

An asymptotic existence result on compressed sensing matrices

DARRYN BRYANT *

PADRAIG Ó CATHÁIN †

*School of Mathematics and Physics,
The University of Queensland, QLD 4072, Australia.*

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Abstract

For any rational number h and all sufficiently large n we give a deterministic construction for an $n \times \lfloor hn \rfloor$ compressed sensing matrix with (ℓ_1, t) -recoverability where $t = O(\sqrt{n})$. Our method uses pairwise balanced designs and complex Hadamard matrices in the construction of ϵ -equiangular frames, which we introduce as a generalisation of equiangular tight frames. The method is general and produces good compressed sensing matrices from any appropriately chosen pairwise balanced design. The (ℓ_1, t) -recoverability performance is specified as a simple function of the parameters of the design. To obtain our asymptotic existence result we prove new results on the existence of pairwise balanced designs in which the numbers of blocks of each size are specified.

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**E-mail:* db@maths.uq.edu.au

†*E-mail:* p.ocathain@gmail.com

1 Introduction

Compressed sensing is an approach to data sampling under the hypothesis that all observations are drawn from a set X of t -sparse vectors in \mathbb{R}^N , where a vector is said to be t -sparse if it has at most t non-zero entries, and $t \ll N$. It has attracted a lot of attention in the statistics, signal processing, optimization and computer science literature [5, 13] (an extensive bibliography is maintained at <http://dsp.rice.edu/cs>). It is often possible to recover arbitrary $x \in X$ with far fewer than N linear measurements. Note that while the measurements are linear, the reconstruction algorithm may be complex and non-linear. We will address computational and algorithmic aspects of our construction in a subsequent paper.

The problem then is to design an $n \times N$ matrix Φ , with n small relative to N , such that the matrix equation $\Phi x = y$ has a unique solution for every t -sparse vector x . This can be phrased in terms of ℓ_0 -minimization¹. Unfortunately, it is known that ℓ_0 -minimization is NP-hard [20], so we expect that efficient polynomial-time algorithms for this problem do not exist. A novelty of compressed sensing lies in the surprising connection between ℓ_0 -minimization and ℓ_1 -minimization [6], which is essentially linear programming, and for which efficient algorithms are available.

Definition 1. A matrix Φ is said to have (ℓ_1, t) -recoverability if and only if for any t -sparse vector x with $\Phi x = y$, the solution of the linear programming problem $\Phi \bar{x} = y$ of minimal ℓ_1 -norm is equal to x .

1.1 Mutual incoherence parameters and the Welch bound

Given a matrix Φ , finding the maximum value of t for which Φ has (ℓ_1, t) -recoverability appears to be a computationally difficult problem. The restricted isometry property (RIP) of Candès, Romberg and Tao is one approach which has proved to be useful in the analysis of probabilistic constructions [6]. For example, their methods can be used to show that $n \times N$ matrices whose entries are drawn independently from a Gaussian distribution have the (ℓ_1, t) -recoverability property with high probability for $t \sim n/\log(N)$, a result which is best possible. Evaluating the RIP parameters of order t for a matrix Φ requires knowledge of the largest and smallest eigenvalues of all principal $t \times t$ -submatrices of the $N \times N$ Gram matrix $\Phi^T \Phi$. While random matrix theory can supply this information for random matrices, applications of RIP have been less successful for deterministic constructions.

Instead, the mutual incoherence parameters (MIP) approach seems to be a standard tool for establishing (ℓ_1, t) -recoverability results for deterministic compressed sensing matrices, see [14]. The mutual incoherence parameter $\text{MIP}(\Phi)$ of a matrix Φ is defined by

$$\text{MIP}(\Phi) = \max_{i \neq j} \frac{|\langle c_i, c_j \rangle|}{\|c_i\| \|c_j\|}$$

for distinct columns c_i and c_j of Φ , where the norm here is ℓ_2 . The following theorem provides the basis for the MIP approach to establishing (ℓ_1, t) -recoverability.

¹Recall that for $p > 0$ the ℓ_p norm of v is $(\sum_{1 \leq i \leq n} |v_i|^p)^{\frac{1}{p}}$. The ℓ_0 ‘norm’ of a vector v is given by the limit as $p \rightarrow 0$ of the norm ℓ_p . It counts the number of non-zero entries in v . While it is not a true norm, it is convenient to abuse notation in order to facilitate a comparison with the ℓ_1 norm.

Theorem 2 (Proposition 1, [4]). *If $\text{MIP}(\Phi) = \mu$, then Φ has (ℓ_1, t) -recoverability for all $t < \frac{1}{2\mu} + \frac{1}{2}$.*

The Welch bound [22] states that if Φ is any $n \times N$ matrix with columns of unit norm, then

$$\text{MIP}(\Phi) \geq \mu_{n,N} = \sqrt{\frac{N-n}{(N-1)n}}. \quad (1)$$

Combining the Welch bound with Theorem 2, we see that if (ℓ_1, t) -recoverability can be established for an $n \times N$ matrix Φ using the MIP approach, then

$$t \leq \frac{\sqrt{n} \sqrt{N-1}}{2 \sqrt{N-n}} + \frac{1}{2}. \quad (2)$$

Provided that $N > \frac{4n-1}{3}$, we have that $t < \sqrt{n}$. It should be emphasised that a matrix Φ may support (ℓ_1, t) -recovery for values of t greater than the bound in (2), but the MIP framework is too coarse a tool to establish this. This shortcoming of the MIP approach is known as the *square-root bottleneck*, and overcoming it requires a fundamentally different approach to estimating (ℓ_1, t) -recoverability.

For restricted ranges of parameters, there are many known constructions for compressed sensing matrices close to the square-root bottleneck. Indeed, the first deterministic construction for compressed sensing matrices by de Vore reached this limit [12]. The only progress towards overcoming the square-root bottleneck is a deep paper of Bourgain et al. [4] in which (ℓ_1, t) -recoverability for $t \sim n^{\frac{1}{2}+\epsilon}$ is achieved. Instead of attempting to exceed this threshold, in this paper we will show that matrices reaching the square-root bottleneck are plentiful and easily constructed.

2 ϵ -equiangular frames

Matrices in which $\text{MIP}(\Phi) = \mu_{n,N}$ (see Equation (1)) are called *equiangular tight frames* (ETFs). We give an overview of some of their properties in this section. We then introduce a generalisation which we call ϵ -equiangular frames. If V is a finite dimensional inner product space and T is a finite subset of V , then T is a frame if and only if T spans V (the term originates in functional analysis, and needs refinement if T is infinite).

A frame T is *tight* if there exists a constant α such that for any $v \in V$

$$\sum_{x \in T} |\langle x, v \rangle|^2 = \alpha |v|_2^2,$$

and is *equiangular* if there exists some $\mu \in \mathbb{R}$ such that for all distinct $x_i, x_j \in T$

$$|\langle x_i, x_j \rangle| = \mu.$$

Note that if Φ is an $n \times N$ ETF, then elementary arguments show that any pair of columns necessarily has inner product $\mu_{n,N}$, see [21] for a survey of ETFs.

It is known that if T is an ETF in V , then $|T| \leq \binom{\dim(V)}{2}$ if V is real, or $|T| \leq \dim(V)^2$ if V is complex. In the case that all entries of T are contained in some proper subfield of \mathbb{C} (e.g. the rationals or a cyclotomic field), number theoretic constraints can be used to

rule out the existence of certain large ETFs [21]. The theory of equiangular tight frames has applications in quantum computing and functional analysis, however we are interested in its application to compressed sensing. By Theorem 2, an $n \times N$ ETF has (ℓ_1, t) -recoverability for all $t \leq \frac{\sqrt{n}}{2} \frac{\sqrt{N-1}}{\sqrt{N-n}} + \frac{1}{2}$, and so reaches the square-root bottleneck.

Now, for any columns c_i and c_j of the matrix Φ , let $\mu(c_i, c_j) = \frac{|\langle c_i, c_j \rangle|}{|c_i||c_j|}$ denote the normalised inner product of the vectors c_i and c_j (unless specified otherwise, all measurements are with the ℓ_2 -norm).

Definition 3. Let Φ be an $n \times N$ frame and let $\mu_{n,N}$ be the Welch bound. We say that Φ is ϵ -equiangular if

$$(1 - \epsilon)\mu_{n,N} \leq \mu(c_i, c_j) \leq (1 + \epsilon)\mu_{n,N}$$

for any two distinct columns c_i and c_j of Φ .

Our interest in ϵ -equiangular frames stems from the following straightforward result.

Proposition 4. *If Φ is an ϵ -equiangular frame, then Φ has the (ℓ_1, t) -recovery property for all $t \leq \frac{\sqrt{n}}{2(1+\epsilon)}$.*

Proof. By definition, $\text{MIP}(\Phi) \leq (1 + \epsilon)\mu_{n,N}$. By Theorem 2, Φ has (ℓ_1, t) -recoverability for all

$$t \leq \frac{\sqrt{(N-1)n}}{2(1+\epsilon)\sqrt{N-n}} + \frac{1}{2} = \frac{1}{2(1+\epsilon)} \sqrt{n + \frac{n(n-1)}{N-n}} + \frac{1}{2}. \quad (3)$$

So the result holds by observing that $\frac{1}{2(1+\epsilon)}\sqrt{n}$ is less than the right side of (3). \square

Remark 5. We observe that $2n \leq N$ implies $\frac{n(n-1)}{N-n} \leq n$, which means that we cannot establish (ℓ_1, t) -recoverability for values of t larger than $\frac{1}{\sqrt{2(1+\epsilon)}}\sqrt{n}$ within the MIP framework. Thus in all cases of interest, the results that we obtain are best possible to within a (small) multiplicative constant.

Now we give a construction of ϵ -equiangular frames. (We give an alternate construction in Section 4.3 which gives ϵ -equiangular frames for $\epsilon < 1$ and generalises a result of Fickus, Mixon and Tremain [15].) We begin by recalling the definition of a pairwise balanced design. Our use of terminology is standard, and consistent with [3], for example.

Definition 6. If V is a set of v points and \mathcal{B} is a collection of subsets of V , called *blocks*, such that each pair of points occurs together in exactly λ blocks for some fixed positive integer λ , then (V, \mathcal{B}) is a *pairwise balanced design*. If each block in \mathcal{B} has cardinality in K , then the notation $\text{PBD}(v, K, \lambda)$ is used. For each point $x \in V$, the *replication number* r_x of x is defined by $r_x = |\{B \in \mathcal{B} : x \in B\}|$.

Construction 7. If (V, \mathcal{B}) is a $\text{PBD}(v, K, 1)$, then let $n = |\mathcal{B}|$ and $N = \sum_{x \in V} r_x$ and define Φ to be the $n \times N$ frame constructed as follows.

- Let A be the transpose of the incidence matrix of (V, \mathcal{B}) : rows of A are indexed by blocks, columns of A by points, and the entry in row B and column x is 1 if $x \in B$ and 0 otherwise.

- For each $x \in V$ let H_x be a (possibly complex) Hadamard matrix of order r_x (see e.g. Section 2.8 of [11]).
- For each $x \in V$, column x of A determines r_x columns of Φ : each zero in column x is replaced with the $1 \times r_x$ row vector $(0, 0, \dots, 0)$, and each 1 in column x is replaced with a distinct row of $\frac{1}{\sqrt{r_x}}H_x$.

By Theorem II.2.6 of [3], the rows of Φ span \mathbb{R}^n , and so Φ is a frame. The substitution of one matrix into another in Construction 7 is similar to the column replacement techniques considered in [8]. Note that column x of A has precisely r_x 1s, which is the number of rows in H_x . A standard counting argument implies that $N = \sum_{x \in V} r_x = \sum_{B \in \mathcal{B}} |B|$, so the number of rows in Φ is the number of blocks in \mathcal{B} and the number of columns is the sum of the sizes of the blocks in \mathcal{B} . These parameters are not directly dependent on v , or on the sizes of individual blocks. Furthermore, we observe that there exist complex Hadamard matrices of order r for each natural number r ; the character table of a cyclic group of order r will suffice for the purposes of our construction. (See [17] for the definition and properties of a character table.)

Lemma 8 shows that while the frames given by Construction 7 are not tight, a small modification makes them so. In Proposition 9 we show that Construction 7 does indeed produce ϵ -equiangular frames.

Lemma 8. *If $c \in \mathbb{R}$, $c > 0$, and Φ is a frame from Construction 7, then any frame Φ' produced by normalising every row of Φ to have length c is tight.*

Proof. Since all rows have equal length, it suffices to verify that distinct rows of Φ' are orthogonal. Consider two rows of Φ' , labelled by blocks B_i and $B_j \in \mathcal{B}$. Both rows are non-zero precisely on the columns labelled by the points in $B_i \cap B_j$. Orthogonality is obvious if $B_i \cap B_j$ is empty, so suppose otherwise. Consider the set of r_x columns labelled by some $x \in B_i \cap B_j$. These contain two entire rows of a Hadamard matrix, say h_i and h_j . The inner product restricted to these columns is of the form $\langle \alpha h_i, \beta h_j \rangle = \alpha \beta \langle h_i, h_j \rangle = 0$. Since the inner product on the set of columns labelled by each point is zero, the result follows. \square

If K is a set of integers, then we denote the maximum element of K by K_{\max} and the minimum element of K by K_{\min} .

Proposition 9. *If (V, \mathcal{B}) is a PBD($v, K, 1$) with $2 \leq K_{\min}$ and $K_{\max} \leq \sqrt{2}(K_{\min} - 1)$, then the frame Φ produced by Construction 7 is 1-equiangular.*

Proof. Recall that Φ is $n \times N$ where $n = |\mathcal{B}|$ and N is the sum of the block sizes (equivalently the sum of the replication numbers of points), and that $\mu_{n,N}$ is the Welch bound for Φ , see Equation (1). Also recall that $\mu(c_i, c_j) = \frac{|\langle c_i, c_j \rangle|}{|c_i||c_j|}$ and that $\text{MIP}(\Phi) = \max \mu(c_i, c_j)$, where the maximum is taken over all pairs of distinct columns of Φ .

It suffices to show that $\mu(c_i, c_j) \leq 2\mu_{n,N}$ for any pair c_i and c_j of columns of Φ . Instead of verifying this inequality directly, we show that $\frac{1}{2} \frac{K_{\max}}{v-1} \leq \mu_{n,N} \leq \text{MIP}(c_i, c_j) \leq \frac{K_{\max}}{v-1}$. This suffices to show 1-equiangularity. (Later, in Proposition 20, we will replace $\frac{K_{\max}}{v-1}$ with another function of the block sizes of (V, \mathcal{B}) and follow a similar argument to that given here.)

1. For each $x \in V$ we have $\frac{v-1}{K_{\max}-1} \leq r_x \leq \frac{v-1}{K_{\min}-1}$. By counting pairs of incident points, we can bound the number n of blocks in \mathcal{B} , as follows:

$$\frac{v(v-1)}{K_{\max}(K_{\max}-1)} \leq n \leq \frac{v(v-1)}{K_{\min}(K_{\min}-1)}.$$

2. We produce upper and lower bounds on $\mu_{n,N}$ in terms of K_{\max} and v . First, using the given **lower** bound for n ,

$$\mu_{n,N} = \sqrt{\frac{N-n}{(N-1)n}} < \sqrt{\frac{1}{n}} < \sqrt{\frac{K_{\max}(K_{\max}-1)}{v(v-1)}} < \frac{K_{\max}}{v-1}.$$

(We use $2 < K_{\max} < v$ to establish the last inequality.)

Under the hypothesis (which fails only in degenerate situations) that $2n-1 \leq N$, we have that $\frac{1}{\sqrt{2n}} \leq \mu_{n,N}$. Using the given **upper** bound on n , the trivial fact that $v > K_{\max}$ and the hypothesis that $\frac{K_{\max}}{\sqrt{2}} \leq K_{\min}-1$, we obtain

$$\mu_{n,N} \geq \sqrt{\frac{K_{\min}(K_{\min}-1)}{2v(v-1)}} \geq \frac{1}{\sqrt{2}} \frac{K_{\min}-1}{v-1} \geq \frac{1}{2} \frac{K_{\max}}{v-1}.$$

3. We have established bounds on $\mu_{n,N}$. To complete the argument it suffices to show that for any choice of i and j , $\mu(c_i, c_j)$ is bounded above by $\frac{K_{\max}}{v-1}$. Clearly, any two columns of Φ labelled by the same point $x \in V$ are orthogonal, and hence have inner product 0. (Note that this implies that Φ cannot be ϵ -equiangular for $\epsilon < 1$.) So it suffices to consider columns labelled by distinct points.
4. If c_i and c_j are columns labelled by distinct points, then there exists a unique row of Φ in which c_i and c_j are both non-zero. Every entry ϕ_{xy} in the matrix Φ satisfies $\sqrt{\frac{K_{\min}-1}{v-1}} \leq |\phi_{xy}| \leq \sqrt{\frac{K_{\max}-1}{v-1}}$, so we have $\frac{K_{\min}-1}{v-1} \leq |\langle \phi_{ik}, \phi_{jk} \rangle| \leq \frac{K_{\max}-1}{v-1}$. In particular, $\mu(c_i, c_j) \leq \text{MIP}(\Phi) \leq \frac{K_{\max}}{v-1}$. (The lower bound is irrelevant due to the presence of orthogonal vectors.) This completes the proof. \square

In the statement of Proposition 9 the constants $\sqrt{2}$ and 1 were chosen to obtain a neat formulation of the theorem. In fact, for the purposes of showing $(\ell_1, O(\sqrt{n}))$ -recoverability, it suffices to show that $\max \mu(c_i, c_j) \leq \alpha \mu_{n,N}$ for some constant α . The structure of Proposition 9 lends itself to easy adaption to other constants.

We demonstrate that designs not meeting the conditions of Proposition 9 can be shown to produce ϵ -equiangular frames for some easily computable value of ϵ . A PBD($v, \{3, 5\}, 1$) with a single block of size 5 exists for all $v \equiv 5 \pmod{6}$ (see Theorem 6.8, [10]). Applying Construction 7 it is easily seen that all non-zero inner products of columns are in the range $\left[\frac{2}{v-1}, \frac{2}{v-3}\right]$, while the Welch bound is very closely approximated by $\frac{2}{\sqrt{(v-4)(v+3)}}$. Thus the matrix obtained from a PBD($v, \{3, 5\}, 1$) via Construction 7 is 1-equiangular for all $v \geq 5$.

We summarise the main results of this section as a theorem.

Theorem 10. *Let K be a set of integers with $2 \leq K_{\min}$ and $K_{\max} \leq \sqrt{2}(K_{\min}-1)$. If there exists a PBD($v, K, 1$) with n blocks in which the sum of the block sizes is N , then there exists an $n \times N$ compressed sensing matrix with the (ℓ_1, t) -recovery property for all $t \leq \frac{\sqrt{n}}{4}$.*

Proof. Construction 7 gives a frame Φ with n rows and N columns, Proposition 9 establishes 1-equiangularity and Proposition 4 guarantees (ℓ_1, t) -recovery for all $t \leq \frac{\sqrt{n}}{4}$. \square

Theorem 10 demonstrates that pairwise balanced designs offer a rich supply of compressed sensing matrices with (ℓ_1, t) -recovery properties close to the square-root bottleneck.

3 Asymptotic existence of compressed sensing matrices

In this section we use results on the existence of PBDs, which we derive from a result of Caro and Yuster on asymptotic existence of certain graph decompositions, to construct compressed sensing matrices. We begin by producing some results, which we believe to be new, on the existence of PBDs in which the number of blocks of each size is specified. This builds on an existing literature [23, 9].

A *decomposition* of a graph G is a set $\mathcal{D} = \{H_1, H_1, \dots, H_n\}$ of subgraphs of G such that $\bigcup_{i=1}^n E(H_i) = E(G)$ and $E(H_i) \cap E(H_j) = \emptyset$ for $1 \leq i < j \leq n$. If \mathcal{F} is a family of graphs and $\mathcal{D} = \{H_1, H_2, \dots, H_n\}$ is a decomposition of G such that each H_i is isomorphic to some graph in \mathcal{F} , then \mathcal{D} is called an \mathcal{F} -decomposition. It is clear that a $\text{PBD}(v, K, 1)$ is equivalent to an \mathcal{F} -decomposition of K_v where $\mathcal{F} = \{K_k : k \in K\}$. Here K_v denotes the complete graph on v vertices. It has an edge joining each pair of distinct vertices.

We shall be using a result of Caro and Yuster [7] on \mathcal{F} -decompositions. Their result uses a theorem of Gustavsson [16] which has not been published in a refereed journal, but two independent proofs have recently been published on arxiv.org by Barber, Kühn, Lo and Osthus [2], and also by Keevash [18]. If H is a graph, then $\text{gcd}(H)$ is defined by $\text{gcd}(H) = \text{gcd}(\{\deg(x) : x \in V(H)\})$ where $\deg(x)$ denotes the degree (in H) of the vertex x . Let $\mathcal{F} = \{H_1, H_2, \dots, H_s\}$. A graph G is said to be \mathcal{F} -list-decomposable if for every list $\alpha_1, \alpha_2, \dots, \alpha_s$ of integers satisfying $\sum_{i=1}^s \alpha_i |E(H_i)| = |E(G)|$, there exists an \mathcal{F} -decomposition of G in which the number of copies of H_i is α_i for $i = 1, 2, \dots, s$.

Theorem 11 ([7], Theorem 1.1). *If \mathcal{F} is any finite family of graphs such that $\text{gcd}(H) = d$ for each $H \in \mathcal{F}$, then there exists a constant $C_{\mathcal{F}}$, depending only on \mathcal{F} , such that K_n is \mathcal{F} -list-decomposable for all n satisfying $n > C_{\mathcal{F}}$ and $d \mid n - 1$.*

We require \mathcal{F} -decompositions where \mathcal{F} consists of a number of complete graphs. Theorem 11 cannot be applied directly in this case because $\text{gcd}(K_k) \neq \text{gcd}(K_l)$ for $k \neq l$. Lemma 13 provides a way around this issue, though first we require some more notation.

Definition 12. Let \mathcal{D} be an $\{F_1, F_2, \dots, F_s\}$ -decomposition of G and let (F_1, F_2, \dots, F_s) be a given ordering of F_1, F_2, \dots, F_s . The *type* of \mathcal{D} is the vector $(\alpha_1, \alpha_2, \dots, \alpha_s)$ where α_i is the number of copies of F_i in \mathcal{D} for $i = 1, 2, \dots, s$. We say that a type $(\alpha_1, \alpha_2, \dots, \alpha_s)$ is $(G, (F_1, F_2, \dots, F_s))$ -feasible if $\sum_{i=1}^s \alpha_i |E(F_i)| = |E(G)|$. A $\{K_{k_1}, K_{k_2}, \dots, K_{k_s}\}$ -decomposition of K_v is a $\text{PBD}(v, K, 1)$ with $K = \{k_1, k_2, \dots, k_s\}$, and in the context of PBDs we shall write $(v, (k_1, k_2, \dots, k_s))$ -feasible rather than $(K_v, (K_{k_1}, K_{k_2}, \dots, K_{k_s}))$ -feasible. When G and (F_1, F_2, \dots, F_s) are clear from context, we may just write *feasible* rather than $(G, (F_1, F_2, \dots, F_s))$ -feasible.

Lemma 13. *Let $K = \{k_1, k_2, \dots, k_s\}$ and let M be an $s \times t$ matrix with non-negative integer entries with rows indexed by $k_1 - 1, k_2 - 1, \dots, k_s - 1$. Further, suppose that for each column*

c of M , the gcd of the row indices of the non-zero entries in c is 1. There exists a constant C such that if $v > C$ and $(\alpha_1, \alpha_2, \dots, \alpha_s)$ is $(v, (k_1, k_2, \dots, k_s))$ -feasible, then there exists a $(v, K, 1)$ -PBD of type $(\alpha_1, \alpha_2, \dots, \alpha_s)$ whenever

$$MX = (\alpha_1, \alpha_2, \dots, \alpha_s)^\top$$

has a solution X in non-negative integers.

Proof. For $j \in \{1, 2, \dots, t\}$ define the graph $F_j = \sum_{i=1}^s m_{ij} K_{k_i}$. Here, m_{ij} is the entry in row i and column j of the given matrix M , and $m_{1j} K_{k_1} + m_{2j} K_{k_2} + \dots + m_{sj} K_{k_s}$ is the union of vertex disjoint complete graphs, where the number of copies of K_{k_i} is m_{ij} for $i = 1, 2, \dots, s$. The hypothesis concerning the columns of M ensures that $\gcd(F_j) = 1$ for all j . Thus, by Theorem 11, there exists a constant C such that for all $v > C$, K_v is $\{F_1, F_2, \dots, F_t\}$ -list-decomposable.

By hypothesis, $MX = (\alpha_1, \alpha_2, \dots, \alpha_s)^\top$ has a solution $(x_1, x_2, \dots, x_t)^\top$. Since the type $(\alpha_1, \alpha_2, \dots, \alpha_s)$ is $(v, (k_1, k_2, \dots, k_s))$ -feasible, it follows that the type (x_1, x_2, \dots, x_t) is $(K_v, (F_1, F_2, \dots, F_s))$ -feasible. Hence there exists an $\{F_1, F_2, \dots, F_t\}$ -decomposition of type (x_1, x_2, \dots, x_t) (because K_v is $\{F_1, F_2, \dots, F_t\}$ -list-decomposable). For $j = 1, 2, \dots, t$, F_j can be decomposed into m_{1j} copies of K_{k_1} , m_{2j} copies of K_{k_2} , and so on. The resulting decomposition of K_v corresponds to PBD($v, K, 1$) of type $(\alpha_1, \alpha_2, \dots, \alpha_s)$. \square

In the case that M is invertible, a PBD of type $(\alpha_1, \alpha_2, \dots, \alpha_s)$ exists whenever $M^{-1}(\alpha_1, \alpha_2, \dots, \alpha_s)^\top$ consists of non-negative integers. If in addition M is unimodular, we need only check non-negativity. Proposition 14 illustrates the utility of Lemma 13.

Proposition 14. *If $k > 3$ is an integer, then there exists a constant C_k such that for every $v > C_k$, there exists a PBD($v, \{k-1, k, k+1\}, 1$) of type $(\alpha_{k-1}, \alpha_k, \alpha_{k+1})$ for every integer solution $(\alpha_{k-1}, \alpha_k, \alpha_{k+1})$ of the following linear program.*

$$\alpha_k \geq \alpha_{k-1} \tag{4}$$

$$\alpha_k \geq \alpha_{k+1} \tag{5}$$

$$\alpha_{k+1} + \alpha_{k-1} \geq \alpha_k \tag{6}$$

$$\alpha_{k-1} \binom{k-1}{2} + \alpha_k \binom{k}{2} + \alpha_{k+1} \binom{k+1}{2} = \binom{v}{2} \tag{7}$$

Proof. Let $K = \{k-1, k, k+1\}$, and let

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Note that M satisfies the requirements of Lemma 13, with constant C_k .

Since M is invertible the system $MX = (\alpha_{k-1}, \alpha_k, \alpha_{k+1})^\top$ is equivalent to

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{k-1} \\ \alpha_k \\ \alpha_{k+1} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Now, M is unimodular, so $(x_1, x_2, x_3)^\top$ is integral when $(\alpha_{k-1}, \alpha_k, \alpha_{k+1})^\top$ is. Clearly, $x_1 = \alpha_x - \alpha_{k+1}$ is positive precisely when inequality (5) is satisfied. Likewise, Inequalities (4) and (6) correspond to the second and third rows of this linear system. It follows that for any integer solution $(\alpha_{k-1}, \alpha_k, \alpha_{k+1})^\top$ of the system of equations (4)-(7), $X^\top = M(\alpha_{k-1}, \alpha_k, \alpha_{k+1})^\top$ consists of non-negative integers. Hence by Lemma 13 there exists a PBD($v, K, 1$) of type $(\alpha_{k-1}, \alpha_k, \alpha_{k+1})$. \square

Now we turn to the construction of compressed sensing matrices. The following lemma follows from an easy manipulation of binomial coefficients, but will be used repeatedly, so we record it here.

Lemma 15. *The number of pairs of edges covered by the union of α_{k-1} vertex disjoint copies of K_{k-1} , α_k vertex disjoint copies of K_k and α_{k+1} vertex disjoint copies of K_{k+1} is $F(\alpha_{k-1}, \alpha_k, \alpha_{k+1}) = \alpha_{k-1} \binom{k-1}{2} + \alpha_k \binom{k}{2} + \alpha_{k+1} \binom{k+1}{2}$. This function obeys the identity $F(\alpha_{k-1} + t, \alpha_k - 2t, \alpha_{k+1} + t) = F(\alpha_{k-1}, \alpha_k, \alpha_{k+1}) + t$.*

Proposition 16. *If $k > 3$ is an integer, then there exists a constant C_k such that for all $n > C_k$, there exists an $n \times kn$ compressed sensing matrix with (ℓ_1, t) -recoverability for all $t \leq \frac{\sqrt{n}}{4}$.*

Proof. By Theorem 10, it is sufficient to construct a PBD($v, K, 1$) with n blocks such that the sum of the block sizes is kn . We show that such designs exist for all sufficiently large n .

Let $K = \{k-1, k, k+1\}$ and suppose that v is sufficiently large that Proposition 14 holds. Then every solution to Equations (4)-(7) corresponds to a PBD($v, K, 1$), (V, \mathcal{B}) , of type $(\alpha_{k-1}, \alpha_k, \alpha_{k+1})$.

Set $n = \alpha_{k-1} + \alpha_k + \alpha_{k+1} = |\mathcal{B}|$. For the moment, we assume that $n \equiv 0 \pmod{12}$ to reduce the amount of notation we need to employ. We discuss the other congruence classes at the end of the argument. We require that the number of columns be kn , that is

$$kn = \alpha_{k-1}(k-1) + \alpha_k k + \alpha_{k+1}(k+1).$$

This is clearly equivalent to the requirement that $\alpha_{k-1} = \alpha_{k+1}$. Note that only k is specified in the statement of the proposition: in addition to the value of $(\alpha_{k-1}, \alpha_k, \alpha_{k+1})$, we are free to choose the value of v . We now have the following simplified system of inequalities for Proposition 14:

$$\alpha_{k-1} \leq \alpha_k \leq 2\alpha_{k-1} \tag{8}$$

$$2\alpha_{k-1} + \alpha_k = n \tag{9}$$

$$\alpha_{k-1}(k^2 - k + 1) + \alpha_k \frac{k^2 - k}{2} = \binom{v}{2} \tag{10}$$

(We have expanded the binomial coefficients and gathered like terms in Equation (10).)

We note that the simultaneous solutions to Equations (8) and (9) are all of the form

$$(\alpha_{k-1}, \alpha_k, \alpha_{k+1}) = \left(\frac{n}{4} + \tau, \frac{n}{2} - 2\tau, \frac{n}{4} + \tau \right) \tag{11}$$

for some $0 \leq \tau \leq \frac{n}{12}$. It suffices to show that there exists a solution to (10) among the vectors of form (11). We demonstrate this via the function $F(\alpha_{k-1}, \alpha_k, \alpha_{k+1})$ of Lemma 15.

As τ ranges over the interval $[0, \frac{n}{12}]$, $F(\alpha_{k-1}, \alpha_k, \alpha_{k+1})$ ranges over the interval $[n\binom{k}{2} + \frac{n}{4}, n\binom{k}{2} + \frac{n}{3}]$. Clearly every integer in this interval has a unique preimage of the form given in Equation (11). This interval is of length $\frac{n}{12} \sim O(n)$. On the other hand, the distance between consecutive triangular numbers of order n (that is numbers of the form $\binom{v}{2}$ for positive integer v) is $O(\sqrt{n})$. We conclude that for sufficiently large n (guaranteed already by our application of Proposition 14) this interval contains many numbers of the form $\binom{v}{2}$. Furthermore, each equation $F(\alpha_{k-1}, \alpha_k, \alpha_{k+1}) = \binom{v}{2}$ in this interval corresponds to solution $(\alpha_{k-1}, \alpha_k, \alpha_{k+1})$ of the linear program of Proposition 14. By construction, the design corresponding to this solution has n blocks and average block size k , establishing the required result in the case that $n \equiv 0 \pmod{12}$.

The general case $n \equiv i \pmod{12}$ requires the introduction of an error term $\iota \equiv -n \pmod{12}$ in Equation (11) which complicates the presented formulae and reduces the range of τ slightly, but does not change the general argument or the conclusion of the theorem. This completes the proof for every integer k and every $n > C_k$. \square

Theorem 17 below is an extension of Proposition 16 to all rational numbers. While Theorem 17 subsumes Proposition 16, we feel that the proof of Proposition 16 illustrates the key concepts without the technical complications of the proof of Theorem 17.

Theorem 17. *If $h > 3$ is a rational number, then there exists a constant C_h such that for all $n > C_h$ there exists an $n \times \lfloor hn \rfloor$ matrix with (ℓ_1, t) -recoverability for all $t \leq \frac{\sqrt{n}}{4}$.*

Proof. The proof here is similar in outline to that of Proposition 16. Proposition 14 will not suffice in this case, we will need to apply Lemma 13 directly. Our notation here is as in Proposition 16. Take k to be the integer closest to h , so $h = k + \epsilon$ for $|\epsilon| \leq \frac{1}{2}$. Take $K = \{k-1, k, k+1\}$. If $\epsilon < \frac{1}{n}$, then Proposition 16 applies. We consider the case $\epsilon \in [\frac{1}{n}, \frac{1}{2}]$ first. By Theorem 10, the existence of a PBD satisfying the following linear system for all sufficiently large n will establish the theorem.

$$\sum_{i \in K} \alpha_i = n \tag{12}$$

$$\sum_{i \in K} i\alpha_i = \lfloor (k + \epsilon)n \rfloor = kn + \lfloor \epsilon n \rfloor \tag{13}$$

For convenience, we write $\sigma = \lfloor \epsilon n \rfloor$. Since $\sum_{i \in K} \alpha_i = n$, Equation 13 is equivalent to $\alpha_{k+1} - \alpha_{k-1} = \sigma$, where by hypothesis $1 \leq \sigma \leq \frac{n}{2}$. We have two linear equations in three unknowns, so solutions are parameterised by a single variable. It is easily verified that one solution is $(0, n - \sigma, \sigma)$ and that a vector in the nullspace is $(1, -2, 1)$. So clearly every solution is of the form $(\alpha_{k-1}, \alpha_k, \alpha_{k+1}) = (\tau, n - \sigma - 2\tau, \sigma + \tau)$. The number of blocks in a design is a non-negative integer, so we require this of any putative solution. Observe that there is a linear order (given by the value of τ) on the solutions and that the extremal elements of this system are $(0, n - \sigma, \sigma)$ and $(\frac{n - \sigma}{2}, 0, \frac{n + \sigma}{2})$. By hypothesis, $\sigma \leq \frac{n}{2}$, so we have at least $\frac{n}{4}$ distinct integer solutions.

Now, for the existence of a PBD, we require that $\sum_{i \in K} \binom{i}{2} \alpha_i = \binom{v}{2}$ has a solution. As in Proposition 16, we consider the function $F(\alpha_{k-1}, \alpha_k, \alpha_{k+1})$ of Lemma 15, supported on the set $(\alpha_{k-1}, \alpha_k, \alpha_{k+1}) = (\tau, n - \sigma - 2\tau, \sigma + \tau)$ where $0 \leq \tau \leq \frac{n}{4}$. We will show that for any choice of σ , there exists a matrix M as in Lemma 13 such that there exists an interval of length

at least $\frac{n}{36}$ on which each solution of the equation $F(\alpha_{k-1}, \alpha_k, \alpha_{k+1}) = \binom{v}{2}$ corresponds to a PBD($v, K, 1$) of type $(\alpha_{k-1}, \alpha_k, \alpha_{k+1})$. Note that, as $\sigma \rightarrow \frac{n}{2}$, the inequalities of Proposition 14 fail to hold on an interval of length $O(n)$.

First we deal with the case where $0 \leq \sigma \leq \frac{n}{4}$. The three inequalities (4), (5) and (6) are equivalent to the requirement $2\sigma + 3\tau \leq n \leq 2\sigma + 4\tau$. Recalling that $\sigma = \lfloor \epsilon n \rfloor$, we have $3\tau \leq (1 - 2\epsilon)n \leq 4\tau$. Solving for τ we obtain $\frac{(1-\epsilon)n}{4} \leq \tau \leq \frac{(1-\epsilon)n}{3}$. This is an interval of length $\frac{(1-\epsilon)n}{12}$, which is of length $O(n)$ for any $0 \leq \epsilon \leq \frac{1}{4}$.

Now we consider $\frac{n}{4} \leq \tau \leq \frac{n}{2}$. Let

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Inverting M as in Proposition 14, we obtain the inequalities

$$\alpha_{k-1} \geq 0, \text{ and } 10\alpha_{k-1} + 2\alpha_{k+1} \geq 2\alpha_k \geq 10\alpha_{k-1} + \alpha_{k+1}.$$

Substituting for σ and τ as given in our parametrisation $(\alpha_{k-1}, \alpha_k, \alpha_{k+1}) = (\tau, n - \sigma - 2\tau, \sigma + \tau)$ of the solution space, we find that we require $16\tau + 4\sigma \geq 2n \geq 15\tau + 3\sigma$. Now, recalling that $\sigma = \lfloor \epsilon n \rfloor$ with $\frac{1}{4} \leq \epsilon \leq \frac{1}{2}$, we solve for τ :

$$\frac{n(1 - 2\epsilon)}{8} \leq \tau \leq \frac{n(2 - 3\epsilon)}{15}.$$

This is an interval of length $\frac{n(1+6\epsilon)}{120}$, where $\epsilon \geq \frac{1}{4}$. It follows that this interval is of length at least $\frac{n}{48} \sim O(n)$.

The density of triangular numbers then establishes the existence of many solutions of $\sum_{i \in K} \binom{i}{2} \alpha_i = \binom{v}{2}$ in the solution space of Equations (12) and (13) for any value of $\epsilon \in [0, \frac{n}{2}]$.

So we have shown that for any $\epsilon \in [0, \frac{1}{2}]$, for any k and for all sufficiently large n , there exists an interval of length $O(n)$ on which every feasible type $(\alpha_{k-1}, \alpha_k, \alpha_{k+1})$ corresponds to a PBD($v, K, 1$) of type $(\alpha_{k-1}, \alpha_k, \alpha_{k+1})$. By construction, each such design has n blocks and the sum of the block sizes is $\lfloor (k + \epsilon)n \rfloor = \lfloor hn \rfloor$. The argument extends to $-\frac{1}{2} \leq \epsilon \leq 0$ by swapping the roles of $k - 1$ and $k + 1$ in the preceding argument. \square

4 Generalisations and modifications

We have introduced methods of some generality for the construction of compressed sensing matrices. In the interests of clarity and brevity, we have sketched only the basic ideas. In this section we give a number of extensions.

4.1 Extending Construction 7 using MUBs

Let $\mathcal{M} = \{M_0, M_1, \dots, M_e\}$ be a set of orthonormal bases of \mathbb{C}^n (written as matrices with the basis vectors as columns). We say that \mathcal{M} is a set of *mutually unbiased bases* (MUBs) if, for any $i \neq j$, all entries of $M_i^\dagger M_j$ have absolute value $\frac{1}{\sqrt{n}}$. Without loss of generality, we take $M_0 = I$, in which case each M_i is a complex Hadamard matrix. We show that a set of

MUBs can be used to increase the number of columns in the matrices given in Construction 7 without any loss in (ℓ_1, t) -recoverability.

Suppose that (V, \mathcal{B}) is a PBD in which all points have replication number r , and let $\{M_0(=I), M_1, \dots, M_e\}$ be a set of MUBs of dimension r . Denote by Φ_i the frame constructed from Construction 7 using M_i throughout. We claim that the frame $[\Phi_1 | \Phi_2 | \dots | \Phi_e]$ is 1-equiangular. To see this, it suffices to consider the inner product of a column from Φ_i with a column from Φ_j . In the case that the columns are labelled by distinct points, the columns share a single non-zero entry, so the inner product is of absolute magnitude at most $\frac{1}{r}$. In the case that the columns are labelled by the same point from V , we have that the inner product is $\frac{1}{r}$, by the definition of the MUBs.

Thus we obtain a frame with the same number of rows as in a naive application of Construction 7, but with e times as many columns. This construction is particularly effective when r is a prime power, as in this case a full set of $r + 1$ MUBs exists [19]. If (V, \mathcal{B}) is a BIBD($v, k, 1$) (that is, a PBD($v, K, 1$) with $K = \{k\}$), then $|\mathcal{B}| = n = \frac{v(v-1)}{k(k-1)}$, and every point has replication number $\frac{v-1}{k-1}$. A direct application of Construction 7 yields a $\frac{v(v-1)}{k(k-1)} \times \frac{v(v-1)}{k-1}$ compressed sensing matrix. Under the assumption that $r = \frac{v-1}{k-1}$ is a prime power and using a set of MUBs, we obtain a $\frac{v(v-1)}{k(k-1)} \times \frac{v(v-1)^2}{(k-1)^2}$ matrix. While in the first case we obtain a ratio $1 : k$ between rows and columns, in the second we obtain a ratio $1 : \frac{k(v-1)}{k-1} > v$, which is a substantial improvement.

Of course, the restriction that all replication numbers are equal is merely a convenience. We are free to replace each Hadamard matrix in Construction 7 with a set of mutually unbiased Hadamard matrices. Little is known about the existence of MUBs when the dimension is not of prime power order, so the practical applications of this observation in the general case may be limited.

4.2 A generalisation of Construction 7 using packings

If V is a set of V points and \mathcal{B} is a collection of subsets of V , then (V, \mathcal{B}) is a *packing* if each pair of points occurs together in at most one block of \mathcal{B} . If each block in \mathcal{B} has cardinality in K , then we denote such a packing by PBD(v, K, q). (If every pair of points is contained in exactly one block we recover the definition of a PBD.) It is easily verified that Construction 7 produces ϵ -equiangular frames when a packing is used in place of a pairwise balanced design, provided that there are no points with replication numbers that are too small.

Proposition 18. *If there exists a PBD($v, K, 1$) with $2 \leq K_{\min}$ and $K_{\max} \leq \sqrt{2}(K_{\min} - 1)$ in which the smallest replication number is at least $r_x \geq \frac{v-1}{\tau(K_{\min}-1)}$, then there exists a compressed sensing matrix with (ℓ_1, t) -recoverability for all $t \leq \frac{\sqrt{n}}{4\tau}$. If the average block size is k , then the ratio of rows to columns is $1 : k$.*

We observe that the existence of dense packings is guaranteed by the Rödl ‘nibble’ (see Section 4.7 of [1], for example). Whilst results depending on Theorem 11 are asymptotic in nature, one can apply the Rödl nibble to obtain a packing of any complete graph with graphs from \mathcal{F} . As $v \rightarrow \infty$, this packing will tend to an \mathcal{F} -decomposition. Thus we can in principal construct compressed sensing matrices of any size using this method, though they will approach Welch-optimality only as $n \rightarrow \infty$.

4.3 An alternative construction for ϵ -equiangular frames

Construction 19. If (V, \mathcal{B}) is a PBD($v, K, 1$), then let $n = |\mathcal{B}|$ and $N = \sum_{x \in V} r_x + 1$ and define Φ to be the $n \times N$ frame constructed as follows.

- Let A be the transpose of the incidence matrix of (V, \mathcal{B}) , defined precisely as in Construction 7.
- For each $x \in V$ let H_x be a (possibly complex) Hadamard matrix of order $r_x + 1$.
- For each $x \in V$, column x of A determines $r_x + 1$ columns of Φ : each zero in column x is replaced with the $1 \times (r_x + 1)$ row vector $(0, 0, \dots, 0)$, and each 1 in column x is replaced with a distinct non-initial row of $\frac{1}{\sqrt{r_x + 1}} H_x$.

Results analogous to those shown for Construction 7 hold also for Construction 19. The main interest of Construction 19 is as a source of ϵ -equiangular frames for $\epsilon < 1$.

Proposition 20. *If (V, \mathcal{B}) is a PBD($v, K, 1$), then Construction 19 produces an $\frac{K_{\max} - K_{\min}}{K_{\min} - 1}$ -equiangular frame.*

Proof. Using the techniques developed in Proposition 9, it can be shown that

$$\frac{K_{\min} - 1}{v + K_{\min} - 2} \leq \frac{1}{r_{\max} + 1} \leq |\langle c_i, c_j \rangle| \leq \frac{k_{\max} - 1}{v + k_{\max} - 2} \leq \frac{k_{\max} - 1}{v + k_{\min} - 2}.$$

We note in particular that the ratio between the upper and lower bounds is $\frac{K_{\max} - 1}{K_{\min} - 1}$.

It is almost immediate from the statement of the Welch bound that $\mu_{n,N} \leq \frac{K_{\max} - 1}{v + K_{\min} - 2}$. We now produce a sharper lower bound for $\mu_{n,N}$ than that given in Proposition 9. Observe that $\mu_{n,N}$ is an increasing function in N and $N \geq nK_{\min}$. Recall also that $n \leq \frac{v(v-1)}{K_{\min}(K_{\min}-1)}$. Then

$$\begin{aligned} \mu_{n,N} &\geq \sqrt{\frac{nK_{\min} - n}{n(nK_{\min} - 1)}} \\ &\geq \sqrt{\frac{K_{\min} - 1}{nK_{\min}}} \\ &\geq \sqrt{\frac{K_{\min} - 1}{K_{\min} \frac{v(v-1)}{K_{\min}(K_{\min}-1)}}} \\ &\geq \sqrt{\frac{(K_{\min} - 1)^2}{v(v-1)}} \\ &\geq \sqrt{\frac{(K_{\min} - 1)^2}{v^2}} \\ &\geq \frac{K_{\min} - 1}{v} \\ &\geq \frac{K_{\min} - 1}{v + K_{\min} - 2} \end{aligned}$$

Using the ratio between the upper and lower bounds on inner products described above and the fact that $\mu_{n,N}$ lies between the bounds, we obtain the following:

$$\frac{K_{\min} - 1}{K_{\max} - 1} \mu_{n,N} \leq |\langle c_i, c_j \rangle| \leq \frac{K_{\max} - 1}{K_{\min} - 1} \mu_{n,N}.$$

Finally, as in the definition of equiangularity, we solve for ϵ , finding it to be the greater of $\frac{K_{\max} - K_{\min}}{K_{\max} - 1}$ and $\frac{K_{\max} - K_{\min}}{K_{\min} - 1}$, completing the proof. \square

We note that in the special case that $K = \{k\}$, we achieve a 0-equiangular frame. Lemma 8 implies that such a frame is tight, and so we obtain an ETF. This is the main result of Fickus et al., see Theorem 1 of [15].

4.4 Adaptations of Theorem 17

In Proposition 16 we showed that in the (n, N) plane, if we choose any ray through the origin (with integer slope k), there exists a constant C_k such that all points of the form (n, kn) at distance at least C_k from the origin correspond to compressed sensing matrices meeting the square-root bottleneck. We generalised this in Theorem 17 to rational slopes, and showed that points close to the ray of form $(n, [kn])$ corresponded to compressed sensing matrices. We give another result in this section which shows that small perturbations to the underlying PBD can be used to obtain $n \times (k + \epsilon)n$ compressed sensing matrices close to the square-root bottleneck.

Proposition 21. *If Φ is an $n \times kn$ compressed sensing matrix with (ℓ_1, t) -recoverability as considered in Proposition 16, then for each $\epsilon \in [\frac{-1}{12k}, \frac{1}{12k}]$ there exists an $n \times [(k + \epsilon)n]$ matrix Φ_ϵ with (ℓ_1, t) -recoverability.*

Proof. First note the elementary identity

$$(2k - 1) \binom{k}{2} = k \binom{k-1}{2} + (k-1) \binom{k+1}{2},$$

which tells us that $2k - 1$ blocks of size k cover the same number of pairs of points as k blocks of size $k - 1$ and $k - 1$ blocks of size $k + 1$. But observe that the sum of the block sizes on the left is $k(2k - 1)$, whereas the sum of the block sizes on the right is $2k^2 - k - 1$. That is, we can reduce the number of columns by 1 by swapping $2k - 1$ blocks of size k for k blocks of size $k - 1$ and $k - 1$ blocks of size $k + 1$. The inverse operation increases the number of blocks by 1.

Now, suppose that Φ is constructed with the maximum possible number of blocks of size k , so Φ is of type $(\alpha_{k-1}, \alpha_k, \alpha_{k+1}) = (\frac{n}{4} + \eta, \frac{n}{2} - 2\eta, \frac{n}{4} + \eta)$, where η is close to zero. Then we can reduce the number of columns approximately $\frac{n}{6} \times \frac{1}{2k-1} \approx \frac{n}{12k}$ times before we reach our lower bound $\frac{n}{3}$ on the number of blocks of size k . Likewise, beginning with the minimum number of blocks of size k , we can increase the number of columns approximately $\frac{n}{12k}$ times. \square

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