Doctoral Thesis

Rogue Traders

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Declaration of Authorship

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Abstract

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Rogue Traders

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Investing on behalf of a firm, a trader can feign personal skill by committing fraud that with high probability remains undetected and generates small gains, but that with low probability bankrupts the firm, offsetting ostensible gains. Honesty requires enough *skin in the game*: if two traders with isoelastic preferences operate in continuous-time and one of them is honest, the other is honest as long as the respective fraction of capital is above an endogenous *fraud threshold* that depends on the trader's preferences and skill. If both traders can cheat, they reach a Nash equilibrium in which the fraud threshold of each of them is lower than if the other one were honest. More skill, higher risk aversion, longer horizons, and greater volatility all lead to honesty on a wider range of capital allocations between the traders.

Chapter 1

Background

1.1 Introduction

The expression 'rogue trader' entered popular culture in 1995, when Nicholas W. Leeson, a trader of an overseas office of Barings Bank in Singapore, made unauthorized bullish bets on the Japanese stock market, concealing his losses in an error account. At first, losses were recovered with a profit, but in the aftermath of the Kobe earthquake they reached $1.4 billion (Brown and Steenbeek, 2001), forcing the 233-year old bank into bankruptcy. Earlier episodes of rogue trading *ante litteram* include the losses of Robert Citron in 1994 for Orange County ($1.7 billion, Kenyon (1997)) and of Toshihide Iguchi in 1983-1995 for Daiwa Bank ($1.1 billion, Iguchi (2014)). The earliest case is possibly the one involving the law firm of Grant & Ward in 1884, which embarrassed former president Ulysses S. Grant, one of the firm's partners (Krawiec, 2000).

Since the demise of Barings Bank, rogue trading episodes have increased in frequency and magnitude. In 2008, Jerome Kerviel, a junior trader at Société Générale who had been exceeding positions limits through fictitious trades to avoid detection, eventually lost $7.6 billion, the largest rogue trading loss in history. In his defense, he claimed that colleagues also engaged in unauthorized trading.[1] Year 2004 sees the fraud by four traders on foreign currency options trading desk - David Bullen, Luke Duffy, Vincent Ficarra, and Gianni Gray - in National Australia Bank, incurring a loss of $160m at the time of exposure as a result of falsifying profit and hiding losses[2].

Most recently in September 2021, Keith A. Wakefield, the former head of the fixed income trading desk at the broker-dealer IFS Securities, was charged by the U.S. Securities and Exchange Commission with unauthorized speculative trading and creating fictitious trading profits of approximately $820,000, leading to the closure of IFS Securities and substantial losses to both IFS Securities and one dozen counter-parties to

[1] See "Bank Outlines How Trader Hid His Activities" https://www.nytimes.com/2008/01/28/business/worldbusiness/28bank.html and "Timeline of events in SocGen rogue trader case" https://www.reuters.com/article/uk-socgen-kerviel-events-timeline-idUKL1885652420080318.

[2] See "Former NAB foreign currency options traders sentenced" https://web.archive.org/web/20070928000543/http://www.asic.gov.au/asic/asic.nsf/byheadline/06-221+Former+NAB+foreign+currency+options+traders+sentenced?openDocument.

Name	Country	Year	Loss	Institution
Robert Citron	USA	1994	$1.7bn[4]	Orange County
Joseph Jett	USA	1994	$74.6m[5]	Kidder, Peabody&Co
Nick Leeson*	Singapore	1995	$1.4bn[6]	Barings Bank
Toshihide Iguchi	Japan	1995	$1.1bn[7]	Resona Holdings
Yasuo Hamanaka	Japan	1996	$1.8bn[8]	Sumitomo Corporation
John Rusnak	USA	2002	$691m[9]	Allied Irish Banks
David Bullen et al.	Australia	2004	$160m[10]	National Australia Bank
Chen Jiulin	Singapore	2005	$550m[11]	China Aviation Oil
Matthew Taylor	USA	2007	$118m[12]	Goldman Sachs
Boris Picano-Nacci	France	2008	$751.5m[13]	Groupe Caisse d'Epargne
Jerome Kerviel	France	2008	$6.9bn[14]	Societe Generale
Kweku Adoboli	UK	2011	$2.2bn[15]	UBS
Keith Wakefield*	USA	2021	$30m[16]	IFS Securities

Table 1.1: Historical and publicly known episodes of rogue traders in ascending order by the year in which the fraud is exposed. The columns from left to right correspond to the name of the traders, the country at which the fraud is committed, the year in which the fraud is exposed, the approximate total loss resulted by the fraud and the institution in which the trader(s) is employed.
*The case resulted in the bankruptcy of the employer.

the trades.[3] Table 1.1 summarizes some historical episodes of rogue traders, whose unauthorized actions were deemed fraudulent by the local judicial system.

[3]See 'SEC Charges Rogue Trader Who Bankrupted His Firm' https://www.sec.gov/news/press-release/2021-205.

[4]See Kenyon (1997).

[5]See 'NASD Panel Sides With Jett, Orders Kidder to Free Funds' https://www.wsj.com/articles/SB851033074547103500.

[6]See Brown and Steenbeek (2001).

[7]See Iguchi (2014).

[8]See 'The Copper King: An Empire Built on Manipulation' https://www.investopedia.com/articles/financial-theory/08/mr-copper-commodities.asp.

[9]See 'Rogue trader Rusnak 'relieved' at AIB settlement' https://www.irishtimes.com/business/rogue-trader-rusnak-relieved-at-aib-settlement-1.2497430.

[10]See 'Former NAB foreign currency options traders sentenced' https://web.archive.org/web/20070928000543/http://www.asic.gov.au/asic/asic.nsf/byheadline/06-221+Former+NAB+foreign+currency+options+traders+sentenced?openDocument.

[11]See 'Former CAO Executive Pleads Guilty' https://www.wsj.com/articles/SB114241939419798794.

[12]See 'Rogue trader gets prison, told to repay Goldman $118 million' https://www.reuters.com/article/us-goldman-trader-sentencing-idUSBRE9B50XQ20131206.

[13]See 'French judge files preliminary charges against trader at Caisse d'Épargne' https://www.nytimes.com/2008/10/30/business/worldbusiness/30iht-30trader.17398216.html.

[14]See 'Bank Outlines How Trader Hid His Activities' https://www.nytimes.com/2008/01/28/business/worldbusiness/28bank.html.

[15]See 'Kweku Adoboli: From 'rising star' to rogue trader' https://www.bbc.com/news/uk-19660659.

[16]See 'SEC Charges Rogue Trader Who Bankrupted His Firm' https://www.sec.gov/news/press-release/2021-205.

Krawiec (2000) provides a concise definition of a rogue trader:

> 'A rogue trader is a market professional who engages in *unauthorized* purchases or sales of securities, commodities or derivatives, often for a financial institution's proprietary trading account.'

By this definition, rogue trading differs from the devastating trading losses where the trading actions with excessive risk-exposure are known and acknowledged by the higher management. Such losses include the cases that are often framed as 'rogue' by the public such as Brian Hunter, whose loss led to the bankruptcy of Amaranth Advisors[17], and Bruno Iksil, who bets incurred a $6bn loss for JPMorgan Chase[18].

It is worth noting that a trader is labelled as a 'rogue trader' only if the waged bets result in a loss, which is then exposed and charged by the regulators. As the failed internal risk management costs both the firm itself and the shareholders, the firms would also face a prosecution. This leads to the employers quietly firing traders who committed frauds.

The rise in rogue trading and its threat to both financial institutions and financial stability has been recognized by the Basel Committee as *operational risk*, defined as *'the risk of loss resulting from inadequate or failed internal processes, people and systems or from external events'* (Committee et al. (2011)). The Capital Accord of Basel II (and Basel III, to be enacted in 2023) include provisions for protection from operational losses: while insurance can cover high-frequency, low-impact events, rogue trading falls squarely in the low-frequency, high-impact category of uninsurable risks, which incur capital charges. Such charges are in turn based on standardized approaches or statistical models, due in part to the absence of consensus on the origin of rogue trading, which is the focus of this work.

The literature on rogue trading is relatively sparse. Most existing work explores the legal (Krawiec (2000) and Krawiec (2009)), social psychological (Wexler (2010)), and regulatory (Moodie (2009)) aspects of rogue trading, and offer a number of hypotheses for mechanisms that may foster malfeasance in trading. Armstrong and Brigo (2019) find that risk measures such as value at risk and expected shortfall are ineffective in preventing excessive risk-taking by traders who are risk-averse while gaining, but risk-loving while losing (this is modelled by S-shaped utilities e.g. $U(x) = x^\gamma \mathbb{1}_{\{x\geq 0\}} - \lambda(-x)^\gamma \mathbb{1}_{\{x<0\}}$ with $\lambda > 0$ and $\gamma \in (0,1)$). Xu, Zhu, and Pinedo (2020) use stochastic control to minimize operational risk through preventative and corrective policies where the operational risks are modelled as exogenous shocks. Xu, Pinedo, and Xue (2017) review the recent literature on operational risk. Contrary to the these studies, the fraud risk arises endogenously in our model so as to investigate what factors motivate a trader to engage in fraudulent activities.

[17] See 'TIMELINE-Amaranth's Brian Hunter settles with U.S. CFTC' `https://www.reuters.com/article/amaranth-settlement-idUSL2N0P413B20140915`.

[18] See "The London Whale" `https://www.bloomberg.com/quicktake/the-london-whale`.

1.1.1 Main takeaways of this work

The starting point of this work is that *'The continued existence of rogue trading [...] presents a mystery for many scholars and industry observers.'* (Krawiec (2000)). *'Operational risk is unlike market and credit risk; by assuming more of it, a financial firm cannot expect to generate higher returns.'* (Crouhy, Galai, and Mark (2004)). In other words, *prima facie* it is hard to reconcile rogue traders' actions with the optimizing behavior of sophisticated rational agents.

We propose a model in which rational, self-interested, risk-averse traders deliberately engage in fraudulent activity that has zero risk premium. While undetected, fraud allows a trader to feign superior returns, ostensibly without additional risk. In reality, higher returns are exactly offset by a higher probability of bankruptcy, thereby creating no value for the firm. Yet, under some circumstances, fraud may be optimal for a trader because, while its benefits are personal, potential bankruptcy costs are shared with other traders. Furthermore, a trader who understands the circumstances leading to others' fraud, can anticipate them and act accordingly, leading to a dynamic Nash equilibrium.

In equilibrium, each trader abstains from fraud as long as the respective share of wealth under management exceeds an endogenous *fraud threshold* that depends on both traders' preferences (risk aversions and average horizon) and investment characteristics (expected returns and volatilities). Thus, a trader must have enough *skin in the game* to remain honest: when the share of managed assets drops below the *fraud threshold*, then the marginal utility of fraudulent trades becomes positive, and a trader cheats as little and as quickly as possible to restore the wealth share to the honesty region. Importantly, such fraudulent activity does not generate extra volatility, so that it cannot be detected by monitoring wealth before bankruptcy occurs.

These results bring several insights. First, our model suggests that rogue trading has an important social component: A sole trader investing all the firm's capital would not engage in fraud because such a trader would bear in full both the costs and the benefits of fraudulent activity (Proposition 2.3.2). Furthermore, the fraud threshold is higher if a trader knows that nobody else is cheating (Lemma 3.4.2 and Theorem 3.5.2).

Second, the model emphasizes the danger that traders with relatively smaller amounts of capital can pose to a financial institution, due to their insufficient stakes in the firm. This concern is confirmed by the cases of the junior trader Jerome Kerviel and Nick Leeson. By reviewing Mr. Leeson's trading record and the investigation reports from Singaporean authorities, Brown and Steenbeek (2001) suggest that he had excluded the error account (meant for traders to settle minor trading mismatches) from the market reports to the headquarter and built up unauthorized speculative position in the early days after assuming the duty at Baring's office in Singapore in 1992.

Third, our comparative statics offer some clues for assessing and mitigating rogue-trading risk. The incidence of fraud is higher in less skilled traders, which means that emphasis on performance evaluation has the indirect benefit of fraud reduction. Fraud also declines significantly as risk aversion increases, suggesting that, *ceteris paribus*, the most fearless traders are also the ones most tempted by fraud, and that the most dangerous combination is found in a trader with *high* risk tolerance and *low* share of managed assets. Somewhat counter-intuitively, a longer horizon does not necessarily imply lower fraud, though fraud eventually declines when the horizon is long enough.

Fourth, our model hints at a subtle trade-off between investment performance and operational risk. Classical portfolio theory implies that diversification can only increase performance, hence the addition of a trader with expertise in a new asset class always improves the risk-return tradeoff. Yet, our results caution that a higher number of traders, each with a lower share of assets under management, may also increase the appeal of fraud for each of them, potentially worsening the firm's risk profile. (The quantitative analysis of the trade-off between diversification and fraud requires very different technical tools, hence is deferred to future research.)

This work offers the first structural model of rogue trading, in which fraud arises from agency issues between traders and their firms. A priori, it is traders' hidden action that enables fraudulent activity. A posteriori, the traders' optimal strategies imply that fraud is both continuous and of finite-variation, which makes it hard to detect even for a hypothetical observer who could continuously monitor traders' wealth.

In the interest of both simplicity and relevance, the model assumes that each trader is compensated with a fraction of trading profits, i.e., contracts are linear. As a result, the fraudulent activity that arises in the model does not stem from nonlinear incentives that may encourage risk-taking (cf. Carpenter (2000)), but merely from the asymmetric opportunity of taking personal credit from fraudulent gains while sharing bankruptcy costs. In this sense, each trader's fraud represents an externality for other traders and the firm, whence overall demand for fraud is socially suboptimal (i.e., nonzero).

At the technical level, our analysis contributes to the financial application of the theory of nonzero-sum stochastic differential games with singular controls. In contrast to single-agent singular stochastic control problems, which date back to the finite fuel problem of Bather and Chernoff (1967), research on singular stochastic differential games is relatively recent: Guo and Xu (2019) generalize the finite fuel problem to an n-player stochastic game and a mean-field game, in which each player minimizes the distance of an object to the center of N objects, while keeping her total amount of controls at minimal. Guo, Tang, and Xu (2018) extend this analysis to a larger class of games with possible moving reflecting boundaries in Nash equilibria. Kwon (2020) analyzes the game of contribution to the common good and discovers Nash equilibria of mixed type i.e. the strategies in equilibrium have both absolute continuous and singular components. De Angelis and Ferrari (2018) establish the connection between a class of stochastic games with singular controls and a certain optimal stopping game, where the underlying state processes differ but the reflective and exit boundaries coincide. Kwon and Zhang (2015) and Ekström, Lindensjö, and Olofsson (2020) study optimal stopping games in which all or one of the players control an exit time that terminates the game. Note that the fraud in Ekström, Lindensjö, and Olofsson (2020) differs from the one considered here, in that their model entails an agent stealing from another one, who seeks to detect fraud and can terminate the game. In these papers, players are forbidden to execute discontinuous actions simultaneously, whereas the present model does not impose such a restriction. In addition, this work provides a structural formulation of Nash equilibrium in the presence of singular controls. Adopting BSDE Techniques, Karatzas and Li (2012) investigate existence and uniqueness of Nash equilibrium in games of control and stopping, while Hamadène and Mu (2015) establish existence for the games without exit but with unbounded drift. Dianetti and Ferrari (2020) employ fixed point methods for the monotone-follower games with submodular costs.

The results in this work also bear a curious analogy with portfolio choice with proportional transaction costs in that, similar to Davis and Norman (1990), the solution to the present model leads to an inaction region, surrounded by two regions in which actions are performed as little as necessary to return to the inaction region. Although the mechanisms underlying the two models are very different, it is worth pointing out the common feature that leads to the common structure. In both cases (and in many other singular control problems), an action is performed only in positive amount (fraud of either trader in this paper, buying or selling in portfoio choice). As a result, the inaction region arises when each action is counterproductive for its agent, while the action regions are visited at their boundaries because costs are linear in the action performed (bankruptcy probability in this paper, trading costs in portfolio choice).

The rest of this thesis is organized as follows. Section 1.2 reviews several applications of the theory of stochastic differential game with singular controls. Chapter 2 describes the model of rogue trading and its rationale. Chapter 3 constructs a Nash equilibrium with two traders and states the main result. Chapter 4 discusses the interpretation of the results and their implications.

1.2 Applications of stochastic differential games with singular controls

This section is devoted to introducing stochastic differential games with singular controls in a brief manner and give some of its applications. Most rigour concerning technical assumptions, regularities and controls' admissibility is foregone in favor of readability.

Assume that the dynamical system $X = (X_t)_{t\geq 0}$ (i.e. the state) follows a stochastic differential equation (SDE) of the form:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t + \kappa(X_{t-})dA_t \tag{1.2.1}$$

with initial data $X_{0-} = x \in \mathbb{R}^d$, where B is a d-dimensional Brownian motion and the coefficients drift μ and diffusion σ are measurable deterministic functions $\mathbb{R}^d \to \mathbb{R}^d$, the coefficient of the last term $\kappa : \mathbb{R}^{d\times l} \to \mathbb{R}^d$, and a progressively measurable process $A = (A_t)_{t\geq 0}$ valued in $\mathbb{R}^l$ of which the components are right-continuous non-decreasing with left limits. Since the process A is of finite variation, dA_t induces a positive measure on $\mathbb{R}$ and the integral of $\kappa(t, X_{t-})dA_t$ can be viewed as the Lebesgue-Stieltjes integral constructed path by path.

An agent can influence the state X by controlling the process A. Since the continuous component A^c of A is not necessarily absolute continuous (that is, the path of A^c can have a component that is a singular function[19]), the control A is said to be a *singular control* (cf. Reppen (2018, section 1.2) for a thorough discussion on the naming of 'singular control'.) If the agent aims at maximizing some reward functional $J(x; A)$ for all admissible control A, then this problem is known as the (stochastic) optimal control problem with singular control. The optimal strategy in this context typically has the characteristic of activating according to local time at some boundary, which is known as a solution to the Skorokhod problem. Such strategy can be identified by two disconnected regions of the state space, an *action* region in which the agent immediately brings the state to the boundary of this region; and a *continuation* region, in the interior of which the agent does not exercise the control at all but on the boundary of which the agent exercise the control minimally so as to reflect the state back into the interior of the region. Also in this thesis, a Skorokhod problem arises- the fraud commitment of a trader occurs only when her/his share of wealth drops to certain threshold (see section 3.3).

To accommodate more agents, we modify the state X by generalizing the coefficient $\kappa : \mathbb{R}^{l\times d\times N} \to \mathbb{R}^d$ and the process $A = (A^1, \dots, A^N)$ be valued in $\mathbb{R}^{l\times N}$, where $N \geq 1$. Now, instead of one agent, there are n agents (labelled by the integers $1, \dots, N$) who can jointly influence the state X via their individual control A^i valued in $\mathbb{R}^d$ for all $i \in \{1, \dots, N\}$. Each agent i attempts to maximize his/her individual reward functional

$$J^i\left(x; (A^1, \dots, A^N)\right) = \mathbb{E}\left[\int_{[0,\infty]} \left(f^i(t, X_{t-}, A_{t-})dt + g^i(t, X_t, A_t)dA_t\right)\right]$$

over all admissible A^i in some admissible set $\mathcal{A}$, guaranteeing the value functions to be well-defined. Note that the reward to a given agent (apart from her/his own action) depends upon the actions of other players indirectly through the values of the state

[19] A standard example is the Cantor function, which is continuous, but not absolutely continuous.

X over time, but also directly as the specific actions taken by the other players may appear explicitly in the expression of the running cost f^i and g^i of agent i. In this setting, the optimality is defined by the well-celebrated notion of Nash equilibrium.

Definition 1.2.1. *A set of admissible controls $A^* = (A^{1,*}, \dots, A^{N,*})$ is said to be a Nash equilibrium if for all $i \in \{1, \dots, N\}$ and any $x \in \mathbb{R}^d$*

$$J^i\left(x; (A^{1,*}, \dots, A^{i-1,*}, A^i, A^{i+1,*}, \dots, A^{N,*})\right) \leq J^i(x; A^*) \text{ for all } A^i \in \mathcal{A}.$$

If all players jointly maximize the same reward functional, i.e. $J^i = J$ for all $i \in \{1, \dots, N\}$ for some functional J, then such game is known as the *cooperative* differential game. In the case when $N = 2$ and $J^1(x; A) + J^2(x; A) = 0$, the game turns into the well-known *zero-sum* stochastic differential game as one player's entire gain is on the entire loss of the other. The examples considered in this section are instead non zero-sum games.

Remark 1.2.2. *The Nash equilibria of stochastic differential games with singular controls may not necessarily be of singular type. Section 1.2.2 presents such an example, where under symmetry of the players, the equilibrium strategies can be absolutely continuous.*

1.2.1 Market share duopoly game

This example considers a fierce market share battle that typically occurs in mature industries with shrinking profit opportunities. For instance, in 1990s the US-telecommunication giants AT&T, MCI and Sprint compete in capturing larger market share by spending heavily on marketing.

Let $d = 1$ and $N = 2$, the two players compete against each other in a mature and saturated market, where the market share of player i ($i \in \{1, 2\}$) denoted by X^i is such that $X^1_t + X^2_t = 1$ a.s. for all $t \geq 0$ and each solves the SDE

$$dX^i_t = \mu_i(X^i_t)dt + \sigma_i(X^i_t)dB_t + dA^i_t - dA^j \text{ for any } t < \tau_1 \wedge \tau_2$$

with $X^i_{0-} = x_i \in [0, 1]$, $\mu_i(x) = -\mu_j(1 - x)$ and $\sigma_i(x) = -\sigma_j(1 - x)$. The control A^i of player i represents the cumulative effort up to time t to increase her/his market share, while the player can also exit the market represented at some stopping time τ_i. Each player i's objective is to maximize the expected cumulative discounted profit (with discount factor $r > 0$)

$$J^i\left(x; (A^i, \tau_i), (A^j, \tau_i)\right) = \mathbb{E}\left[\int_0^{\tau_1 \wedge \tau_2} e^{-rt}\left(\pi_i(X^i_t)dt - k_i(X^i_t) \circ dA^i_t\right) \right.$$
$$\left. + \mathbb{1}_{\{\tau_i \geq \tau_j\}} e^{-r\tau_j} \frac{\pi_i(1)}{r}\right]$$

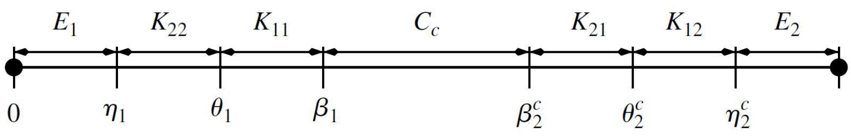


Fig. 1.1: Figure 1 from Kwon and Zhang (2015): Action region partition in Nash equilibrium in view of X^1, where the constants $\beta_2^c := 1 - \beta_2$, $\theta_2^c := 1 - \theta_2$ and $\eta_2^c := 1 - \eta_2$ with β_2, θ_2 and η_2 being player 2's action boundaries in view of X^2.

over all admissible controls (A^i, τ_i), where the term with operator $\circ$ is defined for any $t_1, t_2 \in [0, \infty)$ such that $t_1 < t_2$ by

$$\int_{t_1}^{t_2} e^{-rt} k_i(X_t^i) \circ dA_t^i := \int_{t_1}^{t_2} e^{-rt} k_i(X_t^i) dA_t^{i,c} + \sum_{t \in [t_1, t_2]} e^{-rt} \int_0^{\Delta A_t^i} k_i(X_{t-}^i + u) du,$$

with $A^{i,c}$ being the continuous component of A^i. The function π_i represents the profit rate for occupying a certain market share, k_i is the cost rate of exercising the control effort A^i and the last term $\frac{\pi_i(1)}{r}$ is the perpetual stream of profit after the other player j exits (if instead player i decides to exit then this term vanishes). It is assumed that $\Delta A_t^1 \Delta A_t^2 = 0$ a.s. for all $t \geq 0$.

Under appropriate conditions, Kwon and Zhang (2015, Theorem 2) state a class of Nash equilibria with the structure depicted as in figure 1.1. In equilibrium, if the initial market share of of player 1 starts in

(i) E_1 $(= [0, \eta_1])$: then player 1 exits the competition as the profit is negative when the market share is too low, and the cost of effort to increase market share is also too high;

(ii) K_{11} $(= [\theta_1, \beta_1])$: then player i immediately exercises A^i to boost X^i to β_1 and reflects X^i at β_1 thereafter to keep her/his market share from dropping below β_1;

(iii) C_c: then neither of the participants acts, except when the market share slides to the boundary points β_1 (β_2^c), at which player 1 (2) acts minimally to reflects her/his share towards higher value;

(iv) K_{12} $(= [\theta_2^c, \eta_2^c])$: by an instant effort player i brings X^i to η_2^c (which is the exit zone of player 2) forcing player 2 out of the competition.

The argument applies likewise to player 2 with X^2 being the reference state.

1.2.2 Game of contribution to common good

Two players (i.e. $N = 2$) consider irreversible and costly contribution to a stock of common good. The dynamic of the common state process X_t valued in $\mathbb{R}$ solves the

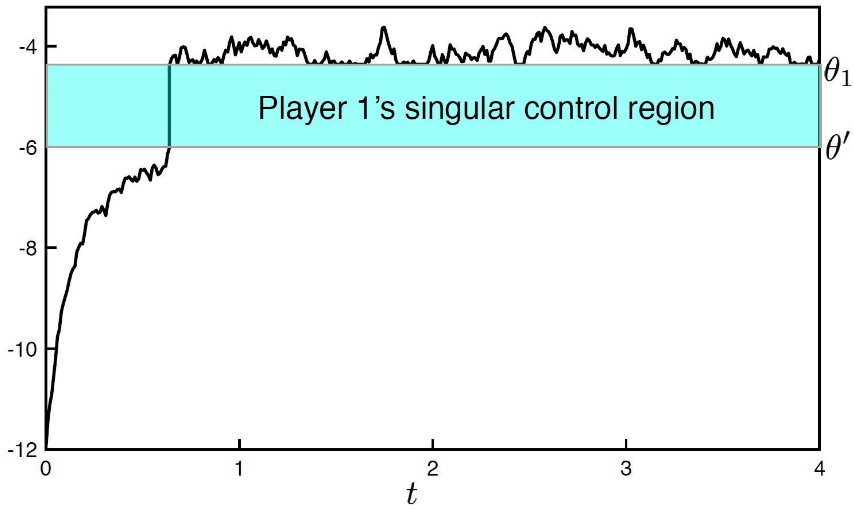


Fig. 1.2: Figure 4 from Kwon (2020): A sample path of $t \mapsto X_t$ in Nash equilibrium: $\mu = -1, \sigma = r = k_1 = k_2 = 1, \rho = 2, x_c = -10$ and the initial data $x < \theta$.

SDE

$$dX_t = \mu(X_t^i)dt + \sigma(X_t)dB_t + dA_t^1 + dA_t^2$$

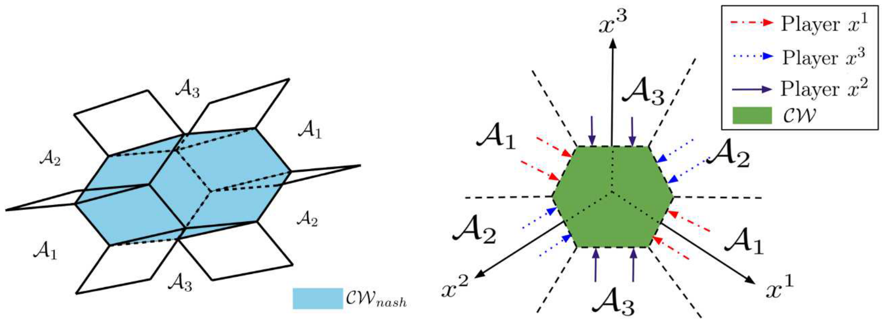
with $X_{0-} = x \in \mathbb{R}$ for all $t \geq 0$, where A^i is the cumulative contribution of player i. Each player i maximizes the the expected discounted (the discount factor $r > 0$) payoff

$$J^i(x; A^1, A^2) = \mathbb{E}\left[\int_0^\infty e^{-rt}\left(\pi_i(X_t)dt - k_i dA_t^i\right)\right]$$

over all admissible controls A^i. The coefficient μ is assumed to be negative to model the deterioration of the common good. Similar to the reward functional in section 1.2.1, π_i represents the profit rate and k_i is the cost rate for increasing the contribution.

An instance of this model is the advertisement expenditures in the stock of goodwill such as the generic advertising on commodities (known as the free rider problem of commodity promotion). For example, the advertising expenditures by Florida on 'Florida grown' orange juice benefit not only the local orange juice industry but also non-Florida orange juice importers (Lee and Fairchild (1988)) and a salmon promotion program conducted by Norway has benefited its international competitors (Kinnucan and Myrland (2003) studies this relation between Norway and the UK). More cases include the Polish diary campaign with the slogan: 'Drink milk, be great!' and the USA campaign for 'Cotton: the fabric of our lives' (cf. Just and Pope (2016)) etc.. The stock of the product's overall goodwill is the common good, a manufacture that does not invest on it *free rides* the benefits from the other manufacture's expenditure. This problem is a sub-class of variable concession games, which is an extension of game of war of attrition (cf. Smith (1974)).

Under appropriate assumptions, Kwon (2020) shows that:



(a) $\mathcal{CW}$ and $\mathcal{A}_i$ $(i = 1, 2, 3)$

(b) $\mathcal{A}_1$ and $\mathcal{CW}$,
a bird's-eye view from $(1, 1, 1)$
to $(0, 0, 0)$

Fig. 1.3: Figure 3 from Guo and Xu (2019): Action region partition in Nash equilibrium when $N = 3$.

(i) when $N = 1$, the optimal strategy of the only player is of the singular type, keeping the state process above a constant threshold;

(ii) when $N = 2$: the existence of a class of Nash equilibria of the regular type, i.e. $A^{i,*}$ is absolute continuous w.r.t. Lebesgue measure and the mixed type, i.e. of both the regular and singular type (for at least one of the players). The numerical example Figure 1.2 illustrates case 2 in which player 1's equilibrium control is of mixed type while player 2's is of regular type, in which the profit rate is given by $\pi(x) = 1 - e^{vx}$ for $x \geq x_c$ and $\pi(x) = \pi(x_c) + (x - x_c)\rho$ for $x < x_c$. The region $[0, \theta']$ is the regular control zone for both players. Upon X reaching θ', player 1 boosts X up to θ^1. Then thereafter, the state is subject to singular control of player 1 reflecting X at boundary θ^1. Note that swapping these strategies between the players is also a Nash equilibrium;

(iii) when $N = 2$ with asymmetry ($\pi_1 \neq \pi_2$ or/and $k_1 \neq k_2$): the only Nash equilibria are of mixed type.

For generalized game(s) with open-loop strategies for $N > 2$ players, we refer readers to Ferrari, Riedel, and Steg (2017).

1.2.3 Fuel follower game

Let the dimension of the state equal number of players, i.e. $d = N$. Each player controls (on the cost of fuel) a single object that is moving in a real line, with the objective of minimizing the distance between the object and the center of all objects while consuming the least amount of fuel. The position of player i ($i \in \{1, \ldots, N\}$) denoted by X^i solves the SDE

$$X_t^i = x_i + B_t^i + A_t^{i,+} - A_t^{i,-},$$

with $X^i_{0-} = x_i$ representing the initial position of player i. Player i can adjust her/his own position higher or lower by controlling $(A_t^{i,+}, A_t^{i,-}) =: A^i$. Notice that the state processes are uncoupled, that is, the position and the control of one player cannot directly affect that of another.

Each player i minimizes the cost functional

$$J^i(x; A) = \mathbb{E}\left[\int_0^\infty e^{-\alpha t}\left[h\left(X_t^i - \frac{\sum_{k=1}^N X_k}{N}\right) dt + dA_t^{i,+} + dA_t^{i,-}\right]\right]$$

over all $A^i = (A^{i,+}, A^{i,-}) \in \mathcal{A}$ where the discount factor $\alpha > 0$, $dA_t^{i,+} + dA_t^{i,-}$ is the fuel usage in an infinitesimal time interval, $X_t^i - \frac{\sum_{k=1}^N X_k}{N}$ is the distance between the position of player i and the center of all players' positions. In other words, each player minimizes her/his distance to the center of all players while trying to consume minimal amount of fuel.

Guo and Xu (2019, Theorem 3) provide a verification theorem for a Nash equilibrium, where each player i has an action region $\mathcal{A}_i$ (at which $\Delta A_t^i \neq 0$ a.s.) and a common no-action region $\mathcal{CW} \subset \mathbb{R}^N$. When the state X is:

(i) in the interior of $\mathcal{A}_i$: player i exercises A^i to instantly push X to the boundary $\partial\mathcal{A}_i$;

(ii) at the boundary $\partial\mathcal{A}_i$: player i pushes minimally such that X is reflected inward along the direction perpendicular to the boundary $\partial\mathcal{A}_i$;

(iii) in the exterior of $\mathcal{A}_i$: player i does not act at all, i.e. $A^i = 0$.

Figure 1.3 illustrates the action regions in Nash equilibrium when $N = 3$. Note that the a player's action region $\mathcal{A}_i$ is divided into two disconnected sub-regions, where in one region the position X^i is pushed up by the control $A^{i,+}$ whereas X^i is pulled down in the other sub-region by the control $A^{i,-}$.

Chapter 2

A Model of Rogue Trading

Most episodes of fraudulent trading share some distinctive features. First, they violate a firm's internal rules or external regulations. Second, fraud often remains concealed and results in modest (relative to the firm's size) gains that are ascribed to the skill of the perpetrator. Third, fraud generates substantial risk without expected return for the firm, and is revealed only when catastrophic losses eventually materialize.

To reproduce these features, it is useful to think of a small fraud as a (forbidden) bet that a trader wages on the whole firm's capital. With a small chance (say, ε), the bet bankrupts the firm, but most of the time (with probability $1 - \varepsilon$) it results in a return of $1/(1 - \varepsilon) \approx 1 + \varepsilon$, which the trader can take credit for. Of course, the bet's overall return for the firm is zero, as $(1 - \varepsilon) \cdot 1/(1 - \varepsilon) + 0 \cdot \varepsilon = 1$. Such asymmetric outcomes (likely small gains against unlikely large losses) are in fact common in both illicit and licit trading strategies (for example, selling deep out-of-the money options), and have attracted the label of 'picking up nickels in front of a steamroller' (Duarte, Longstaff, and Yu (2006)).

Thus, the dilemma of an unscrupulous but profit-driven and risk-averse trader is to what degree to engage in fraud, as cheating too little may forego some easy profits, but cheating too much may result in likely bankruptcy. If one imagines the small fraud above as the outcome of a (heavily biased) coin-toss, the trader essentially ponders how many coins to toss. For example, tossing two coins would generate a likely payoff of $(1 - \varepsilon)^{-2}$ but may also lead to bankruptcy with probability $2\varepsilon - \varepsilon^2$.

If the trader is the sole firm's owner, it is not hard to see that fraud does not pay: when one bears both gains and losses in full, waging fair bets on one's capital merely replaces a payoff with another one, more uncertain but with the same mean – an inferior choice by risk aversion.

In this sense, fraud arises from social interactions, both through the incentives implied by traders' compensation contracts, or by each trader's awareness that others traders in the firm may engage in fraud. The present model focuses on the latter motive by assuming that each trader receives a fixed fraction of individual profits and losses, which is a common arrangement for bonuses with clawback provisions. The model envisages multiple traders: each of them has a mandate to invest a share of the firm's capital in some risky asset with a positive risk premium and is paid with a fraction of the terminal payoff. Thus, each trader's objective is aligned with the firm's. For the sake of tractability and clarity, the paper focuses on the case of two traders.

The moral hazard stems from the asymmetric effects of fraud on a trader's reward: as long as the fraudulent activity is successful, the trader can disguise its revenues as the fruit of personal skill in performing the investment mandate. In reality, such additional revenues merely compensate for the fraudulent bets that the trader wages on the capital of the *whole* firm, rather than personal capital (e.g. exceeding risk limits by either collateralizing firm's asset or assuming excess liabilities). Of course, such bets are possible exactly because they are fraudulent, and are explicitly forbidden by the firm's regulation: they nonetheless exist, due to 'inadequate or failed internal processes, people and systems' embodied in the definition of operational risk (Committee et al. (2011)).

The appeal of fraud – privatizing gains while socializing losses – thus varies with a trader's share of the firm's capital: intuitively, the temptation of fraudulently enriching oneself is much stronger for a small trader, who has little to lose and much to gain from gambling with others' wealth, than for a large trader who has significant skin in the game. For this reason, in the present continuous-time model each trader can cheat with varying intensity in response to changes in one's and others' wealth.

After this informal description, the precise definition of the model follows.

2.1 Investment and fraud

Fix a complete stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the natural filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ of an N-dimensional ($N \geq 1$) Brownian motion $B = (B_t)_{t\geq 0}$, satisfying the usual hypotheses of right-continuity and saturatedness. Denote the σ-algebra $\sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right) \subset \mathcal{F}$ by $\mathcal{F}_\infty$.

Assuming a zero safe rate to ease notation, in the absence of fraud the capital Y^i_t of the i-th trader ($i \in \{1, \dots, N\}$) evolves as

$$dY^{i,x}_t = \mu_i Y^{i,x}_t dt + \sigma_i Y^{i,x}_t dB^i_t, \qquad Y^{i,x}_0 = x_i > 0,$$

reflecting the trader's average ability $\mu_i > 0$ to deliver excess returns with the volatility $\sigma_i > 0$ that the firm's risk management is willing to accept. For simplicity, assume that B^i and B^j are independent for $i \neq j$, which means that traders take uncorrelated risks (for example, one invests in stocks and the other in bonds).

To describe how each trader may engage in fraud by endangering the firm's capital, define the class of increasing processes

$$\mathcal{A} := \Big\{ A = (A_t)_{t\geq 0} : \mathbb{F}\text{-adapted, right-continuous,} \\ \text{non-decreasing, with } A_{0-} = 0 \in \mathbb{R} \Big\}.$$

For $A \in \mathcal{A}$, A_t represents the cumulative amount of 'bets' waged by a trader on the firm's capital up to time t. To understand such a representation, suppose that $A_t = \int_0^t \lambda_s ds$, which means that in the interval $[s, s+ds]$ the trader wages a fair bet that has the probability $\lambda_s ds$ of bankrupting the firm. Because the fraud is illicitly waged on the firm's capital (thereby exceeding the capital $Y^{i,x}$ that the trader has been assigned), if bankruptcy does not occur such fraud yields a profit of $Y_s^S \lambda_s ds$, where $Y^{S,x} := \sum_{k=1}^N Y^{k,x}$ is the total capital of the firm.

Although this description is intuitive, it has a twofold limit: First, it encompasses only the case of fraud with a finite rate λ_s, in that it excludes bursts of rogue trades at any instant. Second, it cannot extend the impact of fraud on bankruptcy probability to arbitrary time interval (note that simply by integration the value of $\int_s^t \lambda_u du$ can exceed 1). For this reason, a more careful but also more technical description is necessary.

To make precise the intuition that how dA_s drives the bankruptcy rate, note first that any $A \in \mathcal{A}$ is right-continuous and of finite variation. Therefore, it has the representation $A_t = A_t^c + \sum_{0\leq s\leq t} \Delta A_s$ for any $t \geq 0$, where $\Delta A_s = A_s - A_{s-}$ and A^c is the continuous part of the process A with $A_0^c = 0$. For a set of fraud process $A = (A^1, \ldots, A^N) \in \mathcal{A}^N$, denote the *total fraud* process by $A_\cdot^S = \sum_{k=1}^N A_\cdot^k$. The *bankruptcy time* is then defined as

$$\tau_A = \inf \left\{ t \geq 0 : A_t^S \geq \theta \right\}, \tag{2.1.1}$$

where θ is an $\mathcal{F}$-measurable exponential random variable with rate 1, independent of the filtration $\mathbb{F}$. (Recall the convention that $\tau_A = +\infty$ on the empty set.)

Before bankruptcy occurs, the wealth of each trader follows the dynamics

$$dY_t^{i,x} = \mu_i Y_t^{i,x} dt + \sigma_i Y_t^{i,x} dB_t^i + Y_{t-}^S d\tilde{A}_t^i, \qquad Y_{0-}^{i,x} = x_i > 0, \quad 1 \leq i \leq N, \tag{2.1.2}$$

where the integral with respect to $\tilde{A}^i$ in (2.1.2) is understood in the Lebesgue-Stieltjes sense, and $\tilde{A}_t^i := A_t^{i,c} + \sum_{0\leq s\leq t} \left(e^{\Delta A_s^i} - 1 \right)$ reflects the fact that the *simple* return of a jump in fraud is not Δ itself but rather $e^{\Delta} - 1$. Proposition 2.3.1 further justifies the choice of the return from jump fraud. Such a distinction is immaterial with continuous fraud because $e^{\Delta} - 1 \sim \Delta$ for Δ close to zero. The following result shows that the pre-bankruptcy wealth (2.1.2) is well-defined, strictly positive with probability 1 and provides an expression in terms of the stochastic exponential (see Definition 2.4.1) for both of the individual trader's and the total firm's wealth.

Lemma 2.1.1. *For any $k \in \{1,\ldots,N\}$, let $r_k(x) = \frac{x_k}{\sum_{i=1}^N x_i}$ for any $x \in \mathbb{R}_+^N$.*

(i) There exists a unique strong solution $Y^x = \left(Y^{1,x},\ldots,Y^{N,x}\right)$ to the SDE (2.1.2) and for all $i \in \{1,\ldots,N\}$, $\mathbb{P}\left(Y_t^{i,x} > 0 \text{ for all } t \geq 0\right) = 1$.

(ii) For all $i \in \{1,\ldots,N\}$ and any $t \geq 0$

$$Y_t^{i,x} = x_i \mathcal{E}\left(\mu_i \cdot + \sigma_i B_\cdot^i + \int_{[0,\cdot]} r_i(Y_{s-}^x)^{-1} d\tilde{A}_s^i\right)_t \quad a.s. \tag{2.1.3}$$

and

$$Y_t^{S,x} = \left(\sum_{k=1}^N x_k\right) \mathcal{E}\left(\sum_{k=1}^N \left(\int_0^\cdot \mu_k r_k(Y_s^x) ds + \int_0^\cdot \sigma_k r_k(Y_s^x) dB_s^k\right) + \tilde{A}_\cdot^S\right)_t \quad a.s. \tag{2.1.4}$$

where $\tilde{A}_\cdot^S = \sum_{k=1}^N \tilde{A}_\cdot^k$.

Proof of Lemma 2.1.1. Denote by I_N an $N \times N$ identity matrix. The SDE (2.1.2) can be written in the vector form,

$$dY_t = \operatorname{diag}(Y_t) dR_t + \operatorname{trace}\left(\operatorname{diag}(Y_{t-})\right) I_N d\tilde{A}_t, \qquad Y_{0-} = x \in \mathbb{R}_+^n, \tag{2.1.5}$$

where $R = (R^1,\ldots,R^N)$ with $R_t^i = \mu_i t + \sigma_i B_t^i$ for any $i \in \{1,\ldots,N\}$ and $\tilde{A} = (\tilde{A}^1,\ldots,\tilde{A}^N)$. The linearity of the coefficients of (2.1.5) implies uniform Lipschitz continuity, hence the existence and uniqueness of a strong solution (cf. Cohen and Elliott, 2015, Theorem 16.3.11).

For any $i \in \{1,\ldots,N\}$, let $Z_t^i = R_t^i + \tilde{A}_t^i$ and $H_t^i = x_i + \int_{[0,t]} \sum_{j \neq i}^N Y_{s-}^j d\tilde{A}_s^i$ for all $t \geq 0$ with $Z_{0-}^i = 0$ and $H_{0-}^i = x_i$. Rewriting (2.1.2) yields

$$Y_t^i = H_t^i + \int_{[0,t]} Y_{s-}^i dZ_t^i.$$

By Jacod, 2006, Theorem 6.8, it follows that

$$Y_t^i = \mathcal{E}(Z^i)_t \left(x_i + \int_{[0,t]} \mathcal{E}(Z^i)_{s-}^{-1} d\bar{H}_s^i\right), \tag{2.1.6}$$

where

$$\bar{H}_t^i = H_t^i - \sum_{0 \leq s \leq t} \frac{\Delta H_s^i \Delta \tilde{A}_s^i}{1 + \Delta \tilde{A}_s^i} = H_t^{i,c} + \sum_{0 \leq s \leq t} \frac{\Delta H_s^i}{1 + \Delta \tilde{A}_s^i} = x_i + \int_{[0,t]} \sum_{j \neq i}^N Y_{s-}^j d\bar{A}_s^i \tag{2.1.7}$$

and $\bar{A}_t^i = A_t^{i,c} + \sum_{0 \leq s \leq t} \left(\frac{\Delta \tilde{A}^i}{1 + \Delta \tilde{A}^i}\right)$ with $\bar{A}_{0-}^i = 0$. Substituting (2.1.7) into (2.1.6) yields

$$Y_t^i = \mathcal{E}(Z^i)_t \left(x_i + \int_{[0,t]} \mathcal{E}(Z^i)_{s-}^{-1} \sum_{j \neq i}^N Y_{s-}^j d\bar{A}_s^i\right). \tag{2.1.8}$$

Let τ_0 be the first exit time of the process $\min_{k\in\{1,\dots,N\}} Y^k$ from $\mathbb{R}_+$, that is, $\tau_0 = \inf\left\{t \geq 0 : \min_{k\in\{1,\dots,N\}} Y^k_t \leq 0\right\}$. Suppose $\mathbb{P}\left(0 \leq \tau_0 < \infty\right) > 0$, then for any $\omega \in \{\omega : 0 \leq \tau_0(\omega) < \infty\}$, there exists $q \in \{1, \cdots, N\}$ such that $Y^q_{\tau_0(\omega)}(\omega) \leq 0$ and $Y^q_{\tau_0(\omega)-}(\omega) \geq 0$ as Y^q is càdlàg. Since $x_q > 0$, (2.1.8) implies

$$\sum_{j\neq q}^{N} Y^j_s(\omega) < 0$$

for some $s < \tau_0(\omega)$, which contradicts the definition of τ_0. This finishes the proof of (i).

We thus may rewrite (2.1.2) and the firm's total pre-bankruptcy wealth Y^S as

$$\begin{aligned} \frac{dY^i_t}{Y^i_{t-}} &= dR^i_t + \frac{Y^S_{t-}}{Y^i_{t-}} d\tilde{A}^i_t, & Y^i_{0-} &= x_i, \\ \frac{dY^S_t}{Y^S_{t-}} &= \sum_{k=1}^{N} \frac{Y^k_t}{Y^S_t} dR^k_t + d\tilde{A}^S_t, & Y^S_{0-} &= \sum_{k=1}^{N} x_k. \end{aligned}$$

An application of Jacod, 2006, Theorem 6.8 yields (ii). □

Incorporating the effect of bankruptcy after τ_A, the final expression for wealth is

$$X^{i,x}_t = \mathbb{1}_{\{t<\tau_A\}} Y^{i,x}_t \quad \text{with} \quad X^{i,x}_{0-} = x_i. \tag{2.1.9}$$

Upon bankruptcy (on the set $t \geq \tau_A(\omega)$), the wealth of all traders vanishes and remains null thereafter, hence the dynamics of the pre-bankruptcy wealths beyond $\tau_A(\omega)$ is irrelevant for the model. Effectively, fraud is described by the stopped process $A^i_{\cdot\wedge\tau_A}$.

Remark 2.1.2. *One may be tempted to define the pre-bankruptcy process as*

$$dY^{i,x}_t = \mu_i Y^{i,x}_t dt + \sigma_i Y^{i,x}_t dB^i_t + Y^S_{t-} dA^i_t$$

where the last term is not modified for the instantaneous fraud ΔA^i. This leads to ill-posedness of the model: Let $N = 1$ and consider a sequence of deterministic and continuous fraud process $A^{\epsilon,n}$, $n = 1, 2, 3, \dots$ given by $A^{\epsilon,n}_t = nt$ for $t \leq \epsilon/n$ and $A^{\epsilon,n}_t = \epsilon$ for $t > \epsilon/n$. Note that $A^{\epsilon,n}$ converges pointwise to $A^\epsilon_t := \epsilon$ for all $t \geq 0$. By the continuity of paths of $A^{\epsilon,n}$ and Lemma 2.1.1, it follows that for all $t \geq 0$ $\lim_{n\to\infty} Y^x_t(A^{\epsilon,n}) = \lim_{n\to\infty} x_i e^{A^{\epsilon,n}_t} \mathcal{E}\left(\mu_i \cdot + \sigma_i B^i_\cdot\right)_t = x_i e^\epsilon \mathcal{E}\left(\mu_i \cdot + \sigma_i B^i_\cdot\right)_t$ a.s., whereas $Y^x_t(\lim_{n\to\infty} A^{\epsilon,n}) = x_i(1+\epsilon)\mathcal{E}\left(\mu_i \cdot + \sigma_i B^i_\cdot\right)_t$. Hence, $\lim_{n\to\infty} Y^x_t(A^{\epsilon,n}) > Y^x_t(\lim_{n\to\infty} A^{\epsilon,n})$ a.s. for all $t > 0$.

2.2 Bankruptcy time

The bankruptcy time τ_A is not an $\mathbb{F}$-stopping time since the random threshold θ is independent of the σ-algebra $\mathcal{F}_\infty$. Thus, to accommodate the wealth process $X^x = (X^{1,x}, \ldots, X^{N,x})$, we expand the filtration $\mathbb{F}$ 'minimally'. To this end, let $\mathbb{H}^A = \left(\mathcal{H}^A_t\right)_{t\geq 0}$ be the natural filtration of the bankruptcy process $\left(\mathbb{1}_{\{\cdot\geq\tau_A\}}\right)_{t\geq 0}$ and define the enlarged filtration $\mathbb{G}^A = \left(\mathcal{G}^A_t\right)_{t\geq 0}$ as $\mathcal{G}^A_t = \bigcap_{s>t}\left(\mathcal{F}_s \vee \mathcal{H}^A_s\right)$, which is the smallest right-continuous filtration containing $\mathbb{F}$ such that τ_A is a stopping time. Such an extension is known as 'progressive filtration enlargement' (cf. Jeulin, 2006 and Jeanblanc and Le Cam, 2009) and is popularly utilized in credit risk modelling.

Let $\bar{A}^S_t = A^{S,c}_t + \sum_{0\leq s\leq t}(1 - e^{-\Delta A^S_s})$ for all $t \geq 0$. Note that $\bar{A}^S \in \mathcal{A}$ and $\bar{A}^S$ differs from the total fraud process $\tilde{A}^S$. In fact, $\bar{A}^S_t \geq \tilde{A}^S_t$ a.s. for all $t \geq 0$ since $1 - e^{-\sum_{i=1}^N \alpha_i} \geq \sum_{i=1}^N (1 - e^{-\alpha_i})$ for any $\alpha \in [0,\infty)^N$, where the equality holds *if and only if* $\alpha \in \{0\}^{N-1} \times [0,\infty)$. Hence, the processes $\bar{A}^S$ and $\tilde{A}^S$ coincide if no simultaneous jump frauds occur at any time $t \geq 0$. The following result shows that $\bar{A}^S$ is the compensator of the bankruptcy process $\mathbb{1}_{\{\cdot\geq\tau_A\}}$.

Lemma 2.2.1 (Doob-Meyer decomposition)**.** *The process $M^A = (M^A_t)_{t\geq 0}$ defined by*

$$M^A_t = \mathbb{1}_{\{t\geq\tau_A\}} - \bar{A}^S_{t\wedge\tau_A}$$

is a uniformly integrable $\mathbb{G}^A$-martingale. Furthermore, $\bar{A}^S_{\cdot\wedge\tau_A}$ is the unique $\mathbb{G}^A$-predictable, integrable and non-decreasing process such that M is a $\mathbb{G}^A$-martingale and $\bar{A}^S_{0-} = 0$.

Proof. Note that the non-decreasing process $\mathbb{1}_{\{\cdot\geq\tau_A\}}$ is a $\mathbb{G}^A$-submartingale. Define the stochastic process $Z = (Z_t)_{t\geq 0}$ by $Z_t := \mathbb{P}(t < \tau_A | \mathcal{F}_t)$. Because $\mathbb{F} \subset \mathbb{G}^A$ and all $\mathbb{F}$-martingales are continuous (by the Martingale Representation Theorem, Karatzas and Shreve, 1998, Theorem 4.2), the dual $\mathbb{F}$-predictable projection of $\mathbb{1}_{\{\cdot\geq\tau_A\}}$ is $1 - Z_\cdot$ by Aksamit and Jeanblanc, 2017, Proposition 3.9 (b). It follows by Aksamit and Jeanblanc, 2017, Proposition 2.15 that the $\mathbb{G}^A$-compensator of τ_A is $\int_{[0,\cdot\wedge\tau_A]} Z^{-1}_{s-} d\left(1 - Z_s\right)$. Then by Lemma 2.2.3 and the Itô formula,

$$\begin{aligned}\int_{[0,t\wedge\tau_A]} Z^{-1}_{s-} d\left(1 - Z_s\right) &= \int_{[0,t\wedge\tau_A]} e^{A^S_{s-}} d\left(-e^{-A^S_{s-}}\right) \\ &= A^{S,c}_{t\wedge\tau_A} + \sum_{0\leq s\leq t\wedge\tau_A}\left(1 - e^{-\Delta A^S_s}\right) \quad \text{a.s. for all } t \geq 0.\end{aligned}$$

□

The occurrence of bankruptcy time τ_A should come as a 'surprise' to the traders. More formally, τ_A *cannot* be $\mathbb{G}^A$-predictable (except for trivial cases). Lemma 2.2.2 shows τ_A is $\mathbb{G}^A$-predictable (i.e. announced by a strictly increasing sequence of $\mathbb{G}^A$-stopping times) *if and only if* either fraud is absent or if any fraudulent trades are performed only at the initial time $t = 0$.

Lemma 2.2.2. *The bankruptcy time τ_A is $\mathbb{G}^A$-predictable if and only if $A^S_{t\wedge\tau_A} = A^S_0$ a.s. for any $t \geq 0$.*

Proof. Suppose τ_A is a $\mathbb{G}^A$-predictable stopping time. It follows that the process $\mathbb{1}_{\{\cdot\geq\tau_A\}}$ is $\mathbb{G}^A$-predictable. Lemma 2.2.1 implies that the $\mathbb{G}^A$-martingale M_t is $\mathbb{G}^A$-predictable. A predictable martingale of finite variation must be constant, hence, $M^A_t = M^A_0$ for any $t \geq 0$. It follows that

$$\mathbb{1}_{\{t\geq\tau_A\}} - \mathbb{1}_{\{0=\tau_A\}} = \bar{A}^S_{t\wedge\tau_A} - \bar{A}^S_0 \text{ a.s.}$$

for any $t \geq 0$. On the event $\{0 < \tau_A < +\infty\}$, for any $t < \tau_A$, $\bar{A}^S_{t\wedge\tau_A} - \bar{A}^S_0 = 0$ a.s. and then $\Delta\bar{A}^S_{\tau_A} = 1$ a.s., which contradicts the fact that $\Delta\bar{A}^S_t = 1 - e^{-A^S_t} < 1$. Hence, $\mathbb{P}(0 < \tau_A < +\infty) = 0$. On the events $\{\tau_A = 0\}$ and $\{\tau_A = \infty\}$, it is clear that $\bar{A}^S_{t\wedge\tau_A} = \bar{A}^S_0$ a.s. (and thus $A^S_{t\wedge\tau_A} = A^S_0$.)

Conversely, let for any $t \geq 0$, $A^S_{t\wedge\tau_A} = A^S_0$ a.s. Then, by definition of bankruptcy, $\tau_A \in \{0, +\infty\}$ a.s. As the events $\{\tau_A = 0\}$ and $\{\tau_A = \infty\}$ are in $\mathcal{G}^A_0$, the sequence τ_n of increasing $\mathbb{G}^A$-stopping times defined by $\tau_n = \mathbb{1}_{\{\tau_A=0\}} + n\mathbb{1}_{\{\tau_A=+\infty\}}$ announces τ_A. $\square$

The following lemma characterizes the conditional probability of bankruptcy time τ_A on filtration $\mathbb{F}$ in relation to total frauds A^S.

Lemma 2.2.3. *The following hold for all $t \geq 0$, $\mathbb{P}$-almost surely:*

(i)

$$\mathbb{P}\left(\tau_A > t|\mathcal{F}_t\right) = e^{-A^S_t}, \tag{2.2.1}$$

(ii)

$$\mathbb{P}\left(\tau_A > t|\mathcal{F}_\infty\right) = \mathbb{P}\left(\tau_A > t|\mathcal{F}_t\right). \tag{2.2.2}$$

Furthermore, any $\mathbb{F}$-martingale is a $\mathbb{G}^A$-martingale (this is known as the immersion property).

Proof. First, we show that

$$\{t < \tau_A\} = \{A^S_t < \theta\}. \tag{2.2.3}$$

On one hand, $\{t < \tau_A\} \subset \{A^S_t < \theta\}$ follows by the definition of τ_A. On the other hand, let $\omega \in \Omega$ be such that $A^S_t(\omega) < \theta(\omega)$. If $\tau_A(\omega) < +\infty$, then $\theta(\omega) \leq A^S_{\tau_A(\omega)}(\omega)$. Hence, $t < \tau_A(\omega)$ because $A^S_t(\omega) < A^S_{\tau_A(\omega)}(\omega)$ and A^S is non-decreasing. If $\tau_A(\omega) = +\infty$, then trivially $t < \tau_A(\omega)$.

Because $\mathbb{P}(\theta > x) = e^{-x}$ for any $x \geq 0$ and θ is independent of $\mathcal{F}_\infty$,

$$\mathbb{P}\left(\tau_A > t|\mathcal{F}_\infty\right) = \mathbb{P}\left(A^S_t < \theta|\mathcal{F}_\infty\right) = e^{-A^S_t}.$$

By the tower property of conditional expectation and $A^S_\cdot$ being $\mathbb{F}$-adapted, (2.2.1) and (2.2.2) follow.

Finally, Aksamit and Jeanblanc, 2017, Lemma 3.8 shows that (2.2.2) implies the immersion property. □

Note that by the immersion property and Lévy's characterization theorem, it follows that B remains a Brownian motion on the enlarged filtration $\mathbb{G}^A$.

2.3 Absence of moral hazard

As waging bets on one's own wealth means bearing the risk in full, and thus, in the absence of investment skill (i.e. $\mu_1 = 0$)[1], yielding a zero risk-premium. In other words, a trader who owns the whole firm ($N = 1$) without legitimate investment has a wealth process that is a $\mathbb{G}^A$-*martingale*.

In principle, one could consider the case of fraud with a negative risk premium (hence, a wealth process that is a $\mathbb{G}^A$-*supermartingale*). This model focuses on the parsimonious case of zero-risk premium, which maximizes the propensity for a trader to cheat. If the risk premium were positive, the bet would be a legitimate investment opportunity, for which the label of "fraud" would not be justified. This is verified by next result, which also justifies the treatment of jumps in the definition of $\tilde{A}$ (i.e. it is the only one consistent with the martingale property for wealth in the absence of skill).

Proposition 2.3.1. *Let $N = 1$, $\mu_1 = 0$ and let $\tilde{A}_t^1 := A_t^{1,c} + \sum_{0\le s\le t} g\left(\Delta A_s^1\right)$ where $g : [0, +\infty) \to [0, +\infty)$ is measurable such that $g(0) = 0$.*

(i) If $A_t^1 = A_t^{1,c}$ a.s. for all $t \ge 0$, or if $g(a) = e^a - 1$, then X^{1,x_1} is a $\mathbb{G}^A$-martingale.

(ii) If X^{1,x_1} is a $\mathbb{G}^A$-martingale for any $A^1 \in \mathcal{A}$, then the jump size function is given by $g(a) = e^a - 1$ for any $a \ge 0$.

Proof of Proposition 2.3.1. Proof of (i): Lemma 2.1.1 (ii) and Cohen and Elliott, 2015, Corollary 15.1.9 yield that for any $t \ge 0$

$$\begin{aligned} X_t^{i,x} &= x_i \mathbb{1}_{\{t<\tau_A\}} \mathcal{E}\left(\sigma_1 B_\cdot^1 + \tilde{A}_\cdot^1\right)_t \\ &= x_i \mathbb{1}_{\{t<\tau_A\}} \mathcal{E}\left(\tilde{A}_\cdot^1\right)_t \mathcal{E}\left(\sigma_1 B_\cdot^1\right)_t \qquad \text{a.s.} \end{aligned}$$

If A^1 has a.s. continuous paths or $g(a) = e^a - 1$, then $\mathcal{E}\left(\tilde{A}_\cdot^1\right)_t = e^{A_t^1}$, and $\mathbb{1}_{\{t<\tau_A\}} e^{A_t^1}$ is a $\mathbb{G}^A$-martingale by Bielecki and Rutkowski, 2004, Lemma 5.1.7. Because the covariation between $\mathbb{1}_{\{t<\tau_A\}} e^{A_t^1}$ and $\mathcal{E}\left(\sigma_1 B_\cdot^1\right)_t$ is zero and $X_t^{1,x_1} = X_{t\wedge\tau_A}^{1,x_1}$ a.s. for all $t \ge 0$, it follows that $X^{i,x}$ is a $\mathbb{G}^A$-martingale.

Proof of (ii): Consider the family A^ξ of strategies indexed by $\xi \ge 0$, defined for $t \ge 0$ by $A_t^\xi = 1_{\{t\ge 1\}}\xi$. By construction $A^\xi \in \mathcal{A}$ for all $\xi \ge 0$. Denote the corresponding

[1]Note that eliminating investment skill is to solely examine the effect of rogue trading to the wealth.

wealth as $X^{1,x_1,\xi}$. By assumption, it is a $\mathbb{G}^A$-martingale for any $\xi \geq 0$. One can factorize it as $X^{1,x_1,\xi} = M_t U_t$, where for any $t \geq 0$

$$M_t = x_1 \mathbb{1}_{\{t<\tau_A\}} e^{A_t^1} \mathcal{E}\left(\sigma_1 B_\cdot^1\right)_t \tag{2.3.1}$$

and

$$U_t = \prod_{0 \leq s \leq t \wedge \tau_A} e^{-\Delta A_s^1} \left(1 + g(\Delta A_s^1)\right) = 1_{t<1} + e^{-\xi}(1+g(\xi))1_{\{t \geq 1\}}.$$

Note that M is a $\mathbb{G}^A$-martingale and by Lemma 2.2.1, the finite variation process U is $\mathbb{G}^A$-predictable. Integration by parts (Aksamit and Jeanblanc, 2017, Proposition 1.16) yields

$$X_t^{1,x_1} = M_t U_t = x + \int_{[0,t]} U_s dM_s + \int_{[0,t]} M_{s-} dU_s.$$

The process $\int_0^\cdot U_s dM_s$ is a $\mathbb{G}^A$-(local)martingale. As the process $\int_0^\cdot M_{s-} dU_s$ is the limit of Riemann-Stieltjes sums, it inherits the $\mathbb{G}^A$-predictability of finite variation from its integrator U. Because $X^{1,x_1,\xi}$ is a $\mathbb{G}^A$-martingale, $\int_0^\cdot M_{s-} dU_s$ is a $\mathbb{G}^A$-local martingale. Then, by Cohen and Elliott, 2015, Lemma 10.3.9. $\int_0^\cdot M_{s-} dU_s$ is a constant. Since $M_{1-} > 0$ with positive probability[2] and

$$\int_0^\cdot M_{s-} dU_s = \begin{cases} 0, & t < 1 \\ M_{1-}(e^{-\xi}(1+g(\xi)) - 1), & t \geq 1 \end{cases},$$

we conclude that $e^{-\xi}(1+g(\xi)) = 1$, for all $\xi \geq 0$. □

The goal of each trader is to maximize expected utility over a random horizon τ, which is an $\mathcal{F}$-measurable exponential random variable with rate $\lambda > 0$, independent of both $\mathcal{F}_\infty$ and θ (hence of the bankruptcy time τ_A). This random horizon models a trader with an open-ended contract, whose mandate is to maximize profits in the long-term. The arrival rate λ summarizes both the traders' impatience and the likelihood that business may end for exogenous reasons (that is, independently of traders' performance).

A trader's attitude to risk is represented by a utility function of power type

$$U^i(x_i) = \frac{x_i^{1-\gamma_i}}{1-\gamma_i}, \quad \text{with} \quad 0 < \gamma_i < 1.$$

In particular, the relative risk aversion parameter γ_i is below one, so that the utility is finite also upon bankruptcy ($x_i = 0$), and the problem is nontrivial. If it were greater

[2] In (2.3.1) all quantities except the indicator function are strictly positive; and $P(\tau_A \geq 1) > 0$ because, in view of (2.2.3), $0 < \mathbb{P}\left(A_1^\xi < \theta\right) \leq \mathbb{P}\left(\cap_{\varepsilon \in (0,1)} \{A_{1-\varepsilon} < \theta\}\right) = \mathbb{P}\left(\cap_{\varepsilon \in (0,1)} \{\tau_A > 1-\varepsilon\}\right)$

or equal to one, then zero wealth would be completely unacceptable ($U(0) = -\infty$) and fraud would disappear. In fact, as shown below (Remark 3.5.3), fraud does vanish as γ_i converges to one.

As anticipated in the description, an important implication of this model is that a rational and strictly risk-averse trader abstains from fraud if no other trader is present. Its significance is to confirm that in this model fraud stems from the ability to share losses but not gains, hence disappears when such sharing disappears. (Note that for this result the assumption of an exponential horizon, made in the rest of the paper, can be dropped.)

Proposition 2.3.2 (No fraud for $N = 1$). *Let $N = 1$, $\kappa \geq 0$ and τ_1 be an $\mathcal{F}$-measurable, a.s. finite random horizon (independent of $\mathcal{F}_\infty$ and θ) such that $\mathbb{E}\left[e^{\left((1-\gamma_1)(\mu_1-\gamma_1\sigma_1^2/2)-\kappa\right)\tau_1}\right] < \infty$.*

If the sole trader maximizes

$$\mathbb{E}\left[e^{-\kappa\tau_1}U^1(X^{1,x_1}_{\tau_1})\right]$$

over all fraud processes $A^1 \in \mathcal{A}$, then $A^{1,\star}$ is optimal if and only if $A^{1,\star}_t = 0$ a.s. for any $t \geq 0$ such that $\mathbb{P}(\tau_1 \geq t) > 0$. In particular:

(i) If τ_1 is unbounded, then $A^{1,\star}_t = 0$ a.s. for all $t \geq 0$.

(ii) If $\tau_1 \leq T^1$ a.s. for some $T^1 > 0$, then $A^{1,\star}_{T^1_-} = 0$. If $\mathbb{P}\left(\tau_1 = T^1\right) > 0$, then also $A^{1,\star}_{T^1} = 0$ a.s.

Proof. By Lemma 2.1.1, trader 1's wealth has the expression

$$X^{1,x_1}_t = \mathbb{1}_{\{t<\tau_A\}} x_1 e^{A^1_t + (\mu_1 - \sigma_1^2/2)t + \sigma_1 B^1_t}, \quad t \geq 0.$$

Then, by Lemma 2.2.3, it follows that

$$\begin{aligned}
\mathbb{E}\left[U^1(X^{1,x_1}_t)|\mathcal{F}_t\right] &= \mathbb{E}\left[\mathbb{1}_{\{t<\tau_A\}}U^1(Y^{1,x_1}_t)|\mathcal{F}_t\right] \\
&= e^{-A^1_t}U^1(Y^{1,x_1}_t) \\
&= \frac{x_1^{1-\gamma_1}}{1-\gamma_1} e^{-\gamma_1 A^1_t + (1-\gamma_1)\left((\mu_1-\sigma_i^2/2)t+\sigma_1 B^1_t\right)} \\
&\leq \frac{x_1^{1-\gamma_1}}{1-\gamma_1} e^{(1-\gamma_1)\left((\mu_1-\sigma_i^2/2)t+\sigma_1 B^1_t\right)}.
\end{aligned}$$

Therefore, by the tower property of conditional expectation,

$$\mathbb{E}\left[U^1(X^{1,x_1}_t)\right] \leq \frac{x_1^{1-\gamma_1}}{1-\gamma_1} e^{(1-\gamma_1)(\mu_1-\gamma_1\sigma_1^2/2)t} \tag{2.3.2}$$

and

$$\mathbb{E}\left[U^1(X^{1,x_1}_t)\right] = \frac{x_1^{1-\gamma_1}}{1-\gamma_1} e^{(1-\gamma_1)(\mu_1-\gamma_1\sigma_1^2/2)t} \quad \text{if and only if} \quad A_t = 0 \quad \text{a.s..} \tag{2.3.3}$$

Let $\mathbb{P}_{\tau_1}$ be the law of τ_1, i.e. $\mathbb{P}_{\tau_1}(U) = \mathbb{P}(\tau_1 \in U)$ for any τ_1-measurable set $U \subset \mathbb{R}_+$. Then, by the law of total probability and the independence of τ_1 from B and θ,

$$\begin{aligned} \mathbb{E}\left[e^{-\kappa\tau_1} U^1(X^{1,x_1}_{\tau_1})\right] &= \int_0^\infty \mathbb{E}\left[e^{-\kappa\tau_1} U^1(X^{1,x_1}_{\tau_1}) | \tau_1 = t\right] d\mathbb{P}_{\tau_1}(dt) \\ &= \int_0^\infty e^{-\kappa t}\mathbb{E}\left[U^1(X^{1,x_1}_t)\right] d\mathbb{P}_{\tau_1}(dt). \end{aligned} \tag{2.3.4}$$

Thus, (2.3.2) implies that

$$\begin{aligned} \mathbb{E}\left[e^{-\kappa\tau_1} U^1(X^{1,x_1}_{\tau_1})\right] &\le \frac{x_1^{1-\gamma_1}}{1-\gamma_1} \int_0^\infty e^{\left((1-\gamma_1)(\mu_1-\gamma_1\sigma_1^2/2)-\kappa\right)t} \mathbb{P}_{\tau_1}(dt) \\ &= \frac{x_1^{1-\gamma_1}}{1-\gamma_1} \mathbb{E}\left[e^{\left((1-\gamma_1)(\mu_1-\gamma_1\sigma_1^2/2)-\kappa\right)\tau_1}\right] \end{aligned} \tag{2.3.5}$$

and, due to (2.3.3) and (2.3.4), the equality

$$\mathbb{E}\left[e^{-\kappa\tau_1} U^1(X^{1,x_1}_{\tau_1})\right] = \frac{x_1^{1-\gamma_1}}{1-\gamma_1} \mathbb{E}\left[e^{\left((1-\gamma_1)(\mu_1-\gamma_1\sigma_1^2/2)-\kappa\right)\tau_1}\right]$$

holds if and only if $P\left(A^1_t = 0\right) = 1$ for all $t > 0$ for which $\mathbb{P}\left(\tau_1 \ge t\right) > 0$. (Suppose, by contradiction, that there exists some $t_0 \ge 0$ for which $\mathbb{P}\left(\tau_1 \ge t_0\right) > 0$, but $\mathbb{P}\left(A^1_{t_0} > 0\right) > 0$. Because A^1 is non-decreasing a.s., for all $t \ge t_0$, $\{A^1_t \ge A^1_{t_0}\}$ is an event of probability one, and (2.3.3) implies that

$$\mathbb{E}\left[U^1(X^{1,x_1}_t)\right] < \frac{x_1^{1-\gamma_1}}{1-\gamma_1} e^{(1-\gamma_1)(\mu_1-\gamma_1\sigma_1^2/2)t}, \quad t \ge t_0,$$

and since $\mathbb{P}\left(\tau_1 \ge t_0\right) > 0$, we have upon integration (cf. (2.3.4)) indeed strict inequality in (2.3.5).) $\square$

Note that this proposition fails if $N \ge 2$ because the coupling term $Y^{S,x}_{t-}$ of (2.1.2) rescinds the martingale property (Proposition 2.3.1) for each trader's wealth in the absence of drift ($\mu_i = 0$). For example, if all but the i-th trader abstain from fraud, then $X^{i,x}$ can become a sub-martingale if the i-th trader cheats: in this case, the wealth processes of other traders become super-martingales as they share the bankruptcy risk from the i-th trader action. As shown in Section 3, engaging in fraud may be optimal depending on traders' shares of capital, risk-aversions, drifts, and volatilities.

2.4 Supplements

Here we recall the definition of the stochastic exponential (cf. Jacod and Shiryaev, 2013, Eq. I.4.62) for general semimartingales:

Definition 2.4.1 (Stochastic exponential)**.** *For any $\mathbb{R}$-valued semimartingale S, the stochastic exponential of S is the process*

$$\mathcal{E}(S)_t := \exp\left(S_t - S_0 - \frac{1}{2}[S^c]_t\right) \prod_{0 \le s \le t} e^{-\Delta S_s}(1 + \Delta S_s) \qquad t \ge 0$$

where $\mathcal{E}(S)_{0-} = 1$ and S^c denotes the continuous part of S.

All stochastic exponentials in this thesis are $\mathbb{P}$-a.s. strictly positive because the jump sizes are bounded away from -1 (cf. Jacod and Shiryaev, 2013, Theorem I.4.61 (c). For finite-variation jumps (as in this paper), note that

$$S_t = S_t^c + \sum_{0 \le s \le t} \Delta S_s,$$

with $S_{0-}^c = 0$, therefore the stochastic exponential simplifies to

$$\mathcal{E}(S)_t = \exp\left(S_t^c - \frac{1}{2}[S^c]_t\right) \prod_{0 \le s \le t} (1 + \Delta S_s).$$

Chapter 3

Nash Equilibrium

While the presentation in the previous section considered an arbitrary number N of traders, the main result in this section focuses on two traders to simplify both the exposition and the proofs.[1] Thus, henceforth $N = 2$ and, for clarity, the indexes $\{a, b\}$ replace $\{1, 2\}$ to identify traders. The wealth processes are denoted by either $X^x(A^a, A^b)$ or X^x (respectively, $Y^x(A^a, A^b)$ or Y^x), depending on the need to specify the fraud process $A = (A^a, A^b)$ in context.

3.1 Preliminary results for value functions

For any $i, j \in \{a, b\}$ such that $i \neq j$ (henceforth abbreviated as *'for any $i \neq j \in \{a, b\}$'*), the goal of trader i is to maximize expected utility on a random horizon τ i.e.,

$$J^i(x; A^i, A^j) := \mathbb{E}\left[e^{-\kappa\tau} U^i\left(X_\tau^{i,x}(A^i, A^j)\right)\right] \tag{3.1.1}$$

over any $A^i \in \mathcal{A}$, where

$$U^i(x_i) = \frac{x_i^{1-\gamma_i}}{1-\gamma_i} \quad \text{with} \quad 0 < \gamma_i < 1.$$

Here $\kappa \geq 0$ is the discount rate and the random horizon τ is independent of $\mathbb{F}$ and θ, and exponentially distributed with rate λ (meaning that $\frac{1}{\lambda}$ represents traders' *average horizon* or, equivalently, λ is their time-preference or impatience rate). Let thus

$$V^i(x; A^j) := \sup_{A^i \in \mathcal{A}} J^i(x; A^i, A^j)$$

be the *value function* for i-th trader given trader j's fraud process A^j and initial wealth $x \in \mathbb{R}_+^2$.

The next assumption stands *throughout the paper*, which is required to ensure that the optimization problem is well-posed: the sum of the impatience rate λ and the discount rate κ needs to be large enough in relation to risk aversion and skill, so that the

[1] A model with N traders implies that relative shares of capital follow a $N-1$ dimensional diffusion, which is considerably simpler in one dimension.

value function remains finite. This parameter restriction depends only on minimum risk aversion and maximum skill. If the impatience rate is unrealistically too small, then the value functions would blow up with only the legitimate investment.

Assumption 3.1.1. *Let $\lambda^\kappa = \kappa + \lambda$, $\lambda^\kappa > (1 - \gamma_a \wedge \gamma_b)(\mu_a \vee \mu_b)$.*

As shown in the next result, the p-moment for $0 < p < 1$ of firm's wealth, which then implies traders' wealth, are finite.

Lemma 3.1.2. *For any $p \in (0,1)$ and $t \geq 0$,*

$$\mathbb{E}\left[\left(X_t^{S,x}\right)^p\right] \leq \left(\sum_{k=1}^N x_k\right)^p e^{pt \max_{k\in\{1,\dots,N\}} \mu_k}. \tag{3.1.2}$$

Proof. Using the expression of $Y^{S,x}$ as in (2.1.4), since the covariation between the terms inside $\mathcal{E}(\cdot)$ in (2.1.4) is zero, by Cohen and Elliott, 2015, Corollary 15.1.9, it follows that

$$Y_t^{S,x} = \left(\sum_{k=1}^N x_k\right) \mathcal{E}\left(\tilde{A}_\cdot^S\right)_t \mathcal{E}\left(\sum_{k=1}^N \mu_k \int_0^\cdot \frac{Y_s^{k,x}}{Y_s^{S,x}} ds\right)_t \mathcal{E}\left(\sum_{k=1}^N \sigma_k \int_0^\cdot \frac{Y_s^{k,x}}{Y_s^{S,x}} dB_s^k\right)_t$$

and, upon rearranging and estimating terms,

$$\begin{aligned} Y_t^{S,x}\mathcal{E}\left(\tilde{A}_\cdot^S\right)_t^{-1} &= \left(\sum_{k=1}^N x_k\right) \mathcal{E}\left(\sum_{k=1}^N \mu_k \int_0^\cdot \frac{Y_s^{k,x}}{Y_s^{S,x}} ds\right)_t \mathcal{E}\left(\sum_{k=1}^N \sigma_k \int_0^\cdot \frac{Y_s^{k,x}}{Y_s^{S,x}} dB_s^k\right)_t \\ &\leq \left(\sum_{k=1}^N x_k\right) \mathcal{E}\left(\left(\max_{k\in\{1,\dots,N\}} \mu_k\right) \sum_{k=1}^N \int_0^\cdot \frac{Y_s^{k,x}}{Y_s^{S,x}} ds\right)_t \mathcal{E}\left(\sum_{k=1}^N \sigma_k \int_0^\cdot \frac{Y_s^{k,x}}{Y_s^{S,x}} dB_s^k\right)_t \\ &= \left(\sum_{k=1}^N x_k\right) e^{(\max_{k\in\{1,\dots,N\}} \mu_k)t} \mathcal{E}\left(\sum_{k=1}^N \sigma_k \int_0^\cdot \frac{Y_s^{k,x}}{Y_s^{S,x}} dB_s^k\right)_t. \end{aligned} \tag{3.1.3}$$

For any $k \in \{1, \dots, N\}$ and any $t \geq 0$, as

$$\exp\left(\frac{\sigma_k^2}{2} \int_0^t \left(\frac{Y_s^{k,x}}{Y_s^{S,x}}\right)^2 ds\right) \leq \exp\left(\frac{\sigma_k^2}{2} t\right) < \infty$$

a.s. so the Novikov's criterion (Revuz and Yor, 2013, Corollary 1.16) holds and hence,

$$\mathcal{E}\left(\sum_{k=1}^N \sigma_k \int_0^\cdot \frac{Y_s^{k,x}}{Y_s^{S,x}} dB_s^k\right)_{t\geq 0}$$

is a true martingale. Taking expectation on both sides of (3.1.3) yields

$$\mathbb{E}\left[Y_t^{S,x}\mathcal{E}\left(\tilde{A}_\cdot^S\right)_t^{-1}\right] \leq \left(\sum_{k=1}^N x_k\right) e^{(\max_{k\in\{1,\dots,N\}} \mu_k)t}. \tag{3.1.4}$$

The stochastic exponential $\mathcal{E}(\tilde{A}^S)_t$ satisfies

$$\begin{aligned}\mathcal{E}(\tilde{A}^S)_t &= e^{\tilde{A}^S_t} \prod_{0\le s\le t} (1+\Delta\tilde{A}^S_s)e^{-\Delta\tilde{A}^S_s} \\ &= e^{A^{S,c}_t} \prod_{0\le s\le t} \left(1+\sum_{k=1}^N \left(e^{\Delta A^k_s}-1\right)\right) \\ &\le e^{A^{S,c}_t} \prod_{0\le s\le t} e^{\sum_{k=1}^N \Delta A^k_s} = e^{A^S_t}, \end{aligned} \tag{3.1.5}$$

where the inequality (3.1.5) follows from $e^{\sum_{k=1}^N y_k} - 1 \ge \sum_{k=1}^N (e^{y_k}-1)$ for any $(y_1,\ldots,y_N)$ in $\mathbb{R}^N_+$. Note that the equality of (3.1.5) is strict *if and only if* the fraud processes $(A^1,\ldots,A^N)$ satisfy

$$\mathbb{P}\left(\Delta A^i_s \Delta A^j_s = 0\right) = 1, \quad 0\le s\le t, \quad \text{for all } i\ne j\in\{1,\ldots,N\}. \tag{3.1.6}$$

By Lemma 2.2.3 (i), for any $t\ge 0$

$$\mathbb{E}\left[X^{S,x}_t\right] = \mathbb{E}\left[\mathbb{1}_{\{t<\tau_A\}} Y^{S,x}_t\right] = \mathbb{E}\left[Y^{S,x}_t \mathbb{E}\left[\mathbb{1}_{\{t<\tau_A\}}|\mathcal{F}_t\right]\right] = \mathbb{E}\left[e^{-A^S_t} Y^{S,x}_t\right]. \tag{3.1.7}$$

Therefore, by the estimate (3.1.5),

$$\mathbb{E}\left[X^{S,x}_t\right] \le \mathbb{E}\left[Y^{S,x}_t \mathcal{E}\left(\tilde{A}^S_\cdot\right)^{-1}_t\right] \tag{3.1.8}$$

and note that the strict equality holds *if and only if* condition (3.1.6) holds.

Finally, it follows by Jensen's inequality, (3.1.8) and (3.1.4) that for any $0<p\le 1$,

$$\mathbb{E}\left[\left(X^{S,x}_t\right)^p\right] \le \left(\mathbb{E}\left[X^{S,x}_t\right]\right)^p \le \left(\sum_{k=1}^N x_k\right)^p e^{pt\max_{k\in\{1,\ldots,N\}}\mu_k}.$$

□

The next result reveals several properties of the reward functional J^i and the value function V^i. Note that the upper-bound of the value function (Lemma 3.1.3 (iii)) follows by a direct application of Lemma 3.1.2.

Lemma 3.1.3. *For any $i\ne j\in\{a,b\}$, $x\in\mathbb{R}^2_+$ and $(A^i,A^j)\in\mathcal{A}^2$,*

(i) $J^i(x;A^i,A^j) = \lambda\mathbb{E}\left[\int_0^\infty e^{-\lambda^\kappa t - A^S_t} U^i(Y^{i,x}_t(A^i,A^j))dt\right]$.

(ii) For any $c>0$, $J^i(cx;A^i,A^j) = c^{1-\gamma_i} J^i(x;A^i,A^j)$.

(iii) $0 < V^i(x;A^j) \le \dfrac{\lambda U^i(x_a+x_b)}{\lambda^\kappa - (1-\gamma_i)(\mu_a\vee\mu_b)}$.

Proof of Lemma 3.1.3. By the independence between τ and $\mathcal{F}_\infty\vee\sigma(\theta)$, the tower property of the conditional expectation, and Lemma 2.2.3, it follows that for any $i\ne j\in$

$\{a, b\}$ and any $A^i, A^j \in \mathcal{A}$,

$$\begin{aligned}
\mathbb{E}\left[e^{-\kappa\tau}U^i\left(X_\tau^{i,x}(A^i,A^j)\right)\right] &= \mathbb{E}\left[\mathbb{E}\left[e^{-\kappa\tau}U^i\left(X_\tau^{i,x}(A^i,A^j)\right)|\mathcal{F}_\infty \vee \sigma(\theta)\right]\right] \\
&= \mathbb{E}\left[\int_0^\infty \lambda e^{-\lambda^\kappa t}U^i(X_t^{i,x}(A^i,A^j))dt\right] \\
&= \lambda\mathbb{E}\left[\int_0^\infty e^{-\lambda^\kappa t}\mathbb{E}\left[\mathbb{1}_{\{t<\tau_A\}}|\mathcal{F}_t\right]U^i(Y_t^{i,x}(A^i,A^j))dt\right] \\
&= \lambda\mathbb{E}\left[\int_0^\infty e^{-\lambda^\kappa t - A_t^S}U^i(Y_t^{i,x}(A^i,A^j))dt\right]
\end{aligned}$$

which proves (i).

Furthermore, the scale-invariance (ii) is an immediate result of $Y^{i,cx}$ and $cY^{i,x}$ being indistinguishable (implied by Lemma 2.1.1 (ii)).

Finally, Tonelli's theorem and Lemma 3.1.2 yield

$$\begin{aligned}
J^i(x; A^i, A^j) &= \frac{\lambda}{1-\gamma_i}\int_0^\infty e^{-\lambda^\kappa t}\mathbb{E}\left[\left(X_t^{i,x}(A^i,A^j)\right)^{1-\gamma_i}\right]dt \\
&\le \frac{\lambda\left(x_a + x_b\right)^{1-\gamma_i}}{1-\gamma_i}\int_0^\infty e^{-\lambda^\kappa t + (1-\gamma_i)(\mu_a \vee \mu_b)t}dt \\
&= \frac{\lambda U^i\left(x_a + x_b\right)}{\lambda^\kappa - (1-\gamma_i)(\mu_a \vee \mu_b)}.
\end{aligned}$$

Taking the supremum over $A^i \in \mathcal{A}$, the proof of (iii) follows. □

By Lemma 3.1.3 (i), we take the approach of focusing on the pre-bankruptcy wealth Y^x as the state processes of the optimization problem; By Lemma 3.1.3 (ii), we exploit the scale-invariance of the value function to reduce the resulting Hamilton-Jacobi-Bellman (HJB) equations to ordinary differential equations (see Section 3.5.1 for the derivation and simplification of the HJB equations).

3.2 Definition of Nash equilibrium

In section 2 we have not specified the information a trader can act on; now we introduce an information structure as follows: For any $i \neq j \in \{a, b\}$, at time t, trader i *acts by* the path of her own wealth $s \mapsto Y_s^{i,x}$ for all $s \in [0, t)$, her colleague trader j's path of wealth $s \mapsto Y_s^{j,x}$ for all $s \in [0, t]$ (so that trader i can respond to trader j's instant wealth change $\Delta Y_t^{j,x}$) and lastly the path of her own fraud $t \mapsto A_s^i$ for all $s \in [0, t)$. To be more formal, we first introduce some notation. For $t \ge 0$, let $\mathcal{D}^+([0, t])$ (resp. $\mathcal{D}^+([0, t))$) denote the set of $\mathbb{R}_+$-valued càdlàg functions on $[0, t]$ (resp. $[0, t)$) with left-limit at $t = 0$ valued in $\mathbb{R}_+$. Analogously, let $\mathcal{D}^\uparrow([0, t])$ (resp. $\mathcal{D}^\uparrow([0, t))$) be the set of $[0, +\infty)$-valued non-decreasing, right-continuous functions on $[0, t]$ (resp. $[0, t)$) with left-limit of 0 at time 0. For any process $Z = (Z_t)_{t\ge0}$, the processes $Z_{[0,t)}$, $Z_{[0,t]}$

denote the restrictions of the paths of Z, to the interval $[0,t)$ and $[0,t]$, respectively, with the left-limit of each at 0.

To construct Nash equilibrium of closed-loop form, we consider a special class of fraud strategies which constitute a trader's responses to the fraudulent activities of the other trader, depending on the latter only through the *wealth* of both traders and one's *own strategy*:

Definition 3.2.1 (Responses via wealth observation). *Let $i \neq j \in \{a,b\}$. The set Λ^i is the collection of response maps $\Psi = (\Psi_t)_{t\geq 0}$ which are, for any $t \geq 0$, measurable maps of the form*

$$\Psi_t : \mathcal{D}^+([0,t)) \times \mathcal{D}^+([0,t]) \times \mathcal{D}^\uparrow([0,t)) \to [0,\infty)$$

such that for any $x = (x_i, x_j) \in \mathbb{R}^2_+$ and any $A^j \in \mathcal{A}$ there exists a unique $A^i \in \mathcal{A}$ satisfying

$$A^i_t = \Psi_t\left(Y^{i,x}_{[0,t)}, Y^{j,x}_{[0,t]}, A^i_{[0,t)}\right), \quad t \geq 0, \quad \mathbb{P} - a.s., \tag{3.2.1}$$

where $(Y^{i,x}, Y^{j,x})$ is the pre-bankruptcy wealth associated with (A^i, A^j).[2]

We are ready to define Nash equilibria in the context of this paper:[3]

Definition 3.2.2 (Nash equilibrium). *A pair $(\Psi^{\star,a}, \Psi^{\star,b}) \in (\Lambda^a, \Lambda^b)$ is called Nash equilibrium if for any $x \in \mathbb{R}^2_+$ there exists a unique pair $(A^{a,\star}, A^{b,\star}) \in \mathcal{A}^2$ such that for any $i \neq j \in \{a,b\}$,*

(i) for any $t \geq 0$, $\mathbb{P}$-almost surely, $A^{i,\star}_t = \Psi^{\star,i}_t\left(Y^{i,x,\star}_{[0,t)}, Y^{j,x,\star}_{[0,t]}, A^{i,\star}_{[0,t)}\right)$, where $(Y^{a,x,\star}, Y^{b,x,\star})$ denotes the wealth associated with $(A^{a,\star}, A^{b,\star})$.

(ii) non-cooperative optimality holds, that is, for any $A^i \in \mathcal{A}$, the response A^j satisfying (3.2.1) with $\Psi^j = \Psi^{\star,j}$ forces A^i to be sub-optimal in that

$$J^i\left(x; A^i, A^j\right) \leq J^i\left(x; A^{i,\star}, A^{j,\star}\right). \tag{3.2.2}$$

We call the pair $(A^{a,\star}, A^{b,\star})$ *equilibrium fraud processes.*

Remark 3.2.3. *Although the readers may be familiar with the popular notion of Nash equilibrium, it is important that a few points to be clarified:*

a A Nash equilbrium $(\Psi^{\star,a}, \Psi^{\star,b}) \in (\Lambda^a, \Lambda^b)$ does not necessarily constitute best-response maps: For any $i \neq j \in \{a,b\}$, for any $A^i \in \mathcal{A}$, it does not in general imply that

$$J^j\left(x; A^{j,\prime}, A^i\right) = \sup_{A^j \in \mathcal{A}} J^j\left(x; A^j, A^i\right), \tag{3.2.3}$$

[2] According to Lemma 2.1.1 (i), $(Y^{i,x}, Y^{j,x})$ is the unique strong solution to the SDE (2.1.2) for the given pair of fraud processes (A^a, A^b) and initial wealth $x \in \mathbb{R}^2_+$.

[3] See Carmona, 2016 for an overview of Nash equilibria in stochastic settings with absolute continuous type controls.

where $A^{j,'}$ satisfying (3.2.1) with $\Psi^j = \Psi^{\star,j}$. Put differently, the response $\Psi^{\star,j}$ of trader j does not have to be optimal in the sense of (3.2.3), but merely enough to deter trader i from deviating from $A^{i,\star}$. Note that if trader i adopts the equilibrium fraud process, i.e. $A^i = A^{i,\star}$, then clearly (3.2.3) holds due to condition (ii) in definition 3.2.2.

b *Furthermore, since the 'deviating' fraud processes $A^i \in \mathcal{A}$ in condition (ii) in Definition 3.2.2 are not restricted to any other form, condition (ii) is stronger than the alternative:*

- *For any $\Psi^i \in \Lambda^i$ such that there exists a unique tuple (A^i, A^j, Y^x) with Y^x being the unique strong solution to the SDE (2.1.2) and $A^i_\cdot = \Psi^i_\cdot\left(Y^{i,x}_{[0,\cdot)}, Y^{j,x}_{[0,\cdot]}, A^i_{[0,\cdot)}\right) \in \mathcal{A}$ and $A^j_\cdot = \Psi^{j,\star}_\cdot\left(Y^{j,x}_{[0,\cdot)}, Y^{i,x}_{[0,\cdot]}, A^j_{[0,\cdot)}\right) \in \mathcal{A}$, $J^i\left(x; A^i, A^j\right) \leq J^i\left(x; A^{i,\star}, A^{j,\star}\right)$.*

In other words, given trader i's inference of trader j's strategy Ψ^j, trader i chooses only among the strategies that lead to the existence of a joint fraud process. Most importantly, condition (ii) implies a stronger Nash equilibrium in the sense that even if trader i can observe trader j's fraud, i.e. A^i is of the form $A^i_\cdot = \Psi_\cdot\left(Y^{i,x}_{[0,\cdot)}, Y^{j,x}_{[0,\cdot]}, A^i_{[0,\cdot)}, A^j_{[0,\cdot)}\right)$ for some mapping Ψ, she does not need to observe the other trader's fraud action to execute her own strategy, as her optimal response $\Psi^i \in \Lambda^i$ to trader j's strategy $\Psi^j \in \Lambda^j$ does not depend on trader j's fraud.

c *Traders realistically cannot have the perfect knowledge of each other's investment skill represented by μ_i and σ_i and the level of risk-aversion, but this does not prevent each other from guessing them and deriving a Nash equilibrium in their own perspective and then executing their equilibrium strategies respectively. Although the accuracy of such estimation can be wildly wrong: A newly joined trader may estimate the existing trader's skill more accurately by observing the latter's past performance, than the existing trader guessing that of the newly joined trader by only looking at her/his resume. Section 4.2 models skill estimation with binomial distribution.*

d *Unlike in other definitions of Nash equilibrium (cf. Carmona, 2016, Definition 5.2), where the sub-optimal condition (3.2.2) would be of the form*

$$J^i\left(x; A^i, A^{j,\star}\right) \leq J^i\left(x; A^{i,\star}, A^{j,\star}\right) \text{ for all } A^i \in \mathcal{A}, \tag{3.2.4}$$

we instead drop the superscript '$\star$' for trader j's fraud process A^j in the left-hand side of (3.2.4). This is because the star symbol merely represents the equilibrium tuple $(Y^{x,\star}(A^{i,\star}, A^{j,\star}), A^{i,\star}, A^{j,\star})$, where $A^{j,\star}_t = \Psi^{\star,j}_t\left(Y^{j,x,\star}_{[0,t)}, Y^{i,x,\star}_{[0,t]}, A^{j,\star}_{[0,t)}\right)$. When trader i deviates with some other fraud process $A^i \in \mathcal{A}$, the resulting tuple $(Y^x(A^i, A^j), A^i, A^j)$ is not indistinguishable with $(Y^{x,\star}(A^{i,\star}, A^{j,\star}), A^{i,\star}, A^{j,\star})$, although trader j's strategy $\Psi^{\star,j}$ remains unchanged.

3.3 Share of wealth and Skorokhod reflection problem

To rigorously define the behavior of each trader cheating as little as necessary so as to keep the personal share of wealth above a certain threshold, it is necessary to examine the processes of share of wealth and then recall the notion of Skorokhod reflection.

For any $x = (x_a, x_b) \in \mathbb{R}_+^2$ and any $i \in \{a, b\}$, define $r_i(x) = \frac{x_i}{x_a + x_b}$, then $r_i(Y_t^x)$ is trader i's share of the firm's capital at time t. Let $W_t^{i,w_i}(A^i, A^j) = r_i\left(Y_t^x(A^i, A^j)\right)$ for any $t \geq 0$ with $W_{0-}^{i,w_i}(A^i, A^j) = r_i(x) = w_i$. For any $i \neq j \in \{a, b\}$, define the coefficient functions (of a single variable)

$$\bar{b}_i(w) := w(1-w)\left(\sigma_j^2(1-w) - \sigma_i^2 w + \mu_i - \mu_j\right), \tag{3.3.1}$$

$$\bar{\sigma}_i(w) := \begin{cases} (\sigma_a w(1-w), -\sigma_b w(1-w)) & \text{if } i = a, \\ (-\sigma_a w(1-w), \sigma_b w(1-w)) & \text{if } i = b, \end{cases} \tag{3.3.2}$$

respectively. Introduce also the processes $\tilde{Q}_t^i = A_t^{i,c} + \sum_{0 \leq s \leq t} q_i(\Delta \tilde{A}_s)$ where $q_i : \mathbb{R}_+^2 \to \mathbb{R}_+$ is defined as $q_i(a_1, a_2) = \frac{a_i}{1 + a_1 + a_2}$. Next result provides the identification of the wealth share $W^{i,w}$ and the SDE (3.3.3).

Lemma 3.3.1. *The following statements hold:*

(i) For any $i \neq j \in \{a, b\}$ and $x \in \mathbb{R}_+^2$, the traders' pre-bankruptcy wealth shares $r_i\left(Y_\cdot^x\right)$ is the unique strong solution to the SDE

$$W_t^{i,w_i} = w_i + \int_0^t \bar{b}_i(W_s^{i,w_i})ds + \int_0^t \bar{\sigma}_i(W_s^{i,w_i})dB_s + \int_{[0,t]} (1 - W_{s-}^{i,w_i})d\tilde{Q}_s^i - \int_{[0,t]} W_{s-}^{i,w_i} d\tilde{Q}_s^j. \tag{3.3.3}$$

with $w_i = r_i(x)$.

(ii) For all $t \geq 0$, $W_t^{i,w_i} \in (0, 1)$ a.s.

Proof. (i): An application of Itô formula shows that the fractions $r_i\left(Y_\cdot^x\right)$ satisfy the SDE (3.3.3) and the uniqueness follows by local Lipschitz continuity of its coefficients (cf. Cohen and Elliott, 2015, Theorem 16.3.11.). Furthermore, the strict positivity of the pre-bankruptcy wealth Y^x (Lemma 2.1.1) proves (ii). □

Now we present the Skorokhod reflection problem in the context of a trader's share of wealth.

Definition 3.3.2 (Skorokhod reflection). *Let $i \neq j \in \{a, b\}$ and $m_i \in (0, 1)$. A $\Psi^{i,m_i} \in \Lambda^i$ is the solution of the (one-sided) Skorokhod reflection problem (henceforth $\boldsymbol{SP^i_{m_i+}}$) if for any $A^j \in \mathcal{A}$ and any $x \in \mathbb{R}_+^2$, the associated pair (A^i, Y^x) to Ψ^{i,m_i} is the unique pair satisfying*

(i) $m_i \leq W_t^{i,w_i}(A^i, A^j) < 1$ a.s. for all $t \geq 0$, and

(ii) $\int_{[0,\infty)} \mathbb{1}_{\left\{W_t^{i,w_i}(A^i,A^j)>m_i\right\}} dA_t^i = 0$ *a.s..*

By (i), $W_t^{i,w_i}(A^{i,\star},A^j)$ almost surely does not take values below m_i for any $t \geq 0$, while (ii) means that, as $A_t^{i,\star}$ increases, $W_t^{i,w_i}(A^{i,\star},A^j)$ can reach m_i but without spending any positive amount of time at this point. Clearly, for any $i \neq j \in \{a,b\}$, W^{i,w_i} being reflected at m_i towards 1 is the same as the other trader's fraction of wealth $W^{j,1-w_i}$ being reflected at $1-m_i$ towards 0, because W^{i,w_i} and $1-W^{j,1-w_i}$ are indistinguishable. Moreover, the uniqueness of the solution to $SP^i_{m_i+}$ is up to identifying the same pair $(A^{i,\star}, Y^x)$.

Per the discussion of the fraud strategy in section 3.2, since trader i at time t acts by observing the path $s \mapsto Y_s^i$ for all $s \in [0,t)$ and $t \mapsto Y_t^j$ for all $s \in [0,t]$, then she can act by the path of 'moment-before trader i's own jump-fraud share of wealth' i.e $s \mapsto r_i\left(Y_{s-}^{i,x}, Y_s^{j,x}\right)$ for all $s \in [0,t]$, which is almost surely semi-continuous with left and right limits.

The next statement provides explicit solution to $SP^i_{m_i+}$ and also establishes the condition (i.e. two traders' respective fraud thresholds cannot cross each other) under which the individual Skorokhod reflections becomes a two-sided Skorokhod reflection, which ultimately allows to derive a Nash equilibrium.

Proposition 3.3.3 (Skorokhod reflection problem)**.** *Let $i \neq j \in \{a,b\}$ and $m_i \in (0,1)$. The parameterized functional $\Psi^{i,m_i} \in \Lambda^i$ with $\Psi_t^{i,m_i} = \Psi_t^{i,c,m_i} + \Psi_t^{i,d,m_i}$:*

$$\left(y^i_{[0,t)}, y^j_{[0,t]}, a^i_{[0,t)}\right) \in \mathcal{D}^+([0,t)) \times \mathcal{D}^+([0,t]) \times \mathcal{D}^\uparrow([0,t)) \to [0,\infty)$$

given by

$$\Psi_t^{i,c,m_i}\left(y^i_{[0,t)}, y^j_{[0,t]}, a^i_{[0,t)}\right) = \frac{1}{1-m_i}\left(\sup_{s\in[0,t]}\left[m_i - w_s^{i-} + (1-m_i)a_s^{i,c} + \sum_{0\leq u<s}\left[m_i - w_u^{i-}\right]^+\right]^+ - \sum_{0\leq s\leq t}\left[m_i - w_s^{i-}\right]^+\right)$$

and

$$\Psi_t^{i,d,m_i}\left(y^i_{[0,t)}, y^j_{[0,t]},, a^i_{[0,t)}\right) = \sum_{0\leq s\leq t} \ln\left[1 + \frac{w_{s-}^{i-}}{1-m_i}\left[\frac{m_i}{w_s^{i-}} - 1\right]\right]^+ \tag{3.3.4}$$

where $w_t^{i-} := r_i(y_{t-}^i, y_t^j)$ *for all* $t \geq 0$ *and* $a^{i,c}$ *denotes the continuous part of* a^i, *solves* $SP_{m_i+}^i$. *The associated unique fraud process* $A_\cdot^{i,\prime}$ *is: for any* $t \geq 0$, $\mathbb{P}$*-a.s.,*

$$\begin{aligned} A_t^{i,c,\prime} &= \Psi_t^{i,c,m_i}\left(Y_{[0,t)}^j, Y_{[0,t]}^j, A_{[0,t)}^{i,\prime}\right), \\ \Delta A_t^{i,\prime} &= \Delta\Psi_t^{i,d,m_i}\left(Y_{[0,t)}^j, Y_{[0,t]}^j, A_{[0,t)}^{i,\prime}\right) \\ &= \ln\left[1 + \frac{r_i(Y_{t-}^i, Y_{t-}^j)}{1-m_i}\left[\frac{m_i}{r_i(Y_{t-}^i, Y_t^j)} - 1\right]\right]^+. \end{aligned} \tag{3.3.5}$$

Furthermore, if $m_a + m_b < 1$, *then there exists a unique tuple* $(Y^x, A^{i,\prime}, A^{j,\prime})$ *such that* $Y^{i,x}$ *is the unique strong solution of the SDE* (2.1.2) *with* $A_\cdot^{i,\prime} = \Psi_\cdot^{i,m_i}\left(Y_{[0,\cdot]}^i, Y_{[0,\cdot]}^j, A_{[0,\cdot)}^{i,\prime}\right)$ *and* $A_\cdot^{j,\prime} = \Psi_\cdot^{i,m_j}\left(Y_{[0,\cdot]}^j, Y_{[0,\cdot]}^i, A_{[0,\cdot)}^{j,\prime}\right)$ *(known as two-sided Skorokhod reflection problem). In this case, the expression of* $\Delta A_t^{k,\prime}$ *simplifies to* $\mathbb{1}_{\{t=0\}}\left[\ln\left(\frac{1-w_k}{1-m_k}\right)\right]^+$ *for all* $k \in \{a, b\}$.

Proof. Fix $x \in \mathbb{R}_+^2$ and let $w_i = r_i(x)$. Note that if $A^i \equiv 0$ then the process $\left(W_t^{i,w_i}(0, A^j)\right)_{t\geq 0}$ with $W_{0-}^{i,w_i}(0, A^j) = w_i \in (0,1)$ satisfies

$$\begin{aligned} W_t^{i,w_i}(0, A^j) = w_i &+ \int_0^t \bar{b}_i(W_s^{i,w_i}(0, A^j))ds + \int_0^t \bar{\sigma}_i(W_s^{i,w_i}(0, A^j))dB_s \\ &- \int_{[0,t]} W_{s-}^{i,w_i}(0, A^j)d\bar{A}_s^j. \end{aligned} \tag{3.3.6}$$

Slightly generalize the SDE (3.3.6) by adding a process $P^i \in \mathcal{A}$: for any $w_i \in (0,1)$ and $t \geq 0$,

$$\begin{aligned} W_t^{i,w_i}(P^i, A^j) = w_i &+ \int_0^t \bar{b}_i(W_s^{i,w_i}(P^i, A^j))ds + \int_0^t \bar{\sigma}_i(W_s^{i,w_i}(P^i, A^j))dB_s \\ &- \int_{[0,t]} W_{s-}^{i,w_i}(P^i, A^j)d\bar{A}_s^j + P_t^i. \end{aligned} \tag{3.3.7}$$

Note that $W_\cdot^{i,w_i}(P^i, A^j)$ is not necessarily bounded above by 1 depending on the process P^i. By De Angelis and Ferrari (2018, Lemma 2.2.), there exists a unique pair $\left(W^{i,w_\cdot}(P^{i,\prime}, A^j), P^{i,\prime}\right)$ such that $W^{i,w_\cdot}(P^{i,\prime}, A^j)$ is the unique strong solution to the SDE (3.3.7) with $P^{i,\prime} \in \mathcal{A}$ given by

$$\begin{aligned} P_t^{i,\prime} &= \sup_{0\leq s\leq t}\left[m_i - w_i - \int_0^t \bar{b}_i(W_s^{i,w_i}(P^{i,\prime}, A^j))ds - \int_0^t \bar{\sigma}_i(W_s^{i,w_i}(P^{i,\prime}, A^j))dB_s\right. \\ &\qquad \left. + \int_{[0,t]} W_{s-}^{i,w_i}(P^{i,\prime}, A^j)d\bar{A}_s^j\right]^+ \\ &= \sup_{0\leq s\leq t}\left[m_i - W_s^{i,w_i}(P^{i,\prime}, A^j) + P_s^{i,\prime}\right]^+ \end{aligned} \tag{3.3.8}$$

a.s. for any $t \geq 0$ satisfying

(i) $m_i \le W_t^{i,w_i}(P^{i,\prime}, A^j) < 1$ a.s. for any $0 \le t < \tau_1$,

(ii) $\int_{[0,\tau_1)} \mathbb{1}_{\left\{W_t^{i,w_i}(P^{i,\prime},A^j)>m_i\right\}} dP_t^{i,\prime} = 0$ a.s.,

where the exit time $\tau_1 = \inf\{t \ge 0 : W_t^{i,w_i}(P^{i,\prime}, A^j) \ge 1\}$.

Let $P^{i,c,\prime}$ denote the continuous part of the process $P^{i,\prime}$. Then there exists a unique process with continuous path $(A_t^{i,c,\prime})_{t\ge 0} \in \mathcal{A}$ such that

$$P_t^{i,c,\prime} = \int_0^t (1 - W_s^{i,w_i}(P^{i,\prime}, A^j))dA_s^{i,c,\prime} \tag{3.3.9}$$

or equivalently

$$\begin{aligned} A_t^{i,c,\prime} &= \int_0^t (1 - W_s^{i,w_i}(P^{i,\prime}, A^j))^{-1} dP_s^{i,c,\prime} \\ &= \int_0^t \left(\mathbb{1}_{\{W_s^{i,w_i}(P^{i,\prime},A^j)>m_i\}} + \mathbb{1}_{\{W_s^{i,w_i}(P^{i,\prime},A^j)=m_i\}}\right) \left(1 - W_s^{i,w_i}(P^{i,\prime}, A^j)\right)^{-1} dP_s^{i,c,\prime} \\ &= \frac{P_t^{i,c,\prime}}{1-m_i}, \end{aligned} \tag{3.3.10}$$

where the third equality follows by condition (ii).

Note that for any $t \ge 0$, $\Delta W_t^{i,w_i}(P^{i,\prime}, A^j) = \Delta P_t^{i,\prime} - W_{t-}^{i,w_i}(P^{i,\prime}, A^j)\Delta \bar{A}_t^j$, a.s. Condition (ii) implies that if $\Delta P_t^{i,\prime}(\omega) > 0$ for some $\omega \in \Omega$, then $W_t^{i,w_i}(P^{i,\prime}, A^j)(\omega) = m_i$ for any $t \in [0, \tau_1)$, which in turn implies that

$$\Delta P_t^{i,\prime}(\omega) = m_i - e^{-\Delta A_t^j(\omega)} W_{t-}^{i,w_i,\prime}(P^{i,\prime}, A^j)(\omega) > 0.$$

Otherwise, if $\Delta P_t^{i,\prime}(\omega) = 0$ for some $\omega \in \Omega$, then $W_t^{i,w_i}(P^{i,\prime}, A^j)(\omega) \ge \tilde{w}_i$. Hence,

$$\Delta P_t^{i,\prime} = \left[m_i - e^{-\Delta A_t^j} W_{t-}^{i,w_i,\prime}(P^{i,\prime}, A^j)\right]^+ \tag{3.3.11}$$

a.s. for any $t \ge 0$.

Let $p^{i,j}(a_i, a_j, w) = \frac{a_i(1+a_j-w)}{(1+a_j)(1+a_i+a_j)}$ for any $(a_i, a_j, w) \in \mathbb{R}_{0,+}^2 \times (0,1)$. Because $a_i \mapsto p^{i,j}(a_i, a_j, w)$ is strictly increasing, for any $t \ge 0$ there exists a unique random variable $\Delta A_t^{i,\prime}$ such that

$$\Delta P_t^{i,\prime} = p^{i,j}\left(\Delta \tilde{A}_t^{i,\prime}, \Delta \tilde{A}_t^j, W_{t-}^{i,w_i}(P^{i,\prime}, A^j)\right), \tag{3.3.12}$$

or, equivalently,

$$\begin{aligned} \Delta \tilde{A}_t^{i,\prime} &= \frac{\Delta P_t^{i,\prime}(\Delta \tilde{A}_t^j + 1)}{1 - \Delta P_t^{i,\prime} - (\Delta \tilde{A}_t^j + 1)^{-1} W_{t-}^{i,w_i}(P^{i,\prime}, A^j)} \\ &= \frac{1}{1-m_i}\left[m_i e^{\Delta A_t^j} - W_{t-}^{i,w_i}(P^{i,\prime}, A^j)\right]^+ \end{aligned} \tag{3.3.13}$$

where the second equality follows by (3.3.11)..

Defining the process $\tilde{A}^{i,'}_t = A^{i,c,'}_t + \sum_{0\le s\le t} \Delta\tilde{A}^{i,'}_s$, and substituting (3.3.9) and (3.3.12) into (3.3.7) shows that $(W^{i,w_i}_t(P^{i,'}, A^j))_{t\ge 0}$ solves the SDE (3.3.3) and hence, we set $W^{i,w_i}_t(A^{i,'}, A^j) = W^{i,w_i}_t(P^{i,'}, A^j)$ for all $t \ge 0$. As for any $(A^i, A^j) \in \mathcal{A}^2$ the solution to the SDE (3.3.3) with initial data $w_i \in (0,1)$ never leaves the interval $(0,1)$ with probability 1 (Lemma 3.3.1 (ii)), it follows that $\tau_1 = +\infty$.

By Lemma 2.1.1 (i) it follows that there exists a unique strong solution $Y^x(A^{i,'}, A^j)$ to the SDE (2.1.2), and Lemma 3.3.1 (i) implies that $r_i\left(Y^x_\cdot(A^{i,'}, A^j)\right)$ and $W^{i,w_i}_\cdot(A^{i,'}, A^j)$ are indistinguishable. Let $W^{i-,w_i}_t(A^{i,'}, A^j) = r_i\left(Y^{i,x}_{t-}, Y^{j,x}_t\right)$ for all $t \ge 0$ and note that $\mathbb{P} - a.s.$

$$W^{i-,w_i}_t(A^{i,'}, A^j) = \frac{Y^{i,x}_{t-}}{Y^{i,x}_{t-} + Y^{j,x}_t} = \frac{Y^{i,x}_{t-}}{Y^{i,x}_{t-} + Y^{j,x}_{t-}} e^{-\Delta A^j_t} = W^{i,w_i}_{t-}(A^{i,'}, A^j) e^{-\Delta A^j_t}. \tag{3.3.14}$$

Also, by (3.3.7) it follows that

$$\begin{aligned} W^{i,w_i}_{t-}(A^{i,'}, A^j) - P^{i,'}_{t-} = w_i + \int_0^t \bar{b}_i(W^{i,w_i}_s(A^{i,'}, A^j))ds + \int_0^t \bar{\sigma}_i(W^{i,w_i}_s(A^{i,'}, A^j))dB_s \\ - \int_{[0,t)} W^{i,w_i}_{s-}(A^{i,'}, A^j) d\bar{A}^j_s, \end{aligned} \tag{3.3.15}$$

and substracting $W^{i,w_i}_{t-}(A^{i,'}, A^j)\Delta\bar{A}^j_t$ from both sides of (3.3.15), it follows by (3.3.14) that

$$\begin{aligned} W^{i-,w_i}_{t-}(A^{i,'}, A^j) - P^{i,'}_{t-} &= w_i + \int_0^t \bar{b}_i(W^{i,w_i}_s(A^{i,'}, A^j))ds + \int_0^t \bar{\sigma}_i(W^{i,w_i}_s(A^{i,'}, A^j))dB_s \\ &\quad - \int_{[0,t]} W^{i,w_i}_{s-}(A^{i,'}, A^j) d\bar{A}^j_s \\ &= W^{i,w_i}_t(A^{i,'}, A^j) - P^{i,'}_t \end{aligned} \tag{3.3.16}$$

a.s. for any $t \ge 0$. It follows by substituting (3.3.16) into (3.3.8) that

$$P^{i,'}_t = \sup_{s\in[0,t]} \left[m_i - W^{i-,w_i}_s(A^{i,'}, A^j) + P^{i,'}_{s-}\right]^+$$

a.s., which in turn yields

$$P^{i,c,'}_t + \sum_{0\le s\le t} \Delta P^{i,'}_s = \sup_{s\in[0,t]} \left[m_i - W^{i-,w_i}_s(A^{i,'}, A^j) + P^{i,c,'}_s + \sum_{0\le u<s} \Delta P^{i,'}_u\right]^+ . \tag{3.3.17}$$

The equality (3.3.14) implies

$$\Delta P^{i,'}_t = \left[m_i - W^{i-,w_i,'}_t(P^{i,'}, A^j)\right]^+ \tag{3.3.18}$$

and together with (3.3.13) it follows that $\Delta\tilde{A}^{i,\prime}_t = \frac{W^{i-,w_i}_{t-}(A^{i,\prime},A^j)}{1-m_i}\left[\frac{m_i}{W^{i-,w_i}_t(A^{i,\prime},A^j)} - 1\right]^+$ a.s..

To complete the proof, substituting (3.3.10) and (3.3.18) into (3.3.17) yields

$$A^{i,c,\prime}_t = \Psi^{i,c,m_i}_t\left(Y^j_{[0,t)}(A^{i,\prime},A^j), Y^j_{[0,t]}(A^{i,\prime},A^j), A^{i,\prime}_{[0,t)}\right)$$

a.s. and by $\Delta\tilde{A}^{i,\prime}_t = e^{\Delta A^{i,\prime}_t} - 1$ it follows that

$$A^{i,d,\prime}_t = \Psi^{i,d,m_i}_t\left(Y^j_{[0,t)}(A^{i,\prime},A^j), Y^j_{[0,t]}(A^{i,\prime},A^j), A^{i,\prime}_{[0,t)}\right).$$

Concerning the second part of the claim, for any process $P^i \in \mathcal{A}$ and $P^j \in \mathcal{A}$ consider the SDE:

$$W^{i,w_i}_t(P^i,P^j) = w_i + \int_0^t \bar{b}_i(W^{i,w_i}_s(P^i,P^j))ds + \int_0^t \bar{\sigma}_i(W^{i,w_i}_s(P^i,P^j))dB_s \\ + P^i_t - P^j_t \tag{3.3.19}$$

By Tanaka, 2002, Theorem 4.1, for any $W^{i,w_i}_{0-}(P^{i,\prime},P^{j,\prime}) = w_i \in (m_i, 1-m_j)$, there exists a unique triplet $(W^{i,w_i}(P^{i,\prime},P^{j,\prime}), P^{i,\prime}, P^{j,\prime})$ with continuous paths such that $W^{i,w_i}(P^{i,\prime},P^{j,\star})$ is the unique strong solution to (3.3.19) $(P^{i,\prime},P^{j,\prime}) \in \mathcal{A}^2$, and with probability 1,

(a) For any $t \geq 0$, $m_i \leq W^{i,w_i}_t(P^{i,\prime},P^{j,\prime}) < 1 - \tilde{w}_j$,

(b) $\int_{[0,\infty)} \mathbb{1}_{\left\{W^{i,w_i}_t(P^{i,\prime},P^{j,\prime})>m_i\right\}} dP^{i,\prime}_t = 0$,

(c) $\int_{[0,\infty)} \mathbb{1}_{\left\{W^{i,w_i}_t(P^{i,\prime},P^{j,\prime})<1-m_j\right\}} dP^{j,\prime}_t = 0$.

Therefore, we can define a unique pair $(A^{i,c,\prime},A^{j,c,\prime}) \in \mathcal{A}^2$ with continuous paths such that

$$P^{i,\prime}_t = \int_0^t (1 - W^{i,w_i}_s(P^{i,\prime},P^{j,\prime}))dA^{i,c,\prime}_s = (1-m_i)A^{i,c,\prime}_t, \tag{3.3.20}$$

$$P^{j,\prime}_t = \int_0^t W^{i,w_i}_s(P^{i,\prime},P^{j,\prime})dA^{j,c,\prime}_s = (1-m_j)A^{j,c,\prime}_t. \tag{3.3.21}$$

Substituting (3.3.20) and (3.3.21) into (3.3.19) reveals that $W^{i,w_i}_t(P^{i,\prime},P^{j,\prime})$ satisfies the SDE (3.3.3) with $A^k = A^{k,c,\prime}$ for any $k \in \{a,b\}$. For $w_i \in (0,1)$, define the processes $A^{i,\prime}_t = A^{i,\prime,c}_t + \left[\ln\left(\frac{1-w_i}{1-m_i}\right)\right]^+$ and $A^{j,\prime}_t = A^{j,\prime,c}_t + \left[\ln\left(\frac{1-w_i}{m_j}\right)\right]^+$. Note that $(A^{i,\prime}_0, A^{j,\prime}_0)$ are the unique functions of w_i such that $W^{i,w_i}_0(A^{i,\prime},A^{j,\prime}) \in [m_i, 1-m_j]$, $A^{i,\prime}_0 \mathbb{1}_{\left\{W^{i,w_i}_0(A^{i,\prime},A^{j,\prime})>m_i\right\}} = 0$ and $A^{j,\prime}_0 \mathbb{1}_{\left\{W^{i,w_i}_0(A^{i,\prime},A^{j,\prime})<1-m_j\right\}} = 0$. Together with the properties (a)–(c),Lemma 2.1.1 (i) and Lemma 3.3.1 (i) it follows by Part (i) of the claim that

$$A^{i,\prime}_t = \Psi^{i,m_i}_t\left(Y^i_{[0,t)}(A^{i,\prime},A^{j,\prime}), Y^j_{[0,t]}(A^{i,\prime},A^{j,\prime}), A^{i,\prime}_{[0,t)}\right)$$

for any $i \neq j \in \{a,b\}$. As for any $k \in \{a,b\}$ and any $t \geq 0$, $\Delta A_t^{k,\prime} = 0$ a.s., the expressions of (3.3.5) simplifies to $\mathbb{1}_{\{t=0\}} \left[\ln\left(\frac{1-w_k}{1-m_k}\right)\right]^+$. $\square$

3.4 Fraud thresholds

This section establishes the existence of the fraud thresholds in Nash equilibrium from the main result (Section 3.5).

For better readability, it is necessary to introduce some notation.

Definition 3.4.1. *For any $i \neq j \in \{a,b\}$, define the parameter-dependent threshold*

$$\hat{w}_i := \frac{-\alpha_i(1-\gamma_i)}{\gamma_i - \alpha_i}, \tag{3.4.1}$$

where

$$\alpha_i := \frac{1}{\sigma^2}\left(k_i - \sqrt{k_i^2 + 2\sigma^2 p_i}\right), \qquad \sigma^2 := \sigma_a^2 + \sigma_b^2,$$
$$p_i := \lambda^\kappa - (1-\gamma_i)\left(\mu_j - \frac{\gamma_i \sigma_j^2}{2}\right), \qquad k_i := \mu_j - \mu_i - \gamma_i \sigma_j^2 + \frac{\sigma^2}{2}.$$

Furthermore, set

$$q_i := \lambda^\kappa - (1-\gamma_i)\left(\mu_i - \frac{\gamma_i \sigma_i^2}{2}\right), \qquad a_i := 1 - \gamma_i - \alpha_i,$$
$$\beta_i := \frac{1}{\sigma^2}\left(k_i + \sqrt{k_i^2 + 2\sigma^2 p_i}\right), \qquad b_i := 1 - \gamma_i - \beta_i.$$

Let $\Delta := \{(w_a, w_b) \in (0,1)^2 : w_a + w_b < 1\}$, for any $i \neq j \in \{a,b\}$ define $F^i : \Delta \to \mathbb{R}$ by

$$\begin{aligned} F^i(w_i, w_j) :=\ & a_i(\alpha_i(1-\gamma_i-w_i) + \gamma_i w_i)(w_j - b_i)\left(\frac{w_i}{1-w_i}\right)^{b_i}\left(\frac{w_j}{1-w_j}\right)^{-\beta_i} \\ & - b_i(\beta_i(1-\gamma_i-w_i) + \gamma_i w_i)(w_j - a_i)\left(\frac{w_i}{1-w_i}\right)^{a_i}\left(\frac{w_j}{1-w_j}\right)^{-\alpha_i} \\ & + (a_i - b_i)(w_i(\alpha_i + \beta_i - 1) - \alpha_i\beta_i) w_j^{\gamma_i}(1-w_j)^{1-\gamma_i}. \end{aligned}$$

Observe that F^i depends on the arrival rate of random horizon and on all of both traders' parameters *but* trader j's risk-aversion γ_j. The next result presents the inequalities of some parameters and identifies the fraud thresholds used in Theorem 3.5.1 below to construct the Nash equilibrium. For any $i \neq j \in \{a,b\}$, let $\hat{\Delta}^i = \{(w_i, w_j) \in (0,1)^2 : w_i < \min(\hat{w}_i, 1-w_j)\} \subset \Delta$.

Lemma 3.4.2 (Existence of thresholds)**.** *The following hold for any $i \neq j \in \{a,b\}$:*

(i) $\alpha_i < 0$, $a_i > 1-\gamma_i$, $\beta_i > 1-\gamma_i$, $b_i < 0$ *and* $\hat{w}_i \in (0, 1-\gamma_i)$.

(ii) *There exists a function* f^i *whose graph satisfies* $\{(f^i(w_j), w_j) : w_j \in (0,1)\} \subset \hat{\Delta}^i$ *and*

$$\left\{(w_i, w_j) \in \Delta : F^i(w_i, w_j) = 0\right\} = \left\{(f^i(w_j), w_j) : w_j \in (0,1)\right\}. \tag{3.4.2}$$

(iii) (a) f^i *is differentiable.*

(b) $\lim_{w_j \uparrow 1} f^i(w_j) = 0$ *and* $\lim_{w_j \downarrow 0} f^i(w_j) = \hat{w}_i$.

(iv) *There exists* $(\tilde{w}_a, \tilde{w}_b) \in \Delta$ *such that*

$$F^a(\tilde{w}_a, \tilde{w}_b) = F^b(\tilde{w}_b, \tilde{w}_a) = 0. \tag{3.4.3}$$

Moreover, such pair $(\tilde{w}_a, \tilde{w}_b)$ *satisfies* $\tilde{w}_k < \hat{w}_k$ *for all* $k \in \{a, b\}$.

Proof of Lemma 3.4.2. Starting with (i), Assumption 3.1.1 implies that

$$p_i > (1-\gamma_i)\left([\mu_i - \mu_j]^+ + \frac{\gamma_i \sigma_j^2}{2}\right) \quad \text{and} \quad q_i > (1-\gamma_i)\left([\mu_j - \mu_i]^+ + \frac{\gamma_i \sigma_i^2}{2}\right), \tag{3.4.4}$$

and inequality (3.4.4) yields

$$\begin{aligned}
&\alpha_i < \frac{1}{\sigma^2}\left(k_i - \sqrt{k_i^2 + \sigma^2(1-\gamma_i)\left(2[\mu_i - \mu_j]^+ + \gamma_i \sigma_j^2\right)}\right) < 0, \qquad a_i > 1-\gamma_i, \\
&\beta_i > \frac{1}{\sigma^2}\left(k_i + \sqrt{k_i^2 + \sigma^2(1-\gamma_i)\left(2[\mu_i - \mu_j]^+ + \gamma_i \sigma_j^2\right)}\right) =: \hat{\beta}_i(\gamma_i), \\
&\hat{w}_i < 1-\gamma_i.
\end{aligned}$$

Note that k_i also depends on γ_i. Because $(\hat{\beta}_i(\gamma_i) - (1-\gamma_i))' > 0$ for any $0 < \gamma_i < 1$, it follows that $\hat{\beta}_i(\gamma_i) - (1-\gamma_i) \geq \hat{\beta}_i(0) - 1 \geq 0$, whence $\beta_i > 1-\gamma_i$ and $b_i < 0$.

Proof of (ii): First, show the inclusion '$\supset$' in (3.4.2). For any $w_j \in (0,1)$,

$$\lim_{w_i \downarrow 0} F^i(w_i, w_j) = \lim_{w_i \downarrow 0} \alpha_i a_i (1-\gamma_i)\left(\frac{1-w_j}{w_j}\right)^{\beta_i}\left(\frac{1-w_i}{w_i}\right)^{-b_i}(w_j - b_i) = -\infty \tag{3.4.5}$$

and

$$\lim_{w_i \uparrow 1-w_j} F^i(w_i, w_j) = -a_i b_i (a_i - b_i)\gamma_i \left(\frac{1-w_j}{w_j}\right)^{1-\gamma_i} > 0. \tag{3.4.6}$$

Next, to show that

$$F^i(\hat{w}_i, w_j) > 0 \quad \text{for any} \quad w_j < 1-\hat{w}_i. \tag{3.4.7}$$

decompose F^i as

$$F^i(\hat{w}_i, w_j) = \frac{\alpha_i b_i(a_i - b_i)(\frac{w_j}{1-w_j})^{-\alpha_i}}{(1-\alpha_i)(\gamma_i - \alpha_i)} l^i(w_j), \tag{3.4.8}$$

where

$$l^i(w_j) = (1-w_j)(1-\alpha_i)^2 \left(\frac{w_j}{1-w_j}\right)^{\gamma_i+\alpha_i} - (1-\gamma_i)^2(a_i - w_j)\left(\frac{\gamma_i(1-\alpha_i)}{-\alpha_i(1-\gamma_i)}\right)^{\gamma_i+\alpha_i}.$$

Note that $\frac{\alpha_i b_i(a_i-b_i)(\frac{w_j}{1-w_j})^{-\alpha_i}}{(1-\alpha_i)(\gamma_i-\alpha_i)} > 0$, so $\operatorname{sgn}\left(F^i(\hat{w}_i, w_j)\right) = \operatorname{sgn}\left(l^i(w_j)\right)$. The first and second derivatives of l^i are

$$l^i_{w_j}(w_j) = (\alpha_i - 1)^2 w_j^{-1}(\alpha_i + \gamma_i - w_j)\left(\frac{w_j}{1-w_j}\right)^{\gamma_i+\alpha_i} + (1-\gamma_i)^2\left(\frac{\gamma_i(1-\alpha_i)}{-\alpha_i(1-\gamma_i)}\right)^{\gamma_i+\alpha_i},$$

and

$$l^i_{w_j w_j}(w_j) = -\frac{a_i(\gamma_i+\alpha_i)(\alpha_i-1)^2}{w_j^2(1-w_j)}\left(\frac{w_j}{1-w_j}\right)^{\gamma_i+\alpha_i} \geq 0 \quad \textit{iff} \quad \alpha_i + \gamma_i \leq 0. \tag{3.4.9}$$

Distinguish now the two cases:

(i) If $\alpha_i + \gamma_i \leq 0$, it follows by (3.4.9) that

$$l^i_{w_j}(w_j) \leq l^i_{w_j}(1-\hat{w}_i) = -a_i\gamma_i^{-1}(\gamma_i - \alpha_i)^2\left(\frac{\gamma_i(1-\alpha_i)}{-\alpha_i(1-\gamma_i)}\right)^{\gamma_i+\alpha_i} < 0,$$

and therefore,

$$l^i(w_j) > l^i(1-\hat{w}_i) = a_i(1-\gamma_i)(\gamma_i-\alpha_i)\left(\frac{\gamma_i(1-\alpha_i)}{-\alpha_i(1-\gamma_i)}\right)^{\gamma_i+\alpha_i} > 0.$$

(ii) If $\alpha_j + \gamma_j > 0$, then

$$\lim_{w_j \downarrow 0} l^i(w_j) = 0.$$

The concavity of l^i (implied by (3.4.9)) in combination with $l^i(1-\hat{w}_i) > 0$ yield

$$l^i(w_j) > 0, \quad w_j < 1 - \hat{w}_i,$$

whence (3.4.7) follows.

Due to (3.4.5),–(3.4.7) the intermediate value theorem implies that for any $w_j \in (0,1)$, there exist $u_i \in \min(\hat{w}_i, 1-w_j)$ such that $F^i(u_i, w_j) = 0$. Thus, there exists a function f^i with graph in $\hat{\Delta}^i$ such that the '$\supset$' containment relation of (3.4.2) holds.

Next, we prove the '$\subset$' containment relation of (3.4.2) by first showing that, for any fixed $w_j \in (0,1)$, if $u_i \in (0, \min(\hat{w}_i, 1-w_j))$ is such that $F^i(u_i, w_j) = 0$, then

$F^i_{w_i}(u_i, w_j) > 0$. The equation $F^i(u_i, w_j) = 0$ expands to

$$\begin{aligned}
a_i(\alpha_i(1-\gamma_i-u_i)+\gamma_i u_i)(w_j-b_i)\left(\frac{1-u_i}{u_i}\right)^{-b_i}\left(\frac{1-w_j}{w_j}\right)^{\beta_i} &= \\
-(a_i-b_i)(u_i(\alpha_i+\beta_i-1)-\alpha_i\beta_i)w_j^{\gamma_i}(1-w_j)^{1-\gamma_i}& \\
+b_i(\beta_i(1-\gamma_i-u_i)+\gamma_i u_i)(w_j-a_i)\left(\frac{u_i}{1-u_i}\right)^{a_i}\left(\frac{w_j}{1-w_j}\right)^{-\alpha_i}& \qquad (3.4.10)
\end{aligned}$$

and differentiating F^i over w_i yields for any $(w_i, w_j) \in \Delta$

$$\begin{aligned}
&-(1-w_i)w_iF^i_{w_i}(w_i,w_j) = \\
&a_i(w_j-b_i)\left(\gamma_i w_i(w_i-b_i-1)+b_i\gamma_i\alpha_i+(1-w_i)(w_i-b_i)\alpha_i\right)\left(\frac{1-w_i}{w_i}\right)^{-b_i}\left(\frac{1-w_j}{w_j}\right)^{\beta_i} \\
&-b_i(a_i-w_j)\left(\gamma_i w_i(1+a_i-w_i)-a_i\gamma_i\beta_i-(1-w_i)(w_i-a_i)\beta_i\right)\left(\frac{w_i}{1-w_i}\right)^{a_i}\left(\frac{w_j}{1-w_j}\right)^{-\alpha_i} \\
&-(a_i-b_i)(\alpha_i+\beta_i-1)(1-w_i)w_i(1-w_j)^{1-\gamma_i}w_j^{\gamma_i}. \qquad (3.4.11)
\end{aligned}$$

Substituting (3.4.10) into (3.4.11) yields that

$$\operatorname{sgn}\left(\partial_{w_i}F^i(u_i,w_j)\right) = \operatorname{sgn}\left(\rho^i(u_i,w_j)l^i(u_i)\right),$$

where

$$\rho^i(w_i,w_j) = -(1-\gamma_i-w_i)(a_i-w_j)\left(\frac{w_i}{1-w_i}\right)^{a_i}\left(\frac{w_j}{1-w_j}\right)^{a_i}+(w_i-\alpha_i)w_j,$$

and

$$l^i(w_i) = -(1-\gamma_i)\alpha_i\beta_i+(\alpha_i\beta_i+\gamma_i(1-\alpha_i-\beta_i))w_i$$

for any $w_i \in \left(0, \min(\hat{w}_i, 1-w_j)\right)$. If $\alpha_i\beta_i+\gamma_i(1-\alpha_i-\beta_i) \geq 0$, then $l^i(u_i) > 0$; if $\alpha_i\beta_i+\gamma_i(1-\alpha_i-\beta_i) < 0$ then, by the inequality $u_i < \hat{w}_i$,

$$l^i(u_i) > \frac{\alpha_i(\alpha_i-1)(1-\gamma_i)\gamma_i}{\gamma_i-\alpha_i} > 0.$$

Hence,

$$\operatorname{sgn}\left(\partial_{w_i}F^i(u_i,w_j)\right) = \operatorname{sgn}\left(\rho^i(u_i,w_j)\right).$$

It is clear that $\rho^i(u_i,w_j) > 0$ if $w_j \geq a_i$. Thus, assume $w_j < a_i$. Since $w_j + u_i < 1$, $w_j \geq a_i$ is satisfied if and only if $w_j < \min\{a_i, 1-u_i\}$. Note that

$$\lim_{w_j \downarrow 0} \rho^i(u_i,w_j) = 0, \qquad (3.4.12)$$

and

$$\rho^i\left(u_i, \min(a_i, 1-u_i)\right) = \begin{cases} a_i(w_i - \alpha_i) > 0, & \text{if} \quad a_i < 1 - u_i \\ \gamma_i a_i > 0, & \text{if} \quad a_i \geq 1 - u_i \end{cases}, \tag{3.4.13}$$

and $\rho^i_{w_j w_j}(u_i, w_j)$ has the same sign as

$$-(a_i - 1)a_i(1 - \gamma_i - u_i)(a_i + w_j)\left(\frac{u_i}{1-u_i}\right)^{a_i}\left(\frac{w_j}{1-w_j}\right)^{a_i} \leq 0 \quad \textit{iff} \quad a_i \geq 1. \tag{3.4.14}$$

If $a_i \geq 1$, then the concavity of ρ^i, (3.4.12) and (3.4.13) yield

$$\rho^i(u_i, w_j) \geq \frac{\rho^i\left(u_i, \min\{a_i, 1-u_i\}\right)}{\min\{a_i, 1-u_i\}} w_j > 0.$$

If $a_i < 1$, then $\left(\frac{u_i}{1-w_j}\right)^{a_i} > \frac{u_i}{1-w_j}$ and $\left(\frac{w_j}{1-u_i}\right)^{a_i} > \frac{w_j}{1-u_i}$ give

$$\begin{aligned} \rho^i(u_i, w_j) &> -(1-\gamma_i - u_i)(a_i - w_j)\left(\frac{u_i}{1-u_i}\right)\left(\frac{w_j}{1-w_j}\right) + (u_i - \alpha_i)w_j \\ &= \frac{w_j \hat{\rho}(u_i, w_j)}{(1-u_i)(1-w_j)}, \end{aligned}$$

where, for any $w_i \in \left(0, \min(\hat{w}_i, 1-w_j)\right)$,

$$\hat{\rho}^i(w_i, w_j) := -w_i(1-\gamma_i - w_i)(a_i - w_j) + (1-w_i)(1-w_j)(w_i - \alpha_i).$$

Taking the partial derivatives with respect to w_i yields

$$\hat{\rho}^i_{w_i}(u_i, w_j) = (1-a_i)(2-\gamma_i - 2u_i - w_j) > 0.$$

(The inequality follows from $u_i + w_j < 1$ and $u_i < \hat{w}_i < 1 - \gamma_i$.) Thus,

$$\partial_{w_i} F^i(u_i, w_j) > 0 \tag{3.4.15}$$

for any $u_i \in (0, \min(\hat{w}_i, 1-w_j))$ is such that $F^i(u_i, w_j) = 0$. Define f^i as follows:

$$f^i(w_j) = \inf\left\{w_i \in \left(0, \min(\hat{w}_i, 1-w_j)\right) : F^i(w_i, w_j) = 0\right\},$$

which is the minimal zero of $w_i \mapsto F^i(w_i, w_j)$. Suppose, by contradiction, that there exist $w_i > f^i(w_j)$ such that $F^i(w_i, w_j) = 0$ and let

$$v_i = \inf\left\{w_i \in \left(f^i(w_j), \min\left(\hat{w}_i, 1-w_j\right)\right) : F^i(w_i, w_j) = 0\right\}, \tag{3.4.16}$$

which is the first zero of $w_i \mapsto F^i(w_i, w_j)$ after $f^i(w_j)$. Then by (3.4.15),

$F^i_{w_i}(f^i(w_j),w_j) > 0$ and $F^i_{w_i}(v_i,w_j) > 0$. By the smoothness of F^i, there exists $\epsilon \in \left(0, \frac{v_i - f^i(w_j)}{2}\right)$ such that $F^i(f^i(w_j)+\epsilon, w_j) > 0$ and $F^i(v_i - \epsilon, w_j) < 0$. However, the intermediate value theorem implies there exists $z_i \in (f^i(w_j)+\epsilon, v_i - \epsilon)$ such that $F^i(z_i, w_j) = 0$, which is impossible in view of definition (3.4.16).

Observe that $(0, \min(\hat{w}_i, 1-w_j)) \subset (0, 1-w_j)$. To establish the claim, it remains to show that $f^i(w_j)$ is the unique solution in the larger domain $(0,1-w_j)$ such that $F^i(f^i(w_j),w_j) = 0$. This facts follows by showing that there does not exist $(w_i,w_j) \in \Delta\backslash\hat{\Delta}^i$ such that $F^i(w_i,w_j) = 0$. Note that $\Delta\backslash\hat{\Delta}^i = \{(w_i,w_j) \in \Delta : w_i \geq \hat{w}_i\}$. By virtue of (3.4.7), it suffices to consider $w_i \in (\hat{w}_i, 1)$.

Differentiating F^i with respect to w_j reveals that $F^i_{w_j}(w_i,w_j)$ has the same sign as

$$\begin{aligned}
&a_i(b_i\beta_i + (1-w_j-\beta_i)w_j)((1-\gamma_i-w_i)\alpha_i + \gamma_i w_i)\left(\frac{1-w_j}{w_j}\right)^{\beta_i}\left(\frac{1-w_i}{w_i}\right)^{-b_i} \\
&- b_i(a_i\alpha_i + w_j(1-w_j-\alpha_i))(\gamma_i w_i + (1-\gamma_i-w_i)\beta_i)\left(\frac{w_j}{1-w_j}\right)^{-\alpha_i}\left(\frac{w_i}{1-w_i}\right)^{a_i} \\
&- (a_i-b_i)(\gamma + (1-2\gamma_i)w_j)((\alpha_i+\beta_i-1)w_i - \alpha_i\beta_i)(1-w_j)^{1-\gamma_i}w_j^{\gamma_i}. \qquad (3.4.17)
\end{aligned}$$

Let $(u_i,u_j) \in \Delta$ be such that $F^i(u_i,u_j) = 0$. Substituting $F^i(w_i,u_j) = 0$ into (3.4.17) yields

$$\operatorname{sgn}\left(F^i_{w_j}(u_i,u_j)\right) = \operatorname{sgn}\left(G^i(u_i,u_j)\right), \qquad (3.4.18)$$

where for any $(w_i,w_j) \in \Delta$,

$$\begin{aligned}
G^i(w_i,w_j) &:= (\gamma_i+\alpha_i)(\beta_i(1-\gamma_i-w_i)+\gamma_i w_i) \\
&\quad - (\gamma_i+\beta_i)(\alpha_i(1-\gamma_i-w_i)+\gamma_i w_i)\left(\frac{1-w_i}{w_i}\right)^{\beta_i-\alpha_i}\left(\frac{1-w_j}{w_j}\right)^{\beta_i-\alpha_i}. \qquad (3.4.19)
\end{aligned}$$

Note that for any $w_i \in (\hat{w}_i, 1)$, $\alpha_i(1-\gamma_i-w_i)+\gamma_i w_i > 0$, and it follows that

$$\operatorname{sgn}\left(G^i_{w_j}(w_i,w_j)\right) = \operatorname{sgn}(\alpha_i(1-\gamma_i-w_i)+\gamma_i w_i) > 0. \qquad (3.4.20)$$

Suppose, by contradiction, that for any fixed $w_i \in (\hat{w}_i, 1)$ there exists $u_j \in (0, 1-w_i)$ such that $F^i(w_i,u_j) = 0$. Let v_j be the smallest u_j, i.e.,

$$v_j = \inf\left\{w_j \in (0,1-w_i) : F^i(w_i,w_j) = 0\right\}. \qquad (3.4.21)$$

Then by (3.4.18) and (3.4.20), $F^i_{w_j}(w_i,v_j) > 0$, implying there exists an $\epsilon \in (0,v_j)$ such that $F^i(w_i, v_j - \epsilon) < 0$. Thus, as for any $w_i \in (\hat{w}_i, 1)$,

$$\lim_{w_j \downarrow 0} F^i(w_i,w_j) = +\infty,$$

there exists an intermediate point $z_j \in (0, v_j - \epsilon)$ such that $F^i(w_i, z_j) = 0$, which contradicts definition (3.4.21), and shows the non-existence of the solution in $\Delta \backslash \hat{\Delta}^i$, hence the inclusion in (3.4.2).

Proof of (iii)a: By (3.4.15), it follows by the implicit function theorem that, in the neighborhood of any point $(w_i^0, w_j^0) \in \Delta$ solving $F^i(w_i^0, w_j^0) = 0$ with the points (w_i, w_j) in this neighborhood satisfying $w_i < \min(\hat{w}_i, 1 - w_j)$, there exists a unique smooth (in fact, analytic) function $f_0^i(w_j)$ satisfying $f_0^i(w_j^0) = w_i^0$ such that $f_0^i(w_j) \in (0, \min(\hat{w}_i, 1 - w_j))$ and $F^i(f_0^i(w_j), w_j) = 0$. Suppose, by contradiction, that f^i is not analytic. Then, then there exists $w_j^0 \in (0,1)$, where the local function $f_0^i(w_j^0) \neq f^i(w_j^0)$. But this implication contradicts uniqueness, and thus $w_j \mapsto f^i(w_j)$ is indeed analytic on the open domain $(0, 1)$.

Proof of (iii)b: Since $0 < f^i(w_j) < 1 - w_j$ for all $w_j \in (0,1)$, the limit $\lim_{w_j \uparrow 1} f^i(w_j) = 0$ follows. Moreover, for any $w_i \in (0, \hat{w}_i)$,

$$\lim_{w_j \downarrow 0} F^i(w_i, w_j) = -\infty, \tag{3.4.22}$$

$$\lim_{w_j \uparrow 1 - w_i} F^i(w_i, w_j) = -\gamma_i a_i b_i (a_i - b_i) \frac{w_i}{1 - w_i}^{1-\gamma_i} > 0. \tag{3.4.23}$$

Thus, by the intermediate value theorem, for any $w_i \in (0, \hat{w}_i)$ there exists $u_j < 1 - w_i$ such that $F^i(w_i, u_j) = 0$ and so there exists a function $g^i : (0, \hat{w}_i) \to (0, 1)$ such that

$$\left\{ (w_i, g^i(w_i)) : w_i \in (0, \hat{w}_i) \right\} \subset \left\{ (w_i, w_j) \in \hat{\Delta}^i : F^i(w_i, w_j) = 0 \right\}.$$

Property (iii)b yields

$$\left\{ (w_i, g^i(w_i)) : w_i \in (0, \hat{w}_i) \right\} \subset \left\{ (f^i(w_j), w_j) : w_j \in (0, 1) \right\},$$

which implies $\sup_{w_j \in (0,1)} f^i(w_j) \geq \hat{w}_i$, equivalently,

$$\max_{w_j (0,1)} f^i(w_j) \geq \hat{w}_i \quad \text{or} \quad \lim_{w_j \downarrow 0} f^i(w_j) \geq \hat{w}_i.$$

On the other hand, $f^i(w_j) < \hat{w}_i$ for all $w_j \in (0, 1)$ implies $\max_{w_j (0,1)} f^i(w_j) < \hat{w}_i$ and by the continuity of f^i, $\lim_{w_j \downarrow 0} f^i(w_j) \leq \hat{w}_i$. Thus, $\lim_{w_j \downarrow 0} f^i(w_j) = \hat{w}_i$.

Proof of (iv): First we establish the existence of $(\tilde{w}_a, \tilde{w}_b) \in \Delta$ such that $F^a(\tilde{w}_a, \tilde{w}_b) = F^a(\tilde{w}_b, \tilde{w}_a) = 0$. For any $i \neq j \in \{a, b\}$, by (iii)b we extend f^i continuously to 0 by setting $f^i(0) = \lim_{w_j \downarrow 0} f^i(w_j) = \hat{w}_i$ and to 1 by setting $f^i(1) = \lim_{w_j \uparrow 1} f^i(w_j) = 0$. Let $\mathcal{D} = (0,1)^2$ and define the function $H : \bar{\mathcal{D}} \to \bar{\mathcal{D}}$ such that $H(w_a, w_b) := \left(f^a(w_b), f^b(w_a) \right)$. Property (ii) implies for any $i \neq j \in \{a, b\}$ and for any $w_j \in [0, 1]$, $f^i(w_j) \in [0, \hat{w}_i] \subset [0, 1]$. (Therefore, H is well-defined.) Since $\bar{\mathcal{D}}$ is compact and H is continuous (due to (iii)a), by Brouwer's fixed point theorem there exists $(\tilde{w}_a, \tilde{w}_b) \in \bar{\mathcal{D}}$ such that $\left(f^a(\tilde{w}_b), f^b(\tilde{w}_a) \right) = (\tilde{w}_a, \tilde{w}_b)$. Next, we show that $(\tilde{w}_a, \tilde{w}_b) \notin \partial \mathcal{D}$. Note that

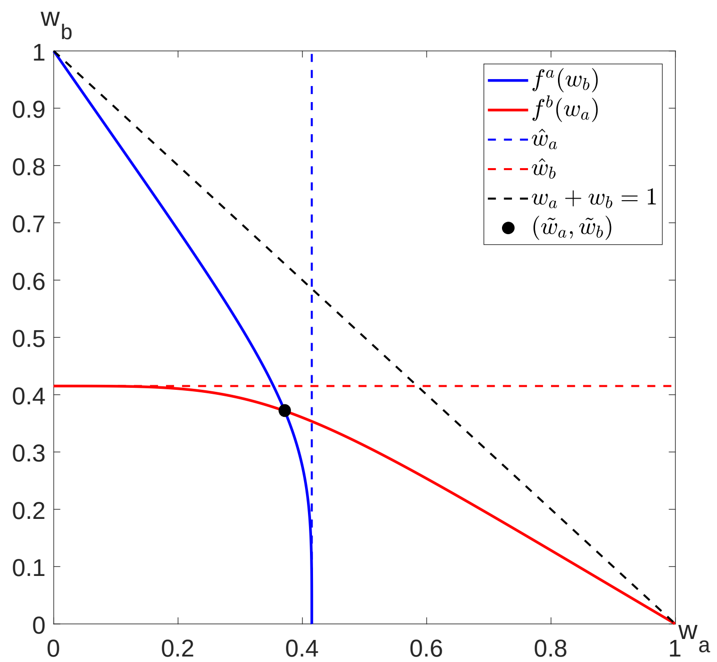


Fig. 3.1: Approximation of the implicit curves $f^a(w_b)$ and $f^b(w_a)$ in Lemma 3.4.2 that satisfy $F^a\left(f^a(w_b), w_b\right) = 0$ for any $0 < w_b < 1$ (vertical axis) and $F^b\left(w_a, f^b(w_a)\right) = 0$ for any $0 < w_a < 1$ (horizontal axis) such that $w_a + w_b < 1$ with parameters $\gamma_a = \gamma_b = 0.5$, $\mu_a = \mu_b = 10\%$, $\sigma_a = \sigma_b = 20\%$ and $\lambda^\kappa = 1/3$.

$\partial\mathcal{D} = \mathcal{D}_1^{a,b} \cup \mathcal{D}_1^{b,a} \cup \mathcal{D}_2^{a,b} \cup \mathcal{D}_2^{b,a}$ where

$$\mathcal{D}_1^{a,b} = \{[0, w_b] : w_b \in [0,1]\}, \qquad \mathcal{D}_1^{b,a} = \{[w_a, 0] : w_a \in [0,1]\},$$
$$\mathcal{D}_2^{a,b} = \{[1, w_b] : w_b \in [0,1]\}, \qquad \mathcal{D}_2^{b,a} = \{[w_a, 1] : w_a \in [0,1]\}.$$

For any $i \neq j \in \{a, b\}$, on $\mathcal{D}_1^{i,j}$, $f^i\left(f^j(0)\right) = f^i(\hat{w}_j) \neq 0$ since $f^i(w_j) \in (0, \hat{w}_i)$ for any $w_j \in (0,1)$; and on $\mathcal{D}_2^{i,j}$, $f^i\left(f^j(1)\right) = f^i(0) \neq 1$ because $f^i(0) = \hat{w}_i$. Hence, $(\tilde{w}_a, \tilde{w}_b) \notin \partial\mathcal{D}$ and so $(\tilde{w}_a, \tilde{w}_b) \in \mathcal{D}$. To this end, by (ii) it follows that $\tilde{w}_i = f^i(\tilde{w}_j) < 1 - \tilde{w}_j$ for any $i \neq j \in \{a, b\}$, thus, $(\tilde{w}_a, \tilde{w}_b) \in \Delta$.

To conclude the proof, note that if a pair $(\tilde{w}_a, \tilde{w}_b) \in \Delta$ is such that $F^a(\tilde{w}_a, \tilde{w}_b) = F^a(\tilde{w}_b, \tilde{w}_a) = 0$, then by (ii), $(\tilde{w}_a, \tilde{w}_b) = \left(f^a(\tilde{w}_b), f^b(\tilde{w}_a)\right) \in \hat{\Delta}^a \cap \hat{\Delta}^b$, meaning that $\tilde{w}_k < \hat{w}_k$ for any $k \in \{a, b\}$. $\square$

Figure 3.1 displays the functions f^a, f^b (identifying the levels sets of F^a and F^b, respectively) and the solution $(\tilde{w}_a, \tilde{w}_b)$ to (3.4.3), where one can observe that f^a and f^b satisfy (ii) and (iii). In this case, the pair $(\tilde{w}_a, \tilde{w}_b)$ satisfying (iv) is unique.

In the following result, we first obtain some properties of the function G^i (given by (3.4.19)); then we find that the function f^i is strictly decreasing.

Lemma 3.4.3 (Monotonicity of f^i)**.** *The following statements hold for any $i \neq j \in \{a, b\}$,*

(i) *If $\alpha_i < -\gamma_i$, then*

(a) *There exists a function $g^i : \left(0, \frac{1+\gamma_i}{2}\right) \to (0, \hat{w}_i)$ such that*

$$\left\{(w_i, w_j) \in \hat{\Delta}^i : G^i(w_i, w_j) = 0\right\} = \left\{(g^i(w_j), w_j) : w_j \in \left(0, \frac{1+\gamma_i}{2}\right)\right\}. \tag{3.4.24}$$

(b) *For any $(w_i, w_j) \in \hat{\Delta}^i$,*

$$G^i(w_i, w_j) \begin{cases} > 0 & \text{if } w_j \geq \frac{1+\gamma_i}{2}, & (3.4.25a) \\ > 0 & \text{if } w_j < \frac{1+\gamma_i}{2} \text{ and } w_i < g^i(w_j), & (3.4.25b) \\ < 0 & \text{if } w_j < \frac{1+\gamma_i}{2} \text{ and } w_i > g^i(w_j). & (3.4.25c) \end{cases}$$

(c) *g^i is differentiable and $\frac{dg^i(w_j)}{dw_j} > 0$ for all $w_j \in \left(0, \frac{1+\gamma_i}{2}\right)$.*

(d) *For any $w_j \in \left(0, \frac{1+\gamma_i}{2}\right)$, $g^i(w_j) > \frac{1-\gamma_i}{2}$ with $\lim_{w_j \uparrow \frac{1+\gamma_i}{2}} g^i(w_j) = \frac{1-\gamma_i}{2}$ and $\lim_{w_j \downarrow 0} g^i(w_j) = \hat{w}_i$.*

(ii) *f^i is strictly decreasing on $(0, 1)$.*

(iii) *Denote by $f^{i,-1}$ the inverse of f^i. For any $(w_i, w_j) \in \hat{\Delta}^i$,*

$$F^i(w_i, w_j) \begin{cases} > 0, & \text{if} \quad w_j > f^{i,-1}(w_i) \\ < 0, & \text{if} \quad w_j < f^{i,-1}(w_i) \end{cases}.$$

Proof. Proof of (i)a and (i)b: Note that $\alpha_i < -\gamma_i$ if and only if $\hat{w}_i > \frac{1-\gamma_i}{2}$. A direct calculation reveals that

$$\lim_{w_i \downarrow 0} G^i(w_i, w_j) = +\infty, \tag{3.4.26}$$

$$\lim_{w_i \uparrow \hat{w}_i} G^i(w_i, w_j) = (\gamma_i + \alpha_i)(\beta_i(1 - \gamma_i - w_i) + \gamma_i w_i) < 0 \quad w_j \in (0, 1 - \hat{w}_i), \tag{3.4.27}$$

$$\lim_{w_i \uparrow 1 - w_j} G^i(w_i, w_j) = \gamma_i(\alpha_i - \beta_i)(1 + \gamma_i - 2w_j) < 0 \quad w_j \in \left(1 - \hat{w}_i, \frac{1+\gamma_i}{2}\right), \tag{3.4.28}$$

$$\lim_{w_i \uparrow 1 - w_j} G^i(w_i, w_j) = \gamma_i(\alpha_i - \beta_i)(1 + \gamma_i - 2w_j) \geq 0 \quad w_j \in \left[\frac{1+\gamma_i}{2}, 1\right), \tag{3.4.29}$$

and G^i has the partial derivative

$$\begin{aligned} G^i_{w_i}(w_i, w_j) &= (\alpha_i + \gamma_i)(\beta_i + \gamma_i) \\ &+ (\gamma_i + \beta_i)\left(\alpha_i - \gamma_i + (\beta_i - \alpha_i)\left(\gamma_i w_i + (1 - \gamma_i - w_i)\alpha_i\right)\right)\left(\frac{(1 - w_i)(1 - w_j)}{w_i w_j}\right)^{\beta_i - \alpha_i} \\ &< 2\gamma_i(\alpha_i - \beta_i) < 0, \end{aligned} \tag{3.4.30}$$

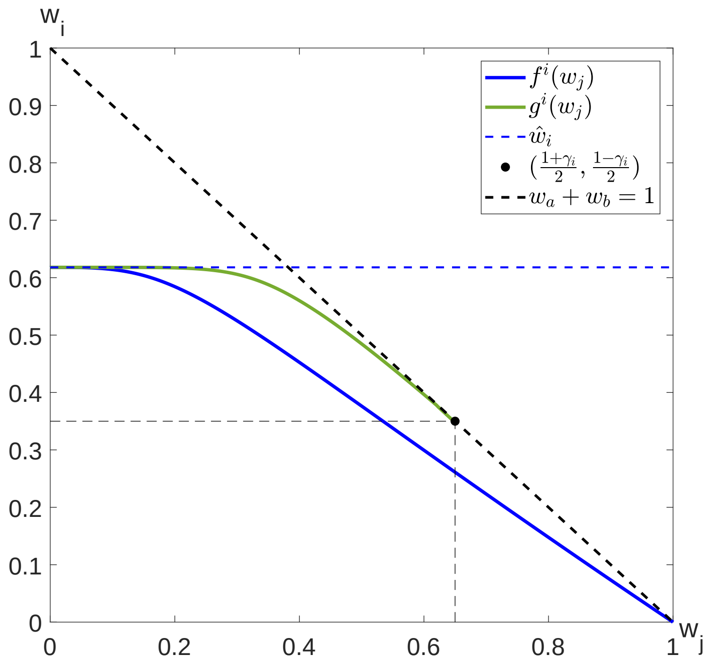


Fig. 3.2: The function $f^i : (0, 1) \to (0, \hat{w}_i)$ (Lemma 3.4.2, in blue) such that $F^i\left(f^i(w_j), w_j\right) = 0$; and the function $g^i : \left(0, \frac{1+\gamma_i}{2}\right) \to (0, \hat{w}_i)$ (Lemma 3.4.3, in green) such that $G^i\left(g^i(w_j), w_j\right) = 0$, with the domain of each in dashed horizontal lines.

where the inequality follows from $w_i < 1 - w_j$ and $\gamma_i w_i + (1 - \gamma_i - w_i)\alpha_i < 0$ (which, in turn, follows from $w_i < \tilde{w}_i$).

Let first $w_j \geq (1 + \gamma_i)/2$: Strict monotonicity (3.4.30) and the limits (3.4.26) and (3.4.29) of equal sign imply inequality (3.4.25a). Let now $w_j < (1 + w_i)/2$. The map $w_i \mapsto G^i(w_i, w_j)$ changes sign on $(0, \min(\hat{w}_i, 1 - w_j)$ due to (3.4.26) and (3.4.27) resp. (3.4.28). Strict monotonicity (3.4.30) therefore implies the existence of a unique $u_i = g^i(w_j) \in (0, \min(\hat{w}_i, 1 - w_j))$ such that $G^i(u_i, w_j) = 0$, and therefore also the inequalities (3.4.25b)–(3.4.25c) must hold. This settles both the proof of (i)b and of (i)a.

Proof of (i)c: The differentiability follows by similar arguments as in the proof of Lemma 3.4.2 (iii)a. To check the monotonicity, note that the implicit function theorem implies for any $w_j \in \left(0, \frac{1+\gamma_i}{2}\right)$,

$$\frac{dg^i(w_j)}{dw_j} = -\frac{G^i_{w_j}(w_i, w_j)}{G^i_{w_i}(w_i, w_j)}\bigg|_{w_i = f^i(w_j)}.$$

Checking the partial derivative $G^i_{w_j}$ and using $w_i < \hat{w}_i$ it follows that

$$\operatorname{sgn}\left(G^i_{w_j}(w_i, w_j)\right) = \operatorname{sgn}(\gamma_i w_i + (1 - \gamma_i - w_i)\alpha_i) < 0,$$

which together with (3.4.30) yields

$$\operatorname{sgn}\left(\frac{dg^i(w_j)}{dw_j}\right) = \operatorname{sgn}\left(-\frac{G^i_{w_j}(w_i,w_j)}{G^i_{w_i}(w_i,w_j)}\Big|_{w_i=f^i(w_j)}\right) < 0.$$

Proof of (i)d: $w_i = \frac{1-\gamma_i}{2}$ satisfies $w_j < 1 - w_i = \frac{1+\gamma_i}{2}$, which implies

$$\begin{aligned} G^i\left(\frac{1-\gamma_i}{2}, w_j\right) &= \frac{1-\gamma_i}{2}(\gamma_i+\alpha_i)(\gamma_i+\beta_i)\left(1-\left(\frac{1+\gamma_i}{1-\gamma_i}\right)^{\beta_i-\alpha_i}\left(\frac{1-w_j}{w_j}\right)^{\beta_i-\alpha_i}\right) \\ &> 0. \end{aligned}$$

Hence, $g^i(w_j) > \frac{1-\gamma_i}{2}$ by the monotonicity of $w_i \mapsto G^i(w_i,w_j)$ (see (3.4.30)). Together with the continuity of g^i and $g^i(w_j) < \min(\hat{w}, 1-w_j)$ ((i)a), it follows that $\lim_{w_j \uparrow \frac{1+\gamma_i}{2}} g^i(w_j) = \frac{1-\gamma_i}{2}$. Finally, as for any $w_i \in \left(\frac{1-\gamma_i}{2}, \hat{w}_i\right)$,

$$\begin{aligned} &\lim_{w_j \downarrow 0} G^i(w_i,w_j) = +\infty, \\ &\lim_{w_j \uparrow 1-w_i} G^i(w_i,w_j) = \gamma_i(\beta_i-\alpha_i)(1-\gamma_i-2w_i) < 0, \end{aligned}$$

and by the intermediate value theorem for any $w_i \in \left(\frac{1-\gamma_i}{2}, \hat{w}_i\right)$ there exists $u_j \in (0, 1-w_i)$ such that $G^i(w_i,u_j) = 0$. By the inclusion $\subseteq$ in (i)a, $g^i(u_j) = w_i$. Because this statement holds for any $w_i \in \left(\frac{1-\gamma_i}{2}, \hat{w}_i\right)$, also $\sup_{w_j \in \left(0, \frac{1+\gamma_i}{2}\right)} g^i(w_j) = \hat{w}_i$. As g^i is strictly bounded from below by $\hat{w}_i$, it follows that $\lim_{w_j \downarrow 0} g^i(w_j) = \hat{w}_i$.

Proof of (ii): The implicit function theorem yields

$$\frac{df^i(w_j)}{dw_j} = -\frac{F^i_{w_j}(w_i,w_j)}{F^i_{w_i}(w_i,w_j)}\Big|_{w_i=f^i(w_j)}, \tag{3.4.31}$$

hence by (3.4.15), (3.4.18) and (3.4.31),

$$\operatorname{sgn}\left(\frac{df^i(w_j)}{dw_j}\right) = -\operatorname{sgn}\left(G^i(f^i(w_j), w_j)\right). \tag{3.4.32}$$

Next, to show that $f^i(w_j)$ is strictly decreasing, distinguish two cases:

(i) $\alpha_i \geq \gamma_i$: Lemma 3.4.2 (i) shows that $G^i(w_i,w_j) > 0$ on $\hat{\Delta}^i$. As Lemma 3.4.2 (ii) implies the graph of f^i satisfies $\left\{(f^i(w_j), w_j) : w_j \in (0,1)\right\} \subset \hat{\Delta}^i$, by (3.4.32) it follows that $\frac{df^i(w_j)}{dw_j} < 0$ for all $w_j \in (0,1)$.

(ii) $\alpha_i < \gamma_i$: By (3.4.25a), $G^i(w_i, w_j) > 0$ on $\hat{\Delta}^i \cap \left\{(w_i, w_j) \in \hat{\Delta}^i : w_j \geq \frac{1+\gamma_i}{2}\right\}$ and by Lemma 3.4.2 (i) it follows that

$$\left\{(f^i(w_j), w_j) : w_j \in \left(0, \frac{1+\gamma_i}{2}\right)\right\} \subset \hat{\Delta}^i \bigcap \left\{(w_i, w_j) \in \hat{\Delta}^i : w_j \geq \frac{1+\gamma_i}{2}\right\}.$$

Hence, by (3.4.32) $\frac{df^i(w_j)}{dw_j} < 0$ for any $w_j \in [\frac{1+\gamma_i}{2}, 1)$. It remains to to check the monotonicity of f^i on the interval $(0, \frac{1+\gamma_i}{2})$ (note that this coincides with the whole domain of g^i).

We proceed by showing $z^i(w_j) := f^i(w_j) - g^i(w_j) \leq 0$ for all $w_j \in (0, \frac{1+\gamma_i}{2})$. Note that since $f^i(w_j) < 1 - w_j$ for any $w_j(0,1)$, then $f^i\left(\frac{1+\gamma_i}{2}\right) < \frac{1-\gamma_i}{2}$. Then by (i)d, it follows that

$$\lim_{w_j \uparrow \frac{1+\gamma_i}{2}} z^i(w_j) = f^i\left(\frac{1+\gamma_i}{2}\right) - \lim_{w_j \uparrow \frac{1+\gamma_i}{2}} g^i(w_j) < 0. \tag{3.4.33}$$

Suppose, by contradiction, that there exist $v_j \in \left(0, \frac{1+\gamma_i}{2}\right)$ such that $z^i(v_j) > 0$. Then, the intermediate value theorem implies that there exists $w_j \in \left(v_j, \frac{1+\gamma_i}{2}\right)$ such that $z^i(w_j) = 0$. Let w_j^* be the first such point, i.e.,

$$w_j^* = \inf\left\{w_j \in \left(v_j, \frac{1+\gamma_i}{2}\right) : z^i(w_j) = 0\right\}.$$

Note that this definition implies

$$z^i(w_j) > 0 \quad \text{for any} \quad w_j \in \left(v_j, w_j^*\right). \tag{3.4.34}$$

By the mean value theorem, there exists $u_j^* \in \left(v_j, w_j^*\right)$ such that

$$\frac{dz^i(w_j)}{dw_j}\Big|_{w_j = u_j^*} = \left(\frac{df^i(w_j)}{dw_j} - \frac{dg^i(w_j)}{dw_j}\right)\Big|_{w_j = u_j^*} < 0.$$

Then, by (i)c it follows that

$$\frac{df^i(w_j)}{dw_j}\Big|_{w_j = u_j^*} < \frac{dg^i(w_j)}{dw_j}\Big|_{w_j = u_j^*} < 0$$

and (3.4.32) implies $G^i\left(f(u_j^*), u_j^*\right) > 0$, which in turn by (3.4.25b) yields $f(u_j^*) < g^i(u_j^*)$, in contradiction to (3.4.34). Hence, $z^i(w_j) \leq 0$ for all $w_j \in (0, \frac{1+\gamma_i}{2})$.

Next, we show that $z^i(w_j) < 0$ almost everywhere (a.e.) on $\left(0, \frac{1+\gamma_i}{2}\right)$. Suppose, by contradiction, that there exists an interval $(a, b) \subset \left(0, \frac{1+\gamma_i}{2}\right)$ such that

$z^i(w_j) = 0$ for any $w_j \in (a, b)$ or, equivalently,

$$f^i(w_j) = g^i(w_j) \quad \text{on} \quad (a, b). \tag{3.4.35}$$

By (i)a it follows that $G^i(f^i(w_j), w_j) = 0$ for any $w_j \in (a, b)$. Then (3.4.32) yields that f^i is a constant on (a, b), which by (3.4.35) in turn implies g^i is a constant on (a, b), an impossibility due to (i)c. Hence, $f^i(w_j) < g^i(w_j)$ almost everywhere on $\left(0, \frac{1+\gamma_i}{2}\right)$. By 3.4.25b and (3.4.32), it follows that $\frac{df^i(w_j)}{dw_j} < 0$ a.e. also on $\left(0, \frac{1+\gamma_i}{2}\right)$, completing the proof.

Proof of (iii): The strict monotonicity (ii) shows the inverse function $f^{i,-1}$ of f^i is well-defined and differentiable. By Lemma 3.4.2 (ii), for any $w_i \in (0, \hat{w}_i)$, $f^{i,-1}(w_i)$ is the unique point in $(0, 1 - w_i)$ such that $F^i(w_i, f^{i,-1}(w_i)) = 0$. Note that $\lim_{w_j \downarrow 0} F^i(w_i, w_j) = -\infty$ and $\lim_{w_j \uparrow 1-w_i} F^i(w_i, w_j) = -\gamma_i a_i b_i (a_i - b_i) \frac{w_i}{1-w_i}^{1-\gamma_i} > 0$. Suppose, by contradiction, that there exists

$$u_j \in \left(0, f^{i,-1}(w_i)\right) \left(\text{resp. } u_j \in \left(f^{i,-1}(w_i), 1 - w_i\right)\right)$$

such that $F^i\left(w_i, u_j\right) > 0$ (resp. $F^i\left(w_i, u_j\right) < 0$). Then, the intermediate value theorem implies that there exists $v_j \in (0, u_j) \subset (0, f^{i,-1}(w_i))$ (resp. $v_j \in (u_j, 1 - w_i) \subset (f^{i,-1}(w_i), 1 - w_i)$) such that $F^i\left(w_i, v_j\right) = 0$, contradicting the uniqueness of $f^{i,-1}(w_i)$ which is a zero of the map $w_j \mapsto F^i(w_i, w_j)$. Hence, $F^i(w_i, w_j) < 0$ for any $w_j \in (0, f^{i,-1}(w_i))$, and $F^i(w_i, w_j) > 0$ for any $w_j \in (f^{i,-1}(w_i), 1 - w_i)$. $\square$

Remark 3.4.4. *To show uniqueness of the fraud thresholds $(\tilde{w}_a, \tilde{w}_b)$, one approach is to check if $f^{a,-1}(w_a) - f^b(w_a)$ is strictly monotone on $[0, \hat{w}_a]$ (or $f^{b,-1}(w_b) - f^a(w_b)$ is strictly monotone on $[0, \hat{w}_b]$). However, differentiating $f^{a,-1}(w_a) - f^b(w_a)$ yields a significantly long expression which is difficult to be simplified and hence, intractable.*

As we have not encountered examples, where uniqueness fails, we conjecture that uniqueness indeed is a feature of the system of equations (3.4.3). Figure 3.3 demonstrates the curves f^a and f^b with varying parameters used in section 4.1, where the thresholds appear to be unique.

3.5 Main result

Without additional assumptions, this section presents the main results of which the heuristic derivation and the proofs are deferred to section 3.5.1 and section 3.5.2,

Theorem 3.5.1 (Nash equilibrium). *For $(\tilde{w}_a, \tilde{w}_b)$ as in Lemma 3.4.2, the pair $(\Psi^{a,\tilde{w}_a}, \Psi^{b,\tilde{w}_b})$ is a Nash equilibrium. The corresponding game values satisfy for any $i \neq j \in \{a, b\}$ and any*

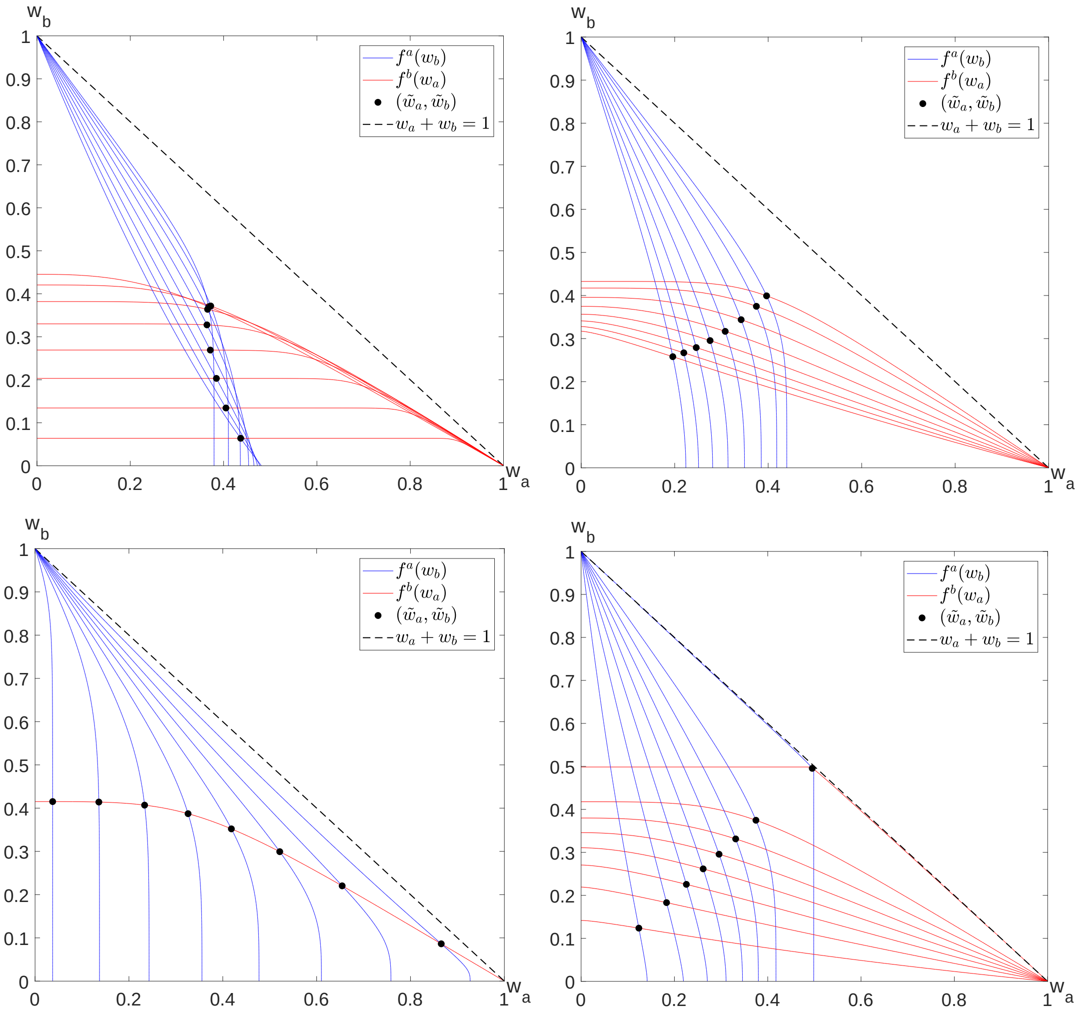


Fig. 3.3: The functions f^a (blue) and f^b (red) (described in Lemma 3.4.2) satisfying $F^a(f^a(w_b), w_b) = 0$ and $F^b(f^b(w_a), w_a)$; and the fraud threshold $(\tilde{w}_a, \tilde{w}_b)$ satisfying $F^a(\tilde{w}_a, \tilde{w}_b) = F^b(\tilde{w}_b, \tilde{w}_a) = 0$ against trader a's expected return (upper-left, $0\% \leq \mu_a \leq 60\%$), volatility (upper-right, $0\% < \sigma_a \leq 100\%$), risk-aversion (bottom-left, $0.1 < \gamma_a < 0.9$), and average horizon (bottom-right, $0 < 1/\lambda^\kappa \leq 20$). Other parameters are $\mu_a = \mu_b = 10\%$, $\sigma_a = \sigma_b = 20\%$, $\gamma_a = \gamma_b = 0.5$, $w_a = 0.5$, $\lambda^\kappa = 1/3$.

$x \in \mathbb{R}^2_+$,

$$V^i(x; A^{j,\star}) = \lambda(x_a + x_b)^{1-\gamma_i} \varphi^i(r_i(x)),$$

where

$$\varphi^i(w) = \begin{cases} c_0^i(1-w)^{-\gamma_i}, & w \in (0, \tilde{w}_i), \quad (3.5.1a) \\ c_1^i w^{\alpha_i}(1-w)^{a_i} + c_2^i w^{\beta_i}(1-w)^{b_i} + \dfrac{U^i(w)}{q_i} & w \in [\tilde{w}_i, 1-\tilde{w}_j), \quad (3.5.1b) \\ c_3^i w^{-\gamma_i}, & w \in [1-\tilde{w}_j, 1), \quad (3.5.1c) \end{cases}$$

with strictly positive constants

$$c_0^i = \frac{-a_i b_i(1-\tilde{w}_i)^{\gamma_i} U^i(\tilde{w}_i)}{q_i\left((\alpha_i+\beta_i-1)\tilde{w}_i - \alpha_i\beta_i\right)},$$
$$c_1^i = \frac{-b_i \tilde{w}_i^{a_i}(1-\tilde{w}_i)^{-a_i}(\gamma_i\tilde{w}_i + (1-\gamma_i-\tilde{w}_i)\beta_i)}{(1-\gamma_i)q_i(\beta_i-\alpha_i)\left((\alpha_i+\beta_i-1)\tilde{w}_i - \alpha_i\beta_i\right)},$$
$$c_2^i = \frac{a_i \tilde{w}_i^{b_i}(1-\tilde{w}_i)^{-b_i}(\gamma_i\tilde{w}_i + (1-\gamma_i-\tilde{w}_i)\alpha_i)}{(1-\gamma_i)q_i(\beta_i-\alpha_i)\left((\alpha_i+\beta_i-1)\tilde{w}_i - \alpha_i\beta_i\right)},$$
$$c_3^i = (1-\tilde{w}_j)^{\gamma_i}\left(c_1^i(1-\tilde{w}_j)^{\alpha_i}\tilde{w}_j^{a_i} + c_2^i(1-\tilde{w}_j)^{\beta_i}\tilde{w}_j^{b_i} + \frac{U^i(1-\tilde{w}_j)}{q_i}\right).$$

The equilibrium fraud processes $\left(A^{a,\star}, A^{b,\star}\right)$ induced by the pair $(\Psi^{a,\tilde{w}_a}, \Psi^{b,\tilde{w}_b})$ are such that: for any $i \neq j \in \{a, b\}$, trader i cheats instantly at time 0 if the initial (at time $0-$) fraction of wealth $r_i(x)$ is in her fraud-region $(0, \tilde{w}_i)$, which then brings her share of wealth immediately up to $\tilde{w}_i$. Thereafter, the fraud is committed 'minimally' to keep the share of wealth $r_i(Y_t^x)$ in trader i's no-fraud-region $[\tilde{w}_i, 1)$. As the same argument applies to trader j, and since trader j's no-fraud-region $[\tilde{w}_j, 1)$, in view of trader i's wealth share, is $(0, 1-\tilde{w}_j]$, then the intersection of both traders' no-fraud-regions $[\tilde{w}_i, 1-\tilde{w}_j)$ represents the common-no-fraud-region. Note that $(\tilde{w}_a, \tilde{w}_b) \in \Delta$ ensures $\tilde{w}_a + \tilde{w}_b < 1$ which in turn guarantees $[\tilde{w}_i, 1-\tilde{w}_j) \subset (0, 1)$.

For the purpose of comparative statics, it is useful to consider the case when only one trader can commit fraud. Indeed, depending on circumstances, access to fraud may be uneven. For instance, Nick Leeson was able to conceal his unauthorized trades because he was allowed to settle his own trades (controlling both the front- and the back-office), a privilege that other traders of the firm did not share. In this regard, assuming that one of the two traders cannot cheat, the other trader achieves optimality by a Skorokhod-type strategy (see Definition 3.3.2) as in Nash equilibrium, but with a different fraud threshold than $\tilde{w}_i$:

Theorem 3.5.2 (Solo Rogue Trader)**.** *For any $i \neq j \in \{a, b\}$, if $A^j \equiv 0$, then the optimal fraud process for trader i is $A_\cdot^{i,\star} = \Psi_\cdot^{i,\hat{w}_i}\left(Y_{[0,\cdot)}^{i,x}, Y_{[0,\cdot)}^{j,x}, A_{[0,\cdot)}^{i,\star}\right)$ and the corresponding value* *function satisfies*

$$V^i(x;0) = \lambda(x_a + x_b)^{1-\gamma_i}\hat{\varphi}^i(r_i(x))$$

for any $x \in \mathbb{R}^2_+$, *where*

$$\hat{\varphi}^i(w) = \begin{cases} s_0^i(1-w)^{-\gamma_i} & \text{on } (0, \hat{w}_i), \quad (3.5.2a) \\ s_1^i w^{\alpha_i}(1-w)^{a_i} + \dfrac{U^i(w)}{q_i} & \text{on } [\hat{w}_i, 1), \quad (3.5.2b) \end{cases}$$

with

$$s_0^i = \frac{a_i(1-\hat{w}_i)^{\gamma_i}}{q_i(\hat{w}_i - \alpha_i)}U^i(\hat{w}_i) > 0 \quad \text{and} \quad s_1^i = \frac{1-\gamma_i-\hat{w}_i}{(1-\gamma_i)q_i(\hat{w}_i-\alpha_i)}\left(\frac{\hat{w}_i}{1-\hat{w}_i}\right)^{a_i} > 0.$$

Remark 3.5.3. *Because* $\hat{w}_i > \tilde{w}_i$ *(Lemma 3.4.2), a sole cheater is less tolerant of decreasing wealth than if there was another cheater present. The fraud region of trader i is indeed smaller in the Nash equilibrium, where both cheat as little as possible (Theorem 3.5.1), so as to keep their proportion of wealth above* $\tilde{w}_i$. *Furthermore,* $\lim_{\gamma_i \uparrow 1} \hat{w}_i = 0$ *follows by* (3.4.1) *and, due to* $\hat{w}_i > \tilde{w}_i$ *then* $\lim_{\gamma_i \uparrow 1} \tilde{w}_i = 0$, *which shows that fraud is undesirable under log-utility for both a sole cheater and dual cheaters because bankruptcy becomes totally unacceptable (i.e.* $\log(0) = -\infty$*).*

As shown in figure 3.4, the presence of an additional cheater reduces the value function of the sole cheater (trader a in this example) across all allocation of initial wealth share from 0% to 100%. Such reduction is most significant when the other trader has much less skin in the game. Moreover, the value function of the sole cheater rises approximately linearly as the initial share of wealth increases; the game value in Nash equilibrium climbs at first and reaches the peak in the fraud-free region, but then declines steadily in the fraud zone of the other trader.

Remark 3.5.4. *One may readily check the following inequalities satisfied by the sole cheater fraud threshold* $\hat{w}_i$ *by viewing it as a function of various parameters.*

a *The better skill, the higher the threshold:* $d\hat{w}_i/d\mu_i > 0$ *as the time effect of the capital accumulation prevails.*

b *The better skill of the other trader or the higher the volatilities, the lower the threshold:* $d\hat{w}_i/d\mu_j$, $d\hat{w}_i/d\sigma_i^2 > 0$ *and* $d\hat{w}_i/d\sigma_j^2 > 0$ *due to the increased frequency of the threshold being visited.*

c *Higher risk-aversion lowers threshold:* $\hat{w}_i/d\gamma_i < 0$ *so as to reduce frauds on worry of the bankruptcy.*

d *Higher impatience rate increases the threshold:* $d\hat{w}_i/d\lambda > 0$ *as the cost of waiting increases.*

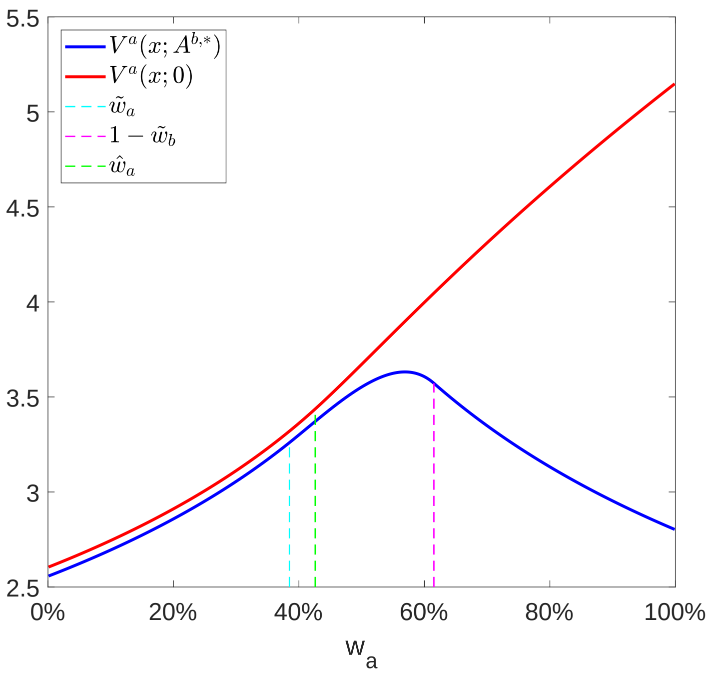


Fig. 3.4: Trader a's game value $V^a(x; A^{b,\star})$ (blue), fraud threshold $\tilde{w}_a$ (dashed line in canyon), trader b's fraud threshold in view of trader a's wealth share $1 - \tilde{w}_b$ (dashed line in purple) as in Theorem 3.5.1, and value function (red) and fraud threshold $\hat{w}_a$ (dashed line in green) as in Theorem 3.5.2, against the initial share of wealth of trader a ($0\% < w_a < 100\%$). Other parameters are $\mu_a = \mu_b = 10\%$, $\sigma_a = \sigma_b = 20\%$, $\gamma_a = \gamma_b = 0.5$, $\lambda = 1/3$, $\kappa = 10\%$, $x_a + x_b = (\lambda + \kappa)^{\gamma_a - 1}$.

3.5.1 Heuristic Derivation

By the linearity of the SDE (2.1.2), we conjecture a singular-type Nash equilibrium in which each trader prevents the wealth process from leaving a region. Let for any $i \neq j \in \{a,b\}$, $\mathcal{C}^j \subset \mathbb{R}^2_+$ be an open set and $\bar{\mathcal{C}}^j := \mathcal{C}^j \cup \partial\mathcal{C}^j$ its closure in $\mathbb{R}^2_+$. Let $\Psi^{j,\mathcal{C}^j} \in \Lambda^j$ be such that for any $A^i \in \mathcal{A}$ the associated pair $(A^{j,\star}, Y^x)$ to $\Psi^{j,\mathcal{C}^j}$ is the unique pair satisfying: $\mathbb{P}$-a.s.,

(i) for any $t \geq 0$, $Y^x_t(A^i, A^{j,\star}) \in \mathbb{R}^2_+ \backslash \mathcal{C}^j$,

(ii) $\int_{[0,\infty)} \mathbb{1}_{\{Y^x_t(A^i,A^{j,\star}) \in \partial\mathcal{C}^j\}} dA^{j,\star}_t = 0$.

In this way, trader j keeps personal wealth inside the region $\mathbb{R}^2_+ \backslash \mathcal{C}^j$ at any time $t \geq 0$ and for any trader i's fraud process $A^i \in \mathcal{A}$. Moreover, if $x \in \mathcal{C}^j$, then trader j cheats instantly so as to bring wealth at time 0 to $\partial\mathcal{C}^j$ and if the state is at $\partial\mathcal{C}^j$ trader j cheats as little as necessary so to keep the wealth process in the interior of $\mathbb{R}^2_+ \backslash \mathcal{C}^j$ (hence, $\bar{\mathcal{C}}^j$ is the *fraud-region* of trader j).

Suppose the optimal fraud process $A^{i,\star} \in \mathcal{A}$ for trader i is such that a.s. for any $t \geq 0$

$$dA^{i,\star}_t dA^{j,\star}_t = 0 \tag{3.5.3}$$

(which means that $A^{i,\star}$ and $A^{j,\star}$ do not simultaneously increase and we will return to this point later) and the value function $V^i(x; A^{j,\star})$ is smooth for any $x \in \mathbb{R}^2_+$. Then for any $x = (x_a, x_b) \in \mathcal{C}^j$, $A^{j,\star}_0 = A^{j,\star}_0(x) > 0$ is such that $Y^x_0 \in \partial\mathcal{C}^j$. By (3.5.3) and Lemma 3.1.3 (i), the game value satisfies, for any $x \in \mathcal{C}^j$ and any $0 \leq \alpha \leq A^{j,\star}_0(x)$

$$\begin{aligned} V^i((x_i,x_j); A^{j,\star}) &= e^{-\alpha}\mathbb{E}\Bigg[\lambda \int_0^\infty e^{-\lambda^\kappa t - A^{i,\star}_t - A^{j,\star}_t} U^i\left(Y^{i,x}_t\right) dt \Bigg| \\ &\qquad\qquad x = (x_i, x_j + (x_a+x_b)(e^\alpha - 1))\Bigg] \\ &= e^{-\alpha} V^i\left((x_i, x_j + (x_a+x_b)(e^\alpha-1)); A^{j,\star}\right). \end{aligned}$$

Since

$$\lim_{\alpha \downarrow 0} \frac{e^{-\alpha}V^i\left((x_i, x_j + (x_a+x_b)(e^\alpha - 1)); A^{j,\star}\right) - V^i((x_i,x_j); A^{j,\star})}{\alpha} = 0,$$

it follows that

$$(x_a + x_b)V^i_{x_j} - V^i = 0 \text{ on } \mathcal{C}^j. \tag{3.5.4}$$

Define the associated differential operator (this is the infinitesimal generator of the uncontrolled pre-bankruptcy process $Y^x(0,0)$),

$$\mathcal{L}\phi(x) = \sum_{k\in\{a,b\}} \mu_k x_k \partial_{x_k}\phi(x) + \frac{1}{2}\sum_{k\in\{a,b\}} \sigma_k^2 x_k^2 \partial^2_{x_k x_k}\phi(x) \tag{3.5.5}$$

for any $\phi \in C^2(\mathbb{R}^2_+)$. For any $x \in \mathbb{R}^2_+\backslash\mathcal{C}^j$, the problem for trader i becomes an optimal (singular) control problem. Treating the triplet $\left(A^i_\cdot, A^j_\cdot, Y^x_\cdot\right)$ as the state process, the dynamic programming principle (see e.g. Fleming and Soner, 2006, Section VIII.2) yields the following quasi-variational inequality (QVI):

$$\max_{\mathbb{R}^2_+\backslash\mathcal{C}^j}\left\{\mathcal{L}V^i - \lambda^\kappa V^i + U^i, (x_a+x_b)V^i_{x_i} - V^i\right\} = 0 \tag{3.5.6}$$

and the verification theorems (cf. Carmona, 2016, Theorem 3.18, Fleming and Soner, 2006, Theorem 4.1, Øksendal and Sulem, 2019, Theorem 8.2) suggest that the set

$$\bar{\mathcal{C}}^i = \left\{x \in \mathbb{R}^2_+\backslash\mathcal{C}^j : (x_a+x_b)V^i_{x_i} - V^i = 0\right\}, \tag{3.5.7}$$

corresponds to the fraud-region of $A^{i,\star}$, so that the optimal cheating strategy for trader i is to only cheat in a region $\mathcal{C}^i \subset \mathbb{R}^2_+$ and prevent the wealth process from leaving the region $\mathbb{R}^2_+\backslash\mathcal{C}^i$ at any time $t \geq 0$. More formally, and similarly to $A^{j,\star}$, $A^{i,\star}$ is such that, a.s. for any $t \geq 0$,

(i) $Y^x_t(A^{i,\star}, A^{j,\star}) \in \mathbb{R}^2_+\backslash(\mathcal{C}^j \cup \mathcal{C}^i)$

(ii) $\int_{[0,\infty)} \mathbb{1}_{\{Y^x_t(A^{i,\star},A^{j,\star})\in\partial\mathcal{C}^i\}} dA^{i,\star}_t = 0$

Here $\mathbb{R}^2_+\backslash(\mathcal{C}^j \cup \mathcal{C}^i)$ is the common no-fraud region. Note that Condition (3.5.3) implies that their fraud-regions do not intersect, that is, $\mathcal{C}^i \cap \mathcal{C}^j = \emptyset$.

Substituting (3.5.7) into (3.5.6), it follows that for any $x \in \mathbb{R}^2_+\backslash\mathcal{C}^j$

$$\mathcal{L}V^i - \lambda^\kappa V^i + U^i = 0 \quad \text{on} \quad \mathbb{R}^2_+\backslash(\mathcal{C}^i \cup \mathcal{C}^j), \tag{3.5.8}$$

$$\mathcal{L}V^i - \lambda^\kappa V^i + U^i < 0 \quad \text{on} \quad \mathcal{C}^i, \tag{3.5.9}$$

$$(x_a+x_b)V^i_{x_i} - V^i < 0 \quad \text{on} \quad \mathbb{R}^2_+\backslash(\mathcal{C}^i \cup \mathcal{C}^j). \tag{3.5.10}$$

Let $\mathcal{L}^i$ be the following differential operator acting on $\varphi \in C^2(\mathbb{R}_+)$,

$$\begin{aligned}\mathcal{L}^i\varphi(w) = (1-\gamma_i)&\left(\mu_i w + \mu_j(1-w) - \frac{\gamma_i}{2}(\sigma_i^2 w^2 + \sigma_j^2(1-w)^2)\right)\varphi(w) \\ &+ \left(\mu_i - \mu_j + \gamma_i(\sigma_j^2(1-w) - \sigma_i^2 w)\right) w(1-w)\varphi_w(w) \\ &+ \frac{\sigma_i^2+\sigma_j^2}{2} w^2(1-w)^2\varphi_{ww}(w).\end{aligned}$$

Now, conjecture that for both traders $k \in \{a, b\}$, the fraud regions in a Nash equilibrium are of the form

$$\mathcal{C}^k = \{x \in \mathbb{R}^2_+ : r_k(x) < m_k\}, \tag{3.5.11}$$

where $m_k \in (0, 1)$ such that $0 < m_a + m_b < 1$. In other words, by cheating, traders prevent their fraction of wealth from going below their critical threshold m_k.

The condition $m_a + m_b < 1$ is equivalent to $C^i \cap C^j = \{x \in \mathbb{R}^2_+ : r_i(x) < m_i \text{ and } r_i(x) > 1 - m_j\} = \varnothing$ (using the equality $r_i(x) + r_j(x) = 1$); and this condition is required because otherwise, there would not exist a corresponding wealth Y^x satisfying (i) i.e. $r_i(Y^x_t) \in [m_i, 1 - m_j]$ a.s. (impossible as $m_i + m_j \geq 1$ and $r_i(Y^x_t) \in (0, 1)$ a.s.), which also justifies the postulation (3.5.3).

Hence, the equilibrium fraud processes $(A^{a,\star}, A^{b,\star})$ and the pre-bankruptcy wealth $Y^x(A^{a,\star}, A^{b,\star})$ together are the associated processes to Ψ^{k,m_k} which solves SP_{m_k+} for all $k \in \{a, b\}$ (see Definition 3.3.2). Note that the scale invariance of J^i (Lemma 3.1.3 (ii)) is inherited by the value function i.e. for any $c > 0$ and any $x \in \mathbb{R}^2_+$,

$$V^i(cx; A^{j,\star}) = c^{1-\gamma_i} V^i(x; A^{j,\star}). \tag{3.5.12}$$

Combining (3.5.12) with Lemma 3.1.3 (i) yield that V^i is of the form

$$V^i(x; A^{j,\star}) = \lambda (x_a + x_b)^{1-\gamma_i} \varphi^i(r_i(x)), \tag{3.5.13}$$

where $\varphi^i(w) = V^i((w, 1 - w); A^{j,\star})$ for any $w \in (0, 1)$. Let $r^i(x) = w$ and substitute (3.5.13) and (3.5.11) into (3.5.8), (3.5.9), (3.5.7), (3.5.10) and (3.5.4), yielding the HJB equations

$$\begin{aligned}
\mathcal{L}^i \varphi^i(w) - \lambda^\kappa \varphi^i(w) + U^i(w) &= 0 \quad \text{on} \quad (m_i, 1 - m_j), && (3.5.14)\\
\mathcal{L}^i \varphi^i(w) - \lambda^\kappa \varphi^i(w) + U^i(w) &< 0 \quad \text{on} \quad (0, m_i), && (3.5.15)\\
(1 - w)\varphi^i_w(w) - \gamma_i \varphi^i(w) &= 0 \quad \text{on} \quad (0, m_i), && (3.5.16)\\
(1 - w)\varphi^i_w(w) - \gamma_i \varphi^i(w) &< 0 \quad \text{on} \quad (m_i, 1 - m_j), && (3.5.17)\\
w\varphi^i_w(w) + \gamma_i \varphi^i(w) &= 0 \quad \text{on} \quad (1 - m_j, 1). && (3.5.18)
\end{aligned}$$

which are the starting point of verification.

Now we seek the candidate solutions to the HJB equations (3.5.14) - (3.5.18). We start by identifying the solutions to (3.5.16), (3.5.18) and (3.5.14). For any $i \neq j \in \{a, b\}$, the unique solutions to the first-order linear ODEs (3.5.16) and (3.5.18) are

$$\varphi^i(w) = c^i_0 (1 - w)^{-\gamma_i} \text{ on } (0, m_i)$$

and

$$\varphi^i(w) = c_3^i w^{-\gamma_i} \text{ on } (1 - m_j, 1)$$

respectively, for some constants $c_0, c_3 \in \mathbb{R}$ to be determined. In order to find the solution to the inhomogeneous second-order linear ODE (3.5.14), first we find the general solution to the second-order homogeneous linear ODE:

$$\mathcal{L}^i \varphi^i(w) - \lambda \varphi^i(w) = 0 \tag{3.5.19}$$

equivalently,

$$g_2^i(w)\varphi_{ww}^i(w) + g_1^i(w)\varphi_w^i(w) + g_0^i(w)\varphi^i(w) = 0,$$

where

$$\begin{aligned}
g_2^i(w) &= \frac{\sigma^2}{2} w^2 (1-w)^2, \\
g_1^i(w) &= \left(\mu_i - \mu_j + \gamma_i \sigma_j^2 - \gamma_i \sigma^2 w\right) w(1-w), \\
g_0^i(w) &= (1-\gamma_i)\left(\mu_j - \frac{\gamma_i \sigma_j^2}{2} + (\mu_i - \mu_j + \gamma_i \sigma_j^2) w - \frac{\gamma^2 \sigma^2}{2} w^2\right) - \lambda^\kappa.
\end{aligned}$$

Using the fact that α_i and β_i are the roots of quadratic equation

$$\frac{\sigma^2}{2} x^2 - k_i x - p_i = 0,$$

one can readily verify that both of the candidates $\rho^{i,1}(w) := w^{\alpha_i}(1-w)^{a_i}$ and $\rho^{i,2}(w) := w^{\beta_i}(1-w)^{-b_i}$ for any $w \in (0,1)$ solve (3.5.19). Since the Wronskian between $\rho^{i,1}$ and $\rho^{i,2}$ satisfies

$$\begin{aligned}
\mathrm{W}(\rho^{i,2}, \rho^{i,2})(w) &= \rho^{i,1}(w)\rho_w^{i,2}(w) - \rho_w^{i,1}(w)\rho^{i,2}(w) \\
&= c_1 c_2 (\alpha_i - \beta_i) w^{\alpha_i + \beta_i - 1} (1-w)^{1 - 2\gamma_i - \alpha_i - \beta_i} (1 - 2w) \\
&\neq 0
\end{aligned}$$

for any $w \in (0,1)$, $\rho^{i,1}$ and $\rho^{i,2}$ are linearly independent. It follows that

$$\rho^i(w) = c_1^i w^{\alpha_i}(1-w)^{a_i} + c_2^i w^{\beta_i}(1-w)^{-b_i}$$

for some constants $c_1, c_2 \in \mathbb{R}$ is the general solution to (3.5.19). Next, note that if neither of the traders engages in fraud, the corresponding reward functional is

$$J^i(x; 0, 0) = \lambda (x_i + x_j)^{1-\gamma_i} \frac{U^i(r_i(x))}{q_i}.$$

Factoring out $\lambda(x_i + x_j)^{1-\gamma_i}$ then note that $\frac{U^i(w)}{q_i}$ is a particular solution to (3.5.14), leading to the solution to (3.5.14)

$$\varphi^i(w) = c_1^i w^{\alpha_i}(1-w)^{a_i} + c_2^i w^{\beta_i}(1-w)^{-b_i} + \frac{U^i(w)}{q_i}.$$

on the fraud-free region $(m_i, 1-m_j)$. To identify the total of 10 unknown constants $m_a, m_b, c_0^a, c_1^a, c_2^a, c_3^a, c_0^b, c_1^b, c_2^b, c_3^b$, we impose the continuity and smooth pasting conditions at m_i and $1-m_j$ for φ^i

$$\begin{aligned}
\varphi^i(m_i-) &= \varphi^i(m_i+), \\
\varphi^i_w(m_i-) &= \varphi^i_w(m_i+), \\
\varphi^i\left((1-m_j)-\right) &= \varphi^i\left((1-m_j)+\right), \\
\varphi^i_w\left((1-m_j)-\right) &= \varphi^i_w\left((1-m_j)+\right),
\end{aligned}$$

which yield 8 equations. In addition, Soner and Shreve, 1989 suggests the twice-continuous-differentiability of the value function for singular control problem with two-dimensional Brownian motion. Since given trader j's fraud strategy of reflecting the wealth share $r_i(Y^x)$ at $1-m_j$, trader i faces an optimal control problem of choosing her/his fraud threshold m_j, so we impose

$$\varphi^i_{ww}(m_i-) = \varphi^i_{ww}(m_i+),$$

mounting to a total of 10 equations. After rearranging the equations, one obtains the expressions for the constants c_0^i-c_3^i in Theorem 3.5.1 and arrives at (m_a, m_b) solves $F^a(w_a, w_b) = 0$ and $F^b(w_b, w_a) = 0$.

The checking of the variational inequalities (3.5.15) and (3.5.17) is rather technical and is deferred to section 3.5.2.

3.5.2 Verification

In this section, we verify that the functions $V^i(x; A^{j,\star})$ in Theorem 3.5.1 and $V^i(x; 0)$ are indeed the value functions for the Nash equilibrium and the case of the solo cheater.

The following result establishes the link between the HJB equations (3.5.16)–(3.5.18) and the optimization problem.

Lemma 3.5.5. *Let $(m_a, m_b) \in \Delta$. For any $i \neq j \in \{a, b\}$ let $\varphi^i \in C^1((0,1))$ be an $\mathbb{R}_+$-valued function satisfying (3.5.16)-(3.5.18). For any $\alpha \geq 0$ and $w \in (0,1)$, set*

$$\tilde{f}^j(\alpha, w) := \left[\ln\left(1 + \frac{1}{1-m_j}\left[m_j e^{\alpha} - (1-w)\right]\right)\right]^+ \tag{3.5.20}$$

and

$$\tilde{h}^i(\alpha, w) := e^{-\alpha - \tilde{f}^j(\alpha,w)}(e^\alpha + e^{\tilde{f}^j(\alpha,w)} - 1)^{1-\gamma_i}\varphi^i\left(\frac{e^\alpha - (1-w)}{e^\alpha + e^{\tilde{f}^j(\alpha,w)} - 1}\right). \tag{3.5.21}$$

Then:

(i) $\tilde{f}^j(\alpha, w) > 0$ *if and only if one of the following inequalities is satisfied:*

  (a) $w > 1 - m_j$,

  (b) $w \leq 1 - m_j$ *and* $\alpha > \ln\left(\frac{1-w}{m_j}\right)$.

(ii) *If* (α, w) *is such that* $\tilde{f}^j(\alpha, w) > 0$ *then* $\partial_\alpha \tilde{h}^i(\alpha, w) < 0$.

(iii) *For any* $\alpha \geq 0$ *and for all* $w \in (0, 1)$,

$$\tilde{h}^i(\alpha, w) - \varphi^i(1-w) \leq 0,$$

and the equality holds for a fixed w *if and only if*

$$\alpha = 0, \qquad \text{if} \quad w \in (m_i, 1), \tag{3.5.22}$$

$$\alpha \leq \ln\left(\frac{1-w}{1-m_i}\right), \qquad \text{if} \quad w \in (0, m_i]. \tag{3.5.23}$$

Proof. Proof of (i): We show the equivalent statement, $\tilde{f}^j(\alpha, w) = 0$ if and only if $w \leq 1 - m_j$ and $\alpha \leq \ln\left(\frac{1-w}{m_j}\right)$. Note that $\tilde{f}^j(\alpha, w) = 0$ if and only if $e^\alpha \leq \frac{1-w}{m_j}$. If $e^\alpha \leq \frac{1-w}{m_j}$, then $\frac{1-w}{m_j} \geq 1$ because $e^\alpha \geq 1$, which, together with $e^\alpha \leq \frac{1-w}{m_j}$, implies that $\alpha \leq \ln\left(\frac{1-w}{m_j}\right)$. The converse implication follows from the monotonicity of the exponential, applied to $\alpha \leq \ln\left(\frac{1-w}{m_j}\right)$.

Proof of (ii): If $\tilde{f}^j(\alpha, w) > 0$, then

$$\frac{e^\alpha - (1-w)}{e^\alpha + e^{\tilde{f}^j(\alpha,w)} - 1} = 1 - m_j$$

and thus $\tilde{h}^i$ simplifies to

$$\tilde{h}^i(\alpha, w) = \frac{1 - (1-w)e^{-\alpha}}{w + (e^\alpha - 1)m_j}\left(\frac{1-m_j}{e^\alpha - (1-w)}\right)^{\gamma_i}\varphi^i(1-m_j).$$

Differentiating $\tilde{h}^i$ with respect to α, and recalling that φ^i is strictly positive, it follows that $\partial_\alpha \tilde{h}^i(\alpha, w)$ has the same sign as

$$g^i(\alpha, w) = (1-w)(w-m_j) - e^{2\alpha}(1+\gamma_i)m_j + e^\alpha\left(2(1-w)m_j - \gamma_i(w-m_j)\right).$$

The inequalities

$$\partial_\alpha g^i(\alpha, w) \leq e^\alpha\left(-\gamma_i(w+m_j) - 2wm_j\right) < 0,$$

and

$$g^i(\alpha, w) \leq g^i(0, w) = -w(\gamma_i + m_j - (1 - w)) < 0,$$

implies that $\partial_\alpha \tilde{h}(\alpha, w) < 0$.

Proof of (iii): All classical solutions of the linear ODEs (3.5.16) and (3.5.18) are of the form

$$C(1-w)^{-\gamma_i} \quad \text{and} \quad Dw^{-\gamma_i}, \quad C, D \in \mathbb{R},$$

respectively. As φ^i is positive,

$$\varphi^i = \begin{cases} C_0(1-w)^{-\gamma_i} & \text{for } w \in (0, m_i), & (3.5.24\text{a}) \\ C_1 w^{-\gamma_i} & \text{for } w \in (1 - m_j, 1), & (3.5.24\text{b}) \end{cases}$$

where $C_0 > 0$ and $C_1 > 0$. Distinguish now three cases:

(i) $w \in (1 - m_j, 1)$: By (i) $\tilde{f}^i(\alpha, w) > 0$ for any $\alpha \geq 0$, thus by (ii), $\tilde{h}^i(\alpha, w) \leq \tilde{h}^i(0, w)$ with equality if and only if $\alpha = 0$. Thus, in conjunction with (3.5.24b),

$$\begin{aligned} \tilde{h}^i(\alpha, w) - \varphi^i(w) &\leq \tilde{h}^i(0, w) - \varphi^i(w) \\ &= \left(\frac{1 - m_j}{w}\right)^{\gamma_i} \varphi^i(1 - m_j) - \varphi^i(w) \\ &= 0, \end{aligned}$$

where the equality holds if and only if $\alpha = 0$.

(ii) $w \in (m_i, 1 - m_j]$: If $\alpha > \ln\left(\frac{1-w}{m_j}\right)$, by (i) $\tilde{f}^i(\alpha, w) > 0$. Therefore, by (ii), $\partial_\alpha \tilde{h}^i(\alpha, w) < 0$ and thus $\tilde{h}^i(\alpha, w) < \tilde{h}^i\left(\ln\left(\frac{1-w}{m_j}\right), w\right)$ for any $\alpha > \ln\left(\frac{1-w}{m_j}\right)$. If $\alpha \leq \ln\left(\frac{1-w}{m_j}\right)$, then by (i) $f^i(\alpha, w) = 0$, and $\tilde{h}^i$ reduces to

$$\tilde{h}^i(\alpha, w) = e^{-\gamma_i \alpha} \varphi^i\left(1 - (1 - w)e^{-\alpha}\right).$$

By (3.5.17),

$$\begin{aligned} \partial_\alpha \tilde{h}^i(\alpha, w) &= -\gamma_i \tilde{h}^i(\alpha, w) + e^{-(1+\gamma_i)\alpha}(1 - w)\varphi^i_w\left(1 - (1 - w)e^{-\alpha}\right) \\ &\leq -\gamma_i \tilde{h}^i(\alpha, w) + \gamma_i e^{-(1+\gamma_i)\alpha}(1 - w)\varphi^i\left(1 - (1 - w)e^{-\alpha}\right) \\ &= \gamma_i e^{-\gamma_i \alpha}\left(e^{-\alpha} - 1\right)\varphi^i\left(1 - (1 - w)e^{-\alpha}\right) \leq 0, \end{aligned}$$

where equality holds if and only if $\alpha = 0$. Hence, for any $\alpha \geq 0$,

$$\tilde{h}^i(\alpha, w) - \varphi^i(w) \leq \tilde{h}^i(0, w) - \varphi^i(w) = 0,$$

with equality if and only if $\alpha = 0$.

(iii) $w \in (0, m_i]$: If $\alpha > \ln\left(\frac{1-w}{m_j}\right)$, then by (i) and (ii) it follows that $\tilde{h}^i(\alpha, w) < \tilde{h}^i\left(\ln\left(\frac{1-w}{m_j}\right), w\right)$ for any $\alpha > \ln\left(\frac{1-w}{m_j}\right)$. If $\alpha \leq \ln\left(\frac{1-w}{m_j}\right)$, then a similar argument as above yields $\tilde{h}^i(\alpha, w) = e^{-\gamma_i \alpha}\varphi^i\left(1-(1-w)e^{-\alpha}\right)$.

If $\alpha \in \left(\ln\left(\frac{1-w}{1-m_i}\right), \ln\left(\frac{1-w}{m_j}\right)\right]$ then $1-(1-w)e^{-\alpha} \in [m_j, 1-m_i)$. By (3.5.17), it analogously follows that $\partial_\alpha \tilde{h}^i(\alpha, w) < 0$, thus, $\tilde{h}^i\left(\ln\left(\frac{1-w}{m_j}\right), w\right) < \tilde{h}^i\left(\ln\left(\frac{1-w}{1-m_i}\right), w\right)$. If $\alpha \leq \ln\left(\frac{1-w}{1-m_i}\right)$, then $1-(1-w)e^{-\alpha} \in [1-m_i, 1)$ and by (3.5.24a) it follows that,

$$\begin{aligned}\tilde{h}^i(\alpha, w) - \varphi^i(w) &= e^{-\gamma_i \alpha}\varphi^i\left(1-(1-w)e^{-\alpha}\right) - \varphi^i(w) \\ &= C_0 e^{-\gamma_i \alpha}\left((1-w)e^{-\alpha}\right)^{-\gamma_i} - C_0(1-w)^{-\gamma_i} = 0.\end{aligned}$$

□

Theorem 3.5.6 (Verification). *Let $(m_a, m_b) \in \Delta$. For any $i \neq j \in \{a, b\}$, let $\varphi^i \in C^1([0,1]) \cap C^2\left((0, 1-m_j)\right)$ be $\mathbb{R}_+$-valued such that φ^i_w is also Lipschitz continuous on $(0,1)$ and satisfies (3.5.14)-(3.5.18). Let $\phi^k(x) := \lambda(x_a + x_b)^{1-\gamma_k}\varphi^k(r_k(x))$ for any $x \in \mathbb{R}^2_+$ and $k \in \{a, b\}$. Then the pair $(\Psi^{a,m_a}, \Psi^{b,m_b})$ is a Nash equilibrium and (ϕ^a, ϕ^b) are the corresponding game values, i.e. for any $i \neq j \in \{a, b\}$*

$$\phi^i(x) = V^i(x; A^{j,\star}).$$

Proof. Let $i \neq j \in \{a, b\}$. We first show that

$$\phi^i(x) \geq \sup_{A^i \in \mathcal{A}} J^i(x; A^i, A^j), \quad x \in \mathbb{R}^2_+. \tag{3.5.25}$$

where A^j satisfies (3.2.1) with $\Psi^j = \Psi^{j,m_j}$. To this end, extend φ^i to $\mathbb{R}$ by setting $\varphi^i(w) := \varphi^i(0)$ for any $w < 0$ and $\varphi^i(w) := \varphi^i(1)$ for any $w > 1$. Let $\xi \in C^\infty(\mathbb{R})$ be a non-negative function, compactly supported in the interval $[-1, 1]$ and such that $\int_{\mathbb{R}} \xi(x)dx = 1$. For any $m \geq 1$, let $\xi_m(w) := \frac{\xi(mw)}{m}$ and define the smooth function through convolution

$$\varphi^{i,m}(w) := \int_{\mathbb{R}} \varphi^i(y)\xi_m(w-y)dy.$$

Since $\mathrm{supp}(\xi_m) \subset [-1/m, 1/m]$, for any Lebesgue measurable function h on $\mathbb{R}$, the value of $(h * \xi_m)(w_0) = \int_{\mathbb{R}} h(y)\xi_m(w_0 - y)dy$ depends only on the values of h in $[w_0 - 1/m, w_0 + 1/m]$. Since φ^i is continuous on $\mathbb{R}$, $\varphi^{i,m}$ converges to φ^i as $m \to \infty$ uniformly on any compact subsets of $\mathbb{R}$. Moreover, as $\varphi^i_w \in C([0,1])$, also $\varphi^{i,m}_w$ converges to φ^i_w on any compact subset of $\mathbb{R}$ (cf. the argument in Fleming and Soner, 2006, Appendix C). For $r > 0$, define the disk $D_r(x) := \{x \in \mathbb{R}^2 : |x| < r\}$ and

$\mathcal{R}_{m,r} := \mathcal{R}_m \cap D_r(0)$, where

$$\mathcal{R}_m := \left\{ x \in \mathbb{R}^2_+ : \min\{x_i, x_j\} > \frac{1}{m} \right\}.$$

Define the exit time $\tau_{m,r} := \inf\{t \geq 0 : Y^x_t \notin R_{m,r}\}$ and the function $\phi^{i,m}(x) := \lambda(x_a + x_b)^{1-\gamma_i}\varphi^{i,m}(r_i(x))$. Applying Itô formula to $e^{-\lambda^\kappa(t\wedge\tau_{m,r})-A^S_{t\wedge\tau_{m,r}}}\phi^{i,m}(Y^x_{t\wedge\tau_{m,r}})$ we obtain, upon taking expectations (using the abbreviation $Y^x_t = Y^x_t(A^i, A^j)$)

$$\begin{aligned}
\phi^{i,m}(x) &= \mathbb{E}\left[e^{-\lambda^\kappa(t\wedge\tau_{m,r})-A^S_{t\wedge\tau_{m,r}}}\phi^{i,m}(Y^x_{t\wedge\tau_{m,r}})\right] \\
&- \mathbb{E}\left[\lambda\int_0^{t\wedge\tau_{m,r}} e^{-\lambda^\kappa s-A^S_s}(Y^{x,S}_s)^{1-\gamma_i}\left(\mathcal{L}^i\varphi^{i,m}(W^{i,w_i}_s) - \lambda^\kappa\varphi^{i,m}(W^{i,w_i}_s)\right)ds\right] \\
&- \mathbb{E}\left[\lambda\int_0^{t\wedge\tau_{m,r}} e^{-\lambda^\kappa s-A^S_s}(Y^{x,S}_s)^{1-\gamma_i}\left((1-W^{i,w_i}_s)\varphi^{i,m}_w(W^{i,w_i}_s) - \gamma_i\varphi^{i,m}(W^{i,w_i}_s)\right)dA^{i,c}_s\right] \\
&+ \mathbb{E}\left[\lambda\int_0^{t\wedge\tau_{m,r}} e^{-\lambda^\kappa s-A^S_s}(Y^{x,S}_s)^{1-\gamma_i}\left(W^{i,w_i}_s\varphi^{i,m}_w(W^{i,w_i}_s) + \gamma_i\varphi^{i,m}(W^{i,w_i}_s)\right)dA^{j,c}_s\right] \\
&- \mathbb{E}\Bigg[\lambda\sum_{0\leq s\leq t\wedge\tau_{m,r}} e^{-\lambda^\kappa s-A^S_{s-}}(Y^{x,S}_{s-})^{1-\gamma_i}\bigg(e^{-\Delta A^S_s}\left(e^{\Delta A^i_s} + e^{\Delta A^{j,\star}_s} - 1\right)^{1-\gamma_i}\varphi^{i,m}(W^{i,w_i}_s) \\
&- \varphi^{i,m}(W^{i,w_i}_{s-})\bigg)\Bigg],
\end{aligned} \tag{3.5.26}$$

where $W^{i,w_i}_t = r_i(Y^x_t)$ for any $t \geq 0$ with $W^{i,w_i}_{0-} = r_i(x) = w_i$.

Since Ψ^{j,m_j} solves $SP^j_{m_j+}$, then by Proposition 3.3.3 that $0 < W^{i,w_i}_t \leq 1 - m_j$ a.s. for any $t \geq 0$. By the continuity of $\mathcal{L}^i\varphi^i$ on $(0, 1-m_j)$, it follows that $\lim_{m\to\infty}\mathcal{L}^i\varphi^{i,m}(w) = \mathcal{L}^i\varphi^i(x)$ for any $w \in (0, 1-m_j)$. Also, since φ^i_w is Lipschitz continuous on $(0,1)$, for any $r > 0$ there exists $M > 0$ such for any $m \in \mathbb{N}$ and for any $x \in \mathcal{R}_{m,r}$, $|(x_i + x_j)\mathcal{L}^i\varphi^{i,m}(r_i(x))| < M$. As

$$\lim_{m\to\infty}\tau_{m,r} = \tau_r := \inf\left\{t \geq 0 : Y^x_t \notin \mathbb{R}^2_+ \cap D_r(0)\right\},$$

dominated convergence implies that, with probability one,

$$\lim_{m\to\infty}\int_0^{t\wedge\tau_{m,r}} e^{-\lambda^\kappa s-A^S_s}(Y^{x,S}_s)^{1-\gamma_i}\mathcal{L}^i\varphi^i_m(W^{i,w_i}_s)ds = \int_0^{t\wedge\tau_r} e^{-\lambda^\kappa s-A^S_s}(Y^{x,S}_s)^{1-\gamma_i}\mathcal{L}^i\varphi^i(W^{i,w_i}_s)ds.$$

Using the fact that $A^{j,c}$ increases only at $1 - m_j$ and the equality (3.5.18), letting $m \to \infty$ and $r \to \infty$ in (3.5.26) ($\lim_{r\to\infty}\tau_r \to \infty$ a.s. as $\partial\mathbb{R}^2_+$ is unattainable for Y^x when $x \in \mathbb{R}^2_+$), we obtain

$$\begin{aligned}
\phi^i(x) = \mathbb{E}\left[e^{-\lambda^\kappa t - A_t^S}\phi^i(Y_t^x)\right] & \\
- \mathbb{E}\left[\lambda\int_0^t e^{-\lambda^\kappa s - A_s^S}(Y_s^{x,S})^{1-\gamma_i}\left(\mathcal{L}^i\varphi^i(W_s^{i,w_i}) - \lambda^\kappa\varphi^i(W_s^{i,w_i})\right)ds\right] & \quad (3.5.27)\\
- \mathbb{E}\left[\lambda\int_0^t e^{-\lambda^\kappa s - A_s^S}(Y_s^{x,S})^{1-\gamma_i}\left((1-W_s^{i,w_i})\varphi_w^i(W_s^{i,w_i}) - \gamma_i\varphi^i(W_s^{i,w_i})\right)dA_s^{i,c}\right] & \\
& \quad (3.5.28)\\
- \mathbb{E}\Bigg[\lambda\sum_{0\le s\le t} e^{-\lambda^\kappa s - A_{s-}^S}(Y_{s-}^{x,S})^{1-\gamma_i}\Big(e^{-\Delta A_s^S}\left(e^{\Delta A_s^i} + e^{\Delta A_s^{j,\star}} - 1\right)^{1-\gamma_i}\varphi^i(W_s^{i,w_i}) & \\
& \quad (3.5.29)\\
- \varphi^i(W_{s-}^{i,w_i})\Big)\Bigg]. &
\end{aligned}$$

Note that by Lemma 3.3.3 and the equality (3.3.14), $\Delta A_t^j = \tilde{f}^j\left(\Delta A_t^i, W_{t-}^{i,w_i}\right)$ a.s., where $\tilde{f}^j$ is given by (3.5.20). Hence, Lemma 3.5.5 (iii) yields

$$\begin{aligned}
\tilde{h}^i\left(\Delta A_t^i, r_i(Y_t^x)\right) - \varphi^i\left(w_i(Y_{t-}^x)\right) &= e^{-\Delta A_t^S}\left(e^{\Delta A_t^i} + e^{\Delta A_t^j} - 1\right)^{1-\gamma_i}\varphi^i\left(r_i(Y_t^x)\right) \\
&\qquad - \varphi^i\left(r_i(Y_{t-}^x)\right) \\
&\le 0
\end{aligned}$$

a.s. for any $t \ge 0$, where $\tilde{h}^i$ is given by (3.5.21). Together with the HJB equations (3.5.14)-(3.5.18) and the fact that $(x_i + x_j)^{1-\gamma_i}U^i(r_i(x)) = U^i(x_i)$ for any $x \in \mathbb{R}_+^2$, it follows that, for any $t \ge 0$

$$\phi^i(x) \ge \mathbb{E}\left[\lambda e^{-\lambda^\kappa t - A_t^S}(Y_t^{x,S})^{1-\gamma_i}\varphi^i(r_i(Y_t^x))\right] + \mathbb{E}\left[\lambda\int_0^t e^{-\lambda^\kappa s - A_s^S}U^i(Y_s^{i,x})ds\right]. \quad (3.5.30)$$

Lemma 2.2.3 implies $\mathbb{E}\left[e^{-A_t^S}(Y_t^{x,S})^{1-\gamma_i}\right] = \mathbb{E}\left[\left(\mathbb{1}_{\{t<\tau_A\}}Y_t^{x,S}\right)^{1-\gamma_i}\right]$. Using the boundedness of φ^i, Lemma 3.1.2, (3.1.5) and Jensen's inequality yield

$$\begin{aligned}
\mathbb{E}\left[e^{-A_t^S}(Y_t^{x,S})^{1-\gamma_i}\varphi^i(r_i(Y_t^x))\right] &\le M\mathbb{E}\left[\left(\mathbb{1}_{\{t<\tau_A\}}Y_t^{x,S}\right)^{1-\gamma_i}\right] \\
&\le M\mathbb{E}\left[\mathbb{1}_{\{t<\tau_A\}}Y_t^{x,S}\right]^{1-\gamma_i} \\
&\le M(x_i + x_j)e^{(1-\gamma_i)t\max_{k\in\{a,b\}}\mu_k}
\end{aligned}$$

for some constant $M \ge \max_{0\le w\le 1}\left|\varphi^i(w)\right|$. Assumption 3.1.1 implies that

$$\lim_{t\to\infty}\mathbb{E}\left[e^{-\lambda^\kappa t - A_t^S}(Y_t^{x,S})^{1-\gamma_i}\varphi^i(r_i(Y_t^x))\right] = 0.$$

Letting $t \to \infty$ for (3.5.30), dominated convergence theorem yields (3.5.25).

Next we improve (3.5.25) by showing that equality indeed holds: If trader i employs the cheating strategy Ψ^{i,m_i}, then by Proposition 3.3.3, $m_i < r_i\left(Y_t^x(A^{i,\star}, A^{j,\star})\right) < 1 - m_j$ a.s. for almost every $t \geq 0$, where $A^{i,\star} = \Psi^{i,m_i}\left(Y^{i,x}_{[0,\cdot)}, Y^{j,x}_{[0,\cdot)}, A^{i,\star}_{[0,\cdot)}\right)$. The process $A^{i,c,\star}$ increases only when W^{i,w_i} is at m_i; hence the term (3.5.28) vanishes by (3.5.16). The jump $\Delta A_t^{i,\star} = \mathbb{1}_{\{t=0\}}\left[\ln\left(\frac{1-w_i}{1-m_i}\right)\right]^+$ is nonzero only when $r_i(x) < m_i$, and such jump brings W_0^{i,w_i} to m_i, thus by Lemma 3.5.5 (iii) the term (3.5.29) vanishes. Therefore, using (3.5.14) for the term (3.5.27) equality in (3.5.30) follows.

Finally, letting t converge to infinity yields

$$\phi^i(x) = J^i(x; A^{i,\star}, A^{j,\star}) = \sup_{A^i \in \mathcal{A}} J^i(x; A^i, A^j).$$

where A^j satisfies (3.2.1) with $\Psi^j = \Psi^{j,m_j}$. □

Lemma 3.5.7. *(i) The constants c_k^i ($k = 0,1,2,3$) in Theorem 3.5.1 are strictly positive.*

(ii) Let $c > 0$, $w^\star \in (0, \hat{w}_i]$ and suppose $f^\star(w) := c(1-w)^{-\gamma_i}$ satisfies

$$\mathcal{L}^i f^\star(w^\star) - \lambda^\kappa f^\star(w^\star) + U^i(w^\star) = 0, \quad w \in (0, w^\star]. \tag{3.5.31}$$

Then $\mathcal{L}^i f^\star(w) - \lambda^\kappa f^\star(w) + U^i(w) < 0$ for any $w \in (0, w^\star)$.

Proof. Proof of (i): First, show that for any $w \in (0, \hat{w}_i]$, $(\alpha_i + \beta_i - 1)w - \alpha_i\beta_i > 0$. If $\alpha_i + \beta_i - 1 > 0$, then clearly $(\alpha_i + \beta_i - 1)w - \alpha_i\beta_i > 0$. If $\alpha_i + \beta_i - 1 < 0$, by the inequalities $w < \hat{w}_i$ and $\beta_i > 1 - \gamma_i$ (Lemma 3.4.2 (i)),

$$\begin{aligned}(\alpha_i + \beta_i - 1)w - \alpha_i\beta_i &> (\alpha_i + \beta_i - 1)\hat{w}_i - \alpha_i\beta_i \\ &= -\frac{\alpha_i(1-\alpha_i)(\beta_i - (1-\gamma_i))}{\gamma_i - \alpha_i} > 0.\end{aligned} \tag{3.5.32}$$

Since $\tilde{w}_i < \hat{w}_i$ (Lemma 3.4.2 (iv)), it follows by (3.5.32) that c_0^i,c_1^i,c_2^i are strictly positive, which in turn implies $c_3^i > 0$.

Proof of (ii): For any $w \in (0, w^\star)$, $\mathcal{L}^i f^\star(w) - \lambda^\kappa f^\star(w) + U^i(w)$ has the same sign as

$$\begin{aligned}l(w) &:= w^{1-\gamma_i}(1-w)^{\gamma_i} - c(1-\gamma_i)\left(p_i - \left(\frac{\sigma^2}{2} - k_i\right)w\right) \\ &= w^{1-\gamma_i}(1-w)^{\gamma_i} - \frac{\sigma^2}{2}c(1-\gamma_i)\left((\alpha_i + \beta_i - 1)w - \alpha_i\beta_i\right).\end{aligned} \tag{3.5.33}$$

The condition 3.5.31 implies $l(w^\star) = 0$, and

$$\lim_{w \downarrow 0} l(w) = \frac{1}{2}c(1-\gamma_i)\sigma^2\alpha_i\beta_i < 0, \tag{3.5.34}$$

$$l_{ww}(w) = -\gamma_i(1-\gamma_i)w^{-1-\gamma_i}(1-w)^{-2+\gamma_i} < 0. \tag{3.5.35}$$

Supposing, by contradiction, that $\sup_{w\in(0,w^\star)} l(w) > l(w^\star) = 0$, by (3.5.34) and the strict concavity (3.5.35), the maximum of l is attained at some $z \in (0, w^\star)$, that is, $\sup_{w\in(0,w^\star)} l(w) = l(z)$. Thus,

$$l_w(z) = z^{-\gamma_i}(1-z)^{\gamma_i-1}(1-\gamma_i-z) + \frac{\sigma^2}{2}c(1-\gamma_i)(1-\alpha_i-\beta_i) = 0. \tag{3.5.36}$$

As $z < \hat{w}_i < 1-\gamma_i$ (Lemma 3.4.2 (i)) plugging (3.5.36) into (3.5.33) yields

$$\begin{aligned} l(z) &= \frac{c(1-\gamma_i)\sigma^2}{2(1-\gamma_i-z)}\left((\alpha_i-1)\gamma_i z + (\gamma_i z + (1-\gamma_i-z)\alpha_i)\,\beta_i\right) \\ &< \frac{c(1-\gamma_i)\sigma^2}{2(1-\gamma_i-z)}(\alpha_i-1)\gamma_i z < 0, \end{aligned}$$

which contradicts $l(z) > 0$. Hence, $\sup_{w\in(0,w^\star)} l(w) \le 0$, which implies $l(w) < 0$ for any $w \in (0, w^\star)$. □

Proof of Theorem 3.5.1:

The proof follows by verifying that the conditions of the Verification Theorem 3.5.6 are met: By construction, for any $i \neq j \in \{a, b\}$, the function φ^i satisfies the ODEs (3.5.14), (3.5.16) and (3.5.18), as well as the smooth pasting conditions

$$\varphi^i(\tilde{w}_i-) = \varphi^i(\tilde{w}_i+), \tag{3.5.37}$$

$$\varphi^i_w(\tilde{w}_i-) = \varphi^i_w(\tilde{w}_i+), \tag{3.5.38}$$

$$\varphi^i_{ww}(\tilde{w}_i-) = \varphi^i_{ww}(\tilde{w}_i+). \tag{3.5.39}$$

As $F^i(\tilde{w}_i, \tilde{w}_j) = 0$, also the following hold:

$$\varphi^i((1-\tilde{w}_j)-) = \varphi^i((1-\tilde{w}_j)+), \tag{3.5.40}$$

$$\varphi^i_w((1-\tilde{w}_j)-) = \varphi^i_w((1-\tilde{w}_j)+). \tag{3.5.41}$$

By construction, $\varphi^i \in C^2((0,\tilde{w}_i)) \cap C^2((\tilde{w}_i, 1-\tilde{w}_j)) \cap C^2((1-\tilde{w}_j, 1))$. The equalities (3.5.37)-(3.5.39) therefore imply $\varphi^i \in C^2(0, 1-\tilde{w}_j)$ and equalities (3.5.40) and (3.5.41) yield $\varphi^i \in C^1(0,1)$.

Due to the finiteness of the limits

$$\lim_{w\downarrow 0} \varphi^i(w) = c_0^i, \quad \lim_{w\uparrow 1} \varphi^i(w) = c_3^i, \quad \lim_{w\downarrow 0} \varphi^i_w(w) = \gamma_i c_0^i, \quad \lim_{w\uparrow 1} \varphi^i(w) = -\gamma_i c_3^i,$$

we may extend the function φ^i to be in $C^1([0,1])$. Furthermore, in view of the finite limits $\lim_{w\downarrow 0} \varphi^i_{ww}(w) = c_0^i\gamma_i(1+\gamma_i)$, $\lim_{w\uparrow 1} \varphi^i_{ww}(w) = c_3^i\gamma_i(1+\gamma_i)$, $\varphi^i_{ww}((1-\tilde{w}_j)+) =$

$c_3^i\gamma_i(1+\gamma_i)(1-\tilde{w}_j)^{-2-\gamma_i}$,

$$\begin{aligned}\varphi^i_{ww}((1-\tilde{w}_j)-) = w^{-2}(1-w)^{-2}\Big(&c_1^i w^{\alpha_i}(1-w)^{a_i}\left(\alpha_i^2-(1-2\gamma_i w)\alpha_i-(1-\gamma_i)\gamma_i w^2\right)\\ &+c_2^i w^{\beta_i}(1-w)^{b_i}\left(\beta_i^2-(1-2\gamma_i w)\beta_i-(1-\gamma_i)\gamma_i w^2\right)\Big)\\ &-\left.\frac{\gamma_i}{q_i w^{1+\gamma_i}}\right|_{w=1-\tilde{w}_j}\end{aligned}$$

and the continuity of φ^i_{ww} on the regions $(0,1-\tilde{w}_j)$ and $(1-\tilde{w}_j,1)$, it follows that $\sup_{w\in(0,1)}\left|\varphi^i_{ww}(w)\right|<\infty$, whence φ^i_w is Lipschitz continuous.

The inequality (3.5.15) follows from $\tilde{w}_i<\hat{w}_i$ in conjunction with Lemma 3.5.7 (i) and (ii). To check the inequality (3.5.17), note that for any $w\in(\tilde{w}_i,1-\tilde{w}_j)$, $(1-w)\varphi^i_w(w)-\gamma_i\varphi^i(w)$ has thee same sign as

$$(w_j-\gamma_i)F^i(\tilde{w}_i,w_j)+\gamma_i a_i b_i l^i(w_j) =: h^i(w_j),$$

where $w_j := 1-w$ (so $w_j\in(\tilde{w}_j,1-\tilde{w}_i)$) and

$$\begin{aligned}l^i(w_j) := (\gamma_i\tilde{w}_i+(1-\gamma_i-\tilde{w}_i)\beta_i)\left(\frac{w_j}{1-w_j}\right)^{-\alpha_i}\left(\frac{\tilde{w}_i}{1-\tilde{w}_i}\right)^{a_i}\\ -(\gamma_i\tilde{w}_i+(1-\gamma_i-\tilde{w}_i)\alpha_i)\left(\frac{1-w_j}{w_j}\right)^{\beta_i}\left(\frac{1-\tilde{w}_i}{\tilde{w}_i}\right)^{-b_i}.\end{aligned}$$

As $\tilde{w}_i<\hat{w}_i$ implies $\gamma_i\tilde{w}_i+(1-\gamma_i-\tilde{w}_i)\alpha_i<0$, it follows that $l^i(w_j)>0$.

By Lemma 3.4.2 (ii) and Lemma 3.4.3 (ii), $(\tilde{w}_i,\tilde{w}_j)\in\left\{(w_i,f^{i,-1}(w_i)) : w_i\in(0,\hat{w}_i)\right\}$. It follows that $f^{i,-1}(\tilde{w}_i)=\tilde{w}_j$, and Lemma 3.4.3 (iii) yields that

$$F^i(\tilde{w}_i,w_j)>0 \quad \text{for any} \quad w_j\in(\tilde{w}_j,1-\tilde{w}_i). \tag{3.5.42}$$

If $w_j\le\gamma_i$, by Lemma 3.4.2 (i) and (3.5.42) it follows that $h^i(w_j)<0$. It remains to check the case $w_j>\gamma_i$: Factoring out $\left(\frac{1-w_j}{w_j}\right)^{1-\gamma_i}$ from h^i yields

$$\operatorname{sgn}\left(h^i(w_j)\right)=\operatorname{sgn}\left(\bar{h}^i(w_j)\right),$$

where

$$\begin{aligned}\bar{h}^i(w_j) := a_i(w_j-b_i-\gamma_i)\left(\gamma_i\tilde{w}_i+(1-\gamma_i-\tilde{w}_i)\alpha_i\right)\left(\frac{1-\tilde{w}_i}{\tilde{w}_i}\right)^{-b_i}\left(\frac{1-w_j}{w_j}\right)^{-b_i}\\ +b_i(a_i+\gamma_i-w_j)\left(\gamma_i\tilde{w}_i+(1-\gamma_i-\tilde{w}_i)\beta_i\right)\left(\frac{\tilde{w}_i}{1-\tilde{w}_i}\right)^{a_i}\left(\frac{w_j}{1-w_j}\right)^{a_i}\\ +(a_i-b_i)(w_j-\gamma_i)(\tilde{w}_i(\alpha_i+\beta_i-1)-\alpha_i).\end{aligned}$$

It follows from $w_j > \gamma_i$ and Lemma 3.4.2 (i) that $w_j - \gamma_i - b_i > 0$, $\gamma_i \tilde{w}_i + (1-\gamma_i - \tilde{w}_i)\alpha_i < 0$ and $a_i + \gamma_i - w_j > 1 - w_j > 0$. The inequalities $\frac{1-\tilde{w}_i}{w_j} > 1$ and $\frac{1-w_j}{\tilde{w}_i} > 1$ imply that

$$\begin{aligned}\bar{h}^i(w_j) &< a_i(w_j - b_i - \gamma_i)\left(\gamma_i \tilde{w}_i + (1-\gamma_i-\tilde{w}_i)\alpha_i\right) \\ &\quad + b_i(a_i+\gamma_i - w_j)\left(\gamma_i\tilde{w}_i + (1-\gamma_i-\tilde{w}_i)\beta_i\right) \\ &\quad + (a_i - b_i)(w_j-\gamma_i)(\tilde{w}_i(\alpha_i+\beta_i-1) - \alpha_i\beta_i) \\ &= a_i b_i (1-\tilde{w}_i - w_j)(\beta_i - \alpha_i) < 0.\end{aligned}$$

Therefore, $(1-w)\varphi^i_w(w) - \gamma_i\varphi^i(w) < 0$ for any $w \in (\tilde{w}_i, 1-\tilde{w}_j)$, proving the inequality (3.5.17). □

In the proof of Theorem 3.5.2 below, we use an auxiliary statement similar to Lemma 3.5.5(iii). The similar, but simpler proof, is omitted for brevity.

Lemma 3.5.8. *For any $i \neq j \in \{a,b\}$ and for any $m_i \in (0,1)$, let $\varphi^i \in C^1((0,1))$ be an $\mathbb{R}_+$-valued function satisfying (3.5.16)–(3.5.17). Then for any $\alpha \geq 0$ and any $w \in (0,1)$, the function*

$$\hat{h}^i(\alpha, w) := e^{-\alpha\gamma_i}\varphi^i\left(1 - e^{-\alpha}(1-w)\right)$$

satisfies

$$\hat{h}^i(\alpha,w) - \varphi^i(1-w) \leq 0,$$

where the equality holds for a fixed w, if and only if

$$\begin{aligned}&\alpha = 0, && \text{if } w \in (m_i, 1), \\ &\alpha \leq \ln\left(\frac{1-w}{1-m_i}\right), && \text{if } w \in (0, m_i].\end{aligned}$$

Proof of Theorem 3.5.2: A direct calculation yields that $\hat{\varphi}^i$ satisfies

$$\mathcal{L}^i\hat{\varphi}^i(w) - \lambda^\kappa\hat{\varphi}^i(w) + U^i(w) = 0, \quad w \in (\hat{w}_i, 1), \tag{3.5.43}$$
$$(1-w)\hat{\varphi}^i_w(w) - \gamma_i\hat{\varphi}^i(w) = 0, \quad w \in (0, \hat{w}_i), \tag{3.5.44}$$

and

$$\hat{\varphi}^i(\hat{w}_i-) = \hat{\varphi}^i(\hat{w}_i+), \tag{3.5.45}$$
$$\hat{\varphi}^i_w(\hat{w}_i-) = \hat{\varphi}^i_w(\hat{w}_i+), \tag{3.5.46}$$
$$\hat{\varphi}^i_{ww}(\hat{w}_i-) = \hat{\varphi}^i_{ww}(\hat{w}_i+). \tag{3.5.47}$$

As $\hat{\varphi}^i \in C^2((0,\hat{w}_i))$ and $\hat{\varphi}^i \in C^2((\hat{w}_i, 1))$, it follows by (3.5.45), (3.5.46) and (3.5.47) that $\hat{\varphi}^i \in C^2((0,1))$. Moreover, Lemma 3.4.2 (i) implies that $s^i_0 > 0$ and $s^i_1 > 0$. Hence, by

Lemma 3.5.7 (ii)

$$\mathcal{L}^i\hat{\varphi}^i(w) - \lambda^\kappa\hat{\varphi}^i(w) + U^i(w) < 0, \quad w \in (0, \hat{w}_i). \tag{3.5.48}$$

Next, we prove that

$$(1-w)\hat{\varphi}^i_w(w) - \gamma_i\hat{\varphi}^i(w) < 0, \quad w \in (\hat{w}_i, 1). \tag{3.5.49}$$

To this end, note that for any $w \in (\hat{w}_i, 1)$,

$$\operatorname{sgn}\left((1-w)\hat{\varphi}^i_w(w) - \gamma_i\hat{\varphi}^i(w)\right) = \operatorname{sgn}\left(l^i(w)\right),$$

where

$$l^i(w) = \frac{1-\gamma_i-w}{(1-\gamma_i)q_i} - s^i_1\left(\frac{1-w}{w}\right)^{a_i}(w-\alpha_i).$$

Also,

$$\lim_{w\downarrow\hat{w}_i} l^i(w) = \lim_{w\downarrow\hat{w}_i} l^i_w(w) = 0, \quad \text{and} \quad \lim_{w\uparrow 1} l^i(w) = -\frac{\gamma_i}{(1-\gamma_i)q_i} < 0,$$

and

$$\operatorname{sgn}\left(l^i_{ww}(w)\right) = \operatorname{sgn}\left((\gamma-\alpha_i)w + \alpha_i(1+a_i)\right).$$

As $w \in (\hat{w}_i, 1)$, it follows that

$$(\gamma-\alpha_i)w + \alpha_i(1+a_i) \in \left((1-\alpha_i)\alpha_i, (1-\alpha_i)(\gamma_i+\alpha_i)\right).$$

Distinguish two cases:

(i) If $\gamma_i + \alpha_i \le 0$, then $l^i_{ww}(w) < 0$ and hence,

$$l^i_w(w) < \lim_{w\downarrow\hat{w}_i} l^i_w(w) = 0.$$

Therefore, an ODE comparison argument yields that $l^i < 0$ on $(\hat{w}_i, 1)$.

(ii) If $\gamma_i + \alpha_i > 0$, then $l^i_{ww}(w) \le 0$ on $\left(\hat{w}_i, \frac{-\alpha_i(1+a_i)}{\gamma_i-\alpha_i}\right]$ and it follows again by an ODE comparison argument that $l^i(w) < 0$ for any $w \in \left(\hat{w}_i, \frac{-\alpha_i(1+a_i)}{\gamma_i-\alpha_i}\right]$. As l^i is strictly convex on the interval $\left(\frac{-\alpha_i(1+a_i)}{\gamma_i-\alpha_i}, 1\right)$, and $l^i\left(\frac{-\alpha_i(1+a_i)}{\gamma_i-\alpha_i}\right) < 0$, $\lim_{w\uparrow 1} l^i(w) < 0$, we conclude that $l^i < 0$ on $\left(\frac{-\alpha_i(1+a_i)}{\gamma_i-\alpha_i}, 1\right)$.

This completes the proof of subclaim (3.5.49). An application of Itô's Lemma to $e^{-\lambda^\kappa t - A^i_t}\hat{\varphi}^i(Y^x_t)$ yields, upon taking expectations (The dependence of Y^x and W^{i,w_i} on

$(A^i, 0)$ is omitted for the sake of brevity.)

$$\begin{aligned}\hat{\varphi}^i(x) &= \mathbb{E}\left[e^{-\lambda^\kappa t - A^i_t}\hat{\varphi}^i(Y^x_t)\right] \\ &- \mathbb{E}\left[\lambda \int_0^t e^{-\lambda^\kappa s - A^i_s}(Y^{x,S}_s)^{1-\gamma_i}\left(\mathcal{L}^i\hat{\varphi}^i(W^{i,w_i}_s) - \lambda^\kappa\hat{\varphi}^i(W^{i,w_i}_s)\right) ds\right] \\ &- \mathbb{E}\left[\lambda \int_0^t e^{-\lambda^\kappa s - A^i_s}(Y^{x,S}_s)^{1-\gamma_i}\left((1 - W^{i,w_i}_s)\hat{\varphi}^i_w(W^{i,w_i}_s) - \gamma_i\hat{\varphi}^i(W^{i,w_i}_s)\right) dA^{i,c}_s\right] \\ &- \mathbb{E}\left[\lambda \sum_{0\le s\le t} e^{-\lambda^\kappa s - A^i_{s-}}(Y^{x,S}_{s-})^{1-\gamma_i}\left(e^{-\gamma_i \Delta A^i_s}\hat{\varphi}^i(W^{i,w_i}_s) - \hat{\varphi}^i(W^{i,w_i}_{s-})\right)\right]. \end{aligned} \tag{3.5.50}$$

By Lemma 3.5.8 and (3.5.44), (3.5.49), the term in the last line of (3.5.50) is non-negative. Using (3.5.43)–(3.5.44) and (3.5.48)–(3.5.49), similar arguments as in the proof of Theorem 3.5.6 yield

$$\hat{\varphi}^i(x) \ge \sup_{A^i \in \mathcal{A}} J^i(x; A^i, 0), \quad x \in \mathbb{R}^2_+. \tag{3.5.51}$$

Finally, using the properties of $\Psi^{i,\hat{w}_i}$ established in Proposition 3.3.3, the equality in (3.5.51) follows. □

3.6 Notes

The heuristic arguments that derived the free boundary problem (3.5.4) and (3.5.6) can easily be generalized to apply for arbitrary number of traders ($N > 2$), yielding that for any $i \in \{1, \dots, N\}$ and any $x \in \mathbb{R}_+^N$,

$$\left(\sum_{k=1}^{N} x_k\right) V^i_{x_j} - V^i = 0 \text{ on } \mathcal{C}^j$$

for all $j \neq i \in \{1, \dots, N\}$ and

$$\max_{\mathbb{R}_+^2 \setminus \cup_{j \neq i}^N \mathcal{C}^j} \left\{ \mathcal{L}^N V^i - \lambda^\kappa V^i + U^i, \left(\sum_{k=1}^{N} x_k\right) V^i_{x_i} - V^i \right\} = 0$$

where the operator $\mathcal{L}^N$ is given by

$$\mathcal{L}^N \phi(x) = \sum_{k=1}^{N} \mu_k x_k \partial_{x_k} \phi(x) + \frac{1}{2} \sum_{k=1}^{N} \sigma_k^2 x_k^2 \partial^2_{x_k x_k} \phi(x).$$

Although the scale invariance property (3.5.12) can reduce the problem by 1 dimension, the reduced problem of dimension $N - 1$ in general remains a challenge to be solved analytically. Recall that when $N = 2$, the system of PDEs reduces to a system of ODEs and the free domains could be tackled by the smooth-pasting technique.

Even if the solution to the HJB equations is obtained, the resulted fraud regions of traders can intersect, but simultaneous burst of frauds by multiple traders may not form a Nash equilibrium. This requires a weaker concept of equilibrium e.g. ϵ-Nash equilibrium (c.f. Benazzoli, Campi, and Di Persio, 2019) for the problem to be solved globally.

Lastly, the major issue arises from the boundary of the fraud region not being a hyperplane of the form $\{x \in \mathbb{R}_+^N : r_i(x) = c \in (0,1)\}$, implying that traders would have to observe the wealth shares across all traders instead of just her/his own share to execute the fraud strategy - somewhat 'unrealistic'. A possible solution to this problem is to reformulate the Nash equilibrium problem to a filtering problem in which traders can only observe some reasonable signal processes and this is deferred to a future research.

Chapter 4

Numerical Examples

4.1 Fraud thresholds and frauds

This section brings to life the theoretical results in Section 3.5 by examining the properties of the Nash equilibrium for concrete parameter values.

Figures 4.1 and 4.2 display the dependence of the fraud thresholds and the average amount of fraud of each trader on model parameters (The numerical scheme for deriving the estimator for average fraud is described in section 4.3), respectively.

A trader's fraud threshold is relatively insensitive to the profitability of personal investments (figure 4.1, upper left), even as such profitability increases from 10% to 60%. The flatness of the threshold, however, does not imply flatness of average fraud, which instead declines rapidly as profitability increases (figure 4.2, upper left). The explanation of this phenomenon lies in the dynamics of relative wealth shares: when one trader's profitability is high, that trader's wealth share tends to increase over time, thereby reaching the fraud threshold less often, hence generating lower fraud.

By contrast, the fraud threshold of the other trader (whose profitability remains constant) rapidly withdraws upwards, meaning that this trader cheats when the respective wealth share falls below a lower threshold. Again, this fact does not imply a decline in the amount of fraud, because such trader's typical wealth share also tends to decline. In fact, figure 4.2 shows that the amount of fraud first increases (up to $\mu_a \approx 40\%$) and then decreases: The initial rise is understood as a *short-term* appropriation, whereby the less skilled trader's higher fraud pilfers the other's profits. The subsequent decline is more akin to a *long-term* appropriation: the less skilled trader recognizes that the other's skill is so high that it is overall more profitable to limit the amount of fraud per unit of time, as to let the other's wealth grow faster, so that future fraud can be even more profitable. Put differently, the less skilled trader establishes a sort of parasite-host relationship with the more skilled trader, thereby avoiding excessive cheating, lest the host perish. Note also that the threshold of the more skilled trader is more sensitive to the honesty (or lack thereof) of the other trader, while the less skilled trader becomes indifferent to the other's honesty when the profitability is sufficiently high.

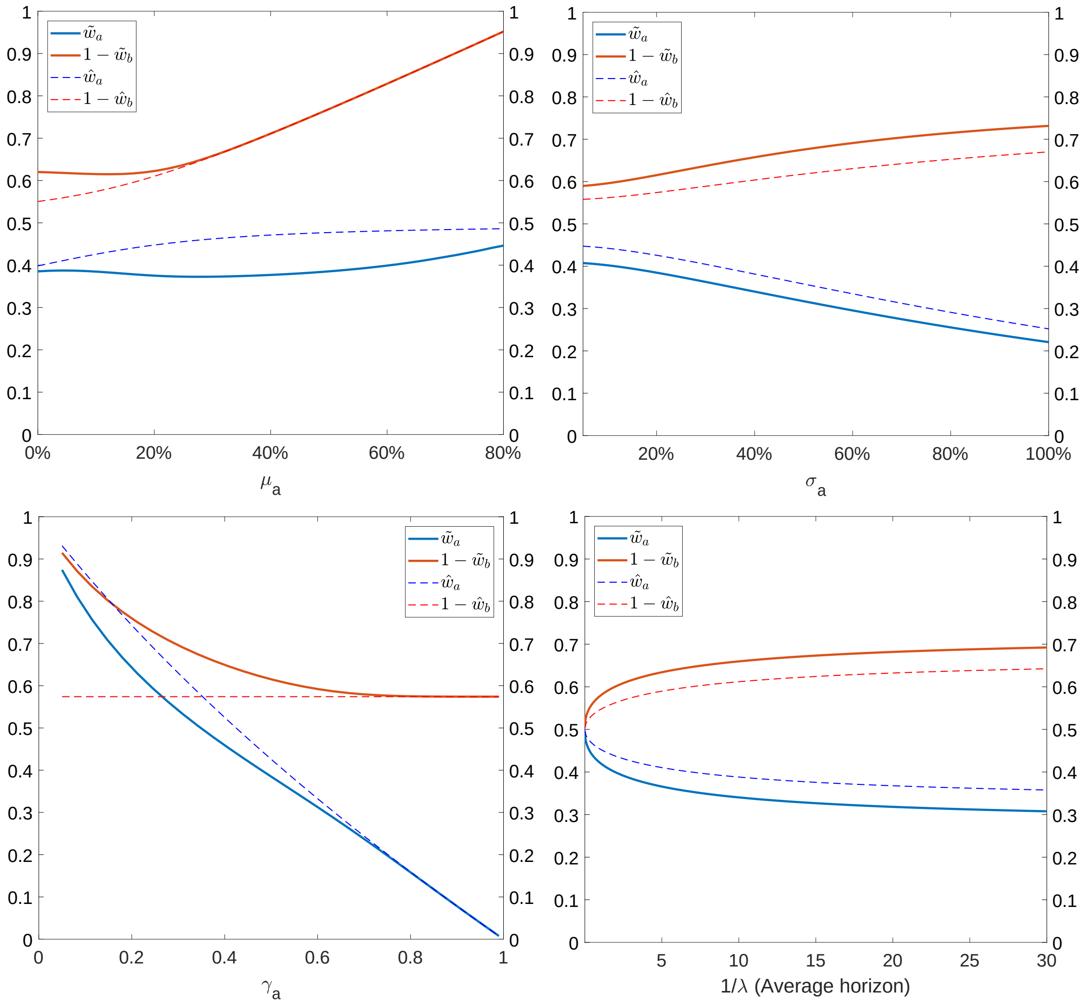


Fig. 4.1: Fraud thresholds for trader a (blue) and b (red), in view of trader a's share of wealth (vertical axis), in Nash equilibrium (solid line), and when the other trader is honest (dashed line), against trader a's expected return (upper-left, $0\% \leq \mu_a \leq 60\%$), volatility (upper-right, $0\% < \sigma_a \leq 100\%$), risk-aversion (bottom-left, $0 < \gamma_a < 1$), and average horizon (bottom-right, $0 < 1/\lambda \leq 20$). Other parameters are $\mu_a = \mu_b = 10\%$, $\sigma_a = \sigma_b = 20\%$, $\gamma_a = \gamma_b = 0.5$, $\lambda = 1/3$, $\kappa = 10\%$.

As the volatility of a trader's investments increases (upper right, figures 4.1 and 4.2), that trader's fraud threshold recedes aggressively, but fraud increases significantly. Increased volatility is qualitatively similar to lower skill, which makes the trader more reliant on fraud to generate profits. Vice versa, the other trader can still rely on a personal payoff with lower volatility, which would be significantly degraded by the additional asymmetry generated by more fraud.

Risk aversion (lower left, figures 4.1 and 4.2) has a major impact on propensity for fraud. Holding the opponent's risk aversion constant at 0.5, as a trader's risk aversion increases from zero to one, the fraud threshold declines very rapidly from one (incessant fraud) to zero (no fraud). Note that, as a fraud threshold declines, the other threshold also declines, not to zero, but to the threshold that assumes the other's honesty. Put differently, a fearless trader's propensity for fraud forces the other, more prudent, trader to withdraw from fraud, as overall risk is already too high. The implication is that, when the two traders have very different risk aversion but similar investment opportunities, it is the least risk averse that has most potential for fraud. Vice versa, when risk aversions are similar, the overall potential for fraud is much lower and is evenly distributed between traders.

Fraud completely disappears with unit risk aversion (i.e. logarithmic preferences). In this case, the dread of bankruptcy is so high that traders abstain from fraud regardless of its potential rewards. Note that this phenomenon stems from the fraud's inherent discontinuity, which always implies a probability, however small, that wealth may vanish. Put differently, for the logarithmic investor the marginal utility of any amount of fraud is infinitely negative, regardless of expected profits.

The average horizon is also an important determinant of fraud (lower right, figures 4.1 and 4.2). Fraud thresholds recede as the horizon increases (λ decreases) and with it the expected reward for delaying fraud. In fact, the average amount of fraud increases sharply up to a horizon of about five years, climbing steadily thereafter and converging eventually. The implication is that: while reducing fraud *per unit of time*, its overall amount in fact increases the most in the medium term - the typical traders turnover in financial institutions.

4.2 Uncertain opponent's skill

In practice, a trader may not have perfect information about the other's investment skill and portfolio risk, but she/he may be able to estimate them. Volatility can be determined rather precisely from frequent (say, daily) observations of wealth history: indeed, in the model, volatility follows directly from the quadratic variation of the logarithmic wealth process, which is insensitive to fraud (as it is a finite-variation process).

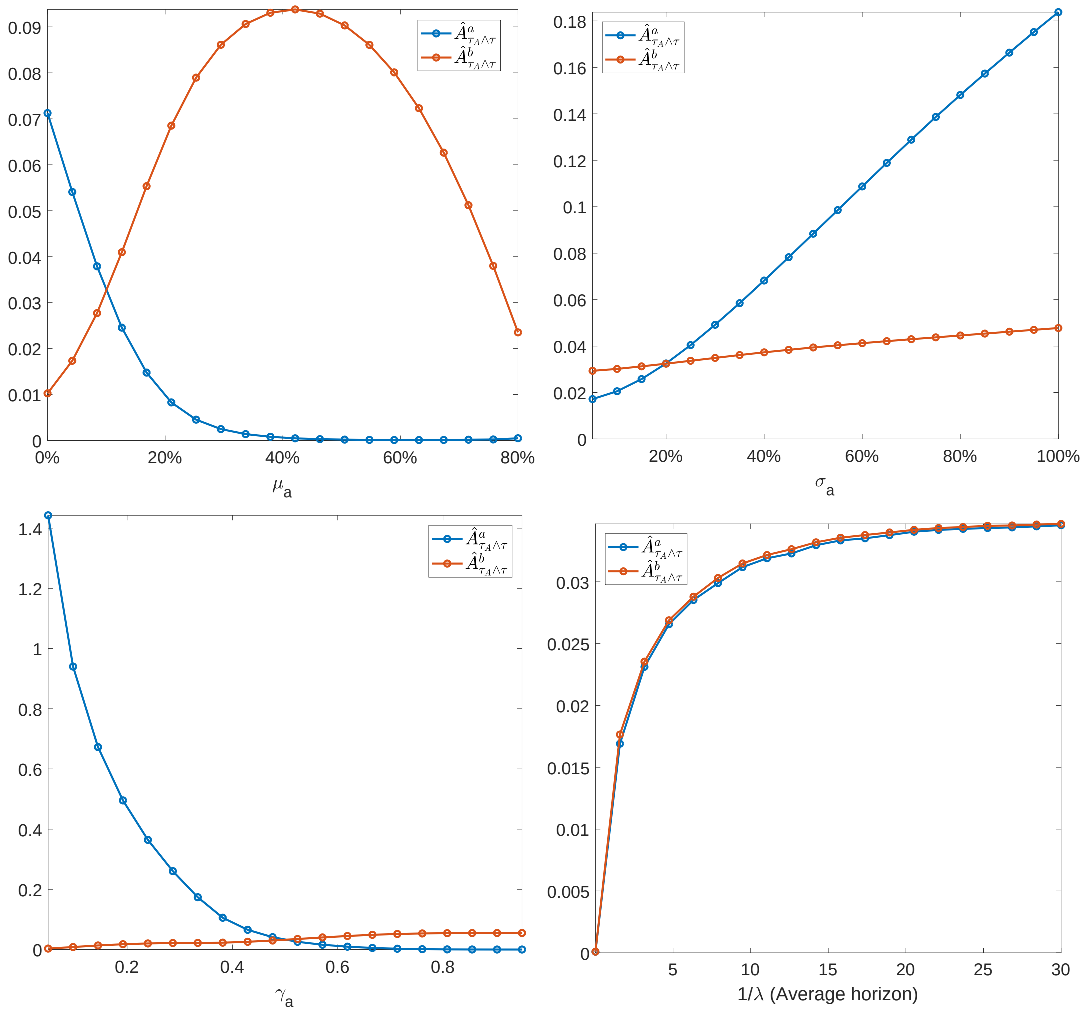


Fig. 4.2: Equilibrium average fraud (vertical axis), up to horizon or bankruptcy, of traders a (blue) and b (red) against trader a's expected return (upper-left, $0\% \leq \mu_a \leq 60\%$), volatility (upper-right, $0\% < \sigma_a \leq 100\%$), risk-aversion (bottom-left, $0.1 < \gamma_a < 0.9$), and average horizon (bottom-right, $0 < 1/\lambda \leq 20$). Results obtained from simulation of 10^4 paths, each with step size $5 \cdot 10^{-4}$. Other parameters are $\mu_a = \mu_b = 10\%$, $\sigma_a = \sigma_b = 20\%$, $\gamma_a = \gamma_b = 0.5$, $w_a = w_b = 0.5$, $\lambda = 1/3$, $\kappa = 10\%$.

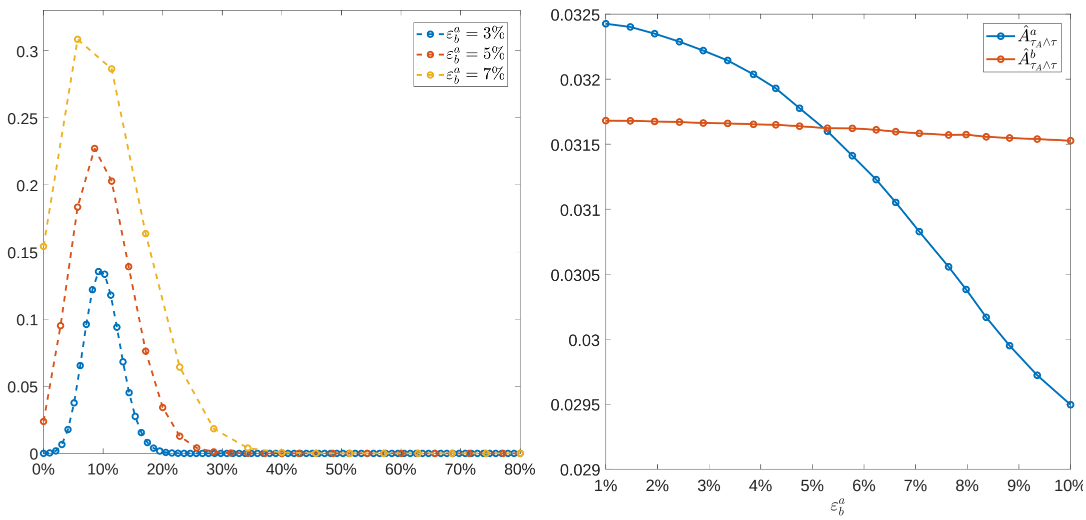


Fig. 4.3: Left-panel: Probability mass function of trader a's estimator $\hat{\mu}_b^a$ with mean 10% and standard deviation ε_b^a (3%, 5% and 7%). Right-panel: Equilibrium average fraud (vertical axis) with estimated drifts, up to horizon or bankruptcy, of traders a (blue) and b (red) against trader a's estimation error ($1\% \leq \varepsilon_b^a \leq 10\%$). Results obtained from simulation of 10^4 paths, each with step size $5 \cdot 10^{-4}$. Other parameters are $\mu_a = \mu_b = 10\%$, $\sigma_a = \sigma_b = 20\%$, $\gamma_a = \gamma_b = 0.5$, $w_a = w_b = 0.5$, $\lambda = 1/3$, $\kappa = 10\%$ and $\hat{\mu}_a^b$ is with mean 10% and standard deviation $\varepsilon_a^b = 5\%$.

The situation is more delicate for the skill μ_j. As Theorem 3.5.1 proves that a rational trader cheats only when the respective wealth share drops below some boundary[1], the cumulative return of the opponent satisfies

$$\frac{dY_t^j}{Y_t^j} = \mu_j dt + \sigma_j dB_t^j + dU_t, \quad Y_0^j > 0,$$

where the continuous, non-decreasing process U, which reflects the contribution of fraud to returns, increases only on the set $\{(t,\omega) : r_j(Y_t^x(\omega)) = w_j\}$, where w_j is the fraud threshold. Thus, the opponent's return includes the contributions of both skill and fraud, but the latter can be removed by excluding the returns that take place near the minimum of r_j. In practice, if the discrete-time observations are $(Y_{t_k}^j)_{0 \leq k \leq n}$, the trader calculates the minimum $\underline{r} = \min_{1 \leq k \leq n} r_j(Y_{t_{k-1}}^x)$, and then estimates the opponent's skill μ_j from the returns

$$\hat{\mu}_j = \frac{1}{m} \sum_{r_j(Y_{t_{k-1}}^x) > \underline{r} + \varepsilon} \left(\frac{Y_{t_k}^j}{Y_{t_{k-1}}^j} - 1 \right) \qquad \text{where } m = \#\{1 \leq k < n : r_j(Y_{t_{k-1}}^x) > \underline{r} + \varepsilon\} \quad (4.2.1)$$

[1] And spending approximately zero time at such boundary.

and the parameter ε is chosen so that the probability that $r_j(Y_t^x)$ reaches $\underline{r}$ between $r_j(Y_{t_{k-1}}^x)$ and $r_j(Y_{t_k}^x)$ is negligible, hence the estimator of μ_j is approximately unbiased.[2]

The large-sample distribution of $\hat{\mu}_j$ is close to normal, but the the trader recognizes that the exact normal distribution is ill-suited to estimate the skill μ_j, which is assumed to be positive and to satisfy Assumption 3.1.1. Instead, a viable alternative distribution that is close to normal while preserving positivity is the binomial distribution, so that trader i can posit that

$$\mu_j \sim \mathrm{Bi}(n_j, p_j),$$

where the parameters n_j and p_j are identified by the first two moments $n_j p_j = \hat{\mu}_j$ and $n_j p_j(1 - p_j) = \hat{v}_j$, where $\hat{v}_j$ is the variance associated to the opponent's skill.[3] Then, the trader can choose a personal cheating threshold that maximizes expected utility for an uncertain opponent's skill with prescribed distribution.

Figure 4.3 helps to understand the impact of uncertainty on the opponent's skill on fraud: the left panel displays the dependence of the probability mass function of the drift estimator, while the right panel plots the average amount of fraud of each trader on the estimation error, holding the opponent's estimator of the trader's drift constant with mean 10% and error 5%. As trader's estimation error of the opponent's skill increases from 1% to 10% (horizontal axis), fraud reduces significantly (approximately 10% with the chosen parameters), while the opponent's behaviour remains nearly constant.

This is because, as the left-panel suggests, the 'wrong' estimates are more likely to be over-estimations; and as the top-left panel of figure 4.1 suggests, overestimation of the opponent's investment skill leads to a significant pull-back of a trader's cheating threshold. The positive correlation between drift-overestimation and fraud-reduction reveals that: a trader's fraud - while exposing the firm to bankruptcy risk - nevertheless can reduce the other trader's fraud by misguiding the other trader to over-estimate one's investment skill.

[2] The probability that a Itô process with diffusion coefficient σ moves from $x > y$ to $z > y$ in Δt time without reaching y is approximately $e^{-2(x-y)(z-y)/(\sigma^2 \Delta t)}$ (Borodin and Salminen, 2002, 1.2.8 p. 252). Thus, choosing $x - y, z - y \approx 2\sigma\sqrt{\Delta t}$, for daily observations such a probability is about $e^{-8} \approx 0.03\%$, corresponding to a frequency of less than one day in ten years ($0.03\% \cdot 252 \cdot 10 \approx 0.8$). Hence, a reasonable choice for ε is two standard deviations of the daily change in wealth share.

[3] A frequentist trader who estimates variance only from returns would choose $\hat{v}_j$ to be their sample variance, i.e., $\frac{1}{m-1} \sum_{r_j(Y_{t_{k-1}}) > \underline{r}+\varepsilon} \left(Y_{t_k}^j / Y_{t_{k-1}}^j - 1 - \hat{\mu}^j \right)^2$. A Bayesian trader may use different estimators for $\hat{\mu}_j$ and $\hat{v}_j$, depending on the relative weight of the prior on the opponent's skill.

4.3 Supplements

In this section, we present the numerical method used in section 4.1 and 4.2 to estimate

$$\mathbb{E}\left[A^{a,\star}_{\tau_A\wedge\tau}\right] \quad \text{and} \quad \mathbb{E}\left[A^{b,\star}_{\tau_A\wedge\tau}\right] \tag{4.3.1}$$

where $A^{a,\star}$ and $A^{b,\star}$ are equilibrium fraud processes in Theorem (3.5.1).

First we simulate the equilibrium state fractional wealth process $(W^{i,\star}_t)_{t\geq 0}$ by an approximation scheme. Recall that, by Lemma 3.3.1, the dynamics of trader i's fractional wealth solves the SDE:

$$W^{i,w_i}_t = w_i + \int_0^t \bar{b}_i(W^{i,w_i}_s)ds + \int_0^t \bar{\sigma}_i(W^{i,w_i}_s)dB_s + \int_{[0,t]}(1-W^{i,w_i}_{s-})d\tilde{Q}^i_s - \int_{[0,t]} W^{i,w_i}_{s-}d\tilde{Q}^j_s. \tag{4.3.2}$$

with $w_i = r_i(x)$ for any $x \in \mathbb{R}^2_+$, $\bar{b}$ is given by (3.3.1) and $\bar{\sigma}$ is given by (3.3.2). Let $P^i_t = \int_{[0,t]}(1-W^{i,w_i}_{s-})d\tilde{Q}^i_s$ and $P^j_t = \int_{[0,t]} W^{i,w_i}_{s-}d\tilde{Q}^j_s$ for all $t \geq 0$.

In contrast to simulate SDEs where both strong and weak convergence schemes are available under mild conditions on the coefficients, similar results are not easy to obtain for reflected SDE where the main difficulty owes to approximate the local time spent on the reflecting boundary. For one-sided reflection, Lepingle, 1995, Theorem 2 demonstrates a strong convergence of order 1/2 by the Euler-Peano scheme and a weaker rate of convergence is shown in the penalization scheme and the Euler scheme (see Liu, 1995 and Słomiński, 1994, respectively). Here, we adopt the numerical scheme for two-sided reflection proposed by Lepingle, 1995 where a weaker result is proven.

Step 1. Fix $h > 0$, let $l_i, l_j \in (0,1)$ be such that $m_i < l_i < 1 - l_j < 1 - m_j$ and let $N^h = \lfloor \frac{\tau}{h} \rfloor$ be the random total number of steps which ends at the largest integer just below τ/h. For $k \in \{0, 1, \ldots, N^h - 1\}$, discretize (4.3.2) as follow

$$\begin{aligned}
W^{i,w_i,h}_0 &= \mathbb{1}_{\{w_i<m_i\}}m_i + \mathbb{1}_{\{m_i\leq w_i\leq 1-m_j\}}w_i + \mathbb{1}_{\{w_i>1-m_j\}}(1-m_j),\\
W^{i,w_i,h}_{k+1} &= m_i \vee \Big((1-m_j) \wedge \Big(W^{i,w_i,h}_k + \bar{b}_i(W^{i,w_i,h}_k)h + \bar{\sigma}_i(W^{i,w_i,h}_k)(B_{(k+1)h} - B_{kh})\\
&\quad + \mathbb{1}_{\{W^{i,w_i,h}_k<l_i\}}[H^{i,h}_{k+1} - (W^{i,w_i,h}_k - m_i)]^+\\
&\quad - \mathbb{1}_{\{W^{i,w_i,h}_k>1-l_j\}}[H^{j,h}_{k+1} + (W^{i,w_i,h}_k - (1-m_j))]^+\Big)\Big)
\end{aligned}$$

where

$$H^{i,h}_{k+1} = \sup_{kh \le s \le (k+1)h} \left\{ -\bar{b}_i(W^{i,w_i,h}_k)(s-kh) - \bar{\sigma}_i(W^{i,w_i,h}_k)(B_s - B_{kh}) \right\},$$
$$H^{j,h}_{k+1} = \sup_{kh \le s \le (k+1)h} \left\{ \bar{b}_i(W^{i,w_i,h}_k)(s-kh) + \bar{\sigma}_i(W^{i,w_i,h}_k)(B_s - B_{kh}) \right\}.$$

Since $\bar{b}_i$ and $\bar{\sigma}_i$ are Lipschitz, Lepingle (1995, Theorem 3) ensures that there exists a constant $C > 0$ (which does not depend on N^h) such that with probability 1 (as N^h is random),

$$\sup_{k\in\{1,\dots,N^h\}} \left(\mathbb{E}\left[\left| W^{i,w_i}_{kh} - W^{i,w_i,h}_k \right|^2 \right] \right)^{1/2} \le Ch^{1/2}.$$

For any $k \in \{1,\dots,N^h\}$, let $G^h_k = (G^{i,h}_k, G^{j,h}_k)$ be a a sequence of i.i.d two-dimensional Gaussian centered random vectors with covariance matrix $hI_{2\times 2}$, and let E_k be a sequence of i.i.d exponential random variables with parameter $\frac{1}{2h}$. Next, define for any $k \in \{0,1,\dots,N^h-1\}$

$$\begin{aligned}
\hat{G}^{i,h}_{k+1} &= -\bar{\sigma}_i(W^{i,w_i,h}_k)G^h_{k+1} - \bar{b}_i(W^{i,w_i,h}_k)h \\
&\quad + \left(|\bar{\sigma}_i(W^{i,w_i,h}_k)|^2 E_{k+1} + (\bar{\sigma}_i(W^{i,w_i,h}_k)G^h_{k+1} + \bar{b}_i(W^{i,w_i,h}_k)h)^2 \right)^{1/2}, \\
\hat{G}^{j,h}_{k+1} &= \bar{\sigma}_i(W^{i,w_i,h}_k)G^h_{k+1} + \bar{b}_i(W^{i,w_i,h}_k)h \\
&\quad + \left(|\bar{\sigma}_i(W^{i,w_i,h}_k)|^2 E_{k+1} + (\bar{\sigma}_i(W^{i,w_i,h}_k)G^h_{k+1} + \bar{b}_i(W^{i,w_i,h}_k)h)^2 \right)^{1/2}.
\end{aligned}$$

Then Lepingle (1995, Theorem 1) implies that conditioned on $W^{i,w_i,h}_k$,

$$(B_{(k+1)h} - B_{kh}, H^{i,h}_{k+1}) \text{ equals } (G^h_{k+1}, \hat{G}^{i,h}_{k+1}/2) \text{ in distribution;}$$
$$(B_{(k+1)h} - B_{kh}, H^{j,h}_{k+1}) \text{ equals } (G^h_{k+1}, \hat{G}^{j,h}_{k+1}/2) \text{ in distribution.}$$

In summary, the steps $W^{i,w_i,h}_k$ is simulated recursively - given a step $W^{i,w_i,h}_k$, the next step $W^{i,w_i,h}_{k+1}$ is obtained by simulating $(G^h_{k+1}, \hat{G}^{i,h}_{k+1}/2)$.

Step 2. Now we approximate $P^{i,c,\star}$ and $P^{j,c,\star}$ by extracting the reflections from the obtained $W^{i,w_i,h}$. For any $k \in \{0,1,\dots,N^h-1\}$, since simultaneous reflections by both upper and lower controls are forbidden (in other words, at most only one reflection can occur at each step), let

$$P^{i,c,h}_{k+1} = \sum_{0\le m\le k+1} \left[W^{i,w_i,h}_{k+1} - W^{i,w_i,h}_k - \bar{b}_i(W^{i,w_i,h}_k)h - \bar{\sigma}_i(W^{i,w_i,h}_k)G^h_{k+1} \right]^+,$$
$$P^{j,c,h}_{k+1} = -\sum_{0\le m\le k+1} \left[W^{i,w_i,h}_{k+1} - W^{i,w_i,h}_k - \bar{b}_i(W^{i,w_i,h}_k)h - \bar{\sigma}_i(W^{i,w_i,h}_k)G^h_{k+1} \right]^-,$$

which account for the intervention excluding that from the drift and the volatility. As a result, for any $k \in \{0, \dots, N^h - 1\}$

$$W_{k+1}^{i,w_i,h} = W_k^{i,w_i,h} + \bar{b}_i(W_k^{i,w_i,h})h + \bar{\sigma}_i(W_k^{i,w_i,h})G_{k+1}^h + P_{k+1}^{i,c,h} - P_{k+1}^{j,c,h}.$$

Then by Lemma 3.3.3, we approximate the cheating processes by

$$A_{k+1}^{i,h} = \left[\ln\left(\frac{1-w_i}{1-m_i}\right)\right]^+ + \frac{P_{k+1}^{i,c,h}}{1-m_i},$$
$$A_{k+1}^{j,h} = \left[\ln\left(\frac{w_i}{1-m_j}\right)\right]^+ + \frac{P_{k+1}^{j,c,h}}{1-m_j}.$$

Step 3. For Monte Carlo simulation, fix sample size $M \in \mathbb{N}$ and let τ_m and θ_m for $m \in \{1, \dots, M\}$ be sequences of i.i.d. random variables having the same distribution as τ and θ, respectively. Then we have M number of total steps $N_m^h = \lfloor \frac{\tau_m}{h} \rfloor$. Let

$$\tau_{A,m}^h = \inf\left\{0 \le k \le N_m^h : \sum_{k \ge 0} A_k^{S,h} \ge \theta_m\right\}$$

be the approximated bankruptcy time, where $A_k^{S,h} = A_k^{i,h} + A_k^{j,h}$ for all $0 \le k \le N_m^h$. Finally, we estimate (4.3.1) by

$$\hat{A}_{\tau_A \wedge \tau}^{i,h} = \frac{1}{M}\sum_{m=1}^{M} A_{\tau_{A,m}^h \wedge \tau_m^h}^{i,h}.$$

We point out two main sources of errors that affect the precision of this approximation: 1. The statistical error, i.e. the difference between $\hat{A}_{\tau_A \wedge \tau}^{i,h}$ and $\mathbb{E}\left[A_{\tau_A^h \wedge \tau^h}^{i,h}\right]$; 2. The discretization error, i.e. the difference between $\mathbb{E}\left[A_{\tau_A^h \wedge \tau^h}^{i,h}\right]$ and $\mathbb{E}\left[A_{\tau_A \wedge \tau}^{i}\right]$.

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