A TLM method for steady-state convection-diffusion: some additions and refinements

Alan Kennedy[1,*,†] and William J. O'Connor[2]

1*School of Mechanical and Manufacturing Engineering, Dublin City University, Glasnevin, Dublin 9, IRELAND*
2*School of Electrical, Electronic and Mechanical Engineering, University College Dublin, Belfield, Dublin 4, Ireland*

SUMMARY

A recent paper introduced a novel and efficient scheme, based on the Transmission Line Modelling (TLM) method, for solving steady-state convection-diffusion problems. This paper shows how this one-dimensional scheme can be adapted to include reaction and source terms and how it can be implemented with non-equidistant nodes. It introduces new ways of calculating the necessary model parameters which can improve the accuracy of the scheme, shows how steady-state solutions can be obtained directly, and compares results with those from two finite difference (FD) methods. While the cost of implementation is higher than for the FD schemes, the new TLM scheme can be significantly more accurate, especially when convection dominates.

INTRODUCTION

The convection-diffusion equation (CDE) describes physical processes in the areas of pollution transport, biochemistry, semiconductor behaviour, heat transfer, and fluid dynamics [1-3]. Most numerical schemes have difficulties accurately modelling convection-dominated systems unless fine meshes are used, resulting in significant computational costs [1-4]. Methods specifically designed to reduce errors in such situations, and thereby allow less fine meshes, are generally less accurate when diffusion dominates. A recent paper presented a novel Transmission Line Modelling (TLM) scheme which can produce accurate results no matter which term dominates [5]. This high level of accuracy is confined to steady-state problems, the scheme exhibiting significant numerical convection and diffusion under transient conditions unless short time steps are used [6].

In many applications the equations to be solved include reaction and/or source terms. Reaction terms describe changes in the concentration of the convecting-diffusing variable due to chemical reactions, for example, or charge-recombination in semiconductors. Examples of the need for source terms are

[**]Correspondence to: Alan Kennedy, School of Mechanical and Manufacturing Engineering, Dublin City University, Glasnevin, Dublin 9, IRELAND
[†]E-mail: alan.kennedy@dcu.ie

the injection of pollutant in pollution modelling or local heat generation in heat transfer problems. The general one-dimensional steady-state convection-diffusion equation then becomes

$$0 = \frac{d}{dx}\left(D(x)\frac{dV}{dx} \right) - v(x)\frac{dV}{dx} - V\frac{dv}{dx} - K(x)V + S(x) \tag{1}$$

where $D(x)$ is the diffusion coefficient, $v(x)$ is the convection velocity, $K(x)$ is the reaction coefficient, and $S(x)$ is the source function. This paper extends the TLM scheme to model such general convection-diffusion problems. It also shows how non-equidistant nodes can be incorporated, allowing the node density to be increased where most needed.

The differential equation governing the voltage along a lossy transmission line (TL), i.e. a pair of conductors, with distributed resistance, capacitance and inductance $R_d(x)$, $C_d(x)$, and $L_d(x)$ respectively (all per unit length and varying with position, x), and with an additional distributed current source, $I_{Cd}(x)$, is, under steady-state conditions (see Appendix),

$$0 = \frac{d}{dx}\left(\frac{1}{R_d(x)C_d(x)}\frac{dV}{dx} \right) - \frac{d}{dx}\left(\frac{1}{C_d(x)} \right)\frac{1}{R_d(x)}\frac{dV}{dx} + \frac{I_{Cd}(x)}{C_d(x)} \tag{2}$$

This will be equivalent to Eq. (1) if the TL properties satisfy

$$D(x) = \left[R_d(x)C_d(x) \right]^{-1} \tag{3}$$

$$Pe(x) := \frac{v(x)}{D(x)} = C_d(x)\frac{d}{dx}\left(\frac{1}{C_d(x)} \right) \tag{4}$$

(where Pe is the Peclet number) and

$$S(x) - V(x)K^*(x) = I_{Cd}(x)C_d(x)^{-1} \tag{5}$$

where $K^*(x) = K(x) + dv/dx$. From Equation (4), it can be seen that a spatial variation in the distributed capacitance can produce a voltage drift analogous to convection.

The lossy TLM method, originally developed by Johns in 1977 [7], models a transmission line, calculating the voltage at a series of points (or "nodes") in space, and thereby indirectly solving the diffusion equation (if there is no spatial variation in C_d). It is well established as a technique for modelling particle diffusion and heat transfer in one, two, and three dimensions [8-10]. Current sources have been added previously to the scheme to model source and reaction terms [8, 11]. This paper shows how they can be used for the same purpose in convection-diffusion models.

Traditional lossy TLM is a straightforward explicit time-domain method [8]. To find a steady-state solution it must be run until transients reduce to an acceptable level [12, 13]. It is shown here how steady-state solutions can be found directly, thus increasing efficiency and avoiding difficulties associated with the time-domain scheme.

The basic convection-diffusion scheme without current sources solves the "non-conservative" form of the CDE without reaction or source terms

$$0 = \frac{d}{dx}\left(D(x)\frac{dV}{dx} \right) - v(x)\frac{dV}{dx} \tag{6}$$

so called because it does not include the second convection-related term (Vdv/dx) in Equation (1). A previous paper has described a method for including this term (and which could also be used to model other reaction terms) [5]. A more accurate technique is proposed here.

To calculate the parameters required for a TLM model it is necessary to solve an ODE that depends on $Pe(x)$ (Equation (4)) and to integrate the result over space. If $Pe(x)$ varies over space then a closed-form solution may not be available and the cost of calculating the parameter values numerically is similar to that of solving the CDE itself. An efficient, but less accurate, alternative is to assume that v and D are both constant over space when deriving the necessary equations [5]. The resulting formulae can then be used to calculate the TLM model parameters at each node using the local values of v and D. An alternative approach which assumes that v and D vary in a piecewise-constant fashion is described below.

TL PROPERTIES

If D and v are both constant over space then so is their ratio, Pe. Solving Equation (4) gives

$$C_d(x) = c\exp(-Pex) \tag{7}$$

where c is a constant. The relationship between the distributed resistance and D is (from Equation (3))

$$R_d(x) = \left[DC_d(x) \right]^{-1} \tag{8}$$

In the 1D TLM scheme, both space (i.e. the length of the TL) and time are divided into finite increments. The method keeps track of voltage pulses that travel along the TL in both directions. To ensure synchronisation of the scheme, the properties of the TL must be such that the time increment corresponds to the pulse travel time between any pair of adjacent nodes (the space increment), and is the same for all pairs [8]. The propagation velocity, u, (i.e. the velocity of any pulse) for a TL depends on the distributed capacitance and inductance, $u(x) = \left[L_d(x)C_d(x) \right]^{-\frac{1}{2}}$, and the impedance of a TL is $Z(x) = \left[L_d(x)C_d^{-1}(x) \right]^{\frac{1}{2}}$. Combining these two equations gives

$$Z(x) = \left[u(x)C_d(x) \right]^{-1} \tag{9}$$

Equation (5) gives $I_{Cd}(x)$ in terms of $S(x)$ and $K^*(x)$

$$I_{Cd}(x) = S(x)C_d(x) + K^*(x)V(x)C_d(x) \tag{10}$$

These equations give the required properties of the TL (i.e. $R_d(x)$, $C_d(x)$, $Z(x)$, and $I_{Cd}(x)$) to model given values of the physical variables $D(x)$, $v(x)$, $K^*(x)$, $S(x)$, and $u(x)$, and can be used to calculate the necessary parameters for the TLM model. The fact that the current source is dependent on $V(x)$ need not significantly affect the complexity of the scheme as will be shown below.

THE TLM METHOD

In one-dimensional TLM a continuous lossy TL is approximated by a series of lossless (i.e. with zero resistance) TL segments joined by pairs of lumped resistors as in Figure 1. The distributed current source is approximated by a current source connected at each node. The voltage along the TL is calculated at these nodes at discrete time steps Δt apart. The distance between adjacent nodes n and n+1 is Δx_{n+1} and the two TL segments in the "connecting line" between these nodes are each $\frac{1}{2}\Delta x_{n+1}$ long. The two resistors in each connecting line are equal, as are the two TL segment impedances. There are other possible configurations but only that shown in Figure 1 is considered here [8, 14, 15].

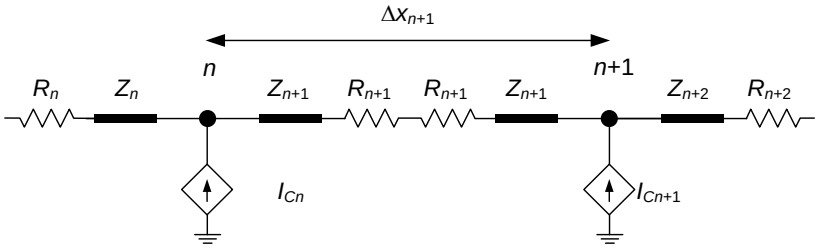

Figure 1: Two nodes, numbered n and n+1, in a TLM network. One conductor of each lossless TL segment is shown, represented by a thick line. The second conductor is connected to ground and is not shown.

Dirac voltage pulses propagating through the network travel unchanged along the lossless TL segments, the properties of which are such that a pulse entering at one end will arrive at the other end $\frac{1}{2}\Delta t$ later. Pulses, therefore, travel between adjacent nodes in one time step. The closer the nodes are together, the higher the propagation velocity must be.

At each time step there are voltage pulses incident at each node, one from the left (*Vil*) and one from the right (*Vir*). These instantaneously raise the "node voltage" (*Vn*) which is common to the lines meeting at the node. The difference between the node voltage and the incident voltages leads to further pulses being scattered from the node, one to the left (*Vsl*) and one to the right (*Vsr*).

The values of *Vn*, along the line and over time, represent the solution of the equation being modelled. Two rules can be used to determine any node voltage in terms of incident pulses. Firstly, the total current entering a node at any time step must equal the total current leaving that node, and so, the current associated with the two incident pulses (the voltage divided by the impedance in each case) plus the current from the current source at node n at time step k must equal the current associated with the pulses scattered from that node

$$\frac{{}_kVil_n}{Z_n} + \frac{{}_kVir_n}{Z_{n+1}} + {}_kI_{Cn} = \frac{{}_kVsl_n}{Z_n} + \frac{{}_kVsr_n}{Z_{n+1}} \tag{11}$$

The second rule is that for each TL segment connected to a node, the voltage scattered into the segment equals the difference between the node voltage and the voltage incident from that segment

$${}_kVsl_n = {}_kVn_n - {}_kVil_n \tag{12}$$

$${}_kVsr_n = {}_kVn_n - {}_kVir_n \tag{13}$$

Using Equations (12) and (13) to replace the scattered voltages in Equation (11) and rearranging gives the node voltage at node n and time step k in terms of the incident voltages and ${}_kI_{Cn}$

$${}_kVn_n = \frac{2\,{}_kVil_n + 2P_n\,{}_kVir_n + Z_n\,{}_kI_{Cn}}{1+P_n} \tag{14}$$

where $P_n = Z_n/Z_{n+1}$ is the "impedance ratio" at node n.

The current added at node n by the lumped current source, ${}_kI_{Cn}$, is a function of the distributed current source, $I_{Cd}(x)$, between nodes $n-1$ and $n+1$ at time step k for the TL being modelled. From Equation (5), this current has two components; one associated with $S(x)$, and one with $K^*(x)V(x, t)$. It is assumed here, for the sake of simplicity, when considering any node n, that $V(x)$ at time step k between nodes $n-1$ and $n+1$ is constant and equal to the node voltage, ${}_kVn_n$. This allows the introduction of two new node parameters, X_n and Y_n, such that

$$Z_n\,{}_kI_{Cn} = X_n - Y_n\,{}_kVn_n \tag{15}$$

It will be shown below how these are related to $I_{Cd}(x)$. Replacing $Z_{nk}I_{Cn}$ in Equation (14) and rearranging gives

$${}_kVn_n = \frac{2\,{}_kVil_n + 2P_n\,{}_kVir_n + X_n}{1+P_n+Y_n} \tag{16}$$

Any pulse leaving a node will arrive at an impedance discontinuity $\frac{1}{2}\Delta t$ later due to the presence of resistors in the network. A fraction (τ, the transmission coefficient, where $0 \le \tau < 1$) travels on, arriving at the adjacent node at the next time step. The remaining fraction ($1-\tau$) is reflected back, arriving at the node from which it originated at the next time step. The incident pulses at node n at time step $k+1$ are therefore

$${}_{k+1}Vil_n = (1-\tau_n)\,{}_kVsl_n + \tau_n\,{}_kVsr_{n-1} \tag{17}$$

$${}_{k+1}Vir_n = (1-\tau_{n+1})\,{}_kVsr_n + \tau_{n+1}\,{}_kVsl_{n+1} \tag{18}$$

where τ_n, the transmission coefficient for connecting line n (i.e. the line between nodes $n-1$ and n), is [8]

$$\tau_n = Z_n/(Z_n + R_n) \tag{19}$$

The model is initiated by setting the two incident voltages at each node equal to half the desired initial node voltage distribution along the line. Equation (16) is used to calculate the resulting node voltages, then Equations (12), (13) give the scattered voltages, and Equations (17), (18) give the incident voltages for the next time step. The only difference between this and the standard lossy TLM method for diffusion is in the equation for the node voltage. Traditionally, if conditions are not time-varying and a steady-state solution is required, then this procedure is repeated until the desired level of convergence is obtained [8, 12, 13]. It will be shown below that a steady-state solution can, instead, be obtained directly.

Before modelling can begin, it is necessary to calculate P_n, X_n, and Y_n, for each node, and τ_n for each connecting line, from the required properties of the TL defined above from the physics being modelled. It has been shown [5] that accurate results are obtained if Z_n and R_n are, respectively, the average of the distributed impedance and half the total resistance of the TL between x_{n-1} and x_n, the locations of the ends of connecting line n. Assuming Pe is constant over space, Equations (7) and (9) give $Z(x)$ and the TL segment impedance is

$$Z_n = \frac{1}{\Delta x_n}\int_{x_n-\Delta x_n}^{x_n} Z(x)dx = \frac{\Delta t \exp\left(Pex_n\right)\left[1-\exp\left(-Pe\Delta x_n\right)\right]}{cPe\Delta x_n^2} \tag{20}$$

since u_n, the propagation speed between the two nodes, must equal $\Delta x_n/\Delta t$. Assuming D is also constant over space, the resistance R_n is

$$R_n = \tfrac{1}{2}\int_{x_n-\Delta x_n}^{x_n} R(x)dx = \frac{\exp\left(Pex_n\right)\left[1-\exp\left(-Pe\Delta x_n\right)\right]}{2cDPe} \tag{21}$$

Substituting these into Equation (19) gives

$$\tau_n = \frac{1}{1+\frac{1}{2}D^{-1}\Delta x_n^2\Delta t^{-1}} \tag{22}$$

If D varies over space then the average of D_n and D_{n+1}, the values at the nodes at either end of the connecting line, can be used to calculate τ_n.

Deriving an equation for Z_{n+1} as above (i.e. by integrating $Z(x)$ between x_n and $x_n+\Delta x_{n+1}$) and using it and Equation (20) to calculate the impedance ratio at node n, $P_n = Z_n/Z_{n+1}$, gives

$$P_n = \frac{\Delta x_{n+1}^2}{\Delta x_n^2}\frac{A_n}{B_n} \tag{23}$$

where

$$A_n = 1-\exp\left(-Pe\Delta x_n\right),\ \ B_n = \exp\left(Pe\Delta x_{n+1}\right)-1 \tag{24}$$

If Pe varies over space then its value at node n can be used when calculating P_n. Note that if $Pe = 0$ at any node then the impedance ratio at that node must be set to $\lim_{Pe\to 0} P_n = \Delta x_{n+1}/\Delta x_n$. Also note

that, if $Pe\Delta x_n$ or $Pe\Delta x_{n+1}$ approach zero, significant round-off errors can occur when calculating A_n and B_n (and similar terms below) and that these can significantly affect the accuracy of the scheme.

There is no physical basis for the use of these equations when D and/or Pe vary spatially. One possibility is to assume that both parameters vary in a piecewise-constant fashion so that $Pe(x) = Pe_n$ and $D(x) = D_n$ between $x_n - \frac{1}{2}\Delta x_n$ and $x_n + \frac{1}{2}\Delta x_{n+1}$. Solving Equation (4) over this space with these settings gives

$$C_{d,n}(x) = c_n \exp(-Pe_n x), \quad x_n - \tfrac{1}{2}\Delta x_n < x < x_n + \tfrac{1}{2}\Delta x_{n+1} \tag{25}$$

Since the distributed capacitance must be continuous, $C_{d,n-1}(x)$ and $C_{d,n+1}(x)$ can also be written in terms of the constant c_n. The distributed impedance can then be found as before. The average impedance of the two transmissions lines at node n can then be calculated giving the impedance ratio as

$$P_n = \frac{Pe_{n+1}\Delta x_{n+1}^2 \alpha_{7,n}}{Pe_{n-1}\Delta x_n^2 \alpha_{8,n+1}} \tag{26}$$

where

$$\begin{gathered} \alpha_{1,n} = \exp\left(-\tfrac{1}{2} Pe_n \Delta x_n\right), \ \alpha_{2,n} = \exp\left(-\tfrac{1}{2} Pe_{n-1}\Delta x_n\right), \ \alpha_{3,n} = (\alpha_{1,n} - 1) Pe_{n-1}, \\ \alpha_{4,n} = (\alpha_{2,n} - 1) Pe_n, \ \alpha_{5,n} = (1 - 1/\alpha_{1,n}) Pe_{n-1}, \ \alpha_{6,n} = (1 - 1/\alpha_{2,n}) Pe_n, \\ \alpha_{7,n} = \alpha_{1,n}\alpha_{4,n} + \alpha_{3,n}, \ \alpha_{8,n} = \alpha_{6,n} + \alpha_{5,n}/\alpha_{2,n} \end{gathered} \tag{27}$$

Calculating the equivalent transmission coefficient gives

$$\tau_n = \frac{2}{2 + \dfrac{\Delta x_n^2}{\Delta t}\left(\dfrac{\alpha_{4,n}/D_{n-1} + \alpha_{5,n}/D_n}{\alpha_{4,n} + \alpha_{5,n}}\right)} \tag{28}$$

Note that the limits of $\alpha_{4,n}$ and $\alpha_{5,n}$, as Pe_n and Pe_{n-1}, respectively, go to zero, are both $\frac{1}{2}\Delta x_n$.

Now X_n depends on the component of I_{Cn} due to $S(x)$, which, in turn, is a function of the component of $I_{Cd}(x)$ due to $S(x)$, $I_{CdS}(x) = S(x)C_d(x)$. At any point, x, on the TL being modelled, the distributed current source sees two impedances in parallel, one between x and the node to the left, $Z_L(x)$, and one between x and the node to the right, $Z_R(x)$. The portion of the current that will flow towards the node to the right is simply $Z_L(x)/(Z_L(x)+Z_R(x))$. Note that, for any point x on connecting line n, $Z_L(x)+Z_R(x) = Z_n\Delta x_n$ (since Z_n is the average impedance of the connecting line). In the TLM model, the current at node n due to $S(x)$ from the lumped current source, I_{CnS}, is the current from the equivalent distributed current source between nodes x_{n-1} and x_n flowing to the right, plus that between nodes x_n and x_{n+1} flowing to the left

$$I_{CnS} = \frac{1}{Z_n \Delta x_n} \int_{x_n - \Delta x_n}^{x_n} I_{CdS}(x) \left[\int_{x_n - \Delta x_n}^{x} Z(x) dx \right] dx + \frac{1}{Z_{n+1}\Delta x_{n+1}} \int_{x_n}^{x_n + \Delta x_{n+1}} I_{CdS}(x) \left[\int_{x}^{x_n + \Delta x_{n+1}} Z(x) dx \right] dx \tag{29}$$

Now $X_n = Z_n I_{CnS}$ so

$$X_n = \frac{1}{\Delta x_n} \int_{x_n-\Delta x_n}^{x_n} I_{CdS}(x) \left[\int_{x_n-\Delta x_n}^{x} Z(x)dx \right] dx + \frac{P_n}{\Delta x_{n+1}} \int_{x_n}^{x_n+\Delta x_{n+1}} I_{CdS}(x) \left[\int_{x}^{x_n+\Delta x_{n+1}} Z(x)dx \right] dx \qquad (30)$$

The distributed current due to the reaction terms is $I_{CdK*}(x, t) = K^*(x)V(x, t)C_d(x)$. Assuming $V(x) = {}_kVn_n$ between nodes $n-1$ and $n+1$ at time step k, this can be rewritten as $I_{CdK*}(x, k) = {}_kVn_nK^*(x)C_d(x)$, $x_n-\Delta x_n < x < x_n-\Delta x_{n+1}$. Now $Y_n = Z_nI_{CnK*}/{}_kVn_n$ and so the relationship between for Y_n and $K^*(x)$ is the same as that between X_n and $S(x)$.

Assuming S and Pe are both constant over space, Equation (30) gives

$$X_n = \frac{S\Delta t}{Pe\Delta x_n}\left[1 - \frac{\Delta x_{n+1}}{\Delta x_n}\frac{A_n}{B_n}\right] \qquad (31)$$

or, if $Pe = 0$, $X_n = \frac{1}{2}S\Delta t(\Delta x_{n+1} + \Delta x_n)/\Delta x_n$. For the case where both Pe and S vary in a piecewise-constant fashion (consistent with that described above)

$$X_n = \Delta t\left(Q_{1,n}S_{n-1} + Q_{2,n}S_n + Q_{3,n}S_{n+1}\right) \qquad (32)$$

where

$$\begin{gathered} Q_{1,n} = \frac{\alpha_{2,n} + \frac{1}{2}Pe_{n-1}\Delta x_n - 1}{\left(Pe_{n-1}\Delta x_n\right)^2}, \quad Q_{3,n} = -\alpha_{7,n}\frac{\alpha_{5,n+1}/Pe_n + \frac{1}{2}Pe_{n+1}\Delta x_{n+1}}{Pe_{n-1}Pe_{n+1}\Delta x_n^2\alpha_{8,n+1}}, \\ Q_{2,n} = \frac{\alpha_{7,n} - \alpha_{4,n} + \frac{1}{2}\alpha_{9,n}}{\alpha_{9,n}Pe_n\Delta x_n} + \alpha_{7,n}\frac{\alpha_{5,n+1} - \alpha_{8,n+1} - \frac{1}{2}\alpha_{9,n+1}}{\alpha_{9,n}Pe_n\Delta x_n\alpha_{8,n+1}} \end{gathered} \qquad (33)$$

and $\alpha_{9,n} = Pe_{n-1}Pe_n\Delta x_n$. If $Pe_n = 0$ for all nodes then this simplifies to

$$\lim_{Pe\to 0} X_n = \tfrac{1}{8}\Delta t\left[\Delta x_n\left(S_{n-1} + 3S_n\right) + \Delta x_{n+1}\left(S_{n+1} + 3S_n\right)\right]\Delta x_n^{-1} \qquad (34)$$

Replacing S with K^* in the equations for X_n derived above gives equivalent equations for Y_n.

DIRECT SOLUTION

The TLM model described above is straightforward to implement. If, however, boundary and other conditions are steady and the steady-state solution is required, the model must be run until it converges. With no current sources, it is stable for large time-steps but these lead to wave-like behaviour in the solution that takes time to settle down [5, 12, 13]. As a result, the number of time-steps required for a given level of accuracy can be highly dependent on Δt. Although the minimum number, corresponding to the optimum Δt value, can be small, there is no efficient way of determining that value for a given problem. This has been shown previously for both diffusion and convection-diffusion schemes [5].

Stability of the transient TLM scheme is guaranteed if all components in the corresponding TL network are passive [8]. Testing of the scheme described here has shown that instability can occur with negative Y_n values (i.e. when the current source at a node is adding a current proportional to the node voltage). This can arise where $K(x)$ and/or dv/dx are negative. By obtaining steady-state solutions directly, such convergence and stability problems can be avoided and the overall computational cost of the scheme reduced.

Equations (12), (13), (17), (18), and (16), can be combined to give both ${}_{k+1}Vil_n$ and ${}_{k+1}Vir_n$ in terms of incident voltages at time-step k

$$ {}_{k+1}Vir_n = (1-\tau_{n+1})\left[\frac{2_kVil_n + 2P_{nk}Vir_n + X_n}{1+P_n+Y_n} - {}_kVir_n\right] + \tau_{n+1}\left[\frac{2_kVil_{n+1} + 2P_{n+1k}Vir_{n+1} + X_{n+1}}{1+P_{n+1}+Y_{n+1}} - {}_kVil_{n+1}\right] \quad (35) $$

$$ {}_{k+1}Vil_n = (1-\tau_n)\left[\frac{2_kVil_n + 2P_{nk}Vir_n + X_n}{1+P_n+Y_n} - {}_kVil_n\right] + \tau_n\left[\frac{2_kVil_{n-1} + 2P_{n-1k}Vir_{n-1} + X_{n-1}}{1+P_{n-1}+Y_{n-1}} - {}_kVir_{n-1}\right] \quad (36) $$

Under steady-state conditions ${}_{k+1}Vil_n = {}_kVil_n = {}_\infty Vil_n$ and ${}_{k+1}Vir_n = {}_kVir_n = {}_\infty Vir_n$ (where ${}_\infty Vil_n$ and ${}_\infty Vir_n$ are the steady-state incident voltages at node n) and so Equations (35) and (36) can be rewritten to give both ${}_\infty Vil_n$ and ${}_\infty Vir_n$ in terms of other steady-state incident voltages

$$ a_{nl\,\infty}Vil_{n-1} + b_{nl\,\infty}Vir_{n-1} - {}_\infty Vil_n + c_{nl\,\infty}Vir_n = d_{nl}\ ,\ 2 \le n \le N \quad (37) $$

$$ a_{nr\,\infty}Vil_n - {}_\infty Vir_n + b_{nr\,\infty}Vil_{n+1} + c_{nr\,\infty}Vir_{n+1} = d_{nr}\ ,\ 1 \le n \le N-1 \quad (38) $$

where

$$ \begin{gathered} a_{nl} = \frac{2\sigma_{2,n}\tau_n}{\sigma_{7,n}}\ ,\ b_{nl} = \frac{a_{nl}}{2}\sigma_{5,n-1}\ ,\ c_{nl} = \frac{2\sigma_{9,n}(1-\tau_n)}{\sigma_{7,n}}\ ,\ d_{nl} = \frac{-c_{nl}\sigma_{10,n} - a_{nl}X_{n-1}}{2}\ , \\ a_{nr} = \frac{2\sigma_{2,n+1}(1-\tau_{n+1})}{\sigma_{8,n}}\ ,\ b_{nr} = \frac{\sigma_{2,n}\sigma_{6,n+1}}{\sigma_{8,n}}\ ,\ c_{nr} = \frac{2\sigma_{9,n+1}\tau_{n+1}}{\sigma_{8,n}}\ ,\ d_{nr} = \frac{-c_{nr}\sigma_{10,n+1} - a_{nr}X_n}{2} \end{gathered} \quad (39) $$

and

$$ \begin{gathered} \sigma_{1,n} = P_n + Y_n\ ,\ \sigma_{2,n} = 1+\sigma_{1,n}\ ,\ \sigma_{3,n} = 1+\sigma_{2,n}\ ,\ \sigma_{4,n} = Y_n + 1,\ \sigma_{5,n} = P_n - \sigma_{4,n}\ ,\ \sigma_{6,n} = \sigma_{3,n}\tau_n\ , \\ \sigma_{7,n} = \sigma_{2,n-1}(\sigma_{6,n} + 2\sigma_{1,n}),\ \sigma_{8,n} = \sigma_{2,n+1}(\sigma_{5,n}\tau_{n+1} + 2\sigma_{4,n}),\ \sigma_{9,n} = P_n\sigma_{2,n-1}\ ,\ \sigma_{10,n} = X_n/P_n \end{gathered} \quad (40) $$

To solve these equations for each node and obtain the steady-state incident voltages (and, from those, the steady-state node voltages) at all nodes, it is necessary to derive equations for ${}_\infty Vil_1$ and ${}_\infty Vir_N$ from the boundary conditions. This paper only considers the implementation of models with Dirichlet boundary conditions and with nodes located at both boundaries. The node voltage at these boundary nodes is simply kept fixed at the desired level [5]. At the left-hand boundary, while there is no need to have a TL segment to the left of node, it is convenient to imagine a steady-state incident voltage, ${}_\infty Vil_1$, that keeps Vn_1 at the desired value, V_L. This must satisfy

$$\frac{2_{\infty}Vil_1 + 2P_1{}_{\infty}Vir_1 + X_1}{1 + P_1 + Y_1} = V_L \tag{41}$$

Since the values of P_1, X_1, and Y_1, are all dependent on the impedance of the imaginary TL segment to the left of node 1, they can in practice be set arbitrarily, thus simplifying this equation. If $P_1 = 1$ and $X_1 = Y_1 = 0$ then ${}_{\infty}Vil_1 + {}_{\infty}Vir_1 = V_L$. Similarly, setting $P_N = 1$ and $X_N = Y_N = 0$ gives ${}_{\infty}Vir_{N.} + {}_{\infty}Vil_{N.} = V_R$, where V_R is the desired node voltage at the right-hand boundary.

Equations (37) and (38), when written for each node and combined with equations for ${}_{\infty}Vil_1$ and ${}_{\infty}Vir_N$, form a pentadiagonal system which can be reduced to tri-diagonal form by Gaussian elimination and solved to find the steady-state incident voltages. Equation (16) can then be used to calculate the steady-state node voltages.

TESTING

Curiously, the equations for the steady-state solution given above (in particular, the coefficients that are functions of τ_n, X_n, and Y_n) depend on the time-step size. While Δt may be chosen arbitrarily, testing shows that the value affects the condition of the system of equations, and therefore, the effect of round-off errors on the solution. If, when the equations are solved, the residual is significant then it may be possible to use a different value of Δt that reduces it. However, testing suggests that there is no straightforward way of choosing such a time step. It would appear to be more efficient to use iterative improvement [16] to increase the solution accuracy if required. For all results presented below, a time step of $\Delta t = 0.0001$ has been used and the TLM equations have been solved using the tri-diagonal matrix algorithm (TDMA) with one step of iterative improvement.

Errors arising from the source term

It has been shown previously that the accuracy of the TLM method with $S(x) = K(x) = 0$ and with $v(x)$ and $D(x)$ both constant over space, is limited only by round-off error [5]. This is not in general true, however, if $S(x) \neq 0$.

Models for CDEs of the form $0 = DV'' - vV' + S(x)$ with boundary conditions $V(0) = V(\lambda) = 0$ have been tested and the errors at each node (defined as $E_n = V_n - V_{tn}$ where V_{tn} is the true value of the solution at node n) have been examined. This has been done for problems where analytical solutions can be found using symbolic maths software. With v and D both constant, the errors are solely a result of the source term and its implementation. It has been found that, if X_n is calculated assuming a piecewise-constant variation in $S(x)$ (i.e. using Equation (32)), then the errors depend on the differences between the values of S_n and the averages of $S(x)$ between $x_n - \frac{1}{2}\Delta x_n$ and $x_n + \frac{1}{2}\Delta x_{n+1}$ for each node. This difference for node n will be referred to here as the *error in* S_n.

In the models tested, the values of S_n were simply set equal to $S(x_n)$, i.e. the value of the function $S(x)$ at the location of node n. In this case, if the node spacing is constant, as Δx approaches 0 the errors in the S_n values approach $E_S(x) = \Delta x^2 \, d^2S/dx^2$. This result is easily shown. The question then arises as to

how the errors in the node voltages are dependent on $E_S(x)$. The relationship has been empirically established and is given here for models with equidistant nodes for 3 situations; where $v = 0$, where v approaches ∞, and where v approaches $-\infty$.

As the Peclet number approaches infinity, the CDE becomes $V' = S(x)/v$. Not surprisingly, therefore, it has been found empirically that the errors in the node voltages approach an integral of the errors in S_n over the length of the model. Testing has shown that if the convection velocity is positive and the node spacing is small then the error in the solution at node n, E_n, approaches $I_L(x_n)/24v$ as v increases, where $I_L(x) = \int_0^x E_S(x)dx$, but cannot fully match it since the errors at both boundaries are zero. As v approaches minus infinity, E_n approaches $I_R(x_n)/24v$ where $I_R(x) = \int_x^\lambda E_S(x)dx$.

Sample results are shown in Figure 2.

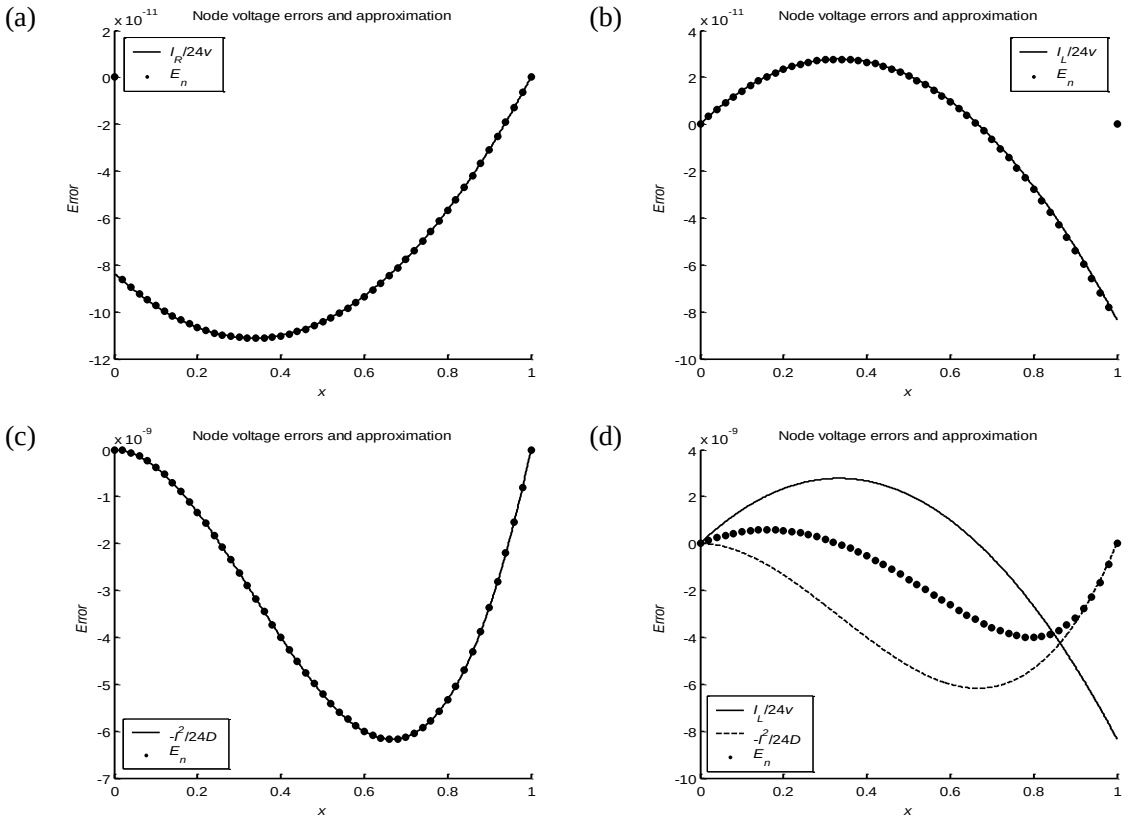

Figure 2: Errors in the solutions of $0 = V'' - vV' + x^2 - x^3$ with $V(0) = V(1) = 0$ for models with 1001 nodes (only every 20th error value is plotted) and with four different convection velocities ($v = -500$ (a), $v = 500$ (b), $v = 0$ (c), and $v = 5$). Note that Equation (32) was used to calculate the X_n values with $S_n = S(x_n)$. In each case the appropriate approximation is also shown. In the case with $v = 5$, the errors lie between those for $v = 0$ and those for $v = \infty$.

As v approaches 0 the CDE becomes $V_{xx} = -S(x)/D$, and the errors in the node voltages approach a double-integral of the errors in S_n. It has been found empirically that E_n approaches $I^2(x_n)/24D$ where

$I^2(x) = \frac{1}{2}\left(-\int_0^x I_L(x) + c_L\, dx + \int_x^\lambda I_R(x) + c_R\, dx \right)$ and where c_L and c_R are constants chosen so that $I^2(0) = I^2(\lambda) = 0$ (since the errors at both boundaries are zero).

It should be noted that if $S(x)$ varies linearly over space then there are no systematic errors in the solution (so long as Δx, v, and D are all constant over space). This clearly fits with the findings presented above since E_{Sn} is then zero for all nodes. If $S(x)$ does not vary linearly then it has been found that choosing S_n values to ensure $E_{Sn} = 0$ for all nodes does not reduce the solution errors to zero and may not give better results than simply letting S_n equal $S(x_n)$.

The solution errors have also been examined for similar models using Equation (31) to calculate X_n (i.e. assuming that $S(x)$ is a constant). In the case where $v = 0$ it has been found that (assuming a constant node spacing), the error at each node can be approximated by $-I^2(x_n)/12D$ (i.e. the errors at each node are approximately -2 times those obtained if Equation (32) is used). This would suggest that the errors could be, at least partially, cancelled out by using an appropriately weighted average of the X_n values obtained using the two equations. For $Pe = 0$ this average is

$$X_n = \frac{1}{12}\Delta t\left[\Delta x_n\left(S_{n-1} + 5S_n\right) + \Delta x_{n+1}\left(S_{n+1} + 5S_n\right)\right]\Delta x_n^{-1} \tag{42}$$

This equation does give significantly more accurate results (an example is given below) but only if Δx is constant. If Δx is not constant, then the errors obtained using Equation (31) are not minus twice those obtained using Equation (32).

As v approaches infinity, E_n (if Equation (31) is used) approaches $I_{LC}(x_n)/12D$ where $I_{LC}(x) = \Delta x^2 \int_0^x \frac{dS}{dx} dx$. As v approaches minus infinity, E_n approaches $I_{RC}(x_n)/12D$ where $I_{RC}(x) = -\Delta x^2 \int_x^\lambda \frac{dS}{dx} dx$. Unlike in the case where $v = 0$, there is no straightforward relationship between these errors and those obtained using Equation (32).

These error relationships were established empirically and are presented as such, with no attempt in this paper to account for their observed features. Note also that they all refer to models with constant node spacing: to date establishing corresponding relationships for models with non-equidistant nodes has not been attempted.

It is clear from these results that the errors due to source terms are of the order of Δx^2. Comparisons are made below with two standard second-order finite difference (FD) schemes, one using centred-differences (CD) and one upwind-difference scheme (UP). The equation for the CD scheme on a non-equidistant mesh is

$$D_n\left(\frac{2V_{n-1}}{\Delta x_n \Delta x_n^*} - \frac{2V_n}{\Delta x_n \Delta x_{n+1}} + \frac{2V_{n+1}}{\Delta x_{n+1}\Delta x_n^*}\right) + v_n^*\left(\frac{\Delta x_{n+1}V_{n-1}}{\Delta x_n \Delta x_n^*} + \frac{(\Delta x_n - \Delta x_{n+1})V_n}{\Delta x_n \Delta x_{n+1}} - \frac{\Delta x_n V_{n+1}}{\Delta x_{n+1}\Delta x_n^*}\right) - K_n^* V_n + S_n = 0 \tag{43}$$

where $v_n^* = v_n - dD(x_n)/dx$ and $\Delta x_n^* = \Delta x_n + \Delta x_{n+1}$. The equation for the UP scheme (for $v(x) \geq 0$) is the same but with the convection term changed to

$$v_n^* \left(-\frac{\Delta x_n V_{n-2}}{\Delta x_{n-1} \Delta x_n^*} + \frac{\Delta x_n^* V_{n-1}}{\Delta x_n \Delta x_{n-1}} - \frac{\left(\Delta x_n + \Delta x_n^*\right) V_n}{\Delta x_n \Delta x_n^*} \right) \tag{44}$$

In the UP scheme the CD equation is used to calculate V_2. Note that for all results presented here, the spatial derivatives of v and D, where required for the two FD schemes and the TLM scheme, were calculated analytically.

Table 1 contains maximum errors (defined as $\max_{1 \leq n \leq N} |E_n|$) in results from models for $0 = V'' - vV' + x^2$, with boundary conditions $V(0) = V(1) = 0$, and with 11 nodes. Results from two TLM schemes are given: one using Equation (31) to calculate X_n (denoted TLM_C) and one using Equation (32) (denoted TLM_{PC}). In both cases $S_n = S(x_n)$ for all nodes. It is clear that the use of Equation (32), rather than the less complex Equation (31), is only justified in this case when the convection term is significant. Testing shows that, in general, the accuracy of TLM increases with v but that of the FD schemes decreases. It should be noted that the significant increase in the accuracy of the UP scheme in this case as v is increased above 20 is due to the presence of a "boundary layer" in the solution and the fact that, with higher v values, this layer is narrower and therefore there are no nodes located within it [17].

As has already been mentioned, and is clear from Table 1, the magnitude of the errors obtained from the TLM_C scheme with $v = 0$ are twice those obtained using the TLM_{PC} scheme (so long as the nodes are equidistant). Using Equation (42) instead to calculate the X_n values reduces the maximum error in this case to 3.52E−16. Testing has shown that the level of improvement obtained by using this equation depends on the nature of the source function.

It should be noted that the TLM_C scheme also uses equations for P_n and τ_n that assume that v and D are constant over space, while the TLM_{PC} scheme assumes that these are piecewise constant. Since in this problem v and D are both constant, these algorithm differences do not affect the results.

v	0	1	3	10	20	100
TLM_{PC}	1.04E−04	1.02E−04	9.34E−05	5.70E−05	3.60E−05	1.33E−05
TLM_C	2.08E−04	1.06E−04	9.62E−05	4.19E−04	5.39E−04	3.37E−04
CD	2.08E−04	8.60E−05	3.04E−04	1.38E−03	2.48E−03	2.34E−03
UP	2.08E−04	7.59E−04	1.86E−03	2.93E−03	2.41E−03	1.82E−04

Table 1: Comparison of the maximum errors obtained from four different schemes with different convection velocities.

An interesting feature of both TLM schemes is that the errors arising from the inclusion of a source term are independent of the boundary conditions (i.e. it is independent of the shape of the solution). So, for example, if $V(1)$ is changed to 1 for the model described above with $v = 10$, the maximum

errors in the two TLM schemes remain equal to 5.70E−05 and 4.19E−04, as given in Table 1, but the maximum errors in the CD and UP schemes increase to 3.32E−02 and 6.75E−02 respectively.

Errors arising from reaction terms and other sources

Depending on which equation is used to calculate the Y_n values, it is assumed that either $K(x)$ is piecewise constant or that $K(x)$ is constant. In general, whichever assumption is used will lead to errors similar to those associated with the implementation of source terms. In either case there will be a second source of errors due the assumption that the solution, $V(x)$, is a constant. Table 2 compares results for three cases, one with a reaction term, one with $v(x)$ varying over space (which has errors due to this variation and also due to the resulting reaction term), and one with $D(x)$ varying over space. All results are for models with 11 equidistant nodes. In general the TLM schemes are more accurate than the FD schemes (especially for higher Peclet numbers) and, not surprisingly, TLM_{PC} is generally the more accurate of the two TLM schemes. One particular exception is Test 3 with $v = 1$ since the solution in that case $V(x) = x$ and, unlike the TLM schemes, the FD methods exhibit no systematic error. Also in Test 3, the TLM_C method is more accurate than the TLM_{PC} method for lower convection velocities. The reason for this is not known but it may be a result of round-off errors in the calculation of the model parameters. Note that testing has shown that all errors are, in general, second order but no further analysis of the errors due to reaction terms or spatial variations in $v(x)$ and/or $D(x)$ has, as yet, been performed.

v	0	1	3	10	20
Test 1: $0 = V'' - vV' - 10V$, $V(0) = 0$, $V(1) = 1$, $N = 11$					
TLM_{PC}	1.43E−03	1.21E−03	8.45E−04	2.30E−04	4.75E−05
TLM_C	1.43E−03	1.21E−03	8.45E−04	2.30E−04	4.75E−05
CD	1.43E−03	1.04E−03	1.11E−03	3.15E−02	1.29E−01
UP	1.43E−03	4.17E−03	1.35E−02	6.99E−02	1.29E−01
Test 2: $0 = V'' - v(x+1)V'$, $V(0) = 0$, $V(1) = 1$, $N = 11$					
TLM_{PC}	2.38E−14	3.06E−05	7.84E−05	4.05E−05	8.40E−07
TLM_C	2.38E−14	2.30E−04	1.01E−03	2.28E−03	1.11E−03
CD	1.42E−15	2.33E−04	7.16E−03	1.17E−01	3.20E−01
UP	1.42E−15	3.52E−03	3.11E−02	1.31E−01	1.32E−01
Test 3: $0 = ((x+1)V')' - vV'$, $V(0) = 0$, $V(1) = 1$, $N = 11$					
TLM_{PC}	1.51E−04	7.11E−05	5.75E−05	1.66E−04	1.61E−04
TLM_C	7.53E−05	3.55E−05	2.86E−05	6.45E−04	1.43E−03
CD	7.53E−05	3.07E−15	2.30E−04	8.51E−03	3.65E−02
UP	5.19E−04	6.66E−16	4.04E−04	2.25E−02	6.76E−02

Table 2: Comparison of errors from models with reaction terms and with spatial variations in v and D.

For a model solving $0 = V'' - 3V' - 10V$ with $V(0) = 0$ and $V(1) = 1$, the error due to the reaction term is only due to the spatial variation in $V(x)$ (and is therefore the same for the two TLM methods). The solution varies slowly near the left-hand boundary but varies rapidly to the right. It would be expected, therefore, that concentrating the nodes towards the right-hand boundary could improve the

accuracy of the solution while not significantly increasing the computational cost. The errors obtained using the TLM, CD, and UP methods with 11 equidistant nodes are 8.45E−04, 1.11E−03, and 1.35E−02 respectively (as given in Table 2). With an exponential variation in the node spacing defined by $\Delta x_{n+1}/\Delta x_n = 0.95$, the equivalent errors are 1.16E−04, 2.83E−03, and 9.74E−03. There appears to be some loss of accuracy in all cases due to the uneven node spacing and this more than cancels out any accuracy gains resulting from the concentration of nodes towards the right-hand boundary in the CD scheme. The accuracy of the TLM results is, however, significantly improved in this example, while that of the UP scheme is improved to a much lesser extent.

All the testing reported above has been for problems where analytical solutions are available. Figure 3 shows the solution obtained from the three schemes (note that TLM in this case refers to TLM_{PC}) for a more general problem, $0 = ((1-\sin(x))V')' - ((5x^2+1)V)' - 10(x-1)^3V + 5\text{erf}(x)$ with $V(0) = 0$, $V(1) = 1$ and with 21 evenly spaced nodes. It also shows how the solution at $x = 0.9$ varies with Δx. It is clear that the TLM results are considerably more accurate than those obtained using the FD methods.

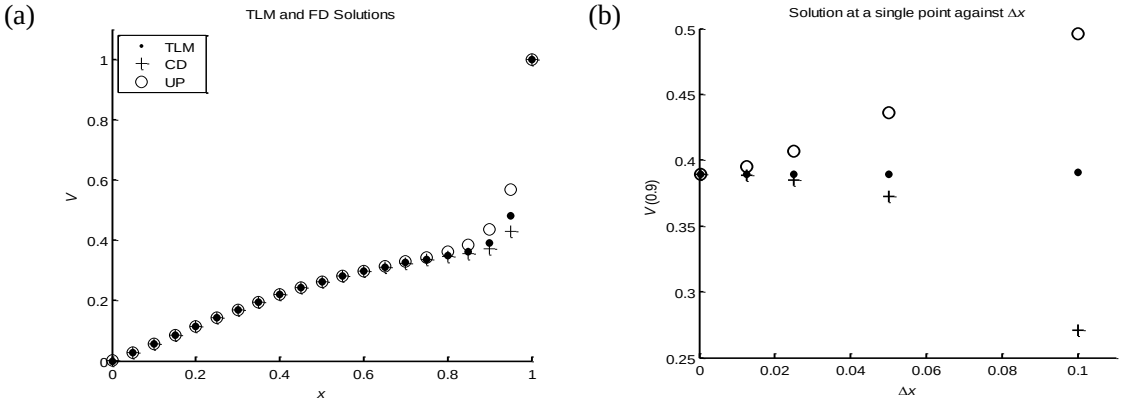

Figure 3: The solution obtained using three schemes with 21 equidistant nodes for a problem with source and reaction terms and with spatial variations in both $v(x)$ and $D(x)$ (a), and the variation in the solution at $x = 0.9$ with Δx (b).

A previous paper [5] proposed two methods for incorporating reaction terms (specifically the Vdv/dx term) in TLM convection-diffusion models. It was shown that the generally more accurate of the two involves adjusting the node voltage to

$$ {}_kVn_n = \left(\frac{2_kVil_n + 2P_{n\,k}Vir_n}{1+P_n} \right)\left(1 - \frac{K_n}{2}\right) \tag{45} $$

but that the accuracy is dependent on Δt [5]. It can be seen that this is equivalent to using Equation (16) with $Y_n = K_n(1+P_n)/(K_n-2)$. Table 3 contains results obtained by repeating Test 1 from Table 2 using this formula for Y_n and with Δt set so as to maximise accuracy. It also shows results obtained using a sub-optimal time step. It can be seen that the method is, in most cases, less accurate than that

presented in this paper and that the accuracy is highly dependent on the time step. It should be noted that, in general, there is no way to determine the optimum time step directly or efficiently.

v	0	1	3	10	20
$0 = V'' - vV' - 10V$					
TLM_{PC}	1.43E−03	1.21E−03	8.45E−04	2.30E−04	4.75E−05
TLM_1	1.43E−03	1.09E−03	8.68E−05	2.04E−03	1.82E−03
TLM_2	1.20E−01	1.05E−01	7.75E−02	2.85E−02	9.02E−03

Table 3: Comparison of errors obtained using the TLM_{PC} scheme and two TLM models using Equation (45) to calculate Y_n, one with the optimum Δt setting (the setting, determined empirically, that minimises errors), TLM_1, and one with the time step set to half that optimum value, TLM_2.

DISCUSSION AND CONCLUDING REMARKS

This paper presents techniques for implementing source and reaction terms in TLM convection-diffusion models, and for solving the steady-state one-dimensional convection-diffusion equation with non-equidistant nodes. The errors in the scheme are on the order of Δx^2. The TLM method has been compared with two second-order finite difference schemes. The accuracy is similar to that of the more accurate of these when the convection term approaches 0. When convection dominates it is significantly more accurate than the FD methods. Unlike with upwind methods, there is no need to treat nodes adjacent to upwind boundaries differently and there is no additional complication if $v(x)$ has variations in sign over space.

Two sets of equations have been derived for calculating the necessary model parameters. The first assumes a constant convection velocity, diffusion coefficient, etc., while the second assumes that these vary in a piecewise-constant fashion. While the equations for the latter are more complex (and therefore more costly to implement), the results obtained using them are, in general, significantly more accurate. The extra effort may not be worthwhile if the convection term is insignificant or if the error in the solution is largely a result of the reaction term (in which case errors arising from the assumption that the solution is constant over space will be similar for the two schemes). Both TLM schemes are more costly to implement than the FD schemes tested but, although no analysis of computational cost has been made here, it is clear that the TLM scheme can be more efficient when the convection term dominates.

The method for implementing the reaction term assumes that the solution is constant over space. It would be possible to alter it so that a piecewise-constant variation is assumed but this would increase the complexity and cost of the scheme.

Care must be taken when calculating the TLM model parameters when $Pe(x)$ is close to zero at any node since round-off errors can then become significant.

Methods have been presented previously for implementing reaction terms. The scheme proposed here, while more complex, is, in general, significantly more accurate.

Traditionally TLM is a time-domain method and, if a steady-state solution is required, it must be run until the solution converges to an acceptable degree. A more efficient method has been presented here for finding a steady-state solution directly. This also avoids potential problems of instability previously noted in TLM models of networks with active elements (e.g. voltage-dependent current sources).

Lossy TLM is well established for diffusion modelling. The equations for the convection-diffusion scheme described here are, in the limit as *Pe* goes to zero, similar to those presented in previous work by other authors. An equation for X_n (i.e. for the implementation of source terms), however, which is a weighted average of those derived assuming $S(x)$ constant and $S(x)$ piecewise constant, would appear to be significantly more accurate than those used previously. The reason for this has not yet been established.

Lossy TLM for diffusion has been successfully extended to two and three dimensions using grids of interconnected transmission lines. There would appear to be no reason why the TLM method for convection-diffusion described here can not be similarly extended.

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APPENDIX

Let $C_d(x)$, $L_d(x)$, and $R_d(x)$ be the distributed capacitance, inductance, and resistance respectively of a non-uniform lossy transmission line at a position x along its length. The transmission line also has a distributed current source supplying a current $I_{Cd}(x)$ per unit length at all points. Over a short length, Δx, these can be represented by lumped components as in Figure 4.

The voltage across the inductor is $(\partial V/\partial x)\Delta x + iR_d(x)\Delta x$. Since, the voltage across an inductance L with current i flowing through it is $V = -L(di/dt)$, then for the distributed TL inductance

$$\frac{\partial V}{\partial x}\Delta x + iR_d(x)\Delta x = -L_d(x)\Delta x\frac{\partial i}{\partial t} \tag{46}$$

Dividing across by $R_d(x)C_d(x)\Delta x$ gives

$$\frac{1}{R_d(x)C_d(x)}\frac{\partial V}{\partial x} + i\frac{1}{C_d(x)} = -\frac{L_d(x)}{R_d(x)C_d(x)}\frac{\partial i}{\partial t} \tag{47}$$

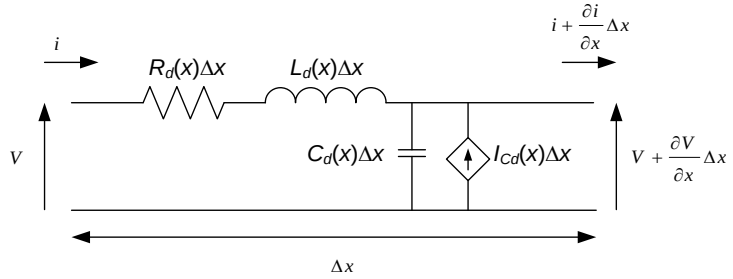

Figure 4: Lossy transmission line segment with current source.

and rearranging Equation (46) gives

$$-i = \frac{1}{R_d(x)}\frac{\partial V}{\partial x} + \frac{L_d(x)}{R_d(x)}\frac{\partial i}{\partial t} \tag{48}$$

The currents at either end of the TL segment are different. The difference, $(\partial i/\partial x)\Delta x$, is the current from the current source minus that charging the capacitor. Since the current charging a capacitor equals CdV/dt, then for the TL segment

$$\frac{\partial i}{\partial x}\Delta x = I_{Cd}(x)\Delta x - C_d(x)\Delta x\frac{\partial V}{\partial t} \tag{49}$$

Differentiating Equation (47) with respect to x and partially expanding the result gives

$$\frac{\partial}{\partial x}\left(\frac{1}{R_d(x)C_d(x)}\frac{\partial V}{\partial x}\right) + \frac{1}{C_d(x)}\frac{\partial i}{\partial x} + \frac{\partial}{\partial x}\left(\frac{1}{C_d(x)}\right)i = -\frac{L_d(x)}{R_d(x)C_d(x)}\frac{\partial^2 i}{\partial x\partial t} - \frac{\partial}{\partial x}\left(\frac{L_d(x)}{R_d(x)C_d(x)}\right)\frac{\partial i}{\partial t} \tag{50}$$

Differentiating Equation (49) with respect to t gives

$$-\frac{\partial^2 i}{\partial x\partial t} = C_d(x)\frac{\partial^2 V}{\partial t^2} - \frac{\partial I_{Cd}(x)}{\partial t} \tag{51}$$

Using Equations (48), (49), and (51), to replace terms in Equation (50), and setting time-derivative terms to zero, yields an equation for the voltage under steady-state conditions

$$0 = \frac{d}{dx}\left(\frac{1}{R_d(x)C_d(x)}\frac{dV}{dx}\right) - \frac{d}{dx}\left(\frac{1}{C_d(x)}\right)\frac{1}{R_d(x)}\frac{dV}{dx} + \frac{I_{Cd}(x)}{C_d(x)} \tag{52}$$