Working with early number algebraically: The Mental Starters Assessment Project

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Hamsa Venkat[1&2], Mike Askew[2] and Mellony Graven[3]

[1]CASTEL, Dublin City University, Faculty of Education, Dublin, Ireland; hamsa.venkat@dcu.ie

[2]Wits School of Education, University of the Witwatersrand, Johannesburg, South Africa;

[3]Rhodes University, Makhanda, South Africa

In this paper, we share features of the Mental Starters Assessment Project (MSAP) teaching and assessment materials from South Africa that integrate early algebraic approaches in an initiative focused on early arithmetic. The features of generalized arithmetic described in the early algebra literature that are incorporated into the MSAP materials are detailed: attention to number relations/properties and base ten structure, pattern, and moves between key representations. The mechanisms through which these features are integrated are: inclusion of open number sentences with a focus on equivalence, verbalisations of equivalence, attention to patterns in calculations in base ten structure, and key representations and their interconnections. The trajectory of outcomes from iterative scaling-up trials suggests that integration of algebraic features alongside arithmetical goals builds mental mathematical skills in ways that are grounded in a relational sense of number.

Keywords: Mental mathematics, generalized arithmetic, strategic calculation, equivalence, South Africa.

Introduction

The Mental Starters Assessment Project (MSAP) in South Africa is ostensibly focused on early number and number sense. Its format is deliberately simple: six units for use in the mental mathematics starters 15-minute section of lessons, with each unit focused on a particular strategy such as bridging through ten, jump strategies or rounding and adjusting. Each unit is comprised of eight interim mental starter activities together with 5-minute pencil and paper assessments to be used before and after teaching the unit, assessments that the class teacher can use to explore the profile of children's mental calculation skills and to support development of these skills within and across units.

Each unit includes attention to the underlying 'rapid recall' facts required for efficient and strategic calculation within that unit's focus strategy. For example, to bridge through ten effectively and strategically, underlying rapid recall facts needed include knowledge of the basic number bonds for making up to the next ten, breaking down the addend on the basis of the previous step, and adding a single digit on to a multiple of ten. Each unit also includes attention to what we describe as 'strategic thinking' with a focus on the underlying number relations involved rather than on calculation. In the bridging through ten unit, for example, this focus is incorporated through the inclusion of items such as: $7 + 5 = 7 + \_\_ + 2$ in the pre- and post-assessments, and inclusion of attention to such 'open' number sentence expressions in the interim lesson starters. The intervention, now a part of national policy, is geared towards a context of limited resources and poor early number outcomes in the early primary grades and beyond. In previous writing, we have described the substance of the MSAP materials and their development/early trial outcomes (Graven & Venkat, 2021), and the outcomes from a national 'familiarization trial' that preceded national rollout (Askew, Graven & Venkat, 2022).

In this paper, our guiding question is: Why and how does the MSAP intervention integrate an algebraic approach? This question arises from the ongoing evidence that early arithmetical experiences rarely foster the algebraic structural awareness that is core to the number sense that underlies strong mental mathematics skills (Arcavi et al., 2017). Our response to this question contributes to the literature base distinguishing arithmetical thinking from algebraic thinking with exemplifications on how a national intervention, ostensibly focused on arithmetic, integrates an algebraic orientation. The particular form of the early algebraic approach taken in the MSAP project is justified in relation to culture and conditions in the South African terrain, outlined in the next section.

South Africa: Context, conditions and evidence on early number sense

The South African primary education context is marked by low performance in mathematics at all levels. Recent studies note substantial lags for the majority of students in relation to national curriculum specifications for mathematics even as early as the end of the first year of schooling, when age-appropriate students are aged 6. Spaull et al. (2022) found that most learners in the two South African provinces studied were unable to correctly answer questions involving addition and subtraction of single digit units. While outcomes have depreciated in the context of Covid, an extensive body of evidence in South Africa has shown the persistent use of counting in ones-based approaches to arithmetic calculations, well beyond the early number ranges in which such approaches can be seen as practical (Schollar, 2008). This evidence flags a key problem: very limited awareness of working with numbers as objectified entities, rather than entities that emanate out of counting processes only. Two corollaries of this problem further exacerbate progress with early number working: a) students' limited awareness of additive part-part-whole relationships, particularly in the context of the 1-10 and 1-20 number ranges; b) limited awareness of the base ten structure of the decimal system, and the ways in which multiples of ten feature as 'benchmarks' in this system that can facilitate number sense and efficient calculation (McIntosh, Reys & Reys, 1992).

In the broader context, there is also evidence of gaps in primary teachers' mathematical content knowledge, and teaching that exhibits limited coherence and connection (Askew et al., 2019). Specifically in relation to early number, teaching that keeps children back in counting in ones approaches through the provision of, and encouragement to use, concrete counting resources has been reported (Ensor et al., 2009). Beyond the school setting, capacity concerns and limited mathematical content knowledge at the level of district personnel providing teaching support have contributed to national interventions focusing on curriculum reform and national workbook provision. District personnel place emphasis on monitoring content coverage to check alignment with the prescribed sequence and pacing, with much less monitoring of the extent of student understanding. Additionally, while classroom resourcing has improved, resources (in terms of manipulatives and digital access) remain limited and average class sizes exceed 35 in all provinces, and classes of 50+ common in urban township schools.

Taken together, this complex of concerns suggests several needs to support the development of mental mathematics and early number sense: a model that dovetails with the broader curriculum coverage structure without disrupting it; an approach that is relatively sparse/simple in its format;

rationales, goals and a model that can be understood by teachers and teacher educators; and realisable in classrooms with large numbers of students and gaps in the knowledge resources of teachers.

Early algebraic approaches to arithmetic

There is now an extensive literature base on early algebra – what constitutes this approach and why the approach is important. Kieran et al.'s (2016) review describes the expansions in focus on algebraic approaches to arithmetic in the last two decades, with generalizing activity often represented at the heart of such approaches. Generalized arithmetic includes an emphasis on number relations and properties of arithmetical operations, and the extent of their generalizability. Kieran and colleagues note that arithmetical work can be considered algebraic in its orientation if: 'its purpose is not on calculation per se but on the representation of a generic example' (p.11). This pointer was useful to us given that one critique of early number pedagogy in South Africa is a preoccupation with producing the correct answer, without attention to the means through which an answer is produced. The disregard for how answers are produced contributes to the ongoing acceptance of counting-in-ones approaches.

Within number relations, a relational conception of the equals sign is critical, rather than operational conceptions that lead to calculation of the 'result'. One approach that has been used to study and develop a relational conception of equivalence within early number work involves open number sentences of different types. Among others, Stephens and Ribeiro (2012) included the use of open number sentences across the four arithmetic operations, using sentences involving one and two unknowns, for example:

$$\_\_ + 17 = 15 + 24 \quad \text{and} \quad 18 + \_\_ = 20 + \_\_$$

In examples like the first sentence, they studied the extent to which students followed operational or relational approaches. Operational approaches involved working out the unknown by calculating the answer to one side and making the other side equivalent to this result. Relational approaches include verbalisation/jotting/symbolizing noting that since 17 is 2 more than 15, the other addend has to be 2 less than 24. The focus in the latter approach is on the behaviour of addition within arithmetic (Schifter, 1999), and how numbers can be adjusted to maintain equivalence without calculating. In the second two unknowns problem type, Stephens and Ribeiro (2012) ask for multiple examples of numbers that make the sentence correct, and then turn to ask about the general relationship between the two unknowns. An alternative route was to ask true/false questions about given expressions involving an equivalence statement, with this approach too, shifting away from a preoccupation with calculating answers (Molina & Ambrose, 2008). The focus across this research base was on number relations in the context of equivalence, with relational understanding premised on comparisons incorporating variation and leading into generalization. Sfard's (1991) writing on the importance of being able to see mathematical objects operationally and structurally underpins these studies. A key problem encountered by students in algebraic working is the need to view expressions holistically, rather than as operations that produce an answer. This body of work pointed to the usefulness of open number sentences for drawing attention to properties such as equivalence and commutativity, but also to the need for supporting students beyond dealing with tasks only in operational terms.

Somewhat less prominent in the generalized arithmetic literature base within algebraic approaches is attention to base ten number relations, as a specific aspect of number relations more generally.

Evidence points to associations between awareness and use of base ten structure and a sense of the relative size of numbers (Ellemor-Collins & Wright, 2009). Several of the mental calculation strategies described in the literature as flexible and efficient (Beishuizen, 1993) are premised on awareness and use of base ten number relations. Bridging through ten, jump strategies, compensation and doubling/halving all rest on fluent working with informal 'base ten thinking' and place value decomposition. Similarly, van den Heuvel-Panhuizen's (2008) trajectory from calculating-by-counting into calculating-by-structuring relies on developing base ten thinking. In the Realistic Mathematics Education-based work, this trajectory is supported through the use of structured resources and representations like bead strings, the rekenrek, and eventually, empty number lines.

While much of this literature sits within the field of early arithmetic, we see the underlying reliance on the properties of operations and patterns associated with base ten relations and number relations more generally as essentially algebraic. This led us to the view that developing an algebraic appreciation of number structure and base ten structure was important in its own terms, but also critical for developing arithmetic proficiency within mental mathematics. Molina and Castro's (2021) study provided us with a useful instance of algebraic attention to decimal structure alongside number relations more generally. This was seen in their inclusion of number sentence tasks asking about the truth of statements such as:

True or False $257 - 34 = 257 - 30 - 4$

Given the South African evidence of lack of movement from calculating-by-counting into calculating-by-structuring, underpinned by base ten thinking-oriented strategies like bridging through ten and jump strategies, attention to base ten structure was a particular imperative for us. Similarly, while patterns form a core focus of attention in the early algebra literature, we found it useful to view the number-oriented attention to patterns seen within decimal structure calculation strings such as: $3 + 5 = 8$; $13 + 5 = 18$; $43 + 5 = 48$ in algebraic terms. This helped us to incorporate attention to generalized expressions (any number with three units that has five units added to it will end up with eight units, leaving the other parts unchanged) within the Mental Starters materials, noting that this would be useful to support efficient production of answers in higher number ranges without reversion to counting in ones.

Early algebraic thinking has also been linked to processes such as moving between representations, given that this involves the abstracting of core relationships and inscribing them into a different form, within the same or a different register (English, 1997).

The literature led us into seeing the usefulness of integrating open number sentences with their emphasis on number relations, including the number relations linked to decimal structure alongside and integrated with a focus on calculation-by-structuring. It was also important to focus on varying the representation of these relations across key relational forms for additive and early multiplicative situations (part-part-whole bar diagrams and empty number line formats for additive relations, and doubling/halving folded dot card images and place value composition/decomposition for doubling/halving situations). In integrating structure-oriented tasks alongside calculation tasks, we were aware of the evidence suggesting that purely arithmetic attention to early number and mental mathematics tends neither to be efficient, in the medium term, for the ongoing learning of number nor for the move into algebra and other parts of mathematics (Irwin & Britt, 2005).

Taking all of these aspects together, we found Hewitt's (2019) recent description of algebraic structure in the context of number working particularly useful for our purposes:

> a way of viewing an object or expression such that it is seen as a combination of recognizable parts along with recognizable patterns which connect those parts together. Such a way of viewing results in an expression of the object in terms of the parts and connections which places the object as a particular example of a more general type. (p.559)

This statement resonated with what we were trying to achieve in the Mental Starters Assessment Project: supporting children to notice, appreciate, and represent number relations and base ten relations, and to use these relations to leverage efficiencies of calculation in the base ten system. In the next section, we detail the ways in which these twin goals of appreciation of number structure/base ten structure, and use of number structure/base ten structure in efficient mental calculation using strategic calculation were designed into the Mental Starters materials.

The Mental Starters Assessment Project

The Mental Starters materials are being rolled out nationally at Grade 3 level in South Africa. The materials (available on the Department of Basic Education website: https://www.education.gov.za/MSAP2022.aspx) are focused on six strategic calculation approaches that, while noted in the mandated curriculum, are not delineated clearly into a learning (and teaching) trajectory in the curriculum specifications, and – as noted earlier – are largely ignored in the ongoing acceptance of counting-in-ones approaches to calculation. The six strategic approach units in the materials are summarized and exemplified in Figure 1. Each strategic approach rests on some aspect or combination of aspects of number properties and base ten structure. For example, jump strategies involve decomposing the addend/subtrahend into its place value component parts, and then jumping forwards/backwards using these decomposed parts, with the patterns in the base ten structure lending support for adding or subtracting multiples of ten to or from a number. Similarly, rounding and adjusting also rests on the use of base ten relationships, with the addend/subtrahend needing to be seen in relation to the multiple of ten they are closest to, with a subsequent compensation step then required.

Within each unit's teaching and assessment materials, three foci were integrated to support improved use of the strategic calculation approach and appreciation of its underlying structure:

- Rapid recall skills (e.g., knowing the result of: adding or subtracting 1, 2, 3, 4, 5, 10 to any number; basic doubles and halves; adding/subtracting multiples of ten to/from any number; bonds of ten; the next/previous multiple of ten)
- Strategic calculation approach (developing facility with using the focal strategic approach for efficient calculating-by-structuring)
- Strategic thinking (developing awareness of the underlying structure of the strategic calculation approach in each unit in open calculation representations; translating or connecting between representations of this structure, e.g., between number sentences and number line images)

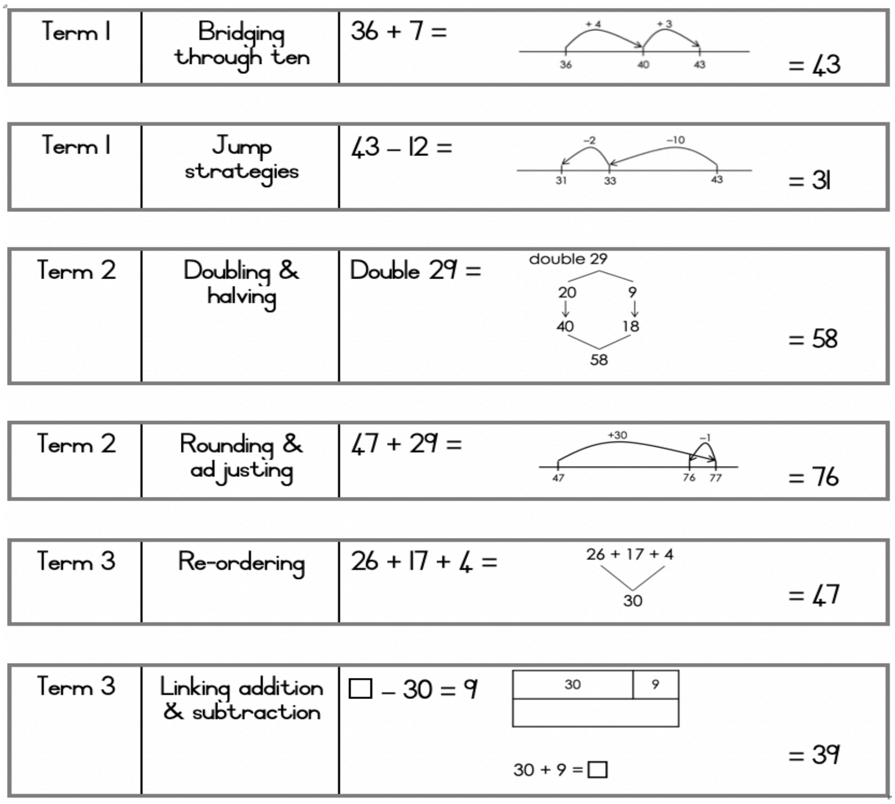


Figure 1: The Mental Starters strategic calculation goals

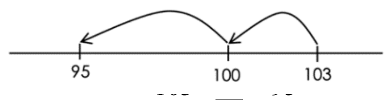
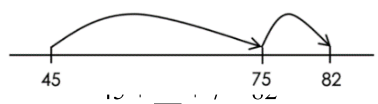
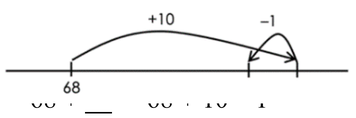
Bridging through ten	94 – __ = 94 – 4 – 2	
Doubling and halving	double 39 = 78 half of 78 = __ 39 + 38 = __ double 39 = 40 + 40 – __	
Jump strategies	61 – 32 = 61 – __ – 2	
Rounding and adjusting		Circle the number sentence that gives the same answer as 80 – 59. 80 + 60 – 1 80 – 60 – 1 80 – 60 + 1 80 + 60 + 1
Re-ordering	99 + 97 + 1 + __ = 200	6 + 98 = 98 + 6
Linking addition and subtraction	27 + 15 = 42 42 – 15 = __ 42 + 15 = 57	Use the three numbers in the sentence below in two different subtraction sentences: 83 + 37 = 120 __ - __ = __ __ - __ = __

Figure 2: Examples of open number algebraic thinking tasks in the MSAP materials

As noted by Stephens and Ribeiro (2012) and others, many of these items may be attempted operationally rather than relationally. An intentional part of our approach to encouraging relational awareness within both teaching and learning in the Mental Starters Assessment Project was to include a limited timed component across the rapid recall item set and the strategic calculating/ strategic thinking (the non-calculation generalized arithmetic questions) item sets, with 2 minutes for 20 rapid recall items and 3 minutes for 10 strategic calculating/strategic thinking items). Within the low stakes, in-class intervention model, the timed element also served to communicate that these items were not oriented towards calculation.

Several increasingly large-scale iterations of the materials have produced promising results. Graven & Venkat (2021), reporting on the initial small-scale trials in two provinces, indicated shifts from 20% mean pre-test score to 32% mean post-test score in the poorer Eastern Cape province (n=65 matched learners), and a parallel 45% to 60% shift in the wealthier Gauteng province on the Bridging Through Ten unit in 2017 (n=134 matched learners). In the 2021 national trial involving all early grade mathematics subject advisers working with a school in their districts, Askew et al. (2022) reported an increase from a mean pre-test score of 28% to a mean post-test score of 44% on the Jump Strategies unit (n=1379 matched learners).

Concluding comments

The purpose of this paper was to show why and how the MSAP intervention integrates an algebraic approach. Preliminary results suggest that this rather sparse model of inclusion of early algebraic approaches embedded alongside strategic calculation is working to support developments in mental mathematics underpinned by growing awareness of number structure and relations. Our sense is that this approach is one avenue to addressing the highly localized and short-term orientations to task completion that many teachers in South Africa and elsewhere work with (Prediger et al, 2022) in ways that address a longer-term trajectory into number structure and into increasingly efficient calculation.

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