A study of students' concept images of inverse functions in Ireland and Sweden

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In this paper we focus on first-year university students' conceptions of inverse function. We present results from two projects, conducted in Ireland and Sweden respectively. In both countries, data were collected through questionnaires, as well as through student interviews in Sweden. We draw on the notion of concept image and describe the components of students' evoked concept images. The students' responses involved e.g. ″reflection″, ″reverse″, and concrete ″examples″, while just a few students gave explanations relating to the definition of inverse functions. We found that the conceptions of inverses as reflections and reverse processes are important and relatively independent of local factors, and the data seemed to suggest that a ″reverse″ conception is linked to an appreciation of injectivity more than a ″reflection″ conception.

The focus of this paper is students' concept images of inverse functions. A well-developed conception of ″function″ is necessary for first-year university mathematics, since it is crucial for the students' understanding of the major ideas of calculus (e.g. Oehrtman, Carlson & Thompson, 2008). Building such a conception is often troublesome for students; it takes a long time and requires effort (see e.g. Pettersson, Stadler & Tambour, 2013). The concept of function is central in much of undergraduate mathematics and there are numerous publications on students' learning of the concept (e.g. Breidenbach, Dubinsky, Hawks & Nichols, 1992; Dubinsky & Wilson, 2013). The concepts of function and its inverse are both essential for representing and interpreting the changing nature of a wide array of situations (Carlson & Oehrtman, 2005) and also to describe the relationships between certain functions such as exponentials and logarithms.

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Despite the volume of research on students' conceptions of function, research focussing on inverse function, as in this study, is rare. Data in this study are drawn from two projects, one in Ireland and one in Sweden. These projects were designed and conducted independently of each other but had much in common; both focussed on the development of conceptual understanding of the concept of function among first year undergraduate students, and both included questions on inverse functions. When two of the authors became aware of each other's work, they saw the possibility of comparing their findings with a view to gaining a richer understanding of students' concept image. A preliminary report using data from these projects was presented at the CERME 9 conference (Breen, Larson, O'Shea & Pettersson, 2015). For the present paper, students' responses to open-ended questions in questionnaires given in both countries have been reanalysed, and interview data from the Swedish project have been included to deepen the analysis. We will describe our methodology in more detail below, but first we will consider three common conceptions of inverse functions, review some previous research relevant for our study, and present the analytical framework used.

Three main conceptions of the inverse function are evident from the literature: the inverse as a reverse process; the inverse as a reflection; the inverse as the result of swapping variables or coordinates (Carlson & Oehrtman, 2005). Before we present the literature on these conceptions, let us briefly explain them mathematically. If $f: X \rightarrow Y$ is a function, then we say that the function $g: Y \rightarrow X$ is its inverse if and only if $g(f(x)) = x$ for all $x \in X$, and $f(g(y)) = y$ for all $y \in Y$. The inverse function g is often written as f^{-1}, and in order for it to exist f must be a bijection. The inverse can be viewed as a *reverse process* since f^{-1} reverses or undoes the effect of f, that is $f^{-1}(f(x)) = x$ for all $x \in X$. If f is a function from R to R, then the graphs of f and f^{-1} are related, and one can be obtained from the other by *reflection* in the line $y = x$. To find a formula for the inverse function in this situation, one method involves solving the equation $y = f(x)$ for x, to get $x = g(y)$, and then *swapping the variables* to get a formula in the form $y = g(x) = f^{-1}(x)$.

To investigate students' knowledge and understanding of the concept of inverse function, Even (1992) gave prospective secondary mathematics teachers an open-ended questionnaire. One of the tasks that the participants were given was to find $(f^{-1} \circ f)(512.5)$. Both the function $f(x) = 2x - 10$ and its inverse were given. Using the idea of the inverse as undoing, the solution is straightforward. However the results showed that several of the students used a chain of calculations to get the answer. Even proposed that the tendency to calculate instead of using the property of undoing may be related to weak conceptual knowledge. However, she concluded that "a solid understanding of the concept of inverse function cannot be limited to an immature conceptual understanding of 'undoing' " (Even, 1992, p. 561), which she claimed may result in incorrect conclusions, e.g. that all functions have inverses.

Oehrtman et al. (2008) found that students who cannot think of a function as a process that may be reversed are restricted to using the procedural tasks of swapping x and y, or reflecting the graph of f across the line $y = x$, to obtain the inverse. Both the swapping and reflection conceptions were considered by Wilson, Adamson, Cox and O'Bryan (2011), and they pointed out that for either of these approaches, one usually does not consider the important issue of the domain of the inverse function being the range of the function and vice versa. They argued that, especially for contextual or real-world problems, these approaches could be confusing for students and the meaning of the result could be hidden. In particular, such approaches may cause problems when the dependent and the independent variable of the function are in different units. Attorps, Björk, Radic and Viirman (2013) also comment on the geometric view. In their study, GeoGebra was used to teach inverse functions in a bid to investigate whether the assistance of technology would contribute to students' understanding of inverse functions. Their results revealed that several students showed an intuitive conception of inverse functions as some kind of reflection, but lacked the full comprehension of why and where the reflection should be performed.

The notion of "inverse" is not uniquely used for inverse functions. In a study reported by Bagley, Rasmussen and Zandieh (2015), students were interviewed about composing a function and its inverse, and also about composing a linear transformation and its inverse. The results demonstrated that the students discussed three distinct mathematical objects, all called inverse and all symbolised with a superscript -1: the multiplicative inverse of a number (i.e. the reciprocal), the functional inverse, and the multiplicative inverse of a matrix. The paper reported on a mix-up of these three and indicated that confusion between the multiplicative inverse and the functional inverse may give students the misconception that $f^{-1} = 1/f$.

The results from a study where high school students were taught about inverse functions (Bayazit & Gray, 2004) concluded that differences in the students' learning were attributable to the teachers' instructional practices. Bayazit and Gray also found that the students who showed a conceptual understanding of the inverse function put particular emphasis on the "one-to-one and onto" conditions. They suggested that helping students to connect the inverse function more explicitly to these conditions, as well as to the concept of function itself, may enhance students' meaningful learning: that is, their development of conceptual understanding and links between sub-concepts as well as their acquisition of procedural skills.

Analytical framework

Participants in our study were asked either to draw a concept map of the notion of an inverse function or to explain in their own words what it means to say that one function is the inverse of another. The aim of these questions was to explore students' conceptions related to inverse functions and we chose to use the analytical framework of *concept image* (Tall & Vinner, 1981) to analyse our data. This framework has proved to be a useful tool (for an overview see Bingolbali & Monaghan, 2008), and is still used in analyses of undergraduate students' conceptions (e.g. Dickerson & Pitman, 2016; Wawro, Sweeney & Rabin, 2011). Tall and Vinner (1981) used the term concept image to denote the cognitive structure associated with a concept. The concept image comprises personal interpretations and understandings of the concept. It includes all the characteristics and processes that the person associates with the concept, and also the mental images such as figures and graphs that the person connects to the concept. The cognitive structure is successively built up through experience with the concept. It may include a formal definition, if known by the student, and also an interpretation of the definition.

The concept image may include conflicting components, but this may not cause the student problems if the contradictory parts are not brought into focus at the same time. For students who have not yet mastered or completely acquired a particular concept, Vinner and Dreyfus (1989) advised that a teacher's knowledge of the different and possibly conflicting cognitive schemes students subscribe to may make the teacher more sensitive to the students' reactions. Thus, in order to improve communication between teachers and students, Vinner and Dreyfus recommended students' concept images of various mathematical objects should be explored.

When a student – for example in a task – meets a concept, parts of his/her concept image will be activated, the activated part is called the *evoked concept image* (Tall & Vinner, 1981). Since it is not possible to directly observe concept images, we need to search for traces of them in utterances from the students. Such traces can only be expected from the evoked concept images. Hence, what is possible to observe are traces of the evoked concept images and through these traces we can discern components of the evoked concept images.

Information about components of students' concept images can be useful for instructors both in the design of courses and tasks, and in interactions with students (Carlson & Oehrtman, 2005; Vinner & Dreyfus, 1989). Therefore, the aim of this study is to identify and describe components of students' evoked concept images. The research question we will address is: What characteristic elements can be found in the evoked concept image of inverse functions of first-year university students?

Methods

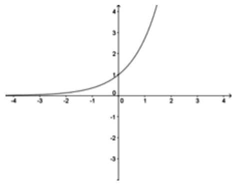
The two projects used different data collection methods: the Irish project considered 65 responses to a written questionnaire, while the Swedish project was of a smaller scale with 11 responses to a written task and interviews with 5 of these students. Although the data collection instruments were designed independently by the Irish and Swedish teams, they bore significant resemblances to each other. We will describe them below.

The Irish project

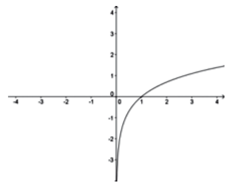
The data from Ireland involved students' responses to one of twelve questions on a concept inventory instrument designed to investigate undergraduate students' understanding of the concept of function. First-year Humanities, Education, and Finance students taking calculus modules (taught by the first and third authors) in two Irish universities were asked to voluntarily complete the inventory at the end of their module. 100 students took the test, 65 of whom answered at least part of Question I (see figure 1).

The mathematics syllabus followed by these students at secondary or post-primary school mentioned inverse functions solely in the context of inverse trigonometric functions and the textbooks did not contain formal definitions or geometric representations of inverses. Inverse functions were initially discussed in both university modules (recall that these were taught by the first and third authors) as reverse processes, and the role of bijectivity in determining whether an inverse exists was identified. A formal definition of inverse [$f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$ for all x in the domain of f and all y in the range of f] was presented. The possibility of adjusting the domain or codomain of a function to make it bijective and thus invertible was discussed and examples illustrating this process for specific functions were worked through with students. The graphs of a function and its inverse as mirror images of each other in the line $y = x$ were also explored. While the algebraic method of finding an inverse was not demonstrated or advocated by the lecturers in these modules, when

I. (a) Look at the graphs (i) and (ii) below. The claim is that graph (ii) represents a function which is the inverse of the function represented by graph (i). Explain what this means and state whether you think the claim is true.

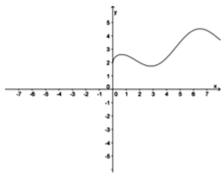


(i)



(ii)

(b) Look at the graph (iii) given below and draw, on the same axes, a graph to represent its inverse if you think there is an inverse. Explain clearly why you think there is an inverse or not an inverse.



(iii)

Figure 1. *The questions in the Irish project (I)*

it was presented in students' responses to the problems assigned it was acknowledged as a legitimate means of finding the inverse.

The Swedish project

The Swedish data were collected by the fourth author through questionnaires and interviews with students following a programme for teacher education for prospective secondary school teachers (school year 7–9 or 10–12). There were 18 students in the relevant cohort, and they participated voluntarily in three questionnaires relating to their understanding of the concept of function (Pettersson et al., 2013). During the first term of their studies, the students enrolled in courses in general education and an introductory course in mathematics. In the second term they studied courses in mathematics only, namely *Vectors and functions*, *History of mathematics*, *Geometry and combinatorics* and *Calculus*. Students who volunteered were interviewed a few days after each questionnaire using their answers in the questionnaires as a starting point. Answers to one question on the second questionnaire and the subsequent interview have been used as data in this paper. This questionnaire was administered to the students in the middle of the second term, towards the end of the course on vectors and functions. Both the questionnaire and the interview dealt with the concept of function in general and included questions on the concept of inverse function. The questionnaire consisted of two open questions on two different pages, thus there was a large space for writing the answers and explanations. The second task, in this paper referred to as S1, focussed on inverse functions (see figure 2). Eleven students answered this question and these answers are used as data for this paper.

S1. Make a mind-map or a concept map with the word inverse function as starting point. Bring in other words/concepts that you can associate with inverse function. Please also indicate how you think that the words/concepts are interrelated.

Figure 2. *The question in the Swedish project (S1)*

The subsequent interview was semi-structured and included questions on functions in general, but also covered inverse functions. The interview questions from which we have used data for this paper were "What is an inverse function?" and "What does it mean for a function to be invertible?" Five students volunteered for interviews.

Inverses are not explicitly included in the mathematics syllabus followed by these students at secondary school although they might be mentioned. At university these students were introduced to inverse functions in the course Vectors and functions. Because of lecture observations we know that the inverse function was defined by $f(a) = b \Leftrightarrow f^{-1}(b) = a$, which also corresponds to the definition in the course literature (Gottlieb, 2002, p. 36). Both algebraic and geometric aspects were mentioned and also that the function needs to be one-to-one for an inverse to exist. The possibility to restrict the domain of the function to ensure it is one-to-one was discussed.

Coding

The analyses of the data started with identifying which elements of the evoked concept images (Tall & Vinner, 1981) the Irish students showed in their answers on I(a) and (b). For each question, the students' responses were coded using a grounded theory approach: that is the responses were read multiple times, codes were assigned, and these codes were then grouped into categories. As an indication of how codes emerged and were then grouped into categories, we can give an example from the Irish data. Some responses were initially coded as "mirror", "symmetry" or "reflection", as these were terms used by the students themselves. Responses with these codes were then placed in a category labelled "reflection" as in each case the student seemed to be attempting to describe an inverse function as a reflection of the original function. The categories that are presented here emerged from the data itself. Findings from existing studies on inverse functions were not used to identify categories when analysing the data collected, the resulting conceptions of inverse were later compared to others reported in the literature.

In the next step, data from the Swedish task S1 were analysed. Since this analysis started later, it was possible to use the categories found in the Irish project as a starting point. Researchers from both countries had access to both sets of data and so efforts could be made to ensure that student responses were categorised similarly in both countries. Except for the existence of initial categories, the Swedish analysis was made using the same principles as the Irish, including the creation of new categories when needed. The Swedish interviews were analysed using the categorisation that emerged from the questionnaires as a starting point. Data from the interviews were then used to further illustrate the categories of evoked concept images traced in the students' answers.

In each country, the responses to the tasks above were coded by one researcher, then checked by another and any disagreements were discussed and resolved, before the agreed codes were grouped into categories. The initial use of the Irish categories in the Swedish analysis strengthens the connections between the two projects. In addition, the sets of categories which emerged in Ireland and Sweden were compared to check for consistency. That is, for categories where the same labels were used in both countries (e.g. reflection, reverse, swap x and y), efforts were made to ensure these categories contained the same type of student responses.

Results

This section first presents overviews of the results from the Irish and the Swedish projects. We will then expand on some elements of the evoked concept images to describe similarities and differences between the students' evoked concept images in the two countries.

Results from the Irish project

We first considered the students' answers to question I(a) above, i.e. the students' explanations as to what it means to say one function is the inverse of another. Sixty-five students made an attempt to answer this part of the question; 4 students explained the concept correctly, 50 students gave an explanation which contained errors or was incomplete, while 11 answered true or false with no explanation. An example of a correct answer was:

> Let $f(x)$ be the function in (i). Let $f^{-1}(x)$ be the function in (ii). The claim states that $f^{-1}(f(x))=x$. It claims that $f^{-1}(x)$ is a reflection of $f(x)$ through the line $y=x$. I agree with the claim.

We categorised the components that arose in the evoked concept images of inverse function of the 54 students whose response contained an explanation; these are shown in table 1. Note that some students were counted more than once here if their answer referred to two or more of the concept image components identified. For example, the response above was counted in both the "definition" and "reflection" categories.

The conception that appeared most frequently in the students' answers is that of "reflection". Seventeen students used the word "opposite" in their explanations; however it was clear that 10 of them used it in the context of reflections and one of them used it in the sense of "reverse". This left 6 responses coded in the "opposite" category. Ten students saw the inverse as a reverse process, while only 5 students volunteered a concept definition of an inverse in answer to this question. There were 10 responses categorised in the "other" category, these include: 6 responses which refer to a feature of the graphs given in I(a), and 1 response "domain and range interchanged". The remaining responses in the "other" category are not mathematically relevant.

We have included both correct and incorrect notions within each component of the concept image in table 1; for instance, although responses

Table 1. *Conceptions emerging in response to I(a)*

Conception	Sample explanations given by students	Total
Reflection	The inverse is that function mirrored through the line $y=x$	39
Reverse	The inverse of a function, it's the function in reverse	10
Opposite	The inverse is the opposite of the original function	6
Definition	$f^{-1}(f(x))=x$ for all x in the domain of f	5
$1/f$	Inverse = $1/f(x)$	3
Swap x and y	When the x and y coordinates swap, e.g. Here the point (1, 2) becomes (2, 1)	1
Example	$f(x)=x^2$, inverse = $x^{1/2}$	1
Other	Graph (i) has a positive slope while graph (ii) has a negative slope	10

from 39 students were categorised as ”reflection”, only 9 students correctly described the line of reflection or symmetry as the line $y = x$.

Results from the Swedish project

We first considered the 11 responses to question S1 – recall that students were asked to make a mind map about the notion of inverse function. As with the Irish data, elements of the concept images evoked were categorised. Table 2 below shows the frequency of each category respectively. Note that some students gave answers which fit into more than one category.

In what follows we will expand on some of these categories to describe similarities and differences between the students' evoked concept images in the two countries. Table 1 showed that the most frequent category in the Irish project was ”reflection”, followed by ”reverse”. In the Swedish project, see table 2, most students' concept maps contained examples of pairs of inverse functions (either given as formulae or as graphs); apart from these the most frequently occurring categories of conceptions again included ”reflection” and ”reverse”.

The Reflection conception

In the Irish project, as earlier mentioned, 9 students in task I(a) described the line of reflection or symmetry as the line $y = x$. One example was the

Table 2. *Conceptions emerging in response to S1*

Conception	Sample explanations given by students	Total
Reflection	Reflection in the line $y = x$	6
Reverse	An inverse function assumes you can "run" the function both ways	4
Opposite	"Opposite" written as a single word in the mind map, without any further explanation	2
Definition	$f(a) = b\,;\, f^{-1}(b) = a$	1
Swap x and y	The reflection takes part in $y = x$, to get a reflection, that is the inverse, we swap x and y	4
Algebraic example	Various examples of pairs of inverse functions such as e^x and $\ln x$; x^2 and $\sqrt{x}$; 2^x and $\log_2 x$	7
Graphical example	Graphs of exponential and logarithmic functions	5
Injective/ invertible	Inverse function can be reflected and get one y value for one x value	3

utterance: ”The inverse of a function is the function given through symmetry in the line $y=x$.” Correspondingly in the Swedish task S1, 3 students mentioned that the reflection is made in the line $y=x$ (or ”around the 45°-axis”) which also was explained in one of the student interviews: ”[...] yes in fact it reflects the graph in the 45 degree slope, if we draw a line 45 degrees as support”. One student also elaborated her answer in S1 by writing ”sometimes the reflection is not a function, and then no inverse exists”.

However, students also suggested other kinds of reflections, which of course were incorrect in the case of inverses. In Ireland, only 9 students (out of 39 in this category) correctly described the line of reflection as the line $y=x$, while 2 students referred specifically to reflection in the origin and 3 to reflection in the x-axis. In Sweden, some students were also unsure on this point as can be seen from this answer in the interview: ”Some kind of reflection. [...] Yes, well, for example, either in the x-axis or in the y-axis or in some other line”.

In the Irish project, a further 6 spoke of a reflection without being specific (for example ”Inverse is a reflection of the function”). There were similar statements written by Swedish students in answer to S1 such as ”the reflection is made in 'a line' ” or ”reflection of a function”.

The Irish project produced more data on conceptions in the reflection category through the analysis of task I(b). There were 58 responses to this question and 10 of these were correct, that is, the students were able to say that the function did not have an inverse and were able to furnish a reason for their answer. However, 47 students attempted to draw an inverse function, even though no inverse exists, with 45 of them sketching a reflection of some sort; table 3 shows the distribution of these attempts.

The students who thought an inverse existed gave a variety of explanations for their answer: for instance, one said that every function has an inverse, while another said that the function was ”defined for its entire domain”. We also saw that the conception of inverse function as a reflection in the line $y = x$ could be misleading with four students making

Table 3. *Answers to I(b)*

Type of reflection	Total
In (0, 0)	22
In x-axis	13
In y-axis	3
In $y = x$	3
In $y = -x$	1
In $y = 2$	3

remarks such as ”There is an inverse as possible to draw line $y = x$ and reflex (sic) images”. Similar examples also appeared in Sweden, for example one student drew the (complete) graph $y = \pm\sqrt{x}$ as an inverse to $y = x^2$.

The Reverse conception

In the Irish project, 10 students' answers were categorised as ”reverse”; for example ”The term 'inverse' means the same word for backwards” and ”The inverse of a function is where you are given the output and are required to solve for a certain input”. There are responses from the Swedish interviews that connect to these, for example ”The inverse function works so that you want to return to, well, you want to return to the x-value you had in the beginning [...]” and

> But isn't that the inverse function which for each, what do you say, if one inserts one x-value and gets one y-value out, then since it is the inverse function, you should be able to insert your y-value and get the x-value out.

Four of the Swedish students on task S1 wrote about the inverse function as some kind of reverse process. Three of these used a form of the verb ”ogöra”, a word that does not exist in Swedish. It is a negation of the verb ”göra”, which means ”do”. Hence ”ogöra” is a direct translation of ”undo” and probably understandable even though it is not a real word. The word ”ogör” is not mentioned in the course literature, but it was introduced to the students by the teacher. The word ”ogör” was also mentioned in the interviews, for example, when one student said: ”[...] well what did he say, the inverse function undoes [ogör] what the first does”.

The injective or one-to-one property

Of the 10 Irish students who answered I(b) correctly, 6 said that the function did not have an inverse as it failed the horizontal line test and 6 said that the function was not one-to-one. Only one student gave another reason as to why the function was not invertible and so the one-to-one property seems worthy of further mention. (Note that some students said the function was not one-to-one and illustrated this using the horizontal line test which accounts for the numbers adding to 13 rather than 10.)

In their mind map responses to S1, only one student mentioned the word ”injective”. She did not mention ”injective” in the subsequent interview, although she referred to properties connected to that concept. Looking at the other interviews, two students also seemed to understand the importance of injectivity to the existence of inverse functions. One of them alluded to injectivity when she explained how the inverse function would swap x- and y-values: "And if I have -2 here and square it, it will give just 4. But if I have 4 and go backwards, I don't know which way I came from and then I cannot invert it."

The terms "injective" and "one-to-one" had not been mentioned in Sweden during the course or in the course literature, which could explain why only one student used the notion of injectivity explicitly.

Examples given by formulae or graphs

The clearest difference between the concept images evoked in the two countries is the frequency with which Swedish students produced answers categorised as "algebraic example" or "graphical example". Three students in the category "graphical example" had given other forms of examples as well. Hence, three out of five students in "graphical example" were also counted in the category "algebraic example".

When giving algebraic examples, in most cases the student gave expressions for two functions, which were each other's inverses. For instance, one Irish student wrote "$f(x)=x^2$, inverse = $x^{1/2}$", while some of the Swedish students volunteered $y=e^x$ and $y=\ln x$; $y=x^2$ and $y=\sqrt{x}$; or the pair $y=2^x$ and $y=\log_2 x$, which was mentioned in the course literature (Gottlieb, 2002).

Examples given in graphical form included exponential and logarithmic functions, quadratic and root function, and linear functions which were each other's inverses. Some of these graphical examples were incorrect, for example the case suggesting the inverse to $y=x^2$ being the complete reflected graph, $y=\pm\sqrt{x}$.

Discussion

The study reported on here concerns students' concept images of the notion of inverse functions. Our data allowed us to explore students' associations with and views of the inverse function concept, and so Tall and Vinner's (1981) concept image framework was appropriate for our analysis.

The Irish data from I(a) highlighted four main components of the evoked concept image of inverse functions: reflection, reverse, opposite, and the definition. These categories were present also in the Swedish data from S1 but with a different emphasis, and alongside other conceptions involving examples (both algebraic and graphical), properties such as injectivity, and the notion of swapping the x- and y-coordinates. We should mention that students frequently displayed more than one conception in their answers, and so the categories we have found here are not disjoint. It has been argued previously (e.g. Tall & Vinner, 1981) that students can hold different (even conflicting) conceptions at the same time especially when building their understanding, and having a rich concept image is useful. Our analysis has shown that although the students in both the Irish and the Swedish groups exhibited a variety of concept images for the notion of inverse function, there was much commonality between the two groups, even though the questions asked and, more importantly, the students' mathematical backgrounds were not the same. This seems to suggest that the conceptions of inverses as reflections and reverse processes are important and relatively independent of local factors.

Relatively few students attempted to present a formal definition of an inverse function in response to either I(a) or S1, despite the fact the concept had been formally defined with both Irish and Swedish students. Vinner and Dreyfus (1989) spoke of a compartmentalisation of students' concept definitions of function and their function concept images, and it would appear that there was also a divide for the students in this study between their definitions of inverses and their inverse concept images. Vinner and Dreyfus (1989) suggested that a student does not necessarily use the definition of a mathematical concept when deciding whether a given mathematical object is an example of the concept or not. Instead, in most cases, he or she decides on the basis of a concept image and, hence, the set of mathematical objects considered by the student to be examples of the concept is not necessarily the same as the set of objects determined by the definition. Taking this perspective and acknowledging the number of students having a reflection component to their concept image, it is perhaps no longer surprising that there were so many incorrect responses to I(b).

It may be that the particular questions asked in the projects had an influence on the elements of evoked concept image that were apparent in the data. For example, the majority of Irish students wrote about reflections when answering I(a), and possibly this was because of the graphs presented. None of them mentioned properties like injectivity in their answers to this question, however 10 of them were able to use this notion to arrive at the correct answer in I(b). Similarly almost all Swedish students (9 of 11) included examples in their answer to S1; it is possible that asking for a mind map encouraged students to do this.

Two main conceptions of inverse function were apparent in our data: inverse functions as reflections of graphs, and as reverse processes. The reflection conception seemed to be more prevalent in Ireland (possibly explained by the type of question asked), while the reverse process conception was more obvious in Sweden (4/11 responses to S1 compared with 10/65 responses to I(a)), especially in the interview data where it was referred to by all five students. This may be because of the influence of the lecturer, who introduced the word ”ogör” [undo] to the class. Another possibility is that the creation of this new word in the Swedish language caught the students' imagination and stayed in their minds because of its novelty value. The notion of ”undoing” can have positive and negative effects, as pointed out by Even (1992); that is, it may contribute to a conceptual understanding of inverse function but may also lead students to believe that all functions have inverses. However, the responses of the students in the interviews seemed to point to this conception as being close to the definition of inverses and linked to an appreciation of the importance of injectivity. In contrast, we found that the reflection conception was associated more with the notion that all functions have inverses than was the case for the reverse conception; for example we saw this in some answers to I(b) such as the response ”there is an inverse as possible to draw line $y = x$ and reflex (sic) images”. The problem may be that unlike the ”reverse” conception, the ”reflection” conception is not linked to the notion of injectivity in students' minds. We would recommend (as did Bayazit & Gray, 2004) that the necessity of injectivity for the existence of an inverse should be emphasised in instruction. In addition, we found that students who described inverses as reflections were often not able to say what they should reflect in and gave a variety of possibilities such as reflection in the axes or reflection in the origin. Similar results were found by Attorps et al. (2013); the students in their study were also unclear about the axis of reflection. We think that more work should be done on the link between students' conceptions of functions and those of inverses. For example, the students who reflected the graph in task I(b) and said that the function had an inverse did not notice that the curve that they drew did not pass the vertical line test. This gives us information about their understanding of graphs of functions, and it may be that this kind of situation is the first meaningful time when students have to appreciate the full definition of function.

The algebraic conception of swapping coordinates reported by Carlson and Oehrtman (2005) and by Wilson et al. (2011) was mentioned by only one Irish student out of 65, as opposed to 4 out of 11 in Sweden. Also the misconception that $f^{-1} = 1/f$ commented on by Bagley et al. (2015) was not an issue for the study reported here with $1/f$ seldom observed in the concept images evoked: only 3 Irish students and no Swedish students referred to $1/f$.

Our study has allowed us to explore undergraduate students' concept images of inverse functions in two different countries; the fact that the elements of these concept images identified in both projects are very similar suggests that our results may be of interest in other educational systems too. The information on the concept images of students regarding the concept of inverse functions could be used by lecturers when teaching this topic; knowledge of the likely conceptions and misconceptions that their students may hold can be a powerful tool in planning instructional activities and may improve communication between lecturers and students.

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