Classifying cocyclic Butson Hadamard matrices

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Abstract

We classify all the cocyclic Butson Hadamard matrices $\mathrm{BH}(n,p)$ of order n over the pth roots of unity for an odd prime p and $np \leq 100$. That is, we compile a list of matrices such that any cocyclic $\mathrm{BH}(n,p)$ for these n, p is equivalent to exactly one element in the list. Our approach encompasses non-existence results and computational machinery for Butson and generalized Hadamard matrices that are of independent interest.

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1 Introduction

We present a new classification of Butson Hadamard matrices within the framework of cocyclic design theory [9, 16]. New non-existence results are also obtained. We extend MAGMA [1] and GAP [13] procedures implemented previously for 2-cohomology and relative difference sets [12, 21, 23] to determine the matrices and sort them into equivalence classes.

Cocyclic development was introduced by de Launey and Horadam in the 1990s, as a way of handling pairwise combinatorial designs that exhibit a special symmetry. It has turned out to be a powerful tool in the study of real Hadamard matrices (see [21] for the most comprehensive classification). A basic strategy, which we follow here, is to use algebraic and cohomological techniques in systematically constructing the designs.

Butson Hadamard matrices have applications in disparate areas such as quantum physics and error-correcting codes. So lists of these objects have value beyond design theory. We were motivated to undertake the classification in this paper as a first step towards augmenting the available data on complex Hadamard matrices (and we did find several matrices not equivalent to any of those in the online catalog [3]).

Specifically, we classify all Butson Hadamard matrices of order n over pth roots of unity for an odd prime p and $np \leq 100$. The restriction to pth roots is a convenience that renders each matrix generalized Hadamard over a cyclic group of order p; for these we have a correspondence with central relative difference sets that enables us to push the computation to larger orders. It must be emphasized that most of the techniques that we present apply with equal validity to generalized Hadamard matrices over any abelian group—but are not valid for Butson Hadamard matrices over kth roots of unity with k composite. Moreover, the tractability of the problem considered in this paper suggests avenues for investigation of other cocyclic designs, such as complex weighing matrices and orthogonal designs.

The paper is organized as follows. In Section 2 we set out background from design theory: key definitions, our understanding of equivalence, and general non-existence results. Section 3 is devoted to an explanation of our algorithm to check whether two Butson Hadamard matrices are equivalent. We recall the necessary essentials of cocyclic development in Section 4. Then in Section 5 we specialize to cocyclic Butson Hadamard matrices. The full classification is outlined in Section 6. We end the paper with some miscellaneous comments prompted by the classification.

For space reasons, the listing of matrices in our classification is not given herein. It may be accessed at [10].

2 Background

Throughout, p is a prime and G, K are finite non-trivial groups. We write ζ_k for $e^{2\pi \mathrm{i}/k}$.

2.1 Butson and generalized Hadamard matrices

A *Butson Hadamard matrix of order n and phase k*, denoted $\mathrm{BH}(n,k)$, is an $n \times n$ matrix H with entries in $\langle \zeta_k \rangle$ such that $HH^* = nI_n$ over $\mathbb{C}$. Here H^* is the usual Hermitian, i.e., complex conjugate transpose.

For n divisible by $|K|$, a *generalized Hadamard matrix* $\mathrm{GH}(n,K)$ *of order n over K* is an $n \times n$ matrix $H = [h_{ij}]$ whose entries h_{ij} lie in K and such that

$$HH^* = nI_n + \frac{n}{|K|}(\textstyle\sum_{x \in K} x)(J_n - I_n)$$

where $H^* = [h_{ji}^{-1}]$, J_n is the all 1s matrix, and the matrix operations are performed over the group ring $\mathbb{Z}K$.

The transpose of a $\mathrm{BH}(n,k)$ is a $\mathrm{BH}(n,k)$; the transpose of a $\mathrm{GH}(n,K)$ is not necessarily a $\mathrm{GH}(n,K)$, except when K is abelian [9, Theorem 2.10.7]. However, if H is a Butson or generalized Hadamard matrix then so too is H^*.

For the next couple of results, see Theorem 2.8.4 and Lemma 2.8.5 in [9] (the former requires a theorem from [18]).

Theorem 2.1. *If there exists a* $\mathrm{BH}(n,k)$*, and* $p_1,\ldots,p_r$ *are the primes dividing* k*, then there exist* $a_1,\ldots,a_r \in \mathbb{N}$ *such that* $n = a_1p_1 + \cdots + a_rp_r$*.*

One consequence of Theorem 2.1 is that $\mathrm{BH}(n,p^t)$ can exist only if $p|n$.

Lemma 2.2. *Let* ω *be a primitive* p*th root of unity. Then* $\sum_{i=0}^{n} a_i\omega^i = 0$ *for* $n < p$ *and* $a_0,\ldots,a_n \in \mathbb{N}$ *not all zero if and only if* $n = p-1$ *and* $a_0 = \cdots = a_n$*.*

Let $C = \langle x\rangle \cong \mathrm{C}_k$ and define $\eta_k : \mathbb{Z}C \to \mathbb{Z}[\zeta_k]$ by $\eta_k\big(\sum_{i=0}^{k-1} c_ix^i\big) = \sum_{i=0}^{k-1} c_i\zeta_k^i$. The map η_k extends to a ring epimorphism $\mathrm{Mat}(n,\mathbb{Z}C) \to \mathrm{Mat}(n,\mathbb{Z}[\zeta_k])$.

Lemma 2.3. *If* M *is a* $\mathrm{GH}(n,\mathrm{C}_k)$ *then* $\eta_k(M)$ *is a* $\mathrm{BH}(n,k)$*; if* M *is a* $\mathrm{BH}(n,p)$ *then* $\eta_p^{-1}(M)$ *is a* $\mathrm{GH}(n,\mathrm{C}_p)$*.*

Proof. The first part is easy, and the second uses Lemma 2.2. □

Thus, a $\mathrm{BH}(n,p)$ is the same design as a $\mathrm{GH}(n,\mathrm{C}_p)$. Butson's seminal paper [4] supplies a construction of $\mathrm{BH}(2^ap^b,p)$ for $0 \le a \le b$.

Example 2.4. For composite n, the Fourier matrix (more properly, Discrete Fourier Transform matrix) of order n is a $\mathrm{BH}(n,n)$ but not a $\mathrm{GH}(n,\mathrm{C}_n)$.

Example 2.5. There are no known examples of $\mathrm{GH}(n,K)$ when K is not a p-group. Indeed, finding a $\mathrm{GH}(n,K)$ with $|K| = n$ not a power of p would resolve a long-standing open problem in finite geometry; namely, whether a finite projective plane always has prime-power order.

2.2 Equivalence relations

Let X, Y be $\mathrm{GH}(n,K)$s. We say that X and Y are *equivalent* if $MXN = Y$ for monomial matrices M, N with non-zero entries in K. If X, Y are $\mathrm{BH}(n,k)$s then they are equivalent if $MXN = Y$ for monomials M, N with non-zero entries from $\langle\zeta_k\rangle$. Equivalence in either situation is denoted $X \approx Y$, whereas if M, N are permutation matrices then X, Y are *permutation equivalent* and we write $X \sim Y$. The equivalence operations defined above are *local*, insofar as they are applied entrywise to a single row or column one at a time. We will not regard taking the transpose or Hermitian as equivalence operations.

If H is a $\mathrm{GH}(n,K)$ then $H \approx H'$ where H' is normalized (its first row and column are all 1s) and thus *row-balanced*: each element of K appears with the same frequency, $n/|K|$, in each non-initial row. Similarly, H' is column-balanced. Unless k is prime, neither property is necessarily held by a normalized $\mathrm{BH}(n,k)$.

2.3 Non-existence of generalized Hadamard matrices

Certain number-theoretic conditions exclude various odd n as the order of a generalized Hadamard matrix; see, e.g., [5, 6, 25]. The main general result of this kind that we need is due to de Launey [6].

Theorem 2.6. *Let K be abelian, and r, n be odd, where r is a prime dividing $|K|$. If a $\mathrm{GH}(n, K)$ exists then every integer $m \not\equiv 0 \bmod r$ that divides the square-free part of n has odd multiplicative order modulo r.*

Remark 2.7. $\mathrm{BH}(n, p)$ do not exist for $(n, p) \in \{(15, 3), (33, 3), (15, 5)\}$.

We shall derive non-existence conditions for cocyclic $\mathrm{BH}(n, p)$ later.

3 Deciding equivalence of Butson Hadamard matrices

In this section we give an algorithm to decide equivalence of Butson Hadamard matrices. The problem is reduced to deciding graph isomorphism, which we carry out using *Nauty* [19]; and subgroup conjugacy and intersection problems, routines for which are available in MAGMA.

3.1 Automorphism groups, the expanded design, and the associated design

The direct product $\mathrm{Mon}(n, \langle\zeta_k\rangle) \times \mathrm{Mon}(n, \langle\zeta_k\rangle)$ of monomial matrix groups acts on the (presumably non-empty) set of $\mathrm{BH}(n, k)$ via $(M, N)H = MHN^*$. The orbit of H is its equivalence class; the stabilizer is its full *automorphism group* $\mathrm{Aut}(H)$.

Example 3.1. ([9, Section 9.2].) Denote the r-dimensional $\mathrm{GF}(p)$-space by V. Then $D = [xy^\top]_{x,y\in V}$ is a $\mathrm{GH}(p^r, \mathrm{C}_p)$, written additively. In fact D is the r-fold Kronecker product of the Fourier matrix of order p (so when $p = 2$ we get the Sylvester matrix). If $r \neq 1$ or $p > 2$ then $\mathrm{Aut}(D) \cong (\mathrm{C}_p \times \mathrm{C}_p^r) \rtimes \mathrm{AGL}(r, p)$.

Let $\mathrm{Perm}(n)$ be the group of all $n \times n$ permutation matrices. The *permutation automorphism group* $\mathrm{PAut}(X)$ of an $n \times n$ array X consists of all pairs $(P, Q) \in \mathrm{Perm}(n)^2$ such that $PXQ^\top = X$. Clearly $\mathrm{PAut}(H) \leq \mathrm{Aut}(H)$. The array X is *group-developed* over a group G of order n if $X \sim [h(xy)]_{x,y\in G}$ for some map h. We readily prove that X is group-developed over G if and only if G is isomorphic to a regular subgroup (i.e., subgroup acting regularly in its induced actions on the sets of row and column indices) of $\mathrm{PAut}(X)$.

The full automorphism group $\mathrm{Aut}(H)$ has no direct actions on rows or columns of H. Rather, it acts on the *expanded design* $\mathcal{E}_H = [\zeta_k^{i+j} H]$ via a certain isomorphism Θ of $\mathrm{Aut}(H)$ onto $\mathrm{PAut}(\mathcal{E}_H)$: see [9, Theorem 9.6.12].

Proposition 3.2 (Corollary 9.6.10, [9])**.** *If H_1 and H_2 are equivalent $\mathrm{BH}(n, k)$s then $\mathcal{E}_{H_1} \sim \mathcal{E}_{H_2}$; therefore $\mathrm{PAut}(\mathcal{E}_{H_1})$ and $\mathrm{PAut}(\mathcal{E}_{H_2})$ are isomorphic as conjugate subgroups of $\mathrm{Perm}(nk)^2$.*

A converse of Proposition 3.2 also holds, which we might use as a criterion to distinguish Butson Hadamard matrices. For computational purposes it is preferable to work with the $(0, 1)$-matrix A_H (the *associated design* of H) obtained from $\mathcal{E}_H$ by setting its non-identity entries to zero. Then we need an analog of Proposition 3.2 for the associated design. Before stating this, we say a bit more about the embedding $\Theta : \mathrm{Mon}(n, \langle\zeta_k\rangle)^2 \to \mathrm{Perm}(nk)^2$. It maps (P, Q) to $(\theta^{(1)}(P), \theta^{(2)}(Q))$ where $\theta^{(1)}$ (resp. $\theta^{(2)}$) replaces each non-zero entry by the permutation matrix representing that entry in the right (resp. left) regular action of $\langle\zeta_k\rangle$ on itself. Denote the image of $\mathrm{Mon}(n, \langle\zeta_k\rangle)^2$ under Θ by $M(n, k)$.

Proposition 3.3. *Let H_1, H_2 be* $\mathrm{BH}(n,k)$*s. We have $H_1 \approx H_2$ if and only if $\mathrm{A}_{H_1} = X\mathrm{A}_{H_2}Y^\top$ for some $(X,Y) \in M(n,k)$.*

Proof. Suppose that $\theta^{(1)}(P)\mathrm{A}_{H_2}\theta^{(2)}(Q)^\top = \mathrm{A}_{H_1}$, and write $\mathcal{E}_{H_i} = \sum_{r\in\langle\zeta_k\rangle} rH_{i,r}$ (so $\mathrm{A}_{H_i} = H_{i,1}$). By Theorem 9.6.7 and Lemma 9.8.3 of [9],

$$H_{1,r} = \theta^{(1)}(P)H_{2,r}\theta^{(2)}(Q)^\top.$$

Therefore $\mathcal{E}_{H_1} = \mathcal{E}_{PH_2Q^*}$ by [9, Lemma 9.6.8]. This implies that $H_1 = PH_2Q^*$. □

We also use the following simple fact.

Lemma 3.4. *Let A, B be subgroups and x, y be elements of a group G. Then either $xA \cap yB = \emptyset$, or $xA \cap yB = g(A\cap B)$ for some $g \in G$.*

We now state our algorithm to decide equivalence of Butson Hadamard matrices H_1 and H_2 of order n and phase k.

1. Compute $G_1 = \mathrm{PAut}(\mathrm{A}_{H_1})$ with *Nauty*.
2. Attempt to find $\sigma \in \mathrm{Perm}(nk)^2$ such that $\sigma\mathrm{A}_{H_1} = \mathrm{A}_{H_2}$.

   If no such σ exists then return `false`.
3. Compute $U = G_1 \cap M(n,k)$ and a transversal T for U in G_1.
4. If there exists $t \in T$ such that $\sigma t \in M(n,k)$ then return `true`;

   else return `false`.

If $H_1 \approx H_2$ then $\sigma G_1 \cap M(n,k) \neq \emptyset$ by Proposition 3.3, so by Lemma 3.4 we must find a t as in step 4. A report of `false` is then correct by Proposition 3.3; a report of `true` is clearly correct. Note that if the algorithm returns `true` then we find an element $\Theta^{-1}(\sigma t)$ mapping H_1 to H_2.

Step 1 is a potential bottleneck, although it remains feasible for graphs with several hundred vertices. Equivalence testing is therefore practicable for many $\mathrm{BH}(n,k)$ that have been considered in the literature.

Example 3.5. The authors of [20] construct a series of $\mathrm{BH}(2p,p)$ but cannot decide whether their matrices are equivalent to those of Butson [4, Theorem 3.5]. Our method, which has been implemented in MAGMA, shows that the $\mathrm{BH}(10,5)$ denoted S_{10} in [20] is equivalent to Butson's matrix in less than $0.1\,\mathrm{s}$ (an explicit equivalence is given at [10]).

4 Cocyclic development

Since our main concern is Butson Hadamard matrices, we recap the essential ideas of cocyclic development solely for this type of design.

4.1 Second cohomology and designs

Let H be a $\mathrm{BH}(n,k)$, and let W be the $k \times k$ block circulant matrix with first row $(0_n, \ldots, 0_n, I_n)$. A regular subgroup of $\mathrm{PAut}(\mathcal{E}_H)$ containing the central element $(W^\top, W)$ is *centrally regular*. By [9, Theorem 14.7.1], $\mathrm{PAut}(\mathcal{E}_H)$ has a centrally regular subgroup if and only if $H \approx [\psi(x,y)]_{x,y\in G}$ for some G and cocycle $\psi : G \times G \to \langle \zeta_k \rangle$; i.e., $\psi(x,y)\psi(xy,z) = \psi(x,yz)\psi(y,z) \ \forall\, x,y,z \in G$. We say that $H \approx [\psi(x,y)]_{x,y\in G}$ is *cocyclic*, with *indexing group G and cocycle ψ*. A cocycle of H is *orthogonal*.

Let U be a finite abelian group and denote the group of all cocycles $\psi : G \times G \to U$ by $Z(G,U)$. Our cocycles are normalized, meaning that $\psi(x,y) = 1$ when x or y is 1. If $\phi : G \to U$ is a normalized map then $\partial\phi \in Z(G,U)$ defined by $\partial\phi(x,y) = \phi(x)^{-1}\phi(y)^{-1}\phi(xy)$ is a *coboundary*. These form a subgroup $B(G,U)$ of $Z(G,U)$, and $H(G,U) = Z(G,U)/B(G,U)$ is the *second cohomology group of G*.

For each $\psi \in Z(G,U)$, the central extension $E(\psi)$ of U by G is the group with elements $\{(g,u) \mid g \in G,\ u \in U\}$ and multiplication given by $(g_1,u_1)(g_2,u_2) = (g_1g_2, u_1u_2\psi(g_1,g_2))$. Conversely, let E be a central extension of U by G, with embedding $\iota : U \to E$ and epimorphism $\pi : E \to G$ satisfying $\ker \pi = \iota(U)$. Choose a normalized map $\tau : G \to E$ such that $\pi\tau = \mathrm{id}_G$. Then $\psi_\tau(x,y) = \iota^{-1}(\tau(x)\tau(y)\tau(xy)^{-1})$ defines a cocycle ψ_τ, and $E(\psi_\tau) \cong E$. Different choices of right inverse τ of π do not alter the cohomology class of ψ_τ.

A $\mathrm{BH}(n,k)$, H, is cocyclic with cocycle ψ if and only if $E(\psi)$ is isomorphic to a centrally regular subgroup of $\mathrm{PAut}(\mathcal{E}_H)$ by an isomorphism mapping $(1,\zeta_k)$ to $(W^\top, W)$. If H is group-developed over G then H is equivalent to a cocyclic $\mathrm{BH}(n,k)$ with cocycle $\psi \in B(G, \langle\zeta_k\rangle)$ and extension group $E(\psi) \cong G \times \mathrm{C}_k$.

Example 4.1. The Butson Hadamard matrix D in Example 3.1 is cocyclic, with indexing group C_p^r and cocycle $\psi \notin B(\mathrm{C}_p^r, \mathrm{C}_p)$ defined by $\psi(x,y) = xy^\top$. Note that ψ is multiplicative and symmetric. If p is odd then $E(\psi) \cong \mathrm{C}_p^{r+1}$. The determination of all cocycles, indexing groups, and extension groups of D would be an interesting exercise; cf. the account for $p = 2$ in [9, Chapter 21].

4.2 Computing cocycles

We compute $Z(G, \langle\zeta_k\rangle)$ by means of the Universal Coefficient theorem:

$$H(G,U) = I(G,U)/B(G,U) \times T(G,U)/B(G,U)$$

where $T(G,U)/B(G,U) \cong \mathrm{Hom}(H_2(G), U)$ and $I(G,U)/B(G,U)$ is the isomorphic image under inflation of $\mathrm{Ext}(G/G', U)$. Here $G' = [G,G]$ and $H_2(G)$ is the Schur multiplier of G.

We describe the calculation of $I(G,U)$ for $U = \langle u\rangle \cong \mathrm{C}_p$ as this is used in a later proof. Let $\prod_i \langle g_iG'\rangle$ be the Sylow p-subgroup of G/G', where $|g_iG'| = p^{e_i}$. Define M_i to be the $p^{e_i} \times p^{e_i}$ matrix whose rth row is $(1, \ldots, 1, u, \ldots, u)$, the first u occurring in column $p^{e_i} - r + 2$. Let N_i be the $|G| \times |G|$ matrix obtained by taking the Kronecker product of M_i with the all 1s matrix. Up to permutation equivalence, the N_i constitute a complete set of representatives for the elements of $I(G,U)/B(G,U)$ displayed as cocyclic matrices. For more detail see [12].

4.3 Shift action

In a search for orthogonal elements of $Z(G, \mathrm{C}_p)$, it is not enough to test a single ψ from each cohomology class $[\mu] \in H(G, \mathrm{C}_p)$: if ψ is orthogonal then $\psi' \in [\mu]$ need not be orthogonal. Horadam [16, Chapter 8] discovered an action of G on each $[\psi]$ that preserves orthogonality, defined by $\psi \cdot g = \psi\partial(\psi_g)$ where $\psi_g(x) = \psi(g, x)$. This 'shift' action induces a linear representation $G \to \mathrm{GL}(V)$ where V is any G-invariant subgroup of $Z(G, \mathrm{C}_p)$, allowing effective computation of orbits in V [11].

4.4 Further equivalences for cocyclic matrices

Equivalence operations preserving cocycle orthogonality, apart from local ones, arise from the shift action or natural actions on $Z(G, \langle\zeta_p\rangle)$ by $\mathrm{Aut}(G) \times \mathrm{Aut}(\mathrm{C}_p)$. The action by $\mathrm{Aut}(\mathrm{C}_p)$ alone furnishes a *global* equivalence operation. Together with the local operations these generate the holomorph $\mathrm{C}_p \rtimes \mathrm{C}_{p-1}$ of $\langle\zeta_p\rangle$ [9, Theorem 4.4.10].

4.5 Central relative difference sets

Theorem 4.2. *There exists a cocyclic* $\mathrm{BH}(n, p)$ *with cocycle* ψ *if and only if there is a relative difference set in* $E(\psi)$ *with parameters* $(n, p, n, n/p)$ *and central forbidden subgroup* $\langle(1, \zeta_p)\rangle$.

Proof. This follows from [9, Corollary 15.4.2] or [22, Theorem 4.1]. □

We explain one direction of the correspondence in Theorem 4.2. Let E be a central extension of $U \cong \mathrm{C}_p$ by G. Say ι embeds U into the center of E, and $\pi : E \to G$ is an epimorphism with kernel $\iota(U)$. Suppose that $R = \{d_1 = 1, d_2, \ldots, d_n\} \subseteq E$ is an $(n, p, n, n/p)$-relative difference set with forbidden subgroup U; i.e., the multiset of quotients $d_i d_j^{-1}$ for $j \neq i$ contains each element of $E \setminus \iota(U)$ exactly n/p times, and contains no element of $\iota(U)$. Since R is a transversal for the cosets of $\iota(U)$ in E, we have $G = \{g_i := \pi(d_i) \mid 1 \leq i \leq n\}$. Put $\tau(g_i) = d_i$. Then $[\psi_\tau(x, y)]_{x,y \in G}$ is a $\mathrm{BH}(n, p)$.

5 Cocyclic Butson Hadamard matrices

Theorem 5.1. *Let* K *be abelian,* $n = |G|$ *be divisible by* $|K|$*,* $\psi \in Z(G, K)$*, and* $H = [\psi(x, y)]_{x,y \in G}$*. Then* H *is a* $\mathrm{GH}(n, K)$ *if and only if it is row-balanced. In that event* H *is column-balanced too.*

Proof. This follows from [16, Lemma 6.6], which generalizes a phenomenon observed for cocyclic Hadamard matrices [9, Theorem 16.2.1]. □

So we begin our classification by searching for balanced cocycles in the relevant $Z(G, \mathrm{C}_p)$. When k is not prime, a cocyclic $\mathrm{BH}(n, k)$ need not be balanced; by [16, Lemma 6.6] again, $[\psi(x, y)]_{x,y \in G}$ for $\psi \in Z(G, \langle\zeta_k\rangle)$ is a $\mathrm{BH}(n, k)$ if and only if each non-initial row sum is zero.

We mention extra pertinent facts about Fourier matrices.

Lemma 5.2. *The Fourier matrix of order* n *is a cocyclic* $\mathrm{BH}(n, n)$ *with indexing group* C_n*. If* n *is odd then it is equivalent to a group-developed matrix.*

Proposition 5.3 ([14])**.** *Every circulant* $\mathrm{BH}(p, p)$ *is equivalent to the Fourier matrix of order* p.

Proposition 5.4 ([15])**.** *For* $p \leq 17$*, the Fourier matrix of order* p *is the unique* $\mathrm{BH}(p, p)$ *up to equivalence.*

5.1 Non-existence of cocyclic Butson Hadamard matrices

As we expect, there are restrictions on the order of a group-developed Butson Hadamard matrix.

Lemma 5.5. *Set* $r_j = \mathrm{Re}(\zeta_k^j)$ *and* $s_j = \mathrm{Im}(\zeta_k^j)$. *A* $\mathrm{BH}(n,k)$ *with constant row and column sums exists only if there are* $x_0, \ldots, x_{k-1} \in \{0, 1, \ldots, n\}$ *satisfying*

$$\left(\sum_{j=0}^{k-1} r_j x_j\right)^2 + \left(\sum_{j=0}^{k-1} s_j x_j\right)^2 = n \tag{1}$$

and $\sum_{j=0}^{k-1} x_j = n$.

Proof. Let H be a $\mathrm{BH}(n,k)$ with every row and column summing to $s = \sum_{j=0}^{k-1} x_j \zeta_k^j = a + bi$. Then

$$nJ_n = J_n H H^* = sJ_n H^* = s\bar{s} J_n$$

implies $n = a^2 + b^2$, which is (1). $\square$

Remark 5.6. If $k = 2$ then (1) just gives that n must be square, which is well-known. If $k = 4$ then n is the sum of two integer squares. As a sample of other exclusions, the following cannot be the order of a group-developed $\mathrm{BH}(n,k)$.

(i) $k = 3$, $n \leq 100$: $6, 15, 18, 24, 30, 33, 42, 45, 51, 54, 60, 66, 69, 72, 78, 87, 90, 96, 99$.

(ii) $k = 5$, $n \leq 25$: $10, 15$.

Some of these orders are covered by general results (see Remark 2.7).

Henceforth p is odd.

Lemma 5.7. *Let* $k = p^t$ *and* $n = p^r m$ *where* $p \nmid m$. *Suppose that* H *is a cocyclic* $\mathrm{BH}(n,k)$ *with indexing group* G *such that* G/G' *has a cyclic subgroup of order* p^r. *Then any cocycle* $\psi \in I(G, \mathrm{C}_k)$ *of* H *is in* $I(G, \mathrm{C}_k)^p$.

Proof. (Cf. [16, Corollary 7.44].) By Subsection 4.2, $\psi = \psi_1 \partial\phi$ for some ψ_1 inflated from $Z(G/G', \mathrm{C}_k)$ and map ϕ. Assume that $\psi_1 \notin I(G, \mathrm{C}_k)^p$. Then $[\psi_1(x,y)]_{x,y \in G}$ has a row with m occurrences of ζ_k and every other entry equal to 1. Label this row a. Now

$$\begin{aligned}\prod_{y\in G} \partial\phi(a,y) &= \left(\prod_{y\in G} \phi(a)^{-1}\right)\left(\prod_{y\in G} \phi(y)^{-1}\right)\left(\prod_{y\in G} \phi(ay)\right)\\ &= \phi(a)^{-n} \in \langle \zeta_k^p \rangle.\end{aligned}$$

So, if we multiply along row a of $[\psi(x,y)]_{x,y\in G}$ then we get an element of $\langle \zeta_k \rangle \setminus \langle \zeta_k^p \rangle$. But this is a contradiction. For suppose that $\sum_{i=0}^{k-1} c_i {\zeta_k}^i = 0$. Since the kth cyclotomic polynomial $\sum_{i=0}^{p-1} \mathrm{x}^{i(p^{t-1})}$ divides $\sum_{i=0}^{k-1} c_i \mathrm{x}^i$, we have $c_j = c_{p^{t-1}+j} = \cdots = c_{(p-1)p^{t-1}+j}$, $0 \leq j \leq p^{t-1} - 1$. It is then straightforward to verify that $\prod_{i=0}^{k-1} \zeta_k^{ic_i} \in \langle \zeta_k^p \rangle$. $\square$

Corollary 5.8. *If* n *is* p*-square-free then a cocyclic* $\mathrm{BH}(n,p)$ *is equivalent to a group-developed matrix.*

Proof. Let G be the indexing group of a cocyclic $\mathrm{BH}(n,p)$. Either p divides $|G'|$ or Lemma 5.7 applies, and thus $I(G, \mathrm{C}_p) = B(G, \mathrm{C}_p)$. Also $\mathrm{Hom}(H_2(G), \mathrm{C}_p) = 1$ by [17, Theorem 2.1.5]. $\square$

Proposition 5.3 then yields

Corollary 5.9. *A cocyclic* $\mathrm{BH}(p,p)$ *is equivalent to the Fourier matrix of order* p.

Remark 5.10. By Remark 5.6 and Corollary 5.8, for $(n,p)=(10,5)$ or $p=3$ and $n\in\{6,24,30\}$, there are no cocyclic $\mathrm{BH}(n,p)$ at all (so Butson's construction [4] is not cocyclic). Furthermore, a cocyclic $\mathrm{BH}(12,3)$, $\mathrm{BH}(21,3)$, $\mathrm{BH}(20,5)$, or $\mathrm{BH}(14,7)$ is equivalent to a group-developed matrix.

5.2 Existence of cocyclic $\mathrm{BH}(n,p)$, $np\le 100$

The table below summarizes existence of matrices in our classification.

$p \setminus \frac{n}{p}$	1	2	3	4	5	6	7	8	9	10	11
3	F	NC	E	E	N	S_2	S_1	NC	E	NC	N
5	F	NC	N	S_1							
7	F	S_1									

Table 1: Existence of $\mathrm{BH}(n,p)$

N: no Butson Hadamard matrices by Remark 2.7.

NC: no cocyclic Butson Hadamard matrices by Remark 5.10.

E: cocyclic Butson Hadamard matrices exist. See Section 6.

S_1: no cocyclic Butson Hadamard matrices according to a relative difference set search.

S_2: no cocyclic Butson Hadamard matrices according to an orthogonal cocycle search.

F: the Fourier matrix is the only Butson Hadamard matrix by Proposition 5.4 (or Corollary 5.9).

Remark 5.11. There are non-cocyclic $\mathrm{BH}(6,3)$ and $\mathrm{BH}(10,5)$ by [4]. Non-existence of cocyclic $\mathrm{BH}(6,3)$ is established by computer in [16, Example 7.4.2].

We relied on computation of relative difference sets only for parameter values that we could not settle otherwise. Nevertheless, those calculations were not onerous. The search for a relative difference set with parameters $(14,7,14,2)$ ran in under an hour; the test for an $\mathrm{RDS}(20,5,20,4)$ took about a day, with most of the time being spent on C_{100}. We note additionally that there are theoretical obstructions to the existence of an $\mathrm{RDS}(21,3,21,7)$: the system of diophantine *signature equations* that such a difference set must satisfy does not admit a solution [24].

6 The full classification

The only cases left to deal with are $(n,p)\in\{(9,3),(12,3),(27,3)\}$. In this section we discuss our complete and irredundant classification of such $\mathrm{BH}(n,p)$.

Our overall task splits into two steps. We first compute a set of cocyclic $\mathrm{BH}(n,p)$ containing representatives of every equivalence class. Then we test equivalence of the matrices produced. Since our method for the second step was given in Section 3, and the orders involved pose no computational difficulties, we say nothing further about this step. Two complementary methods were used for the first step: checking shift orbits for orthogonal cocycles, and constructing relative difference sets. See Subsections 4.2 and 4.3; also, we refer to [21, Section 6], which discusses a classification of cocyclic Hadamard matrices via central relative difference sets. The algorithm for constructing the difference sets in this paper is identical to the one there, and was likewise carried out using M. Röder's GAP package *RDS* [23].

Example 6.1. Table 2 lists the number t of orthogonal elements of $Z(G, \mathrm{C}_3)$ for $|G| = 9$ or 12.

G	C_9	C_3^2	C_{12}	$\mathrm{C}_3 \rtimes \mathrm{C}_4$	$\mathrm{Alt}(4)$	D_6	$\mathrm{C}_2^2 \times \mathrm{C}_3$
t	18	144	0	288	48	0	96

Table 2: Counting orthogonal elements of $Z(G, \mathrm{C}_3)$

If $|G| \in \{6, 15, 18\}$ then $t = 0$.

6.1 $\mathrm{BH}(9, 3)$.

There are precisely three equivalence classes of cocyclic $\mathrm{BH}(9, 3)$.

One class contains $\mathrm{BH}(3, 3) \otimes \mathrm{BH}(3, 3)$, which has indexing group C_3^2 and cocycle that is not a coboundary. Some matrices H_1 in this class are group-developed over C_3^2. No H_1 has indexing group C_9. See Examples 3.1 and 4.1.

Another equivalence class contains group-developed matrices with indexing group C_9. No matrix H_2 in this class has indexing group C_3^2; hence the cocycles of H_2 are all coboundaries by Lemma 5.7. This class is not represented in [3], but happens to be an example of the construction in [7] (cf. [2]). A representative is the circulant with first row $(1, 1, 1, 1, \zeta_3, \zeta_3^2, 1, \zeta_3^2, \zeta_3)$.

The third class contains matrices $H_3 \approx H_2^*$ that are cocyclic with indexing group C_9. Again, H_3 is equivalent to a circulant, does not have indexing group C_3^2, all of its cocycles are coboundaries, and it is not in [3].

By Proposition 3.2, $\mathrm{PAut}(\mathcal{E}_{H_2}) \cong \mathrm{PAut}(\mathcal{E}_{H_3})$, and this is solvable. We described $\mathrm{PAut}(\mathcal{E}_{H_1})$ in Example 3.1.

6.2 $\mathrm{BH}(12, 3)$.

Each cocyclic $\mathrm{BH}(12, 3)$ is equivalent to a group-developed matrix (Remark 5.10) over one of $\mathrm{C}_3 \rtimes \mathrm{C}_4$, $\mathrm{C}_2^2 \rtimes \mathrm{C}_3$, or $\mathrm{C}_2^2 \times \mathrm{C}_3$. There are just two equivalence classes, which form a Hermitian pair. The automorphism groups have order 864.

This is the only order n in our classification which is not a prime power and for which cocyclic $\mathrm{BH}(n, p)$ exist.

6.3 $\mathrm{BH}(27, 3)$.

Predictably, order 27 was the most challenging one that we faced in our computations. An exhaustive search for orthogonal cocycles was not possible, so this order was classified by the central relative difference sets method.

There are sixteen equivalence classes of cocyclic $\mathrm{BH}(27, 3)$ in total. Some are Kronecker products of cocyclic $\mathrm{BH}(9, 3)$ with the unique $\mathrm{BH}(3, 3)$, but the majority are not of this form. Each matrix is equivalent to its transpose. There are two classes that are self-equivalent under the Hermitian; the rest occur in distinct Hermitian pairs.

Except for the generalized Sylvester matrix, whose automorphism group as stated in Example 3.1 is not solvable, the automorphism group of a $\mathrm{BH}(27, 3)$ has order $2^a 3^b$.

Every non-cyclic group of order 27 is an indexing group of at least one $\mathrm{BH}(27, 3)$. There are no circulants.

7 Concluding comments

It is noteworthy that all matrices in our classification are equivalent to group-developed ones (nontrivial cohomology classes appear too). This may be compared with [21], which features many equivalence classes not containing group-developed Hadamard matrices. Also, while there exist circulant $\mathrm{BH}(p^r, p)$ for all odd p and $r \leq 2$ [2, 7], we have not yet found a circulant $\mathrm{BH}(n, p)$ when n is not a p-power.

A few composition results should be given. Let $\psi_i \in Z(G_i, \mathrm{C}_k)$ for $i = 1, 2$, and define $\psi \in Z(G_1 \times G_2, \mathrm{C}_k)$ by $\psi((a, b), (x, y)) = \psi_1(a, x)\psi_2(b, y)$. It is not hard to show that $\psi \in B(G_1 \times G_2, \mathrm{C}_k)$ if and only if ψ_1, ψ_2 are coboundaries.

Lemma 7.1. *Suppose that H_i is a cocyclic $\mathrm{BH}(n_i, k)$ with cocycle ψ_i, $1 \leq i \leq 2$. Then $H_1 \otimes H_2$ is a cocyclic $\mathrm{BH}(n_1 n_2, k)$ with cocycle ψ.*

Corollary 7.2. *For $a \geq 1$, $b \geq a$, and $G \in \{\mathrm{C}_3 \rtimes \mathrm{C}_4, \mathrm{C}_2^2 \rtimes \mathrm{C}_3, \mathrm{C}_2^2 \times \mathrm{C}_3\}$, there exists a group-developed $\mathrm{BH}(2^{2a}3^b, 3)$ with indexing group $G^a \times \mathrm{C}_3^{b-a}$.*

Corollary 7.2 was proved by de Launey [8, Corollary 3.10], albeit only for indexing groups $\mathrm{C}_2^{2a} \times \mathrm{C}_3^b$.

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