A New Method for the Determination of the Locking Range of Oscillators

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***Abstract*— A time-domain method for the determination of the injection-locking range of oscillators is presented. The method involves three time dimensions: the first and the second are warped time scales used for the free-running frequency and the external excitation, respectively and the third is to account for slow transients to reach a steady-state regime. The locking range is determined by tuning the frequency of the external excitation until the oscillator locks. The locking condition is determined by analyzing the Jacobian matrix of the system. The method is advantageous in that the computational effort is independent of the presence of widely separated time constants in the oscillator. Numerical results for a Van Der Pol oscillator are presented.**

I. Introduction

All free-running oscillators can exhibit the phenomenon of injection locking when perturbed by an external signal whose frequency is close to the free-running frequency. This phenomenon whereby the natural frequency of the oscillator changes to be identical to the external frequency is found in all branches of science.

In many fields of circuit design, the injection locking phenomenon is exploited for beneficial purposes. RF Phase-Locked Loops (PLL) are widely used in wireless communication applications such as frequency synthesis. In the feedback loop of a frequency synthesizer, a frequency prescaler is usually employed to divide the frequency by a fixed number. While static dividers offer a wide bandwidth, Injection Locked Frequency Dividers consume significantly less power [1], [2] and therefore are preferable for low-power applications. Coupled array oscillators for phase control in quasi-optical power combining also involve the technique of injection locking. Thus an accurate and efficient technique for the determination of injection locking phenomena is important for today's circuit design industry [1], [2], [16], [17]. Various analytical and numerical approaches have been suggested for predicting the locking range, *e.g.* the classic Adler approach in [14] and more recently [17]–[19].

Many early circuits to illustrate the injection-locking phenomena in oscillators were built using vacuum tubes [13]. In this contribution, the focus is on the accurate determination of the locking range of such an oscillator using a multitime scale model.

Simulation techniques involving multiple timescales have recently been applied to great effect for circuits with widely separated time constants. References [3]–[11] are some examples. Roychowdhury *et al.* proposed the term Multitime Partial Differential Equation (MPDE) when these techniques are applied to circuits involving straightforward envelope modulation [3], and Warped MPDE (WaMPDE) for the techniques used to analyze oscillators or circuits with FM signals [4]. An even more general approach for the analysis of oscillators termed GeMPDE was also presented [4], [6]. In this paper we use these names and a similar notation. Mixed frequency-time methods involving Harmonic Balance in the two periodic dimensions have been proposed [10], [11] to analyze PLL circuits.

In this work, a time-domain formulation of the WaMPDE in conjunction with the bisection method is employed to identify the input frequencies to an oscillator at which locking occurs. Since an injection-locked oscillator is an autonomous system synchronized to an external signal, the oscillator autonomous solution is considered in a warped time scale. The input signal introduces a second warped time scale. The warped time scales enable a slow variation of the input and the local frequency. A third dimension is employed for the transient evolution. The method is advantageous in that it is independent of the time constants of the oscillator. Only a few transient time-steps are required regardless of how many periods are required to lock the oscillator. Brute force transient simulation would be prohibitive as many thousands of periods are required to lock an oscillator with a high Q factor and each period requires many small time steps for accurate simulation. Also for small injections close to the free-running frequency, identification of locking from time-domain waveforms becomes very difficult.

Section II reviews the WaMPDE technique. Section III describes solving for the locking range of an injection-locked oscillator using the WaMPDE. Section IV illustrates the method of solution with a simulation of a forced Van Der Pol oscillator.

II. Warped Multitime Partial Differential Equation Approach

Suppose that an autonomous nonlinear circuit is described by:

$$\dot{x} = f(x) + b(t)\ . \tag{1}$$

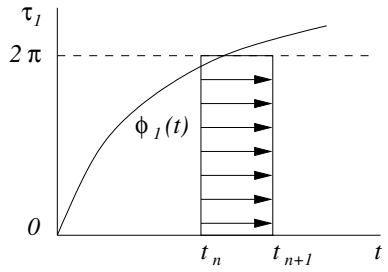

Fig. 1. Relation between τ_1, t and $\phi_1(t)$

where $b(t)$ is the excitation vector, $x(t)$ is a vector of state variables, and f is a nonlinear function. In the WaMPDE approach [4], the variations of the state variables are decomposed in several time dimensions. Eq. (1) is transformed into the following partial differential equation:

$$\sum_{i=1}^{p}\left(\omega_i(t)\frac{\partial \hat{x}}{\partial \tau_i}\right)+\frac{\partial \hat{x}}{t}=f(\hat{x})+\hat{b}(\tau_1,\ldots,\tau_p,t)\,, \qquad (2)$$

where $p+1$ is the number of time dimensions being considered, and $\hat{x}$ and $\hat{b}$ denote the multi-dimensional variables corresponding to x and b, respectively. $\tau_1,\ldots,\tau_p$ are warped time scales and t is the time scale of the original equation. The local angular frequency in the i^{th} dimension is denoted ω_i. The phase (ϕ_i) at each warped time dimension is determined as follows

$$\phi_i(t)=\int_0^t \omega_i(t)dt\,.$$

It was shown in [4] that

$$x(t)=\hat{x}(\phi_1(t),\phi_2(t),\ldots,\phi_p(t),t)\,.$$

Figure 1 shows the relation between τ_1, t and $\phi_1(t)$ for a case with $p=1$. The rectangle with arrows indicates how the PDE in Eq. (2) is integrated along τ_1 and t.

The WaMPDE can be solved using exclusively time-domain approaches or mixed time-frequency domain methods. In any case, periodic boundary conditions must be set for $\tau_1,\ldots,\tau_p$. For instance, for τ_1 this condition can be expressed as

$$\hat{x}(\tau_1,\tau_2,\ldots,t)=\hat{x}(\tau_1+2\pi,\tau_2,\ldots,t)\,.$$

In addition, in order to obtain a unique solution it is necessary to impose a 'phase condition' for each warped time scale with unknown local frequency [4]. This is because there are infinite solutions of Eq. (2) that only differ in the phase in the $\tau_1,\ldots,\tau_p$ time scales. The consequence of this is that after discretization (in frequency or time domain) there are more unknowns than equations (p extra unknowns if all ω_i are unforced). The phase condition is a set of extra equations that are used to make the Jacobian matrix of the system to be full rank. When Eq. (2) is solved in the frequency domain it is a common practice to impose the phase condition by forcing a particular phase in one of the state variables, *e.g.* the imaginary part is set to zero. This is equivalent to selecting one of the infinite solutions that only differ in the phase. In time domain, imposing the phase condition is more difficult as the phase of the state variables is not directly available. The phase condition used in this work is discussed in the following Section.

III. Determination of the Locking Range

The WaMPDE in Eq. (2) is used with three timescales to determine the locking range of an oscillator. Eq. (2) thus takes the form:

$$\omega(t)\frac{\partial \hat{x}}{\partial \tau_1}+\omega_{in}\frac{\partial \hat{x}}{\partial \tau_2}+\frac{\partial \hat{x}}{\partial t}=f(\hat{x})+b(t)\,. \qquad (3)$$

The oscillator autonomous solution is considered in the τ_1 time scale. The input signal introduces a second time scale — τ_2 and ω_{in} is the input frequency. τ_1 and τ_2 are warped time scales to enable the slow variation of the input and the local frequencies. A third dimension, t, is employed for real time. Eq. (3) may be solved with time-domain, mixed time-frequency domain methods or methods based on other basis functions such as wavelets. In this contribution, a time-domain method is presented. The derivatives with respect to the warped variables are evaluated using the five-point centered difference formula [20]:

$$\frac{\partial \hat{x}_k}{\partial \tau_1}\approx\frac{x_{k-2}-8x_{k-1}+8x_{k+1}-x_{k+2}}{12h}, \qquad (4)$$

where h is the time-step in the τ_1 scale and k denotes the index in the same scale. The Backward Euler (BE) rule is used in the t time scale to calculate the transient evolution. One useful characteristic of the BE rule is that it introduces artificial damping if the time step is large. This helps to attenuate undesired oscillations along the t dimension.

The additional equation required to solve for the local frequency (the 'phase condition') is implemented in this work by making one sample of the derivative of one of the state variables equal to zero:

$$\frac{\partial \hat{x}_0}{\partial \tau_1}=0\,. \qquad (5)$$

This condition is guaranteed to be satisfied by oscillatory and non-oscillatory solutions if they have a continuous derivative. The derivative only has to be continuous for the state variable where the phase condition is imposed.

On replacement of the time-domain derivatives in Eq. (3) with Eq. (4), use of the BE rule and the periodic boundary conditions, the resulting nonlinear algebraic equations may be solved in conjunction with Eq. (5) using Newton's method [15], [20].

The procedure to determine the lower limit of the locking range is described next. The process is started by running a transient simulation with the following initial condition: the free-running oscillation is set along τ_1 and no oscillations along τ_2. The frequency of the excitation is set to a value known not to lock the oscillator. Eq. (3), along with the boundary conditions is then solved for several time-steps along t until the steady-state condition is achieved. In order to minimize the number of time steps necessary to achieve the steady-state condition (along t), the time step size in this dimension is exponentially increased. When the difference

between two steps along t is small enough, the analysis is switched from a 3-D transient analysis to a 2-D steady-state analysis by making the derivatives with respect to t equal to zero. This procedure results in an autonomous solution along τ_1, if it exists. A similar idea was used in Reference [9] to avoid unstable equilibrium points. At each time-step, but especially when solving for the steady-state condition, the Jacobian matrix that is employed in Newton's method is noted. If the column that corresponds to the local frequency is near zero, this implies the system is becoming independent of the local frequency because there are no more oscillations along τ_1 (to see the proof of this refer to [5]). Thus, an autonomous solution along τ_1 is no longer possible and only the forced oscillations are present. Locking has happened. Note that the solution of the locked oscillator is not calculated because at this condition the Jacobian matrix is singular and so the Newton iterations are stopped. Thus, the method relies on the fact that the 'initial guess' for the Newton method that solves the 2-D steady-state solution is very close to the solution.

After the steady-state solution is obtained (assuming no locking), the input frequency (ω_{in}) is progressively incremented. For each increment, a transient analysis is performed and followed by a steady-state analysis. If the oscillator is deemed locked, further analysis may be necessary to determine the starting point of the locking range to a certain degree of accuracy. The bisection method [20] is used in this work. Therefore, the frequency step by which ω_{in} is incremented is halved and the procedure is re-started with the initial conditions from the last frequency point at which there was no locking. This is very important because if the system is initialized with no oscillations along τ_1, a non-physical locked solution may be obtained (in addition to a singular Jacobian matrix). This progressive halving of the frequency-step increment is repeated until the lower limit of the locking range is attained to a certain tolerance.

A similar procedure is subsequently performed to determine the upper limit of the locking range.

IV. Simulation of an Injection Locked Van Der Pol Oscillator

A forced Van Der Pol Oscillator is employed to model a microwave negative oscillator and to illustrate injection locking phenomena [12]. The phase dynamics of the Van Der Pol Oscillator are the same as the Adler Equation [14] used to describe the phase dynamics of the injection locked microwave negative oscillator. The standard Van Der Pol equation has the following form:

$$-V_{inj}\sin\omega_{in}t = \ddot{V}_x - \varepsilon(\alpha - V_x^2)\dot{V}_x + \omega_0^2 V_x \, .$$

In this work the following parameters are used: $\varepsilon = 0.2$, $\alpha = 1$, $\omega_0 = 1$ and $V_{inj} = 0.7$. This oscillator exhibits widely separated time constants when excited with a frequency near the edge of the locking range. Fig. 2 shows that when excited with a frequency of 0.714 rad/s (which is known to lock the oscillator), the steady state is not yet achieved after a simulation time equal to 1000 periods of the free-running oscillation.

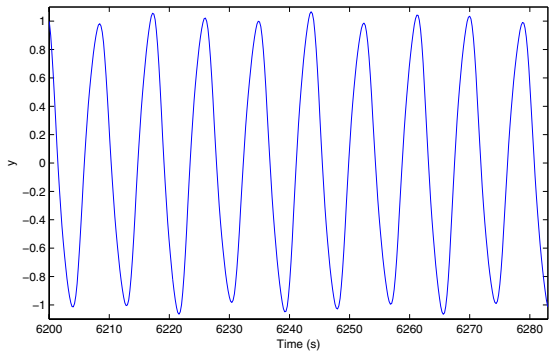

Fig. 2. Steady state is not yet achieved after 1000 oscillations

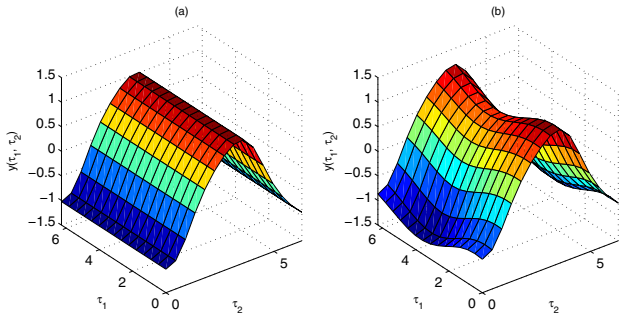

Fig. 3. Steady state: (a) $\omega_{in} = 0.714$ rad/s and (b) $\omega_{in} = 0.712$ rad/s

A transient analysis of the oscillator using the WaMPDE approach with three time scales as described in Section III is presented now. Each step along t involves the calculation of a bi-dimensional steady-state problem with a grid with 15 points in each dimension, τ_1 and τ_2. Fig. 3 show the steady state solutions for two different values of ω_{in}. Only 27 steps along t were necessary to simulate an elapsed time of 75000 periods and obtain the locked steady-state of Fig. 3a. The locking condition can also be observed in Fig. 4. The Jacobian column approaches zero if the oscillator locks. Note that the time to reach the steady state increases as ω_{in} approaches the locking range limit.

The algorithm to find the locking range is demonstrated now. To determine the lower limit of the frequency range, the initial angular frequency is set to 0.7 rad/s with an initial frequency step of 0.1 rad/s and a tolerance of 0.001 rad/s. After 64 iterations (this includes rejected time steps) the lower locking limit of 0.714 rad/s is determined. Fig. 5 shows the variation of the forcing frequency and the norm of the Jacobian column as a function of the iteration number. If no locking is detected, the frequency is increased. Each time that locking is detected, a lower frequency is tried and the initial conditions are reset to the last known oscillatory solution. That is the reason for the peaks observed in the norm of the Jacobian column.

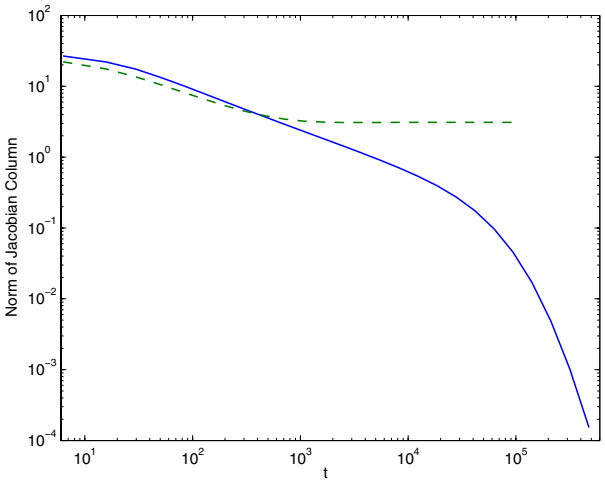

Fig. 4. Comparison of the norm of the Jacobian column with $\omega_{in} = 0.714$ rad/s (continuous line) and $\omega_{in} = 0.712$ rad/s (dashed line)

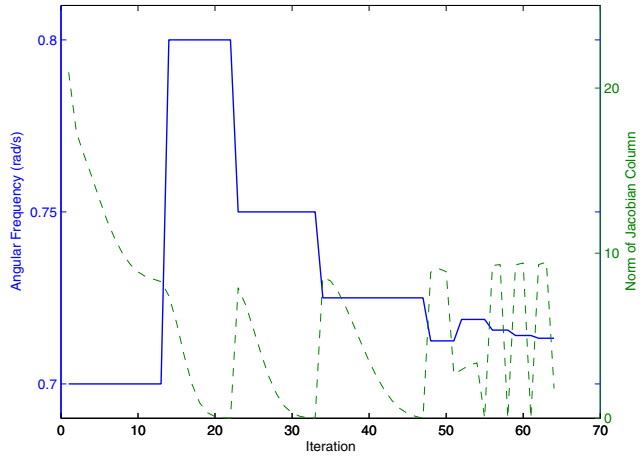

Fig. 5. Local frequency and norm of Jacobian column (dashed line) vs. iteration number

A similar procedure was followed to find the upper locking limit which is equal to 1.217 rad/s.

V. Conclusions

The Warped Multitime scale approach is employed to simulate an injection-locked Van Der Pol oscillator. The technique enables a fast and accurate method of determining the locking range and is applicable to any kind of oscillator that can be described by a set of differential and algebraic equations since no assumptions about the oscillator are made except that it has locked and unlocked states. This is an important contribution as the determination of the Injection Locked Range for circuits exploiting oscillator injection locking phenomena is important in design and analysis work.

It was shown that the time to determine the locking condition using the proposed method is independent of the presence of widely separated time constants in the system being analyzed.

The proposed method works in time domain. Thus adaptive grids could be used and this should result in a very efficient method for strongly nonlinear systems. That and the analysis of more complex oscillators will be the subject of future research.

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