

so combining these two results, the desired limit is obtained. □

Chapter 5

Discrete Equations with Rapidly Decaying Solutions

5.1 Introduction

In Chapter 3, we determined the precise asymptotic behaviour of

$$x'(t) = -f(x(t)) + g(t), \quad t > 0 \tag{5.1.1}$$

for equations in which the solution of the unperturbed equation is rapidly varying in the sense that

$$\lim_{t\to\infty} \frac{F^{-1}(\lambda t)}{F^{-1}(t)} = 0, \quad \text{for} \quad \lambda > 1, \tag{5.1.2}$$

and the function f is asymptotic at zero to an odd and asymptotic preserving function. Stochastic equations were also studied. In Chapter 4, we saw that the split–step backward Euler method with a *constant step size* is able to recover the convergence to zero of the solution of (5.1.1).

We now show that the SSBE method can also recover the precise asymptotic behaviour established in Chapter 3. We do this in the case when $f(x)/x \to 0$ as $x \to 0$ only. We prove results for small, moderate and large perturbations, and in each case a close analogue of the continuous time result is established. We also show, in the case that the perturbations are rapidly decaying and positive that a limit result i.e.,

$$\lim_{n\to\infty} \frac{F(x_h(n))}{nh} \text{ exists}$$

is obtainable for the split step scheme and that the split step scheme has a positive solution, both of which properties are faithfully inherited from the continuous time problem.

In Chapter 6 we will show to what degree asymptotic behaviour of the SDE is recovered by the split step scheme. However, we should note that much of the groundwork is done in this chapter. As in Chapter 4, we have often proven results in this chapter in a way that allows us to handle both deterministic or stochastic perturbations. As a consequence, most of the theorems in Chapter 6 will be corollaries of results in this chapter, with only relatively modest adjustments being needed to deal with the randomness.

Looking at the results in this Chapter, in Chapter 4 and in Chapter 6, it can be seen that a common viewpoint has been taken concerning the performance of the numerical scheme. Roughly, the viewpoint is this. The perturbations in the continuous deterministic or stochastic equations (i.e. g and σ) are *sufficiently well behaved* that the asymptotic behaviour of their quadrature recovers their original asymptotic behaviour in continuous time. Granted that this is the case, the split step method recovers the continuous time asymptotic behaviour of the underlying ODE or SDE. Our belief is that, if the perturbations were not so well behaved, and that more sophisticated quadrature rules were needed to recover their asymptotic behaviour, the changes needed to the proofs for the asymptotic behaviour of the split step scheme *would not require fundamental revisions*. To put it another way, we believe that we have managed to largely decouple the problem of studying the dynamics of the split step scheme from the problem of recovering the asymptotic behaviour of the perturbations. We expand on this point a little more in Chapter 7.

5.2 Problems for Superlinear Equations with Constant Step Sizes

Before dealing with the main topic of this chapter, which is perturbed equations in the case when $f(x)/x \to 0$ as $x \to 0$ (or when $f'(x) \to 0$ as $x \to 0$), we wish to reflect on why we do not attempt to capture rates for perturbed equations using constant step sizes in the case when $f(x)/x \to \infty$ as $x \to 0$ (or when $f'(x) \to \infty$ as $x \to 0$). This is because we do not expect to be able to get the correct rate of decay when $f(x)/x \to \infty$ as $x \to 0$ with a constant step size. In order to do this, vanishing step sizes must likely be taken. This justifies the results we prove in Chapter 4 for the SSBE scheme and for the tail martingales, which are sufficiently flexible to allow for varying step sizes. We remark more on proposed numerical methods for vanishing step sizes in Chapter 7.

Let us examine the problems in recovering the rate for non–constant step sizes by considering an unperturbed equation. Let f be continuous on $[0, \infty)$ with $f(0) = 0$ and $f(x) > 0$ for $x > 0$. Here, as usual, F is defined by

$$F(x) = \int_x^1 \frac{1}{f(u)}\, du, \quad x > 0 \tag{5.2.1}$$

and $F(x) \to \infty$ as $x \to 0^+$.

Consider now the implicit scheme

$$x_h(n+1) = x_h(n) - hf(x_h(n+1)), \quad n \geq 0; \quad x_h(0) = \xi > 0. \tag{5.2.2}$$

Notice that this scheme is precisely the split step scheme in the case of zero perturbations. Suppose now that $f(x)/x \to \infty$ as $x \to 0$. Here $x_h(n)$ approximates $x(nh)$, where x solves the differential equation

$$x'(t) = -f(x(t), \quad t > 0; \quad x(0) = \xi.$$

The continuous solution of this differential equation is positive and decreasing on $[0, \infty)$ with $x(t) \to 0$ as $t \to \infty$. Moreover, the rate of decay is superexponential in the sense that

$$\lim_{t\to\infty} \frac{x'(t)}{x(t)} = -\infty, \text{ or } \lim_{t\to\infty} \frac{1}{t} \log x(t) = -\infty.$$

Many of the properties of the solution of the differential equation are recovered by the implicit scheme in (5.2.2). For any solution $(x_h(n))_{n\geq 0}$ of (5.2.2), we still have $x_h(n) \downarrow 0$ as $n \to \infty$ and $x_h(n) > 0$ for all $n \geq 0$. Indeed, it can be shown that

$$\lim_{t\to\infty} \frac{1}{nh} \log x_h(n) = -\infty$$

which mirrors the superexponential decay in the ODE. However, as we now show, when $f'(x) \to \infty$ as $x \to 0^+$, the nonlinear Liapunov exponent present in the continuous equation, namely

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 1, \tag{5.2.3}$$

is *not recovered* by (5.2.2), where F is given by (5.2.1).

Theorem 5.2.1. *Suppose that $f \in C([0, \infty); [0, \infty))$ and $f(x) > 0$ for $x > 0$ and $f(0) = 0$. Suppose also that f' is continuous on $(0, \infty)$ with $f'(x) \to \infty$ as $x \to 0^+$. Let x_h be any solution of (5.2.2) and suppose that F defined by (5.2.1) obeys $F(x) \to \infty$. Then $(x_h(n))_{n\geq 0}$ is a positive monotone decreasing sequence with $x_h(n) \to 0$ as $n \to \infty$ and obeys*

$$\lim_{t\to\infty} \frac{F(x_h(n))}{nh} = 0. \tag{5.2.4}$$

Note that (5.2.4) means that the rate of decay in the solution of the ODE is significantly underestimated, in the sense that the numerical scheme predicts a zero Liapunov like exponent, whereas the true exponent is strictly positive. Therefore, in these cases, we need to take a step size which tends to zero as the simulated solution $x_h(n)$ tends to zero. For more details consult Colgan [29].

Proof of Theorem 5.2.1. To prove (5.2.4), note that $x_n = x_{n+1} + hf(x_{n+1})$, so that

$$F(x_{n+1}) - F(x_n) = \int_{x_{n+1}}^{x_{n+1}+hf(x_{n+1})} \frac{1}{f(u)}\,du.$$

Therefore, if we can show that

$$\lim_{x\to 0^+} \int_x^{x+hf(x)} \frac{1}{f(u)}\,du = 0, \tag{5.2.5}$$

it will follow that $F(x_{n+1}) - F(x_n) \to 0$ as $n \to \infty$, from which (5.2.4) holds. Let $\epsilon \in (0, 1 \wedge h)$ be arbitrary and fixed. For the integral, notice that f is increasing on $(0, \delta)$ for some $\delta > 0$. We take arguments of functions which are always smaller than δ to make use of this monotonicity. Let $\delta' + \epsilon f(\delta') = \delta$. Then using the monotonicity of f, and taking $x \in (0, \delta')$, we get

$$\begin{aligned}
\int_x^{x+hf(x)} \frac{1}{f(u)}\,du &= \int_0^h \frac{f(x)}{f(x+\alpha f(x))}\,d\alpha \\
&= \int_0^\epsilon \frac{f(x)}{f(x+\alpha f(x))}\,d\alpha + \int_\epsilon^h \frac{f(x)}{f(x+\alpha f(x))}\,d\alpha \\
&\le \epsilon + (h-\epsilon)\frac{f(x)}{f(x+\epsilon f(x))}.
\end{aligned}$$

Now, for every $x \in (0, \delta')$, there is an $\xi_{\epsilon,x} \in [x, x+\epsilon f(x)]$ such that $f(x+\epsilon f(x)) = f(x) + f'(\xi_{\epsilon,x})\epsilon f(x)$. Since $\xi_{\epsilon,x} \to 0$ as $x \to 0^+$, and $f'(\xi_{\epsilon,x}) \to \infty$ as $x \to 0^+$ it follows that $f(x+\epsilon f(x))/f(x) \to \infty$ as $x \to 0^+$. Therefore

$$\limsup_{x\to 0^+} \int_x^{x+hf(x)} \frac{1}{f(u)}\,du \le \epsilon.$$

Since $\epsilon > 0$ was chosen arbitrarily, and the left hand side is ϵ–independent, letting $\epsilon \to 0^+$ gives (5.2.5), as required. □

5.3 Main Results

In the last chapter, we saw that the solution $x_h(n)$ of the split step method

$$x_h(0) = \zeta; \tag{5.3.1a}$$

$$x_h^*(n) = x_h(n) - hf(x_h^*(n)), \quad n \ge 0; \tag{5.3.1b}$$

$$x_h(n+1) = x_h^*(n) + \gamma_h(n+1), \quad n \ge 0, \tag{5.3.1c}$$

can be written in terms of an auxiliary sequence $z_h(n)$ in the case that

$$\lim_{n\to\infty}\sum_{j=0}^{n-1}\gamma_h(j+1) \text{ exists and is finite.} \tag{5.3.2}$$

In this case, we introduce the sequence

$$\Gamma_h(n) = -\sum_{j=n}^{\infty}\gamma_h(j+1), \quad n\geq 0 \tag{5.3.3}$$

and writing $z_h(n) = x_h(n) - \Gamma_h(n)$ for $n \geq 0$, we have that z_h obeys the difference equation

$$z_h(n+1) = z_h(n) - hf\left(z_h(n+1)+\Gamma_h(n)\right), \quad n\geq 0. \tag{5.3.4}$$

Notice that z_h is the analogue of the function $z(t) = x(t) - \Gamma(t)$ and that Γ_h is the analogue of the function

$$\Gamma(t) = -\int_t^{\infty} g(s)\,ds, \quad t\geq 0 \tag{5.3.5}$$

used to study the differential equation (5.1.1). Notice also that z obeys a differential equation very similar in structure to the discrete equation (5.3.4), namely

$$z'(t) = -f(z(t)+\Gamma(t)), \quad t\geq 0. \tag{5.3.6}$$

Since the study of the asymptotic behaviour of the solution x of (5.1.1) in Chapter 3 was mainly carried out by analysing z through the differential equation (5.3.6), our approach here is to determine the asymptotic behaviour of x_h solving (5.3.1) through the discrete analogue z_h of z using (5.3.4), which is the discrete analogue of equation (5.3.6).

In Chapter 4, we showed that when $\Gamma_h(n)\to 0$ as $n\to\infty$, then $x_h(n)$ and $z_h(n)\to 0$ as $n\to\infty$. In that case, the following limits inferior, motivated by corresponding limits for the functions Γ, z and x in the continuous case, can be introduced:

$$\lambda_\Gamma(h) = \liminf_{n\to\infty}\frac{F(|\Gamma_h(n)|)}{nh}, \tag{5.3.7}$$

$$\lambda_x(h) = \liminf_{n\to\infty}\frac{F(|x_h(n)|)}{nh}, \tag{5.3.8}$$

$$\lambda_z(h) = \liminf_{n\to\infty}\frac{F(|z_h(n)|)}{nh}, \tag{5.3.9}$$

where the function F is defined as usual by

$$F(x) = \int_x^1 \frac{1}{f(u)}\,du, \quad x>0 \tag{5.3.10}$$

and $F(x)\to\infty$ as $x\to 0^+$. We demonstrate that when the "internal" perturbation Γ

decays to zero so rapidly that
$$\lambda_\Gamma(h) \geq 1,$$
and the solution of (5.3.4) tends to zero as $n \to \infty$, the asymptotic behaviour of (4.1.6) is preserved in the following sense.

Theorem 5.3.1. *Let f obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3) where the function φ in (3.2.4) and (3.2.3) is in $C^1(0,\infty)$ and obeys $\varphi'(x) \to 0$ as $x \to 0$. Let F be the function defined by (5.3.10) with $F(x) \to \infty$ as $x \to 0^+$. Suppose further that γ_h obeys (5.3.2) and that Γ_h defined (5.3.3) obeys*
$$\lambda_\Gamma(h) \geq 1, \tag{5.3.11}$$
where $\lambda_\Gamma(h)$ is defined by (5.3.7). Then any solution x_h of (5.3.1) obeys
$$\lambda_x(h) \geq 1.$$
where $\lambda_x(h)$ is defined by (5.3.8).

The following converse theorem will be useful in establishing that the existence of a $\lambda_x(h)$ is essentially contingent on the existence of a $\lambda_\Gamma(h)$.

Theorem 5.3.2. *Let f obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3) where the function φ in (3.2.4) and (3.2.3) is in $C^1(0,\infty)$ and obeys $\varphi'(x) \to 0$ as $x \to 0$. Let F be the function defined by (5.3.10) with $F(x) \to \infty$ as $x \to 0^+$. Let x_h be a solution of (5.3.1) with*
$$\lambda_x(h) > 0.$$
where $\lambda_x(h)$ is defined by (5.3.8). Then γ_h obeys (5.3.2) and Γ_h defined (5.3.3) obeys
$$\lambda_\Gamma(h) \geq \lambda_x(h),$$
where $\lambda_\Gamma(h)$ is defined by (5.3.7).

We next consider the case when the perturbations decays slowly in the following theorem:

Theorem 5.3.3. *Let f obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3) where the function φ in (3.2.4) and (3.2.3) is in $C^1(0,\infty)$ and obeys $\varphi'(x) \to 0$ as $x \to 0$. Let F be the function defined by (5.3.10) with $F(x) \to \infty$ as $x \to 0^+$. Suppose further that γ_h obeys (5.3.2) and that Γ_h defined (5.3.3) obeys $\lambda_\Gamma(h) \in (0,1)$ where $\lambda_\Gamma(h)$ is defined by (5.3.7). Then any solution x_h of (5.3.1) obeys*
$$\lambda_x(h) = \lambda_\Gamma(h),$$
where $\lambda_x(h)$ is defined by (5.3.8).

Very slow decay is dealt with by the next result.

Theorem 5.3.4. *Let f obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3) where the function φ in (3.2.4) and (3.2.3) is in $C^1(0,\infty)$ and obeys $\varphi'(x) \to 0$ as $x \to 0$. Let F be the function defined by (5.3.10) with $F(x) \to \infty$ as $x \to 0^+$. Suppose further that γ_h obeys (5.3.2) and that Γ_h defined (5.3.3) obeys $\lambda_\Gamma(h) = 0$ where $\lambda_\Gamma(h)$ is defined by (5.3.7). Then any solution x_h of (5.3.1) obeys*

$$\lambda_x(h) = 0,$$

where $\lambda_x(h)$ is defined by (5.3.8).

Hence we arrive at the following result.

Theorem 5.3.5. *Let f obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3) where the function φ in (3.2.4) and (3.2.3) is in $C^1(0,\infty)$ and obeys $\varphi'(x) \to 0$ as $x \to 0$. Let F be the function defined by (5.3.10) with $F(x) \to \infty$ as $x \to 0^+$. Suppose further that γ_h obeys (5.3.2) and that Γ_h defined (5.3.3) and that $\lambda_\Gamma(h)$ is defined by (5.3.7). Let x_h be any solution of (5.3.1). Then*

(i) $\lambda_\Gamma(h) \in [0,1]$ *implies* $\lambda_x(h) = \lambda_\Gamma(h)$;

(ii) $\lambda_\Gamma(h) > 1$ *implies* $\lambda_x(h) \geq 1$

(iii) *If* $\sum_{j=0}^{n} \gamma_h(j+1)$ *does not converge, and* $x_h(n) \to 0$ *as* $n \to \infty$, *then* $\lambda_x(h) = 0$

Clearly, these discrete results are very closely related to the continuous results in Chapter 3.

We now investigate a special case in which the perturbed ODE (5.1.1) and its SSBE (5.3.1) have exactly the same asymptotic behaviour. Suppose

$$g \in C([0,\infty);(0,\infty) \text{ is a decreasing function with } g \in L^1(0,\infty). \tag{5.3.12}$$

Write

$$\gamma_h(n+1) = hg(nh), \quad n \geq 0. \tag{5.3.13}$$

Then $\sum_{j=0}^{\infty} \gamma_h(j) < +\infty$, and note that for the function Γ in (5.3.5) is well–defined along with the Liapunov–like exponent

$$\lambda_\Gamma = \liminf_{t\to\infty} \frac{F\left(\int_t^\infty g(s)\,ds\right)}{t}.$$

With the choice of γ_h, we have

$$\Gamma_h(n) = -\sum_{j=n}^{\infty} hg(jh), \quad n \geq 0 \tag{5.3.14}$$

We now show that the asymptotic behaviour of the discrete approximation Γ_h of Γ is recovered for some important classes of perturbation g.

Lemma 5.3.1. *Let f obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3) where the function φ in (3.2.4) and (3.2.3) is in $C^1(0,\infty)$ and obeys $\varphi'(x) \to 0$ as $x \to 0$. Let F be the function defined by (5.3.10) with $F(x) \to \infty$ as $x \to 0^+$.*

(i) Suppose g is continuous, positive, decreasing and in $L^1(0,\infty)$. Then Γ in (5.3.5) is well–defined and define

$$\lambda_\Gamma = \liminf_{t\to\infty} \frac{F\left(|\Gamma(t)|\right)}{t}. \tag{5.3.15}$$

If Γ_h is given by (5.3.14) then $\lambda_\Gamma(h)$ defined by (5.3.7) obeys

$$\lambda_\Gamma(h) = \lambda_\Gamma. \tag{5.3.16}$$

(ii) Suppose g is continuous, positive and in $L^1(0,\infty)$. Suppose further that g obeys

$$\lim_{t\to\infty} \max_{0\le s\le T} \left| \frac{g(t-s)}{g(t)} - 1 \right| = 0 \quad \textit{for all } T > 0, \tag{5.3.17}$$

and (5.3.15) holds. Then (5.3.16) holds.

The property (5.3.17) implies that g decays to zero *subexponentially*, in the sense that $\lim_{t\to\infty} e^{\epsilon t} g(t) = +\infty$ for all $\epsilon > 0$.

Proof. We consider first the proof of part (i). Let $a_n := \int_{nh}^{(n+1)h} g(s)\,ds$. If this holds, since

$$\frac{|\Gamma(nh)|}{|\Gamma_h(n)|} = \frac{\int_{nh}^\infty g(s)\,ds}{\sum_{j=n}^\infty hg(jh)} = \frac{\sum_{j=n}^\infty a_j}{\sum_{j=n}^\infty hg(jh)} \tag{5.3.18}$$

Next, as g is decreasing we have $hg((j+1)h) \le a_j \le hg(jh)$. Therefore

$$\sum_{j=n+1}^\infty hg(jh) = \sum_{j=n}^\infty hg((j+1)h) \le \sum_{j=n}^\infty a_j \le \sum_{j=n}^\infty hg(jh).$$

Thus from the last member of the inequality, we have $|\Gamma(nh)| \le |\Gamma_h(n)|$. But from the first member we have $|\Gamma_h(n+1)| \le |\Gamma(nh)|$. Since F is decreasing, we get

$$\frac{F(|\Gamma(nh)|)}{nh} \ge \frac{F(|\Gamma_h(n)|)}{nh}, \quad \frac{F(|\Gamma_h(n+1)|)}{(n+1)h} \cdot \frac{(n+1)h}{nh} \ge \frac{F(|\Gamma(nh)|)}{nh}.$$

Taking the liminf as $n \to \infty$ in the first inequalities gives $\lambda_\Gamma \ge \lambda_\Gamma(h)$, and in the second gives $\lambda_\Gamma(h) \ge \lambda_\Gamma$. Combining these inequalities gives $\lambda_\Gamma(h) = \lambda_\Gamma$, as required.

We consider next the proof of part (ii). Let a_n be defined as in the proof of part (i). We will presently show that $a_n \sim hg(nh)$ as $n \to \infty$. If this holds, since (5.3.18) is true holds, by Toeplitz lemma (see e.g., [74, Ch. 3, IV, Lemma 1]) we have we have $|\Gamma(nh)| \sim |\Gamma_h(n)|$ as $n \to \infty$. Since $F \in RV_0(0)$, it follows that $F(|\Gamma(nh)|) \sim F(|\Gamma_h(n)|)$ as $n \to \infty$. So

$$\lambda_\Gamma(h) = \liminf_{n\to\infty} \frac{F(|\Gamma_h(n)|)}{nh} = \liminf_{n\to\infty} \frac{F(|\Gamma(nh)|)}{nh} = \lambda_\Gamma,$$

as claimed. It remains to show $a_n \sim hg(nh)$ as $n \to \infty$. Note that $g((n+1)h) \sim g(nh)$ by applying (5.3.17) with $t = (n+1)h$ and $T = h$. Hence

$$\begin{aligned}\frac{a_n}{hg((n+1)h)} - 1 &= \int_{\alpha=0}^{1} \frac{g(nh+\alpha h)}{g((n+1)h)}\, d\alpha - 1 \\ &= \int_{\alpha=0}^{1} \left[\frac{g((n+1)h - (1-\alpha)h)}{g((n+1)h)} - 1\right] d\alpha.\end{aligned}$$

Thus

$$\begin{aligned}\left|\frac{a_n}{hg((n+1)h)} - 1\right| &\le \max_{0\le\alpha\le1} \left|\frac{g((n+1)h - (1-\alpha)h)}{g((n+1)h)} - 1\right| \\ &= \max_{0\le s\le h} \left|\frac{g((n+1)h - s)}{g((n+1)h)} - 1\right|.\end{aligned}$$

This tends to 0 as $n \to \infty$ by (5.3.17). Hence $a_n \sim hg((n+1)h) \sim hg(nh)$ as $n \to \infty$. □

This lemma leads automatically to the following result which clearly connects the behaviour of solutions of the ODE (5.1.1) and solutions of the SSBE (5.3.1).

Theorem 5.3.6. *Let f be increasing and obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3) where the function φ in (3.2.4) and (3.2.3) is in $C^1(0,\infty)$ and obeys $\varphi'(x) \to 0$ as $x \to 0$. Let F be the function defined by (5.3.10) with $F(x) \to \infty$ as $x \to 0^+$. Suppose further that g is continuous, positive, decreasing and in $L^1(0,\infty)$. Define*

$$\lambda_\Gamma = \liminf_{t\to\infty} \frac{F\left(\int_t^\infty g(s)\, ds\right)}{t}.$$

Let x be any solution of (5.1.1) and x_h be any solution of (5.3.1) with γ_h given by (5.3.13).

(i) If $\lambda_\Gamma \ge 1$, then

$$\lim_{t\to\infty} \frac{F(x(t))}{t} = 1, \quad \lim_{n\to\infty} \frac{F(x_h(n))}{nh} = 1.$$

(ii) If $\lambda_\Gamma \in [0,1]$, then

$$\liminf_{t\to\infty} \frac{F(x(t))}{t} = \lambda_\Gamma, \quad \liminf_{n\to\infty} \frac{F(x_h(n))}{nh} = \lambda_\Gamma.$$

All of this can be read off from existing results, apart from the second limit in (i). We now address this matter.

Theorem 5.3.7. *Suppose $f \in C([0,\infty);[0,\infty))$ obeys $f(0)=0$, $f(x)>0$ for $x>0$, f is increasing and is asymptotic to a function φ in $C^1(0,\infty)$ with $\varphi'(x)\to 0$ as $x\to 0^+$. Let $g\in C([0,\infty);(0,\infty))$. Let F be the function defined by (5.3.10) with $F(x)\to\infty$ as $x\to 0^+$. Suppose further that $x_h(0)=\zeta>0$, and let x_h be any solution of (5.3.1) with γ_h given by (5.3.13). Then $x_h(n)>0$ for all $n\geq 0$ and*

$$\limsup_{n\to\infty}\frac{F(x_h(n))}{nh}\leq 1.$$

Proof. We first show, with $\gamma_h(n+1)=hg(nh)$, that $x_h(n)>0$ for all $n\geq 0$. This is true for $n=0$ by hypothesis. Make the n–th level induction hypothesis

$$x_h(n)>0. \tag{5.3.19}$$

Suppose (5.3.19) is true. Then $x_h^*(n)\in(0,x_h(n))$ because $x_h^*(n)=x_h(n)-hf(x_h^*(n))$. But

$$x_h(n+1)=x_h^*(n)+\gamma_h(n+1)=x_h^*(n)+hg(nh)>x_h^*(n)>0,$$

so (5.3.19) holds at level $n+1$. Since $x_h(0)=\zeta>0$ by hypothesis, we have $x_h(n)>0$ for all $n\geq 0$. Next, let $y_h(0)=\zeta/2$ and define y_h by

$$y_h(n+1)=y_h(n)-hf(y_h(n+1)),\quad n\geq 0. \tag{5.3.20}$$

Let $y_h^*(n):=y_h(n+1)$, so that (y_h,y_h^*) obeys $y_h^*(n)=y_h(n)-hf(y_h^*(n))$, so $y_h(n+1)=y_h^*(n)$. Define $F_h(x):=x+hf(x)$ for $x\geq 0$. Since $h>0$ and f is increasing, F_h is increasing and has an increasing inverse. Note that $x_h^*(n)=F_h^{-1}(x_h(n))$ and $y_h^*(n)=F_h^{-1}(y_h(n))$ for all $n\geq 0$ by construction. Make the n–th level induction hypothesis

$$x_h(n)>y_h(n). \tag{5.3.21}$$

Note also $x_h(0)>y_h(0)$, so the hypothesis is true for $n=0$. If (5.3.21) is true at level n, then

$$x_h(n+1)=x_h^*(n)+\gamma_h(n+1)>x_h^*(n)>y_h^*(n)=y_h(n+1),.$$

so (5.3.21) is true at level $n+1$. Therefore

$$x_h(n)>y_h(n)\qquad\text{for all } n\geq 0. \tag{5.3.22}$$

It remains to show that

$$\lim_{n\to\infty}\frac{F(y_h(n))}{nh}=1. \tag{5.3.23}$$

Write $y_h(n) = y_h(n+1) + hf(y_h(n+1))$. Then by the mean value theorem

$$F(y_h(n)) = F(y_h(n+1)) + F'(\tilde{y}_n)hf(y_h(n+1))$$

where $\tilde{y}_n \in [y_h(n+1), y_h(n+1) + hf(y_h(n+1))]$. Thus

$$F(y_h(n)) = F(y_h(n+1)) - h\frac{f(y_h(n+1))}{f(\tilde{y}_n)}.$$

Now, write $\tilde{y}_n = y_h(n+1) + \theta_n hf(y_h(n+1))$, so that $\theta_n \in [0,1]$ and define

$$\lambda_n := \frac{\varphi(\tilde{y}_n)}{\varphi(y_h(n+1))}.$$

Notice, because $f(x) \sim \varphi(x)$ as $x \to 0^+$, that if $\lambda_n \to 1$ as $n \to \infty$, then $f(\tilde{y}_n)/f(y_h(n+1)) \to 1$ as $n \to \infty$ because $y_h(n) \to 0$ as $n \to \infty$, and $\tilde{y}_n \to 0$ as $n \to \infty$. Therefore, if $\lambda_n \to 1$ as $n \to \infty$

$$F(y_h(n+1)) - F(y_h(n)) = h\frac{f(y_h(n+1))}{f(\tilde{y}_n)} \to h \quad \text{as} \quad n \to \infty.$$

From this limit (5.3.23) follows automatically. Finally, by the mean value Theorem

$$\begin{aligned}\lambda_n &= \frac{\varphi\left(y_h(n+1) + \theta_n hf(y_h(n+1))\right)}{\varphi(y_h(n+1)} \\ &= \frac{\varphi(y_h(n+1)) + \varphi'(\bar{y}_n)\theta_n hf(y_h(n+1))}{\varphi(y_h(n+1))},\end{aligned}$$

where $\bar{y}_n \in [y_h(n+1), \tilde{y}_n]$. Since $\bar{y}_n \to 0$ as $n \to \infty$, $\varphi'(\bar{y}_n) \to 0$ as $n \to \infty$. Also, $f(y_h(n+1))/\varphi(y_h(n+1)) \to 1$ as $n \to \infty$ and $\theta_n \in [0,1]$ so $\lambda_n \to 1$ as $n \to \infty$. Hence the proof of (5.3.23) is complete. But by (5.3.22), we have $F(x_h(n)) < F(y_h(n))$ for all $n \geq 0$. Thus by (5.3.23), we have

$$\limsup_{n\to\infty} \frac{F(x_h(n))}{nh} \leq 1,$$

as claimed. □

5.4 Proofs

5.4.1 Proof of Theorem 5.3.1

The proof consists of three main steps; the second step is the most difficult to achieve.

Step 1: $\lambda_\Gamma(h) \geq 1$ implies $\Lambda_z(h) = \limsup_{n\to\infty} \frac{F(|z_h(n)|)}{nh} \geq 1$.

Step 2: $\lambda_\Gamma(h) \geq 1$ implies $\lambda_z(h) = \liminf_{n\to\infty} \frac{F(|z_h(n)|)}{nh} \geq 1$.

Step 3: $\lambda_\Gamma(h) \geq 1$ and $\lambda_z(h) \geq 1$ implies $\lambda_x(h) \geq 1$.

Lemma 5.4.1. *Let f obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3) where the function φ in (3.2.4) and (3.2.3) is in $C^1(0,\infty)$ and obeys $\varphi'(x) \to 0$ as $x \to 0$. Let F be the function defined by (5.3.10) with $F(x) \to \infty$ as $x \to 0^+$. Suppose further that γ_h obeys (5.3.2) and that Γ_h defined (5.3.3) and that $\lambda_\Gamma(h)$ is defined by (5.3.7) obeys $\lambda_\Gamma(h) \geq 1$. Let z_h be any solution of (5.3.4). Then*

$$\Lambda_z(h) := \limsup_{n\to\infty} \frac{F\left(|z_h(n)|\right)}{nh} \geq 1.$$

Proof. Suppose

$$M = \limsup_{n\to\infty} \frac{F\left(|z_h(n)|\right)}{nh} \geq 1.$$

Then, by supposition we have that $F(|\Gamma_h(n)|)/nh > 1-\epsilon$ for all $n \geq N_1(\epsilon)$ and $F\left(|z_h(n+1)|\right)/nh < M+\epsilon$ for all $n \geq N_2(\epsilon)$ and as $M < \lambda_\Gamma(h)$ we suppose $\epsilon > 0$ is so small that $M+\epsilon < 1-\epsilon$. Now

$$\limsup_{n\to\infty} \frac{F\left(|z_h(n+1)|\right)}{nh} = \limsup_{n\to\infty} \frac{F\left((|z_h(n+1)|\right)}{(n+1)h}\frac{(n+1)h}{nh} = M.$$

Hence $|z_h(n+1)| > F^{-1}\left((M+\epsilon)nh\right)$ and $|\Gamma_h(n)| < F^{-1}\left((1-\epsilon)nh\right)$ for all $n \geq N_3(\epsilon) = \max\left(N_1(\epsilon), N_2(\epsilon)\right)$. Also, for $n \geq N_3(\epsilon)$

$$\left|\frac{\Gamma_h(n))}{z_h(n+1)}\right| < \frac{F^{-1}\left((1-\epsilon)nh\right)}{F^{-1}\left((M+\epsilon)nh\right)}.$$

Since F^{-1} is rapidly varying, we have $\mu_h(n) = 1 + \Gamma_h(n)/z_h(n+1) \to 0$ as $n \to \infty$. Therefore

$$\frac{z_h(n) - z_h(n+1)}{f(z_h(n+1))} = \frac{f(\mu_h(n)z_h(n+1)}{f(z_h(n+1))} \to h, \quad n \to \infty.$$

Therefore we may write

$$z_h(n) - z_h(n+1) = \tilde{h}_n f(z_h(n+1)), \quad h_n \to h, \quad n \to \infty.$$

Now we may use the same argument as used to deduce (5.3.23) from (5.3.20) to get

$$\lim_{n\to\infty} \frac{F\left(|z_h(n)|\right)}{nh} = 1.$$

But this implies

$$1 > M = \limsup_{n\to\infty} \frac{F\left(|z_h(n)|\right)}{nh} = \lim_{n\to\infty} \frac{F\left(|z_h(n)|\right)}{nh} = 1,$$

a contradiction. Hence if $z_h(n) > 0$, we arrive at a contradiction. Similarly, a contradiction results when $z_h(n) < 0$ for all $n \geq N_5(\epsilon)$. Therefore, the assumption $M < 1$ is false, which implies that we must have

$$\limsup_{n\to\infty} \frac{F(|z_h(n)|)}{nh} > 1,$$

as required. □

The next result is the most crucial (and most complicated) one.

Lemma 5.4.2. *Let f obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3) where the function φ in (3.2.4) and (3.2.3) is in $C^1(0,\infty)$ and obeys $\varphi'(x) \to 0$ as $x \to 0$. Let F be the function defined by (5.3.10) with $F(x) \to \infty$ as $x \to 0^+$. Suppose further that γ_h obeys (5.3.2) and that Γ_h defined (5.3.3) and that $\lambda_\Gamma(h)$ is defined by (5.3.7) obeys $\lambda_\Gamma(h) \geq 1$. Let z_h be any solution of (5.3.4). Then $\lambda_z(h)$ defined by (5.3.9) obeys*

$$\lambda_z(h) \geq 1.$$

Proof. Define

$$\Phi_\epsilon := \liminf_{x\to 0^+} \frac{\varphi((1-\epsilon)x)}{\varphi(x)},$$

and, by hypothesis we have $\lim_{\epsilon\to 0^+} \Phi_\epsilon = 1$. Since φ is increasing, if $\eta < \epsilon < 1$ then $\Phi_\epsilon \leq \Phi_\eta \leq 1$. Hence for every $\eta \in (0,1)$, there is is $\tilde{x}_2(\eta,\epsilon) > 0$ such that

$$\frac{\varphi((1-\epsilon)x)}{\varphi(x)} > \Phi_\epsilon - \eta, \quad x < \tilde{x}_2(\eta,\epsilon).$$

Take $\eta = \eta(\epsilon) = \Phi_\epsilon \epsilon/2$, so that for $x < x_2(\epsilon) := \tilde{x}_2(\eta(\epsilon),\epsilon)$

$$\frac{\varphi((1-\epsilon)x)}{\varphi(x)} > \Phi_\epsilon(1-\epsilon/2).$$

We now construct for each $\epsilon \in (0,1)$ sufficiently small an integer $N^*(\epsilon)$. Since $z_h(n) \to 0$ as $n \to \infty$, $\Gamma_h(n) \to 0$ as $n \to \infty$ and $f(x) \sim \varphi(x)$ as $x \to 0$, there is $N_0(\epsilon) > 0$ such that

$$1 - \frac{\epsilon}{2} < \frac{f(z_h(n+1) + \Gamma_h(n))}{\varphi(z_h(n+1) + \Gamma_h(n))} < 1 + \frac{\epsilon}{2}, \quad n \geq N_0(\epsilon).$$

Since by the hypothesis $\lambda_\Gamma(h) \geq 1$, we have that for all $\epsilon > 0$ sufficiently small that there is an $N_1(\epsilon) > 0$ such that $|\Gamma_h(n)| < \Phi^{-1}(\Phi_\epsilon(1-\epsilon/2)(n+1)h)$ for $n \geq N_1(\epsilon)$, or

$$-\Phi^{-1}(\Phi_\epsilon(1-\epsilon/2)(n+1)h) < \Gamma_h(n) < \Phi^{-1}(\Phi_\epsilon(1-\epsilon/2)(n+1)h), \quad n \geq N_1(\epsilon).$$

Let $N_1'(\epsilon) = \max(N_0(\epsilon), N_1(\epsilon))$.

Next as Φ^{-1} is rapidly varying, for every $\epsilon \in (0,1)$ sufficiently small there is an $x_3(\epsilon) > 0$ such that

$$\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)x\right) < \frac{\epsilon}{2}\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)^2 x\right), \quad x > x_3(\epsilon).$$

Let $N_2(\epsilon)$ be so large that $(n+1)h > x_3(\epsilon)$ for all $n \geq N_2(\epsilon)$. Then

$$\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)(n+1)h\right) < \frac{\epsilon}{2}\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)^2(n+1)h\right), \quad n \geq N_2(\epsilon). \qquad (5.4.1)$$

Since $\Phi^{-1}(t) \to 0$ as $t \to \infty$, for every $\epsilon \in (0,1)$ sufficiently small there exists an $N_3(\epsilon)$ such that $\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)^2 nh\right) < x_2(\epsilon)$ for all $n \geq N_3(\epsilon)$.

Since $\Phi^{-1}(\lambda(\epsilon)nh) \to 0$ as $n \to \infty$, and $\varphi(x)/x \to 0$ as $x \to 0^+$, it follows that there is $N_4(\epsilon) > 0$ such that

$$\frac{\varphi\left(\Phi^{-1}(\lambda(\epsilon)nh)\right)}{\Phi^{-1}(\lambda(\epsilon)nh)} < \frac{\epsilon}{4h}, \quad n \geq N_4(\epsilon). \qquad (5.4.2)$$

Now, let $N''(\epsilon) > \max_{j=0,1,2,3,4} N_j(\epsilon)$.

Define next $\lambda(\epsilon) = \Phi_\epsilon(1-\epsilon/2)^2$. Since $\Phi(x) \sim F(x)$ as $x \to 0$, by Lemma 5.4.1 we have

$$\limsup_{n\to\infty} \frac{\Phi\left(|z_h(n)|\right)}{nh} \geq 1.$$

It therefore follows for every $\epsilon \in (0,1)$ sufficiently small that there exists a sequence $n_j(\epsilon) \nearrow \infty$ such that for all $j \geq j^*(\epsilon)$ we have

$$\frac{\Phi\left(|z_h(n_j)|\right)}{n_j h} > \lambda(\epsilon),$$

or

$$|z_h(n_j)| < \Phi^{-1}\left(\lambda(\epsilon)n_j h\right).$$

Now, take $j^{**}(\epsilon)$ such that $n_{j^{**}(\epsilon)} > N''(\epsilon)$. Then, for $n_{j^{**}(\epsilon)} > N''(\epsilon)$, we have

$$|z_h(n_{j^{**}(\epsilon)})| < \Phi^{-1}\left(\lambda(\epsilon)n_{j^{**}(\epsilon)}h\right). \qquad (5.4.3)$$

Now, define $N^*(\epsilon) := n_{j^{**}(\epsilon)}$. Note that $N^*(\epsilon) > N''(\epsilon)$.

Most of the rest of the proof is devoted to showing that

$$\text{If } |z_h(n)| \leq \Phi^{-1}(\lambda(\epsilon)nh) \quad \text{for some} \quad n \geq N^*(\epsilon),$$
$$\text{then } |z_h(n+1)| \leq \Phi^{-1}(\lambda(\epsilon)(n+1)h). \qquad (5.4.4)$$

If (5.4.4) holds, it becomes relatively straightforward to secure the desired asymptotic behaviour. Clearly, by (5.4.3), for $n = N^*(\epsilon)$ the first statement in (5.4.4) holds.

Therefore $|z_h(n+1)| \leq \Phi^{-1}(\lambda(\epsilon)(n+1)h)$. By induction, it is immediate that

$$|z_h(n)| \leq \Phi^{-1}(\lambda(\epsilon)nh), \quad \text{for all } n \geq N^*(\epsilon).$$

Hence $\Phi(|z_h(n)|) \geq \lambda(\epsilon)nh$ for all $n \geq N^*(\epsilon)$. Letting $n \to \infty$ yields

$$\liminf_{n\to\infty} \frac{\Phi(|z_h(n)|)}{nh} \geq \lambda(\epsilon).$$

Thus by letting $\epsilon \to 0^+$, and using $\lambda(\epsilon) \to 1$ as $\epsilon \to 0^+$, we have that

$$\liminf_{n\to\infty} \frac{\Phi(|z_h(n)|)}{nh} \geq 1,$$

as needed.

To prove (5.4.4), we force a contradiction. Suppose, contrary to (5.4.4) that

$$\text{There exists } n \geq N^*(\epsilon) \text{ such that } |z_h(n)| \leq \Phi^{-1}\left(\lambda(\epsilon)nh\right), \\ \text{and } |z_h(n+1)| > \Phi^{-1}\left(\lambda(\epsilon)(n+1)h\right). \quad (5.4.5)$$

Our goal will be to show that (5.4.5) is impossible.

We first consider the case that

$$\text{There exists } n \geq N^*(\epsilon) \text{ such that } |z_h(n)| \leq \Phi^{-1}\left(\lambda(\epsilon)nh\right), \\ \text{and } z_h(n+1) > \Phi^{-1}\left(\lambda(\epsilon)(n+1)h\right). \quad (5.4.6)$$

Later we consider the case when

$$\text{There exists } n \geq N^*(\epsilon) \text{ such that } |z_h(n)| \leq \Phi^{-1}\left(\lambda(\epsilon)nh\right), \\ \text{and } z_h(n+1) < -\Phi^{-1}\left(\lambda(\epsilon)(n+1)h\right). \quad (5.4.7)$$

Let us now show that (5.4.6) is impossible. Start with the estimate

$$0 = z_h(n+1) - z_h(n) + hf\left(z_h(n+1) + \Gamma_h(n)\right) > hf\left(z_h(n+1) + \Gamma_h(n)\right) \\ + \Phi^{-1}\left(\lambda(\epsilon)(n+1)h\right) - \Phi^{-1}\left(\lambda(\epsilon)nh\right). \quad (5.4.8)$$

Since $n \geq N^*(\epsilon) > N'(\epsilon)$, we have that

$$z_h(n+1) + \Gamma_h(n) \\ > \Phi^{-1}\left(\Phi_\epsilon\left(1-\frac{\epsilon}{2}\right)^2(n+1)h\right) - \Phi^{-1}\left(\Phi_\epsilon\left(1-\frac{\epsilon}{2}\right)(n+1)h\right) > 0,$$

because Φ^{-1} is decreasing. Therefore

$$\begin{aligned} &f\left(z_h(n+1)+\Gamma_h(n)\right) \\ &> \left(1-\frac{\epsilon}{2}\right)\varphi\left(z_h(n+1)+\Gamma_h(n)\right) \\ &> \left(1-\frac{\epsilon}{2}\right)\varphi\left(\Phi^{-1}\left(\lambda(\epsilon)(n+1)h\right)-\Phi^{-1}\left(\Phi_\epsilon\left(1-\frac{\epsilon}{2}\right)(n+1)h\right)\right). \end{aligned} \tag{5.4.9}$$

Thus by (5.4.8) and (5.4.9) we have

$$\begin{aligned} 0 &> \Phi^{-1}\left(\lambda(\epsilon)(n+1)h\right)-\Phi^{-1}\left(\lambda(\epsilon)nh\right) \\ &+ h(1-\epsilon/2)\varphi\left(\Phi^{-1}\left(\lambda(\epsilon)(n+1)h\right)-\Phi^{-1}\left(\Phi_\epsilon\left(1-\epsilon/2\right)(n+1)h\right)\right). \end{aligned} \tag{5.4.10}$$

Hence by the mean value theorem there exists a $\theta_n \in [0,1]$ such that

$$\begin{aligned} 0 &> \left(\Phi^{-1}\right)'\left(\Phi_\epsilon(1-\epsilon/2)^2(n+\theta_n)h\right)\Phi_\epsilon(1-\epsilon/2)^2h \\ &+ h(1-\epsilon/2)\varphi\left(\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)^2(n+1)h\right)-\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)(n+1)h\right)\right) \\ &= -\varphi\left(\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)^2(n+\theta_n)h\right)\right)\Phi_\epsilon(1-\epsilon/2)^2h \\ &+ h(1-\epsilon/2)\varphi\left(\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)^2(n+1)h\right)-\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)(n+1)h\right)\right). \end{aligned}$$

This yields

$$\begin{aligned} &\Phi_\epsilon(1-\epsilon/2)\varphi\left(\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)^2(n+\theta_n)h\right)\right) \\ &\qquad > \varphi\left(\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)^2(n+1)h\right)-\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)(n+1)h\right)\right). \end{aligned}$$

Hence as $\theta_n \in [0,1]$ we have

$$\begin{aligned} &\Phi_\epsilon(1-\epsilon/2)\varphi\left(\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)^2nh\right)\right) \\ &\qquad > \varphi\left(\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)^2(n+1)h\right)-\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)(n+1)h\right)\right). \end{aligned} \tag{5.4.11}$$

Notice that the inequality (5.4.11) is a simpler consequence of (5.4.5), as it does not depend on Γ_h, z_h or θ_n.

Since $N^*(\epsilon) > N_2(\epsilon)$, for $n \geq N_2(\epsilon)$, by combining (5.4.1) with (5.4.11), we get

$$\begin{aligned} &\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)^2(n+1)h\right)-\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)(n+1)h\right) \\ &\qquad > (1-\epsilon/2)\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)^2(n+1)h\right) > 0, \end{aligned}$$

and so as φ is increasing, we have that

$$\varphi\left(\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)^2(n+1)h\right)-\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)(n+1)h\right)\right) > \varphi\left((1-\epsilon/2)\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)^2(n+1)h\right)\right).$$

Thus as $n \geq N^*(\epsilon)$ we have

$$\Phi_\epsilon(1-\epsilon/2)\varphi\left(\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)^2 nh\right)\right) > \varphi\left((1-\epsilon/2)\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)^2(n+1)h\right)\right). \quad (5.4.12)$$

This estimate is simpler than (5.4.11); however, if we could strip away the prefactor on the left hand side, by monotonicity of φ, a simpler consequence of (5.4.12) could emerge. The next estimates acheive this.

Since $N^*(\epsilon) > N_3(\epsilon)$, for $n \geq N^*(\epsilon)$ we have

$$\varphi\left((1-\epsilon)\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)^2 nh\right)\right) > \Phi_\epsilon(1-\epsilon/2)\varphi\left(\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)^2 nh\right)\right). \quad (5.4.13)$$

Hence by (5.4.12) and (5.4.13)

$$\varphi\left((1-\epsilon)\Phi^{-1}\left(\lambda(\epsilon)nh\right)\right) > \varphi\left((1-\epsilon/2)\Phi^{-1}\left(\lambda(\epsilon)(n+1)h\right)\right).$$

Since φ is increasing, we get for $n \geq N^*(\epsilon)$ that

$$(1-\epsilon)\Phi^{-1}\left(\lambda(\epsilon)nh\right) > (1-\epsilon/2)\Phi^{-1}\left(\lambda(\epsilon)(n+1)h\right). \quad (5.4.14)$$

The next estimates seek to equalise the arguments in inequalities resulting from (5.4.14). In particular, we wish to end up with a multiple of nh in the argument on the right hand side. To do this, note by the mean value theorem that there exists a $\theta_n \in [0,1]$ such that

$$\begin{aligned}
&\Phi^{-1}\left(\lambda(\epsilon)(n+1)h\right) \\
&= \Phi^{-1}\left(\lambda(\epsilon)nh\right) + \left(\Phi^{-1}\right)'\left(\Phi_\epsilon(1-\epsilon/2)^2(n+\theta_n)h\right)\Phi_\epsilon(1-\epsilon/2)^2 h \\
&= \Phi^{-1}\left(\lambda(\epsilon)nh\right) - \varphi\left(\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)^2(n+\theta_n)h\right)\right)\Phi_\epsilon(1-\epsilon/2)^2 h.
\end{aligned}$$

Thus from (5.4.14), for $n \geq N^*(\epsilon)$ we have

$$(1-\epsilon)\Phi^{-1}\left(\lambda(\epsilon)nh\right) > (1-\epsilon/2)\Phi^{-1}\left(\lambda(\epsilon)nh\right) - (1-\epsilon/2)^3\Phi_\epsilon h\varphi\left(\Phi^{-1}\left(\lambda(\epsilon)(n+\theta_n)h\right)\right).$$

Rearranging this inequality gives

$$(1-\epsilon/2)^3\Phi_\epsilon h\varphi\left(\Phi^{-1}\left(\lambda(\epsilon)(n+\theta_n)h\right)\right) > \frac{\epsilon}{2}\Phi^{-1}\left(\lambda(\epsilon)nh\right)$$

for $n \geq N^*(\epsilon)$. Since $\theta_n \geq 0$, φ is increasing and Φ^{-1} is decreasing, we have

$$(1-\epsilon/2)^3\Phi_\epsilon h\varphi\left(\Phi^{-1}\left(\lambda(\epsilon)nh\right)\right) > \frac{\epsilon}{2}\Phi^{-1}\left(\lambda(\epsilon)nh\right).$$

Hence, we have, as $\Phi_\epsilon \leq 1$

$$\frac{\varphi\left(\Phi^{-1}(\lambda(\epsilon)nh)\right)}{\Phi^{-1}\left(\lambda(\epsilon)nh\right)} > \frac{\epsilon/2}{(1-\epsilon/2)^3\Phi_\epsilon h} > \frac{\epsilon}{2h} \tag{5.4.15}$$

for $n \geq N^*(\epsilon)$. Since $N^*(\epsilon) > N_4(\epsilon)$, it now follows that (5.4.15) and (5.4.2) are contradictory. Hence we have arrived at a contradiction to (5.4.6).

Now we wish to examine (5.4.7). That is, we want to show that if there is an $N^*(\epsilon)$ such that

There exists $n \geq N^*(\epsilon)$ such that $|z_h(n)| \leq \Phi^{-1}\left(\lambda(\epsilon)nh\right)$,

and $z_h(n+1) < -\Phi^{-1}\left(\lambda(\epsilon)(n+1)h\right)$,

a contradiction will again result.

Let $n \geq N^*(\epsilon)$ be such that the last inequality holds. Then we have $-\Phi^{-1}\left(\lambda(\epsilon)nh\right) \leq z_h(n) \leq \Phi^{-1}\left(\lambda(\epsilon)nh\right)$. Thus, as $N^*(\epsilon) > N_1'(\epsilon)$, we have

$$z_h(n+1) + \Gamma_h(n) < -\Phi^{-1}\left(\lambda(\epsilon)(n+1)h\right) + \Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)(n+1)h\right) < 0,$$

because $\lambda(\epsilon) < \Phi_\epsilon(1-\epsilon/2)$. Therefore, as φ is odd and increasing, we have

$$\begin{aligned}
&\varphi\left(z_h(n+1) + \Gamma_h(n)\right) \\
&< \varphi\left(-\Phi^{-1}\left(\lambda(\epsilon)(n+1)h\right) + \Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)(n+1)h\right)\right) \\
&= -\varphi\left(-\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)(n+1)h\right) + \Phi^{-1}\left(\lambda(\epsilon)(n+1)h\right)\right).
\end{aligned}$$

Since $N^*(\epsilon) > N_1'(\epsilon)$, $f(z_h(n+1)+\Gamma_h(n))/\varphi(z_h(n+1)+\Gamma_h(n)) > 1-\epsilon/2$ and $\varphi(z_h(n+1)+\Gamma_h(n)) < 0$, then we have $f(z_h(n+1)+\Gamma_h(n)) < (1-\epsilon/2)\varphi(z_h(n+1)+\Gamma_h(n))$. Hence

$$\begin{aligned}
f\left(z_h(n+1) + \Gamma_h(n)\right) &< (1-\epsilon/2)\varphi\left(z_h(n+1) + \Gamma_h(n)\right) \\
&< -(1-\epsilon/2)\varphi\left(-\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)(n+1)h\right) + \Phi^{-1}\left(\lambda(\epsilon)(n+1)h\right)\right).
\end{aligned}$$

Thus

$$\begin{aligned}
&-\Phi^{-1}\left(\lambda(\epsilon)(n+1)h\right)+\Phi^{-1}\left(\lambda(\epsilon)nh\right)\\
&> z_h(n+1)-z_h(n)\\
&= -hf\left(z_h(n+1)+\Gamma_h(n)\right)\\
&> h(1-\epsilon/2)\varphi\left(-\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)(n+1)h\right)+\Phi^{-1}\left(\lambda(\epsilon)(n+1)h\right)\right).
\end{aligned}$$

Rearranging this inequality, we get

$$\begin{aligned}
0 &> \Phi^{-1}\left(\lambda(\epsilon)(n+1)h\right)-\Phi^{-1}\left(\lambda(\epsilon)nh\right)\\
&\qquad + h(1-\epsilon/2)\varphi\left(-\Phi^{-1}\left(\Phi_\epsilon(1-\epsilon/2)(n+1)h\right)+\Phi^{-1}\left(\lambda(\epsilon)(n+1)h\right)\right).
\end{aligned}$$

This inequality is precisely (5.4.10). Since this inequality leads to a contradiction, it follows that it is impossible for there to be an $n \geq N^*(\epsilon)$ such that $|z_h(n)| \leq \Phi^{-1}(\lambda(\epsilon)nh)$ and $z_h(n+1) \geq -\Phi^{-1}\left(\lambda(\epsilon)(n+1)h\right)$, and (5.4.7) is impossible. Since both (5.4.6) and (5.4.7) are impossible, (5.4.5) is impossible, and we have the desired contradiction. The proof is therefore complete. □

Lemma 5.4.3. *Let f obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3) where the function φ in (3.2.4) and (3.2.3) is in $C^1(0,\infty)$ and obeys $\varphi'(x) \to 0$ as $x \to 0$. Let F be the function defined by (5.3.10) with $F(x) \to \infty$ as $x \to 0^+$. Suppose further that γ_h obeys (5.3.2) and that Γ_h defined (5.3.3) and that $\lambda_\Gamma(h)$ is defined by (5.3.7) obeys $\lambda_\Gamma(h) \geq 1$. Let x_h be any solution of (5.3.1). Then $\lambda_x(h)$ defined by (5.3.8) obeys*

$$\lambda_x(h) \geq 1.$$

Proof. Let x_h be any solution of (5.3.1). Since $\Gamma_h(n) \to 0$ as $n \to \infty$, $\lambda_\Gamma(h)$ is defined by (5.3.7) is well defined, and by hypothesis obeys $\lambda_\Gamma(h) \geq 1$. Define $z_h(n) = x_h(n) - \Gamma_h(n)$ for $n \geq 0$. Then z_h obeys (5.3.4). Therefore, by Lemma 5.4.2, we have that $\lambda_\Gamma(h) \geq 1$ implies $\lambda_z(h) \geq 1$.

Therefore for every $\epsilon \in (0,1)$ we have that $F\left(|z_h(n)|\right) \geq (1-\epsilon)nh$ for all $n \geq N_1(\epsilon)$ and $F\left(|\Gamma_h(n)|\right) \geq (1-\epsilon)nh$ for all $n \geq N_2(\epsilon)$. Let $N_3(\epsilon) = \max\left(N_1(\epsilon), N_2(\epsilon)\right)$. Then $|z_h(n)| \leq F^{-1}\left((1-\epsilon)nh\right)$ and $|\Gamma_h(n)| \leq F^{-1}\left((1-\epsilon)nh\right)$ for all $n \geq N_3(\epsilon)$. Now $x_h(n) = z_h(n) + \Gamma_h(n)$, so $|x_h(n)| \leq 2F^{-1}\left((1-\epsilon)nh\right)$ for all $n \geq N_3(\epsilon)$. Thus $1/2|x_h(n)| \leq F^{-1}\left((1-\epsilon)nh\right)$, so $F\left(1/2|x_h(n)|\right) \geq (1-\epsilon)nh$. Therefore

$$\liminf_{n\to\infty} \frac{F\left(1/2|x_h(n)|\right)}{nh} \geq 1.$$

Since $x_h(n) \to 0$ as $n \to \infty$ and $F \in RV_0(0)$, we have

$$\liminf_{n\to\infty} \frac{F\left(|x_h(n)|\right)}{nh} = \liminf_{n\to\infty} \frac{F\left(|x_h(n)\right)}{F\left(1/2|x_h(n)|\right)} \frac{F\left(1/2|x_h(n)|\right)}{nh} \geq 1,$$

as required. □

5.4.2 Proof of Theorem 5.3.2

Proof. By hypothesis for every $\epsilon \in (0,1)$ there is $N_1(\epsilon) > 0$ such that

$$|x_h(n)| < \Phi^{-1}\left(\lambda_x(h)(1-\epsilon)(n+1)h\right), \quad n \geq N_1(\epsilon). \tag{5.4.16}$$

By the form of the split step method (5.3.1), $|x_h^*(n)| \leq |x_h(n)|$. Moreover, we note that (a) if $x_h(n) > 0$ then $0 < x_h^*(n) < x_h(n)$; (b) if $x_h(n) < 0$ then $x_h(n) < x_h^*(n) < 0$; and (c) if $x_h(n) = 0$ then $x_h^*(n) = 0$. Thus $|x_h^*(n)| \leq \Phi^{-1}\left(\lambda_x(h)(1-\epsilon)(n+1)h\right)$ for all $n \geq N_1(\epsilon)$. From putting the first equation in (5.3.1) into the second, we get

$$x_h(n+1) - x_h(n) = -hf\left(x^*(n)\right) + \gamma_h(n+1), \quad n \geq 0,$$

and summing on both sides and telescoping on the left yields

$$x_h(n+1) = x_h(0) - h\sum_{j=0}^{n} f\left(x_h^*(j)\right) + \sum_{j=0}^{n}\gamma_h(j+1). \tag{5.4.17}$$

Therefore, as $x_h(n) \to 0$ as $n \to \infty$, if $\sum_{j=0}^{n} f\left(x_h^*(j)\right)$ tends to a finite limit as $n \to \infty$, then $\sum_{j=0}^{n}\gamma_h(j+1)$ tends to a finite limit as $n \to \infty$. Now, since $f(x) \sim \varphi(x)$ as $x \to 0$ for every $\epsilon \in (0,1)$ there exists an $x_1(\epsilon) > 0$ such that

$$1-\epsilon < \frac{f(x)}{\varphi(x)} < 1+\epsilon, \quad |x| < x_1(\epsilon).$$

Moreover, there is an $N_2(\epsilon)$ such that $|x_h^*(j)| < x_1(\epsilon)$ for all $j > N_2(\epsilon)$. Let $N_3(\epsilon) = \max\left(N_1(\epsilon), N_2(\epsilon)\right)$. If $x_h^*(n) > 0$ then $0 < f\left(x_h^*(n)\right) < (1+\epsilon)\varphi\left(x_h^*(n)\right)$ and so $f\left(|x_h^*(n)|\right) < (1+\epsilon)\varphi\left(|x_h^*(n)|\right)$ for all $n \geq N_3(\epsilon)$. Also, if $x_h^*(n) < 0$ then $1+\epsilon > f\left(x_h^*(n)\right)/\varphi\left(x_h^*(n)\right) > 1-\epsilon$, and $\varphi\left(x_h^*(n)\right) < 0$. Thus $(1+\epsilon)\varphi\left(x_h^*(n)\right) < f\left(x_h^*(n)\right) < (1-\epsilon)\varphi\left(x_h^*(n)\right) < 0$ which implies that $|f\left(x_h^*(n)\right)| < (1+\epsilon)|\varphi\left(x_h^*(n)\right)| = (1+\epsilon)\varphi\left(|x_h^*(n)|\right)$. Either way we get that $|f\left(x_h^*(n)\right)| < (1+\epsilon)\varphi\left(|x_h^*(n)|\right) < (1+\epsilon)\varphi\left(\Phi^{-1}\left(\lambda_x(h)(1-\epsilon)(n+1)h\right)\right)$ for all $n \geq N_3(\epsilon)$. Hence

$$|hf\left(x_h^*(n)\right)| \leq h(1+\epsilon)\left(\varphi\circ\Phi^{-1}\right)\left(\lambda_x(h)(1-\epsilon)(n+1)h\right), \quad n \geq N_3(\epsilon).$$

Let $t \in [nh, (n+1)h]$. Then as Φ^{-1} is decreasing, and φ increasing, we have

$$\begin{aligned}\left(\varphi\circ\Phi^{-1}\right)\left(\lambda_x(h)(1-\epsilon)nh\right) &\geq \left(\varphi\circ\Phi^{-1}\right)\left(\lambda_x(h)(1-\epsilon)t\right)\\ &\geq \left(\varphi\circ\Phi^{-1}\right)\left(\lambda_x(h)(1-\epsilon)(n+1)h\right).\end{aligned}$$

Integrating gives

$$h\left(\varphi\circ\Phi^{-1}\right)(\lambda_x(h)(1-\epsilon)nh) \geq \int_{nh}^{(n+1)h}\left(\varphi\circ\Phi^{-1}\right)(\lambda_x(h)(1-\epsilon)t)\,dt$$
$$\geq h\left(\varphi\circ\Phi^{-1}\right)(\lambda_x(h)(1-\epsilon)(n+1)h).$$

Hence we have for $n \geq N_3(\epsilon)$

$$|hf(x_h^*(n))| \leq (1+\epsilon)\int_{nh}^{(n+1)h}\left(\varphi\circ\Phi^{-1}\right)(\lambda_x(h)(1-\epsilon)t)\,dt. \tag{5.4.18}$$

But

$$\sum_{n=0}^{\infty}\int_{nh}^{(n+1)h}\left(\varphi\circ\Phi^{-1}\right)(\lambda_x(h)(1-\epsilon)t)\,dt = \int_{t=0}^{\infty}\left(\varphi\circ\Phi^{-1}\right)(\lambda_x(h)(1-\epsilon)t)\,dt$$
$$= \frac{1}{\lambda_x(h)(1-\epsilon)}.$$

Thus $|hf(x_h^*(n))|$ is summable, and hence $\sum_{j=0}^{n}\gamma_h(j+1) \to \gamma^* \in (-\infty,\infty)$ as $n\to\infty$. We write $\sum_{j=0}^{\infty}\gamma_h(j+1) = \gamma^*$. Next, letting $n\to\infty$ in (5.4.17) yields

$$0 = x(0) - \sum_{j=0}^{\infty} hf(x_h^*(j)) + \sum_{j=0}^{\infty}\gamma_h(j+1).$$

Subtracting (5.4.17) gives, for $n \geq 0$,

$$x_h(n+1) = \sum_{j=n+1}^{\infty} hf(x_h^*(j)) + \Gamma_h(n+1).$$

Hence

$$x_h(n) = \sum_{j=n}^{\infty} hf(x_h^*(j)) + \Gamma_h(n), \quad n\geq 1. \tag{5.4.19}$$

Therefore

$$|\Gamma_h(n)| \leq |x_h(n)| + \sum_{j=n}^{\infty}|hf(x_h^*(j))|.$$

Therefore for $n \geq N_3(\epsilon)$ using (5.4.16) and (5.4.18)

$$|\Gamma_h(n)| \leq \Phi^{-1}(\lambda_x(h)(1-\epsilon)(n+1)h)$$
$$+ (1+\epsilon)\int_{nh}^{\infty}\left(\varphi\circ\Phi^{-1}\right)(\lambda_x(h)(1-\epsilon)t)\,dt$$
$$= \Phi^{-1}(\lambda_x(h)(1-\epsilon)(n+1)h)$$
$$+ (1+\epsilon)\int_0^{\Phi^{-1}((1-\epsilon)nh)}\varphi(x)\frac{1}{\varphi(x)}\frac{1}{\lambda_x(h)(1-\epsilon)}\,dx.$$

Thus for $n \geq N_3(\epsilon)$

$$|\Gamma_h(n)| \leq \Phi^{-1}\left(\lambda_x(h)(1-\epsilon)(n+1)h\right) + \frac{1+\epsilon}{\lambda_x(h)(1-\epsilon)}\Phi^{-1}\left(\lambda_x(h)(1-\epsilon)nh\right).$$

Now, by monotonicity of Φ^{-1}, we get

$$|\Gamma_h(n)| < \left(1 + \frac{1+\epsilon}{\lambda_x(h)(1-\epsilon)}\right)\Phi^{-1}\left(\lambda_x(h)(1-\epsilon)nh\right), \quad n \geq N_3(\epsilon).$$

Since we can take $\epsilon \in (0, 1/2)$, then for $n \geq N_3(\epsilon)$ we have

$$|\Gamma_h(n)| < \left(1 + \frac{3}{\lambda_x(h)}\right)\Phi^{-1}\left(\lambda_x(h)(1-\epsilon)nh\right).$$

Therefore

$$\Phi\left(\frac{1}{1+3/\lambda_x(h)}|\Gamma_h(n)|\right) > \lambda_x(h)(1-\epsilon)nh, \quad n \geq N_3(\epsilon).$$

Since $\Phi \in RV_0(0)$ and

$$\liminf_{n\to\infty} \frac{\Phi\left(\frac{1}{1+3/\lambda_x(h)}|\Gamma_h(n)|\right)}{nh} \geq \lambda_x(h),$$

we have that

$$\liminf_{n\to\infty} \frac{\Phi\left(|\Gamma_h(n)|\right)}{nh} \geq \lambda_x(h).$$

Finally, since $\Phi(x) \sim F(x)$ as $x \to 0^+$, and $\Gamma_h(n) \to 0$ as $n \to \infty$, therefore

$$\lambda_\Gamma(h) = \liminf_{n\to\infty} \frac{\Phi\left(|\Gamma_h(n)|\right)}{nh} \geq \lambda_x(h),$$

as claimed. □

5.4.3 Proof of Theorem 5.3.3

Like earlier results, the proof can divided into several parts. However, the second step below is once again the most difficult to prove.

Step 1: $\lambda_\Gamma(h) \in (0,1)$ implies $\Lambda_z(h) \geq \lambda_\Gamma(h)$.

Step 2: $\lambda_\Gamma(h) \in (0,1)$ implies $\lambda_z(h) \geq \lambda_\Gamma(h)$.

Step 3: $\lambda_\Gamma(h) \in (0,1)$ implies $\lambda_x(h) \geq \lambda_\Gamma(h)$.

Lemma 5.4.4. *Let f obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3) where the function φ in (3.2.4) and (3.2.3) is in $C^1(0,\infty)$ and obeys $\varphi'(x) \to 0$ as $x \to 0$. Let F be the function defined by (5.3.10) with $F(x) \to \infty$ as $x \to 0^+$. Suppose further that γ_h obeys (5.3.2) and that Γ_h defined (5.3.3) and that $\lambda_\Gamma(h)$ is defined by (5.3.7) obeys $\lambda_\Gamma(h) \in (0,1)$.*

Let z_h be any solution of (5.3.4). Then

$$\Lambda_z(h) := \limsup_{n\to\infty} \frac{F\left(|z_h(n)|\right)}{nh} \geq \lambda_\Gamma(h).$$

Proof. Suppose

$$M := \limsup_{n\to\infty} \frac{F\left(|z_h(n)|\right)}{nh} < \lambda_\Gamma(h).$$

Therefore

$$\limsup_{n\to\infty} \frac{F\left(|z_h(n+1)|\right)}{nh} = \limsup_{n\to\infty} \frac{F\left(|z_h(n+1)|\right)}{(n+1)h} \frac{(n+1)h}{nh} = M.$$

Since $M < \lambda_\Gamma(h)$ we suppose $\epsilon > 0$ is so small that $M + \epsilon < \lambda_\Gamma(h) - \epsilon$. Then, by supposition we have that $F\left(|\Gamma_h(n)|\right)/nh > \lambda_\Gamma(h) - \epsilon$ for all $n \geq N_1(\epsilon)$ and $F\left(|z_h(n+1)|\right)/nh < M + \epsilon$ for all $n \geq N_2(\epsilon)$. Hence $|z_h(n+1)| > F^{-1}\left((M+\epsilon)nh\right)$ and $|\Gamma_h(n)| < F^{-1}\left((\lambda_\Gamma(h) - \epsilon)nh\right)$ for all $n \geq N_3(\epsilon) := \max\left(N_1(\epsilon), N_2(\epsilon)\right)$. Also, for $n \geq N_3(\epsilon)$

$$\left|\frac{\Gamma_h(n)}{z_h(n+1)}\right| < \frac{F^{-1}\left((\lambda_\Gamma(h) - \epsilon)nh\right)}{F^{-1}\left((M+\epsilon)nh\right)}.$$

Since F^{-1} is rapidly varying, we have $\mu(n) := 1 + \Gamma_h(n)/z_h(n+1) \to 1$ as $n \to \infty$. Thus as $|z_h(n+1)| > 0$ for $n \geq N_3(\epsilon)$ we have

$$z_h(n+1) = z_h(n) - \tilde{h}_n \varphi\left(z_h(n+1)\right).$$

where

$$\tilde{h}_n = h\frac{f\left(z_h(n+1)\mu(n)\right)}{\varphi\left(z_h(n+1)\right)} = h\frac{f\left(z_h(n+1)\mu(n)\right)}{\varphi\left(z_h(n+1)\mu(n)\right)} \cdot \frac{\varphi\left(z_h(n+1)\mu(n)\right)}{\varphi\left(z_h(n+1)\right)}.$$

Thus $\tilde{h}_n \to h$ as $n \to \infty$, because $f(x) \sim \varphi(x)$ as $x \to 0$, φ is asymptotic preserving and $\mu(n) \to 1$ as $n \to \infty$. For $n \geq N_4(\epsilon)$, $\tilde{h}_n > 0$ and so $|z_h(n+1)| < |z_h(n)|$ and $z_h(n)$ has the same sign for all $n \geq N_4$. Suppose $z_h(n) > 0$ for all $n \geq N_4$, then

$$z_h(n+1) = z_h(n) - \tilde{h}_n \varphi\left(z_h(n+1)\right), \quad n \geq N_4, \quad \tilde{h}_n > 0; \quad z_h(N_4) > 0$$

and we have $z_h(n+1)/z_h(n) \to 1$ as $n \to \infty$. Therefore we have $z_h(n) > 0$ for all $n \geq N_4(\epsilon)$, and with $h_n^* = \tilde{h}_n \varphi\left(z_h(n+1)\right)/\varphi\left(z_h(n)\right)$ we get

$$z_h(n+1) = z_h(n) - h_n^* \varphi\left(z_h(n)\right).$$

Since φ is asymptotic preserving and $z_h(n+1) \sim z_h(n)$ as $n \to \infty$, we have $h_n^* \to h$ as

$n \to \infty$. Then by the mean value theorem there is an $\theta_n \in [0,1]$ such that

$$\Phi\left(z_h(n+1)\right) = \Phi\left(z_h(n)\right) + \Phi'\left(z_h(n) - h_n^*\theta_n\varphi\left(z_h(n)\right)\right) \cdot -h^*(n)\varphi\left(z_h(n)\right).$$

Thus

$$\Phi\left(z_h(n+1)\right) - \Phi\left(z_h(n)\right) = h_n^* \frac{\varphi\left(z_h(n)\right)}{\varphi\left(z_h(n) - h_n^*\theta_n\varphi\left(z_h(n)\right)\right)}.$$

Since $\theta_n \in [0,1]$, $h_n^* \to h$ as $n \to \infty$ and $\varphi(x)/x \to 0$ as $x \to 0^+$ then we get

$$\lim_{n\to\infty} \frac{z_h(n) - h_n^*\theta_n\varphi\left(z_h(n)\right)}{z_h(n)} = 1.$$

Therefore as φ is asymptotic preserving, we have $\Phi\left(z_h(n+1)\right) - \Phi\left(z_h(n)\right) \to h$ as $n \to \infty$. Hence

$$\lim_{n\to\infty} \frac{\Phi\left(z_h(n)\right)}{nh} = 1.$$

Therefore

$$\lim_{n\to\infty} \frac{F\left(|z_h(n)|\right)}{nh} = 1,$$

and so

$$1 > \lambda_\Gamma(h) > \limsup_{n\to\infty} \frac{F\left(|z_h(n)|\right)}{nh} = \lim_{n\to\infty} \frac{F\left(|z_h(n)|\right)}{nh} = 1,$$

a contradiction. Hence in the case that $z_h(n) > 0$, we arrive at a contradiction. Similarly, a contradiction ensues when $z_h(n) < 0$ for all n sufficiently large. Therefore, the assumption $\Lambda_z(h) < \lambda_\Gamma(h)$ is false, which implies that we must have $\Lambda_z(h) \geq \lambda_\Gamma(h)$ as claimed. □

Lemma 5.4.5. *Let f obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3) where the function φ in (3.2.4) and (3.2.3) is in $C^1(0,\infty)$ and obeys $\varphi'(x) \to 0$ as $x \to 0$. Let F be the function defined by (5.3.10) with $F(x) \to \infty$ as $x \to 0^+$. Suppose further that γ_h obeys (5.3.2) and that Γ_h defined (5.3.3) and that $\lambda_\Gamma(h)$ is defined by (5.3.7) obeys $\lambda_\Gamma(h) \in (0,1)$. Let z_h be any solution of (5.3.4). Then $\lambda_z(h)$ defined by (5.3.9) obeys*

$$\lambda_z(h) \geq \lambda_\Gamma(h).$$

Proof. Since $F(x) \sim \Phi(x)$ as $x \to 0^+$ and $z_h(n) \to 0$ as $n \to \infty$, it suffices to show that

$$\liminf_{n\to\infty} \frac{\Phi\left(|z_h(n)|\right)}{nh} \geq \lambda_\Gamma(h).$$

By Lemma 5.4.4

$$\limsup_{n\to\infty} \frac{F\left(|z_h(n)|\right)}{nh} =: \Lambda_z(h) \geq \lambda_\Gamma(h). \tag{5.4.20}$$

Since $\lambda_\Gamma(h) \in (0,1)$, for all $\epsilon \in (0,1)$ sufficiently small, by hypothesis we have that

$\Phi(|\Gamma_h(n)|)/(n+1)h > \lambda_\Gamma(h) - \epsilon/4$ for all $n \geq N_1(\epsilon)$. Hence

$$|\Gamma_h(n)| < \Phi^{-1}((\lambda_\Gamma(h) - \epsilon/4)(n+1)h), \quad n \geq N_1(\epsilon).$$

Since $\Gamma_h(n) \to 0$ and $z_h(n) \to 0$ as $n \to \infty$, for every $\epsilon \in (0,1)$ sufficiently small there is an $N_0(\epsilon)$ such that

$$1 - \frac{\epsilon}{8} < \frac{f(z_h(n+1) + \Gamma_h(n))}{\varphi(z_h(n+1) + \Gamma_h(n))} < 1 + \frac{\epsilon}{8}, \quad n \geq N_0(\epsilon).$$

Since Φ^{-1} is rapidly varying, for every $\epsilon \in (0,1)$ sufficiently small, there is an $N_2(\epsilon)$ such that

$$\frac{\Phi^{-1}((\lambda_\Gamma(h) - \epsilon/4)(n+1)h)}{\Phi^{-1}((\lambda_\Gamma(h) - \epsilon)(n+1)h)} < \frac{\epsilon}{8}, \quad n \geq N_2(\epsilon).$$

Let

$$\Phi_{\epsilon/8} := \liminf_{x \to 0^+} \frac{\varphi((1-\epsilon/8)x)}{\varphi(x)},$$

and thus $\Phi_{\epsilon/8} \to 1$ as $\epsilon \to 0^+$. Now, for all $\eta > 0$ there exists $\tilde{x}_4(\eta, \epsilon)$ such that $x < \tilde{x}_4(\eta, \epsilon)$ implies $\varphi((1-\epsilon/8)x)/\varphi(x) > \Phi_{\epsilon/8} - \eta$. Take $\eta(\epsilon) = \Phi_{\epsilon/8}\epsilon/8$. Let $x_4(\epsilon) := \tilde{x}_4(\eta(\epsilon), \epsilon)$: then we have

$$\varphi((1-\epsilon/8)x) > \Phi_{\epsilon/8}(1-\epsilon/8)\varphi(x), \quad x < x_4(\epsilon).$$

Now, there is $N_2'(\epsilon) \in \mathbb{N}$ such that

$$\Phi^{-1}((\lambda_\Gamma(h) - \epsilon)(n+1)h) < x_4(\epsilon), \quad n \geq N_2'(\epsilon).$$

Next, let $\psi(x) := (\varphi \circ \Phi^{-1})(x)$. Then

$$\frac{\psi'(x)}{\psi(x)} = -\varphi'\left(\Phi^{-1}(x)\right).$$

Therefore, as $\Phi^{-1}(x) \to 0$ as $x \to \infty$ and $\varphi'(x) \to 0$ as $x \to 0$, we have $\lim_{x\to\infty} \psi'(x)/\psi(x) = 0$. Thus

$$\lim_{x \to \infty} \frac{\psi(x + h(\lambda_\Gamma(h) - \epsilon))}{\psi(x)} = 1$$

and so with $x = (\lambda_\Gamma(h) - \epsilon)nh$ we have

$$\lim_{n \to \infty} \frac{\psi((\lambda_\Gamma(h) - \epsilon)(n+1)h)}{\psi((\lambda_\Gamma(h) - \epsilon)nh)} = 1.$$

Therefore, for every $\eta > 0$ there is $\tilde{N}_3(\eta, h, \epsilon) \in \mathbb{N}$ such that for all $n \geq \tilde{N}_3(\eta, h, \epsilon)$

$$\frac{\psi((\lambda_\Gamma(h) - \epsilon)(n+1)h)}{\psi((\lambda_\Gamma(h) - \epsilon)nh)} > 1 - \eta.$$

Take $\eta = \epsilon/8$, $N_3(\epsilon, h) = \bar{N}_3(\epsilon/8, h, \epsilon)$. Then

$$\frac{(\varphi \circ \Phi^{-1})\left((\lambda_\Gamma(h) - \epsilon)(n+1)h\right)}{\varphi\left(\Phi^{-1}\left((\lambda_\Gamma(h) - \epsilon)nh\right)\right)} > 1 - \frac{\epsilon}{8}, \quad n \geq N_3(\epsilon, h).$$

Let $N'(\epsilon) > N_0(\epsilon) \vee N_1(\epsilon) \vee N_2(\epsilon) \vee N_2'(\epsilon) \vee N_3(\epsilon)$.

By (5.4.20) it follows that for every $\epsilon > 0$ sufficiently small there is a sequence $n_j \nearrow \infty$ such that $F\left(|z_h(n_j)|\right)/n_j h > \Lambda_z(h) - \epsilon \geq \lambda_\Gamma(h) - \epsilon$. Hence

$$|z_h(n_j)| \leq \Phi^{-1}\left((\lambda_\Gamma(h) - \epsilon)n_j h\right). \tag{5.4.21}$$

Let $N^*(\epsilon) = \min\{n_j(\epsilon) : n_j(\epsilon) > N'(\epsilon)\}$. Note that $N^*(\epsilon) > N'(\epsilon)$

Most of the rest of the proof is devoted to showing that

$$\text{If } n \geq N^*(\epsilon) \text{ and } |z_h(n)| \leq \Phi^{-1}\left((\lambda_\Gamma(h) - \epsilon)nh\right) \quad \text{then} \quad |z_h(n+1)| \leq \Phi^{-1}\left((\lambda_\Gamma(h) - \epsilon)(n+1)h\right). \tag{5.4.22}$$

If (5.4.22) holds, it is relatively straightforward to secure the desired asymptotic behaviour. Clearly, by (5.4.21) for $n = N^*(\epsilon)$ the first statement in (5.4.22) holds. Therefore $|z_h(n+1)| \leq \Phi^{-1}\left((\lambda_\Gamma(h) - \epsilon)(n+1)h\right)$. By induction, it is immediate that

$$|z_h(n)| \leq \Phi^{-1}((\lambda_\Gamma(h) - \epsilon)nh), \quad \text{for all } n \geq N^*(\epsilon).$$

Hence $\Phi(|z_h(n)|) \geq (\lambda_\Gamma(h) - \epsilon)nh$ for all $n \geq N^*(\epsilon)$. Letting $n \to \infty$ yields

$$\liminf_{n\to\infty} \frac{\Phi(|z_h(n)|)}{nh} \geq \lambda_\Gamma(h) - \epsilon.$$

Thus by letting $\epsilon \to 0^+$, we have that

$$\lambda_z(h) = \liminf_{n\to\infty} \frac{\Phi(|z_h(n)|)}{nh} \geq \lambda_\Gamma(h),$$

as needed.

Suppose that (5.4.22) does not hold. There are two possibilities: we first consider the case that

$$\text{There exists } n \geq N^*(\epsilon) \text{ such that } |z_h(n)| \leq \Phi^{-1}\left((\lambda_\Gamma(h) - \epsilon)nh\right) \quad \text{and} \quad z_h(n+1) > \Phi^{-1}\left((\lambda_\Gamma(h) - \epsilon)(n+1)h\right). \tag{5.4.23}$$

The other possibility that leads to a contradiction of (5.4.22) is

$$\text{There exists } n \geq N^*(\epsilon) \text{ such that } |z_h(n)| \leq \Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)nh\right) \\ \text{and} \quad z_h(n+1) < -\Phi^{-1}\left((\lambda_{\Gamma(h)}-\epsilon)(n+1)h\right), \quad (5.4.24)$$

and we eliminate this possibility later.

First, we show that (5.4.23) leads to a contradiction. Since $z_h(n+1) = z_h(n) - hf\left(z_h(n+1)+\Gamma_h(n)\right)$ we have that

$$\begin{aligned} 0 &= z_h(n) - z_h(n+1) - hf\left(z_h(n+1)+\Gamma_h(n)\right) \\ &< \Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)nh\right) - \Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+1)h\right) \\ &\quad - hf\left(z_h(n+1)+\Gamma_h(n)\right). \end{aligned}$$

Next, as $N^*(\epsilon) > N_0(\epsilon) \vee N_1(\epsilon)$ we have

$$\begin{aligned} &z_h(n+1)+\Gamma_h(n) \\ &\qquad > \Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+1)h\right) - \Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon/4)(n+1)h\right) > 0. \end{aligned}$$

Thus

$$\begin{aligned} f\left(z_h(n+1)+\Gamma_h(n)\right) &> (1-\epsilon/8)\varphi\left(z_h(n+1)+\Gamma_h(n)\right) \\ &> \left(1-\frac{\epsilon}{8}\right)\varphi\left(\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+1)h\right) - \Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon/4)(n+1)h\right)\right). \end{aligned}$$

Hence

$$\begin{aligned} &-\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+1)h\right) + \Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)nh\right) \\ &\quad - h\left(1-\frac{\epsilon}{8}\right)\varphi\left(\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+1)h\right) - \Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon/4)(n+1)h\right)\right) \\ &\qquad > 0. \quad (5.4.25) \end{aligned}$$

By the mean value theorem, there is $\theta_n \in [0,1]$ such that

$$\begin{aligned} &\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+1)h\right) - \Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)nh\right) \\ &= (\Phi^{-1})'\left((\lambda_\Gamma(h)-\epsilon)(n+\theta_n)h\right)(\lambda_\Gamma(h)-\epsilon)h \\ &= -\varphi\left(\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+\theta_n)h\right)\right)(\lambda_\Gamma(h)-\epsilon)h. \end{aligned}$$

Thus

$$\varphi\left(\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+\theta_n)h\right)\right)(\lambda_\Gamma(h)-\epsilon)$$
$$> (1-\epsilon/8)\varphi\left(\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+1)h\right)-\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon/4)(n+1)h)\right)\right).$$

Since $\varphi\circ\Phi^{-1}$ is decreasing, we have

$$\frac{\lambda_\Gamma(h)-\epsilon}{1-\epsilon/8}\varphi\left(\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)nh\right)\right)$$
$$> \varphi\left(\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+1)h\right)-\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon/4)(n+1)h)\right)\right).$$

Hence, as $n\geq N^*(\epsilon)>N_2(\epsilon)$ and φ is increasing

$$\frac{\lambda_\Gamma(h)-\epsilon}{1-\epsilon/8}\varphi\left(\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)nh\right)\right)$$
$$> \varphi\left((1-\epsilon/8)\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+1)h\right)\right).$$

Since $n\geq N^*(\epsilon)>N_2'(\epsilon)\vee N_2(\epsilon)$, we have

$$\begin{aligned}&\frac{\lambda_\Gamma(h)-\epsilon}{1-\epsilon/8}\varphi\left(\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)nh\right)\right)\\ &> \varphi\left((1-\epsilon/8)\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+1)h\right)\right)\\ &> \Phi_{\epsilon/8}(1-\epsilon/8)\varphi\left(\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+1)h\right)\right).\end{aligned}$$

Hence

$$\frac{\lambda_\Gamma(h)-\epsilon}{(1-\epsilon/8)^2}\frac{1}{\Phi_{\epsilon/8}} > \frac{\varphi\left(\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+1)h\right)\right)}{\varphi\left(\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)nh\right)\right)}.$$

Since $N^*(\epsilon)>N_3(\epsilon,h)$, we have for all $\epsilon>0$ sufficiently small that

$$\frac{\lambda_\Gamma(h)-\epsilon}{(1-\epsilon/8)^2}\frac{1}{\Phi_{\epsilon/8}} > 1-\frac{\epsilon}{8}$$

or $\lambda_\Gamma(h)>(1-\epsilon/8)^3\Phi_{\epsilon/8}+\epsilon=:\mu(\epsilon)$. But $\lim_{\epsilon\to0^+}\mu(\epsilon)=1$. Thus for all $\epsilon<\epsilon_0$, $\mu(\epsilon)>\lambda_\Gamma(h)+1/2(1-\lambda_\Gamma(h))>\lambda_\Gamma(h)>\mu(\epsilon)$, a contradiction. Therefore, the statement (5.4.23) is false. Hence

$$\text{If } n\geq N^*(\epsilon) \text{ and } |z_h(n)|\leq\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)nh\right)$$
$$\text{then}\quad z_h(n+1)\leq\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+1)h\right). \tag{5.4.26}$$

Next, we suppose that (5.4.24) holds and show that this produces a contradiction.

Since $n \geq N^*(\epsilon)$, if (5.4.24) holds we have

$$-z_h(n) \leq \Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)nh\right), \quad z_h(n+1) < -\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+1)h\right).$$

Since $-hf\left(z_h(n+1)+\Gamma_h(n)\right) = z_h(n+1) - z_h(n)$, we have

$$\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+1)h\right) - \Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)nh\right) \\ - hf\left(z_h(n+1)+\Gamma_h(n)\right) < 0. \quad (5.4.27)$$

Since $N^*(\epsilon) > N_1(\epsilon)$, $|\Gamma_h(n)| < \Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon/4)(n+1)h\right)$ and so

$$z_h(n+1)+\Gamma_h(n) \\ < -\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+1)h\right) + \Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon/4)(n+1)h\right) < 0.$$

Thus as φ is odd and increasing, we have that

$$\begin{aligned} &\varphi\left(z_h(n+1)+\Gamma_h(n)\right) \\ &< \varphi\left(-\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+1)h\right) + \Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon/4)(n+1)h\right)\right) \\ &= -\varphi\left(\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+1)h\right) - \Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon/4)(n+1)h\right)\right). \end{aligned}$$

Hence

$$- \varphi\left(z_h(n+1)+\Gamma_h(n)\right) > \\ \varphi\left(\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon/4)(n+1)h\right) - \Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+1)h\right)\right). \quad (5.4.28)$$

Also, as $N^*(\epsilon) > N_0(\epsilon)$, we have

$$1 - \frac{\epsilon}{8} < \frac{f\left(z_h(n+1)+\Gamma_h(n)\right)}{\varphi\left(z_h(n+1)+\Gamma_h(n)\right)},$$

so as $z_h(n+1)+\Gamma_h(n) < 0$, we have

$$h(1-\epsilon/8)\varphi\left(z_h(n+1)+\Gamma_h(n)\right) > hf\left(z_h(n+1)+\Gamma_h(n)\right)$$

and so

$$- h(1-\epsilon/8)\varphi\left(z_h(n+1)+\Gamma_h(n)\right) < -hf\left(z_h(n+1)+\Gamma_h(n)\right). \quad (5.4.29)$$

Combine (5.4.27), (5.4.28) and (5.4.29) to get

$$\begin{aligned}
&\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)nh\right)-\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+1)h\right)\\
&>-hf\left(z_h(n+1)+\Gamma_h(n)\right)\\
&>-h(1-\epsilon/8)\varphi\left(z_h(n+1)+\Gamma_h(n)\right)\\
&>h(1-\epsilon/8)\varphi\left(\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon/4)(n+1)h\right)-\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+1)h\right)\right).
\end{aligned}$$

This precisely (5.4.25) in the earlier part of the proof and we have already shown that (5.4.25) leads to a contradiction. Therefore the supposition (5.4.24) is false. Therefore

$$\begin{aligned}
&\text{If } n\geq N^*(\epsilon) \text{ and } |z_h(n)|\leq \Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)nh\right)\\
&\qquad\qquad \text{then}\quad z_h(n+1)\geq -\Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+1)h\right).
\end{aligned} \tag{5.4.30}$$

Since we have already shown (5.4.26) i.e.,

$$\begin{aligned}
&\text{If } n\geq N^*(\epsilon) \text{ and } |z_h(n)|\leq \Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)nh\right)\\
&\qquad\qquad \text{then}\quad z_h(n+1)\leq \Phi^{-1}\left((\lambda_\Gamma(h)-\epsilon)(n+1)h\right).
\end{aligned}$$

it follows that by combining (5.4.26) and (5.4.30) that we have (5.4.22) holds, as required. □

We are now in position to prove Theorem 5.3.3.

Proof. By hypothesis, for every $\epsilon>0$ sufficiently small, there is an $N_1(\epsilon)$ such that $F\left(|\Gamma_h(n)|\right)/nh>\lambda_\Gamma(h)-\epsilon$ for all $n\geq N_1(\epsilon)$. Let $z_h(n)=x_h(n)-\Gamma_h(n)$ for $n\geq 0$, where x_h is any solution of (5.3.1). Then z_h is a solution of (5.3.4). Thus, by virtue of Lemma 5.4.5, for every $\epsilon>0$ sufficiently small, there is an $N_2(\epsilon)$ such that $F\left(|z_h(n)|\right)/nh>\lambda_\Gamma(h)-\epsilon$ for all $n\geq N_2(\epsilon)$. Let $N_3(\epsilon)=\max\left(N_1(\epsilon),N_2(\epsilon)\right)$. Then we have

$$|z_h(n)|<F^{-1}\left((\lambda_\Gamma(h)-\epsilon)nh\right),\quad |\Gamma_h(n)|<F^{-1}\left((\lambda_\Gamma(h)-\epsilon)nh\right)$$

for all $n\geq N_3(\epsilon)$. Hence $|x_h(n)|\leq 2F^{-1}\left((\lambda_\Gamma(h)-\epsilon)nh\right)$ for all $n\geq N_3(\epsilon)$ and thus $F\left(1/2|x_h(n)|\right)\geq(\lambda_\Gamma(h)-\epsilon)nh$ for all $n\geq N_3(\epsilon)$. Therefore

$$\liminf_{n\to\infty}\frac{F\left(\frac{1}{2}|x_h(n)|\right)}{nh}\geq\lambda_\Gamma(h).$$

Since $F\in RV_0(0)$, we have $\lambda_x(h)\geq\lambda_\Gamma(h)>0$. On the other hand, by Theorem 5.3.2, we have that $\lambda_x(h)\leq\lambda_\Gamma(h)$. Combining these inequalities yields $\lambda_x(h)=\lambda_\Gamma(h)$ as claimed. □

5.4.4 Proof of Theorem 5.3.4

Proof. Suppose, contrary to the claim of the theorem, that $\lambda_x(h) > 0$. Then by Theorem 5.3.2 we have $\lambda_\Gamma(h) \geq \lambda_x(h) > 0$. But $\lambda_\Gamma(h) = 0$ by hypothesis, a contradiction. Therefore $\lambda_x(h) \leq 0$. Also, if $x_h(n) \to 0$ as $n \to \infty$ then $\lambda_x(h) \geq 0$. Hence $\lambda_x(h) = 0$ as required. □

5.4.5 Proof of Theorem 5.3.5

To prove part (i), notice that the case $\lambda_\Gamma(h) \in (0, 1)$ has been dealt with in Theorem 5.3.3. The case when $\lambda_\Gamma(h) = 0$ is the subject of Theorem 5.3.4. In the case when $\lambda_\Gamma(h) = 1 > 0$, we have from Theorem 5.3.2 that $\lambda_x(h) \leq \lambda_\Gamma(h) = 1$. On the other hand, from Theorem 5.3.1 we have that $\lambda_x(h) \geq 1$. Combining the inequalities, we get $\lambda_x(h) = 1$. This completes the proof of part (i).

The proof of part (ii) is precisely Theorem 5.3.2 restricted to the case when $\lambda_\Gamma(h) > 1$.

For the proof of part (iii), by hypothesis we have that $x_h(n) \to 0$ as $n \to \infty$ and that $\sum_{j=0}^{n} \gamma_h(j+1)$ does not converge. Then $\lambda_x(h)$ is well–defined and non–negative. Suppose $\lambda_x(h) > 0$. Then by Theorem 5.3.2 we have that

$$\lim_{n\to\infty} \sum_{j=0}^{n} \gamma_h(j+1) \text{ exists and is finite.}$$

But this contradicts our hypothesis. Therefore, we cannot have $\lambda_x(h) > 0$, so $\lambda_x(h) = 0$ as claimed.

Chapter 6

Numerical Analysis of Stochastic Differential Equations

6.1 Introduction

We will now use the results from the last chapter to analyse the asymptotic behaviour of the discretisation of the stochastic differential equation

$$dX(t) = -f(X(t))\,dt + \sigma(t)\,dB(t).$$

The discretisation, of course, is the uniform step split step backward Euler method

$$X_h(0) = \zeta; \tag{6.1.1a}$$

$$X_h^*(n) = X_h(n) - hf(X_h^*(n)), \quad n \geq 0; \tag{6.1.1b}$$

$$X_h(n+1) = X_h^*(n) + \sqrt{h}\sigma(nh)\xi(n+1), \quad n \geq 0. \tag{6.1.1c}$$

where $h > 0$ is the step size, and $(\xi(n))_{n\geq 1}$ is a sequence of independent and identically distributed normal random variables.

Our proof relies on arguing pathwise and applying the deterministic results from the last chapter path–by–path. To do this, three ingredients are required. The first is to show that the sequence γ_h defined by

$$\gamma_h(n+1) := \sqrt{h}\sigma(nh)\xi(n+1), \quad n \geq 0$$

obeys

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} \gamma_h(j+1) \quad \text{exists and is finite a.s.} \tag{6.1.2}$$

By virtue of the martingale convergence theorem, this will be true if the sequence $(\sqrt{h}\sigma(nh))_{n\geq 0} \in \ell^2(\mathbb{N})$. This can be achieved by appropriate integrability and regularity conditions on σ.

The second, if the convergence holds, is to determine the asymptotic behaviour of

$$\Gamma_h(n) := -\sum_{j=n}^{\infty} \gamma_h(j+1)$$

This can be achieved using the results of Chapter 4 on the "tail martingale"

$$\sum_{j=n}^{\infty} \sigma_j \xi(j+1)$$

where

$$\sigma_n := \sqrt{h}\sigma(nh),$$

so the asymptotic behaviour of the SSBE scheme can be checked. Of course, the third and final part of the analysis is to see if, when the corresponding continuous time hypotheses are imposed, the SSBE scheme recovers the same asymptotic behaviour. It turns out that this can be done, whether the perturbations are small or large.

6.2 Martingale properties

For continuous time results, we asked that $\sigma \in L^2(0,\infty)$ so that

$$\lim_{t\to\infty} \int_0^t \sigma(s)\, dB(s) \text{ exists and is finite, a.s.}$$

and to get the asymptotic behaviour of $\int_t^\infty \sigma(s)\, dB(s)$ we also asked that

$$\int_t^\infty \sigma^2(s)\, ds > 0, \quad t \ge 0.$$

Suppose

$$\sigma \in L^2(0,\infty), \quad \sigma^2 \text{ is decreasing}, \quad \int_t^\infty \sigma^2(s)\, ds > 0, \quad t \ge 0. \tag{6.2.1}$$

Then

$$\sigma_n \in \ell^2(\mathbb{N}). \tag{6.2.2}$$

This is because for $N \ge 1$

$$\begin{aligned}\sum_{j=0}^{N} \sigma_j^2 &= h\sigma^2(0) + \sum_{j=0}^{N-1} h\sigma^2\left((j+1)h\right) \\ &\le h\sigma^2(0) + \sum_{j=0}^{N-1} \int_{jh}^{(j+1)h} \sigma^2(s)\, ds = h\sigma^2(0) + \int_0^{Nh} \sigma^2(s)\, ds.\end{aligned}$$

Since σ^2 is decreasing, and $t \mapsto \int_t^\infty \sigma^2(s)\,ds$ is positive for all $t \geq 0$, it follows that $\sigma^2(t) > 0$ for all $t \geq 0$. Hence $\sigma_n^2 = h\sigma^2(nh) > 0$ for all n. Therefore

$$\sum_{j=n}^{\infty} \sigma_j^2 > 0, \quad n \geq 0. \tag{6.2.3}$$

Since $\sigma_n \in \ell^2(\mathbb{N})$, it now follows that (6.1.2) is true, so we may define a.s.

$$\tilde{S}_n := \sum_{j=n}^{\infty} \sigma_j \xi(j+1),$$

(where now $\tilde{S}_n \to 0$ as $n \to \infty$ a.s.) and

$$\Sigma_n = \sqrt{2 \sum_{j=n}^{\infty} \sigma_j^2 \log\log \frac{1}{\sum_{j=n}^{\infty} \sigma_j^2}}, \quad n \geq 1.$$

Since (6.2.2) and (6.2.3) hold, by Lemma 4.4.2, we have that

$$\limsup_{n\to\infty} \frac{|\tilde{S}_n|}{\Sigma_n} \leq 1, \quad \text{a.s.} \tag{6.2.4}$$

Note that

$$\tilde{S}_n = \sum_{j=n}^{\infty} \sqrt{h}\sigma(jh)\xi(j+1)$$

and

$$\Sigma_n = \sqrt{2 \sum_{j=n}^{\infty} h\sigma^2(jh) \log\log\left(\frac{1}{\sum_{j=n}^{\infty} h\sigma^2(jh)}\right)}.$$

6.3 The Split Step Scheme

We are now in a position to study the asymptotic behaviour of the split step scheme.

Assume, as in the continuous case that f be asymptotic preserving, and asymptotic to an odd and increasing function φ with $\varphi'(x) \to 0$ as $x \to 0^+$. The last hypothesis was employed in the last chapter in place of the weaker sublinearity hypothesis $f(x)/x \to 0$ as $x \to 0$. Suppose the solution of the unperturbed ODE

$$y'(t) = -f(y(t)), \quad t \geq 0; \quad y(0) = 1. \tag{6.3.1}$$

is rapidly varying. Let σ be decreasing and in $L^2(0,\infty)$. Let X be a continuous solution of the stochastic differential equation

$$dX(t) = -f(X(t))\,dt + \sigma(t)\,dB(t), \quad t \geq 0. \tag{6.3.2}$$

Since σ is decreasing, the sequence $\sigma_j := \sqrt{h}\sigma(jh)$ is in $\ell^2(\mathbb{N})$. This implies that

$$\lim_{n\to\infty}\sum_{j=0}^{n-1}\sqrt{h}\sigma(jh)\xi(j+1) \quad \text{tends to a finite limit, a.s.}$$

Therefore, we have that any solution of the SSBE scheme $(X_h(n))_{n\geq 0}$ obeys $X_h(n)\to 0$ as $n\to\infty$, a.s.

The convergence of the above sum ensures that the random variable

$$\lambda_\sigma(h) = \liminf_{n\to\infty}\frac{F\left(\left|\sum_{j=n}^{\infty}\sqrt{h}\sigma(jh)\xi(j+1)\right|\right)}{nh}$$

is well-defined. Now defining

$$\lambda_X(h) = \liminf_{n\to\infty}\frac{F\left(|X_h(n)|\right)}{nh},$$

we have that $\lambda_\sigma(h)\geq 1$ a.s. implies $\lambda_X(h)\geq 1$ a.s. Also, if $\lambda_\sigma(h)\in(0,1)$ a.s. then $\lambda_X(h)=\lambda_\sigma(h)$ a.s. and if $\lambda_\sigma(h)=0$ a.s. then $\lambda_X(h)=0$ a.s. We now show that whenever

$$\Sigma(t) = \sqrt{2\int_t^\infty \sigma^2(s)\,ds\log_2\frac{1}{\int_t^\infty\sigma^2(s)\,ds}}$$

and

$$L_\Sigma := \lim_{t\to\infty}\frac{F(\Sigma(t))}{t} \tag{6.3.3}$$

is well-defined, then $\lambda_\sigma(h)=L_\Sigma$ a.s. Therefore, the sharp sufficient conditions we have explored in Chapter 3 under which the rate of decay of the SDE (6.3.2) are known will also give rise to the same rates of the decay in the solution of the split step backward Euler method.

Suppose now that Ł$_\Sigma$ in (6.3.3) is well–defined. Define

$$\Sigma_n = \sqrt{2\sum_{j=n}^{\infty}h\sigma^2(jh)\log_2\frac{1}{\sum_{j=n}^{\infty}\sigma^2(jh)}}$$

By the discrete iterated logarithm bound in (6.2.4), we have

$$\limsup_{n\to\infty}\frac{\left|\sum_{j=n}^{\infty}\sqrt{h}\sigma(jh)\xi(j+1)\right|}{\Sigma_n}\leq 1, \quad \text{a.s.}$$

Thus, as F is in $RV_0(0)$,

$$\liminf_{n\to\infty}\frac{F\left(\left|\sum_{j=n}^{\infty}\sqrt{h}\sigma(jh)\xi(j+1)\right|\right)}{F(\Sigma_n)}\geq 1, \quad \text{a.s.}$$

Therefore, we have

$$\lambda_\sigma(h) = \liminf_{n\to\infty} \frac{F\left(\left|\sum_{j=n}^{\infty} \sqrt{h}\sigma(jh)\xi(j+1)\right|\right)}{F(\Sigma_n)} \cdot \frac{F(\Sigma_n)}{nh}$$

and so

$$\lambda_\sigma(h) \geq \liminf_{n\to\infty} \frac{F(\Sigma_n)}{nh}. \tag{6.3.4}$$

Since σ^2 is decreasing, then we have

$$h\sigma^2\left((j+1)h\right) \leq \int_{jh}^{(j+1)h} \sigma^2(s)\,ds \leq h\sigma^2(jh)$$

and thus

$$\sum_{j=n}^{\infty} h\sigma^2\left((j+1)h\right) \leq \int_{nh}^{\infty} \sigma^2(s)\,ds \leq \sum_{j=n}^{\infty} h\sigma^2(hj).$$

Define $H(x) = \sqrt{2x \log_2 \frac{1}{x}}$. Since H is increasing for $x \in (0,\delta)$ and $\delta > 0$ sufficiently small, for all n sufficiently large, we get

$$H\left(\sum_{j=n}^{\infty} h\sigma^2((j+1)h)\right) \leq H\left(\int_{nh}^{\infty} \sigma^2(s)\,ds\right) \leq H\left(\sum_{j=n}^{\infty} h\sigma^2(jh)\right).$$

Therefore $\Sigma_{n+1} \leq \Sigma(nh) \leq \Sigma_n$. Since F is decreasing, we get

$$\frac{F(\Sigma_{n+1})}{nh} \geq \frac{F\left(\Sigma(nh)\right)}{nh} \geq \frac{F(\Sigma_n)}{nh}. \tag{6.3.5}$$

Consider the second inequality in (6.3.5); letting $n \to \infty$ and using (6.3.3), we get

$$\limsup_{n\to\infty} \frac{F(\Sigma_n)}{nh} \leq \lim_{n\to\infty} \frac{F\left(\Sigma(nh)\right)}{nh} = L_\Sigma.$$

Now, consider the first inequality in (6.3.5), letting $n \to \infty$ and using (6.3.3), we get

$$\begin{aligned}\liminf_{n\to\infty} \frac{F(\Sigma_{n+1})}{(n+1)h} &= \liminf_{n\to\infty} \frac{F(\Sigma_{n+1})}{nh} \cdot \frac{nh}{(n+1)h} \\ &= \liminf_{n\to\infty} \frac{F(\Sigma_{n+1})}{nh} \geq \lim_{n\to\infty} \frac{F\left(\Sigma(nh)\right)}{nh} = L_\Sigma.\end{aligned}$$

Hence

$$\lim_{n\to\infty} \frac{F(\Sigma_n)}{nh} = L_\Sigma. \tag{6.3.6}$$

Therefore

$$\lambda_\sigma(h) \geq L_\Sigma \quad \text{a.s.} \tag{6.3.7}$$

Now, suppose $L_\Sigma \geq 1$. Then, by (6.3.7), $\lambda_\sigma(h) \geq 1$ a.s., and so $\lambda_X \geq 1$ a.s. Thus we have proven the following theorem.

Theorem 6.3.1. *Let f be increasing and obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3) where the function φ in (3.2.4) and (3.2.3) is in $C^1(0,\infty)$ and obeys $\varphi'(x) \to 0$ as $x \to 0$. Let F be the function defined by (5.3.10) with $F(x) \to \infty$ as $x \to 0^+$. Let σ be a continuous, positive and decreasing function in $L^2(0,\infty)$, and suppose further that*

$$L_\Sigma = \lim_{t\to\infty} \frac{F\left(\Sigma(t)\right)}{t} \geq 1.$$

Then

(i) Any continuous solution X of (6.3.2) obeys

$$\lambda_X = \liminf_{t\to\infty} \frac{F\left(|X(t)|\right)}{t} \geq 1, \quad a.s.$$

(ii) Any solution X_h of (6.1.1) obeys

$$\liminf_{n\to\infty} \frac{F\left(|X_h(n)|\right)}{nh} \geq 1, \quad a.s.$$

For any value of L_Σ, our argument shows that $\lambda_\sigma(h) \geq L_\Sigma$ a.s. we prove next when $L_\Sigma \in (0,1)$, that $\lambda_\sigma(h) = L_\Sigma$ a.s., provided that σ obeys some extra regularity conditions. Let us suppose now that σ obeys

$$\lim_{n\to\infty} \frac{h\sigma^2(nh)}{\sum_{j=n}^\infty h\sigma^2(jh)} = 0. \tag{6.3.8}$$

Then, as $\sigma \in L^2(0,\infty)$, we have the exact iterated logarithm law

$$\limsup_{n\to\infty} \frac{\left|\sum_{j=n}^\infty \sqrt{h}\sigma(jh)\xi(j+1)\right|}{\Sigma_n} = 1, \quad \text{a.s.}$$

Hence, as F is slowly varying and deceasing

$$\liminf_{n\to\infty} \frac{F\left(\left|\sum_{j=n}^\infty \sqrt{h}\sigma(jh)\xi(j+1)\right|\right)}{F(\Sigma_n)} = 1, \quad \text{a.s.}$$

Then, by (6.3.6)

$$\lambda_\sigma(h) = \liminf_{n\to\infty} \frac{F\left(\left|\sum_{j=n}^\infty \sqrt{h}\sigma(jh)\xi(j+1)\right|\right)}{F(\Sigma_n)} \cdot \frac{F(\Sigma_n)}{nh} = L_\Sigma, \quad \text{a.s.}$$

The condition (6.3.8) now suffices if we want to state a theorem comparing the asymptotic behaviour if the SDE (6.3.2) and the SSBE. However, we observe that (6.3.8) will hold when σ^2 enjoys an appropriate continuous subexponential property. This is a quite reasonable supposition, because, as we now show $t \mapsto \int_t^\infty \sigma^2(s)\,ds$ is bounded above and below by subexponential functions when $L_\Sigma \in (0,1)$. First notice, that for every $\epsilon \in (0,1)$ there is $T(\epsilon) > 0$ such that $L_\Sigma(1-\epsilon)t < F(\Sigma(t)) < L_\Sigma(t+\epsilon)t$ for all $t \geq T(\epsilon)$. Since F is decreasing and H^{-1} is increasing, we have

$$\left(H^{-1} \circ F^{-1}\right)(L_\Sigma(1-\epsilon)t) > \int_t^\infty \sigma^2(s)\,ds > \left(H^{-1} \circ F^{-1}\right)(L_\Sigma(1+\epsilon)t)\,. \tag{6.3.9}$$

Now F^{-1} is subexponential, and $H^{-1} \in RV_0(2)$. Therefore, we have that

$$\lim_{t\to\infty} \frac{H^{-1}\left(F^{-1}(t-\tau)\right)}{H^{-1}\left(F^{-1}(t)\right)} = 1$$

because of the subexponential property $F^{-1}(t-\tau)/F^{-1} \to 1$ as $t \to \infty$ and H^{-1} is asymptotic preserving. Thus $H^{-1} \circ F^{-1}$ is subexponential. Hence the upper and lower bound in (6.3.9) are subexponential, as claimed.

A condition on σ which gives (6.3.8) is

$$\lim_{t\to\infty} \frac{\sigma^2(t)}{\int_t^\infty \sigma^2(s)\,ds} = 0. \tag{6.3.10}$$

To see this, note that σ^2 decreasing yields, as before,

$$\int_{nh}^\infty \sigma^2(s)\,ds \leq \sum_{j=n}^\infty h\sigma^2(jh).$$

Hence by (6.3.10)

$$\lim_{n\to\infty} \frac{h\sigma^2(nh)}{\sum_{j=n}^\infty h\sigma^2(jh)} \leq \lim_{n\to\infty} \frac{h\sigma^2(nh)}{\int_{nh}^\infty \sigma^2(s)\,ds} = 0$$

which gives (6.3.8). The condition (6.3.10) implies that $\rho(t) := \int_t^\infty \sigma^2(s)\,ds$ is subexponential, because

$$\lim_{t\to\infty} \frac{\rho'(t)}{\rho(t)} = \lim_{t\to\infty} \frac{-\sigma^2(t)}{\int_t^\infty \sigma^2(s)\,ds} = 0.$$

Thus, our theorem for slowly decaying perturbation can be stated.

Theorem 6.3.2. *Let f be increasing and obey (3.2.1), (3.2.2), (3.2.4) and (3.2.3) where the function φ in (3.2.4) and (3.2.3) is in $C^1(0,\infty)$ and obeys $\varphi'(x) \to 0$ as $x \to 0$. Let F be the function defined by (5.3.10) with $F(x) \to \infty$ as $x \to 0^+$. Let σ be a continuous, positive and decreasing function in $L^2(0,\infty)$, and suppose further that*

$$\lim_{t\to\infty} \frac{\sigma^2(t)}{\int_t^\infty \sigma^2(s)\,ds} = 0$$

and

$$L_\Sigma = \lim_{t\to\infty} \frac{F(\Sigma(t))}{t} \in (0,1).$$

Then

(i) Any continuous solution X of (6.3.2) obeys

$$\liminf_{t\to\infty} \frac{F\left(|X(t)|\right)}{t} = L_\Sigma, \quad a.s.$$

(ii) Any solution X_h of (6.1.1) obeys

$$\liminf_{n\to\infty} \frac{F\left(|X_h(n)|\right)}{nh} = L_\Sigma, \quad a.s.$$

Of course, we are able to tackle the case when $L_\Sigma = 0$ and when $\sigma \notin L^2(0,\infty)$ as well, but merely give the last two theorems as representative results.

Chapter 7

Future Work

7.1 Introduction

In this short chapter, we explore some of the directions in which the work in this thesis might develop, or discuss briefly some work that has been done during the period of research, but not included in the final thesis.

In fact, the topics we expand on are:

- how one might proceed with discretisation when f is superlinear (in the sense that $f'(x) \to \infty$ as $x \to 0$, for instance);
- how one might attempt to recover in discrete time the type of Hartman–Wintner established in Chapter 2;
- how one might tackle the quadrature of the perturbation if it were more irregular than the monotone or subexponential cases considered here;
- how large perturbations or growth problems might be tackled in continuous time.

7.2 Split Step Method and superlinearity

In Chapter 2, we considered the rates of decay of continuous ODEs and SDEs when f is asymptotic preserving. To obtain the Hartman–Winter results in Chapter 2, f should be sublinear. For Hartman–Grobman results, in Chapter 3, we do not restrict attention to the sublinear case, and we can have $f(x)/x \to \infty$ as $x \to 0$.

On the side of numerical methods, we show in Chapter 4 that convergence to the equilibrium of the unperturbed problem is recovered by the split–step method without restricting the nonlinearity. However, in Chapter 5 when we wish to study rates of decay we only cover the case when f is sublinear. We leave the situation when f is superlinear (in the sense that $f(x)/x \to \infty$ as $x \to 0^+$) to one side, apart from showing that constant step sizes would not recover the rates of decay.

We do not believe however, that this is due to any particular technical barrier; indeed, although the work in not included in this thesis, much of the details have been successfully worked out, particularly if we reimpose the condition that f is in $RV_0(1)$, which we have successfully managed to remove from continuous time theorems in Chapters 2 and 3, for example. In this situation, results can be proven very much along the lines of Chapter 5, using comparison and induction arguments, and obtaining the correct estimates using proof by contradiction (or "time of the first breakdown") arguments.

The main problem that one can face is that the step size should no longer be taken constant, but rather should vary with the iteration count n (and indeed vanish as $n \to \infty$ if $f(x)/x \to \infty$ as $x \to 0$). This is reasonable in view of the fact that unperturbed problem with a constant step size would not recover the correct rate of decay, as we showed at the start of Chapter 5. In the deterministic case, we choose h_n deterministically (and independently of the state) in such a manner that it would successfully discretise the unperturbed ODE. It is, incidentally, an interesting question as to whether more slowly decaying perturbations may need a less computationally expensive scheme: this is not implausible, because the slow decay in the perturbation may offset the affect of the strong nonlinearity.

In the stochastic case, we would again make this time step deterministic so that the discretisation stochastic of the Itô integral preserves the Gaussianity of the integral, which arises from the fact that the diffusion coefficient is a deterministic function. One reason to do this is that the discrete martingale results which apply the Gaussian case form Chapter 4 can be applied. However, on a less technical level it also seems more satisfying to preserve the finite–dimensional distributions of the stochastic terms if this can be achieved, and not to introduce state–dependence into the stochastic terms in the discretisation, especially as this state–dependence was explicitly absent in the original SDE.

We mention briefly here the approach dealing with the case when

$$\lim_{x \to 0^+} \frac{f(x)}{x} = +\infty.$$

We now take a variable step size $h_n > 0$ such that $\sum_{j=0}^{\infty} h_j = +\infty$; define

$$t_n = \sum_{j=0}^{n-1} h_j$$

so that $t_{n+1} - t_n = h_n$ and h_n is the step size between time t_n and t_{n+1}. Hence the split-step scheme (SSBE) is now given by

$$x_h(0) = \zeta; \tag{7.2.1a}$$

$$x_h^*(n) = x_h(n) - h_n f(x_h^*(n)), \quad n \geq 0; \tag{7.2.1b}$$

$$x_h(n+1) = x_h^*(n) + \gamma_h(n+1); \quad n \geq 0. \tag{7.2.1c}$$

In this case $\gamma_h(n+1)$ approximates $\int_{t_n}^{t_{n+1}} g(s)ds$. We define

$$\Gamma_h(n) := -\sum_{j=n}^{\infty} \gamma_h(j+1)$$

and $z_h(n) = x_h(n) - \Gamma_h(n)$ so that $z_h(n)$ obeys

$$z_h(n+1) = z_h(n) - h_n f(z_h(n+1) + \Gamma_h(n)), \quad n \geq 0.$$

We want to choose h_n well. In view of the fact that the perturbation is present, and thinking also about the discretisation of SDE, we will not in the first instance seek to make the step size h_n state-dependent. One reason for this, in the SDE case at least, is that we can take

$$\gamma(n+1) = \sqrt{h_n}\sigma(t_n)\xi(n+1)$$

where $(\xi(n))_{n\geq 0}$ is a sequence if independent and identically distributed $N(0,1)$ random variables to approximate

$$I(n+1) := \int_{t_n}^{t_{n+1}} \sigma(s)dB(s).$$

If the h_n's are deterministic, the approximation shares important properties of the stochastic integral; namely that it is Gaussian distributed, and independent of its past values (i.e, $\gamma(n+1)$ is independent of $\gamma(j)$, $j \leq n$ and $I(n+1)$ independent of $I(j)$, $j \leq n$).

The other important question is how rapidly should $h_n \to 0$ as $n \to \infty$. If there is no perturbation, we already have an answer, supplied in the PhD thesis of Colgan [29]. We take

$$h_n = \frac{\delta(x_h(n))x_h(n)}{f(x_h(n))},$$

where $\delta(x) \to \Delta \in [0,1)$ as $x \to 0$. Therefore, our plan is to consider the mesh for the perturbed equation by using the mesh prescribed for the unperturbed equation. This means that we compute $(y(n))_{n\geq 0}$ by

$$y_{n+1} = y_n(1 - \delta(y_n)), \quad n \geq 0; \quad y_0 = 1,$$

and take

$$h_n = \frac{\delta(y_n)y_n}{f(y_n)}.$$

This is the h_n we use in the spilt–step scheme for the perturbed equation in the case when $f(x)/x \downarrow \infty$. We note that already it is known that $\sum_{j=0}^{\infty} h_j = +\infty$, so the convergence of the split step scheme to zero is assured by our earlier remarks.

It should be noted that although the scheme to generate x_h is semi-implicit, the scheme that generates the sequence (h_n) is explicit, and therefore fast.

7.3 Numerical schemes to recover Hartman–Wintner Theorems

As already pointed out in Chapter 2, the discretisation of unperturbed equations do not necessarily recover the exact rate of decay when $f(x)$ is not $o(x/\log\frac{1}{x})$ as $x \to 0^+$. Therefore, we have not attempted for perturbed equations in Chapter 5 or 6 to prove results which relate $x_h(n)$ (or $X_h(n)$) to $F^{-1}(nh)$ directly. In order for this to be done, it would seem that it would first be necessary to improve the situation regarding the unperturbed problems, possibly by means of a more sophisticated discretisation method, or by taking vanishing time steps.

There is one situation, however, where Hartman–Wintner and numerical results may agree, and that is in the case that F^{-1} is asymptotic preserving. Part of the research carried out during work on this thesis involved proving discrete analogues of results in Appleby and Patterson, presented in [11], in the case when $f \in \text{RV}_0(\beta)$ for $\beta > 1$. In this case, as F^{-1} is in $\text{RV}_\infty(-1/(\beta-1))$, it is asymptotic preserving, and Hartman–Wintner type results can be proven for constant step size split step methods. However, we have not included this work in the thesis, partly owing to length restrictions, and partly because we believe that both the discrete and continuous results can be generalised and simplified using the approach based on asymptotic preserving functions presented in Chapter 3.

One possible solution for perturbed equations is to embrace non–constant step sizes, and to show that the results in Chapter 2 could be recovered with sufficient computational effort. Another is to try to establish the class of perturbed problems to which the constant step size methods would still give satisfactory results. We note that this can be strictly larger than the class of equations in which the nonlinearity f gives rise to an asymptotic preserving F^{-1}.

7.4 Quadrature of the Perturbation

We now expand on the discussion of this matter started in Chapter 6. Recall from that discussion that if one examines the discrete results in this thesis, a common pattern often occurs. We suppose that the (uniform time step) discretisation gives a good asymptotic approximation of quantities like $\int_t^\infty g(s)\,ds$ or $\int_t^\infty \sigma(s)\,dB(s)$. Given that this good approximation is achieved, our theorems usually state that the solution of the split step method with constant step size recovers the asymptotic behaviour of the corresponding ODE. For example, a continuous result of the type

$$\liminf_{t\to\infty} \frac{F(|\Gamma(t)|)}{t} = \lambda_\Gamma \in (0,1) \quad \text{implies} \quad \liminf_{t\to\infty} \frac{F(|X(t))}{t} = \lambda_\Gamma$$

has the discrete analogue

$$\liminf_{n\to\infty} \frac{F(|\Gamma_h(n)|)}{nh} = \lambda_\Gamma(h) \quad \text{implies} \quad \liminf_{n\to\infty} \frac{F(|X_h(n)|)}{nh} = \lambda_\Gamma(h).$$

We have shown in this thesis that when Γ (or perhaps g) has an additional nice property (for instance g may be decreasing, or subexponential), then one can show that the asymptotic behaviour of $\Gamma_h(n)$ and $\Gamma(nh)$ match sufficiently well in the sense that $\lambda_\Gamma(h) = \lambda_\Gamma$ or if $\lambda_\Gamma \geq 1$, then $\lambda_\Gamma(h) \geq 1$. However, if g is more irregular, it might not be the case that $\lambda_\Gamma(h)$ and λ_Γ will agree. In this situation, it seems that one would have to make a more careful discretisation which compensates for rapid change in g. It is then a new problem to show that this time-varying time stepping will still preserves the asymptotic behaviour in the solution of the split step equation. This should still be possible in principle: as seen in Chapter 4, convergence at least can still be secured if the time step is forced to vary, provided that the sum of the time steps still diverges.

Restricting attention to the deterministic case for a moment, and considering time points t_n generated by $t_{n+1} = t_n + h_n$ it would appear that a key property is to make the discretisation in such a way that

$$\gamma_h(n+1) \sim \int_{t_n}^{t_n+h_n} g(s)\,ds \text{ as } n \to \infty,$$

because such a method would replace the integrability of g with the summability of γ. Since F is usually asymptotic preserving (due to the asymptotic preserving property of f), we should therefore expect by Toeplitz lemma to get

$$F(|\sum_{j=n}^{\infty} \gamma_h(j+1)|) \sim F(|\int_{t_n}^{\infty} g(s)\,ds|), \text{ as } n \to \infty,$$

and therefore, when $t_n \to \infty$ as $n \to \infty$, we should get

$$\frac{F(|\sum_{j=n}^{\infty} \gamma_h(j+1)|)}{t_n} \sim \frac{F(|\int_{t_n}^{\infty} g(s)\,ds|)}{t_n}, \text{ as } n \to \infty.$$

The right hand side looks at a quantity along a sequence of times which has been assumed to have good limiting properties in continuous time. Therefore, we should check that our approximation also can be extended to continuous time by simple interpolation, and make adjustments to the mesh to ensure that this can be achieved.

In the case that g is positive and in C^1 with derivative bounded by a (known) increasing function μ, say, we can write $\gamma_h(n+1) = h_n g(t_n)$ and obtain conditions on h_n for which

$$\lim_{n\to\infty} \frac{\int_{t_n}^{t_n+h_n} g(s)\,ds}{h_n g(t_n)} = 1,$$

and which allow h_n and therefore t_n to be constructed.

It is an obvious observation that good progress might also be made in the stochastic case, with σ^2 replacing g in the above discussion.

7.5 Asymptotic behaviour with large perturbations

The Liapunov exponent results in Chapter 3 do not give precise behaviour for the solution of the perturbed ODE when the perturbations are large in the sense that

$$\lambda_\Gamma = \liminf_{t\to\infty} \frac{F(|\Gamma(t)|)}{t} = 0.$$

Indeed one can at best conclude that that $\lambda_x = 0$. This opens up the question as to how one might get more precise asymptotic information, and this surely starts by a more precise assumption on the asymptotic behaviour of Γ (or perhaps when $g \notin L^1(0,\infty)$ and neither Γ nor λ_Γ are well–defined).

Also, although the results in this thesis are confined to solutions tending to a finite limit, these methods in this work are equally suited to dealing with growing solutions. It is therefore an interesting programme of work to see whether the asymptotic behaviour could be worked out in continuous time (and in the discrete case also) when g or Γ are "large" relative to solutions of unperturbed equations, and also to consider unbounded solutions. At this moment, however, our initial investigations rely on pointwise conditions on g. This leads to results in a different direction and type to those in this thesis, where we have tried to determine asymptotic behaviour of solutions of equations using integral, rather than pointwise conditions, on g. Results in this direction for a more limited class of equations can be seen in [13] or for functional differential equations with perturbations in [3].

Thus, at this moment, apart from considerations as to the length of this work, the results are of a sufficiently different type to those in this thesis to exclude them from our presentation.

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