An Investigation into the Capacities of Pre-service Post-primary Mathematics Teachers to Effectively Teach Problem-solving

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Declaration:

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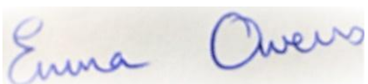
Signed:  (Candidate) ID No.: 12301346 Date: 01/08/2023

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Table of Abbreviations

Abbreviation	Meaning
PSMT	Pre-service mathematics teachers
DCU	Dublin City University
ITE	Initial Teacher Education
MKT	Mathematical knowledge for teaching
CK	Content knowledge
PK	Pedagogical knowledge
NCCA	National Council for Curriculum and Assessment
PDST	Professional Development Service for Teachers
CCSS	Common Core State Standards
PSCK	Problem-solving content knowledge
PPSK	Pedagogical problem-solving knowledge
IMB	Indiana Mathematics Belief
MPSR	Mathematical Problem Solving Rubric
NCTM	National Council of Teachers of Mathematics
DES	Department of Education
SEC	State Examinations Commission

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ABSTRACT

Title: An investigation into the capacities of pre-service post-primary mathematics teachers to effectively teach problem-solving

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Problem-solving has always been a part of mathematics, but the formal study of problem-solving has a short history. There is widespread agreement that the development of students' problem-solving capabilities is a main goal of mathematics instruction with emphasis on problem-solving in curricula nationally and internationally. In Ireland, problem-solving is specifically mentioned in post-primary curricula. However, according to the PISA and TIMSS results, it appears that students in Ireland have a lower performance in translating real-world situations into mathematical representations than in applying procedures. Since teachers play a key role in students' problem-solving, the aim of this research was to investigate and develop the capacities of pre-service post-primary mathematics teachers (PSMTs) to effectively teach mathematical problem-solving. These capacities involve: knowledge of problems, knowledge of problem-solving, knowledge of problem-posing, and affective factors and beliefs. This research focuses on designing a university module to investigate and develop these capacities in PSMTs in Ireland. Based on these capacities, specific instruments were developed for the intervention, namely: mathematical task classification instruments and rubric, mathematical problem generation and reformulation instruments, implementation of taught strategies rubric, and open-ended affective questions. These instruments were implemented alongside pre-existing instruments: the Indiana Mathematics Belief scale (Kloosterman & Stage, 1992), a mathematical problem-solving proficiency rubric (Oregon, 2010), and 'Think Aloud' interviews. This intervention was developed and implemented with four cohorts of PSMTs over four years. Our findings include that PSMTs: demonstrate adequate ability to communicate reasoning and use representations while problem-solving but have difficulty in reflecting on their solution; have difficulty in posing mathematical problems; and text based tasks are often misclassified as mathematical problems. In relation to the affective domain, the beliefs stated by the PSMTs about problem-solving, in some instances, contradicted the affective factors expressed while problem-solving. These findings have led to refinements of the module mentioned above, and to suggestions for further developments.

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CHAPTER 1: INTRODUCTION

1.1 Background

Problem-solving has a long standing history of being at the heart of mathematics (Cockcroft, 1982), but the formal study of problem-solving has a shorter history being brought to the forefront by the work of Polya (1945). Within the study of problem-solving, there are disparities between the definitions offered by different researchers for the terms *problems* and *problem-solving* (Schoenfeld, 1992). Despite this confusion, there are characteristics which are common to the definitions offered by researchers who are held in high regard. As we will discuss in detail in Section 2.1 below, this leads us to describe mathematical problem-solving as mathematical activity with the following three key characteristics:

- Problem-solving includes a goal;
- It is not immediately clear to the problem-solver how to achieve the goal;
- The problem-solver must organise prior knowledge to generate reasoning towards achieving the goal.

Problem-solving is essential in a real-life context, in terms of economic competitiveness, as problem-solving is a mathematical skill which is fundamental to STEM disciplines (STEM Education Review Group, 2016). Similarly, Gainsburg and English (2016) outline that problem-solving skills are imperative in a wide range of areas of employment. Given the important role that problem-solving holds in real-life situations, it is no surprise that problem-solving features prominently in mathematics education curricula both nationally and internationally (Cheng, 2001; DfE, 2013; NCTM, 2000; Pehkonen, 2007) .

There are multiple factors which contribute to successful problem-solving (Lesh & Zawokewski, 2007). Among these factors are: mathematical content knowledge (Lester & Kehle, 2003), problem-solving strategies, and affect (Lesh & Zawokewski, 2007). There is an interconnected nature in these factors, as outlined by Schoenfeld (1992), who acknowledges that heuristics can support problem-solvers in unfamiliar situations, but are not a replacement for mathematical content knowledge. However, heuristics are viewed as beneficial as they are not limited to specific content knowledge (Carson, 2007). The impact of the affective domain on problem-solving behaviours is widely acknowledged (Lester & Kroll, 1993; Mason et al., 2011; Schoenfeld, 1983). These influences can be both positive and negative depending on the beliefs or the emotions expressed by the problem-solver. If a problem-solver has a negative attitude towards mathematics, then the cognitive resources available to the person can be impeded (Schoenfeld, 1983). Similarly, negative feelings experienced while problem-solving can result in the problem-solver making no further progress (Mason et al., 2011; McLeod, 1988). However, in relation to positive attitudes, which are evidence of a growth mindset (Boaler, 2016), there is an increased likelihood of persistence when faced with difficulties during a problem-solving attempt (Dweck, 2008). To gain control over the consequences of both positive and negative feelings, it is vital for problem-solvers to become aware of their emotions (McLeod, 1988). As previously mentioned, these factors are not independent of each other, much like the interdependence of the components required for mathematical proficiency as outlined by Kilpatrick et al., (2001).

Taking into consideration that these factors impact students' problem-solving behaviours, it is important to also recognise the role of the teacher. The role of the teacher should not be underestimated, "teachers are among the most powerful influences in learning" (Hattie, 2012, p. 18). Teachers decide how problem-solving is incorporated into their teaching of mathematics, despite it being a key element of the mathematics curriculum (Lester, 2013). Not only does the teacher choose the approach which they undertake to teach problem-solving, but the teacher's beliefs about problem-solving have a direct impact on their students (Marcou & Philippou, 2005). It is essential for mathematics teachers to be experienced in problem-solving, and have a comprehensive understanding of what successful problem-solving involves (Lester, 2013). Teachers have the responsibility of choosing the mathematical tasks which their students have exposure to. A teacher's ability to generate new problems and reformulate previous tasks is influential in positively enhancing students' problem-solving ability (Chapman, 2015). When selecting which problems to use, careful consideration must be given to the learning that students will engage with and also the misconceptions that may occur (Lubienski, 1999). Teachers must account for the prior knowledge of their students when selecting problems, in order to build upon their understanding of mathematical concepts (Depaepe et al., 2013) and make connections (when appropriate) between different topics (Zbiek et al., 2010). It is important for teachers to recognise that problems allow students to use various strategies and thus the teacher must be competent in perceiving the implications of the different strategies and assessing the validity of the approach (Chapman, 2009).

Continuing with the role of the teacher in students' problem-solving, the teacher's opinions about the theory of learning are of significance. *Constructivism* is viewed positively in mathematics education (Lerman, 1989) as the learner is placed at the centre of the learning, where there is active engagement in the building of knowledge (Pritchard, 2009). Within a constructivist classroom, the teacher assumes a facilitator role, where they seek to guide students through their learning (NCCA, 2005). It is the aim for students to maintain autonomy over their problem-solving strategies (Carpenter et al., 1999) and through collaboration with peers, share ideas (Chapman, 1999, p. 199). A constructivist approach is favoured and promoted within an Irish mathematics education context (NCCA, 2005, 2013).

1.2 Research Context

1.2.1 Research Problem

Problem-solving holds a key position in mathematics curricula in Ireland. Post-primary education in Ireland is split into two cycles: Junior Cycle is the first three years and Senior Cycle is the last two years of study with an optional year after the Junior Certificate examinations. At Junior Cycle level, problem-solving is one of the six elements that is evident in each of the strands of the mathematics syllabus. Similarly, at Senior Cycle level, problem-solving is prominent in each of the strands of the mathematics syllabus. While problem-solving holds such a distinguished position in the curricula, it has been reported that post-primary Irish students are falling short in problem-solving. In the PISA (2012) report, it was shown that Irish students were above the OECD average in applying mathematical concepts but were less capable of mathematising real-world problems. The TIMSS (2015) report showed that Irish students were most proficient in procedural memorised tasks whereas they had a low average in tasks classified as requiring problem-solving skills. In the PISA (2018) report, it was seen that there was no significant change in the mean scores of Irish students compared to the scores in 2012 showing overall similar trends (McKeown et al., 2019).While problem-solving has a prominent role in the post-primary curricula, the teaching of problem-solving is regarded as complex and challenging (Zimmermann, 2016).The teacher is responsible for creating the situation, proposing the problem, and asking questions to create the opportunity for students to learn (Felmer & Perdomo-Díaz, 2016, p. 289). Initial teacher education programmes (ITE) are acknowledged by the Teaching Council of Ireland (2017) as an important stage in teachers' development. The main goal of ITE must be to develop the teaching potential of PSMTs which will ultimately lead to improvement in the learning of mathematics in schools (Lerman et al., 2009). The primary concern of ITE is mathematical knowledge (Liljedahl et al., 2009) however learning to teach mathematics involves a combination of both theoretical and practical knowledge and skills (Lerman, 2009) . Within ITE programmes, it is recommended for pre-service teachers to experience solving and posing problems (Zimmermann, 2016). This is echoed by Felmer and Perdomo-Diaz (2016) who state that it is essential for teachers to experience struggling with a problem to understand how their students may feel while problem-solving. It is important to recognise that PSMTs do not enter ITE programmes without any pre-existing experiences and beliefs about teaching mathematics knowledge (Liljedahl et al., 2009). It is reported that these beliefs are often contradictory to what research describes as good practice meaning that it is incredibly important for ITE programmes to reshape these beliefs to promote effective mathematics teaching knowledge (Liljedahl et al., 2009).

There has been extensive research conducted regarding students' problem-solving and the pedagogical approaches but there has not been a focus on teachers as problem-solvers (Felmer & Perdomo-Díaz, 2016, p. 289). Similar to how general mathematical ability is not enough for effective mathematics teaching (Ball et al., 2008), problem-solving ability does not encapsulate effective teaching of problem-solving (Chapman, 2015). Effective teaching of mathematical problem-solving is challenging (Lubienski, 1999), and it requires an understanding of the complex network of interconnected knowledge (Chapman, 2015).

1.2.2 Research Aim and Objectives

Based on a review of the literature, this research aimed to investigate and develop the capacities of pre-service post-primary mathematics teachers to effectively teach problem-solving. The development of a university module which focused on mathematical problem-solving was at the centre of this study. Participants undertaking this module as part of their university programme would graduate as qualified teachers of post-primary mathematics in Ireland. The purpose of this module was the development of the pre-service teachers' capacities to teach problem-solving.

These capacities included: understanding the nature of problems, problem-solving proficiency, problem-solving posing and the affective domain. The facilitation of this module was conducted in a constructivist manner.

The following objectives were identified based on the aims of the study:

- Conduct a review of the literature to gain an understanding of the capacities required to teach problem-solving and inform the intervention design .
- Develop a theoretical framework to inform the design of the university module.
- Implement a university module for pre-service post-primary mathematics teachers to focus on the development of problem-solving proficiency, understanding of the nature of problems, problem posing, and the affective domain.
- Investigate the capacities of pre-service post-primary mathematics teachers' problem-solving proficiency, understanding of the nature of problems, problem posing, and the affective domain.
- Collect quantitative and qualitative data to investigate the effectiveness of different aspects of the module, and adapt the module based on these data.

1.3 Research Questions

As outlined above, conducting a thorough literature review was the first key objective of the study, focusing on the research problem presented in Section 1.2. From this review, research questions were developed to directly address the research problem. These research questions informed all aspects of the research conducted in this study. As described above, it is clear that problem-solving is a central component of the post-primary mathematics curricula in Ireland. For students to develop their problem-solving skills and be considered 'good' problem-solvers (Lester & Kehle, 2003), the teacher plays an integral role (Lester, 2013). While the curricula display the learning objective of improving students' problem-solving skills, it is the responsibility of the teacher to hold the capacities required to effectively teach mathematical problem-solving. Considering the desired capacities which are provided by Chapman (2015), this research aims to provide an insight into initial teacher education (ITE) programmes which strive to support and develop prospective teachers' abilities to effectively teach problem-solving.

Through the consideration of the capacities which are deemed to be essential (Chapman, 2015), the main research questions were formulated. The research questions specifically focused on the preparation for pre-service mathematics teachers (PSMTs) within a university setting: Dublin City University (see Section 4.7).

The research questions specifically focused on the preparation for pre-service mathematics teachers (PSMTs) within a university setting: Dublin City University (see Section 4.7). In particular, the research questions relate to the capacities held by teachers, and a module for which the learning outcomes address the development of these capacities.. ITE programmes are viewed as a stage of utmost importance in teachers' development (Teaching Council of Ireland, 2017). During these programmes, there is the opportunity for prospective teachers to be challenged on their beliefs about teaching and learning that they will ultimately bring into their professional practice (Teaching Council of Ireland, 2017). This study resides in the body of work which focuses on developing teachers' knowledge in a university setting (see Section 2.5.1). The capacities that are focused on in this study do not include capacities which are specific to classroom settings that are, therefore, beyond the scope of the study.

This study sought to address the following research questions:

Question 1: What do pre-service teachers understand a mathematical problem to be?

A. Are PSMTs proficient at classifying mathematical tasks?
B. Does an adaptation of the intervention that focuses on providing a rationale for task-classification lead to enhancement of PSMTs' capacities in task-classification?

Question 2: Are pre-service teachers proficient in problem-solving?

A: How does PSMTs' problem-solving proficiency change over the duration of the intervention?

B: Did the ongoing adaptations of the intervention lead to a greater enhancement of the problem-solving capacities of successive cohorts of PSMTs?

C: Are taught strategies implemented while problem-solving throughout the different iterations of the intervention?

Question 3: What are pre-service teachers' capacities in relation to problem posing?

A: How do pre-service teachers' capacities in relation to problem posing change over the duration of the intervention?

B: Did the ongoing adaptations of the intervention lead to a greater enhancement of the problem-posing capacities of successive cohorts of PSMTs?

Question 4: What beliefs and affective factors do pre-service teachers hold regarding problem-solving?

A: How does the affective domain of one cohort of PSMTs change over the course of the final iteration of the intervention?

1.4 Research Methods

The methodology used in this study is a mixed methods approach, as there is a combination of both qualitative and quantitative methods in the collection of data (Creswell, 2012). The use of multiple methods produces a broader understanding of the research problem at hand than a single method would produce (Creswell, 2012). Combining methods allows the researcher to triangulate the findings, meaning that confidence in the validity of the research is increased (Cohen et al., 2007). The module was evaluated using the four elements of Shapiro's model (Shapiro, 1987). The methodology of this study is described in detail in Chapter 4 with the evaluation described in Chapter 8.

1.5 Research Phases

This study consisted of five phases which are presented in Figure 1. Each phase informed the next phase of the study and was influenced by the literature review in Phase 1. The research phases are further explained in Section 4.6, the relevant literature is detailed in Chapter 2 with an outline of the methodology in Chapter 3.

The first three objectives were addressed in Phase 1 of the study. This involved the literature review, the development of a theoretical framework, and an assessment of how the module addressed the different capacities as outlined in the theoretical framework (Chapman, 2015). Phase 1 involved establishing the first version of the module which was an object of the study. . The other two objectives of the study were addressed in the other phases of the study which involved the implementation of the module with four cohorts of PSMTs.

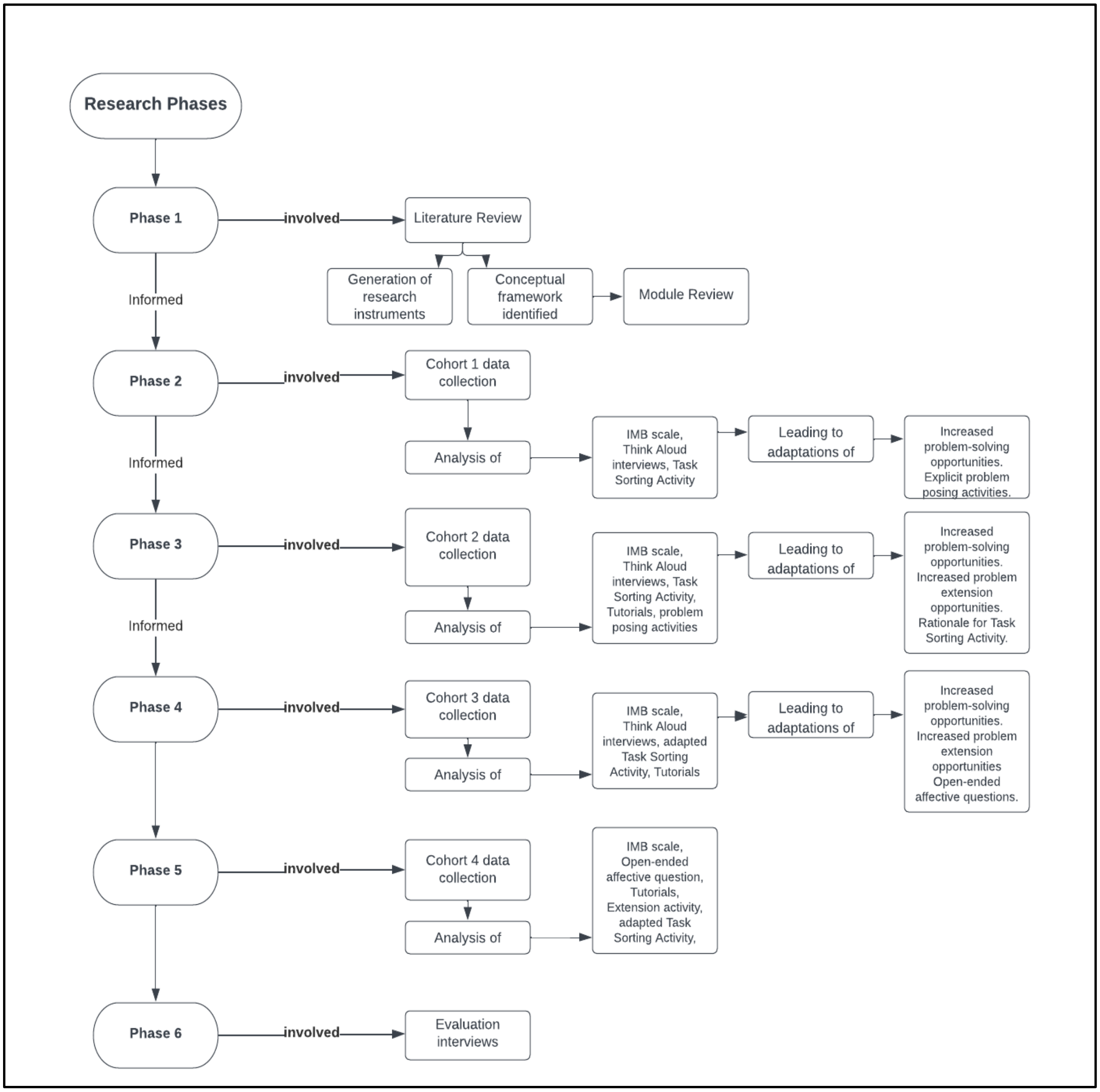

Figure 1: Research Phases

Each phase, shown above, was informed by the relevant literature. Table 1 below presents the literature that was most relevant to each phase of the research.

Phase		Most Prominent Literature
1	Literature Review	(Chamberlin, 2008; Chapman, 2015; Kilpatrick et al., 2001; Lester, 2013; Lester & Kehle, 2003; J. Mason et al., 2011; McLeod, 1988; Polya, 1945; A. Schoenfeld, 1992; Shiel & Kelleher, 2017; Teaching Council of Ireland, 2017; Vygotsky & Cole, 1978)
2 -5	Data Collection	(Chapman, 2015; Felmer & Perdomo-Díaz, 2016; Kloosterman & Stage, 1992; J. Mason et al., 2011; Oregon, 2011; Silber & Cai, 2017; E. Silver, 1995)
	Data Analysis	(Cohen et al., 2007, 2011; J. W. Creswell, 2012; Docktor et al., 2016; Kloosterman & Stage, 1992; Oregon, 2011; Thomas, 2006)
6	Intervention Evaluation	(O'Meara, 2011; Prendergast, 2011; Shapiro, 1987)

Table 1: Most relevant literature of each phase of research

1.6 Research Contributions

This study focuses on studying the capacities of PSMTs for teaching problem solving in the context of a university module, and developing this module to better develop these capacities using an Action Research approach. Research instruments were designed and developed to assess the capacities of PSMTs to effectively teach problem-solving. The iterations of this study involved the adaptation and generation of a university module which develops the capacities of PSMTs to effectively teach problem-solving.

The first element of the research involved investigating the PSMTs' capacities in relation to identifying and constructing mathematical problems. The PSMTs' shortcomings in relation to this capacity were identified and conclusions were drawn from the analysis of the activities. The next element of the research focused on the problem-solving capacity of the PSMTs. Triangulation of methods were utilised to investigate the PSTMs' problem-solving proficiency. Alongside this, a rubric was developed to assess the level of implementation of the Rubric Writing approach. The final element of the research involved the triangulation of methods to explore the affective domain of the PSMTs. The beliefs and the affective factors of the PSMTs were investigated and the level of understanding of the affective domain was increased through the use of triangulation. The conclusions drawn from each of the research elements contribute to the field of mathematics education research. The research contributions are described in detail in Section 8.3.

1.7 Overview of the Thesis

This research comprises eight chapters including this introductory chapter. The outline of the chapters will now be presented.

Chapter 2 gives a review of the literature on problem-solving in mathematics education. This includes the definition of mathematical problem-solving and the role of mathematical problem-solving in school curricula at both national and international levels. The characteristics of 'good' problem solvers are discussed along with the factors affecting students as problem solvers. The role of the teacher in teaching mathematical problem-solving and the preparation of mathematics teachers in general and for teaching mathematical problem-solving are discussed. The role of problem-solving outside the classroom is also analysed. The final section of the chapter is an overview of constructivism.

Chapter 3 gives an overview of frameworks that have been developed around the capacities required to effectively teach mathematical problem-solving. The framework utilised in this study, is described in detail (Chapman, 2015).

Chapter 4 provides an overview of the mixed-methods approach which was utilised in this study (Creswell, 2012). This includes a detailed description of the instruments used to address each of the research questions. The use of action research (Cohen et al., 2018) in the development of the university module is described. The methods of analysis are described in detail for each instrument. The participants of the study are described. The use of triangulation (Cohen et al., 2007) is discussed along with the presentation of limitations of the study.

Chapter 5 describes the university module that this study took place in. The learning activities undertaken by each cohort of participants are outlined, along with the overarching learning objectives of the module. The lecture content and instruments used in the module are described. Details of the adaptations made in each iteration of the module are outlined, including the lecture content and activities completed as part of the intervention.

Chapter 6 reports on the results of the data collection methods for the research questions. This chapter provides the results for the analysis of the data collection instruments which were outlined in Chapter 4.

Chapter 7 discusses the findings outlined in Chapter 6. The findings are discussed in the context of the research questions posed in Section 1.3.

Chapter 8 concludes the thesis by discussing the evaluation of the module, the contributions of this study to this field of research, and provides recommendations for future research arising from the findings.

CHAPTER 2: LITERATURE REVIEW

This chapter presents the literature that has been reviewed for this study. Focusing on the teaching of mathematical problem-solving, this study begins with describing the existing definitions of problem-solving and states the understanding of these definitions, based on this literature. Following on from this, the role of problem-solving in curricula both in Ireland and on an international level is discussed. This discussion gives an overview of the key role that problem-solving holds in education systems worldwide and shows why the development of teachers' teaching of problem-solving is critical. The characteristics of 'good' problem-solvers are outlined, followed by the factors affecting students as problem-solvers. It is important to identify these factors as teachers need to take these into consideration and strive towards characteristics of 'good' problem solvers in order to teach problem-solving effectively. Next, the role of the teacher is discussed with a focus on how the mathematics teachers are prepared in initial teacher education. The role of the teacher in the teaching of problem-solving is described in relation to the teachers' view of problem-solving, the selection of problems, problem posing, and teachers' understanding of what constitutes mathematical problem-solving. The influence of teachers' identity on how they teach is discussed and how identity can be formed and reformed. Given that this study focuses on the teaching of mathematical problem-solving, it is vital to outline the characteristics which influence the teaching of problem-solving. This study is based in a university setting, so it is important to highlight how teacher education programmes influence prospective teachers. Following on from the importance placed on problem-solving in curricula, the key role that problem-solving plays outside of a school setting is discussed. This highlights how problem-solving skills extend beyond the classroom and ultimately influence economies, showing the importance of effective teaching of problem-solving. While the earlier sections of this literature review revolve around the affecting factors on students' problem-solving, the role of the teacher and the preparation of mathematics teachers, the final section provides an overview of a theory of learning - constructivism. A constructivist approach was utilised in this study and is advocated as having a positive influence in the teaching of mathematical problem-solving.

2.1 What is mathematical problem-solving?

The central focus of this thesis is the development of capacities for teaching problem-solving. Thus, we begin our literature review by analysing the answers which researchers and practitioners have provided to the question of what precisely is meant by mathematical problem-solving. The study of mathematical problem-solving, and its role in mathematical education generally, has a long history. Problem-solving has always been a part of mathematics, but the formal study of problem-solving has a shorter history with a prominent role in the earliest stages taken by Polya (1945). Schoenfeld (1985, p. 69) states that Polya's work on problem-solving "is held in high regard by both mathematicians and mathematics educators". According to Stanic and Kilpatrick (1989), problems have held a fundamental role in the school mathematics curriculum, but problem-solving has not. Cockcroft (1982, p. 73) claims that "the ability to solve problems is at the heart of mathematics"

Acknowledging Polya's efforts to put problem-solving at the centre of mathematical instruction, Schoenfeld (1992) attests that there is a wide variety of meanings for the terms "problems" and "problem-solving": this has been highlighted more recently by Lester (2013). With the focus on developing problem solving, there is a lack of consensus regarding the actual definition of problem-solving (Stanic & Kilpatrick, 1989: Chamberlin, 2008). Problem-solving is defined as the ability to apply mathematical knowledge to various situations (Cockcroft, 1982). Schoenfeld (2013, p. 3) provides the following general definition for problem-solving; *"trying to achieve some outcome, when there was no known method (for the individual trying to achieve that outcome) to achieve it".* A more detailed definition of problem-solving is given by Lesh and Zawojewski (2007, p.760). They state that *"problem solving is defined as the process of interpreting a situation mathematically, which usually involves several iterative cycles of expressing, testing and revising mathematical interpretations - and of sorting out, integrating, modifying, revising, or refining clusters of mathematical concepts from various topics within and beyond mathematics"*. These above definitions are just three of the definitions that have been proposed for problem-solving. It is clear that there are both variations and also similarities amongst the definitions.

This variety was studied by Chamberlin (2008), who applied a Delphi technique protocol (Cohen et al., 2007, p.309) to attempt "to ascertain what mathematical problem-solving is in the primary and secondary mathematics classroom" (Chamberlin, 2008). Twenty participants (experts on mathematical problem-solving in the classroom) considered a list of 38 components of the processes and characteristics of problem-solving. This list was developed through the participants being asked what they considered to be the definition of problem-solving. This generated qualitative data which was then presented using a Likert Scale. After two rounds of consideration, in which participants rated the components as always, sometimes, rarely or never being a component of problem-solving, consensus among the group emerged on just 21 of these 38 statements. The participants agreed on the following as some of the characteristics of mathematical problem solving: 1) cognition was always evident; 2) problem-solving tasks cannot be solved straight away; 3) the problem-solver will seek a goal as they attempt the problem-solving task; and 4) a mathematical problem can be solved using more than one approach. However, it is interesting to note that, amongst the 17 statements that the participants *did not* agree on, were characteristics of problem solving that have been widely accepted as key characteristics by renowned researchers in mathematical problem solving. Consensus was not met by the participants in this study on: novelty in a problem; the idea that problem solving tasks involve non-routine, open-ended, or unique situations; and metacognition. These are just four of the statements that were rejected by the participants. The rejection of metacognition as a characteristic of problem-solving contradicts the previous research that promotes its inclusion as a key component of problem-solving (Schoenfeld, 1992; Lester et al., 1989). Schoenfeld (1992) describes metacognition as one's own knowledge about one's own cognitive processes. He highlights that metacognitive ability plays an essential part in problem-solving, and he notes that this is the structure that allows problem-solvers to dismantle more challenging problems into subtasks, prioritize and order the importance of each subtask and then complete each subtask in sequential order. Lester et al. (1989) suggest that improving metacognitive ability should be concurrent with the learning of mathematical content and is most effective when it takes place in the context of specific mathematical ideas.

Acknowledging the lack of clarity on the definition of problem-solving, but recognising the need for clear terms, this study highlights three contributions towards a characterization of mathematical problem-solving. First is Lester's observation (2013) that among the many different perspectives on problem-solving, there appears to be agreement that there must be a goal, a problem solver and the lack of a means of immediately attaining the goal. Second is the statement of the key learning outcomes associated with problem-solving as presented in the NCCA syllabus document:

"Students should be able to investigate patterns, formulate conjectures, and engage in tasks in which the solution is not immediately obvious, in familiar and unfamiliar contexts (NCCA, 2017, p. 10)."

Finally, we mention the characterization offered in Lester & Kehle (2003):

"Successful problem solving involves coordinating previous experiences, knowledge, familiar representations and patterns of inference, and intuition in an effort to generate new representations and related patterns of inference that resolve some tension or ambiguity (i.e., lack of meaningful representations and supporting inferential moves) that prompted the original problem-solving activity" (Lester & Kehle, 2003, p. 510).

Based on these three observations, we adopt the following perspective on problem-solving:

- Problem-solving includes a goal,
- it is not immediately clear to the problem-solver how to achieve the goal,
- the problem-solver must organise prior knowledge to generate reasoning towards achieving the goal.

These points are referred to as the Three Key Characteristics in the remainder of this report. The Three Key Characteristics are among those that Chamberlin (2008) reports to be consistent with the characteristics of problem-solving on which there was consensus.

From the Three Key Characteristics of problem-solving, the following definition of a *mathematical problem* is provided:

- There is a goal,
- It is not immediately clear how to achieve the goal,
- Prior knowledge is required to generate reasoning towards achieving the goal.

This definition aligns with the definition of a mathematical problem provided by Schoenfeld (1992, p.74): '*If one has access to a solution schema for a mathematical task, that task is an exercise and not a problem'*. This emphasises that a problem has a level of ambiguity on how to achieve the goal.

2.2 Mathematical problem-solving in school curricula

The following section describes the position of mathematical problem-solving in school curricula in Ireland and then at an international level.

2.2.1 Ireland

The lack of consensus described above must be set against the widespread acknowledgement of the importance of problem solving in the mathematics curriculum (Conway & Sloane, 2006). To take a local view, mathematical problem solving occupies a privileged position in the Irish post-primary mathematics syllabus. Problem solving is identified as one of the six elements of the Unifying Strand of the Junior Cycle syllabus (for students aged 12-15 years) that over-arches the four content strands of the syllabus. Likewise, mathematical problem solving is highlighted under the 'Being Numerate' heading of the Junior Cycle Key Skills, and constitutes one of the 24 'Statements of Learning' of the Junior Cycle (NCCA, 2017).

Similarly, problem solving is mentioned in each strand of the Senior Cycle of the Irish post-primary syllabus (for students aged 15-18 years). The Professional Development Service for Teachers (PDST) is a support service for teachers in Ireland (PDST, 2022). The PDST recognises the importance of problem-solving as a skill that needs to be developed and encourages teaching and learning through problem-solving and provides resources on problem-solving. While problem-solving is at the centre point of the Junior Cycle and Leaving Certificate curricula, evidence has emerged showing that there are shortcomings in Irish students' problem-solving abilities. The State Examinations Commission (SEC) is responsible for the development, assessment, accreditation and certification of post-primary education at both Junior Cycle and Leaving Certificate levels in Ireland. In a report conducted by the Chief Examiner (SEC, 2015) on Leaving Certificate mathematics, many observations were made in relation to problem solving. The Chief Examiner is responsible for providing analysis on performance (SEC, 2022). The role of the Chief Examiners' Report is to review the performance of candidates in state examinations and provides information on the standard of answers given in these exams. The report highlighted that many candidates did not succeed when they were expected to apply knowledge in an unfamiliar context. It was also noted that knowledge appeared to be compartmentalised, as candidates struggled with questions that incorporated knowledge from multiple strands of the curriculum (SEC, 2015). This may indicate a lack of proficiency in problem-solving, as problem-solving sometimes involves the ability to link knowledge from different mathematical topics (Kilpatrick *et al.*, 2001). There was also evidence that the candidates lacked initiative when faced with a challenging question that could be approached in numerous ways (SEC, 2015). The difficulties highlighted here are not particular to the one cohort of students, as a study conducted by Faulkner (2021) shows. In this study, it was found that undergraduate students from a range of programmes involving mathematics performed significantly better in questions that required procedural skills than problem-solving skills. Given that these students represent a wide variety of mathematical entry standards, and a range of disciplines, it is evident that problem-solving skills require further attention in post-primary education in Ireland.

In a report conducted by Shiel and Kelleher (2017), information from both the PISA 2012 and TIMSS 2015 reports regarding Irish students' problem-solving competencies was presented. The PISA (2012) report found that Irish students performed above the OECD average at the process of applying mathematical concepts and the process of interpreting solutions back to the original problem. However, the report highlighted that students in Ireland were less capable in the process of translating real-world problems into mathematical representations that are productive in solving problems relative to other problem-solving processes. This shows that the students had the most difficulty with mathematising problems and were more comfortable with applying procedures. The TIMSS test showed that Irish students demonstrated most proficiency in tasks that required recall of memorised facts, carrying out pre-learned procedures, and retrieval of information from mathematical representations such as tables or charts. It was also noted in the TIMSS test that Irish students' mean result was significantly lower in the *Applying* elements of the test. This assessed students' ability to apply mathematics to a variety of situations. Problem-solving is seen to be a key component of *Applying* (Shiel & Kelleher, 2017). The problems in this domain of the test were set both in real-life situations and in mathematical contexts. A low mean performance in this domain of the test shows that students had more difficulty with tasks that demanded the student to select a suitable strategy and implement it, than with tasks which called for the recall of memorised steps.

While problem-solving is at the core of the mathematics curricula in Ireland, there appear to be discrepancies between the aims of the inclusion of problem-solving and its incorporation in classrooms. Textbooks are used as a main resource by teachers of mathematics with O'Keeffe (2011) stating that textbooks are used by over 75 percent of teachers on a daily basis in classrooms.

Research has shown that Irish textbooks contain very few problems (O'Sullivan, 2017). On the analysis of over 7000 tasks from three popular senior cycle textbooks, O'Sullivan (2017) found that there is a need for more novelty in textbook tasks; communicating mathematically is neglected in the tasks; and he suggests that there is a need for an increase in the level of cognitive demand from the tasks. In a review of post-primary mathematics education conducted by the NCCA (2005), it was found that when compared to other countries, Irish textbooks contain more procedural tasks. The work of O'Sullivan (2017) shows that this is still the current situation. As a result of a reliance of teachers on these procedural textbooks, they state that a procedural approach to problem-solving is happening in classrooms. In 2008, a new syllabus called 'Project Maths' was implemented in 24 pilot post-primary schools in Ireland which was then implemented in phases in all schools from September 2010. This revised syllabus increased emphasis on problem-solving, with specific focus on the development of the skills of "explanation, justification, and communication" (SEC, 2015, p.3). In a report outlining the impact of Project Maths conducted by Jeffes et. al (2013), it was found that while there is some evidence of change in teaching practices, traditional approaches remain prominent. It is important to note that at the time of this report, Project Maths had been very recently introduced (Jeffes et al., 2013, p.4). This includes heavy reliance on textbooks where students practise what was learnt in class in the form of practising questions in class and/or as homework. Additionally, in a study which focused on investigating teachers' perceptions of Project Maths, five years after its introduction, it was found that while most teachers were supportive of the curriculum, the teachers expressed difficulty in implementing the curriculum (Johnson et al., 2019).

2.2.2 International

Looking at problem-solving at an international level, it is evident that problem-solving plays a key role in mathematics education. For example, in Singapore, in order to increase the development of students' mathematical abilities, in particular problem-solving, the "Singapore Mathematics Curriculum Framework" was put in place at all grade levels in their education system. The purpose of this curriculum is to emphasise the application of mathematics in practical situations and real-life problems (Cheng, 2001).

In Finland, problem-solving has been at the core of their mathematics curricula for over twenty years (Pehkonen, 2007). Problem-solving is a key objective of the mathematics curriculum and is emphasised in the textbooks that are heavily used in Finnish schools. The development of mathematical thinking, learning mathematical concepts, and problem-solving methods are the purpose of mathematics instruction in Finland, according to the National Core Curriculum for Basic Education (Mullins et al., 2016).

Problem-solving is also prominent in the English mathematics curriculum. Problem-solving is outlined as one of the key aims of the curriculum (DfE, 2013). This aim states that throughout their education, students should be able to apply mathematics to both routine and non-routine problems. These problems should be of increasing difficulty and include opportunities to break problems down into manageable steps. It also mentions that problem-solving should enable students to make connections between different mathematical content areas, thereby developing fluency. This overarching aim of problem-solving runs throughout the mathematics curriculum and is specifically referenced in each of the different topics (DfE, 2013). Thus, mathematical problem solving is recognized and valued as a central part of post-primary mathematics education, both nationally and internationally.

The Common Core State Standards (CCSS) for mathematics are a set of standards which include both content and processes and outline what students should be able to do at the end of each grade of school in the USA. Within the CCSS, standards of mathematical practice are outlined for educators to strive towards. One of these standards is “Make Sense of Problems and Persevere in solving them” (NCTM, 2010, p. 6). This involves students firstly understanding the meaning of problems. Students should develop the following skills: identifying an entry point to a problem, form plans, form conjectures, use different representations, monitor progress during the problem-solving process, and finally check solutions. This standard is applicable for each grade of the school system in the USA and is also specifically highlighted in each strand of mathematics.

Looking beyond the examples of the countries above, problem-solving is explicitly assessed in the PISA test (Perkins & Shiel, 2014). Within the PISA test, there are six levels of problem-solving proficiency that students’ problem-solving skills and competencies can be described (Perkins & Shiel, 2014). This reflects that (at least for OECD countries) there is effectively world-wide opinion that problem-solving is important.

2.3 Characteristics of problem-solvers

Given that the focus of this study is the development of capacities for teaching problem-solving, it is important to outline what characterises ‘good’ problem solvers. When investigating the differences between ‘good’ and ‘bad’ problem solvers, Lester & Kehle (2003, p.507) found that the good problem solvers had greater mathematical knowledge than the poor problem solvers, and the knowledge that they had was “well connected and composed of rich schemas”. In this context, a schema can be described as an individual’s conceptual structure of knowledge (Skemp, 1989). Good problem solvers went beyond the superficial features of the problem that the poor problem solvers seemed to focus on, and they delved into the structural features of the problem. This highlights that it is the manner in which the problem solver uses their knowledge, rather than the quantity of knowledge, that differentiates between accomplished and poor problem solvers. A continuum of the characteristics of the different levels of problem-solvers is provided by Muir et al., (2008, p. 230) in Figure 2. The comprehensive development of content, problem-solving strategies, higher-order thinking and affect, all play a role, of varying degrees, in the overarching development of expertise in problem solving (Lesh & Zawokewski, 2007).

	Naive	Routine	Sophisticated
Behaviours	Employs coping strategies such as manipulating numbers	Implements strategy in a systematic manner	Generates own strategies
	Relies on one or two strategies	Does not adapt or switch strategies if one is not working	Willing to use a variety and combination of strategies
	Metacognitive thinking not displayed in written or verbal communication	Metacognitive thinking displayed verbally	Metacognitive thinking evident in written and verbal responses
	Errors occur at any and/or all 4 stages of problem solving	No attempt usually made to verify solution	Scores highly for each of the 4 stages of Polya's heuristic plan and verifies solution
	Cannot articulate having solved a similar problem before	Can identify a similar problem, but not necessarily on the basis of mathematical structure	Identifies similar problems according to their mathematical structure
	Written communication is usually inadequate	Written and verbal communication is usually clear	Scores highly for both written and verbal communication
	Often uses the same method to solve all problems	Focuses on one way to solve the problem	Identifies alternative ways to solve problems
	Equates confidence with achieving the answer quickly	Often expresses a lack of confidence in problem solving ability	Displays confidence in problem solving ability

Figure 2: Characteristics of problem-solvers continuum (Muir et al., 2008, p.230)

The four stages which are referred to in this continuum are the stages outlined by Polya (1945). These stages will be discussed in detail in Section 5.3.1.

2.4 The factors affecting students as problem-solvers

To be effective in their teaching, teachers need to be aware of and take into consideration the factors affecting students as problem -solvers.

Lester et al., (1989) suggest that students must regularly attempt to solve problems of various types over a prolonged period of time in order to improve their problem-solving performance. In addition to this practice, specific skills are needed to effectively solve problems. These skills include the students' mathematical knowledge, metacognitive elements, and the knowledge and employment of heuristics. Alongside these skills, positive beliefs and dispositions are viewed as important elements in problem solving (Kloosterman & Stage, 1992).

Lester (2013) states that for a problem solver to be successful, the individual must have a strong content knowledge, flexibility in using various strategies, the experience necessary in learning how to solve problems, a substantial ability in recognizing and constructing patterns, and importantly, the intuition required. In mathematical problem solving, the solver needs to be able to change the problem or their perspective so that their prior mathematical knowledge can be applied to solve it, implying that mathematical problem solving has two phases: exploring the problem and trying to implement mathematical knowledge; and finding pieces of information relevant to the problem (Nunokawa, 2005).

2.4.1 Mathematical Knowledge

Ball, Thames and Phelps (2008) outline that there is a specific type of knowledge needed by mathematics teachers that is not required by other professions. They state that general mathematical ability does not supply the full range of skills and knowledge that are required for mathematics teaching. Chapman (2015) reiterates this point by suggesting that the mathematical understanding needed by mathematicians is different to the conceptual understandings required by teachers. This type of knowledge is described by the term Mathematical Knowledge for Teaching' (MKT). MKT is broken into distinct categories focusing on the areas of content knowledge (CK) and pedagogical knowledge (PK). Pedagogical content knowledge (PCK) combines these two sections (Depaepe et al., 2013) and was initially defined by Shulman (1987) as "that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding". Sibbald (2009) describes PCK as "the intersection of pedagogy and content”. Similarly, Inoue (2009) defines PCK as “an integrated synthesis of subject matter content knowledge and pedagogical knowledge”. Ball et al. (2001) describes PCK as the teacher’s ability to think in various ways about specific mathematical concepts and to make connections between different mathematical ideas meaning that CK is an important prerequisite for PCK. Grossman (1990) suggests that there are four sections that play a key role in a teacher’s PCK: knowledge of students’ understanding; knowledge of curriculum; knowledge of instructional strategies; and knowledge of purposes for teaching. Baumert et al. (2010) point out that a strong knowledge of mathematics does not necessarily mean that they would be effective teachers of mathematics, however, teachers who do not have such knowledge will undoubtedly be limited in the professional capabilities. PCK is not something that is gained by studying mathematics at an advanced level, but it is attained by teaching experience (Ball et al., 2001). Chapman (2015) states that what mathematics teachers know is equally important to how they know it and how they can organise it for teaching.

Ball et al. (2008, p. 400) state that “teaching requires knowledge beyond that being taught to students”. This includes understanding various possible meanings for operations that students may have. The goal of teaching mathematics is to develop students’ fluency with a compressed knowledge by using teachers’ decompressed knowledge. Teachers’ decompressed knowledge involves unpacking mathematical concepts, skills and procedures that are within their compressed mathematical knowledge in order to develop students’ knowledge (Zopf, 2010). The use and understanding of sophisticated mathematical ideas and procedures by students is the end goal (Ball et al., 2008). This can only be accomplished if the teacher has the skills to expose students to certain content and make it accessible (Ball et al., 2008).

Ball et al. (2008) outlines that it is advantageous to distinguish between PCK and other sections of teachers’ knowledge to identify the types of knowledge that are required to teach mathematics effectively. It is well documented that knowledge specific to teachers is required to teach mathematics effectively, however the question that we must confront is: what specific knowledge is required for teaching problem solving? (Baumert et al., 2010; Hill et al., 2005).

Ball et al., (2008) outlines the need for the teacher to have a deep understanding of the content in the school curriculum. This enables them to guide students in problem solving, answer the students' questions, and efficiently and correctly evaluate the students' work. Ball et al., (2008) state that when planning for teaching problem- solving the teacher must understand the mathematics in the students' curriculum, be able to solve the work they assign, and use notation and terms correctly. They found that valuable time was lost, and it was detrimental to instruction when the teacher made calculation errors, got stuck when solving a problem on the whiteboard, or mispronounced terms. Ball et al., (2008) suggest that this planning does not always take place due to time constraints to cover the content of the curriculum, meaning that there is a lack of rich problem-solving experiences for the students.

2.4.2 Mathematical proficiency

Kilpatrick et al., (2001) suggest that there are five components of *mathematical proficiency* which they describe as the term that encompasses what they "believe is necessary for anyone to learn mathematics successfully" (p. 116). These five components (strands) are: 1) *conceptual understanding* – comprehension of mathematical concepts, operations and relations, 2) *procedural fluency* – skills in carrying out procedures flexibly, accurately, efficiently, and appropriately, 3) *strategic competence* – ability to formulate, represent, and solve mathematical problems, 4) *adaptive reasoning* – capacity for logical thought, reflections, explanation and justification, 5) *productive disposition* – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (p. 116). Kilpatrick et al. (2001) argue that these components are not isolated from one another, and they stress that "the five strands are interwoven and interdependent in the development of proficiency in mathematics" (p. 116). This means that it is not sufficient to focus on individual strands to develop proficiency, but rather concentrate on all strands.

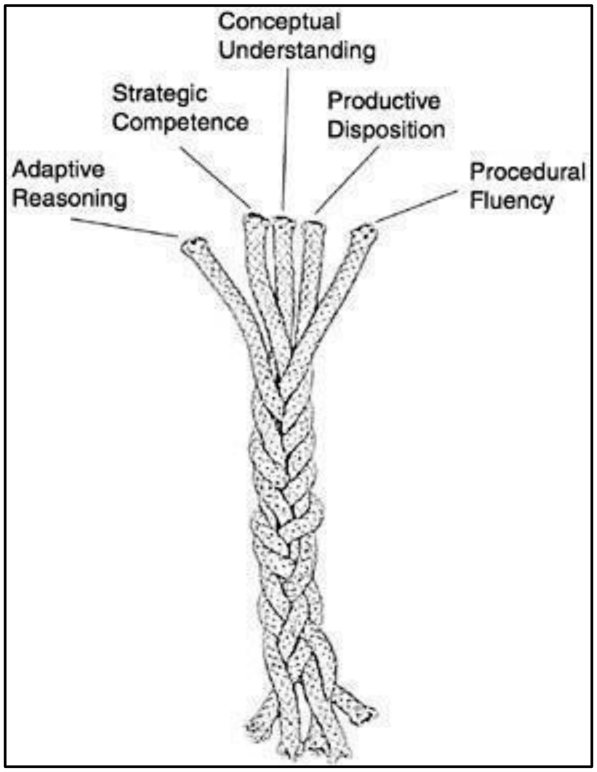

Figure 3: Kilpatrick et al., (2001, p.117)

Conceptual understanding is described as "an integrated and functional grasp of mathematical ideas" (Kilpatrick et al., 2001, p. 199). A strong conceptual understanding is useful in a variety of contexts as the knowledge is organised in a systematic way. This is consistent with the study by Lester and Kehle (2003) on 'good' problem-solvers, as discussed above. Kilpatrick et al. (2001) maintain that this schema of knowledge allows for easier connection between what an individual already knows and the generation of something new. Furthermore, they suggest that if a method is truly understood then it is unlikely for it to be applied incorrectly when being used. This shows the connection between conceptual understanding and retention. Procedural knowledge is the "knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently" (p.121). There is a strong link between conceptual understanding and procedural fluency as it is important to understand that it is possible to solve a wide array of problems using the same procedure. Kilpatrick et al. (2001) continue that the study of algorithms can show that mathematics is highly structured, and a well-developed procedure can complete a multitude of routine tasks. It is suggested that skills are necessary to learn mathematical concepts and using procedures can aid the development of this understanding. They state that it is difficult for students to enhance their problem-solving ability or their understanding of mathematical concepts, without ample procedural fluency. Strategic competence is described as “the ability to formulate mathematical problems, represent them, and solve them” (p.124). They relate this strand to problem-solving that occurs in real life and highlight that outside of school, problems are not usually clearly outlined, and it is sometimes difficult to deduce what precisely is the problem. This means that it is necessary to be competent in formulating problems along with problem-solving. When representing a problem mathematically, the individual must be able to identify the essential components of the problem and separate the unimportant components. Kilpatrick et al. (2001) state that to become a proficient problem solver, it is necessary to be skilful in forming mental representations, notice mathematical connections, and create non-routine methods when appropriate. The development of such novel approaches is dependent on the understanding of the relationship between the quantities in the problem and the individual’s routine procedural fluency. Adaptive Reasoning “refers to the capacity to think logically about the relationship among concepts and situations” (p.129). This reasoning is developed from analysis of alternatives and is justifiable. The key component of adaptive reasoning is the ability to justify the validity of the work. A single justification of a procedure is not acceptable. A procedure must be justified through a variety of problems to enhance proficiency. Kilpatrick et al. (2001) state that “adaptive reasoning interacts with the other strands of proficiency, particularly during problem-solving” (p.130). It is adaptive reasoning that determines the validity of the proposed approach after the formulation, representation and heuristics have all been selected. Finally, productive disposition “refers to the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics” (p.131). If an individual is to develop the four aforementioned components, then it is important for them to believe that with significant effort they are capable of understanding mathematics. Kilpatrick et al. (2001), note that as students’ skills in solving novel problems improve, their attitude towards mathematics improves. This confidence positively influences the individuals' procedural fluency and their adaptive reasoning abilities (Kilpatrick et al., 2001).

2.4.3 Metacognition

Schoenfeld (1992) describes metacognition as one's own knowledge about one's own cognitive processes. He highlights that metacognitive ability plays an essential part in problem-solving, as it allows problem solvers to dismantle more challenging problems into subtasks, to prioritise and order the importance of each subtask and then to complete each subtask in sequential order. Lester et al., (1989) suggest that improving metacognitive ability should be concurrent with the learning of mathematical content and is most effective when it takes place in the context of specific mathematical ideas. They outline that metacognition instruction in particular, in association with problem-solving instruction, is most effective when it is presented in a systematic manner under the direction of a teacher. Schoenfeld (1992) suggests that by continually asking students certain questions - such as "What are you doing? Why are you doing it?" - teachers can help students to develop their metacognitive abilities. He claims further that this can be achieved despite students' initial discomfort in answering questions.

2.4.4 Heuristics

Schoenfeld (1980) states that a variety of heuristic approaches are necessary if one is to be successful at problem-solving. Heuristics is described as a general strategy, independent of any subject topic, that helps in the understanding of the problem and allows the problem solver to use their knowledge to solve it. While heuristics are useful for problem-solvers, heuristics themselves are not enough to ensure problem-solving proficiency (Schoenfeld, 1992). The key element of heuristics is that they require mathematical knowledge in order to aid problem-solvers (Schoenfeld, 1992). He continues that while heuristics can aid problem-solvers in unfamiliar situations, heuristics cannot replace subject matter knowledge. There needs to be adequate detail in the heuristics in order for the heuristics to be beneficial (Schoenfeld, 1992), which is a reason that Schoenfeld (1992) gives for the varying results of heuristics in research.

Schoenfeld (1992) highlights that although textbooks contain 'problem-solving' sections, students are given (so-called) problems to practise, after being shown a strategy. He outlines that when problem-solving strategies are taught in this way, they are no longer heuristics but rather algorithms, meaning that students will not develop their problem-solving skills as there is no ambiguity in how the students should approach the problem. This strategy removes the unknown which essentially means that the task at hand is no longer a problem. The teacher has the choice to teach the students the procedure for solving the selected problem, but they are not promoting the development of problem-solving skills. (Schoenfeld, 1992). Schoenfeld (1992) advocates that heuristics should not be initially exposed to students at college level, but throughout students' mathematics education.

While teaching problem-solving, teachers must have a deep understanding of the strategy they choose to use and the metacognition that is associated with it (Chapman, 2015). To do this, teachers must have the ability to analyse the underlying reasons for employing a certain strategy and use this knowledge to solve the encountered problems (van den Kieboom, 2013)

While conducting research into the differences between proficient and less accomplished student problem solvers, Muir et al. (2008) found that the students who had the greatest level of success were able to identify alternate strategies and use previous mathematical encounters to generate their own strategies. This study consisted of a selection of twenty 6^{th} grade students from five different primary schools. Although students who solely relied on memorised routine problem-solving methods did have a degree of success, the lack of verification and justification of the employed strategies demonstrated the lack of understanding and flexibility. Muir et al refer to the reliance of the students on the teacher considering that the majority of the 'below average' students, all yielding from the same school, were dependent on previously-learned strategies showing a lack of creativity. If the students were not made accountable for creating their own strategies and engagement in higher order thinking is not required (Willoughby, 1990), then it is probable that the students would hold the belief that mathematics consists of applying rules provided by the teacher.

2.4.5 Affective domain

It has been widely reported that the affective domain is an important contributor to problem-solving behaviour (Lester & Kroll, 1993). The affective domain includes attitudes, feelings and emotions. Identity is defined as "the embodiment of an individual's knowledge, beliefs, values, commitments, intentions, and affect as they relate to one's participation within a particular community of practice; the ways one has learned to think, act, and interact" (Philipp, 2007, p. 308). Identity consists of a combination of personal features and social features which combine in a construct that includes factors such as knowledge, beliefs, and values amongst others (Chapman, 2015). Beliefs impact on problem-solving performance since beliefs contain their subjective knowledge about self, mathematics and the topics dealt with, in particular mathematical tasks (Lester & Kroll, 1993). The beliefs of teachers have a direct influence on the beliefs of their students because when many students are confronted with a mathematics problem, they have low motivation to work through the problem and depend on extrinsic motivation (Marcou & Philippou, 2005) which primarily comes from the teacher. When a problem is presented to students, they may not feel comfortable admitting their shortcomings in mathematical knowledge and attempt to use inappropriate methods and guessing in order to cover up their deficiency (Seldon & Seldon, 1997). Schoenfeld (1983) highlighted the existence of beliefs that lies behind students' behaviour when attempting mathematical problems. Furthermore, he outlines that the cognitive resources available to students when learning are categorically related to the students' beliefs around what they consider useful in learning maths. If the beliefs deter rather than promote understanding, a large segment of stored information is made inaccessible to the individual (Schoenfeld,1983).

McLeod (1988) set out to provide a theoretical framework for investigating the affective factors that are associated with problem-solving. McLeod defines affect as a term used to represent "all of the feelings that seem to be related to mathematics learning" (p.135). He highlights that a variety of emotions can be expressed while a person is trying to solve a non-routine mathematical problem. When failure to reach a solution occurs, he states that the emotions can include frustration and panic. These emotions can become increasingly intense over a prolonged period of time, particularly for novice problem solvers with little experience of problem-solving.

McLeod outlines the work of Mandler (1984, as cited in McLeod, 1988, p.135) who states that emotion can arise from the interruption of a person's plan. Mandler describes that a person's plan arises through the activation of a schema that creates an action sequence to reach completion of the plan. He states that when an interruption takes place, this sequence can no longer take place resulting in the production of emotions such as frustration or surprise. The emotions can be interpreted as either positive or negative. These interruptions of planned sequences can be described as "discrepancies between what we expect and what we perceive".

As stated in the Three Key Characteristics of mathematical problem-solving, it is not immediately clear to the problem-solver how to achieve the goal of the problem. McLeod highlights that this ambiguity in how to approach a non-routine mathematical problem is precisely the situation that Mandler (1984, as cited in McLeod, 1988) describes. This results in an interruption occurring, leading to emotions arising. McLeod (1988) explains that the reaction to problem-solving can be different in every person with both negative and positive emotions being possible results to the same interruption. He explains that when the majority of a student's mathematical experience involves doing routine exercises, then the inevitable consequence to interruptions during problem-solving is intense emotional reactions.

McLeod (1988) identifies four major factors involved in the research of the affective domain and problem-solving. These are; 1) magnitude and direction of the emotion, 2) duration of the emotion, 3) level of awareness of the emotion, 4) level of control of the emotion.

The first factor identified is *magnitude and direction* of emotion. Magnitude describes the varying levels of intensity of affective influence on problem-solving. Direction refers to the positive or negative nature of the influence. McLeod (1988) notes that frustration is the most common reaction which is a result of getting stuck while doing a problem. It is not only negative reactions that occur, there are also positive reactions. These are particularly evident after achieving an 'Aha!' moment during their problem-solving attempt.

The second factor is *duration* of the emotion. McLeod (1988) outlines that the affective reactions during problem-solving are usually short in duration, although they can be intense. In response to the intensity of the negative reaction, students have difficulty in persevering with a problem and tend to quit. The students who do persevere with the problem and who experience positive reactions to making progress in their attempt, appear to go to negative emotions when feeling stuck and then return to positive emotions again.

The next factor is the *level of awareness* of emotion. McLeod (1988) suggests that there is often an unawareness of the influence of emotions on problem-solving processes amongst problem solvers. Problem solvers are frequently unaware of their automatic responses. One such automatic response is quitting in response to being stuck on a problem rather than accepting being stuck as a normal part of the problem-solving process. McLeod (1988) advocates that awareness of emotional reactions may help problem solvers control their automatic responses and therefore increase the possibility for success.

The final factor is the *level of control* of the emotion. It is highlighted by McLeod (1988) that different students will have different levels of control over their emotions. In relation to negative emotions, students could benefit from controlling their emotions to overcome frustration from feeling stuck, resulting in them persisting with the problem and not quitting. Similarly, in relation to positive emotions, students could benefit from controlling the emotion of joy from reaching a solution. Instead of reacting to these positive emotions as a signal to stop, problem-solvers should use these as a cue to review their work and try to find an alternative method to verify their solution (McLeod, 1988). Supporting this opinion is Mason et al. (2011) who state that a positive attitude and self-image is essential to overcome being stuck, whereas negativity prohibits progress. Avoiding becoming stuck is not the goal, but rather becoming stuck and overcoming it will provide the experience necessary to overcome being stuck on future occasions (Mason et al., 2011).

2.5 The Role of the Teacher

The teacher plays a critical role in students' learning. According to Hattie (2012, p.18), "teachers are among the most powerful influences in learning". Hattie explains that the teacher's view of their role is vital. It is important for teachers to view their role as evaluators of the effect that they have on their students' learning. Hattie suggests that while it is important what teachers do, it is most important that the teachers can effectively review the impact their actions have on their students' learning. It is when a teacher adopts an evaluator role that students are more likely to achieve higher levels of achievement (Hattie, 2012). There is an acceptance that learners are different and face different challenges. In response to this, teachers need to have the skills to provide students with a variety of learning strategies to build students' understanding of a concept. When learning is not occurring, the teacher needs to know how to intervene to change the approach towards the goal. For these interventions to be effective, the teacher needs to be aware of the following: learning intentions; when a student fulfils the learning intentions; understand students' prior knowledge; and provide challenges to students to ensure development. The teacher is responsible for creating an environment that will foster students' learning. This environment needs to accept errors and use errors as learning opportunities. Teachers provide feedback to students when mistakes are made that will promote their learning. The teacher needs to be competent in a variety of learning strategies to create this environment. Along with this, the teacher needs to have an awareness if the strategy is working or not and subsequently react (Hattie, 2012).

2.5.1 Preparation of mathematics teachers

Adler *et al*. (2005) highlight that due to the increase in the awareness of mathematics as an important life skill, there is a need for quality teaching of mathematics. They advocate that high-quality teaching of mathematics is not only important where mathematics is a specialised subject, but where it is a general requirement. In response to this demand, teacher preparation and teacher education programmes are vital (Adler *et al.*, 2005). Teacher education programmes need to promote prospective teachers' understanding of complex classroom situations and the wide variety of students' needs (Darling-Hammond, 2006). Along with increasing awareness and responses to students' needs, it is important that teacher education programmes include core knowledge and subject content. These need to be incorporated in a way that prospective teachers can use them in the practical setting of a classroom (Darling-Hammond, 2006). Teacher education programmes are viewed as a critical stage in teachers' development (Teaching Council of Ireland, 2017). During teacher education programmes, prospective teachers' beliefs regarding teaching and learning should be considered and challenged which they will bring forward into their professional practice (Teaching Council of Ireland, 2017).

Much research has been conducted in a university setting involving prospective mathematics teachers, particularly in relation to 'mathematical knowledge for teaching' (MKT). MKT may be described as the mathematical knowledge that is required to carry out the work of teaching mathematics (Hill et al., 2005). This mathematical knowledge includes the pedagogical aspects and the content.

An example of this research is the study conducted by Superfine and Wagreich (2009). This research involved the development of a course for pre-service teachers to undertake as part of their teacher training aimed at developing MKT. This course included a range of tasks and activities for the PSTs to complete. These activities gave the PSTs the opportunity to develop their abilities to 'engage with explaining, representing, and understanding and reacting to mathematical thinking that is different from their own' (Superfine and Wagreich, 2009, p.28).

An example of this type of research in an Irish university is a study conducted by Prendergast *et al.* (2014). The participants in this study were pre-service mathematics teachers from two different programmes of study. The aim of the study was to deepen the relational understanding and pedagogical beliefs in teaching post-primary mathematics. This was done through the delivery of a weekly programme that was each approximately 90 minutes in duration. The programme focused on exposing the pre-service mathematics teachers (PSMTs) to a constructivist role of the teacher rather than an instructional role. The authors noted that with the introduction of the 'Project Maths' syllabus in Ireland, there is a need for the teacher to make connections between different elements of mathematics. This highlights the need for well-developed MKT. The study was based in the university setting and the authors state that, while it would be beneficial to observe the PSMTs in a teaching environment in order to evaluate the programme, it is not feasible to do so. However, feedback along with pre- and post-diagnostic examinations demonstrated statistically significant increase in the PSMTs' relational understanding.

Along with the study of MKT with prospective mathematics teachers in a university setting, research has also been conducted in this environment in relation to developing problem-solving competencies of PSMTs. Guberman and Leiken (2013) conducted a study on a problem-solving course undertaken by PSTMs as part of their teacher education programme. This course aimed at developing both the competencies and conceptions that are associated with solving problems. The course included the PSMTs completing activities which involved implementing different problem-solving strategies and techniques. Within these activities it was encouraged that the PSMTs try to find multiple ways to solve the problems. It was found that the PSMTs developed their problem-solving skills through the demonstration of more advanced problem-solving strategies. It was evident that the PSMTs used fewer trial-and-error strategies and connections with mathematical concepts were more prominent.

Another study about teaching problem-solving that was conducted in a university setting was by Fritzlar (2006). This study involved the development of a complex computer programme which was used with prospective mathematics teachers. The programme was made up of a database of recordings from grade 4 and grade 5 pupils from 50 classes. The pupils in the recordings were trying to solve a given mathematical problem. These recordings made up a virtual type of lesson which gave the prospective teachers opportunities to choose an intervention. The programme allowed for the prospective teachers to revise their chosen intervention if it did not prove effective. The prospective teachers could run the computer programme multiple times allowing for the use of different teaching methods, goals and classes. The aim of this computer programme was to help prospective teachers gain an understanding of the complexity of teaching problem-solving.

2.5.2 Teaching Problem-solving

While problem-solving is now part of the mathematics curriculum, the teachers' view on problem-solving as either an end result or a means of teaching, affects the manner in which problem-solving is conducted in the classroom (Lester, 2013). Furthermore, the teachers' perspectives of the role of problem-solving in teaching mathematics influences their approach to teaching problem solving. The two main distinguished approaches are 'teaching via problem solving' where mathematical procedures and concepts are learned through problem-solving, and 'teaching for problem-solving' where the end result is being proficient in problem-solving (Lester, 2013). Lester (2013, p.249) states that strong content knowledge, flexibility in using various strategies, experience in solving problems, significant experience in recognizing and constructing patterns are required by prospective teachers who aim to teach mathematics either *for* or *via* problem-solving. The manner in which a teacher behaves while teaching either *for* or *via* problem-solving is pre-determined by the decisions made by the teacher before entering the class and is also greatly influenced by the teacher's knowledge (Lester, 2013). The planning undertaken by the teacher is a key factor in the instruction of problem-solving that takes place in the class and this planning may be influenced by previous experience and reflection of teaching the lesson to other students and could be in a particular sequential order of teaching actions (Lester, 2013). Neither approach is seen to have better credentials than the other, but both should be combined and have a place in the development of problem-solving skills (DiMatteo & Lester, 2010; Stein & Silver, 2003). According to Lester (2013), mathematics teachers do not necessarily need to be expert problem solvers, but should have an extensive understanding of what successful problem-solving entails, and should be experienced in problem-solving.

According to Schroeder and Lester (1989), teaching for problem-solving entails a teacher focusing on how the mathematics that is being taught is applicable to finding the solution of both routine and non-routine problems. This is demonstrated by the teachers exposing the students to different mathematical concepts and supplying the students with multiple variations of routine and non-routine problems, giving the students ample opportunities to apply the mathematical content they were shown to the problems. Lubienski (1999) maintains that it is not easy to teach through problem-solving: this requires teachers to carefully plan the problems they select to use and consider the ways in which students will learn from them and also the misconceptions that may occur. Polya (1945) states that for a teacher to teach mathematics well, they must not only understand it to the same extent as the students do, but they must also be able to misunderstand it as well as their students do. He continues that if a teacher's statement can be considered in more than one way, then some students will understand it one way and some students understand it the other, resulting in various results generating different outcomes of the teaching.

Lester (2013) states that teachers who adopt a teaching via problem-solving approach must be skilled at selecting good problems, listening to and observing students while they undertake a problem, and asking the right questions. In addition to this, Lester identifies the need to be competent at problem-solving methods, making these methods accessible to students. Lester extends that it is important for teachers to also ensure that students are challenged by the chosen problems, which are not made straightforward by the help of the teacher, who should understand when to probe the students and when to resist making comments. Lester identifies that it is not sufficient for teachers to be proficient in these attributes, but they must also know when to act and predict the implications of these actions.

In research conducted by Lubienski (1999), it was evident that teachers have different interpretations of teaching through problem-solving. One teacher expected that through interesting problems, students would learn important mathematical concepts. This teacher wanted the students to assume responsibility for their own learning and make sense of the problems for themselves without giving the students direct instruction. However, this view of teaching through problem-solving differs from another teacher in this study who analysed problems before going into class so there was a clear plan that guided the students' thinking about the problems, to ensure that the students learned what was intended. This planning is vital for ensuring that the mathematical concept within the problem is understood by the students and discussion of mathematics with colleagues can help teachers think about how particular problems can guide students' learning (Lubienski, 1999).

The strategies employed by teachers must be selected to meet the needs of the students. Strategies are used to develop the understanding of the students and each student will have different previous experiences and knowledge (Depaepe et al., 2013). This demonstrates the strong connection between instructional strategies and the knowledge of students' misconceptions, which are both central to the teachers' PCK (Tirosh, 2000). The work of Carpenter, Moser, and Bebout (1988) found that teachers' knowledge of their students made a positive contribution to their problem-solving teaching. Different students will employ various strategies when attempting a problem and the teacher must be equipped to perceive the implications of the different approaches and understand why some are valid or not and explain this to the students to develop their ideas (Chapman, 2009). When teaching problem-solving, teachers require both the content knowledge and pedagogical knowledge to make connections between the different topics of mathematics and present these to their students when the opportunities arise (Zbiek et al., 2010). By having a deep understanding of the content, the teacher is capable of helping students who find an area difficult, by directing them to use different representations in problem-solving and ask the students to justify the different approaches and overcome the misconceptions. Even when a teacher has the ability to approach problems using different methods or representations, they must have the mathematical and pedagogical knowledge to convey this to their students and use this knowledge to validate their students' methods (Zbiek et al., 2010). An understanding of how students tend to interpret and use representations of particular topics is a key element of PCK. Knowledge of the intersections of the subject matter and pedagogy are built up over time by teachers as they teach the same topics to different groups of students. This experience develops the teachers' ability to make the connections between different concepts and predict the difficulties that students will have with particular topics (Ball et al., 2001). However, possessing this interweaving content and pedagogical knowledge may not always suffice in the actual practice of teaching. Flexibility of knowledge is an important attribute of the teacher to deal with an unanticipated proposed strategy or question by a student. This knowledge is specific to teaching (Ball et al., 2001).

2.5.3 Problem-posing

Mathematical problem posing is the 'creation of mathematical problems, generated from given situations or from a pre-existing question' (Silber & Cai, 2017, p. 163). It is outlined by Ellerton and Clarkson (1996) that imagination is necessary for the advancement of mathematics which occurs through the asking of new questions and the reviewing of old questions. One such advancement of mathematics is the posing of a mathematical problem as an idealisation of real-world phenomena (Christou et al., 2005). In a general sense, problem posing plays an integral role in mathematics in a variety of ways (Silber & Cai, 2017). Problem posing can occur in response to real-life situations with the aim of finding a solution. Similarly, problem posing can occur in the form of self-reflection in the problem-solving process whereby the person can pose a similar problem to the given problem or pose problems as a tool for evaluating their work (Silber & Cai, 2017). Problem posing has many benefits, such as the improvement of students' thinking and problem-solving skills, an improved attitude towards mathematics, and the development of students' understanding of mathematical concepts (Christou et al., 2005). Thus, given the important role that problem posing plays in both the advancement of mathematics and the benefits to students, it is essential for problem posing to be highly valued in mathematics education (NCTM, 2000).

It is the responsibility of the teacher to pose problems in the classroom meaning the teacher must have a well-developed knowledge of problem posing. A teacher's ability to generate new problems and reformulate previous ones plays a key role in positively enhancing students' problem-solving ability (Chapman, 2015). It is vital that the problems are meaningful and that the students can identify that the property in the question is not trivial, so there is a substantial reason for attempting to solve the problem (Nunokawa & Fukuzawa, 2002). The teacher needs to select appropriate problems that will link the prior mathematical knowledge of the students to the problem encountered and gives the students an opportunity to reorganise their knowledge into new methods. The problems selected must highlight to the students the limitations of their knowledge but also optimise their use of it, to formulate the new mathematical knowledge that the teacher expects of them (Nunokawa & Fukuzawa, 2002). From the literature, it is evident that some teachers struggle in problem posing (Chapman, 2015; Inoue, 2009). Both pre-service and practising teachers created problems that were predictable, ill-formulated, undemanding, and unsolvable to various degrees when building upon previous mathematical problems (Silver et al., 1996). In a study conducted by Chapman (2012), it was shown that preservice teachers had similar difficulty in posing open-ended problems.

Problems are used in classrooms for a variety of beneficial purposes, such as promoting mathematical thinking and gaining insight into students' understanding of mathematics (Silber & Cai, 2017). Other benefits of mathematical problems include the connection that problem-solving brings between theory and practise through the use of real-world applications (Carson, 2007). Problem-solving also allows connections to be made between different mathematical concepts, and the problem-solver can identify similarities amongst problems that initially appear to be different on a conceptual level (Carson, 2007). It is typical for mathematicians to pose problems to themselves while they work on a mathematical problem. The purpose of this manner of posing problems is the evaluation of their work and reflection on their approach. Posing problems allows the mathematician to test the generalisation of their approach or act as an aid to make progress in their attempt (Silber & Cai, 2017). The benefit of posing problems in this situation is dependent on the problems posed, and the skill of posing mathematical problems is a challenge in learning to teach mathematics (Crespo, 2003).

2.5.4 Teachers' identity

Teacher identity plays an integral role in the teaching of problem-solving. The teaching of problem-solving does not simply rely on the techniques employed, but it "comes from the identity and integrity of the teacher" (Palmer, 1998, p. 149) meaning "we teach who we are" (p.2). Identity consists of a combination of personal features and social features which combine in a construct that includes factors such as knowledge, beliefs and values amongst others (Chapman, 2015). As determined by Van Zoest & Bohl (2005), beliefs incorporate values and conceptions which regularly provide explanations for responding in certain ways to particular types of knowledge in different situations. Furthermore, Vahasantanen (2015, p. 3) extends that teachers' professional identity incorporates the "individual's current professional interests, views on teaching and on the students' learning, and future prospects". Hodgen (2011) specifically connects identity with mathematics knowledge in teaching.

Initial teacher education programmes have a strong influence on teachers' identity and ultimately their teaching. Ruohotie-Lyhty & Moate (2016) establish that preservice teachers' identity was not associated with the specific teacher education programme at the end of the second year, but was more connected to the greater professional community. This observation is in line with a study undertaken by Kayi - Aydar (2015) which found that in an attempt to compose their own professional identity, prospective teachers distanced themselves from teaching tutors. Collectively these outline that identity is an incredibly personal feature of a professional and in order to preserve and develop an individual's identity, the teacher must relate themselves to the greater professional community and focus on their beliefs and interactions within it. However, it is documented that newly qualified teachers find it difficult to distance themselves from their prior experiences and not continue to replicate the teaching methods and practices of their own teachers. It is necessary for the new teachers to align themselves with the current education expectations (Ball, 1990). While a number of teachers engage with their professional community, many remain independent and teach in a similar manner to what they were taught and rely on the textbook for guidance. Van Zoest and Bohl (2005) comment that newly qualified teachers experience a period of time where they do not belong to any specific professional community related to mathematics education, while they are progressing from their education institution to their school of employment. As a result of leaving their university, their only support received is from their new school. This may be beneficial or not to the teacher in terms of beliefs regarding reform, depending on the influencing mathematics department's attitudes. From the literature, it is documented that university-based reform-oriented modules are successful in having an extensive impact (Ball, 1990; Ensor, 2001; Van Zoest & Bohl, 2005).

A teacher's identity can give an insight into their opinions of educational reform and professional development (Ruohotie-Lyhty & Moate, 2016). Identity is in a constant state of change due not only the teacher's own "values, experiences and beliefs" (Duff, 2012), but also external factors including environment and social expectations and demands. It is not possible to understand the development of identity without taking into consideration the environmental setting of the individual, as identity can be viewed as both social and context dependent (Ruohotie-Lyhty & Moate, 2016). Educational reform demands that teachers re-evaluate their identity in response to changes. Pre-service teachers are advised to question their beliefs in relation to education to support their professional development. This practice continues throughout their teaching career and as reform occurs in the curriculum and surrounding educational system, teachers must adapt their beliefs and attitudes to implement the changes expected in their teaching. Some teachers do not acknowledge the requisite to redefine their teaching methods in respect to mathematics education reform as it is considered either too large of a change from their beliefs about teaching (Smith, 1996) or that it does not justify the effort required from their comfort zone.

There is a responsibility on the part of teachers to understand the beliefs that students must hold in order to be successful in problem-solving. Kloosterman and Stage (1992, p.113) outline that to alter negative beliefs, success must be experienced by students who feel that they cannot solve time-consuming mathematics problems. Similarly, problems that require common sense are more beneficial to reform the beliefs of students who feel that all mathematical problems can be solved by applying memorised rules. It is important to note that "beliefs don't change easily" especially if the individual is content with the way that they view mathematics and "themselves as learners of mathematics". If students object to learning about the reason 'why' and 'how' mathematical solutions come about, it is important that teachers do not reinforce unfavourable beliefs by taking the easier choice of skipping over the understanding. The students' beliefs need to contain: an appreciation for requirement of understanding a concept and that it is worthwhile spending time doing working on this; that word problems are a key component of mathematics; that effort will improve mathematical ability; the limitations of rote memorisation; and that they are capable of success in solving time-consuming problems. It is the partial responsibility of the teacher to support and develop these positive beliefs of students.

Lampert (1990) further highlights the role of the teacher in influencing students' beliefs, stating that students share a typical opinion that mathematics is about certainty, completing questions quickly, and achieving correct answers. These beliefs are reinforced when confirmed by the teacher. The students believe that mathematics entails the rules given by the teacher, followed, remembered, then applied in the future. Mason (2003) found that students became increasingly confident that the application of routine procedures is not sufficient in solving all problems, as they progressed through secondary school. This could be because of experiencing progressively difficult word problems that tested their understanding of the concept and perhaps the influence of their teachers. Likewise, the way in which teachers present mathematical content, the type of task they set, assessment strategies, the goal they expect of students to achieve, have a significant influence on the development of students' beliefs (Pehkonon, cited in Mason, 2003).

2.6 Understanding in problem-solving

The term understanding is widely associated with problem-solving, but it is important to note that there are multiple elements to this term. An example of the complex role understanding plays in mathematics, is process- concept theory (Gray & Tall, 1994). Procedural aspects of mathematics are viewed as routine manipulation of mathematical objects while conceptual aspects are more complex in nature (Gray & Tall, 1994). Conceptual knowledge is an interconnected relationship between pieces of information and is flexible while procedural is rigid (Gray & Tall, 1994). However, procedural aspects build the foundations in developing, through understanding, towards flexible thinking (Gray & Tall, 1994). The Action, Process, Object, and Schema (APOS) theory is regularly conducted in studies which are concerned with students' understanding of mathematical topics (Oktac, Trigueros, & Romo, 2019). The APOS theory involves a basic structure being applied to a previous mental structure (action), which is then repeated and reflected on (process). These processes then in turn become Objects which actions can be applied to. A schema is then created which is a collection of connected structures (Oktac, Trigueros, & Romo, 2019). Relational understanding and Procedural understanding are two elements of understanding which can have a significant impact on how problem-solving is incorporated in the teaching of mathematics. Muir et al., (2008) identifies the differences between the term 'relational understanding', meaning that the problem solver understands both what to do and why to do it, and the term 'procedural understanding' meaning that the problem solver is unable to justify why the procedure works, although they are applying it.

The work of Skemp (1989) has been highly influential in relation to the study of understanding. Skemp (1989, p. 4) describes two disparities that can occur in teaching mathematics: 1. "Pupils whose goal is to understand instrumentally, taught by a teacher who wants them to understand relationally, 2. The other way about." He maintains that in the first situation students do not care for the explanations or the preparation for the next learning block, which in this circumstance is frustrating for the teacher. The sole focus of these students is getting the solution of the question, everything else is ignored once this is achieved. A problem with this is that the students are not prepared for questions that do not abide exactly to the learnt rule, meaning they will get it wrong. He claims that the second situation is more damaging to the student than the first. Skemp describes the importance of both instrumental and relational understanding. Instrumental understanding provides quick and correct answers meaning visible and immediate rewards. This promotes a feeling of success the importance of which, he claims, should not be underrated. In contrast to this, he states that relational understanding makes new tasks more accessible to students. He also highlights that although there is more to learn in relational understanding, what is learnt is longer lasting, as connections can be made between topics. Skemp describes relational mathematics as a person developing a schema from which they have the ability to create numerous plans for getting from any starting point to any finishing point within their schema.

Skemp (1989) suggests that there are reasons why a teacher might choose pursuit of instrumental understanding: 1. A particular procedure is all that is needed by the students and the relational understanding would be too time consuming; 2. The students will be assessed in the content of examinations, but the relational understanding is difficult to comprehend; 3. The students must encounter the skill in a different subject before it can be fully understood; 4. A new teacher to the school is surrounded by an environment that is based on instrumental mathematics teaching. Similarly, he justifies why teachers may choose a relational understanding approach and how it differs from instrumental: 1. The means become independent of the desired end goal; 2. It becomes intrinsically rewarding to build up a schema within a certain area of knowledge; 3. Confidence in one's own ability is increased the more complete their schema, meaning less reliance on external help to find ways to solve the problem; 4. A schema is constantly growing, resulting in a greater recognition of possibilities. This highlights the impact the choice of the teacher regarding teaching for relational versus instrumental understanding can have on the level of problem-solving incorporated in the classroom (Skemp, 1989).

2.7 Problem-solving outside of the classroom

2.7.1 Real-life problems

Problem-solving skills are not solely relavant to education systems, but also to employment. According to a review conducted by the STEM Education Review Group (2016), it is paramount for economic competitiveness that there is a large quantity of graduates of STEM subjects. They state that mathematics underpins all of the STEM disciplines and problem-solving is a key skill within all of these disciplines. The reliance on STEM graduates to maintain economic competitiveness is not unique to Ireland, but is applicable to all knowledge-based economies.

Lesh and Zawojewski (2007) state that there is a growing realisation that there is a discrepancy between the skills that are highlighted as important in curricula and the abilities and understandings that are needed to succeed after school. These differences consist of the inability to connect and interlink knowledge from various topics of mathematics and to create and adjust mathematical models. Lobato (2003) and Hohensee (2014) outline that it is difficult to transfer problem-solving competencies from one situation to another. Incorporating mathematics learning in a real-life situation has been proposed as a solution to this dilemma (Gainsburg & English, 2016). Gainsburg and English (2016) provide a succinct review of the literature which highlight the requirement of employees in the 21st century to have a high level of problem-solving skills and the necessity of these skills in vast areas of employment due to developments such as technological advancements. The demand for effective problem solvers as employees essentially calls for schools to teach problem-solving (Gainsburg & English, 2016). However, in a study undertaken by Vershaffel et al. (1997) it was found that pre-service teachers had a strong propensity to exclude real-world knowledge when undertaking a challenging word problem. This study found that both first and third year pre-service teachers were disappointing in their inclusion of real-world knowledge when tackling a non-routine problem. As a teacher's beliefs have a significant impact on their teaching behaviours, this suggests further that the teacher's students will fail to learn the ability to use mathematics in real-world situations (Verschaffel et al., 1997). These authors (op. cit., p.358) conclude that in order for students to make connections between problem-solving in school mathematics and real-world situations, it is necessary to support prospective teachers in constructing 'the proper concepts, skills and beliefs' that are required by realistic modelling of problematic situations and for the realistic interpretations of the solutions of these problems. Similarly, Verschaffel et al. (2020) highlight that there remains an issue in applying real-life knowledge to mathematical problems.

2.7.2 A Simplistic View of real-life problems

Lester (2013) claims that conceptualising problems in a simplistic view may be a contributing factor to the ineffectiveness of instruction in improving students' problem-solving ability. A simplistic view may be described as a naive view of problem-solving, whereby mathematical content can be learned separately from its applications with no connection between new and old concepts. Teachers who adhere to this view have the tendency to explore problem-solving and applications of mathematics only after the relevant mathematical concepts and skills have been exposed to the students in decontextualized settings. Students carry out low cognitive demand procedural tasks prior to considering problems and application (Lester, 2013). In this view, problem-solving has three steps: the problem is posed in a realistic setting which the problem solver then translates into abstract mathematical terms; the problem solver then works on this mathematical representation to find a mathematical solution of the problem; and finally this solution is translated back in terms of the initially posed problem (Lester, 2013, p. 254). The perspective of the teachers who use this approach is that the ability of translating real-world problems into mathematical problems is a priority of their teaching. The use of this approach means that when less straightforward problems are introduced, it is not sufficient for the problem solver to apply previously memorised procedures to solve the problem. These non-routine problems require more complicated processes including, 'planning, selecting strategies, identifying sub-goals, choosing or creating appropriate representations, conjecturing, and verifying that a solution has been found' (Lester, 2013, p.255). Verschaffel et al. (1997, p.339) state that problems in mathematics classrooms are 'artificial, puzzle-like tasks' that are independent of real-world knowledge and that students do not need to consider the realistic contexts in which the problems are taking place. Lesh and Zawojewski (2007) claim that there is an increasing consensus among mathematics educators that there is a significant divide between the low-level skills that are currently components of curricula and the type of skills and understandings that are required for success after school. It is suggested by Lesh and Zawojewski that mathematics is prominent in certain careers such as architecture or engineering, whereas in everyday living activities, mathematics as currently taught in school has little contribution. Lesh and Zawojewski cite the work of ethnographers (such as Hall (1999) and Gainsburg (2003)) who maintain that the mathematics that is used in certain, mathematics dominant careers, makes use of school mathematics. However, in these careers, mathematics does not appear as straightforward application of procedures and rules. The problems encountered by such professionals require them to interpret the problem and write it in terms of mathematical notation before creating models within the constraints of the problem. The representation and interpretation of the solutions often appear in different formats including charts, graphs, written directions, etc (Lesh & Zawojewski, 2007).

2.7.3 A new perspective

Lester (2013, p.255) states that there is an alternative view to his previously discussed simplistic view, which also operates at two different levels: 'the everyday world of problems and the abstract world of mathematical concepts, symbols and operations'. The main difference between this view and the previous is that the mathematical processes are in a dynamic state, for example, are continuously being learned rather than already learned. Within this view, the problem solver must identify relationships between the steps in the mathematical process and the consequential actions on the elements in the problem. This means that the problem solver has an understanding and can develop the ability to create abstract written records of their actions. Within this view, the problem solver can move between the context of the real-world problem and the mathematical context. Consequently, if the problem solver cannot remember the correct procedure to use, they are capable of using the information provided to determine what to do. Although Lester states that this is an improvement on the simplistic view of problem-solving, he criticises this new perspective as it does not take into consideration the metacognitive activity of the problem solver. Lester states that continuous comparison of conclusions between the original context of the problem and the mathematical representations should take place throughout the process of problem-solving, which is a crucial component of success in solving complex mathematical problems. Lester continues that the amount of comparison that is needed throughout solving a problem distinguishes between routine and non-routine problems. The below diagram is a representation of the complex mathematical activity that takes place when problem-solving.

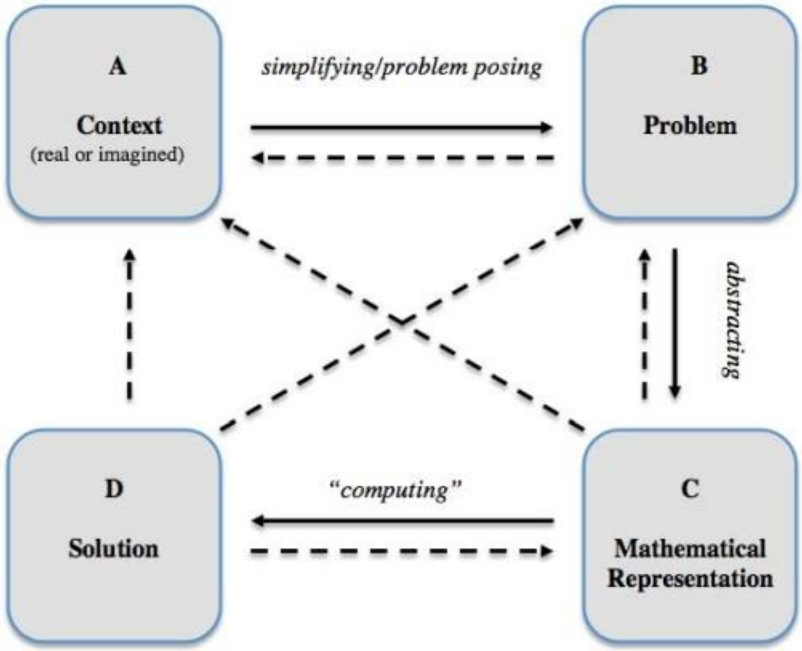

Figure 4: A model of complex mathematical activity (Lester, 2013, p.258)

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2.8 Constructivism

Constructivism is a theory of learning, the central claim of which is that "knowledge is actively constructed by the cognizing subject, not passively received from the environment" (Lerman, 1989). This places the learner at the centre of the learning, actively engaged in building their knowledge (Pritchard, 2009). Constructivists take the following view: 'We learn best when we actively construct our own understanding' (Pritchard, 2009). Given the involvement of the learner, constructivism is positively viewed within mathematics education (Lerman, 1989). Within the US, constructivism has been associated with moving mathematics education in a positive direction as learners are actively involved in their own learning and a focus is placed on conceptual understanding (Schmittau, 2004).

The roots of constructivism are attributed to the work of Piaget (Lerman, 1989; Pritchard, 2009; Sjøberg, 2010). Piaget's work focused on the development of knowledge and questioned how new knowledge is constructed (Sjoberg, 2007). The work of Piaget is mostly associated with cognitive constructivism (Pritchard, 2009). Noddings (1990, p. 9) describes how cognitive constructivists believe that 'knowledge is constructed and that the instruments of construction include cognitive structures that are either innate or are themselves products of development construction'. Within general constructivist theory, there are also different theories such as radical constructivism (von Glasersfeld, 1990) and social constructivism (Vygotsky & Cole, 1978).

Radical constructivism has been described by von Glasersfeld (1995, p. 18) as follows: 'It starts from the assumption that knowledge, no matter how it be defined, is in the heads of persons, and that the thinking subject has no alternative but to construct what he or she knows on the basis of his or her own experience'. Radical constructivism holds that there is not one correct fixed method for teaching, but rather advocates for teachers to use their imagination at opportunities.

Social constructivism is associated with the work of Vygotsky (1978) which involves focusing on understanding the social and cultural conditions for learning (Sjoberg, 2007). The emphasis of social constructivism is on the 'interaction between the learner and others' (Pritchard, 2009, p.24) through the collaborative nature of learning (Sjoberg, 2007). The interaction with others may be in the form of dialogue with a peer or a more knowledgeable person (Pritchard, 2009). A major attribute of Vygotsky's work was the *zone of proximal development* which he described as (Vygotsky & Cole, 1978, p. 86);

> "the distance between the actual developmental level as determined by independent problem-solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers."

This demonstrates the idea that by working with peers or through guidance from a tutor, learning can be maximised. From working independently only, the learner does not have the opportunity to reach their full potential. From interacting and cooperating with others in their environment, internal development processes are activated and these processes become part of the learners' 'independent development achievement', once they are internalised (Vygotsky & Cole, 1978, p. 90).

While there are differences between constructivist views, there is a general agreement on the following (Noddings, 1990, p. 10):

1. All knowledge is constructed. Mathematical knowledge is constructed, at least in part, through a process of reflective abstraction.

2. There exist cognitive structures that are activated in the processes of construction. These structures account for the construction; that is, they explain the result of cognitive activity in roughly the way a computer program accounts for the output of a computer.

3. Cognitive structures are under continual development. Purposive activity induces transformation of existing structures. The environment presses the organism to adapt.

4. Acknowledgment of constructivism as a cognitive position leads to the adoption of methodological constructivism.

a. Methodological constructivism in research develops methods of study consonant with the assumption of cognitive constructivism.

b. Pedagogical constructivism suggests methods of teaching consonant with cognitive constructivism.

The NCCA (2005) advocates for a constructivist approach to be adopted by teachers in order to develop relational understanding. This development is done by giving students the opportunity to explore concepts, reflection, and activities. The NCCA (2013) continues to promote the constructivist approach at both Junior Cycle and Leaving Certificate level. Students' Leaving Certificate mathematics experience should be a constructivist approach to allow students to understand their mathematical experiences and solve problems that they may come across in their lives. Similarly, a constructivist approach is recommended at Junior Cycle with the new Junior Cycle mathematics specification has an increased focus on students actively learning in mathematics (NCCA, 2013).

According to Gresalfi & Lester (2009) one of the aims of constructivist teachers is to help students become proficient in selecting appropriate procedures in solving problems. In a constructivist-oriented problem-solving class, students are expected to collaborate and share ideas which are supported by the teacher (Chapman, 1999) and students create their own strategies for solving a problem (Carpenter et al., 1999). It is important that while students have a sense of autonomy over their problem-solving strategies, teachers need to identify when they should intervene if a student is pursuing an approach that is completely unsuitable or if a student is stuck (Chapman, 2015).

Guidelines have emerged for teachers from De Corte et al., (1996) to support the dispositional view and constructivist approach to mathematics as follows:

- Induce and support constructive, cumulative, and goal-oriented acquisition processes in students.
- Enhance students' self-regulation of their own learning processes.
- Embed learning as much as possible in authentic contexts that are rich in resources and offer ample opportunities for interaction and collaboration.
- Allow for the flexible adaptation of instructional and emotional support, taking into account individual differences among students.
- Facilitate the acquisition of general learning strategies and problem-solving skills embedded within the mathematics curriculum.

Teaching mathematical problem-solving involves the teachers' view of problem-solving in their classroom as described in Section 2.5.2. Despite the different views that teachers may have, teaching mathematical problem-solving involves selecting problems (Lester, 2013), guiding students through problems or allowing them to devise their own strategies (Lubienski, 1999), selecting appropriate strategies to meet students' needs (Depaepe et al., 2013), and building mathematical proficiency (Kilpatrick et al., 2001) (see Section 2.4.2). The key component of teaching problem-solving is the development of students' learning and problem-solving skills.

This coincides with a constructivist approach which puts the learner at the centre and focuses on integrating new knowledge to gain a deeper understanding.

2.9 Summary

The review of the literature started with the exploration of the different definitions that are offered for a ‘mathematical problem’ and mathematical problem-solving. The ‘Three Key Characteristics’ of problem-solving were outlined along with the criteria for a mathematical problem. The prominent role that problem-solving plays in mathematics education both in Ireland and at an international level was discussed. As problem-solving is of explicit importance in the curricula, the role of the teacher is of equal importance. The different aspects of teaching mathematical problem-solving are outlined, along with the preparation of mathematics teachers. Looking beyond the classroom, the application of problem-solving in real-life contexts is considered. Finally, the suitability of the constructivist approach and problem-solving is illustrated. The decisions made in this study were grounded in the literature and will be described in detail in Chapter 4.

CHAPTER 3: THEORETICAL FRAMEWORK

This chapter provides an overview of frameworks that describe the capacities required for the effective teaching of problem-solving. The framework is necessary to provide a structure for the study and specifically outline the capacities required by teachers, which would then be investigated in this study. The chapter will conclude with a detailed description of the framework that is employed in this study.

3.1 Overview of different frameworks

Researchers have developed different frameworks for investigating the different capacities needed to effectively teach problem-solving.

3.1.1 Lester’s framework

Lester (2013) outlines a framework for research on mathematics teaching that is also applicable for research on problem-solving instruction. This framework is made up of four categories: 1) Non-classroom factors, 2) Teacher planning, 3) Classroom processes, and 4) Instructional outcomes. The first category, non-classroom factors, includes the teachers' and students' knowledge, beliefs, attitudes, emotions, and dispositions. This category also includes *task features* - the characteristics of the mathematical task that is the focus of a problem-solving activity. These task features include: syntax, content, context, structure, and process (Lester, 2013). *Syntax* refers to the arrangement of words and symbols in the task. *Content* features refer to mathematical meanings in the problem whereas *context* features refer to the non-mathematical features such as the format of the presentation of information. *Structure* refers to the logical-mathematical properties such as the particular representations of the task. *Process* refers to the interaction between the problem-solver and the task whereby the task determines a particular pathway to achieving a solution. The second category, teacher planning, includes the decisions made by the teacher before, during, and after instruction. These decisions involve, amongst others, instructional materials and teaching methods. Classroom processes involve both the teacher's and students' knowledge and affect, and their respective behaviours. Lester identifies that during instruction both the teacher's and students' behaviours and thinking are aimed at achieving predetermined goals. Within this framework Lester notes that teachers' thinking is restricted to the facilitation of students' problem solving, not general instruction. The final category, instructional outcomes, includes three types of outcomes; student outcomes, teacher outcomes, and incidental outcomes. Student outcomes include both the immediate and long-term effects on student learning. An example of this is a change in a student's ability to apply a problem-solving strategy (Lester, 2013, p.270). Along with a change in skill, change in students' beliefs and attitudes towards problem-solving and confidence as problem-solver is also an element of student outcome. Teacher outcomes include changes to beliefs and attitudes, planning, and classroom behaviour. Lester highlights that it is reasonable to suggest that experience affects teacher outcomes as each problem-solving situation that a teacher is involved in results in changes. Incidental outcomes are possible outcomes that may potentially arise such as increased performance or increase in parental interest.

3.1.2 Active Learning Framework

A study which focused primarily on problem posing was conducted by Ellerton (2013). In this study, a framework called the Active Learning Framework is presented which shows the position of problem posing in mathematics classrooms. Ellerton outlines that while problem-solving has been at the forefront of curriculum and classroom practices, there has been little focus placed on problem-posing. Yet, without a problem being posed, problem-solving cannot take place. This study took place in a university setting with 154 pre-service middle-school mathematics teachers taking part. There is an assumption among pre-service mathematics teachers that they will be able to use external resources to source mathematical problems for their students (Ellerton, 2013). The participants of this study undertook two types of problem posing activities. The first activity was referred to as 'routine problem-posing' activity. This activity involved participants working in groups to solve given problems. Next, they were asked to create other problems that had a similar structure to the given problem but had different contexts. Activity two was referred to as 'project problem-posing' activity. Here the participants worked in pairs to solve a given problem. After this, each pair had to create two problems which encompassed the same mathematical concept as the given problem but different context. Each pair had to present their problems to the class and ask the class to solve one of their created problems. The class was encouraged to share their strategies and solutions. Finally, as part of this activity each pair had to provide multiple solutions to each problem along with reflections of their experience. In addition to the two problem-posing activities, the participants completed a questionnaire about their attitudes towards problem-posing. Each response was given a numerical value from 0 to 5. The author notes that the problems created by the pre-service teachers often had imperfections in logic or wording. Ellerton advocates that the skill of problem-posing should be developed in teacher education programmes for teachers to then promote problem-posing in their own students. It is highlighted that the ability to pose mathematical problems "lies at the heart of understanding and developing mathematical ideas – yet it is an under-utilised tool in mathematics teaching and learning" (Ellerton, 2013, p.100). While this framework clearly focuses on problem posing, the diagram shows that there are other capacities that teachers need to possess. The framework implicitly suggests that teachers must have the ability to select and pose problems for students, and support students in their problem-solving attempt which requires understanding of the mathematical concept and prediction of student solutions. Furthermore, teachers must support students in students' problem posing by providing structures to students.

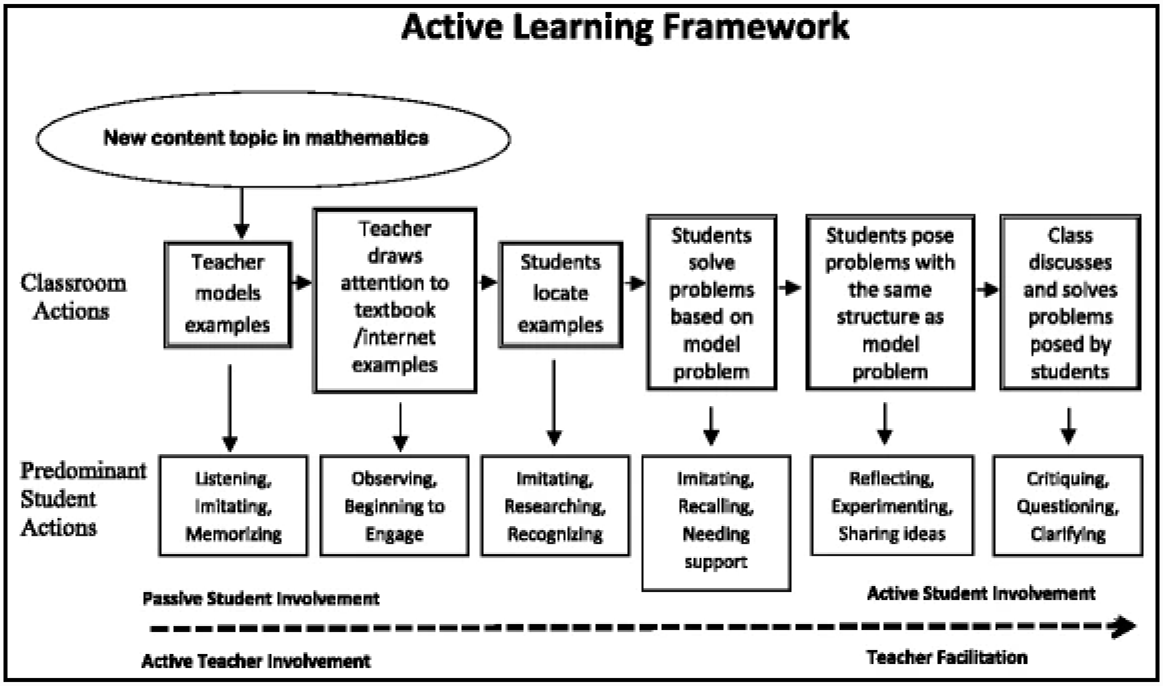


Figure 5: Active Learning Framework (Ellerton, 2013)

3.1.3 F-TAPS framework

A study was conducted in an Irish university which aimed at developing and implementing a framework for teaching and assessing problem-solving (F-TAPS) in mathematics (Guerin, 2017). This study involved conducting an intervention with pre-service post-primary mathematics teachers. The aim of this intervention was to develop the problem-solving ability of the participants and increase the participants' awareness of mindset, metacognition, and problem-solving models. The intervention lasted six weeks in duration. The intervention involved the pre-service teachers working in groups of three or four to discuss and solve problems. Before moving onto the next problem, each member of the group must understand the solution or proof. It is the role of the instructor to check the progress of the groups and rearrange members, if necessary, to ensure groups are working at a similar rate. The instructor may offer suggestions to a group on how to improve through questioning. The groups had not previously seen the problems and were given a whiteboard that they could work at as a group. Each problem was designed to be possible to solve within one class period. In addition, problems were assigned to be completed individually as homework. Assessments were completed pre and post the intervention evaluating the problem-solving process of the participants. The F-TAPS consisted of six components: knowledge, intrinsic motivation, mathematical thinking, mathematical proficiency, assessment, and teaching strategies (Guerin, 2017). The first component, knowledge, consisted of participants being presented with different heuristics (Mason et al., 2011; Polya, 1945; Schoenfeld, 1980) along with Tall's (2008) framework. The participants were exposed to problems of a variety of contexts and conceptual components and for a selected number of these problems, participants were asked to complete a metacognitive journal entry. The component, *Intrinsic motivation*, involved mindsets, intuition, and interest. Participants were given a presentation on mindsets at the first class of the intervention which gave information on fixed and growth mindsets and how to develop them. When completing problems, participants were asked to present, explain and defend their solutions and share these solutions with peers. In the pre-assessment stage of the intervention, participants were asked to state their interests. The problems that were selected for the intervention were based on the development of the participants' *mathematical thinking* and *mathematical proficiency.* The assessment element of the framework involved assessing the participants' solutions for demonstration of mathematical proficiency, problem-solving skills and competencies based on Polya's (1945) heuristic. The final component of this framework is the teaching strategy which used the Modified Moore Method as the mode of instruction (Guerin, 2017).

The phases of the problem -solving process are: reading and understanding, planning and solving, and solution and checking (Guerin, 2017). Improvements were seen in all three areas. There was no time limit given for the pre and post-tests in problem-solving. Analysis of the mean time spent by the participants engaging in the pre and post-tests showed a statistically significant increase suggesting an increase in perseverance. It is interesting to note that three participants completed the pre and post-tests but did not take part in the intervention. All three of these participants demonstrated lower scores in all three phases of problem solving and a decrease in the time spent engaged in the problems. The study also investigated the mindsets of the pre-service teachers which were also evaluated pre- and post-intervention using a questionnaire. The results showed that there was a statistically significant increase in the mindset score of the participants. This study also involves research of problem-solving capacities of pre-service mathematics teachers in a university setting.

3.2 Chapman's framework

To identify what capacities teachers need to teach problem-solving effectively, Chapman (2015) conducted an extensive review of the literature with research articles dating from 1920 to 2015. Chapman (2015) identifies that teachers need to have knowledge of teaching problem-solving, and not just problem-solving abilities in order to teach problem-solving effectively. From this review of the literature, Chapman states that there are three main categories that make up the mathematical problem-solving knowledge for teaching. These components are: 1) Problem-solving content knowledge (PSCK), 2) Pedagogical problem-solving knowledge (PPSK), and 3) Affective factors and beliefs. These three components are made up of six different capacities. Chapman's framework is central to this thesis, and so we review these components in some detail.

Problem-solving Content Knowledge (PSCK)

PSCK is made up of the following three capacities: knowledge of problems, knowledge of problem-solving, and knowledge of problem-posing.

Knowledge of problems

This describes the need for teachers to understand the nature of problems. This understanding is an influential factor in the teacher's ability to select and design mathematical problems. Consistent with the Three Key Characterizations highlighted above, Chapman notes that a teacher should see problems as mathematical tasks that do not have a clear solution. Chapman maintains that when selecting problems, the teacher must be aware of the potential impact of the characteristics of the problems on their students. For example, Silver (1985) shows that students experience greater difficulty with multistep problems than one-step problems. Chapman outlines that it is important for teachers to know different types of problems to effectively teach problem-solving. An example of research into the development of the pre-service teachers' knowledge of problems is a study conducted by Arbaugh and Brown (2004). This study focused on developing the pre-service teachers' understanding of the thinking a mathematical task requires. Chapman (2015) outlines the work of other researchers who found that secondary school students find abstract problems considerably more difficult than concrete problems (Caldwell & Goldin, 1987), and that students have difficulty in understanding the text of a word problem more than the actual solution (Lewis & Mayer, 1987). Since problem-solving is specifically identified as a key element across all strands of the Irish post-primary mathematics curricula, then it is crucial for mathematics teachers in Ireland to understand precisely what is meant by a mathematical problem.

Knowledge of problem-solving

Teachers should be proficient in problem solving and in understanding the nature of approaches to problem solving. Chapman (2015) outlines that it is important for teachers to be proficient in problem-solving in order for them to: understand students' solutions, understand the implications of different approaches, and be able to identify whether or not these approaches will be effective. Problem-solving proficiency allows teachers to make connections between the mathematics in the students' various solutions to the same problem. While problem-solving proficiency is seen to be important for assisting students, it does not mean that teachers must be expert problem-solvers but rather be experienced in a variety of problems and understand what successful problem-solving involves (Lester, 2013). It is suggested that there are five components of mathematical proficiency: 1) conceptual understanding, 2) procedural fluency, 3) strategic competence, 4) adaptive reasoning, and 5) productive reasoning (Kilpatrick et al., 2001). Chapman compares the relationship between these components as they apply to problem-solving proficiency and the skills that Schoenfeld (1985) outlines are crucial for successful problem solving. Schoenfeld (1985) proposes that: sufficient resources and competency in using them, heuristic strategies, metacognitive control, and appropriate beliefs are necessary for successful problem solving. Kilpatrick et al., (2001) states that the components of mathematical proficiency are not one-dimensional and are interdependent. Chapman develops this idea and proposes that since mathematical proficiency is interwoven, then problem-solving proficiency is too. Problem-solving proficiency is defined as "what is necessary for one to learn and do genuine PS successfully"

(Chapman, 2015, p.9). She suggests that to support students in developing their problem-solving proficiency, teachers must be able to solve the problems and also understand the elements associated with the development of problem-solving proficiency. Chapman (2015) also notes that one aspect of the nature of problem solving involves understanding that problem solving is not only a process, but a way of thinking. She suggests that to teach for problem-solving proficiency, the teacher must be aware of the many problem-solving models that exist and understand these to know what the student must do and the thinking that must occur during the process to work towards a solution.

Knowledge of Problem Posing

Problem-posing is defined as the generation of new problems and the re-formulation of given problems (Silver, 1994). From a teaching perspective, this capacity involves a teacher's ability to generate new problems and to adapt existing problems for their students' needs. It is important for teachers to have this skill as it can have a positive influence on students' mathematical thinking and therefore improve their problem-solving ability (English, 1997, cited in Chapman, 2015). Chapman outlines findings by Silver et al. (1996) which found that both practising and preservice teachers created problems that were either predictable, undemanding, ill-formulated or unsolvable, when trying to make expansions on given problems. Similarly, Ellerton (2013) found that problems posed by prospective mathematics teachers contained mistakes in wording or logic. It is beneficial for prospective mathematics teachers to experience problem-posing during teacher education programmes to subsequently increase the probability of them incorporating problem-posing in their classrooms. Within an Irish context, textbooks play a significant role in mathematics teaching (O'Sullivan, 2017). However, on analysis of the post-primary textbooks that are used, 74% of the tasks were classified as 'not novel'. Additionally, the majority of the exercises that were analysed in terms of mathematical reasoning required imitative and algorithmic reasoning. The tasks would suggest that post-primary students would get insufficient exposure to non-routine problems through textbooks with 89% of tasks solvable using imitative reasoning (O'Sullivan, 2017). Since the majority of textbooks are centred around routine tasks, it is imperative that PSMTs have developed problem-posing skills as they would not have the resource of textbooks to draw upon for problems.

Pedagogical Problem-solving Knowledge

The two capacities that make up PPSK are the knowledge of students as problem-solvers, and the knowledge of instructional practices.

Knowledge of students as problem solvers

Chapman suggests that in order for a teacher to help a student develop problem-solving skills, the teacher needs to have knowledge of students as problem solvers. This knowledge includes understanding the difficulties students have while problem-solving and the characteristics of successful problem solvers. Chapman highlights that it is important for the teacher to be able to interpret the difficulties from the perspective of the student to gain understanding of the students' needs. Chapman advises that teachers should be aware of the characteristics of 'good' problem solvers and the heuristics employed and dispositions of these solvers to help them promote these behaviours in their students. It has been shown that a teacher's understanding of the way that students think improves the quality of their teaching of problem solving (Maher et al., 2014).

Knowledge of problem-solving instruction

This capacity involves teachers' knowledge of instructional strategies and metacognition to support problem-solving proficiency. Teachers need to understand the implications that the teaching strategy that they choose to use has on students' problem-solving proficiency. Chapman (2015) highlights the work of Kilpatrick (1985) who outlines the drawbacks of showing the students a method and then the students practising similar methods compared to the advantages of the teacher employing a constructivist role in the classroom. A teacher who adopts a constructivist role involves teaching via problem solving. This is where mathematical procedures and concepts are learned through problem solving rather than where the learning outcomes focus on being proficient in problem solving (Lester, 2013). Lester (2013) states that teachers who teach via problem solving must be skilled at selecting good problems, listening to and observing students while they undertake a problem, and asking the right questions. In addition to this, Lester identifies the need to be competent at problem solving methods and making these accessible to students. Lester further points out that it is important for teachers to also ensure that students are challenged by the chosen problems and that these are not rendered straightforward by the teacher's input. Teachers must understand when to probe the students and when to resist making comments (Brodie, 2009; Ollin, 2008). According to Gresalfi & Lester (2009), one of the aims of constructivist teachers is to help students become proficient in selecting appropriate procedures in solving problems. During problem-solving instruction, the teacher needs to be competent in their strategy to overcome some of the following challenges: deciding when and how to intervene and assist a student while retaining the students' ownership of their work; what to do when a student is spending a considerable amount of time with an unproductive approach. In addition, it is essential for teachers to identify if different approaches are productive or not and understand why so.

The final category identified by Chapman covers affective factors and beliefs.

Affective Factors and Beliefs

In considering the role of affective factors and beliefs in a teacher's overall capacity to teach problem-solving, Chapman (2015) refers to Polya (1962) who noted the importance of a teacher's positive attitude in aiding students in problem solving. The teaching of problem solving does not simply rely on the techniques employed but it "comes from the identity and integrity of the teacher" (Palmer, 1998, p. 149) meaning "we teach who we are" (p.2). As summarised by Van Zoest and Bohl, (2005), beliefs incorporate conceptions that are based on values, which regularly provide explanations for responding in certain ways to particular types of knowledge in different situations. Lester and Kroll (1993) declare that the affective domain is an important contributor to problem solving behaviour. The affective domain includes attitudes, feelings and emotions. Beliefs impact on problem solving performance since beliefs contain their subjective knowledge about self, mathematics and the topics dealt with in particular mathematical tasks (Lester and Kroll, 1993). As mentioned above, productive disposition is highlighted as one of the five interrelated strands that mathematical proficiency consists of (Kilpatrick et al., 2001). Philipp (2007) suggests that if degree programmes that prospective teachers undertake are to develop mathematical proficiency then productive disposition must be included in order to ensure that graduates will create a positive mathematical learning environment for their students.

The understanding of the students' own beliefs is required by the teacher. Furthermore, the teacher must try to develop the students' belief in their ability, their appreciation of understanding concepts and of the importance of the role of word problems as part of mathematics, and a belief that effort will improve their mathematical ability (Kloosterman and Stage, 1992). Dweck (2017) highlights that there is an increasing amount of research that identifies mindsets as a key contributor of students' academic performance in mathematics. She outlines two types of mindsets: 1) a fixed mindset, believing that intelligence or ability is a fixed trait, and 2) a growth mindset, believing that abilities can be developed. Dweck states that students who have a growth mindset are at a significant advantage to students who are of a fixed mindset. In research conducted by Dweck, it was found that students with a growth mindset cared more about learning and also demonstrated a greater belief in the influence of effort on their grades than students with a fixed mindset. Similarly, it was found that those having a growth mindset reacted in a more positive manner to setbacks than those with a fixed mindset.

The teacher plays a crucial role in the creation of a growth mindset classroom as they make the decisions in relation to challenging and supporting the students to the right level (Boaler, 2016). Students benefit from these classrooms as they gain confidence in their mathematical abilities. The first decision that a teacher needs to make is the tasks that the students will work on. These tasks should be open-ended and 'teach important mathematics, inspire interest, and encourage creativity' (Boaler, 2016, p. 115). The selected tasks should be complex problems which produce success and failure, but students remain positive throughout. These tasks should be at a level of difficulty that allows students to make connections and promote understanding. If students are working on closed questions, then a fixed mindset is promoted through the idea that maths is a fixed subject with right-or-wrong solutions (Boaler, 2016). To promote a growth mindset in students, teachers must be aware of the influence that their words have. Teachers should always encourage students to instil positive messages. While it is more obvious for teachers to communicate positively to students who are motivated, it is even more important for teachers to encourage positive expectations to students who appear unmotivated or who struggle. The praise that the teacher offers should be centred around good thinking, hard work, and persistence (Boalar, 2016). When teachers select open-ended tasks, it is probable that students will experience elements of struggle. While it is the role of the teacher to help students progress, teachers need to be aware that by helping too much they are reducing the cognitive demands of the task resulting in students missing learning opportunities. Boaler acknowledges that teachers have to use their professional judgement to decide when students can tolerate more struggle without getting discouraged. However, the overall advice for teachers is that not helping students is 'often the best help we can give them' (Boaler, 2016, p. 179).

Chapman states that it is important to recognise that the three categories described above - problem-solving content knowledge, pedagogical problem-solving content knowledge, and affective factors and beliefs - are interdependent, and understanding the connections between them is as important as the knowledge of each component. Chapman (2015) refers to the definitions for problem-solving provided by Polya (1945) and the NCTM (2000) who outline that problem-solving is the engagement in a task for which the solution method is not known in advance and it is not immediately clear how to achieve the aim of the task. These definitions align with the Three Key Characteristics of a problem.

Table 2 below gives an overview of key points of the different frameworks described above.

Lester (2013)	Ellerton (2013)	Guerin (2017)	Chapman (2015)
Non- classroom factors Teacher planning Classroom processes Instructional outcomes	Scale of student activity Listening and imitating Observing Researching and recalling Experimenting and sharing Critiquing and clarifying	Knowledge Intrinsic motivation Mathematical thinking Mathematical proficiency Assessment Teaching strategies	Problem-solving Content Knowledge Pedagogical problem-solving Affective factors and beliefs

Table 2: Overview of key points for different frameworks

When comparing the frameworks provided by Chapman (2015) and Lester (2013), both frameworks account for both in-classroom and outside of classroom elements. With the exception of the framework developed by Ellerton (2013), each of the frameworks explicitly involve the affective domain and problem-solving proficiency. The framework outlined by Ellerton (2013) focuses on problem-solving and problem-posing which are also included in the framework outlined by Chapman (2015). The frameworks provided by Ellerton (2013) and Lester (2013) are both situated in classroom settings while the framework provided by Guerin (2017) is based in a university setting. The framework outlined by Chapman aligns with the capacities outlined in the other frameworks mentioned above (Ellerton, 2013; Guerin, 2017; Lester, 2013). Chapman's framework provides capacities which are based in a classroom setting and also, capacities which are applicable to a university setting. Based on a review of nearly 100 years of research, Chapman's framework may be viewed as comprehensive, covering all aspects of teachers' work in relation to teaching and assessing problem solving. In addition, it aligns with the 'Three Key Characteristics' of problem-solving. This study will be based on Chapman's framework.

CHAPTER 4: METHODOLOGY

The overall purpose of this chapter is to outline the way that this research was conducted. It will begin by describing the research assumptions upheld by the researcher, the research problem, and the research questions. Next the overall methodology will be described, followed by the methods of data collection and analysis that were employed by the researcher to address the research questions.

4.1 Research Assumptions

In the context of qualitative research, a paradigm is described by Guba and Lincoln (1994, p. 107) as a "set of basic beliefs…(which) represent a worldview that defines, for its holder, the nature of the 'world', the individual's place in it, and the range of possible relationships to that world and its parts". Each paradigm consists of four components: ontology, epistemology, methodology, and methods (Scotland, 2012). The ontological and epistemological assumptions influenced the methodology and methods employed in a study (Scotland, 2012). A research paradigm, and the associated assumptions, form the foundations of a study.

Ontological assumptions are centred around the "nature of reality and the nature of things" (Cohen et al., 2011). As described by Guba and Lincoln (1994), this involves questioning the form of reality and what can be known about it. There are two possible views that can be taken regarding ontological assumptions: subjective and objective (Bahari, 2010). Subjectivism is associated with the belief that "social life is the product of social interactions and beliefs of the social actors (Bahari, 2010, p. 23). As a result of these social interactions, social phenomena are continually revised (Saunders et al., 2012). In contrast, objectivism is associated with the view that social phenomena and social actors exist independently of each other (Saunders et al., 2012).

Epistemological assumptions are concerned with knowledge (Bahari, 2010; Cohen et al., 2011; Saunders et al., 2012). This involves how knowledge is formed, how it is acquired, and how knowledge is communicated to other people (Cohen et al., 2011). There are two main epistemological paradigms: positivism and interpretivism. The underlying beliefs of a positivist view is that the "social world exists externally" (Bahari, 2010, p. 23). A positivist perspective is scientific, where the view is that science offers the clearest ideal of knowledge (Cohen et al., 2007). This results in a greater emphasis being placed on quantifiable observations and statistical analysis (Saunders et al., 2012). Epistemology is a key feature of both theory and practice of mathematics education (Ernest, 2012). A positivist stance can cause substantial difficulty when studying human nature in situations such as studying teaching and learning (Cohen et al., 2011). In contrast to this, interpretivists view knowledge as being subjective to a person, meaning that the construction of knowledge is dependent on personal experiences and interpretations (Cohen et al., 2011; Saunders et al., 2012). Learning and understanding mathematics is influenced by the learning environment (Steinbring, 2006) and the construction of knowledge is also influenced by social interactions (Steinbring, 2006). The social interactions are not just between the student and teacher, but student collaborations are an integral part of developing knowledge (Vygotsky & Cole, 1978). Regardless of the epistemological view held by a researcher, the methods and methodologies employed in a study are underpinned by epistemology (Ernest, 2012), which is established on the ontological position (Scotland, 2012).

This study undertakes an interpretivist epistemological position which is founded on a subjectivist ontological perspective which upholds the view that the construction of knowledge is personal and is influenced by social interactions and the environment. Considering this view, the facilitation of the module was conducted using a constructivist approach.

4.2 Research problem

Problem-solving plays a prominent role in the post-primary mathematics curriculum in Ireland (NCCA, 2013, 2017). However, the results of PISA and TIMSS show that students in Ireland had a lower performance in translating real-world situations into mathematical representations (an important aspect of problem solving) than in applying mathematical procedures (a capacity that exists separately to the higher–order thinking skills associated with problem solving) see (Shiel & Kelleher, 2017). Teaching impacts on performance, so it is important to study the effectiveness of this teaching. In this study we focus on the capacities that teachers need to effectively teach problem-solving. This research addresses the development of capacities for teaching problem solving amongst pre-service, post-primary mathematics teachers. A key concern is what these capacities are: what do mathematics teachers need to know, what skills do they need to have, and what are the attitudes that they should hold in order to be effective teachers of mathematical problem solving?

The teacher plays a key role in relation to problem-solving in a mathematics classroom (Lester, 1994). It has been shown that external influential factors such as the teaching environment and teachers' beliefs result in many curricular concepts being taught in different ways (Rickard, 1996). The teacher's decision to include problem-solving within their teaching of the curriculum correlates positively with the degree of subject-matter knowledge of mathematics of the teacher, and their proficiency in connecting mathematics and problem-solving (Rickard, 1996). While problem-solving is now part of the mathematics curriculum, the teacher's view on problem-solving as either an end result or a means of teaching affects the manner in which problem-solving is conducted in the classroom (Lester, 2013). Furthermore, the teachers' perspectives of the role of problem solving in teaching mathematics influences their approach to teaching problem solving. The two main distinguished approaches are ‘teaching via problem solving’ where mathematical procedures and concepts are learned through problem solving, and ‘teaching for problem solving’ where the end result is being proficient in problem solving (Lester, 2013). Neither approach is seen to have better credentials than the other but both should be combined, and both have a place in the development of problem solving skills (DiMatteo & Lester, 2010; Stein & Silver, 2003). According to Lester (2013), mathematics teachers do not necessarily need to be expert problem solvers but should have an extensive understanding of what successful problem-solving entails, and should be experienced in problem solving.

Teacher education programmes are viewed as a critical stage in teachers’ development (Teaching Council of Ireland, 2017). When entering teacher education programmes, future teachers possess ideologies on what being a teacher entails, based primarily on their previous experiences (Taguchi, 2007) . The beliefs of pre-service teachers surrounding the importance of problem solving in teaching mathematics play a key role in the way in which students learn, therefore determining their practice as future teachers and the future of their students’ problem-solving exposure (Kayan & Cakiroglu, 2008). Kayi-Aydar (2015) highlights the fact that the prospective teachers’ identity is not based on the attributes they desire to possess when teaching once qualified. However, a teacher’s identity is greatly influenced by their experiences in formal teacher education processes and is reflected in their teaching (Chapman, 2014). During teacher education programmes, prospective teachers’ beliefs regarding teaching and learning should be considered and challenged: they will bring these beliefs forward into their professional practice (Teaching Council of Ireland, 2017).

This research focussed on developing the capacities of PSMTs to effectively teach problem-solving in order to equip them with the skills needed to develop problem-solving skills of post-primary students in Ireland.

4.3 Research Aims

The aim of this research was to investigate and develop the capacities of pre-service mathematics teachers to effectively teach mathematical problem-solving. This involved focusing on the capacities of: knowledge of problems, knowledge of problem-solving, knowledge of problem-posing, affective factors and beliefs as outlined by Chapman (2015) as the aspects that need to be considered for teaching mathematical problem-solving. These capacities are discussed in the next section. To develop these capacities, an intervention was conducted in the form of a university initial teacher education module. This module also forms part of the focus of this thesis: a complementary aim of the research is to determine how the capacities mentioned above can be developed in the university setting - that is, away from the school setting. This will be discussed in Chapter 6.

4.4 Framework

4.4.1 Theoretical Framework

As discussed in detail in Chapter 3, the theoretical framework of this study is provided by Chapman (2015) with a particular focus on four capacities: *knowledge of problems, knowledge of problem-solving, knowledge of problem posing,* and *affective factors and beliefs* (Chapman, 2015).

4.4.2 Action Research

Action research is defined as 'a collaborative transformative approach with joint focus on rigorous data collection, knowledge generation, reflection and distinctive action/change elements that pursue practical solutions' (Piggot-Irvine et al., 2015, p. 548). The purpose of action research is to allow practitioners to develop areas of study through evaluation in order to improve and direct decision making and practice (Corey, 1953). There are multiple characteristics of action research, some of which include: enhancing the competencies of participants, is undertaken directly in situ, uses feedback from data in an ongoing cyclical process, includes evaluation and reflection, and is methodological eclectic (Cohen et al., 2018). This study focused on evaluating and developing the PSMTs' capacities to effectively teach problem-solving (Chapman, 2015), which involved continuously adapting the module the the PSMTs were undertaking. The adaptations to the module were based on the action research process as outlined by Cohen et al. (2018, p.450): 1) Identification of the area of study/problem, 2) discussion between interested parties, 3) review of the literature, 4) review initial problem, 5) section of research methods, 6) choice of evaluation methods, 7) implementation of intervention, 8) interpretation of data and drawing of inferences. In this study, the research problem was identified (Section 1.2.1) and a review of the literature was conducted (Chapter 2). The review of the literature informed the discussions between the researcher and research supervisor when making decisions regarding the methodology and research instruments. The research methods for this study was a mixed methods approach as discussed in Section 4.7. The results of the qualitative and quantitative research instruments were analysed and these inferences formed the basis of the adaptations to the module.

4.5 Research Questions

Research questions were developed to address the research problem outlined in Section 4.1. These research questions informed the research conducted throughout this study. As we have seen, a high level of importance is placed on problem solving within the specifications for mathematics in post-primary education in Ireland. Likewise, the teacher plays a key role in supporting their students' development of problem-solving capacities, and Chapman (2015) has described the capacities that teachers themselves must hold in order to carry out this work. With these points in mind, this research aims to provide an insight into the inputs required in teacher education programmes that will support teachers' abilities to teach problem solving effectively.

We consider these issues in relation to pre-service mathematics teachers in Dublin City University (see Section 4.7). The main research questions are motivated by considering the capacities identified by Chapman (2015) as being important elements in the teaching of problem-solving. The focus of this study will be on the elements of preparation for teaching that take place solely within the university setting. Lester (2013) states that it is teacher education programmes that have the most potential to influence teachers' skills in teaching problem-solving effectively. This study resides in the body of work which focuses on developing teachers' knowledge in a university setting (see Section 2.7). The two capacities that are in the component of PPSK both involve the interaction of PSMTs with students in a classroom setting. These components are beyond the scope of this study.

The questions that emerged were as follows:

Question 1: What do pre-service teachers understand a mathematical problem to be?

A: Are PSMTs proficient at classifying mathematical tasks?

B: Does an adaptation of the intervention that focuses on providing a rationale for task-classification lead to enhancement of PSMTs' capacities in task-classification?

Question 2: Are pre-service teachers proficient in problem-solving?

A: How does PSMTs' problem-solving proficiency change over the duration of the intervention?

B: Did the ongoing adaptations of the intervention lead to a greater enhancement of the problem-solving capacities of successive cohorts of PSMTs?

C: Are taught strategies implemented while problem-solving throughout the different iterations of the intervention?

Question 3: What are pre-service teachers' capacities in relation to problem posing?

A: How do pre-service teachers' capacities in relation to problem posing change over the duration of the intervention?

B: Did the ongoing adaptations of the intervention lead to a greater enhancement of the problem-posing capacities of successive cohorts of PSMTs?

Question 4: What beliefs and affective factors do pre-service teachers hold regarding problem-solving?

A: How does the affective domain of one cohort of PSMTs change over the course of the final iteration of the intervention?

Research Question	Capacity

Question 1	Knowledge of problems
Question 2	Knowledge of problem-solving
Question 3	Knowledge of problem-posing
Question 4	Affective factors and beliefs

Table 3: Association of each research question with the relevant capacity

Question 1 focuses on the *Knowledge of problems* capacity. This research question aims to investigate the PSMTs understanding of what the term 'mathematical problem' means. Question 1B seeks to investigate if the requirement of the provision of a rationale enhances the PSMTs' capacity to classify tasks. The correct understanding of this definition is vital for teachers as they select and design mathematical tasks that are indeed problems.

Research Question 2 investigates the *Knowledge of problem-solving* capacity. The problem-solving proficiency of a teacher has been identified as an important aspect in the teaching of problem-solving (Chapman, 2015). Along with teachers' own proficiency, it is highlighted that teachers should be aware of problem-solving heuristics and understand the process of these strategies. A key part of this study involves tracking the evolution of these capacities over the course of the module and over different iterations of the module.

Question 3 investigates the *Knowledge of problem posing* capacity. This capacity involves teachers' ability to generate new problems along with reformulating given problems.

Question 4 involves investigating the *Affective factors and beliefs* capacity. This question focuses on investigating the prospective teachers' beliefs about problem-solving rather than the beliefs of students. As stated above, the beliefs of a teacher play an integral role in the learning environment created for students with regards to problem-solving (Philipp, 2007). A key part of this study involves tracking the evolution of these capacities over the course of the module. This is necessary to evaluate the impact of the module on the participants' capacities.

Throughout this document, we will attend to each of the research questions and discuss the respective research conducted.

4.6 Research Phases

The research phases of the study are shown in Figure 6 below. There are six phases of the study with four iterations of data collection. Action research was utilised in this study meaning that in each iteration of the module, data was collected and informed changes to the module for successive cohorts.

Phase 1: A comprehensive review of the literature about the teaching of problem-solving was conducted with particular focus on the role of problem-solving in the post-primary mathematics curricula in Ireland. The review of the literature led to the identification of conceptual frameworks which have been used in the research of problem-solving. The theoretical framework for this research was provided by Chapman (2015). The module was pre-existing to the study so a module review also formed part of Phase 1. The decision was made to focus on the capacities of pre-service mathematics teachers in relation to teaching mathematical problem-solving, and how these can be developed in the university setting. This decision was made as initial teacher education programmes offer the opportunity for conducting an intervention to challenge beliefs and build understanding (Teaching Council of Ireland, 2017). Similarly, all PSMTs in this study had experience of the post-primary mathematics curriculum in Ireland where problem-solving is integral. This decision resulted in excluding capacities which involved a classroom setting.

Phase 2: Phase 1 informed the generation of the research instruments which would be used in this study. The first iteration of the data collection was conducted. Cohort 1 of participants completed the Indiana Mathematics Belief scale (IMB) scale, a voluntary focus group completed one 'Think Aloud' interview, a 'Task Sorting Activity' was completed, and tutorial worksheets were completed.

Phase 3: Phase 3 involved the second iteration of data collection. Cohort 2 and cohort 3 completed the IMB scale. A focus group from Cohort 2 completed two 'Think Aloud' interviews each at the beginning and the end of the module. The 'Task Sorting' activity was completed. Three activities were completed to investigate the PSMTs' knowledge of problem posing. The problems in the first interview were changed based on observations from Phase 2. Tutorial worksheets were completed.

Phase 4: Phase 4 involved the third iteration of data collection. Cohort 3 completed the IMB scale. A focus group from Cohort 2 completed two 'Think Aloud' interviews each at the beginning and the end of the module. The problems in the first interview were changed based on observations from Phase 2. The 'Task Sorting' activity was adapted from Phase 2 and Phase 3 to provide more information about the rationale for the participants' choices. This adapted activity was piloted with a group of in-service post-primary mathematics teachers. Tutorial worksheets were completed. Changes were made to the research instruments to accommodate restrictions due to Covid-19.

Phase 5: Phase 5 involved the fourth iteration of data collection. Cohort 4 completed the IMB scale. Cohort 4 completed three mathematical problems of which, two of the three problems were used in 'Think Aloud' interviews in previous iterations. Coinciding with the three mathematical problems, participants completed open-ended affective questions. Cohort 4 completed three extension problems to investigate their knowledge of problem posing.

Phase 6: Post-module interviews were conducted with a voluntary focus group of participants of Cohort 4 to assess the effectiveness of the intervention.

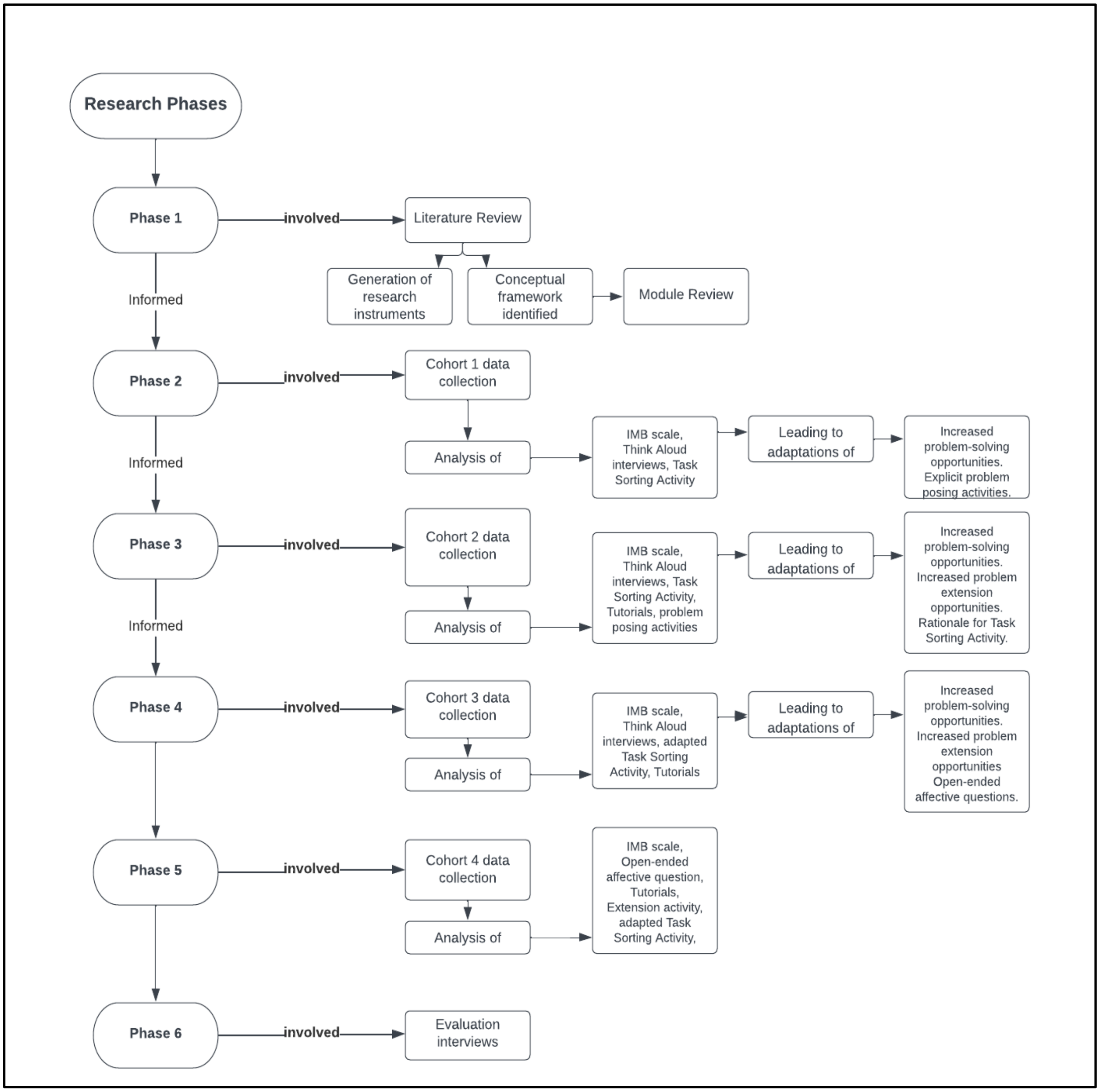


Figure 6: Research Phases

4.7 Mixed Methods

The overall methodology for this study is a mixed methods approach. Creswell & Garrett (2008) describe mixed methods as an approach to inquiry that involves the researcher connecting quantitative and qualitative data in some way in order to make a 'unified understanding of a research problem' (Creswell & Garrett, 2008, p.2). In order to answer research questions, a mixed methods approach involves the researcher collecting, analysing and then combining qualitative and quantitative data. They note that the use of a mixed methods approach is common in social, behavioural, and health sciences. A review of studies that identified as using a mixed methods approach was conducted by Tashakkori & Creswell (2007). They found that, to be classified as mixed methods, at least two of the following need to be involved: 1) Two types of research question (at least one which requires the collection of qualitative data and at least one that requires the collection of quantitative data); 2) the manner in which research questions are developed (preplanned or participatory); 3) two types of sampling procedures; 4) two types of data collection procedures; 5) two types of data; 6) two types of data analysis; and 7) two types of conclusions (Tashakkori and Creswell 2007, p.4). Creswell (2012) notes that a mixed methods approach produces a greater understanding of the research problem than a singular quantitative or qualitative method would provide.

Thomas (2006) defines the characteristics of a mixed methods approach as the use of: 1) quantitative and qualitative methods within the same research study, 2) a clear account of how the quantitative and qualitative aspects of the research relate to each other, with a greater emphasis on how triangulation is used, 3) a research design that specifies the sequencing and priority that is given to the qualitative and quantitative elements of data collection and analysis, and 4) pragmatism as the philosophical underpinning for the research.

From the above descriptions of the characteristics of mixed methods, the integration of quantitative and qualitative data appears to be the core characteristic. This study involves research questions that require the collection of qualitative data that is open-ended, and also quantitative data.

4.7.1 Qualitative

Qualitative research is described by Strauss and Corbin (1994, p. 17) as "any kind of research that produces findings not arrived at by means of statistical procedures or other means of quantification". Qualitative data can be collected through multiple forms including but not limited to, interviews, field notes, observation and reports (Cohen et al., 2011). Creswell (2012) notes that qualitative forms of data collection involve open-ended questions so that participants are not constrained by the researcher and have freedom to voice their own thoughts. This research produces qualitative data which is then analysed. Cohen (2011, p. 537) describes qualitative data analysis as ‘organizing, accounting for and explaining the data’. This involves analysing the data through the viewpoint of the participants' definitions and observing categories, patterns, and recurring themes. While qualitative research typically deals with fewer participants than quantitative research, the data collected tends to be richer and provide greater detail than quantitative data (Cohen et al., 2011).

Qualitative data allows the researcher to (Cohen et al., 2011, p. 258):

- Describe, portray, summarise
- Discover patterns,
- Generate themes
- Understand individuals and idiographic features
- Understand groups and nomothetic features
- Prove or demonstrate,
- Explore and test
- Discover commonalities, differences, and similarities
- Examine the application and operation of the same issues in a different context.

The researcher's decision on the purpose of the qualitative data collection influences the analysis of the data. Creswell (2012, p.237) provides the following image as a description for qualitative data analysis.

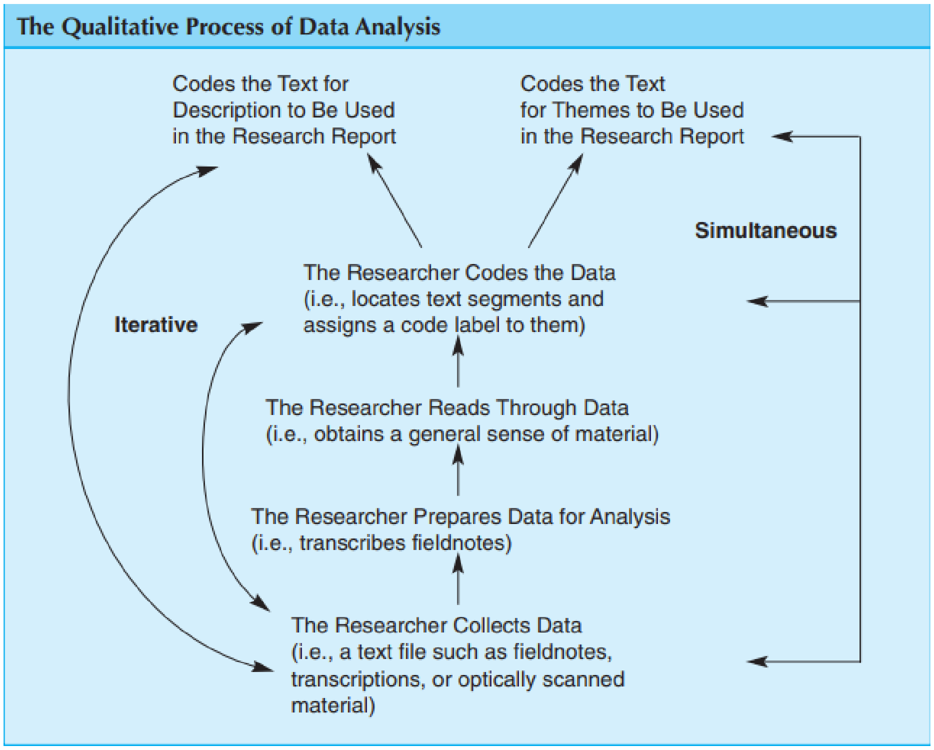


Figure 7: Qualitative data analysis (Creswell, 2012)

The steps in this image show that after data collection, the data is prepared for analysis which in some cases involves transcription. Next the researcher reads through the data, divides the data, and begins to code any recurring patterns that are present. This is an iterative process which results in the production of categories or themes which share similar characteristics defined by the researcher. The analysis of qualitative data involves interpretation of the data by the researcher (Cohen et al., 2011). Creswell (2012) describes this interpretation as the researcher making judgements of the description for different themes or categories which represent the data collected. The qualitative data in this study included 'Think Aloud' interviews (Research Question 2), open-ended affective questions (Research Question 4), and problem-posing activities (Research Question 3).

4.7.2 Quantitative research

Quantitative research is described as scientific investigation which involves experiments and methods which focus on quantifying performance (Hoy & Adams, 2015). This type of research involves using statistical analysis of numerical data (Cohen et al., 2011) whereby information is presented in numerical form and is quantified and expressed in statistical terms (Golafshani, 2003). Analysis of quantitative data involves inputting numerical data into a statistical software package. The researcher then has the opportunity to use descriptive statistics or inferential statistics to obtain conclusions about the population (Cohen et al., 2011)**.** Quantitative research has a range of advantages such as a large capacity for generalizability and quantification of large data sets (Mat Roni et al., 2020). The quantitative elements of this study included: the Indiana Mathematics Belief scale (Research Question 4), the Mathematical Problem-Solving Rubric (Research Question 2)

4.7.3 Triangulation

Triangulation is defined as "the use of two or more methods of data collection in the study of some aspect of human behaviour" (Cohen et al., 2011, p. 141). It is noted that triangulation offers the researcher increased confidence in the validity of their research. This increased confidence is achieved when the outcomes of different methods correspond to each other (Cohen et al., 2011). In addition to this, triangulation also allows researchers to ascertain a greater understanding of the problem (Thurmond, 2001). Triangulation may lead to three types of results: converging results, complementary results, and contradictions (Flick, 2009, p. 450). If the results are completely convergent, then one method may have been more appropriate. However, if the results of different methods complement or contradict to a certain degree, it is recommended to look for theoretical explanations of where the differences come from and what these differences say about the research (Flick, 2009). Triangulation can take the form of (Thurmond, 2001, pp. 255–257):

- *Data triangulation:* the use of both structured and unstructured techniques to collect data from different groups;
- *Data analysis triangulation:* the use of two or more methods for analysing data.
- *Investigator triangulation:* input from multiple researchers in order to reduce potential bias and in turn increase credibility;
- *Methodological triangulation:* the combination of both qualitative and quantitative approaches within the same study;
- *Theoretical triangulation:* the employment of more than one theoretical hypothesis in order to reduce the possibility of alternative explanations.

This study employs a mixed method approach with the use of both qualitative and quantitative data, the use of different data analysis methods, and data collection from different cohorts of participants. Data triangulation is evident in this study through the utilisation of interviews, surveys, scenario activities and written problem-solving attempts. This research utilises methodological triangulation in the form of between-methods triangulation through the employment of both qualitative and quantitative methods. Investigator triangulation was employed in this research through the research supervisor independently analysing samples of the qualitative data to ensure reliability and reduce potential bias.

4.8 Setting

The researcher is based in Dublin City University in Ireland. Our focus is on pre-service teachers who will qualify as post-primary mathematics teachers in the Irish education system. The education system in Ireland at post-primary level is broken into two cycles. Junior Cycle has a duration of three years and is for students typically aged 12-15 years old. At the end of the Junior Cycle, students undertake state examinations called the Junior Certificate. After Junior Cycle, there is an optional year called 'Transition year'. Students then progress onto Senior Cycle, which is a two-year programme. At the end of the two years, students complete the Leaving Certificate exam. The results from the Leaving Certificate are converted into a points system from which college places are allocated. It is a requirement for all students to study mathematics at both Junior Certificate and Leaving Certificate. In the Irish education system, subjects are divided into different levels that students then select depending on ability. At Senior Cycle, there are three levels in mathematics: foundation level, ordinary level, and higher level.

4.9 Participants

The participants in this study are pre-service mathematics teachers (PSMTs) undertaking a concurrent initial teacher education programme. The participants are students of two different programmes of study. One group of students are in their first year of third-level education and the other is comprised of second-year students. Both cohorts were taking a module that includes the study (and practice) of mathematical problem-solving. Graduates of the relevant programmes are qualified to teach mathematics to Leaving Certificate level in Ireland, and typically go on to do so. Thus, preparing the PSMTs for the task of teaching problem-solving is a key concern of the programme team. Graduates of the first programme will be qualified to teach mathematics to all levels in post-primary schools. Within the second programme, there is an option to choose two out of three subjects for which graduates are then qualified to teach. All graduates of this second programme will be qualified to teach mathematics to Junior Cycle higher level, and graduates of this programme who select mathematics as a core subject will be qualified to teach mathematics at all levels in post-primary schools. The vast majority of these students do indeed select mathematics as one of their qualification subjects. Both of these programmes are concurrent teacher education programmes which means that graduates are fully qualified to teach at post-primary level in Ireland without further university education. All the participants completed their second-level education in the Irish system, and thereby completed the Leaving Certificate curriculum.

Data collection has taken place over four years involving four cohorts of students. Each cohort was a mixture of PSMTs from two different programmes and was undertaking a module as part of their third-level education focusing on problem-solving. Within this module, the PSMTs were exposed to a Rubric Writing approach (Mason et al., 2011) to problem-solving. This approach is discussed in detail in Chapter 5. The four cohorts of participants in this study will now be presented.

Cohort 1

The first cohort of research subjects was drawn from a set of 38 students. Participants were recruited on a voluntary basis at the beginning of the module. They were given a plain language statement about the overall study and had the option to remove themselves from the study at any point. A total of 38 participants volunteered to make up this cohort. 21 of these were first-year students and 17 were second-year students. The module, which was part of the PSMTs' university programme, consisted of 10 lectures and 8 workshops which focused on problem-solving. Both the lectures and workshops were 50 minutes in duration. All 38 students gave permission for their tutorial work to be collected and analysed as part of the study. Nine participants volunteered as a focus group to take part in interviews investigating Research Question 1 and 2.

Cohort 2:

The second cohort of research subjects was drawn from a set of 55 students. Participants were recruited on a voluntary basis at the beginning of the module. They were given a plain language statement about the overall study and had the option to remove themselves from the study at any point. A total of 55 volunteered to participate in the study. 36 of these were first-year students and 19 were second-year students. The module, which was part of the PSMTs' university programme, consisted of 10 lectures and 8 workshops which focused on mathematical problem-solving. Both the lectures and workshops were 50 minutes in duration.

Cohort 3:

The third cohort of research subjects was drawn from a set of 49 students. Participants were recruited on a voluntary basis at the beginning of the module. They were given a plain language statement about the overall study and had the option to remove themselves from the study at any point. A total of 49 participants volunteered to make up this cohort. Due to Covid-19, the module was delivered wholly online using the Zoom platform in the form of live lectures, breakout rooms for workshops and supplementary content.

Cohort 4:

The fourth cohort of research subjects was drawn from a set of 50 students. Participants were recruited on a voluntary basis at the beginning of the module. They were given a plain language statement about the overall study and had the option to remove themselves from the study at any point. A total of 50 participants volunteered to make up this cohort. The module consisted of 10 lectures and 8 workshops which focused on mathematical problem-solving.

4.10 Data Collection

Table 4 below outlines the instruments used to collect data relevant to each research question. The cohort of participants involved in each iteration of each instrument is also outlined.

Research question	Capacity	Instrument	Cohort involved
Question 1	Knowledge of problems	Task Sorting activity	Cohort 1, 2 Adapted after Cohort 2, Cohort,3,4
Question 2 A	Knowledge of problem-solving	'Think Aloud' interviews	Cohort 1,2,3
Question 2 A	Knowledge of problem-solving	Tutorial worksheets	Cohort 1,2,3
Question 2 A	Knowledge of problem-solving	Three mathematical problems	Cohort 4
Question 2 C	Knowledge of problem-solving	'Think Aloud' interviews	Cohort 1,2,3
Question 3	Knowledge of problem-posing	Activity 1 Activity 2 Activity 3 Extension problems	Cohort 2 Cohort 4
Question 4	Affective factors and beliefs	IMB scale	Cohort 1,2,3,4
Question 4	Affective factors and beliefs	Open-ended affective questions	Cohort 4
	Module Effectiveness	Post-module interviews	Cohort 1 and 4

Table 4: Outline of research instruments for each research question

The instruments mentioned in Table 4 are described in detail below along with the analysis process of each instrument. Any changes made to the methods of data collection between cohorts will also be discussed for the relevant research question.

4.10.1 Question 1: What do pre-service teachers understand a mathematical problem to be?

A: Are PSMTs proficient at classifying mathematical tasks?

B: Does an adaptation of the intervention that focuses on providing a rationale for task-classification lead to enhancement of PSMTs' capacities in task-classification?

This research question was developed in relation to the Knowledge of Problems capacity outlined by Chapman (2015). Chapman states that to effectively teach for problem-solving proficiency, the teacher must be able to both select and design problems. This means that teachers must understand the characteristics of a problem so they can decide whether a mathematical task is a problem or an exercise. The following activity was designed to investigate the PSMTs' understanding of a mathematical problem.

Methodology:

The activity was designed in the form of a worksheet, the 'Task Sorting Activity'. This worksheet consisted of a selection of tasks. The participants were asked to decide whether each of the tasks would be classified as either an exercise or a problem. The participants had three options to choose from: *Exercise, Problem*, or *Not Sure*. Prior to completing this activity, the participants were introduced to the difference in definition of the terms problem: the task presents the person working on it with a clear goal; it is not immediately clear to the person how to achieve the goal; the person must organise prior mathematical knowledge to generate reasoning towards achieving the goal, and exercise: the task presents the person working on it with a clear goal;the person has access to (or knowledge of) a procedure that they can follow to complete the task; the person must organise prior knowledge to generate reasoning towards achieving the goal. The definitions for each were provided on the worksheet, and had been discussed previously in lectures. . The problems were taken from the NRich (NRich, 2019) website and from secondary school textbooks (see Appendix B). The exercises were taken from secondary school textbooks. To categorise the tasks, both researchers independently compared the task to the following two criteria of a problem: 1) there is a goal, 2) it is not clear how to reach the goal. The researchers agreed on all but two of the categorisations, which were subsequently resolved through a discussion resulting in a second round of independent categorisations of the tasks (O'Connor & Joffe, 2020) . The inter-rater reliability was 0.90. The questions that the researchers initially disagreed on were Question 7 and Question 11 which were then agreed upon on discussion.

This activity was based on the definition of a mathematical problem and the PSMTs' ability to identify the characteristics of a mathematical task that would correspond to a problem. This understanding of the nature of problems is vital for teachers' ability to select and design mathematical problems (Chapman, 2015). It is outlined by the NCTM (2000) that this selection of problems is difficult as it requires analysis of the characteristics of the tasks and anticipation of the mathematical ideas involved in the problems (NCTM, 2000). As stated by Yeo (2007), teachers must be able to differentiate between types of mathematical tasks in order to develop students' mathematical thinking since different types of tasks evoke different skills and thinking. The tasks which were included for classification in the 'Task Sorting Activity' comprised of a variety of tasks which we will now discuss.

A procedural task, which may also be called a routine task, involves the person applying previously learnt skills to a situation whereby it is immediately clear how to solve it (Yeo, 2007). While this may initially be a problem for a student who has not yet been taught the skill, with practice this type of task is procedural for the student (Yeo, 2007). While procedural tasks allow students to develop their proficiency in certain procedures, this type of task does not develop students' understanding of connections between topics and an understanding of the procedures that they are applying (Schoenfeld, 1988). Procedural tasks may also be considered as 'closed tasks' where there is a clear goal and one specific correct answer. There may be multiple methods to achieve this goal, for example, finding the roots of a quadratic equation has multiple approaches to finding a solution such as factorising or the quadratic formula. These methods are procedural and do not require mathematical thinking (Yeo, 2007). If a mathematical task requires some higher-level of thinking but can be solved using a procedure, it can be argued to be in the 'grey area' between procedural and problem-solving tasks (Yeo, 2007).

Mathematical tasks which involve words, can be referred to as 'word problems' however, they may not actually be mathematical problems but rather share the same characteristics as procedural tasks (Yeo, 2007). Moschkovich (2002) identifies that 'word problems' traditionally used in schools provide the opportunity for students to practise learnt procedures and methods rather than genuine problem-solving. Both procedural tasks and 'word problems' do not promote the development of new knowledge but are more appropriate for the consolidation of methods and facts (Orton & Frobisher, 2004) . Since 'word problems' are not necessarily mathematical problems, the tasks which were included in the 'Task Sorting Activity' which were text-based but classified by the researcher as an exercise are referred to as 'wordy questions'.

Following the completion of this activity by Cohort 1 and Cohort 2, the 'Task Sorting Activity' was adapted (see Appendix C). A pilot of this adapted activity was conducted with a group of six in-service post-primary mathematics teachers. In the revised activity, a rationale for each selection was also required of the participants. This involved reducing the number of questions from twenty questions to ten questions to accommodate the additional requirement of the rationale. The questions which were continued from the original to the adapted activity, were selected based on the results of Cohort One and Cohort Two along with the inclusion of a variety of tasks. In the selection of tasks for both activities, a variety of the types of tasks were included. In the selection of tasks for the adapted activity, this variety was maintained while reducing the number of questions. Additionally, the tasks which were retained had a variety in the level of success of the two initial cohorts. The new tasks were introduced to increase the variety of tasks, such as the addition of an investigative task which is neither an exercise or a problem. Participants were asked to explain why they chose the response *Exercise*, *Problem*, or *Not Sure* in relation to each mathematical task. The original task was adapted to include a rationale for classification to gain a deeper understanding into the PSMTs' understanding of the nature of a mathematical problem. To gain more information on the PSMTs' understanding of the definition of a mathematical problem, it was appropriate to ask for a rationale to provide an insight into what characteristics in the given tasks prompted their classification of the tasks. The rationale also provided an insight into what structures in the given tasks provoked classification such as structures or syntax of the tasks. In the selection of the tasks for the original Task Sorting Activity and the adapted version, it was ensured that participants would definitely have the prior knowledge required to ensure that lack of prior knowledge is not justification for classifying the task as a problem. Each task involved prior knowledge that would be obtained during the course of post-primary mathematics education.

4.10.2 Question 2: Are pre-service teachers proficient in problem-solving?

A: How does PSMTs' problem-solving proficiency change over the duration of the intervention?

B: Did the ongoing adaptations of the intervention lead to a greater enhancement of the problem-solving capacities of successive cohorts of PSMTs?

This research question was developed to investigate the mathematical problem-solving proficiency of PSMTs. As outlined above, Chapman (2015) highlighted problem-solving proficiency as an important capacity to effectively teach problem-solving. It is suggested that problem-solving proficiency is necessary for the teacher to be able to: understand students' approaches to a problem, predicting the outcome of these approaches, and understanding unusual solutions. Lester (2013) highlights that teachers should have a strong understanding as to what constitutes successful problem-solving and should be experienced problem solvers. To address this research question, two data-collection activities were designed and planned. These were tutorial sheets and semi-structured interviews.

Methodology:

First activity: Tutorial sheets.

As part of the module, the PSMTs are required to complete tutorial sheets. This involved a combination of working in groups and individually on different problems. The problems which were analysed were completed individually and selected by the researcher. These problems and the intervals between each problem for each cohort of PSMTs is outlined in Section 5.5.2. The tutorial sheets included problems and also gave recommendations for the use of Mason's Rubric Writing. Tutorials from specific weeks, whereby the problems were completed individually, were then analysed using a problem-solving proficiency rubric created by the University of Oregon (Oregon, 2011) (Appendix E). The problems which were attempted in groups were not included in the analysis. The problems which were analysed are: *Four-Legged Lawnmower* (Cohort 1, Cohort 2, Cohort 3, Cohort 4), *Professor on an Escalator* (Cohort 1, Cohort 2, Cohort 3), *Threaded Pins* (Cohort 2), *Pitstop* (Cohort 4), and *Cube on a Ladder* (Cohort 4). *Pitstop* (Problem 4) and *Cube on a Ladder* (Problem 5) were undertaken by the PSMTs in Cohort 2 and Cohort 3 who participated in the 'Think Aloud' interviews. These problems were selected for analysis as they assessed different aspects of problem-solving skills such as the possibility of using different representations and multiple strategies.

A rubric is a suitable tool for assessing problem-solving as it provides a comprehensive assessment of mathematical learning (Rosli et al., 2013). Rubrics offer an insight into a person's mathematical development (Rosli et al., 2013). In a research situation, rubrics can assess behaviour in an authentic problem-solving situation and include criteria for success (Docktor et al., 2016). Rubrics allow researchers to attain details on how participants solve problems compared to simple grading for correct answers (Hull et al., 2013). When using rubrics to gain this insight, it is important for the person to only consider what is written and avoid assuming correct thought processes that are not written (Docktor, 2009). It is common for rubrics to focus on general features of problem-solving (Docktor et al., 2016) which are independent of specific pedagogical approaches making the rubrics applicable for multiple situations (Hull et al., 2013).

Rubrics have been used extensively to gain an insight into behaviours while problem solving. One such study was conducted by Abdullah et al. (2010) in which a rubric was developed to evaluate 53 students' use of Polya's problem-solving heuristic, mathematical communication and teamwork. This rubric had scores on a scale of 1 to 4 points with 4 points indicating the maximum score. A maximum score was allocated to problem-solving attempts which demonstrated a complete use of Polya's problem-solving heuristic.

Another study was conducted by Docktor (2009) which focused on developing a valid rubric for assessing problem-solving in physics. This rubric consisted of a five-point scale under five headings. In this study, several rounds of testing were conducted with university students who attempted problems in different areas such as calculus-based mechanics and algebra-based mechanics. It was found that there was evidence of each heading in the students' written problem-solving attempts. Within this study, a focus group of participants also completed problems in a

‘Think Aloud’ manner in order to assess the extent to which the written responses correspond with the students’ self-reported thought processes.

To assess the PSMTs’ problem-solving proficiency, specific written problem-solving attempts which were individually completed during tutorials were analysed using a mathematical problem-solving rubric (MPSR) (Oregon, 2011) (see Section 5.5.2 for details of the selection of problem solutions analysed and for the timing of the relevant tutorials in the module). This rubric has been used to assess problem-solving proficiency of highly-able post-primary students (Fitzsimons, 2021).

The MPSR consisted of five headings with a maximum score of 6 points per heading meaning the maximum total score was 30. A score of 0 points was only possible if no attempt had been made. The headings of the MPSR are: *making sense of the task, representing and solving the task, communicating reasoning, accuracy, reflecting and evaluating.* Each heading will now be described:

- *‘Making Sense’:* Interpret the concepts of the task and translate them into mathematics.
- *‘Representing and Solving the Task’*: Use models, pictures, diagrams, and/or symbols to represent and solve the task situation and select an effective strategy to solve the task.
- *Communicating Reasoning:* Coherently communicate mathematical reasoning and clearly use mathematical language.
- *Accuracy:* Support the solution/outcome
- *Reflecting and Evaluating:* State the solution/outcome in the context of the task. Defend the process, evaluate and interpret the reasonableness of solution/outcome.

The MPSR was selected as the most appropriate instrument to measure the PSMTs’ problem-solving proficiency. Problem-solving proficiency involves a combination of the components of mathematical proficiency (see Section 2.4.2) and the components of successful problem-solving as discussed in Section 0. The headings of the MPSR above align with the components of mathematical proficiency as outlined by Kilpatrick et al. (2001) and also with some aspects of the

Rubric Writing approach to problem-solving (Mason et al., 2011) which was a central component of the module (see Chapter 4).

The first heading *Making Sense* involves interpreting mathematical concepts in the task corresponding to the *conceptual understanding* component of mathematical proficiency. The identification of what the question is asking aligns with the *Entry phase* of the Rubric Writing approach (Mason et al., 2011) and Polya's (1945) model for problem-solving.

The next heading, *Representing*, is consistent with the *strategic competence* strand of mathematical proficiency. Both of these involve the representation of the information in the problem through the use of models or diagrams and also the selection of an appropriate strategy. The use of diagrams and planning of a strategy are both characteristics of successful problem-solving (Mason et al., 2011; Polya, 1945).

The *Reflecting* heading corresponds to the *adaptive reasoning* strand of mathematical proficiency as both refer to the problem-solver reflecting on their work and also evaluating and justifying their attempt. The process of reflecting on a problem-solving attempt is regarded as having utmost importance in improving problem-solving proficiency (Mason et al., 2011). Another aspect of *adaptive reasoning* is 'logical thought' which can be seen in the *Communication* heading of the MPSR. The communication of logical reasoning is specifically outlined in the MPSR.

The final strand of mathematical proficiency, *Productive disposition* (Kilpatrick et al., 2001), is not addressed in the MPSR. However, through the use of 'Think Aloud' interviews as a second instrument in assessing problem-solving proficiency, this can be attended to.

Second activity:

Participants were recruited on a voluntary basis from Cohort 1, Cohort 2, and Cohort 3 while undertaking the module mentioned above. The PSMTs were interviewed on a one-on-one basis by the researchers. The interview consisted of the participants being asked to solve two problems in a 'Think Aloud' manner. To test the problem-solving proficiency of the pre-service teachers, qualitative analysis is required as the data is in text form. Interviews were not conducted with Cohort 4 because a greater focus was put on the collection of data from the MPSR. While interviews provide valuable information, it is difficult to use interviews with a greater number of participants (Docktor et al., 2016). Using a rubric allows a researcher to gain a greater insight into a large number of participants' problem-solving behaviours that would not be possible to assess through simple grading (Docktor et al., 2016)

A focus group of nine PSMTs from Cohort one volunteered to participate in an interview. The participants were asked to solve the problems following a 'Think Aloud' protocol (Salkind, 2010). Cowan (2019, p.1) describes the 'Think Aloud' process as "a voluntary activity in which learners, having been asked to tackle a relevant task, talk their thoughts out aloud, while engaging with the task". Cowan states that this approach can be beneficial in gaining a reliable insight into students' learning. An example of a study which used the 'Think Aloud' method was a study conducted by Rosenweig et al. (2011). The aim of this study was to investigate the processing differences during mathematical problem-solving between students of different academic ability. While the 'Think Aloud' method may alter the participants' problem-solving approach due to the environment in which they are working on a problem, it is a method that is widely used in education research to gather data on working memory of participants (Charters, 2003). The data on working memory shows the cognitive processing that occurs during the task (Young, 2005).

The interviews with the PSMTs were audio-recorded and transcribed. The interviews ranged in duration from 04:54 to 31.36 minutes. The length of time was determined by the participants. The interviewer only intervened if the participant was stuck for a considerable length of time. The interviews came to an end when the participants had nothing further to add to their attempt.

Interviews were conducted with nine participants from Cohort 1. These interviews were conducted in week six of the module. Prior to the interviews, the participants experienced two weeks of the module which explicitly focused on problem-solving. At this point in the module, the PSMTs' had experience of using all aspects of the Rubric Writing approach in problems both individually and in groups. Subsequently, interviews with PSMTs from Cohort 2 and Cohort 3 were completed upon completion of the module in addition to early-stage interviews. These participants were given two problems: Problem One dealing with probability and Problem Two with geometry and trigonometry (see Appendix F). Both problems were taken from the NRICH website (NRICH, 2019) where the problems are organised by age categories with the difficulty of the problems measured on a scale of 1-3 stars (3 stars being most difficult). Problem One is classified as a 2-

star short probability problem, appropriate for students aged 14-16. Problem Two is classified as a 3-star short trigonometry problem, appropriate for students aged 14-16. The PSMTs could be expected to use these problems or similar in their own teaching and should therefore have the capacity to make a good attempt at solving them.

Participants from Cohort 2 completed two rounds of interviews. The first interview was conducted at the start of week five of the module, whereby the participants would have received one week of lectures that specifically focused on problem-solving (see Appendix G). While PSMTs in Cohort 1 and Cohort 3 had received two weeks of problem-solving instruction prior to interviews, all cohorts had been exposed to the key elements of the Entry Phase. This change in timing was unavoidable due to availability of the PSMTs who volunteered to participate in the interviews. Five participants completed this round of interviews. Again, the participants were asked to complete two problems. Problem One is classified as a 2-star short numbers problem, appropriate for students aged 11-14. Problem Two is classified as a 3-star short trigonometry problem, appropriate for students aged 14-16. The second interview was conducted one week after the completion of the module. Three of these five participants completed the second interview. The problems were not the same as the problems in the interview conducted near the beginning of the module. In order to match the problems in level of difficulty, s problems which were graded at the same difficulty level were chosen in each iteration. Problem One was classified as a 3-star short ratio, proportion and rate of change problem, appropriate for students aged 14-16. Problem Two was classified as a 3-star short trigonometry problem, appropriate for students aged 14-16.

Participants from Cohort 3 completed the same problems as participants in Cohort 2 in two rounds of interviews. Five participants from Cohort 3 volunteered to do both interviews. The first interview was completed in week 5 of the module after two weeks of instruction on problem-solving. The second interview was completed one week after the completion of the module. We acknowledge that there is a difference in the number of weeks of instruction that the PSMTs had received before conducting the interviews, so it is not a case of directly comparing like for like. However, key aspects of the Rubric Writing approach had been learnt so we were satisfied that the learning experience was similar meaning all cohorts could be considered together. The interview participants were volunteers, and it was due to their availability that dictated the timing of the interviews.

In all cases, the interviews were conducted in a semi-structured manner where participants were asked to complete the problems in a ‘Think Aloud’ manner. It is stated by Jacobse and Harskamp (2012) that the ‘Think Aloud’ protocol provides information on the metacognitive processes that occur while problem-solving. This is supported by Depaepe et al., (2013) who outline that ‘Think Aloud’ protocols are used to investigate the metacognitive skills in successful and unsuccessful problem solvers. An advantage of the ‘Think Aloud’ protocol is that the information presented is immediate and gives insight into the metacognitive processes as they occur. This is seen to be more beneficial than retrospective interviews as the information is not distorted by memory (Jacobse & Harskamp, 2012). Schoenfeld (1985) states that the analysis of ‘Think Aloud’ transcripts allows for the assessment of the quality of the decisions in a problem attempt and how these decisions impact on attaining a solution. It has been shown that data derived from ‘Think Aloud’ situations may not represent the true behaviours while problem-solving due to the environment for which the participants attempt the problem in (Schoenfeld, 1985). To overcome this possibility a combination of methodologies for assessing problem-solving behaviours is required (Schoenfeld, 1985), hence the implementation of the MPSR alongside the interviews.

Mason (2002) highlights the need for rigorous planning when preparing for interviews. Mason states that if the interviewer is unwilling to plan in advance, then they must be capable of thinking quickly, ensuring their questions and conversation stay aligned to the research questions in order to generate relevant data. Mason suggests that this can be difficult to do while maintaining a pleasant social setting. As a result of this, Mason states that it is not possible to gather data in a completely unstructured manner in a qualitative interview because the contributions of the interviewer can impact the generated data (Mason, 2002).

Before conducting the problem-solving interviews, the two problems were attempted by the researcher using a variety of mathematical approaches in order to identify the strategies that the participants might pursue. From this, it was possible to try and identify any misconceptions the participants may have had and thereby question them on their approach. A list of questions was written out ensuring that they were not leading questions. These questions were: 1) Why did you choose that approach? 2) Why did you decide to restart? 3) Why did you draw a diagram? 4) Why did you decide to stop? The purpose of these questions was to gain an insight into the thought process of the PSMTs in choosing their approach, choosing to use certain representations, and their decision to stop or restart the question.

The participants were verbalising their thoughts and actions throughout the interviews. The interviewer only directly questioned the participant if they were stuck for a considerable amount of time without making any comment or they had exhausted their strategy. When each participant decided to finish their attempt, they were asked why they had selected the strategy they did and why they felt they were stuck in order to identify the rationale behind the selected approach. As Mason (2002) states, transcripts are only a partial record of the interview as they merely record the verbal interactions. Thus, the written work produced by the PSMTs during the protocol was retained for analysis.

Analysis

First activity:

The tutorials of Cohort 1, Cohort 2 and Cohort 3 were evaluated using the mathematical problem-solving rubric (MPSR) provided by the University of Oregon (2011) . Participants in Cohort 4 completed mathematical problems at three different stages throughout the module. The decision was made to focus on using the MPSR to assess the PSTMs' problem-solving proficiency in Cohort 4 and not conduct interviews. The MPSR allowed for the assessment of a greater number of problem attempts than interviews (Docktor et al., 2016) which provided a greater insight into the problem-solving proficiency of the PSMTs across a broader range of topics. Two problems attempted by Cohort 1 were analysed: 'Professor on an Escalator' in Week Seven after two weeks of problem-solving instruction and practice, and 'Four-Legged Lawnmower' in Week Nine after four weeks of problem-solving instruction and practice. Based on the results of Cohort 1 and observations by the researcher in consultation with the facilitator, additional time was dedicated to problem-solving within the module. This allowed for the addition of more problems for analysis in Cohort 2. Three problems attempted by Cohort 2 were analysed: 'Professor on an Escalator' in Week Five after two weeks of problem-solving instruction and practice, 'Four-Legged Lawnmower' in Week Six after three weeks of problem-solving instruction and practice, and 'Threaded Pins' in Week Eight after five weeks of problem-solving instruction and practice. Due to the time constraints of a shortened university semester, it was not possible to conduct three problems for analysis with Cohort 3. Two problems attempted by Cohort 3 were analysed: 'Professor on an Escalator' in Week Four after three weeks of problem-solving instruction and practice, and 'Four-Legged Lawnmower' in Week Seven after five weeks of problem-solving instruction and practice. The PSMTs in Cohort 4 returned to the twelve week module length as experienced by Cohort 1 and Cohort 2. Based on the findings of previous iterations of the module, the time dedicated to problem-solving was increased for Cohort 4. Three problems attempted by Cohort 4 were analysed: 'Four-Legged Lawnmower' in Week Two after two weeks of problem-solving instruction and practice, 'Petrol Pitstop' in Week Six after five weeks of problem-solving instruction and practice, and 'Ladder Leaning on a Cube' in Week Ten after eight weeks of problem-solving instruction and practice. The breakdown of the instruction received by each cohort of PSMTs is detailed in Section 5.5.2.

The first problem attempted by Cohort 4 was sourced from Mason et al. (2011) and the other two were selected from NRich (2019) (see Appendix G). 48 participants completed problem one, 40 participants completed problem two, and 34 participants completed problem three. These written attempts were then analysed using the MPSR as described above. The rubric grading was provided to each participant after each attempt. However, the numerical values were omitted from the rubric so that the participants would focus on how they could improve rather than on the score.

All of the scoring from the MPSR was inputted into statistical software and analysed to find statistical differences. The data collected from Cohort 4 was analysed using SPSS to find the mean and standard deviation of each of the five headings of the MPSR. Cohen's d was calculated to determine the effect of the module on the participants' problem-solving proficiency (Cohen, 1977). The following criteria were used to analyse Cohen's d for each of the headings of the MPSR:

- 0 - 0.2 = weak effect
- 0.21- 0.5 = modest effect
- 0.51 - 1.0 = moderate effect
- > 1 = strong effect

(Cohen et al., 2011, p. 521)

Given that the number of participants was greater than 30, the data was assumed to be normally distributed and therefore it was possible to use parametric tests on the data (Ghasemi & Zahediasl, 2012). One-tailed t-tests were used to investigate if there were statistically significant differences between the means of the MPSR headings of the three problems. The one-tailed t-test was used since the data collections involved the same group of participants and the data was assumed to be normally distributed ($n>30$) (Cohen et al., 2011).

Second activity:

The analysis of the interview transcripts involved the iterative process of coding, comparing, and grouping the data with similarities to construct categories (Jones & Alony, 2011). This process starts with reading the transcripts multiple times to identify themes and keywords that appeared regularly. Next, systematic characterization of each piece of the data into categories was done. It is important to identify if categories shared similar properties and combined these. This process was repeated until clear distinctions were made between the categories and no piece of data could be associated with more than one category. Watling & Lingard (2012) state that it is important to focus on small units of data such as sentences within transcripts in the initial coding phase. This ensures that the smaller details are given sufficient attention which in the second phase of coding will build onto broader categories. This iterative process was also conducted by a second researcher and the categorisation of the data from both researchers was compared and produced an inter-coder reliability of 84%.

4.10.3 Question 2C: Are taught strategies implemented while problem-solving throughout the different iterations of the intervention?

The aim of this research question is to investigate if taught strategies are implemented by students who were undertaking the module in problem-solving which will be further described in Chapter 5. Mason's Rubric Writing approach was the main strategy taught to students during the module and students were also given the opportunity to practise the use of this approach in tutorials. It has been shown that it is beneficial for problem solvers to implement taught problem-solving strategies when attempting problems (Mason et al., 2011; Muir et al., 2008; Schoenfeld, 1992).

This research question seeks to investigate if the students employed the use of Mason's approach to aid their problem-solving attempts or if elements of the approach were evident. Schoenfeld (1980) states that a variety of heuristic approaches are necessary if one is to be successful at problem solving. The PSMTs were exposed to different heuristics in the course of the module but particular focus is put on the Rubric Writing approach (Mason et al., 2011). The selection of this heuristic is further discussed in Chapter 5. Heuristics is described as a general strategy, independent of any subject topic, that helps in the understanding of the problem and allows the problem solver to use their knowledge to solve it. Implementing heuristics, selecting strategies based on the problem solver's decision on what way to approach the problem and the recognition to abandon this when it does not work are key traits in problem solving (Garofalo and Lester, 1985). In a study conducted by Schoenfeld (1992) it was found that if the problem solver did not recognize that the wrong decision, such as strategy selection, was made and subsequently reversed then failure was imminent.

Analysis

The data collected from the 'Think Aloud' interviews mentioned above (see Section 4.10.2) were analysed using a different approach for this research question. The combined interview data and written work was rated in terms of the degree to which these evidenced implementation of the Rubric Writing approach of Mason et al., (2011). This was done by both researchers independently and then compared and revised as necessary. The analysis of the *Entry Phase* of Mason's Rubric Writing Approach was done by implementing a three-point grading scale. The descriptors for the three-point grading system are as follows: 0 points are awarded for no evidence where there is no referral to any elements of the *Entry Phase*. 2 points are awarded for significant evidence of the elements of the *Entry Phase*. This entails explicit structured use of *Introduce*, *I Know*, and *I Want.* 1 point is awarded for some use of the *Entry Phase* elements but with limited structure.

Depaepe et al. (2013) implemented a two-point grading system in a study which aimed to investigate students' application of heuristics. In the analysis of students' problem-solving response sheets, 0 points were awarded if the written response had no evidence of the application of a heuristic. 1 point was awarded if there was evidence of one of the eight characteristics of the heuristic that the researchers were interested in.

A three-point grading system was used in a study conducted by Nicol and Bragg (2009). This study was concerned with pre-service teachers' capacity to pose open-ended problems. Along with each posed problem, the pre-service teacher had to make a mathematical statement about the intended learning outcome of the problem. Each statement was graded on the following criteria: 0 points for no link between the statement and the problem, 1 point for a partial link between the statement and the problem, and 2 points for a strong link between the statement and the problem.

While the study conducted by Depaepe et al. (2013) can be seen as similar to this study through a shared goal in investigating the evidence of the application of a heuristic, it was decided that a three-point grading system was more suitable than a two-point grading system. A three-point grading system as seen in Nicol and Bragg (2009) offered a greater representation of the level of application of a heuristic through the identification of a partial use of the heuristic or an explicit use of the heuristic.

To identify evidence of applying these different aspects of the *Entry Phase,* we used the following characterizations (Mason et al., 2010). Mason describes *'I Want'* as directing attention to the task at hand or deciding what needs to be done in order to solve the problem. Evidence that we looked for was that the participant (re)stated precisely the goal of the problem (for example, the value of a positive integer *N* in Problem 1, and the value of a distance *x* in Problem 2). *'I Know'* refers to selecting all the relevant information given in the problem and identifying any associated mathematical concepts that are likely to be relevant. Evidence of *I Know* was only considered early on in the attempt for it to be relevant to the *Entry Phase. I Know* refers to when the participant recorded relevant information given in the problem and/or stated related mathematical facts. '*Introduce'* includes the introducing notation, organising the *I Know* elements, and representing information in the question through the use of tables, charts, and diagrams. Evidence of *Introduce* in this study was identified as the drawing of diagrams, the introduction of notation, and constructions within the given diagrams.

4.10.4 What are pre-service teachers' capacities in relation to problem posing?

A: How do pre-service teachers' capacities in relation to problem posing change over the duration of the intervention?

B: Did the ongoing adaptations of the intervention lead to a greater enhancement of problem-posing capacities of successive cohorts of PSMTs?

It is highlighted that problem posing is an important capacity for the teaching of problem solving (Chapman, 2015). According to Silver (1995, p. 69) "problem posing has long been identified as an important aspect of mathematical activity". Problem posing includes the ability to generate new problems and also adapt existing problems to cater for students' needs. It is important for teachers to have the capacity to generate diverse and meaningful problems in order to support the development of their students' problem-solving proficiency (Chapman, 2015). In addition to generating problems, teachers need to be proficient in analysing problems through the lens of anticipating the mathematical ideas that are involved in the problem along with the potential questions that their students may ask (NCTM, 2000). This analysis of the appropriateness of a problem by a teacher for a specific student is crucial for teachers (NCTM, 2000). According to Silver (1995), problem posing can occur in three different scenarios: a problem is generated from a specific situation, in the form of an extension which occurs after solving a problem and the problem-solving context is presented in a new situation, and posing problems during the problem-solving process whereby the problem solver changes their goal intentionally. Crespo (2003) highlights that while great focus has been placed on teachers' ability to solve problems, there has not been the same level of attention paid to their ability to construct and pose mathematical problems.

We will now discuss some studies which investigated prospective teachers' mathematical problem-posing proficiency.

Crespo (2003) conducted a study with pre-service elementary teachers as part of a mathematics teacher education program. The aim of the study was to investigate how pre-service teachers pose mathematical problems to pupils. The participants posed written mathematical problems to pupils who attempted the problems in a written form. These solutions were then returned to the preservice teachers. There was no verbal interaction between the teacher and pupils. The posed problems were analysed based on: the problem type, problem features, adaptation type, problem's source, associated questions and scaffolds, and associated reflections (Crespo, 2003, p. 250).

Another study involving pre-service elementary teachers was conducted by Chapman (2012) who aimed to investigate pre-service teachers' ability to pose a variety of mathematical problems. This study involved posing problems under various conditions such as generating a problem similar to a given problem, generating an open-ended problem, generating a problem to incorporate specific mathematics concepts, modifying a problem, and other situations. The analysis of the posed problems involved coding the problems through identifying the types and nature of the problems.

It was found that some of the problems posed were ill-formulated, lacked sufficient information, or were not mathematical in nature. However, it was noted that these poorly posed problems were not designed to be so intentionally, but rather the participants were not aware of these elements of the problems that they posed (Chapman, 2012).

The final study that we will discuss was conducted by Silber and Cai (2017). This involved investigating pre-service teachers' posing of problems under two conditions: free posing condition and structured posing. A *free posing* situation involves posing problems for an open mathematical situation. A *structured posing* situation involves posing problems based on a given situation or match the problem to a specified operation such as a specific solution. The analysis of the problems posed by the pre-service teachers involved a three-step approach. First, the problems were coded as a mathematical question that could be solved using arithmetic and mathematical reasoning. Responses not coded as meeting this criterion were coded as statements and nonmathematical questions. Next, the mathematical problems were coded as solvable or not solvable. A problem that was considered 'not solvable' either demonstrated insufficient information to find a solution or contained contradictory information. The final step assessed the complexity of the posed problems which had been classified as mathematical and solvable (Silber & Cai, 2017).

In our research, the problem posing capacity of PSMTs in Cohort 2 and Cohort 4 was investigated. Three different activities were designed and implemented to investigate the problem posing capacity of the PSMTs. These activities were first introduced for Cohort 2, which consisted of 55 participants. The first activity, Activity One, focused on the PSMTs' ability to select problems. Activity two focused on the generation of problems for a specific situation. Activity three focused on the reformulation of problems and open-ended problems. The activities were completed at intervals of one week in Week 10, Week 11, and Week 12 of the module. Participants in Cohort 4 completed extension problems which will be further discussed below. Due to the changes in the university semester in response to Covid-19, it was not possible to investigate PSMTs' problem posing proficiency with Cohort 3. As a result of Covid-19 the duration of the university semester was shortened meaning that the instrument used for the investigation of problem posing had to be revised for Cohort 4. We will now discuss the three activities completed by Cohort 2 followed by the extension activity completed by Cohort 4.

Methodology

Activity One:

This activity focused on investigating the ability of the PSMTs to be able to select a task that would constitute a problem for a student who possesses certain categorical characteristics. For this activity, the PSMTs would need to consider the definition of a problem which meets the criteria of the Three Key Characteristics. The participants were given 13 scenarios (see Appendix H) which each outlined the following information: the year the student was in school, their level of study (higher or ordinary level), and their topics of prior knowledge. A mathematical task was then stated (one for each of the 13 scenarios). The participants were asked to decide if the task was a problem for the hypothetical students described and to justify their answer.

This activity coincides with Research Question 1 but there is the addition of the PSMT needing to consider the described student along with the definition of a mathematical problem. This activity aligns with the NCTM (2000) who states that the selection of problems requires both the characteristics of the tasks and also an understanding of the mathematical concepts involved in the problem. Activity One specifically outlines the prior knowledge of the student meaning that the PSMTs need to consider the mathematical concepts in the given task and decide based on the prior knowledge of the students and the Three Key Characteristics of problem-solving whether the task would be considered a problem or not.

The tasks included in Activity One consisted of a variety of mathematical tasks which were all text-based questions. As outlined in Section 4.10.1, mathematical tasks involving words which can be referred to as 'word-problems' are not necessarily mathematical problems (Yeo, 2007). Each task included in Activity One was classified as a mathematical problem or not a problem independently by the researcher and the research supervisor. The criteria for the classification involved the definition of a mathematical problem, and comparison of the mathematical concepts involved in the task and the prior knowledge and level of study of the student outlined in each task. The independent classifications were all in agreement with the exception of Task 12, for which both considered the task to be borderline. This task was included in the analysis to gain an understanding of the PSMTs' understanding in tasks which are not clearly distinguished. Through discussion both agreed to classify this task as a mathematical problem.

Activity two:

The second activity used the same scenarios as the previous activity. However, the participants were asked to *create* a problem that would be suitable for the hypothetical student (see Appendix I). Activity Two focused on the generation of new problems across a range of mathematical topics which is an important skill for teachers to have (Chapman, 2015).

The participants were given an explicit situation that the problem needed to be based on. This situation involved the level of study of the student and the prior mathematical knowledge of the student. The PSMTs were free to create a problem on any topic which would be suitable for the prior knowledge stated. An example of the scenarios is:

Create a mathematical problem that is suitable for each of the described students below.

> *It is 2nd year ordinary level class who have completed trigonometry, algebra, indices, factorising and geometry.*

Activity three:

The third activity focused on the reformulation of existing problems (see Appendix J). This activity firstly involved the participants attempting to create a problem from an exercise. Next, participants were given problems to reformulate so that specific mathematical concepts were addressed. Finally, the participants were asked to reformulate a mathematical task to make an open-ended problem. The reformulation of given tasks into problems meeting a range of criteria (specific topic, open-ended) gave a broader overview of the PSMTs' ability to reformulate mathematical tasks than just focusing on one type of task reformulation.

The reformulation of an exercise into a mathematical problem is a key skill for teachers as textbooks, which are a heavily relied on resource (Jeffes et al., 2013), are predominantly procedural tasks (O'Sullivan, 2017; Zhu & Fan, 2006). Crespo (2003) highlights that routine tasks can be reformulated into engaging mathematical tasks. Reformulating tasks to address specific topics is a valuable skill to demonstrate links between different mathematical topics. The final question in Activity Three involved investigating the PSMTs' ability to reformulate a given task into an open-ended problem. This type of reformulation aligned with Chapman (2012) and Nicol and Bragg (2009) where open-ended problems were a particular focus of the studies.

Analysis

Activity One:

The analysis of the PSMTs' responses for Activity One involved comparing the PSMTs' classification of the tasks with the researchers' classification. The rationale provided by the PSMTs were then analysed to determine the reasons for which the PSMTs classified the task as a problem for the specified student or not.

Activity Two:

The problems posed by the PSMTs were analysed in a two-step process. The first step involved comparing the posed problem to the definition of a mathematical problem. The posed problems were classified as '*Achieved'* or *'Not achieved'* where *Achieved* were the problems that met the criteria of a mathematical problem. The criteria for assessing the task posed by the PSMTs as a mathematical problem is based on the Three Key Characteristics of problem-solving. If the posed task had no ambiguity in how to approach the task, then it was deemed to be procedural. This involved tasks which could be solved using previously-learnt procedures or formulae. The second step of the analysis involved analysing the characteristics of the posed problems which were classified as *Not Achieved*. The problems were classified as '*not solvable'* if there was insufficient information or the mathematical information was contradictory (Silber & Cai, 2017). Similarities and differences in the posed problems were coded using an inductive approach (Thomas, 2006).

Activity Three:

The analysis of the posed problems consisted of a two-step process. The first step involved comparing the posed problem to the definition of a mathematical problem. The posed problems were classified as '*Achieved'* or *'Not achieved'* where *Achieved* were the problems that met the criteria of a mathematical problem. The second step of the analysis involved analysing the characteristics of the posed problems which were classified as *Not Achieved*. Similarities and differences in the posed problems were coded using an inductive approach (Thomas, 2006).

Cohort 4: Extension task

We will now discuss the problem posing that was completed by Cohort 4 which took the form of the extension of problems. As outlined in Section 4.10.2, the PSMTs in Cohort 4 completed three mathematical problems which were analysed over the duration of the module. Each problem involved a different mathematical topic. Upon completion of each of these problems, the PSMTs were asked to extend the given problem in the form of generating a new mathematical problem. The posing of a problem in the manner of extending an attempted problem is considered by Silver (1995) as one of the forms of problem posing. The extension of a problem aligns with heuristics such as Polya (1945) and Mason et al., (2011) which was at the centre point of the module. It is argued by Prestage and Perks (2007) that extending a problem can give the person an insight into different approaches to mathematics and connections can become apparent between mathematical topics.

Analysis

The analysis of the posed problems consisted of a two-step process. The first step involved comparing the posed problem to the definition of a mathematical problem. The posed problems were classified as '*Achieved'* or *'Not achieved'* where *Achieved* were the problems that met the criteria of a mathematical problem. The second step of the analysis involved analysing the characteristics of the posed problems which were classified as *Not Achieved*. Similarities and differences in the posed problems were coded using an inductive approach (Thomas, 2006) .

4.10.5 Question 4: What beliefs and affective factors do pre-service teachers hold regarding problem solving?

A: How does the affective domain of one cohort of PSMTs change over the course of the final iteration of the intervention?

This research question investigates the beliefs of PSMTs in relation to problem-solving. Identity and beliefs are highlighted by Chapman (2015) as one of the key capacities that influences the teaching of problem-solving. Having appropriate beliefs play an important role in terms of successful problem-solving (Lester, 2013; Marcou & Philippou, 2005; Philipp, 2007; A. Schoenfeld, 1985). Similarly, the beliefs of the teacher about the role of problem-solving also clearly influences the way that problem-solving is incorporated in the classroom (Andrews & Xenofontos, 2015; Kloosterman & Stage, 1992; Lester, 2013).

Methodology

The quantitative instrument that was used to investigate this research question was the Indiana Mathematics Belief Scale (IMB) which was developed by Kloosterman and Stage (1992) (see Appendix L). Kloosterman and Stage created an instrument to measure the beliefs of secondary school and college students. This instrument was extensively analysed with pilot scales repeatedly developed. The validity of the instruments was ensured using statistical software to assess the internal consistency reliability. The IMB has subsequently been used to investigate students' beliefs (Mason, 2003; Prendergast et al., 2018).

This instrument consists of five scales with six items in each scale totalling in thirty items overall. Each item provides a statement, and respondents select their responses on a five-point Likert scale (strongly agree,...,strongly disagree). Of these thirty items, twelve questions have a negative valence, meaning that eighteen have a positive valence. When administering the survey, the questions were rearranged so that no consecutive statements were from the same category. Each cohort of PSMTs completed the survey in Week 1 of the module. There was no time limit given to the participants. In tabulating the results, the scales are reversed (where necessary) so that in every case, a higher mark corresponds to a more positive disposition. The following scores were applied (appropriately adjusted for negative valence questions): 1 = strongly disagree, 2 = disagree, 3 = undecided, 4 = agree, 5 = strongly.

Each of the scales are described below:

1) I can solve time-consuming mathematics problems

The first scale involved investigating the person's perceived ability to solve time-consuming mathematics problems. According to Schoenfeld (1985), students who give up on any problem which cannot be completed in five minutes or less believe that problems must be solvable in five minutes or less. In addition to this, students will have difficulty in college-level mathematics courses if they have no motivation to solve problems (Kloosterman & Stage, 1992).

2) There are word problems that cannot be solved with simple step-by-step procedures

This scale references the use of procedural skills and formulae to solve problems. Given that the definition of a mathematical problem involves having no clear path or procedure readily available, it is evident that a problem solver must be motivated to solve problems for which there are no memorised procedures to employ (Charles and Lester, 1982). Kloosterman and Stage (1992) determined that it was important to develop this scale in order to investigate the associated beliefs.

3) Understanding concepts is important in mathematics

This scale measures the level to which the respondent believes in the importance of understanding mathematical concepts. This involves understanding why an answer is correct, and how the solution was obtained. In order to solve problems, it is essential to have a knowledge and understanding of concepts that are beyond the basic rules of that specific mathematical topic (Carson, 2007). Problems require the application and combination of concepts meaning that students have the opportunity to see similarities and patterns in problems thus increasing their conceptual understanding (Carson, 2007). High scores on this scale are associated with motivation to learn and solve mathematical problems (Kloosterman & Stage, 1992).

4) *Word problems are important in mathematics*

This scale involves investigating the respondents' beliefs about the importance of word problems compared to computational or procedural skills. It has been shown that those who believe that computational skills are more important than word problems will be less motivated to solve problems (Kloosterman & Stage, 1992).

5) Effort can increase mathematical ability

Like the other scales, belief that effort can increase mathematical ability is associated with motivation in problem-solving. This belief is the central tenet of those who believe in the importance of having a growth mathematical mindset (Dweck, 2008). This scale is used to provide an insight into the respondents' attitude towards their ability to improve their mathematical skills by putting in effort (Kloosterman & Stage, 1992). These beliefs were chosen specifically for secondary school and college level students in relation to their motivation to learn to solve mathematical problems (Kloosterman and Stage, 1992).

Note that Scales 1, 3 and 5 measure the beliefs of the respondent as a learner of mathematics while Scales 2 and 4 measure the beliefs about mathematics.

Kloosterman and Stage (1992) highlighted that, from the analysis of the results from Scale 5, it was evident that there were various interpretations of the meaning of the term 'word-problem'. The multiple interpretations of 'word-problem' is similar to the many definitions offered for a mathematical problem. The definition of the term is inconsistent amongst teachers and textbooks and students may have had different exposure to various meanings of the word resulting from taking various courses and having different teachers (Kloosterman & Stage, 1992). Thus, the meaning of 'word-problem' must firstly be clarified in order to collect valid data and secondly for teachers to thoroughly grasp what teaching word-problems entails. The ambiguity associated with the term 'word-problem' was discussed previously in Section 4.9.1. The meaning of a 'word problem' was clearly explained to the PSMTs before completing the survey. This need for clarity on the definitions of a 'word-problem' echoes the need for a clear definition of what is meant by problem-solving which as discussed in Section 2.1 has consequences for researchers, teachers and teacher-educators (Chamberlin, 2008; Lesh & Zawokewski, 2007; Lester, 2013).

A mixed methods approach was utilised to investigate the PSMTs' beliefs about problem-solving. The IMB provides an insight into the PSMTs' overarching beliefs about problem-solving but it was important to also investigate the PSMTs during the problem-solving process. The qualitative element of the methodology for this research question involved participants answering an open-ended question directly after attempting a mathematical problem (Felmer & Perdomo-Díaz, 2016). The combination of both the qualitative and quantitative produces a greater understanding of the PSMTs' beliefs (Creswell, 2012) through methodological triangulation (Thurmond, 2001). The qualitative instrument will now be discussed.

To gain an insight into the affective factors the PSMTs held during the problem solving process, they were asked to answer an open-ended question directly after attempting a mathematical problem (Felmer & Perdomo-Díaz, 2016). The participants were asked to describe how they felt at three different stages of their problem-solving attempt, namely the start of the problem, the middle of the problem, and the end of the problem (see Appendix K). The students were prompted to refer to how they felt if they were stuck and if they were making progress. This was done at three different points during the module: Week Two, Week Six, Week Ten, while participants worked on three different mathematical problems. The first problem involved the topic of area, the second problem involved number and algebra, and the third problem involved the topic of trigonometry.

Analysis

The results from the four iterations of the IMB scale were inputted into statistical software (SPSS) in order to analyse the data for statistical differences. Individual items were scored by assigning the number 1 to the least positive response (strongly disagree if the item was positively worded and strongly agree if the item was negatively worded) and so on up to the number 5 for the most positive response (strongly agree if the item was positively worded and strongly disagree if the item was negatively worded). A total score for the six items in each scale was determined by adding the scores for each of the six items on the scale meaning there was a maximum possible score of 30 for each scale (Kloosterman & Stage, 1992). Descriptive statistics - mean and standard deviation - for each scale for each cohort were produced. Cronbach's alpha was also calculated to investigate the reliability of each scale.

The responses to the qualitative question were analysed using a general inductive approach (Thomas, 2006) which is evident in grounded theory (Strauss & Corbin, 1994). Since the initial development of grounded theory, it has come to play a large role in education research (Chong & Yeo, 2015). This involved the coding and identification of categories from iterative reading of the raw data by the researcher. Thomas (2006) outlines that the following stages are involved in the process of inductive analysis: 1) the reading of data in detail to get an understanding of the themes that are present, 2) the identification and defining of categories which involves an upper level that are related to the research aims and a lower level that emerges from in vivo coding, 3) iterative readings of the data and classifications of categories to narrow down the number of categories with clear definition between them. This approach was selected because it involves the development of categories directly from the data which eliminates researchers from strategically selecting certain examples that confirm the theory that they have chosen and negating other data that denies the theory (Strauss & Corbin, 1994).

Inductive analysis was implemented on each of the open-ended questions: namely how the respondent felt at the start of the problem, how the respondent felt if making progress or if stuck, and how the respondent felt at the end of the problem. The statements were analysed individually, and were classified using inductive analysis (Thomas, 2006). Next, the set of statements made by each student were classified and then categorised.

The inductive analysis was initially conducted to categorise statements made by participants. Following on from this, the number of participants were categorised into the categories that were identified in the initial inductive analysis of the statements. The results of this analysis will be discussed in Section 6.5.2.

4.11 Research Issues

The ethics of this research will be discussed in this section along with any limitations that exist in this study. Additionally, the reliability and validity of the data collection and the instruments used for each research question will now be discussed.

4.11.1 Ethics

Prior to the commencement of the project, ethical approval was sought through the submission of an application for an expedited review to Dublin City University. Ethical approval was granted by Dublin City University Research Ethics Committee (Appendix N). The application included both the qualitative and quantitative instruments and the plain language statement (PLS) and informed consent form (Appendix A). The PLS was given to all prospective participants prior to the beginning of the project. Within the PLS it was clearly stated the participation was voluntary and withdrawal was possible at any stage of the project. Participants had the opportunity to ask questions before deciding whether to take part or not. It was made clear to participants that choosing to opt out of the study would not have any effect on their university module.

4.11.2 Data Collection and Analysis

Research Question 1: The ‘Task Sorting’ activity was based on the definition of a mathematical problem (Section 4.10.1). The adapted version was piloted with in-service post-primary mathematics teachers. The tasks were classified by the researcher and research supervisor independently to reduce potential bias.

Research Question 2A: The pre-existing MPSR as outlined in Section 4.10.2 (Appendix E) was used consistently for Cohorts 2, 3, and 4. The interviews were conducted by the researcher who was independent of the facilitation of the module and the interviews followed a semi-structured protocol (Cohen et al., 2007). To ensure descriptive validity (Maxwell, 1992), care was taken to transcribe the interviews correctly and statements were not taken in isolation in order to ensure that the context of the statement was misconstrued. The analysis of a sample of the interview transcripts was independently coded by the researcher and research supervisor.

Research Question 2C: The scoring rubric was developed based on the literature of Mason’s Rubric Writing approach (Mason et al., 2011) (Appendix D). A sample of the transcripts were scored independently by the research supervisor to ensure validity.

Research Question 3: The three activities were developed based on the literature relevant to problem selection, posing and reformulation. The classification of tasks in Activity One (Section 4.10.4) were classified by the researcher and research supervisor independently to reduce potential bias. The criteria for the classification of a mathematical problem were based on the definition of a mathematical problem.

Research Question 4: The reliability of the quantitative IMB scale was calculated using Cronbach’s alpha which calculated the internal consistency of the items of the scale (Cohen et al., 2011). The qualitative open-ended questions were developed based on the literature and were conducted with Cohort 4.

The results for each of the data collection methods will be discussed in detail in Chapter 6.

4.11.3 Researcher Distance

The researcher was involved in the analysis of the collected data but was not involved in the implementation of the module. The module lectures were delivered by the research supervisor and the tutorials were conducted by different lecturers who were independent of the study. The researcher conducted all interviews which were then transcribed and analysed using appropriate research methods.

4.11.4 Validity

Validation in research is the process of evaluating logical arguments and scientific evidence that supports claims (Taylor, 2013, p. 2). Validity is necessary for both quantitative and qualitative research (Cohen et al., 2011). There are many forms of validity of which some types are pertinent to this research, namely; descriptive validity, generalizability, statistical conclusion validity, and internal validity (Maxwell, 1992). We will now discuss how each of these are considered and evident in this study.

Descriptive validity refers to the ‘factual accuracy’ of the research (Maxwell, 1992, p. 285) meaning that the results were not distorted in their selection process and gave a true representation of the data (Cohen et al., 2011). The researcher remained objective (Section 4.11.3) and the instruments used in the study were based on the literature (Section 4.11.2).

Generalisability is the view that the research is applicable in other similar situations (Cohen et al., 2011). In particular, *internal generalisability* (Maxwell, 1992) is possible with this study through the implementation of this module with other groups of pre-service or in-service mathematics teachers. The scoring rubric which was developed to assess the level of implementation of a problem-solving heuristic is applicable to any person solving a mathematical problem. The qualitative open-ended question in Research Question 4 is not only appropriate for PSMTs but it can be used to gain an insight into the beliefs of any person while attempting a mathematical problem. The 'Task Sorting' activity in Research Question 1 and the three activities which were developed for Research Question 3 could be completed by pre-service or in-service mathematics teachers to gain an understanding of their understanding of a mathematical problem and their ability to pose mathematical problems.

Statistical Conclusion Validity is the validity of the 'claims made based on the strength of statistical results' (Taylor, 2013, p. 65). Statistical tests were used to evaluate the quantitative data collected. The use of one-tailed t-tests and their justification was described in Section 4.10.2. Additionally, as presented in Chapter 6, Cohen's d was used to examine the effect size of the module.

Internal validity means the extent to which the results are accurate of the raw data (Godwin et al., 2003). As outlined above (Section 4.11.4), peer evaluation of the data and triangulation were utilised to ensure internal validity (Cohen et al., 2011). The results of this study are outlined in Chapter 5 with examples of the raw data given as support.

4.11.5 Reliability

Reliability means "dependability, consistency and replicability over time, over instruments and over groups of respondents" (Cohen et al., 2011, p. 199). To achieve reliability, research must yield similar results if carried out in a similar context with similar respondents (Cohen et al., 2011). In terms of this study, the module for each cohort of PSMTs was conducted in the same manner over the same time period with the exception of Cohort 3. As described in Chapter 3, the module for Cohort 3 was delivered online over a ten week period. However, despite this change of delivery method, the content of the module remained the same except for the implementation of Research Question 3 instruments. As described above, the instruments used to address each of the research questions were based on the relevant literature.

4.12 Limitations of Research

As discussed above, each decision made over the course of this research was considered and based on the literature. While this was the case, it is recognised that there are limitations to this research.

Module constraints: The study was based in a university module meaning that the time allocation of the module was outside the control of the researcher. As a result of Covid-19 there was a change to the number of weeks allocated to the semester. These changes impacted Cohort 3 and Cohort 4. A second impact of Covid-19 restrictions was the method of delivery of the module which took place in an online capacity for Cohort 3 whereas it was in person for the three other cohorts. The online delivery of the module meant that the data collection instruments needed to be adapted and in the case of Research Question 3, data collection was not possible due to the shortened length of the semester.

It is a central feature of action research that the intervention changed from cohort to cohort meaning that the PSMTs in each cohort had slightly different experiences. Due to the changes in the number of weeks focusing on problem-solving for each cohort, the timing of data collection points varied between cohorts and the problems attempted by the PSMTs were changed (Section 5.5.2).

Time constraints: The module consisted of 3 hours of teaching per week for the duration of the university semester. The data collection of problem-solving tutorials took place in a one hour session meaning that the PSMTs were constrained to this time frame to complete their attempt. Similarly, in the case of Cohort 4, time may have been an influential factor in the completion of the open-ended affective questions and extension task. This resulted in a varying response rate for the extension task (Research Question 4).

Participants: It has to be acknowledged that the PSMTs are at the very earliest stage of their professional formation as mathematics teachers. However, while undertaking the module/intervention (all PSMTs involved), or in previous semesters (PSMTs in their second year of ITE programme), they also completed modules on Introduction to Teaching and Microteaching, and so were beginning this process of professional formation. Due to a variety of constraints, the students took the module which forms the intervention at this early stage of their studies, and so the project was tied to what we have called PSMTs at an early stage. With a strong focus on the development of the intervention as well as the developing capacities of the PSMTs over the course of the module, the limits of the study were necessarily drawn at the timeframe of the module (and shortly afterwards), rather than extending into subsequent years of the degree programme.

Participants of the 'Think ALoud' interviews volunteered to participate meaning that it is possible that they are better disposed to problem-solving. However, there is no actual evidence of this.

Facilitation of the module: The researcher was not present for the implementation of the module. However, the lecturers of the modules were experienced in problem-solving and the implementation of a constructivist approach.

4.13 Summary

This chapter began by outlining the research problem and the aim of the study - an investigation into the capacities of PSMTs to effectively teach problem-solving. Rationale for the selection and design of data collection instruments were outlined along with detailed description for each of the instruments. Additionally, the procedures for the methods of analysis were discussed and the results for each of these are wholly presented in Chapter 6. The setting, participants and the facilitation of the study were described. The university module for which this study was situated, will now be described in greater detail in Chapter 5.

CHAPTER 5: MODULE

5.1 Setting

This research focused on a particular module in an Irish university which pre-service post-primary mathematics teachers undertook as part of a concurrent initial teacher education programme. This module was centred around mathematical problem-solving with a particular focus on preparing PSMTs for the effective teaching of problem-solving. The capacities outlined in Section 3.2 were the basis on which the module learning outcomes and learning activities were developed and implemented.

5.2 Facilitator

The facilitator of the module was the research supervisor, with the exception of Cohort 11, and different tutors contributed to the teaching of tutorials across the different years. The facilitator assumed a constructivist perspective. Constructivists view learning "as a process of actively exploring new information and constructing meaning from the new information by linking it to previous knowledge and experience" (Alesandrini & Larson, 2002, p. 118). They state that the role of the teacher in a constructivist classroom is to act as a facilitator guiding learners through their own learning. Constructivism is the approach which the NCCA advocates for within the Irish mathematics curricula (see Section 2.9). It is important for PSMTs to experience constructivism since teachers who have not previously experienced a constructivist classroom themselves, may not be equipped to create a constructivist classroom for their students (Alesandrini & Larson, 2002). The role of the teacher in a constructivist classroom which involves problem-solving requires a high level of subject-matter understanding due to the various ways in which problems can be explored (Windschitl, 1999). To respond to the actions of students, it is the role of the facilitator to use strategies such as scaffolding, and modelling. *Scaffolding* is described by Hammond (2001) as support structures that teachers provide to students in order for learners to develop their understanding of new concepts and develop new abilities that the students would not be able to accomplish on their own. This support includes the way in which teachers sequence their activities and the quality of the teacher's guidance challenges which ultimately aim to challenge the students (Hammond, 2001). Blum and Borromeo Ferri (2009) describe mathematical modelling as the translation of real world situations into mathematics and vice versa. Mathematical modelling requires the teacher to recognise the balance between students' independence and teachers' guidance (Blum & Borromeo Ferri, 2009). Another strategy that the facilitator may use is questioning to guide student thinking (Hmelo-Silver & Barrows, 2006). This must be done while retaining the students' autonomy over the problem (Windschitl, 1999).

Throughout the module there was a mixture of group and individual work employed while working on problems, with an emphasis on the former. Group work is especially appropriate in problem-solving in the following situations: exploring new problem-solving strategies, new mathematics topics, or when the focus is specifically on the problem-solving process (Lester, 2013). Problem-solving in groups lends itself to sharing and expanding on ideas (Stacey, 1992) which could not be recognised by an individual alone (Mason et al., 2011).

5.3 Heuristics

A key aspect of the module involved PSMTs developing their problem-solving skills. Felmer and Perdomo-Diaz (2016) state that in order to teach problem-solving, teachers must experience it themselves. Given that the participants will ultimately be teaching students, Mason (1992, p. 4) states that in order for one 'to appreciate the struggles of another, it is necessary to struggle oneself, and to be able to re-enter that struggle'. This shows both the importance of PSMTs experiencing problem-solving as part of their ITE, and the importance of knowing strategies to help progress in the problem-solving process.

A key aspect of the module was the use of heuristics as part of developing PSMTs' problem-solving proficiency. A heuristic is a set of guidelines which a person can utilise in various situations (Carson, 2007). It is argued that heuristics are useful for solving new problems as they are not limited to specific content knowledge. In contrast, algorithms are only for specific purposes and do not require understanding of a concept but only memory and routine application (Carson, 2007). The benefits of heuristics are further developed in Section 2.4.4.

Below we will discuss three heuristics for solving mathematical problems developed by Polya (1945), Krulik & Rudnick (1988), and Mason et al., (2011).

5.3.1 Polya's model

The first heuristic is Polya's model (1945) which has four stages: 1) Understand the problem, 2) Make a plan, 3) Carry out the plan, 4) Look back.

Understand the problem involves reading the question carefully and deciphering what is clearly required. Questions that are appropriate for the problem-solver to ask at this state include; '*What are the data? What is the unknown? Introduce notation?'* (p.37). The next step, *Make a plan*, involves linking how various parts of the information are connected and how the unknown is linked to the given information. The problem-solver here plans which calculations, constructions, or computations which can be performed to obtain the unknown. *Carry out the plan* follows on from the previous step which involves using prior knowledge to select an approach. The final step of Polya's model is *Look back* which involves the problem-solver looking back at the completed solution, reviewing it, and discussing it. Polya advocates that looking back at how a solution was achieved consolidates their knowledge. Within this stage, the problem solver should also reconsider the path they took to achieve their solution. This stage would ultimately improve problem-solving skills. Questions that Polya recommends to improve understanding of a solution at this stage are; '*Can you derive the result differently?, Can you check the result?, Can you check the argument?' (p.46).*

5.3.2 Krulik and Rudnick (1980)

The next heuristic that we will discuss is by Krulik & Rudnick (1980). The first step of Krulik & Rudnick's model is *Read*. This involves the problem-solver identifying what the problem is by noting keywords, restating the problem in different language, or asking themselves what the question is asking. The next step, *Explore*, involves relating the problem to prior knowledge and using representations such as diagrams. From here, the problem-solver must *Select a strategy*, whereby the problem-solver makes a decision about how they could solve the problem. The problem-solver then proceeds to *Solve the problem* by applying the chosen strategy. Finally, the problem-solver verifies their solution and identifies other methods for reaching the solution. This is the *Review and Extend* step.

5.3.3 Rubric Writing Approach

The final heuristic that we will discuss is a Rubric Writing approach created by Mason et al., (2010). This Rubric Writing approach (which may be described as a problem-solving heuristic) provides structured guidelines to promote the introduction of diagrams and notation, to draw upon prior knowledge, and to focus on metacognition through the reviewing of work. Mason's Rubric Writing approach gives guidance to problem solvers on how to approach a problem. There are three phases to this approach: 1) Entry phase, 2) Attack phase, and 3) Review phase.

This Rubric Writing approach encourages the problem solver to extract what the problem is asking them to do and what relevant prior knowledge they have; to introduce notation or to mathematise the problem. By learning to execute the Entry phase, the solver creates a platform from which a strategy can be implemented. The Entry phase proposes three questions: 1) What do I want?, 2) What do I know?, and 3) What can I introduce?. These three questions give guidance to the solver on how to determine a starting point in their problem-solving attempt. The question 'What do I want?' essentially ensures that the student fully understands the problem and promotes key details in the problem. The question 'What do I know?' aids the student to identify the information presented in the question and select prior knowledge that may be useful to them. The 'What can I introduce?' question allows the student to select a different representation of the information given or use notation to mathematise the information.

The next phase, the Attack phase, includes trying different approaches, specialising and generalising, and getting stuck. It is in this phase that the essential mathematical activity takes place (Mason et al., 2011).

The Review phase happens when a solution has been reached or the problem solver has exhausted their approach. This phase includes three steps: 1) Check the resolution, 2) reflect on the key ideas and key moments, and 3) extend to a wider context (Mason et al., 2011, p.36).

Table 5 gives an outline of the three heuristics described above.

Polya (1945)	Krulik & Rudnick (1980)	Mason et al. (2011)
- Understand the problem - Make a plan - Carry out a plan - Looking back	- Read and think - Explore and plan - Select a strategy - Find an answer - Reflect and extend	Entry Phase - Understand - Introduce Attack Phase - Plans formulated - Plans tried out Review phase - Check - Reflect - Extend

Table 5: Outline of heuristics (Krulik & Rudnick, 1988; Mason et al., 2011; Polya, 1945)

There are similarities between the three of these heuristics. Each of these heuristics begin with the problem-solver reading the question and identifying what the goal of the question is. Next, the problem-solver should make a plan to solve the problem relating the question to their prior knowledge and selecting a strategy. Both Polya (1945) and Mason et al., (2011) specifically outline the step of carrying out the plans whereas Krulik & Rudnick (1980) focuses on finding an answer. The final step of all three of these heuristics is reflecting on the problem attempt. This module adopted the Rubric Writing approach to problem solving (Mason et al., 2011) as each phase gave specific questions to aid the problem-solver in making progress and the structured approach lends itself to instruction. This approach has been used extensively in different situations such as undergraduate mathematics courses, high school, and teacher preparation courses (Mason et al., 2011).

Figure 8 below is a visual representation of Mason's Rubric Writing Approach.

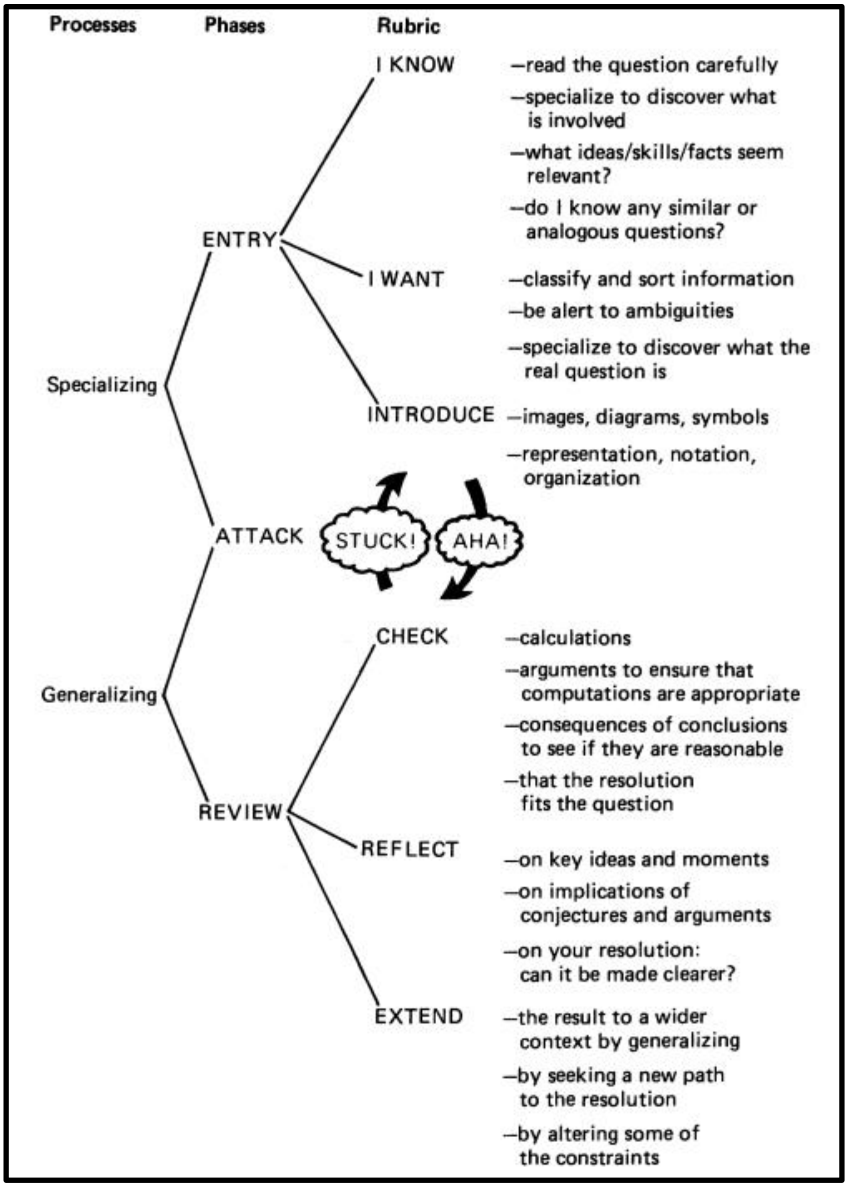


Figure 8: Rubric Writing approach (Mason et al., 2011, p.44)

There is an alignment between elements of the Rubric Writing approach (Mason et al., 2011) and the capacities outlined by Chapman (2015) that are required to effectively teach mathematical problem-solving. Chapman (2015) outlines that teachers' understanding of problem-solving heuristics is important for teachers to understand the thinking process involved and the stages that a student goes through while attempting to reach a solution. The *Entry Phase* and *Attack Phase* of the Rubric Writing approach aligns with the capacity of *knowledge of problem-solving*. The use of representations, rational thinking using prior knowledge, conjecturing, specialising and generalising are all components of problem-solving proficiency (Chapman, 2015) and are explicit elements of the Rubric Writing approach (Mason et al., 2011). Similarly, the ‘extend’ element of the *Review stage* of the Rubric Writing approach aligns with the capacity; *knowledge of problem posing*. Chapman (2015) states that the affective domain is a crucial capacity for the teaching of problem-solving. While the Rubric Writing approach does not specifically address the affective domain, the ‘Aha!’ moments and ‘Stuck’ moments are evident. Mason et al., (2011) discuss how the affective domain influences and also changes in these moments of the problem-solving process.

5.4 Nature of problems - identification of problems vs exercise

There was an explicit focus in the module on distinguishing between a mathematical problem and other mathematical tasks. The definition of a mathematical problem was explored, and these characteristics were discussed (see Section 2.1). To further develop the PSMTs’ understanding of the nature of a mathematical problem, prior to the ‘Task Sorting Activity’, different types of mathematical tasks were examined with the aim of identifying distinguishing features of the tasks.The ‘Task Sorting’ activity assessed the PSMTs’ understanding of the nature of a mathematical problem and on completion of the activity, a whole class discussion ensued to allow the PSMTs to express a rationale for their classification. The discussions from Cohort One and Cohort Two showed a limited understanding which prompted the requirement of a rationale in the adapted Task Sorting Activity in order to gain an insight into the PSMTs’ reasoning for classifications. The need for this was agreed upon during meetings between the researcher and the research supervisor/facilitator in which the effectiveness of this aspect of the intervention was discussed. The discussion also served the purpose of clarifying any misconceptions and allowing for the debate of different features of the mathematical tasks.

5.5 Module Overview

5.5.1 Learning outcomes

The learning outcomes (LO) of the module were as follows:

Module Learning Outcomes	
LO1	Gain a deeper insight into the nature of mathematical problem solving including the analysis of mathematical tasks for problem-solving
LO2	Develop proficiencies in problem solving and in the teaching of problem solving
LO3	Learn how to use mathematical language correctly and develop an appreciation of the importance and nature of proof in mathematics
LO4	Learn a variety of proof techniques
LO5	Develop an awareness of the concepts of growth and fixed mindsets and how these impact the learning of mathematics
LO6	Discuss key historical milestones in the development of mathematics
LO7	Learn about different ways that maths is present in our culture

Table 6: Learning outcomes of the module

Outcomes LO1-LO5 - those of relevance to the current project - were addressed by all four cohorts of students. Cohorts 1 and 2 also addressed LO6 and LO7: these play no role in the current project. The discussion below relates to activities relevant to LO1-LO5. Cohorts 3 and 4 attended a reduced version of the module due to the Covid 19 pandemic, and did not address LO6 or LO7. Cohort 3 attended the module wholly online.

5.5.2 Overview of Module for Each Iteration

The module on which this study is focused on a pre-existing module in the PSMTs' initial teacher education programme. In each iteration of the module, students attended three hours of classes per week. Based on the analysis of the research instruments, the literature, and the lecturer's observations, the module was adapted for each cohort of PSMTs in an action research design (Cohen et al, 2018). Below, a week-by-week breakdown of the module for each cohort is described. As some of the elements of the module are not relevant to this study, some content is not associated with any of the research questions. The problems which were completed individually by the PSMTs and analysed using the MPSR are underlined.

Cohort 1: Learning activities were described as two lectures and one tutorial per week. Lectures comprised a blend of direct input by the lecturer, with time for students to begin on different activities, principally working on problems. In relation to problem solving, the direct input involved discussion of the nature of mathematical problems and problem solving, the role of problem solving in the Irish curriculum, problem posing and a drawn out discussion of Mason's Rubric Writing approach, following Mason et al. (2011). Thus, students were introduced sequentially and on a week-by-week basis to the different elements of the Rubric Writing approach, and completed problem-tasks associated with learning how to implement each successive element. The strand of the module dealing with mathematical proof stemmed from this, with different proof techniques being presented as problem-solving tools. Lecture time was also used for direct input on mindsets and the history of mathematics. In tutorials, students worked in small groups on assigned problems, with the tutor adopting a constructivist 'guide on the side' approach to teaching. Tutorial exercise sets were designed to enable the students to build their capacities in implementing the Rubric Writing approach, so that in the early weeks of the modules, there was an emphasis on the Entry phase of Rubric Writing, and in later weeks, the tutorial sets explicitly required students to design extension tasks, in line with the Extend component of the Review stage of Rubric Writing. Students were encouraged to apply the Rubric Writing approach to tutorial sets that addressed different proof strategies (e.g., mathematical induction, the pigeonhole principle, working backwards and forwards).

Input on mathematical mindsets, and more generally, productive dispositions for problem solving was provided in a number of plenary sessions. At the beginning of the module, students completed the Indiana Beliefs Survey (Kloosterman & Stage, 1992). The results of the survey were analysed (See Section 6.4.1) and were used to inform elements of the lecturer's direct input during classes.

The purpose here was to highlight ‘negative’ attitudes, and to point out how these can hinder learning, or how they do not provide an accurate reflection of genuine problem solving.

Feedback from the Kloosterman and Stage survey showed the need to address affective issues. Two activities were included in the module to address this: the pre-existing Mindsets input/activity, and the new Red Hair problem activity. While these are not the direct subject of any research question, they are likely to be relevant to Research Question 4. They are mentioned only for completeness as they are relevant to LO1-LO5

In a separate session, students were introduced to Dweck’s ideas on fixed and growth mindsets (Dweck, 2017) and Boaler’s application of those ideas to mathematical learning (Boaler, 2016).As a follow-on task, the students then completed a very short online course developed by Boaler on mathematical mindsets: (https://www.edx.org/course/how-to-learn-math-for-students-2?utm_source=sailthru&utm_medium=email&utm_campaign=triggered_shareit).

Finally, one session was spent on the Blonde Hair Problem (localised to the Red Hair Problem). This activity was devised by Andreas and Gabriel Stylianides as an intervention of short duration that, by presenting students with a memorable, positive problem solving experience, addresses four counterproductive mathematical beliefs (Stylianides & Stylianides, 2014). These are that any problem that can be solved can be solved quickly; that perseverance is not a central aspect of mathematical problem solving; that mathematical problems are identifiable by the presence of numbers and other recognisable mathematical objects and that mathematical problem solving is not rewarding or enjoyable.

The tables below outline the instruction received by each cohort of PSMTs. The tutorial problems which were attempted individually and subsequently analysed using the MPSR are underlined. The data collected for each research question is also outlined. As this university module was pre-existing to this study, some of the content listed is not relevant to a specific research question.

Cohort 1			
Week	**Content Title**	**Description of Content**	**Relevant Research Question [data collected]**

1	Mathematical Mindset	Kloosterman & Stage (1992) IMB survey Growth mindset versus Fixed mindset (Dweck) Mathematical Mindset Course (Boalar, 2016)	RQ 4 [IMB survey data]
2	Mathematical Language	Using correct notation and symbols Types of errors in communication using mathematical language Familiarisation in the meaning of mathematical terminology.	
3	Mathematical Language	Sets and statements Sets and propositions Theorems/axioms (2 worksheets)	
4	Mathematical Language	Conditional statements Contrapositive, converse and inverse, qualifiers	
5	Problem-solving	Introduction to mathematical thinking (Fiekler, 2007) Taking a structured approach	RQ2 [Think Aloud interview]
6	Problem-solving	Rubric Writing Approach (Mason et al., 2010) What is a mathematical problem? Task Sorting Activity	RQ2 RQ1 [Think Aloud interview] [Task Sorting]
7	Problem-solving	Rubric Writing Approach (Mason et al., 2010) 'Humour in accounting' 'Patchwork', 'Professor on an Escalator'	RQ2
8	Problem-solving	Rubric Writing; Palindromes problem	RQ2
9	Problem-solving	'Four-Legged Lawnmower' problem Mathematical Proofs; mathematical statements	RQ2
10	Problem-solving	Proof by Contrapositive Argument	
11	History of Mathematics		
12	History of Mathematics	Poster presentation	

Table 7: Module breakdown for Cohort 1

Cohorts 2 : From the results of the research instruments, it was recognised by the researcher that there was a need for PSMTs to spend more time developing problem-solving skills (Schoenfeld, 2013). It was recognised by the researcher and supervisor that problem posing did not have an explicit position in the first iteration of the module. It is evident that both pre-service and in-service teachers experience difficulties with posing problems (Chapman, 2015; Inoue, 2009; Silver et al., 1996). These recognitions led to the restructuring of the module for Cohort 2 and subsequent cohorts.

Students attended one two-hour workshop and one additional one-hour session, which was used for a variety of purposes. In the workshop, a reduced amount of time was spent on direct input by the lecturer, allowing students more time to work on problems and thereby become proficient in the Rubric Writing approach. Thus, essentially all workshop time was spent working in groups, with the lecturer acting as facilitator (either in plenary or working with groups on a one-to-one basis). In addition to working on problems, time was spent in these sessions on task-sorting activities (see Section 4.10.1) and extending problems and devising scenario problems (see Section 4.10.4). The one-hour sessions were used for a variety of purposes including direct input on the nature of problems, problem-solving in the curriculum, classification of problem types and for module assessment tasks. The content on mindsets was included as for Cohort 1.

Cohort 2			
Week	Content	Description of Content	Relevant Research Question **[data collected]**
1	Mathematical Mindset	Growth mindset versus Fixed mindset (Dweck) Mathematical Mindset Course (Boalar, 2016)	RQ4 [IMB survey data]
2	Mathematical Language	Definitions of mathematical terms Importance of accuracy in mathematical language Reading Proofs Set notation	

3	Mathematical Language	Set notation Sets and Propositions	
4	Problem-Solving	Introduction to problem-solving Mathematical Thinking (Mason et al., 2010; Fiekler, 2007) Rubric Writing Approach - ‘Patchwork’	RQ2
5	Problem-Solving	Rubric Writing Approach - <u>‘Professor on an Escalator’</u> ‘Palindromes’/ Being STUCK ‘Hailstones Sequence’ Red Hair Problem (Stylianides & Stylianides, 2014)	RQ2 [Think Aloud interview]
6	Problem-solving	<u>‘Four-legged Lawnmower’</u> Proofs - mathematical statements Proof by Exhaustion Working forwards, working backwards	
7	Problem-solving	Proof by Contrapositive Arguments/ Contradiction/ Pigeonhole/ Induction	
8	Problem-solving	Proof tasks ‘Threaded Pins’	
9	Nature of Problems	Definition of mathematical problems, exercises, other types of tasks. Role of problem-solving in Junior Cycle and Senior Cycle curricula. Task Sorting activity	RQ1 [Task Sorting]
10	Problem Posing	Different types of mathematical problems and	RQ3

		adaptations of tasks. Activity One - problem posing	[Activity One problem posing]
11	Problem Posing	Poster Presentations Adapting mathematical tasks as problems Activity Two	RQ3 [Activity Two problem posing]
12	Problem Posing	Activity Three	RQ3 [Activity Three problem posing]

Table 8: Module breakdown for Cohort 2

Cohort 3: For Cohort 3, who completed this module in the first semester of the academic year 2020-21, delivery was wholly online. The workshops were held in an online synchronous format using the Zoom platform, attempting to replicate the key features of the live classroom insofar as was possible. Thus, plenary discussion was followed by students working in small groups in breakout rooms. The lecturer moved from room to room, checking progress, questioning students and providing prompts as would be done in the face-to-face setting. This was supplemented by recorded content, and by a bi-weekly tutorial. The emphasis in the tutorial (for which the class was split in two, giving c. 25 students in each tutorial) was on spending more time working on the problems encountered during the workshops. Essentially the same teaching materials (lecture notes, exercises and problems) were used for Cohort 3 as were used for Cohorts 2 and 4. The content on mindsets was moved to smaller online synchronous settings (three groups of 15-17 students), providing the opportunity for plenary review of students' work.

Cohort 3			
Week	Content	Description of Content	Relevant Research Question **[data collected]**
1	Affective domain Problem-solving	Kloosterman & Stage (1992) IMB survey Red Hair Problem (Stylianides & Stylianides, 2014)	RQ4 [IMB survey data] RQ2

		Introduction to Mathematical Thinking (Mason & Watson, 1998)	
2	Problem-solving	Warehouse' problem, 'Bookshelf' problem, and 'Paperstrip' problem (Mason et al., 2011) Rubric Writing (Mason et al., 2011)	RQ2
3	Nature of mathematical problems Affective Domain	Purpose of mathematics and types of mathematical tasks (Breen & O'Shea), exercises and problems. Adapted Task Classification activity Mathematical Mindset Course (Boalar, 2016)	RQ1 [Task Sorting] RQ4
4	Problem-solving	Rubric Writing (Mason et al., 2011) 'Patchwork/Palindromes' Professor on an Escalator	RQ2
5	Problem-solving	Rubric Writing Approach 'Palindrome'	RQ2 [Think Aloud interview]
6	Affective Domain	Maths Anxiety (Finlayson, 2014; Beilock & Willinghaus, 2014) Book Report (Mathematical Literature)	RQ4
7	Problem-solving	Proof Methods 1 Moving forwards, moving backwards. 'Tethered Lawnmower' Problem-solving assignment	
8	Problem-solving	Proof Methods 2; Mathematical Language. Pigeonhole Principle, Induction	

9	Problem-solving	Proof Methods 3; Contrapositive Argument	
10		History Project Problem-solving assignment	

Table 9: Module breakdown for Cohort 3

Cohort 4: Cohort 4 reverted back to the same in-person structure as experienced by Cohort 2. There was further time dedicated to focus on problem-solving than on previous iterations of the module due to the removal of other aspects of the module which were outside the scope of this study. These changes were made based on the analysis of previous iterations of the module and the literature.

Cohort 4			
Wee k	**Content**	**Description of Content**	**Relevant Research Question [data collected]**
1	Orientation week	Introduction to module; Rubric Writing approach	
2	Problem-solving Affective Domain	'Warehouse' problem and 'Paperstrip' problem (Mason et al., 2011) IMB instrument (Kloosterman & Stage, 1992) Problem 1: 'Four-Legged Lawnmower' and open-ended affective question	RQ 2 RQ4 RQ2 RQ3 RQ4
3	Nature of mathematical problems	Purpose of mathematics and types of mathematical tasks (Breen & O'Shea), exercises and problems. Adapted Task Classification activity	RQ1 [Task Sorting]
4	Affective domain Problem-solving	Red Hair Problem (Stylianides & Stylianides, 2014) Rubric Writing (Mason et al., 2011) 'Bookshelf/Patchwork/Palindromes'	RQ4 RQ2 RQ3
5	Problem-solving	Rubric Writing (Mason et al., 2011) Mathematical Mindset Course (Boalar, 2016)	RQ2 RQ4

	Affective Domain		
6	Problem-solving Affective Domain	Problem 2: Pitstop Problem and open-ended affective question	RQ2 RQ3 RQ4
7	Problem-solving Nature of Problems Problem posing	Rubric Writing; Being STUCK. 'Cut out/ Threaded Pins' (Mason et al., 2011) Nature of problems; Open vs Closed, Levels of Cognitive Demand, Mathematical Structures, Syntax. Task analysis.	RQ2 RQ1 RQ3
8	Problem-solving	Stuck; 'Threaded Pins' Rubric Writing; Conjectures. 'Circles and Spots'.	RQ2
9	Problem-solving Nature of Problems	Conjectures Nature of problems - task selection for teaching purposes.	RQ2 RQ3
10	Problem-solving Affective Domain	Patterns Problem 3: 'Leaning on a Cube' and open-ended affective domain	RQ2 RQ2 RQ3 RQ4
11	Problem-solving	Problem-solving assignment	RQ2
12		No classes	

Table 10: Module breakdown for Cohort 4

5.5.3 Assessment

For all cohorts, assessment of the problem-solving components of the module comprised two categories of task.

For Cohorts 2 and 4, each of the weekly workshops concluded with a brief task for students to complete to be submitted either by the end of the workshop, or on Loop (the DCU implementation of Moodle - our VLE). These tasks typically involved completing one aspect of a problem-solving task, implementing the Rubric Writing approach. Second, more substantial problem-solving tasks were assigned in weeks 7 and 10 of the (12-week) module. These were focussed on assessing students' ability to implement the Rubric Writing approach. The assessment schedule for Cohorts

1 and 3 was very similar, with marks for the weekly workshop tasks substituted by marks for tutorial participation. The weekly workshop tasks were, correspondingly, scaled back. For all cohorts, there was also a mindsets assessment task which involved completing the online course mentioned above.

CHAPTER 6: RESULTS

The purpose of this chapter is to present the results of the study which will then be discussed in detail in Chapter 7. The results obtained from the different research instruments that were utilised to address the research questions, as outlined in Section 4.5, will be laid-out with discussion of the results in the consecutive chapter. This chapter will begin with the Task Classification Activity, the quantitative results of the Tutorial worksheet, the qualitative results of the 'Think Aloud' interviews, the quantitative results of the Indiana Mathematics Beliefs scale, and the qualitative results of the open-ended question.

6.1 Research Question 1: What do pre-service teachers understand a mathematical problem to mean?

The PSMTs completed an activity which asked them to classify mathematical tasks as a *problem*, an *exercise*, or *other/not sure*. As outlined in Section 4.10.1, this activity consisted of participants from Cohort 1 and Cohort 2 classifying twenty mathematical tasks (see Appendix B). Cohort 1 and Cohort 2 completed the same activity which consisted of the same mathematical tasks to be classified. An adapted version of this activity was completed by Cohort 3 and Cohort 4 which consisted of classifying ten mathematical tasks and giving a rationale for the classification (see Appendix C). The tasks were classified independently by the researcher and the primary research supervisor. The classifications were then discussed and agreed upon. From the independent classification of the tasks, two tasks had different classifications. These were Task 7 and Task 11. After a discussion, the researchers agreed on the classifications as outlined in Table 7.

Before doing this activity, all participants had received instruction on what constitutes a mathematical problem and a mathematical exercise. The participants were provided with the following definitions based on the Three Key Characteristics as discussed in Section 2.1:

A mathematical tasks is a problem if:

- The tasks presents the person working on it with a clear goal;
- It is not immediately clear to the person how to achieve the goal;
- The person must organise prior mathematical knowledge to generate reasoning towards achieving this goal.

A mathematical task is an exercise if:

- The task presents the person working on it with a clear goal;
- The person has access to (or knowledge of) a procedure that they can follow to complete the task;
- The person must organise prior knowledge to generate reasoning towards achieving the goal.

Table 7 below shows the classifications for each of the mathematical tasks for the activity completed by Cohort 1 and Cohort 2 (Appendix B). ‘P’ represents a mathematical problem and ‘E’ represents a mathematical exercise.

Task	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Classification	P	P	E	E	E	E	P	E	P	P	P	E	E	P	E	P	P	P	P	P

Table 11: Correct classification of the mathematical tasks in activity completed by Cohort 1 and Cohort 2.

The classification of tasks by Cohort 1 are shown in Figure 9. This activity was completed by 45 participants in Cohort 1.

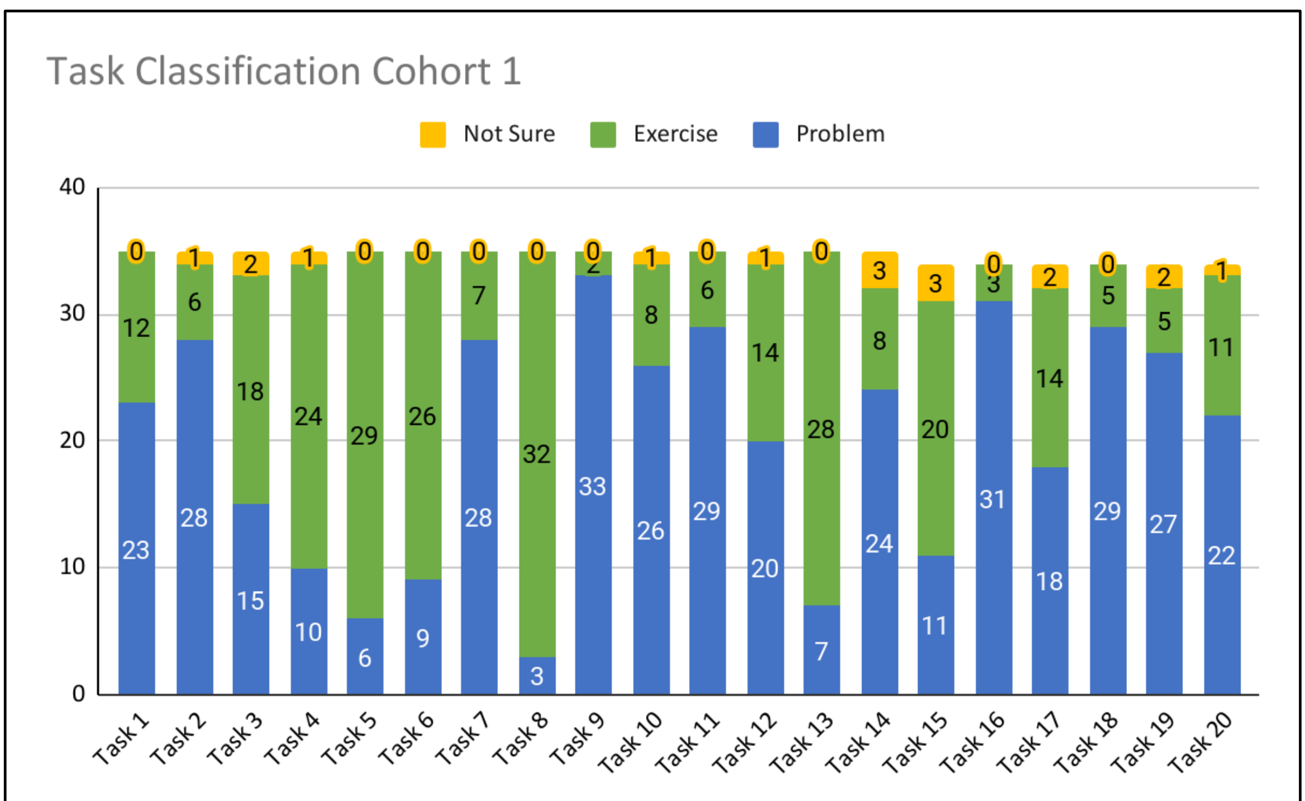


Figure 9: Cohort 1 Classification of Tasks: Cohort 1 Figure 10 below shows the percentage of participants who correctly classified the mathematical tasks. A correct classification is one which agrees with that arrived at by the researchers.

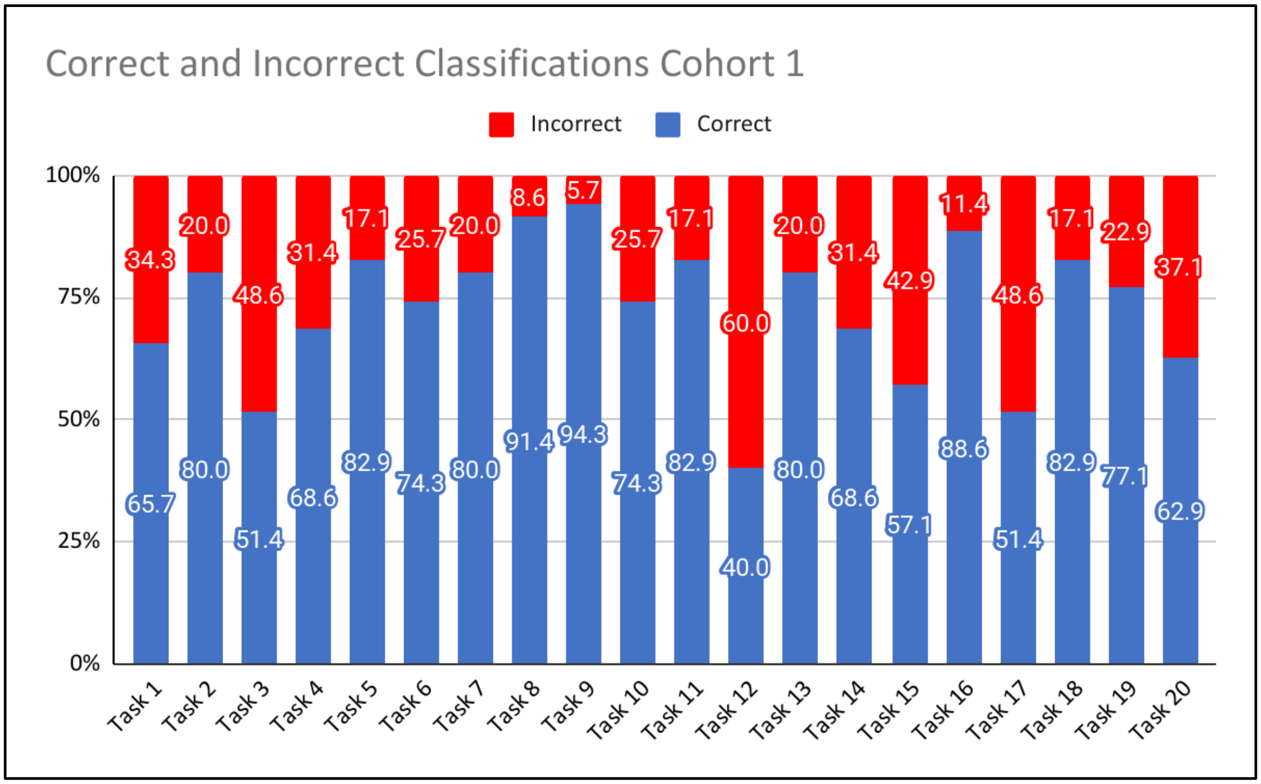


Figure 10: Percentage of correct classifications of tasks by Cohort 1

Figure 11 shows the classification of the mathematical tasks by Cohort 2. The number of participants who completed this activity was 48.

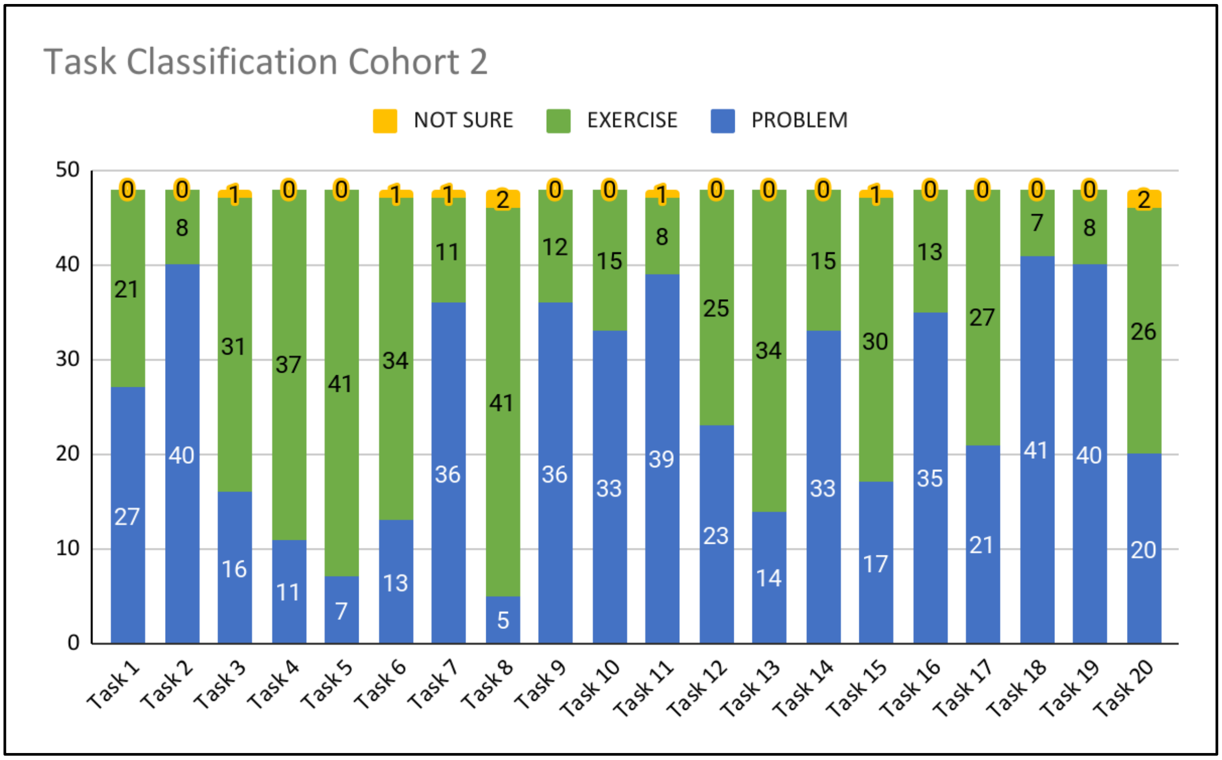


Figure 11: Classification of tasks by Cohort 2

Figure 12 outlines the percentage of accuracy by participants in Cohort 2 in classifying the mathematical tasks.

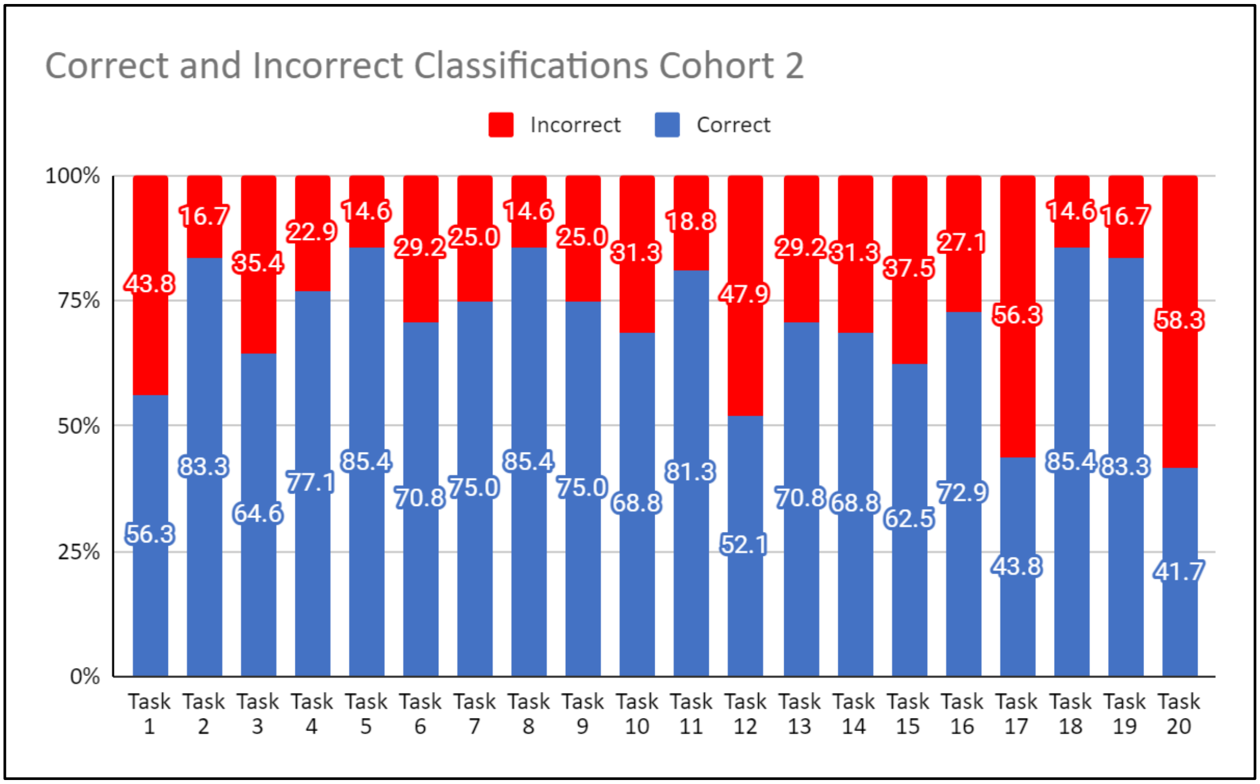


Figure 12: Percentage of correct classifications by Cohort 2

The overall success rate was 72.7% for Cohort 1 and 70.2% success rate for Cohort 2. On analysis of the results of the 'Task Sorting' activity, a threshold mark of 55% success rate by one of the cohorts, was used to determine which tasks to further discuss below . The tasks meeting these criteria will now be highlighted, and further discussed in the next chapter.

Task 12 had a success rate of 40% from Cohort 1 and 52% from Cohort 2.

Task 12: *Shane wins a sum of money on a scratch card. He decides to invest €700 in a bank that offers an interest rate of 8%. How much will Shane have at the end of the two years? He then decides to invest €400 in a bank that offers an interest rate of 2% for a further 3 years. How much interest will he make from the €400?*

This task was also included in the adapted version of the activity and is referred to as Task 7.

Task 3 was classified as an exercise as there is a clear goal which is to calculate the income tax, there is a procedure available to achieve an answer, and the participants would have prior knowledge of this type of task from Junior Cycle curriculum (SEC, 2016). Although tasks similar to this are regularly completed as part of post-primary mathematics, it is interesting to note that only 43.8% of Cohort 2 correctly classified it as an exercise. Potential confusion regarding the use of words in tasks and the classification of these tasks is discussed in Section 4.10.1.

Task 3: *Niamh has an annual salary of €48000. She has a standard cut-off point of €34000 and a tax credit of €4600. If the standard rate of income tax is 20% and the higher rate is 42%, find how much income tax she pays.*

Task 17 had a success rate of 51.4% from Cohort 1 and only 43.8% success rate from Cohort 2. This task was sourced from NRich (2019) and is classified as a problem and it was independently classified as a problem by the researchers. However, the participants disagreed with this classification.

Similar to Task 17, Task 20 was sourced from NRich (2019) and is classified as a problem, and it was independently classified as a problem by the researchers. While 62.9% of the PSMTs in Cohort 1 agreed with this classification, only 41.7% of PSMTs in Cohort 2 agreed with the classification as a *problem.*

Task 20: *In how many whole numbers between 100 and 999 is the middle digit equal to the sum of the other two digits?*

The Task Classification Activity was adapted for Cohort 3 and Cohort 4. This involved reducing the number of questions from twenty to ten. Additionally, participants were asked to provide a rationale for their classification of each task. This rationale was included to give further information on the PSMTs' understanding of what a mathematical problem is. Below Table 12 highlights the questions which were common to both activities.

Cohort 1 and Cohort 2		Cohort 3 and Cohort 4
Wood Pile	*Rory Lemonade*	Workshop handshakes
Garden Fence	*Scratch Card*	Quadratic Equation
Niamh Salary	Pet Survey	Power Patterns

Functions	Audience	Centre of Circle
Interest Rates	*Ratio money*	*Escalator*
Midpoint	Clown Hats	Similar Triangles
Phone Mast	Mean Number	*Scratch Card*
Cone Sphere	Truth and Lies	*Rory Lemonade*
Stephen's Bike	*Escalator*	*Garden Fence*
Test Scores	Middle Digit	*Ratio Money*

Table 12: Tasks in original and adapted Task Sorting Activity

Table 13 below shows the classifications for each of the mathematical tasks for the activity completed by Cohort 3 and Cohort 4 (Appendix C). 'P' represents a mathematical problem and 'E' represents a mathematical exercise.

Task	1	2	3	4	5	6	7	8	9	10
Classification	P	E	P	E	P	E	E	P	P	E

Table 13: Correct classification of the mathematical tasks in activity completed by Cohort 3 and Cohort 4

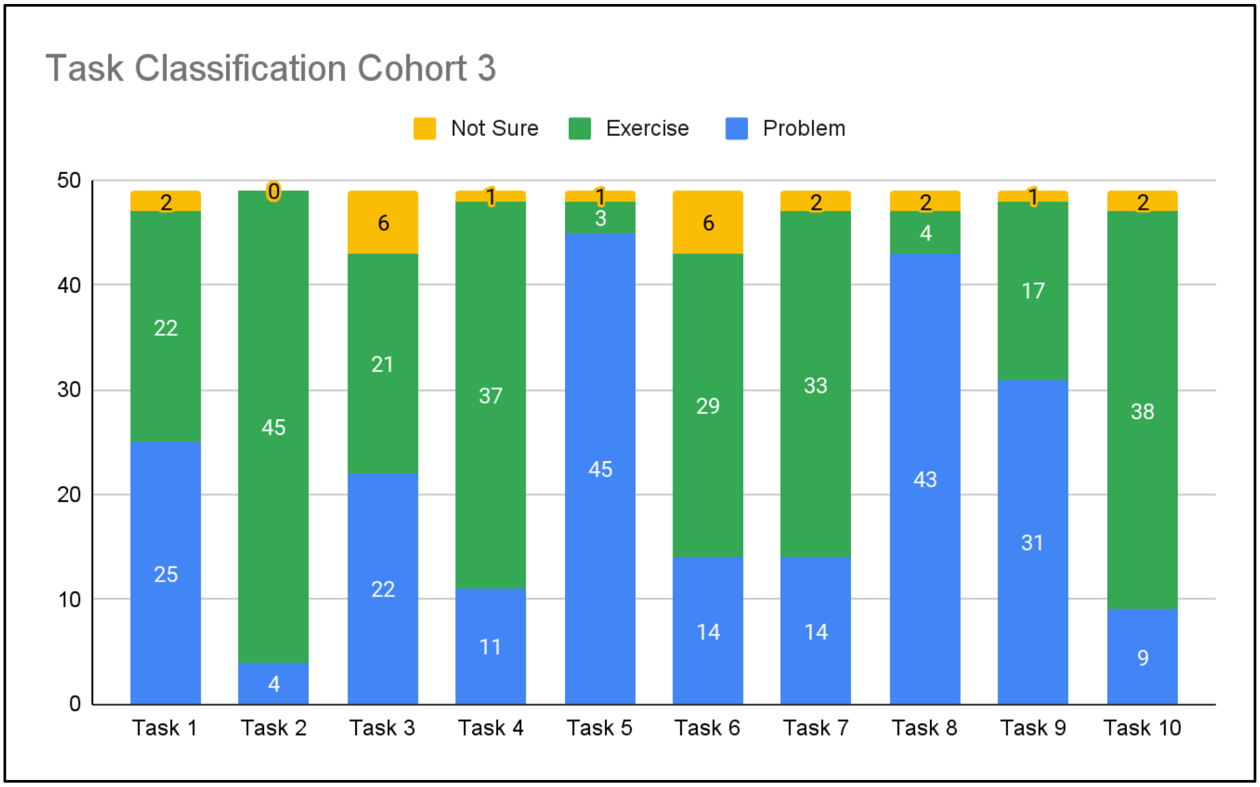


Figure 13: Classification of tasks by Cohort 3

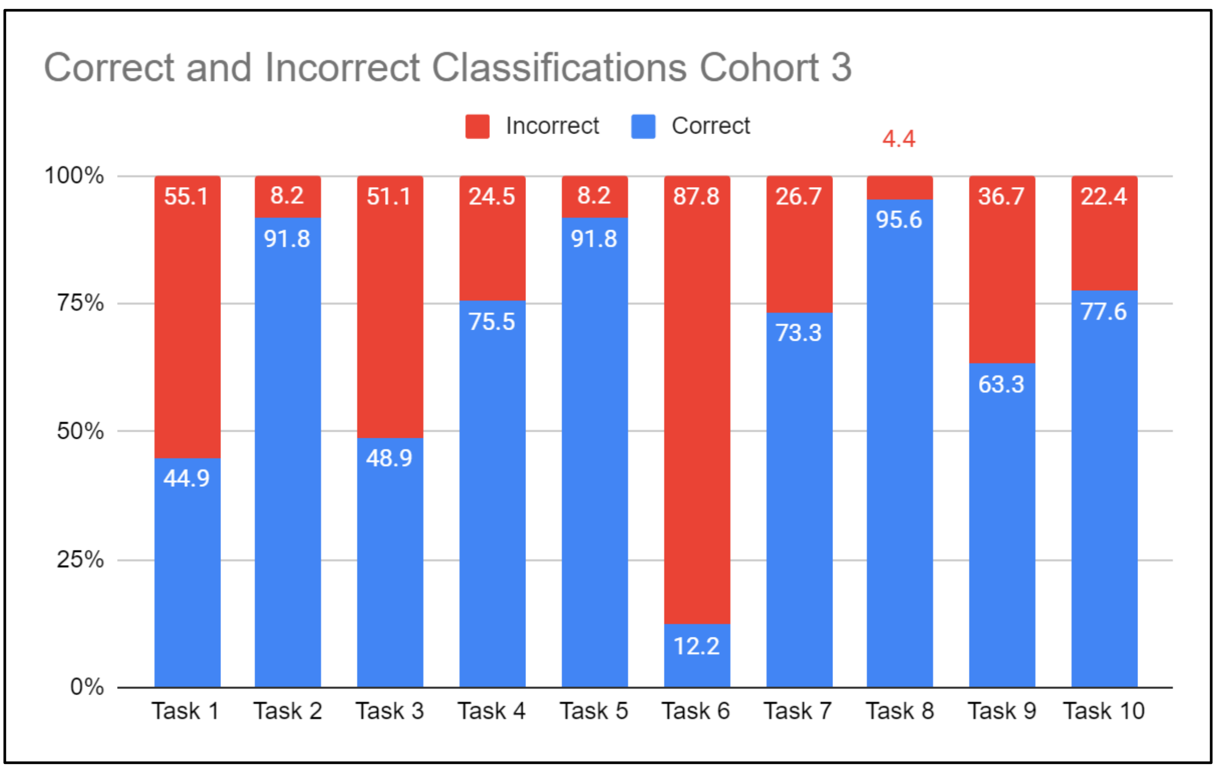


Figure 14: Percentage of correct classifications by Cohort 3

3

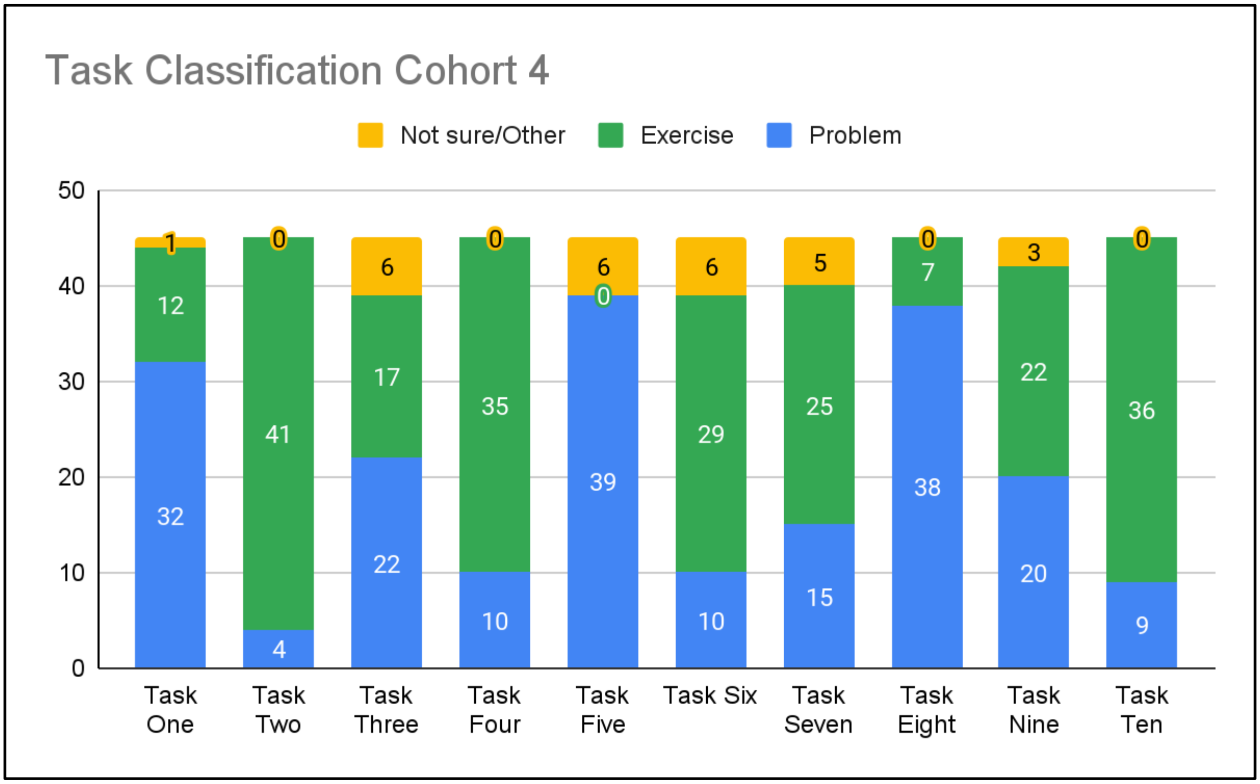


Figure 15: Classification of tasks by Cohort 4

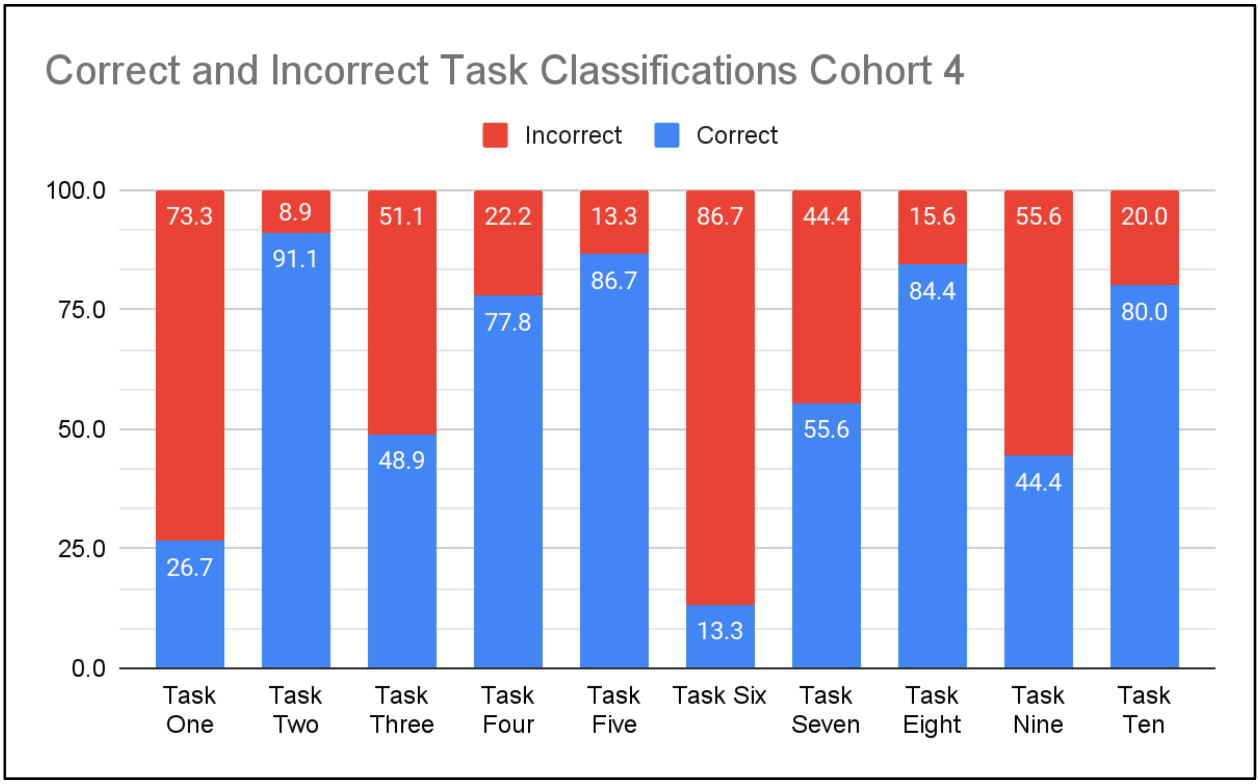


Figure 16: Percentage of correct classifications by Cohort 4

The overall success rate of Cohort 3 was 67.5% and the overall success rate of Cohort 4 was 60.9%. Cohort 4 had a relatively low success rate in Task 7 with a score of 55.6% which was similar to the findings of Cohort 1 and Cohort 2. However, a higher percentage of participants, 73.3%, in Cohort 3 classified Task 7 as an exercise.

Task 9, which was classified as an exercise, was classified as a *problem* or *other* by 36.7% of the participants in Cohort 3 and by 55.6% of the participants in Cohort 4. Task 9 would share similar characteristics with Task 7 as being a 'wordy' question.

Task 6 showed the lowest scores by both Cohort 3 and Cohort 4 with accuracy of 12.2% and 13.3% respectively. This task was classified as *Other* due to it not having the characteristics of either a *problem* or an *exercise.*

Task 6*: In Junior Cycle maths, you learnt about similar triangles and their properties.*

(a) What do you think might be meant by similar quadrilaterals? Give some examples of pairs of quadrilaterals which are similar, and pairs which are not.

(b) Which properties of similar triangles will also apply to similar quadrilaterals? Which properties won't?

This task was classified as *Other* by the researchers. The goal of the question is subjective to the person, meaning that the goal is not clear. While prior knowledge is required, there is no procedure to achieving a solution, and it also does not require reasoning to achieve a solution. It is an investigative task and does not meet the criteria of a mathematical problem or an exercise.

Adapted Task Classification Activity

The adapted version of the Task Classification Activity was analysed using a grading rubric (see Appendix D). This grading rubric involved awarding 0, 1, or 2 points to the response regarding each of the three criteria outlined in the definition of a problem or an exercise. For the goal criterion; 2 points were awarded for identifying that the task had a goal and for naming that goal, 1 point was awarded for identifying that the task had a goal, and 0 points was awarded for no mention of a goal. For the immediately clear criterion; 2 points were awarded for identifying whether it was immediately clear or not on how to reach the goal, and the characteristic of the task that made it immediate or not. 1 point was awarded for identifying whether it was immediately clear or not on how to reach the goal. 0 points were awarded for no mention of the immediately clear criterion of the task. Finally, for the prior knowledge criterion; 2 points were awarded for making explicit reference to the prior knowledge required to reach a solution and outlines, if applicable, different approaches and the associated prior knowledge necessary to achieve a solution. 1 point was awarded for reference to prior knowledge necessary to reaching a solution. 0 points were awarded for no reference of prior knowledge. This resulted in a maximum score of 6 points for each task. There were 49 participants in Cohort 3 and 45 participants in Cohort 4 who completed this activity. The activity consisted of classifying ten mathematical tasks.

Table 14 shows the mean score, the modal score, and the standard deviation in each of the three criteria for Cohort 3. The maximum score in each of the criteria is 2 points.

n=475	**Goal**	**Immediately Clear**	**Prior Knowledge**
Mean	0.1	0.8	0.5
Mode	0	1	0
SD	0.38	0.59	0.68

Table 14: Descriptive statistics of the rationale for classifications by participants in Cohort 3.

The charts below will outline the percentage of participants in Cohort 3 who achieved the relevant scores. Quotes demonstrating the different scores are outlined under each of the criteria. For the scores of 0 points, the quotes provided are the total responses of the participant and not an excerpt. These show no reference to the criteria in question. Firstly, the total scores are presented which had a maximum of six points followed by the percentage scores for each of the three criteria of the grading rubric.

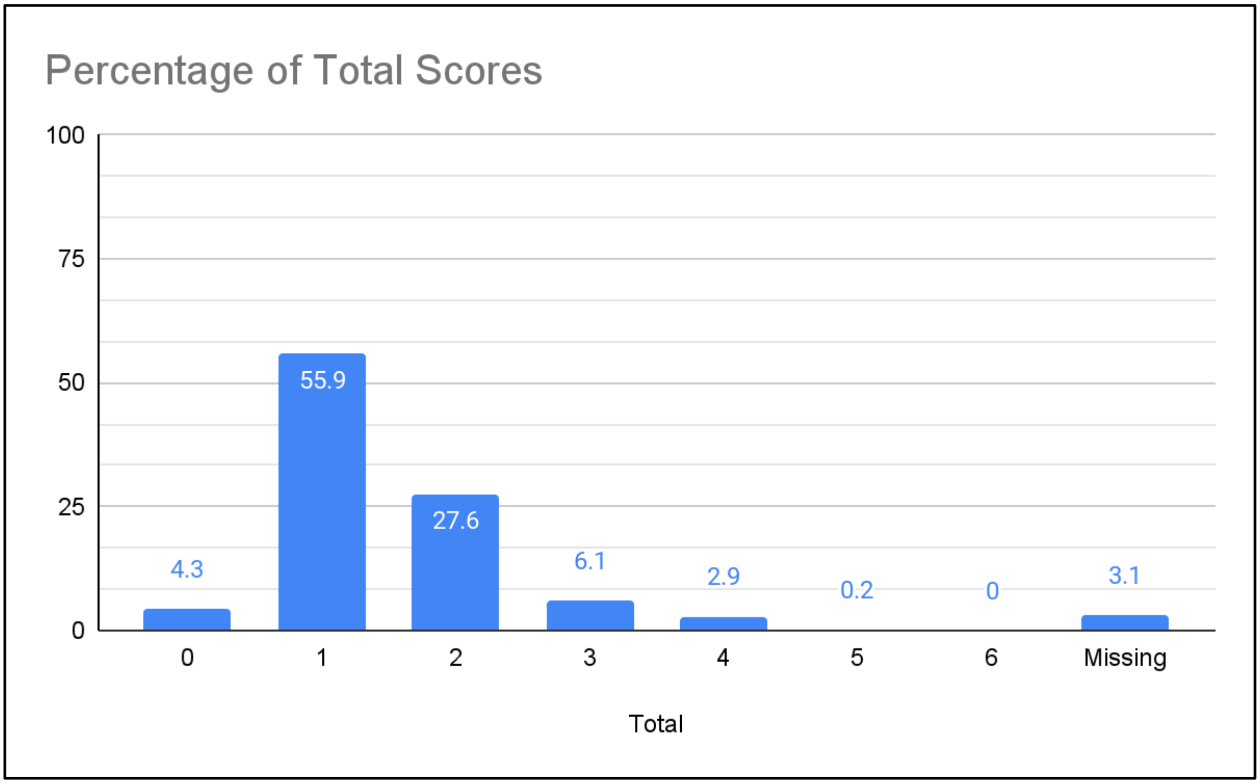


Figure 17: Percentage of achieved rationale total scores by Cohort 3

Below there are quotes selected from Cohort 3 which are representative of the different scores of the total scores. There is no quote in relation to a score of 6 points given that no participant achieved this maximum score.

0 points: *Given all relevant information.*

1 point: *No clear formula to follow.*

2 points: *No clear way to answer this, it requires other knowledge.*

3 points: *They have to find the ratio by putting it into a fraction and then dividing. A relatively clear task.*

3 points: *Clear goal, we don't know what to do immediately, requires thinking and mathematical reasoning to find the answer.*

4 points: *This is an exercise as you just use the two points to find the midpoint which is the center of the circle, it is immediately clear how to reach the goal.*

5 points: *This is an exercise as you are asked to solve the quadratic equation so the procedure to solve this is known , and it is immediately clear to me how to solve this.*

Looking at the example given for 1 point, there was no reference to the goal of the task or to the prior knowledge required to reach a solution. 1 point was awarded for the criterion of 'immediately clear' to the PSMT making reference to *no clear* formula.

In comparison, the example given for 5 points shows that the PSMT, identifies the goal of the question is to solve the quadratic equation (2 points), the fact that it is immediately clear is recognised with the use of a procedure (2 points), and the reference to procedures from prior knowledge is identified (1 point). However, the PSMT does not explicitly state the prior knowledge required or possible approaches.

The example receiving 3 points gained 1 point for identifying that there is a clear goal, however, the goal of the task is not stated. 2 points were awarded for identifying that the goal is not immediately clear, and that reasoning is required implying there is no procedure available to them. There is no mention of the prior knowledge required to reach a solution.

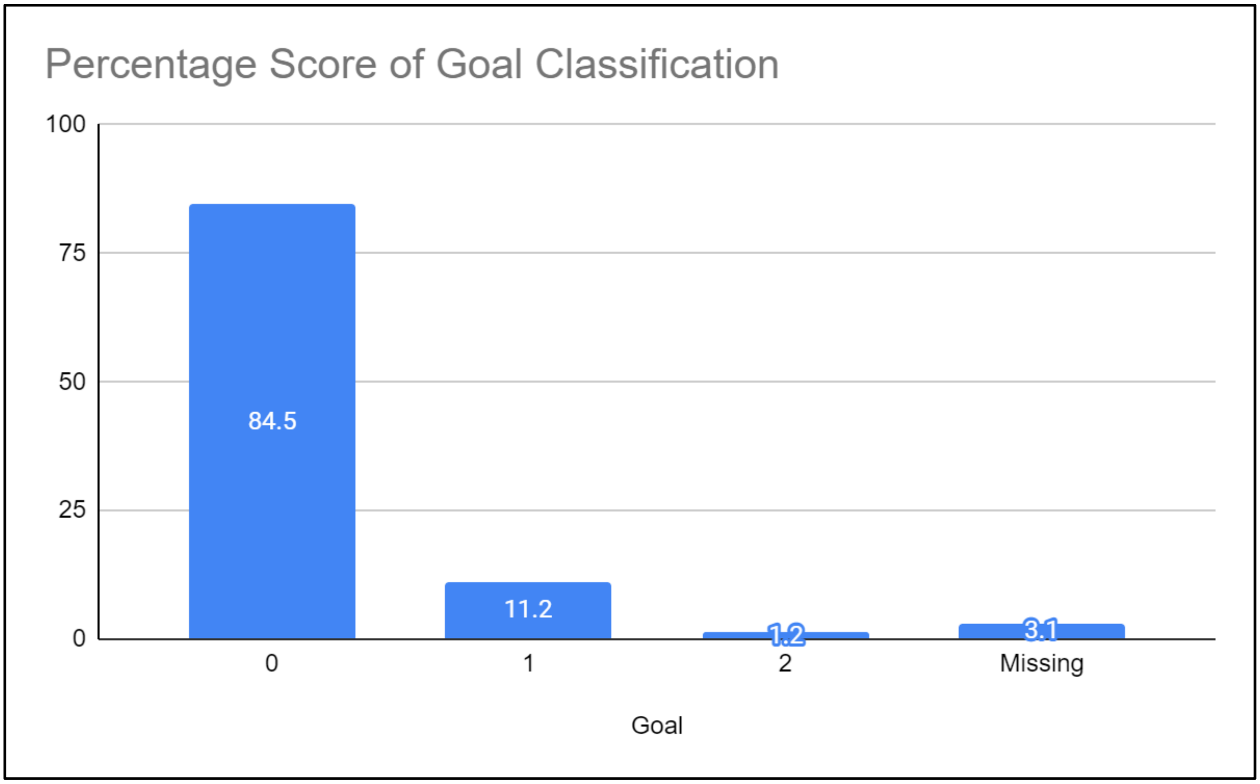


Figure 18: Percentage of achieved Goal identification scores by Cohort 3

Below there are quotes selected from Cohort 3 which are representative of the three different grading points of the rubric in relation to identifying if there is a goal.

0 points: *There is a procedure to follow*

1 point: *Set goal, clear instruction, one way to answer*

2 points: *This is an exercise as you are asked to solve the quadratic equation so the procedure to solve this is known, and it is immediately clear to me how to solve this*

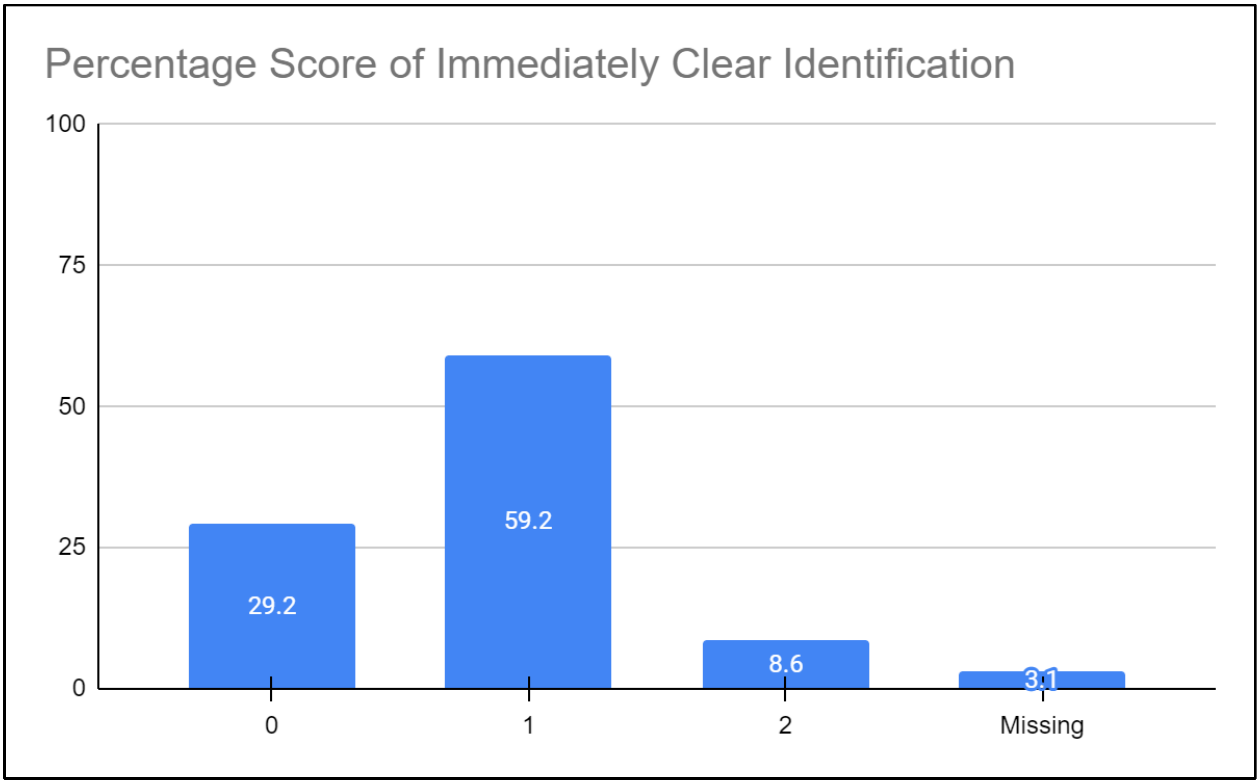


Figure 19: Percentage of achieved Immediately Clear identification scores by Cohort 3

Below there are quotes selected from Cohort 3 which are representative of the three different grading points of the rubric in relation to identifying if it is *Immediately Clear*.

0 points: *It has a clear goal.*

1 point: *There is no procedure, the student uses reasoning to find the solution.*

2 points: *It is clear what is being asked here (total number of handshakes), and they can see a procedure (each person shaking hands).*

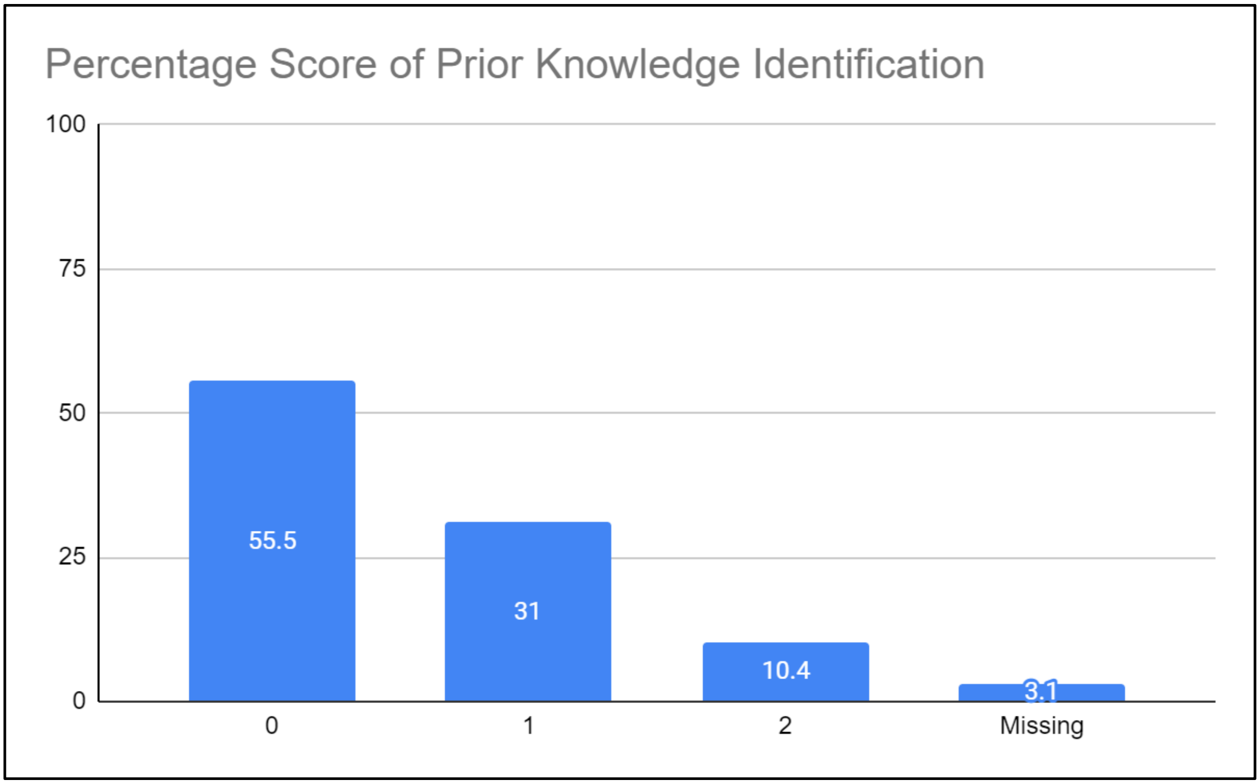


Figure 20: Percentage of achieved Prior Knowledge identification scores by Cohort 3

Below there are quotes selected from Cohort 3 which are representative of the three different grading points of the rubric in relation to *Prior Knowledge*.

0 points: *The task presents a clear goal but its not obvious how to achieve this goal.*

1 *You must use prior mathematical knowledge in order to proceed and try find the answer*

2 points: *Have to recognise the relationship between Pythagoras theorem and the information given. Also need to be apply the information in order to solve it.*

2 points: *Using mathematical knowledge (specifically Pythagoras' theorem) to solve the length of the fence.*

The charts below will outline the percentage of participants in Cohort 4 who achieved the relevant scores. Quotes demonstrating the different scores are outlined under each of the criteria.

Given that there were 45 participants in Cohort 4 and ten tasks in this activity, there were a total of 450 classifications and rationale. There were 7 instances where there was no rationale provided for the choice of classification.

Table 15 shows the mean score, the modal score, and the standard deviation in each of the three criteria for Cohort 4. The maximum score in each of the criteria is 2 points.

n=443	**Goal**	**Immediately Clear**	**Prior Knowledge**
Mean	0.2	0.9	0.7
Mode	0	1	0
SD	0.44	0.62	0.7

Table 15: Descriptive statistics of the rationale for classifications by participants in Cohort 4.

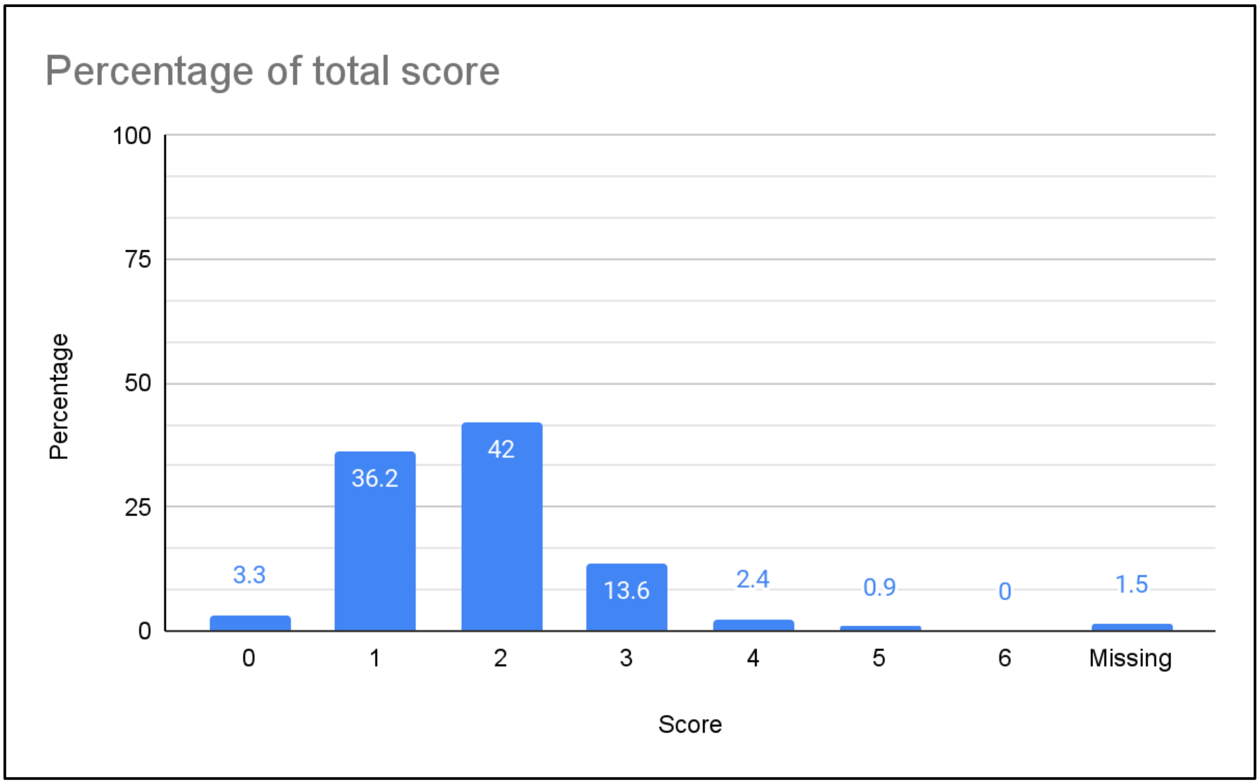


Figure 21: Percentage of achieved total scores by Cohort 45 points: *This is an exercise. part a) uses a person's prior knowledge of similar triangles and is told to use that knowledge to apply to a different shape making it clear how to solve. Part b) also tells you to use prior knowledge of similar triangles and to apply it to quadrilaterals. Showing a clear way of completing it.*

4 points: *Clear goal - find the centre of the circle but isn't told how to find it - has to use prior knowledge*

3 points: *I feel this is an exercise as I think it would be reasonably easy to solve by using methods such as the -b formula or the guide method.*

3 points: *It isn't immediately clear. Must apply prior knowledge about triangles to try and figure out the task*

2 points: *Has a clear goal- but not told how to achieve this goal*

1 point: *To solve this problem it is not immediately clear how to achieve the answer*

0 points: *I feel like I have a rough idea on how to solve this but I'm not certain.*

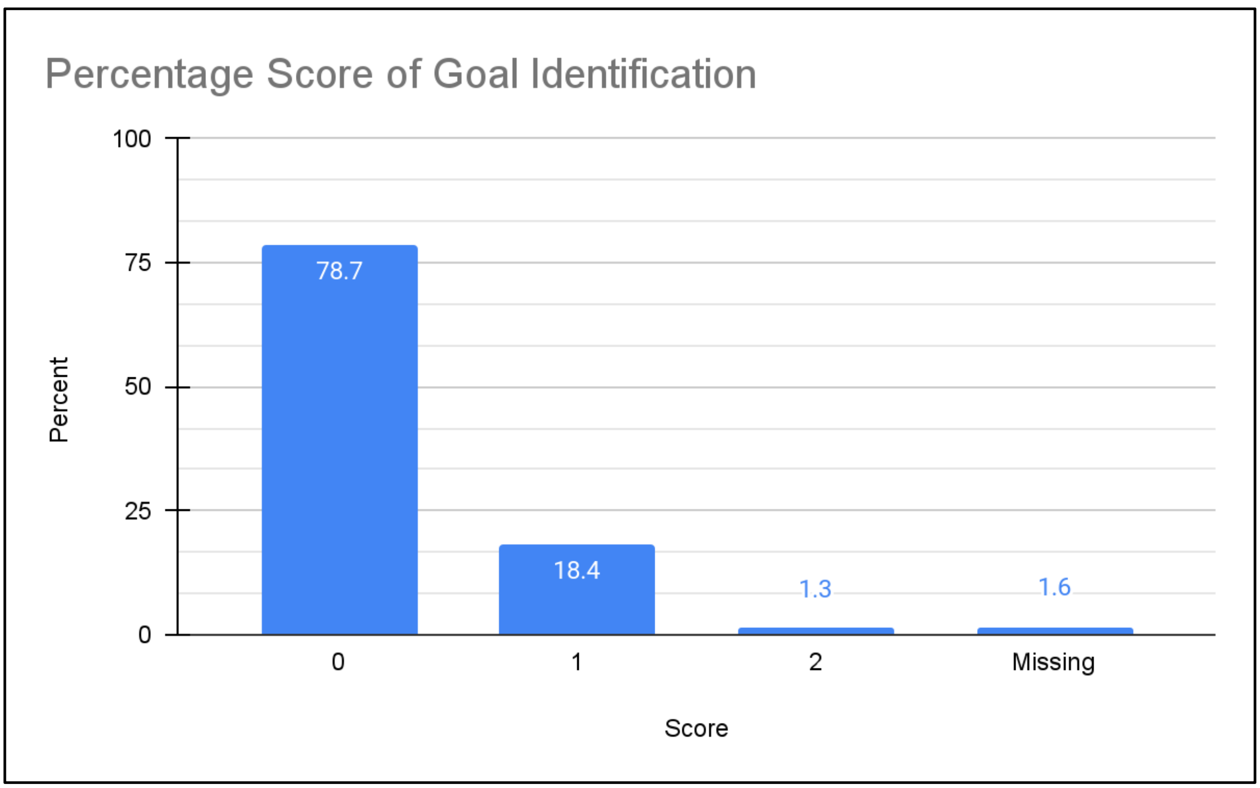


Figure 22: Percentage of achieved Goal identification scores by Cohort 4

0 points: *Not immediately clear what to do, need to think about prior mathematical knowledge*

1 point: *Clear goal. Have access to procedure in order to get an answer.*

2 points: *Clear goal - find the centre of the circle but isn't told how to find it - has to use prior knowledge*

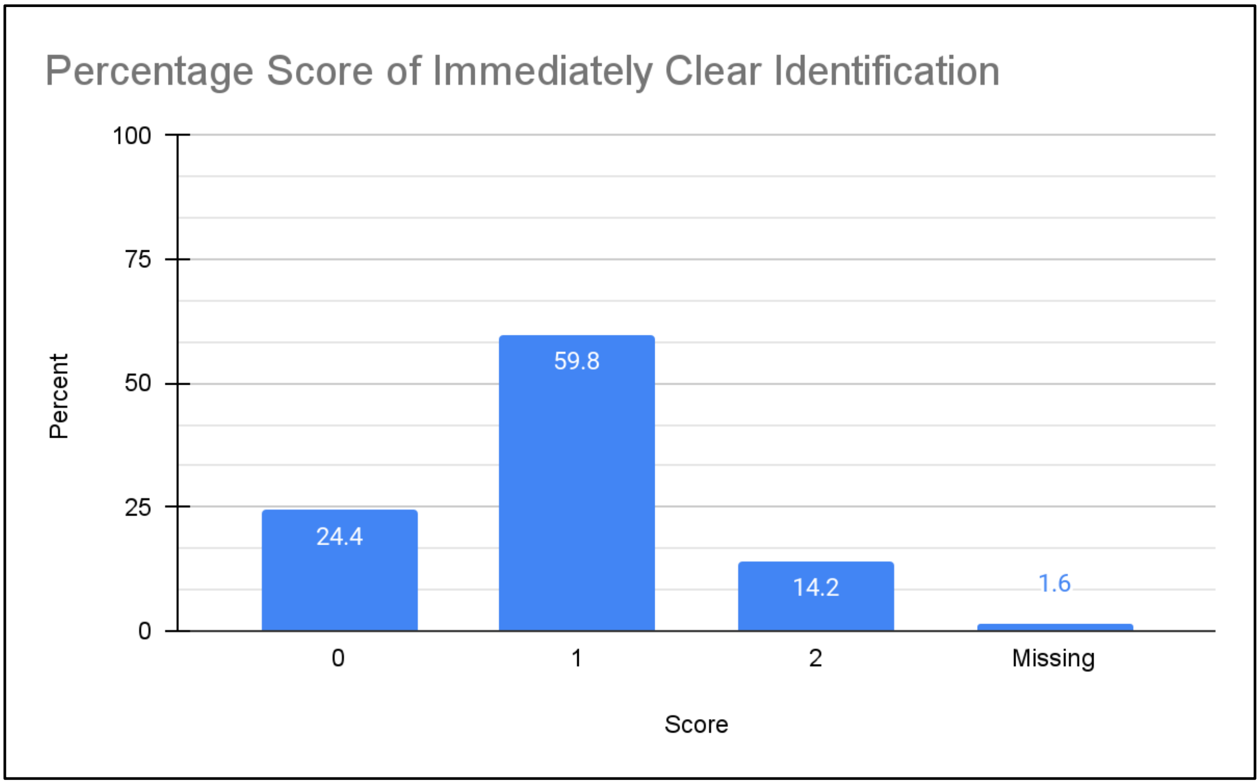


Figure 23: Percentage of achieved Immediately Clear identification scores by Cohort 4

0 points: *There's clear goals. You would have to use prior knowledge to achieve the goal*

1 point: *This question relies on a formula to be solved therefore is an exercise*

2 points: *I think it's a problem as you don't immediately know the solution, you can't supply a certain formula to it. You just have to figure it out.*

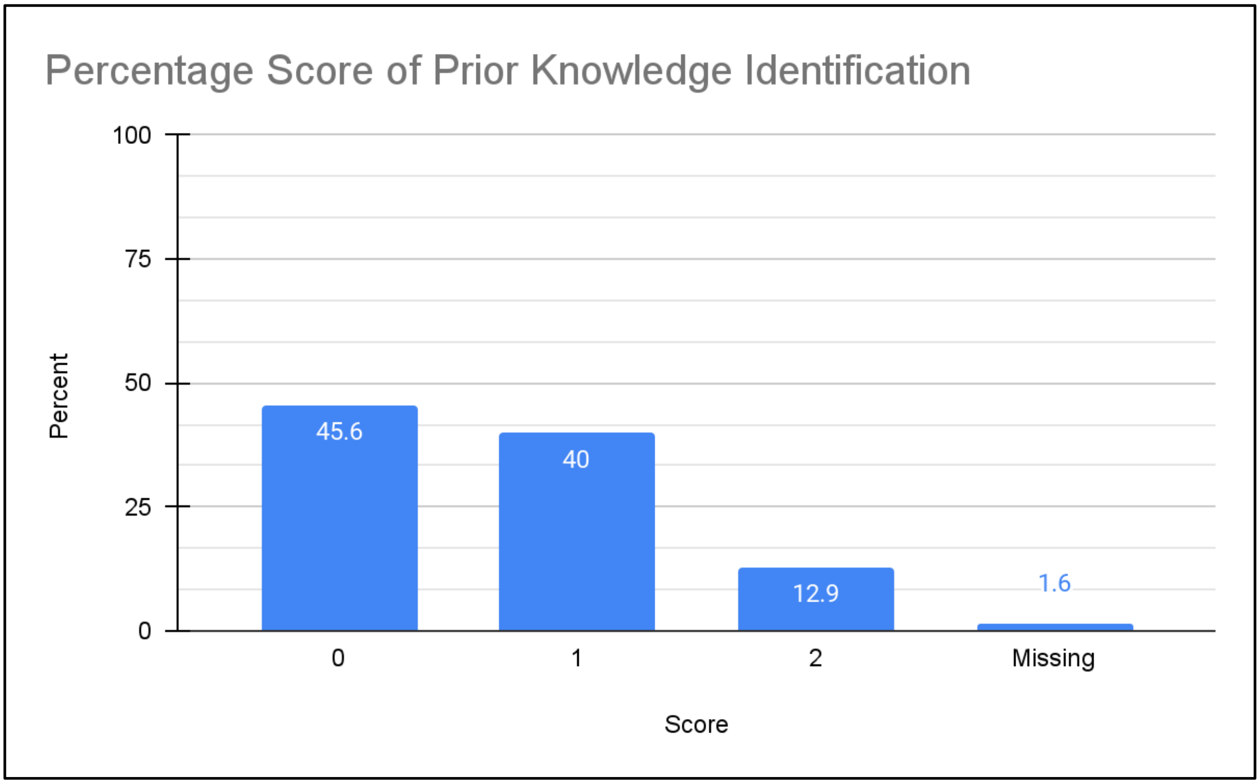


Figure 24: Percentage of achieved Prior Knowledge identification scores by Cohort 4

0 points: *There is a set procedure to solve this exercise*

1 point: *You must use prior mathematical knowledge in order to proceed and try find the answer*

2 points: *Clear method of solving question. Money divided by combined age x individual age.*

6.1 Question 2 (a): Are pre-service teachers proficient at problem-solving?

6.1.1 Results of the MPSR

To investigate the PSMTs' mathematical problem-solving proficiency, the PSMTs' attempts at solving the problems that were undertaken in the tutorials as part of the module (see Section 4.10.2) were analysed using a mathematical problem-solving rubric (MPSR) (Oregon, 2011) (Appendix E). The MPSR consisted of five headings with a maximum score of 6 points per heading meaning the maximum total score was 30. A score of 0 points was only possible if no attempt had been made. This rubric has been used in other studies to assess problem-solving proficiency (Fitzsimons, 2021). The headings of the MPSR are: making sense of the task, representing and solving the task, communicating reasoning, accuracy, reflecting and evaluating.

The MPSR scores correspond to the following descriptors:

Score	Descriptor
1	Minimal
2	Underdeveloped
3	Partially developed
4	Adequately
5/6	Thoroughly/ enhanced/ extensions

Table 16: The descriptor relevant to the points of the MPSR

There were 25 participants in Cohort 1 who completed both tutorial problems. From Cohort 2, 44 participants completed the first problem, 38 participants completed the second problem, and 42 participants completed the third problem. However, to ensure consistency in the results, 23 participants completed all three problems, so these problem attempts were analysed. From Cohort 3, 30 PSMTs completed both tutorial problems. In Cohort 4, 50 participants completed all three problems. The timing of when each cohort completed each problem is outlined in Section 5.5.2. The results from the module tutorials from all four cohorts are displayed in Table 12 below.

Cohort 1		**Making sense**	**Representing**	**Communicating**	**Accuracy**	**Reflecting**
C1: n=25 Problem 1	Mean	2.88	2.68	2.76	1.84	1.08
	SD	0.332	0.476	0.436	0.554	0.277
C1: n=25 Problem 2	Mean	3.04	2.92	3	2.2	1.12
	SD	0.2	0.4	0.289	0.5	0.332
Cohort 2		**Making sense**	**Representing**	**Communicating**	**Accuracy**	**Reflecting**
C2: n=23 Problem 1	Mean	2.38	2.7	2.48	1.91	1.09
	SD	0.491	0.417	0.593	0.596	0.417
C2: n=23 Problem 2	Mean	2.83	2.91	2.87	2.7	1.09
	SD	0.491	0.458	0.458	0.635	0.288
C2: n=23 Problem 3	Mean	3	2.87	2.87	1.87	1.22
	SD	0.674	0.593	0.626	0.757	0.671
Cohort 3		**Making sense**	**Representing**	**Communicating**	**Accuracy**	**Reflecting**
C3: n=30 Problem 1	Mean	2.8	2.57	2.67	1.67	1.53
	SD	0.997	0.898	0.844	1.295	1.22
C3: n=30 Problem 2	Mean	3.8	3.6	3.6	3.4	2.8
	SD	1.031	1.037	1.221	1.221	1.375
Cohort4		**Making sense**	**Representing**	**Communicating**	**Accuracy**	**Reflecting**
C4: n=50 Problem 1	Mean	3.4	3.26	3.26	2.85	1.79
	SD	0.614	0.675	0.675	0.78	0.832
C4: n=50 Problem 2	Mean	3.5	3.33	3.4	2.88	1.6
	SD	0.555	0.73	0.591	0.822	1.057
C4: n=50	Mean	2.97	2.97	2.94	1.88	1.21

Problem 3	SD	0.627	0.577	0.694	1.008	0.592

Table 17: Descriptive statistics of MPSR for Cohort 1, Cohort 2, Cohort 3, and Cohort 4

Cohort 1

Two problems completed by Cohort 1 at two different stages during the module, Week Seven and Week Nine, were analysed (see Section 5.5.2). One-tailed t-tests were conducted to determine if there were statistically significant differences between the mean scores of each problem. The null hypothesis stated that the mean score did not increase over the course of the module, and was tested at a 95% confidence level. P-values below 0.05 meant that the null hypothesis was rejected meaning there were statistically significant increases in the mean scores.

	Mean Problem 1	Mean Problem 2	P value	Null hypothesis
Making Sense	2.88	3.04	<0.001	Reject
Representing	2.68	2.92	0.003	Reject
Communicating	2.76	3	<0.001	Reject
Accuracy	1.84	2.2	<0.001	Reject
Reflecting	1.08	1.12	0.276	Fail to reject

Table 18: One-tailed t-test results for Cohort 1

As outlined in Section 4.8.2.3 Cohen’s d was calculated on the differences between the three problems to investigate the size of the effect of the module on the PSMT’s problem-solving proficiency. The following criteria were used to analyse Cohen’s d for each of the headings of the MPSR (Cohen et al., 2011, p. 521):

- 0 - 0.2 = weak effect
- 0.21- 0.5 = modest effect

- 0.51 - 1.0 = moderate effect
- > 1 = strong effect

Table 14 below displays Cohen's d for Cohort 1 and the effect size for each of the headings of the MPSR.

Heading	**Cohen's d**	**Effect Size**
Making Sense	0.8	Moderate
Representing	0.6	Moderate
Communicating	0.831	Moderate
Accuracy	0.72	Strong
Reflecting	0.121	Weak

Table 19: Results of Cohen's d for Cohort 1

Focusing on the headings for which the t-tests allowed for the rejection of the null hypothesis, it is evident that there was a moderate to strong effect. The *Reflecting* heading showed a weak effect.

Cohort 2

Three problems attempted by Cohort 2 were analysed: 'Professor on an Escalator' in Week Five after two weeks of problem-solving instruction and practice, 'Four-Legged Lawnmower' in Week Six after three weeks of problem-solving instruction and practice, and 'Threaded Pins' in Week Eight after five weeks of problem-solving instruction and practice. One-tailed t-tests were conducted to determine if there were statistically significant differences between the mean scores of each problem. The null hypothesis stated that the mean score did not increase over the course of the module, and was tested at a 95% confidence level. P-values below 0.05 meant that the null hypothesis was rejected meaning there were statistically significant increases in the mean scores.

Problem 1	**Mean Problem 1**	**Mean Problem 2**	**P value**	**Null hypothesis**

Making Sense	2.83	2.83	0.01	Reject
Representing	2.7	2.91	<0.001	Reject
Communicating	2.48	2.87	<0.001	Reject
Accuracy	1.91	2.7	<0.001	Reject
Reflecting	1.09	1.09	0.48	Fail to reject
Problem 3	**Mean Problem 1**	**Mean Problem 3**	**P value**	**Null hypothesis**
Making Sense	2.48	3	0.12	Fail to reject
Representing	2.7	2.87	0.45	Fail to reject
Communicating	2.48	2.87	0.003	Reject
Accuracy	1.91	1.87	0.8	Fail to reject
Reflecting	1.09	1.22	0.186	Fail to reject

Table 20: One-tailed t-test results for Cohort 2

The results of the one-tailed t-tests as presented in Table 20 for Problem 2, showed $p<0.05$ for each of the headings, except for *Reflection,* resulting in the null hypothesis being rejected. This means that it was rejected that there was no increase in the mean scores in the headings. The t-tests for Problem 3, showed that it was not possible to reject the null hypothesis for all of the headings, with the exception of *Communicating*. It is interesting to note that, *Reflecting* was the only heading which showed no significant difference in the mean scores in both problems.

Table 21 below, displays Cohen's d for Cohort 3 and the effect size for each of the headings of the MPSR.

Problem 2		
Heading	**Cohen's d**	**Effect Size**
Making Sense	0	Weak
Representing	0.51	Moderate
Communicating	0.85	Moderate
Accuracy	1.237	Strong
Reflecting	0.01	Weak
Problem 3		
Heading	**Cohen's d**	**Effect Size**
Making Sense	0.26	Modest
Representing	0.37	Modest
Communicating	0.62	Moderate
Accuracy	0.35	Modest
Reflecting	0.19	Weak

Table 21: Results of Cohen's d for Cohort 2

The heading of *Accuracy* showed a strong effect in problem 2. The *Reflecting* heading showed a weak effect in both problems. It is evident that while it was not possible to reject the null hypothesis for *Making Sense*, *Representing*, and *Accuracy* in problem 3, there was a modest effect. *Communicating* was the heading which demonstrated the greatest effect in problem 3.

Cohort 3

Two problems attempted by Cohort 3 were analysed: 'Professor on an Escalator' in Week Four after three weeks of problem-solving instruction and practice, and 'Four-Legged Lawnmower' in Week Seven after five weeks of problem-solving instruction and practice. One-tailed t-tests were conducted to determine if there were statistically significant differences between the mean scores of each problem. The null hypothesis stated that the mean score did not increase over the course of the module, and was tested at a 95% confidence level. P-values below 0.05 meant that the null hypothesis was rejected meaning there were statistically significant increases in the mean scores. The results of the one-tailed t-tests are shown below in Table 17.

	Mean Problem 1	**Mean Problem 2**	**P value**	**Null hypothesis**
Making Sense	2.8	3.8	<0.001	Reject
Representing	2.57	3.6	<0.001	Reject
Communicating	2.67	3.6	<0.001	Reject
Accuracy	1.67	3.4	<0.001	Reject
Reflecting	1.53	2.8	<0.001	Reject

Table 22: One-tailed t-test results for Cohort 3

The results of the one-tailed t-tests as presented in Table 22, showed $p<0.001$ for each of the tests so the null hypothesis was rejected meaning that there were significant increases found between the mean scores.

Table 18 below displays Cohen's d for Cohort 3 and the effect size for each of the headings of the MPSR.

Heading	**Cohen's d**	**Effect Size**

Making Sense	0.97	Moderate
Representing	0.99	Moderate
Communicating	0.762	Moderate
Accuracy	1.417	Strong
Reflecting	0.924	Moderate

Table 18: Results of Cohen's d for Cohort 3

From the analysis there appeared to be a moderate effect on each of the headings except for *Accuracy* for which there was a strong effect.

Cohort 4

Three problems attempted by Cohort 4 were analysed: 'Four-Legged Lawnmower' in Week Two after two weeks of problem-solving instruction and practice, 'Petrol Pitstop' in Week Six after five weeks of problem-solving instruction and practice, and 'Ladder Leaning on a Cube' in Week Ten after eight weeks of problem-solving instruction and practice. One-tailed t-tests were conducted to determine if there were statistically significant differences between the mean scores of each problem. The null hypothesis stated that the mean score did not increase over the course of the module, and was tested at a 95% confidence level. P-values below 0.05 meant that the null hypothesis was rejected meaning there were statistically significant increases in the mean scores.

Problem 2	**Mean Problem 1**	**Mean Problem 2**	**P value**	**Null hypothesis**
Making Sense	3.4	3.5	0.131	Fail to reject
Representing	3.26	3.33	0.288	Fail to reject
Communicating	3.26	3.4	0.071	Fail to reject
Accuracy	2.85	2.88	0.424	Fail to reject

Reflecting	1.79	1.6	0.131	Fail to reject
Problem 3	**Mean Problem 1**	**Mean Problem 3**	**P value**	**Null hypothesis**
Making Sense	3.4	2.97	<0.001	Reject
Representing	3.26	2.97	0.003	Reject
Communicating	3.26	2.94	0.006	Reject
Accuracy	2.85	2.88	0.424	Fail to reject
Reflecting	1.79	1.6	0.131	Fail to reject

Table 23: One-tailed t-test results for Cohort 4

As we see from the results of the one-tailed t-tests as presented in Table 23, the null hypothesis was not rejected in all cases in Problem 2 even though there was an increase in the mean score for each heading. In the case of Problem 3, the null hypothesis was rejected in three instances and the null hypothesis was not rejected in the other two instances.

As outlined in Section 4.8.2.3 Cohen's d was calculated on the differences between the three problems to investigate the size of the effect of the module on the PSMT's problem-solving proficiency.

Problem 2		
Heading	**Cohen's d**	**Effect Size**
Making Sense	0.18	Weak
Representing	0.09	Weak

Communicating	0.237	Modest
Accuracy	0.03	Weak
Reflecting	0.18	Weak
Problem 3		
Heading	**Cohen's d**	**Effect Size**
Making Sense	0.69	Moderate
Representing	0.5	Modest
Communicating	0.46	Modest
Accuracy	0.03	Weak
Reflecting	0.18	Weak

Table 24: Results of Cohen's d for Cohort 4

The results for Cohen's d showed that there was a weak effect in all headings in Problem 2, except for *Communicating* for which there was a modest effect. In Problem 3, the heading with the greatest effect was *Making Sense.* In both Problem 2 and Problem 3, there was a weak effect in *Accuracy* and *Reflecting.*

The results of Cohort 4, for each heading of the MPSR, will now be presented individually in detail.

Making Sense

The *Making Sense* heading of the MPSR involves interpreting concepts of the task and translating them into mathematics. A score of 6 points involves the use of interpretations or translations which are thoroughly developed and enhanced through connections to other mathematical ideas. Figure 25 below shows the percentage of participants who achieved each score for each of the three problems.

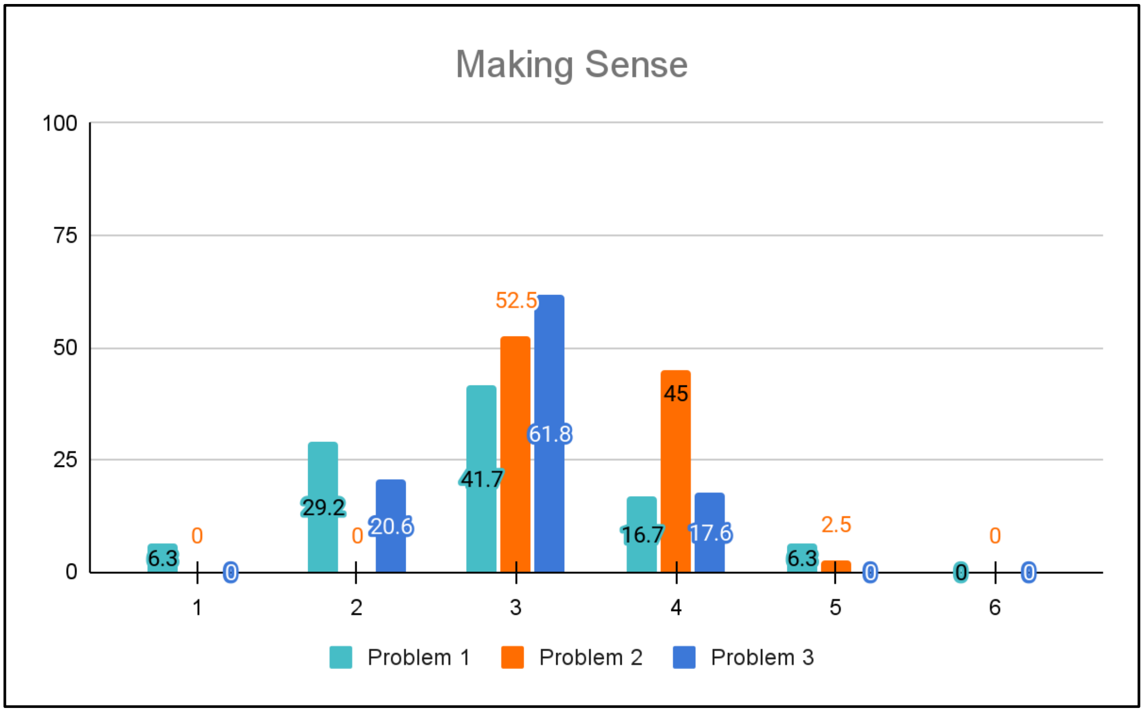


Figure 25: Frequency of each score in the Making Sense heading

The mean scores of *Making Sense* in Problem 1 and Problem 2 of 3.4 and 3.5 respectively suggest that the PSMTs' problem-solving attempts were *partially developed*. However, there was a decrease in Problem 3 with a mean score of 2.97.

Representing and Solving the Task

This heading of the MPSR involves the PSMTs use of models, pictures, diagrams, and/or symbols to represent and solve the task situation or to select an effective strategy for solving the task. A maximum score in this heading involves using complex strategies or representations which are enhanced through comparisons with other representations or generalisations. Figure 26 below shows the percentage of participants who achieved each score for each of the three problems.

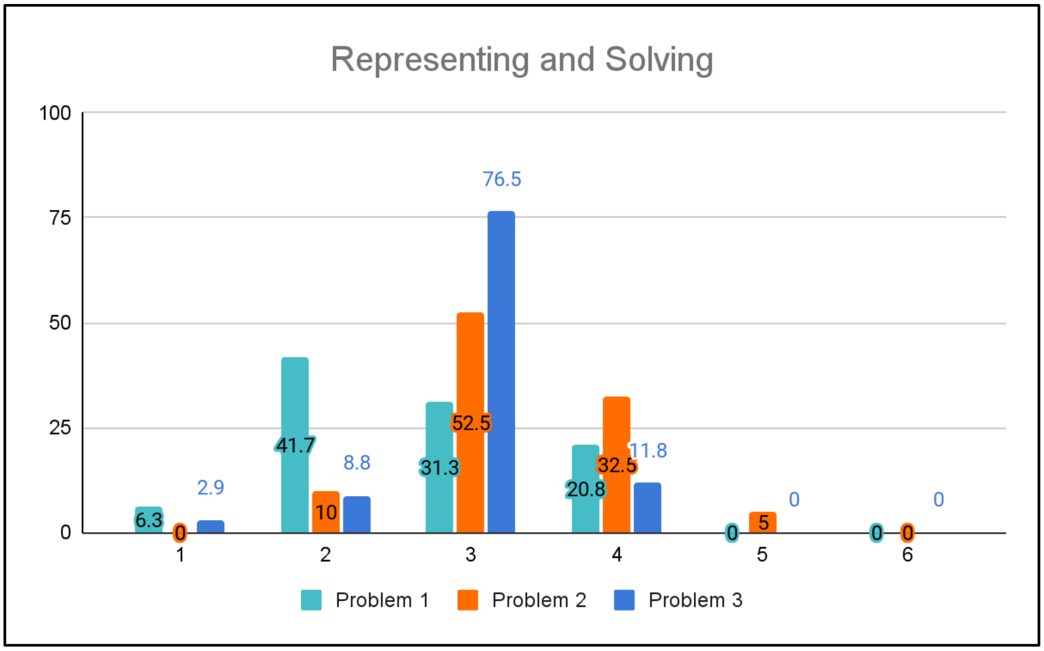


Figure 26: Frequency of each score in the Representing and Solving heading

There was a decrease in the mean score between each of the problems with the mean scores of 3.26, 3.33, and 2.98 for Problem 1, Problem 2, and Problem 3 respectively. 41.7% of PSMTs in Problem 1 scored 2 points indicating *underdeveloped* strategy or use of representations. By Round 3 this had reduced to 8.8% of PSMTs. There was an increase from 31.3% in Problem 1 to 76.5% in Problem 3 of PSMTs scoring 3 points representing *partially developed* representations and strategies. In Problem 2, 5% of PSMTs achieved a score of 5 points.

Communicating

This heading of the MPSR involves communicating mathematical reasoning and clearly using mathematical language. A maximum of 6 points is awarded for the use of mathematical language and communication of reasoning which are enhanced with graphics or examples to aid the reader of the mathematical attempt. Figure 27 below shows the percentage of participants who achieved each score for each of the three problems.

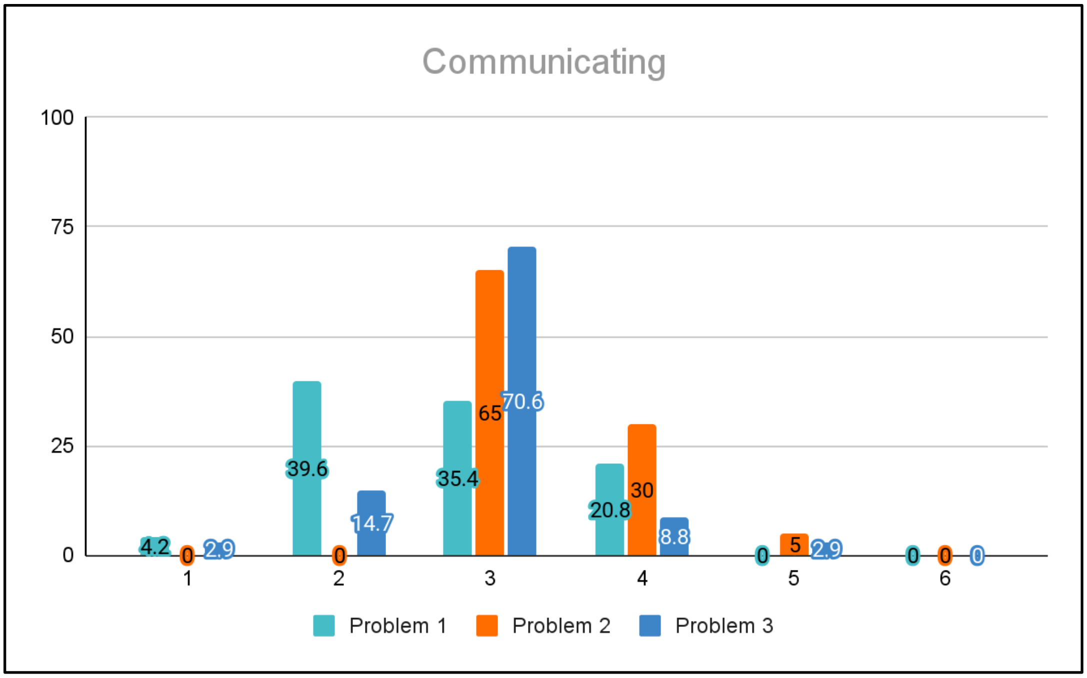


Figure 27: Frequency of each score in the Making Sense heading

The majority of the PSMTs' work in Problem 1 was *underdeveloped* achieving a score of 1 point. This increased with over half of the PSMTs achieving a score of 3 points in Problem 2 and Problem 3. This corresponds with PSMTs *partially displaying* or *not leading to a solution*. In Problem 2 and Problem 3, some PSMTs' use of mathematical language and communication included the use of graphics or examples to communicate their reasoning.

Accuracy

This heading rewards participants for achieving a correct solution. A correct solution and justification amount to a score of 4 points while a score of 5 or 6 points requires extensions and generalisations beyond achieving a solution. Figure 26 below shows the percentage of participants who achieved each score for each of the three problems.

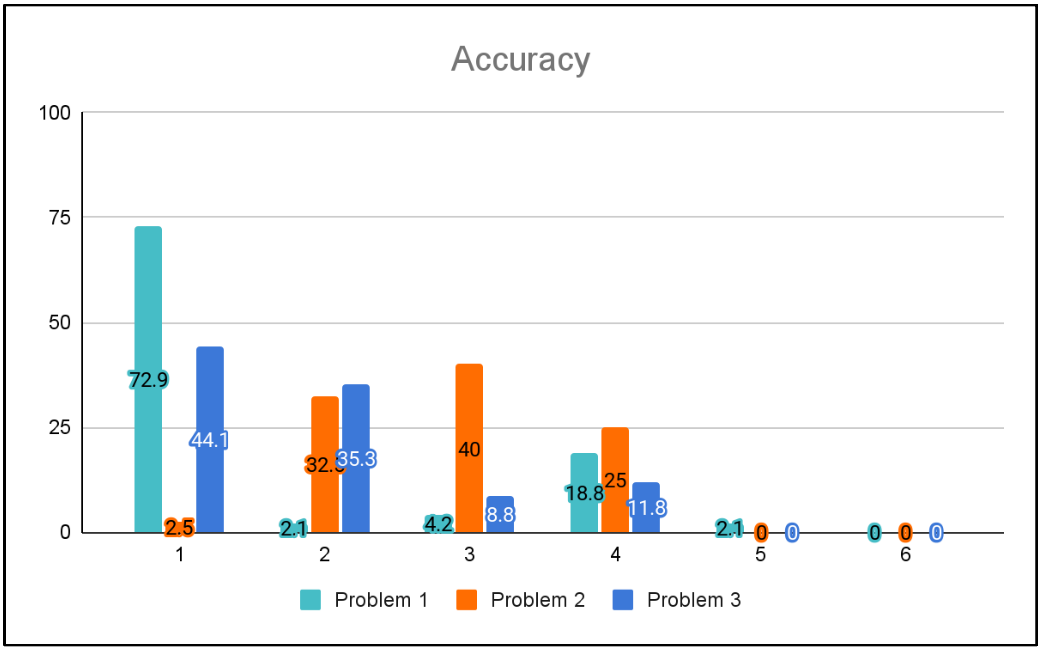


Figure 28: Frequency of each score in the Accuracy heading

The majority of PSMTs in Problem 1 had either an incorrect or incomplete solution leading to a score of 1 point. This changed in Problem 2 with the majority of PSMTs achieving a score of 3 points meaning solutions were partially complete, partially correct or incorrect due to minor errors. One PSMT achieved a score of 5 points in Problem 1 through the use of extensions from a correct solution. There was an increase in the percentage of PSMTs achieving a score of 4 points in Problem 2 which involves attaining a justified correct solution supported by work.

Reflecting

This heading involves stating the solution within the context of the task and justifying the solution by interpreting the reasonableness of the solution. A score of 5 points involves reworking the task using a different method and a score of 6 points is awarded for reworking the task, evaluating the effectiveness of the different approaches, and considering other possible solutions. Figure 29 below shows the percentage of participants who achieved each score for each of the three problems.

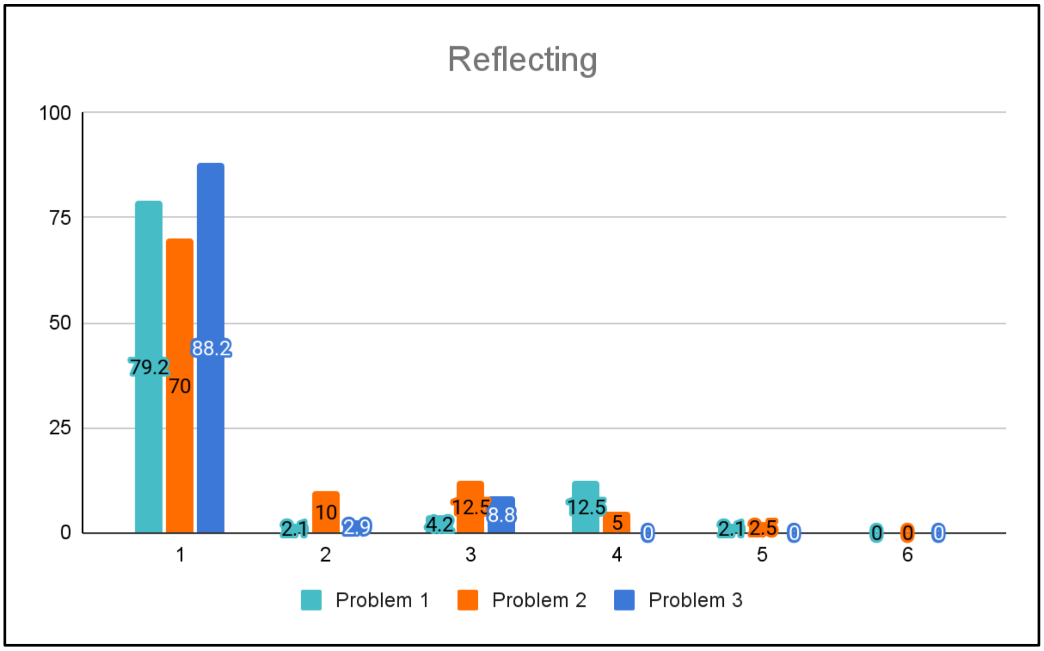


Figure 29: Frequency of each score in the Reflecting heading

This heading had the lowest mean score across each of the three problems. The mean score of the Reflecting heading decreased between each of the problems. The majority of the PSMTs scored 1 point in each of the three problems meaning that minimal justification was conducted by the PSMTs.

6.1.2 Interviews:

Interviews were conducted with PSMTs from Cohort 1, Cohort 2, and Cohort 3, while they were undertaking the module. Participants were recruited on a voluntary basis which is a possible limitation (see Section 4.12). The interviews involved the PSMTs attempting two mathematical problems in a 'Think Aloud' manner (Cowan, 2019). One set of interviews was conducted with Cohort 1, and two sets of interviews (after limited experience of the Rubric Writing Approach and at the end of the module) were conducted with Cohort 2 and with Cohort 3. The details of the problems included in these interviews are outlined in Section 4.10.2. The transcripts of the interviews were analysed using an inductive approach which involved the iterative process of coding, comparing, and grouping the data with similarities to construct categories (Jones & Alony, 2011). The transcripts were systematically analysed by identifying categories with similar properties and combining these to lead to the final categories (Watling & Lingard, 2012). The inductive analysis (see Section 4.10.2), led to the identification of five main themes (or categories) common to all three cohorts. These categories are: *Introduce*, *Productive reasoning, Unproductive reasoning, Resilience,* and *Identity.* Analysis of the interviews from Cohort Two and Cohort Three found that there was evidence of participants questioning themselves. This is referred to as *Productive Questioning* and is viewed as a sub-category of *Productive Reasoning.* Revision of the transcripts of Cohort 1 were done in order to identify if *Productive Questioning* was evident, and it was not found to be so. These categories are discussed below.

Participants 1-9 were in Cohort One, participants 10-14 were in Cohort Two, and participants 15-19 were in Cohort Three. The excerpts below exemplify each category: in these, Px/Cy refers to Participant x of Cohort y. Table 25 shows the number of occurrences of each theme in the interviews. Columns 1 and 2 indicate the relevant cohort interview and problem respectively.

Cohort	**Problem Number**	**Introduce**	**Productive reasoning**	**Unproductive reasoning**	**Resilience**	**Identity**	**Productive Questioning**
1	1	0	23	8	2	8	0
[N=9]	2	34	38	19	25	22	0
2 Pre	3	2	21	4	3	2	7
[N=5]	2	24	26	1	5	14	31
2 Post	4	4	16	0	2	3	6
[N=3]	5	9	12	4	2	5	8
3 Pre	3	3	10	4	0	2	0
[N=5]	2	3	8	8	3	5	4
3 Post	4	5	9	12	5	6	1

[N=5]	5	9	12	8	4	3	5
	Total	93	175	68	51	70	62

Table 25: Occurrence of themes count for every problem by the three different cohorts

Introduce refers to the introduction by the problem-solver of diagrams, constructions within given diagrams, and notation. Mason et al (2011) highlights that the introduction of diagrams and appropriate notation plays a key role in organising information when problem-solving. Examples of participants' use of *Introduce* include:

P 10/ C2: "ok if I set x as time, told travels, x +20 +y"

P11/C2: "So I am going to start by drawing a picture."

The *Productive Reasoning* category includes statements made or actions taken by the participants that promote progress towards a solution of the problem. This category includes the interpreting of information given in the question, use of prior knowledge, specialising and generalising. This category is distinguished from the *Introduce* category by the fact that the participants are organising the given information, making connections with their prior knowledge, or using specialising as a pathway forwards. While the elements of the *Introduce* category could lead to progress, it is not definite.

P13/C2: "well we know that ¼ is more than ⅕ and less than ⅓."

P15/C3: ""So then I would use Pythagoras to look at the top triangle."

As stated above, *Productive Questioning* was evident in the interviews of both Cohort Two and Cohort Three. This category refers to the participant questioning themselves on their work, their chosen strategy, or how to proceed. This *Productive questioning* is seen as a sub-category of *Productive reasoning* as the questioning helped participants towards achieving a solution.

P13/C2: "Can I find the distances from the courtyard that would be helpful? … Is there a way to make right angled triangles to help?"

P12/C2: "So his average speed overall was 93.5km/h. How does that help?"

Unproductive reasoning involves actions or statements which do not help (or even constrict) the problem-solver from progressing or being successful. This includes procedural errors, making assumptions, misconceptions, and persisting with a line of reasoning despite previously stating that it is incorrect.

P16/C3: "we assume that he's at his average speed for almost an hour".

P18/C3: "I'm just going to have to guess 3.37 and I don't even know why."

Resilience includes statements that reflect a participant learning from mistakes, demonstrating a willingness to restart or try a new strategy, and demonstrating a positive response when faced with difficulty.

P13/C2: "So what is some other ways?"

P18/C3: ""I'm just writing down I'm stuck. I'm writing down where I'm stuck. I'm trying to, I don't know how to find a formula to find A the time after."

Statements that indicate a participants' self-belief and confidence make up the *Identity* category. This involves the affective domain which is seen as an important influence on problem-solving behaviour (Lester & Kroll, 1993).

P12/C2: "I just hope I'm on the right path here. […] I'll see where it goes."

P2/C1: "without a calculator, I'm not really good at doing maths in my head […] It's not going to go nicely."

6.2 Question 2 C: Are taught strategies implemented while problem-solving throughout the different iterations of the intervention?

Mason et al., (2011) advocates that the Entry phase is essential in building towards an effective Attack stage of the problem-solving process. He identifies that while people regularly expect to jump straight to the final solution from just reading the question at hand, it is not usually conducive to achieving a solution. The main elements that need to occur in the Entry phase are: deciphering from the information given in the question what is being asked, and preparing for the Attack phase by introducing notation or deciding a strategy. Mason (2011, p.27) states that there are three questions that should be considered in the Entry phase:

- What do I know?
- What do I want?
- What can I introduce?

The I Know element consists of identifying what the person knows from the question and also what the person knows from previous experiences. This involves reading the question through the lens of analysing the information given and deciding what is important. From this the person should write down relevant ideas. Mason et al., (2011) states that while it may seem obvious to 'read the question carefully', it is common for people to try to immediately jump straight into their problem-solving attempt resulting in the question being misunderstood. The *I Want* element of the Entry phase involves the problem-solver identifying what they need to do to either find a solution or proving a statement to be true. Mason et al., (2011) states that *I Want* also involves careful reading of the given question as misinterpretations of the question are possible.

In contrast to the *I Know* and *I Want* elements of the Entry phase, *Introduce* does not involve extracting information from the question but rather adding to the information. This could involve the implementation of notation, organisation, and/or representation. These are described by Mason et al., (2011, p.33) as:

- Notation: choosing what to give a name to, and what name to give
- Organisation: recording and arranging what you *know*
- Representation: choosing elements that are easier to manipulate and substituting these for the elements in the question.

Mason et al., (2011) describes how diagrams can help the problem-solver to decide the key features involved in the question and that appropriate representations can change a question that appears to be difficult into an easy one.

The analysis of the *Entry Phase* of Mason's Rubric Writing Approach was done through the implementation of the following grading system: 0 points (no evidence); 1 point (limited evidence); 2 points (strong evidence). This grading was carried out on the three elements of the *Entry Phase* namely; *Introduce*, *I Know,* and *I Want*. The maximum score for the use of the *Entry* phase was 6 points.

Statements in the *I Know* and *I Want* categories were graded as +1 or +2 only when they clearly appeared in the Entry Phase of the problem solving activity – prior to the Attack Phase. Many such statements involved students repeating or highlighting the information given in the problem. In some cases, there were instances of *I Want* and *I Know* statements being made after the Entry Phase of the problem, for example, if a participant went back and stated what they knew or wanted then this was viewed as a rationale for the approach and work done, rather than used as a starting point.

The participants were asked to attempt two mathematical problems In a 'Think Aloud' manner. The transcripts of these interviews were then analysed using the grading system mentioned above. Nine participants in Cohort 1 completed one round of interviews during the module where they had been exposed to Mason's Rubric Writing approach and the participants had also completed tutorials that explicitly focused on implementing this approach. Five participants from Cohort 2 completed an interview and five participants from Cohort 3 completed an interview. Three of the five participants from Cohort 2 also completed a second interview at the end of the module. All five participants from Cohort 3 completed a second interview at the end of the module. At the point of the first interview, all PSMTs had experience of using the full Rubric Writing Approach. The analysis of the interviews focused specifically on the Entry Phase as the PSMTs would have had most experience of this stage when completing the first interview. Problem One involved probability and was attempted by Cohort 1 . It was decided by the researcher to discontinue Problem One based on the results of the analysis whereby *Introduce* scored 0 points. This was due to the nature of the question and the relatively limited opportunity for PSMTs to include this aspect in their approach. Problem Two involved trigonometry and was attempted by all three cohorts. Problem Three involved Number and was attempted by Cohort 2 and Cohort 3. In the post module interviews, participants from Cohort 3 and Cohort 4 attempted Problem Four and Problem Five. Problem Four involved algebra, speed, distance and time. Problem Five involved trigonometry. The selection for the problems involved ensuring that the mathematical content knowledge required was appropriate for PSMTs who had completed their post-primary education in Ireland. Additionally, the selected problems were from a different strands of mathematics. All the problems used are attached in Appendix F and Appendix G.

Participants 1-9 were in Cohort 1, participants 10-14 were in Cohort 2, and participants 15-19 were in Cohort 3. The excerpts below exemplify each category: in these, Px/Cy refers to Participant x of Cohort y. Table 1 shows the number of occurrences of each theme in the interviews. Columns 1 and 2 indicate the relevant cohort interview and problem respectively.

The tables below will firstly go through the results of the first interview of the three cohorts. These will outline the participants' individual scoring and the number of occurrences of each score for each element of the Entry phase. Next, the tables will show the results of the second interview conducted by Cohort 2 and Cohort 3. As with the first interview, these will outline the participants' individual scoring and the number of occurrences of each score for each element of the Entry phase. The results of the scoring for each problem are outlined in Table 26 and the percentage of scores of each element of the Entry phase per problem is then outlined in Table 27.

Table 26 below shows the results of each participant using the grading system.

Pre module interviews

	Problem	**Introduce**	**Want**	**Know**	**Total**
P1/C1	**1**	0	1	1	2
P1/C1	**2**	1	0	0	3
P2/C1	**1**	0	0	1	1
P2/C1	**2**	0	1	1	2
P3/C1	**1**	0	2	2	4
P3/C1	**2**	0	0	1	1
P4/C1	**1**	0	2	2	4
P4/C1	**2**	2	2	2	6
P5/C1	**1**	0	0	0	0
P5/C1	**2**	0	0	1	1
P6/C1	**1**	0	1	2	3
P6/C1	**2**	1	0	1	2
P7/C1	**1**	0	1	1	2
P7/C1	**2**	1	1	1	3
P8/C1	**1**	0	1	2	3
P8/C1	**2**	2	1	0	3
P9/C1	**1**	0	1	1	2
P9/C1	**2**	2	0	2	4
P10/C2	**3**	0	1	2	3
P10/C2	**2**	2	2	1	5
P11/C2	**3**	2	0	2	4
P11/C2	**2**	2	0	1	3
P12/C2	**3**	0	0	2	2
P12/C2	**2**	0	0	1	1
P13/C2	3	0	0	2	2
P13/C2	2	2	0	2	4

P14/C2	3	0	0	2	2
P14/C2	2	2	0	1	3
P15/C3	3	2	0	2	4
P15/C3	2	0	1	1	2
P16/C3	3	2	0	2	4
P16/C3	2	2	0	0	2
P17/C3	3	0	0	2	2
P17/C3	2	0	1	2	3
P18/C3	3	0	0	2	2
P18/C3	2	2	0	1	3
P19/C3	3	0	2	1	3
P19/C3	2	2	1	1	4

Table 26: Scores using Entry phase rubric of each participant

Table 27 below shows the frequency of each score for each element of the *Entry* phase in pre module interviews.

P1/C1			
Score	Introduce	Want	Know
0	9	1	2
1	0	4	5
2	0	4	2
P2/C1			
Score	Introduce	Want	Know
0	3	5	2
1	3	3	5
2	3	1	2
P2/C2			
Score	Introduce	Want	Know

0	1	4	0
1	0	0	4
2	4	1	2
P3/C2			
Score	Introduce	Want	Know
0	4	4	0
1	0	1	0
2	1	0	5
P2/C3			
Score	Introduce	Want	Know
0	2	2	1
1	0	3	3
2	3	0	1
P3/C3			
Score	Introduce	Want	Know
0	3	4	0
1	0	0	1
2	2	1	4

Table 27: Frequency of each score in the pre-module interviews

An example of a score of 2 points for the *Introduce* category is P13/C2: "*So ok I am drawing out triangles and I'm going to draw the picture out and put dotted lines to split it up into different sections as in the diagram. It's just easier to draw it out."* Here the PSMT drew out a diagram as an aid to starting the problem and also constructed lines to break the diagram up.

Along with drawing a diagram, notation is also an element of the *Introduce* element of the *Entry* phase. An example of scoring 2 points for the introduction of notation to form equations was Px/C2: "*x+y=250 km, t1+t2 = 160 minutes."*

A score of 2 points in the *I Know* element is exemplified by P4/C1: "*So we know that there's five, ten, eleven, ok, so rectangles, parallel lines, opposite sides are the same length. So if I find the length I should be able to get it. So what else do I know?...OK so...I'm obviously not going to reuse angles because there's no angles in the question. So it'll have to be about the sides and can't use Pythagoras because there's no ninety degree angles."* Here this PSMT identifies the information given in the question, the lengths of the sides, and also relates prior knowledge. The PSMT identifies the characteristics of a rectangle and what information would be needed to use the previously learned Pythagoras' theorem. This PSMT also questions themselves on what other information they can take from the question or from their prior knowledge: *"So what else do I know?"*.

A score of 1 point in the *I Know* element was allocated to PSMTs who did not explicitly state the information in the question or prior knowledge relevant to the task. An example of a score of 1 point was P9/C1: *"So the probability it's white is just N over twenty plus N".* This PSMT makes reference to their knowledge of probability but does not relate this to the information given in the question.

A score of 2 points in the *I Want* element of the *Entry* phase is exemplified by P10/C2: "*I want to know x. Distance from the well to the fourth corner."* The participant initially reads the question aloud twice and then identifies what the question is asking them to find. This shows that the PSMT read the question carefully in order to understand the information given and also the goal of the question.

To achieve a score of 1 point in *I Want,* a participant must make reference to what they are trying to achieve but not specifically state what they want to achieve. For example, P17/C3: "*So how would I get to find the distance of x through the well and the rectangle".* This PSMT questions how they would go about finding the value for 'x' but does not specifically state that they want to find 'x' and what 'x' represents in the context of the question.

Below, Table 28 shows the score of each participant for each element of the Entry phase in the post module interviews.

	Problem	Introduce	Want	Know	Total
P10/C2	4	2	0	2	4
P10/C2	5	2	2	1	5
P12/C2	4	2	1	2	5
P12/C2	5	2	1	2	5
P13/C2	4	1	0	0	1
P13/C2	5	2	2	0	4
P15/C3	4	2	0	1	3
P15/C3	5	2	0	1	3
P16/C3	4	0	0	1	1
P16/C3	5	2	1	1	4
P17/C3	4	0	0	0	0
P17/C3	5	0	0	1	1
P18/C3	4	2	0	1	3
P18/C3	5	0	0	0	0
P19/C3	4	2	0	2	4
P19/C3	5	2	2	1	5

Table 28: The scores of each participant in the post module interviews

Table 29 below, shows the frequency of scores in each element of the Entry phase in the post-module interviews.

P4/C2			
Score	Introduce	Want	Know
0	0	2	1
1	1	1	0
2	2	0	2

P5/C2			
Score	Introduce	Want	Know
0	0	0	1
1	0	1	1
2	3	2	1
P4/C3			
Score	Introduce	Want	Know
0	2	5	1
1	0	0	3
2	3	0	1
P5/C3			
Score	Introduce	Want	Know
0	2	3	1
1	0	1	4
2	3	1	0

Table 29: Frequency of scores for each element of the Entry phase in the post-module interviews

An example of a PSMT achieving a score of 2 points in the *Introduce* category is P16/C3: "*So I would probably start by doing a quick diagram of it...I would put x as the, as the, it would be the hypotenuse of the big triangle. That's the one I'm trying to work out*." This PSMT demonstrates introducing a diagram and also notation through the use of the variable of 'x' to represent the hypotenuse. A score of 1 point was awarded to participants who made reference to introducing elements at the start of a problem. For example, PC/C2: "*Trying to work out how to write this as an equation to solve it. So I can write it in terms of x's time maybe."* This PSMT states how they could use notation to help them create equations, but this PSMT does not pursue this thought in their attempt.

An example of a PSMT relating their prior knowledge to the question given and achieving a score of 1 point was P19/C3: *"Well it has to be less than 4 anyway because it's not straight up"*. To be awarded a score of 2 points, the PSMTs needed to explicitly state the information given in the question and also relevant prior knowledge. For example, PA/C2: "*Ok so I am going to start by taking 20 minutes away from 3 hours to get the total time he was driving and that is 2 hours and 40 minutes. So he drove a total of 250 and he did 80km/h beforehand and 100km after. So distance is speed x time so the total distance is speed1 times time1 + speed2 times time2."* This PSMT related their prior knowledge of speed and distance along with outlining the numerical values given in the question.

For the *I Want* element, PC/C2 received a score of 2 points for the following reason: "*So I'm going to set the length on the ground as x and the distance from the ground to the top of the ladder as y…So I want the hypotenuse and I want y."* This PSMT reads the question aloud twice and decides from the information what the question is asking them to find.

A score of 1 point for the *I Want* element is exemplified by PB/C2: *"Ok I need to find the total height of the ladder which is greater than 1 because it is taller than the cube".* This PSMT outlines what they want to find in order to make progress in their problem-solving attempt; however, this PSMT does not state what the overall goal of the question is.

The following Table 30 outlines the percentage of scores achieved for each element of the *Entry* phase per problem attempted by the PSMTs.

Problem 1	Score	Introduce	Want	Know
n=9	0	100	11.1	22.2
	1	0	44.4	55.6
	2	0	44.4	22.2
Problem 2	Score	Introduce	Want	Know

n=19	0	31.6	57.9	15.8
	1	15.8	31.6	63.2
	2	52.6	10.5	21.1
Problem 3 n=10	Score	Introduce	Want	Know
	0	70	80	0
	1	0	10	10
	2	30	10	90
Problem 4 n=8	Score	Introduce	Want	Know
	0	25	87.5	25
	1	12.5	12.5	37.5
	2	62.5	0	37.5
Problem 5 n=8	Score	Introduce	Want	Know
	0	25	25	25
	1	0	62.5	62.5
	2	75	12.5	12.5

Table 30: Percentage of scores achieved for each element of the Entry phase per problem

Table 31 below gives an overview of the overall percentage of PSMTs who achieved a score of 0, 1 or 2 points for the implementation of the *Entry* phase of Mason's Rubric Writing Approach,

Score	**Introduce**	**Want**	**Know**
0	48.1	55.6	16.7

1	7.4	25.9	48.1
2	44.4	18.5	35.2

Table 31: Overall percentage of PSMTs achieving each score

These results are displayed in Figure 30 below:

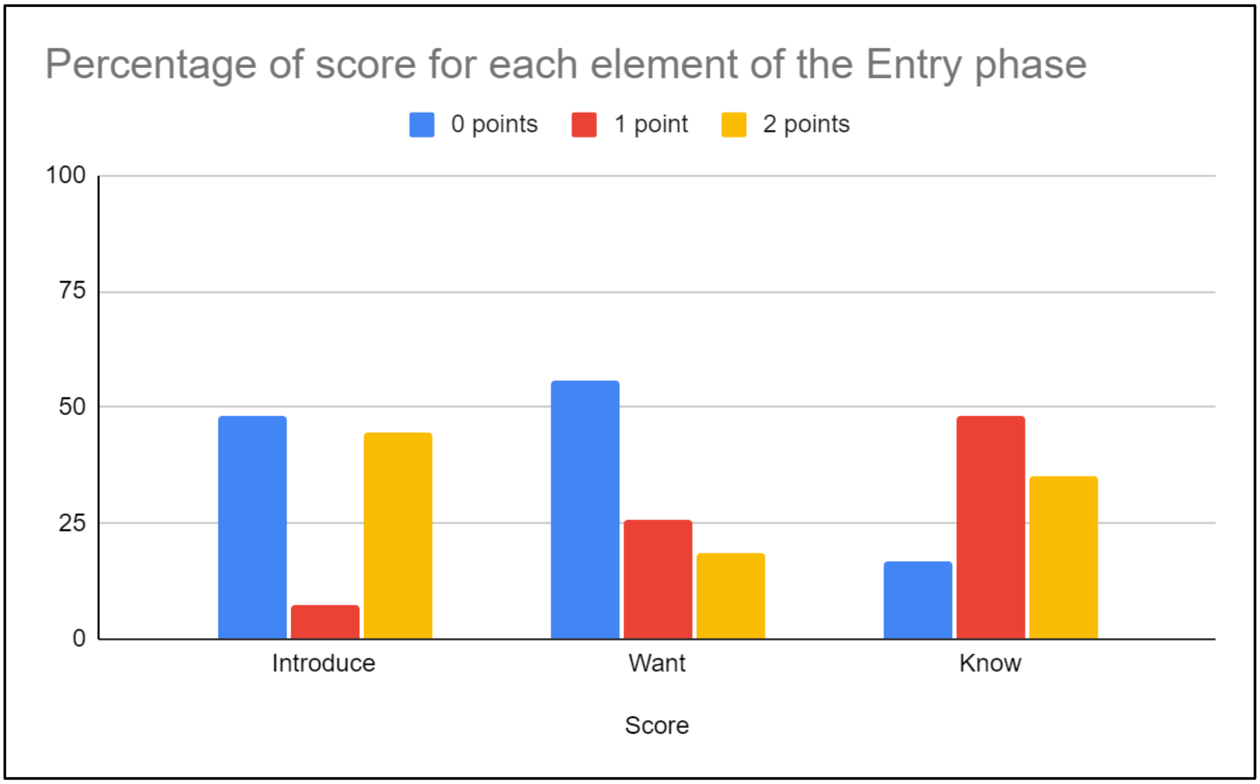


Figure 30: Overall percentage of PSMTs achieving each score

From the overall scores, it is clear a score of 0 points is most common in the *I Want* element. This is also the case in Problem 2, Problem 3, and Problem 4. A score of 1 point was most common in the *I Know* category overall and also in the case of all the individual problems except for Problem 3. In Problem 3, 90% of the PSMTs achieved a score of 2 points in *I Know*. In the overall scoring, *Introduce* had the most common score of 0 points. Although Problem 2, Problem 4 and Problem 5 had a most common score of 2 points in the *Introduce* element, all the PSMTs scored 0 points in Problem 1 in the *Introduce* element.

6.3 Question 3: What are pre-service teachers' capacities in relation to problem posing?

PSMTs in Cohort 2 completed three activities which focused on their ability to pose mathematical problems. This involved the ability to select suitable problems (Activity One), the ability to generate mathematical problems (Activity Two), and finally the ability to reformulate mathematical tasks (Activity Three). PSMTs in Cohort 4 completed three extension tasks. This involved the PSMTs attempting a problem (see Section 4.10.2) and then extending the problem. Extending a problem means that, on completion of a problem, the problem-solver generates a new problem based on the completed problem (Mason et al., 2011). This generation of a new problem through extensions, enables the problem-solver to enhance their understanding of the solution to the previous problem and highlight unexpected features of the problem (Brown & Walter, 2005). Each activity completed by Cohort 2 will be discussed followed by the extension problems completed by Cohort 4.

6.3.1 Activity One:

This activity focused on investigating the ability of the PSMTs to be able to select a task that would be an appropriate problem for a specified student. While this activity focuses on distinguishing between tasks, it is different to the 'Task Sorting' activity, which was developed for Research Question 1, as this present activity involves considering the situation of the learner. For this activity, the PSMTs would need to consider the definition of a problem which meets the criteria of the Three Key Characteristics. Additionally, the PSMTs would need to consider the presented learner and the prior knowledge which they have. The participants were given 13 scenarios (see Appendix H) which each outlined the following information: the year the student was in school, their level of study (higher or ordinary level), and their topics of prior knowledge. A mathematical task was then stated. The participants were asked to decide if the task was a problem for the described students and to justify their answer. These tasks were independently classified by the researcher and supervisor of this study. The two classifications were all in agreement bar one task, task 12, for which both persons considered the task to be borderline. Two years after the initial classification of the tasks, the research supervisor repeated the classification of the tasks. These classifications aligned with the initial classification. Through discussion both agreed to classify this task as not a mathematical problem. The independent classification of these tasks are present in Table 32.

Task	1	2	3	4	5	6	7	8	9	10	11	12	13
Classification	Yes	No	No	Yes	Yes	Yes	Yes	Yes	No	No	Yes	No	No

Table 32:Classification of tasks as a mathematical problem by the researchers

The classification of tasks as problems for the described student by the PSMTs is presented in Figure 31 below.

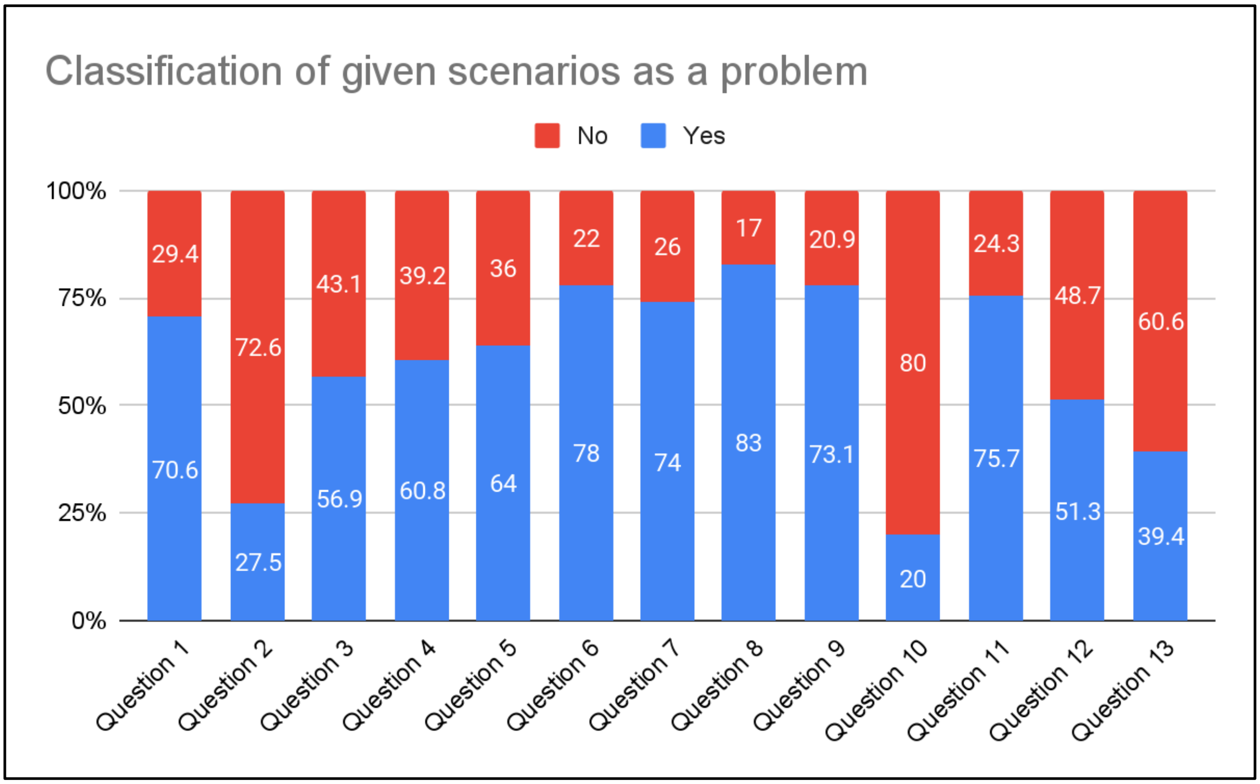


Figure 31: Percentage of Activity One tasks classified as a problem by PSMTs

The alignment of the PSMTs' classification of the tasks to the researchers' classification is presented in Figure 32.

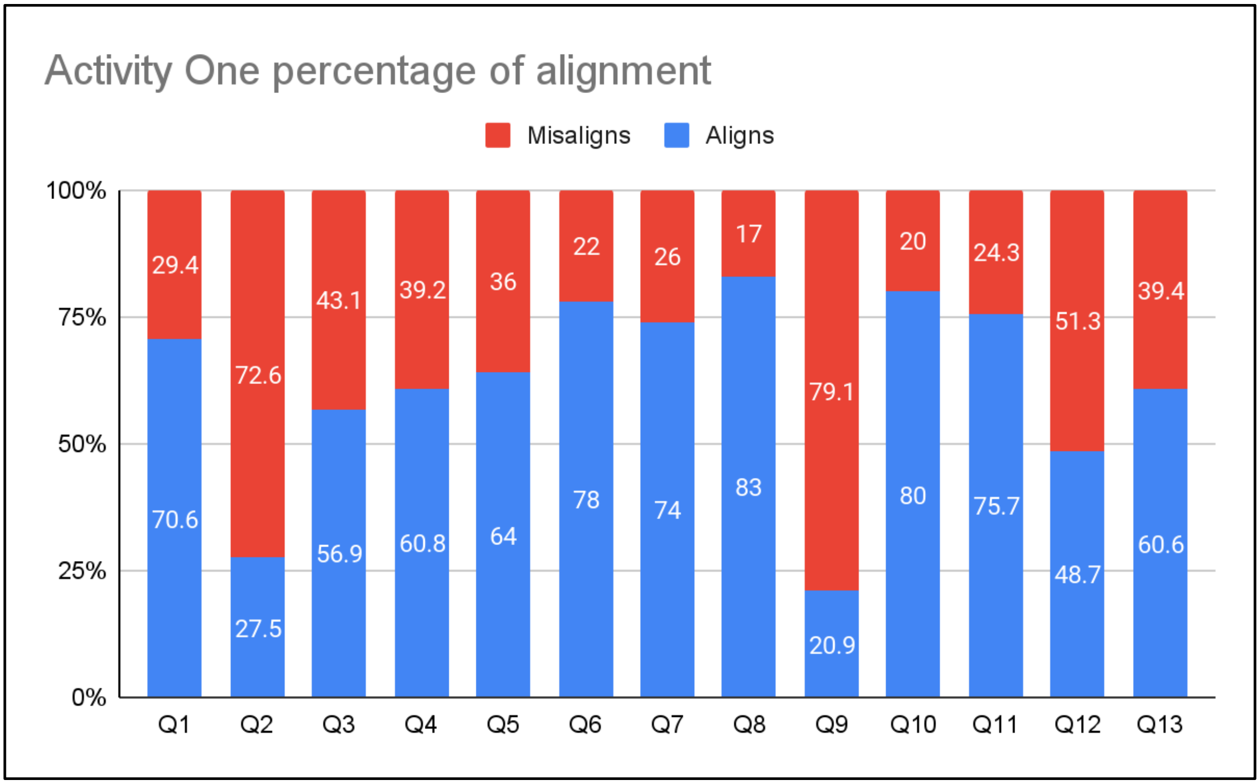


Figure 32: Activity 1 percentage of alignment of PSMTs' classification of tasks

Overall, there was an average of 63.5% alignment between the PSMTs' classification of tasks with the researchers' classification. There was a relatively low level of alignment of classification in question 3 and question 9.

Question 3:

> *A projectile is launched directly upwards and its height, H metres above the ground, is given by* $H = 20t - 5t^2$ *where t is the time in seconds. After how many seconds will the projectile be 20m above the ground?*

Question 9:

> *IQ scores are normally distributed with a mean of 100 and a standard deviation of 15. Isaac has taken an IQ test and scored 132. His friend Eoin has remarked that Isaac's score is in the top 5% of all IQ scores. Isaac disagrees and says that he is in the top 2.5%. Is Eoin's remark correct? Explain your reasoning.*

Both of these tasks were classified as not mathematical problems by the researchers but 60.8% in question 3 and 79.1% in question 9 did not align with this classification. Both questions are 'wordy' questions and have a clear goal. The text of both questions provides a straightforward path with no reasoning or interpretation required. Below are examples of quotes from PSMTs who classified question 3 as a problem.

1. *"The problem isn't visual enough and therefore the students would find working out a word problem such as this one, hard".*
2. *"As the subbing in for H would cause confusion and there are 2 values to be solved".*
3. *"Contains two variables".*

Quote 1 indicates that words signify that this question is a problem, and a diagram or model would be necessary in order to classify this question as not a problem. Both quote 2 and quote 3 indicate that the number of variables in this question would make it a problem. An example of a PSMT who classified question 3 as not a problem is presented below. This extract shows that the PSMT identified that the question involves a procedure that students would have previous knowledge of:

> *"Simple substitution and factorising".*

The results of the PSMTs' classification of Question 12 showed that 48.7% aligned with the researchers' classification and 51.3% misaligned. It is interesting to note that when independently classifying the tasks, question 12 was the only question which required discussion to settle on a classification. Through discussion it was identified that there is a clear goal and although mathematical reasoning is required, there is a clear path with variables clearly given in the question. Thus question 12 was classified as not a problem.

Question 12:

> *Annie is y years old. Her sister is twice as old as her. Their mother is 25 years older than Annie's sister's age. The total of all three ages is 80. How old is Annie?*

Quotes 4 -7 exemplify the PSMTs who considered this question to not be a problem which aligned with the classification. The PSMTs express that it is clear what the person needs to form a linear algebraic equation in order to achieve the solution.

4. *"This is a simple algebraic expression and is clear what is needed to be done"*
5. *"This is an exercise as the students need to write three terms then equate the sum of the term to 80 to find y. As they have covered algebra this should be doable".*
6. *"I feel this is basic enough and is clear how to work out how old Annie is".*
7. *"They have covered enough in algebra to be able to form an equation = 80 and solve it".*

However, in opposition to the PSMTs who stated that forming equations were straightforward, some PSMTs considered the forming of equations as grounds to classify the question as a problem.

8. *"They have to construct their own equation. Should be able to solve but may find difficult".*

Another reason that PSMTs identified as a reason to classify it as a problem was the inclusion of more than one piece of information which involved the person analysing the text of the question to form equations.

9. *"It involves a lot of thought, a few different methods and you need to solve more than one piece of information".*
10. *"Question is abstract and requires thought. Question is not as simple as it appears - students must analyse the wording of the question carefully".*

It was also stated that time would constitute a reason to classify this question as a problem.

11. *"I feel this question would take time to do".*

There was a high level of alignment with question 8 being classified as a problem.

Question 8:

Paula, Henry and Maria are triplets. Henry can paint a room by himself in 3 hours. Paula can paint the room by herself in 4 hours. Maria can paint the room by herself in 6 hours. If they all work together and don't get in each other's' way, how long will the job take?

There was strong consensus amongst the rationale given by the PSMTs as it was not clear on how to approach this question to reach a solution.

12. *"No clear instruction"*
13. *"The answer is not visible from the start and it involves some developing".*
14. *"It is not clear how long it'd take together. It is not clear how to come to the solution".*

This consensus coincided with the researchers' position which was that there was a clear goal to the question but it was not clear on how to achieve the goal.

6.3.2 Activity Two:

As discussed in Section 4.10.4, Activity Two consisted of the generation of 13 mathematical problems (see Appendix I). The PSMTs' were given thirteen scenarios and were asked to generate a mathematical task that would constitute a problem for a student who possesses certain categorical characteristics.

The mathematical tasks that were generated by the PSMTs were analysed to identify if they met the criteria of the Three Key Characteristics of a problem. *Achieved* refers to tasks which were classified as a mathematical problem and *Not Achieved* refers to tasks which were classified as not meeting the criteria of a mathematical problem. The results are displayed below in Figure 33.

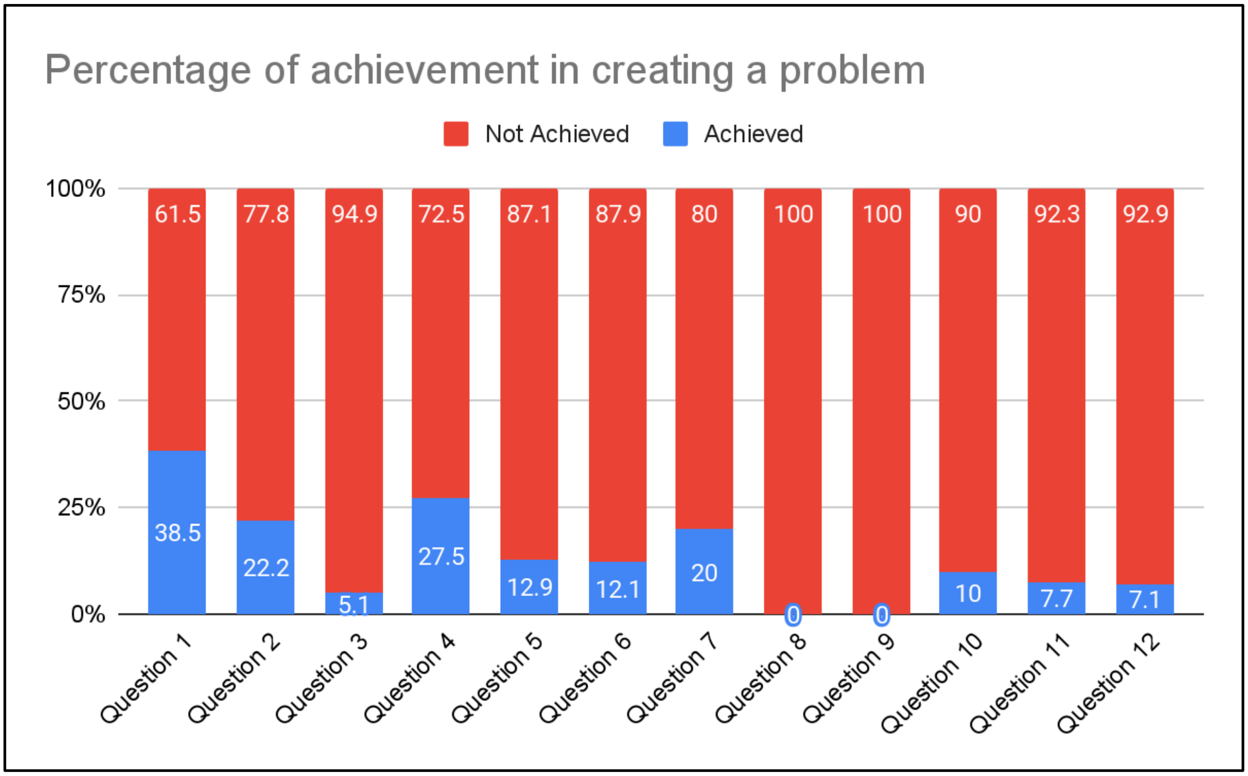


Figure 33: Percentage of achievement in generating a mathematical problem

The mathematical tasks that were generated by the PSMTs were then analysed to identify the reasons for which they did not meet the criteria of the Three Key Characteristics. This resulted in the following categories being identified: *procedural*, *not solvable,* and *not solvable due to missing information.* These categories are described below.

- *Procedural:* Included in this category were tasks which were immediately clear on how to proceed or tasks that could be solved using a previously learnt procedure.
- *Not solvable:* These tasks were not possible to find a solution. This included tasks where real-life contexts of the task were not possible, miscomprehension of the applicability of mathematics to real-life contexts, tasks which had ambiguity in what the goal of the task was, or tasks that were ill-formulated meaning that a solution could not be achieved.
- *Not solvable dues to missing information:* This category involved tasks where the PSMT had left out information that was crucial to achieving a solution meaning that it was not possible to solve.

The analysis of the results in Figure 34 and examples of each category are presented below.

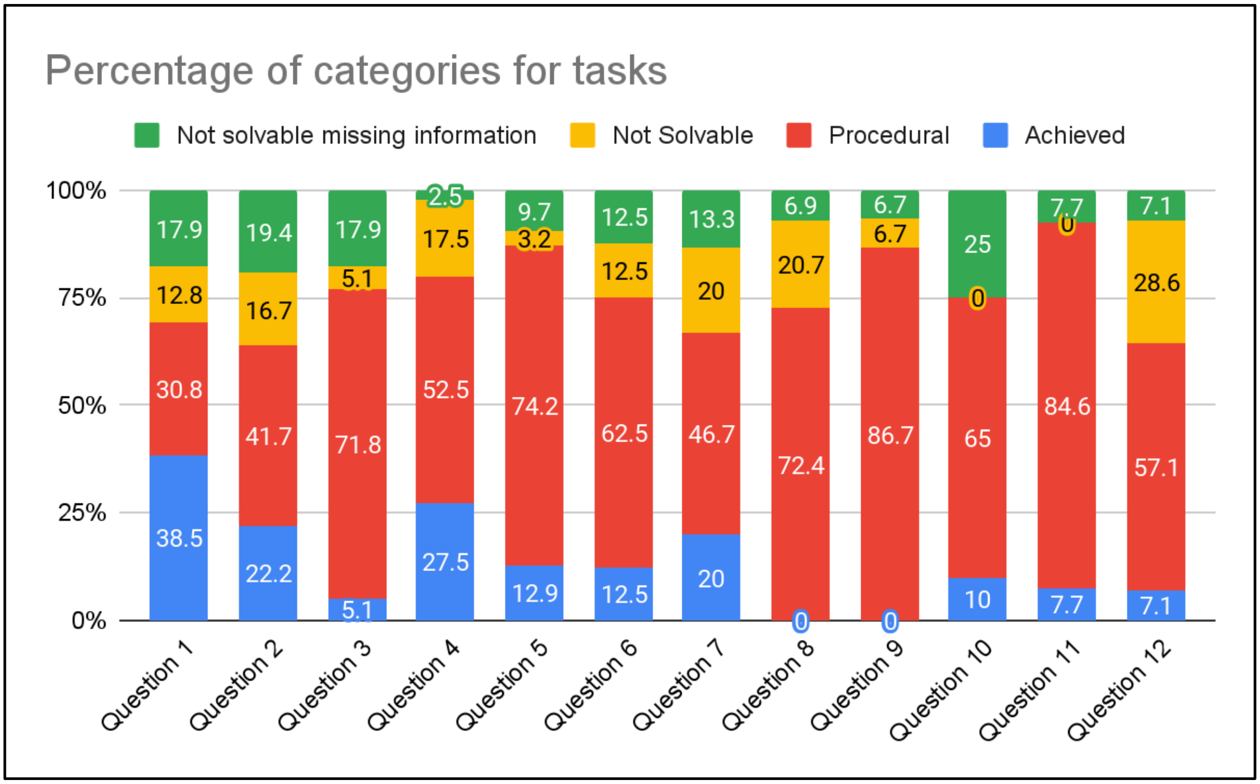


Figure 34: Percentage of generated tasks classified in each category.

As seen in the results above, *Achieved*, had a low success rate in particular in Question 8 and Question 9 with 0% and in Question 10 within which just 10% of the PSMTs generated a mathematical problem. Two examples of tasks which were categorised as mathematical problems are given below.

> *"The diameter of the base of a cone is 7cm, the height is twice the circumference. What is the volume of the cone?"*

> *"A driver drove 90km per hour. He sneezed for half a second and his eyes were closed. How far did he travel in the half second?"*

With the exception of Question 1, the majority of the generated tasks for each question were classified as *Procedural.* Two examples of such tasks are given below.

> *"In a right angled triangle with sides of 4cm and 5cm find the hypotenuse."*

> *"In an equation* $3x^2 + 11x - 14 = 0$ *find x."*

In the first example, the task requires the use of Pythagoras' theorem, and it is immediately clear how to achieve a solution. This immediacy is also seen in the second example which requires the use of *-b formula* or factorising method to achieve a solution.

The *Not solvable* category consisted of tasks which were not possible to solve.

This task was categorised as *Not solvable* due to the wording of the task. The task asks for an 'isosceles' triangle to be constructed with 'one' angle of 60 degrees[1]. However, given the definition of an isosceles requiring two sides and two angles of equal length and through the use of a 60 degree, the constructed triangle would be an equilateral triangle and not an isosceles triangle.

> *"Construct an isosceles triangle with one angle being 60 degrees."*

The following task was classified as *Not solvable* as it is not possible to find the midpoint of a line. The PSMT would need to ask for the midpoint of a line segment.

"There are two lines: 2x +y=9 and x+4y = 16. Find the slope of the two lines and the midpoint of the two lines."

The following two examples are of tasks which were *Not solvable* due to a misunderstanding of probability. That is, probability theory does not allow for a prediction of the number of free kicks that *will* be scored but rather allows the statement of the probability of a specific number of free kicks that will be scored.

"I have a 4/5 chance of scoring a free kick. If I take 17 free kicks how many will I score?"

Additionally, this task is not possible as the solution is not a whole number, (13.6) and it is not possible to score such an amount of free kicks showing that the real-life context of the task is no taken into consideration.

[1] Even though an equilateral triangle has two equal angles, it does not have exactly two equal angles so is therefore not considered a special case of an isosceles triangle (Junior Cycle specification, 2018).

Similarly, in the task below, the solution is not a whole number, and it is not possible given that the question asks for the number of people involved.

> *"A school election was conducted. 37% of students voted for rep A. 23% voted for rep B and 40% voted for rep C. If 157 people voted for B and C, how many voted for A and how many voted?"*

Examples of the tasks classified as *Not solvable due to missing information* are presented below.

In the first example, the PSMT does not include the shape of the water tank which is crucial to being able to achieve a solution.

> *"A water tank has a radius of 5 metres and can hold 20m3 of water. If water flows out of the tank at a rate of 0.25m3 a second how long will it take for the tank to empty?"*

In the next example the PSMT does not provide enough information, the third side, in order for a solution to be achieved.

> *A right angled triangle with hypotenuse of 5+2xcm and opposite of 7-x cm. Find the value of x.*

6.3.3 Activity Three:

Activity Three focused on reformulation of given mathematical tasks (Appendix J). As discussed in Section 4.10.4, Activity Three consisted of five mathematical tasks. Question 1 involved the PSMTs reformulating a given exercise into a problem. Question 2a), Question 2b), and Question 3 involved reformulating given mathematical tasks to problems addressing given mathematical topics. Question 4 involved the PSMTs reformulating a given problem into an open problem. The analysis of Activity Three consisted of comparing the PSMTs' attempts to the Three Key Characteristics. Question 2a), Question 2b), and Question 3 also consisted of comparing the PSMTs' attempts to the Three Key Characteristics and whether their responses addressed the specified mathematical topic.

The analysis of the reformulation tasks involved two stages. The PSMTs' tasks were firstly compared to the Three Key Characteristics, and classified as *Achieved* (met the criteria) or not achieved. The tasks which did not meet the criteria were further analysed to produce categories identifying the reasons for which they were not classified as a mathematical problem. This resulted in the following categories: *not solvable, not a problem,* and *unsolvable due to missing information*. The category of '*did not address the topic*' was applicable to Question 2a), Question 2b) and Question 3. The category of '*not open*' was applicable to Question 5 only. The following are explanations of the categories mentioned above:

- *Unsolvable:* The mathematical task posed was not possible to solve, or there was ambiguity as to what the question was asking due to the poor phrasing of the task.
- *Not a problem:* The task posed did not meet the criteria of the Three Key Characteristics of a mathematical problem.
- *Unsolvable due to missing information:* The task posed did not contain enough information to allow a problem-solver to solve it.
- *Did not address the topic:* The mathematical task did not address the specified topic required in the given question.
- *Not open:* The task posed did not meet the criteria of being an open problem.

Figure 35 shows the percentage of PSMTs who were successful or not successful in the reformulation of the mathematical tasks.

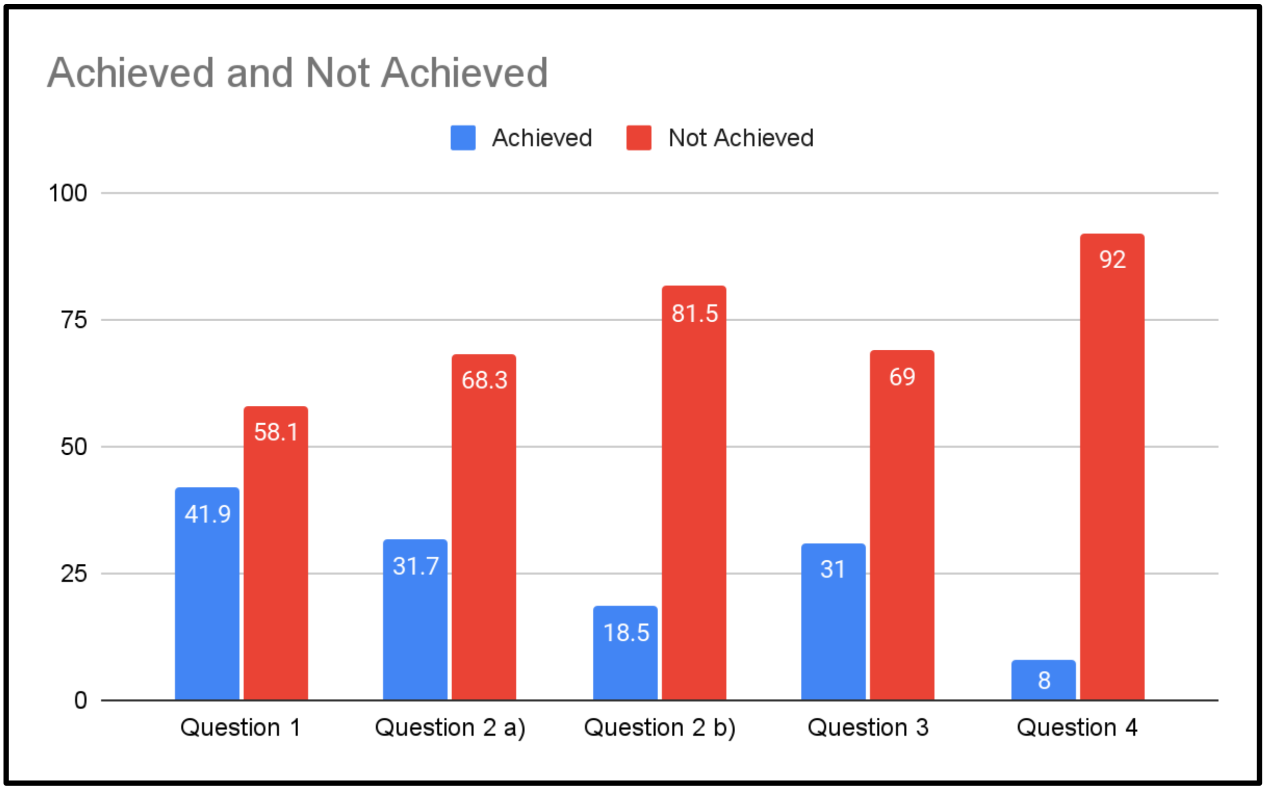


Figure 35: The percentage of PSMTs successful or unsuccessful in reformulating the mathematical tasks

From the analysis of the data it appears that the majority of the PSMTs were unsuccessful in reformulating each of the mathematical tasks. The PSMTs were least successful in reformulating the given task into an open problem. The analysis of each question will now be individually discussed.

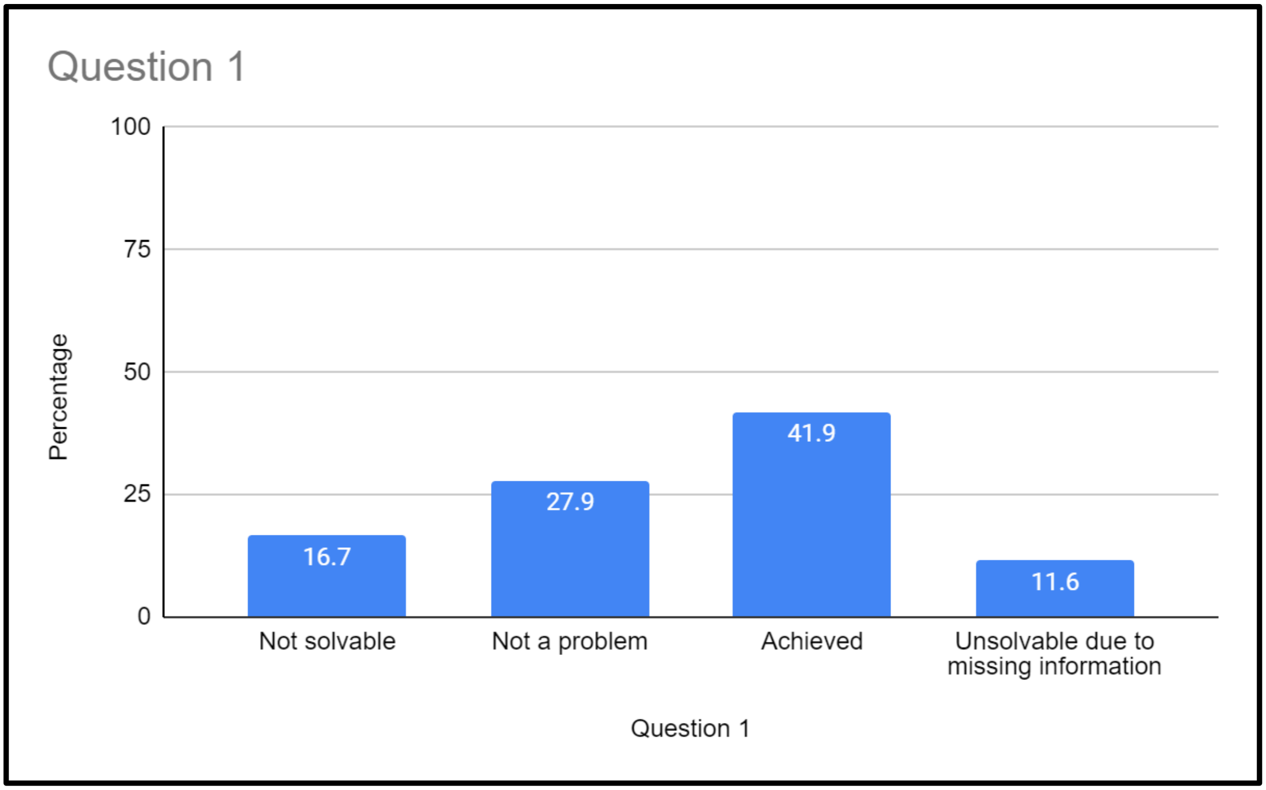


Figure 36: The percentage of tasks in each category for Question 1

Question 1 had the highest success rate out of the five questions with a rate of 41.9% which involved reformulating a given mathematical exercise into a problem. Of the reformulated tasks, 28.3% were not solvable and 27.9% of the tasks did not meet the criteria of a mathematical problem.

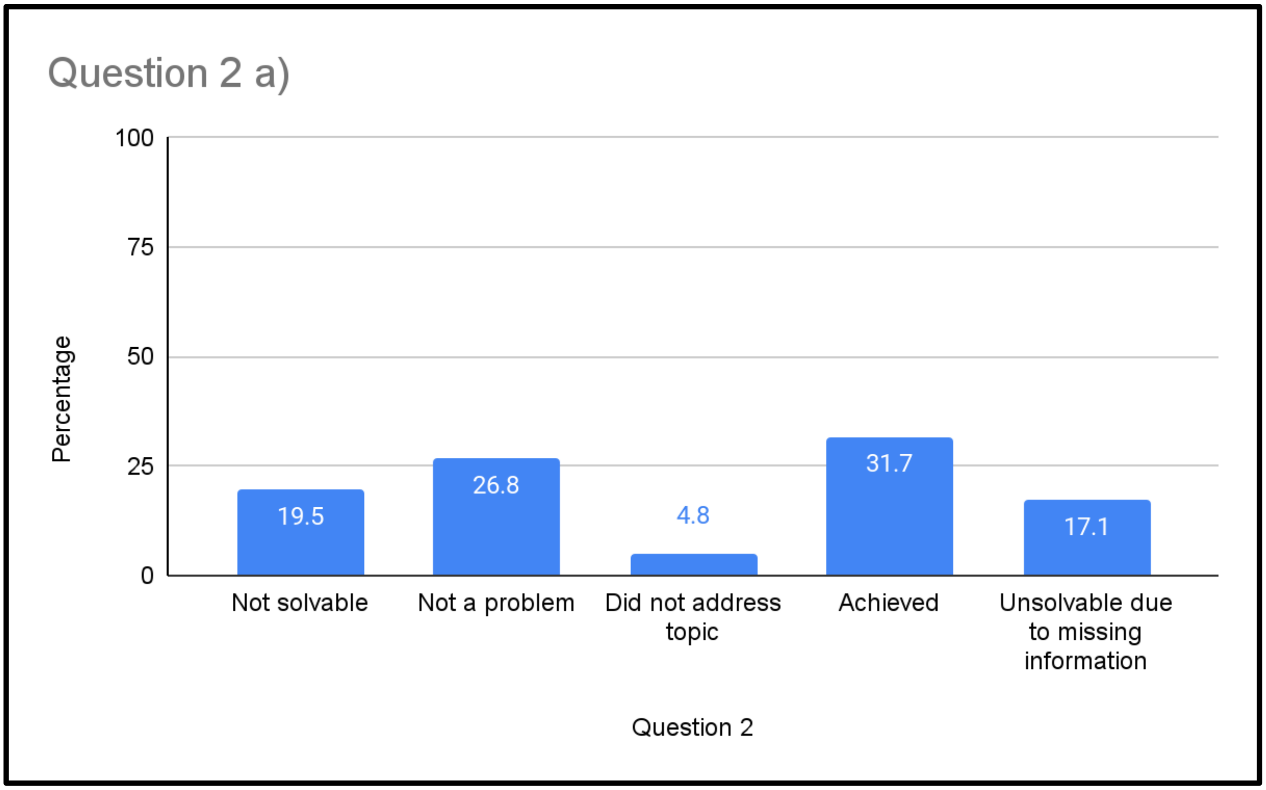


Figure 37: The percentage of tasks in each category for Question 2a)

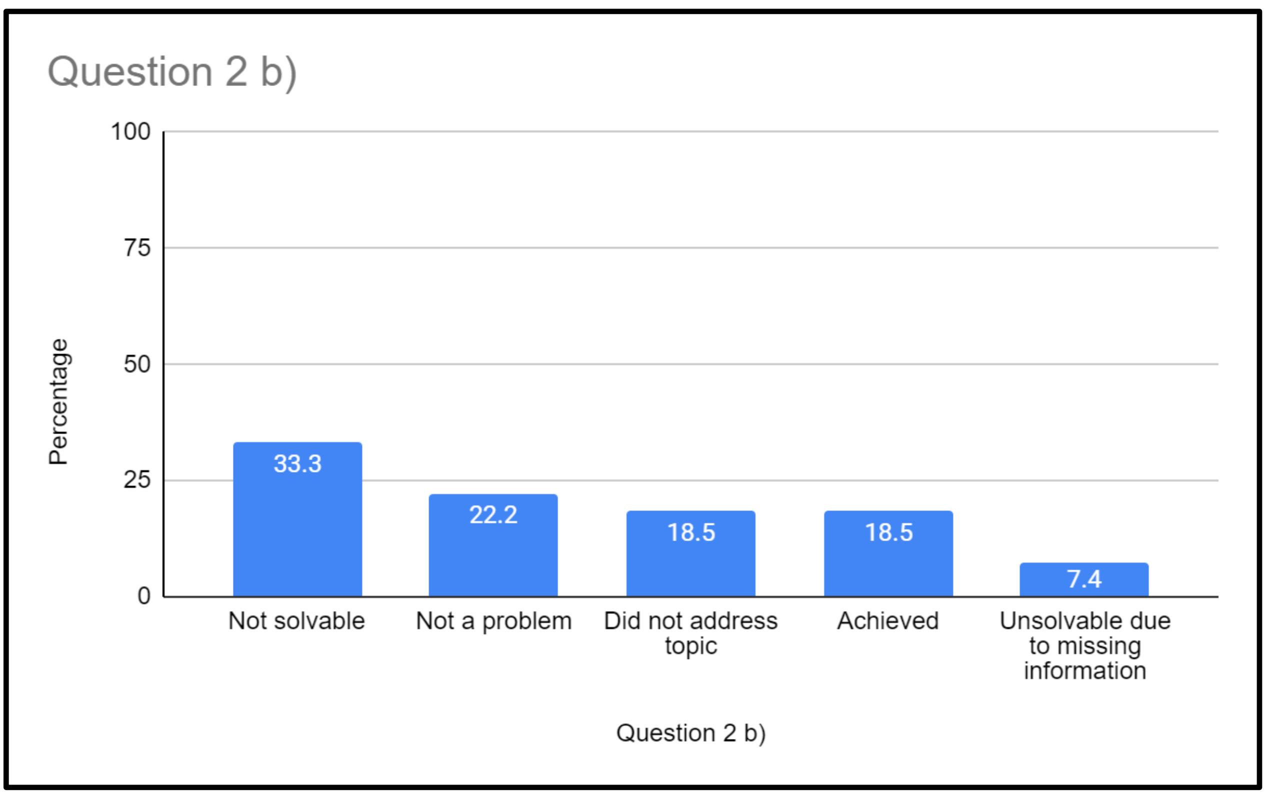


Figure 38: The percentage of tasks in each category for Question 2b)

Question 2a) and Question 2b) involved reformulating mathematical tasks to problems involving the mathematical topic of fractions. The majority of PSMTs were successful in achieving this with 31.7% in Question 2a) but this was not the case in Question 2b) with just 18.5% being successful. This coincides with an increase in PSMTs who were unsuccessful due to not addressing the required topic stated in the question. Question 2a) required the knowledge of fractions while Question 2b) specifically required the division of fractions.

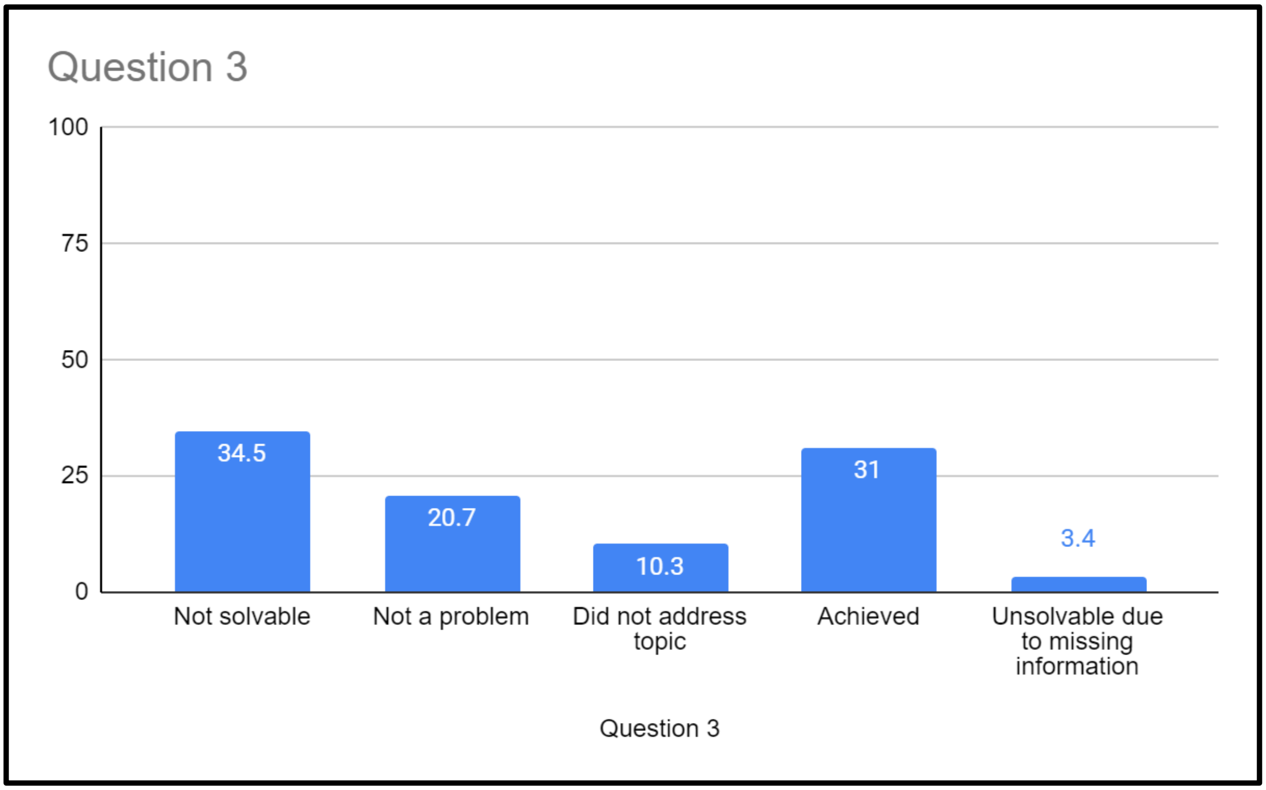


Figure 39: The percentage of tasks in each category for Question 3.

Question 3 specifically asked the PSMTs to reformulate a problem to involve knowledge of ratio. Overall, only 31% of PSMTs were successful in reformulating the problem to involve ratio with 10.3% reformulating the problem but did not address the topic of ratio.

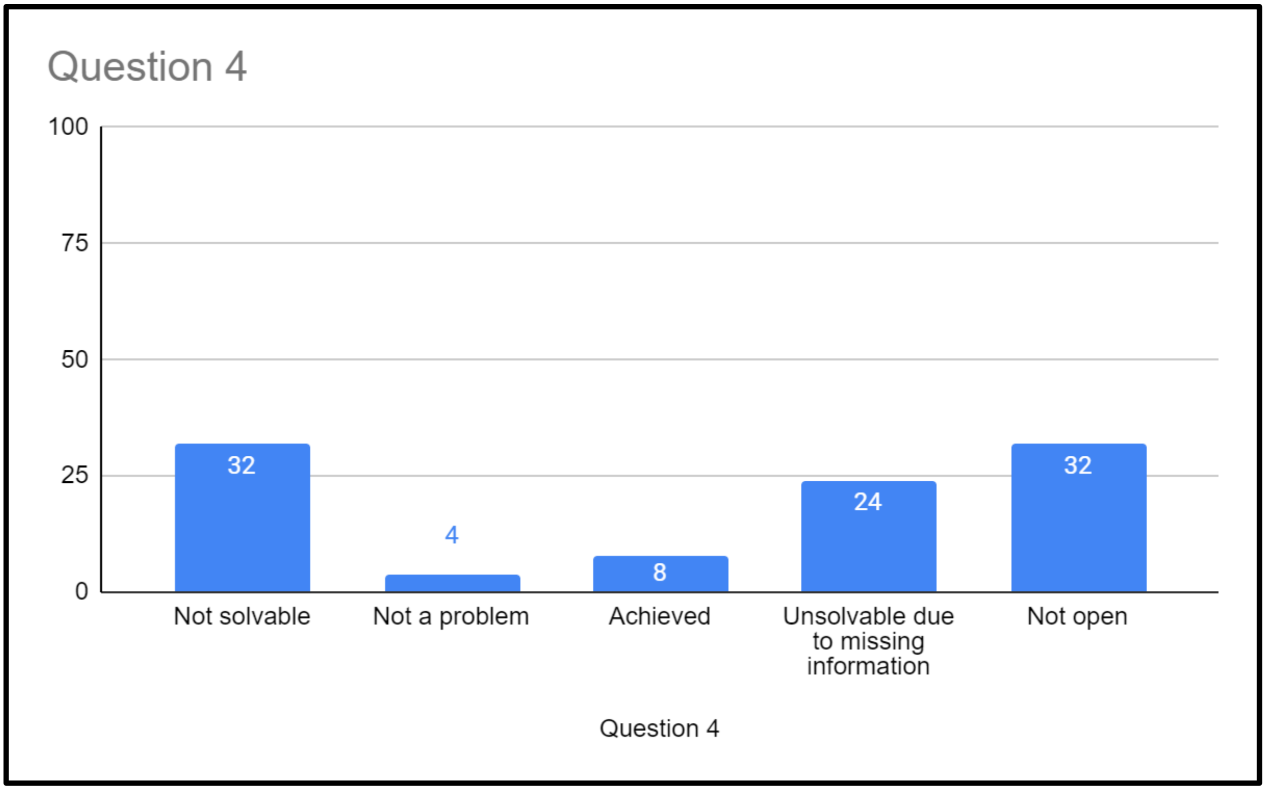


Figure 40: The percentage of tasks in each category for Question 4

From the categorisation of the reformulated mathematical tasks in Question 4, there was an equal percentage of tasks that were categorised as *Not solvable* and *Not open* with 32% each. Only 8% of the PSMTs were successful in reformulating an open problem.

6.3.4 Extension of problems by Cohort 4

To investigate the PSMTs' ability to extend a problem, the PSMTs were asked, on completion, to extend the problems (Appendix M). After their attempt to solve a problem on three different occasions, the PSMTs were asked to extend the given question. Since the PSMTs were asked to extend the problem after attempting to solve the original problem, time was a factor that may have influenced the number of PSMTs who completed the extension task. In Problem One, 12 PSMTs completed the extension task, 33 PSMTs in Problem Two, and 22 PSMTs in Problem Three.

The extension problems were analysed firstly if the PSMTs' extension problems met the criteria of a *problem* or if they were an *exercise*. The results of this analysis are displayed below in Figure

41. It is clear that the majority of each extension task did not meet the criteria of a mathematical problem.

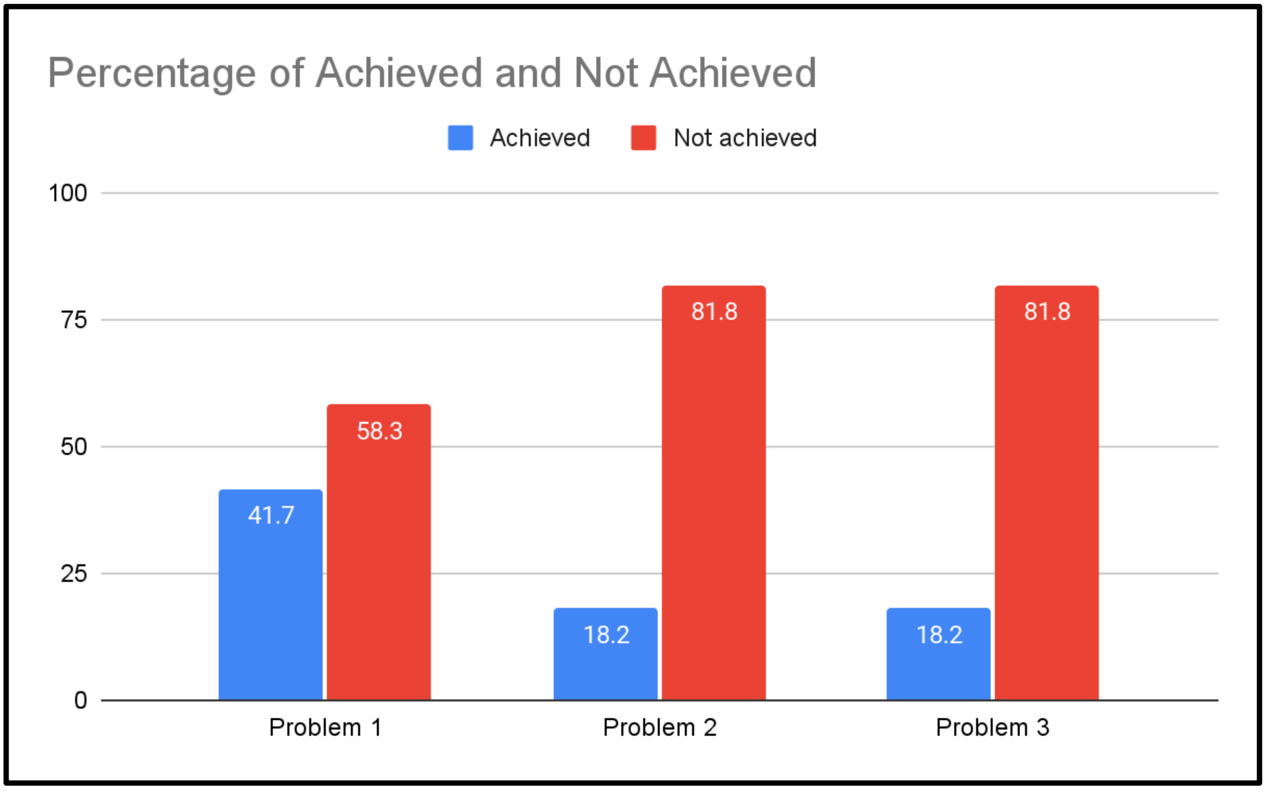


Figure 41: Percentage of achievement in generating a problem as an extension

The extension tasks were then analysed to categorise the reasons for which they did not meet the criteria of a mathematical problem. There were three categories for the extensions that were not categorised as *problems*. The first category was *unsolvable*. This included mathematical tasks that were not possible to solve due to missing information, or poor phrasing which resulted in ambiguity as to what the task was actually asking the problem solver to do. The next category was *procedural* which involved the extension task being immediately clear on how to solve using a known procedure. The category *Measurement Change* involved extension tasks for which the PSMT only changed the units of measurement or given numbers in the original problem. The final category *Achieved* were extension tasks that were possible to solve and met the criteria of a mathematical problem.

The results of the categorisation for each of the extension tasks are displayed below in Figure 42, Figure 43, and Figure 44.

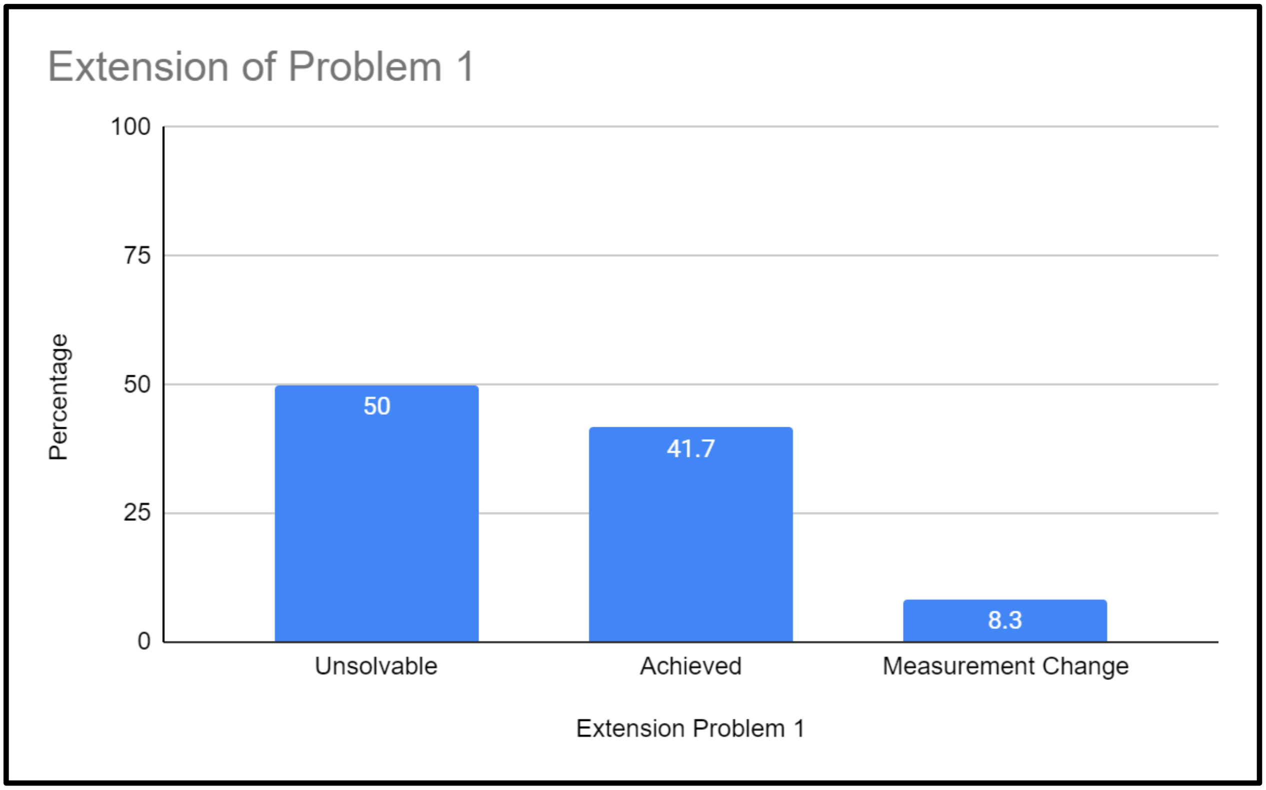


Figure 42: Categorisation of extension tasks for Problem 1

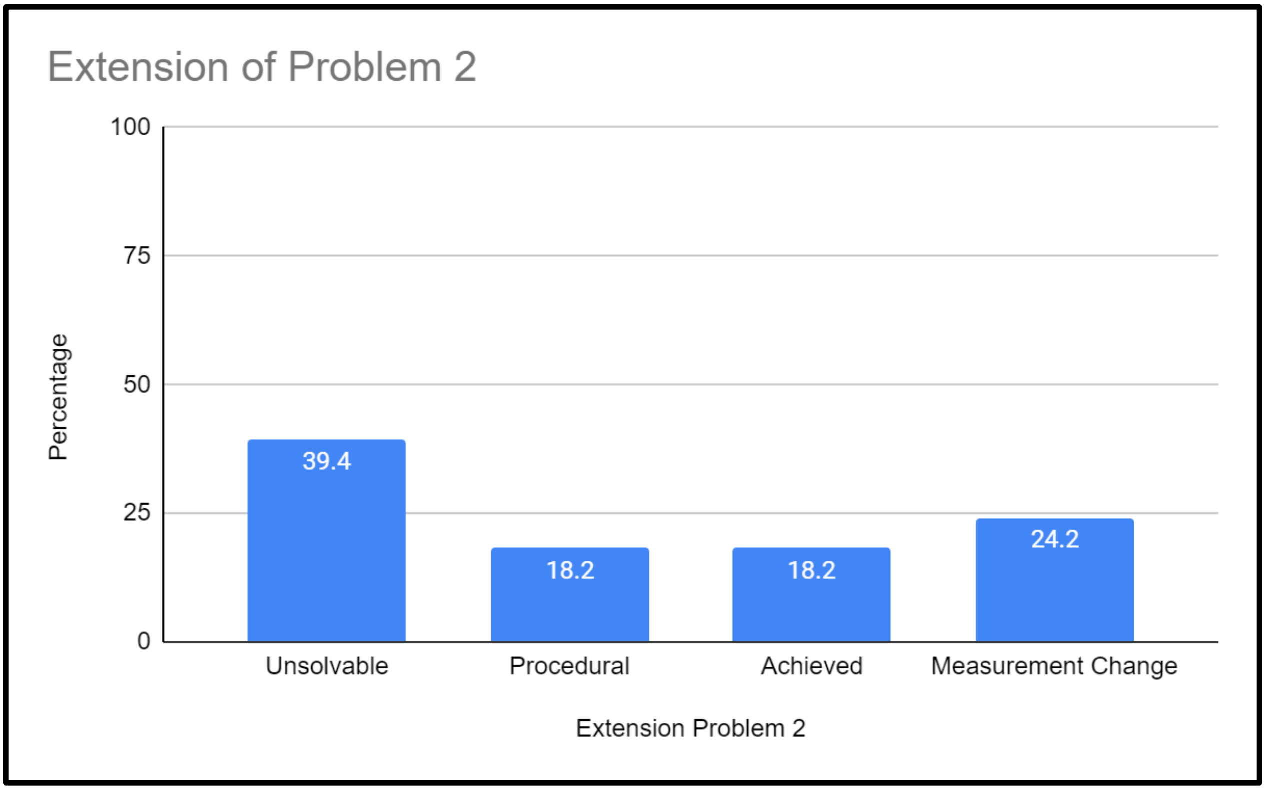


Figure 43: Categorisation of extension tasks for Problem 2

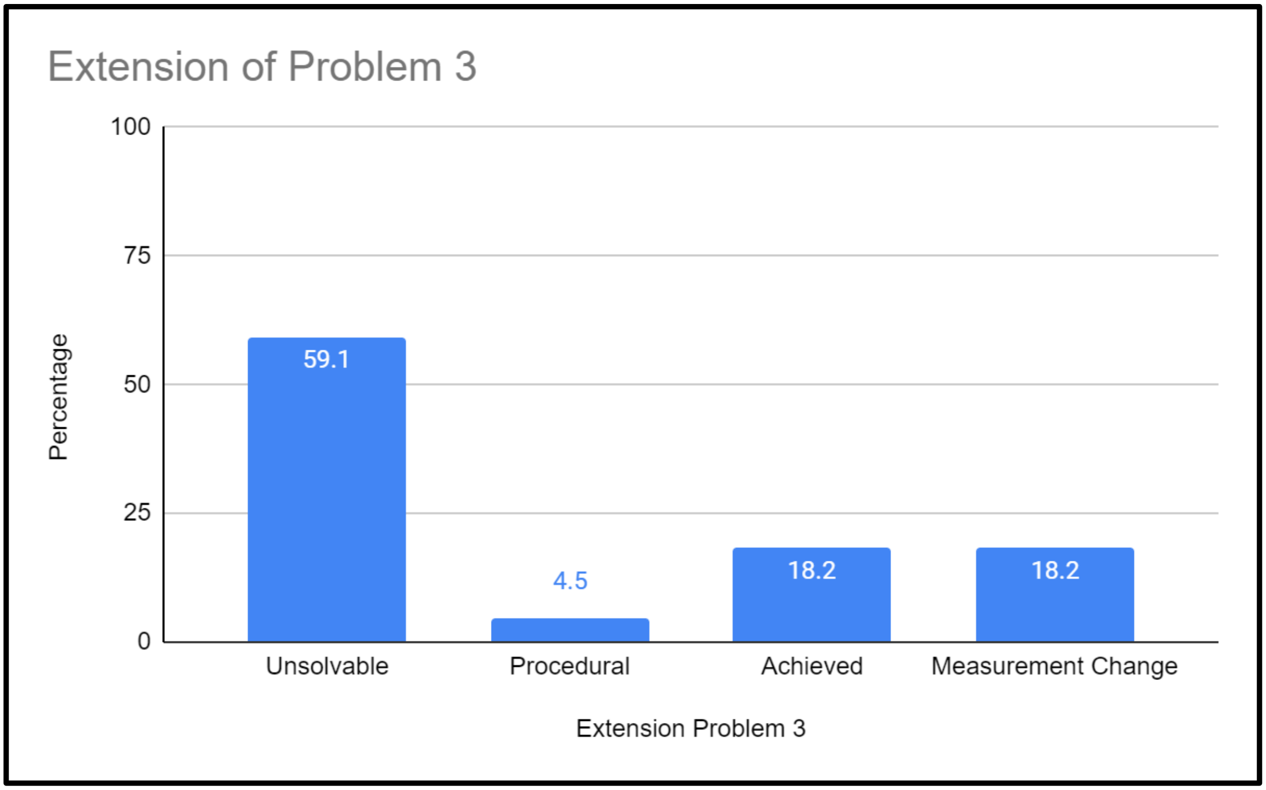


Figure 44: Categorisation of extension tasks for Problem 3

From each of the figures above, the majority of extensions for each of the problems were categorised as *Unsolvable.* There was a decrease between the percentage of tasks that were classified as achieving the criteria of a mathematical problem between Problem 1 and Problem 2 which then remained at the same level for Problem 3.

Example of Problem 1:

A goat is tethered by a 6m rope to the outside corner of a shed measuring 4m by 5m in a grassy field. There is a water trough attached to the shed. Find out the furthest position it can be from the goat, with the goat still able to drink from it.

This task was classified as *unsolvable* as there is not enough information given about the size or shape of the water trough. The language of the task suggests that there is one particular answer for the task, which suggests that assumptions cannot be made about the size or shape of the water trough.

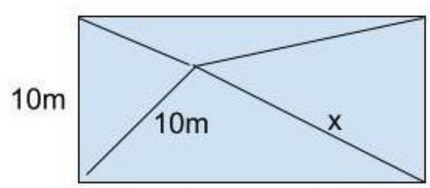
An example for Problem 2 is given below:

Richard took part in 180km cycling race. His average speed was 40km/h throughout the race. During the race he fell and had to stop for some time. Overall, it took him 4.5 hours. How long was he off the bike for?

This is not possible as there is not enough information given regarding the parameters of the question. Extra information such as his speed before or after the fall would need to be given in order to calculate a specific time off the bike.

A final example of a task missing information is a task created as an extension to Problem 3:

There is a well dug in the ground in a courtyard. One side of the courtyard is 10m. From one corner to the well is 10 m. Find the length of x from the corner to the well.



This task was classified as unsolvable as there was not enough information given in the question to solve it.

6.4 Question 4: What beliefs and affective factors do pre-service teachers hold regarding problem solving?

To investigate the beliefs and affective factors of the PSMTs, two instruments were used. The Indiana Mathematics Beliefs scale (IMB) (Kloosterman & Stage, 1992) provides the quantitative element of this research question. An open-ended affective question activity, adapted from Felmer and Perdomo-Diaz (2016), was the qualitative instrument used.

6.4.1 Indiana Mathematics Beliefs scale

The Indiana Mathematics Beliefs scale (IMB) was implemented with four cohorts of participants (Kloosterman & Stage, 1992) (see Appendix L). As outlined in Section 4.10.5, this instrument consisted of five scales with six items in each scale totalling in thirty items overall. A brief description of each scale as described by Kloosterman and Stage (1992) is now provided.

- *I can solve time-consuming mathematics problems*

This involves a persons' perception of their ability to solve time-consuming problems.

- *There are word problems that cannot be solved with simple step-by-step procedures*

This scale references the use of procedural skills and formulae to solve problems.

- *Understanding concepts is important in mathematics*

This scale measures the level to which the respondent believes in the importance of understanding mathematical concepts.

- *Word problems are important in mathematics*

This scale involves investigating the respondents' beliefs about the importance of word problems compared to computational or procedural skills.

- *Effort can increase mathematical ability*

This scale is used to provide an insight into the respondents' attitude towards their ability to improve their mathematical skills by putting in effort.

Of these thirty items, twelve questions were with a negative valence and eighteen with a positive valence. In tabulating the results, the scales are reversed (where necessary) so that in every case, a higher mark corresponds to a more positive disposition. Each item was graded in a Likert- scale fashion whereby the following numbers indicated the respondents' level of agreement or disagreement with each item; 1 = strongly disagree, 2 = disagree, 3 = undecided, 4 = agree, 5 = strongly agree.

The results for the four cohorts are displayed in tables below.

Cohort 1	Mean [Max 30]	SD	Cronbach's Alpha
Difficult Problems	22.9	2.44	0.59
Steps	17.2	2.66	0.49
Understanding	20.9	2.31	0.64
Word Problems	19.2	2.39	0.42
Effort	26.7	2.744	0.95

Table 33: Results of IMB scale for Cohort 1

Cohort 2	Mean	SD	Cronbach's Alpha
Difficult Problems	22.73	2.88	0.65
Steps	16.56	2.75	0.46
Understanding	21.47	2.596	0.77
Word Problems	19.33	3.111	0.67

Effort	27.03	2.619	0.89

Table 34: Results of IMB scale for Cohort 2

Cohort 3	Mean	SD	Cronbach's Alpha
Difficult Problems	17.1	0.729	0.64
Steps	16.8	0.84	0.48
Understanding	19.4	0.712	0.73
Word Problems	14.3	0.775	0.58
Effort	25.5	0.73	0.91

Table 35:Results of IMB scale for Cohort 3

Cohort 4	Mean	SD	Cronbach's Alpha
Difficult Problems	13.4	0.707	0.6
Steps	16.0	0.684	0.49
Understanding	20.7	0.602	0.69
Word Problems	20.5	0.682	0.61
Effort	19.9	0.719	0.86

Table 36: Results of IMB scale for Cohort 4

	Cohort 1 (n=30)	**Cohort 2 (n=44)**	**Cohort 3 (n=30)**	**Cohort 4 (n=47)**	**Total (n=151)**	**Overall mean**
Difficult Problems	22.9	22.7	17.1	13.4	76.1	19.025
Steps	17.2	16.6	16.8	16.0	66.6	16.65
Understanding	20.9	21.5	19.4	20.7	82.5	20.625
Word Problems	19.2	19.3	14.3	20.5	73.3	18.325

Effort	26.7	27.0	25.5	19.9	99.1	24.775

Table 37: Combined results of IMB scale

The analysis of the results of the IMB from the four cohorts of participants is discussed below. The combined results of the four cohorts of the IMB showed that the scale with the highest mean score was *Effort* with a score of 24.85/30. This scale had the highest mean score in Cohorts 1, 2, and 3 and was third highest in Cohort 4 . The scale which had the lowest overall mean was *Steps* with a score of 16.645/30. This scale had the lowest mean score in Cohorts 1,2, and 3 with a similarly low score in Cohort 4. . *Difficult Problems* had a particularly low mean in Cohort 4.

6.4.2 Qualitative study of the affective domain: results

The qualitative element of this research involved participants answering an open-ended question after attempting a mathematical problem. The participants were asked to describe how they felt at three different stages of their problem-solving attempt, namely; the start of the problem, the middle of the problem, and the end of the problem. The students were prompted to refer to how they felt if they were stuck and if they were making progress (see Appendix K). This was done at three different points during the module while participants worked on three different mathematical problems. The PSMTs' responses were analysed in two different ways: the first being the categorisation of respondents at each of the three stages of the problem, and secondly, each statement was categorised for each of the three stages. We will present firstly the overall categorisation of respondents' feelings, and then present in detail the categorisation of the statements.

Results for categorisations of respondents

From the general inductive analysis of the qualitative data, the following categorisations of the respondents at the three different stages of the problem were identified. Analysis of students' comments relating to the start and the middle of the generated the following categories: *neutral*, *negative, positive, both positive and negative*. Analysis of the end of the problem resulted in the following categories being identified: *answer positive, answer negative, process positive, process* *negative, negative answer but positive process, both answer and process positive,* and *both answer and process negative*. The term ‘positive’ refers to statements of a positive disposition, the term ‘negative’ refers to statements of a negative disposition, and the term ‘neutral’ refers to statements which are neither positive nor negative. Each category, including examples of respondents’ statements, will now be discussed. SP1 indicates the start of Problem One, MP1 indicates the middle of Problem One, and EP1 indicates the end of Problem One. This notation is also used for problem two and problem three.

Neutral: Respondents in this category demonstrated neither a negative nor positive disposition.

SP2: *I felt alright at the start*

MP3: *When making progress and when stuck I felt the question was lacking information*

MP3: *I felt normal when making progress*

Negative: Respondents in this category demonstrated a negative disposition. Negative feelings in this context include feelings such as; annoyance, overwhelmed, and anxiety to name a few. Examples of these are:

SP1: *At the start I didn’t feel confident in my knowledge*

SP1: *At the start of the attempt, it felt a little overwhelming with all the information given and trying to find a quick solution*

SP2: *I felt overwhelmed at the beginning, I realised this was the first time I had no idea how to begin a problem*

MP1: *When I was stuck, I felt a bit annoyed that I couldn’t come the solution or how to the necessary steps to get the solution*

Positive: Respondents in this category are positive in disposition. Feelings which demonstrated a positive disposition included statements such as happiness, enjoyment, contentment, or confidence. Examples of these are:

MP1: *As I began to get ideas and attempted the question I began to really enjoy the work.*

MP1: *As I made progress I became more confident that my strategy was working*

SP2: *I felt confident before attempting this problem as when I read the question I could clearly understand what it was asking me to find.*

.

MP3: *When I started to make progress I had more confidence in my ability to solve the problem.*

Both positive and negative: This category accounted for demonstrations of both negative and positive feelings. Examples of these are:

MP1: *I became stuck a number of occasions and although it was a bit frustrating I was still very engrossed in the problem to keep going*

EP1: *At the end I was happy however worried about the mess I made by scribbling over failed attempts*

MP2: *I felt good and confident when I was making progress but when I got stuck I actually got quite frustrated as I knew what I had to do but I just wasn't sure as to how I was to do it.*

MP3: *At the start I felt confident attempting the question but quickly realized it wasn't as easy as I had thought.*

The categorisations of respondents at the end of the problems were as follows:

Answer positive: This category refers to respondents who express positive feelings towards the answer that they reached for the mathematical problem.

EP1: *When I got my final answer I was confident I was correct*

EP1: *I enjoyed getting an answer which I was happy with, it gave me a sense of achievement*

EP3: *I felt confident at end of my work because the simultaneous equations method lead me to a nice round number and I simply checked my answer by plugging it into my calculator.*

Answer negative: This category refers to respondents who expressed negative feelings towards the answer that they reached for the mathematical problem.

EP1: *I didn't solve this problem in the end so I felt annoyed at myself.*

EP1: *I wasn't happy with my final answer*

EP2: *At the end of the problem I didn't feel like I got a proper answer to me my answer was not logical*

EP3: *I feel unhappy with my effort as I did not get the correct answer.*

Process positive: This category refers to respondents who expressed positive feelings towards their problem-solving attempt. This included statements that referred positively towards the approach taken, how they overcame difficulties, or strategies used.

EP1: *I didn't get the question fully finished but overall I am happy with my steps and attempt*

EP1: *I felt satisfied that I had approached the question correctly*

EP3: *Very proud of myself as I took a different approach to solving this problem and it worked.*

Process negative: This category refers to respondents who expressed negative feelings towards their problem-solving attempt. This included statements that referred negatively towards the approach taken, how they struggled to overcome difficulties, mistakes made or strategies used.

> EP3: *I got stuck and couldn't figure out what to do or ways to move forward as a way of finding the answer I just subbed in random numbers*

Negative answer but positive process: This category refers to respondents who expressed negative feelings towards the answer that they reached for the mathematical problem but refer positively towards their problem-solving attempt.

> EP1: *Unsure as to whether I got it right or not but happy with my attempt overall as it was answered to the best of my ability*

> EP2: *At the end of my work I felt annoyed that I did not find a clear path, but satisfied that I threw all of my triangle knowledge at this problem.*

Both answer and process positive: This category refers to respondents who expressed positive feelings towards the answer that they reached for the mathematical problem and positively towards their problem-solving attempt.

> EP1: *I was glad with my work on the problem at the end as I felt I was able to simplify the question and work it out.*

> EP3: *I felt happy and relieved at the end as I'm confident with my answer and glad I gave it a good attempt.*

> EP3: *I kept going and gathered a solution I was satisfied with.*

> EP3: *Overall I enjoyed doing the problem and felt confident in my ability throughout.*

Both answer and process negative: This category refers to respondents who expressed negative feelings towards the answer that they reached for the mathematical problem and negatively towards their problem-solving attempt.

EP3: *The work was incomplete and I was unhappy with my performance.*

Table 38 shows the number of respondents who were classified in the previously mentioned categories at the start, middle and end of each problem. SP1 indicates the start of problem one, MP1 indicates the middle of problem one, and EP1 indicates the end of problem one. This notation is also used for problem two and problem three. Missing values are present due to respondents not differentiating between the relevant stage of the problem-solving attempt.

Classification	**SP 1**	**MP 1**	**EP 1**	**SP 2**	**MP 2**	**EP 2**	**SP 3**	**MP 3**	**EP 3**	**Total**
Neutral	10	4	2	6	3	3	4	11	1	44
Negative	16	12	4	14	15	2	11	14	2	90
Positive	16	16	3	11	8	1	20	9	0	84
Both Positive and Negative	1	9	1	1	6	1	5	6	0	30
Answer positive	0	0	5	0	0	6	0	0	7	18
Answer negative	0	0	9	0	0	10	0	0	12	31
Process positive	0	0	8	0	0	3	0	0	1	12
Process negative	0	0	1	0	0	0	0	0	1	2
Negative answer but positive process	0	0	6	0	0	3	0	0	2	11
Both answer and process positive	0	0	4	0	0	1	0	0	8	13
Both answer and process negative	0	0	0	0	0	0	0	0	2	2
Missing values	7	9	7	18	18	20	10	10	14	113

Total	50	50	50	50	50	50	50	50	50	

Table 38: Classification of respondents

Results for Statements

The second set of analysis that was conducted focused on the categorisation of statements made by each of the PSMTs. This was conducted using an inductive approach, which involved iterative coding and grouping of themes. The statements of each respondent in each of the three problems were analysed using an inductive approach (Thomas, 2006). This resulted in the categorisation of statements at the start, the middle, and at the end of each of the three problems. The categories that were identified are described below along with a count for the relevant categories. Due to the nature of the inductive approach, categories outside of the affective domain were evident in the PSMTs' responses. While the PSMTs were asked specifically to outline their feelings, other information was also given. Acknowledging that these fall outside of the affective domain, these categories are presented in a shaded format. When analysing the statements, some statements were applicable to more than one category meaning some statements were counted more than once.

There were categories that were evident in all three of the phases (the start, the middle, and the end of problem sections), namely: *Positive*, *Negative*, *Stuck*, and *Uncertain*.

- The *Positive* category included statements which demonstrated positive feelings. This included statements which referred to confidence, enjoyment, motivation, satisfaction, and other positive feelings.
- The *Negative* category included statements which demonstrated negative feelings. This included statements referring to negative feelings including feeling anxious, frustrated, annoyance, panic and other negative feelings.
- The *Stuck* category included statements where the participant referred to being stuck and not making progress.
- The *Uncertain* category included statements which expressed uncertainty, confusion, and reluctance.

While these categories were applicable to each of the three stages of the problem, the feelings were dependent on the stage of the problem.

Start

The analysis of the statements referring to the start of the problems identified the following categories: Positive, Negative, Visualization, Level of difficulty, Uncertain, Stuck, and Prior Knowledge. These categories will now be described with examples of statements applicable to each category.

Positive (n=41)

Statements in this category included statements which demonstrated positive feelings. This included statements which referred to confidence, competency in ability, and positive adjectives such as *optimistic* and *excited.*

> P2: "*I felt motivated as I felt that the question was solvable"*
>
> P3: *I felt confident before attempting this problem as when I read the question I could clearly understand what it was asking me to find. No hard language was used in the wording.*

Negative (n=20)

Statements in this category included statements which demonstrated negative feelings.

> P2:*At the start I felt overwhelmed and wasn't 100% sure where to start*
>
> P1: *I was anxious at the beginning*
>
> P3: *close to impossible...didn't know where to start*

Visualisation (n=17)

These statements included reference to drawing a diagram or visualising the problem.

> P2: *I felt the need to draw a quick diagram.*
>
> P1: *I drew a diagram and it clicked in my head what I should do*

Level of difficulty (n=7)

Statements in this category included statements referring to the level of difficulty of the problem. These statements were of both a positive and negative nature.

> P3: *I knew it was going to be a hard problem to solve*

Uncertain (n=22)

These statements included reference to confusion, uncertainty, unsure.

> P1: *Didn't know where to start*
>
> P2: *At the start a little confused as there was a lot of words*

Stuck (n=3)

> *P2: I was stuck for a few minutes trying to figure out where to go next or what to even do.*
>
> P2: *I was stuck from the start.*

Prior knowledge (n=12)

These statements involved respondents mentioning their use of prior mathematical knowledge.

P2: *I felt good as I wrote down what I know...there seemed to be a lot of information*

P3: *Wrote down previous learned formulas*

Table 39 below shows the categorisation of statements at the start of the three problems

Start	P1	P2	P3	Total
Positive	16	12	13	41
Negative	5	4	11	20
Visualisation	16	1	0	17
Level of difficulty	3	0	4	7
Uncertain	7	11	3	21
Stuck	0	3	0	3
Prior knowledge	0	10	2	11

Table 39: Categorisation of statements at the start of the three problems

The figures below show the categories statements that were included in each category. The negative category is in red, and the positive category is in green.

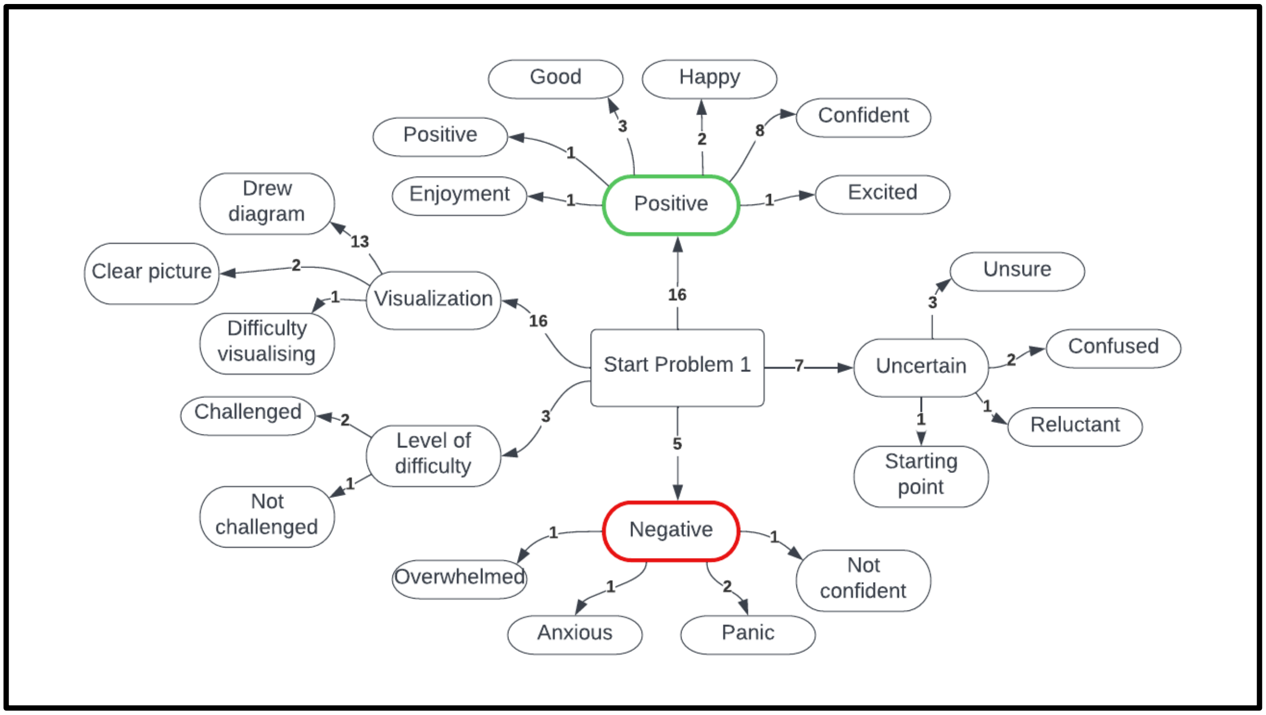


Figure 45: Categorisation of statements at the start of Problem 1

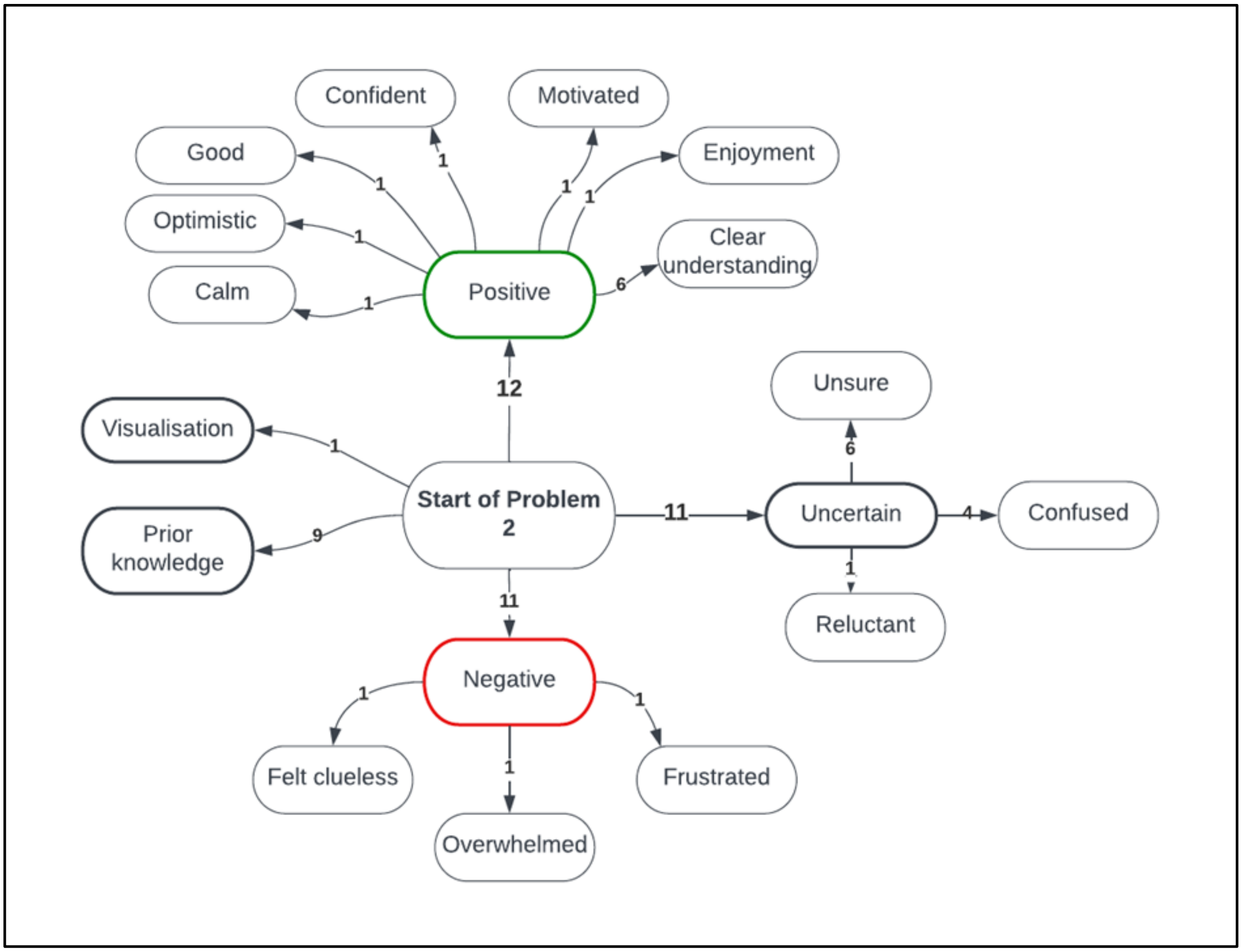


Figure 46: Categorisation of statements at the start of Problem 2

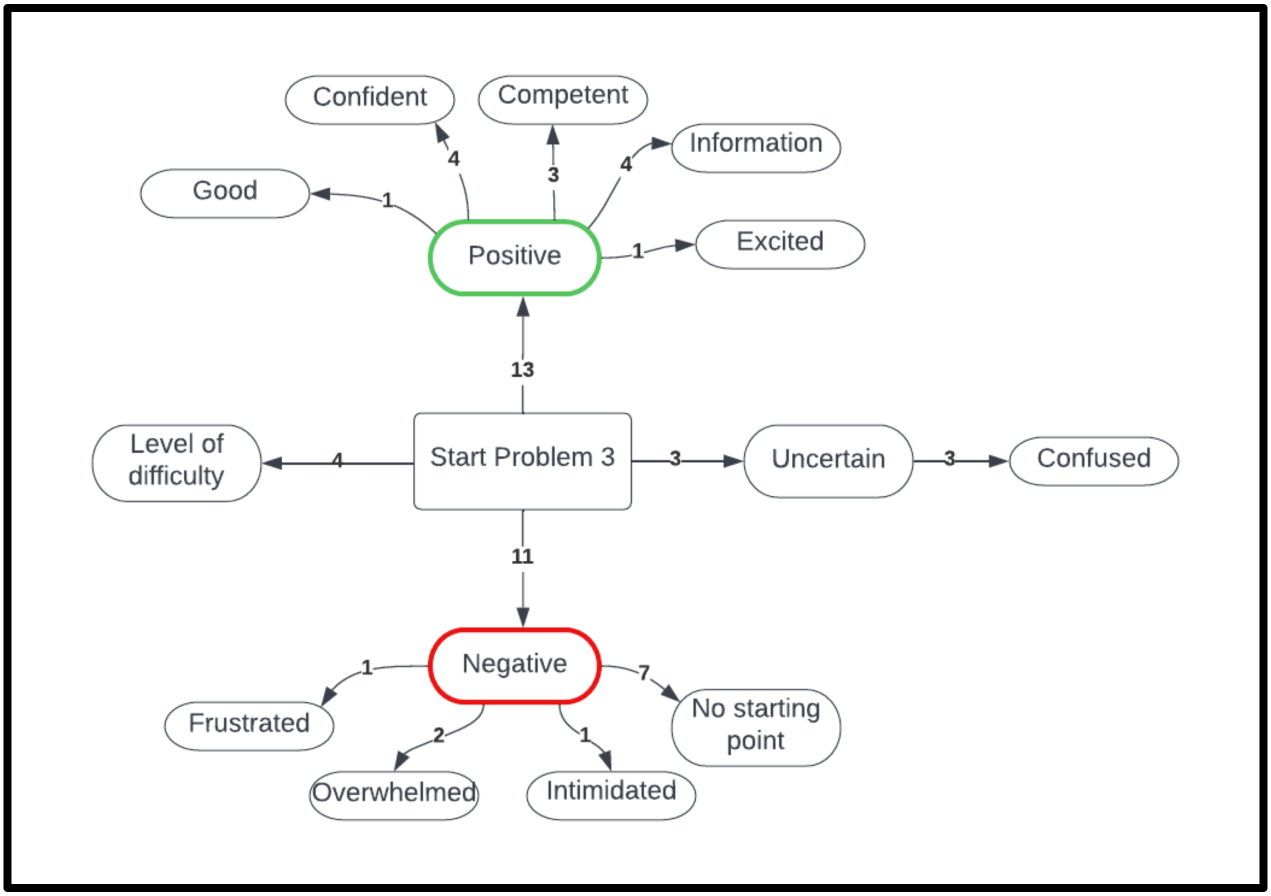


Figure 47: Categorisation of statements at the start of Problem 3

Middle of Problems;

From the analysis of the data, the following categories were identified: Positive, Negative, Uncertain, Challenged, Making Progress, and Stuck. Both the *Making Progress* category and the *Stuck* category had three subsections: positive statements, negative statements, and neutral statements. These categories will now be described with examples of statements applicable to each category.

Table 40 below shows the categorisation of statements at the middle of the three problems. The participants were asked to refer to how they felt if making progress and how they felt if stuck. This table shows that 'Making Progress' and 'Stuck' had three subsections: positive statements, negative statements, and neutral statements.

Middle	P1	P2	P3	Total
Positive	2	0	4	6
Negative	5	6	2	13
Uncertain	0	1	2	3
Challenged	0	0	2	2
Making progress total	23	17	7	47
Making progress positive	18	8	5	31
Making progress negative	0	4	0	4
Making progress neutral	5	5	2	12
Stuck total	19	14	16	49
Stuck positive	2	4	5	11
Stuck negative	12	5	5	22
Stuck neutral	5	6	6	17

Table 40: Categorisation of statements at the middle of the three problems

Positive (n=6)

P1: *Didn't give up and kept trying.*

P3: *I enjoyed the problem.*

Negative (n=13)

P2: *As I began to work it became more difficult, I felt frustrated as I did not know what approach to take.*

P2: *Felt completely lost.*

P3: *Felt annoyed I couldn't see where I was going.*

Uncertain (n=3)

These statements involved references to expressions such as uncertainty in how to proceed, doubt in approach or ability.

P2: Unsure what to do.

Challenged (n=2)

P3: *Challenging to come up with a new path*

Making progress positive (n=31)

P2: *I felt good when making progress as I could see what I was doing.*

P2: *When I started to make progress I had more confidence in my ability to solve the problem*

P3: *Making progress always feel rewarding and it felt great being able to give the question a good attempt*

Making progress negative (n= 4)

P2: *When I was making progress there was times I felt I was getting nowhere.*

P2: *I tried to figure it out but was getting nowhere*

P3: *I thought on different occasions I was making progress*

Making progress neutral (n=12)

P1: *Thought I was making progress*

P2: *I started to make progress.*

Stuck positive (n=11)

P2: *It felt good to be going somewhere and when I got stuck I waited a minute took a deep breath and looked at the question they asked.*

Stuck negative (n= 22)

P3: *As soon as I was stuck I began to panic*

P2: *I felt frustrated and confused when I got stuck on the questions as no matter how long I spend on it I just couldn't figure it out.*

P3: *I got stuck I actually got quite frustrated as I knew what I had to do but I just wasn't sure as to how I was to do it*

Stuck neutral (n=17)

These statements involved the respondents highlighting that they got stuck at the middle stage of the problem. These statements did not refer to either positive or negative feelings towards getting stuck.

P3: *I wasn't long getting stuck with no solution to get unstuck*

The figures below show the categories of the statements and description of the statements that were included in each category.

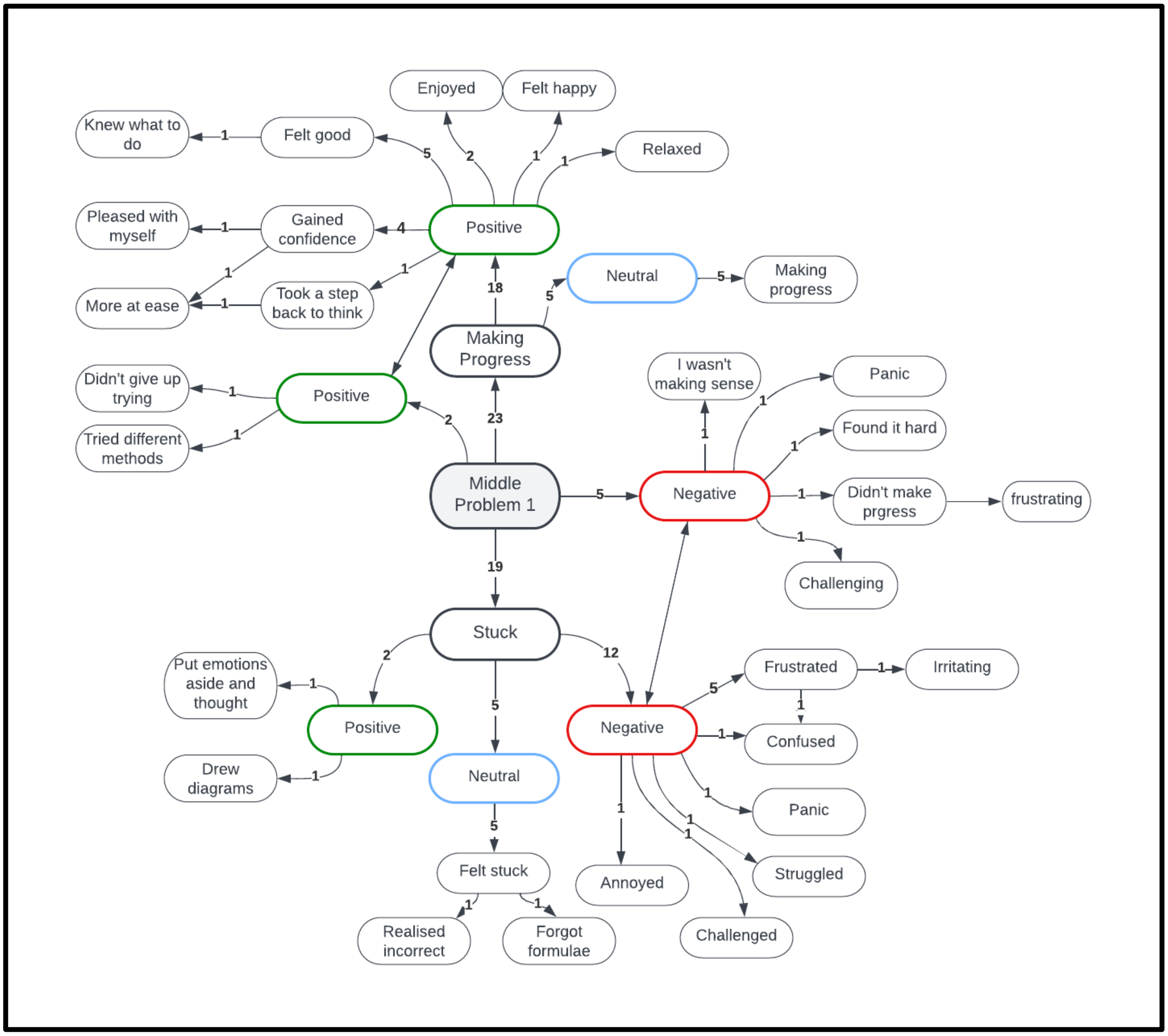


Figure 48: The classification of statements at the middle of Problem 1

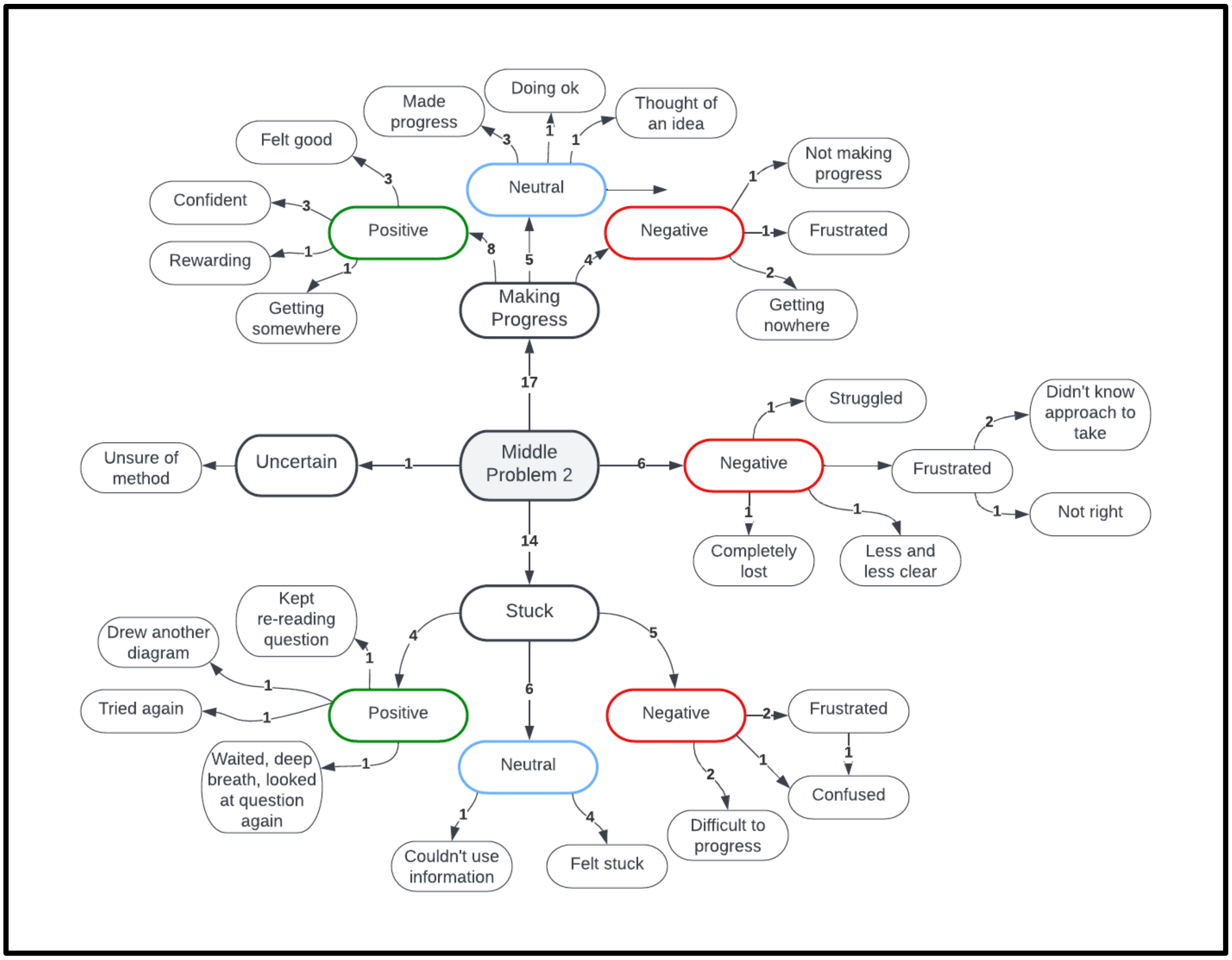


Figure 49: The classification of statements at the middle of Problem 2

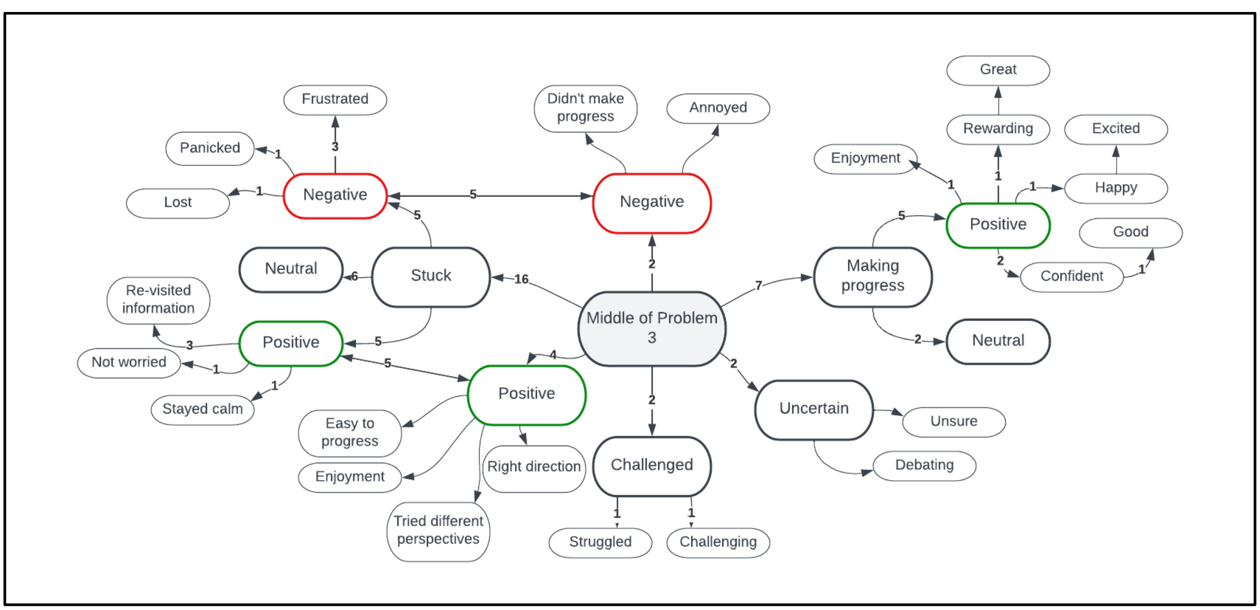


Figure 50: The classification of statements at the middle of Problem 3

End

The categories that were identified for statements referring to the *End* of each problem were: Positive, Negative, Time related, and Uncertain. The inductive analysis showed that the statements were process orientated and answer orientated. These categories will now be described with examples of statements applicable to each category.

Table 41 below shows the categorisation of statements at the end of the three problems.

End Answer	**P1**	**P2**	**P3**	**Total**
Positive	15	12	5	32
Negative	5	12	8	25
Time related	0	0	0	0
Uncertain	2	4	0	6
End Process	**P1**	**P2**	**P3**	**Total**
Positive	15	2	6	23
Negative	0	3	5	8
Time related	0	0	0	0
Uncertain	0	0	0	0
End Neutral	**P1**	**P2**	**P3**	**Total**
Positive	6	4	0	10
Negative	6	8	3	17
Time related	5	5	3	13
Uncertain	2	0	0	2
End Total	**P1**	**P2**	**P3**	**Total**

Positive	32	18	11	61
Negative	11	23	16	50
Time related	5	5	3	13
Uncertain	4	4	1	9

Table 41: Categorisation of statements at the start of the three problems

Positive, Negative, Time related, and *Uncertain* were the four overarching categories that were identified in relation to the End of the three problems. From the analysis it was evident that statements were answer focused, process focused, or neither. The statements that did not allude to achieving an answer or the problem-solving process were categorised as *neutral*. Table 41 above shows the number of statements in each of the four categories under the subcategories of *End Process, End Answer*, and *End Neutral*.

End Answer

Positive (n=32)

P1: *I enjoyed getting an answer which I was happy with*

P2: *I felt good as I think I solved it and got the right answer.*

P3: *I was happy that I was able to find it and confident in my abilities.*

Negative (n=25)

P2: *I wasn't satisfied with the result as I couldn't solve it.*

P2: *I felt quite frustrated at the end as I did not get an answer*

P3: *Disappointed not to solve it after spending quite a while brainstorming different possible solutions.*

P3: *Unsatisfied and annoyed at the end of my work as I couldn't solve it.*

Uncertain (n=6)

P2: *At the end I arrived at an answer but am still a little unsure on what is the correct answer.*

P2: *At the end then I did not know if my answer was correct or incorrect and I tried to check if it was right but again I was not confident.*

End Process

Positive (n=23)

P1: *I was happy with my work at the end and I liked knowing I had made a good attempt and progress*

P2: *I felt I gave the problem my best attempt that I could of and if I approached it again I would of used the same method.*

P3: *I am confident I have attempted this problem to the best of my ability.*

P3: *Relieved that I had made some sort of valid attempt*

Negative (n=8)

P3: *I felt annoyed that I did not find a clear path*

P3: *The work was incomplete and I was unhappy with my performance.*

There were no statements in the category of *Time related* or *Uncertain.*

End Neutral

Positive (n=10)

P2: *I felt relieved and satisfied*

P2: *I felt confident at the end of my work*

P1: *Happy I didn't doubt myself*

Negative (n=17)

P1: *Frustrated and very annoying*

P2: *I wasn't happy. I felt like I should've done better.*

P2: *I felt the same at both the start and finish of this problem, which was lost.*

P3: *Unsatisfied*

Time related (n=13)

P1: *I felt if I had a bit more time I would have been happier with my work.*

P3: *Feel like I would need more time on it as I have not completed the question.*

P2: *Near the end I rushed it*

The following statement is an example of one that was counted in more than one category. The start of the statement was categorised as *Negative Process* but the second part of the statement refers to the time limitations.

P3 : *Dissatisfied with my work at this time but confident if given more time I would eventually solve the problem.*

Uncertain (n=2)

P1: *Doubt my skills and what I know*

The figures below show the categories of the statements and description of the statements that were included in each category.

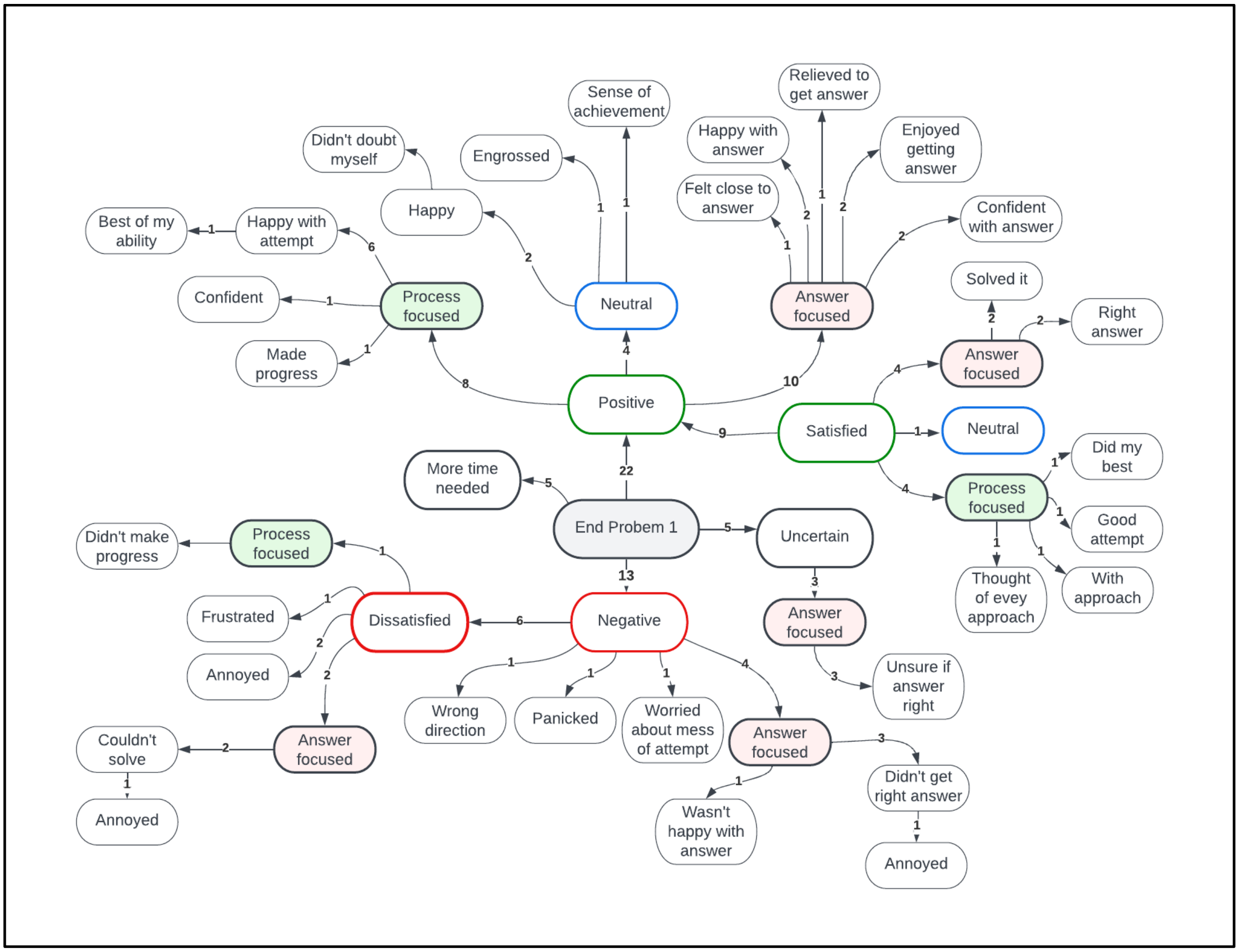


Figure 51: The categorisation of statements at the end of Problem 1

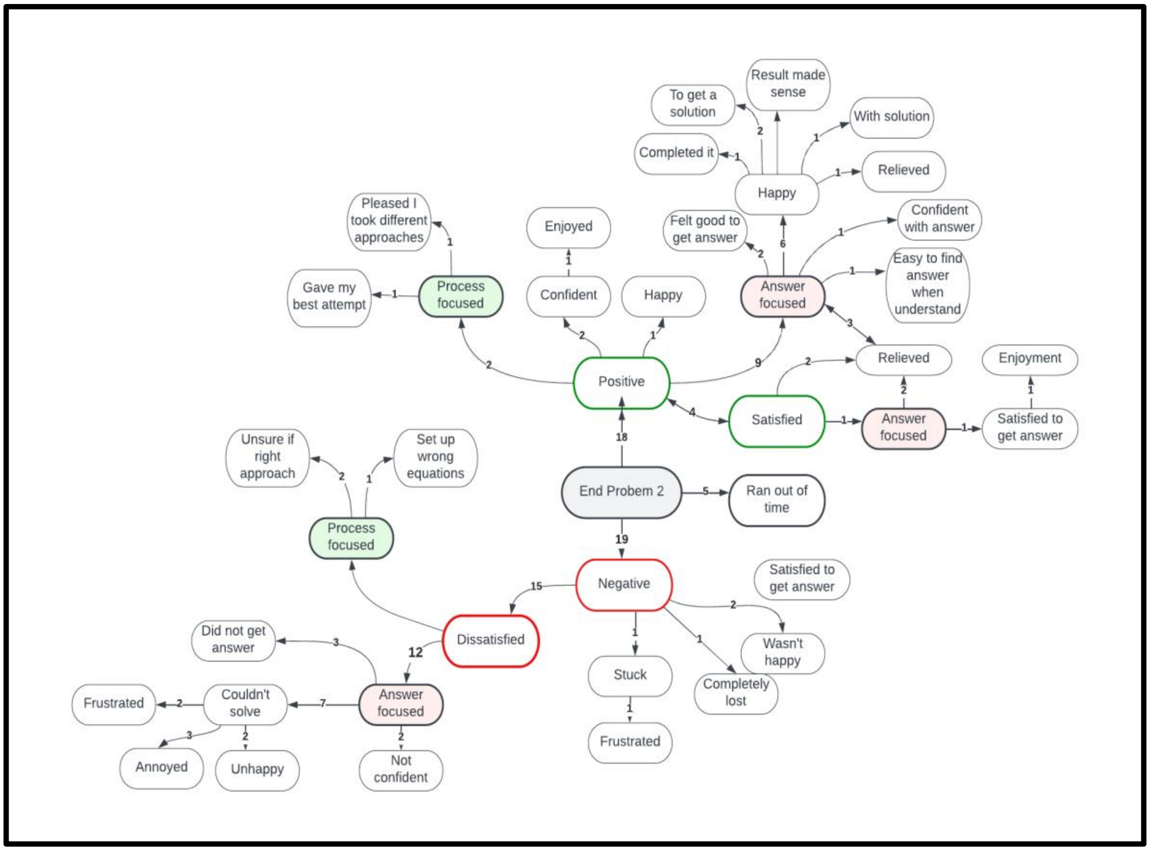


Figure 52: The categorisation of statements at the end of Problem 2

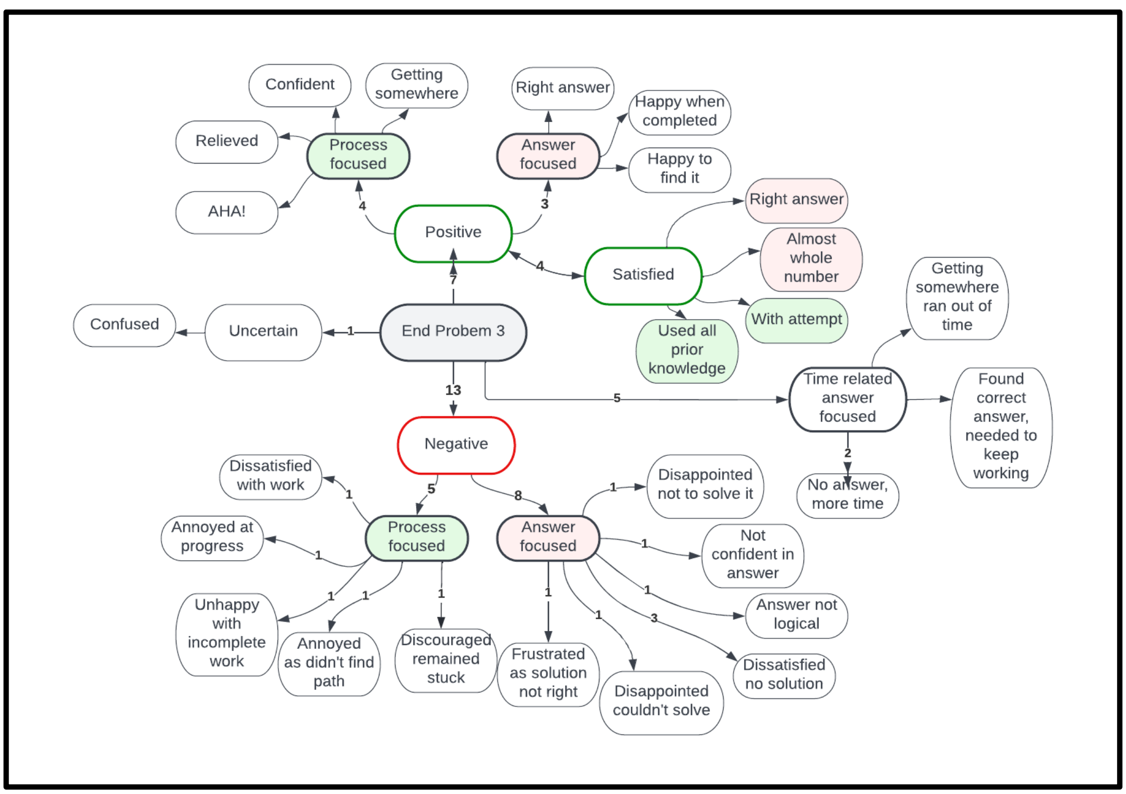


Figure 53: The categorisation of statements at the end of Problem 3

6.5 Summary

The purpose of this chapter was the presentation of results which were collected to address the questions. The chapter began with a presentation of the results of the 'Task Sorting' activity and the rationale behind the classification of tasks. Next, the results of the analysis of the PSMTs' problem-solving attempts using the MPSR were displayed. In addition to the MPSR, the results of the analysis of the 'Think Aloud' interviews which were used to investigate the PSMTs' problem-solving proficiency were outlined. The next section involved presenting the results of the analysis of the 'Think Aloud' interviews for the implementation of a Rubric Writing approach (Mason et al., 2011), which was a central part of the module as outlined in Chapter 5. Finally, the chapter concluded with the results of the two instruments associated with investigating the affective domain: the Indiana Mathematics Belief scale, and the open-ended affective questions. The next chapter will focus on the discussion of the results in relation to addressing the relevant research question.

CHAPTER 7: DISCUSSION

The purpose of this chapter is to discuss the results that were presented in Chapter 6. The findings for each research question will be described and discussed in relation to the literature. A summary of the findings will be presented at the end of each research question.

7.1 Research Question 1: What do pre-service teachers understand a mathematical problem to mean?

A: Are PSMTs proficient at classifying mathematical tasks?

As previously discussed in Section 4.10.1, PSMTs' ability to distinguish between a mathematical problem and other mathematical tasks was tested using a 'Task Sorting Activity'. The participants of this study received instruction on the definition of a mathematical problem and the characteristics which make a task a problem. These defining characteristics were repeatedly referred to throughout the module that they were undertaking. Cohort 1 and Cohort 2 completed the original ‘Task Sorting Activity’, and Cohort 3 and Cohort 4 completed the adapted version which required a rationale for the classification. This section will discuss the analysis of the results for each cohort of PSMTs to probe their understanding of a mathematical problem.

The understanding of the nature of mathematical problems is an important capacity for teachers to have: they need to be proficient in selecting and designing mathematical problems (Chapman, 2015). This section will focus specifically on the capacity of the PSMTs to identify mathematical problems through task classification activities. The process of designing mathematical problems will be addressed later in Research Question 3: this forms a complementary aspect of the overall understanding of problems.

Teachers’ ability to select appropriate problems is vital to supporting students’ problem-solving skills (Chapman, 2015; NCTM, 1991). While the selection of problems is important, the choosing, using, and adapting of problems is difficult (NCTM, 2000). There are many interesting mathematical problems available but these problems may not be suitable for developing the mathematical tasks that are appropriate at a particular point in time in a classroom (NCTM, 2000). When selecting a mathematical task, teachers need to be proficient in analysing tasks and “anticipating the mathematical ideas that can be brought out by working on the problem, and anticipating students’ questions” (NCTM, 2000, p. 53). The ability to select problems is a key skill for teachers as there is a correlation between the selection of problems and the quality of mathematics instruction (Son & Kim, 2015). Since there is a reliance on textbooks among mathematics teachers in Ireland (O’Keeffe, 2011; O’Sullivan, 2017), it is important for teachers to have the skill to classify mathematical tasks in order to differentiate between mathematical problems and exercises (Yeo, 2007). The effects of the selection of tasks by the teacher on students’ learning are illustrated in Figure 54 below, which was adapted from Stein et al., (1996) by Son & Kim (2015);

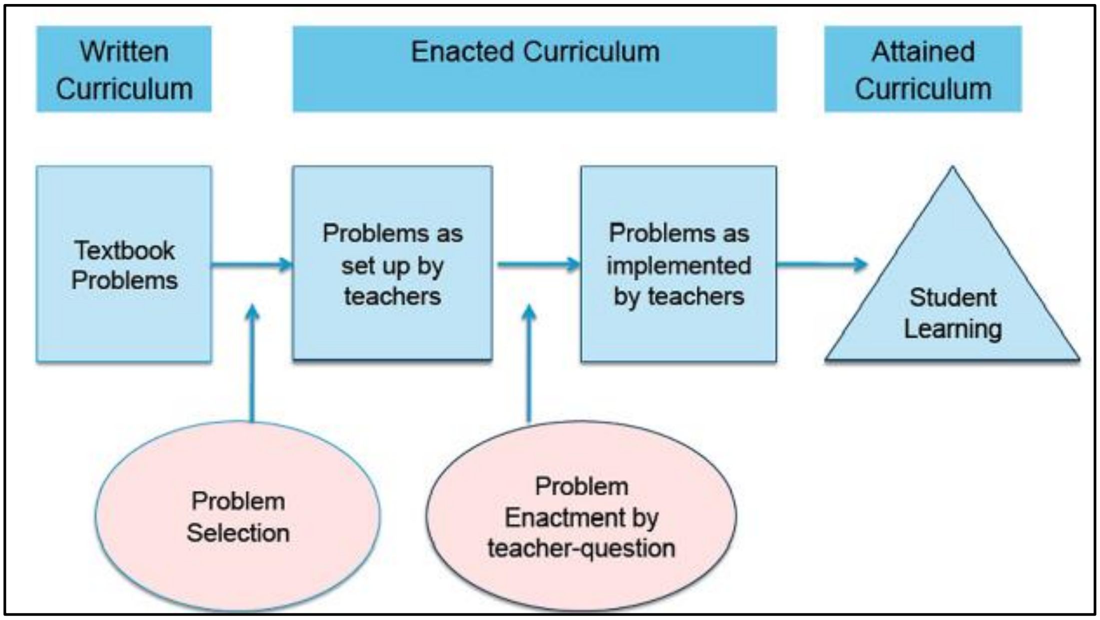


Figure 54: The effects of problem selection on students' learning (Son & Kim, 2015)

The PSMTs in Cohort 1 and Cohort 2 completed a 'Task Sorting Activity' which involved the classification of mathematical tasks as an *Exercise*, a *Problem*, or *Not Sure*. The PSMTs in Cohort 3 and Cohort 4 completed an adapted activity which involved a reduced number of tasks to classify but the PSMTs were required to provide a rationale for their classification. These tasks were independently classified by the researcher and the research supervisor. These classifications were then compared and agreed upon. The level of 'achievement' of the PSMTs refers to the level of agreement between the PSMTs' classification and the researchers' classification.

The analysis of the task classification activities revealed a similar level of achievement in Cohort 1, Cohort 2, and Cohort 3 (72.7%, 70.2%, and 67.5% respectively). Cohort 4 showed the lowest level of achievement in classifying the tasks (60.9%). As described in Section 4.10.1, this activity consisted of a variety of mathematical tasks.

It is noteworthy that two tasks which were misclassified as problems by a large proportion of PSMTs in each of the four cohorts, could be considered 'wordy questions.

One question was Task 3:

Niamh has an annual salary of €48000. She has a standard cut-off point of €34000 and a tax credit of €4600. If the standard rate of income tax is 20% and the higher rate is 42%, find how much income tax she pays.

Another task which had a low level of achievement was Task 12:

Shane wins a sum of money on a scratch card. He decides to invest €700 in a bank that offers an interest rate of 8%. How much will Shane have at the end of the two years? He then decides to invest €400 in a bank that offers an interest rate of 2% for a further 3 years. How much interest will he make from the €400?

Both of these tasks have a clear goal and there is a procedure for both tasks on how to reach a solution. This type of task, which has a clear goal and a straightforward and familiar procedural method, is typical of questions presented in Junior Cycle textbooks. However, the results showed that there was not a strong consensus that these would be classified as exercises. The results for Task 3 showed that 51.4% of Cohort 1 and 64.6% of Cohort 2 classified it as an exercise. The results for Task 12 showed that 40% of Cohort 1 and 52.1% of Cohort 2 classified it as an exercise. One possible reason for this would be that these two tasks are represented in text form, and this could potentially cause the PSMTs to identify the tasks as word problems. Word problems are "verbal descriptions of problem situations" where translation to mathematics is required and mathematical operations are needed to arrive at a solution (Verschaffel et al., 2000, p. ix). While these two tasks are verbal descriptions, they do not meet the criteria of a mathematical problem which the PSMTs were aware of. This illustrates that there may be a tendency for the PSMTs to classify a mathematical task which is presented in word form as a mathematical problem without considering the definition of a mathematical problem. This aligns with the findings of a study conducted by Crespo (2003) who found that pre-service teachers tended to choose problems without understanding or exploring the mathematical elements of the problem.

The potential association of text with mathematical problems further supports the claim that Chapman (2015) makes, which is that teachers need exposure to different types of problems to have a true understanding of what mathematical problems are (Chapman, 2015).

The adapted activity, which was completed by Cohort 3 and Cohort 4, required the PSMTs to provide a rationale for their classification of the tasks. The rationale for each classification was analysed using a grading rubric (see Appendix D). This grading rubric involved awarding 0, 1, or 2 points to the response regarding each of the three criteria outlined in the definition of a problem or an exercise. This resulted in a maximum score of 6 points for each task. The mean scores were generated by all responses not just from correctly classified responses to give an overview of the rationale for both incorrect and correct responses. A high mean score for the first criteria, a *clear goal*, demonstrates the PSMTs' correctly identifying the goal of the task. A high mean score for, *Immediately Clear*, constitutes PSMTs identifying that problems have unclear paths, while exercises have clear paths. A high mean score for the third criteria, *Prior Knowledge*, demonstrates the PSMTs ability to identify the prior knowledge necessary to reach a solution of the problem.

The results from Cohort 3 showed that whether a path was *Immediately Clear* had the highest mean (0.79) and highest mode (1 point), signalling that this was the characteristic on which the PSMTs mostly based their classification. The maximum potential score was 6 points but the majority of PSMTs scored 1 point (55.9%). One PSMT scored 5 points for the following rationale;

> *This is an exercise as you are asked to solve the quadratic equation so the procedure to solve this is known, and it is immediately clear to me how to solve this.*

The PSMT identifies the goal and the prior knowledge required to solve it along with it being immediately clear which means that it is an exercise and not a problem.

Similar to Cohort 3, the PSMTs in Cohort 4 demonstrated that the *Immediately Clear* element of a task was the defining characteristic for their classifications. *Immediately clear* had the highest mean (0.9) and highest mode (1 point). In both Cohort 3 and Cohort 4, the identification of a *goal* and *prior knowledge* had a mode of 0 points showing that the PSMTs did not consider the goal of the task or the prior knowledge required in their classification of the tasks. The overall scores for Cohort 4 showed that the majority (42%) scored 2 points out of a maximum of 6. This was an improvement on the results from Cohort 3. When analysing the rationales provided by the PSMTs, it was found that the identification of the *goal* of the question had the lowest percentage of PSMTs achieving the maximum score of 2 points in both Cohort 3 and Cohort 4. In addition to this low achievement of maximum scores, 84.5% of PSMTs in Cohort 3 and 78.7% of PSMTs in Cohort 4 achieved 0 points in *goal* identification. In a review of the definitions for a mathematical problem that have been offered over the past decades, Lester (2013) highlights that there is agreement that a mathematical problem has a goal and that it is not immediately clear to the problem-solver how to attain this goal. While it is positive that the PSMTs in this study were able to identify whether a path to a solution was immediately clear, it is concerning that there was little focus put on the *goal* of the question and how this goal would contribute to the classification of the tasks.

The highest percentage of PSMTs achieving 2 points was in *Prior knowledge* for PSMTs in Cohort 4 and in *immediately clear* for PSMTs in Cohort 3. A score of 2 points in *prior knowledge* constituted of PSMTs recognising the knowledge required to solve the task. In the case of an exercise, this prior knowledge would tie in with *Immediately Clear,* as a procedure or previously learned formula or method would be sufficient to reach a solution. An example of this was question 10 whereby the method to follow was explicitly stated in the question:

> *$g: x \rightarrow ax^2 + bx + 1$ is a function defined on R. If g(1) = 0 and g(2) = 3, write down two equations in a and b. Solve these equations to find the values of a and b.*

Question 10 was successfully classified as an exercise by 80% of PSMTs in Cohort 3 and 77.6% in Cohort 4. The high success rate of PSMTs correctly classifying this task as an exercise is a positive finding as tasks such as this do not offer students the opportunity to develop high-level cognitive processes (Son & Kim, 2015). The ability to identify this as not being a problem indicates that the PSMTs would not confuse this type of task with tasks which would be considered cognitively demanding and that promote reasoning and thinking in their students (Son & Kim, 2015).

B. Does an adaptation of the intervention that focuses on providing a rationale for task-classification lead to enhancement of PSMTs' capacities in task-classification?

The adapted Task Sorting activity required the PSMTs in Cohort 3 and Cohort 4 to provide a rationale for their classifications. There were five tasks which were common to both the original and adapted activity. The task number is identified as (X/Y) where X represents the original 'Task Sorting' activity and Y represents the task number in the adapted activity. Table 42 represents the level of alignment of the PSMTs' responses with the classifications in the tasks that were common in both activities.

Question	C1	C2	C3	C4
Garden Fence (2/9)	80	83.3	63.3	44.4
Rory Lemonade (11/8)	82.9	81.3	95.6	84.4
Scratch Card (12/7)	40	73.3	73.3	55.6
Ratio Money (15/10)	57.1	62.5	77.6	80
Escalator (19/5)	77.1	83.3	91.8	86.7
Average	67.42	76.74	80.32	70.22

Table 42: Level of alignment of responses to common tasks in Task Sorting activities

Cohort 3 was the first cohort of PSMTs asked to provide a rationale for their classification of the tasks. In four of the five tasks it was evident that there was an increase in the level of alignment in the classifications. Despite the decrease seen in the level of alignment of the 'Garden Fence' task, there was an overall improvement in the average classification in Cohort 3 demonstrating that the requirement of a rationale may increase the PSMTs' proficiency in classifying mathematical tasks. Based on these results, Cohort 4 were also required to provide a rationale for their classification of the tasks. Similar to Cohort 3, it was apparent that there was a decrease in the level of alignment for 'Garden Fence'. A possible reason for this decline is the provision of a rationale requiring the PSMTs to carefully consider the characteristics of the task. This task was classified as a mathematical problem by the researcher as it is not immediately clear on how to reach a solution and multiple steps are required. The average level of alignment in Cohort 4 (70.22%) was lower than that of Cohort 3 (80.32%). Given that 'Garden Fence' was an outlier to the other tasks, the average level of alignment was repeated excluding this task. This is displayed in Table 43 below.

Question	C1	C2	C3	C4
Rory Lemonade (11/8)	82.9	81.3	95.6	84.4
Scratch Card (12/7)	40	73.3	73.3	55.6
Ratio Money (15/10)	57.1	62.5	77.6	80
Escalator (19/5)	77.1	83.3	91.8	86.7
Average	64.28	75.1	84.58	76.68

Table 43: Average level of alignment without outlier task

While the exclusion of 'Garden Fence' resulted in an increase in the average level of alignment for Cohort 4, it was still evident that there was a decrease overall from the results of Cohort 3. Table 44 below displays the average level of alignment of Cohort 1 and Cohort 2 combined, and the average level of alignment of Cohort 3 and Cohort 4 combined. However, the results of Cohort 3 and Cohort 4 showed an increase in the level of alignment suggesting that the provision of a rationale for classifications may improve the PSMTs' classification of mathematical tasks. This is an area which requires further investigation

Question	C1	C2	Average C1 and C2	C3	C4	Average C3 and C4
Garden Fence (2/9)	80	83.3	81.65	63.3	44.4	53.85
Rory Lemonade (11/8)	82.9	81.3	82.1	95.6	84.4	90
Scratch Card (12/7)	40	73.3	56.65	73.3	55.6	64.45
Ratio Money (15/10)	57.1	62.5	59.8	77.6	80	78.8
Escalator (19/5)	77.1	83.3	80.2	91.8	86.7	89.25

Average	67.42	76.74	72.08	80.32	70.22	75.27

Table 44: Average level of alignment for C1,C2 and C3,C4

While it is evident that Cohort 4 did not have a similar increase in the level of alignment that Cohort 3 demonstrated, the average scores of the PSMTs who were required to provide a rationale were all higher than the average score of those who did not provide a rationale.

7.1.1 Summary of Findings

There was a similar average level of alignment in the classification of tasks and the predetermined classifications between Cohort 1 and Cohort 2 in the 'Task Sorting' activity and between Cohort 3 and Cohort 4 in the adapted version. (C1=72.7%, C2= 70.2%, C3= 67.5%, C4= 60.9%). It was evident that questions which were text-based displayed a high level of misclassification with 'wordy' questions being classified as problems despite the tasks not meeting the criteria of a mathematical problem. The tendency for PSMTs to refrain from exploring the mathematical elements of tasks is consistent with the findings of Crespo (2003). It was evident from the rationale given by Cohort 3 and Cohort 4, that the *immediately clear* element of the task was the characteristic that the PSMTs largely based their classification on. This demonstrates that the PSMTs considered the clarity of the path to reach a solution as the consideration of greatest importance when distinguishing between different types of mathematical tasks.

7.2 Research Question 2 : Are pre-service teachers proficient in problem-solving?

Problem-solving proficiency is important since problem-solving activates creative thinking and gives the opportunity to make connections between mathematical concepts and develop understandings (NCCA, 2013). As stated by Chapman (2015), problem-solving is not just a process but it is a way of thinking. We must be alert to this when discussing the problem-solving proficiencies of the PSMTs: we need to pay attention both to the processes that they employ, but also consider how they evidence 'ways of thinking' in their approach to problem-solving. Problem-solving proficiency is described as what is necessary for a person to learn and do problem-solving successfully (Chapman, 2015, p. 20). This involves a combination of the components of mathematical proficiency (see Section 2.4.2) and the components of successful problem-solving as discussed in Section 2.4. While teachers do not need to be expert problem solvers (Lester, 2013; Schoenfeld, 1982), they need to be proficient in order to comprehend varied approaches and misconceptions (Lester, 2013; Zbiek et al., 2010). To assess the PSMTs' problem-solving proficiency, two methods were employed; the analysis of problem-solving attempts using a Mathematical Problem Solving Rubric (MPSR) and 'Think Aloud' interviews.

Research Question 2A: How does PSMTs' problem-solving proficiency change over the duration of the intervention?

7.2.1 Mathematical Problem Solving Rubric (MPSR)

First, we will discuss the results of the MPSR analysis of the mathematical problems completed by the PSMTs as presented in Section 6.2.1. As described previously, the MPSR consists of five headings: Making Sense, Representing, Communicating, Accuracy, and Reflecting. Each heading had a maximum score of 6 points. One-tailed t-tests were used to determine whether statistically significant differences had occurred in students' scores between the problems completed over the duration of the module and the null hypothesis (H0 = no increase in the mean scores) was tested. To determine the effect size of the intervention on PSMTs' problem-solving scores, Cohen's d was calculated for each cohort of participants (Cohen et al., 2007). Two problems were analysed for Cohort 1, three problems were analysed in Cohort 2, two problems were analysed in Cohort 3, and three problems were analysed in Cohort 4 as outlined in Section 5.5.2

Making Sense

This heading corresponds to the PSMTs' ability to interpret the concepts of the task and translate them into mathematics (Oregon, 2011). The PSMTs in each cohort achieved the highest mean score under the 'Making Sense' heading in all problems with the exception of the second problem completed by Cohort 2. It was only under the 'Making Sense' heading that the PSMTs achieved a modal score of 4 points in any of the cohorts. In Cohort 1, the modal score remained constant at 3 points but the one-tailed t-tests showed that there was a significant increase in the mean scores. In Cohort 2, the modal score remained at 3 points for each problem with the one-tailed t-tests failing to reject the null hypothesis. Along with this, Cohen's d showed that there was a weak effect with a score of 0 for problem 2 as there was no change in the mean. Problem 3 showed a modest effect and it was not possible to reject the null hypothesis. This implies that there was no significant increase in the mean scores throughout the module and that the PSMTs' 'Making sense' skills remained at a partially developed level. The PSMTs in Cohort 3 completed two problems and the results showed that the modal score increased from *partially* developed (3 points) in the first problem to *adequately* developed (4 points) in the second problem along with an increase in the mean score (2.8 to 3.8). This improvement in scores is supported by the one-tailed t-tests which rejected the null hypothesis, and Cohen's d showed that there was a moderate effect. The highest mean score achieved by the PSMTs under any individual heading was seen under the 'Making Sense' heading, with a score of 3.5 in problem 2 by Cohort 3. In Cohort 4, the one-tailed t-tests rejected the null hypothesis for significance in the mean score for problem 3, however, it failed to reject significant increase in the mean score for problem 2. Cohen's d for 'Making Sense' in problem 2 and problem 3 for Cohort 4, was weak and moderate respectively.

Reflecting

This heading corresponds to the PSMTs' ability to state the solution or outcome in the context of the task and defend the process, evaluate, and interpret the reasonableness of the solution (Oregon, 2011). The PSMTs in each cohort scored the lowest mean under the 'Reflecting' heading and the scoring of the lowest mean was consistent across each of the problems for all four cohorts. There was no instance for which the mean achieved by the PSMT for 'Reflecting' was greater than 2 meaning that the PSMTs showed minimal or underdeveloped workings in relation to evaluating and interpreting the reasonableness of the solution obtained. It appears that there was a minimal improvement in this area of problem-solving proficiency as the one-tailed t-tests showed that there was not a significant increase between the means in Cohort 1, Cohort 2 and Cohort 4. This is echoed by the fact that the modal score of each problem for these cohorts was 1 point. Cohen's d showed a weak effect in this heading for these three cohorts. However, in Cohort 3, the one-tailed t-tests and Cohen's d showed that there was a significant increase and moderate effect in the

‘Reflecting’ heading with the modal score increasing from 1 point to 2 points. Despite an increase, this still leaves the cohort with a very low level of achievement under this heading.

Communicating

Throughout the module the PSMTs were advised to clearly communicate their reasoning through the use of mathematical language to justify and support their work. In only one instance, under the heading of ‘Communicating’ did the PSMTs achieve a modal score less than 3 points meaning that the communication skills of the PSMTs were either partially effective or adequately effective in allowing the reader to coherently follow the solution. Cohen’s d showed that the module had a moderate effect on the communication of mathematical reasoning of the PSMTs in Cohort 1, Cohort 2, and Cohort 3 , and in Cohort 4 there was a modest effect.

Representing

The heading ‘Representing’ corresponds to the use of both visual diagrams or models and the selection of an effective strategy to solve the task (Oregon, 2011). The modal score of 3 points achieved by Cohort 1, Cohort 2 and Cohort 4 remained constant throughout each problem attempt. There was an increase in the modal score in Cohort 3 from a score of 2 points to 3 points under the ‘Representing’ heading. This improvement in Cohort 3, was supported by the one-tailed t-test which rejected the null hypothesis that there was no significant increase between the mean scores. Similarly, Cohen’s d showed a moderate effect. Given that there was no increase in the modal score and there was minimal change in the mean scores in problem 1 in Cohort 4, the one-tailed t-tests failed to reject the null hypothesis and Cohen’s d showed that there was a modest effect in the PSMTs’ use of visual representations and selection of strategies.

Accuracy

The heading ‘Accuracy’ refers to achieving a correct solution and providing a justification for the solution. The results of Cohort 2 show that the modal score increased from 2 points in the first problem to 3 points in the second problem before returning to 2 points in the third problem. This pattern was similar to the change in mean scores, with a rise followed by a fall. A score of 2 points indicated an incorrect or incomplete solution or a solution that was correct but was not supported by mathematical reasoning. The increase to 3 points showed that the solutions were partially correct or incorrect due to minor errors. Cohort 3 saw the greatest change in accuracy scores with an initial modal score of 1 point increasing to 3 points. A score of 1 point showed that the PSMTs did not achieve a clearly identified solution. This improvement was supported by the one-tailed t-tests which rejected the null hypothesis meaning that there was a significant increase in the mean scores. In comparison to this, Cohen's d showed that the module had a weak effect. The modal score increased from 2 points in problem 1 to 3 points in both problem 2 and problem 3. One PSMT achieved a score of 5 points in Problem 1 through the use of extensions from a correct solution. There was an increase in the percentage of PSMTs achieving a score of 4 points in Problem 2 which involves attaining a *justified correct solution supported by work* (Oregon, 2011).

7.2.2 'Think Aloud' Interviews

As described in Section 4.10.2, the focus groups of PSMTs from Cohorts 1,2, and Cohort 3 completed semi-structured interviews wherein they attempted mathematical problems in a 'Think Aloud' manner (Cowan, 2019). The analysis of the transcripts of these interviews, using a general inductive approach, led to the identification of the following categories: *Introduce, Productive reasoning, Unproductive reasoning, Resilience,* and *Identity*. A subcategory of *Productive reasoning* was also identified - *Productive questioning*. We will now discuss the results for each of the identified categories (see Section 6.2.2).

Introduce

For the purpose of this study, the category *Introduce* includes: introducing notation, drawing diagrams, and adapting given diagrams. Mason et al. (2011) emphasise the importance of problem-solving in organising information by introducing diagrams or charts. Likewise, it is often useful to use appropriate mathematical notation (Polya, 1945). The importance of the *Introduce* phase is highlighted by its explicit appearance in Mason's Rubric Writing approach to problem-solving. The number of actions by the participants that correspond to Mason's 'Introduce' phase led to the identification of this as a separate category. Mason et al. (2011) suggest that *Introduced* elements enable the problem-solver to extract the key information in the question. It may give a starting point by helping the problem-solver to interpret the information, and identify mathematical features or objects that may be of use in solving the problem. Although there were problems which provided the PSMTs with a diagram, many opted to redraw it themselves. An example of a participant drawing a diagram followed by an example of a participant introducing notation are given below:

> *"I'll draw it out in front of me so there's the well, so I have 5, 10 and X".*

> *"ok if I set x as time, told travels, x +20 +y"*

Within this category, extensions of given diagrams also feature prominently. Participants drew diagrams with a view to manipulating them. Participants also introduced notation which is acknowledged as a feature that frequently underpins successful problem-solving (Krulik & Rudnick, 1988; Mason et al., 2011; Polya, 1945). *Introduce* occurred a total of 93 times which was the second highest category after *Productive reasoning.* The *Introduce* category will be further discussed in Section 7.3. It is positive to see that *Introduce* is prominent in the PSMTs' work as it was an element of the Rubric Writing approach (Mason et al., 2011) that was explicitly taught throughout the course of the module along with the fact that the introduction of notation and/or diagrams as a means to start a problem (Krulik & Rudnick, 1988; Mason et al., 2011; Polya, 1945). The use of different elements of *Introduce* signal towards proficiency as outlined in the conceptual framework through the use of heuristics. We will consider below the degree to which appropriate application of the 'Introduce' phase led to progress towards a solution of the problems.

Productive Reasoning

The category of *Productive Reasoning* refers to statements made, or actions taken by the participants that are deemed to represent progress towards a solution of the problem at hand. *'Introduce*' can be thought of as a special category of *'Productive Reasoning*' as the elements of *Introduce* act as a starting point for a problem-solving attempt. The distinction between statements that were categorised as *Introduce* and *Productive Reasoning* is that elements of *Introduce* are at the beginning of the problem-solving attempt. The distinguishing features of *Introduce* are; (a) the idea that something (diagram, notation,…) has been introduced and (b) the idea that this action somehow represents the starting point in a line of reasoning.

Interpreting information given in the question is included in this category. Mason et al. (2010) claim that at the beginning of an attempt to solve a problem, it is important for the individual to decipher the information given in the question to extract information of importance and then to write down what they consider relevant. (Polya, 1962) identifies the questions: “Could you restate the problem?, Could you restate it still differently?”, that an individual should consider when starting a problem. Mason et al. (2010) also point out that it is crucial for the individual to clearly understand what the question is asking in order to be able to then decide a strategy to implement. This is emphasised in Mason’s ‘Entry Phase’ of ‘What do I want? What do I know?’.

The organisation of prior knowledge is also an element of this category. This includes examples of the PSMTs relating prior knowledge from previous experiences to make progress on a problem, either directly or by refining their strategy.

> *“Probability of either one is given as the number of them present, divided by total”.*

> *“I’ll have to add up to 180 degrees”.*

Additionally, prior knowledge can be used to both select or eliminate possible strategies. An example of this is;

> *“Can’t use Pythagoras because there’s no right angles”.*

The PSMTs demonstrated their ability to apply their proficiency in mathematical procedures, their ability to specialise and (from this) to generalise. Specialisation is considered an element of the ‘Attack Phase’ as described by Mason et al. (2011) and can be described as using several examples to help understand the problem posed. It can be useful in helping the problem solver to get an idea of what the solution should be. By specialising numerous times, the individual may be able to notice a pattern which could then in turn progress onto generalisation and ultimately lead to a solution (Mason et al., 2011). The final element of this category is the PSMT’s ability to relate their work to the question. It is a reflective practice by which the individual looks at their progress and determines whether it is leading to a solution of the problem, whether it is of relevance, or whether it is not productive.

The PSMTs demonstration of components of the *Productive reasoning* category showed that it was the category with the highest number of occurrences in total. Similarly, it was the highest individual category for all cohorts with the exception of Problem 4 in the post-module interview of Cohort 3, within which *Unproductive reasoning* outnumbered *Productive reasoning.* Overall, this is a positive finding showing that PSMTs displayed characteristics of problem-solving proficiency. The organisation of prior knowledge to make links amongst concepts is indicative of mathematical proficiency which is integral to problem-solving (Kilpatrick et al., 2001).

Productive questioning was evident in the interviews of both Cohort 2 and Cohort 3. A revision of the transcripts of Cohort 1 was done in order to identify if *Productive Questioning* was evident, and it was not found to be so. This category refers to the participant questioning themselves on their work towards a solution, their chosen strategy, or how to proceed. *Productive questioning* is seen as a subcategory of *Productive reasoning* as the questioning helped participants towards achieving a solution. Self-questioning is described as a form of structured metacognition which aids the person to understand the task at hand, plan, and reflect on the strategy application as they progress through the task (Kramarski et al., 2010). Self-questioning allows the problem-solver to retain control over their behaviours, feelings, and learning (King, 1992). The use of self-questioning by a problem-solver positively contributes to developing the attitude of thinking mathematically (Mason et al., 2011). Mason et al., (2011) outline that questions arise from the noticing of a change and the generation of a tension to which a question corresponds. The examples of self-questions that the PSMTs demonstrated align with the metacognitive questioning which is a part of the IMPROVE method developed by (Mevarech & Kramarski, 1997). This form of metacognitive questioning aligns with Mason et al. (2011) description of a person noticing a change and posing a question about said tension. The questions asked by the PSMTs were examples of ‘strategic questions’ and ‘connection questions’(Mevarech & Kramarski, 1997, p. 370). These questions focused on the specific mathematical concepts and principles and also related the task at hand to previously encountered problems. Not only is metacognition a key skill for problem-solving, it is also vital for teachers to have an understanding of the role of metacognition in students’ learning (Chapman, 2015).

Unproductive Reasoning

Unproductive Reasoning involves actions or statements which do not help (or which may even constrict) the problem-solving from progressing or being successful. Making incorrect assumptions, procedural errors, misconceptions (e.g. using Pythagoras' Theorem for non-right-angled triangles), and persisting with a line of reasoning despite having explicitly acknowledged its erroneous nature all belong in this category. It is noted by Boaler (2016) that mindsets have an effect on achievement. Students with a growth mindset are more likely to be persistent, unlike students who have a fixed mindset, who are more inclined to give up. In relation to mistakes and errors, it has been shown that people with a growth mindset possess a greater awareness of errors and are therefore more likely to try and fix them (Stein & Silver, 2003). This signifies that mistakes can be productive if the person has a growth mindset. The statements that were classified in this category are unproductive in the short term meaning that they will not help solve the problem at hand. However, if the PSMTs possess traits of a growth mindset, the statements in this category may be productive in the long term if reflected on by the PSMTs. Mason et al., (2011) highlight the role of being '*Stuck*' in their problem-solving rubric and suggest ways to overcome it. Although the participants had been exposed to and utilised this rubric, some participants stopped immediately once they felt that they were stuck. An example of a PSMT stopping their attempt after noticing a mistake is:

> *C1/P3: "I really don't know".*

In contrast, some participants noticed that they had made a mistake yet persevered with their method despite knowing that they were incorrect.

> C1/*P4: "I'll just keep going because this seems definitely wrong…would give me* a minus *number under my square root which is not real".*

This continuation with a known incorrect method given the context of the question exemplifies *Unproductive reasoning.*

Resilience

Resilience occurred a total of 51 times in the PSMTs' interviews, which was the lowest scoring category. According to Kooken et al. (2013), resilience is defined as "an orientation to produce a positive response when faced with a negative situation or difficulty in learning mathematics". There are five main components of mathematical resilience: 1) having a growth mindset which is demonstrated through actions such as learning from mistakes; 2) meta-cognition, which is shown through reflection on answers and problem-solving processes: 3) possessing the capability to adapt by demonstrating the willingness to restart or try new approaches; 4) having a sense of purpose by seeking meaning in their learning; and 5) interpersonal aspects of learning such as viewing asking questions as a positive rather than an admission of a lack of knowledge (Morris et al., 2014). PSMTs demonstrated many instances of noticing their mistakes and, as a result, adapted and proceeded with new strategies. It is a positive finding that there was evidence of PSMTs displaying characteristics of resilience as it is indicative of a growth mindset, adaptability, and meta-cognitive awareness. However, it is important to note that 25/51 occurrences of *Resilience* occurred in Problem 2 by Cohort 1, meaning that there were only 26 occurrences across the other nine problems attempted. This is an area that might need to be focused on in order to further improve upon PSMTs resilience which would ultimately lead to increased problem-solving proficiency.

Identity

As outlined previously, it has been widely acknowledged that the affective domain has an influence on problem-solving behaviours (Andrews & Xenofontos, 2015; Lester & Kroll, 1993; Schoenfeld, 1992). Given the prominence of the affective domain, it was not surprising that this category, *Identity*, was apparent in the PSMTs' problem-solving attempts. *Identity* is defined as "the embodiment of an individual's knowledge, beliefs, values, commitments, intentions, and affect as they relate to one's participation within a particular community of practice; the ways one has learned to think, act, and interact" (Philipp, 2007, p.259).

Beliefs around various areas of mathematics, including beliefs about oneself in connection to mathematical learning and problem-solving, lie amongst the components required to obtain a mathematical disposition. This desired positive mathematical disposition would enable students to access inert knowledge to aid them in approaching problems with the facility to utilise their prior knowledge (De Corte et al., 2000). Schoenfeld (1992) similarly outlines that the cognitive resources available to students when learning are related to the students' beliefs around what they consider useful in learning maths. In total, there were 70 occurrences of *Identity* and only 7 of these were classified as reflecting a positive disposition or attitude. An example of a statement expressing a positive disposition, and hence relating to the *Identity* category was:

> *C3/P15: "So I'm going to get, I feel like I'm nearly getting there, that I may be possibly onto something".*

This represents the PSMT's belief that they are progressing in their attempt and shows confidence in their approach. It is clear that the negative statements significantly outweighed the positive statements, meaning that the PSMTs are possibly limiting their ability to undertake the problem (De Corte et al., 2000; Lester & Kroll, 1993; Schoenfeld, 1992). One such limitation is the lack of confidence impeding PSMTs' mathematical proficiency since confidence positively influences individuals' procedural fluency and adaptive reasoning abilities (Kilpatrick et al., 2001).

As discussed above, two methods were used to investigate the PSMTs' problem-solving proficiency: assessment of written problem-solving attempts using the MPSR, and 'Think Aloud' interviews conducted by focus groups from the different cohorts of PSMTs. When comparing the results of both data sets, it is evident that certain attributes are present in both while other attributes are only evident in one method. This combined approach allowed for a more holistic overview of the PSMTs' problem-solving proficiency.

The identification of the *Introduce* category in the interviews aligned with the *Representing* heading of the MPSR. The use of notation, diagrams, or constructions were evident in both data sets, which is a positive finding since this is positively associated with a successful problem-solving approach (Krulik & Rudnick, 1988; Mason et al., 2011; Polya, 1945). Similarly, *Representing* was evident in the 'Think Aloud' interviews through the use of prior knowledge to select an effective strategy. In the overall analysis of the tutorials, the majority of selections of a strategy were *partially effective*. The selection of a strategy in *Representing* was also present in the *Resilience* category of the interviews in the form of adapting and trying new approaches which is a positive finding: this adaptation is shown to have a positive impact on proficiency in research (Muir et al., 2008). While PSMTs demonstrated elements of *Representing*, it is important to note that the modal score was 3 points meaning that it was *partially developed,* leaving further room for improvement in the PSMTs' selection of strategy. While conducting research into the differences between proficient and less accomplished student problem-solvers, Muir et al. (2008) found that the students who had the greatest level of success were able to identify alternate strategies and use previous mathematical encounters to generate their own strategies.

A further alignment between the MPSR and the 'Think Aloud' interviews was seen through interpreting information in the given question which was a characteristic of *Productive reasoning* in the 'Think Aloud' interviews and the *Making Sense* heading of the MPSR. This involves the PSMTs interpreting the information in the question and mathematising the information. In the interviews this was seen through the PSMTs identifying what they know, what they want to achieve, and forming mathematical language which is part of the *Entry* phase (Mason et al., 2011).

On further comparison of the MPSR results and the 'Think Aloud' interviews, there was evidence of *Unproductive reasoning* in the data collections. The components of *Unproductive reasoning;* the selection of an inappropriate strategy, making assumptions, and procedural errors, are all seen in the low scoring range of each element of the MPSR. It was not possible from the analysis of the written problems attempts to assess the affective domain but in the 'Think Aloud' interviews, the category of *Identity* was prominent.

On analysis of the results from both sets of data, the following similarities were apparent:, prior knowledge is used to interpret the information in the task and select a strategy, metacognition is evident in the use of self-questioning, and resilience is also evident.

On analysis of the 'Think Aloud' interviews, it was not evident that the PSMTs reflected on their problem-solving attempt, meaning that there was little evidence of the PSMTs checking the reasonableness of their solution in the context of the question or checking their solution using an alternative approach. The MPSR analysis showed that *Reflecting* had the lowest mean both overall and for each cohort. The two data sets provide an insight into the PSMTs' thoughts while problem-solving and also their written work. The results of both sets of data show that it is evident that the PSMTs do not demonstrate reflection on their work, in either the written form or cognitively. This lack of reflective practice is of concern as 'reflecting is possibly the most important activity for improving mathematical thinking' (Mason et al., 2011, p. 38). Reflection involves identifying key ideas and asking in what situation the strategies or new experience could also be applied (Mason et al., 2011). The importance of reflection for building knowledge based on experiences is similarly acknowledged by Polya (1945).

7.2.3 Summary of Findings

The analysis of the PSMTs' problem-solving proficiency using the MPSR showed that the PSMTs were most proficient in interpreting concepts of a task and translating them into mathematics over any other area of the MPSR (Oregon, 2011) . The area that the PSMTs showed least proficiency was in checking and interpreting the reasonableness of their solution in context of the task. There was a minimal increase in this area across each of the cohorts. However, in all other aspects, it was seen that the module may have had a moderate to strong effect on the PSMTs' problem-solving proficiency.

The 'Think Aloud' interviews gave a further insight into the PSMTs' problem-solving process through the verbal account of thinking rather than the analysis of written work alone. The PSMTs demonstrated the use of introducing visual diagrams or notation in their problem attempts. This is consistent with the Rubric Writing approach (Mason et al., 2011) which was at the core of the module and also aligns with the *Representing* heading of the MPSR. It was evident from the MPSR analysis that the PSMTs used representations or notations which were 'partially effective' (Oregon, 2011). Within the theme *Productive reasoning,* it was promising to see that the PSMTs organised their prior knowledge to make links amongst concepts indicative of mathematical proficiency which is integral to problem-solving (Kilpatrick et al., 2001). Similarly, the presence of *productive questioning* is a positive finding as it allows problem solver to retain control over their behaviours, feelings and learning (King, 1992) and can ultimately lead to the development of an attitude of thinking mathematically (Mason et al., 2011). While it is positive to see that *Productive reasoning* outnumbered *Unproductive reasoning,* it is concerning that there were instances in which some PSMTs continued with an approach which they were aware was incorrect and they stopped immediately once stuck. However, there was evidence that some PSMTs demonstrated a growth mindset (Boaler, 2016) in the form of noticing a mistake and adopting a new strategy. These instances were representative of statements within the *Resilience* category and indicated signs of problem-solving proficiency. The 'Think Aloud' interviews presented information about the *Identity* of the PSMTs which could not be seen through the analysis of the written work. While there were statements of a positive disposition, these only accounted for 10% of the statements. A positive disposition would enable the PSMTs to access inert prior knowledge which could help them in their problem-solving attempt (De Corte et al., 2000). It is clear that the negative statements outweigh the positive, meaning that there is a limitation on the PSMTs' ability to successfully approach the problem at hand (De Corte et al., 2000; Lester & Kroll, 1993; Schoenfeld, 1992). The presence of a negative disposition can adversely influence the PSMTs procedural fluency and adaptive reasoning abilities which are essential for problem-solving proficiency (Chapman, 2015; Kilpatrick et al., 2001).

Research Question 2B: B: Did the ongoing adaptations of the intervention lead to a greater enhancement of the problem-solving capacities of successive cohorts of PSMTs?

In order to investigate the PSMTs' problem-solving proficiency, the PSMTs' attempts at solving problems were analysed using a mathematical problem-solving rubric (MPSR) (Oregan, 2011) (Appendix E). These problems were completed individually by the PSMTs in tutorial settings. Two problems attempted by Cohort 1 were analysed: 'Professor on an Escalator' in Week Seven after two weeks of problem-solving instruction and practice, and 'Four-Legged Lawnmower' in Week Nine after four weeks of problem-solving instruction and practice. Based on the results of Cohort 1 and observations by the researcher in consultation with the facilitator, additional time was dedicated to problem-solving within the module. This allowed for the addition of more problems for analysis in Cohort 2. Three problems attempted by Cohort 2 were analysed: 'Professor on an Escalator' in Week Five after two weeks of problem-solving instruction and practice, 'Four-Legged Lawnmower' in Week Six after three weeks of problem-solving instruction and practice, and 'Threaded Pins' in Week Eight after five weeks of problem-solving instruction and practice. Due to the time constraints of a shortened university semester, it was not possible to conduct three problems for analysis with Cohort 3. Two problems attempted by Cohort 3 were analysed: 'Professor on an Escalator' in Week Four after three weeks of problem-solving instruction and practice, and 'Four-Legged Lawnmower' in Week Seven after five weeks of problem-solving instruction and practice. The PSMTs in Cohort 4 returned to the twelve week module length as experienced by Cohort 1 and Cohort 2. Based on the findings of previous iterations of the module, the time dedicated to problem-solving was increased for Cohort 4 as outlined in Section 5.5.2. Three problems attempted by Cohort 4 were analysed: 'Four-Legged Lawnmower' in Week Two after two weeks of problem-solving instruction and practice, 'Petrol Pitstop' in Week Six after five weeks of problem-solving instruction and practice, and 'Ladder Leaning on a Cube' in Week Ten after eight weeks of problem-solving instruction and practice.

From the results of the MPSR with Cohort 1, it was evident that there was no significant difference in the *Reflecting* heading along with a weak effect size. This impacted the decision to increase the time allocated in the module to problem-solving for Cohort 2. In the module there was a strong emphasis on the implementation of the Rubric Writing approach (Mason et al., 2011). Key elements of the Review stage of the Rubric are 'check' and 'reflect' which align with the *Reflecting* heading of the MPSR. This involves checking the appropriateness or effectiveness of the solution using alternative approaches while also checking the solution for errors (Mason et al., 2011). Additionally, the 'reflect' element of the Rubric Writing approach requires the identification of key mathematical ideas for the solution which can contribute to evaluating the effectiveness of the solution. By increasing the PSMTs' in Cohort 2 problem-solving experience, it was anticipated that they would have greater opportunities to develop their proficiencies in the *Reflecting* heading. From the analysis of the three problems undertaken by Cohort 2, it was evident that there was not a significant difference in their proficiency in the *Reflecting* heading. Similar to Cohort 1, the effective size was weak for *Reflecting.* The scores on the *Reflecting* heading for Cohort 3, showed that there was a significant difference in the mean scores along with a moderate effect size. The PSMTs in this cohort had one week extra problem-solving experience when completing the first problem than the previous two cohorts. This is interesting to note as it was anticipated that the PSMTs in Cohort 2 would have seen a greater improvement in scores between the first and third problem as it was over a longer time period of instruction. Recognising that *Reflecting* was an area for which the PSMTs score relatively poorly, PSMTs in Cohort 4 had increased time dedicated to problem-solving and utilising the Rubric Writing approach in the module. From the analysis of the three problems, it was found that *Reflecting* was the heading of the MPSR with the lowest mean score, showed no statistically significant difference in the mean scores, and had a weak effect size. It was seen that the majority of PSMTs scored 1 point in each of the three problems indicating that there was minimal evidence of the elements of *Reflecting*. This is a disappointing finding given the prominence of the Review phase in the module and signals that further developments need to be considered in this area.

Across each of the four cohorts the modal score under the heading *Making Sense* remained at 3 points with the exception of problem 2 completed by Cohort 3. A score of 3 points in this heading corresponds to *partially developing* or *partially displaying* interpretations of the given task into mathematics. The *Making Sense* category is consistent with the Entry Phase of the Rubric Writing approach which was a key feature of the problem-solving aspect of the module. Within the Entry Phase, it is expected that the problem-solver will identify what 'I Want' - the goal of the question, and 'I Know' - the given information in the question and the relevant prior knowledge (Mason et al., 2011). From the increased experience of problem-solving across each cohort of PSMTs, it is interesting to note that there was only one instance where the modal score increased to *adequately displayed*. The number of weeks of instruction did not appear to have a consistent effect on the level of effectiveness of the module on the *Making Sense* heading.

The heading *Representing* assessed the PSMTs' proficiency in using models, diagrams, or other representations along with selecting an effective strategy to solve the task. This heading aligns with the *Introduce* element of the Entry phase of the Rubric Writing approach (Mason et al., 2011) in terms of using different representations. The selection of an effective strategy aligns with the *I Know* element of the *Entry* phase through the identification of prior knowledge to select a strategy and the implementation of such is an element of the Attack phase of the Rubric. As mentioned above, the successive cohorts of PSMTs had increased experience of problem-solving using the Rubric Writing approach. From the analysis of the problems attempted by Cohort 1, the modal score was found to be 3 points meaning that the selected strategy or representations were partially developed. Despite the increased problem-solving experience, the mode for Cohort 2 was also 3 points and had similar mean scores as Cohort 1 (C1: 2.68, 2.92; C2: 2.7, 2.91, 2.87). However, it was found that in Cohort 3 there was a greater increase in the Representing heading between problem 1 and problem 2 to a mean score of 3.6. This was the highest score across any of the cohorts indicating that the increase in instruction and problem-solving practice positively influenced the PSMTs' performance in Representing. This result indicated that the increased time experienced by the PSMTs in problem-solving had a positive impact so it was anticipated that with

Cohort 4, who again had further time allocated in the module to problem-solving, would demonstrate similar improvement. Unfortunately, this was not the case as the PSMTs in Cohort 4 demonstrated a decrease in scores from problem 1 to problem 3. However, it is important to note that the PSMTs in Cohort 4 had the highest starting mean score and an increase from problem 1 to problem 2 was evident.

While the focus of the Rubric Writing approach (Mason et al., 2011) is on developing the problem-solving process, the Accuracy heading of the MPSR aligns with elements of this approach. The maximum score in this heading involves reaching a correct solution and enhancing it through extensions, connections, generalisations, and/or asking new questions leading to new problems (Oregon, 2010). Generalising is a key component of the Attack Phase while extensions and generating new problems are components of the Review Phase. In Cohort 1 it was evident that there was a strong effect on the Accuracy heading with a significant difference in the mean scores between the two problems. This was also apparent between the first and second problem completed by Cohort 2 and Cohort 3, indicating that the experience of using the Rubric Writing approach and attempting multiple problems had a positive effect on the PSMTs' accuracy. However, between problem 2 and problem 3 attempted by Cohort 2, there was no significant difference found between the mean scores and there was a modest effect size. This suggests that the greatest increase in the Accuracy heading in the earlier stages of the module. While each of the previous cohorts of PSMTs demonstrated a strong effect size at some point during the module, it was disappointing to see that in Cohort 4 there was a weak effect and no significant difference between the mean scores. Cohort 4 experienced more problem-solving instruction and practice than previous cohorts suggesting that this increase did not directly positively impact performance in accuracy.

7.3 Research Question 2C : Are taught strategies implemented while problem-solving?

As discussed in Chapter 5, the PSMTs were exposed to problem-solving strategies over the course of the module, and they had the opportunity to practise implementing these strategies.

The teaching of heuristics has been shown to have a positive effect on problem-solving proficiency (Depaepe et al., 2013). The focus of heuristics in the module undertaken by the PSMTs is described in detail in Chapter 3. In order to investigate the PSMTs' implementation of the taught heuristic, the semi-structured 'Think Aloud' interviews were analysed for evidence of the implementation of Mason's Rubric Writing approach (Mason et al., 2011). These interviews were completed by participants in Cohort 1, Cohort 2 and Cohort 3. Mason et al., (2011) highlight that it is rarely possible to reach a solution of a problem by simply reading the question a number of times, but the implementation of the *Entry phase* prepares for an effective plan or strategy to be carried out. The analysis of the *Entry Phase* of Mason's Rubric Writing Approach was done through the implementation of the following grading system: 0 points (no evidence); 1 point (limited evidence); 2 points (strong evidence). This grading was carried out on the three elements of the *Entry Phase* namely; *Introduce*, *I Know*, and *I Want*. In Section 6.3, the results are presented, and they will now be discussed.

The analysis of the post-module 'Think Aloud' interviews showed that overall, the element *Introduce* had the highest percentage of 2 points with 68.75% of PSMTs achieving 2 points . At the same time, 25 % of PSMTs scored 0 points in the *Introduce* element. A score of two points for *Introduce* would consist of the explicit use of representations through the drawing of diagrams, tables or constructions within given diagrams or the introduction of notation for unknown variables. A score of 1 point was awarded for limited evidence of the use of the above, and a score of 0 points was awarded for no evidence of these elements. This result demonstrates that the PSMTs either explicitly used notation or diagrams at the start of their problem-solving attempt or did not use them at all. Polya (1945) advocates for the introduction of diagrams or suitable notation as a useful step in approaching a problem. The introduction of these elements is part of the first step, *Understanding the problem* in Polya's (1945) heuristic. Similarly, Mason et al., (2011) outline that a benefit of constructing diagrams or tables is the organisation of the given information in a systematic manner. This organisation allows for the essential information to be extracted from the question allowing for a clearer understanding of how to proceed with a possible path (Mason et al., 2010). The introduction of notation or diagrams can make a problem which initially appears to be difficult to approachable by changing it to a new context (Mason et al., 2011). In comparison, in the first round of interview, of the PSMTs in Cohort 2 and Cohort 3, 41.3% scored 2 points and 50.8% scored 0 points. The results of this study show that while over half of the PSMTs used

Introduce competently as an approach to starting a problem, some of the PSMTs did not use these elements at all.

It was disappointing to find that overall, 49.67 % of the problem-solving attempts in the first round of interviews and 56.25% of the problem-solving attempts in the second round of interviews were classified as scoring zero points as they showed no evidence of PSMTs stating what the goal of the question was. In addition to this, the *I want* element of the *Entry* phase was the element with the lowest percentage of 2 points being scored. This low percentage of PSMTs achieving 2 points and the high percentage achieving 0 points is a concern, as failure to identify what the task is specifically asking can cause difficulty with achieving a solution to a problem (Mason et al., 2010). In some of the problems that were attempted by the PSMTs, the goal (what 'I want') is explicit in the task: Example 1: 'How high is the ladder above the ground?' (Appendix G) or Example 2: 'What is the value of n?' (Appendix F). It could be argued that it is somewhat artificial to write this down as the *I Want* statement, but the PSMTs were instructed explicitly to relate the question to the context of the question to clearly identify what is being asked. Looking at Example 2, it is obvious that the goal of the question is to find the value of 'n', but in the context of the question, the person must look at the given information and identify what 'n' means, which in this case is the number of white socks. The appropriate *I Want* statement would not be 'I want to find the value of n' but rather a statement similar to: 'I want to find the value of n, which is the number of white socks in the drawer'. Mason et al., (2010) explain that while finding out what the question is really asking, it is common for problem-solvers to miss the step of carefully reading the question. This ultimately leads to rushing into an approach with the real question being missed (Mason et al., 2010). The importance of reading the question to identify the goal of the task is supported by Polya (1945) who suggests that identifying; 'What is the unknown?' 'What is the data?' and 'What is the condition?' can focus the problem-solver on the relevant information and goal. The high percentage of PSMTs not identifying what they *want* to find coincides with the high percentage of PSMTs who did not identify the goal of the problems when classifying mathematical tasks as part of Research Question 1, as seen in Section 6.1.

It was positive to find that the *I Know* element had a low percentage of 0 points being scored (12.6 7%) in the first interview and 25% in the second interview. This element was the lowest of the three elements to score zero points in both rounds of interviews. *I Know* was also the element with the highest percentage of PSMTs scoring 1 point in both interviews (42.9 % and 50%). A score of 1 point in *I Know* would constitute a PSMT making reference to what information they know from the question or what prior knowledge they have that is relevant. There is room for further improvement in the number of PSMTs stating what information the question presents and stating the relevant prior knowledge they possess, which equates to a score of 2 points. Before proceeding with a possible strategy to solve a given problem, it is important for the problem solver to have a full understanding of that problem (Schoenfeld, 1982). This is echoed by the NCTM (2000), who state that effective problem-solvers ensure that they read the problem carefully in order to understand the problem at hand. In terms of Mason's Rubric Writing approach, this understanding would align with the PSMTs demonstrating what they *know* from the information given in the question and also what they *want* to find.

Schoenfeld (1985) explains that 'good problem-solvers' are able to relate previous experiences of tasks to tasks which only appear to have minute similarities. This enables these problem-solvers to create approaches to unfamiliar problems (Schoenfeld, 1985). The relating of previous experiences to the problem at hand corresponds to the PSMTs outlining their prior knowledge as an element of *I know*. In a study conducted by Schoenfeld (1985), it was found that there was a substantial increase in students' generation of approaches to problems which were similar to ones they had previously encountered after exposure to and practice of heuristics. However, this increase in approaches through the use of heuristics was reduced when the students attempted problems that were less similar to previously experienced problems. The overall results of the present study did not represent a continuous improvement in the use of heuristics as seen in Schoenfeld's (1985) study, but this could be due to the problems which were undertaken by the PSMTs.

We will now discuss the use of heuristics on the individual problems which were attempted by the PSTMs. The problems were from a variety of topics; Problem One involved probability, Problem

Two involved trigonometry, Problem Three involved fractions, Problem Four involved ratio, proportion, and rates of change, and Problem Five involved trigonometry. A study conducted by Artzt and Armour-Thomas (1992) aimed to investigate the implementation of heuristics by students. They acknowledge that the different types of problems involved were a variable of the study and that the type of problem may have affected the problem-solving approaches that were used (Artzt & Armour-Thomas, 1992). Given that this section of the study focuses on the implementation of a taught strategy, problems from different mathematical strands were purposefully chosen to get an overall view of the PSMTs' implementation of a strategy rather than focusing on one topic. The analysis of the use of heuristics over a variety of problems is similar to a study conducted by Depaepe et al., (2013).

One particular problem did not show the highest percentage of PSMTs achieving 2 points in each of the three elements of the *Entry phase*. Problem One, which involved probability, generated the highest percentage of PSMTs (44.4%), achieving 2 points for *I want.* Problem Three, which involved fractions, showed the highest percentage of PSMTs achieving 2 points (90%) and also the lowest percentage of PSMTs achieving 0 points (10%). This demonstrates that PSMTs were proficient in identifying the information given in the question and outlining their prior knowledge in relation to the topic of fractions. Problem Four, which involved the number strand, showed the highest percentage of PSMTs achieving 0 points in both *I want* (87.5%) and *I know* (25%). This shows a particularly high percentage of PSMTs not identifying what the exact goal of the problem is.

Both Problem Two and Problem Five involve trigonometry. Problem Two was completed in the pre-interviews and Problem Five was completed in the post-interviews. Problem Five showed the highest percentage of PSMTs scoring 2 points (75%), and also the lowest percentage scoring 0 points (25%) for *Introduce*. In Problem Two, 52.6% of PSMTs achieved 2 points for *Introduce* which shows that there was an improvement between the percentage achieving 2 points between Problem Two and Problem Five. Problem solvers approaching these two trigonometry problems would benefit from introducing diagrams and notation. The increase in the introduction of diagrams is similar to the findings of a study conducted by Verschaffel et al., (1999). There was little change in the scores achieved by PSMTs in Problem Two and Problem Five for *I Know*. It is positive to see that there was a decrease in the percentage of PSMTs achieving 0 points and an increase in the percentage of PSMTs achieving 1 point. The improvement in both the *Introduce* and *I want* elements between Problem Two and Problem Five shows that there was increased implementation of the heuristics which the PSMTs had continuous exposure to and practice of during the module. This improvement corresponds to the findings of the study conducted by Depaepe et al., (2013), who found that students applied heuristics more frequently when they were engaged in a classroom environment which focused on what, how, and why to use a heuristic.

7.3.1 Summary of Findings

From the analysis of the 'Think Aloud' interviews for the implementation of *Entry Phase* (Mason et al., 2011) it was found that the PSMTs either explicitly used notation or diagrams at the start of their problem solving attempt or did not use them at all. It was disappointing to find that there was no evidence of the PSMTs stating the '*I want'* element in the majority of the problem attempts. The '*I want'* statement allows the problem solver to identify the goal of the question by carefully reading the question (NCTM, 2000) and hopefully avoid rushing into an approach which does not correspond to the actual goal of the problem (Mason et al., 2011). It was positive to see that there was a low percentage (12.67%) of PSMTs who showed no evidence of referring to their relevant prior knowledge. This indicates that the PSMTs are able to relate previous experiences to unfamiliar situations which is a characteristic of a 'good problem solver' as outlined by Schoenfeld (1985). Problem Two and Problem Five were the only problems which were based on the same topic: trigonometry. There was an improvement in the scores achieved by the PSTMs in relation to both the *Introduce* and *I want* elements indicating that there was increased implementation of the heuristics. Given that both of these questions were based on the same mathematical content, the evidence indicates that the increased application of the heuristic could be attributed to the engagement of the PSMTs in the heuristic in the module lectures and tutorials (Depaepe et al., 2013).

7.4 Research Question 3: What are pre-service teachers' capacities in relation to problem posing?

A: How do pre-service teachers' capacities in relation to problem posing change over the duration of the intervention?

7.5 B: Did the ongoing adaptations of the intervention lead to a greater enhancement of the problem-posing capacities of successive cohorts of PSMTs?

As discussed previously, problem-solving plays a central role in mathematics education, meaning that problem posing should also hold a prominent role (Christou et al., 2005). Leung (2013, p. 2) states that, 'at the heart of problem-solving often lies a great problem'.

While problem-solving has been given much attention, the consideration given to problem posing has not been proportional to its importance (Leung, 2013). In a study conducted by Ellerton (2013), it was found that most pre-service teachers preferred problem-solving to problem posing, participants felt that it was more difficult to create a problem than solve a similar problem, and most pre-service teachers expressed a want for more opportunities to create problems. It was not surprising that a strong correlation was found between students who enjoyed creating problems and students who expressed a desire for more problem posing opportunities (Ellerton, 2013). Adopting the perspective that the purpose of mathematics education is to prepare students to engage in mathematical activities, the importance of problem posing is significant (Silber & Cai, 2017).

As discussed in Section 4.10.4, activities were conducted with participants in Cohort 2 and Cohort 4 to investigate the PSMTs' capacities to pose mathematical problems. Silver and Cai (2017) conducted a study in a university setting which aimed to investigate pre-service mathematics teachers' capacity to pose mathematical problems for both an open mathematical situation (free problem posing) and a given mathematical situation (structured problem). They advocate that by engaging the pre-service teachers in problem posing they are encouraged to think about how they would pose problems for their future students (Silber & Cai, 2017). We will now discuss the results of the problem posing activities that were completed by Cohort 2 and Cohort 4 as presented in Section 6.4. The results of the three activities completed by Cohort 2 will be discussed first followed by the results of the three activities completed by Cohort 4.

7.5.1 Activity One

PSMTs in Cohort 2 completed three activities. The first activity, Activity One (see Appendix H), involved investigating the ability of the PSMTs to select a task that would be an appropriate problem for a specified student. For this activity, the PSMTs would need to consider the definition of a problem which meets the criteria of the Three Key Characteristics. The participants were given 13 scenarios (see Appendix H), which each outlined the following information: the year the student was in school, their level of study (higher or ordinary level), and their topics of prior knowledge. A mathematical task was then stated. The participants were asked to decide if the task was a problem for the described students and to justify their answer. These tasks were independently classified by the researcher and supervisor of this study.

The overall results of Activity One showed that there was a 63.5% alignment between the PSMTs' classification of the tasks and the researchers' classification. There were particular questions which contributed to this low level of alignment in classifications. The analysis of Question 3 classifications showed 39.2% alignment, and the analysis of Question 9 showed just 20.9% alignment. These two questions were classified by the researchers as *not a problem* given that there is a clear goal and it is clear how to achieve the goal with no reasoning or interpretation required. These two questions were both 'wordy' questions where they contained text and a story to provide context. However, the inclusion of words alone does not constitute a problem. This was further discussed in Section 4.9.4. Word problems are 'mathematical problems presented in the context of a story or real-life scenario' (Adams, 2003, p. 790). Within a 'word problem', the mathematical nature of the problem is not immediately apparent to the reader, and the reader must be able to identify the information given in the text to solve the problem (Adams, 2003). As seen in Chapter 6, an example of a PSMT referring to the words in the question implying a mathematical problem along with the need for a diagram is given below.

> *"The problem isn't visual enough and therefore the students would find working out a word problem such as this one, hard".*

In contrast to this, a PSMT classified Question 3 as *not a problem* and provided the following rationale;

> *"Simple substitution and factorising".*

This PSMT looked beyond the words given in the text and took into consideration the mathematical nature of the task and the mathematics required to achieve a solution, which in this instance would be considered to be procedural.

The results of Question 12 showed that there was nearly an even split in the classifications with 48.7% stating that it was a problem and 51.3% stating that it was not a problem. This question was considered a problem by both researchers while independently classifying the tasks. However, both researchers acknowledged that it could be seen as borderline. Through debate about the definition of a mathematical problem, it was decided that it met the criteria. This question was purposefully included in this activity as it required the PSMTs to think carefully about the definition of a mathematical problem, compared to only giving tasks for which it is extremely clear as to their nature. Given that problems are subjective to the problem-solver, consideration of the described student needs to be taken into account. These considerations were represented in the rationale provided by the PSMTs, as seen in Section 6.4.1.

7.5.2 Activity Two

The second activity completed by Cohort 2 used the same scenarios as the previous activity. However, the participants were asked to *create* a problem that would be suitable for the hypothetical student (see Appendix I). The PSMTs were given the year of study of the hypothetical student and the topics of prior knowledge. To be categorised as a mathematical problem the Three Key Characteristics were considered. As part of the module, the PSMTs had experience of looking at the concept of what is meant by a mathematical problem. The tasks which did not meet these criteria were subsequently analysed.

Overall, there was a low success rate, with an overall average of 13.6% success in posing a mathematical problem. The highest success rate was seen in Question 1, with a success rate of 38.5%. However, there was a 0% success rate for both Question 8 and Question 9. The tasks which were classified as *Not Achieved* were subsequently analysed. This led to the identification of three categories of tasks; *procedural, not solvable*, and *not solvable due to missing information.*

The category referring to missing information aligns with problems which would be categorised as *ill-formulated problems* (Schoenfeld, 1987). The category *not solvable* refers to the generated tasks that were impossible to solve for any reason. Examples which were seen in the PSMTs' posed problems of impossible tasks were tasks which were mathematically impossible, real-life contexts were not considered, or the wording did not make sense. Leung (2013) states that posed problems can be impossible or insufficiently specified. The problems posed by the PSMTs in the *not solvable* category were illustrative of the problems described by Leung (2013).

The tasks which were classified as *procedural* accounted for the majority of the posed tasks in each question, with the exception of Question 1. The criteria for categorising tasks as *procedural* was that there was an immediate clear path on how to reach a solution and/or could be solved using previously learnt procedures. The overwhelming majority of procedural tasks align with the findings of a study conducted by Crespo (2003). In that study, Crespo (2003) aimed to investigate Canadian pre-service primary school teachers' problem posing capacities. This was conducted in a university whereby a course which exclusively focused on mathematical problem posing ran twice a week for 11 weeks. It was found that at the start of the programme, the tasks posed were straightforward and could be translated with ease into a computational task (Crespo, 2003). The difficulties encountered by PSMTs in posing problems are also reported on by Isik and Kar (2012), who found that the problems posed relating to equations were simple in nature. This is not unique to pre-service teachers: in-service teachers also have a tendency to decrease the cognitive demand of tasks that they pose to their students (Stein et al., 1996).

In a study conducted by Silber and Cai (2017), the problem posing of pre-service teachers was investigated relating to free and structured mathematical problem-posing conditions. The participants of this study were pre-service elementary teachers who completed written activities on problem posing. *Free posing* situations involve posing a problem for an open mathematical situation. In contrast, *structured posing* situations involve posing a mathematical problem based on a predetermined situation and operations (Stoyanova & Ellerton, 1996). The written problems were analysed in three stages. Firstly, the problems were identified as 'non-math questions, math questions, and statements' (Silber & Cai, 2017, p.172). Next, the 'math questions' were identified as solvable or non-solvable. Lastly, the problems in the 'solvable' category were analysed based on mathematical complexity. The category 'non-solvable' aligns with the category *Not solvable* which was identified in this study. Silber & Cai (2017) found that nearly all problems posed by the pre-service teachers were solvable mathematical problems. This is a contrast to the results found in this present study which showed a low percentage of PSMTs being successful in posing a problem. The PSMTs were given specific conditions for which they should consider when posing a problem, making Activity Two a semi-structured posing situation as defined by Silber and Cai (2017). A semi-structured posing situation is a situation where the person is given a specific situation for which they base their posing of a mathematical problem on (Christou et al., 2005).

7.5.3 Activity Three

The third activity focused on the reformulation of existing problems (see Appendix J). This activity first involved the participants attempting to create a problem from an exercise. Next, participants were given problems and asked to reformulate them so that *specified* prior knowledge would be needed to solve the problem. Finally, the participants were asked to reformulate a mathematical task to make an open-ended problem. This activity was completed by 55 participants in Cohort 2. Activity Three aligns with a *structured problem posing situation* whereby participants are asked to reformulate or adapt given problems (Christou et al., 2005). The problems posed by the PSMTs were analysed and categorised as *achieved* or *not achieved*. For a posed problem to be categorised as *achieved*, the problem needed to align with the Three Key Characteristics and also have sufficient information for a problem-solver to solve it (Leung & Silver, 1997). The problems which were categorised as *not achieved* were further analysed and classified into the following categories;

- *Unsolvable:* The mathematical task posed was not possible to solve, or there was an unhelpful ambiguity[2] as to what the question was asking due to the poor phrasing of the task.

[2] An 'unhelpful ambiguity' is uncertainty in the goal of the question due to missing information which differs from an open problem where different interpretations and multiple solutions are possible.

- *Not a problem:* The task posed did not meet the criteria of the Three Key Characteristics of a mathematical problem.
- *Unsolvable due to missing information:* The task posed did not contain enough information to allow a problem-solver to solve it.
- *Did not address the topic:* The mathematical task did not address the specified topic required in the given question.
- *Not open:* The task posed did not meet the criteria of being open.

The category *not open* was only applicable to Question 5 and *did not address the topic* was only applicable to Question 2a) and Question 2b). Problems categorised as *not solvable* align with the definition provided by Leung (2013).

From the analysis of the results outlined in Section 6.4.3, it was seen that Question 1 yielded the highest success rate. This involved reformulating a given mathematical task into a mathematical problem. This generation of a problem from a given situation is similar to a study conducted by Leung and Silver (1997) who investigated prospective elementary teachers' ability to pose an arithmetic problem from a presented story. It was found by O'Sullivan (2017) that Irish mathematics textbooks contain very few problems, and most tasks are routine. In a report conducted by Jeffes et. al., (2013), it was found that there is a heavy reliance on textbooks by teachers in Ireland. Given this lack of easily accessible mathematical problems in textbooks, it is essential for teachers to be able to reformulate exercises as a way of including problem-solving in their classrooms. It was found that while the reformulation of an exercise into a mathematical problem had the highest success rate, this stood at just 41.9%. 27.9% of the tasks posed did not meet the criteria of a mathematical problem as there was an immediate path on how to proceed with the task posed. This shows that PSMTs demonstrated that they were not competent in reformulating a given exercise into a problem which would need greater reasoning and understanding to make progress. The remainder of the posed tasks were not possible to solve for reasons such as missing key information, the phrasing of the task, or the mathematical nature of the task.

The analysis of Question 2a) and Question 2b) generated the inclusion of the *did not address the topic* category, as PSMTs were asked to reformulate given tasks to involve a specified topic. The majority of problems posed in both of these questions were found to be impossible to solve. This was due to either insufficient information given in the task, or the information given in the posed problem being mathematically incorrect, meaning a plausible task was not formed. Research into the generation of problems for specific topics has been conducted for a range of mathematical topics (Greer & McCann, 1991; Leung & Silver, 1997). There was a greater level of success in Question 2a) which required the topic of fractions (31.7%) than there was for Question 2b) which more specifically required the use of division of fractions (18.5%). There was an increase in the number of problems posed that did not address the specified topic, which shows that PSMTs may have difficulty generating a problem for a particular part of a mathematical topic than a broader topic for which they have greater experience. In a study conducted by Toluk-Ucar (2009), it was found that while pre-service teachers were competent and confident in their knowledge of and use of fractions, the problems that they posed contained misconceptions and inaccuracies. The relatively low level of success of the PSMTs in this study in posing problems specifically about fractions may signal that there is a lack of understanding of this topic.

Question 4 specifically required the PSMTs to generate an open-ended problem. This question had the lowest level of success, with just 8% successfully posing an open-ended problem. An open-ended problem is defined as a mathematical problem which does not have a single answer but a range of solutions which can all be justified (Silver, 1994). It is argued that open-ended problems can provide students with important experience in interpreting problems and generating multiple solutions (Foong, 2000; Silver, 1994). The PSMTs were given a mathematical task for which they were asked to generate an open-ended problem. Overall, 56% of the problems were not solvable with 24% of these problems not having sufficient information to be solved. There was a high proportion of the posed problems that did not meet the criteria of being open at 32%. This result was disappointing, with just 8% of the problems being classified as an open-ended problem. A study conducted by Nicol and Bragg (2009) investigated pre-service teachers' ability to design open-ended problems. This was done in two universities, one in Canada and the other in Australia, during mathematics education courses. In a survey completed by the participants, it was found that pre-service teachers found designing open-ended problems difficult but despite this, the analysis of the problems posed, 97% of the problems were open-ended in nature. The participants of that study did not have previous explicit training in problem posing at the point of this data collection. This result is drastically different to the result of the PSMTs in this study who had experience of extending mathematical problems and had received instruction on the characteristics of different mathematical problems. However, the participants in the study conducted by Nicol and Bragg (2009) focused specifically on open-ended problems, with the participants spending time studying open-ended problem posing. They state that despite the initial uncertainty expressed by the pre-service teachers, creating open-ended problems is achievable for pre-service teachers (Nicol & Bragg, 2009).

7.5.4 Extension Task

The PSMTs in Cohort 4 completed three extensions of mathematical problems. As described in Section 4.10.4, the PSMTs in Cohort 4 completed a mathematical problem on three different occasions (Appendix M). After their problem-solving attempt, they were explicitly asked to generate an extension of the problem. The PSMTs had previous exposure to the idea and practice of extending mathematical problems from the Rubric Writing approach (Mason et al., 2011). In the time between each extension task, the PSMTs had increased experience of using the Rubric Writing approach (Mason et al., 2011) as outlined in Section 5.5.2.

It is argued by Prestage and Perks (2007) that pre-service teachers can gain an appreciation of the connections that can be made between different mathematical topics through the multiple possibilities that exist when adapting and extending a problem. Problem posing can occur in different ways, one of which is the generation of a problem after solving or attempting to solve a problem (Silver, 1994). This extension of a given problem can be seen in heuristics such as Polya (1945), who advocates the 'Looking back' phase of problem-solving.

Similarly, it is seen in the Rubric Writing approach developed by Mason et al., (2011) that extension plays an important role. This Rubric Writing approach was prominent in the module undertaken by the PSMTs. There are three stages in the *Review* phase, one of which is *Extend.* The extending of the given problem may be through questioning or changing the form of the given problem and asking ‘What if…?’ (Mason et al., 2011, p. 39). Another form of extending problems can be through removing assumptions. There is not one single possible extension of an interesting problem, but there are usually several extensions that would broaden the scope of the problem. It is argued that the extending of problems promotes mathematical thinking as it engages the person in the question (Mason et al., 2011). The changing of the givens in a problem can offer the experience of different approaches to mathematics and make links between mathematical topics (Prestage & Perks, 2007).

From the analysis of the extension problems completed by Cohort 4, four categories were identified: *unsolvable*, *achieved*, *measurement change*, and *procedural.* Problems classified as *unsolvable* are ill-formulated (Schoenfeld, 1987), due to being impossible to solve or providing insufficient information to enable a solution to be found (Leung, 2013). Problems in the *measurement change* category refer to the extension problems posed by the PSMTs which were the same as the given problem, but the measurement or numbers had been the only change to the original problem. *Achieved* corresponds to problems which met the criteria of the Three Key Characteristics, and *procedural* refers to posed tasks that had an immediate clear path to reach a solution.

The analysis of the results showed that the majority of the problems posed were classified as *not solvable* (P1 = 50%, P2 = 39.4%, P3 = 59.1%). This is a disappointing result as there was an increase in the percentage of problems posed by the PSMTs which were classified as *not solvable* between Problem One and Problem Three. It is apparent that the PSMTs’ problem posing proficiency, in terms of generating solvable mathematical tasks, did not improve between Problem One and Problem Three despite a decrease in *not solvable* problems seen in Problem Two. It was seen that Problem Two had the highest number of problems classified as *measurement change* with a score of 24.2%. This problem involved creating equations involving distance and time. Problem One showed the highest success rate at 41.7% and the lowest level of problems classified as *measurement change*. This problem involved the topic of area and PSMTs used a variety of shapes for their extension problems. In contrast to Problem Two, the PSMTs demonstrated greater proficiency in posing problems for the topic of area rather than the number strand of mathematics.

A possible reason for this may be a difference in understanding of the mathematical concepts as a solid grasp of the mathematical concepts in a particular area is required in order to pose a non-trivial solvable problem (Ellerton, 2013).

Research Question 3B: Did the ongoing adaptations of the intervention lead to a greater enhancement of the problem-posing capacities of successive cohorts of PSMTs?

The results of the analysis of the tasks posed by the PSMTs in Cohort 2 in Activity Two and Activity Three showed that the PSMTs experienced difficulty in posing genuine mathematical problems. Prior to completing these activities, the PSMTs in Cohort 2 had experienced five weeks of instruction and practice of problem-solving, and extending as part of the Rubric Writing approach (Mason et al., 2010), mathematical problems. The PSMTs had received specific instruction regarding the nature of mathematical problems. Additionally, the PSMTs had completed the ‘Task Sorting’ activity which involved classifying mathematical tasks and subsequently participated in a discussion regarding the nature of these tasks. Despite this, it was evident that the PSMTs posed tasks which were procedural or unsolvable.

PSMTs in Cohort 4 were asked to pose a problem, based on a problem they had just attempted, in Week Two, Week Six, and Week Ten of the module. Recognising the difficulties demonstrated by Cohort 2, adaptations were made to the module for Cohort 4 specifically focusing on problem posing. Firstly, overall the PSMTs in Cohort 4 experienced seven weeks of problem-solving prior to the ultimate problem posing activity whereas Cohort 2 experienced five weeks.

In Activity Two completed by Cohort 2, the majority of tasks posed, with the exception of Question 1, were classified as procedural. This suggests that the PSMTs did not fully grasp the meaning of a mathematical problem. This prompted the inclusion of additional lectures focusing on the nature of a mathematical problem with Cohort 4. Prior to the completion of Problem Two, Cohort 4 had experienced four weeks of problem-solving, including one week focusing on the classification of mathematical tasks. Within this time, the PSMTs had also completed the adapted Task Sorting activity which required the inclusion of a rationale for classification before discussion. The tasks posed by Cohort 4 in relation to Problem Two, showed a lower rate of procedural tasks (18.2%) than those posed by Cohort Two in Activity Two.

Further adaptations were made to the module for Cohort 4 based on the results of Cohort 2. Prior to completion of the extension of Problem Three, Cohort 4 had experienced seven weeks of instruction on problem-solving, two weeks more than Cohort 2 experienced. Within these additional weeks, Cohort 4 received instruction on the nature of problems in relation to the level of cognitive demand, the syntax of the question, open or closed type of questions, and the mathematical structures in the question. Cohort 4 also received instruction on selecting tasks specifically for particular teaching situations. Within these additional instruction periods, the PSMTs had the opportunity to analyse mathematical tasks. These additions were made in the module in view to further the PSMTs' understanding of a mathematical problem in response to the challenges encountered by Cohort 2 to pose mathematical problems rather than procedural tasks. While it may be seen as a positive that there was a decrease in the procedural tasks posed in Problem Three (4.5%), the lowest across Cohort 2 and Cohort 4, there was a high number of posed tasks classified as unsolvable (59.1%). This suggests that the changes to the module had a positive impact on reducing the procedural tasks posed by the PSMTs but also highlights that the PSMTs need to further develop their ability to pose mathematical problems that are solvable.

7.5.5 Summary of Findings

Activity One demonstrated that there was a considerable level of disagreement between the PSMTs' classification of the tasks and the researchers' classifications. Two questions in particular were influential in lowering the level of alignment to 63.5% and these two questions were both deemed to be 'wordy' questions. As discussed in Section 4.10.4 the inclusion of words in a task does not constitute a problem but the criteria of a mathematical problem must still be considered (Adams, 2003).

Focussing on the PSMTs' capacity to pose problems, it was clear from Activity Two and Activity Three that the PSMTs had difficulty in doing so. Activity Two was developed so that the PSMTs would pose problems in a semi-structured manner (Silber & Cai, 2017) and Activity Three was a structured problem posing situation (Christou et al., 2005). For the posed tasks to be considered as *achieved* the problem needed to align with the Three Key Characteristics and also have sufficient information for a problem solver to solve it (Leung & Silver, 1997). There was a very low level of success (*achieved* = 13.6%) in Activity Two. The tasks were classified as 'not achieved' for the following reasons: *procedural* (Crespo, 2003), *not solvable* (Leung, 2013; Silber & Cai, 2017), and *missing information* (Schoenfeld, 1987). The majority of the tasks posed were considered to be *procedural* which aligned with the findings of Crespo (2003). The difficulty that the PSMTs demonstrated in posing mathematical problems is not unique to this study (Isik & Kar, 2012) and is also experienced by in-service mathematics teachers (Stein et al., 1996). Activity Three was centred around the reformulation of mathematical tasks to problems. Question 1 showed the highest level of success with 41.9% of responses considered a problem indicating that while the majority of the PSMTs were not competent in reformulating an exercise into a problem, there was some evidence of success. There was a low level of success of the PSMTs in this study posing problems specifically about fractions which may signal that there is a lack of understanding of fractions (Toluk-Uçar, 2009). The question which had the overall lowest level of achievement was Question 4 which asked the PSMTs to pose an open-ended problem (8%). A considerably higher level of achievement was experienced in the 'extension' activity completed by Cohort 4 (P1 =50%, P2 =39.4%, P3= 59.1%). Although they are different cohorts of participants, it is obvious that there is a far greater level of achievement in posing extension problems than posing problems in semi-structured or structured situations. A potential reason for this is that the PSMTs who completed the extension problems had completed a problem and had spent time working on and exploring the mathematical content. According to Ellerton (2013) in order to pose a non-trivial solvable problem, a solid grasp of the mathematical concepts in a particular area is required.

7.6 Research Question 4: What beliefs and affective factors do pre-service teachers hold regarding problem solving?

7.6.1 Indiana Mathematics Belief scale

To investigate the beliefs and affective factors that the PSMTs hold regarding problem-solving, both quantitative and qualitative instruments were utilised. PSMTs from each cohort completed the Indiana Mathematics Belief scale, which was designed by Kloosterman & Stage (1992), as described in 4.10.5. The qualitative element was conducted with Cohort 4, whereby PSMTs were asked to describe how they felt at each stage of their problem-solving attempt: namely the start of the problem, while making progress or stuck, and at the end of the problem. This was done on three different occasions as the PSMTs undertook the problems described in Section 4.10.5. Firstly, the results of the IMB will be discussed, followed by the qualitative element.

The combined results of the four cohorts of the IMB showed that the scale with the highest mean score was *Effort* with a score of 24.85/30. This scale had the highest mean score in Cohorts 1, 2, and 4 and was second highest in Cohort 3. This shows that participants positively agree that effort and working hard can have a positive impact on mathematical ability (Kloosterman & Stage, 1992). This finding is in line with the findings of a study conducted by Prendergast et al., (2018) which involved investigating Irish secondary school students' beliefs about mathematical problem-solving through the implementation of the IMB scale. In this study (Prendergast et al., 2018), the IMB was distributed to nine secondary schools with a total of 975 questionnaires completed and returned. It is interesting to note that the results of the study conducted by Prendergast et al. (2018) could be seen to be representative of the same group of participants that were involved in this study since participants in this study all completed post-primary education in Ireland of which many of the PSMTs had completed months prior to entering the ITE programme. . Similar to the results found in this study, the scale with the highest mean (24.035) was the *Effort* scale. This is a positive finding as there is an implication that participants demonstrate aspects of a growth mindset. Dweck (2008) states that students who have a growth mindset are at a significant advantage to students who are of a fixed mindset. In research conducted by Dweck (2008, p. 4), it was found that students with a growth mindset cared more about learning and also demonstrated a greater belief in the influence of effort on their grades than students with a fixed mindset. Similarly, it was found that those with a growth mindset reacted in a more positive manner to setbacks than those with a fixed mindset (Dweck, 2008, p. 4).

For the PSMTs in the present study, the scale which had the lowest overall mean was *Steps*, with a score of 16.645/30. This scale had the lowest mean score in each of the four cohorts of participants. This result is of concern as it is indicative of the belief that rote learning and procedures are adequate to solve mathematical problems. This lowest mean aligns with the lowest mean found in other studies (Kloosterman & Stage, 1992; L. Mason, 2003; Prendergast et al., 2018).

Jonassen (1997) distinguishes between ‘well-structured problems’ and ‘ill- structured problems’ and the relevance that they have in their application to solving problems in everyday life. Ill-structured problems are problems which are typically not convergent (one single known solution) with predictable solutions. They are problems for which several topics are interwoven and are more representative of problems encountered in everyday life and are not constrained by the content studied in classrooms. In contrast to this, well-structured problems are based on specific content studied in classrooms which require the application of familiar concepts and rules to solve them. These problems have a single solution and are written in a way that is familiar to the learner (Jonassen, 1997). He maintains that well-structured procedural style tasks have little relevance to solving problems in everyday life. This shows that the PSMTs’ low score in this scale is of concern as procedural tasks can negatively influence students’ ability to problem-solve (Jonassen, 1997). Boalar (2015) describes the difference in students’ problem-solving performance who learnt mathematical facts through memorization or other strategies. It was found that students who learnt through other strategies performed significantly better than the students who memorised the facts. Although both sets of students solved the problems at the same speed, the strategy group of students demonstrated an increased ability in transferring knowledge to new problems.

While it has been demonstrated above why the low score in the *Steps* scale is of concern, it is also important to try to understand the factors that may influence this result. As outlined in Section 2.2.1, Irish students’ scores in the PISA (2012) assessment were higher in applying procedures than in translating real-world problems into mathematical representations. This result is mirrored by the results of the TIMSS (2015) test which showed that Irish students were more proficient in tasks which required replicating previously learnt procedures, using memorised facts, and/or using information from tables or charts. These higher scores in the procedural elements of mathematics coincided with relatively low scores in tasks which required problem-solving skills (Shiel & Kelleher, 2017). In addition to this, an analysis of post-primary mathematics textbooks showed that the tasks which students would have exposure to lacked novelty and mathematical communication (O’Sullivan, 2017). This shows that the PSMTs would have experience of mostly procedural mathematical tasks with less exposure to novel mathematical problems leading to increased ability in applying previously learnt steps and facts. Given this increased ability in using memorised steps and experience of procedural tasks, the PSMTs may understand that these steps are sufficient to solve a mathematical task but not be aware of the actual nature of a mathematical problem. As discussed in Section 7.1, there may be an underlying misconception among the PSMTs' understanding of the term 'word problem' with the inclusion of text compared with genuine mathematical problems. This potential confusion may indicate that the PSMTs' do not fully understand the genuine meaning of a mathematical problem.

The scale of *Understanding* had the second highest mean (20.625) overall, was second highest in Cohorts 3 and 4, and third highest mean in Cohorts 1 and 2. A high mean in this scale is indicative of importance placed on conceptual understanding, which incorporates understanding why an algorithm or a procedure works in the present context (Kloosterman & Stage, 1992). It is interesting to note that the *Understanding* scale scored significantly higher than the *Steps* scale meaning that there is a disparity between the PSMTs' views that understanding of concepts is important and that problems can be solved using step-by-step procedures. The PSMTs demonstrated a high score on the scale of *Understanding* which highlights their belief that understanding a concept is more important than following methods or implementing procedures without making connections to the context of the question. However, the low mean score for the *Steps* scale demonstrates the belief that previously learnt formulae or procedures are sufficient for solving problems. This shows that there is a contradiction between the PSMTs' beliefs about the necessity to understand mathematical concepts and the role of step-by-step procedures in problem-solving.

The mean score of *Understanding* in this study is lower than the mean scores found by Prendergast et al., (2018) and by Mason (2003) with scores of 22.71 and 24.82 respectively. The study conducted by Mason (2003) focused on using the IMB scale to investigate Italian high-school students' (n = 599) beliefs about mathematical problem-solving. Prendergast et al., (2018) found that Junior Cycle students demonstrated stronger levels of agreement than Senior Cycle students in the importance of understanding concepts. The authors hypothesise that this could be due to an increased focus on obtaining a correct answer in examinations. The PSMTs in this study had all completed the post-primary education system in Ireland and therefore had completed the Leaving

Certificate examinations. There is a potential that this answer-focused examination has contributed to lesser importance being placed on conceptual understanding. While the Project Maths curriculum, which was experienced by all of the PSMTs in this study, places emphasis on problem-solving skills, there still appears to be room for improvement in the PSMTs' opinions on the importance of understanding mathematical concepts.

The scale *Difficult problems* involves assessing the PSMTs' beliefs about their ability to solve time consuming problems. The overall mean result of this scale (19.025) was lower than the results of the studies of Mason (2003) and Prendergast et al., (2018). It is interesting to note that the low mean of Cohort 4 in this scale, 13.37, negatively influenced the overall mean. This is concerning as it indicates that participants in Cohort 4 had a very negative perception of their ability to solve time- consuming problems. A possible contributor to this may be the examination time pressures that the PSMTs experienced in the post-primary education system in Ireland. Schoenfeld (1988) notes that if students have only experienced mathematical tasks which take a few minutes to complete, then it is likely that when faced with problems that promote mathematical thinking and take time, students will believe that they cannot reach a solution. As seen in the literature, Irish post-primary mathematics textbooks (O'Sullivan, 2017) do not contain many opportunities for students to experience time consuming problems so there is a potential that this lack of exposure increases PSMTs' beliefs that they cannot solve problems which take a considerable amount of time. In a study conducted by Schoenfeld (1985) involving over 200 high school students, it was found that the average amount of time that students felt should be spent on a homework problem was 2.2 minutes, and if a problem took longer than approximately 12 minutes, then that problem would be deemed impossible to solve. While much has changed in mathematics education from this study to the present, it is evident that a negative association between time and ability to solve problems spans a long timeframe. This focus on speed is discussed by Boalar (2015), who states that the idea that students who are fast at maths are strong in maths is a damaging misconception. She further describes how timed tests negatively impact students' ability to perform through the reduction in the ability of the working memory section of the brain (Boaler, 2015). This can lead to anxiety occurring and ultimately cause students to develop a negative opinion of mathematics (Boaler, 2015). This combination of timed tests, such as the State examinations, along with an exam focused classroom philosophy raises the question as to whether the current system in Ireland is impacting students' perception of their ability to solve time consuming problems. This feature also arises later in the analysis of the qualitative element, which we will further discuss.

The final scale that we will now discuss is *Word problems*. This scale assesses the PSMTs' beliefs about the importance of word problems and the role of computational skills. The results of this scale showed that *Word problems* had the second lowest mean score of 18.325. Mathematical tasks can be in a variety of forms such as simple equations or 'word problems' (Hsu, 2013). These tasks can all be classified based on the level of cognitive demand that they require (Silver et al., 2000). Tasks of low cognitive demand require memorisation, or the use of a rote learnt procedure to achieve a successful solution. These tasks involve the use of computational skills and focus on the recall of facts. High cognitive demand tasks involve a deeper understanding of mathematical concepts in order to process more complex information. These latter tasks allow students to explore and interpret mathematical concepts, whereas low cognitive demand tasks do not provoke this level of thought in students (Hsu, 2013). However, if teachers implement the tasks of higher cognitive demand through using closed questions with fixed answers, then students miss the opportunity to explore the task themselves (Hsu, 2013). The results of the IMB scale in this study are concerning given that the overall mean score was 18.325 out of a maximum score of 30. This indicates that the PSMTs do not place significant importance on word problems compared to computational skills. This may indicate that the PSMTs may subsequently reduce their students' opportunities to experience tasks of higher cognitive demand. In the development of the IMB scale, it was noted that there was a low reliability of the *Word problem* scale compared to the other scales, possibly due to the inconsistent understanding of the term '*Word problem'* (Kloosterman & Stage, 1992).

7.6.2 Qualitative study of the affective domain:

As previously mentioned, there was a qualitative element completed by Cohort 4 as part of this research question. This involved the PSMTs writing how they felt after attempting a mathematical problem as outlined in Section 4.10.5. These results will now be discussed. The data was analysed in two ways; we analysed the statements individually, and classified them using inductive analysis. We then classified the set of statements made by each student, and categorised each set. So 'statements' are categorised as in Section 6.5.2, and 'PSMTs' are likewise categorised.

Lewis (2017) conducted a study that aimed to investigate pre-service primary teachers' emotions while performing a mathematical problem. The participants of this study were undertaking a course involving the affective domain and problem solving as part of their teacher education programme. Part of this course involved completing a questionnaire about their affective disposition before and after attempting mathematical problems. A focus group of participants completed the research element of this study. The focus group was shown a mathematical task and then asked to complete a questionnaire which involved: grading the difficulty of the tasks on a 5-point scale from easy to difficult, how they felt about the task, what they were thinking and why (Lewis, 2017, p. 2). On completion of the mathematical task, they completed a second questionnaire outlining how well they thought their problem-solving attempt had gone, asked to describe their emotions, thoughts and feelings throughout the problem-solving process, and then specifically asked about their most negative emotion and how they responded to this emotion (Lewis, 2017).

With the aim of investigating the affective domain of students while solving mathematical problems, DeBellis and Goldin (2006) conducted recorded interviews with individual students across two years. Each interview consisted of a student attempting several mathematical tasks including non-routine problems. The analysis of the videos included inferring emotions from facial movements using predefined codes and then aligning the spoken word in the transcripts with the codes (DeBellis & Goldin, 2006).

The PSMTs' responses at the start of the problem showed that there was a higher level of positivity than negativity. The number of positive statements was comparable to the number of respondents categorised as positive meaning that it was not the case that few PSMTs made a lot of positive statements. This is a positive finding as PSMTs stated feelings such as; *Confidence, Feeling Calm, Excitement, Competent,* and *Optimism*. These positive feelings align with the positive feelings identified by Lewis (2017) who conducted a study investigating the emotions of prospective primary teachers associated with the performance of a problem-solving task. Lewis (2017) found that there was an association between positive feelings expressed at the start and the perception of the task being easy to solve. Similarly, the categorisation of the feelings expressed by the PSMTs as positive are consistent with the positive feelings outlined by Felmer and Perdomo- Diaz (2016). Felmer and Perdomo- Diaz (2016) conducted a study investigating how novice secondary mathematics teachers felt while problem-solving. Their findings showed that overall, the majority of the participants were more positive than negative. Positive feelings such as those mentioned above demonstrate that the PSMTs are not restricting their access to cognitive resources which would aid their problem-solving attempt (Schoenfeld, 1983). Similarly, positive feelings motivate the problem solver to engage with and explore the problem (DeBellis & Goldin, 2006).

While the majority of PSMTs were positive at the start of the problems, some displayed a negative disposition. Statements such as; *Feeling Overwhelmed, Frustrated, Anxious, Panic,* and *Clueless*, were evident in the PSMTs' responses. The categorisation of the feelings expressed by the PSMTs as negative are consistent with the negative feelings outlined by Felmer and Perdomo- Diaz (2016). While negative beliefs can impede a person's ability to access knowledge (Schoenfeld, 1983), *frustration* can lead to two different outcomes. Frustration can cause a problem-solver to change their approach which can ultimately be a positive outcome towards achieving the goal of the problem (DeBellis & Goldin, 2006). However, frustration can also develop into anxiety which can lead to avoidance in continuation with the problem (DeBellis & Goldin, 2006). The feeling of *anxiousness* signals towards anxiety which can have an effect on mathematical performance through negative implications on attention and motivation (Campbell, 2004). Along with this, high levels of anxiety can cause a person to become *overwhelmed* as they try to acquire information (Campbell, 2004).

In relation to the middle of the problem attempt, the PSMTs were explicitly asked to describe how they felt if they were *making progress*. The PSMTs were given the opportunity to write about *making progress* but this was not a compulsory activity. The majority of statements were classified as positive, with PSMTs making statements about confidence, reward, and feeling good with a clear path forward. McLeod (1988) identifies that positive reactions are synonymous with making progress in their attempt and are particularly evident when achieving an 'Aha!' moment. The

PSMTs' demonstration of confidence when making progress is not a surprise as making progress in problems develops confidence (Mason et al., 2011).

Additionally, the PSMTs were prompted to describe how they felt if they were *Stuck* in the middle of the problem. As with *making progress*, the PSMTs were offered to write about being *stuck* as an option but it was not compulsory. There were a total of 49 statements which referred to being stuck, of which 22 were negative statements and 11 were positive statements. The remainder of the statements were classified as neutral statements, as they displayed neither positive nor negative disposition. The statements which were classified as positive include statements such as staying calm, looking at the question again, and putting emotions aside to think. These statements demonstrate that some of the PSMTs had a positive attitude towards their ability to overcome the difficulty. This is a positive finding as overcoming being stuck builds up a reserve of confidence that can help in future situations where the person is stuck again (Mason et al., 2011). However, the majority of the responses to being *stuck* indicated a negative disposition with double the number of negative to positive statements. In particular, frustration was the most common response, which aligns with the work of McLeod (1988). McLeod (1988) highlights that negative reactions to getting stuck can cause a person to quit, which is an area of concern given the majority of PSMTs' responses to being *stuck* were negative. However, the PSMTs were aware of their emotions which they provided in a written response meaning that this awareness of emotion could enable them to control these emotions to overcome the frustration (McLeod, 1988). The benefit of the recognition of being stuck allows the person the opportunity to gain control of emotions by accepting the situation which can then allow them to progress (Mason et al., 2011).

The analysis of the statements made referring to the end of the problem showed that the majority of statements related to achieving or not achieving an answer. The PSMTs specifically referred to finding 'an answer' to the question rather than achieving a full rich solution. From the 150 responses regarding the end of each of the three problems, 84/150 referred to achieving or not achieving an 'answer'. Of these 84 answer-focused responses, 18 were of a positive nature with PSMTs stating; happiness, satisfaction, or feeling good for reaching a solution. There was a greater number of negative statements with 31 negative comments in relation to not being able to achieve an answer. The negative statements included frustration, disappointment, dissatisfaction, and annoyance at not achieving a solution. The correspondence between failure to reach a solution and negative emotions is common (McLeod, 1988). Similar to the control of emotion that is necessary when stuck, the control of positive emotions when reaching a solution is discussed by McLeod (1988). He highlights that the reaction to positive emotions upon reaching a solution can indicate completion of the problem and deter the person in reviewing their work and finding alternative methods for verification.

It is evident that the PSMTs expressed positive emotions mostly in relation to *making progress* and achieving an answer. This corresponds to the findings of a study conducted by Lewis (2017, p.4) who states that 'making progress and getting right answers are seen as a vital condition of satisfaction'. However, it is important to note that there were cases where PSMTs displayed positive emotions in relation to their problem-solving process. This included PSMTs referring to satisfaction with their approach and working to the best of their ability. When focusing on the number of PSMTs who commented on their approach, there was greater evidence of a positive process rather than a negative process. Forty out of the 150 respondents referred to the problem-solving process, with just four of these 40 responses reporting negative feelings towards their problem-solving process. On the other hand, 44 out of 150 responses reported negative feelings toward achieving an answer. This shows that there may be a stronger negative association with not achieving an answer than there is with the problem-solving process. We also note that the number of participants that referred to the problem-solving process declined between each of the problems. Simultaneously, there was an increase in the number of participants who referred to achieving or not achieving an answer.

From the results of the IMB the scale, *Understanding* had the second highest overall mean with a score of 20.61/30. A high score in this scale demonstrates that there is a greater value placed on understanding a mathematical concept rather than on achieving a correct answer. This involves the understanding of why an answer is correct and how a procedure works (Kloosterman & Stage, 1992). This suggests that the PSMTs place a greater importance on understanding than on achieving an answer. This is in contrast to the results of the qualitative data which showed that there was a greater focus on achieving an answer than on the process. A potential reason for this is participants reporting what they believe they should say rather than their actual behaviours. Lerman et al., (2002) outlines that there is a gap between what people say they do and what they actually do in practice. This gap between theory and practice appears to be a 'global phenomenon' (Maben et al., 2006), meaning that it is not specific to the PSMTs in this situation.

The first scale of the IMB involved identifying PSMTs' beliefs about their ability to solve time consuming problems. Relatively high mean scores were seen in Cohort 1 and Cohort 2 (22.9/30 and 22.7/30 respectively), which demonstrates that these PSMTs expressed confidence in their ability to solve time-consuming problems. There was a decline in the mean scores for Cohort 3 and Cohort 4 (17.1/30 and 13.4/30 respectively). Focusing on Cohort 4 specifically, despite the low mean score in the IMB, there was evidence in the analysis of the PSMTs' responses of confidence in their ability to solve time-consuming problems. One such statement was;

> "*I felt if I had a bit more time I would have been happier with my work*"

Time related statements occurred 13 times in response to the end of the problems. All of the statements express the PSMTs' belief that they could have improved their attempt if they had more time. While this does not negate the fact that Cohort 4 had the lowest mean score in this scale in the IMB, there is evidence that PSMTs in Cohort 4 have confidence in their ability to solve time-consuming problems.

Research Question 4A: How does the affective domain of one cohort of PSMTs change over the course of the final iteration of the intervention?

PSMTs in all four cohorts completed the IMB (Kloosterman & Stage, 1992) at the start of each iteration of the module. Three problems attempted individually by Cohort 4, in Week Two, Week Six, and Week Ten asked the PSMTs to describe how they felt at three different stages of their problem-solving attempt, namely; the start of the problem, the middle of the problem, and the end of the problem. The students were prompted to refer to how they felt if they were stuck and if they were making progress (see Appendix K). This was done at three different points during the module while participants worked on three different mathematical problems in Week Two, Week Six, and Week Ten. In the time period between the first problem and the third problem, the PSMTs experienced eight weeks of instruction and practice of problem-solving. The mathematical mindset course (Boalar, 2016) which was completed by all cohorts, and the qualitative open-ended questions which was only completed by Cohort 4, could have increased their awareness of their own emotions (Felmer & Perdomo-Diaz, 2016) The PSMTs' responses were analysed in two different ways: the first being the categorisation of respondents at each of the three stages of the problem, and secondly, each statement was categorised for each of the three stages.

From the categorisation of respondents it was positive to see that there was an increase in the positive respondents at the start of the problems from the first problem to the third problem (16-11-20) (that is, for the first problem-solving activity, 16 respondents were categorised as positive, 11 for the second activity and 20 for the third). This coincided with a decrease in the number of negative respondents across the three problems (16-14-11). This suggests that the PSMTs' opinions on their ability to approach the start of a problem positively changed over the duration of the module. This indicates that the PSMTs were more motivated to engage with the problem (DeBellis & Goldin, 2006) and increased their access to cognitive resources (Schoenfeld, 1983).

Focusing on the middle of the problems, there was a decrease in the number of positive respondents (16-8-9) from the first problem to the other two problems while there was an increase in the number of negative respondents (12-15-14)

From the results of the categorisation of statements for the middle of the problems, it is interesting to see that the number of statements relating to being stuck in the middle of the problem remained similar across the three problems (19-14-16) while the number of statements referring to making progress reduced considerably (23-17-7). This reduction may be considered a negative outcome and could potentially be due to the focus on achieving a solution rather than the process. It is positive to see that there was a decrease in the number of negative statements related to being stuck across the three problems (12-5-5) while there was a small increase in the number of positive statements (2-4-5). A possible explanation for this reduction may be the experience of the open-ended affective questions leading to an increased awareness of emotions meaning an increase in control of emotions (McLeod, 1988). Another possible explanation may be that over the time period between each problem, the PSMTs had opportunities to attempt multiple problems meaning they had greater experience in being stuck, and had received instruction in recognising being stuck and possible ways to overcome it. The Rubric Writing approach was reinforced throughout the module with being stuck at a focal point of the Attack Phase. A key element of being stuck is recognising the emotions and gaining control through the acceptance of the position meaning that progress can then be made (Mason et al., 2010).

Focusing on the end of the problems, it is concerning to see that over the duration of the module the number of respondents who were positive about their problem-solving process decreased (18-7-11). This included the PSMTs who were positive, negative, or neutral towards achieving an answer. Over the duration of the module, there was a strong emphasis on the problem-solving process through the use of the Rubric Writing approach (Mason et al., 2010). In the second analysis of the statements made by the PSMTs, it was found that there was also a decrease in the positive statements made by the PSMTs in relation to the problem-solving process (15-2-6). In terms of the PSMTs' feelings towards achieving or not achieving a solution, there was an increase in the number of respondents who were positive towards their answer (9-7-15) while the number of negative respondents remained relatively constant (15-13-16). Evidently, the PSMTs were increasingly inclined to express feelings based on their perceived ability to reach a solution rather than their problem-solving process. This trend was similar seen in the statement analysis, where across the three problems, the statements regarding answers outnumbered the statements regarding the problem-solving process. Considering that the Understanding scale in the IMB scored the highest mean in Cohort 4, there appears to be a disparity between the PSMTs' focus on achieving an answer and the problem-solving process. While it is not surprising that the PSMTs expressed feelings about their answers at the end of the problem given that a solution is seen to be the ultimate goal of a problem-solving attempt, it is disappointing to see that there was not a substantial increase in the focus on the problem-solving process.

7.6.3 Summary of Findings

It was encouraging to find that *Effort* had a relatively high mean score across each of the cohorts and had the highest mean score overall (24.85/30). This is indicative of a growth mindset amongst the PSTMs through demonstrating the belief that working hard can impact mathematical ability (Dweck, 2008; Kloosterman & Stage, 1992). Amid other benefits of a growth mindset, one such advantage is reacting more positively to setbacks than those with a fixed mindset (Dweck, 2008, p. 4). However, in the responses to the open-ended affective questions it was evident that just 11 out of the 49 statements referring to the feelings experienced while *Stuck* were of a positive disposition. This demonstrates that some of the PSMTs had a positive attitude towards overcoming difficulties in their problem-solving attempt which is imperative to building confidence for future situations (Mason et al., 2011). However, there was double the number of negative responses than positive. This does not mean that these PSMTs necessarily have a fixed mindset but an awareness of the negative emotion that is common when stuck can help the problem solver control and overcome the emotion (McLeod, 1988).

The issue of time was apparent in the PSMTs' responses to the open-ended affective questions. In these responses the PSMTs expressed positive emotions towards running out of time from the view that they would have furthered their attempt if they had more time. In all of the references to time, the PSMTs stated that they felt that they could have progressed if they had more time implying that they did not object to time consuming problems. This finding is in contradiction to the low mean score of Cohort 4 in the *Difficult* scale of the IMB. A low mean score in this scale indicates that Cohort 4 have a negative perception about their ability to solve time consuming problems (Kloosterman & Stage, 1992). This suggests that PSMTs consider time consuming problems to be impossible to solve (Schoenfeld, 1985). However, looking at the qualitative data is it noticeable that there are exceptions within this cohort whereby they did not display a negative disposition to time consuming problems while they undertook the problem.

The responses from the open-ended affective questions revealed that the PSMTs' feelings were more dependent on achieving or not achieving an answer at the end of the problems (84/150 responses) rather than the problem-solving process (40/150 responses). Although there was evidence of an association between achieving an answer and positive feelings (Lewis, 2017), some PSMTs displayed positive emotions in relation to their problem solving process. This included PSMTs referring to satisfaction with their approach, and working to the best of their ability. When focusing on the number of PSMTs who commented on their approach, there was greater evidence of positive emotions rather than negative emotions. Looking at the number of responses of a negative disposition, the responses referring to achieving a solution was ten times the number referring to the problem-solving process. This shows that there may be a stronger negative association with not achieving an answer than there is with the problem-solving process. As stated by McLeod (1988) it is usual for negative emotions to be consistent with failing to reach a solution. From the results of the IMB scale it was found that the scale of *Understanding* had the second highest mean score (20.61/30). A high score in this scale signals the belief that understanding a concept is more important than using methods without making connections to the context of the question (Kloosterman & Stage, 1992). However, in the open-ended affective questions it is clear that the majority of statements were answer focused rather than focused on their problem-solving attempt. A potential reason for this could be that the PSMTs were influenced by their experience of post-primary mathematics exams wherein there is a strong focus on achieving the correct answer (Prendergast et al., 2018).

CHAPTER 8: EVALUATION, CONCLUSION, THESIS CONTRIBUTIONS AND FUTURE WORK

This chapter commences with a description of the evaluation of the module. Following on from this, the main findings of the research will be collated leading to the main conclusions that can be drawn from each of the research questions. The contributions that this study makes to the field of mathematics education will be discussed followed by the author's recommendations for possible future research.

8.1 Module Evaluation

The final phase of this research was the evaluation of the module through evaluation interviews which were conducted with a focus group of PSMTs (n=8). The results from these interviews will be discussed in detail in Section 8.1.5 below. The four key aspects of evaluating the module were (Shapiro, 1987):

- *Treatment effectiveness*
- *Treatment integrity*

- *Social validity*
- *Treatment acceptability*

These four parameters were initially used in the field of psychology (Shapiro, 1987b) but have been used in maths education settings to evaluate interventions (Fitzsimons, 2021; O'Meara, 2011; Prendergast, 2011). Each of these parameters will be discussed in relation to this present study followed by an overview of the results of the evaluation interviews.

8.1.1 Treatment Effectiveness

The effectiveness of an intervention involves the evaluation of any change that occurred as a result of the intervention in relation to the amount of change, the immediacy of change, the strength of change that occurs, and generalisation (Shapiro, 1987). The PSMTs' problem-solving proficiency improved according to the analysis of their problem-solving attempts using the MPSR (Section 6.2.1). This was endorsed by the respondents of the evaluation who stated that they felt that their problem-solving skills or the structure of their approach improved throughout the module. In view of the affective domain, the evaluation interviews showed that over the duration of the module, the PSMTs gained an understanding of the effects of emotions on problem-solving behaviours. A statement which encompasses this recognition is:

> *Probably because he [the facilitator] gave us such great questions that I can kind of understand why other people will get frustrated with like maybe easier problems that I don't find hard*. (PSMT 4)

The immediacy of change refers to the speed of which the intervention has an impact (Shapiro, 1987). The focus on the speed of change, does not however, mean that results which are more gradual are of any less importance or value than results which are more immediate (Shapiro, 1987b). The change in the PSMTs' problem-solving proficiency and problem posing capacity is outlined in Chapter 6.

The *treatment effectiveness* is also dependent on the generalisation of the intervention (Shapiro, 1987b). This generalisation is subject to these four conditions; time, setting, person, and behaviour (Shapiro, 1987b). The intervention in this study could be conducted in any teacher education programme for pre-service mathematics teachers or indeed in-service teachers in the form of continuous professional development. The facilitator would need to have a strong understanding of the different capacities outlined as crucial for the effective teaching of problem-solving. The replicability of this study was discussed further in Section 4.11.5.

8.1.2 Treatment integrity

Treatment integrity involves the 'extent to which a specified treatment is actually implemented in the manner prescribed' (Shapiro, 1987a, p. 292). This is of utmost importance to ensure that replicable results are obtained when the intervention is repeated. Each iteration of the module in this study was conducted in the same manner whereby the facilitator undertook a constructivist approach (Vygotsky & Cole, 1978). The validity of the study is integral to the integrity of the intervention (Prendergast, 2011). The steps taken to ensure validity in this study were outlined in Section 4.11.4.

8.1.3 Social Validity

As described by Shapiro (1987, p.293), social validity 'refers to the evaluation of treatment by consumers'. In order to evaluate the *Social Validity,* considerations need to be given to the 'appropriateness of the intervention, the importance of the outcome and the significance of the intervention goals' (O'Meara, 2011, p. 267). The appropriateness of this study is outlined in Chapter 4, where each decision made in every phase of the study was grounded in the literature and appropriate to the context of the study. In addition to this, the researcher is an experienced post-primary mathematics teacher. This experience was brought to bear in assessing the suitability of the appropriate mathematical problems that the PSMTs were expected to attempt, and so ensuring that they were of a level for which prior knowledge of post-primary mathematics was sufficient to reach a solution.

The collection of PSMTs' opinions of the module were collected through the evaluation interviews.

Overall, all the responses were of a positive nature. Five of the respondents made explicit reference to their satisfaction with the module being relevant to them as prospective teachers. An example of such a response is;

> *It was really useful for I suppose student teachers who would be kind of teaching students how to solve problems. And so from that perspective, it was really useful*. (PSMT 5)

It was interesting that two of the respondents demonstrated satisfaction with the module due the difference between the module and their experience of post-primary mathematics;

> *It was very different to like secondary school classes and we were constantly working on problems*. (PSMT 3)

The PSMTs were asked if their own problem-solving approach had improved over the duration of the module and the following responses were given. Six of the eight respondents said that they felt that their problem-solving skills had improved. One respondent said:

> *I wouldn't call myself a problem-solving kinda person so when we started off, I wasn't great at doing them and then, like, I think it was broken down quite well. So, like, everyone*

> *could understand what they were trying to solve and where to start, where to keep going and where to stop and kind of think.* (PSMT 6)

The other two respondents stated that while they didn't think that their problem-solving skills had improved in terms of reaching a solution, they felt that the structure of their solution method had improved. For example,

> *Going to be a teacher like you always tell your students to show every step of your workings…the way I did things did improve because I became more explicit with what I was doing instead of just writing something down and it helps. Then when I was going back to look at stuff, I knew exactly what I was doing beforehand.* (PSMT 7)

When asked if the module had increased their understanding of what a 'mathematical problem means' all the respondents were positive.

> *I would have a better understanding of it, which would be helpful being a teacher to be able to go, oh, I'm going to give you this exercise now or give them a problem to actually work through and think about*.

This example shows the PSMT's understanding of the difference between the cognitive demands that an exercise or a problem place on the problem solver. Again, reference to post-primary education was made by a respondent:

> *I suppose it was something I never really considered before. Like the difference between an exercise and a problem because obviously in schools, you spend a lot of time doing exercises, and then suddenly you get to a problem and you're like, God, how do I do this? I've only ever done exercise*. (PSMT 2)

The overarching theme that was evident in the responses was the relevance of the module to teaching post-primary mathematics. As seen in the examples above, this was in relation to the content of the module, but the facilitation of the module was also mentioned.

> *He knew when to like to stand back and let us give it a go ourselves and when he saw that nobody was getting the answer, he was able to come in and drop hints… a little clue that would trigger something in our minds…I was keeping an eye to see how I would do that myself in the classroom.* (PSMT 2)

In reference to actually teaching problem-solving, the PSMTs also expressed a desire for more understanding of how to approach students with a negative disposition towards problem-solving.

The importance of investigating PSMTs' capacities to teach problem-solving is rooted in the role that problem-solving plays in mathematics education both nationally and internationally which was thoroughly discussed in Chapter 2. As problem-solving is a key component of the curricula in Ireland, it is essential that mathematics teachers are proficient in the capacities to effectively teach problem-solving. Given that the PISA and TIMSS reports suggest that post-primary students in Ireland are less proficient in problem-solving than procedural skills it is evident that the need for accomplished teaching of problem-solving.

8.1.4 Treatment Acceptability

Treatment acceptability is linked to *social validity*, but it concerns specifically the degree to which participants like the intervention (Shapiro, 1987a). *Treatment Acceptability* is an important parameter to consider in research, as studies which are viewed as objectionable by the participants may fail (Prendergast, 2011). The level for which the participants like the intervention can be influenced by the following four elements (Shapiro, 1987a).

Immediacy and degree of change - This refers to the level and speed of change (if any) that occurred. This is an element that is also considered in *Treatment Effectiveness*. The immediacy of change for each of the research questions is discussed in Chapter 6. The mathematical problem-solving rubric was used to investigate the change in PSMTs' problem-solving proficiency. Cohen's d was used to calculate the effectiveness of the intervention (Cohen et al., 2007). The change in the PSMTs' affective domain was investigated using the open-ended affective question and reported on in Section 7.5.

Effort of implementation - This element consists of the effort that is required by both the researcher and the participants in the conducting of the study. The study was embedded in a university module, there was no extra effort required from the PSMTs other than from those who volunteered to participate in interviews.

Theoretical orientation - The theoretical framework of this study is described in depth in Chapter 3. Additionally, the development and use of any data collection instruments were rooted in this framework, and more broadly, in the relevant literature. The rationale for the implementation of any of the instruments and any decisions made regarding each phase of the study is described in Chapter 4.

Intervention facilitator - The main facilitator of the intervention was the research supervisor with tutorials conducted by different facilitators all of which had teaching experience and were undertaking PhDs in the field of mathematics or mathematics education. The researcher conducted the interviews and was independent of the facilitation of the module. The experience and expertise of the facilitators were important characteristics as it was evident in the evaluation interviews that the PSMTs were satisfied with the facilitation of the module.

The evaluation interviews showed positive responses regarding the PSMTs' view of the module overall. In response to the question; *What did you think of the module?*; all PSMTs' made positive statements. There was a particular emphasis on satisfaction with the module through the perspective of helping them learn to teach problem-solving. This was a positive response as this encapsulates the overall aim of this study.

Other components to be considered to ensure *acceptability* include (O'Meara, 2011, p. 270);

- *Time and Cost of the intervention* - this study was conducted within a pre-existing university module where participation in the study was optional and consisted of no extra workload outside the requirements of the module with the exception of interviews. The PSMTs who volunteered to participate in any of the interviews, completed them outside of the predefined class times. These interviews were conducted by the researcher and were time consuming.

- *Method of delivery* - as outlined in Chapter 5 this study was conducted as part of a pre-existing university module.

- *Effectiveness and Integrity* - this component has previously been discussed in Section 8.1.1and Section 8.1.2.

- *Possible side effects* - from the evaluation interviews and throughout the study there did not appear to be any negative side effects. In the planning of the study, we considered side effects that may occur and mitigated these. For example, giving the PSMTs difficult problems could cause them to withdraw from the associated learning activity but we took an approach based on group work where participation was encouraged, and all contributions were valued.

- *Understanding of Intervention* - the PSMTs' understanding of the aim of the module, which was the applicability of the module learning outcomes to the classroom, was clearly evident in the responses from the evaluation interviews (Section 8.1.5).

8.1.5 Evaluation Interviews

To evaluate the module, a focus group of eight PSMTs volunteered to participate in semi-structured interviews. The aim was to obtain a reasonably informal snapshot of PSMTs' opinions on the module. The PSMTs, through the research instruments, had provided the core data for the study. The aim with these interviews was to hear the PSMTs' perspectives on the final version of the module as a whole - both in terms of what they would mention unprompted, and on questions (as below) that probe the central issues of this thesis. The interviews were not conducted with other cohorts of PSMTs, as the aim was to gain an insight into the final version of the module. We see this as sitting in parallel to the study: in particular, this interview is not considered to be part of the overall methodology applied to seek answers to our research questions. These interviews were conducted after the completion of the module and consisted of six main questions. The questions, and their purpose will now be discussed:

Question 1: What did you think of the module?

This open-ended question was asked to gain an insight into the PSMTs' initial thoughts about the module such as; whether they found it useful or not; or if they enjoyed it or not.

Question 2: Do you think the module helped you to improve your overall approach to problem-solving?

While the data from the MPSR was analysed showing any changes in the PSMTs' problem-solving proficiency, this question aimed to get an understanding of the PSMTs' perception of their problem-solving approach. The question specifically focused on addressing their approach and not on their ability to reach a solution. The purpose of this question was also to gain an insight into the level for which the Rubric Writing approach had an impact on the PSMTs' approach.

Question 3: What did you think of the input of the lecturer during the module?

This question was asked to evaluate the PSMTs' perception of the facilitation of the module.

Question 4: Do you think the module helped you to improve your understanding of what a mathematical problem is?

A key component of the module was to develop the PSMTs' understanding of the nature of a mathematical problem. As described in Chapter 4, the 'Task Sorting' activity was designed to assess the PSMTs' ability to classify mathematical tasks as problems or exercises. This question gave an overview of these PSMTs' views on the effectiveness of the module in developing their understanding of what a mathematical problem means.

Question 5: What was your opinion of the group work?

A key component of the module facilitation was the utilisation of group work. The purpose of asking this question was to investigate how the PSMTs felt about group work while problem-solving.

Question 6: Did the module help you understand frustration while problem-solving?

On review of the results of the open-ended affective question (Research Question 4), the feeling of *frustration* was prominent when *Stuck*. The emotion of *frustration* can have a negative impact on the progression in a mathematical problem (Mason et al., 2011). The aim of this question was to investigate whether the module had given the PSMTs the tools to overcome this emotion in order to progress a mathematical problem attempt.

The transcripts of the interviews were analysed using a general inductive approach (Thomas, 2006). The results of this analysis are presented below in Figure 55.

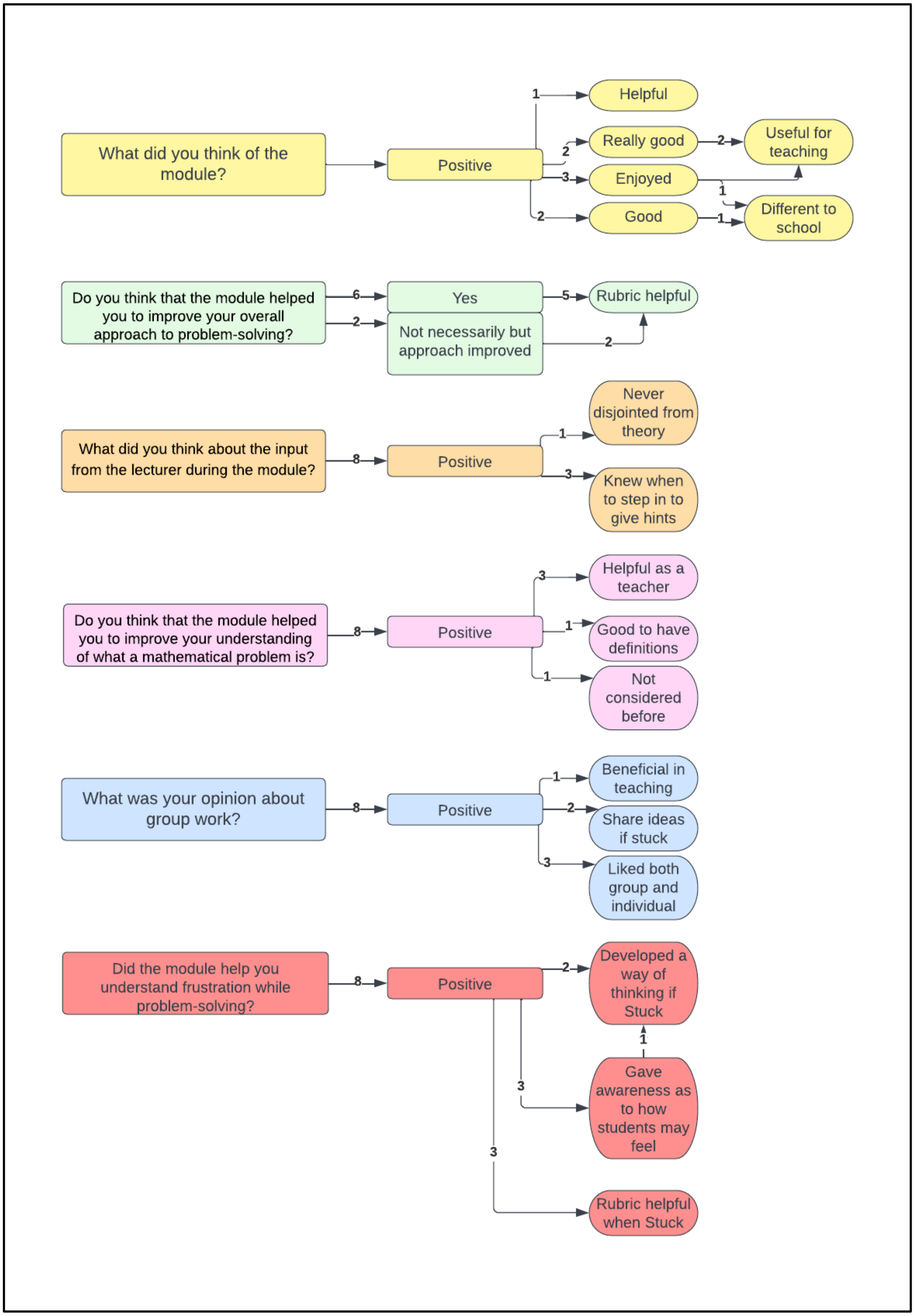


Figure 55: Results of the evaluation interviews

8.2 Summary of Work

The aim at the centre of this study was to investigate pre-service post-primary mathematics teachers' capacity to effectively teach mathematical problem-solving. In order to investigate these capacities, data were collected from four cohorts of PSMTs who were undertaking a university module as part of their teacher education programme. A summary of the findings will now be presented.

8.2.1 Overall summary of findings

This section will present a summary of the overall findings of the PSMTs' capacities to teach mathematical problem-solving and how the interim findings influenced the adaptation of the module.

It was found that when solving problems, the PSMTs are most accomplished in communicating their reasoning and using representations in their attempts. The introduction of notation and diagrams is commonly used at the start of problems as a tool to organise the information. This is a key feature of the Entry Phase of the Rubric Writing approach. On the other hand, it was evident that PSMTs demonstrated difficulty in reflecting on their problem-solving attempt. The improvement during the module in this area was marginal, with significant room for improvement. While this is a feature of the Review Phase of the Rubric Writing approach, further practice needs to be undertaken by the PSMTs in doing this as an integral part of their problem-solving.

The PSMTs demonstrated difficulty in posing problems. It was apparent that the PSMTs struggled to pose mathematical tasks that would be classified as a problem. Many of the posed tasks were deemed to be procedural with a clear path to reaching a solution. Secondly, many of the tasks were not solvable, meaning that PSMTs did not consider the viability of the solution. This suggests that the PSMTs need further development of their reformulation and problem posing skills. It also highlights the need for the PSMTs to develop their specialised content knowledge – that aspect of mathematical content knowledge that is specific to the task of teaching mathematics. When classifying tasks, text based tasks caused confusion for some PSMTs. It was evident that when mathematical tasks were text based, there was a tendency among the PSMTs to classify the task as a problem despite not meeting the criteria of such. However, there was a relatively high level of success in classifying other types of tasks.

In relation to the affective domain, the PSMTs expressed a positive association between the influence of effort on mathematical ability. This was replicated by some PSMTs in the open-ended questions, whereby they expressed positive feelings towards their ability to overcome being *Stuck*. However, there was double the amount of negative feelings towards being *Stuck*, implying that these PSMTs may have a fixed mindset. Despite this, some PSMTs demonstrated resilience by responding positively towards their ability to continue with the problem if they had more time. However, this was in contradiction to the results of the Indiana Mathematics Belief scale, where the PSMTs displayed a negative perception about their ability to solve time consuming problems.

The influence of achieving or not achieving an answer was prominent in the feelings expressed by the PSMTs. There was a strong emphasis on achieving an answer rather than the problem-solving process. Despite the problem-solving process (through the use of heuristics) being at the forefront of the module, the PSMTs were more concerned with achieving a solution than on the process. In the survey, however, the PSMTs did express the belief that understanding a concept is more important than using methods without connection to the concepts. This shows that the PSMTs' overarching beliefs are positive towards understanding but there is still a way to go before their focus is on both the problem-solving process and on achieving an answer.

Throughout the project, the module was adapted in response to the interim findings. Such adaptations included providing the PSMTs with more information about mindsets and the effect of a growth or fixed mindset on problem-solving (Boaler, 2015). Similarly, there was additional lecture material included around mathematical anxiety in students. This was done to build the PSMTs' awareness of the relationship between the affective domain and problem-solving. In terms of building the PSMTs' understanding of the nature of a mathematical problems, there was an increased focus on the characteristics of different types of mathematical tasks. Discussions took place around the classification of tasks including, the level of cognitive demand, the syntax of the task, the spectrum of closed to open tasks. In light of the findings above, it is evident that the module needs to be adapted further to develop the PSMTs' problem posing abilities. Similarly, an increased focus on text-based mathematical tasks should be included in the module. While the module specifically focused on the Rubric Writing approach, it is apparent that *reflecting* is an area that needs to be further developed. These considerations have allowed for the refinement of the module to develop the capacities of the PSMTs.

8.2.2 Summary of findings between the research questions

This study utilised a mixed-methods approach (Creswell, 2012) which allowed for methodological triangulation (Thurmond, 2001). Looking across the analysis of the data collected particular to each research question, there was evidence of some similar findings. These findings will now be discussed.

The analysis of the rationale for the classification of tasks in the 'Task Sorting' activity which were provided by Cohort 3 and Cohort 4 showed that there was a high percentage of PSMTs who achieved zero points for the identification of the *goal* of the task with scores of 84.5% and 78.7% respectively. Despite the variety of definitions for a *problem*, there is a consensus in the literature that the problem solver will seek a *goal* while attempting a problem solving task (Chamberlin, 2008). The lack of identification of a goal in the 'Task Sorting' activity rationale is consistent with the analysis of the 'Think Aloud' interviews with respect to the implementation of the Rubric Writing approach (Mason et al., 2011). The identification of the goal of the question is apparent in the *I want* element of the *Entry phase.* The analysis of the interviews showed that 55.6% of the PSMTs scored zero points through not presenting evidence of stating the goal of the question. The lack of identification of the goal of a problem is of concern as the problem solver may focus on the irrelevant information in the task and rush into a particular approach which could have been avoided if care was taken to identify the goal of the task (Mason et al., 2011; Polya, 1945).

Continuing with the analysis of the rationale for the 'Task Sorting' activity, it was found that the PSMTs based their classification of a task on the *Immediately Clear* characteristic. This signals that the PSMTs analysed the given task to determine if it was possible to use a previously learnt method or procedure to solve the task along with no ambiguity as to how to approach the task. However, in Research Question 3, the tasks produced by the PSMTs were *Procedural* rather than the requested mathematical problem. In Activity Two, it was found that with the exception of Question 1, the majority of the tasks posed by the PSMTs were classified as *Procedural*, meaning that there was an immediate clear path on how to reach a solution and/or could be solved using previously learnt procedures. While the generation of *Procedural* tasks aligns with the findings of other studies involving pre-service teachers (Crespo, 2003; Isik & Kar, 2012; Stein et al., 1996), it is interesting to find that the PSMTs are able to identify the immediacy of tasks but have difficulty in generating tasks that are *problems* and not procedural.

On analysis of the IMB scale, it was found that the *Steps* scale had the lowest overall mean (16.645/30) where 30 was the maximum possible score on each scale. This scale assessed the PSMTs' beliefs about rote learning and procedures for solving mathematical problems (Kloosterman & Stage, 1992). A low score in this scale is indicative of the belief that rote learning and procedures are sufficient for solving problems. This view on procedures along with the high volume of procedural tasks generated in place of genuine problems may signal that PSMTs are unsure of the nature of genuine problems.

Next, we consider together the results of the *Word Problem* IMB scale, the 'Task Sorting' activity (RQ1), and Activity One (RQ3). The results of the *Word problem* scale showed that there was a relatively low mean score (18.325/30) indicating that the PSMTs do not place great importance on word problems compared to computational skills. It was found in the 'Task Sorting' activity that questions which would be considered 'wordy questions' were categorised incorrectly as problems by the majority of PSMTs. This was also the case in Activity One which focused on the selection of a task for a specified student. In this Activity there was a low level of achievement in classifying word based tasks as 'not a problem' despite the tasks not meeting the criteria of a mathematical problem. It is possible that both the low mean score and incorrect classifications of 'wordy questions' as problems may be a result of a misunderstanding of the meaning of a 'word problem'. The presence of words in a mathematical question does not necessarily mean that the task is a problem. Verschaffel et al., (2000, p. ix) defines a word problem as "verbal descriptions of problem situations" which require the translation from words to mathematics as the starting point for the problem solving process. If the PSMTs consider the presence of words to mean a problem, then the characteristics of a mathematical problem are not considered. This potential misassociation of words and a problem signals that teachers need exposure and experience of different types of problems to understand the true meaning of a problem (Chapman, 2015).

To investigate the problem-posing capacity of PSMTs in Cohort 4, they were asked to extend three problems. Although the analysis of these posed problems showed that the majority were classified as ‘unsolvable’, there was a certain level of achievement amongst the problems posed (P1=41.7%, P2=18.2%, P3 = 18.2%). From the analysis of the PSMTs’ (in Cohort 4) problem-solving attempts using the MPSR, it was found that no PSMT scored a maximum score in the ‘Accuracy’ heading. A maximum score would be achieved through enhancing the solution by extensions and/or asking new questions leading to new problems (Oregon, 2011). From comparing these two results, it is apparent that the PSMTs have the ability to attempt to pose problems, but they do not do so unconsciously without direct instruction. However, the low levels of achievement in the extension activities show that the PSMTs had difficulty in posing mathematical problems which is similar to the findings of other studies conducted on pre-service teachers’ problem posing capacity (Isik & Kar, 2012; Stein et al., 1996). The PSMTs had explicit instruction and experience of using the Rubric Writing approach (Mason et al., 2011) which promoted the extension of problems in the *Review* phase of the rubric. This experience further supports the claim that PSMTs pose problems when specifically required rather than as a natural progression in their problem-solving.

Next, we consider the themes which were identified in the analysis of the ‘Think Aloud’ interviews. *Identity* was identified as a prominent theme which corresponds to the focus of Research Question 4. It was not surprising that *Identity* was classified as a theme from the inductive analysis (Thomas, 2006) considering the widely acknowledged influence that the affective domain has on problem-solving behaviours (Andrews & Xenofontos, 2015; Lester & Kroll, 1993; A. Schoenfeld, 1992). The aim of Research Question 4 was to establish the beliefs and affective factors that PSMTs hold in relation to problem-solving. A positive mathematical disposition, that is, positive beliefs around areas of mathematics and beliefs about oneself in connection to mathematical learning and problem-solving, enables students to access their prior knowledge (De Corte et al., 2000).

Another theme which was identified through the inductive analysis of the ‘Think Aloud’ interviews was *Introduce.* This involved PSMTs demonstrating the use of diagrams, constructions within given diagrams and notation. The identification of the *Introduce* category in the interviews aligned with both the *Representing* heading of the MPSR and the Rubric Writing approach (Mason et al., 2011) which was at the core of the module. In the analysis of the interview transcripts, for the purpose of evaluating the level of implementation of this heuristic, it was found that element *Introduce* had the highest percentage (44%) of achieving maximum points for explicit use when compared to other elements of the entry phase. Simultaneously, 48.1% of PSMTs scored 0 points for showing no evidence of *Introduce* elements. The findings of the *Representing* heading of the MPSR showed that the majority of the PSMTs were *partially effective* in their use of *Introduce* elements. Altogether the combination of these findings show that some PSMTs use representations, diagrams, and/or notation comprehensively in their problem-solving attempts. However, there is scope for improvement in the PSMTs' use of these elements which are positively associated with a successful problem-solving approach (Krulik & Rudnick, 1988; Mason et al., 2011; Polya, 1945).

8.3 Research Contributions

This study contributes to the field of mathematics education by investigating and drawing conclusions relating to the question of how adaptations can be introduced to improve pre-service mathematics teachers, in the university setting, for the task of teaching mathematical problem solving. In particular, we investigated and identified shortcoming in PSMTs' capacities in relation to identifying and constructing mathematical problems, and we assessed the level of implementation of a Rubric Writing approach while problem-solving. We triangulated methods to investigate the PSMTs' problem-solving proficiency. Similarly, triangulation of mixed-methods were used to investigate the beliefs and affective factors of the PSMTs in relation to problem-solving.

This study adds to the research that has previously been conducted in an Irish university setting with pre-service mathematics teachers in relation to the teaching of problem-solving (Guerin, 2017). Research instruments were designed and developed to assess the capacities of PSMTs to effectively teach problem-solving. The iterations of this study amounted to the adaptation and generation of a university module which develops the capacities of PSMTs to effectively teach problem-solving.

By designing and implementing mathematical task classification activities and problem posing activities, we were, to the best of our knowledge, the first research team to assess Irish PSMTs' abilities in relation to these capacities. A key component of this study was to assess PSMTs' understanding of the nature of a mathematical problem. Problem-solving is a priority in mathematics education in Ireland. Therefore, a teacher's capacity to distinguish between mathematical tasks is very important. From the literature, it is evident that textbooks hold a strong position in the provision of mathematical tasks in post-primary schools in Ireland (O'Keeffe, 2011). Within the textbooks themselves, there is a lack of opportunities for students to explore novel tasks (O'Sullivan, 2017). The selection and identification of mathematical problems, along with distinguishing between mathematical problems and other tasks, is unique to this study and provides a body of evidence regarding pre-service mathematics teachers' ability to classify mathematical tasks. In relation to the problem-posing tasks, the difficulties experienced by the PSMTs in posing problems corroborate the findings of international studies of both in-service and pre-service mathematics teachers. However, given that there has been no previous studies conducted within an Irish context, this study provides information about the ability of PSMTs to pose mathematical problems.

Acknowledging the multi-faceted nature of problem-solving proficiency, this study utilised the combination of a rubric and 'Think Aloud' interviews to triangulate data of quite different types and thereby develop a richer understanding of the students' capacities in this regard. The use of the rubric allowed for the assessment of the pre-service teachers' problem-solving proficiency which is an important influencer in a teacher's teaching of problem-solving (Lester, 2013). The rubric encompassed the interconnecting components required for problem-solving proficiency (Chapman, 2015; Kilpatrick et al., 2001). However, the 'Think Aloud' interviews allowed for an insight into the PSMTs' thought processes which may not have been evident in the written work such as rationale for decisions regarding strategy selection, or thought processes while stuck. Using these two methods of assessment of problem-solving proficiency, provides a broader overview than one method alone (Cohen et al., 2007).

The study incorporated two different methods for the assessment of the affective domain which allowed for identification of the beliefs that PSMTs hold regarding the role of problem-solving and the affective factors that are evident in their problem-solving attempt. The open-ended affective questions were adapted from the work of Felmer and Perdomo-Diaz (2016) and applied to investigate into the specificity of the feelings of the PSMTs and the situation which caused them to arise. This instrument is a tool which could be adopted by any study which is interested in the affective factors of people while problem-solving. The second purpose of the open-ended affective questions would be to allow the problem-solver to use this point of reflection to gain an awareness of their emotions and ascertain the trigger for the occurrence. Awareness of these emotions can allow the problem-solver to eventually control these emotions (Mason et al., 2011; McLeod, 1988). The use of the IMB scale provides a base of knowledge for pre-service teachers' beliefs about the nature of mathematics and about how mathematics is learned (Kloosterman & Stage, 1992). Combining this data with the open-ended affective questions, it was possible to gain an extra level of understanding into the results of the IMB scale through considering more subtle aspects of the results. The triangulation of these two methods allowed for an overview of the PSMTs' beliefs regarding mathematics and an insight into the affective factors during a problem-solving attempt. Beliefs and feelings are elements of the affective domain which has an influence on both problem-solving (Andrews & Xenofontos, 2015; Lester & Kroll, 1993; McLeod, 1988; Schoenfeld, 1983), and teaching problem-solving (Chapman, 2015; Marcou & Philippou, 2005).

A key feature of the university module was the use of the Rubric Writing approach as a support in solving problems (Mason et al., 2011). The PSMTs received some direct instruction on the use the use of heuristics as a tool to help problem-solvers approach problems (Schoenfeld, 1992), and experience of using the Rubric Writing approach (Mason et al., 2011). In this study, a rubric was developed to assess the PSMTs' implementation of this heuristic in an authentic problem-solving situation (Docktor et al., 2016). The rubric provides a structure for evaluating the level for which the heuristic is implemented and gives details of the problem-solving attempt (Hull et al., 2013). This rubric could be used in situations where the level of implementation of the Rubric Writing approach is of interest.

The overarching contribution of this study is the module, designed for pre-service mathematics teachers, focussing especially on the teaching of mathematical problem-solving. Addressing the capacities required by teachers to effectively teach problem-solving (Chapman, 2015), instruments were developed and adapted to evaluate the PSMTs' problem posing skills, their understanding of the nature of a mathematical problem, their problem-solving proficiency, and the affective domain elements associated with their work on mathematical problem-solving. The lecture content was influenced by the results of each respective instrument. The ultimate goal of the intervention and development of the module, was to support the development of post-primary students' mathematical problem-solving skills (Perkins & Shiel, 2014; Shiel & Kelleher, 2017), the skills which are highlighted as having of utmost importance in mathematics education (Department of Education and Skills (DES), 2011; NCCA, 2017).

8.4 Recommendations

- The content of the module could be adapted as a continuous professional development course for in-service mathematics teachers to undertake. In particular, the activities which involve the classification of mathematical tasks and problem posing activities could be used as tools to enhance teachers' understanding of the nature of a mathematical problem. Additionally, this course would offer teachers guidance, and the opportunity to reformulate tasks into mathematical problems.

- As previously mentioned, the participants in this study were in the first or second year of their respective university programmes. If it was possible, within the university, the participants could further benefit from the module if it was in closer proximity to their teaching placement in a school. The participants could then relate their classroom experiences, and interactions with students, to the module activities and content.

- When selecting problems for the participants to attempt, careful consideration should be given to the variety of topics that are involved in the problems and a range of different representations of problems. Exposure to different representations of problems can build flexibility in approaches. Secondly, a combination of group work and individual work should be incorporated into problem-solving experiences.

- The affective domain has a strong influence in successful problem-solving, yet there is a lack of guidance for students in post-primary education on its impact. The open-ended affective questions could be used by teachers with their students, in order to gain an insight into their students' feelings as they attempt mathematical questions, and also raise the students' awareness of their own feelings. Correlations between feelings and performance could then be identified, and consequently structures (such as heuristics) could be used to allow the student to develop their problem-solving attempt.

8.5 Future Work

There are multiple avenues for which the researcher envisages that further research could take. Some possible research routes are as follows:

- The analysis of the 'Think Aloud' interviews showed the presence of the theme of *Resilience*. Future research, focusing specifically on the *resilience* of pre-service mathematics teachers, is required.
- A longitudinal study exploring the longevity of the module in the practical setting of a classroom would provide information on pre-service teachers' application of the capacities which were developed throughout the module.
- Further study of pre-service and in-service teachers' ability to select and pose mathematical problems is required.
- Future work is necessary to extend the instruments and module content to in-service mathematics teachers as both a resource for the teachers themselves, and as a form of research to get an overview of the capacities of in-service teachers to effectively teach problem-solving.

There is a need for problem posing to be recognised as a key skill in both pre-service and in-service mathematics teachers. Problem-solving is identified as a key still in both Junior Cycle and Senior Cycle curricula. However, there is no mention of problem posing in the policies for either ITE programmes (Teaching Council of Ireland, 2020) or the curricular subject requirements (Teaching Council of Ireland, 2017). Future work needs to be conducted to establish the importance of the capacity to pose mathematical problems.

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APPENDICES

10.1 Appendix A

Plain Language Statement

Title of study: An investigation into the capabilities required by teachers to effectively teach problem-solving.

The study is being conducted through the School of Mathematical Sciences by;

Emma Owens; emma.owens3@mail.dcu.ie, and Brien Nolan; brien.nolan@dcu.ie.

This project investigates the skills and knowledge required by teachers to effectively teach problem-solving. Involvement in this project will require participants to:

(i) complete a survey which is designed to examine their beliefs about mathematics and about mathematical problem-solving (30 minutes);

(ii) complete work during tutorials on mathematical problem-solving which will be studied by the researchers;

(iii) participate in interviews related to different aspects of mathematical problem-solving. The interviews will be audio-recorded and will entail the participant being asked to attempt two problems and to "think aloud", describing exactly what they are thinking while attempting the problems (20 minutes approximately).

Through the link with taught programmes in DCU, the project will enhance the quality of mathematics teacher education programmes in the university. It will ultimately impact on the teaching and learning of mathematics in Irish schools through the greater capacities that the participants will develop in the area of teaching mathematical problem-solving.

For the protection of data all data will be anonymized for analysis and will be stored in DCU at all times. However, confidentiality of information provided is subject to legal limitations. The data collected will destroyed one year after the awarding of PhD.

The involvement in this project is voluntary and at any stage of the project participants can withdraw if desired.

If participants have concerns about this study and wish to contact an independent person,

please contact: **The Secretary, Dublin City University Research Ethics Committee, c/o Research and Innovation Support, Dublin City University, Dublin 9. Tel 01-7008000, e-mail: rec@dcu.ie**

Consent form:

Title of study: An investigation into the capabilities required by teachers to effectively teach problem-solving.

The study is being conducted through the School of Mathematical Sciences by;

Emma Owens; emma.owens3@mail.dcu.ie, and Brien Nolan; brien.nolan@dcu.ie.

The purpose of this study is the investigate the skills and knowledge that are required by teachers to effectively teach mathematical problem-solving.

Participant – please complete the following (Circle Yes or No for each question)

I have read the Plain Language Statement (or had it read to me)	Yes/No
I understand the information provided	Yes/No
I have had an opportunity to ask questions and discuss this study	Yes/No
I have received satisfactory answers to all my questions	Yes/No
I am aware that my interview will be audiotaped	Yes/No
I am aware that I will complete an anonymous questionnaire	Yes/No
I am aware that I will be observed during tutorials doing problems	Yes/No
I am aware that my work will be collected and analysed from tutorials	Yes/No
I am aware that I may withdraw from this study at any point.	Yes/No
I am aware that all data will be anonymised for analysis and that and that confidentiality of information provided is subject to legal limitations	Yes/No
I am aware that data will be destroyed one year after the awarding of PhD	Yes/No

Signature:

I have read and understood the information in this form. My questions and concerns have been answered by the researchers, and I have a copy of this consent form. Therefore, I consent to take part in this research project

Participants Signature:__________________________________

Name in Block Capitals:__________________________________

Witness:____

Date:____

10.2 Appendix B

Task Sorting Activity Cohort 1 and Cohort 2

Instructions:

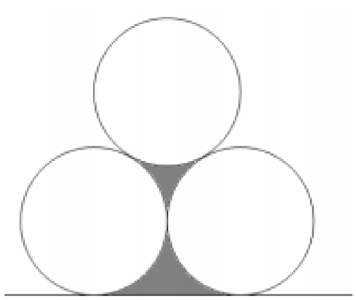
In the following questions, please tick whether you think it is an example of a *Problem*, an *Exercise*, or *Not Sure*. You do not need to work out the solution to any of the questions.

Task 1.

Wood pile perimeter

The diagram shows three touching circles each of radius 5cm and a line which touches two of them.

Find the total length of the perimeter of the two shaded shapes.



Problem Exercise Not Sure

Task 2.

A garden has the shape of a right-angled triangle with sides of length 30, 40 and 50. A straight fence goes from the corner with the right-angle to a point on the opposite side, dividing the garden into two sections which have the same perimeter.

How long is the fence?

Problem Exercise Not Sure

Task 3.

Niamh has an annual salary of €48000. She has a standard cut-off point of €34000 and a tax credit of €4600. If the standard rate of income tax is 20% and the higher rate is 42%, find how much income tax she pays.

Problem Exercise Not Sure

Task 4.

g:x -> ax^2+ bx+1 is a function defined on R. If g(1) =0 and g(2) =3, write down two equations in *a* and *b.* Solve these equations to find the values of *a* and *b*.

Problem Exercise Not Sure

Task 5.

1. Find the interest earned on each of the following;

i) €700 for 2 years at 8% ii) €400 for 3 years at 2.5%

Problem Exercise Not Sure

Task 6.

Find the midpoint of the line segment joining (-3,4) and (3,7). On which axis does the midpoint lie?

Problem Exercise Not Sure

Task 7.

Ryan and Emily are estimating the height of a phone mast. Ryan stands 15m from the mast and measures the angle of elevation to the top as 60°. Emily stands 25m from the mast and measures the angles of elevation to the top as 46°. Can they both be correct? Discuss.

Problem Exercise Not Sure

Task 8.

Stephen has a counter device on his bike. It counts the number of revolutions his wheel has made. His wheels are 40cm in diameter. i) Stephen cycles to his grandmother's house. The counter reads 1989. How far away from does his grandmother live? ii) How many revolutions does his wheel have to make to travel 1km?

Problem Exercise Not Sure

Task 9.

A cone has the same base radius as the radius of a sphere. If the volumes of the cone and the sphere are equal, by what factor is the height of the cone larger than its base radius?

Problem Exercise Not Sure

Task 10.

The marks of 36 students in third-year are given below:

31	49	52	79	39	49	51	76	83	91	23	45	63
	27	56	53	98	78	98	73	52	61	64	72	95
	32	42	56	52	83	64	63	52	69	62	45	62

Copy and complete the grouped frequency table below:

Marks	1-20	21-40	41-60	61-80	81-100
Number of students					

Problem Exercise Not Sure

Task 11.

Rory has some 2-litre bottles of lemonade. Sarah has some 3-litre bottles of lemonade. Altogether they have 27 litres. Rory has 6 bottles more than Sarah. How many bottles does each person have?

Problem Exercise Not Sure

Task 12.

Shane wins a sum of money on a scratch card. He decides to invest €700 in a bank that offers an interest rate of 8%. How much will Shane have at the end of the two years? He then decides to invest €400 in a bank that offers an interest rate of 2% for a further 3 years. How much interest will he make from the €400?

Problem Exercise Not Sure

Task 13.

In a survey of 40 households, 22 had a dog and 16 had a cat. If 8 households had a both a cat and a dog, represent this information on a Venn Diagram and write down how many households had neither.

Problem Exercise Not Sure

Task 14.

When a speaker gave a talk, 6% of the audience slept through the whole thing. 22% of the audience stayed awake and heard the entire talk.

Of the rest of the audience, half of them heard $\frac{2}{3}$ of the talk, and half of them heard $\frac{1}{3}$ of the talk.

What was the average proportion of the talk that people heard?

Problem Exercise Not Sure

Task 15.

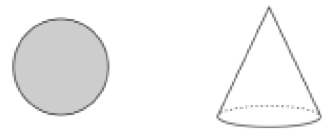
Adam is 12 years old and Emer is 8 years old. €5400 is divided between them in the ratio of their ages. How much does each receive?

Problem Exercise Not Sure

Task 16.

Clown hats
Charlie is making clown hats from a circular piece of cardboard.

The circumference of the base of each hat equals its slant height, which in turn is equal to the radius of the piece of cardboard. What is the maximum number of hats that Charlie can make from the piece of cardboard?



Problem Exercise Not Sure

Task 17.

The mean of a sequence of 64 numbers is 64.
The mean of the first 36 numbers is 36.

What is the mean of the last 28 numbers?

Problem Exercise Not Sure

Task 18.

Mr Ross always tells the truth on Thursdays and Fridays but always tells lies on Tuesdays. On the other days of the week he tells the truth or tells lies, at random. For seven consecutive days he was asked what his name was, and on the first six days he gave the following answers, in order: John, Bob, John, Bob, Pit, Bob. What was his answer on the seventh day?

Problem Exercise Not Sure

Task 19.

> Every day, Aimee goes up an escalator on her journey to work. If she stands still, it takes her 60 seconds to travel from the bottom to the top. One day the escalator was broken so she had to walk up it. This took her 90 seconds.
>
> How many seconds would it take her to travel up the escalator if she walked up at the same speed as before while it was working?

Problem Exercise Not Sure

Task 20.

> In how many whole numbers between 100 and 999 is the middle digit equal to the sum of the other two digits?

Problem Exercise Not Sure

10.3 Appendix C

Task Sorting Activity Cohort 3 and Cohort 4

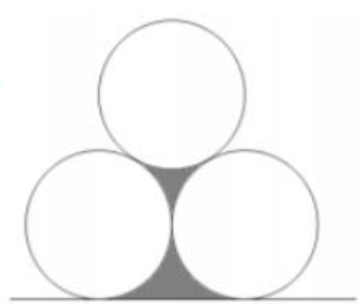
Task Classification

Instructions:

Classify each task below as a either a *Problem*, or an *Exercise.* Choose the option *Not Sure/Other* where appropriate. Give a reason for your classification of each task.

Question 1:

> **Wood pile perimeter**
> The diagram shows three touching circles each of radius 5cm and a line which touches two of them.
>
> Find the total length of the perimeter of the two shaded shapes.



Problem *Exercise* *Not Sure/Other*

Reason:____
____
____
____

Question 2:

Stephen has a counter device on his bike. It counts the number of revolutions his wheel has made. His wheels are 40cm in diameter. i) Stephen cycles to his grandmother's house. The counter reads 1989. How far away from does his grandmother live? ii) How many revolutions does his wheel have to make to travel 1km?

Problem *Exercise* *Not Sure/Other*

Reason:____
____
____
____

Question 3:

A cone has the same base radius as the radius of a sphere. If the volumes of the cone and the sphere are equal, by what factor is the height of the cone larger than its base radius?

Problem *Exercise* *Not Sure/Other*

Reason:____
____
____
____

Question 4:

In a survey of 40 households, 22 had a dog and 16 had a cat. If 8 households had both a cat and a dog, represent this information on a Venn Diagram and write down how many households had neither.

Problem *Exercise* *Not Sure/Other*

Reason:____

____

____

____

Question 5:

Fred flew to Melbourne, Australia. The flying time to Melbourne, which is 11 hours ahead of Britain, was 21 hours. Fred's flight left London at 11.30am on Tuesday. What time was it in Melbourne when Fred's flight arrived?

Problem *Exercise* *Not Sure/Other*

Reason:____

____

____

____

Question 6:

The marks of 36 students in third-year are given below:

49	52	79	39	49	51	76	83	91	23	45	63
27	56	53	98	78	98	73	52	61	64	72	95
32	42	56	52	83	64	63	52	69	62	45	62

Copy and complete the grouped frequency table below:

Marks	1-20	21-40	41-60	61-80	81-100
Number of students					

i) How many students scored between 21 and 60 inclusive?
ii) What is the modal number of the class?

Problem *Exercise* *Not Sure/Other*

Reason:____

____

____

____

Question 7:

Ryan and Emily are estimating the height of a phone mast. Ryan stands 15m from the mast and measures the angle of elevation to the top as 60°. Emily stands 25m from the mast and measures the angles of elevation to the top as 46°. Can they both be correct? Discuss.

Problem *Exercise* *Not Sure/Other*

Reason:____

Question 8:

Mr Ross always tells the truth on Thursdays and Fridays but always tells lies on Tuesdays. On the other days of the week he tells the truth or tells lies, at random. For seven consecutive days he was asked what his name was, and on the first six days he gave the following answers, in order: John, Bob, John, Bob, Pit, Bob. What was his answer on the seventh day?

Problem *Exercise* *Not Sure/Other*

Reason:____

Question 9:

If a flight from Dublin leaves at 08:50 and arrives in London at 10:05 find the speed of the plane if the distance from Dublin to London is 464km.

Problem *Exercise* *Not Sure/Other*

Reason:____

Question 10:

$g: x \rightarrow ax\ 2 + bx + 1$ is a function defined on R. If g(1) =0 and g(2) =3, write down two equations in a and b. Solve these equations to find the values of a and b.

Problem *Exercise* *Not Sure/Other*

Reason:____
____
____
____

10.4 Appendix D

Task Sorting Rubric

Task Sorting Rubric – Problem/Exercise			
Points	**The person outlines whether there is a goal**	**It is not immediately clear to the person how to reach the goal**	**The person must organize prior knowledge to generate reasoning towards achieving the goal**
0	Makes no reference to identifying if there is a goal or not.	The person makes no reference to identifying if it is immediately clear how to reach the goal or not. The person makes no reference to having or not having access to a procedure that allows them to immediately complete the task.	Makes no reference to using prior knowledge necessary to achieve a solution.
1	Makes reference to identifying if there is a goal or not.	The person makes reference to identifying if it is immediately clear how to reach the goal or not. The person makes reference to having or not having access to a procedure that allows them to immediately complete the task.	Makes reference to using prior knowledge necessary to achieve a solution.
2	Identifies if there is a goal and if applicable outlines what the goal is.	The person explicitly states if it is immediately clear or not how to reach the goal. The person states if they have or do not have access to a procedure that allows them to immediately complete the task. If applicable the procedure is stated.	Makes explicit reference to the prior knowledge necessary to achieve a solution. If applicable, outlines different approaches to achieving a solution and the prior knowledge needed for these approaches.

10.5 Appendix E

MPSR (Oregon, 2011)

Process Dimensions	**6/ 5	4	3	*2 / 1
Making Sense of the Task *Interpret the concepts of the task and translate them into mathematics.*	The interpretation and/or translation of the task are • thoroughly developed and/or • enhanced through connections and/or extensions to other mathematical ideas or other contexts.	The interpretation and translation of the task are • adequately developed and • adequately displayed.	The interpretation and/or translation of the task are • partially developed, and/or • partially displayed.	The interpretation and/or translation of the task are • underdeveloped, • sketchy, • using inappropriate concepts, • minimal, and/or • not evident.
Representing and Solving the Task *Use models, pictures, diagrams, and/or symbols to represent and solve the task situation and select an effective strategy to solve the task.*	The strategy and representations used are • elegant (insightful), • complex, • enhanced through comparisons to other representations and/or generalizations.	The strategy that has been selected and applied and the representations used are • effective and • complete.	The strategy that has been selected and applied and the representations used are • partially effective and/or • partially complete.	The strategy selected and representations used are • underdeveloped, • sketchy, • not useful, • minimal, • not evident, and/or • in conflict with the solution/outcome.
Communicating Reasoning *Coherently communicate mathematical reasoning and clearly use mathematical language.*	The use of mathematical language and communication of the reasoning are • elegant (insightful) and/or • enhanced with graphics or examples to allow the reader to move easily from one thought to another.	The use of mathematical language and communication of the reasoning • follow a clear and coherent path throughout the entire work sample and • lead to a clearly identified solution/outcome.	The use of mathematical language and communication of the reasoning • are partially displayed with significant gaps and/or • do not clearly lead to a solution/outcome.	The use of mathematical language and communication of the reasoning are • underdeveloped, • sketchy, • inappropriate, • minimal, and/or • not evident.
Accuracy *Support the solution/outcome.*	The solution/outcome is correct and enhanced by • extensions, • connections, • generalizations, and/or • asking new questions leading to new problems.	The solution/outcome given is • correct, • mathematically justified, and • supported by the work.	The solution/outcome given is • incorrect due to minor error(s), or • a correct answer but work contains minor error(s) • partially complete, and/or • partially correct	The solution/outcome given is • incorrect and/or • incomplete, or • correct, but ○ conflicts with the work, or ○ not supported by the work.
Reflecting and Evaluating *State the solution/outcome in the context of the task.* *Defend the process, evaluate and interpret the reasonableness of the solution/outcome.*	Justifying the solution/outcome completely, the student reflection also includes • reworking the task using a different method, • evaluating the relative effectiveness and/or efficiency of different approaches taken, and/or • providing evidence of considering other possible solution/outcomes and/or interpretations.	The solution/outcome is stated within the context of the task, and the reflection justifies the solution/outcome completely by reviewing • the interpretation of the task • concepts, • strategies, • calculations, and • reasonableness.	The solution/outcome is not stated clearly within the context of the task, and/or the reflection only partially justifies the solution/outcome by reviewing • the task situation, • concepts, • strategies, • calculations, and/or • reasonableness.	The solution/outcome is not clearly identified and/or the justification is • underdeveloped, • sketchy, • ineffective, • minimal, • not evident, and/or • inappropriate.

10.6 Appendix F

Cohort 1 problems for interviews

<u>Question 1</u>

There are 20 black socks and n white socks in a drawer.

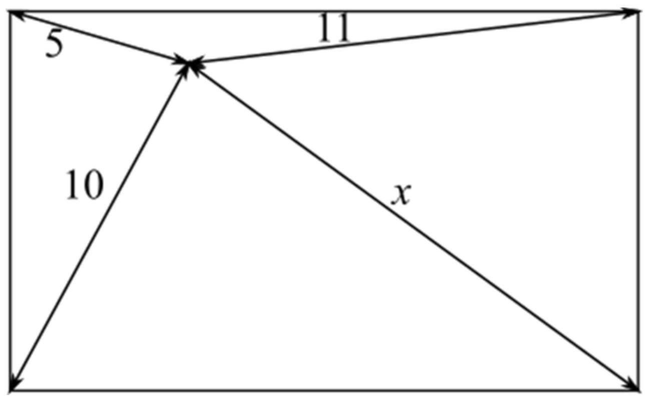
When a sock is taken from the drawer, the probability that it is white is 1+n.

What is the value of n?

Question 2

A well is dug in a courtyard. The distances from the well to three of the corners are 10 metres, 5 metres and 11 metres, as shown in the diagram below.

Find the distance from the well to the fourth corner.



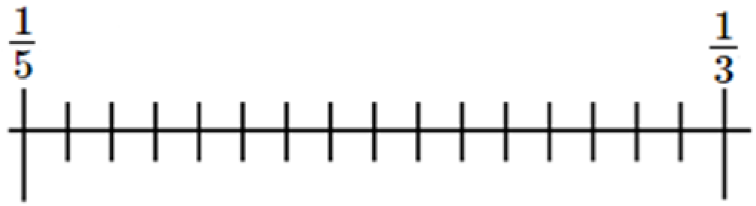
10.7 Appendix G

10.7.1 Cohort 2 and Cohort 3 problems for pre-module interviews

Question 1

The fractions 13 and 15 have been placed on the number-line shown.

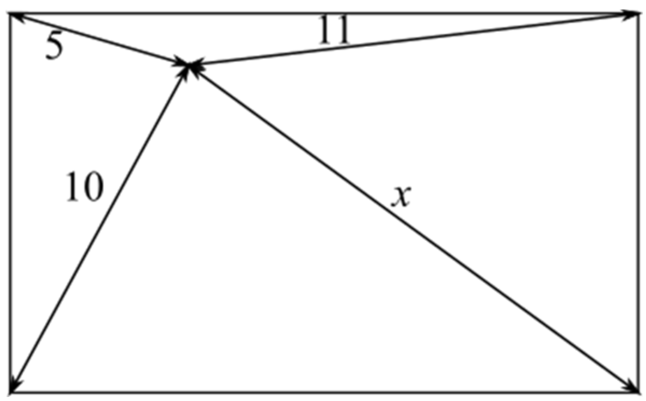
At which position should the fraction 14 be placed?



Question 2

A well is dug in a courtyard. The distances from the well to three of the corners are 10 metres, 5 metres and 11 metres, as shown in the diagram below.

Find the distance from the well to the fourth corner.



10.7.2 Cohort 2 and Cohort 3 problems for post-module interviews

Question 1

Roberto drove a total distance of 250 km.

The whole journey took him 3 hours, including a 20 minute stop to get petrol.

Before he stopped for petrol, his average speed was 80 km per hour.

After his petrol stop, his average speed was 100 km per hour.

How long did Roberto drive for after his petrol stop?

Question 2

A cube (1 metre by 1 metre by 1 metre) has one face on the ground and one face pushed up against a vertical wall.

A 4 metre long ladder is leaning against the wall, just touching the top edge of the cube.

How high is the top of the ladder above the ground?

10.8 Appendix H

Activity One: Scenario Problems?

1) It is a 2nd year higher level class who have completed trigonometry, algebra, geometry and applied measure.

When a solid sphere of radius 6cm is dropped into a cylinder partly filled with water, the level of the water rises Hcm. If the diameter of the cylinder is 16cm, find the value of H.

Is this a problem for these students?

Explain your answer;

2) It is 3rd year higher level class who have completed trigonometry, algebra, indices, factorising and geometry.

A car-hire company charges €x per day to hire a car and then adds a charge of €y for each kilometre travelled.

John hires a car for 2 days and travels 250km. He is charged €200.

Emer hires a car for 5 days and travels 620km. She is charged €498.

Write two equations in x and y and solve them to find the charge per day and the charge for each kilometre.

Is this a problem for these students?

Explain your answer;

3) It is 2nd year ordinary level class who have completed trigonometry, algebra, indices, factorising and geometry.

A projectile is launched directly upwards and its height, H metres above the ground, is given by $H = 20t - 5t^2$ *where t is the time in seconds. After how many seconds will the projectile be 20m above the ground?*

Is this a problem for these students?

Explain your answer;

4) It is a 2nd year ordinary level class who have completed algebra, probability and trigonometry.

Drainage pipes were being laid along the diagonal of a rectangular field. The field has dimensions of 35m and 64m. At what angle, to the shorter sides of the paddock, were the pipes laid?

Is this a problem for these students?

Explain your answer;

5) It is a 2nd year higher level class who have completed algebra, ratio, fractions, probability and measure.

A farmer is putting a new chicken run up against a brick wall. He has 20m of wire to put around the run. If he makes a rectangular run, what is the biggest area that he can enclose?

Is this a problem for these students?

Explain your answer;

6) It is a 5th year higher level group who have completed algebra, trigonometry and geometry.

Hannah looks at her watch and notices the hands of her watch are perpendicular at 3pm and at 9pm. She wonders how many times a day are the hands of her perpendicular to each other?

Is this a problem for these students?

Explain your answer;

7) It is a 2nd year higher level class who have completed algebra, geometry and trigonometry.

Tennis balls are often sold in tubes of 3. Which is greater; the height of the tube; the distance around the tube, or are they the same?

Is this a problem for these students?

Explain your answer;

8) It is a 2nd year higher level group who have completed algebra, geometry, ratio, decimal, fractions and percentages.

Paula, Henry and Maria are triplets. Henry can paint a room by himself in 3 hours. Paula can paint the room by herself in 4 hours. Maria can paint the room by herself in 6 hours. If they all work together and don't get in each others' way, how long will the job take?

Is this a problem for these students?

Explain your answer;

9) It is an ordinary level 5th year class who have completed probability, algebra, statistics and geometry.

IQ scores are normally distributed with a mean of 100 and a standard deviation of 15. Isaac has taken an IQ test and scored 132. His friend Eoin has remarked that Isaac's score is in the top 5% of all IQ scores. Isaac disagrees and says that he is in the top 2.5%. Is Eoin's remark correct? Explain your reasoning.

Is this a problem for these students?

Explain your answer;

10) It is an ordinary level 5th year class who have completed probability, algebra, statistics and geometry.

The experimental probability of scoring from a penalty in the World Cup is 0.77. If 46 penalties are awarded during the next World Cup, how many would you expect to be missed? (to the nearest whole number)

Is this a problem for these students?

Explain your answer;

11) It is an ordinary level 5th year class who have completed probability, algebra, statistics and geometry.

A stadium has a section of red seating in one of its stands. The first and second rows contain two red seats each. The third and fourth rows contain three red seats each. This pattern continues for all other rows in the section. There are 100 rows in the sections. The table below gives the pattern for the first nine rows. How many red seats are in the 98th row?

Row number	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
Number of red seats	*2*	*2*	*3*	*3*	*4*	*4*	*5*	*5*	*6*

Is this a problem for these students?

Explain your answer;

12) It is an ordinary level 3rd year class who have completed probability, algebra, statistics, arithmetic and geometry.

Annie is y years old. Her sister is twice as old as her. Their mother is 25 years older than Annie's sister's age. The total of all three ages is 80. How old is Annie?

Is this a problem for these students?

Explain your answer;

13) It is a higher level 3rd year class who have completed probability, arithmetic, algebra, statistics and geometry.

A 15- year old loan is drawn down for €250000. The rate of interest is 5.3% per annum compound interest. How much interest will have been charged after 10 years if no repayment is made in the 10 years? (to the nearest cent)

Is this a problem for these students?

Explain your answer;

10.9 Appendix I

Activity Two:

Scenarios

Create a mathematical problem that is suitable for each of the described students below.

1. It is a 2^{nd} year higher level class who have completed trigonometry, algebra, geometry and applied measure.

2. It is 3^{rd} year higher level class who have completed trigonometry, algebra, indices, factorising and geometry.

3. It is 2^{nd} year ordinary level class who have completed trigonometry, algebra, indices, factorising and geometry.

4. It is a 2^{nd} year ordinary level class who have completed algebra, probability and trigonometry.

5. It is a 2^{nd} year higher level class who have completed algebra, ratio, fractions, probability and measure.

6. It is a 5^{th} year higher level group who have completed algebra, trigonometry and geometry.

7. It is a 2^{nd} year higher level class who have completed algebra, geometry and trigonometry.

8. It is a 2^{nd} year higher level group who have completed algebra, geometry, ratio, decimal, fractions and percentages.

9. It is an ordinary level 5^{th} year class who have completed probability, algebra, statistics and geometry.

10. It is an ordinary level 5^{th} year class who have completed probability, algebra, statistics and geometry.

11. It is an ordinary level 5^{th} year class who have completed probability, algebra, statistics and geometry.

12. It is an ordinary level 3rd year class who have completed probability, algebra, statistics, arithmetic and geometry.

13. It is a higher level 3rd year class who have completed probability, arithmetic, algebra, statistics and geometry.

10.10 Appendix J

Activity Three:

1) Consider this mathematical task:

At the Olympic Games, a swimmer completed the 1500m freestyle race in 15 minutes. Express this speed in km/hr.

Construct a *mathematical problem* based on the same real-world scenario as this task.

2) a) Reformulate the following problem so that it requires the knowledge of fractions.

12 children share 4 packets of biscuits, each contains 15 biscuits, how many does each child get?

b) Reformulate the following problem so that it requires the division of fractions.

12 children share 4 packets of biscuits, each contains 15 biscuits, how many does each child get?

3) Reformulate the following problem so that it requires knowledge of ratio.

A farmer is putting a new chicken run up against a brick wall. He has 20m of wire to put around the run. The run must be at least 2m deep. What is the longest possible length of the run?

4) Reformulate the following mathematical task so that it is open-ended.

John and Mark play for the same soccer team. John scored 0 goals in the 3 cup games that he played. Mark scored 1 goal in the 7 cup games that he played. John scored 5 goals in the 7 league games he played. Mark scored 3 goals in the 3 league games that he played. Each game lasted 90 minutes. Who scored the most goals?

10.11 Appendix K

Qualitative: open-ended affective questions

An important part of the process of learning to solve mathematical problems is to reflect on the thoughts and feelings that you have while working on those problems. To that end, please answer this question:

How did you feel while working on this problem?

- Please try to include in your answer;
- How you felt at the start of your attempt;
- How you felt when you were making progress and/or when you were stuck;
- How you felt at the end of your work on the problem

10.12 Appendix L

Indiana Mathematics Belief scale (IMB) (Kloosterman & Stage, 1992)

Tick the relevant box for each statement	**Strongly Disagree**	**Disagree**	**Uncertain**	**Agree**	**Strongly Agree**
Maths Problems that take a long time don't bother me.					
Learning to do word problems is mostly a matter of memorizing the right steps to follow.					
A person who doesn't understand why an answer to a maths problem is correct hasn't really solved the problem.					
Computational skills are useless if you can't apply them to real life situations.					
Word problems are not a very important part of mathematics.					
I can get smarter at maths if I try hard.					
I'm not very good at solving maths problems that take a while to figure out.					
Hard work can increase one's ability to do maths.					
Learning to word problems is mostly a matter of memorizing the right steps to follow.					
Working can improve one's ability in mathematics.					
It's not important to understand why a mathematical procedure works as long as it gives a correct answer.					
Computational skills are of little value if you can't use them to solve word problems.					
Ability in maths increases when one studies hard.					
If I can't solve a maths problem quickly, I quit trying.					
Most word problems can be solved by using the correct step-by-step procedure.					
Getting a right answer in maths is more important than understanding why the answer works.					
Word problems can be solved without remembering formulas.					
Maths classes should not emphasize word problems.					
If I can't do a maths problem in a few minutes, I probably can't do it at all.					
Learning computational skills is more important than learning to solve word problems.					

In addition to getting a right answer in mathematics, it is important to understand why the answer is correct.					
Memorizing steps is not that useful for learning to solve word problems.					
I feel I can do maths problems that take a long time to complete.					
By trying hard, one can become smarter in maths.					
A person who can't solve word problems really can't do maths.					
Time used to investigate why a solution to a maths problem works is time well spent.					
I find I can do hard maths problems if I just hang in there.					
There are word problems that just can't be solved by following a predetermined sequence of steps.					
I can get smarter at maths by trying hard.					
It doesn't really matter if you understand a maths problem if you get the right answer.					

10.13 Appendix M

Problem 1:

A goat is tethered by a 6 metre rope to the outside corner of a shed measuring 4 metres by 5 metres in a grassy field. What area of grass can the goat graze?

Problem 2:

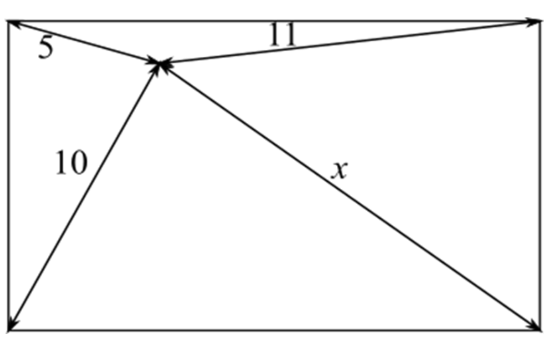
Roberto drove a total distance of 250 km. The whole journey took him 3 hours, including a 20 minute stop to get petrol. Before he stopped for petrol, his average speed was 80 km per hour. After his petrol stop, his average speed was 100 km per hour.

How long did Roberto drive for after his petrol stop?

Problem 3:

A well is dug in a courtyard. The distances from the well to three of the corners are 10 metres, 5 metres and 11 metres, as shown in the diagram below.

Find the distance from the well to the fourth corner.



10.14 Appendix N
Ethics approval

Ollscoil Chathair Bhaile Átha Cliath
Dublin City University

DCU

Ms Emma Owens,

School of Mathematical Science

5th December 2018

REC Reference: **DCUREC/2018/188**

Proposal Title: **An investigation into the capabilities required by teachers to effectively teach problem-solving**

Applicant(s): **Ms Emma Owens, Brien Nolan**

Dear Colleagues,

Further to expedited review, the DCU Research Ethics Committee approves this research proposal.

Materials used to recruit participants should note that ethical approval for this project has been obtained from the Dublin City University Research Ethics Committee.

Should substantial modifications to the research protocol be required at a later stage, a further amendment submission should be made to the REC.

Yours sincerely,

Donal O Gorman

Dr Dónal O'Gorman
Chairperson

DCU
Research & Innovation

10.15 Appendix O

Introduce	Productive Reasoning	Unproductive Reasoning	Identity	Resilience

Problem One

1. So, petrol stop. Roberto drove a total distance of 250.
2. Whole journey, total time is 3 hours that's with 20 minute stop.
3. Before he stopped for petrol his average speed 80km/h.
4. Ok eh, so (30 seconds)...
5. 3 hours is 180minutes minus 20 minutes is 160 minutes so that's his total driving time.
6. Then, we say 80x +100y = 160 minute time ehm (24 seconds)
7. 4x +5y =18 so 4x= 18 - 5y
8. x= (18-5y)/4 and sub that back in, and do 4 times ((18 - 5y)/4) +5y =18 and let me get my calculator…
9. Oh no, that just cancels that, that part won't work because everything cancels.
10. Hmm (15 seconds)
11. 150 divided by 80 (INAUDIBLE 4 mins 30)
12. Ok 60….(53 seconds)
13. I'm going to have to come back to that question
14. REVISITED AFTER ATTEMPTING QUESTION 2
15. So what do we know? We know the total distance, the total time.
16. We know the total time driving is 160 (8 seconds)
17. We can throw...80/100 is ⅘
18. So when he was driving 100km/h he was driving 20% of distance in that time frame.
19. So we split up into (37 seconds)
20. Use the 250 and 250km over 160 minutes is 1.56 but that doesn't really tell me anything.
21. That's just km/min. It doesn't really give a lot of information.
22. If we did 80/180 and 100/1. (Inaudible) percent that would mean the 100 by the 180, 100/1 would be 5.5%.
23. If we did 160 minutes by 55.5%.
24. Oh wait nevermind
25. 160 by 55.5% is 88.8minutes which is just 1 hour 28.8minutes.
26. I probably got both of them wrong but..

Problem Two

1. In the second one, so a cube is 1 metre by 1 metre by 1 metre with one side facing the ground and one is pushed up against a vertical wall. A 4 metre long ladder is leaning up against the wall touching the top edge of the cube so that would mean the diagram would look like that. (Drawing out diagram 20 seconds)
2. A 4metre long ladder is leaning against the wall touching the top edge of the cube, and the bottom of it..(10 seconds)
3. There could be two answers (12 seconds)
4. That's 1 metre. That's a bad diagram so I'll have to draw it again. (25 seconds)
5. So that's 1 metre and that's 4 metres. That's 1.
6. That's one metre there. (46 seconds)
7. How high is the top of the ladder above the ground?
8. So we're looking for there...(50 seconds)
9. So that's the top section after it touches the box is 4 - x that would be x. (65 seconds)

10. Well it has to be less than 4 anyway because it's not straight up. (36 seconds)
11. I could just take a guess at it but I don't like doing that. (6 seconds)
12. I'm just going to have to guess 3.37 and I don't even know why. It's a weird question.
13. I'm going to go back to the first question and have a look.

Problem Two

Ok so we have a cube against a wall. Draw a diagram. A cube has one face on the ground and one face against the wall. 1,1,1,1. I don't think I'm able do the drawing in 3D so I'll just leave it as that. A 4m long ladder was leaning against the wall just touching the top edge of the cube. How high is the top of the ladder above the ground? Ok so, it's touching that there. So this is 4m, 4m ladder. Ok so where are my right angles? My right angles, in the cube bottom corner, outside the cube on the ground and the vertical wall. I need, I know (15 seconds) Pythagoras. I know there is a lot of right angled triangles. Ok I need to find the total height of the ladder which is greater than 1 because it is taller than the cube. I need to find this side here, call that s. So my answer is 1+s. To find s, right hold on now. To find s, starting with this triangle here what do I know? (wait 12 seconds). Right, this right angled triangle here is (wait 10 seconds) Are they similar triangles? Are they similar triangles? (10 seconds) That's 4, that is 4m how do i find that side? How do I find…(wait 28 seconds) So if I find the 1+s, the hypotenuse of that is 4, this is f, 4. So I would try and go where? Right angle triangle where one side is 1+s and the hypotenuse is 4 and the bottom is 1+f. I also have a triangle where, I also have a triangle where, that is, which f, 1 and that's 4-call that something else. 4-d. Ok if they are similar triangles then, right where can I go for two…(wait 15seconds) I have two variables in all of those. 1+s/1 =4/4-d. Ok stuck. I need to get f, if I find f I can get all of these. How do I do that?

Ok I'll read the question again. (wait 20 seconds) that is 4. That's 4, d, not 4. How do I find that? That is (wait 15 seconds) that's 4-d. Pythagoras $1^2+f^2=d^2$. $f^2=d^2-1$. f=square root of d^2-1. What if I, can I sub that back in? $1^2+(d^2-1)^2=d^2$. $1+d^2$..that's no good. Let's go for the..(26 seconds) Ok that small triangle at the top, s,1,4-d. $s^2+1^2=(4-d)^2$. $s^2=(4-d)^2-1^2$. s= square root $(4-d)^2-1$. Ok so I've got that in terms of d. $(4-d)^2-1$. I need to redraw that again. So, 1,1,1. So, my big triangle is 1+square root $(4-d)^2-1$. The bottom side is 1+square root of d^2-1 and then the hypotenuse is 4. Right so I'll put that into the theorem of Pythagoras. $(4-d)^2-1$, this is going to be messy, $+(4-d)^2-1$ =16. So that's going to be, $16-4d+d^2$…I have it down to one variable but it is a mess. Right so that's going to be square root of $(d^2-8d+15)$ + 1+...inaudible, 1+2 times square root of…$+d^2-8d+15$… I just hope I'm on the right path here. If I get that out I'll see where it goes. Ok 1+square root of d^2-1. D^2-1. What are the factors of that? Ok start a new page 1+2(square root($d^2-8d+15$)) +...+1+2(square root of d^2-1)=16
Ok what are our like terms? -1, 1,16. -1 and 16. Ok they cancel. So i've a zero the far side, that's ok, 15+1 is 16. Right so they cancel. 2 times $d^2-8d+15$ + 2 times(d) = 0. 15 is gone. This is a mess. I've two d^2. That's 2 so I can divide across by two… that leaves me with...the root of d^2-1 = 0. Bloody hell. That's the difference of two squares here, can I eh...am I on the right path? I'm done I think.