The use of mathematical tasks to develop mathematical thinking skills in undergraduate calculus courses – a pilot study

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Mathematical thinking is difficult to define precisely but most authors agree that the following are important aspects of it: conjecturing, reasoning and proving, making connections, abstraction, generalization and specialization. In order to develop mathematically, it is necessary for learners of mathematics not only to master new mathematical content but also to develop these skills. However, undergraduate courses in Mathematics tend to be described in terms of the mathematical content and techniques students should master and theorems they should be able to prove. It would appear from such descriptions that students are expected to pick up the skills of (advanced) mathematical thinking as a by-product. Moreover, recent studies have shown that many sets of mathematical tasks produced for students at the secondary-tertiary transition emphasize lower level skills, such as memorization and the routine application of algorithms or procedures. In this paper we will consider some suggestions from the literature as to how mathematical thinking might be specifically fostered in students, through the use of different types of mathematical tasks. Efforts were made to interpret these recommendations in the context of a first undergraduate course in Calculus, on which large numbers of students may be enrolled. This itself constrains to some extent the activities in which the teachers and learners can engage. The tasks referred to here are set as homework problems on which students may work individually or collaboratively. We will report preliminary feedback from the students with whom such tasks were trialled, describing the students' reactions to these types of tasks and their understanding of the purposes of the tasks.

Keywords: mathematical thinking, tasks

Mathematical thinking

Many authors agree that the mathematical practices and thinking to be encouraged in learners of mathematics should mirror the practices of professional mathematicians. However, there are many different definitions and interpretations of the term 'mathematical thinking'. For instance, Hyman Bass (2005) speaks about the mathematical practices or habits of mind of research mathematicians and argues that these practices such as experimentation, reasoning, generalization, the use of definitions and the use of mathematical language can be fostered at any stage in the education system. Mason and Johnston-Wilder (2004) propose that questions posed to students draw on the following words "exemplifying, specializing, completing, deleting, correcting, comparing, sorting, organizing, changing, varying, reversing, altering, generalizing, conjecturing, explaining, justifying, verifying, convincing, refuting" (2004,109) as they believe these words denote the processes and actions that mathematicians employ when they pose and tackle mathematical problems.

Moreover, the Mathematics Learning Study Committee of the US National Research Council (Kilpatrick et al 2001) uses the notion of 'mathematical proficiency' to describe how people learn mathematics successfully. They believe that this has five interwoven strands which should be encouraged and developed together: conceptual understanding; procedural fluency; strategic competence (the ability to formulate and solve mathematical problems); adaptive reasoning (capacity for logical thought, reflections and justification); productive disposition (seeing mathematics as worthwhile and being confident in one's own abilities) (2001, 116).

However, there is evidence to suggest that at undergraduate level, courses often focus on procedural fluency only, to the detriment of the other strands of mathematical proficiency. Dreyfus (1991) asserts that most students learn a large number of standardised procedures in their mathematics courses but not the 'working methodology of the mathematician' (28). He says:

> They have been taught the products of the activity of scores of mathematicians in their final form but they have not gained insight into the processes that have led mathematicians to create these products (1991, 28).

He claims that this lack of insight means that students have knowledge but are not in a position to use it in unfamiliar situations.

Studies classifying tasks

Recent studies have undertaken work investigating the types of tasks that are assigned to students as homework or appear on examinations. Boesen, Lithner and Palm (2010) considered tasks from Swedish national second level high stakes examinations and used textbooks to classify them according to how familiar they were to students, claiming that exposure to familiar tasks alone affects students' ability to reason and so influences student learning. They found that often no conceptual understanding was needed to solve familiar tasks. They used the terms 'imitative reasoning' and 'creative mathematically founded reasoning' to characterize the types of reasoning that students might use to solve problems. Imitative reasoning involves using memorization or well-rehearsed procedures, while creative reasoning is novel reasoning with arguments to back it up, anchored in appropriate mathematical foundations.

Bergqvist (2007) also analyzed 16 examinations from introductory courses in Calculus in four Swedish universities and found that 70% of the exam questions could be solved using imitative reasoning alone and that 15 of the 16 examinations could be passed without using creative reasoning.

Pointon and Sangwin (2003) developed a mathematical question taxonomy in order to undertake a classification of undergraduate course-work questions – this taxonomy is illustrated in Table 1. Successful completion of tasks following 1-4 of the table are deemed characteristic of 'adoptive learning' in which students behave as 'competent practitioners', engaging in an essentially reproductive process requiring the application of well-understood knowledge in bounded situations. While questions in classes 4-8 typically require higher cognitive processes such as creativity, reflection, criticism; and would be characterized as 'adaptive learning', requiring students to behave as 'experts'.

Pointon and Sangwin (2003) used their taxonomy to classify a total of 486 course-work and examination questions used on two first year undergraduate mathematics courses, finding that 61.4% of all questions inspected related to class 2 of Table 1 while only 3.4% of questions related to classes 6-8. They concluded that:

(i) the vast majority of current work may be successfully completed by routine procedures or minor adaption of results learned verbatim and (ii) the vast majority of questions asked may be successfully completed without the use of higher skills (2003, 8).

Adoptive Reasoning	Adaptive Reasoning
1. Factual recall	5. Prove, show, justify - (general argument)
2. Carry out a routine calculation or algorithm	6. Extend a concept
3. Classify some mathematical object	7. Construct an instance
4. Interpret situation or answer	8. Criticize a fallacy

Table 1: Mathematical question taxonomy of Pointon and Sangwin (2003)

Frameworks for tasks

Others have created frameworks to help educators create tasks that would foster and assess aspects of mathematical thinking. Focusing on fostering conceptual understanding, Swan (2008) created a framework of five task types for use at second level. They are classifying mathematical objects, interpreting multiple representations, evaluating mathematical statements, creating problems, and analyzing reasoning and solutions. He believes that students should be encouraged to talk and write about mathematical ideas, and that teachers should emphasize reasoning and not 'answer-getting'. Earlier, Schoenfeld (1992) created a framework for balanced assessment in an NSF-funded project in which he considered the dimensions under which mathematical tasks could be measured: content (including procedure and technique, representations and connections); thinking processes; student products; mathematical point of view; diversity; circumstances of performance; pedagogics-aesthetics. The emphasis is on balance and Schoenfeld recognizes that any one task could not foster all types of thinking, for example, but that when a set of tasks is being designed one should aim to cover as many different dimensions as possible. Mason and Johnston-Wilder (2004) also advocate a ``mixed economy'' (6) in which learners are given a variety of types of tasks to develop mathematical thinking.

Sample tasks for first calculus

In an effort to move away from tasks involving imitative reasoning only and following the advice of Swan (2008) and Schoenfeld (1992), the authors designed a set of tasks for use in undergraduate Calculus courses. These tasks asked students to generalize and specialize, generate examples, make conjectures, reason, make decisions, explore, make connections, and reflect. Some examples of the tasks used and their relation to the frameworks are shown below:

1. Suppose $g(x)$ is an odd function. Is $h(x) = 1/g(x)$ odd? Justify your answer.

    (Swan - classifying mathematical objects; Pointon and Sangwin - classify and justify.)

2. Determine whether the reasoning used in the following is satisfactory, giving reasons to support your answers.

   Conjecture: Suppose a and b are real numbers such that $(a+b)^2 > a^2 + b^2$. Then $a > 0$ and $b > 0$.

   Proof: If $a > 0$ and $b > 0$ then $ab > 0$. Thus, $(a+b)^2 = a^2 + b^2 + 2ab > a^2 + b^2$.

   (Swan - analysing reasoning and solutions; Schoenfeld - analyse, decision and justification.)

3. Does every rational function have a vertical asymptote? Explain.

   (Swan - evaluating mathematical statements; Schoenfeld - reflect, explore.)

4. Give an example of the following:

   a. A function f which is continuous at $x = 5$.

   b. A function f which is not continuous at $x=5$ because $f(5)$ is not defined.

   c. A function f which is not continuous at $x = 5$ because $Lim_{x \to 5} f(x)$ does not exist.

   d. A function f which is not continuous at $x = 5$ because

   $Lim_{x \to 5} f(x) \neq f(5)$.

   (Pointon and Sangwin - Construct an instance; Schoenfeld - explore, choose.)

Methodology

The tasks were trialled with first year Mathematics students at the National University of Ireland, Maynooth (NUIM) and St Patrick's College, Drumcondra (SPD) in the first semester of the 2010/11 academic year. The class at NUIM consisted of 180 first year students. These students were either first year Finance students who were strongly encouraged by their department to take Mathematical Studies or first year Arts students who chose to study Mathematical Studies along with two other Arts subjects. The SPD class consisted of 49 students. These were first year students undertaking either a BEd (Primary) or BA (Humanities) degree who had chosen to study Mathematics to degree level. All students were taking a first course in Differential Calculus – in NUIM this is a one semester course, while in SPD it runs through two semesters. There was a large variation in the mathematical backgrounds of students in both groups.

Each assignment contained some procedural questions as well as at least one question designed or selected with the task frameworks in mind. Five problem sets were assigned during the first semester at SPD, and students were asked to complete them before their tutorial session. The students were twice required to submit solutions to a non-routine problem as part of the continuous assessment for the module. At NUIM, homework was assigned seven times in the semester. Students were required to submit solutions to all questions on each problem set, however only one question per assignment was graded and students were not aware in advance which would be graded. From the seven problem sets, four traditional and three non-routine questions were corrected.

Student reaction to a selection of the tasks was collected using a questionnaire during the last week of the semester. In each institution, students were asked to

comment on pairs of tasks which comprised one traditional question and one non-routine question on the same topic. They were asked to comment on the purpose of each task, whether the task contributed to their knowledge and understanding, and on the differences between the traditional and the unfamiliar tasks. Participation in the survey was voluntary and anonymous. In total 101 students completed the questionnaire - 27 at SPD and 74 at NUIM.

Students' reactions to tasks

We will first consider the students' performance on the tasks. At SPD, students submitted solutions to Tasks 1 and 2. 75% of students there gave a sound argument in response to Task 1. About half of the group realised that the conjecture in Task 2 is false but only 8.5% also realised that the proof given addressed the converse of the statement. A further 43% of students considered the proof in isolation and commented on its correctness. At NUIM, the mean score on the routine problems (65.7%) was significantly higher than the mean score on the non-routine questions (59.7%). Task 4 was one of the questions selected for grading. Almost all students were able to give examples of functions in a) and b) but about 30% had problems with parts c) and d).

At SPD, students were asked to comment on a pairs of tasks on even and odd functions. The first task was routine, the second (Task 1 here) non-routine. The majority of respondents remarked that the non-routine task was more challenging or required more thinking than the routine one. One student said:

> The first task was more basic going over the skills learnt in lectures while the second task involved us thinking more about what exactly we did, without direct examples. Had to use our knowledge to solve an unfamiliar problem. (SPD 9)

At NUIM, students were asked to comment on the differences between two tasks on rational functions. Again the majority of students felt that the non-routine task (Task 3) was more challenging and required more thinking or understanding. Some students felt that Task 3 involved opinion or theory and was not 'mathematical':

> Problem 1 [routine task] was mathematically based, and the other [Task 3] was theory based. (NUIM 26)

Students at both institutions felt that both routine and non-routine tasks deepened their knowledge and understanding but a larger proportion agreed that this was true of the non-routine tasks. The exception to this was Task 3. 41% of respondents at NUIM reported that this task deepened their understanding of rational functions while 69% said the same of a procedural task on the same topic. When asked to comment on the purposes of the non-routine tasks, students at both institutions mentioned the notions of the task as a means of gaining knowledge or understanding; as a test; or as practice. Some also spoke of working independently:

> To allow us to tackle problems without the guidance of a lecturer to test our knowledge and understanding of the topic. (SPD1)

Conclusion

The students in this study seem to have a mature understanding of the purposes of the tasks. The non-routine assignment questions have challenged them and they have reported that these questions require more thinking and understanding than the tasks with which they are familiar. Selden, Selden, Hauk and Mason (2000) investigated the ability of second year university calculus students to solve non-routine problems. They found that more than half of their students could not solve any problems even

though on a separate test although they had demonstrated that they were familiar with the techniques needed. They recommended that lecturers should 'scatter throughout a course a considerable number of problems for students to solve without first seeing very similar worked examples' (150). We have tried to follow this advice, aided by the task frameworks mentioned earlier. These frameworks have enabled us to use a greater variety of question types. The students have recognised that these tasks are different from the traditional ones that they are used to. The tasks have helped to get some students thinking in a new way and may even have begun to change students' view of mathematics.

Acknowledgement

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