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Siún Nic Mhuirí[1] and Patrick Neary[2]

[1]Institute of Education, Dublin City University; Siun.NicMhuiri@dcu.ie

[2]Institute of Education, Dublin City University

The teaching of multi-digit division often involves a procedural focus on the traditional digit-based long division algorithm. This paper charts three students' pathways through a unit on long division in which students' mathematical modelling of problems was foregrounded, and multiple solutions were elicited. Students were supported to share and discuss solutions and to trial the methods of their peers. Analysis shows that a wide variety of division strategies were successfully used by students, and that preferred strategies differed. Implications for teaching and further research are addressed.

Keywords: Invented methods, long division, multiple solutions, students, student choice.

Introduction

This paper arises from an ongoing collaboration between a university-based academic (lead author) and a practising primary teacher (second author) as part of a funded project. A unit of work on division was developed with the aim of publishing this as a resource for other teachers (available on maths4all.ie). The unit of work was enacted in the second author's classroom and data was collected to evaluate the efficacy of the teaching sequence and pedagogy enacted. Previously, we categorised students' solution of division problems (Neary & Nic Mhuirí, 2021). With respect to the five domains discussed in TWG19 at CERME12 (Mosvold et al., 2022), our previous research on this data focused largely on the mathematics. In this paper, we extend that work by adopting an approach which looks at how individual students' solutions developed over the course of the teaching unit.

Literature Review and Context

It is generally accepted that mathematical proficiency involves more than procedural understanding. The strands of mathematical proficiency proposed by Kilpatrick et al. (2001), where procedural fluency is understood to be interwoven with conceptual understanding, strategic competence, adaptive reasoning and productive disposition, are widely referenced. In spite of this, many teachers prioritise procedural fluency as they tend to view mathematical learning as largely concerned with memorising number facts and mastery of traditional computational algorithms (Schulz, 2018). Teachers with this perspective are unlikely to value the multiple solution strategies that arise from students' own modelling of problems, and may emphasize teacher demonstration instead. It is widely accepted within the Realistic Mathematics Education movement (van Putten et al., 2005) and beyond (Lynch & Star, 2014) that it is developmentally appropriate to accept multiple strategies from primary students, who should be encouraged to solve problems using intuitive mathematical methods before progressing to more complex, abstract methods. To date, much of the research on multiple solutions has focused on how classroom discussion of multiple solutions might productively be organised (Lynch & Star, 2014), often with a focus on single lessons and individual problems. We argue, that research over a longer term is necessary to understand how students' strategies change over a series of lessons to give insight into other aspects of the student experience. In particular, we are interested in exploring to what extent children's agency, authority and identity (Schoenfeld, 2014) are developed alongside their mathematical knowledge, when children are encouraged to develop or choose their own solution pathways.

The local context for this study involves a changing primary curriculum, which emphasizes mathematical modelling as a key pedagogical practice (NCCA, 2022). While mathematical modelling has been understood in various ways in the literature, within the Irish policy context it is presented as involving students engaging in a meaning-making process, where they organise and make sense of problems and select ways to communicate and represent their mathematical ideas. The teaching sequence at the centre of this research was devised to use mathematical modeling to teach the topic of long division, with the understanding that an emphasis on children's mathematical modeling was likely to result in multiple solutions strategies. As children engage in mathematical modelling of problem situations, they develop their own models with both conceptual and procedural components (Lesh & Harel, 2003). On a conceptual level, a model describes how elements of a system relate to each other, but it may also have accompanying procedures for accomplishing goals. In relation to multi-digit division, the models that children develop will arise from reasoning about relations between numbers, and reasoning about relations between operations (Schulz, 2018). The first intuitive strategies for division are generally repeated addition and/or subtraction. The multiplicative relationship between the dividend and the divisor is recognised in more complex strategies, with advanced strategies generally decomposing or adapting the dividend and/or divisor to create easier calculations from which the final solution can be derived (Schulz, 2018). Division strategies are also often categorised according to the ways in which students create multiples of the divisor (chunking) to be subtracted from the dividend (van Putten et al., 2005). Low-level chunking refers to using doubling or small multiples, while high-level chunking refers to subtracting higher multiples or chunks, such as multiples of ten or more times the divisor.

Methods

In this paper, teaching is conceived to be a social, cultural and political practice, which mediates expectations for education across the three domains of educational purpose: qualification, subjectification and socialisation (Biesta & Stengel, 2016). Teacher, child (and researchers) are active agents with the capacity to endorse or resist the social practices that they are both subjected and contributing to. We aimed to enact and investigate a particular type of teaching which, although it is endorsed in relevant policy documents, is not necessarily enacted in many Irish classrooms. In relation to Biesta and Stengel's (2016) domains of educational purpose, we aimed to foster students' agency and mathematical authority (subjectification) in the context of the teaching of long division (qualification). Further, particular ways of working, e.g., sharing and discussing solutions collaboratively, emphasis on reasoning, were intentionally planned and enacted (socialisation).

This study used a teaching experiment to investigate a hypothetical learning trajectory. In hypothetical learning trajectories, sequences of instructional activities are designed in ways that are expected to support students in moving from their current levels of thinking to the desired goals (Simon, 1995). The overarching goal of the teaching sequence was for children to develop efficient strategies for completing multi-digit division calculations that made sense to them. Table 1 presents an overview of the tasks and pedagogical approach. When the long division algorithm was presented by the teacher in lesson 3, it was presented as an alternative, rather than a superior method. The teaching approach adopted aligns with the recommendations of Schulz (2018) who argues that it is not just number facts, but also strategies that have to be derived and memorized. Data collected includes students' written solution strategies and reflections. Students were invited to provide a short, written reflection in response to a question posed by the teacher at the end of each lesson. Questions prompted children to reflect on strategies used, or to select and justify their preference of strategy.

Table 1: Overview of teaching sequence, including tasks and pedagogical approach

Lesson	Extension	Goals
Lesson 1-Strawberreis A punnet of strawberries holds 23 berries. How many punnets can be filled from a basket containing 115 strawberries?	228 ÷ 38 176 ÷ 36 279 ÷ 54 375 ÷ 17	Children work in groups to invent methods for solving the word problems, i.e., for sharing a quantity into groups of a specific size. Children's solution strategies are shared and discussed. They are encouraged to employ one another's approaches. Children reflect and chose preferred strategies to employ in subsequent lessons.
Lesson 2- Library A school library has 719 books. How many shelves, each holding 24 books, will be needed to display the books?	416 ÷ 15 786 ÷ 19 805 ÷ 22 751 ÷ 45	
Lesson 3- Car Transporter A car transporter delivered 216 cars over 18 trips. How many cars arrived on each trip?	Explore 'Mandeep's method' – the long division algorithm.	Continuation of the previous approach with the introduction of a new strategy, the long division algorithm.
Lesson 4- Long Division Solve 389 ÷ 17 using 'long division' and at least one other way.	537 ÷ 24 738 ÷ 31 827 ÷ 36 417 ÷ 16	Children are encouraged to employ both invented and informal methods of division. They are encouraged to justify their solutions, regardless of the strategies adopted. Children are asked to compare different solution strategies they adopt, to reflect, and to choose preferred strategies. They are encouraged to justify their choices.
Lesson 5- Going Round in Circle (Nrich) A railway line has 27 stations on a circular loop. If I fall asleep and travel through 312 stations, where will I end up in relation to where I started? Which station will I end up at?	If it is midday now, will it be light or dark in 539 hours? What time will it be?	

We are guided by the research questions: (i) what strategies did children use as they engaged in this sequence of activities? (ii) how did these change over the course of the teaching experiment? (iii) how do individual students' strategy profiles compare to each other? We analysed the reflections and written work of three students to write profiles of each student it would be impossible to address the data of the full class within the constraints of this paper. To analyse this data, different division strategies were assigned a code; repeated addition (RA); repeated addition using multiples of the divisor (MRA); repeated subtraction (RS); repeated subtraction using multiples of the divisor (MRS); high-level chunking (HLC); missing factor (MF), decomposition of the divisor (DD); short division algorithm (SD): long division algorithm (LD). These strategies align with those identified in the literature (c.f., Schulz, 2018), where MRA and MRS can be considered as examples of low-level chunking (van Putten et al., 2005). Missing factor strategies involve children trying to determine the missing factor in multiplication sentences to solve division problems, e.g., determine the missing factor in 12 x __ = 216 to identify the solution to 216 ÷ 12. To solve using missing factors, students generally started by estimating and then solved via trial-and-improvement. In this data, DD involved decomposing the divisor into single-digit factors and dividing each of the factors into the dividend in turn, for example, solving 216 ÷ 12, by dividing 216 by 4 and dividing that answer by 3.

Findings

Table 2 gives an overview of the strategies used across the five-lesson unit of work. This data shows that in most cases, multiple methods were trialled by students in individual lessons. This arises because solution strategies were discussed in a whole-class setting and students were encouraged to trial each other's methods. This tables evidences progression from informal to formal methods. While it appears that all students initially relied on repeated addition or subtraction, all students eventually tried, and articulated a preference for, more advanced strategies. It also shows variability in students' preferred solution strategies as students express preferences that are different from each other. These preferences also evolved as they engaged in the unit of work. Interestingly, not all division strategies identified by Schulz (2018) arose in this data (or in the data of the larger class group).

Table 2: Overview of division strategies used in each lesson, with students' preferred method as identified in written reflection shown in bold (if present)

	Child 1	Child 2	Child 3
Lesson 1	**RA**	**MF,** RA, RS,	MF, RA, **RS**
Lesson 2	HLC, **DD**	**MF**	**MF,** MRA, RS (errors)
Lesson 3	DD, HLC, LD, RA	HLC, LD, MF	MRA, LD
Lesson 4	**DD**, HLC, LD	HLC, **LD**	**LD,** MRA
Lesson 5	DD, LD, SD	HLC, LD	LD

Child 1

This student used a wide range of strategies across lessons. Most notable is his 'invention' of the decomposition of the divisor method, an approach, which was used in all but one lesson. In this method, the divisor is decomposed into single-digit factors and one factor is divided into the dividend, and the result is divided by the second factor. Figure 1 shows two examples of the child's workings, and highlights the mathematical complexity of this method when dealing with remainders. Where the divisor does not divide evenly into the dividend, decomposing the divisor is not the most suitable or efficient method for calculation. Despite this, the student persisted with the method and identified it as his preferred approach in two out of the three reflections where a preference was stated. In fact, a sense of ownership and mathematical authority can be identified in his written reflections: “Tomorrow I will use *my own way* or use adding” (lesson 2); “I would use *my* way to solve a divisor sum tomorrow” (lesson 4), emphasis added. Analysis of this students’ work shows that he can competently perform other division strategies, including higher level chunking and the long division algorithm. In the final lesson of the sequence, he successfully used the recording strategy for short division in the context of a long division problem (312 ÷ 27). This provides evidence of mathematical confidence and creativity. It also shows an understanding that recording methods are not fixed for particular operations or examples. Instead, they can be used flexibly for the problem-solver’s purpose.

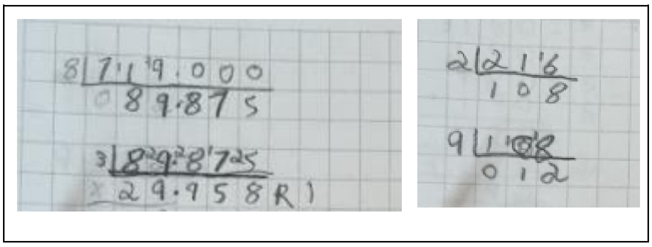

Figure 1: Sample work from Child 1. Examples of decomposition of the divisor method being used to solve (i) 719 ÷ 24 and (ii) 216 ÷ 18 from lessons 2 and 3 respectively

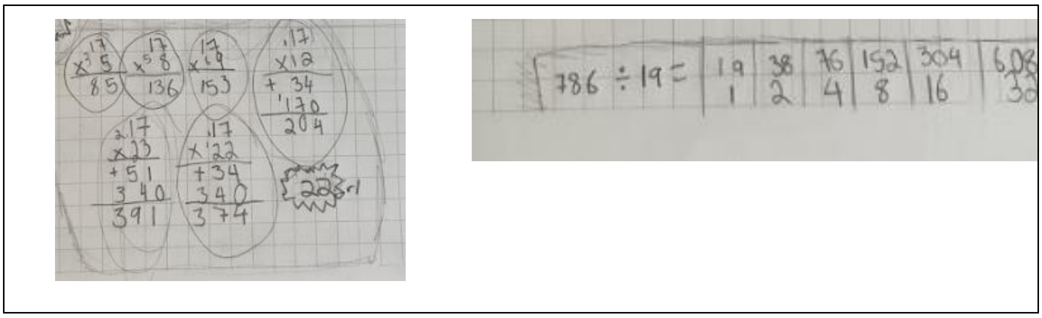

Figure 2: Sample work from Child 2. Attempts at the missing factor strategy

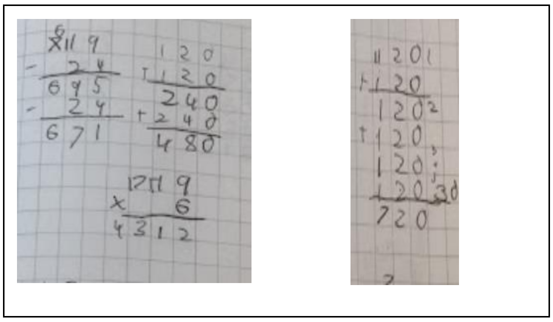

Figure 3: Examples of incomplete or incorrect solutions attempts for 719 ÷ 24 by Child 3 in lesson 2

Child 2

This child also used a range of strategies. Of note, is that an initial preference for the missing factor strategy gives way to an expressed preference for the long division algorithm toward the end of the unit of work. This may be because of difficulty experienced in estimating. Note how initial estimates were a long way from the desired dividend in figure 2. Despite this, the student's reflection displays strong understanding of the method: "My favourite way is (named student's) way. You multiply the divisor by an estimated number" (lesson 1). This also highlights the impact of sharing methods and encouraging students to trial the methods of their peers. In lesson 2, this student also notes that, "when I am doing this tomorrow I will either estimate and multiply or subtract". Children were permitted to use calculators to support their estimations and were shown how to generate lists of multiples by doubling (figure 2). Schulz (2018) identifies the missing factor strategy as more advanced than chunking strategies because it aims to utilize multiplicative relationships between the dividend and the divisor. In later lessons, this child competently performed calculations using higher-level chunking and the long division algorithm. When performing the long division algorithm, she continued to compile lists of multiples to support her estimations. Her reflections on lesson 4 expressed a preference for using the long division algorithm, but did not give a reason.

Child 3

This student also solved problems in a number of different ways. In contrast to students who exhibited some consistency in preferred approaches, this student expressed a preference for three different methods on three different occasions. Some of this inconsistency may be related to difficulties experienced when completing calculations. In the first lesson, the student's work shows that he successfully completed calculations using repeated addition, repeated subtraction and the missing factor approach. The lesson 1 reflection refers to repeated subtraction: "My favourite way is (named child's) way. It's a little long but it's much easier as well because you take away, find out how many packets. So easy." The work for lesson 2 shows evidence of attempts to use repeated subtraction. In figure 3, it can be seen that the student subtracted two lots of 24 from 719 before trialling a different method. It may be that he realised subtraction of single multiples of 24 would take too long. The student then appears to have explored repeated addition of multiples of 24 (5 x 24 = 120). He correctly identified that 6 lots of 120 is equivalent to 30 lots of 24 and is equal to 720. Unfortunately, he did not appear to have been able to adjust this calculation to reach the desired dividend of 719, or any rate, the correct answer was not recorded. While this was not the first task to include remainders, it appears likely that in this instance the remainder made it challenging for him to calculate accurately. There is also evidence to indicate that he trialled the missing factor approach, but multiplied the dividend rather than the divisor (with some calculation errors here too). His reflection for this lesson notes: "I will estimate multiplication using my calculator." Based on the available data, it is hard to be definitive about the reasons for this statement, but it is clear that the missing factor method was shared and discussed in the whole class (Child 2 also refers to it in her reflection). The student successfully completed division questions using MRA and the long division algorithm in lesson 3. It is noteworthy, that the successfully completed examples here did not include examples with remainders, though lesson 4 tasks did, and these were completed successfully using both methods. The reflection on lesson 4 states, "tomorrow I will pick the long division way. It is more interesting easier you use your calculator two (sic.)". As explained above, children used calculators to generate initial estimates and check calculations. In the final lesson, the student used the long division algorithm exclusively, and successfully.

Discussion

Importantly, all students competently completed division calculations at the end of the teaching sequence. Thus, the content or qualification objectives (Biesta & Stengel, 2016) were met. Our analysis highlights differences in students' preferred approaches and in how these evolved over the course of the unit of work. The evidence suggests that the teaching approach adopted allowed these students to develop understanding of division in ways that also supported their agency, authority and mathematical identities (Schoenfeld, 2014). In other words, the goals related to subjectification (Biesta & Stengel, 2016) were also met. We acknowledge that the evidence is more robust for some students than others. It is clear from his reflections that Child 1 experimented with different methods and displayed ownership of *his* method. While there is no evidence that the other students invented methods of their own, the data shows that they trialed the methods of their peers and made choices about which methods to pursue. Viewed in this way, the 'failure' of child 3 to produce an accurate solution in lesson 2 can be viewed more positively as the judicious choice to experiment with more appropriate methods. As such, we see value in this way of working for students at all achievement levels. In lessons which use mathematical modeling and explore multiple solution methods, there are multiple ways to succeed (NCCA, 2022). The fact that the teacher did not specify that a particular method must be used allows space for student agency in selecting preferred approaches.

The tasks (word problems and numerical examples) were chosen with care. For example, quotients were relatively large in lesson 2 compared to lesson 1, with the aim of creating the need for more efficient methods than repeated addition and subtraction. There is some evidence in the work of child 3, that these larger dividends, combined with division situations which have a remainder, proved challenging. Although the five lessons described here, were preceded by three lessons which explored short division, it is likely that some students would benefit from a longer teaching sequence in which they had more time to develop competence with each strategy. This is also relevant in the context of building up a strategy repertoire. Competence in numerical calculation has been characterized as the ability to draw on a strategy repertoire to select an appropriate strategy for the problem context (Verschaffel et al., 2009). In this study, students were at the initial stages of exploring strategies but had not yet progressed to consider for what numerical examples particular strategies would be preferred. A longer teaching sequence would have allowed more time for such considerations. In addition, with the exception of the long division algorithm, this unit of work featured methods that were invented by the students themselves. Future work will consider the potential of the teacher introducing strategies when they do not arise naturally. We believe that this may be particularly important for classes who are not used to working in a flexible way with number computations. We note that the relative success of this project is related to a number of other elements, which are not captured in our data. In particular, a classroom culture of sense making in mathematics that had been built up over time, with attention to productive math talk community norms (NCCA, 2022).

In summary, this paper charted three students' diverse paths through a series of five lessons. This analysis shows, that while different strategies were used and preferred by students, all students were capable of solving division problems in ways that made sense to them. We note that working productively with multiple solutions is challenging and suggest that further attention is needed to investigate (i) how children's preferred methods evolve over time and (ii) how teachers can work with multiple strategies over time to develop children's mathematical thinking. This paper gives some insight into these questions, but more focused work over a range of mathematical topics is needed.

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